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DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

Mathematics II **Advanced Calculus (2AC)**

Vector Functions and Parametric Curves I

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0.1 Introduction

The study of vector functions in connection with parametric curves is so crucial for the vector analysis very useful in physical sciences. For instance, the description of the motion of an object is done through vector functions such as the displacement, the velocity and the acceleration. In this chapter, we will study operations on vector functions such as finding the domain, the limit at a given number, the continuity, the differentiability, the integration. Then we will move to vector functions a tool to describe parametric curves.

0.2 Definition of a vector function of a single variable

A vector function of a single variable is a function of the type $\vec{r} : [a, b] \to \mathbb{R}^n$ $(n \geq 2)$ defined by

$$\boxed{\vec{r}(t) := (r_1(t), r_2(t), \ldots, r_n(t)) \quad \forall t \in [a, b]}$$

where the functions $r_i : [a, b] \to \mathbb{R}$ are called **component functions of** $\vec{r}$.

Notation:

$$\boxed{\vec{r} = (r_1, r_2, \ldots, r_n)}\,.$$

Question: How do I visualize a vector function?
Answer: In the lectures!

Definition 0.1 Let $\vec{r} : [a, b] \to \mathbb{R}^n$ be a vector function. Then the set $\mathcal{C}$ of the locus points $\vec{r}(t)$, $t \in [a, b]$ is called the **curve described by the vector function** $\vec{r}$. In other words the curve $\mathcal{C}$ is the image of the vector function $\vec{r}$. Some textbooks identify the curve $\mathcal{C}$ to the vector function $\vec{r}$ and often use the abuse of language: "let $\vec{r} : [a, b] \to \mathbb{R}^n$ be the curve..."

We would prefer to rather use: "**let $\mathcal{C}$ be a curve described by the vector function $\vec{r}$...**"

In these lecture notes we will be studying only curves in $\mathbb{R}^2$ and $\mathbb{R}^3$!

0.3 Operations on vector functions

Proposition 0.2 *Let $\vec{u}$, $\vec{v} : [a,b] \to \mathbb{R}^n$ and let $\lambda : [a,b] \to \mathbb{R}$. Then, the functions $\vec{u}+\vec{v}$, $\lambda\vec{u}$, $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ for $n = 2, 3$ are defined as:*

(i) $(\vec{u}+\vec{v})(t) := \vec{u}(t)+\vec{v}(t)$ *for every* $t \in [a,b]$;

(ii) $(\lambda\vec{u})(t) := \lambda(t)\vec{u}(t)$ *for every* $t \in [a,b]$;

(iii) $(\vec{u}\cdot\vec{v})(t) := \vec{u}(t)\cdot\vec{v}(t)$ *for every* $t \in [a,b]$;

(iv) $(\vec{u}\times\vec{v})(t) := \vec{u}(t)\times\vec{v}(t)$ *for every* $t \in [a,b]$.

0.3.1 The domain of a vector function

The domain of a vector function $\vec{r} = (r_1, r_2, \ldots, r_n)$ is the set of the numbers t such that all the component functions r_i are defined at t. That is, the domain of $\vec{r}$ is the intersection of the domains of the component functions r_i. Notation

$$\boxed{\mathrm{Dom}(\vec{r}) = \mathrm{Dom}(r_1) \cap \mathrm{Dom}(r_2) \cap \ldots \cap \mathrm{Dom}(r_n)}.$$

Example 0.3 Find the domain of the vector function

$$\boxed{\vec{r}(t) := \left(\ln(t^2-3), \sqrt{5-t}, e^t\right)}.$$

Solution: We find the domain of each component function and then get their intersection. The function $\ln(t^2-3)$ is defined for every $t \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$. The function $\sqrt{5-t}$ is defined for every $t \in (-\infty, 5]$ while the function e^t is defined every where. So, the function $\vec{r}$ has domain

$$\boxed{\mathrm{Dom}(\vec{r}) = \left[(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)\right] \cap (-\infty, 5] = (-\infty, -\sqrt{3}) \cup (\sqrt{3}, 5]}.$$

We will now embark in the section on the notions of limit, continuity and differentiability of vector functions. As we will see, all these operations are componentwise.

