A/B Testing and Beyond

Designed Experiments for Data Scientists

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Week 4

Wednesday September 27th, 2017

Outline

- Recap
- Experiments with Two Conditions
  - Evaluating Assumptions
    - Welch's t-test
    - Randomization tests
    - χ^2-tests
  - A discussion of "peeking"
- Experiments with Multiple Condition
  - Comparing means
  - Comparing proportions
  - The multiple comparison problem

RECAP

Recap

- Experiments with Two Conditions
  - Comparing Means
    - The two-sample *t*-test
    - Power analysis and sample size calculations
  - Comparing Proportions
    - The Z-test for proportions
    - Power analysis and sample size calculations

EXPERIMENTS WITH TWO CONDITIONS

Recall

When comparing means…

- We assume the response variable of interest is measured on a continuous scale

- But this methodology is also commonly applied when the response variable is discrete with a large support set

- We assume that the n_j response measurements in condition $j = 1,2$ follow a normal distribution:

$$Y_{ij} \sim N(\mu_j, \sigma^2)$$

for $i = 1,2, \dots, n_j$

Recall

When comparing means…

To formally decide whether $\mu_1 = \mu_2$ or $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$, we test one or more of the following:

$$H_0: \mu_1 = \mu_2 \text{ vs. } H_A: \mu_1 \neq \mu_2$$

$$H_0: \mu_1 \leq \mu_2 \text{ vs. } H_A: \mu_1 > \mu_2$$

$$H_0: \mu_1 \geq \mu_2 \text{ vs. } H_A: \mu_1 < \mu_2$$

Evaluating Assumptions

When comparing means…

When testing these hypotheses using the standard *t*-test we saw last week, we make two key assumptions:

- We assume the variances in the two conditions are equal
- We assume that the response measurements follow a normal distribution

When these assumptions are not valid we require alternative approaches

Evaluating Assumptions

When $\sigma_1^2 \neq \sigma_2^2$

In this situation we alter our test statistic, and instead of using

$$t = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

we use

$$t = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}}}$$

where $\hat{\sigma}_j^2$ is the sample variance in condition j.

Evaluating Assumptions

When $\sigma_1^2 \neq \sigma_2^2$

However, this version of the test statistic no longer follows a *t*-distribution exactly

It only approximately follows a *t*-distribution with

$$\nu = \frac{\left(\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_2^2}{n_2}\right)^2}{\frac{(\hat{\sigma}_1^2/n_1)^2}{n_1 - 1} + \frac{\hat{\sigma}_2^2/n_2}{n_2 - 1}}$$

degrees of freedom.

Evaluating Assumptions

When $\sigma_1^2 \neq \sigma_2^2$

Thus a conclusion is drawn by comparing the observed value of t with the null distribution: $t_{(\nu)}$ where the degrees of freedom ν are shown on the previous slide.

Note that p-values are calculated as usual.

The *t*-test carried out in this way is called Welch's *t*-test

We can perform this test in `R` by setting `var.equal = FALSE` in the `t.test()` function

Evaluating Assumptions

But how do we know if $\sigma_1^2 \neq \sigma_2^2$ and that Welch's t-test is appropriate?

We could formally decide by testing the following hypothesis:

$$H_0\colon \sigma_1^2 = \sigma_2^2 \text{ vs. } H_A\colon \sigma_1^2 \neq \sigma_2^2$$

- If H_0 can be rejected, we should use Welch's t-test
- If H_0 cannot be rejected, we should use Student's t-test

Evaluating Assumptions

The hypothesis on the previous slide is typically tested using an *F*-test.

This test is so named because the null distribution of the test statistic follows an *F*-distribution.

Let's take a brief detour to familiarize ourselves with this distribution…

Evaluating Assumptions

The *F*-distribution

A $Y \sim F(\nu_1, \nu_2)$ random variable has PDF given by

$$f(y) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1}{\nu_2} y\right)^{-\frac{\nu_1+\nu_2}{2}}$$

for $y \geq 0$ and where ν_1 and ν_2 are positive integers.

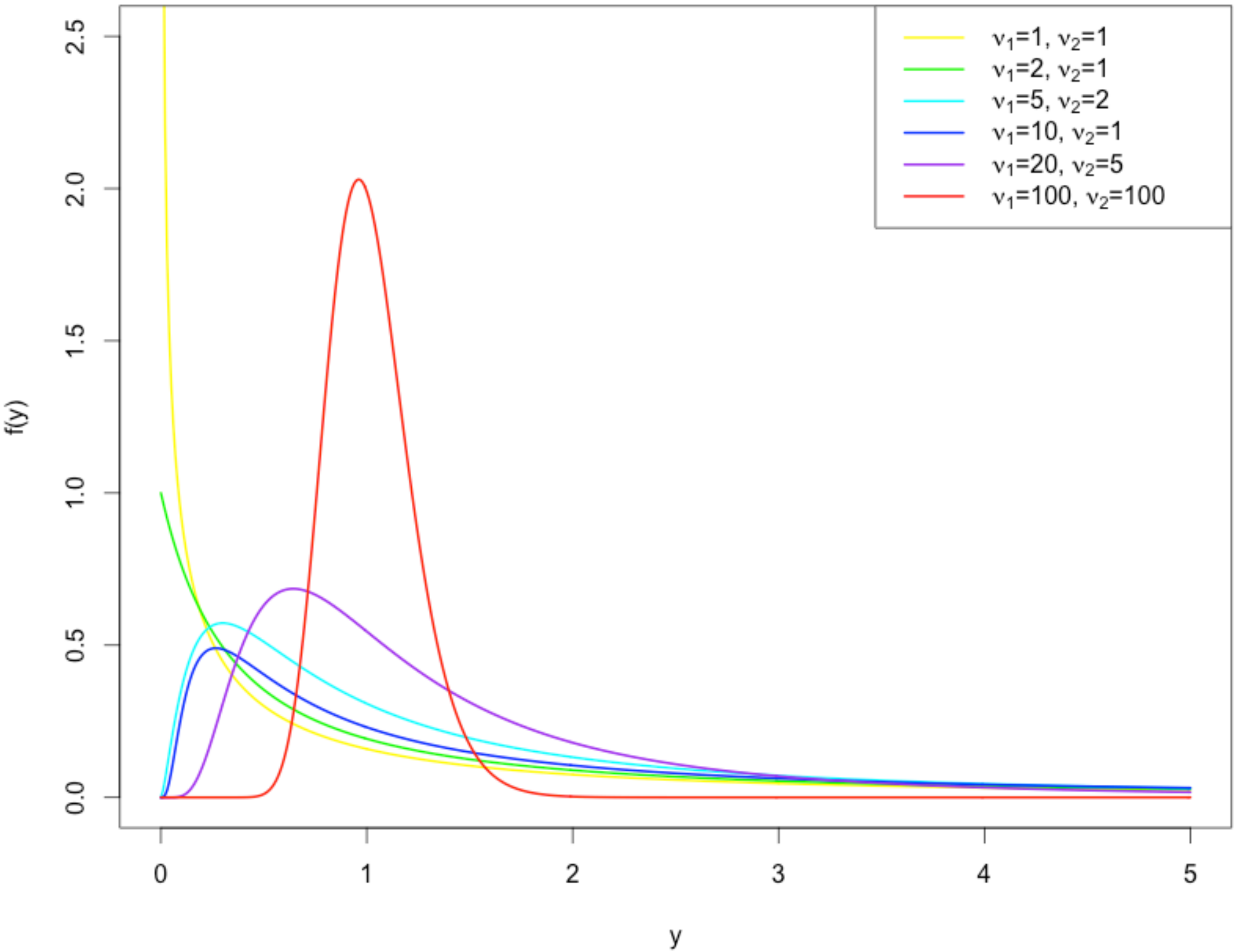
In R:

$P(Y \leq q) =$ `pf(q, df1 = v1, df2 = v2)`

$P(Y \leq q) =$ `1-pf(q, df1 = v1, df2 = v2)`

Evaluating Assumptions

The *F*-distribution



Evaluating Assumptions

The *F*-test for variances

$$H_0: \sigma_1^2 = \sigma_2^2 \text{ vs. } H_A: \sigma_1^2 \neq \sigma_2^2$$

In this situation we assume $Y_{ij} \sim N\left(\mu_j, \sigma_j^2\right)$ for $i = 1,2, \ldots, n_j, j = 1,2$ which means that

$$\frac{\left(n_j - 1\right)\hat{\sigma}_j^2}{\sigma_j^2} \sim \chi^2_{(n_j - 1)}$$

and

$$T = \frac{\hat{\sigma}_1^2 / \sigma_1^2}{\hat{\sigma}_2^2 / \sigma_2^2} \sim F(n_1 - 1, n_2 - 1)$$

Evaluating Assumptions

The *F*-test for variances

We use T (from the previous slide) as our test statistic whose observed value is

$$t = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$$

Since, if H_0 is true, $\sigma_1^2/\sigma_2^2 = 1$.

We compare this value to the $F(n_1 - 1, n_2 - 1)$ distribution to determine it extremity

Evaluating Assumptions

The *F*-test for variances

The p-value is calculated to be

$$\text{p} - \text{value} = P(T \geq t) + P(T \leq 1/t)$$

where $T \sim F(n_1 - 1, n_2 - 1)$

Notice that this calculation is slightly different from other two-sided p-values we've calculated

This arises because "at least as extreme" means something different in this setting

Note that this test can be performed in `R` using the `var.test()` function

Evaluating Assumptions

What if Y_{ij} is not normally distributed?

