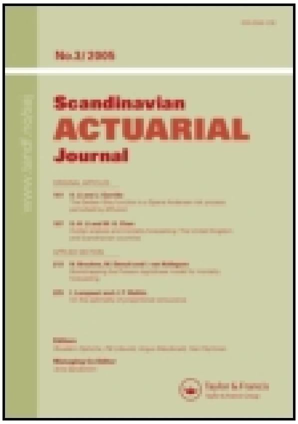

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Original Article

An extension of the Whittaker–Henderson method of graduation

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We present an outline and historical summary of the Whittaker–Henderson method of graduation (or data smoothing), together with an extension of the method in which the graduated values are obtained by minimising a Whittaker–Henderson criterion subject to constraints. Examples are given, using data for the global average temperature anomaly and for a set of share prices, in which the proposed method appears to give good results.

Keywords: Time series; Data smoothing; Life tables; Hodrick–Prescott filter; Global average temperature anomaly

1. Introduction

Many techniques for smoothing data have been developed. We consider in this paper the Whittaker–Henderson method of graduation, which is frequently used in practice, especially by North American actuaries for the construction of life tables. The method originates in the work of Bohlmann (1899) and Whittaker (1923), and contributions to the theory were made by Henderson (1924, 1925) and Aitken (1925), among others.

We consider a sequence of n observations $u_1, u_2, \ldots, u_n$, which are usually indexed by time or age. The problem of graduation is to find a corresponding sequence of graduated values $v_1, v_2, \ldots, v_n$, which is believed to be a better representation of the underlying unknown true values. The Whittaker–Henderson method gives the graduated values by minimising the quantity:

$$Q = \sum_{i=1}^{n} (u_i - v_i)^2 + \lambda \sum_{i=1}^{n-3} (\Delta^3 v_i)^2 \tag{1.1}$$

where $\Sigma_{i=1}^{n} (u_i - v_i)^2$ is a measure of goodness of fit (or fidelity) to the original data and $\Sigma_{i=1}^{n-3} (\Delta^3 v_i)^2$ is a measure of smoothness. The parameter λ is a positive constant chosen to reflect the relative importance being placed on goodness of fit and smoothness. As $\lambda \to 0$

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the graduated values converge to the original data, and as $\lambda \to \infty$ the graduated values converge to a polynomial of degree 2, which has maximum smoothness.

More generally, one may minimise the quantity:

$$Q = \sum_{i=1}^{n} w_i(u_i - v_i)^2 + \lambda \sum_{i=1}^{n-p} (\Delta^p v_i)^2 \tag{1.2}$$

where $\{w_i\}$ are positive weights associated with the original data and p is a positive integer (with $p < n$). In the present article, we shall assume that $w_i = 1$ for all i.

The case $p = 2$ is also called the Hodrick–Prescott filter; see Hodrick & Prescott (1997), who used it on data subject to business cycles. According to Weinert (2007), the Hodrick–Prescott case ($p = 2$) is frequently used in practice.

We mention that the Whittaker–Henderson method is, of course, one of many smoothing methods. Other approaches include Autoregressive Moving Average (ARMA) and Autoregressive Integrated Moving Average (ARIMA) techniques, adjusted averages (see Borgan (1979)), mathematical formulae (see Forfar *et al.* (1988)), splines (see McCutcheon & Eilbeck (1977)) and Bayesian methods (see Dellaportas *et al.* (2001)). A Kalman filter approach (see Kalman (1960), Durbin & Koopman (2001)) may also be used if the underlying rates follow a Markov process with a known generator; if this is unknown it may be possible to use an adaptive filter, but we shall not go into this topic in the present article.

2. Historical outline

The Whittaker–Henderson method of graduation has its origins in the work of Bohlmann (1899), who provided a solution for the case $p = 1$. The method is conventionally credited to Whittaker and Henderson, although from a historical point of view it may be argued that it should be called the Bohlmann–Whittaker–Henderson method.

Whittaker (1923) gave an approximate solution for $p = 3$, with normal equations treated as a sixth-order difference equation with six boundary conditions (see Whittaker (1923), Whittaker & Robinson (1924) for further details).