Exercise 0.4 Find the domain of each vector function:

(a) $\vec{r}(t) := (\ln(6-t), \sin(t-1), \sqrt{t+1})$; **Answer:** $\mathrm{Dom}(\vec{r}) := [-1, 6)$;

(b) $\vec{u}(t) := \left(e^{-\frac{\pi}{t}}, \sqrt{8-t^2+2t}, \dfrac{1}{t+1}\right)$; **Answer:** $\mathrm{Dom}(\vec{u}) := [-2,-1) \cup (-1, 0) \cup (0, 4]$.

0.3.2 Limit - continuity - differentiability

Let $I \subset \mathbb{R}$ be an interval and let $\vec{r} = (r_1, r_2, \ldots, r_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a vector function and let $t_0 \in \mathbb{R}$. We say that $\vec{r}$ has limit at t_0 if each component r_i has limit at t_0 and

$$\boxed{\lim_{t\to t_0} \vec{r}(t) = \left(\lim_{t\to t_0} r_1(t), \lim_{t\to t_0} r_2(t), \ldots, \lim_{t\to t_0} r_n(t)\right)}. \tag{1}$$

Remark 0.5 In a more rigorious setting (as you will study in the second semester module 2RA), the number t_0 in (1) should be an accumulation point of the domain I. You are not required to know the definition of an accumulation point at this stage!

As an example, find the limit

$$\lim_{t\to 0}\left(\frac{\sin(t^2)}{3t}, \frac{1}{t}\int_0^t 2\cos(s^2)ds, t^2\sin\left(\frac{1}{t}\right)\right). \tag{2}$$

We first find

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(t^2)}{3t} &= \lim_{t\to 0}\frac{\sin(t^2)}{t^2}\times\frac{t^2}{3t} = \lim_{t\to 0}\frac{\sin(t^2)}{t^2}\times\frac{t}{3}\\ &= \left[\lim_{t\to 0}\frac{\sin(t^2)}{t^2}\right]\times\lim_{t\to 0}\frac{t}{3} = 1\times 0 = 0\,.\end{aligned}$$

We could also use L'Hôpital's (L'H's) rule!

Now to find $\lim_{t\to 0}\frac{1}{t}\int_0^t 2\cos(s^2)ds$, we set $f(t) := \int_0^t 2\cos(s^2)ds$ which is differentiable from the fundamental theorem of calculus (FTC) with $f'(t) = 2\cos(t^2)$. So,

$$\lim_{t\to 0}\frac{1}{t}\int_0^t 2\cos(s^2)ds = \lim_{t\to 0}\frac{f(t)-f(0)}{t} = f'(0) = 2\cos(0^2) = 2\,.$$

To find $\lim_{t\to 0} t^2\sin\left(\frac{1}{t}\right)$, we write

$$-1 \le \sin\left(\frac{1}{t}\right) \le 1 \quad \forall t\ne 0 \quad \Rightarrow \quad -t^2 \le t^2\sin\left(\frac{1}{t}\right) \le t^2 \quad \forall t \ne 0\,.$$

Hence, $\lim_{t\to 0} t^2\sin\left(\frac{1}{t}\right) = 0$ by the squeeze theorem. Therefore,

$$\boxed{\begin{aligned}&\lim_{t\to 0}\left(\frac{\sin(t^2)}{3t}, \frac{1}{t}\int_0^t 2\cos(s^2)ds, t^2\sin\left(\frac{1}{t}\right)\right)\\ =& \left(\lim_{t\to 0}\frac{\sin(t^2)}{3t}, \lim_{t\to 0}\frac{1}{t}\int_0^t 2\cos(s^2)ds, \lim_{t\to 0} t^2\sin\left(\frac{1}{t}\right)\right) = (0,2,0)\end{aligned}}$$

Remark 0.6 In the 2021 class test, almost 95% of students failed the question of finding the limit:

$$\boxed{\lim_{t\to 0}\left(\frac{1}{t}\int_{t^2}^{-\sin t}\cos(s^2)ds, \arccos\big(2-|t-2|\big)\right)}\,.$$

Definition 0.7

- We say that $\vec{r} = (r_1, r_2, \ldots, r_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ is continous at $t_0 \in I$ if $\lim_{t\to t_0}\vec{r}(t) = \vec{r}(t_0)$ that is, each component r_i is continuous at t_0.
- We say that $\vec{r} = (r_1, r_2, \ldots, r_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ is differentiable at $t_0 \in I$ if each component r_i is differentiable at t_0 and

$$\boxed{\vec{r}\,'(t_0) = \big(r'_1(t_0), r'_2(t_0), \ldots, r'_n(t_0)\big)}\,.$$