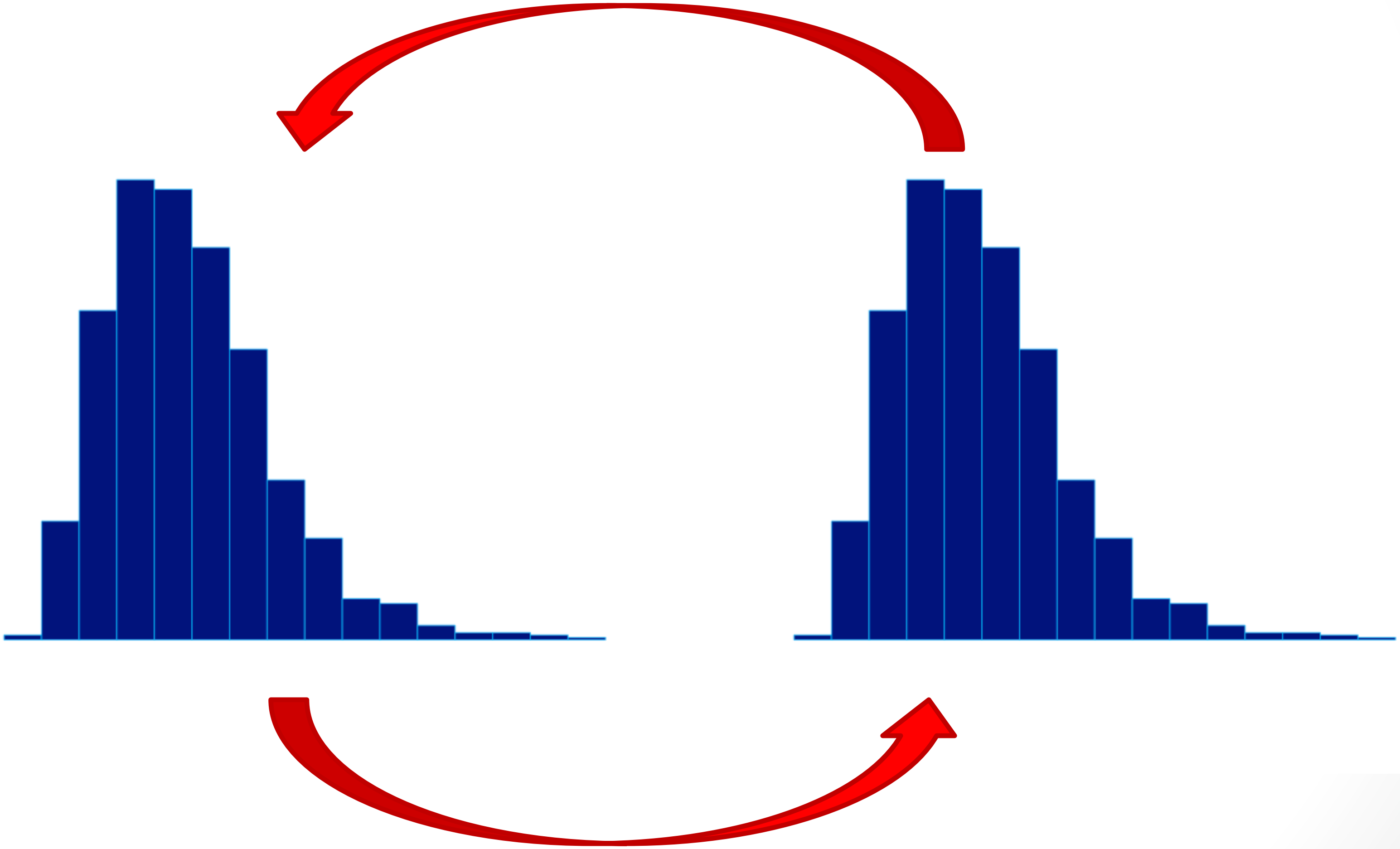
In this case we ran use a permutation or randomization test

Both of these approaches are nonparametric resampling techniques

The motivating idea behind both of these is that, if H_0 is true, any random rearrangement of the data is **equally likely to have been observed**

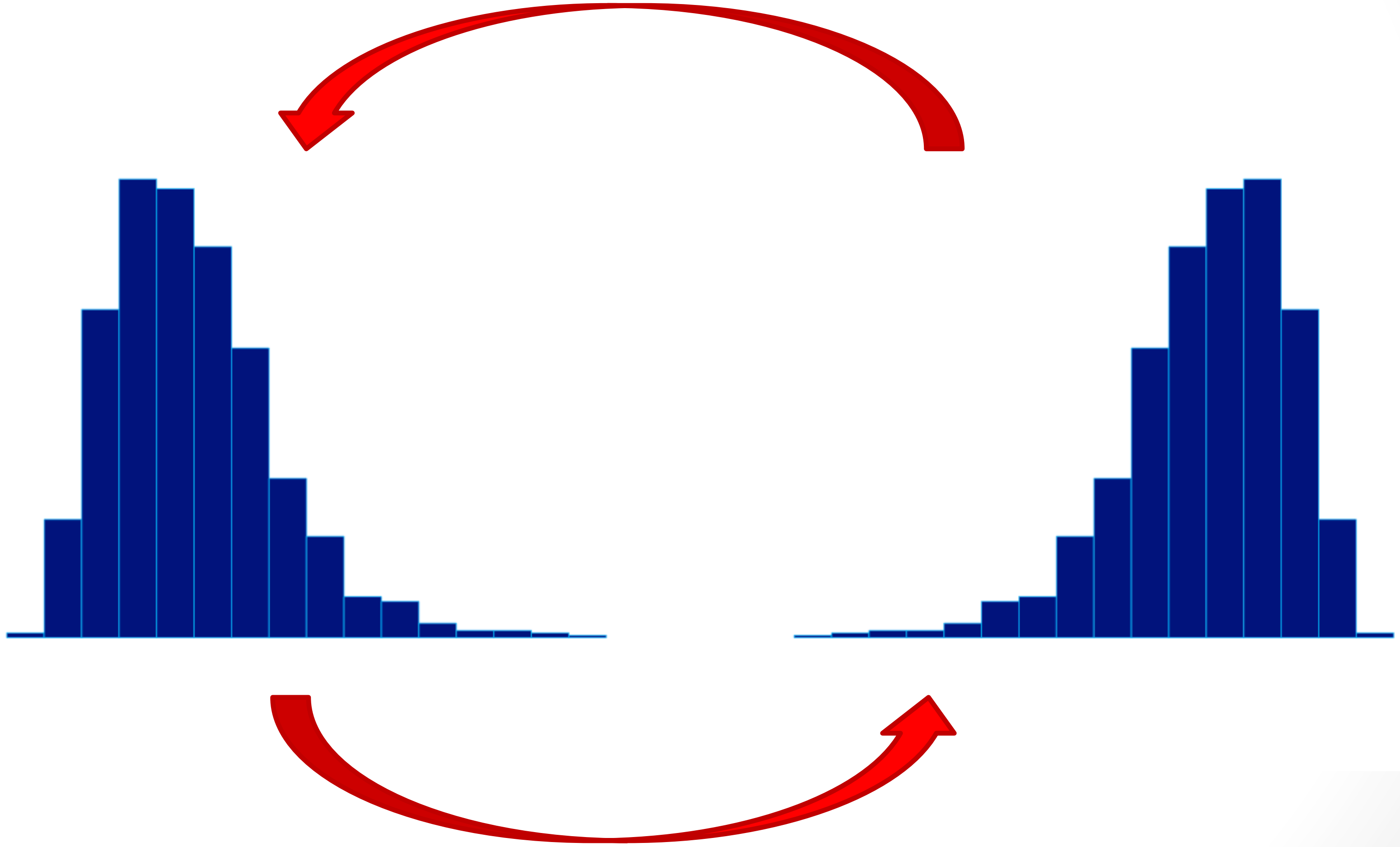
Evaluating Assumptions

Permutation and Randomization Tests



Evaluating Assumptions

Permutation and Randomization Tests



Evaluating Assumptions

Permutation and Randomization Tests

- Note that there are

$$\binom{n_1 + n_2}{n_1} = \binom{n_1 + n_2}{n_2}$$

arrangements of the $n_1 + n_2$ observations into two groups (of size n_1 and n_2)

- A **true permutation test** calculates the test statistic on the original sample ($t = \hat{\mu}_1 - \hat{\mu}_2$) and every permuted sample (t^*)
- The distribution of t^* is taken to be the null distribution of the test statistic and the extremity of t is evaluated in this context

Evaluating Assumptions

Permutation and Randomization Tests

- However, even with reasonably small samples,
$$\binom{n_1+n_2}{n_1} = \binom{n_1+n_2}{n_2}$$
is a very large number

- If $n_1 = n_2 = 50$, then there are 1.09×10^{29} possible samples to consider

- So the permutation test can be computationally expensive

- As an alternative we use a **randomization test** which investigates a large number of resamples, as opposed to all possible permutations

Evaluating Assumptions

Permutation and Randomization Tests

The algorithm for carrying out a randomization test is as follows:

1. Collect response observations in each condition $\left\{y_{1j}, y_{2j}, \ldots, y_{n_j j}\right\}$ for $j = 1,2$
2. Calculate the test statistic $t = \hat{\mu}_1 - \hat{\mu}_2 = \bar{y}_1 - \bar{y}_2$ on the original sample
3. Resample the data without replacement so that n_1 observations are randomly associated with a resampled ‘condition 1’ and n_2 observations are randomly associated with a resampled ‘condition 2’

Evaluating Assumptions

Permutation and Randomization Tests

4. Calculate the value of the test statistic, labeled t^*, on this resample
5. Repeat steps 3 and 4 a few thousand times
6. Compare t to the distribution which is derived from the resampled values of t^* and calculate the p-value

The p-values of this test are calculated empirically and the calculation depends on whether H_A is one- or two-sided.

