SYNTHESIS OF THE PITCH CURVE DESIGN FOR NONCIRCULAR GEAR BASED ON BERNOULLI'S LEMNISCATE

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ABSTRACT*: A novel method for designing pitch curves of noncircular gears is proposed. The mathematical models, including unclosed and closed pitch curves, are established based on Bernoulli's lemniscate. In particular, according to the meshing principle of noncircular gears, two methods for calculating the noncircular driven gear pitch curve conjugated with the noncircular driving gear pitch curve based on Bernoulli's lemniscate are proposed. Several typical examples are implemented in MATLAB, and simulation results demonstrate the feasibility and validity of the proposed design method for noncircular gears pitch curves.*

KEYWORDS*: noncircular gear, pitch curve design, bernoulli's lemniscate, mathematical model*

1 INTRODUCTION

Noncircular gears, which transfers variable speed and torque from one noncircular driving gear to another noncircular driven gear, have been used in a variety of mechanical equipments and automatic instruments: flow meters, packaging machines, textile machines, and aeronautical ring dampers (Litvin, 2004; Stefano, 2012; Arakelian, 2010; Terada, 2012). However, due to the complexity of design and manufacturing for noncircular gears, early researches on noncircular gears focus on elliptical gears and their geometric and manufacturing properties (Chang, 1996; Figliolini, 2003; He, 2004; Litvin, 2007; Lin, 2012; Zhang, 2014).

Recently, with the development of modern computer and CNC machine tools, the design and application of noncircular gears have been expanded in mechanical industry. Particularly, the mathematical methods of the noncircular gears with arbitrary shape pitch curves can be proposed by some researchers. Such as, noncircular gears based on Bézier and B-spline nonparametric curves have been obtained by Hector, Salvador and Liuisa (2007). A mathematical model of the noncircular gears based on spiral of Archimedes have been established by Yao (2013). A design method of noncircular gears with steepest rotation pitch curves have been obtained by Zhang and Fan (2016). Noncircular gears with minimal rotary inertia and mean kinetic energy characteristics have been respectively obtained by Zhang and Fan (2018; 2019). A general form of gears known as noncircular gears that can transfer periodic motion with variable speed through their irregular shapes and eccentric rotation centers has been proposed by Xu, Fu and et al (2020). Moreover, some modification methods of pitch curves for *N*-lobed noncircular gears has been proposed by Zhang and Fan (2021, 2022).

Furthermore, in order to extend the application and transmission performance of noncircular gears, the paper presents a novel mathematical model of the noncircular gears design with the pitch curves based on Bernoulli's lemniscate. The design methods of the unclosed and closed pitch curves can also be presented. Two calculating methods of noncircular driven gear pitch curve conjugated with the noncircular driving gear pitch curve based on Bernoulli's lemniscate are developed according to the meshing principle of noncircular gears. transmission performance of the proposed noncircular gears can also be analyzed. Several mathematical examples are implemented in MATLAB to design the pitch curves based on Bernoulli's lemniscate, and to demonstrate the feasibility and validity of the proposed design method for noncircular gears pitch curves.

2 DESIGN AND ANALYSIS OF THE UNCLOSED PITCH CURVE FOR NONCIRCULAR GEAR BASED ON BERNOULLI'S LEMNISCATE

2.1 Design of the noncircular gear pitch curve based on Bernoulli's lemniscate

The polar equation $r_b(\theta)$ of Bernoulli's lemniscate can be established as (Department of Mathematics, Tongji University, 2014):

$$r_b(\theta)=\begin{cases} r_{b1}(\theta)=a\sqrt{\sin 2\theta},\ \theta\in[0,\ \frac{\pi}{2}] \\ r_{b2}(\theta)=a\sqrt{\sin 2\theta},\ \theta\in[\pi,\ \frac{3\pi}{2}] \end{cases} \quad (1)$$

where a ($a > 0$) is called Bernoulli's lemniscate design parameter, polar angle θ is measured counterclockwise from the positive direction x-axis

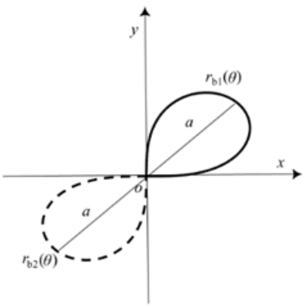


Fig. 1 The graph of Bernoulli's lemniscate $r_b(\theta)$

As shown in Fig. 1, this is a graph of Bernoulli's lemniscate, we know that the curve $r_{b2}(\theta)$ ($\theta \in [\pi, 3\pi/2]$) can be obtained by rotating the curve $r_{b1}(\theta)$ ($\theta \in [0, \pi/2]$) counterclockwise by π radian. Particularly, because the noncircular gear possesses the characteristic of rotation, so the curve $r_{b1}(\theta)$ ($\theta \in [0, \pi/2]$) can be as the research object of the pitch curve design for noncircular gear based on Bernoulli's lemniscate. Because the pitch curve of noncircular gear must be greater than zero, as shown in Fig. 2, the unclosed pitch curve $r(\theta)$ for noncircular gear based on Bernoulli's lemniscate should satisfy the following equation:

$$\begin{cases} r(\theta)=a\sqrt{\sin 2\theta},\theta\in[\theta_1,\ \theta_2] \\ a>0,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \end{cases} \quad (2)$$

where θ_1 and θ_2 are the polar angles corresponding to the left and right boundary points of the pitch curve, respectively.

