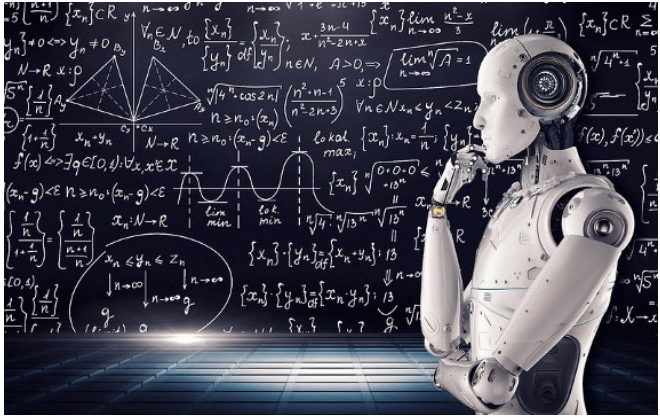
# Common Machine Learning Algorithms



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# CONTENTS

# LINEAR MODELS (LM)

# LINEAR MODELS – FUNCTIONALITY

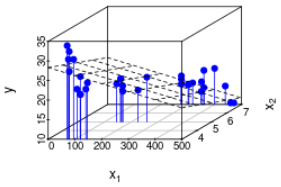
SUPERVISED | REGRESSION | CLASSIFICATION | PARAMETRIC | WHITE-BOX

**General idea** Represent target as function of linear predictor $\boldsymbol{\theta}^T \mathbf{x}$

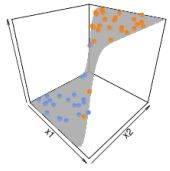
**Hypothesis space**

$\mathcal{H} = \left\{ f : \mathcal{X} \to \mathbb{R} \mid f(\mathbf{x}) = \phi(\boldsymbol{\theta}^\top \mathbf{x}) \right\}$, with suitable transformation $\phi(\cdot)$, e.g.,

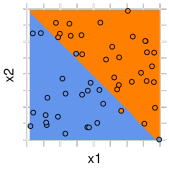
- Identity $\phi(\boldsymbol{\theta}^\top \mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x}$ $\Rightarrow$ **linear regression**
- Logistic sigmoid function $\phi(\boldsymbol{\theta}^T \mathbf{x}) = \frac{1}{1+\exp(-\boldsymbol{\theta}^T \mathbf{x})} =: \pi(\mathbf{x} \mid \boldsymbol{\theta})$ $\Rightarrow$ **(binary) logistic regression**
  - Probability $\pi(\mathbf{x} \mid \boldsymbol{\theta}) = \mathbb{P}(y = 1 \mid \mathbf{x})$ of belonging to one of two classes
  - Separating hyperplane via decision rule (e.g., $\hat{y} = 1 \Leftrightarrow \pi(\mathbf{x}) > 0.5$)




Linear regression hyperplane




Logistic function for bivariate input and loss-minimal $\boldsymbol{\theta}$




Corresponding separating hyperplane

# LINEAR MODELS – FUNCTIONALITY

**Empirical risk**

- **Linear regression**
  - Typically, based on **quadratic** loss: $\mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(y^{(i)} - f\left(\mathbf{x}^{(i)} \mid \boldsymbol{\theta}\right)\right)^2$
    $\Rightarrow$ corresponding to ordinary-least-squares (OLS) estimation
  - Alternatives: e.g., **absolute** or **Huber** loss (both improving robustness)
- **Logistic regression:** based on **Bernoulli/log/cross-entropy** loss
  $\Rightarrow \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) = \sum_{i=1}^{n} -y^{(i)} \log\left(\pi\left(\mathbf{x}^{(i)}\right)\right) - (1 - y^{(i)}) \log\left(1 - \pi\left(\mathbf{x}^{(i)}\right)\right)$

**Optimization**

- For **OLS**: analytically with $\hat{\boldsymbol{\theta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$
  (with $\mathbf{X} \in \mathbb{R}^{n \times p}$ : matrix of feature vectors)
- For **other loss functions**: numerical optimization

**Hyperparameters** None

# LINEAR MODELS – PRO'S & CON'S

**Advantages**

- \+ **Simple and fast** implementation
- \+ **Analytical** solution
- \+ **Cheap** computation
- \+ Applicable for any **dataset size**, as long as number of observations $\gg$ number of features
- \+ Flexibility **beyond linearity** with polynomials, trigonometric transformations etc.
- \+ Intuitive **interpretability** via feature effects
- \+ Statistical hypothesis **tests** for effects available