Evaluating Assumptions

Permutation and Randomization Tests

$$H_0: \mu_1 = \mu_2 \text{ vs. } H_A: \mu_1 \neq \mu_2$$

- p-value = the proportion of resampled test statistics $t^* \geq |t|$ or $t^* \leq -|t|$

$$H_0: \mu_1 \leq \mu_2 \text{ vs. } H_A: \mu_1 > \mu_2$$

- p-value = the proportion of resampled test statistics $t^* \geq t$

$$H_0: \mu_1 \geq \mu_2 \text{ vs. } H_A: \mu_1 < \mu_2$$

- p-value = the proportion of resampled test statistics $t^* \leq t$

Evaluating Assumptions

Permutation and Randomization Tests

Note that we have introduced these tests in the context of comparing means, but they are appropriate for the comparison of any metric that might be compared between two conditions

$$H_0: \theta_1 = \theta_2 \text{ vs. } H_A: \theta_1 \neq \theta_2$$

$$H_0: \theta_1 \leq \theta_2 \text{ vs. } H_A: \theta_1 > \theta_2$$

$$H_0: \theta_1 \geq \theta_2 \text{ vs. } H_A: \theta_1 < \theta_2$$

In this more general case our test statistic is simply calculated to be $t = \hat{\theta}_1 - \hat{\theta}_2$

Evaluating Assumptions

Randomization Test Example

Suppose Niantic is experimenting with two different promotions within Pokémon Go

- The first involves giving users 200 free Pokécoins
- The second involves giving users a 50% discount on in-app shop purchases

What they are interested in is whether, relative to providing no promotion, either of these strategies lead to users spending more of their own money in the shop.

Evaluating Assumptions

Randomization Test Example

To investigate this, a small experiment with $n_1 = n_2 = n_3 = 100$ users is performed in which

- Condition 1 (control): users receive no promotion
- Condition 2: users receive 200 free Pokécoins
- Condition 3: users receive a 50% in the shop

For each user, the amount of real money (in $) that they spend in the 30 days following the experiment is recorded.

