Lecture 7: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

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Vector-Valued Functions and Motion in Space. Overview.

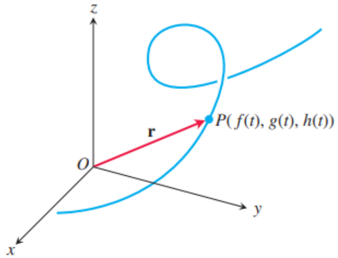
- In the next lectures I will introduce the calculus of **vector-valued functions**.
- The **domains** of these functions are **sets of real numbers**, as before, but their **ranges** consist of **vectors instead of scalars**.
- When a vector-valued function changes, the change can occur in both magnitude and direction, so the **derivative is itself a vector**.
- The **integral** of a vector-valued function is also **a vector**.
- We use the calculus of these functions to **describe the paths and motions of objects** moving in a plane or in space, so their velocities and accelerations are given by vectors.
- I also will introduce new concepts that **quantify** the way that the path of an object moving in space can **twist and turn**.

Curves in Space and Their Tangents

• When a particle moves through space during a time interval I, we think of the **particle's coordinates** as functions defined on I:

$x = f(t),\ y = g(t),\ z = h(t), t \in I$ (Eq. 1)

• The points $(x, y, z) = ((t), g(t), h(t)),\ t \in I$, make up the curve in space that we call the **particle's path**.



• The equations and interval in Eq. 1 **parametrize the curve**.

• A curve in space can also be represented in **vector form**. The vector

$$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the **particle's position vector**.

• The functions f, g, and h are the **component functions** (or components) of the position vector.

• We think of the particle's **path as the curve traced by r** during the time interval I.

Limits

• The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and let $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has **limit L** as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

• If $L = L_1 i + L_2 j + L_3 k$, then it can be shown that $\lim_{t \to t_0} r(t) = L$ precisely when $\lim_{t \to t_0} f(t) = L_1$, $\lim_{t \to t_0} g(t) = L_2$, and $\lim_{t \to t_0} h(t) = L_3$

• The equation

$$\lim_{t \to t_0} r(t) = \left(\lim_{t \to t_0} f(t) \right) i + \left(\lim_{t \to t_0} g(t) \right) j + \left(\lim_{t \to t_0} h(t) \right) k$$

provides a practical way to calculate limits of vector functions.

Continuity

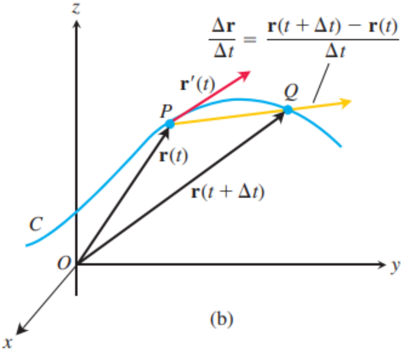
• We define continuity for vector functions the same way we define continuity for scalar functions defined over an interval.

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Derivatives and Motion

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

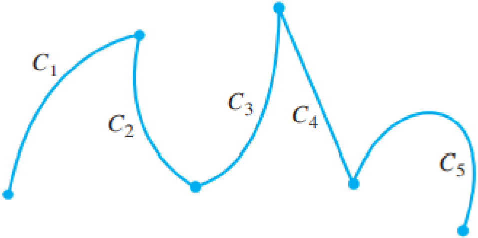


(b)

- As $\Delta t \to 0$, the point Q approaches the point P along the curve C.
- In the limit, the vector $\frac{\overrightarrow{PQ}}{\Delta t}$ becomes the tangent vector $r'(t)$.
- A vector function r is differentiable if it is **differentiable** at every point of its domain.

Derivatives and Motion

• The curve traced by r is **smooth** if dr/dt is continuous and never 0, that is, if f, g, and h have continuous first derivatives that are not simultaneously 0.
• On a smooth curve, there are no sharp, corners or cusps.
• A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth**.



Motion

DEFINITIONS If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Motion

Example 1

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j} + 5\cos^2 t\boldsymbol{k}$. Sketch the velocity vector $\boldsymbol{v}(7\pi/4)$.