Furthermore, as shown in Fig. 3, in order to ensure that the noncircular gear rotates from a polar angle equal to zero, Eq. (2) can be rearranging to

$$\begin{cases} r(\theta)=a\sqrt{\sin 2(\theta+\theta_1)},\theta\in[0,\ \theta_2-\theta_1] \\ a>0,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \end{cases} \quad (3)$$

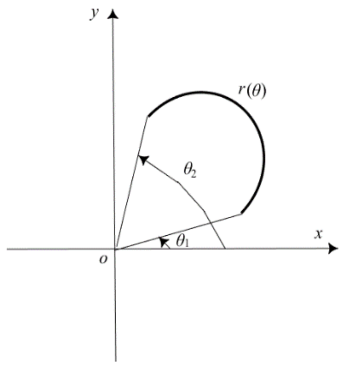


Fig. 2 Unclosed pitch curve $r(\theta)$ for noncircular gear based on Bernoulli's lemniscate

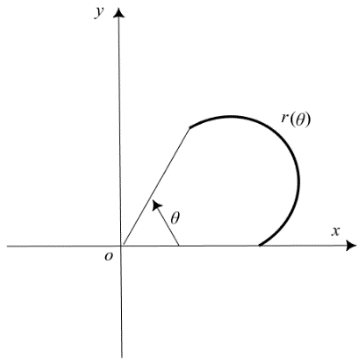


Fig. 3 The improved unclosed pitch curve $r(\theta)$ for noncircular gear that rotates from a polar angle equal to zero

According to Eq. (3) and differential calculus, the first derivative $r'(\theta)$ and the second derivative $r''(\theta)$ of the improved unclosed pitch curve $r(\theta)$ based on Bernoulli's lemniscate can be expressed as, respectively:

$$\begin{cases} r'(\theta)=a\dfrac{\cos 2(\theta+\theta_1)}{\sqrt{\sin 2(\theta+\theta_1)}},\theta\in[0,\ \theta_2-\theta_1] \\ a>0,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \end{cases} \quad (4)$$

$$\begin{cases} r''(\theta)=\dfrac{-a}{\sin 2(\theta+\theta_1)\sqrt{\sin 2(\theta+\theta_1)}},\theta\in[0,\ \theta_2-\theta_1] \\ a>0,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \end{cases} \quad (5)$$

Referring to Eq. (4) and Eq. (5), we know that the first derivative $r'(\theta)$ and the second derivative $r''(\theta)$ possess the following constraints:

$$\begin{cases} \begin{cases} r'(\theta)\neq 0 \\ r''(\theta)>0 \end{cases},\ \theta\in[0,\ \theta_2-\theta_1] \\ a>0,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \end{cases} \quad (6)$$

According to the meshing principle of noncircular gears, the pitch curves $r_e(\theta_e)$ and $r_i(\theta_i)$ of the external meshing and internal meshing noncircular driven gear conjugated with the improved unclosed pitch curve $r(\theta)$ can be calculated as, respectively:

$$\begin{cases} \begin{cases} r_e(\theta_e) = A_e - r(\theta) \\ \theta_e = \int_0^{\theta} \dfrac{r(\theta)}{A_e - r(\theta)} d\theta \end{cases} \\ \theta \in [0,\ \theta_2 - \theta_1],\ \theta_e \in [0,\ \theta_{emax}] \end{cases} \tag{7}$$

$$\begin{cases} \begin{cases} r_i(\theta_i) = A_i + r(\theta) \\ \theta_i = \int_0^{\theta} \dfrac{r(\theta)}{A_i + r(\theta)} d\theta' \end{cases} \\ \theta \in [0,\ \theta_2 - \theta_1],\ \theta_i \in [0,\ \theta_{imax}] \end{cases} \tag{8}$$

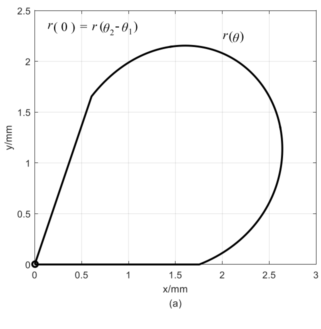
where A_e and A_i are respectively center distances of external meshing and internal meshing noncircular gears, θ_{emax} and θ_{imax} are respectively the maximum of the external meshing and internal meshing polar angles θ_e and θ_i, and they can be calculated as:

$$\theta_{emax} = \int_0^{\theta_2-\theta_1} \frac{r(\theta)}{A_e - r(\theta)} d\theta \tag{9}$$

$$\theta_{imax} = \int_0^{\theta_2-\theta_1} \frac{r(\theta)}{A_i + r(\theta)} d\theta \tag{10}$$

2.2 Mathematical examples of the pitch curve design for noncircular gear based on Bernoulli's lemniscate

According to Eq. (3), two mathematical design examples of this improved unclosed pitch curve $r(\theta)$ for noncircular gear based on Bernoulli's lemniscate are depicted in Fig. 4(a) and Fig. 4(b), their polar equations and design parameters are listed in Table 1.



