Solutions for Tutorial Problem Sheet 2, October 6.

(Vectors and the Geometry of Space, Vector-Valued Functions and Motion in Space.)

Problem 1. Find a plane through the points $P_1(1,2,3)$, $P_2(3,2,1)$ and perpendicular to the plane

$$4x - y + 2z = 7.$$

Solution:

A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$; choosing $P_1(1, 2, 3)$ as a point on the plane $\Rightarrow (-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$ is the desired plane

Problem 2. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

Solution:

The point $P(1, 0, 0)$ is on the first plane and $S(10, 0, 0)$ is a point on the second plane $\Rightarrow \overrightarrow{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{9}{\sqrt{1+4+36}} \right| = \frac{9}{\sqrt{41}}$, which is also the distance between the planes.

Problem 3. The planes $3x + 6z = 1$ and $2x + 2y - z = 3$ intersect in a line.

a) Show that the planes are orthogonal.

b) Find equations for the line of intersection.

Solution:

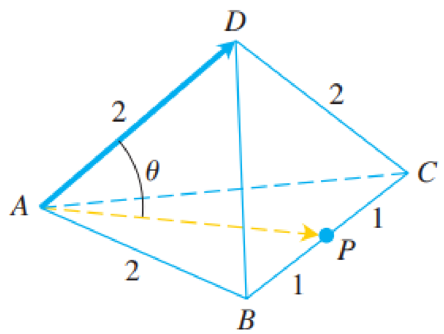
(a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2 = (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the plans are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases} \Rightarrow \begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases} \Rightarrow 15x + 12y = 19 \Rightarrow x = 0$ and $y = \frac{19}{12}$ $\Rightarrow \left(0, \frac{19}{12}, \frac{1}{6}\right)$ is a point on the line we seek. Therefore, the line is $x = -12t$, $y = \frac{19}{12} + 15t$ and $z = \frac{1}{6} + 6t$.

Problem 4. Consider a regular tetrahedron of side length 2.

a) Use vectors to find the angle θ formed by the base of the tetrahedron and any one of its other edges.

b) Use vectors to find the angle θ formed by any two adjacent faces of the tetrahedron. This angle is commonly referred to as a dihedral angle.



Solution:

(a) Place the tetrahedron so that A is at $(0, 0, 0)$, the point P is on the y-axis, and ΔABC lies in the xy-plane. Since ΔABC is an equilateral triangle, all the angles in the triangle are $60°$ and since AP bisects $BC \Rightarrow \Delta ABP$ is a $30°-60°-90°$ triangle. Thus the coordinates of P are $\left(0, \sqrt{3}, 0\right)$, the coordinates of B are $\left(1, \sqrt{3}, 0\right)$, and the coordinates of C are $\left(-1, \sqrt{3}, 0\right)$. Let the coordinates of D be given by (a, b, c). Since all of the faces are equilateral triangles $\Rightarrow$ all the angles in each of the triangles are $60° \Rightarrow \cos(\angle DAB) = \cos(60°) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{|\overrightarrow{AD}||\overrightarrow{AB}|} = \frac{a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow a + b\sqrt{3} = 2$ and $\cos(\angle DAC) = \cos(60°)$

$= \frac{\overrightarrow{AD}\cdot\overrightarrow{AC}}{|\overrightarrow{AD}||\overrightarrow{AC}|} = \frac{-a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow -a + b\sqrt{3} = 2.$ Add the two equations to obtain: $2b\sqrt{3} = 4 \Rightarrow b = \frac{2}{\sqrt{3}}.$

Substituting this value for b in the first equation gives us: $a + \left(\frac{2}{\sqrt{3}}\right)\sqrt{3} = 2 \Rightarrow a = 0.$ Since

$|\overrightarrow{AD}| = \sqrt{a^2+b^2+c^2} = 2 \Rightarrow 0^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + c^2 = 4 \Rightarrow c = \frac{2\sqrt{2}}{\sqrt{3}}.$ Thus the coordinates of D are $\left(0, \frac{2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right).$

$\cos\theta = \cos(\angle DAP) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AP}}{|\overrightarrow{AD}||\overrightarrow{AP}|} = \frac{2}{2\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow 57.74°$

(b) Since ΔABC lies in the xy-plane $\Rightarrow$ the normal to the face given by ΔABC is $\mathbf{n}_1 = \mathbf{k}$. The face given by ΔBCD is an adjacent face. The vectors $\overrightarrow{DB} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ and $\overrightarrow{DC} = -\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ both lie in the plane containing ΔBCD. The normal to this plane is given by $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \end{vmatrix} = \frac{4\sqrt{2}}{\sqrt{3}}\mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}.$

The angle θ between two adjacent faces is given by $\cos\theta = \cos(\angle DAP) = \frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2/\sqrt{3}}{(1)\left(6/\sqrt{3}\right)}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°.$

Problem 5. Show that the vector-valued function

$$r(t) = (2i + 2j + k) + \cos t\left(\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j\right) + \sin t\left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right)$$

describes the motion of a particle moving in the circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.

Solution:

Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, **u** and **v** are orthogonal unit vectors with $\mathbf{u}\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $\left(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1\right)$ the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.