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## 3.2 BINARY SEARCH TREES

---

- ▶ *BSTs*
- ▶ *ordered operations*
- ▶ *deletion*



## 3.2 BINARY SEARCH TREES

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- ▶ *BSTs*
- ▶ *ordered operations*
- ▶ *deletion*

# Binary search trees

**Definition.** A BST is a **binary tree** in **symmetric order**.

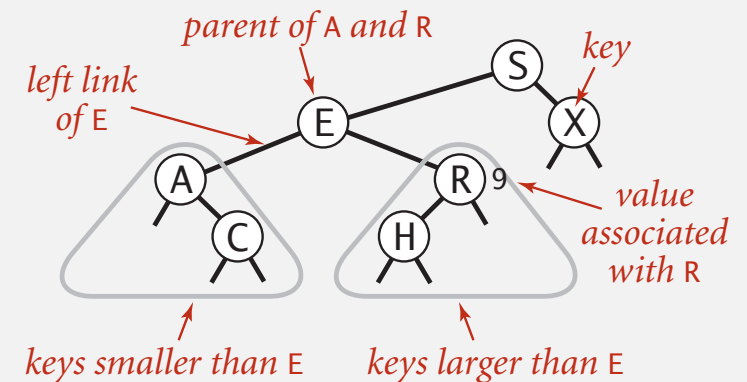
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



**Symmetric order.** Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



# BST representation in Java

**Java definition.** A BST is a reference to a root Node.

A Node is comprised of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

↑ smaller keys      ↑ larger keys

```
private class Node
{
    private Key key;
    private Value val;
    private Node left, right;
    public Node(Key key, Value val)
    {
        this.key = key;
        this.val = val;
    }
}
```

Key and Value are generic types; Key is Comparable



## BST implementation (skeleton)

---

```
public class BST<Key extends Comparable<Key>, Value>
{
```

```
    private Node root;
```

← root of BST

```
    private class Node
    { /* see previous slide */ }
```

```
    public void put(Key key, Value val)
    { /* see next slides */ }
```

```
    public Value get(Key key)
    { /* see next slides */ }
```

```
    public void delete(Key key)
    { /* see next slides */ }
```

```
    public Iterable<Key> iterator()
    { /* see next slides */ }
```

```
}
```

# Binary search tree demo

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**Search.** If less, go left; if greater, go right; if equal, search hit.

**successful search for H**

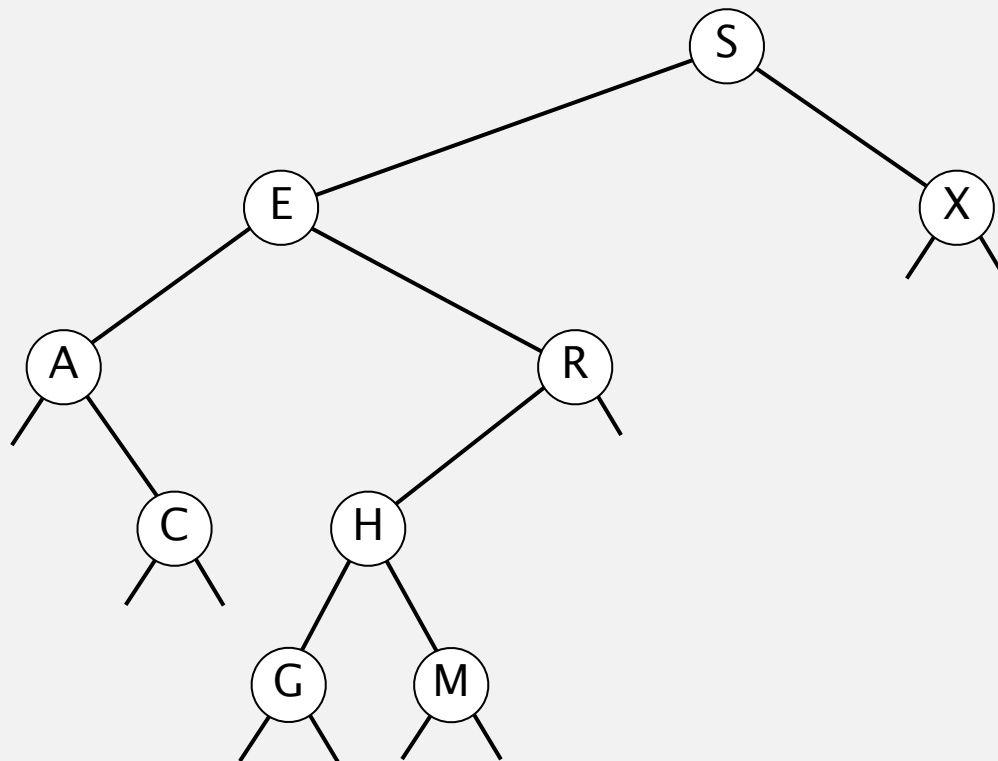


# Binary search tree demo

---

**Insert.** If less, go left; if greater, go right; if null, insert.

**insert G**



## BST search: Java implementation

---

**Get.** Return value corresponding to given key, or null if no such key.

```
public Value get(Key key)
{
    Node x = root;
    while (x != null)
    {
        int cmp = key.compareTo(x.key);
        if (cmp < 0) x = x.left;
        else if (cmp > 0) x = x.right;
        else if (cmp == 0) return x.val;
    }
    return null;
}
```