Whittaker noted that:

(1) the value of λ should be chosen according to the relative importance of goodness of fit and smoothness;
(2) unlike the situation for adjusted-average graduation (see, for example, Herbert & Scott (2006)), there is no 'problem of the tails' to be dealt with; and
(3) the method ensures that there is no distortion of the data if u_i is a polynomial of degree not exceeding 2. (This is the Conservation of Moments Theorem, discussed below.)

Henderson (1924) developed Whittaker's method by solving the difference equation (for $p = 1, 2, 3$), ignoring the boundary conditions and compensating for them later. Aitken (1925) gave an exact solution to the difference equations with the boundary conditions. The work of Henderson and Aitken led to formulae used much in practice, and an account of the method is given in a paper by Joseph (1952). Henderson (1925) used a matrix approach, much used and extended by recent writers.

There are many generalisations and extensions of the Whittaker–Henderson method of graduation in the literature. Chan *et al*. (1984, 1986) show that the graduation problem can be formulated as a linear or quadratic programming problem in some cases. Lowrie (1993) presents an extension of the Whittaker–Henderson method to graduation with constraints and mixed differences, and Verall (1993) describes a Bayesian and state space formulation. Extensions of the Whittaker–Henderson method to cover the graduation of multi-dimensional data are considered by Knorr (1984), Lowrie (1993) and Dermoune *et al*. (2009).

3. Solution of the problem using matrices

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^T$. It may be easily shown that:

$$\Delta^p v_i = \sum_{j=i}^{i+p} (-1)^{p+j-i} \binom{p}{j-i} v_j \qquad (1 \le i \le n-p)$$

Hence,

$$\Delta^p v_i = \sum_{j=i}^{n} K_{ij} v_j \qquad (1 \le i \le n-p)$$

where K is the $(n-p) \times n$ differencing matrix defined by

$$K_{ij} = \begin{cases} (-1)^{p+j-i} \binom{p}{j-i} & \text{for } i = 1, 2, \ldots, p, \; j = i, i+1, \ldots, i+p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the Whittaker–Henderson criterion is:

$$Q = (\boldsymbol{v} - \boldsymbol{u})^T \, (\boldsymbol{v} - \boldsymbol{u}) + \lambda (K\boldsymbol{v})^T \, (K\boldsymbol{v}) \tag{3.1}$$

Let $B = I + \lambda K^T K$, which is a symmetric and positive-definite matrix.

B is invertible since it is positive-definite, and hence

$$Q = (\boldsymbol{v} - B^{-1}\boldsymbol{u})^T B (\boldsymbol{v} - B^{-1}\boldsymbol{u}) + c \tag{3.2}$$

where c is a constant. The minimum occurs when

$$\boldsymbol{v} = B^{-1}\boldsymbol{u}. \tag{3.3}$$

This result is, of course, well known (see Greville (1974) for an algebraic proof).

4. Numerical aspects and the choice of λ

Henderson (1925) was the first to solve the problem using matrices. His idea was to decompose the matrix B into LDL^T where L is a lower-triangular matrix and D is a positive diagonal matrix. This decomposition is, in fact, very efficient although the calculations are laborious when done without a computer for large n and many values of λ. Further information on this method is given by Martin *et al.* (1965).

Chan *et al.* (1982) gave proofs of some properties of the solution using matrix notation. They also mentioned that the computation of the optimal solution can be time consuming, especially when large matrices are considered.

MacLeod (1989) showed that the number of operations needed to evaluate $\boldsymbol{v}$ is proportional to n rather than n^3, as had been previously thought.

Weinert (2007) presented numerically efficient algorithms for the case $p = 2$.

The generalised cross-validation (GCV) method, introduced by Craven & Wahba (1979) for spline smoothing, was adapted by Brooks *et al.* (1988) for use in Whittaker–Henderson graduation. According to this method, λ is chosen to minimise the GCV score, i.e.:

$$\mathrm{GCV} = \frac{1}{n} \cdot \frac{(\boldsymbol{u} - \boldsymbol{v})^T (\boldsymbol{u} - \boldsymbol{v})}{(1 - \mathrm{Tr}(B^{-1})/n)^2} \tag{4.1}$$

where Tr denotes the matrix trace.

