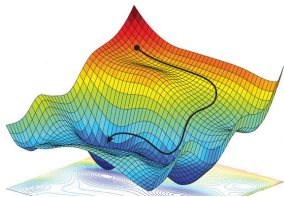


Introduction to Machine Learning

Advanced Regression Losses



Learning goals

- Know the Huber loss
- Know the log-barrier loss
- Know the ϵ -insensitive loss
- Know the quantile loss
- Know the Cauchy loss

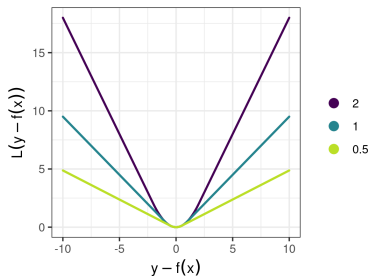
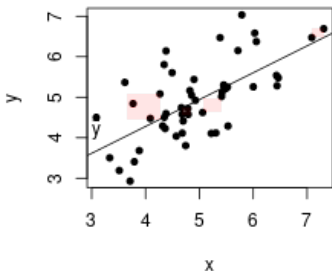
ADVANCED LOSS FUNCTIONS

- Advanced loss functions are designed to achieve special properties (e.g., robustness and smoothness for the Huber or Cauchy loss).
- Furthermore, special loss functions are necessary in certain applications.
- Examples:
 - Quantile loss: Overestimating a clinical parameter might not be as bad as underestimating it.
 - Log-barrier loss: Extremely under- or overestimating demand in production would put company profit at risk.
 - ϵ -insensitive loss: A certain amount of deviation in production does no harm, larger deviations do.
- Sometimes a custom loss must be designed specifically for the given application.
- Some learning algorithms use specific loss function, e.g., the hinge loss for SVMs.

HUBER LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} \frac{1}{2}(y - f(\mathbf{x}))^2 & \text{if } |y - f(\mathbf{x})| \leq \delta \\ \delta|y - f(\mathbf{x})| - \frac{1}{2}\delta^2 & \text{otherwise} \end{cases}, \quad \delta > 0$$

- Piece-wise combination of $L1$ and $L2$ loss
- Analytic properties: convex, differentiable, robust
- Combines advantages of $L1$ and $L2$ loss: differentiable + robust



HUBER LOSS

Risk minimizer:

- There is no closed-form solution for the risk minimizer.
- However, the risk minimizer for the Huber loss is a **trimmed mean**:
The risk minimizer is the (conditional) mean of values between two (conditional) quantiles. The location of the quantiles depends on the distribution as well as the value of δ .

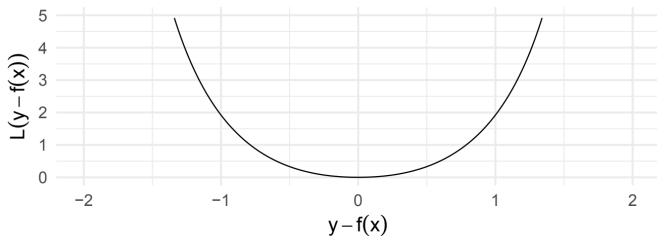
Optimal constant model:

- Similarly, there is no closed-form solution for the optimal constant model.
- Numerical optimization methods are necessary.
- The “optimal” solution can only be approached to a certain degree of accuracy via iterative optimization.

LOG-BARRIER LOSS

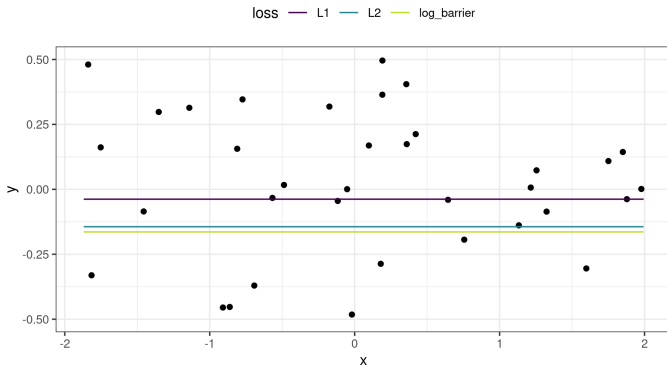
$$L(y, f(\mathbf{x})) = \begin{cases} -a^2 \cdot \log\left(1 - \left(\frac{|y-f(\mathbf{x})|}{a}\right)^2\right) & \text{if } |y - f(\mathbf{x})| \leq a \\ \infty & \text{if } |y - f(\mathbf{x})| > a \end{cases}$$

- Behaves like $L2$ loss for small residuals.
- We use this if we don't want residuals larger than a at all
- No guarantee that the risk minimization problem has a solution.
- Plot shows log-barrier loss for $a = 2$:



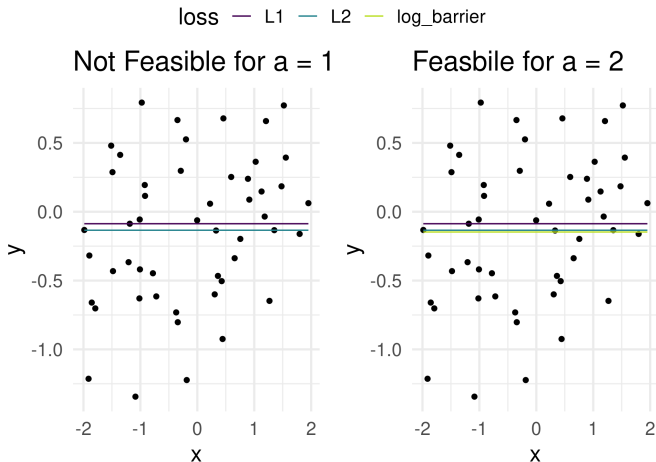
LOG-BARRIER: OPTIMAL CONSTANT MODEL

- Similarly to the Huber loss, there is no closed-form solution for the optimal constant model $f(\mathbf{x}) = \theta$ w.r.t. the log-barrier loss.
- Again, numerical optimization methods are necessary.