**Cost.** Number of compares is equal to 1 + depth of node.



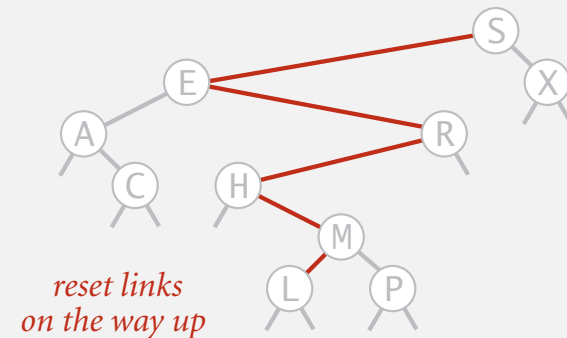
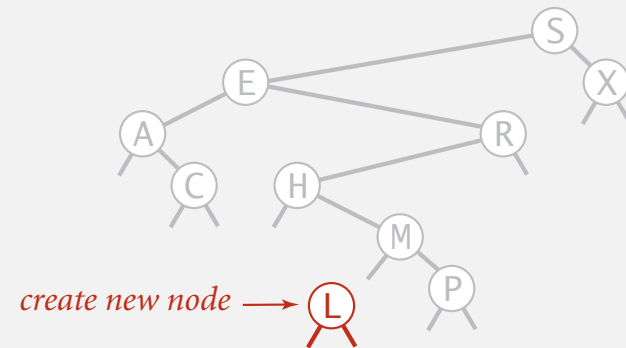
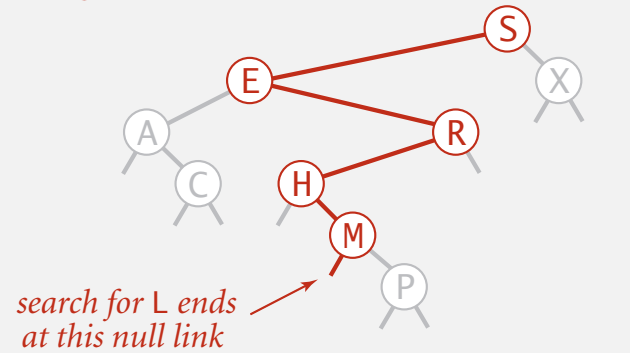
# BST insert

**Put.** Associate value with key.

Search for key, then two cases:

- Key in tree  $\Rightarrow$  reset value.
- Key not in tree  $\Rightarrow$  add new node.

inserting L



Insertion into a BST

## BST insert: Java implementation

---

**Put.** Associate value with key.

```
public void put(Key key, Value val)
{ root = put(root, key, val); }

private Node put(Node x, Key key, Value val)
{
    if (x == null) return new Node(key, val);
    int cmp = key.compareTo(x.key);
    if (cmp < 0)
        x.left = put(x.left, key, val);
    else if (cmp > 0)
        x.right = put(x.right, key, val);
    else if (cmp == 0)
        x.val = val;
    return x;
}
```

concise, but tricky,  
recursive code;  
read carefully!

**Cost.** Number of compares is equal to 1 + depth of node.

# Tree shape

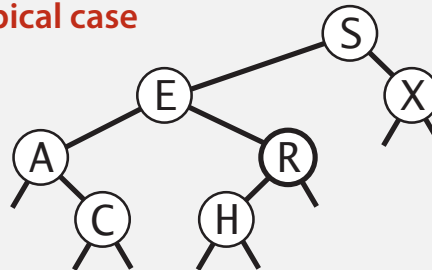
---

- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to  $1 + \text{depth of node}$ .