**Disadvantages**

- — **Nonlinearity** of many real-world problems
- — Further restrictive **assumptions**: linearly independent features, homoskedastic residuals, normality of conditional response
- — **Sensitivity** w.r.t. outliers and noisy data (especially with L2 loss)
- — Risk of **overfitting** in higher dimensions
- — Reature **interactions** must be handcrafted, so higher orders practically infeasible
- — No handling of **missing** data

Simple, highly interpretable method suited for linear problems, but with strong assumptions, practical limitations, and tendency to overfit

# LINEAR MODELS – REGULARIZATION

### Idea

- Unregularized LM: risk of **overfitting** in high-dimensional space with only few observations
- **Goal**: find compromise between model fit and generalization by adding **penalty term**

### Regularized empirical risk

- Empirical risk function **plus complexity penalty** $J(\boldsymbol{\theta})$, controlled by shrinkage parameter $\lambda > 0$: $\mathcal{R}_{\text{reg}}(\boldsymbol{\theta}) := \mathcal{R}_{\text{emp}}(\boldsymbol{\theta}) + \lambda \cdot J(\boldsymbol{\theta})$.
- Popular regularizers
  - **Ridge** regression: L2 penalty $J(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_2^2$
  - **LASSO** regression: L1 penalty $J(\boldsymbol{\theta}) = \|\boldsymbol{\theta}\|_1$

### Optimization under regularization

- **Ridge**: analytically with $\hat{\boldsymbol{\theta}}_{\text{Ridge}} = (\mathbf{X}^\top \mathbf{X} + \lambda \boldsymbol{I})^{-1} \mathbf{X}^\top \mathbf{y}$
- **LASSO**: numerically with, e.g., (sub-)gradient descent

# LINEAR MODELS – REGULARIZATION

### Choice of regularization parameter

- Standard hyperparameter optimization problem
- E.g., choose $\lambda$ with minimum mean cross-validated error (default in R package `glmnet`)

### Ridge vs. LASSO

- **Ridge**
  - Overall smaller, but still dense $\theta$
  - Suitable with many influential features present, handling correlated features by shrinking their coefficients equally
- **LASSO**
  - Actual variable selection
  - Suitable for sparse problems, ineffective with correlated features (randomly selecting one)
- Neither overall better – compromise: **elastic net**
  $\rightarrow$ weighted combination of Ridge and LASSO regularizers

# LINEAR MODELS – PRACTICAL HINTS

## Implementation

- **R:**
  - **Unregularized:** `mlr3` learner `LearnerRegrLM`, calling `stats::lm()` / `mlr3` learner `LearnerClassifLogReg`, calling `stats::glm()`
  - **Regularized:** `mlr3` learners `LearnerClassifGlmnet` / `LearnerRegrGlmnet`, calling `glmnet::glmnet()`
- **Python:** `LinearRegression` from package `sklearn.linear_model`, package for advanced statistical parameters `statsmodels.api`

# LINEAR SUPPORT VECTOR MACHINES (SVM)

# LINEAR SVM – FUNCTIONALITY

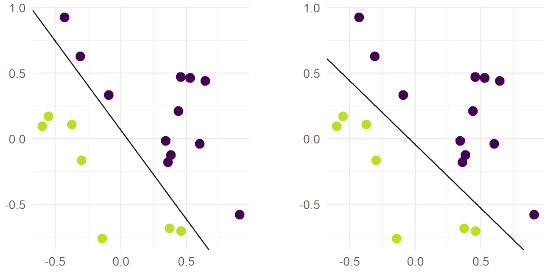
SUPERVISED | CLASSIFICATION | PARAMETRIC | BLACK-BOX

**General idea**

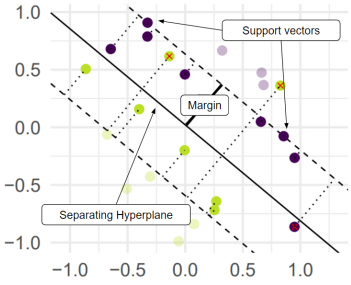
- Find linear decision boundary (**separating hyperplane**) that best separates classes
  - **Hard-margin** SVM: maximize distance (**margin** $\gamma > 0$) to closest members (**support vectors, SV**) on each side of decision boundary
  - **Soft-margin** SVM: relax separation to allowing margin violations $\rightarrow$ maximize margin while minimizing violations
- 3 types of training points
  - **non-SVs** with no impact on decision boundary
  - **SVs** located exactly on decision boundary
  - **margin violators**

