A/B Testing and Beyond

Designed Experiments for Data Scientists

Week 6

Wednesday October 11th, 2017

Outline

- Analyzing Factorial Experiments
  - Continuous Responses
  - Binary Responses
- Two-Level Factorial Experiments
  - 2^k Factorial Experiments
  - 2^{k-p} Fractional Factorial Experiments

RECAP

Recap

- Primer on linear regression
- Experiments with Multiple Conditions
  - Comparing means
  - Comparing proportions
  - The multiple comparison problem
- Experiments with Multiple Factors
  - Factorial vs. One-factor-at-a-time
  - Designing a factorial experiment

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Multivariate Experiments

Designing a Factorial Experiment

The design is conceptually simple:

- Pick your design factors
- Pick their levels
- Your experimental conditions are all of the different combinations of these factors' levels

If you have k factors with $m_1, m_2, \ldots, m_k$ levels, respectively, the corresponding factorial experiment will have

$$M = m_1 m_2 \cdots m_k$$

experimental conditions

Multivariate Experiments

Designing a Factorial Experiment

Once units have been assigned to each condition, the response variable is measured on all of them

Using the collected data we

(1) Identify which factors are influential, and

(2) Identify which combination of factors is optimal

To do (1) we will apply regression techniques

To do (2) we will use two sample *t*-, *Z*- or χ^2-tests

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

We discuss these concepts in the context of the following example:

Suppose, again, Instagram is experimenting with ads to understand their influence on user engagement.

Again we assume the response variable (Y) is session duration (measured in minutes)

But now we assume we have two design factors

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

Factor 1: Ad Frequency

- None (coded as 0)
- 7:1 (coded as 1)
- 4:1 (coded as 2)
- 1:1 (coded as 3)

Factor 2: Ad Type

- Photo (coded as 1)
- Video (coded as 2)

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

Factor 1: Ad Frequency

- None (coded as 0)
- 7:1 (coded as 1)
- 4:1 (coded as 2)
- 1:1 (coded as 3)

Factor 2: Ad Type

- Photo (coded as 1)
- Video (coded as 2)

This leads to 4x2 = 8 unique conditions

Assume we randomize n=1000 units to each and measure Y

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

Frequency: None Type: Photo	Frequency: None Type: Video
Frequency: 7:1 Type: Photo	Frequency: 7:1 Type: Video
Frequency: 4:1 Type: Photo	Frequency: 4:1 Type: Video
Frequency: 1:1 Type: Photo	Frequency: 1:1 Type: Video

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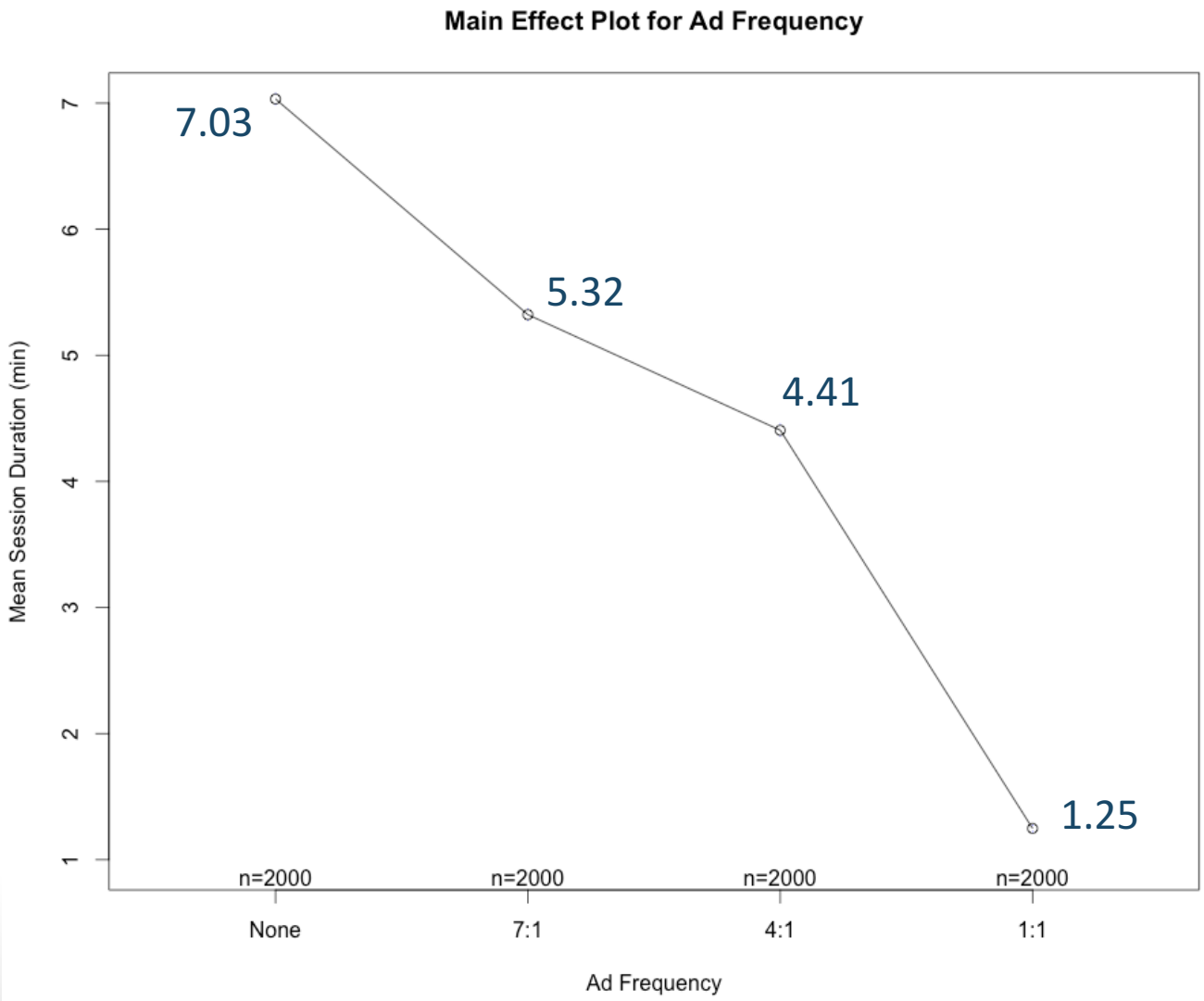
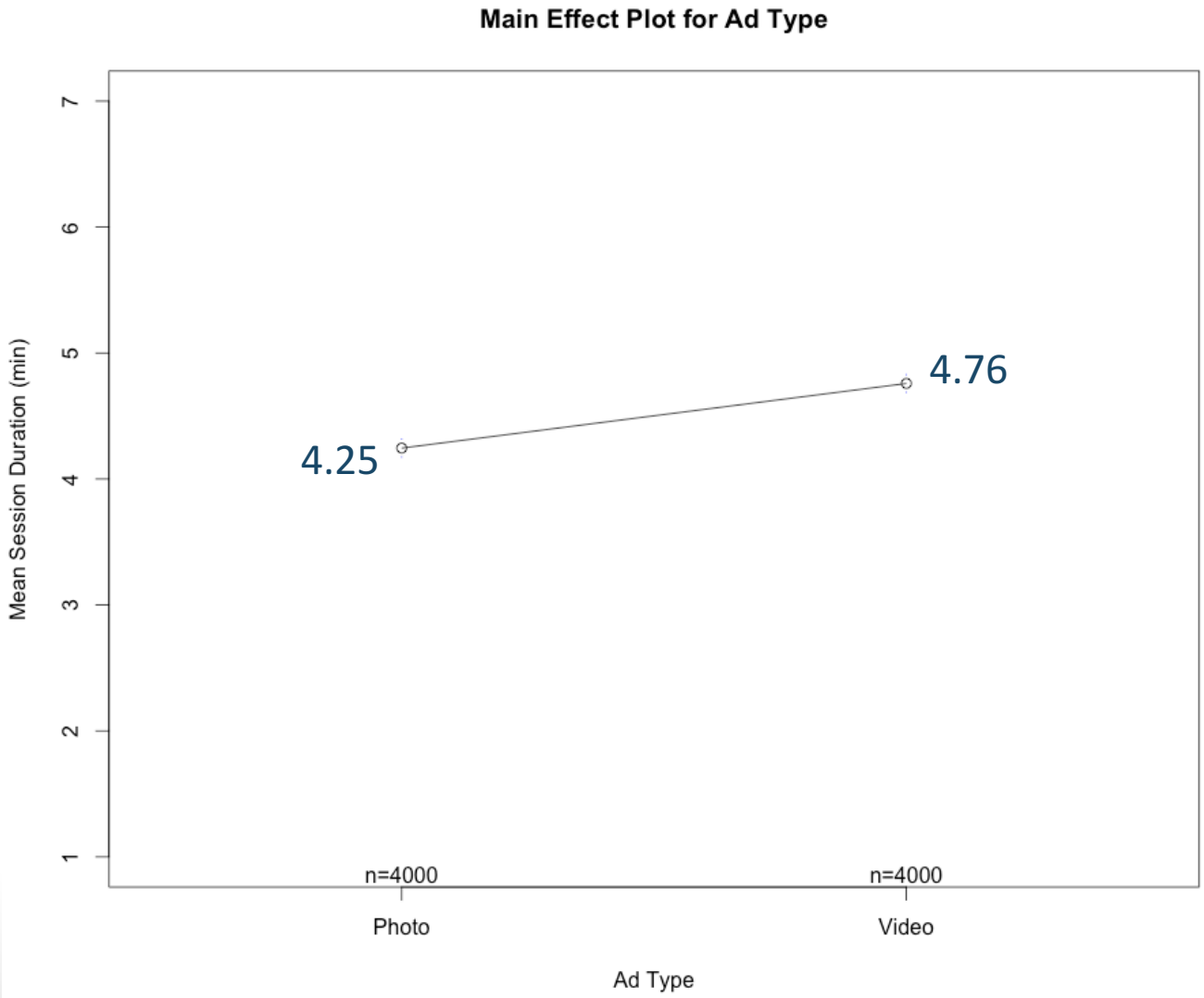
Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y



Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y



Multivariate Experiments

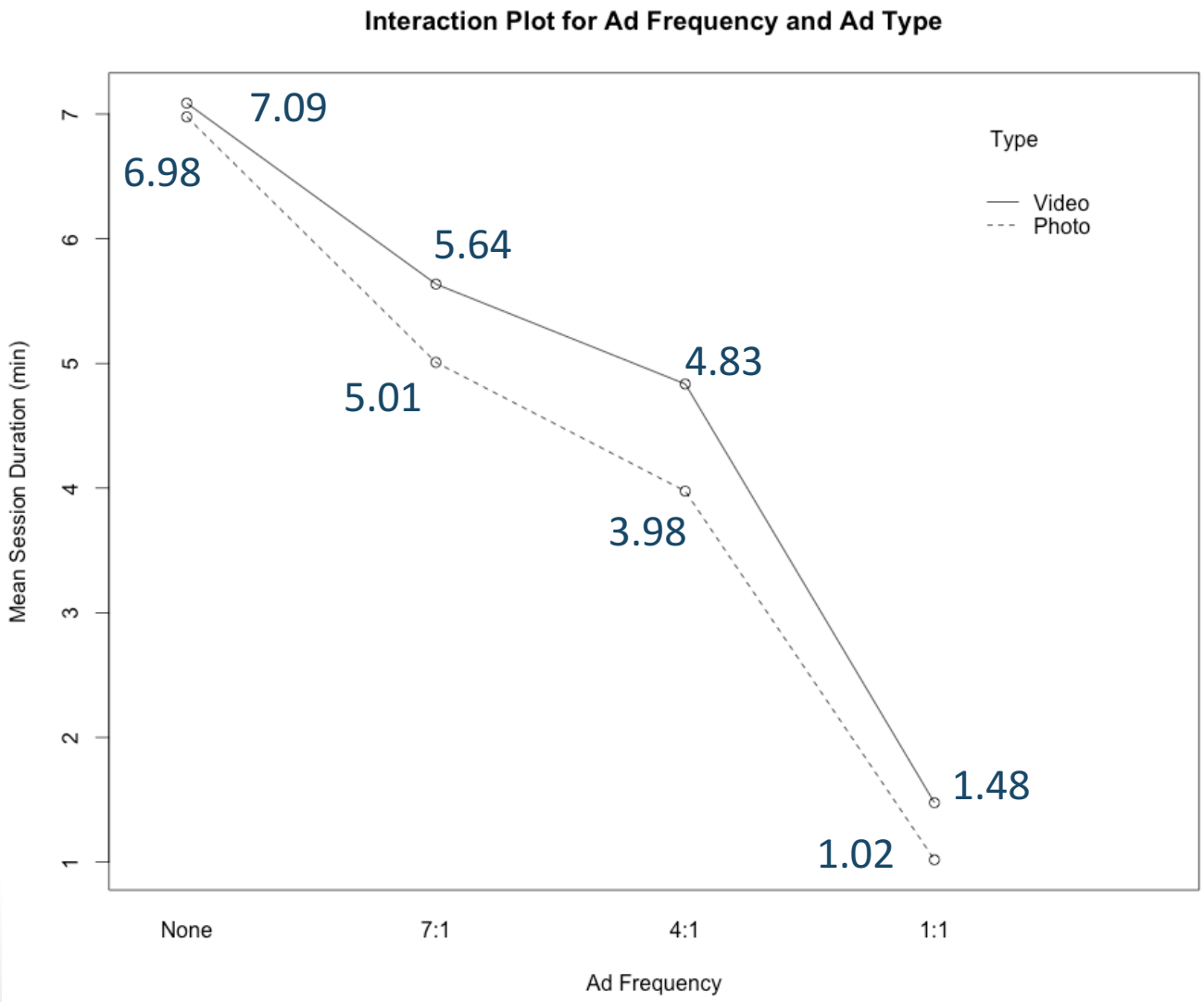
Analyzing a Factorial Experiment – Continuous Y

The main effect plots tell us:

- Session duration decreases as ad frequency increases
- Session duration is slightly longer for video ads vs. photo ads
- The influence of ad frequency is larger than the influence of ad type

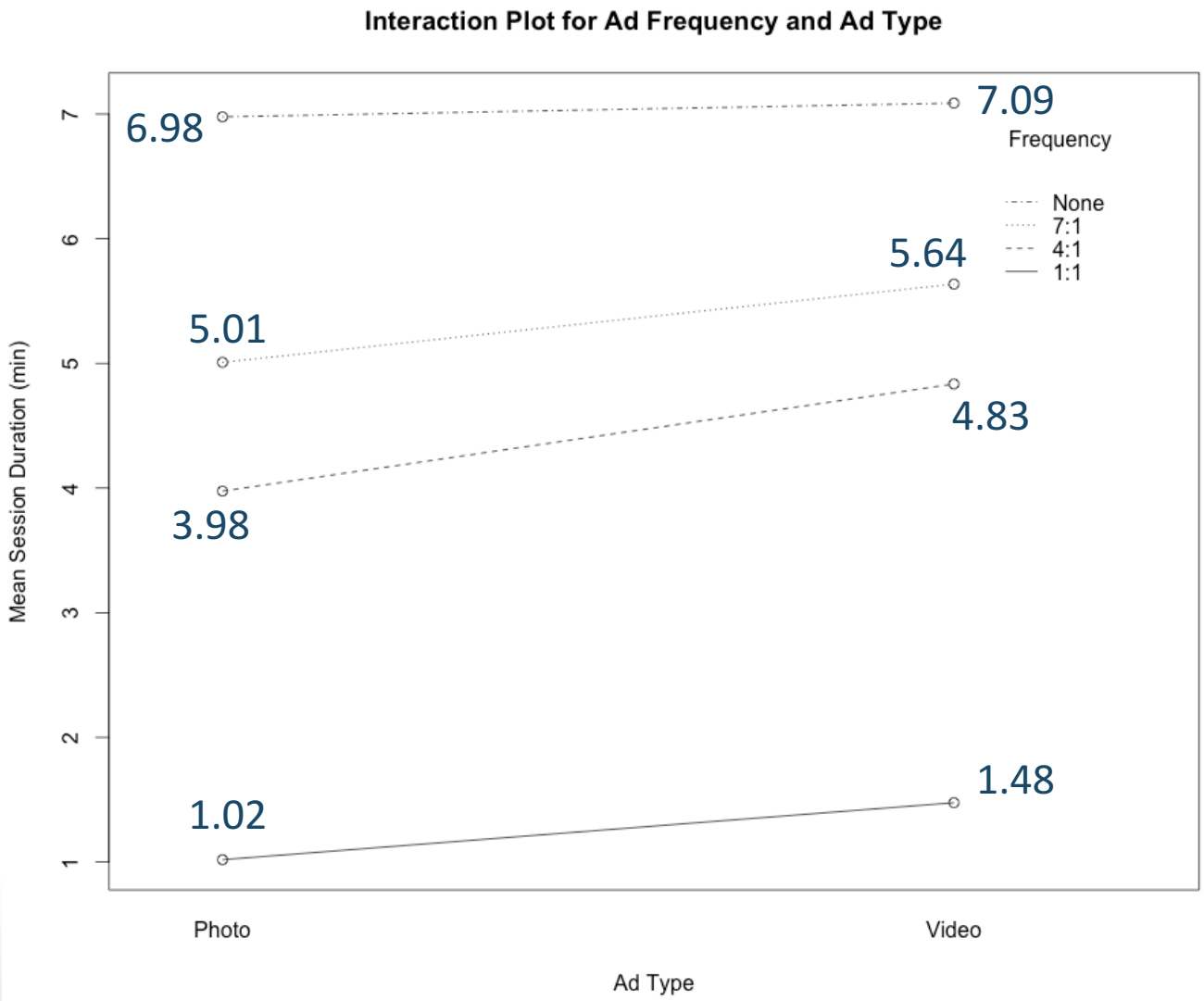
Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y



Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y



Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

The interaction effect plots tell us:

- The effect of ad frequency is not quite the same for both ad types
- The effect of ad type is not quite the same for all ad frequencies
- Thus an interaction is present