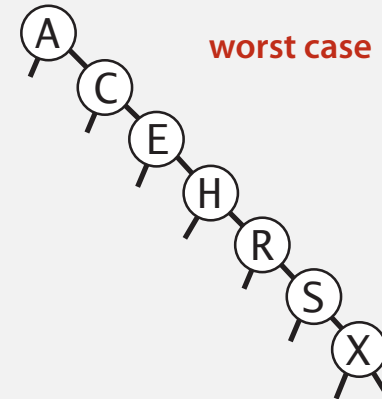
best case



typical case



worst case

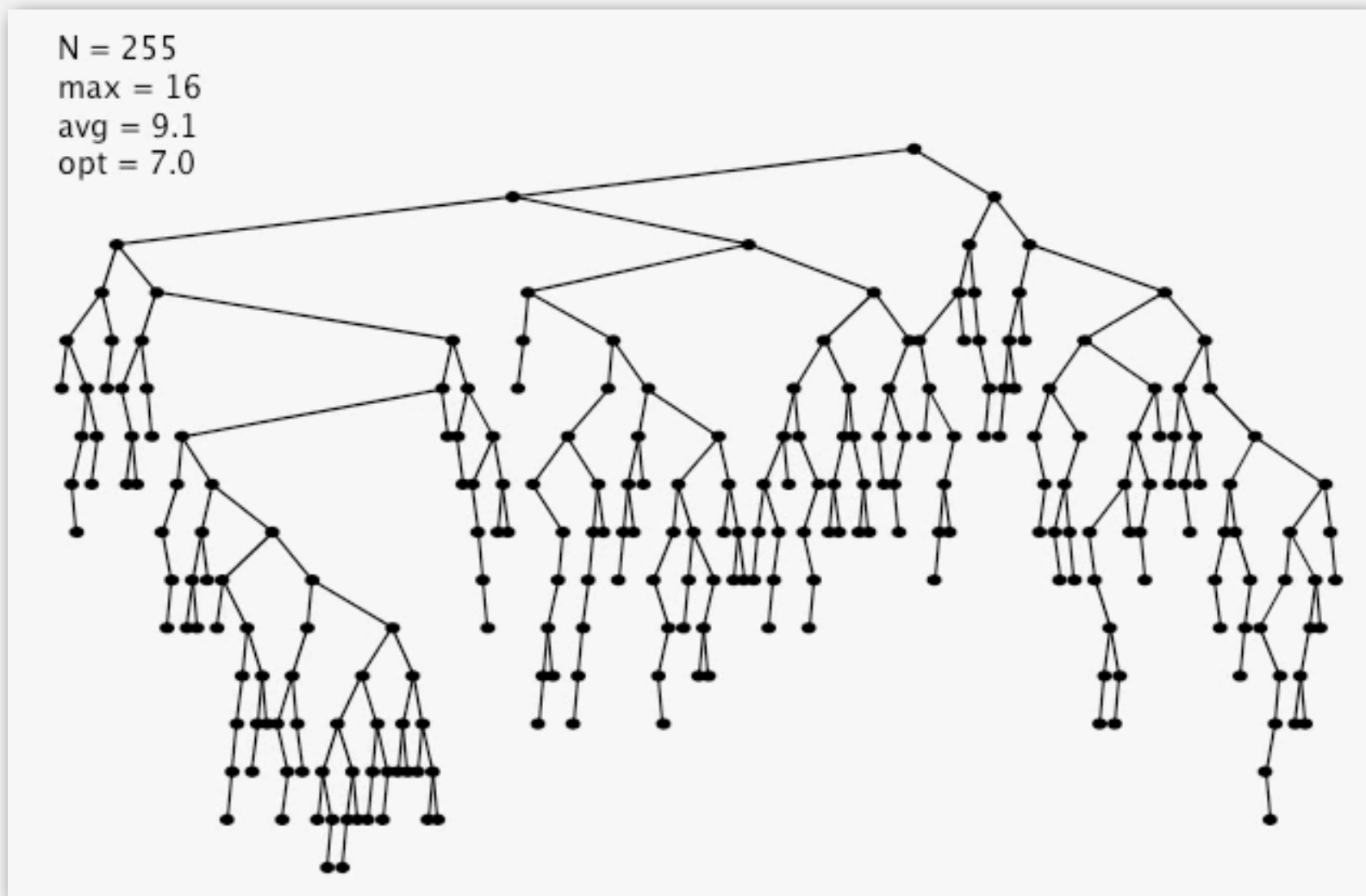


**Remark.** Tree shape depends on order of insertion.

## BST insertion: random order visualization

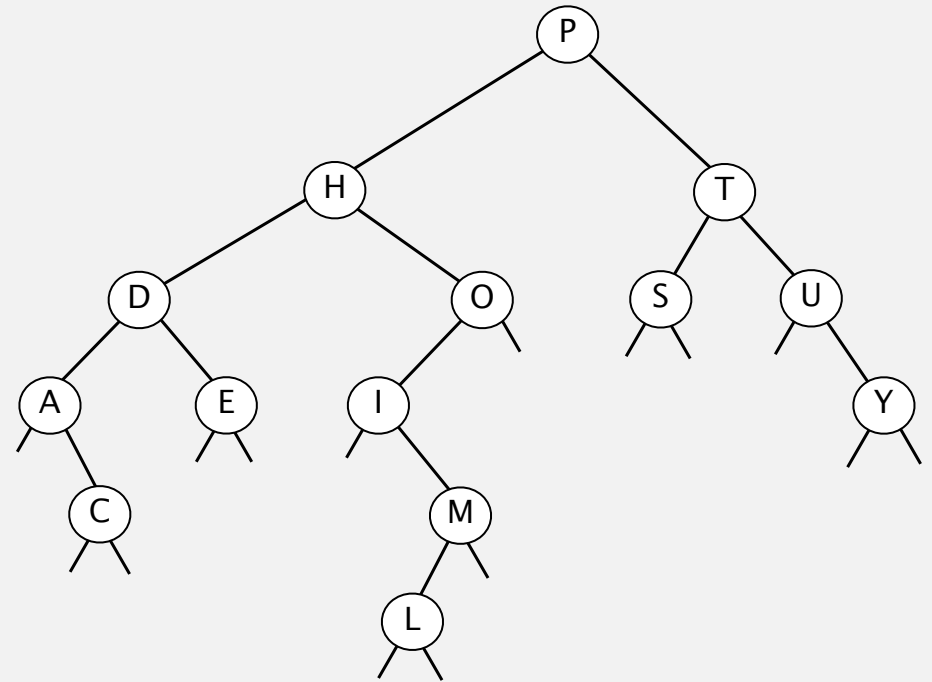
---

Ex. Insert keys in random order.



# Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



**Remark.** Correspondence is 1-1 if array has no duplicate keys.

## BSTs: mathematical analysis

---

**Proposition.** If  $N$  distinct keys are inserted into a BST in **random** order, the expected number of compares for a search/insert is  $\sim 2 \ln N$ .

**Pf.** 1-1 correspondence with quicksort partitioning.

**Proposition.** [Reed, 2003] If  $N$  distinct keys are inserted in random order, expected height of tree is  $\sim 4.311 \ln N$ .

### How Tall is a Tree?

Bruce Reed  
CNRS, Paris, France  
reed@moka.ccr.jussieu.fr

#### ABSTRACT

Let  $H_n$  be the height of a random binary search tree on  $n$  nodes. We show that there exists constants  $\alpha = 4.31107\dots$  and  $\beta = 1.95\dots$  such that  $\mathbf{E}(H_n) = \alpha \log n - \beta \log \log n + O(1)$ . We also show that  $\text{Var}(H_n) = O(1)$ .

**But...** Worst-case height is  $N$ .

(exponentially small chance when keys are inserted in random order)

## ST implementations: summary

---

implementation	guarantee		average case		ordered ops?	operations on keys
	search	insert	search hit	insert		
sequential search (unordered list)	N	N	N/2	N	no	equals()
binary search (ordered array)	lg N	N	lg N	N/2	yes	compareTo()
BST	N	N	1.39 lg N	1.39 lg N	next	compareTo()



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## 3.2 BINARY SEARCH TREES

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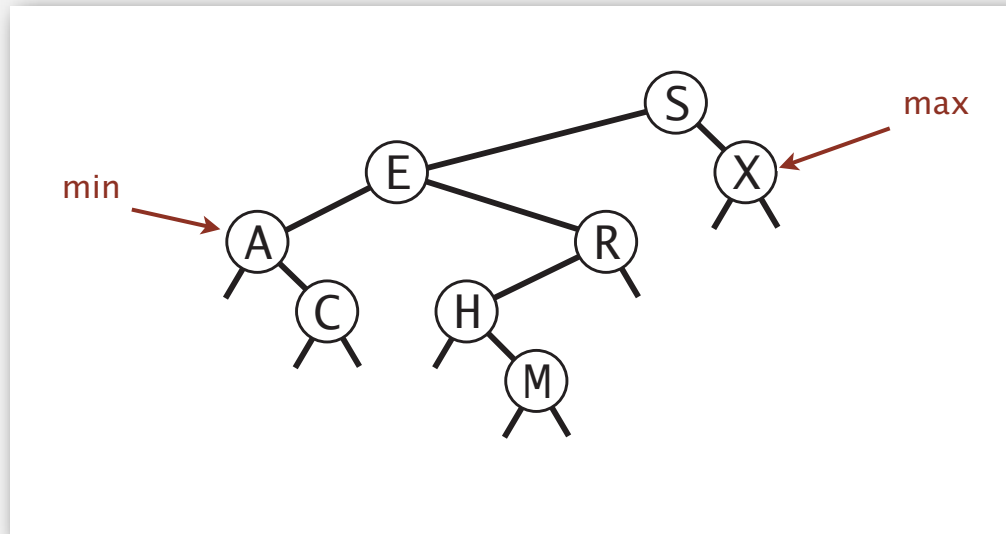
- ▶ *BSTs*
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# Minimum and maximum

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**Minimum.** Smallest key in table.