**Hypothesis space** $\mathcal{H} = \left\{ f : \mathcal{X} \to \mathbb{R} \mid f(\mathbf{x}) = \boldsymbol{\theta}^\top \mathbf{x} + \theta_0 \right\}$ separater intercept notwendig?

# LINEAR SVM – FUNCTIONALITY




Hard-margin SVM: margin is maximized by boundary on the right




Soft-margin SVM with margin violations

## Dual problem

$$\max_{\boldsymbol{\alpha} \in \mathbb{R}^n} \quad \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y^{(i)} y^{(j)} \left\langle \mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right\rangle$$

$$\text{s.t.} \quad 0 \leq \alpha_i \leq C \;\; \forall i \in \{1, \ldots, n\} \;\; (C = \infty \text{ for hard-margin SVM}),$$

$$\sum_{i=1}^{n} \alpha_i y^{(i)} = 0$$

# LINEAR SVM – FUNCTIONALITY

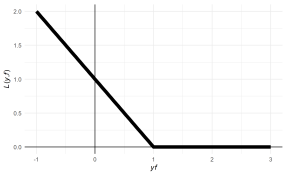
**Empirical risk**

Soft-margin SVM also interpretable as **L2-regularized ERM**:

$$\frac{1}{2}\|\boldsymbol{\theta}\|^2 + C\sum_{i=1}^{n} L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right)$$

with

- $\|\boldsymbol{\theta}\| = 1/\gamma$,
- $C > 0$: penalization for missclassified data points
- $L(y, f) = \max(1 - yf, 0)$: **hinge** loss
  $\rightarrow$ other loss functions applicable (e.g., **Huber** loss)




**Optimization**

- Typically, tackling **dual** problem (though feasible in corresponding primal) via **quadratic programming**
- Popular: **sequential minimal optimization** $\rightarrow$ iterative algorithm based on breaking down objective into bivariate quadratic problems with analytical solutions

**Hyperparameters** Cost parameter $C$

# LINEAR SVM – PRO'S & CON'S

**Advantages**

+ Often **sparse** solution
+ Robust against overfitting (**regularized**); especially in high-dimensional space
+ **Stable** solutions, as non-SV do not influence decision boundary

**Disadvantages**

− **Costly** implementation; long training times
− **Limited scalability** to larger data sets **??**
− Confined to **linear separation**
− Poor **interpretability**
− No handling of **missing** data

CHECK

Very accurate solution for high-dimensional data that is linearly separable

# LINEAR SVM – PRACTICAL HINTS

**Preprocessing**

Features must be rescaled before applying SVMs.

**Tuning**

Cost parameter $C$ must be tuned and has strong influence on resulting separating hyperplane.

**Implementation**

- **R:** `mlr3` learners `LearnerClassifSVM` / `LearnerRegrSVM`, calling `svm()` from `libsvm`
- **Python:** `sklearn.svm.SVC` from package `scikit-learn` / package `libSVM`

# NONLINEAR SUPPORT VECTOR MACHINES

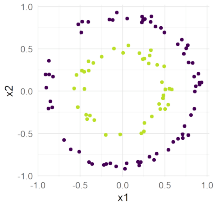
# NONLINEAR SVM – FUNCTIONALITY

SUPERVISED | CLASSIFICATION | NON-PARAMETRIC | BLACK-BOX

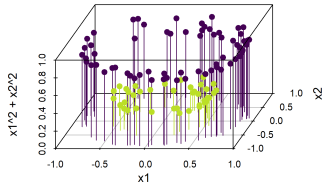
**General idea**

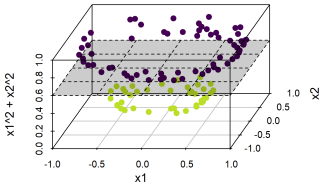
- Move **beyond linearity** by mapping data to transformed space where they are linearly separable
- **Kernel trick** (based on Mercer's theorem, existende of RKHS):
  - Replace two-step operation feature map $\phi \rightsquigarrow$ inner product by **kernel** $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, s.t. $\langle \phi(\mathbf{x}), \phi(\tilde{\mathbf{x}}), = \rangle \, k(\mathbf{x}, \tilde{\mathbf{x}})$
  - No need for explicit construction of feature maps; very fast and flexible