Evaluating Assumptions

Randomization Test Example

$\hat{\mu}_1 = 10.74$

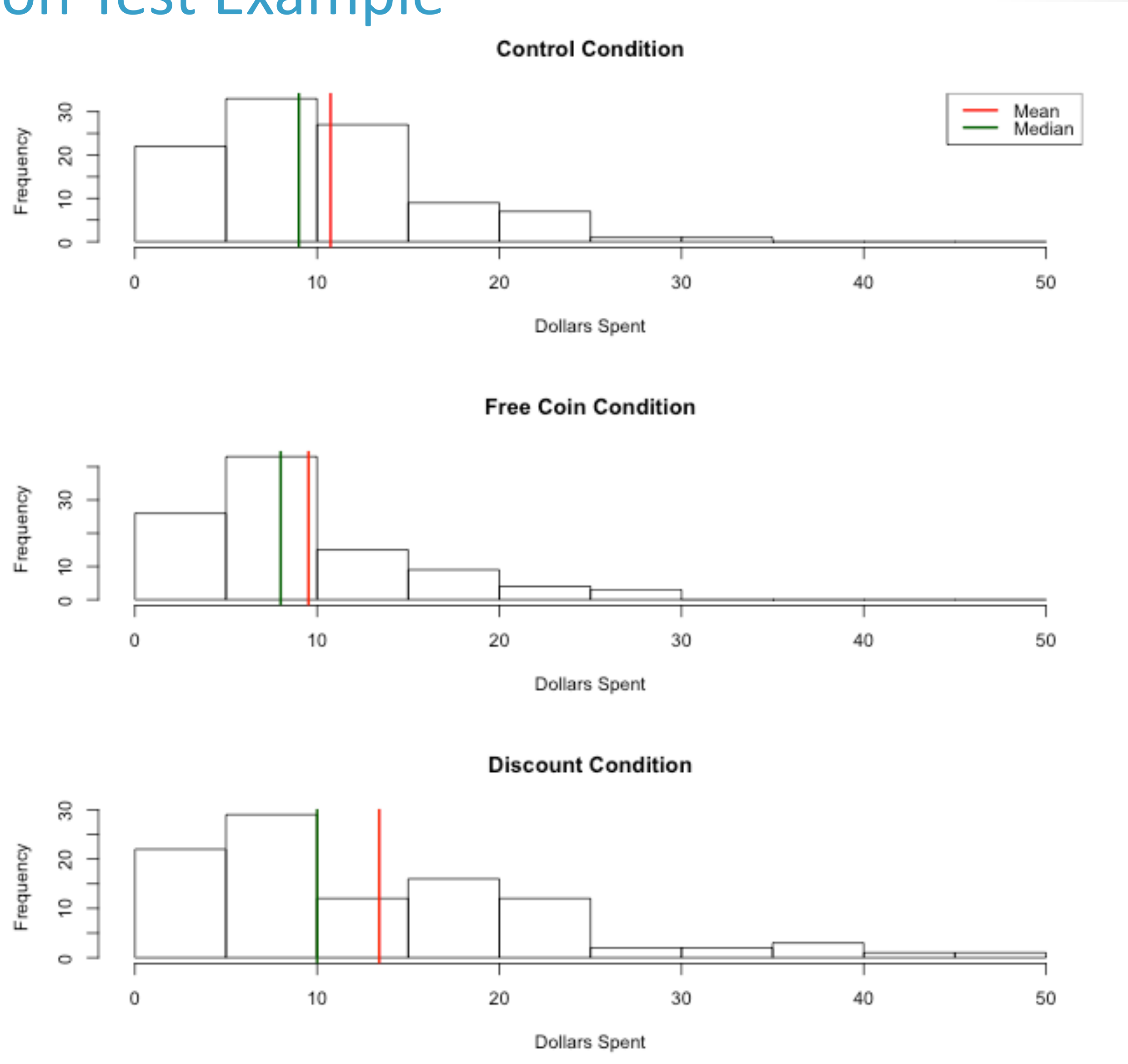
$\check{\mu}_1 = 9$

$\hat{\mu}_2 = 9.53$

$\check{\mu}_2 = 8$

$\hat{\mu}_3 = 13.41$

$\check{\mu}_3 = 10$

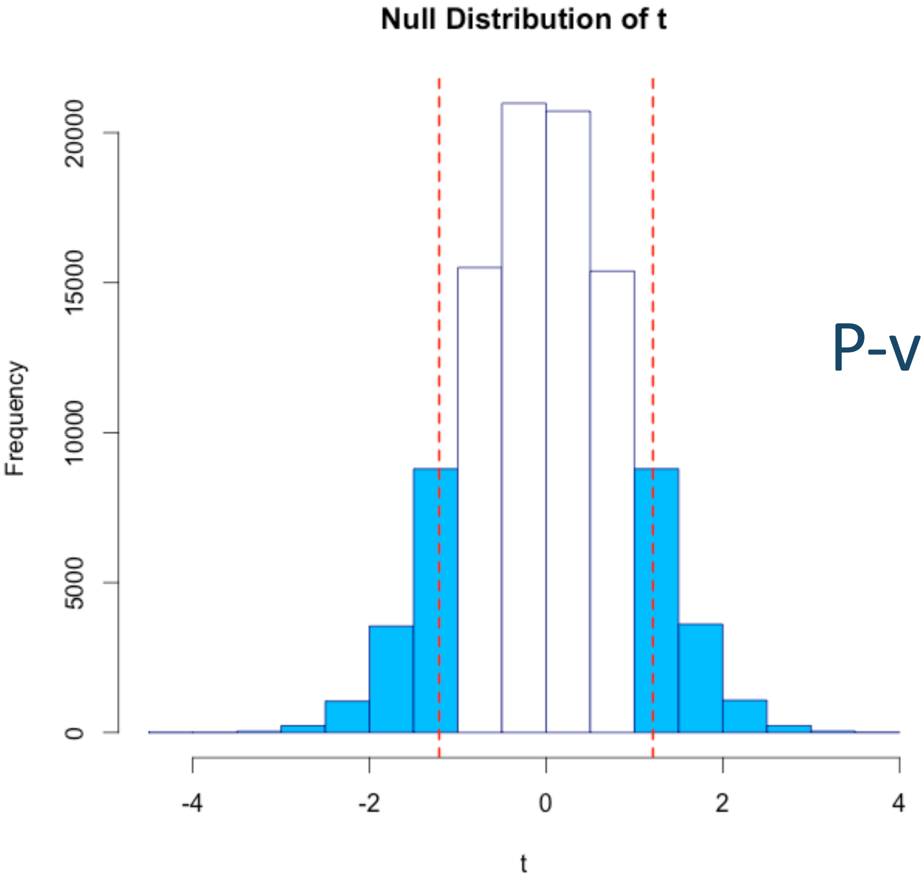


Evaluating Assumptions

Randomization Test Example

Control vs. Free Coins

$$H_0: \mu_1 = \mu_2 \text{ vs. } H_A: \mu_1 \neq \mu_2$$



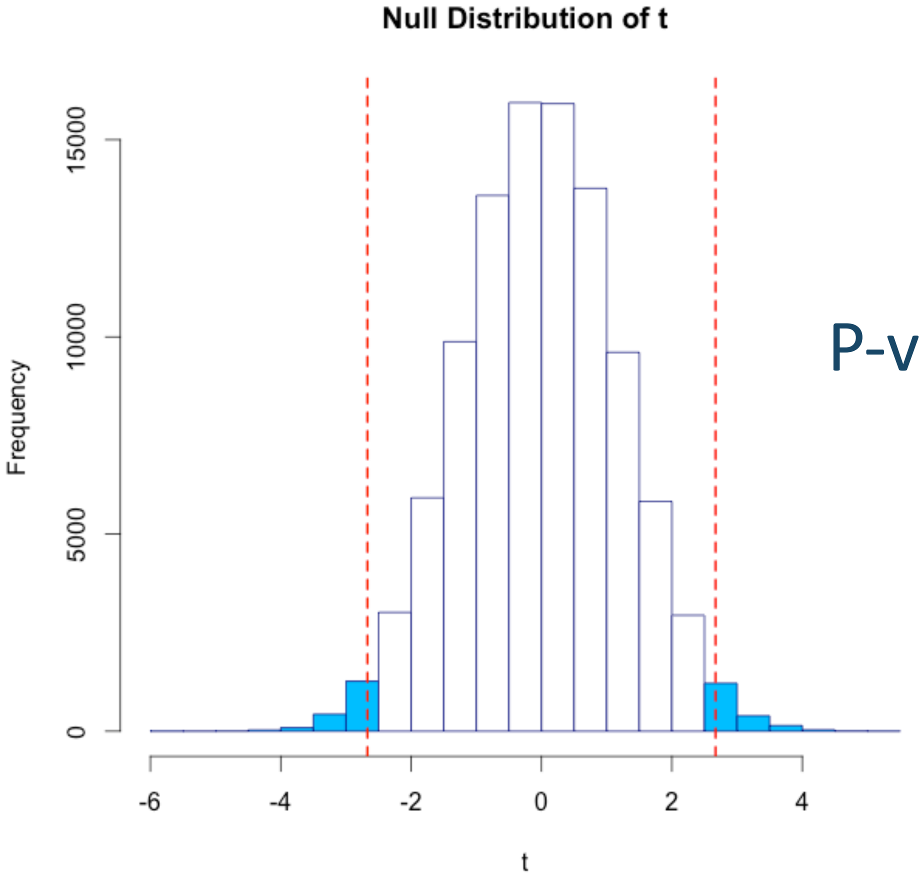
P-value = 0.1882

Evaluating Assumptions

Randomization Test Example

Control vs. Discount

$$H_0 : \mu_1 = \mu_3 \text{ vs. } H_A : \mu_1 \neq \mu_3$$



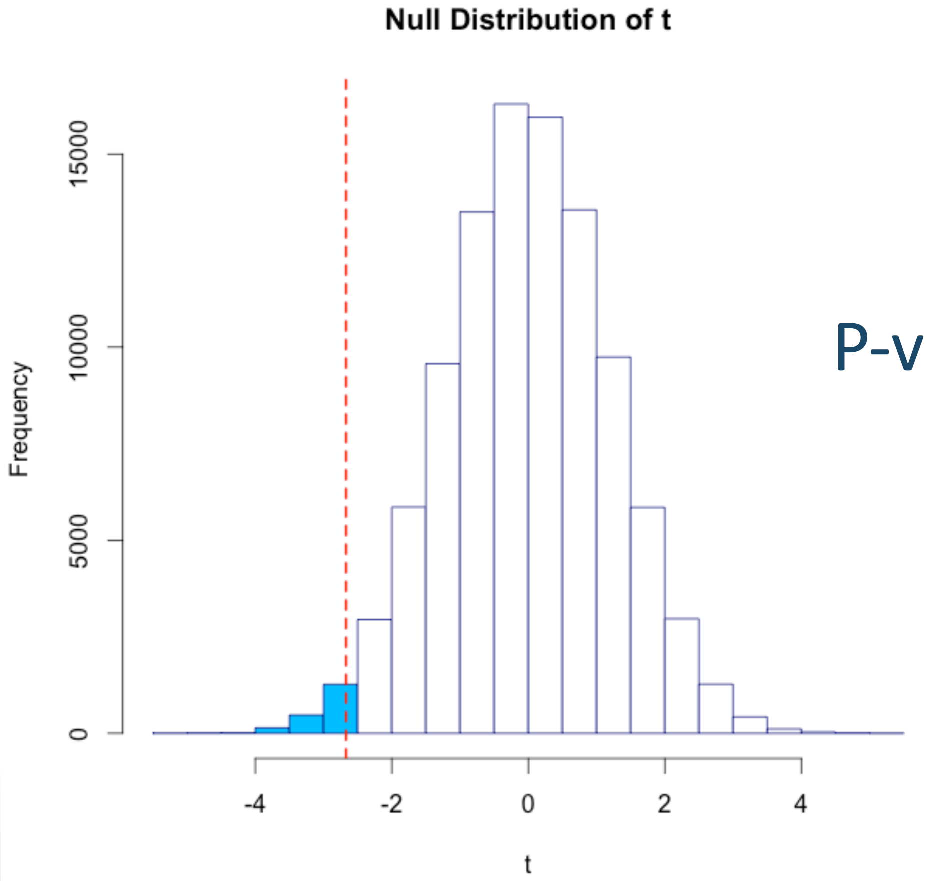
P-value = 0.02515

Evaluating Assumptions

Randomization Test Example

Control vs. Discount

$$H_0: \mu_1 \geq \mu_3 \text{ vs. } H_A: \mu_1 < \mu_3$$



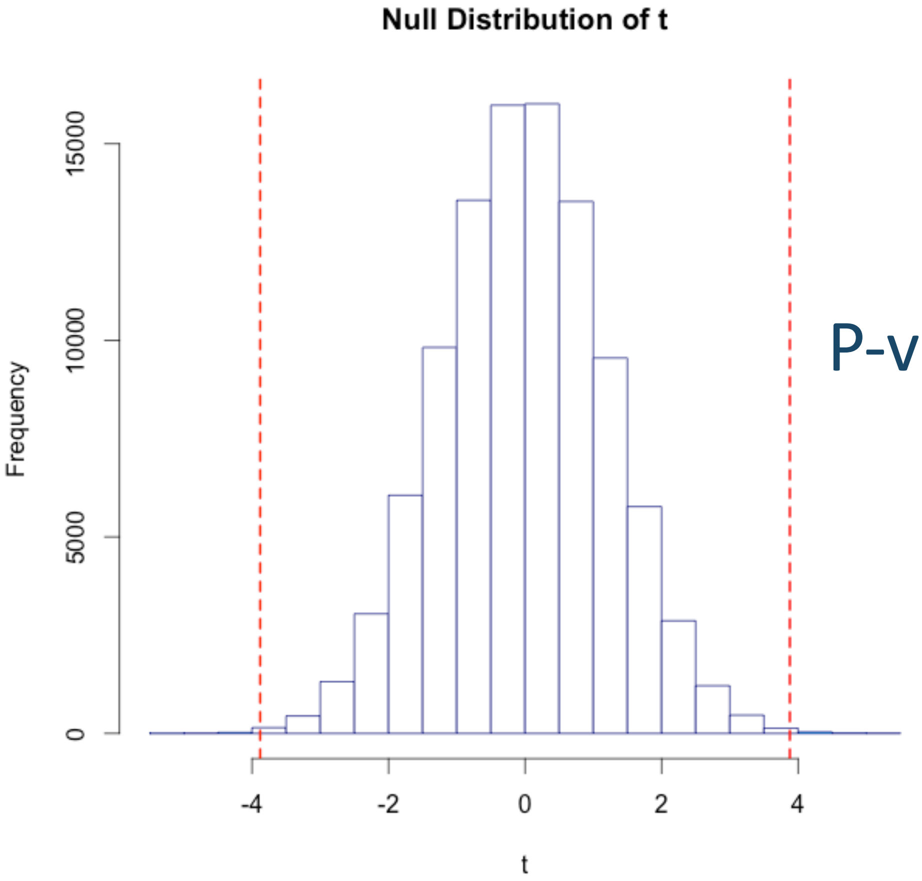
P-value = 0.01349

Evaluating Assumptions

Randomization Test Example

Free Coins vs. Discount

$$H_0: \mu_2 = \mu_3 \text{ vs. } H_A: \mu_2 \neq \mu_3$$



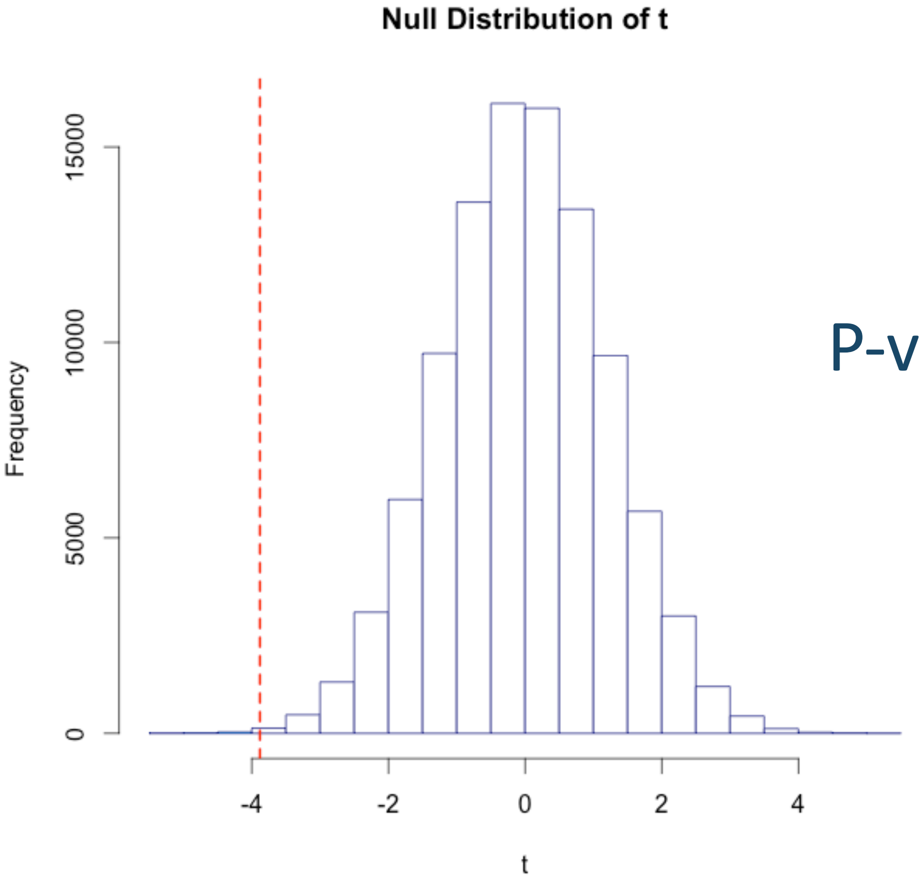
P-value = 0.00112

Evaluating Assumptions

Randomization Test Example

Free Coins vs. Discount

$$H_0: \mu_2 \geq \mu_3 \text{ vs. } H_A: \mu_2 < \mu_3$$



P-value = 0.00055

Recall

When comparing proportions…

Very often the response variable in an A/B test is binary, indicating whether an experimental unit did, or did not, perform some action of interest