**Maximum.** Largest key in table.



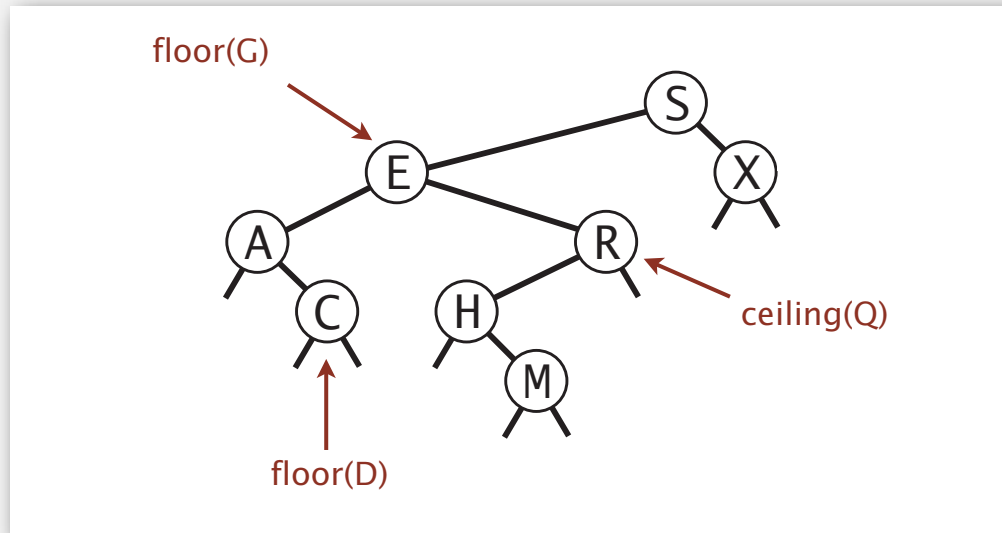
**Q.** How to find the min / max?

# Floor and ceiling

---

**Floor.** Largest key  $\leq$  a given key.

**Ceiling.** Smallest key  $\geq$  a given key.



**Q.** How to find the floor / ceiling?

# Computing the floor

**Case 1.** [ $k$  equals the key at root]

The floor of  $k$  is  $k$ .

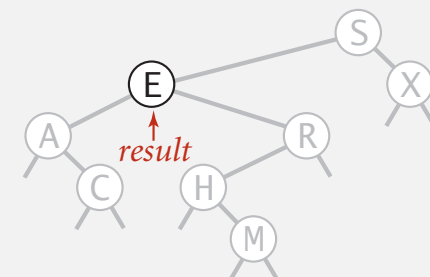
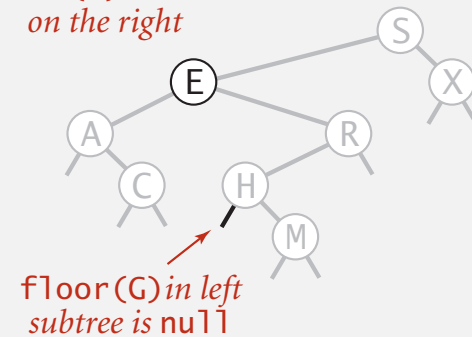
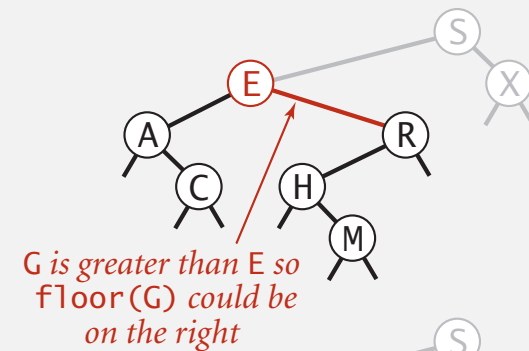
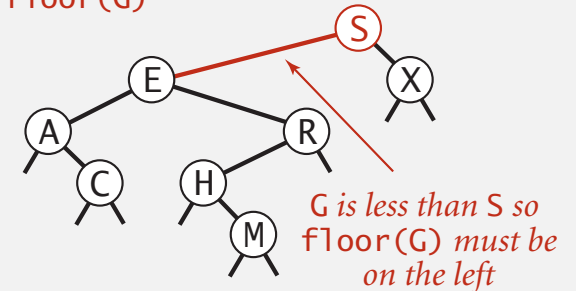
**Case 2.** [ $k$  is less than the key at root]

The floor of  $k$  is in the left subtree.

**Case 3.** [ $k$  is greater than the key at root]

The floor of  $k$  is in the right subtree  
(if there is any key  $\leq k$  in right subtree);  
otherwise it is the key in the root.

finding floor( $G$ )



# Computing the floor

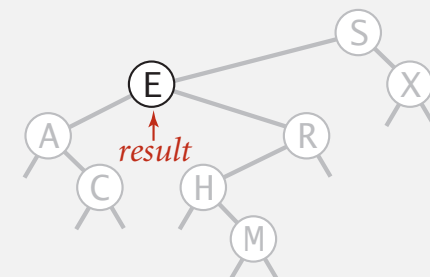
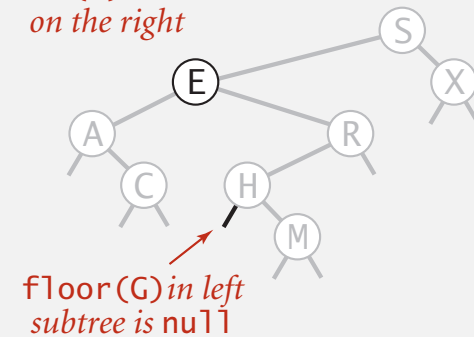
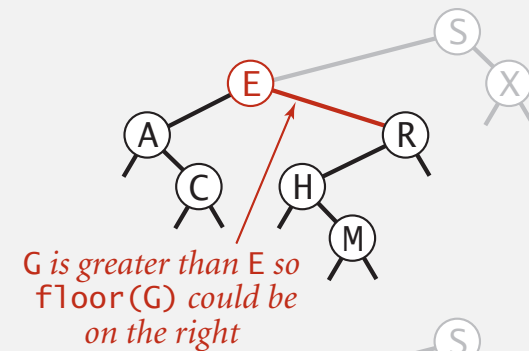
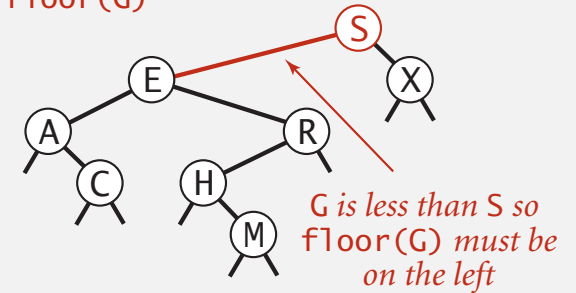
```
public Key floor(Key key)
{
    Node x = floor(root, key);
    if (x == null) return null;
    return x.key;
}
private Node floor(Node x, Key key)
{
    if (x == null) return null;
    int cmp = key.compareTo(x.key);

    if (cmp == 0) return x;

    if (cmp < 0) return floor(x.left, key);

    Node t = floor(x.right, key);
    if (t != null) return t;
    else return x;
}
```

