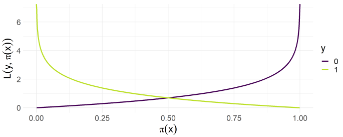
Introduction to Machine Learning

Bernoulli Loss



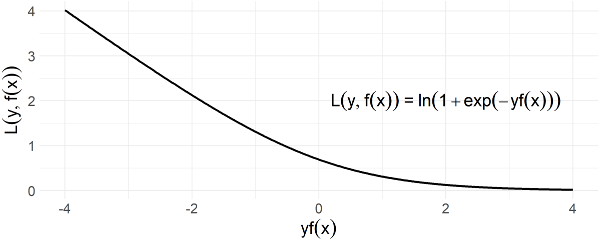
Learning goals

- Know the Bernoulli loss and related losses (log-loss, logistic loss, Binomial loss)
- Derive the risk minimizer
- Derive the optimal constant model
- Understand the connection between log-loss and entropy splitting

BERNOULLI LOSS

$$\begin{aligned} L(y, f(\mathbf{x})) &= \ln(1 + \exp(-y \cdot f(\mathbf{x}))) \quad \text{for } y \in \{-1, +1\} \\ L(y, f(\mathbf{x})) &= -y \cdot f(\mathbf{x}) + \log(1 + \exp(f(\mathbf{x}))) \quad \text{for } y \in \{0, 1\} \end{aligned}$$

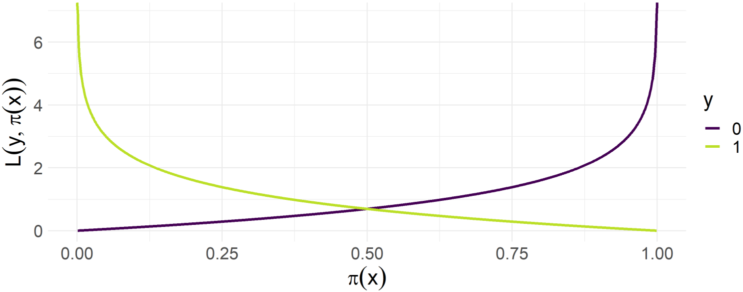
- Two equivalent formulations for different label encodings
- Negative log-likelihood of Bernoulli model, e.g., logistic regression
- Convex, differentiable
- Pseudo-residuals (0/1 case): $\tilde{r} = y - \frac{1}{1+\exp(-f(\mathbf{x}))}$
  Interpretation: $L1$ distance between 0/1-labels and posterior prob!



BERNOULLI LOSS ON PROBABILITIES

If scores are transformed into probabilities by the logistic function $\pi(\mathbf{x}) = (1 + \exp(-f(\mathbf{x})))^{-1}$ (or equivalently if $f(x) = \log\left(\frac{\pi(\mathbf{x})}{1-\pi(\mathbf{x})}\right)$ are the log-odds of $\pi(\mathbf{x})$), we arrive at another equivalent formulation of the loss, where y is again encoded as $\{0, 1\}$:

$$L(y, \pi(\mathbf{x})) = -y \log\left(\pi(\mathbf{x})\right) - (1 - y) \log\left(1 - \pi(\mathbf{x})\right).$$



BERNOULLI LOSS: RISK MINIMIZER

The risk minimizer for the Bernoulli loss defined for probabilistic classifiers $\pi(\mathbf{x})$ and on $y \in \{0, 1\}$ is

$$\pi^*(\mathbf{x}) = \eta(\mathbf{x}) = \mathbb{P}(y = 1 \mid \mathbf{x} = \mathbf{x}).$$

