

Solution 1:

A fair die is rolled at the same time as a fair coin is tossed. Let A be the number on the upper surface of the die and let B describe the outcome of the coin toss, where

$$B = \begin{cases} 1, & \text{head,} \\ 0, & \text{tail.} \end{cases}$$

Two random variables X and Y are given by $X = A + B$ and $Y = A - B$, respectively.

- (a) Calculate the entropies $H(X)$ and $H(Y)$, the conditional entropies $H(Y|X)$ and $H(X|Y)$, the joint entropy $H(X, Y)$ and the mutual information $I(X; Y)$.

Solution:

Let a, b, x , and y denote the realisations of the random variables A, B, X , and Y , respectively. Each event (a, b) is associated with exactly one event (x, y) and the probability for such an event is given by

$$p_{AB}(a, b) = p_{XY}(x, y) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Consequently, we obtain for the joint entropy

$$\begin{aligned} H(X, Y) &= - \sum_{x, y} p_{X, Y}(x, y) \log_2 p_{X, Y}(x, y) = -12 \cdot \frac{1}{12} \log_2 \frac{1}{12} \\ &= \log_2 12 \\ &= 2 + \log_2 3 \end{aligned}$$

Below we list the possible values of the random variables X and Y , the associated events (a, b) , and the probability masses $p_X(x)$ and $p_Y(y)$.

x	events (a, b)	$p_X(x)$	y	events (a, b)	$p_Y(y)$
1	(1, 0)	1/12	0	(1, 1)	1/12
2	(2, 0), (1, 1)	1/6	1	(1, 0), (2, 1)	1/6
3	(3, 0), (2, 1)	1/6	2	(2, 0), (3, 1)	1/6
4	(4, 0), (3, 1)	1/6	3	(3, 0), (4, 1)	1/6
5	(5, 0), (4, 1)	1/6	4	(4, 0), (5, 1)	1/6
6	(6, 0), (5, 1)	1/6	5	(5, 0), (6, 1)	1/6
7	(6, 1)	1/12	6	(6, 0)	1/12

The random variable $X = A + B$ can take the values 1 to 7. The probability masses $p_X(x)$ for the values 1 and 7 are equal to 1/12, since they correspond to exactly one event. The probability masses for the values 2 to 6 are equal to 1/6, since each of these values corresponds to two events (a, b) . An analogue result is obtained for the random variable $Y = A - B$.

The marginal entropies are given by

$$\begin{aligned} H(X) &= - \sum_x p_X(x) \log_2 p_X(x) \\ &= -2 \cdot \frac{1}{12} \log_2 \frac{1}{12} - 5 \cdot \frac{1}{6} \log_2 \frac{1}{6} \\ &= \frac{1}{6} \cdot (\log_2 4 + \log_2 3) + \frac{5}{6} \cdot (\log_2 2 + \log_2 3) \\ &= \frac{7}{6} + \log_2 3 \end{aligned}$$

and for Y

$$\begin{aligned}
 H(Y) &= - \sum_y p_Y(y) \log_2 p_Y(y) \\
 &= -2 \cdot \frac{1}{12} \log_2 \frac{1}{12} - 5 \cdot \frac{1}{6} \log_2 \frac{1}{6} \\
 &= \frac{1}{6} \cdot (\log_2 4 + \log_2 3) + \frac{5}{6} \cdot (\log_2 2 + \log_2 3) \\
 &= \frac{7}{6} + \log_2 3
 \end{aligned}$$

We can determine the conditional entropies using

$$H(X|Y) = H(X, Y) - H(Y) = 2 + \log_2 3 - \frac{7}{6} - \log_2 3 = \frac{5}{6}$$

$$H(Y|X) = H(X, Y) - H(X) = 2 + \log_2 3 - \frac{7}{6} - \log_2 3 = \frac{5}{6}$$

The mutual information $I(X; Y)$ can be determined according to

$$I(X; Y) = H(X) - H(X|Y) = \frac{7}{6} + \log_2 3 - \frac{5}{6} = \frac{1}{3} + \log_2 3$$

or

$$I(X; Y) = H(Y) - H(Y|X) = \frac{7}{6} + \log_2 3 - \frac{5}{6} = \frac{1}{3} + \log_2 3$$

(b) Show that, for independent discrete random variables X and Y ,

$$I(X; X + Y) - I(Y; X + Y) = H(X) - H(Y)$$

Solution:

Using the definition of mutual information for discrete random variables, $I(X; Y) = H(Y) - H(Y|X)$, we can write

$$\begin{aligned}
 I(X; X + Y) - I(Y; X + Y) &= H(X + Y) - H(X + Y|X) - H(X + Y) + H(X + Y|Y) \\
 &= H(X|Y) - H(Y|X) \\
 &= H(X) - H(Y).
 \end{aligned}$$

The first step follows from the fact that modifying the mean of a pmf doesn't change the entropy. For the second step, we used the fact that the conditional entropy $H(X|Y)$ is equal to the marginal entropy $H(X)$ for independent random variables X and Y .