Remark 0.8 If the vector function $\vec{r}\colon [a,b] \to \mathbb{R}^n$ is differentiable at $t_0 \in (a,b)$ then we have

$$\underbrace{\boxed{\vec{r}(t) \approx \vec{r}(t_0) + (t-t_0)\vec{r}\,'(t_0) \quad \text{for } t \text{ close enough to } t_0}}_{\textbf{This is called the linear approximation of } \vec{r} \textbf{ around } t_0} \tag{3}$$

Some properties of the derivative of a vector function.

Proposition 0.9 *Let $\vec{u}$, $\vec{v} : [a,b] \to \mathbb{R}^n$ be differentiable functions and let $\lambda : [a,b] \to \mathbb{R}$. Then, the functions $\vec{u}+\vec{v}$, $\lambda\vec{u}$, $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ for $n = 2, 3$ are differentiable with:*

(i) $(\vec{u}+\vec{v})'(t) := \vec{u}\,'(t) + \vec{v}\,'(t)$ *for every* $t \in [a,b]$*;*

(ii) $(\lambda\vec{u})'(t) := \lambda'(t)\vec{u} + \lambda(t)\vec{u}\,'(t)$ *for every* $t \in [a,b]$*;*

(iii) $(\vec{u}\cdot\vec{v})'(t) := \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$ *for every* $t \in [a,b]$*;*

(iv) $(\vec{u}\times\vec{v})'(t) := \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$ *for every* $t \in [a,b]$.

Exercises 0.10

(a) If $\vec{u} : [a,b] \to \mathbb{R}^n$ is differentiable and $\|\vec{u}(t)\| = 1$ for every $t \in [a,b]$, then show that $\vec{u}(t) \perp \vec{u}\,'(t)$.

(b) Let $\vec{r}\colon [a,b] \to \mathbb{R}^3$ be a twice differentiable vector function. Show that if

$$\vec{u}(t) := \vec{r}(t) \times \vec{r}\,'(t) \quad \forall t \in [a,b],$$

then

$$\boxed{\vec{u}\,'(t) \perp \vec{r}\,''(t) \quad \forall t \in [a,b]}\,. \tag{4}$$

Hint: Use (iv) in Proposition 0.9 to find $\vec{u}\,'(t)$ and use some properties of the cross-product..

Definition 0.11 The integral of $\vec{r}$ over the interval $[a,b]$ is defined as

$$\boxed{\int_a^b \vec{r}(t)dt = \left(\int_a^b r_1(t)dt,\ \int_a^b r_2(t)dt,\ \ldots,\ \int_a^b r_n(t)dt\right)}\,. \tag{5}$$

Example 0.12 Find the integral $\displaystyle\int_0^1 \left(\frac{t-1}{t^2+1},\ te^t, t\sqrt{2-t^2}\right)$.
According to (5) we get

$$\int_0^1 \left(\frac{t-1}{t^2+1},\ te^t, t\sqrt{2-t^2}\right) dt = \left(\int_0^1 \frac{t-1}{t^2+1}\,dt,\ \int_0^1 te^t dt,\ \int_0^1 t\sqrt{2-t^2}dt\right).$$

Now

$$\int_0^1 \frac{t-1}{t^2+1}\,dt = \int_0^1 \left(\frac{t}{t^2+1} - \frac{1}{t^2+1}\right)dt = \left[\frac{1}{2}\ln(t^2+1) - \arctan t\right]_0^1 = \frac{1}{2}\ln 2 - \frac{\pi}{4}\,,$$
$$\int_0^1 te^t dt = \left[te^t\right]_0^1 - \int_0^1 e^t dt = e - (e-1) = 1\,,$$
$$\int_0^1 t\sqrt{2-t^2}dt = \left[-\frac{1}{3}\left(2-t^2\right)^{3/2}\right]_0^1 = \frac{1}{3}\left[2\sqrt{2}-1\right].$$