Weinert (2007) proposed an algorithm to compute the GCV score in MATLAB. Garcia (2010) recently provided an algorithm, also written in MATLAB, that does not involve the determination of B at each iterative step of the minimisation, thus reducing the time needed for the computation.

We remark that the GCV method does not always give a unique solution in the practical range of values of λ.

5. The theorem of conservation of moments

Whittaker (1923) proved that, for $p = 3$, the sums of the u_i's and their first and second moments are equal the sums of the v_i's and their first and second moments, respectively.

This result is called the Theorem of Conservation of Moments, and we shall give a proof for general p. (It is not claimed that this result is original.)

Lemma 5.1 *For fixed p (where $p < n$), the formula*

$$\sum_{k=0}^{p} (-1)^k \binom{p}{k} k^m = 0 \tag{5.1}$$

is true for $m = 0, 1, 2, \ldots, p-1$.

Proof. This result is true for $m = 0$ by the binomial theorem.

Now,

$$(1-x)^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k} x^k \text{ (for all } x) \ .$$

Differentiate m times (where $m \le p-1$) to obtain

$$(-1)^m p(p-1)\dots(p-m+1)(1-x)^{p-m} = \sum_{k=m}^{p} (-1)^k \binom{p}{k} x^{k-m} k(k-1)\dots(k-m+1).$$

On setting $x=1$, we have

$$0 = \sum_{k=0}^{p} (-1)^k \binom{p}{k} k(k-1)\dots(k-m+1) \text{ (for } 1 \le m \le p-1) \ . \tag{5.2}$$

Suppose that Formula (5.1) is true for m up to $t-1$. Formula (5.2) shows that

$$\begin{aligned} 0 &= \sum_{k=0}^{p} (-1)^k \binom{p}{k} [k^t + \text{ terms with powers up to } t-1] \\ &= \sum_{k=0}^{p} (-1)^k \binom{p}{k} k^t + 0. \end{aligned}$$

Thus Formula (5.1) is true for m up to t and result is true by (finite) induction. □

THEOREM 5.2 (*Theorem of Conservation of Moments*)
The moments of orders 0, 1, 2, ..., $p-1$ *are the same for graduated values as for the original data. That is,*

$$H\boldsymbol{u} = H\boldsymbol{v} \tag{5.3}$$

where

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1^1 & 2^1 & \cdots & n^1 \\ \vdots & \vdots & & \vdots \\ 1^{p-1} & 2^{p-1} & \cdots & n^{p-1} \end{bmatrix}.$$

Proof. We need to show that

$$H\boldsymbol{u} = H\boldsymbol{v}$$

which is true if $H = HB^{-1}$ and thus if $H = HB$. It is therefore sufficient to show that $HK^T = 0$.

For $i = 1, 2, \dots, p$, $j = 1, 2, \dots, p$, the (i, j)th element of HK^T is

$$\begin{aligned} &\sum_{r=j}^{j+p} r^{i-1} (-1)^{p+r-j} \binom{p}{r-j} \\ &= \sum_{k=0}^{p} (k+j)^{i-1} (-1)^{p+k} \binom{p}{k}. \end{aligned}$$

On expanding $(k+j)^{i-1}$ by the binomial theorem and applying Lemma (5.1), we see that the (i, j)th element of HK^T is zero. □

6. Whittaker–Henderson graduation with side conditions

Using Theorem 5.2, we see that if u_i is a polynomial of degree $\leq p-1$, the graduation has no effect.

We now consider how to extend this property: in particular, we might wish the graduation to have no effect when u_i is a polynomial of degree p. This leads to the following extended version of the Whittaker–Henderson method of graduation.

THEOREM 6.1 *Let H be an $a \times n$ matrix ($a \leq n$) of full rank. The vector $\boldsymbol{v}^*$ which minimises $Q = (\boldsymbol{v} - B^{-1}\boldsymbol{u})^T B(\boldsymbol{v} - B^{-1}\boldsymbol{u})$ subject to the side condition,*

$$H\boldsymbol{v} = H\boldsymbol{u} \tag{6.1}$$

is

$$\boldsymbol{v}^* = B^{-1}H^T(HB^{-1}H^T)^{-1}H(I - B^{-1})\boldsymbol{u} + B^{-1}\boldsymbol{u}. \tag{6.2}$$

Proof. Let $\boldsymbol{r} = \boldsymbol{v} - B^{-1}\boldsymbol{u}$. We must minimise $\boldsymbol{r}^T B\boldsymbol{r}$ subject to $H\boldsymbol{r} = H(I - B^{-1})\boldsymbol{u}$.