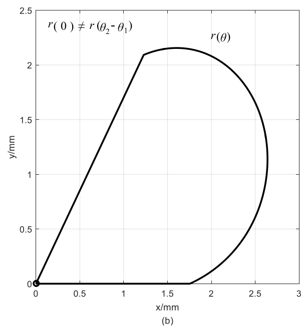


Fig. 4 Two mathematical design examples of this improved unclosed pitch curve *r*(*θ*) based on Bernoulli's lemniscate

Table 1 The polar equations of the improved unclosed pitch curve r(θ) and its design parameters

Design parameters	The polar equations of the improved unclosed pitch curve $r(\theta)$
$\begin{cases} a = 3 \\ \theta_1 = \frac{\pi}{18},\ \theta_2 = \frac{8\pi}{18}, \end{cases}$ $r(0) = r(\theta_2 - \theta_1)$	$\begin{cases} r(\theta) = 3\sqrt{\sin 2(\theta + \frac{\pi}{18})}, \\ \theta \in [0,\ \frac{7\pi}{18}] \end{cases}$ Referring to Fig. 4(a)
$\begin{cases} a = 3 \\ \theta_1 = \frac{\pi}{18},\ \theta_2 = \frac{7\pi}{18}, \end{cases}$ $r(0) \neq r(\theta_2 - \theta_1)$	$\begin{cases} r(\theta) = 3\sqrt{\sin 2(\theta + \frac{\pi}{18})}, \\ \theta \in [0,\ \frac{6\pi}{18}] \end{cases}$ Referring to Fig. 4(b)

Referring to Eq. (7) ~ Eq. (8), according to the meshing principle of noncircular gears, there are two ways to calculate the pitch curves $r_e(\theta_e)$ and $r_i(\theta_i)$ conjugated with the unclosed pitch curve $r(\theta)$ based on Bernoulli's lemniscate for noncircular gears, the calculation flow chart is shown in Fig. 5.

According to the different design requirements of the practical applications of noncircular gears, the designer can design and calculate the pitch curve of the noncircular driven gear by choosing one of the above two methods depicted in Fig. 5. Referring to Fig. 5 and Eq. (7) ~ Eq. (10), take the pitch curve shown in Fig. 4(a) as an example, the pitch curves $r_e(\theta_e)$ and $r_i(\theta_i)$ can be designed and depicted in Fig. 6 and Fig. 7, their corresponding design parameters are listed in Table 2.

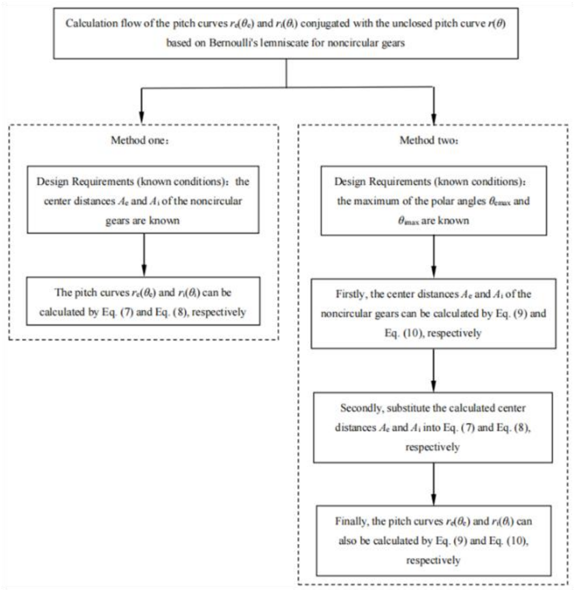


Fig. 5 The calculation flow of the pitch curves $r_e(\theta_e)$ and $r_i(\theta_i)$ conjugated with the unclosed pitch curve $r(\theta)$ based on Bernoulli's lemniscate

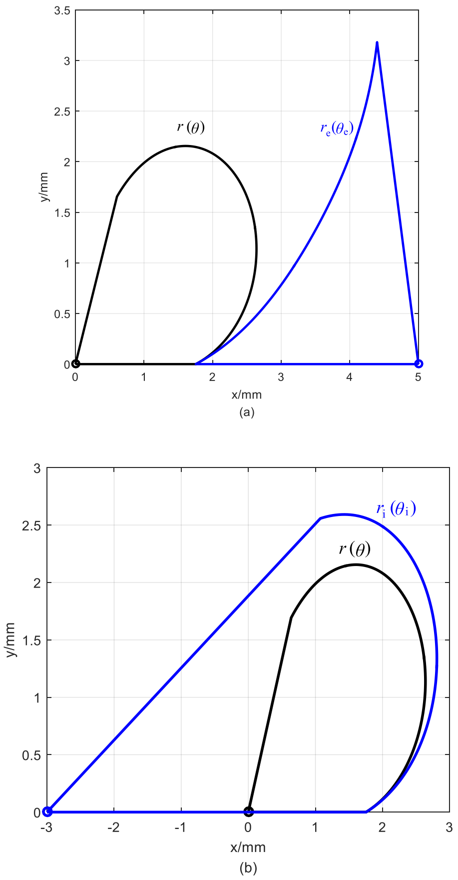


Fig. 6 Unclosed noncircular gear pitch curves calculated by the method one of Fig. 5