**Hypothesis space** $\mathcal{H} = \left\{ f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i y^{(i)} k(\mathbf{x}^{(i)}, \mathbf{x}) + \theta_0 | \; \theta_0, \alpha_i \in \mathbb{R} \right\}$




Data are not linearly separable in original space.




Mapping to 3D space allows for linear separation with hyperplane.

# NONLINEAR SVM – FUNCTIONALITY

**Dual problem**

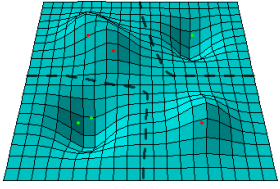
**Kernelize** dual (soft-margin) SVM problem, replacing all inner products $\left\langle \mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right\rangle$ by kernels:

$$\max_{\boldsymbol{\alpha}} \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y^{(i)} y^{(j)} k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}), \quad \text{s.t.} \quad 0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} \alpha_i y^{(i)} = 0.$$

**Hyperparameters** Cost $C$ of margin violations, kernel hyperparameters (e.g., width of RBF kernel)

**Interpretation as basis function approach**

- **Representer theorem:** dual soft-margin SVM problem can be expressed through
  $\boldsymbol{\theta} = \sum_{j=1}^{n} \beta_j \phi\left(\mathbf{x}^{(j)}\right)$
- Sparse, weighted sum of **basis functions** with $\beta_j = 0$ for non-SVs
- **Local** model with smoothness depending on kernel properties



RBF kernel as mixture of Gaussian basis functions, forming bumpy, nonlinear decision surface to discern red and green points.

# NONLINEAR SVM – PRO'S & CON'S

**Advantages**

- \+ high **accuaracy**
- \+ can learn **nonlinear decision boundaries**
- \+ often **sparse** solution
- \+ robust against overfitting (**regularized**); especially in high-dimensional space
- \+ **stable** solutions, as the non-SV do not influence the separating hyperplane

**Disadvantages**

- — **costly implementation**; long training times
- — does not scale well to **larger data sets ??**
- — only **linear separation** → possible with nonlinear SVMs which are explained in the following slides.
- — poor **interpretability**
- — **not easy tunable** as it is highly important to choose the right kernel
- — No handling of **missing** data

CHECK

nonlinear SVMs perform very well for nonlinear separable data, but are hard to interpret and need a lot of tuning.

# NONLINEAR SVM – PRACTICAL HINTS

### Popular kernels

- **Linear** kernel: dot product of given observations $\rightarrow k(\mathbf{x}, \tilde{\mathbf{x}}) = \mathbf{x}^\top \tilde{\mathbf{x}}$
- **Polynomial** kernel of degree $d \in \mathbb{N}$: monomials (i.e., feature interactions!) up to $d$-th order $\rightarrow k(\mathbf{x}, \tilde{\mathbf{x}}) = \left(\mathbf{x}^\top \tilde{\mathbf{x}} + b\right)^d, \; b \geq 0$
- **RBF** kernel: infinite-dimensional feature space, in theory allowing for perfect separation of all finite datasets $\rightarrow k(\mathbf{x}, \tilde{\mathbf{x}}) = \exp\left(-\gamma \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2\right)$ with bandwidth parameter $\gamma > 0$

### Tuning

- High sensitivity w.r.t. hyperparameters, especially those of kernel $\rightarrow$ **tuning** very important
- For RBF kernels, use **RBF sigma heuristic** to determine bandwidth

### Implementation

- **R:** `mlr3` learners `LearnerClassifSVM` / `LearnerRegrSVM`, calling `e1071::svm()` (interface to `libSVM`)
- **Python:** `sklearn.svm.SVC` from package `scikit-learn` / package `libSVM`