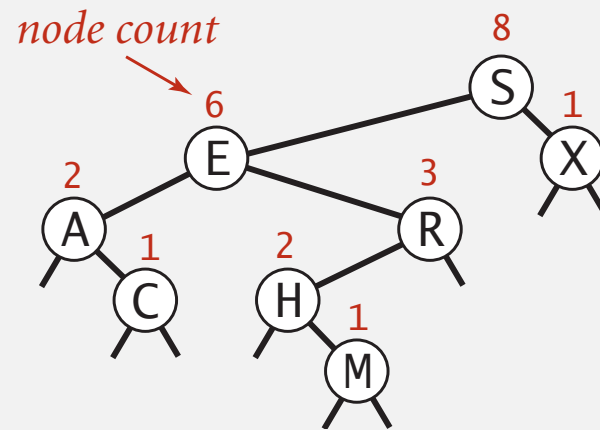
finding floor(G)



## Subtree counts

---


In each node, we store the number of nodes in the subtree rooted at that node; to implement `size()`, return the count at the root.



**Remark.** This facilitates efficient implementation of `rank()` and `select()`.

## BST implementation: subtree counts

```
private class Node
{
    private Key key;
    private Value val;
    private Node left;
    private Node right;
    private int count;
}
```




number of nodes in subtree

```
public int size()
{ return size(root); }
```

```
private int size(Node x)
{
    if (x == null) return 0;
    return x.count;
}
```

ok to call  
when x is null

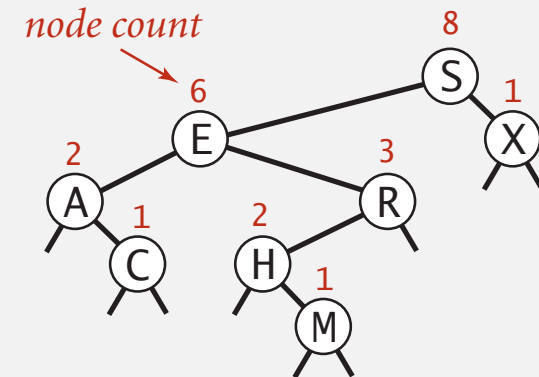


```
private Node put(Node x, Key key, Value val)
{
    if (x == null) return new Node(key, val, 1);
    int cmp = key.compareTo(x.key);
    if (cmp < 0) x.left = put(x.left, key, val);
    else if (cmp > 0) x.right = put(x.right, key, val);
    else if (cmp == 0) x.val = val;
    x.count = 1 + size(x.left) + size(x.right);
    return x;
}
```

# Rank

**Rank.** How many keys  $< k$ ?

Easy recursive algorithm (3 cases!)



```
public int rank(Key key)
{ return rank(key, root); }
```

```
private int rank(Key key, Node x)
{
```

```
    if (x == null) return 0;
    int cmp = key.compareTo(x.key);
    if (cmp < 0) return rank(key, x.left);
    else if (cmp > 0) return 1 + size(x.left) + rank(key, x.right);
    else if (cmp == 0) return size(x.left);
```

```
}
```