Hence,

$$\boxed{\begin{aligned}\int_0^1 \left(\frac{t-1}{t^2+1}, te^t, t\sqrt{2-t^2}\right) dt &= \left(\int_0^1 \frac{t-1}{t^2+1} dt, \int_0^1 te^t dt, \int_0^1 t\sqrt{2-t^2} dt\right) \\ &= \left(\frac{1}{2}\ln 2 - \frac{\pi}{4}, 1, \frac{1}{3}\left[2\sqrt{2}-1\right]\right)\end{aligned}}. \tag{6}$$

0.4 Parametric curves

We recall that given a vector function $\vec{r} : [a, b] \to \mathbb{R}^n$, we define the curve $\mathcal{C}$ described by the vector function $\vec{r}$ as the locus of the points $\vec{r}(t)$ with $t \in [a, b]$. In other words, $\mathcal{C}$ is the image of the vector function $\vec{r}$. Some textbooks are using the terminology the **trace of the vector function** $\vec{r}$ to indicate the curve $\mathcal{C}$ described by the vector function $\vec{r}$.

Notice that a given curve $\mathcal{C}$ can be described by different vector functions. In others words, a given curve can have different parametrizations. In fact, the two vector functions $\vec{r}_1 : [0, 2\pi] \to \mathbb{R}^2$ and $\vec{r}_2 : [0, 4\pi] \to \mathbb{R}^2$ defined by

$$\boxed{\vec{r}_1(t) := (\cos t, \sin t) \quad \forall t \in [0, 2\pi] \quad \text{and} \quad \vec{r}_2(t) := (\cos t, \sin t) \quad \forall t \in [0, 4\pi]} \tag{7}$$

describe the **same** curve $\mathcal{C}$ which is the circle centered at the origin with radius 1.

Definition 0.13 (Change of parametrization)
Let $\mathcal{C}$ be a curve described by the vector function $\vec{r} : I \to \mathbb{R}^n$ where I is an interval (bounded or unbounded) in $\mathbb{R}$. We say that we operate a change of parametrization of the curve $\mathcal{C}$ when we find a new function $\phi : J :\to I$ continuous and strictly increasing from an other interval J into the interval I and defined by $\phi(\tau) := t$ for every $\tau \in J$ such that the vector function $\vec{u} : J \to \mathbb{R}^n$ defined by $u(\tau) := \vec{r}(\phi(\tau))$, describes the same curve $\mathcal{C}$. In many practical situtations, such a function ϕ is even differentiable (with of course $\phi'(\tau) > 0$).

Definition 0.14 Let $\mathcal{C}$ be the curve described by the vector function $\vec{r} : [a, b] \to \mathbb{R}^n$. We say that:

(1) The curve $\mathcal{C}$ is **regular** or **smooth** if

- (i) $\vec{r}$ is continuously differentiable i.e., $\vec{r}$ is differentiable with its derivative $\vec{r}\,'$ continuous;
- (ii) $\vec{r}\,'(t) \neq \vec{0}$ for every $t \in (a, b)$ (Notice that $\vec{r}\,'$ can be 0 at the endpoints a and b).

(2) The curve $\mathcal{C}$ is **simple** if $\vec{r}(t) \neq \vec{r}(s)$ for every $s, t \in (a, b)$ with $s \neq t$, which is equivalent to saying that $\vec{r}(t) = \vec{r}(s)$ if and only if $s = t$ or $s, t \in \{a, b\}$. In words, this means that **the curve $\mathcal{C}$ does not cross itself.**

(3) The curve $\mathcal{C}$ is **closed** if $\vec{r}(a) = \vec{r}(b)$, where $\vec{r}(a)$ is called the initial or starting point while $\vec{r}(b)$ is called the terminal or final point.

(4) The curve $\mathcal{C}$ is **piecewise regular** if $\mathcal{C}$ is a union of a finite number of regular curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ with $\mathcal{C}_i \cap \mathcal{C}_{i+1} = \{\text{single point}\}$ for $i = 1, \ldots, k-1$. A piecewise regular curve is often called a **path**.

Example 0.15

(a) The curve $\mathcal{C}$ (circle centered at the origin and radius 1) described by $\vec{r}_1 : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\boxed{\vec{r}_1(t) := (\cos t, \sin t) \qquad \forall t \in [0, 2\pi]}$$

is simple and closed.