The solution of this problem is

$$\boldsymbol{r}^* = B^{-1}H^T(HB^{-1}H^T)^{-1}H(I - B^{-1})\boldsymbol{u}$$

(see Borgan (1979), Herbert & Scott (2006) for details).

Hence,

$$\boldsymbol{v}^* = B^{-1}H^T(HB^{-1}H^T)^{-1}H(I - B^{-1})\boldsymbol{u} + B^{-1}\boldsymbol{u}. \qquad \square$$

Application Let $p = 3$ and choose H to be

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1^1 & 2^1 & \cdots & n^1 \\ \vdots & \vdots & & \vdots \\ 1^3 & 2^3 & \cdots & n^3 \end{bmatrix}.$$

The Whittaker–Henderson graduation with side conditions will ensure that the moments of the u_i's and v_i's agree up to order 3.

Of course, we could also ensure that this condition holds by using $p = 4$ rather than $p = 3$, but this would alter the smoothness criterion.

In the next section, we shall give two examples of the extended Whittaker–Henderson method. It was decided to use the same value of λ for the extended Whittaker–Henderson method as for the original method (with the same value of p).

7. Illustrative examples

EXAMPLE 1

Data *The global average temperature anomaly (in hundredths of* 1°C*) for the years 1989–2009 inclusive, as given by the Met. Office Hadley Centre (Kennedy 2010).*

The parameter λ was chosen to minimise the GCV score, and it was found that $\lambda = 97$ for $p = 2$ and $\lambda = 1160$ for $p = 3$.

Table 1 presents the values of the raw data (u), values graduated by the Whittaker–Henderson method for $p = 2$ (v_2), values graduated by the extended Whittaker–Henderson method with $p = 2$ (v_2^*), and values graduated by the Whittaker–Henderson method for $p = 3$ (v_3). Tables of differences are also provided. All calculations were carried out in Maple® Version 12.02.

Figure 1 gives a graphical representation of u, v_2, v_2^* and v_3.

EXAMPLE 2

Data *Monthly Prices (in \$) of a share in eBay Inc. on the NASDAQ Stock Market from January 2009 to August 2010 inclusive, as given in Yahoo! Finance UK web page (2010).*

We chose $p = 2$. The parameter λ was chosen to minimise the GCV score, and it was found that the optimal value is $\lambda = 30$.

Table 2 presents the values of the raw data (u), values graduated by the Whittaker–Henderson method for $p = 2$ (v_2), and values graduated by the extended Whittaker–Henderson method with $p = 2$ (v_2^*). Tables of second differences are also provided.

Figure 2 gives a graphical representation of u, v_2 and v_2^*.

8. Conclusions

We have summarised the theory of the Whittaker–Henderson method of data smoothing and have put forward an extension of the method. The extended method of graduation

Table 1. Values of the original data and graduated values for the global average temperature anomaly, 1989–2009.