# $K$-NEAREST NEIGHBORS ($K$-NN)

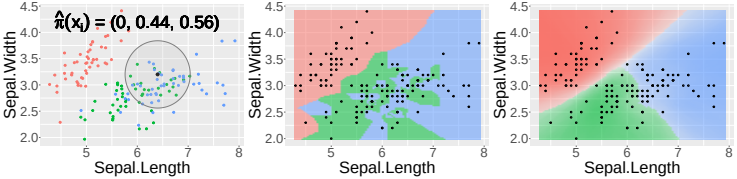
# $K$-NN – FUNCTIONALITY

SUPERVISED | REGRESSION | CLASSIFICATION | NON-PARAMETRIC | WHITE-BOX

## General idea

- Rationale: **similarity** in feature space $\rightsquigarrow$ similarity in target space w.r.t. some similarity/distance **metric**
- **Prediction** for $\mathbf{x}$: construct $k$**-neighborhood** $N_k(\mathbf{x})$ from $k$ points closest to $\mathbf{x}$ in $\mathcal{X}$, then predict
  - (weighted) mean target for **regression**: $\hat{y} = 1/\left(\sum_{i:\mathbf{x}^{(i)} \in N_k(\mathbf{x})} w_i\right) \sum_{i:\mathbf{x}^{(i)} \in N_k(\mathbf{x})} w_i y^{(i)}$
  - most frequent class for **classification**: $\hat{y} = \arg\max_{\ell \in \{1,\ldots,g\}} \sum_{i:\mathbf{x}^{(i)} \in N_k(\mathbf{x})} \mathbb{I}\left(y^{(i)} = \ell\right)$
- No distributional or functional **assumptions**
- **Nonparametric** behavior: parameters = training data; no compression of information

**Hyperparameters** Neighborhood **size** $k$ (locality), **distance** measure




*Left*: Neighborhood for exemplary observation in `iris`, $k = 50$
*Right*: Prediction surfaces for $k = 1$ and $k = 50$

# $K$-NN – PRO'S & CON'S

**Advantages**

- \+ **Easy** to explain and implement
- \+ Applicable to both regression and classification tasks
- \+ No functional **assumptions** – therefore (in theory) able to model data situations of **arbitrary complexity**
- \+ No **training** period; no **optimization** required
- \+ Constant evolvement with **new data**
- \+ Ability to learn **non-linear** decision boundaries

**Disadvantages**

- — Sensitivity w.r.t. **noisy** or **irrelevant** features and outliers due to utter reliance on distances
- — Bad performance when feature **scales** not consistent with importance
- — Heavily afflicted by **curse of dimensionality**
- — No handling of **missing** data
- — Poor handling of data **imbalances** (worse for large $k$)
- — High **memory** consumption of distance computation

Easy and intuitive for small, well-behaved datasets with meaningful feature space distances

# $K$-NN – PRACTICAL HINTS

## Popular distance measures

- Numerical features: typically, **Minkowski** distances $d(\mathbf{x}, \tilde{\mathbf{x}}) = \|\mathbf{x} - \tilde{\mathbf{x}}\|_q = \left(\sum_j |x_j - \tilde{x}_j|^q\right)^{\frac{1}{q}}$
  - $q = 1$: **Manhattan** distance $\rightarrow d(\mathbf{x}, \tilde{\mathbf{x}}) = \sum_j |x_j - \tilde{x}_j|$
  - $q = 2$: **Euclidean** distance $\rightarrow d(\mathbf{x}, \tilde{\mathbf{x}}) = \sqrt{\sum_j (x_j - \tilde{x}_j)^2}$
- In presence of categorical features: **Gower** distance
- **Custom** distance measures applicable
- Optional **weighting** to account for beliefs about varying feature importance

## Implementation

- **R:** `mlr3` learners `LearnerClassifKKNN` / `LearnerRegrKKNN`, calling `kknn::kknn()`
- **Python:** `KNeighborsClassifier` / `KNeighborsRegressor` from package `scikit-learn`

