Quantum representation of $C_2 \times D_4$

We proved that every finite group has its quantum representation. Also, we have presented two approaches to transform an arbitrary finite group G into a set of quantum circuits. The first way that can be considered "classic to quantum" is to construct quantum circuits from a classically computed complex representation. The second way is more "quantum to classic" in that we compose a variational quantum algorithm (VQA) [1] with the group presentation. The absolute presentation of $C_2 \times D_4$ is

$$\begin{aligned}\langle a, b, c | a^2 = b^2 = c^4 = (bc)^2 = 1, ab = ba, \\ ac = ca, a \neq e, b \neq e, ab \neq e, c^2 \neq e\rangle,\end{aligned} \tag{1}$$

where $ab = ba$ can be converted into $(ab)^2 = e$ with $a^2 = b^2 = e$ and $ac = ca$ can be converted into ac^3ac with $a^2 = c^4 = e$.

For the first method, we use its classical representations on Dockchitser's site [2], which are

$$\rho(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

$$\rho(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3}$$

and

$$\rho(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

In this case, the representation is already unitary, but the dimension is 3, and $\rho(a)$ is identical to I_3 with a global phase. To have a unitary matrix that can be transformed into a quantum circuit, we can directly sum an I_1 after each representation. They become

$$U(a) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

$$U(b) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

and

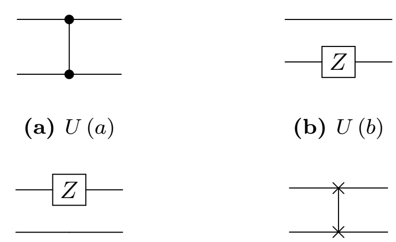
$$U(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

In particular, we have

$$U\left(bc^2\right) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

which is a Z gate on the first qubit and

$$U\left(bc^3\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$



(a) $U(a)$ **(b)** $U(b)$ **(c)** $U\left(bc^2\right)$ **(d)** $U\left(bc^3\right)$

FIG. 1: Quantum circuits of the group $C_2 \times D_4$.

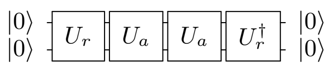
which is a $SWAP$ gate acting on both qubits. Since $U(c) = U\left(bc^2\right) U\left(bc^3\right)$, $U(c)$ can be obtained by applying a Z gate on the first qubit then $SWAP$ gate. Other elements can be turned into circuits with FIG. 1.

For the second method, we can construct a set of variational circuits, as shown in FIG. 2, where quantum generators are trained with the same ansatz illustrated in FIG. 3 with

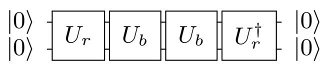
$$R_y(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{10}$$

This ansatz is particularly chosen for three reasons. Firstly, it can not be trained into identity. Thus the result can not be a trivial representation. In the worse case, when every parameter equals 0, we will have $U(a) = U(b) = U(c) \neq I_4$. They form a faithful representation of C_2, a subgroup of $C_2 \times D_4$. Secondly, the components of the full unitary matrix are real numbers, and the problem of the global phase will be less disturbing. Thirdly, similar architecture has been widely applied in previous VQA research [3–5].

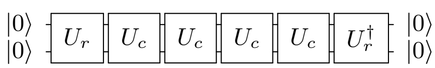
We perform a numerical simulation with the library Qibo [6], and the final result is shown in TABLE I. The code is provided on GitHub [7]. Our readers are encouraged to test this method with a different finite group or a different ansatz. After obtaining the trained parameters, we can use the circuits in FIG. 4 to verify that the quantum representation is faithful.



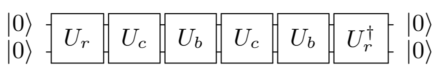
(a) $a^2 = e$



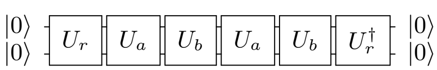
(b) $b^2 = e$



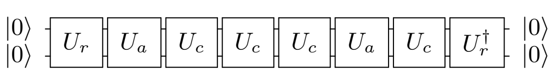
(c) $c^4 = e$



(d) $(bc)^2 = e$



(e) $(ab)^2 = e$



(f) $ac^3ac = e$

FIG. 2: Six variational circuits corresponding to six relations defined in Eq. (1). Notice that only 2 qubits are physically needed instead of 12 since the classical optimization of each loop is performed after the measurement of the quantum state.