Table 2. The polar equations and corresponding design parameters of the pitch curves re(θe) and ri(θi) conjugated with the unclosed pitch curve r(θ) based on Bernoulli's lemniscate shown in Fig. 4(a)

Noncircular driving gear pitch curve $r(\theta)$ based on Bernoulli's lemniscate	Known conditions	Calculated design parameters	Noncircular driven gear pitch curves $r_e(\theta_e)$ and $r_i(\theta_i)$ conjugated with the noncircular driving gear pitch curve $r(\theta)$
$r(\theta)=3\sqrt{\sin 2(\theta+\frac{\pi}{18})}$ $\theta\in[0,\frac{7\pi}{18}]$ (Referring to Fig. 4(a))	$A_e=5$	$\theta_{emax}=1.3846$	$\begin{cases} r_e(\theta_e)=5-r(\theta) \\ \theta_e=\int_0^\theta \frac{r(\theta)}{5-r(\theta)}d\theta \end{cases}$, $\theta_e\in[0, 1.3846]$ (Referring to Fig. 6(a))
	$A_i=3$	$\theta_{imax}=0.5650$	$\begin{cases} r_i(\theta_i)=3+r(\theta) \\ \theta_i=\int_0^\theta \frac{r(\theta)}{3+r(\theta)}d\theta \end{cases}$, $\theta_i\in[0, 0.5650]$ (Referring to Fig. 6(b))
	$\theta_{emax}=\frac{7\pi}{18}$	$A_e=5.3020$	$\begin{cases} r_e(\theta_e)=5.3020-r(\theta) \\ \theta_e=\int_0^\theta \frac{r(\theta)}{5.3020-r(\theta)}d\theta \end{cases}$, $\theta_e\in[0,\frac{7\pi}{18}]$ (Referring to Fig. 7(a))
	$\theta_{imax}=\frac{4\pi}{18}$	$A_i=1.9315$	$\begin{cases} r_i(\theta_i)=1.9315+r(\theta) \\ \theta_i=\int_0^\theta \frac{r(\theta)}{1.9315+r(\theta)}d\theta \end{cases}$, $\theta_i\in[0,\frac{4\pi}{18}]$ (Referring to Fig. 7(b))

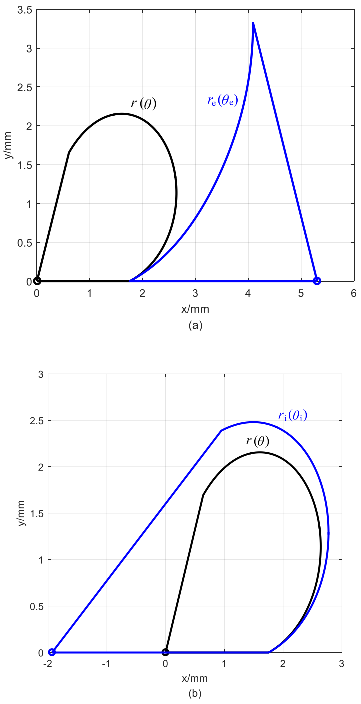


Fig. 7 Unclosed noncircular gear pitch curves calculated by the method two of Fig. 5

2.3 Characteristic analysis of the transmission ratio for noncircular gears with the pitch curve based on Bernoulli's lemniscate

Referring to Eq. (3) and Eq. (7) ~ Eq. (8), the transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ of external meshing and internal meshing noncircular gears can be expressed as:

$$\begin{cases} i_e(\theta)=\int_0^{\theta_2-\theta_1}\dfrac{a\sqrt{\sin 2(\theta+\theta_1)}}{A_e-a\sqrt{\sin 2(\theta+\theta_1)}}d\theta \\ s.t.\begin{cases} a>0,\ 0<\theta_1<\theta_2<\dfrac{\pi}{2} \\ \forall\theta\in[0,\ \theta_2-\theta_1],\ A_e-a\sqrt{\sin 2(\theta+\theta_1)}>0 \end{cases} \end{cases} \tag{11}$$

$$\begin{cases} i_i(\theta)=\int_0^{\theta_2-\theta_1}\dfrac{a\sqrt{\sin 2(\theta+\theta_1)}}{A_i+a\sqrt{\sin 2(\theta+\theta_1)}}d\theta \\ s.t.\begin{cases} a>0,\ 0<\theta_1<\theta_2<\dfrac{\pi}{2} \\ \forall\theta\in[0,\ \theta_2-\theta_1],\ A_i+a\sqrt{\sin 2(\theta+\theta_1)}>0 \end{cases} \end{cases} \tag{12}$$