The same curve $\mathcal{C}$ (circle centered at the origin and radius 1) described by $\vec{r}_2 : [0, 4\pi] \to \mathbb{R}^2$ defined by

$$\boxed{\vec{r}_2(t) := (\cos t, \sin t) \qquad \forall t \in [0, 4\pi]}$$

is closed but not simple because for instance, $\vec{r}_2\left(\frac{\pi}{2}\right) = (0, 1) = \vec{r}_2\left(\frac{5\pi}{2}\right)$.

(b) The curve $\mathcal{C}$ described by the vector function $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\boxed{\vec{r}(t) := (t^3 - t, t^2 - 1) \qquad \forall t \in \mathbb{R}}$$

is neither closed nor simple since $\vec{r}(-1) = (0, 0) = \vec{r}(1)$.

The curve $\mathcal{C}$ is called **strophoid** and is very famous in Geometry. You can read more on Wikipedia about the geometric process through which the strophoid is generated.

(c) However, we consider the piece $\mathcal{C}_1$ of the curve $\mathcal{C}$ described by $\vec{r}_1 := \vec{r}|_{[-1,1]}$ the restriction of the vector function $\vec{r}$ to the interval $[-1, 1]$, i.e.,

$$\boxed{\vec{r}_1(t) := (t^3 - t, t^2 - 1) \qquad \forall t \in [-1, 1]}\,.$$

The piece $\mathcal{C}_1$ is a closed simple curve and is called the **loop of the strophoid**.

0.4.1 On sketching a parametric curve

Let $\mathcal{C}$ be a curve described by the vector function $\vec{r} : [a, b] \to \mathbb{R}^2$ defined by

$$\vec{r}(t) := (f(t), g(t)) \quad \forall t \in [a, b]\,.$$

What is the best way of sketching the curve $\mathcal{C}$ on the xy-plane?

The mistake that students often make for such a task is to give successive values to the parameter t, plot the corresponding points and join them with arcs. Unfortunately, this approach often leads to the wrong shape of the curve.

The right approach consists in the following:

- Write the parametric equations of the curve as

$$\boxed{x = f(t) \qquad \text{and} \qquad y = g(t) \quad \text{for} \quad t \in [a, b]}\,.$$

- Eliminate the parameter t from the parametric equations in order to obtain the cartesian equation of the curve $\mathcal{C}$, which often gives you an idea of the shape of the curve.

- Identify the x- and y-intercepts from the cartesian equation and finally sketch the curve.

Try sketching the strophoid using those steps!

0.4.2 On the use of polar coordiantes

Remark 0.16 So far, we described curves using the cartesian coordinate system. However, it is sometimes more convenient to describe curves in other coordinate systems. For instance, the equation of a curve in polar coordinates reads as

$$\boxed{r = r(\theta) \quad \text{for} \quad \theta_0 \le \theta \le \theta_1}$$

called the **polar equation** of the curve which is converted in cartesian coordinates as

$$\boxed{x = r(\theta)\cos\theta \quad \text{and} \quad y = r(\theta)\sin\theta \quad \text{for} \quad \theta_0 \le \theta \le \theta_1}.$$

If the function $\theta \to r(\theta)$ is of class $\mathcal{C}^1$ i.e., that is $\theta \to r(\theta)$ is differentiable with continuous derivative, then curve $\mathcal{C}$ is regular if and only if

$$\boxed{\big[r(\theta)\big]^2 + \big[r'(\theta)\big]^2 > 0 \qquad \forall\theta \in (\theta_0, \theta_1)}. \tag{8}$$

Remark 0.17 In the 2021 class test, almost 90% of students failed the question:

Let $\mathcal{C}$ be the curve whose polar equation (in polar coordinates r, θ) is given by

$$r(\theta) = \theta^2 - 1, \qquad \frac{\pi}{3} \le \theta \le \frac{\pi}{2}.$$

(a) Find a vector function describing the curve $\mathcal{C}$.

(b) Show that the curve $\mathcal{C}$ is regular and simple.

(c) Find the length of the curve $\mathcal{C}$.

Exercise 0.18 Sketch the curve $\mathcal{C}$ in Example 0.15(b) described by the vector function $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\boxed{\vec{r}(t) := (t^3 - t, t^2 - 1) \qquad \forall t \in \mathbb{R}}$$

Hint: First write the parametric equations of the curve $\mathcal{C}$, then its cartesian equation.