Year	u	Δ^3	v_2	Δ^2	v_2^*	Δ^2	v_3	Δ^3
1989	9.5	11.3	11.501173	−0.020631	10.636881	−0.023728	11.716749	0.001911
1990	24.8	27.5	13.225630	0.078062	12.579208	0.070125	12.949862	−0.004483
1991	19.8	−16.8	14.929456	0.226967	14.497808	0.213458	14.462738	−0.023782
1992	5.8	3.1	16.711345	0.263383	16.486532	0.244282	16.257289	−0.046972
1993	10.3	−30.9	18.720200	0.212993	18.688715	0.188751	18.329033	−0.067132
1994	16.5	64.4	20.992439	0.116290	21.135180	0.087647	20.654188	−0.080679
1995	27.5	−45.4	23.477671	0.061054	23.770396	0.028910	23.185780	−0.091334
1996	12.4	−34.4	26.079192	−0.135205	26.493258	−0.169872	25.856679	−0.087495
1997	35.6	64.5	28.741768	−0.260761	29.245031	−0.296946	28.586205	−0.075210
1998	51.7	−4.6	31.269138	−0.175689	31.826932	−0.212380	31.283025	−0.072078
1999	26.3	−28.7	33.535747	−0.165212	34.111888	−0.201397	33.859642	−0.071583
2000	23.9	4.9	35.626669	−0.275629	36.184463	−0.310295	36.240847	−0.063087
2001	39.9	2.3	37.552378	−0.361843	38.055641	−0.393986	38.354562	−0.047920
2002	45.6	10.2	39.202459	−0.382103	39.616525	−0.410747	40.129210	−0.030801
2003	45.9	−16.1	40.490696	−0.346598	40.783421	−0.370840	41.501689	−0.015520
2004	43.1	−16.1	41.396831	−0.293534	41.539572	−0.312635	42.424094	−0.002660
2005	47.4	−8.7	41.956367	−0.184350	41.924882	−0.197858	42.865619	0.003870
2006	42.7	28.8	42.222370	−0.070242	41.997557	−0.078179	42.810745	0.004166
2007	40.2		42.304022	0.022175	41.872374	0.019077	42.256812	
2008	31.2		42.315433		41.669012		41.207690	
2009	44.5		42.349018		41.484727		39.667546	

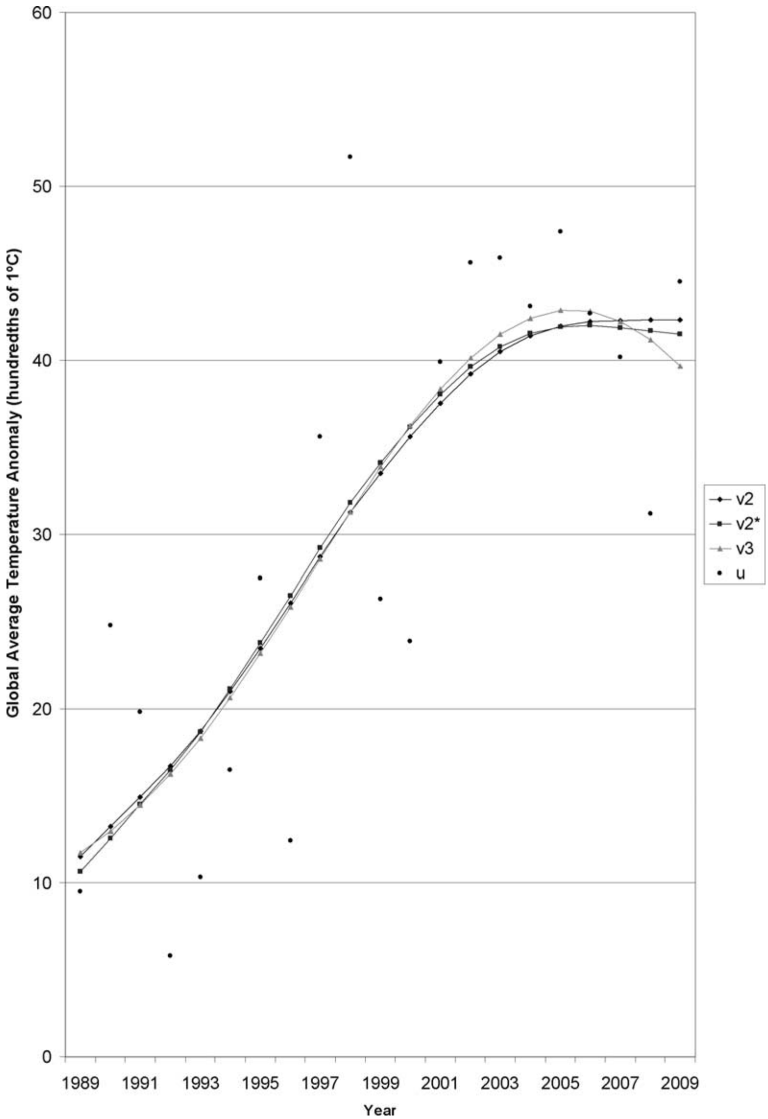

Figure 1. Graphical representation of the original and graduated values for the global average temperature anomaly.

appears to be successful in two practical applications, giving results approximately between those for the original Whittaker–Henderson method for $p=2$ and $p=3$ in Example 1, and what appears to provide a better fitted curve in Example 2. Further applications of the extended Whittaker–Henderson method may be given in a future article.

