Lecture 11: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

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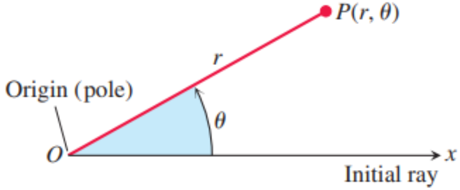
Velocity and Acceleration in Polar and Cylindrical coordinates

• We derive equations for velocity and acceleration in polar and cylindrical coordinates.

• We introduce **polar, cylindrical and spherical coordinates** and their relation to Cartesian coordinates.

• They are useful in **describing the paths of particles** moving in space.

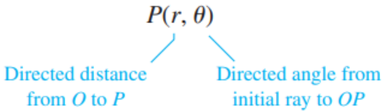
Definition of Polar Coordinates



• To define polar coordinates, we first **fix an origin** O (called the pole) and an **initial ray** from O.

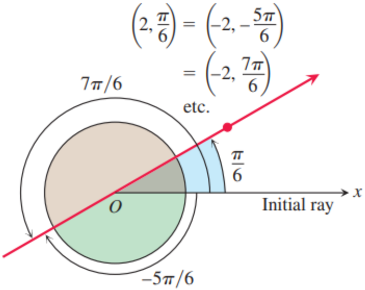
• Usually the **positive x-axis** is chosen as the initial ray.

• Then each point P can be located by **assigning to it a polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as



Definition of Polar Coordinates

- As in trigonometry, θ is **positive** when measured **counterclockwise** and **negative** when measured **clockwise**.
- The **angle** associated with a given point is **not unique**.
- While a point in the plane has just **one pair of Cartesian coordinates**, it has **infinitely many pairs of polar coordinates**.
- For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2, \theta = \pi/6$.
- It also has coordinates shown in Figure.

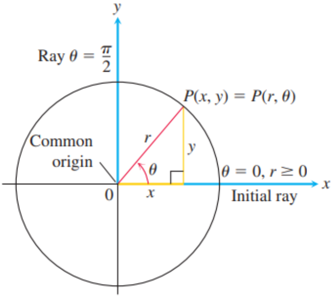


Relating Polar and Cartesian Coordinates

- When we use both polar and Cartesian coordinates in a plane, we place $\theta = \pi/2, r > 0$, becomes the positive y-axis.
- The two coordinate systems are then related by the following equations

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

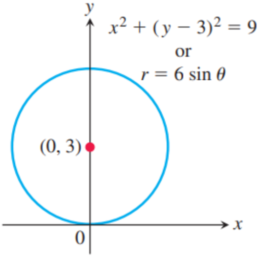


Example 1

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

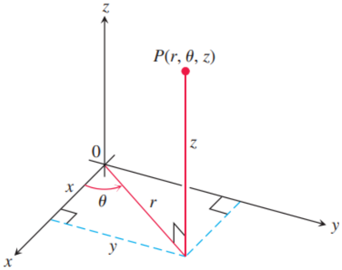
Solution:

We apply the equations relating polar and Cartesian coordinates:



$$
\begin{aligned}
x^2 + (y-3)^2 &= 9 \\
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2,\ y = r\sin\theta \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}
$$

Cylindrical Coordinates



- We obtain **cylindrical coordinates** for space by combining **polar coordinates** in the xy-plane with the usual **z-axis**.
- This assigns to every point in space one or more coordinate **triples** of the form (r, θ, z), as shown in Figure.
- Here we require $r \geq 0$.
- The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

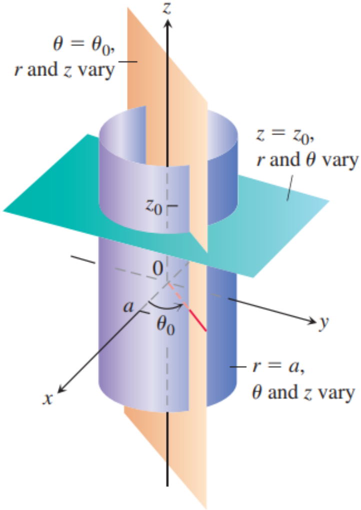
Cylindrical Coordinates

• In cylindrical coordinates, the **equation** $r = a$ describes not just a circle in the xy-plane but an **entire cylinder** about the z-axis.

