

Exercise 1: Lasso, Subdifferentials

Optimization routines for the Lasso use coordinate gradient descent, but instead of using gradients, they resort to subdifferentials. We now try to understand in more detail what subdifferentials are:

- (a) Recall that the Taylor approximation of first order of a function $f(x)$ at point x_0 is

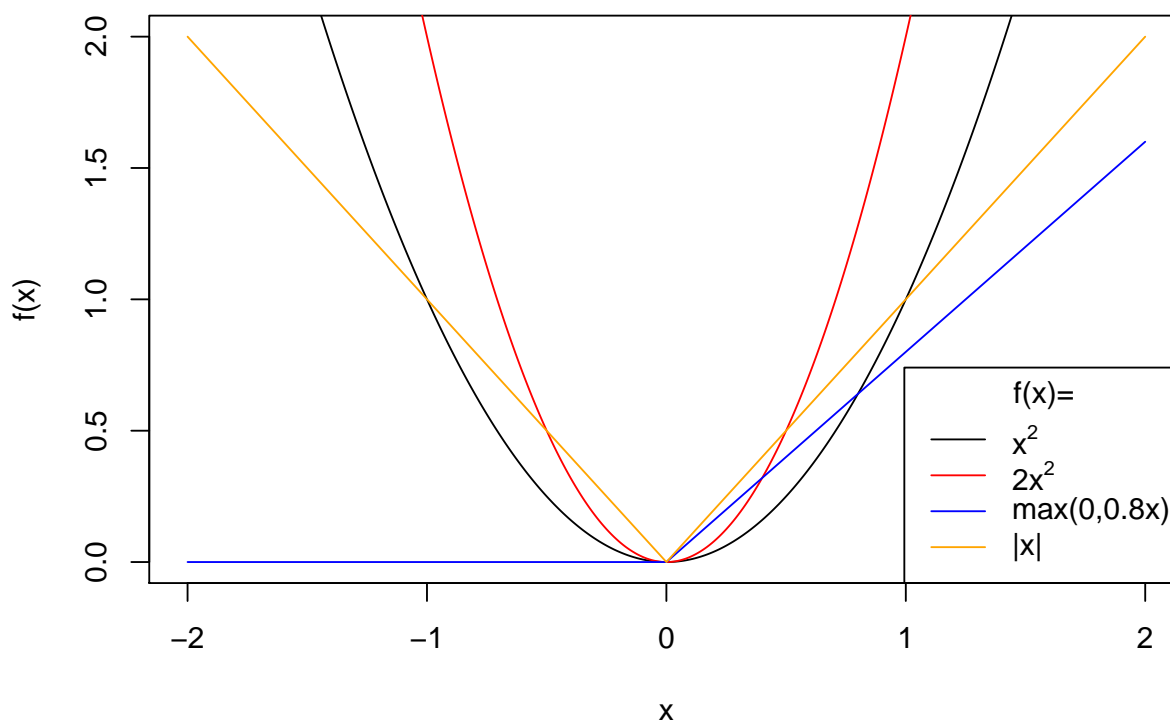
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

On the other hand, a differentiable function f is said to be convex on an interval \mathcal{I} if and only if

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

for all points $x, x_0 \in \mathcal{I}$.

- (i) What conclusion can we therefore draw if we approximate a convex function with a Taylor approximation of first order?
- (ii) Visualize such an approximation for different values x_0 for one of the following convex functions on $\mathcal{I} = [-2, 2]$.



- (b) A subdifferential of f is a set of values $\check{\nabla}_{x_0} f$ defined as

$$\check{\nabla}_{x_0} f = \{g : f(x) \geq f(x_0) + g \cdot (x - x_0) \forall x \in \mathcal{I}\}.$$

Every scalar value $g \in \check{\nabla}_{x_0}$ is said to be a subgradient of f at x_0 . Does a subdifferential have any parallels to the previous question? How can we interpret g ?

- (c) We can make use of subdifferentials for convex but non-differentiable loss functions like the one induced by the Lasso. It holds that:

A point x_0 is the global minimum of a convex function $f \Leftrightarrow 0$ is contained in the subdifferential $\check{\nabla}_{x_0} f$.

We can define a subdifferential at point x_0 also as a non-empty interval $[x_l, x_u]$ where the lower and upper limit is defined by

$$x_l = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}, \quad x_u = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

These resemble the limits of the derivative $\partial f / \partial x$ evaluated at a point very close to x_0 when coming from the left or right side, respectively.

- (i) Derive $\check{\nabla}_{x_0} f$ for $f(x) = |x|$ at $x_0 = 0$.
 - (ii) Is 0 a global minimum? Explain.
 - (iii) What is the subdifferential of the Lasso penalty $\lambda \sum_{j=1}^p |\theta_j|$? Hint: $\check{\nabla}_{x_0}(f + g) = \check{\nabla}_{x_0} f + \check{\nabla}_{x_0} g$. Also, a subdifferential of a constant function is 0 and at any other differentiable point x_0 , the subdifferential is equal to the gradient.
- (d) Derive the subdifferential for the Lasso problem

$$\mathcal{R}_{reg} = n^{-1} \sum_{i=1}^n (y^{(i)} - x_1^{(i)} \theta_1 - x_2^{(i)} \theta_2)^2 + \lambda \sum_{j=1}^2 |\theta_j|$$

w.r.t. θ_2 , i.e., for an $L1$ -regularized linear model with two linear features x_1 and x_2 .