Proof: We can write the risk for binary y as follows:

$$\mathcal{R}(f) \quad = \quad \mathbb{E}_x \left[L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x}))\right],$$

with $\eta(\mathbf{x}) = \mathbb{P}(y = 1 \mid \mathbf{x} = \mathbf{x})$ (see chapter on the 0-1-loss for more details).
For a fixed $\mathbf{x}$ we compute the point-wise optimal value c by setting the derivative to 0:

$$\begin{aligned}
\frac{\partial}{\partial c} \left(- \log c \cdot \eta(\mathbf{x}) - \log(1 - c) \cdot (1 - \eta(\mathbf{x}))\right) &= 0 \\
-\frac{\eta(\mathbf{x})}{c} + \frac{1 - \eta(\mathbf{x})}{1 - c} &= 0 \\
\frac{-\eta(\mathbf{x}) + \eta(\mathbf{x})c + c - \eta(\mathbf{x})c}{c(1 - c)} &= 0 \\
c &= \eta(\mathbf{x}).
\end{aligned}$$

BERNOULLI LOSS: RISK MINIMIZER

The risk minimizer for the Bernoulli loss defined on $y \in \{-1, 1\}$ and scores $f(\mathbf{x})$ is the point-wise log-odds:

$$f^*(\mathbf{x}) \quad = \quad \ln\left(\frac{\mathbb{P}(y \mid \mathbf{x} = \mathbf{x})}{1 - \mathbb{P}(y \mid \mathbf{x} = \mathbf{x})}\right).$$

The function is undefined when $P(y \mid \mathbf{x} = \mathbf{x}) = 1$ or $P(y \mid \mathbf{x} = \mathbf{x}) = 0$, but predicts a smooth curve which grows when $P(y \mid \mathbf{x} = \mathbf{x})$ increases and equals 0 when $P(y \mid \mathbf{x} = \mathbf{x}) = 0.5$.

Proof: As before we minimize

$$\begin{aligned} \mathcal{R}(f) &= \mathbb{E}_x \left[L(1, f(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(-1, f(\mathbf{x})) \cdot (1 - \eta(\mathbf{x}))\right] \\ &= \ln(1 + \exp(-f(\mathbf{x})))\eta(\mathbf{x}) + \ln(1 + \exp(f(\mathbf{x})))(1 - \eta(\mathbf{x})). \end{aligned}$$

BERNOULLI LOSS: RISK MINIMIZER

For a fixed **x** we compute the point-wise optimal value *c* by setting the derivative to 0:

$$\begin{aligned}
\frac{\partial}{\partial c} \ln(1 + \exp(-c))\eta(\mathbf{x}) + \ln(1 + \exp(c))(1 - \eta(\mathbf{x})) &= 0 \\
-\frac{\exp(-c)}{1 + \exp(-c)}\eta(\mathbf{x}) + \frac{\exp(c)}{1 + \exp(c)}(1 - \eta(\mathbf{x})) &= 0 \\
-\frac{\exp(-c)}{1 + \exp(-c)}\eta(\mathbf{x}) + \frac{1}{1 + \exp(-c)}(1 - \eta(\mathbf{x})) &= 0 \\
-\eta(\mathbf{x}) + \frac{1}{1 + \exp(-c)} &= 0 \\
\eta(\mathbf{x}) &= \frac{1}{1 + \exp(-c)} \\
c &= \ln\left(\frac{\eta(\mathbf{x})}{1 - \eta(\mathbf{x})}\right)
\end{aligned}$$

BERNOULLI: OPTIMAL CONSTANT MODEL

The optimal constant probability model $\pi(\mathbf{x}) = \theta$ w.r.t. the Bernoulli loss for labels from $\mathcal{Y} = \{0, 1\}$ is:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\theta} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} y^{(i)}$$

Again, this is the fraction of class-1 observations in the observed data. We can simply prove this again by setting the derivative of the risk to 0 and solving for θ.

BERNOULLI: OPTIMAL CONSTANT MODEL

The optimal constant score model $f(\mathbf{x}) = \theta$ w.r.t. the Bernoulli loss labels from $\mathcal{Y} = \{-1, +1\}$ or $\mathcal{Y} = \{0, 1\}$ is:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\theta} \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \ln \frac{n_+}{n_-} = \ln \frac{n_+/n}{n_-/n}$$

where n_- and n_+ are the numbers of negative and positive observations, respectively.