Furthermore, rearranging Eq. (11) ~ Eq. (12) to:

$$\begin{cases} i_e(\theta)=\int_0^{\theta_2-\theta_1}\dfrac{\sqrt{\sin 2(\theta+\theta_1)}}{P_e-\sqrt{\sin 2(\theta+\theta_1)}}d\theta \\ s.t.\begin{cases} P_e=\dfrac{A_e}{a}>(\sqrt{\sin 2(\theta+\theta_1)})_{\max} \\ \theta\in[0,\ \theta_2-\theta_1],\ a>0,\ 0<\theta_1<\theta_2<\dfrac{\pi}{2} \end{cases} \end{cases} \tag{13}$$

$$\begin{cases} i_i(\theta)=\int_0^{\theta_2-\theta_1}\dfrac{\sqrt{\sin 2(\theta+\theta_1)}}{P_i+\sqrt{\sin 2(\theta+\theta_1)}}d\theta \\ s.t.\begin{cases} P_i=\dfrac{A_i}{a}>0,\ \theta\in[0,\ \theta_2-\theta_1] \\ a>0,\ 0<\theta_1<\theta_2<\dfrac{\pi}{2} \end{cases} \end{cases} \tag{14}$$

The transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ of external meshing and internal meshing noncircular gears are depicted in Fig. 8 ~ Fig. 9, and its design parameters are listed in Table 3. Referring to Fig. 8, when the value range of the polar angle θ is fixed, the transmission ratio function $i_e(\theta)$ of external meshing noncircular gears has the following characteristics: i) The transmission ratio function $i_e(\theta)$ becomes smoother with the increase of the parameter P_e. ii) When the parameter P_e is close to 1, the variation trend of the transmission ratio function $i_e(\theta)$ is more obvious. iii) The transmission ratio function $i_e(\theta)$ reaches its maximum when the polar angle θ is equal to $\pi/2-\theta_1$.

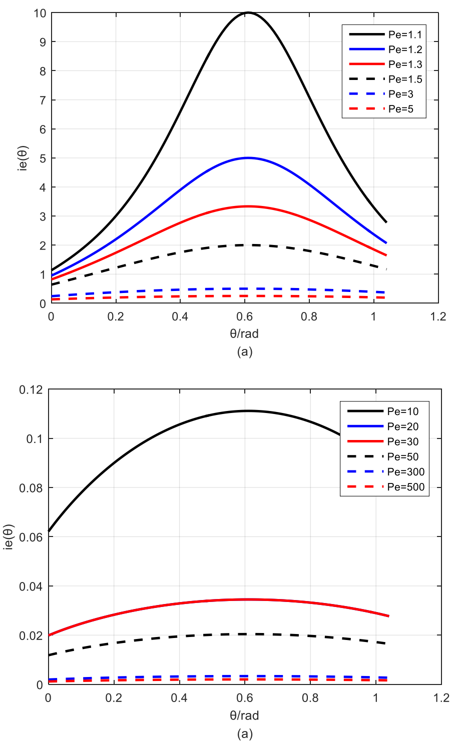


Fig. 8 The transmission ratio functions $i_e(\theta)$ of external meshing noncircular gears with different design parameters P_e

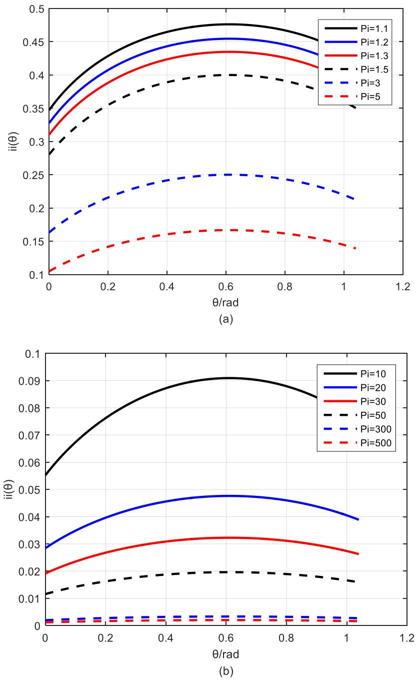


Fig. 9 The transmission ratio functions $i_i(\theta)$ of external meshing noncircular gears with different design parameters P_i

Referring to Fig. 9, when the value range [0, π/3] of the polar angle θ is fixed, the transmission ratio function $i_i(\theta)$ of internal meshing noncircular gears has the following characteristics: i) The transmission ratio function $i_i(\theta)$ becomes smoother with the increase of the parameter P_i. ii) The transmission ratio function $i_i(\theta)$ reaches its maximum when the polar angle θ is equal to $\pi/2-\theta_1$.