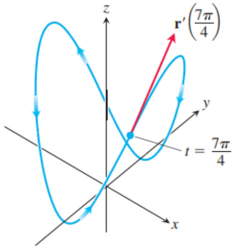
Solution

The velocity and acceleration vectors at time t are

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k}\\ &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k},\\ \mathbf{a}(t) &= \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},\end{aligned}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$



When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 5\mathbf{k}, \quad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}, \quad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

Differentiation Rules

• Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the **same form** as the rules for differentiating scalar functions.

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

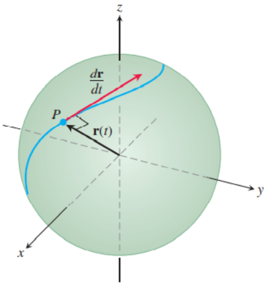
5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector Functions of Constant Length

• When we track a **particle moving on a sphere** centered at the origin shown on Fig., the **position vector has a constant length** equal to the radius of the sphere.
• The velocity vector dr/dt, **tangent to the path of motion**, is tangent to the sphere and hence perpendicular to r.
• This is always the case for a **differentiable vector function of constant length: The vector and its first derivative are orthogonal**.



$$\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2 \qquad |\mathbf{r}(t)| = c \text{ is constant.}$$

$$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = 0 \qquad \text{Differentiate both sides.}$$

$$\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \qquad \text{Rule 5 with } \mathbf{r}(t) = \mathbf{u}(t) = \mathbf{v}(t)$$

$$2\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0.$$

Example 2

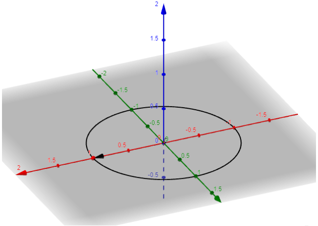
Equation $r(t) = \cos(2t)i + \sin(2t)j, t \geq 0$ describes the motion of a particle having the unit circle $x^2 + y^2 = 1$ path. Answer the following questions about the particle's "dynamics".

i) Does the particle have constant speed? If so, what is its constant speed?

ii) Is the particle's acceleration vector always orthogonal to its velocity vector?

iii) Does the particle move clockwise or counterclockwise around the circle?

iv) Does the particle begin at the point (1, 0)?



Example 2. Solution.

$\mathbf{v}(t) = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4\cos 2t)\mathbf{i} - (4\sin 2t)\mathbf{j};$

(i) $|\mathbf{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2 \Rightarrow$ constant speed;

(ii) $\mathbf{v}\cdot\mathbf{a} = 8\sin 2t\cos 2t - 8\cos 2t\sin 2t = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

Direction of motion:

- We will look at $r'(t) \times r''(t)$. This vector is perpendicular to the plane.
- We choose a right framework in 3D, it is parallel to third unit vector k.
- If its third k coordinate is positive, the motion is counterclockwise, otherwise it is clockwise.
- This is because $r(t)$ always points to the side where the center of curvature is located.

$r'(t) \times r''(t) =$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin 2t & 2\cos 2t & 0 \\ -4\cos 2t & -4\sin 2t & 0 \end{vmatrix} = 0i + 0j + (8\sin^2 2t + 8\cos^2 2t) = 8$$

Example 3

An object is moving with position function $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$, $-3 \le t \le 3$, where distances are measured in feet and time is measured in seconds.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$.

(b) Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.

(c) When is the object's speed minimized?

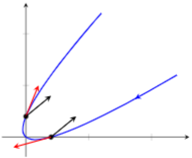
Solution:

(a) Taking derivatives, we find

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}''(t) = \langle 2, 2 \rangle.$$

Note that acceleration is constant.

(b) $\vec{v}(-1) = \langle -3, -1 \rangle$, $\vec{a}(-1) = \langle 2, 2 \rangle$; $\vec{v}(1) = \langle 1, 3 \rangle$, $\vec{a}(1) = \langle 2, 2 \rangle$. These are plotted with $\vec{r}(t)$



(a)

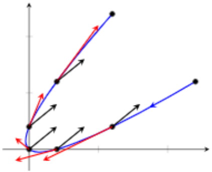
Example 3. Solution.

We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2 \rangle$ direction and is now pointing up and to the right. In Figure 12.3.1(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity. Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21 \rangle$, which is nearly parallel to $\langle 2, 2 \rangle$.

(c) The object's speed is given by

$$\|\vec{v}(t)\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s. The graph in Figure 12.3.1(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.



(b)