LOG-BARRIER: OPTIMAL CONSTANT MODEL

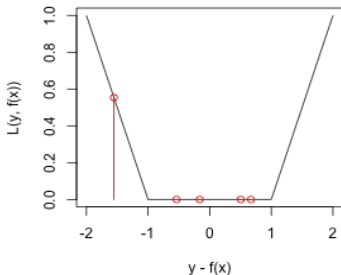
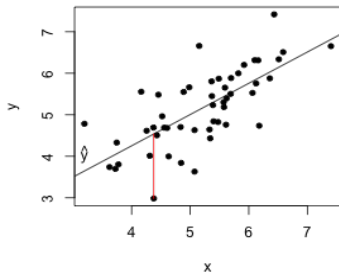
Note that the optimization problem has no (finite) solution if there is no way to fit a constant where all residuals are smaller than a .



ϵ -INSENSITIVE LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} 0 & \text{if } |y - f(\mathbf{x})| \leq \epsilon \\ |y - f(\mathbf{x})| - \epsilon & \text{otherwise} \end{cases}, \quad \epsilon \in \mathbb{R}_+$$

- Modification of $L1$ loss, errors below ϵ accepted without penalty.
- Properties: convex and not differentiable for $y - f(\mathbf{x}) \in \{-\epsilon, \epsilon\}$.



ϵ -INSENSITIVE LOSS: OPTIMAL CONSTANT

What is the optimal constant model $f(\mathbf{x}) = \theta$ w.r.t. the ϵ -insensitive loss $L(y, f(\mathbf{x})) = |y - f(\mathbf{x})| \mathbb{1}_{\{|y - f(\mathbf{x})| > \epsilon\}}$?

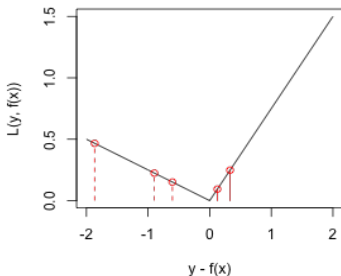
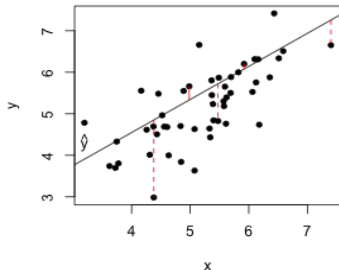
$$\begin{aligned}\hat{\theta} &= \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^n L(y^{(i)}, f(\mathbf{x}^{(i)})) \\ &= \arg \min_{\theta \in \mathbb{R}} \sum_{i \in I_\epsilon} |y^{(i)} - \theta| - \epsilon \\ &= \arg \min_{\theta \in \mathbb{R}} \sum_{i \in I_\epsilon} |y^{(i)} - \theta| - \sum_{i \in I_\epsilon} \epsilon \\ &= \text{median} \left(\left\{ y^{(i)} \mid i \in I_\epsilon \right\} \right) - |I_\epsilon| \cdot \epsilon\end{aligned}$$

with $I_\epsilon := \{i : |y^{(i)} - f(\mathbf{x}^{(i)})| \leq \epsilon\}$.

REGRESSION LOSSES: QUANTILE LOSS

$$L(y, f(\mathbf{x})) = \begin{cases} (1 - \alpha)(f(\mathbf{x}) - y) & \text{if } y < f(\mathbf{x}) \\ \alpha(y - f(\mathbf{x})) & \text{if } y \geq f(\mathbf{x}) \end{cases}, \quad \alpha \in (0, 1)$$

- Is an extension of $L1$ loss (equal to $L1$ for $\alpha = 0.5$).
- Weights either positive or negative residuals more strongly.
- $\alpha < 0.5$ ($\alpha > 0.5$) penalty to over-estimation (under-estimation)
- Also known as **pinball loss**.



REGRESSION LOSSES: QUANTILE LOSS

What is the optimal constant model $f(\mathbf{x}) = \theta$ w.r.t. the quantile loss?

$$\hat{\theta} = \arg \min_{\theta \in \mathbb{R}} \sum_{i=1}^n L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right)$$

$$\Leftrightarrow \hat{\theta} = \arg \min_{\theta \in \mathbb{R}} \left\{ (1 - \alpha) \sum_{y^{(i)} < \theta} |y^{(i)} - \theta| + \alpha \sum_{y^{(i)} \geq \theta} |y^{(i)} - \theta| \right\}$$

$$\Leftrightarrow \hat{\theta} = Q_{\alpha}(\{y^{(i)}\})$$

where $Q_{\alpha}(\cdot)$ computes the empirical α -quantile of $\{y^{(i)}\}, i = 1, \dots, n$.

CAUCHY LOSS

$$L(y, f(\mathbf{x})) = \log\left(\frac{1}{2}(x/c)^2 + 1\right), \quad c \in \mathbb{R}$$

- Particularly robust toward outliers (controllable via c).
- Analytic properties: differentiable, robust, but not convex!