$$Y_{ij} = \begin{cases} 1 \text{ if unit } i \text{ in condition } j \text{ does action} \\ 0 \text{ if unit } i \text{ in condition } j \text{ doesn't do action} \end{cases}$$

for $i = 1,2, \dots, n_j, j = 1,2$

We define $\pi_j = P\left(Y_{ij} = 1\right)$ to be the probability that a unit in condition j performs the action of interest

Recall

When comparing proportions…

The goal of the experiment, then, is to decide whether $\pi_1 = \pi_2, \pi_1 > \pi_2$ or $\pi_1 < \pi_2$

We do this formally by testing hypotheses of the form

$$H_0{:}\,\pi_1 = \pi_2 \text{ vs. } H_A{:}\,\pi_1 \neq \pi_2$$

$$H_0{:}\,\pi_1 \leq \pi_2 \text{ vs. } H_A{:}\,\pi_1 > \pi_2$$

$$H_0{:}\,\pi_1 \geq \pi_2 \text{ vs. } H_A{:}\,\pi_1 < \pi_2$$

Evaluating Assumptions

When comparing proportions…

When testing these hypotheses using the *Z*-test we saw last week, we make one key assumption:

- The validity of the method relies on the results of the Central Limit Theorem
- These results, in turn, rely on the assumption that the sample sizes n_1 and n_2 are suitably large
- As a rule of thumb, this method is not valid unless $n_j\pi_j \geq 10$ and $n_j(1-\pi_j) \geq 10$ for $j = 1,2$

In this case we require an alternative approach

Evaluating Assumptions

χ^2-test of Independence

- This test is typically used as a test for 'no association' between two categorical variables
- Here we test the independence of the binary outcome (whether a unit performs the action of interest) and the particular condition they are in
- If the likelihood of performing the action is the same in each condition (i.e., $\pi_1 = \pi_2$) then the response and conditions are not associated
- As such, this test is useful for testing hypotheses regarding $\pi_1 = \pi_2$, $\pi_1 > \pi_2$ or $\pi_1 < \pi_2$

Evaluating Assumptions

χ^2-test of Independence

- The information pertinent to this test can be summarized in a 2x2 contingency table.
- As a concrete example, consider the data from the Optimizely Example last week

		Condition		
		Original	Redesign	
Conversion	Yes	280	399	679
	No	8592	8243	16835
		8872	8642	17514

Evaluating Assumptions

χ^2-test of Independence

We can write this table more generally as

		Condition		
		1	2	
Conversion	Yes	$O_{1,1}$	$O_{1,2}$	O_1
	No	$O_{0,1}$	$O_{0,2}$	O_0
		n_1	n_2	$n_1 + n_2$

where

- $O_{1,j}$ and $O_{0,j}$ respectively represent the observed number of conversions and non-conversions in condition $j = 1{,}2$, and
- O_1 and O_0 represent the overall number of conversions and non-conversions

Evaluating Assumptions

χ^2-test of Independence

If $\pi_1 = \pi_2 = \pi$ then we would expect the conversion rate in each condition to be the same

Pooled estimates of $\hat{\pi}$ and $1 - \hat{\pi}$ are given by

$$\hat{\pi} = \frac{O_1}{n_1+n_2} \text{ and } 1 - \hat{\pi} = \frac{O_0}{n_1+n_2}$$

With these we can calculate the expected number of observations in each cell of the contingency table:

$$E_{1,j} = n_j\hat{\pi} \text{ and } E_{0,j} = n_j(1 - \hat{\pi})$$

for $j = 1,2$

Evaluating Assumptions

χ^2-test of Independence

The expected frequencies can also be summarized in a contingency table:

		Condition		
		1	2	
Conversion	Yes	$O_{1,1}$	$O_{1,2}$	O_1
	No	$O_{0,1}$	$O_{0,2}$	O_0
		n_1	n_2	$n_1 + n_2$

Note that the margin totals do not change.

The χ^2-test formally compares the what was observed and what is expected under the null hypothesis

Evaluating Assumptions

χ^2-test of Independence

The expected frequencies associated with the Optimizely Example are:

		Condition		
		1	2	
Conversion	Yes	343.96	335.04	679
	No	8524.04	8306.96	16835
		8872	8642	17514

Clearly these don’t match what was observed, but we will use the χ^2-test to formally decide whether the observed and expected frequencies are significantly different

Evaluating Assumptions

χ^2-test of Independence

The test statistic that compares the observed count in each cell to the corresponding expected count, and is defined as

$$T = \sum_{l=0}^{1} \sum_{j=1}^{2} \frac{\left(O_{l,j} - E_{l,j}\right)^2}{E_{l,j}}$$

Assuming H_0 is true, T approximately follows a $\chi^2_{(1)}$ distribution

- As a rule of thumb, this approximation may be very poor unless the observed and expected cell frequencies are all greater than 5

Evaluating Assumptions

χ^2-test of Independence

Conclusions about the test are drawn with p-values in according with the following:

$$H_0: \pi_1 = \pi_2 \text{ vs. } H_A: \pi_1 \neq \pi_2$$

- p-value = $P(T \geq t)$

$$H_0: \pi_1 \leq \pi_2 \text{ vs. } H_A: \pi_1 > \pi_2$$

- p-value = $1 - P(T \geq t)/2$

$$H_0: \pi_1 \geq \pi_2 \text{ vs. } H_A: \pi_1 < \pi_2$$

- p-value = $P(T \geq t)/2$

Evaluating Assumptions

χ^2-test of Independence

Returning to the Optimizely Example, the observed test statistic is calculated to be $t = 25.0755$ and so $P(T \geq t) = 5.52 \times 10^{-7}$

The p-values associated with the three tests are

$$H_0: \pi_1 = \pi_2 \text{ vs. } H_A: \pi_1 \neq \pi_2$$

- p-value = 5.52×10^{-7}

$$H_0: \pi_1 \leq \pi_2 \text{ vs. } H_A: \pi_1 > \pi_2$$

- p-value = 0.9999997

$$H_0: \pi_1 \geq \pi_2 \text{ vs. } H_A: \pi_1 < \pi_2$$

- p-value = 2.76×10^{-7}

The Trouble with Peeking

- The phenomenon whereby you regularly check the results of the experiment before it finishes is known as “peeking”
- This may be tempting, and in some cases impossible to avoid
- Sometimes “peeking” is even a good thing (e.g., to ensure the experiment is not negatively impacting other important metrics)
- The problem, however, arises when, as a result of peeking, you decide to end the experiment early.

The Trouble with Peeking

- Often you might feel pressure to stop the experiment once you see a significant result
- What’s the problem? The results tell us that a winner has been found, right?

Wrong

- Well, maybe, but by stopping the experiment early you have not observed enough data to be confident in this conclusion.

The Trouble with Peeking

- Just because the results suggest a winner at one point in time does not mean that the results won't change as more data is collected.

- I might peek at my experiment now and see that condition 1 is significantly out-performing condition 2, but if I peek again in an hour I might find that the opposite is true

- Only until you have observed the pre-specified amount of data should you be sure of your conclusions.

The Trouble with Peeking

- When you stop the experiment you are rejecting the null hypothesis
- Which means you might be making a Type I error
- And by stopping the experiment early the chances you make a Type I error are much higher than the prespecified value of α
- **After all, we did power analyses and sample size calculations for a reason**

The Trouble with Peeking

- By stopping the experiment at all, you are rejecting the null hypothesis

- Which means you might be making a Type I error

- And by stopping the experiment early the chances you make a Type I error are much higher than the prespecified value of α

- After all, we did power analyses and sample size calculations for a reason

The Trouble with Peeking

To illustrate the dire consequences of peeking and ending an experiment early, consider the following simulation.