3 DESIGN AND ANALYSIS OF THE CLOSED PITCH CURVE FOR NONCIRCULAR GEAR BASED ON BERNOULLI'S LEMNISCATE

3.1 Design of the closed noncircular gear pitch curve based on Bernoulli's lemniscate

Referring to Eq. (3) and Fig. (3), in order to obtain the closed noncircular gear pitch curve $R(\theta)$ based on Bernoulli's lemniscate by rotating about the center of rotation, the improved unclosed pitch curve $r(\theta)$ should satisfy the following constraints:

$$\begin{cases} \theta_2-\theta_1=\dfrac{2\pi}{N} \\ r(0)=r(\dfrac{2\pi}{N}) \end{cases},\ 0<\theta_1<\theta_2<\frac{\pi}{2} \tag{15}$$

where N is the number of lobes of the closed noncircular gear pitch curve $R(\theta)$ and it's an integer greater than 4.

Table 3. The design parameters of the transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ depicted in Fig. 8 ~ Fig. 9

Design parameters		The transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$
$\begin{cases}\theta_1=\dfrac{\pi}{18},\ \theta_2=\dfrac{7\pi}{18}\\ \theta_2-\theta_1=\dfrac{\pi}{3}\end{cases}$	$P_e=1.1,\ 1.2,\ 1.3,\ 1.5,\ 3,\ 5$ $10,\ 20,\ 30,\ 50,\ 300,\ 500$	$i_e(\theta)=\int_0^{\frac{\pi}{3}}\dfrac{\sqrt{\sin 2(\theta+\theta_1)}}{P_e-\sqrt{\sin 2(\theta+\theta_1)}}d\theta$ (Referring to Fig. 8)
	$P_i=1.1,\ 1.2,\ 1.3,\ 1.5,\ 3,\ 5$ $10,\ 20,\ 30,\ 50,\ 300,\ 500$	$i_i(\theta)=\int_0^{\frac{\pi}{3}}\dfrac{\sqrt{\sin 2(\theta+\theta_1)}}{P_i+\sqrt{\sin 2(\theta+\theta_1)}}d\theta$ (Referring to Fig. 9)

Along with Eq. (3), in order ensure that the equation $r(0)=r(\dfrac{2\pi}{N})$ is true, polar angles θ_1 and θ_2 should satisfy the following equation:

$$\theta_2=\frac{\pi}{2}-\theta_1,\ 0<\theta_1<\theta_2<\frac{\pi}{2} \tag{16}$$

Furthermore, rearranging Eq. (15) to:

$$\theta_1=\frac{\pi}{4}-\frac{\pi}{N},\ N>4 \tag{17}$$

Then, the polar equation of the closed noncircular gear pitch curve $R(\theta)$ based on Bernoulli's lemniscate can be expressed as:

$$\begin{cases} R(\theta)=\begin{cases} r_1(\theta)=a\sqrt{\sin 2(\theta+\theta_1)},\theta\in[0,\ \dfrac{2\pi}{N}] \\ r_2(\theta)=r_1(\theta-\dfrac{2\pi}{N}),\theta\in[\dfrac{2\pi}{N},\ \dfrac{4\pi}{N}] \\ \vdots \\ r_k(\theta)=r_1(\theta-\dfrac{2(k-1)\pi}{N}),\theta\in[\dfrac{2(k-1)\pi}{N},\ \dfrac{2k\pi}{N}] \end{cases} \\ s.t.\ a>0,\ N>4,\ \theta_1=\dfrac{\pi}{4}-\dfrac{\pi}{N},\ \ k=1,2,\cdots,N \end{cases} \tag{18}$$

Referring to Eq. (7) ~ (10), in order to ensure that the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$ satisfy the closure conditions of the external meshing and internal meshing noncircular gears conjugated with the closed noncircular gear pitch curve $R(\theta)$, the polar equations and constraint conditions of the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$ can be calculated as, respectively:

$$\begin{cases} R_e(\theta_e) = A_e - R(\theta) \\ \theta_e = \int_0^\theta \dfrac{R(\theta)}{A_e - R(\theta)} d\theta \end{cases}, \ \theta \in [0,\ 2\pi],\ \theta_e \in [0,\ 2\pi] \tag{19}$$

$$\begin{cases} R_i(\theta_i) = A_i + R(\theta) \\ \theta_i = \int_0^\theta \dfrac{R(\theta)}{A_i + R(\theta)} d\theta \end{cases}, \ \theta \in [0,\ 2\pi],\ \theta_i \in [0,\ 2\pi] \tag{20}$$

$$\frac{2\pi}{N_e} = \int_0^{\frac{2\pi}{N}} \frac{R(\theta)}{A_e - R(\theta)} d\theta \tag{21}$$

$$\frac{2\pi}{N_i} = \int_0^{\frac{2\pi}{N}} \frac{R(\theta)}{A_i + R(\theta)} d\theta \tag{22}$$

where N_e and N_i are respectively the numbers of lobes of the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$, and both N_e and N_i are integers.

3.2 Mathematical examples of the pitch curve design for noncircular gear based on Bernoulli's lemniscate

According to Eq. (16), A mathematical design example of the closed noncircular gear pitch curve $R(\theta)$ based on Bernoulli's lemniscate are depicted in Fig. 10, its polar equation and design parameters are listed in Table 4.