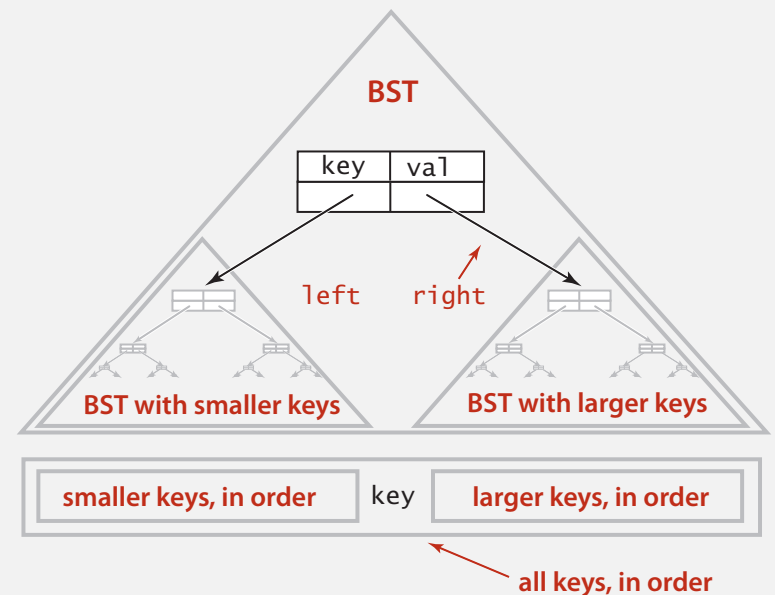


# Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
    if (x == null) return;
    inorder(x.left, q);
    q.enqueue(x.key);
    inorder(x.right, q);
}
```

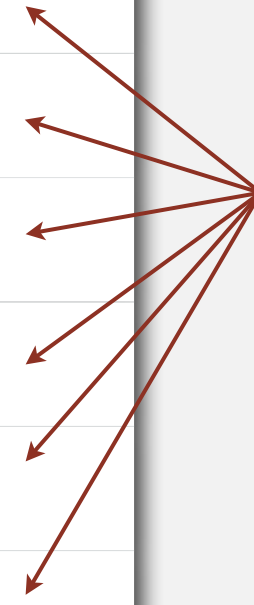


**Property.** Inorder traversal of a BST yields keys in ascending order.

# BST: ordered symbol table operations summary

---

	sequential search	binary search	BST
search	$N$	$\lg N$	$h$
insert	$N$	$N$	$h$
min / max	$N$	$1$	$h$
floor / ceiling	$N$	$\lg N$	$h$
rank	$N$	$\lg N$	$h$
select	$N$	$1$	$h$
ordered iteration	$N \log N$	$N$	$N$



$h$  = height of BST  
(proportional to  $\log N$   
if keys inserted in random order)

order of growth of running time of ordered symbol table operations



## 3.2 BINARY SEARCH TREES

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- ▶ *BSTs*
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## ST implementations: summary

---

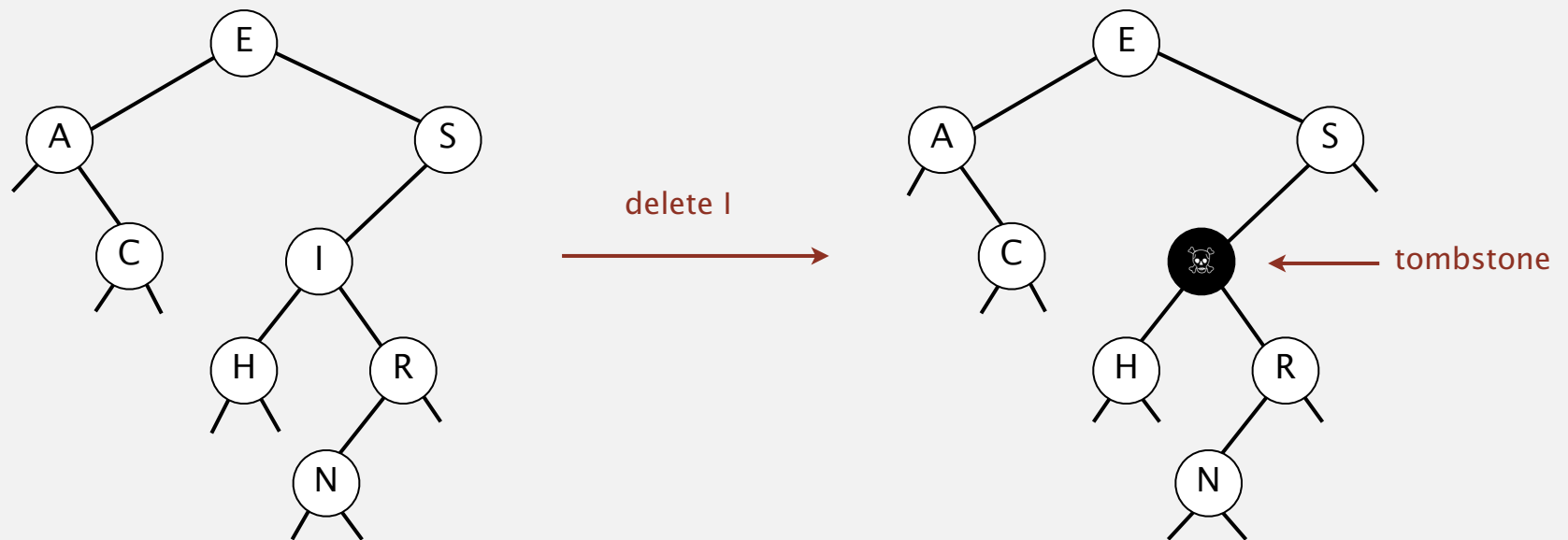
implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	equals()
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	compareTo()
BST	N	N	N	1.39 lg N	1.39 lg N	???	yes	compareTo()

Next. Deletion in BSTs.

# BST deletion: lazy approach

To remove a node with a given key:

- Set its value to null.
- Leave key in tree to guide search (but don't consider it equal in search).



**Cost.**  $\sim 2 \ln N'$  per insert, search, and delete (if keys in random order), where  $N'$  is the number of key-value pairs ever inserted in the BST.

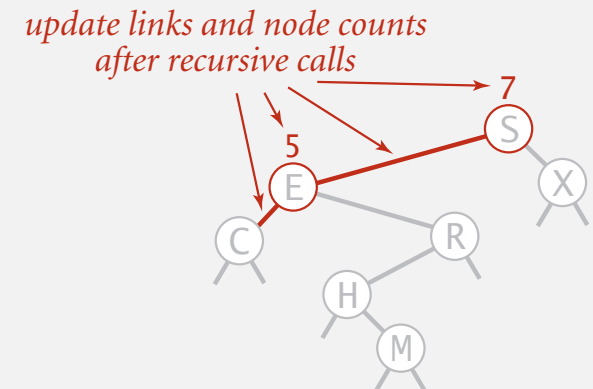
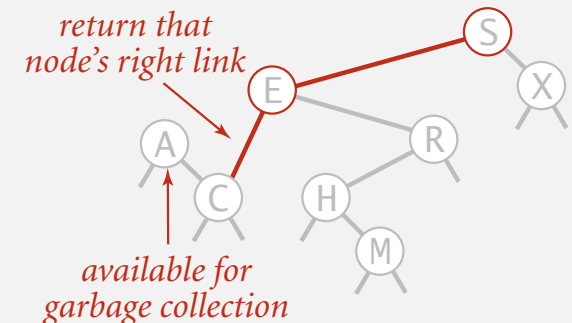
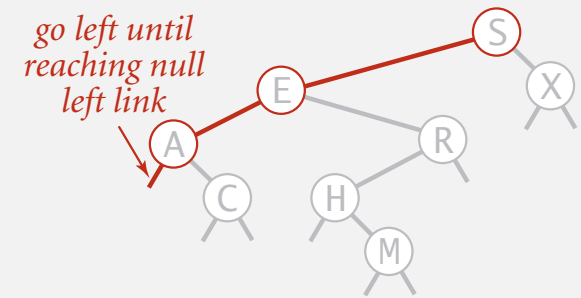
**Unsatisfactory solution.** Tombstone (memory) overload.

# Deleting the minimum

## To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