The set-up:

- $n_1 = n_2 = 1000$ data points are drawn independently from the $N(0,1)$ distribution
- The observations are used to test

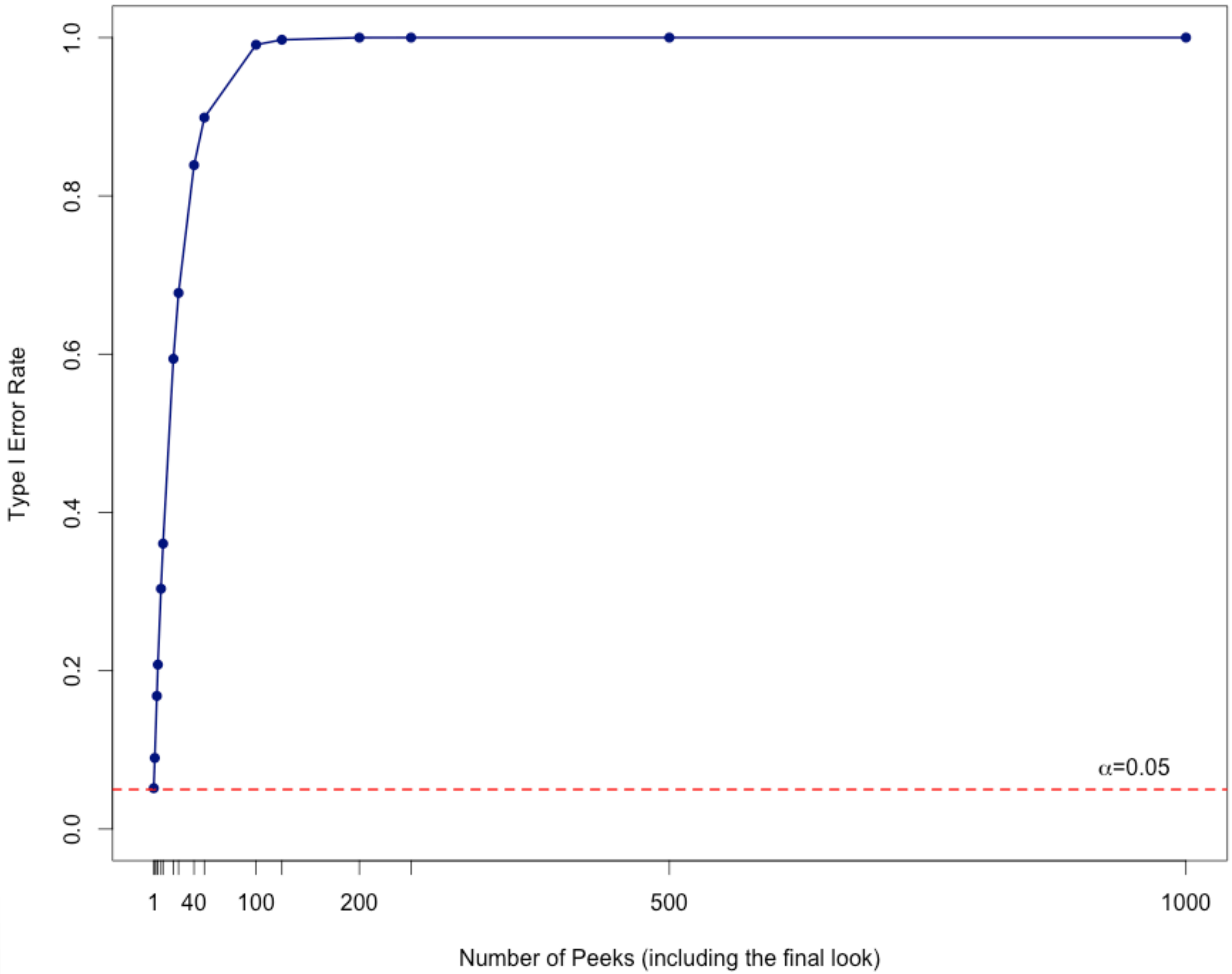
$$H_0: \mu_1 = \mu_2 \text{ vs. } H_A: \mu_1 \neq \mu_2$$

- Because $\mu_1 = \mu_2 = 0$ we should not reject H_0 very often (no more than $\alpha \times 100\%$ of the time)

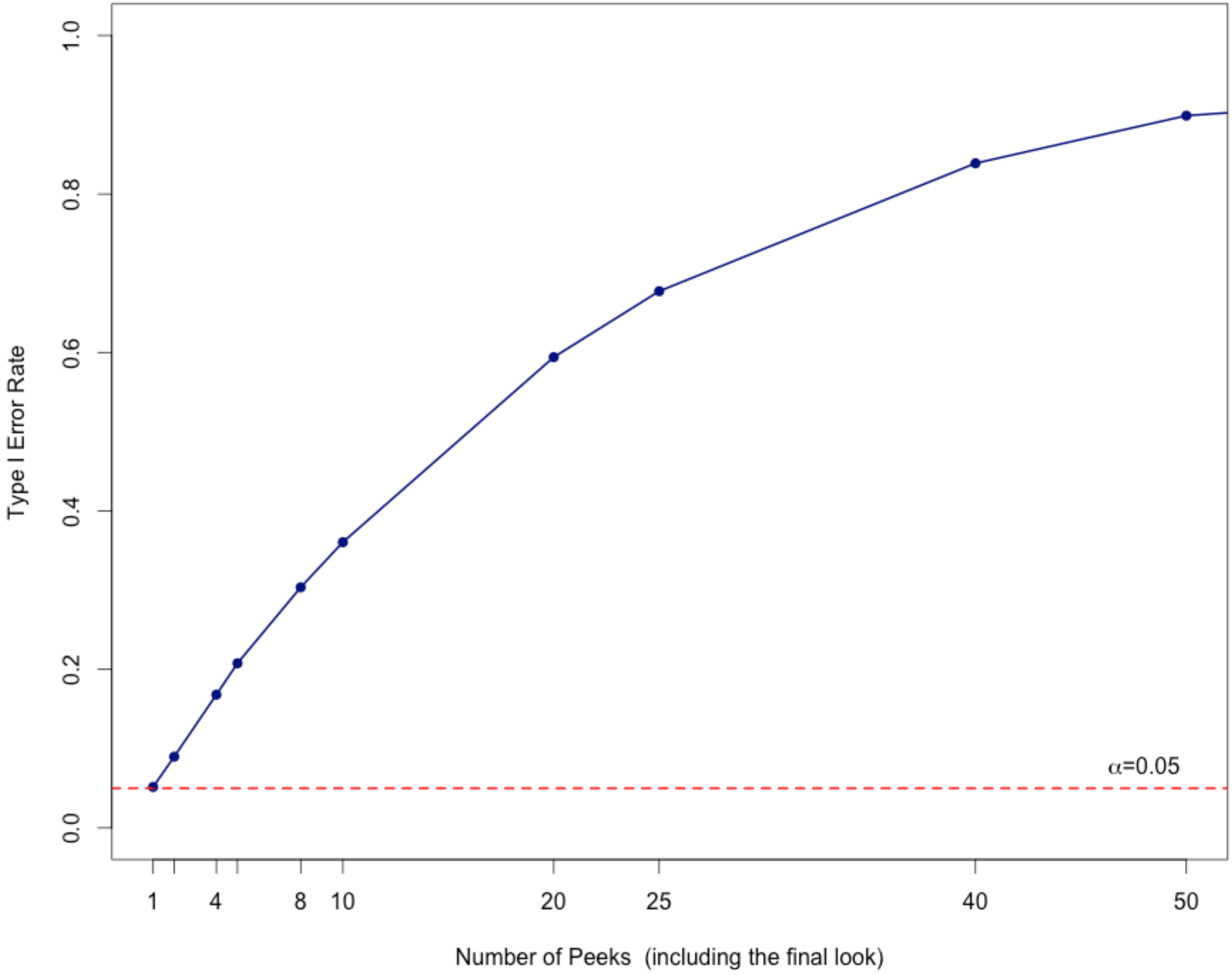
The Trouble with Peeking

- To study the consequences of peeking, we peek – **and end the experiment if a significant result is indicated** – at regular intervals
- We then calculate the Type I Error rate by observing how often an experiment is ended before all 1000 data points are observed in each condition
- We find that by peeking often enough committing a Type I error becomes a certainty

The Trouble with Peeking



The Trouble with Peeking



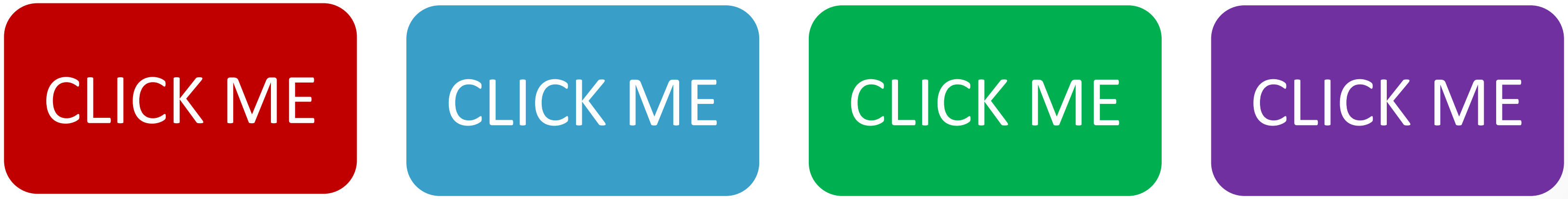
The Trouble with Peeking

- Note that sequential analysis and sequential testing are statistical disciplines devoted to devising statistically sound methods for performing **repeated significance tests** as more data becomes available.
- Essentially, these techniques that allow you to peek and end an experiment early without increasing Type I error rates.
- However, without adopting one of these techniques, peeking (and ending experiments early) should be avoided at all costs.

EXPERIMENTS WITH MULTIPLE CONDITIONS

Comparing Multiple Conditions

- We now consider the design and analysis of an experiment consisting of multiple experimental conditions i.e., an A/B/n Test
- Like an A/B test, the goal is to decide which condition is optimal with respect to some metric of interest – but now we have several conditions



- Given several options, which one is best?