# CLASSIFICATION & REGRESSION TREES (CART)

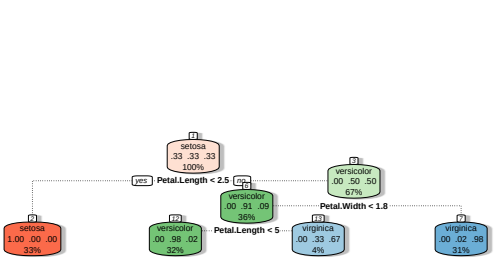
# CART – FUNCTIONALITY

SUPERVISED | REGRESSION | CLASSIFICATION | NONPARAMETRIC | WHITE-BOX | FEATURE SELECTION

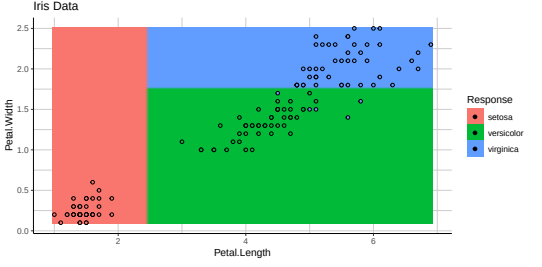
**General idea**

- Starting from root node containing all data, perform repeated **binary splits**, thereby subsequently dividing input space into $T$ **rectangular partitions** $Q_t$
  - In each step, find **optimal split** (feature-threshold combination) $\rightarrow$ greedy search
  - Assign same response $c_t$ to all observations in terminal region $Q_t$
- Splits based on node **impurity**, equivalently interpretable as **ERM**

**Hypothesis space** $\mathcal{H} = \left\{ f(\mathbf{x}) : f(\mathbf{x}) = \sum_{t=1}^{T} c_t \mathbb{I}(\mathbf{x} \in Q_t) \right\}$




Classification tree for `iris` data after 3 splits




Corresponging prediction surface with axis-aligned boundaries

# CART – FUNCTIONALITY

**Empirical risk**

- Calculated for each potential terminal node $\mathcal{N}_t$ of a split
- In general, compatible with arbitrary losses – typical choices:
  - $g$-way classification:
    - **Brier score** $\mathcal{R}(\mathcal{N}_t) = \sum_{(\mathbf{x},y) \in \mathcal{N}_t} \sum_{k=1}^{g} \left(\mathbb{I}(y = k) - \pi_k(\mathbf{x})\right)^2$ $\rightarrow$ **Gini** impurity
    - **Bernoulli** loss $\mathcal{R}(\mathcal{N}_t) = \sum_{(\mathbf{x},y) \in \mathcal{N}_t} \sum_{k=1}^{g} \mathbb{I}(y = k) \cdot \log(\pi_k(\mathbf{x}))$ $\rightarrow$ **entropy** impurity
  - Regression: **quadratic** loss $\mathcal{R}(\mathcal{N}_t) = \sum_{(\mathbf{x},y) \in \mathcal{N}_t} (y - c_t)^2$

**Optimization**

- **Exhaustive** search over all split candidates, choice of risk-minimal split
- In practice: limit number of candidates, use tricks to avoid combinatorial explosion

**Hyperparameters** **Complexity**, i.e., number of leaves $T$ (controlled indirectly)

# CART – PRO'S & CON'S

**Advantages**

- **Easy** to understand & visualize
- Highly **interpretable**
- Built-in **feature selection**
- Applicable to **non-numerical** features
- Automatic handling of **missings**
- **Interaction** effects between features naturally included, even of higher orders
- **Fast** computation and good scalability
- High **flexibility** (custom split criteria or leaf-node prediction rules)

**Disadvantages**

- Rather **poor generalization** when used stand-alone
- High **variance**/**instability**: strong dependence on training data
- Substantial risk of **overfitting**
- Not well-suited for modeling **linear** relationships
- **Bias** toward features with **many categories**

Simple, good with feature selection and highly interpretable, but not the best predictor

# CART – PRACTICAL HINTS

**Complexity control**

- Unless interrupted, splitting continues until we have 1 observation per leaf node (costly + overfitting)
- Limit tree growth
  - **Early stopping:** stop growth prematurely
    $\rightarrow$ hard to determine good stopping point before actually trying all combinations
  - **Pruning:** grow to large size and cut back in risk-optimal manner

**Bagging / boosting** As CART are highly **instable** predictors on their own, they are typically used as base learners in bagging (random forest) or boosting ensembles.

**Implementation**

- **R:** `mlr3` learners `LearnerClassifRpart` / `LearnerRegrRpart`, calling `rpart::rpart()`
- **Python:** `DecisionTreeClassifier` / `DecisionTreeRegressor` from package `scikit-learn`
- Complexity controlled via tree depth, minimum number of observations per node, maximum number of leaves, minimum risk reduction per split, ...