```
public void deleteMin()  
{ root = deleteMin(root); }  
  
private Node deleteMin(Node x)  
{  
    if (x.left == null) return x.right;  
    x.left = deleteMin(x.left);  
    x.count = 1 + size(x.left) + size(x.right);  
    return x;  
}
```

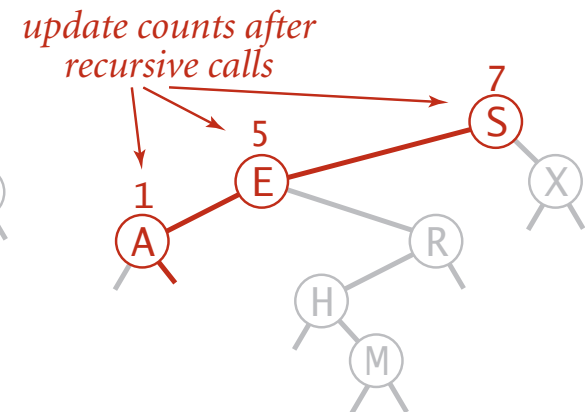
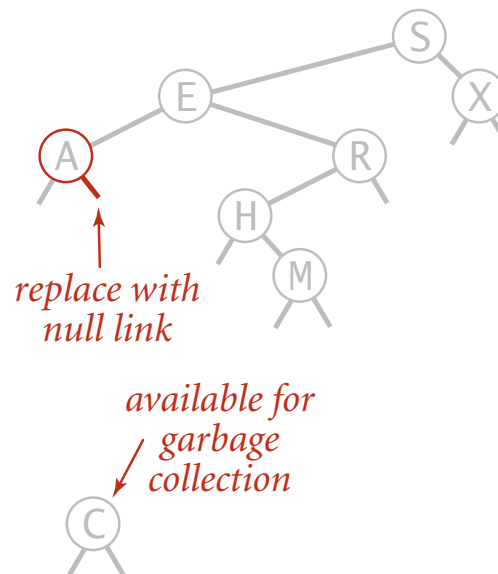
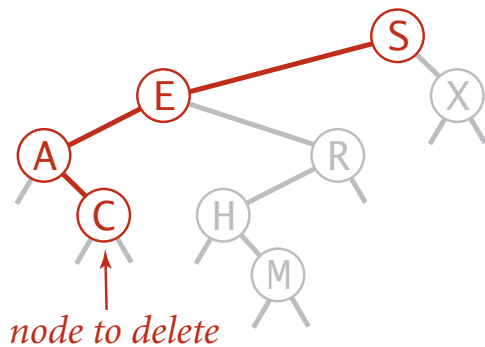


# Hibbard deletion

To delete a node with key  $k$ : search for node  $t$  containing key  $k$ .

**Case 0.** [0 children] Delete  $t$  by setting parent link to null.

deleting C



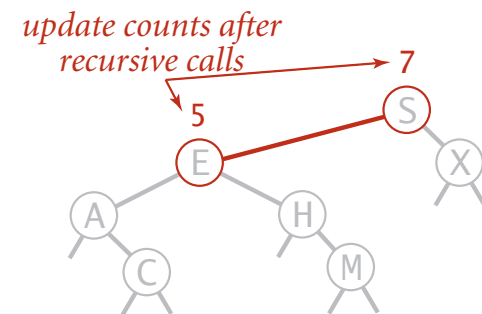
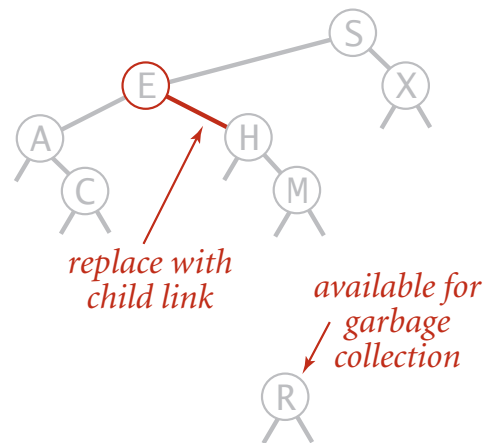
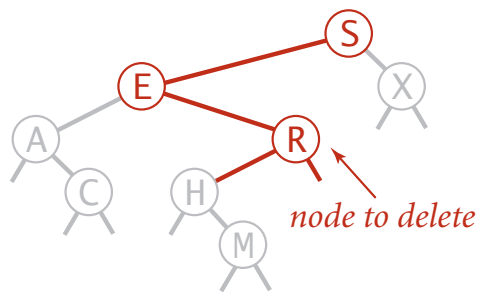


# Hibbard deletion

To delete a node with key  $k$ : search for node  $t$  containing key  $k$ .

**Case 1.** [1 child] Delete  $t$  by replacing parent link.

deleting R

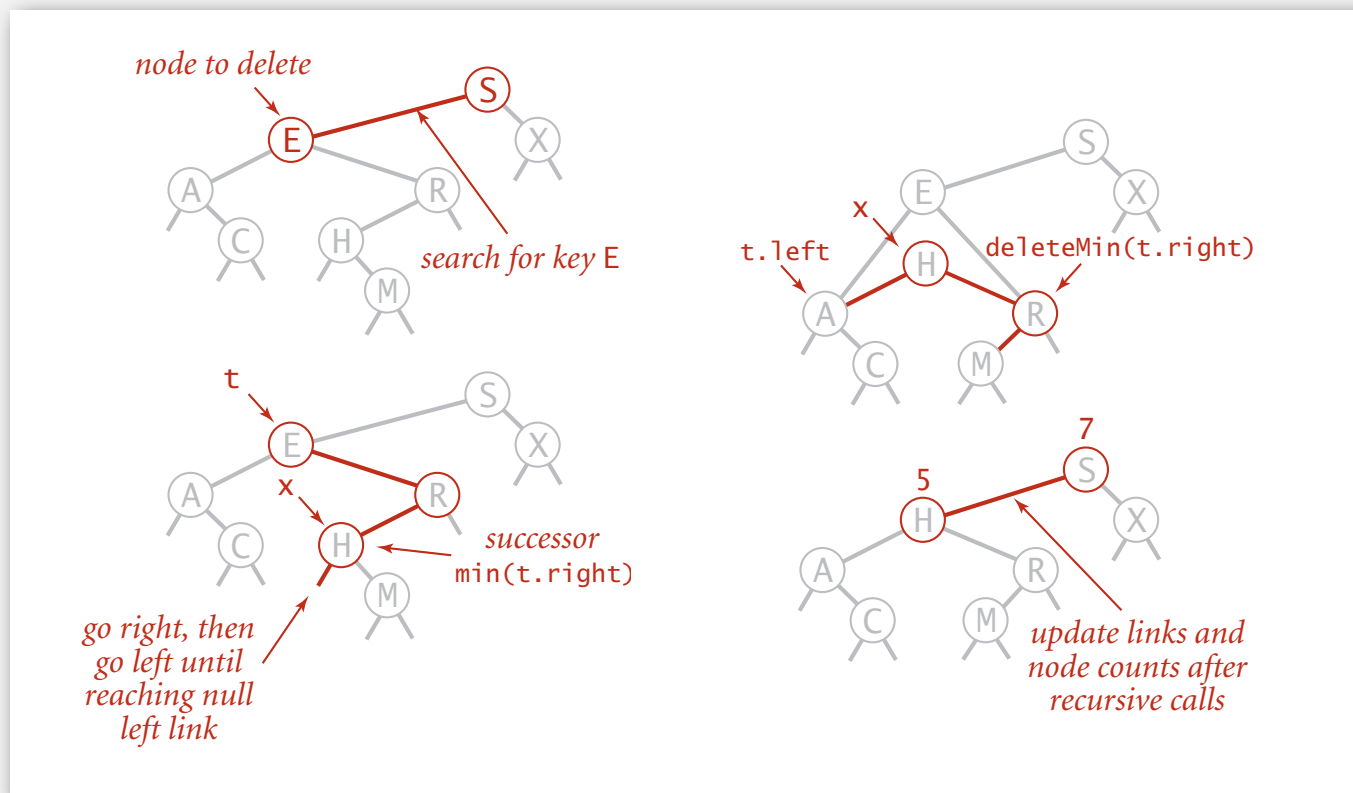


# Hibbard deletion

To delete a node with key  $k$ : search for node  $t$  containing key  $k$ .

## Case 2. [2 children]

- Find successor  $x$  of  $t$ .
  - Delete the minimum in  $t$ 's right subtree.
  - Put  $x$  in  $t$ 's spot.
- ←  $x$  has no left child  
← but don't garbage collect  $x$   
← still a BST



# Hibbard deletion: Java implementation

---