Comparing Multiple Conditions

Designing a multi-condition test:

- Choose your response variable (y)
- Choose a metric θ that summarizes the response
- Choose a design factor and m levels to experiment with
- Choose $n_1, n_2,\ldots, n_m$ – the number of units to assign to each condition

Comparing Multiple Conditions

Data Collection:

- Randomly assign n_j units to condition $j = 1,2, \ldots, m$
- Measure the response (y) on each unit and summarize the measurements with the metric of interest θ in each of the conditions and hence obtain

$$\hat{\theta}_1,\ \hat{\theta}_2, \ldots,\ \hat{\theta}_m$$

Goal:

- Identify the optimal condition

Comparing Multiple Conditions

In order to identify the optimal condition, we simply need to do a series of pairwise comparisons using two-sample tests

- t-tests, Z-tests, and χ^2-tests may be used for this purpose

However, while identifying the optimal condition is the ultimate goal, it is prudent to first decide whether a difference exists, at all, between the conditions

Comparing Multiple Conditions

To answer this question formally, we may test a hypothesis of the form

$$H_0: \theta_1 = \theta_2 = \cdots = \theta_m \text{ vs. } H_A: \theta_j \neq \theta_k$$

for some $j \neq k$

Next we discuss how to test this hypothesis in the cases that the metric of interest is either a

- Mean, or a
- Proportion (rate)

Comparing Multiple Means

The Linear Regression *F*-test

Here interest lies in testing the hypothesis

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_m \text{ vs. } H_A: \mu_j \neq \mu_k$$

for some $j \neq k$.

This may be done with the *F*-test associated with an appropriately defined linear regression model.

Specifically, we adopt the following model:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{m-1} x_{i,m-1} + \epsilon_i$$

Comparing Multiple Means

The Linear Regression *F*-test

In this model

- $Y_i \sim N(\mu_j, \sigma^2)$ represents the response observation for unit $i = 1,2, \dots, N = \sum_{j=1}^{m} n_j$.
- Each x_{ij} is a dummy (indicator) variable taking on the value 1 if unit i is in condition j, and 0 otherwise
- $\epsilon_i \sim N(0, \sigma^2)$ represents the random error term for unit i
- The β's are unknown regression parameters

Comparing Multiple Means

The Linear Regression *F*-test

The parameter β_0 is interpreted as the expected response value when $x_1 = x_2 = \cdots = x_m = 0$

In other words, β_0 is the expected response value in condition m

We can also show that $\beta_0 + \beta_j$ is the expected response value in condition $j = 1,2, \dots, m - 1$

Comparing Multiple Means

The Linear Regression *F*-test

As such

$$\begin{gathered}\mu_1 = \beta_0 + \beta_1 \\ \mu_2 = \beta_0 + \beta_2 \\ \mu_3 = \beta_0 + \beta_3 \\ \vdots \\ \mu_{m-1} = \beta_0 + \beta_{m-1} \\ \mu_m = \beta_0\end{gathered}$$

and

$$\mu_1 = \mu_2 = \cdots = \mu_m$$

if and only if

$$\beta_1 = \beta_2 = \cdots = \beta_m = 0$$

Comparing Multiple Means

The Linear Regression *F*-test

So testing

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_m \text{ vs. } H_A: \mu_j \neq \mu_k$$

for some $j \neq k$

is equivalent to testing

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_m = 0 \text{ vs. } H_A: \beta_j \neq 0$$

for some $j = 1,2, \dots, m$

This latter test is corresponds to the *F*-test for overall significance in a linear regression model

Comparing Multiple Means

Example: Candy Crush

Candy Crush is experimenting with three different versions of in-game “boosters”:

- The lollipop hammer
- The jelly fish
- The color bomb

Users are randomized to one of these three conditions ($n_1 = 121, n_2 = 135, n_3 = 117$) and they receive (for free) 5 boosters corresponding to their condition.

Let μ_j represent the average length of game play in condition $j = 1,2,3$.

Comparing Multiple Means

Example: Candy Crush

While interest ultimately lies in finding the booster condition that maximizes use engagement, (i.e., has the largest μ_j) we will first decide whether any difference at all exists between the conditions:

$$H_0: \mu_1 = \mu_2 = \mu_3 \text{ vs. } H_A: \mu_j \neq \mu_k$$

for some $j \neq k$

To do so, we fit an "appropriately defined linear regression model". The results are shown on the next slide.

Comparing Multiple Means

Example: Candy Crush

```
Call:
lm(formula = time ~ factor(booster), data = candy)
Residuals:
    Min        1Q     Median     3Q      Max
-2.84231 -0.69476 0.02617 0.65326 2.76681
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      5.01281   0.08664   57.859 <2e-16 ***
factor(booster)2 1.17528   0.11931   9.851  <2e-16 ***
factor(booster)3 4.88279   0.12357   39.515 <2e-16 ***
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
Residual standard error: 0.953 on 370 degrees of freedom
Multiple R-squared: 0.8216,Adjusted R-squared: 0.8206
F-statistic: 851.9 on 2 and 370 DF, p-value: < 2.2e-16
```

Comparing Multiple Means

Example: Candy Crush

From this output we see that $\hat{\beta}_0 = 5.0128$, $\hat{\beta}_1 = 1.1753$ and $\hat{\beta}_2 = 4.8828$

This means that the average length of game play in each condition is estimated to be

- $\hat{\mu}_1 = 5.0128$ minutes in the lollipop hammer condition
- $\hat{\mu}_2 = 6.1881$ minutes in the jelly fish condition
- $\hat{\mu}_3 = 9{:}8956$ minutes in the color bomb condition

Comparing Multiple Means

Example: Candy Crush

The p-value associated with the *F*-test for overall significance in a linear regression model is less than 2.2×10-16 which provides very strong evidence against H_0

Thus we conclude that the average length of game play is not the same for each of the boosters.

To determine which booster is optimal – the one that maximizes game play duration – we must use a series of pairwise t-tests

Comparing Multiple Proportions

χ^2-test of Independence

Here interest lies in testing the hypothesis

$$H_0 : \pi_1 = \pi_2 = \cdots = \pi_m \text{ vs. } H_A : \pi_j \neq \pi_k$$

for some $j \neq k$.

This may be done with the same χ^2-test of independence that we discussed in the $m = 2$ case

Yes, it generalizes!

Comparing Multiple Proportions

χ^2-test of Independence

In the case of m conditions we have a $2 \times m$ contingency table:

		Condition				
		1	2	$\cdots$	m	
Conversion	Yes	$O_{1,1}$	$O_{1,2}$	$\cdots$	$O_{1,m}$	O_1
	No	$O_{0,1}$	$O_{0,2}$	$\cdots$	$O_{0,m}$	O_0
		n_1	n_2	$\cdots$	n_m	$\sum_{j=1}^{m} n_j$

- $O_{1,j}$ and $O_{0,j}$ respectively represent the observed number of conversions and non-conversions in condition $j = 1,2, \ldots, m$
- O_1 and O_0 represent the overall number of conversions and non-conversions

Comparing Multiple Proportions

χ^2-test of Independence

If $\pi_1 = \pi_2 = \cdots = \pi_m = \pi$ then we would expect the conversion rate in each condition to be the same

Pooled estimates of $\hat{\pi}$ and $1 - \hat{\pi}$ are given by

$$\hat{\pi} = \frac{O_1}{\sum_{j=1}^{m} n_j} \text{ and } 1 - \hat{\pi} = \frac{O_0}{\sum_{j=1}^{m} n_j}$$

With these we can calculate the expected number of observations in each cell of the contingency table:

$$E_{1,j} = n_j\hat{\pi} \text{ and } E_{0,j} = n_j(1 - \hat{\pi})$$

for$j = 1,2, \ldots , m$

Comparing Multiple Proportions

χ^2-test of Independence

The expected frequencies can also be summarized in a contingency table:

		Condition				
		1	2	⋯	m	
Conversion	Yes	$E_{1,1}$	$E_{1,2}$	⋯	$E_{1,m}$	O_1
	No	$E_{0,1}$	$E_{0,2}$	⋯	$E_{0,m}$	O_0
		n_1	n_2	⋯	n_m	$\sum_{j=1}^{m} n_j$

Note that the margin totals do not change.

Again, the χ^2-test formally compares the what was observed and what is expected under the null hypothesis

Comparing Multiple Proportions

χ^2-test of Independence

The test statistic that compares the observed count in each cell to the corresponding expected count, and is defined as

$$T = \sum_{l=0}^{1} \sum_{j=1}^{m} \frac{\left(O_{l,j} - E_{l,j}\right)^2}{E_{l,j}}$$

Assuming H_0 is true, T approximately follows a $\chi^2_{(m-1)}$ distribution

- As a rule of thumb, this approximation may be very poor unless the observed and expected cell frequencies are all greater than 5

Comparing Multiple Proportions

Example: Nike SB

- Suppose that Nike is running an ad campaign for Nike SB, their skateboarding division
- The ad campaign involves $m = 5$ different video ads being shown in Facebook newsfeeds
- In these five video conditions there are $n_1 =$ 5014, $n_2 =$ 4971, $n_3 =$ 5030, $n_4 =$ 5007, and $n_5 =$ 4980 users, respectively
- The videos in these conditions are viewed 160, 95, 141, 293, and 197 times yielding watch rates:

$$\hat{\pi}_1 = 0.03, \hat{\pi}_2 = 0.02, \hat{\pi}_3 = 0.03,$$

$$\hat{\pi}_4 = 0.06, \hat{\pi}_5 = 0.04$$

Comparing Multiple Proportions

Example: Nike SB

The observed contingency table is

		Condition					
		1	2	3	4	5	
View	Yes	160	95	141	293	197	886
	No	4854	4876	4889	4714	4783	24116
		5014	4971	5030	5007	4980	25002