0.4.3 Tangent line to a regular curve

For a better illustration we take $n = 2$. So, let $\mathcal{C}$ be a regular curve described by the vector function $\vec{r} : [a, b] \to \mathbb{R}^2$. We want to write the parametric equations of the tangent line to the curve $\mathcal{C}$ at the point $(x_0, y_0) = \vec{r}(t_0)$ for some $t_0 \in (a, b)$. So, the function $\vec{r}$ is differentiable at the number t_0 with $\vec{r}\,'(t_0) \ne (0, 0)$.
We consider the secant lines (L_t) passing throught the point (x_0, y_0) and the running points $\vec{r}(t)$ for $t \in (a, b)$. Each secant line has direction which is conveniently chosen as

$$\boxed{\frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0}}. \tag{9}$$

Now, we ask the question: **What happens to the family of secant lines as t approaches the number t_0?**

It follows from the differentiability of the function $\vec{r}$ at the number t_0 that the directions of the secant lines L_t in (9) approach the vector $\vec{r}\,'(t_0)$ which is the direction of the new line passing through the point (x_0, y_0) called the **tangent line to the curve $\mathcal{C}$ at the point** (x_0, y_0).

$$\boxed{\vec{r}\,'(t_0) \textbf{ is the direction of the tangent line to the curve } \mathcal{C} \textbf{ at } \vec{r}(t_0)}\,. \tag{10}$$

Now, having the direction $\vec{r}\,'(t_0) = (r\,'_1(t_0), r\,'_2(t_0))$ of the tangent line to the curve $\mathcal{C}$ at the point $(x_0, y_0) = \vec{r}(t_0)$, we write the parametric equations of that line are

$$\boxed{x = x_0 + \lambda\, r\,'_1(t_0), \qquad y = y_0 + \lambda\, r\,'_2(t_0), \qquad \lambda \in \mathbb{R}}\,. \tag{11}$$

Remark 0.19 Notice that the notion of regularity or smoothness of a curve geometrically means means that not only the curve has a tangent line at every point but also that the tangent varies contiunuously as we move along the curve.

- The **unit tangent vector of $\mathcal{C}$ at** $\vec{r}(t)$ is defined as

$$\boxed{\vec{T}(t) := \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}\,. \tag{12}$$

Example 0.20 Let $\mathcal{C}$ be the curve described by the vector function $\vec{r}\colon [0, 5\pi/4] \to \mathbb{R}^2$ defined by $\vec{r}(t) := (-t + 2\sin(2t), \cos(3t) + \cos(2t))$. Write down the parametric equations of the tangent line to the curve $\mathcal{C}$ at the point $\big(2 - \frac{\pi}{4}, -\frac{1}{\sqrt{2}}\big)$.

Solution: It is easy to see that the given point $\big(2 - \frac{\pi}{4}, -\frac{1}{\sqrt{2}}\big)$ corresponds to $t = \dfrac{\pi}{4}$. Hence, the direction of the tangent line to $\mathcal{C}$ at the point $\big(2 - \frac{\pi}{4}, -\frac{1}{\sqrt{2}}\big)$ is given by $\vec{r}\,'(\frac{\pi}{4})$. We get

$$\vec{r}\,'(t) = (-1 + 4\cos(2t), -3\sin(3t) - 2\sin(2t)) \quad \Rightarrow \quad \vec{r}\,'\Big(\frac{\pi}{4}\Big) = \Big(-1, -\frac{3}{\sqrt{2}} - 2\Big)\,.$$

So, the tangent line has parametric equations

$$\boxed{x = 2 - \frac{\pi}{4} - \lambda, \quad y = -\frac{1}{\sqrt{2}} - \lambda\Big[\frac{3}{\sqrt{2}} + 2\Big], \qquad \lambda \in \mathbb{R}}\,. \tag{13}$$

0.4.4 The arclength

Let $\mathcal{C}$ be the curve described by the vector function $\vec{r}: [a, b] \to \mathbb{R}^2$. We want to find the length $L(\mathcal{C})$ of the curve $\mathcal{C}$.
We consider a partition P of the interval $[a, b]$:

$$\boxed{P: \quad a = t_0 < t_1 < \ldots < t_{k-1} < t_k = b}$$

with each subinterval $[t_{i-1}, t_i]$ having length $\Delta t_i = t_i - t_{i-1}$. To the partition P, there corresponds the following points on the curve $\mathcal{C}$

$$\boxed{\vec{r}(t_0),\, \vec{r}(t_1),\, \ldots,\, \vec{r}(t_{k-1}),\, \vec{r}(t_k)}\,.$$