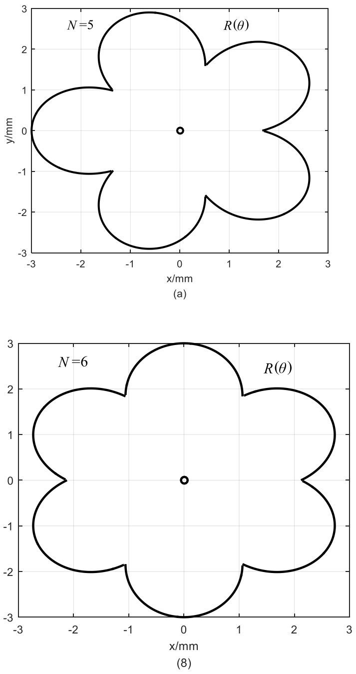


Fig. 10 Two mathematical design examples of the closed noncircular gear pitch curve *R*(*θ*) based on Bernoulli's lemniscate

Table 4. The polar equations of the closed noncircular gear pitch curve R(θ) and its design parameters

Design parameters	Calculated parameter	The polar equations of the closed noncircular gear pitch curve $R(\theta)$
$\begin{cases} a=3 \\ N=5 \end{cases}$	$\theta_1 = \frac{\pi}{4} - \frac{\pi}{5} = \frac{\pi}{20}$	$R(\theta) = \begin{cases} r_1(\theta) = 3\sqrt{\sin 2(\theta + \frac{\pi}{20})},\ \theta \in [0,\ \frac{2\pi}{5}] \\ r_2(\theta) = r_1(\theta - \frac{2\pi}{5}),\ \theta \in [\frac{2\pi}{5},\ \frac{4\pi}{5}] \\ \vdots \\ r_5(\theta) = r_1(\theta - \frac{8\pi}{5}),\ \theta \in [\frac{8\pi}{5},\ 2\pi] \end{cases}$ (Referring to Fig. 10(a))
$\begin{cases} a=3 \\ N=6 \end{cases}$	$\theta_1 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$	$R(\theta) = \begin{cases} r_1(\theta) = 3\sqrt{\sin 2(\theta + \frac{\pi}{12})},\ \theta \in [0,\ \frac{\pi}{3}] \\ r_2(\theta) = r_1(\theta - \frac{\pi}{3}),\ \theta \in [\frac{\pi}{3},\ \frac{2\pi}{3}] \\ \vdots \\ r_6(\theta) = r_1(\theta - \frac{5\pi}{3}),\ \theta \in [\frac{5\pi}{3},\ 2\pi] \end{cases}$ (Referring to Fig. 10(b))

Referring to Eq. (18) ~ Eq. (20), take the pitch curve shown in Fig. 10(a) as an example, the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$ can be designed and depicted in Fig. 11, their corresponding design parameters are listed in Table 5.

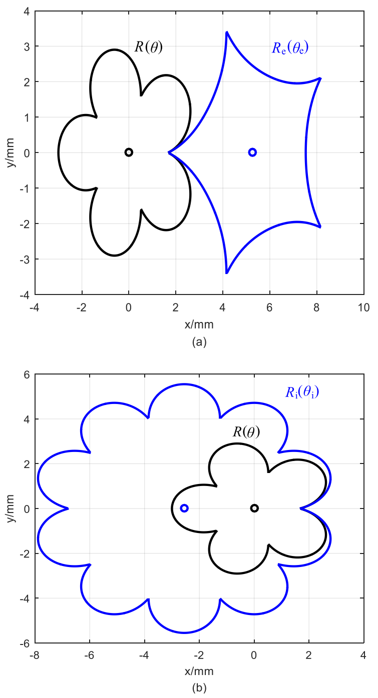


Fig. 11 Closed noncircular gear pitch curves based on Bernoulli's lemniscate

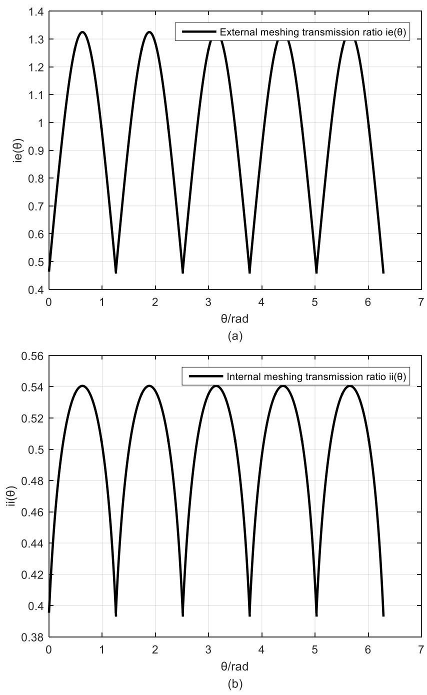


Fig. 12 Transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ of closed external meshing and internal meshing noncircular gears