To formally decide whether the main and interaction effects are significant, we use linear regression

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

Linear regression models used for this purpose should contain

- Indicator variables for each factor; the number of indicators for a particular factor is equal to the number of levels of that factor, minus 1.
  - This allows us to evaluate main effects
- k-way products of the indicator variables for the k different factors.
  - This allows us to evaluate interaction effects

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

The linear regression model appropriate for the Instagram factorial example is

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i1} x_{i4} + \beta_6 x_{i2} x_{i4} + \beta_7 x_{i3} x_{i4} + \epsilon_i$$

where

- $x_{i1} = 1$ if unit i is in the 7:1 condition
- $x_{i2} = 1$ if unit i is in the 4:1 condition
- $x_{i3} = 1$ if unit i is in the 1:1 condition
- $x_{i4} = 1$ if unit i is in the video condition

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

Main effects become irrelevant in the context of interaction, and so it is common practice to first decide whether the interaction effect is significant

Note that $\beta_5 = \beta_6 = \beta_7 = 0$ removes the interaction terms from the model and so a test of

$$H_0: \beta_5 = \beta_6 = \beta_7 = 0 \text{ vs. } H_A: \beta_j \neq 0$$

formally tests whether the interaction effect is significant for at least one of $j = 5,6,7$

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

If the interaction effect is significant (i.e., we do not reject H_0) we must be careful to only draw conclusions regarding the effect of one factor in the context of the levels of the other factor

However, if the interaction effect is not significant (i.e., we do not reject $\beta_5 = \beta_6 = \beta_7 = 0$) we may use the **reduced** main effects model:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \epsilon_i$$

which can be used to evaluate the significance of main effects

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

The expected response, based on this model, in each of the "photo" conditions is shown below.

Frequency	Expected Response
None	$E[Y_i \mid x_{i1} = x_{i2} = x_{i3} = 0, x_{i4} = 0] = \beta_0$
7:1	$E[Y_i \mid x_{i1} = 1, x_{i4} = 0] = \beta_0 + \beta_1$
4:1	$E[Y_i \mid x_{i2} = 1, x_{i4} = 0] = \beta_0 + \beta_2$
1:1	$E[Y_i \mid x_{i3} = 1, x_{i4} = 0] = \beta_0 + \beta_3$

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

The expected response, based on this model, in each of the "video" conditions is shown below.

Frequency	Expected Response
None	$E[Y_i \mid x_{i1} = x_{i2} = x_{i3} = 0, x_{i4} = 1] = \beta_0 + \beta_4$
7:1	$E[Y_i \mid x_{i1} = 1, x_{i4} = 1] = \beta_0 + \beta_1 + \beta_4$
4:1	$E[Y_i \mid x_{i2} = 1, x_{i4} = 1] = \beta_0 + \beta_2 + \beta_4$
1:1	$E[Y_i \mid x_{i3} = 1, x_{i4} = 1] = \beta_0 + \beta_3 + \beta_4$

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

- Notice that the expectations in each row are identical if $\beta_1 = \beta_2 = \beta_3 = 0$
- Thus, ad frequency does not significantly influence the response if $\beta_1 = \beta_2 = \beta_3 = 0$
- We formally test whether the main effect of ad frequency is significant by testing

$$H_0: \beta_1 = \beta_2 = \beta_3 = 0 \text{ vs. } H_A: \beta_j \neq 0$$

for at least one of $j = 1,2,3$

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

- Notice that the expected response for photo vs. video ads becomes the same if $\beta_4 = 0$
- Thus, ad type does not significantly influence the response if $\beta_4 = 0$
- We formally test whether the main effect of ad type is significant by testing

$$H_0: \beta_4 = 0 \text{ vs. } H_A: \beta_4 \neq 0$$

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

- All of these hypothesis tests correspond to simultaneously setting a subset of the β's equal to zero
- Thus, each of these tests generates a reduced model with fewer terms than the corresponding full model
- In each case we compare the full and reduced models to decide if they seem significantly different – rejecting H_0 if they do
- This is done formally with a partial *F*-test

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

- The partial *F*-test compares the mean squared errors between the full and reduced models (similar to the F-test for overall significance in a linear regression)

- The test statistics and p-values associated with this test are provided in standard linear regression ANOVA output like `anova()` in R.

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

```
lm(formula = Time ~ Frequency * Type)
Residuals:
   Min     1Q    Median   3Q     Max
-3.7276 -0.5474 -0.0020 0.5499 4.4332
Coefficients:
                      Estimate Std.Error t value  Pr(>|t|)
(Intercept)            6.97785  0.02824  247.104 < 2e-16 ***
Frequency7:1          -1.96929  0.03994  -49.312 < 2e-16 ***
Frequency4:1          -3.00204  0.03994  -75.173 < 2e-16 ***
Frequency1:1          -5.95856  0.03994 -149.206 < 2e-16 ***
TypeVideo              0.10993  0.03994    2.753 0.00592 **
Frequency7:1:TypeVideo 0.51768  0.05648    9.166 < 2e-16 ***
Frequency4:1:TypeVideo 0.74924  0.05648   13.266 < 2e-16 ***
Frequency1:1:TypeVideo 0.34731  0.05648   6.150 8.14e-10 ***
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
Residual standard error: 0.893 on 7992 degrees of freedom
Multiple R-squared: 0.8497,Adjusted R-squared: 0.8496
F-statistic: 6455 on 7 and 7992 DF, p-value: < 2.2e-16
```

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Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

```
Analysis of Variance Table
Response: Time
                  Df Sum Sq  Mean Sq   F value        Pr(>F)
Frequency          3  35353  11784.3  14778.187  < 2.2e-16 ***
Type               1    527    527.3    661.318  < 2.2e-16 ***
Frequency:Type     3    149     49.8     62.398  < 2.2e-16 ***
Residuals       7992   6373      0.8
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
```

Multivariate Experiments

Analyzing a Factorial Experiment – Continuous Y

The p-values in the ANOVA table are sufficiently small so we conclude:

- Ad frequency has a significant main effect
- Ad type has a significant main effect
- The interaction between these factors is also significant

This means that both factors should be considered when trying to optimize session duration.

To determine which condition is optimal we can use a series of pairwise t-tests

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LOGISTIC REGRESSION – A PRIMER

Logistic Regression

- Linear regression is an effective method of modeling the relationship between a single response variable (Y), and one or more explanatory variables ($x_1, x_2, \ldots, x_p$)
- However, ordinary linear regression is appropriate for continuous response variable – particularly when $Y \sim N(\mu, \sigma^2)$
- When a response variable is binary, i.e. $Y \sim BIN(1, \pi)$, we use logistic regression instead

Logistic Regression

- In the context of regression models we call

$$\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

  the linear predictor

- In ordinary linear regression we relate the expected response to this linear predictor as follows:

$$E\left[Y|x_1, x_2, \ldots, x_p\right] = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

- But in logistic regression, the expected value of Y is $\pi \in [0,1]$, and it's unrealistic to constrain the linear predictor to the [0,1] interval

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Logistic Regression

- So instead of equating the linear predictor to $E[Y]$, in logistic regression we relate them through a link function that maps [0,1] to the real numbers
- The link function chosen for logistic regression is the logit function

$$\log\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

- Inverting this gives

$$E[Y|x_1, x_2, \ldots, x_p] = \frac{e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}}{1 + e^{\beta_0+\beta_1 x_1+\cdots+\beta_p x_p}}$$

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Logistic Regression

- β_0 is interpreted as the log-odds that $Y = 1$ when $x_1 = x_2 = \cdots = x_p = 0$
- Thus e^{β_0} is interpreted as the odds that $Y = 1$ (i.e., the odds that a unit performs the action of interest) when $x_1 = x_2 = \cdots = x_p = 0$
- It can also be shown that β_j, for $j = 1,2, \ldots, p$, is a log-odds ratio
- Specifically, e^{β_j} is interpreted as the odds ratio comparing the odds that $Y = 1$ when $x_j = x$ vs. when $x_j = x + 1$

Logistic Regression

- Parameter estimation in a logistic regression model is typically carried out use maximum likelihood estimation
- As a consequence tests for individual regression coefficients, such as

$$H_0: \beta_j = 0 \text{ vs. } H_A: \beta_j \neq 0$$

  are carried out using *Z*-tests
- To test whether several β's are simultaneously zero, and hence to compare the **full** model to a **reduced** one we use likelihood ratio tests

Logistic Regression

- The test statistic for the likelihood ratio test is referred to as the deviance and is defined as

$$\Lambda = 2[\log \text{likelihood}(\text{full model}) - \log \text{likelihood}(\text{reduced model})]$$

$$= 2\log\left(\frac{\text{likelihood}(\text{full model})}{\text{likelihood}(\text{reduced model})}\right)$$

- If H_0 is true (i.e., there is no difference between the two models) this approximately follows a χ^2 distribution with l degrees of freedom
- l is the number of restrictions imposed by H_0

Logistic Regression

- In the previous example the design of the experiment dictated the following linear predictor

$$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_1 x_4 + \beta_6 x_2 x_4 + \beta_7 x_3 x_4$$