Table 2. Values of the original data and graduated values for the monthly prices per eBay share from January 2009 to August 2010.

Month/year	u	Δ^2	v_2	Δ^2	v_2^*	Δ^2
01/2009	12.02	2.84	10.897828	0.037406	10.548786	0.035473
02/2009	10.87	2.22	12.458255	0.021870	12.203814	0.017203
03/2009	12.56	−2.76	14.056089	−0.043536	13.894316	−0.051020
04/2009	16.47	−1.64	15.675791	−0.082468	15.602020	−0.092451
05/2009	17.62	4.61	17.251958	−0.109132	17.258705	−0.121125
06/2009	17.13	−3.23	18.745657	−0.189652	18.822939	−0.203136
07/2009	21.25	0.57	20.130223	−0.232845	20.266048	−0.247351
08/2009	22.14	−2.79	21.325138	−0.248876	21.506021	−0.264006
09/2009	23.6	3.53	22.287208	−0.221148	22.498643	−0.236570
10/2009	22.27	−3.14	23.000402	−0.217766	23.227259	−0.233189
11/2009	24.47	0.43	23.492448	−0.181799	23.719305	−0.196929
12/2009	23.53	0.51	23.766727	−0.153724	23.978162	−0.168229
01/2010	23.02	3.95	23.859208	−0.153621	24.040090	−0.167106
02/2010	23.02	−7.14	23.797964	−0.179451	23.933789	−0.191444
03/2010	26.97	0.52	23.583099	−0.092384	23.660381	−0.102367
04/2010	23.78	0.57	23.188783	0.014390	23.195530	0.006906
05/2010	21.41	3.1	22.702083	0.078094	22.628311	0.073427
06/2010	19.61	0.92	22.229772	0.054473	22.067999	0.052540
07/2010	20.91		21.835555		21.581114	
08/2010	23.13		21.495811		21.146769	

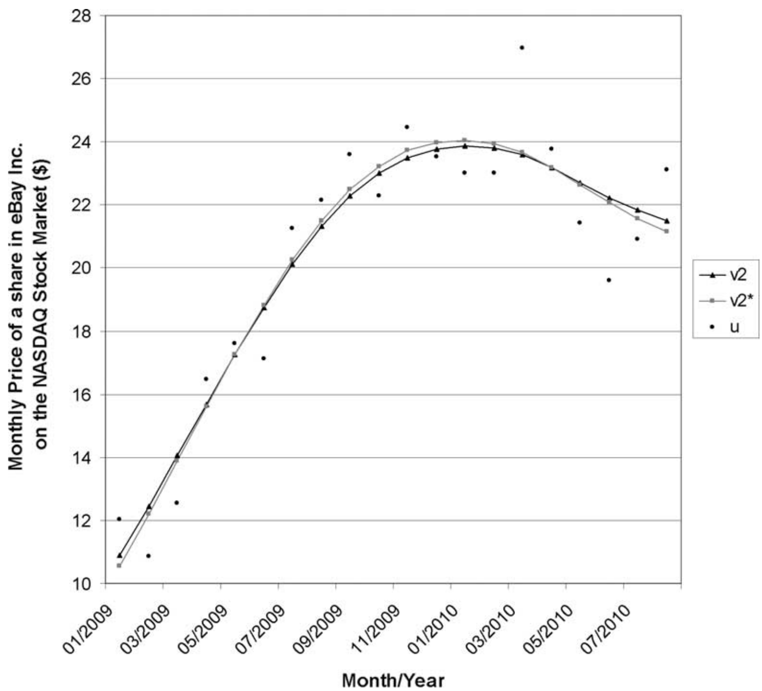

Figure 2. Graphical representation of the original and graduated values of the monthly eBay share price.

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