Table 5. The polar equations and corresponding design parameters of the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$ conjugated with the closed pitch curve $R(\theta)$ based on Bernoulli's lemniscate shown in Fig. 10(a)

The closed noncircular driving gear pitch curve $R(\theta)$ based on Bernoulli's lemniscate	$R(\theta)=\begin{cases} r_1(\theta)=3\sqrt{\sin 2(\theta+\frac{\pi}{20})},\ \theta\in[0,\ \frac{2\pi}{5}] \\ r_2(\theta)=r_1(\theta-\frac{2\pi}{5}),\ \theta\in[\frac{2\pi}{5},\ \frac{4\pi}{5}] \\ \vdots \\ r_5(\theta)=r_1(\theta-\frac{8\pi}{5}),\ \theta\in[\frac{8\pi}{5},\ 2\pi] \end{cases}$, Referring to Fig. 8(a)	
Known conditions	$N_e=5$	$N_i=10$
Calculated design parameters	$A_e=5.2645$	$A_i=2.55$
Noncircular driven gear pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$ conjugated with the noncircular driving gear pitch curve $R(\theta)$	$\begin{cases} \begin{cases} R_e(\theta_e)=5.2645-R(\theta) \\ \theta_e=\int_0^{\theta}\frac{R(\theta)}{5.2645-R(\theta)}d\theta \end{cases} \\ s.t.\ \theta\in[0,\ 2\pi],\ \theta_e\in[0,\ 2\pi] \end{cases}$ (Referring to Fig. 11(a))	$\begin{cases} \begin{cases} R_i(\theta_i)=2.55+R(\theta) \\ \theta_i=\int_0^{\theta}\frac{R(\theta)}{2.55+R(\theta)}d\theta \end{cases} \\ s.t.\ \theta\in[0,\ 2\pi],\ \theta_i\in[0,\ 2\pi] \end{cases}$ (Referring to Fig. 11(b))
Transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ of closed external meshing and internal meshing noncircular gears	$\begin{cases} i_e(\theta)=\frac{R(\theta)}{5.2645-R(\theta)} \\ \theta\in[0,\ 2\pi] \end{cases}$ (Referring to Fig. 12(a))	$\begin{cases} i_i(\theta)=\frac{R(\theta)}{2.55+R(\theta)} \\ \theta\in[0,\ 2\pi] \end{cases}$ (Referring to Fig. 12(b))

Fig. 12 are the transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ of closed external meshing and internal meshing noncircular gears depicted in Fig. 11. Referring to Fig. 12, the transmission ratio functions $i_e(\theta)$ and $i_i(\theta)$ possess the transmission characteristics described in Section 2.3 in a rotation period. Therefore, the proposed design method of the pitch curves based on Bernoulli's lemniscate can be used to design the noncircular gears with the above transmission characteristics.

4 CONCLUDING REMARKS

The unclosed and closed pitch curves for noncircular gears based on Bernoulli's lemniscate are obtained successfully in this paper. In particular, two calculating methods for noncircular driven gear pitch curve conjugated with the pitch curve based on Bernoulli's lemniscate are proposed. According to the different design requirements of the practical applications of noncircular gears, the designer can design and calculate the pitch curve of the noncircular driven gear by choosing one of the above two methods. The transmission ration characteristics of noncircular gears are analyzed and summarized. Several numerical examples are implemented to clarify the design process of the pitch curve based on Bernoulli's lemniscate, and to demonstrate the feasibility and validity of the proposed design method. This novel method builds a solid foundation for further researches on other particular geometrical and mechanical properties of noncircular gears.

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7 NOTATION

The following symbols are used in this paper:

a = Bernoulli's lemniscate design parameter;

$i_e(\theta)$, $i_i(\theta)$ = transmission ratio functions of the external meshing and the internal meshing noncircular gears, respectively;

$r(\theta)$ = the unclosed pitch curve for noncircular gear based on Bernoulli's lemniscate;

$r_b(\theta)$ = polar equation of Bernoulli's lemniscate;

$r_e(\theta_e)$, $r_i(\theta_i)$ = pitch curves of the external meshing and the internal meshing noncircular driven gear conjugated with the improved unclosed pitch curve $r(\theta)$

A_e, A_i = center distances of the external meshing and the internal meshing noncircular gears, respectively;

N= the number of lobes of the closed noncircular gear pitch curve $R(\theta)$;

N_e, N_i = numbers of lobes of the pitch curves $R_e(\theta_e)$ and $R_i(\theta_i)$, respectively;

$R(\theta)$ = the closed noncircular gear pitch curve based on Bernoulli's lemniscate;

$R_e(\theta_e)$, $R_i(\theta_i)$ = pitch curves satisfied the closure conditions of the external meshing and internal meshing noncircular gears conjugated with the closed noncircular gear pitch curve $R(\theta)$

θ = polar angles of noncircular gear pitch curve;

θ_e, θ_i = the external meshing and the internal meshing polar angles, respectively;

θ_{emax}, θ_{imax} = the maximum of the external meshing and internal meshing polar angles θ_e and θ_i, respectively