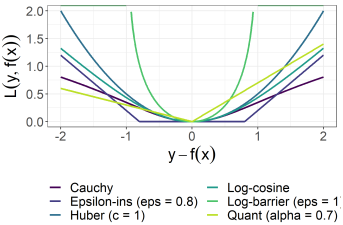
Introduction to Machine Learning

Advanced Regression Losses



Learning goals

- Know the Huber loss
- Know the log-cosh loss
- Know the Cauchy loss
- Know the log-barrier loss
- Know the ϵ-insensitive loss
- Know the quantile loss

ADVANCED LOSS FUNCTIONS

Special loss functions can be used to estimate non-standard posterior components, to measure errors in a custom way or are designed to have special properties like robustness.

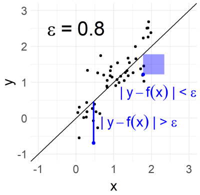
Examples:

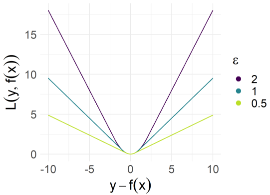
- Quantile loss: Overestimating a clinical parameter might not be as bad as underestimating it.
- Log-barrier loss: Extremely under- or overestimating demand in production would put company profit at risk.
- ϵ-insensitive loss: A certain amount of deviation in production does no harm, larger deviations do.

HUBER LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} \frac{1}{2}(y - f(\mathbf{x}))^2 & \text{if } |y - f(\mathbf{x})| \leq \epsilon \\ \epsilon|y - f(\mathbf{x})| - \frac{1}{2}\epsilon^2 & \text{otherwise} \end{cases}, \quad \epsilon > 0$$

- Piece-wise combination of $L1/L2$ to have robustness/smoothness
- Analytic properties: convex, differentiable (once)





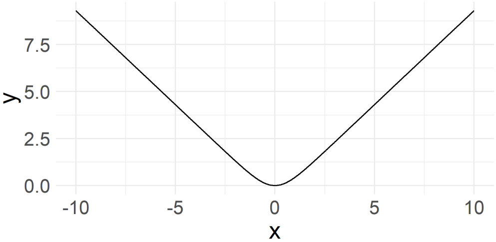
- Risk minimizer and optimal constant do not have a closed-form solution. To fit a model numerical optimization is necessary.
- Solution behaves like **trimmed mean**: a (conditional) mean of two (conditional) quantiles.

LOG-COSH LOSS

$$L(y, f(\mathbf{x})) = \log\left(\cosh(|y - f(\mathbf{x})|)\right)$$

- Logarithm of the hyperbolic cosine of the residual.
- Approximately equal to $0.5(|y - f(\mathbf{x})|)^2$ for small $\mathbf{x}$ and to $|y - f(\mathbf{x})| - \log 2$ for large $\mathbf{x}$, meaning it works mostly like $L2$ loss but is less outlier-sensitive.
- Has all the advantages of Huber loss and is, moreover, twice differentiable everywhere.

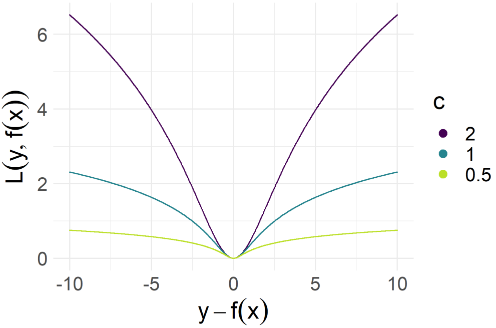
LOG-COSH LOSS



CAUCHY LOSS

$$L(y, f(\mathbf{x})) = \frac{c^2}{2} \log\left(1 + \left(\frac{|y - f(\mathbf{x})|}{c}\right)^2\right), \quad c \in \mathbb{R}$$

- Particularly robust toward outliers (controllable via c).
- Analytic properties: differentiable, but not convex!



LOG-BARRIER LOSS

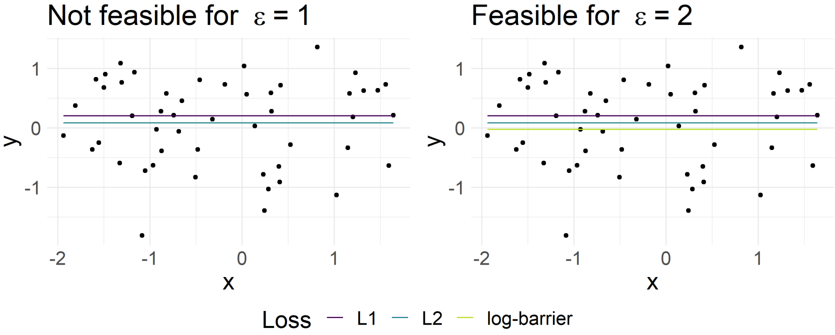
$$L\left(y, f(\mathbf{x})\right) = \begin{cases} -\epsilon^2 \cdot \log\left(1 - \left(\frac{|y-f(\mathbf{x})|}{\epsilon}\right)^2\right) & \text{if } |y - f(\mathbf{x})| \leq \epsilon \\ \infty & \text{if } |y - f(\mathbf{x})| > \epsilon \end{cases}$$

- Behaves like *L*2 loss for small residuals.
- We use this if we don't want residuals larger than ϵ at all.
- No guarantee that the risk minimization problem has a solution.
- Plot shows log-barrier loss for $\epsilon = 2$:

LOG-BARRIER LOSS

LOG-BARRIER LOSS

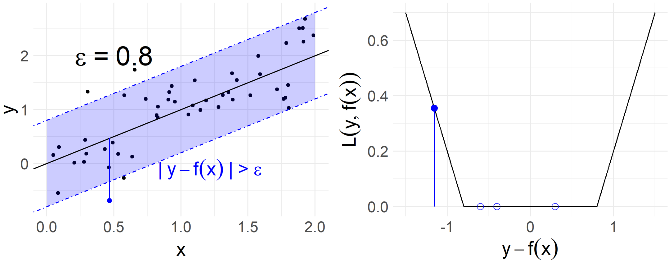
- Note that the optimization problem has no (finite) solution if there is no way to fit a constant where all residuals are smaller than a.



ϵ-INSENSITIVE LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} 0 & \text{if } |y - f(\mathbf{x})| \le \epsilon \\ |y - f(\mathbf{x})| - \epsilon & \text{otherwise} \end{cases}, \quad \epsilon \in \mathbb{R}_+$$

- Modification of $L1$ loss, errors below ϵ accepted without penalty.
- Used in SVM regression.
- Properties: convex and not differentiable for $y - f(\mathbf{x}) \in \{-\epsilon, \epsilon\}$.



QUANTILE LOSS / PINBALL LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} (1-\alpha)(f(\mathbf{x}) - y) & \text{if } y < f(\mathbf{x}) \\ \alpha(y - f(\mathbf{x})) & \text{if } y \geq f(\mathbf{x}) \end{cases}, \quad \alpha \in (0, 1)$$

- Extension of $L1$ loss (equal to $L1$ for $\alpha = 0.5$).
- Weights either positive or negative residuals more strongly.
- $\alpha < 0.5$ ($\alpha > 0.5$) penalty to over-estimation (under-estimation)
- Risk minimizer is (conditional) α-quantile (median for $\alpha = 0.5$).