- And a test of

$$H_0: \beta_5 = \beta_6 = \beta_7 = 0$$

was used to formally evaluate whether the interaction effect is significant

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Logistic Regression

- If the design of the experiment was identical, except the response variable was binary, the logistic regression model would be

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i1} x_{i4} + \beta_6 x_{i2} x_{i4} + \beta_7 x_{i3} x_{i4}$$

and

$$H_0: \beta_5 = \beta_6 = \beta_7 = 0$$

would be tested using a likelihood ratio test whose null distribution is the χ^2 distribution with $l = 3$ degrees of freedom, since H_0 involves 3 β's

Logistic Regression

- Any easy way to remember the degrees of freedom in a likelihood ratio is test is

$$l = p - q$$

  where p is the number of regression coefficients in the full model and q is the number of regression coefficients in the reduced model

- We will see an application of logistic regression to analysis of factorial experiments in the context of a two-level factorial experiment

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TWO-LEVEL FACTORIAL EXPERIMENTS

TWO-LEVEL FACTORIAL EXP'S

- Factorial experiments are an informative and efficient means of exploring several factors
- **Advantage:** every possible combination of factor levels is tested
  - We do not risk missing an optimal combination
- **Disadvantage:** every possible combination of factor levels is tested
  - Such experiments can become very large
- **Compromise:** two-level factorial experiments

TWO-LEVEL FACTORIAL EXP'S

Two-Level Factorial Experiments

- When investigating k factors, two-level factorial experiments are the smallest possible factorial experiments
- Such experiments are typically used for factor screening
- Pareto Principle: only *a vital few* factors are important relative to the *trivial many*
- The purpose of a screening experiment is to identify this small number of influential factors

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TWO-LEVEL FACTORIAL EXP'S

Two-Level Factorial Experiments

Here we discuss two particular types of two-level factorial experiments for investigating k factors:

- 2^k factorial experiments
  - These investigate each of the unique 2^k conditions

- 2^{k-p} fractional factorial experiments
  - These investigate just a *fraction* of the unique 2^k conditions

2^k FACTORIAL EXPERIMENTS

Designing 2^k Factorial Experiments

Step 1: Choose k factors that are expected to influence the response in some way

Step 2: Choose two levels for each factor to experiment with

- It's important to choose levels that provide the largest opportunity for an influential factor to be noticed
- Levels should be chosen that are quite different from one another; even a very influential factor may not appear to be influential if the factor levels are too similar.

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2^k FACTORIAL EXPERIMENTS

Designing 2^k Factorial Experiments

Step 3: The experimental conditions are defined to be the unique combination of these factors' levels

- There will be 2^k of them

Step 4: Assign experimental units to each of the 2^k conditions

- For ease of notation, we assume that the experiment is balanced and n units are assigned to each condition
- The sample size n can be determined by power analyses based on two-sample tests that account for the multiple comparison problem

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

Analysis in this context is performed using regression:

- Linear regression (when Y is continuous)
- Logistic regression (when Y is binary)

In either case the linear predictor of the model has

- $\binom{k}{1} = k$ main effect terms
- $\binom{k}{2}$ two-factor interaction terms
- $\vdots$
- $\binom{k}{k} = 1$ k-factor interaction term

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

The linear predictor associated with a 2^3 factorial experiment is

$$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3$$

For two-level designs we code the levels of the x's using ± 1's where -1 corresponds to the *low* level of the factor and +1 corresponds to the *high* level

Note that when a factor is categorical, the labels *low* and *high* are arbitrary

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

But when a factor is numeric, the *low* and *high* levels correspond to the small and large values of the factor

- In this case the translation from natural units to coded units can be done using the following formula

$$x_C = \frac{x_N - (x_H + x_L)/2}{(x_H - x_L)/2}$$

where x_H and x_L correspond to the high and low values of the factor as recorded in the natural units and x_N is any value of the factor in the natural units

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

- Using the ± 1 coding, each experimental condition can be identified by a unique combination of plus and minus ones
- The design of the experiment can be displayed in what is known as a design matrix
- Consider the design matrix for a 2^3 factorial experiment
  - Rows correspond to unique conditions
  - Columns indicate, for a given condition, which level each factor should be set at

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

- Using the ± 1 coding, each experimental condition can be identified by a unique combination of plus and minus ones
- The design of the experiment can be displayed in what is known as a design matrix
- Consider the design matrix for a 2^3 factorial experiment
  - Rows correspond to unique conditions
  - Columns indicate, for a given condition, which level each factor should be set at

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

Condition	Factor 1	Factor 2	Factor 3
1	-1	-1	-1
2	+1	-1	-1
3	-1	+1	-1
4	+1	+1	-1
5	-1	-1	+1
6	+1	-1	+1
7	-1	+1	+1
8	+1	+1	+1

- Design matrices provide a prescription for running 2^k factorial experiments

2^k FACTORIAL EXPERIMENTS

Analyzing 2^k Factorial Experiments

- Using the data collected from such a study we fit a linear or logistic regression model
- Apart from this difference, the models are similar in that
  - they are based on exactly the same linear predictor
  - we can evaluate the significance of main and interaction effects by performing tests concerning individual or multiple β's (although the specific tests that are used differ in the two settings)

2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

Montgomery (2017) presents an example in which a credit card company ran a 2^4 factorial experiment to test new ideas of improving conversion rates on credit card offers.

Factor	Low (-)	High (+)
Annual fee (x_1)	Current	Lower
Account-opening fee (x_2)	No	Yes
Initial interest rate (x_3)	Current	Lower
Long-term interest rate (x_4)	Low	High

The 2^4=16 combinations of these levels defined 16 credit card offers which were each sent to $n =$ 7500 people

Condition	x_1	x_2	x_3	x_4	Sign-ups	Conversion Rate
1	-1	-1	-1	-1	184	2.45%
2	+1	-1	-1	-1	252	3.36%
3	-1	+1	-1	-1	162	2.16%
4	+1	+1	-1	-1	172	2.29%
5	-1	-1	+1	-1	187	2.49%
6	+1	-1	+1	-1	254	3.39%
7	-1	+1	+1	-1	174	2.32%
8	+1	+1	+1	-1	183	2.44%
9	-1	-1	-1	+1	138	1.84%
10	+1	-1	-1	+1	168	2.24%
11	-1	+1	-1	+1	127	1.69%
12	+1	+1	-1	+1	140	1.87%
13	-1	-1	+1	+1	172	2.29%
14	+1	-1	+1	+1	219	2.92%
15	-1	+1	+1	+1	153	2.04%
16	+1	+1	+1	+1	152	2.03%

2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

Using the data we fit a logistic regression model with the following linear predictor:

$$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{34} x_3 x_4 + \beta_{123} x_1 x_2 x_3 + \beta_{124} x_1 x_2 x_4 + \beta_{134} x_1 x_3 x_4 + \beta_{234} x_2 x_3 x_4 + \beta_{1234} x_1 x_2 x_3 x_4$$

We do so using the `glm()` function in `R` with a logit link function. The model summary is shown on the next slide.

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```
Coefficients:
             Estimate Std. Error  z value   Pr(>|z|)
(Intercept) -3.739697   0.019342 -193.347   < 2e-16 ***
x1           0.080845   0.019342    4.180   2.92e-05 ***
x2          -0.106211   0.019342   -5.491   3.99e-08 ***
x3           0.058248   0.019342    3.011   0.00260 **
x4          -0.108086   0.019342   -5.588   2.29e-08 ***
x1:x2       -0.055164   0.019342   -2.852   0.00434 **
x1:x3       -0.004794   0.019342   -0.248   0.80426
x2:x3       -0.006967   0.019342   -0.360   0.71868
x1:x4       -0.013178   0.019342   -0.681   0.49566
x2:x4        0.010625   0.019342    0.549   0.58280
x3:x4        0.038079   0.019342    1.969   0.04899 *
x1:x2:x3    -0.009646   0.019342   -0.499   0.61799
x1:x2:x4     0.010629   0.019342    0.550   0.58265
x1:x3:x4    -0.002543   0.019342   -0.131   0.89539
x2:x3:x4    -0.020946   0.019342   -1.083   0.27885
x1:x2:x3:x4 -0.009496   0.019342   -0.491   0.62347
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
(Dispersion parameter for binomial family taken to be 1)
Null deviance: 26854 on 119999 degrees of freedom
Residual deviance: 26741 on 119984 degrees of freedom
AIC: 26773
Number of Fisher Scoring iterations: 6
```

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2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

- All main effects are significant
- Two two-factor interactions are significant
- All three-factor interactions are insignificant
- All four-factor interactions are insignificant

This suggests that a reduced model with

$$\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{123} = \beta_{124} = \beta_{134} = \beta_{234} = \beta_{1234} = 0$$

may be appropriate. The summary of this model is shown on the next slide

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```
Coefficients:
             Estimate Std. Error  z value   Pr(>|z|)
(Intercept) -3.73961    0.01934  -193.316   < 2e-16 ***
x1           0.08214    0.01920     4.279   1.88e-05 ***
x2          -0.10834    0.01920    -5.644   1.66e-08 ***
x3           0.05886    0.01916     3.072   0.00212 **
x4          -0.11068    0.01916    -5.777   7.61e-09 ***
x1:x2       -0.05706    0.01920    -2.972   0.00296 **
x3:x4        0.04051    0.01916     2.115   0.03447 *
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
(Dispersion parameter for binomial family taken to be 1)
Null deviance: 26854 on 119999 degrees of freedom
Residual deviance: 26744 on 119993 degrees of freedom
AIC: 26758
Number of Fisher Scoring iterations: 6
```