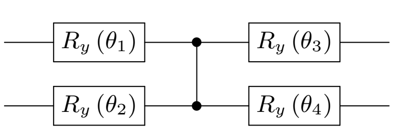
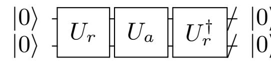
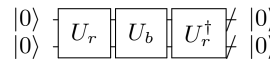


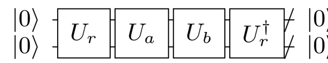
FIG. 3: Variational ansatz



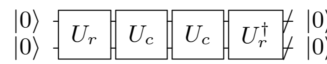
(a) $a \neq e$



(b) $b \neq e$



(c) $ab \neq e$



(d) $c^2 \neq e$

FIG. 4: Four circuits to verify that the quantum representation is faithful.

	θ_1	θ_2	θ_3	θ_4
U_a	5.84551571	-8.06312288	-12.12870095	-17.06961836
U_b	16.17214308	13.9280257	-9.88895781	4.92153015
U_c	13.69949734	-8.06312283	2.00846589	33.19586423

TABLE I: One numerical outcome of the VQA. U_r is also prepared with the ansatz in FIG. 3 with random parameters. Our readers can try a more general U_r. The classical optimizer is the genetic algorithm *CMA* [8].

Furthermore, by studying the final values of the VQA, we can construct an analytical relation between the parameters as shown in TABLE II. It is not evident to deduce such an interpretation for a more complex group. However, we can settle certain parameters and train the circuits iteratively. For example, in TABLE. I, θ_2 for U_a and U_c are approximate, then they can be treated as the same parameter for the second training. Likewise, θ_4 for U_a and U_c have a difference of 16π, which can be considered the same for the next iteration. Although it is complicated to expand the full matrices, the numerical simulation demonstrates that this analytical relation satisfies the circuits in FIG. 2 and FIG. 4 with one arbitrary parameter ϕ. By applying $\phi = 0$, the full unitary matrices to represent a, b and c are given in Eq. (11 - 13). It can be verified numerically that they form a faithful representation of $C_2 \times D_4$ [7]. In this way, classical representations of the finite group can be reconstructed from the output of the VQA.

	θ_1	θ_2	θ_3	θ_4
U_a	$3\pi-\phi$	3ϕ	$\phi-\pi$	$4\pi-3\phi$
U_b	ϕ	$3\phi-\pi$	$2\pi-\phi$	$3\pi-3\phi$
U_c	$\frac{\pi-2\phi}{2}$	3ϕ	$\frac{\pi+2\phi}{2}$	$4\pi-3\phi$

TABLE II: One possible analytical interpretation of the numerical result.

$$U_{\phi=0}\left(a\right)=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\tag{11}$$

$$U_{\phi=0}\left(b\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\tag{12}$$

$$U_{\phi=0}\left(c\right)=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&-1\end{pmatrix}\tag{13}$$

[1] Marco Cerezo, Andrew Arrasmith, Ryan Babbush, Simon C Benjamin, Suguru Endo, Keisuke Fujii, Jarrod R McClean, Kosuke Mitarai, Xiao Yuan, Lukasz Cincio, et al. Variational quantum algorithms. *Nature Reviews Physics*, 3(9):625–644, 2021.

[2] Tim Dokchitser. https://people.maths.bris.ac.uk/ matyd/GroupNames/index.html.

[3] Carlos Bravo-Prieto, Josep Lumbreras-Zarapico, Luca Tagliacozzo, and José I Latorre. Scaling of variational quantum circuit depth for condensed matter systems. *Quantum*, 4:272, 2020.

[4] Jonathan Romero, Jonathan P Olson, and Alan Aspuru-Guzik. Quantum autoencoders for efficient compression of quantum data. *Quantum Science and Technology*, 2(4):045001, 2017.

[5] Carlos Bravo-Prieto. Quantum autoencoders with enhanced data encoding. *Machine Learning: Science and Technology*, 2(3):035028, 2021.

[6] Stavros Efthymiou, Sergi Ramos-Calderer, Carlos Bravo-Prieto, Adrián Pérez-Salinas, Diego García-Martín, Artur Garcia-Saez, José Ignacio Latorre, and Stefano Carrazza. Qibo: a framework for quantum simulation with hardware acceleration. *Quantum Science and Technology*, 7(1):015018, dec 2021.

[7] Ruge Lin. https://github.com/GoGoKo699/group-to-circuits.

[8] Nikolaus Hansen, yoshihikoueno, ARF1, Kento Nozawa, Luca Rolshoven, Matthew Chan, Youhei Akimoto, brieglhostis, and Dimo Brockhoff. Cma-es/pycma: r3.2.0, February 2022.