This again shows a tight (and unsurprising) connection of this loss to log-odds.

Proving this is also a (quite simple) exercise.

BERNOULLI-LOSS: NAMING CONVENTION

We have seen three loss functions that are closely related. In the literature, there are different names for the losses:

$$
\begin{aligned}
L(y, f(\mathbf{x})) &= \ln(1 + \exp(-yf(\mathbf{x}))) \quad \text{for } y \in \{-1, +1\} \\
L(y, f(\mathbf{x})) &= -y \cdot f(\mathbf{x}) + \log(1 + \exp(f(\mathbf{x}))) \quad \text{for } y \in \{0, 1\}
\end{aligned}
$$

are referred to as Bernoulli, Binomial or logistic loss.

$$L(y, \pi(\mathbf{x})) = -y \log\left(\pi(\mathbf{x})\right) - (1 - y) \log\left(1 - \pi(\mathbf{x})\right) \quad \text{for } y \in \{0, 1\}$$

is referred to as cross-entropy or log-loss.

We usually refer to all of them as **Bernoulli loss**, and rather make clear whether they are defined on labels $y \in \{0, 1\}$ or $y \in \{-1, +1\}$ and on scores $f(\mathbf{x})$ or probabilities $\pi(\mathbf{x})$.

BERNOULLI LOSS MIN = ENTROPY SPLITTING

When fitting a tree we minimize the risk within each node $\mathcal{N}$ by risk minimization and predict the optimal constant. Another approach that is common in literature is to minimize the average node impurity $\text{Imp}(\mathcal{N})$.

Claim: Entropy splitting $\text{Imp}(\mathcal{N}) = \sum_{k=1}^{g} \pi_k^{(\mathcal{N})} \log \pi_k^{(\mathcal{N})}$ is equivalent to minimize risk measured by the Bernoulli loss.

Note that $\pi_k^{(\mathcal{N})} := \frac{1}{n_\mathcal{N}} \sum\limits_{(\mathbf{x},y)\in\mathcal{N}} [y = k]$.

Proof: To prove this we show that the risk related to a subset of observations $\mathcal{N} \subseteq \mathcal{D}$ fulfills

$$\mathcal{R}(\mathcal{N}) = n_\mathcal{N} \text{Imp}(\mathcal{N}),$$

where *I* is the entropy criterion $\text{Imp}(\mathcal{N})$ and $\mathcal{R}(\mathcal{N})$ is calculated w.r.t. the (multiclass) Bernoulli loss

$$L(y, \pi_k(\mathbf{x})) = \sum_{k=1}^{g} [y = k] \log \left(\pi_k(\mathbf{x})\right).$$

BERNOULLI LOSS MIN = ENTROPY SPLITTING

$$
\begin{aligned}
\mathcal{R}(\mathcal{N}) &= \sum_{(\mathbf{x},y)\in\mathcal{N}} \sum_{k=1}^{g} [y=k] \log \pi_k(\mathbf{x}) \overset{(*)}{=} \sum_{k=1}^{g} \sum_{(\mathbf{x},y)\in\mathcal{N}} [y=k] \log \pi_k^{(\mathcal{N})} \\
&= \sum_{k=1}^{g} \log \pi_k^{(\mathcal{N})} \underbrace{\sum_{(\mathbf{x},y)\in\mathcal{N}} [y=k]}_{n_{\mathcal{N}} \cdot \pi_k^{(\mathcal{N})}} \\
&= n_{\mathcal{N}} \sum_{k=1}^{g} \pi_k^{(\mathcal{N})} \log \pi_k^{(\mathcal{N})} = n_{\mathcal{N}} \text{Imp}(\mathcal{N}),
\end{aligned}
$$

where in $^{(*)}$ the optimal constant per node $\pi_k^{(\mathcal{N})} = \frac{1}{n_{\mathcal{N}}} \sum\limits_{(\mathbf{x},y)\in\mathcal{N}} [y=k]$

was plugged in.