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2^k FACTORIAL EXPERIMENTS

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may be appropriate. The summary of this model is shown on the next slide

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2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

The formal test of

$$H_0: \beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = \beta_{123} = \beta_{124} = \beta_{134} = \beta_{234} = \beta_{1234} = 0$$

and hence a comparison of the reduced model to the full model involves a likelihood ratio test using

- $\log \text{likelihood}(\text{full}) = -13370.45$
- $\log \text{likelihood}(\text{reduced}) = -13371.91$

Which give a deviance statistic of $\Lambda = 2.9244$ and p-value of

$$P(T \geq 2.9244) = 0.9672$$

where $T \sim \chi^2_{(9)}$

2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

- Thus we do not reject H_0 which implies that the reduced model adequate
- Therefore, only the main effects and two of the two-factor interactions are significant
- The next two slides provide main effect and interaction effect plots for each of the significant effects

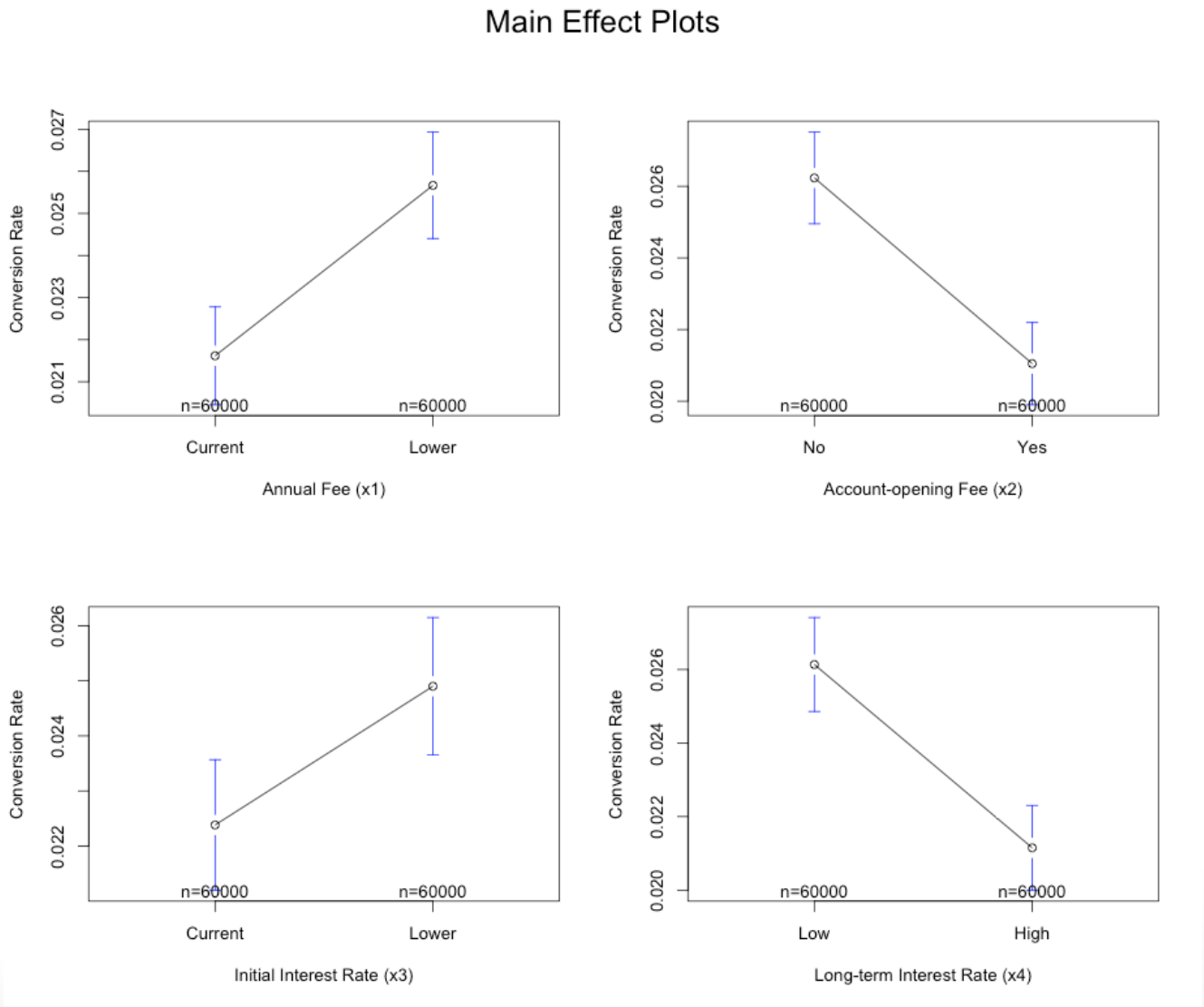
2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers

- The main effect plots suggest that:
  - lower annual fees, no account-opening fee, lower initial interest rates and lower long-term interest rates are all associated with increased conversion rates
- The interaction effect plots suggest that:
  - lower annual fees are associated with high conversion rates when there is no account-opening fee; when there is an account opening fee, the annual fee is not as influential
  - as long as the long-term interest rate is low, the initial interest rate doesn't really matter

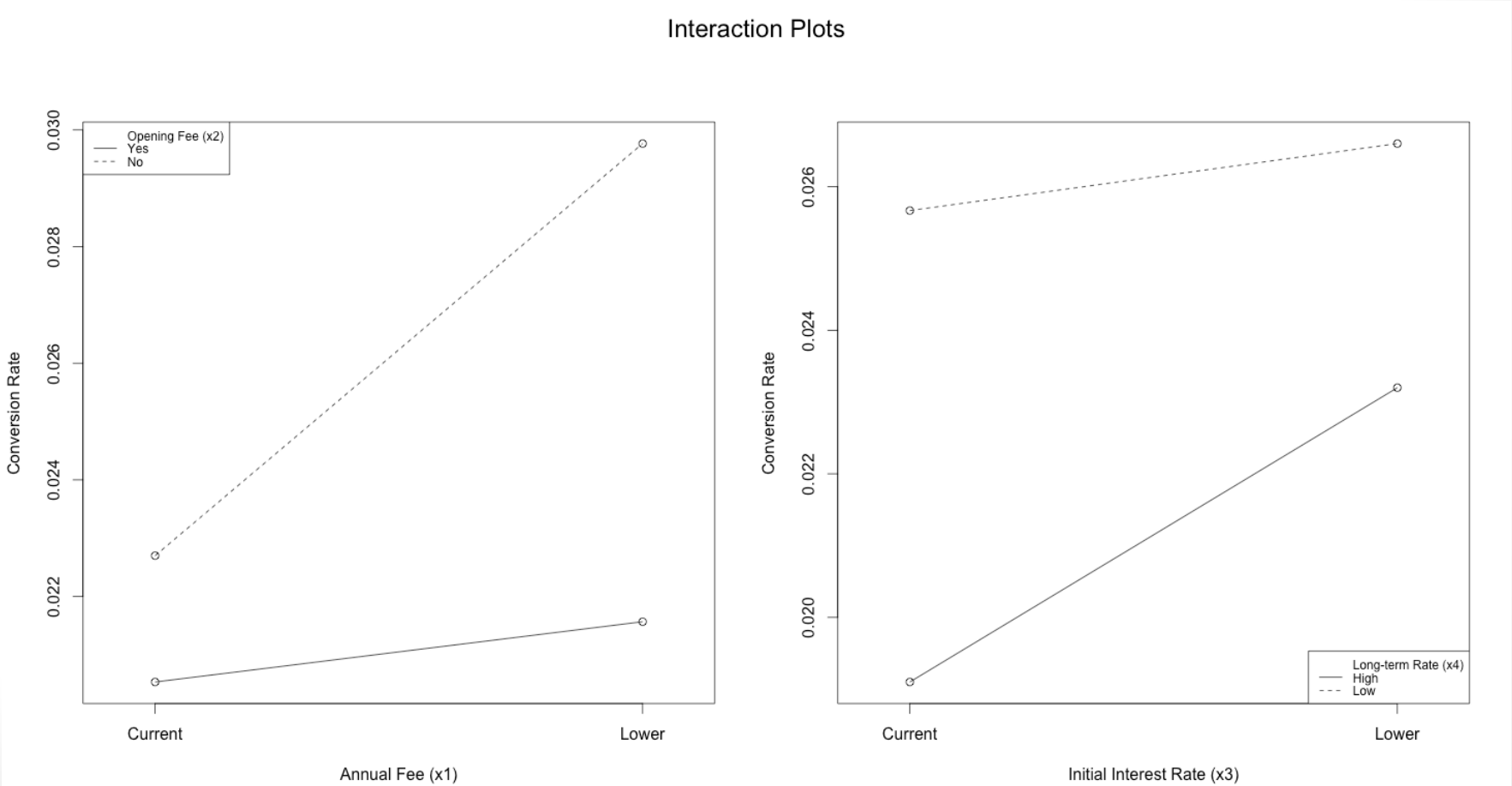
2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers



2^k FACTORIAL EXPERIMENTS

Example: Conversion on Credit Card Offers



2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- 2^k factorial experiments are a useful special case of a general factorial experiment
  - They minimize the number of levels being investigated, and hence reduces the overall number of experimental conditions

- BUT they still investigate **all possible** combinations of the factor levels – which can be a lot!
  - With $k = 8$ factors the 2^k factorial experiment has 256 conditions

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- Alternatively we could use a 2^{k-p} fractional factorial experiment which also investigates k factors, but with just a fraction of the conditions
- Rather than performing 2^k conditions, we perform 2^{k-p} specially selected conditions which still allow us to estimate main effects and potentially important interaction effects
- **With these experiments we can investigate a relatively large number of factors with a relatively small number of conditions**
- However, we sacrifice the ability to separately estimate *all* main and interaction effects

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

Motivation: the linear predictor for a 2^k factorial experiment consists of

- $\binom{k}{1} = k$ main effect terms
- $\binom{k}{2}$ two-factor interaction terms
- $\vdots$
- $\binom{k}{k} = 1$ k-factor interaction term

That's a total of $\sum_{i=1}^{k} \binom{k}{i} = 2^k - 1$ terms

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- Of these $2^k - 1$ terms, only $k + \binom{k}{2}$ of them are main effects and two factor interactions – the remaining correspond to higher order interaction terms

- If $k = 8$, there are 8 main effects, 28 two-factor interactions and 219 higher order interactions, many of which are likely to be insignificant

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- Principle of effect sparsity: in the presence of several factors, variation in the response is likely to be driven by a **small number of main effects and low-order interactions**
- Thus, it is typically a waste of resources to estimate these higher order interaction terms
- It is a better use of these resources is to estimate the main effects and low-order interactions of a larger number of factors
- So how do we do this?

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2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

First let's discuss p:

- Investigating k factors in a full factorial experiments takes 2^k conditions
- If we'd like to investigate k factors in half as many conditions, we use a 2^{k-1} experiment
- If we'd like to investigate k factors with just a quarter of the conditions, we use a 2^{k-2} experiment
- In general, if we'd like to investigate k factors in $(1/2)^p$ as many conditions, we use a 2^{k-p} experiment

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2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- If a full factorial approach requires 2^k conditions and we only want 2^{k-p}, we need to choose **which** 2^{k-p} conditions to experiment with

- For instance, if $k = 5$ and $p = 2$, then the goal is to investigate 5 factors in $2^3 = 8$ conditions (where normally 32 conditions would be required with the full factorial approach).

- The question is, among these 32 conditions, which 8 do we choose to for the 2^{5-2} fractional design?

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Condition	A	B	C	D	E
1	-1	-1	-1	-1	-1
2	+1	-1	-1	-1	-1
3	-1	+1	-1	-1	-1
4	+1	+1	-1	-1	-1
5	-1	-1	+1	-1	-1
6	+1	-1	+1	-1	-1
7	-1	+1	+1	-1	-1
8	+1	+1	+1	-1	-1
9	-1	-1	-1	+1	-1
10	+1	-1	-1	+1	-1
11	-1	+1	-1	+1	-1
12	+1	+1	-1	+1	-1
13	-1	-1	+1	+1	-1
14	+1	-1	+1	+1	-1
15	-1	+1	+1	+1	-1
16	+1	+1	+1	+1	-1

Condition	A	B	C	D	E
17	-1	-1	-1	-1	+1
18	+1	-1	-1	-1	+1
19	-1	+1	-1	-1	+1
20	+1	+1	-1	-1	+1
21	-1	-1	+1	-1	+1
22	+1	-1	+1	-1	+1
23	-1	+1	+1	-1	+1
24	+1	+1	+1	-1	+1
25	-1	-1	-1	+1	+1
26	+1	-1	-1	+1	+1
27	-1	+1	-1	+1	+1
28	+1	+1	-1	+1	+1
29	-1	-1	+1	+1	+1
30	+1	-1	+1	+1	+1
31	-1	+1	+1	+1	+1
32	+1	+1	+1	+1	+1

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- To answer this, we consider and extended version of the design matrix associated with a full 2^3 factorial experiment

Condition	A	B	C	AB	AC	BC	ABC
1	-1	-1	-1	+1	+1	+1	-1
2	+1	-1	-1	-1	-1	+1	+1
3	-1	+1	-1	-1	+1	-1	+1
4	+1	+1	-1	+1	-1	-1	-1
5	-1	-1	+1	+1	-1	-1	+1
6	+1	-1	+1	-1	+1	-1	-1
7	-1	+1	+1	-1	-1	+1	-1
8	+1	+1	+1	+1	+1	+1	+1

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- When it comes to fitting a regression model, each column in this matrix is used to estimate the corresponding effect a particular effect
- For instance:
  - the AB column is used to estimate β_{AB}, the interaction effect between factors A and B
  - the ABC column is used to estimate β_{ABC}, the interaction effect between factors A, B and C
- Now recall the effect sparsity principle: if an interaction is likely to be negligible, why not use its column to dictate the levels of an extra factor?

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

For example:

- Let's use the ± 1's in the ABC column as a prescription for when to run D at its low and high levels

- Let's use the ± 1's in the BC column as a prescription for when to run E at its low and high levels

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

Condition	A	B	C	AB	AC	E=BC	D=ABC
1	-1	-1	-1	+1	+1	+1	-1
2	+1	-1	-1	-1	-1	+1	+1
3	-1	+1	-1	-1	+1	-1	+1
4	+1	+1	-1	+1	-1	-1	-1
5	-1	-1	+1	+1	-1	-1	+1
6	+1	-1	+1	-1	+1	-1	-1
7	-1	+1	+1	-1	-1	+1	-1
8	+1	+1	+1	+1	+1	+1	+1

- Here we say that D and ABC are aliased and E and BC are aliased
- 'D=ABC' and 'E=BC' are called the design generators

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2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- When two terms are aliased, their effects become confounded
- For instance the D=ABC column estimates the ABC interaction **and** the main effect of D
- So the coefficient β_{ABC} quantifies the joint effects of the ABC interaction and the main effect of factor D
- Thus, we cannot separately estimate the main effect of D from the ABC interaction effect

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- Thus confounding results from aliasing a new main effect with an existing interaction
- As such, it is important to think carefully about **which** interaction to choose as an alias
- It is best to avoid aliasing a new factor with an interaction that is likely to be significant (since separately estimating significant effects is desirable)
- So high order interaction terms (that are unlikely to be significant) are good choices for aliases

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- This notion is quantified by the resolution of the fractional factorial design
- A design is of resolution R if main effects are aliased with interaction effects involving at least $R - 1$ factors
- In the design we've been discussing main effects are aliased with two- and three-factor interactions
- Thus it is a resolution III experiment denoted by

$$2^{5-2}_{III}$$

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- In general, higher resolution designs are to be preferred over lower resolution designs.
- For instance, resolution IV and V designs are to be preferred over a resolution III designs
- In these cases main effects will not be confounded with two-factor interactions
- Since two-factor interactions are typically important, it is best if their effects are not confounded with main effects

2^{k-p} FRACTIONAL FACTORIAL

Designing 2^{k-p} Fractional Factorial Experiments

- The resolution of a fractional factorial experiment is determined by two things:
  1. The degree of fractionation desired (i.e., the size of p relative to k)
  2. The design generators chosen for aliasing
- The degree of fractionation is typically determined by resource constraints – how many conditions can you manage?
- Given the degree of fractionation (p) we typically choose design generators to maximize resolution

2^{k-p} FRACTIONAL FACTORIAL

Analyzing 2^{k-p} Fractional Factorial Experiments

- The analysis of these fractional factorial experiments is based on regression models
  - Linear regression (if Y is continuous)
  - Logistic regression (if Y is binary)

- In fact, the analysis is not very different from what we saw in the credit card example
  - We perform individual and simultaneous hypothesis tests to compare full and reduced models
  - This allows us to evaluate the significance of various main and interaction effects

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2^{k-p} FRACTIONAL FACTORIAL

Analyzing 2^{k-p} Fractional Factorial Experiments

The wrinkle:

- Here the effects estimated in these models are confounded with other effects
- So we can't be 100% certain that a given effect is due to say a main effect, or perhaps the interaction it is aliased with
- But, if the resolution is high, we hope that important effects are aliased with high-order interactions (that are likely negligible)
- This provides confidence that significant effects are not due to the high-order interactions

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

Chehalem is a winery in Newberg Oregon that regularly uses experiments to develop and refine wine recipes. Montgomery (2017) discusses a 2_{IV}^{8-4} fractional factorial experiment that was used to investigate $k = 8$ factors with just 16 conditions.

The goal of the experiment was to evaluate and quantify the influence of several factors on the quality of the wine. The response variable here is a tasting score provided subjectively by $n = 5$ taste-testers.