Let $\mathcal{P}$ be the polygonal line wth vertices $\vec{r}(t_i)$. The length of $\mathcal{P}$ is

$$\boxed{L(\mathcal{P}) = \sum_{i=1}^{k} \|\vec{r}(t_i) - \vec{r}(r_{i-1})\| \approx \sum_{i=1}^{k} \|\vec{r}\,'(t_i)\|\,\Delta t_i \quad \text{(from the linear approximation in (3))}}\,.$$

Intuitively, the number $L(\mathcal{P})$ can be used as an approximation of the length of the curve. However, one realises that such an approximation is as optimal as the maximal length $\Delta\ell$ of the segment lines in the polygonal line is small. Therefore the length of the curve $\mathcal{C}$ is

$$\boxed{L(\mathcal{C}) := \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{k} \|\vec{r}(t_i) - \vec{r}(t_{i-1})\| = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{k} \|\vec{r}\,'(t_i)\|\,\Delta t_i = \int_a^b \|\vec{r}\,'(t)\|\,dt} \tag{14}$$

where the second "=" follows from the fact that $\|\vec{r}(t_i) - \vec{r}(t_{i-1})\| \approx \|\vec{r}'(t_i)\|\,\Delta t_i$ (see (3)) and the last "=" follows from the fact $\displaystyle\sum_{i=1}^{k} \|\vec{r}\,'(t_i)\|\,\Delta t_i$ is the Riemann sum of the continuous function $t \in [a,b] \to \|\vec{r}\,'(t)\|$.
So,

$$\boxed{L(\mathcal{C}) := \int_a^b \|\vec{r}\,'(t)\|\,dt} \tag{15}$$

The definition of $L(\mathcal{C})$ can be made more rigourous using some well-known tools or techniques of Mathematical Analysis.

Example 0.21 Find the length of the given curve.

(i) $\mathcal{C}$ described by the vector function $\vec{r}$; $[0, 2\pi] \to \mathbb{R}^3$ defined by $\vec{r}(t) = (2\sin t, -t\sqrt{5}, 2\cos t)$.
We get

$$\boxed{\vec{r}\,'(t) = (2\cos t, \sqrt{5}, -2\sin t) \quad \Rightarrow \quad L(\mathcal{C}) = \int_0^{2\pi} \|\vec{r}\,'(t)\|\,dt = \int_0^{2\pi} 3dt = 6\pi}\,.$$

(ii) $\mathcal{C}$ described by the vector function $\vec{r}$; $[0, 1] \to \mathbb{R}^3$ defined by $\vec{r}(t) = (t\sin t, t\cos t)$.
We get

$$\vec{r}\,'(t) = (\sin t + t\cos t, \cos t - t\sin t) \quad \Rightarrow \quad \|\vec{r}'(t)\| = \sqrt{1+t^2} \quad \Rightarrow \quad L(\mathcal{C}) = \int_0^1 \sqrt{1+t^2}dt\,.$$

Hence,

$$\boxed{L(\mathcal{C}) = \int_0^1 \sqrt{1+t^2}\,dt = \int_0^{\frac{\pi}{4}} \sec^3(\theta)\,d\theta = \textbf{(To evaluate like in MAM1000W!!)}}\,.$$

Remark 0.22 (The length of a piecewise regular curve)
If $\mathcal{C}$ is **piecewise regular**, i.e., $\mathcal{C}$ is a union of a finite number of regular curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ with $\mathcal{C}_i \cap \mathcal{C}_{i+1} = \{\text{single point}\}$ for $i = 1, \ldots, k-1$, then the length of $\mathcal{C}$ is given by

$$\boxed{L(\mathcal{C}) := \sum_{i=1}^{k} L(\mathcal{C}_i)}\,,$$

where $L(\mathcal{C}_i$ is the length of the regular piece $\mathcal{C}_i$ given according to (15).

The next set of notes will be on the arclength reparameterization, the curvature and the motion of objects!!!