• The **z-axis** is given by $r = 0$.

• The equation $\theta = \theta_0$ describes the **plane that contains the z-axis** and makes an angle θ_0 with the positive x-axis.

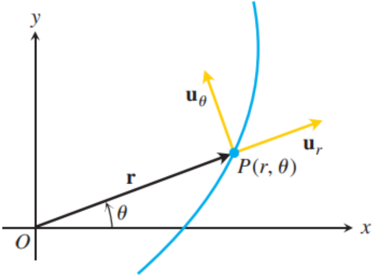
• And, just as in rectangular coordinates, the equation $z = z_0$ describes a **plane perpendicular to the z-axis**.



Motion in Polar and Cylindrical Coordinates

• When a particle at $P(r, \theta)$ moves along a curve in the polar coordinate plane, we express its position, velocity, and acceleration in terms of the **moving unit vectors**

$$u_r = (\cos\theta)i + (\sin\theta)j, \quad u_\theta = -(\sin\theta)i + (\cos\theta)j,$$



• The length of vector $\mathbf{r}$ is the positive polar coordinate r of the point P.
• Thus, u_r is $\mathbf{r}/|\mathbf{r}|$.
• The vector u_r points along the position vector $\overrightarrow{OP}$, so $\mathbf{r} = r\mathbf{u}_r$.
• The vector u_θ , orthogonal to u_r, points in the direction of increasing u.

Motion in Polar and Cylindrical Coordinates

- From equations u_r and u_θ follow that

$$\frac{d\mathbf{u}_r}{d\theta} = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$$

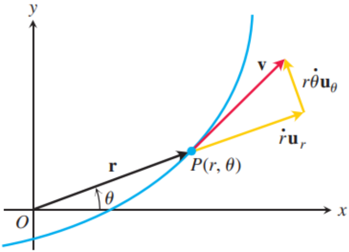
$$\frac{d\mathbf{u}_\theta}{d\theta} = -(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r.$$

- When we differentiate u_r and u_θ with respect to t to find how they change with time, the Chain Rule gives

$$\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta}\dot{\theta} = \dot{\theta}\mathbf{u}_\theta, \qquad \dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta}\dot{\theta} = -\dot{\theta}\mathbf{u}_r.$$

- Hence, we can express the velocity vector in terms of u_r and u_θ as

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{u}_r) = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$



- The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta).$$

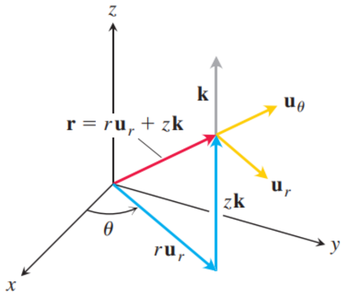
- Evaluating $\dot{u}_r$ and $\dot{u}_\theta$ and separating components, the equation for acceleration in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ becomes

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta.$$

- To extend these equations of motion to space, we add $z\mathbf{k}$ to the right-hand side of the equation $\mathbf{r} = r\mathbf{u}_r$. Then, in these cylindrical coordinates, we have

Position:	$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$	
Velocity:	$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}$	(3)
Acceleration:	$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$	

Position vector and basic unit vectors in cylindrical coordinates.



- The vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ make a right-handed frame in which

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \quad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \quad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta.$$

Example 2

Given $r = \frac{1}{\theta}$ and $\frac{d\theta}{dt} = t^2$, find the velocity and acceleration vectors in terms of u_r and u_θ.