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

Factor	Low (-)	High (+)
Pinot Noir clone (A)	Pommard	Wadenswil
Oak type (B)	Allier	Troncais
Age of barrel (C)	Old	New
Yeast/Skin contact (D)	Champagne	Montrachet
Stems (E)	None	All
Barrel toast (F)	Light	Medium
Whole cluster (G)	None	10%
Fermentation Temperature (H)	Low (75°F max)	High (92°F max)

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Condition	A	B	C	D	E	F	G	H	Avg. Rating
1	-1	-1	-1	-1	-1	-1	-1	-1	9.6
2	+1	-1	-1	-1	-1	+1	+1	+1	10.8
3	-1	+1	-1	-1	+1	-1	+1	+1	12.6
4	+1	+1	-1	-1	+1	+1	-1	-1	9.2
5	-1	-1	+1	-1	+1	+1	+1	-1	9.0
6	+1	-1	+1	-1	+1	-1	-1	+1	15.0
7	-1	+1	+1	-1	-1	+1	-1	+1	5.0
8	+1	+1	+1	-1	-1	-1	+1	-1	15.2
9	-1	-1	-1	+1	+1	+1	-1	+1	2.2
10	+1	-1	-1	+1	+1	-1	+1	-1	7.0
11	-1	+1	-1	+1	-1	+1	+1	-1	8.8
12	+1	+1	-1	+1	-1	-1	-1	+1	2.8
13	-1	-1	+1	+1	-1	-1	+1	+1	4.6
14	+1	-1	+1	+1	-1	+1	-1	-1	2.4
15	-1	+1	+1	+1	+1	-1	-1	-1	9.2
16	+1	+1	+1	+1	+1	+1	+1	+1	12.6

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- Because the response variable is continuous we use linear regression for this analysis
- Because only 2^4=16 conditions were used, we can only fit a model with 16 regression coefficients
- In the context of a full 2^4 factorial experiment this corresponds to a model with
  - 4 main effects
  - 6 two-factor interactions
  - 4 three-factor interactions
  - 1 four-factor interaction

```
Coefficients:
            Estimate Std. Error  t value   Pr(>|t|)
(Intercept) 8.5000    0.2658     31.985    < 2e-16 ***
A           0.8750    0.2658      3.293    0.001619 **
B           0.9250    0.2658      3.481    0.000906 ***
C           0.6250    0.2658      2.352    0.021772 *
D          -2.3000    0.2658     -8.655    2.27e-12 ***
A:B        -0.3500    0.2658     -1.317    0.192532
A:C         1.3000    0.2658      4.892    7.07e-06 ***
B:C         0.4500    0.2658      1.693    0.095261 .
A:D        -0.8750    0.2658     -3.293    0.001619 **
B:D         1.2250    0.2658      4.610    1.98e-05 ***
C:D         0.3750    0.2658      1.411    0.163063
A:B:C       1.5750    0.2658      5.927    1.35e-07 ***
A:B:D      -0.3000    0.2658     -1.129    0.263168
A:C:D      -1.0000    0.2658     -3.763    0.000367 ***
B:C:D       1.1000    0.2658      4.139    0.000104 ***
A:B:C:D     0.4750    0.2658      1.787    0.078613 .
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
Residual standard error: 2.377 on 64 degrees of freedom
Multiple R-squared: 0.7873,Adjusted R-squared: 0.7374
F-statistic: 15.79 on 15 and 64 DF, p-value: 4.547e-16
```

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2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- Notice this output does not involve the factors E, F, G or H – it only directly references factors A, B, C and D

- However, because of the confounding associated with the aliasing in this experiment
  - BCD interaction estimate corresponds to E's main effect
  - ACD interaction estimate corresponds to F's main effect
  - ABC interaction estimate corresponds to G's main effect
  - ABD interaction estimate corresponds to H's main effect

```
Coefficients:
            Estimate Std. Error  t value   Pr(>|t|)
(Intercept) 8.5000    0.2658     31.985    < 2e-16 ***
A           0.8750    0.2658      3.293    0.001619 **
B           0.9250    0.2658      3.481    0.000906 ***
C           0.6250    0.2658      2.352    0.021772 *
D          -2.3000    0.2658     -8.655    2.27e-12 ***
E           1.1000    0.2658      4.139    0.000104 ***
F          -1.0000    0.2658     -3.763    0.000367 ***
G           1.5750    0.2658      5.927    1.35e-07 ***
H          -0.3000    0.2658     -1.129    0.263168
A:B        -0.3500    0.2658     -1.317    0.192532
A:C         1.3000    0.2658      4.892    7.07e-06 ***
A:D        -0.8750    0.2658     -3.293    0.001619 **
A:E         0.4750    0.2658      1.787    0.078613 .
A:F         0.3750    0.2658      1.411    0.163063
A:G         0.4500    0.2658      1.693    0.095261 .
A:H         1.2250    0.2658      4.610    1.98e-05 ***
---
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
Residual standard error: 2.377 on 64 degrees of freedom
Multiple R-squared: 0.7873,Adjusted R-squared: 0.7374
F-statistic: 15.79 on 15 and 64 DF, p-value: 4.547e-16
```

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- All of the main effects – except H (fermentation temperature) – are significant
- Factors D, E, F, G (yeast/skin contact, stems, barrel toast, whole cluster) are most influential
- AC, AH and AD interactions are also significant
- Because of aliasing and confounding, it is equivalent to conclude that the DF, FG and EG interactions are significant
- Because factors D, E, F and G are most influential, it is likely that the DF, FG and EG interactions are responsible for the significant effect

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- Note that partial *F*-test of

$$H_0: \beta_H = \beta_{AB} = \beta_{AE} = \beta_{AF} = \beta_{AG} = 0$$

which compares the full model above to the one that is reduced by H_0 has an associated p-value of

$$P(T \geq 2.2124) = 0.06375$$

where $T \sim F(5, 64)$

- Thus we do not reject H_0 and we conclude that all factors other than H are significantly influential, and the DF, FG and EG interactions are statistically significant