And the expected contingency table is

		Condition					
		1	2	3	4	5	
View	Yes	177.68	176.16	178.25	177.43	176.48	886
	No	4836.32	4794.84	4851.75	4829.57	4803.52	24116
		5014	4971	5030	5007	4980	25002

Comparing Multiple Proportions

Example: Nike SB

- The observed value of the test statistic for this test is $t = 129.1761$ and the corresponding p-value is 5.84×10-27 and so there is strong evidence again H_0
- As such, we conclude that the likelihood that someone “views” a video is not the same for all of the videos
- To determine which video is optimal – the one with the highest likelihood of viewing – we must use a series of pairwise *Z*-tests or χ^2-tests

The Multiple Comparison Problem

As we saw in the previous two examples, the null hypothesis of overall equality is often rejected

In these cases a family of follow-up pairwise comparisons are necessary to determine which condition(s) is (are) optimal

Statistically we know how to do this

However, when doing multiple comparisons, it is important to recognize that the overall Type I Error rate associated with this family of tests is inflated

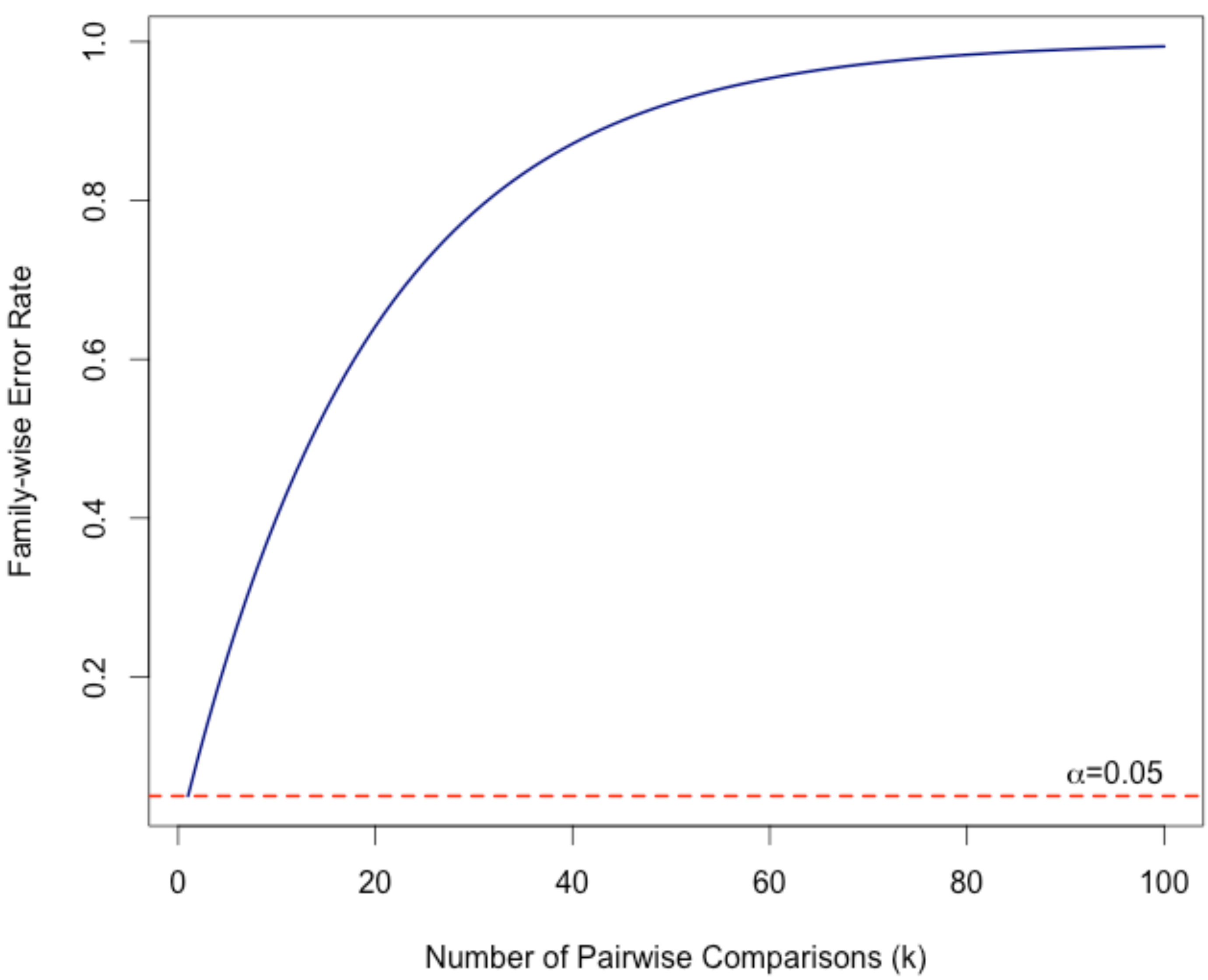
The Multiple Comparison Problem

This problem – where a series of independent hypothesis tests lead to an inflated family-wise error rate – is known as the multiple comparison or multiple testing problem.

It can be shown that if a family of k hypothesis tests, each with significance level α, the family-wise error rate is

$$1 - (1 - \alpha)^k$$

The Multiple Comparison Problem



The Multiple Comparison Problem

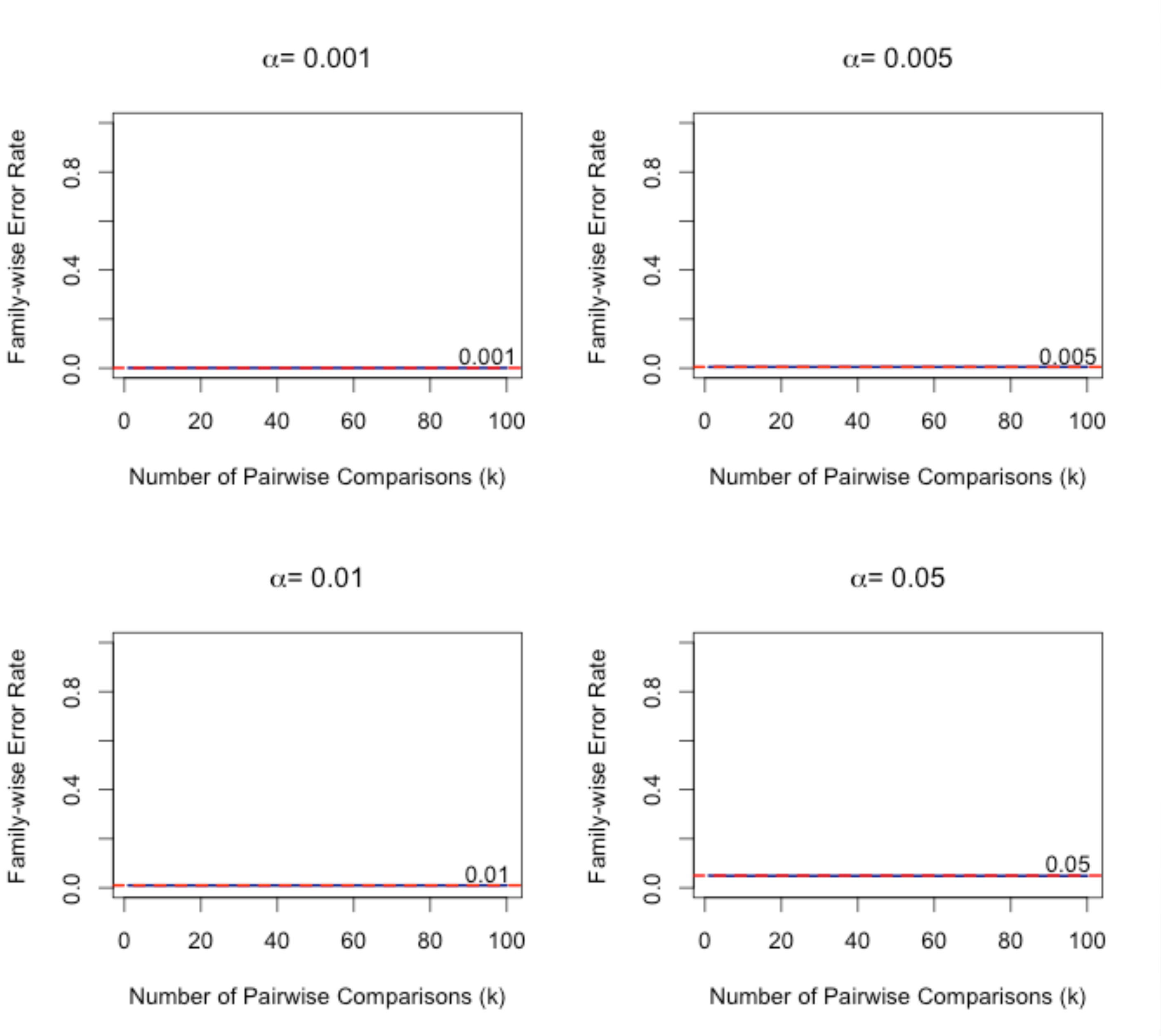
We combat this problem with the Bonferroni correction

- With this correction we test each of the k hypothesis tests at a significance level α/k, if maintaining an error rate of α is of interest
- Doing so yields a family-wise error rate of

$$1 - \left(1 - \frac{\alpha}{k}\right)^k$$

which, for typical values of α is approximately equal to α

The Multiple Comparison Problem



The Multiple Comparison Problem

So what does this mean for sample size calculations and power analyses?

The sample size formulas we derived previously did not account for this multiple comparison problem

In order to do so, when performing a power analysis, use α/k and not α as the significance level in the sample size calculations

See you next week!