Solution:

$$\frac{d\theta}{dt} = t^2 = \dot{\theta} \Rightarrow \ddot{\theta} = 2t, \;\; r = \frac{1}{\theta} \Rightarrow \dot{r} = \frac{-\dot{\theta}}{\theta^2} = \frac{-t^2}{\theta^2} \Rightarrow \ddot{r} = \frac{2\theta\left(\dot{\theta}\right)^2 - \theta^2\ddot{\theta}}{\theta^4} = \frac{2\theta\left(t^2\right)^2 - \theta^2(2t)}{\theta^4} = \frac{2t\left(t^3-\theta\right)}{\theta^3};$$

$$\vec{\mathbf{v}}(t) = \left(\frac{-t^2}{\theta^2}\right)\vec{\mathbf{u}}_r + \left(\frac{1}{\theta}\right)\left(t^2\right)\vec{\mathbf{u}}_\theta = \frac{-t^2}{\theta^2}\vec{\mathbf{u}}_r + \frac{t^2}{\theta}\vec{\mathbf{u}}_\theta; \;\; \vec{\mathbf{a}}(t) = \left(\left(\frac{2t\left(t^3-\theta\right)}{\theta^3}\right) - \left(\frac{1}{\theta}\right)\left(t^2\right)^2\right)\vec{\mathbf{u}}_r + \left(\left(\frac{1}{\theta}\right)(2t) + 2\left(\frac{-t^2}{\theta^2}\right)\left(t^2\right)\right)\vec{\mathbf{u}}_\theta$$

$$= \frac{t\left(2t^3 - 2\theta - t^3\theta^2\right)}{\theta^3}\vec{\mathbf{u}}_r + \frac{2t\left(\theta - t^3\right)}{\theta^2}\mathbf{u}_\theta$$

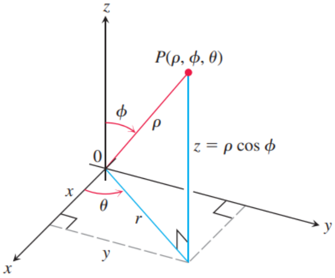
Spherical Coordinates

• **Spherical coordinates** locate points in space with **two angles** and **one distance**, as shown in Figure.

• The first coordinate, $\rho = |\overrightarrow{OP}—$, is the **point's distance from the origin** and is never negative.

• The second coordinate, ϕ, is the **angle** $\overrightarrow{OP}$ **makes with the positive z-axis**.

• It is required to lie in the interval $[0, \pi]$. The third coordinate is the **angle** θ as measured in cylindrical coordinates.



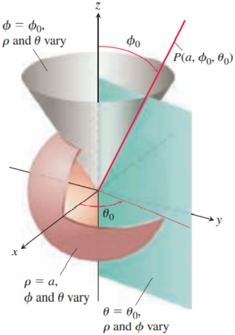
DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin $(\rho \geq 0)$.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates.

Spherical Coordinates

• On maps of the earth, θ is related to the **meridian** of a point on the earth and ϕ to its **latitude**, while r is related to **elevation above the earth's surface**.

• The equation $\rho = a$ describes the **sphere** of radius a centered at the origin.
• The equation $\phi = \phi_0$ describes a single **cone** whose vertex lies at the origin and whose axis lies along the z-axis.
• The equation $\theta = \theta_0$ describes the **half-plane** that contains the z-axis and makes an angle θ_0 with the positive x-axis.



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

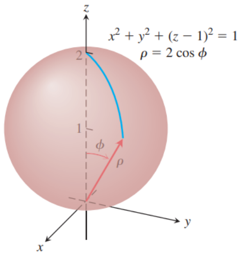
$$
\begin{aligned}
r &= \rho \sin \phi, \qquad & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\
z &= \rho \cos \phi, \qquad & y &= r \sin \theta = \rho \sin \phi \sin \theta, \\
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. & &
\end{aligned}
\tag{1}
$$

Example 3

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

Solution:

We use Equations (1) to substitute for x, y, and z:



$$x^2 + y^2 + (z-1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1 \qquad \text{Eqs. (1)}$$

$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi. \qquad \rho > 0$$

- The angle ϕ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole, the angle θ does not appear in the expression for ρ, reflecting the symmetry about the z-axis