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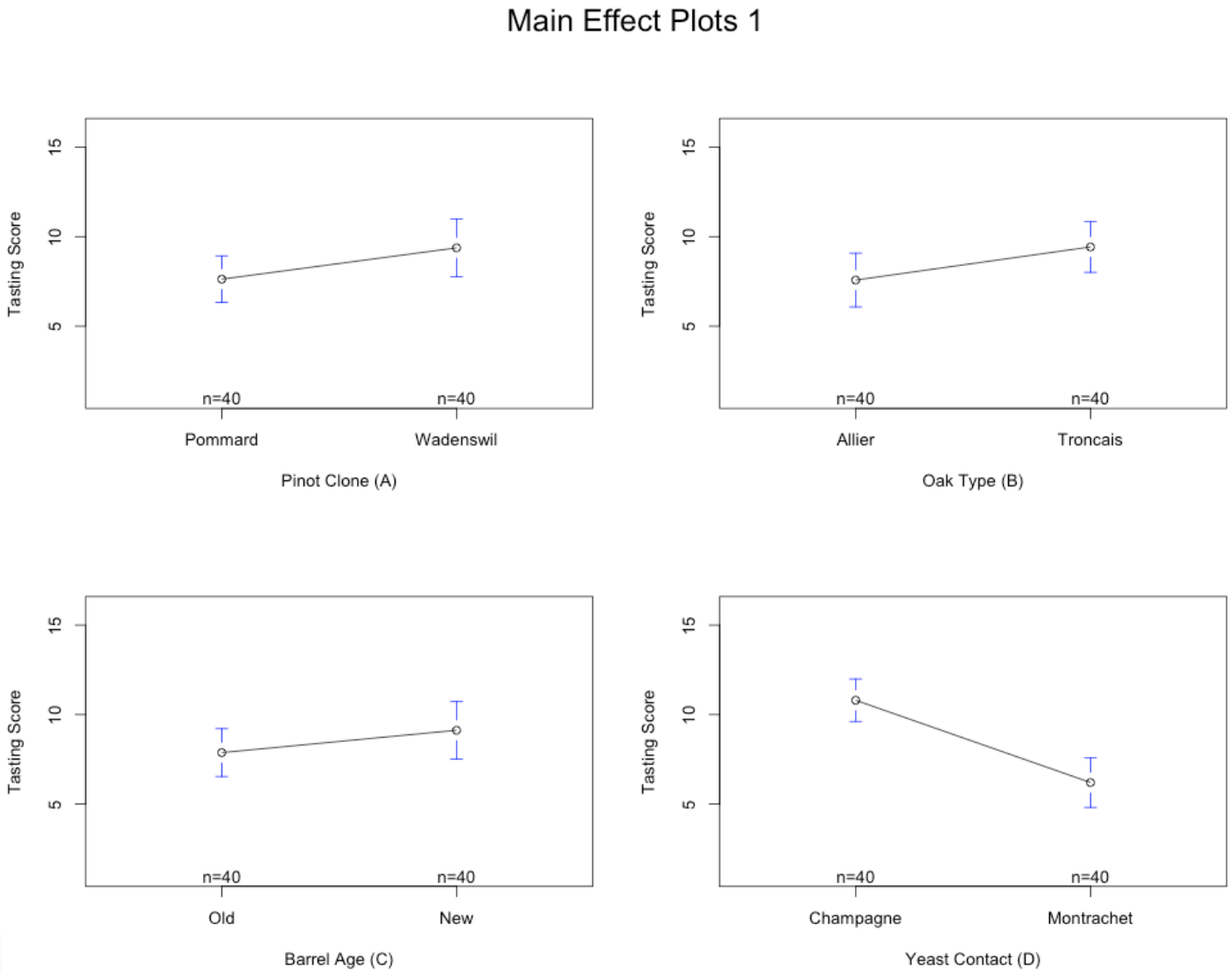
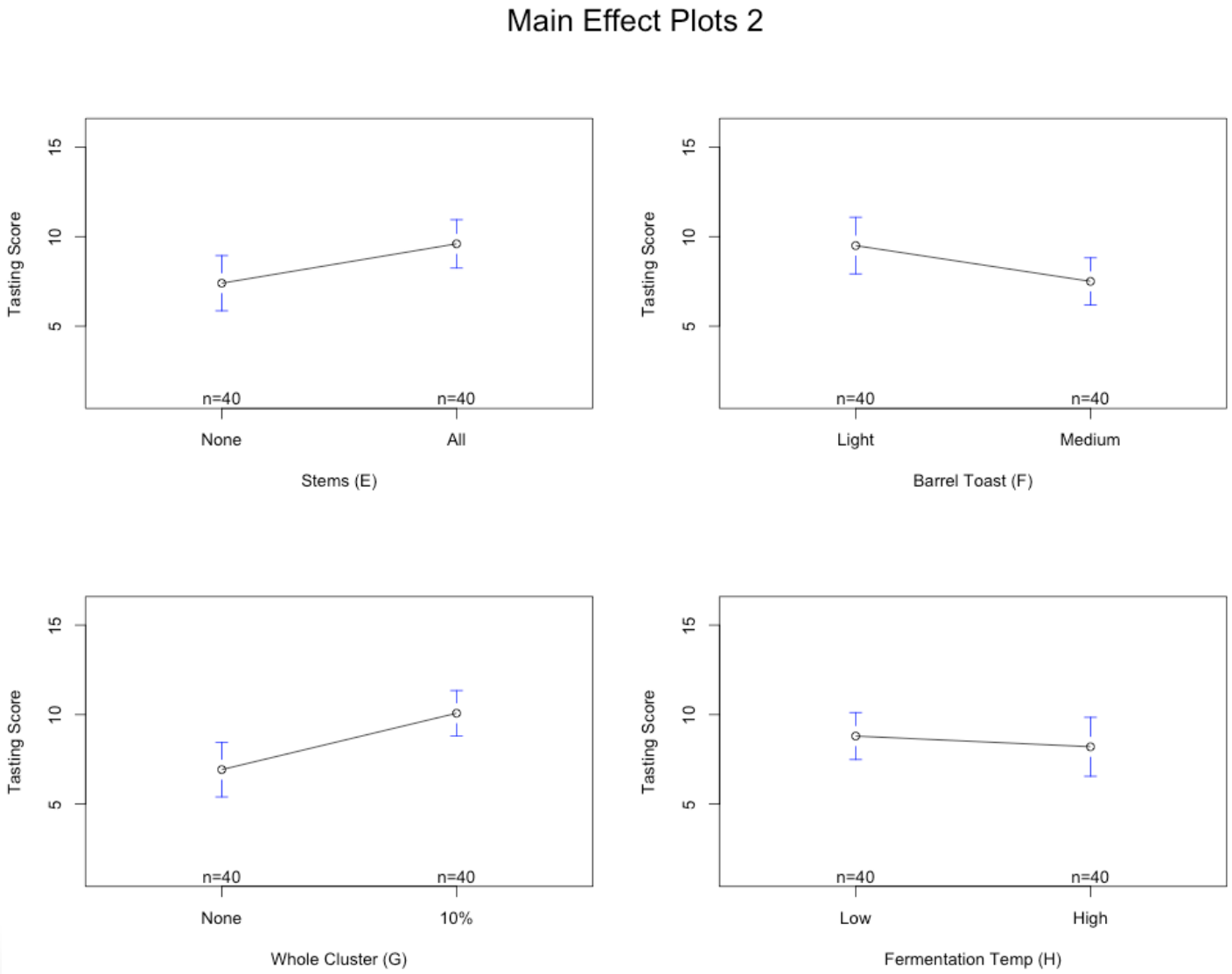
2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment



2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment



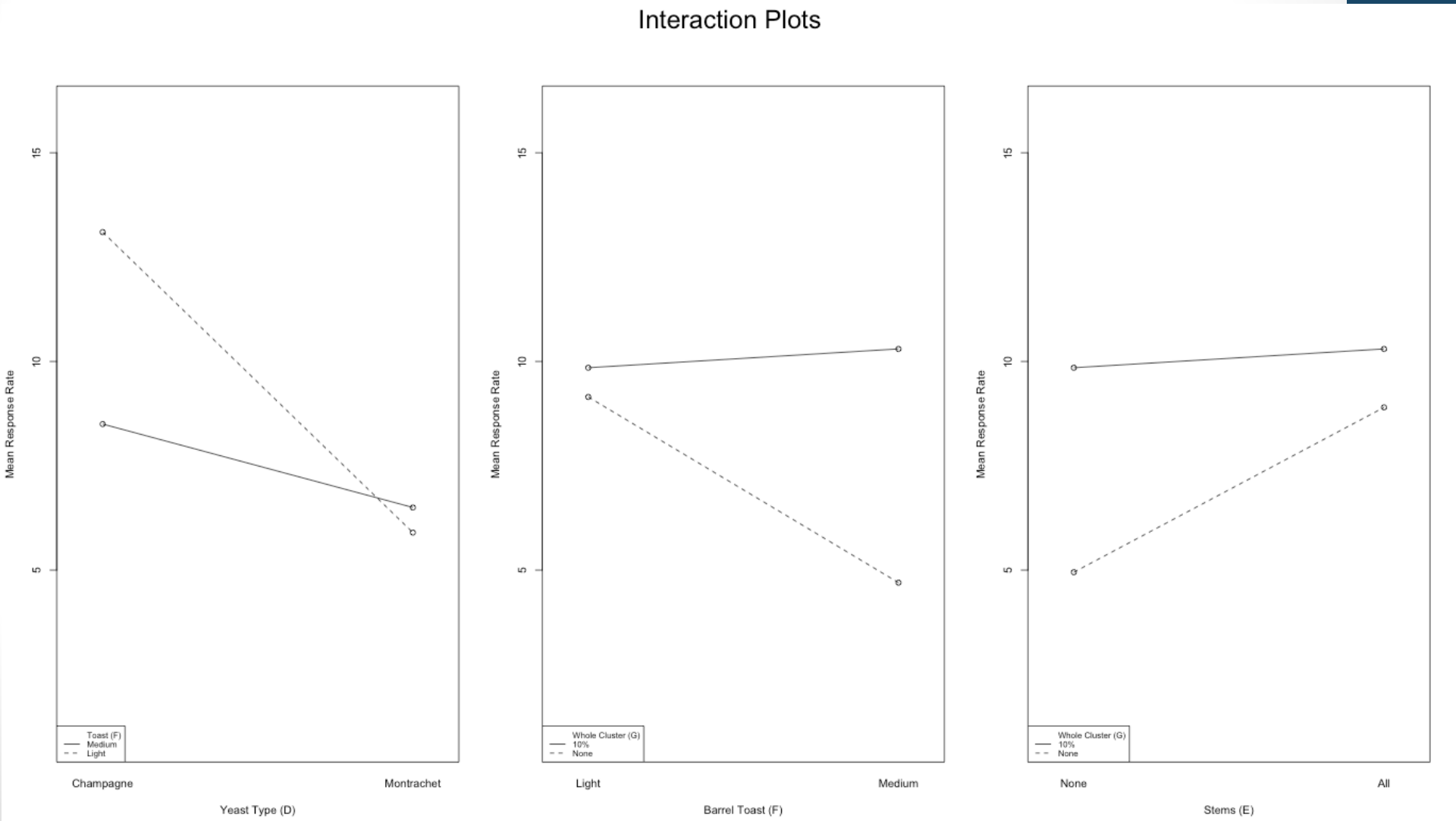
2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- The main effect plots suggest that:
  - yeast type (D) and the amount of whole clusters (G) used during fermentation are most important, with no whole clusters and Montrachet yeast producing a better tasting Pinot Noir
  - medium barrel toast (F) and no stems (E) also seem to correspond to a better tasting wine

2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment



2^{k-p} FRACTIONAL FACTORIAL

Example: The Chehalem Experiment

- The interaction effect plots suggest that:
  - If yeast type is Montrachet, the level of barrel toasting doesn't matter much, but if yeast type is Champagne, a medium barrel toast is best.
  - If barrel toast is chosen to be medium, then not including any whole-clusters is best
  - If using none of the stems, then it is also best not to include any whole-clusters

Take Home Exercises

Using R or Python, formally do the pairwise comparisons to find the optimal condition in each of the examples presented here. Be sure to account for the multiple comparison problem.

See you next week!

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