```
public void delete(Key key)
{ root = delete(root, key); }
```

```
private Node delete(Node x, Key key) {
```

```
    if (x == null) return null;
```

```
    int cmp = key.compareTo(x.key);
```

```
    if (cmp < 0) x.left = delete(x.left, key);
```

```
    else if (cmp > 0) x.right = delete(x.right, key);
```

```
    else {
```

```
        if (x.right == null) return x.left;
```

```
        if (x.left == null) return x.right;
```

```
        Node t = x;
```

```
        x = min(t.right);
```

```
        x.right = deleteMin(t.right);
```

```
        x.left = t.left;
```

```
    }
```

```
    x.count = size(x.left) + size(x.right) + 1;
```

```
    return x;
```

```
}
```

← search for key

← no right child

← no left child

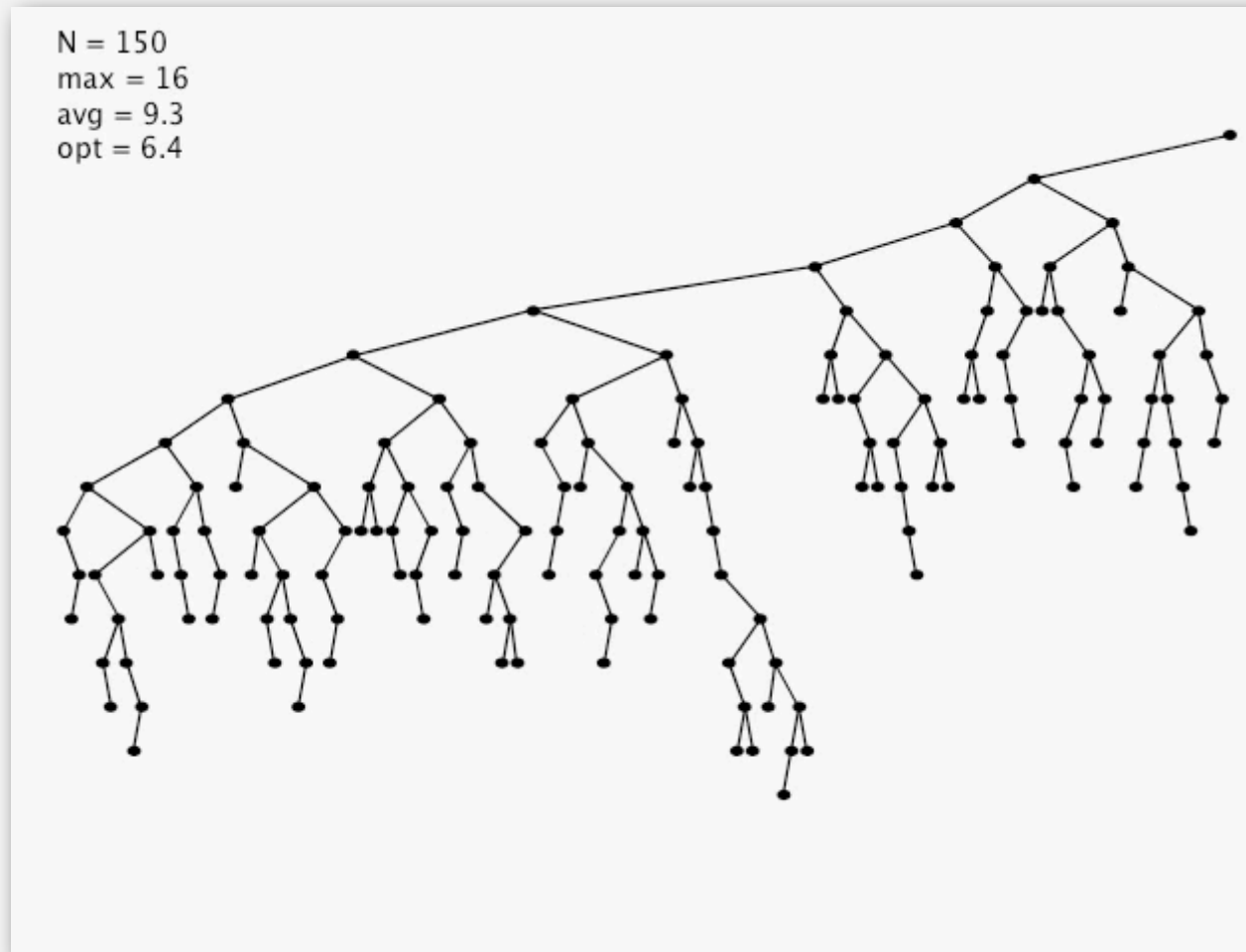
← replace with  
successor

← update subtree  
counts

# Hibbard deletion: analysis

---

Unsatisfactory solution. Not symmetric.

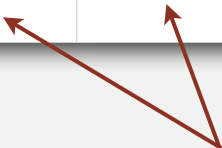


Surprising consequence. Trees not random (!)  $\Rightarrow \sqrt{N}$  per op.  
Longstanding open problem. Simple and efficient delete for BSTs.

# ST implementations: summary

---

implementation	guarantee			average case			ordered iteration?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	N/2	N	N/2	no	equals()
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	compareTo()
BST	N	N	N	1.39 lg N	1.39 lg N	$\sqrt{N}$	yes	compareTo()



other operations also become  $\sqrt{N}$   
if deletions allowed

Next lecture. **Guarantee** logarithmic performance for all operations.



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