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MA3077 (DLI) Operational Research

Lecture 8 – Mixed-integer linear problems

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Recap and plan of the day

Summary: so far we learnt:

- how to model linear programming problems and solve them in Matlab,
- how to analyze them using Farkas' lemma,
- the role (and interpretation) of weak and strong duality and sensitivity analysis.

Today: A first glimpse into mixed-integer programming following loosely Ch. 3 of OptArt and Ch. 9 of the book by Hillier and Lieberman.

A&E unit – problem statement

Scenario: Evening shift doctors in the A&E department of a hospital work five consecutive days and have two consecutive days off.

Their five days of work can start on any day of the week and the schedule rotates indefinitely.

The hospital requires the following minimum number of doctors working each day:

Mon	Tue	Wed	Thu	Fri	Sat	Sun
45	50	61	40	60	50	26

Finally, no more than 35 doctors can start their five working days on the same day.

Question: Find a rota to minimize the number of A&E doctors employed.

A&E unit – mathematical model

Let denote the how many doctors start on Monday, ..., Sunday. Then, we can model the problems as follows:

This is a (mixed-)integer linear programming problem.

Solving this example in Matlab with intlinprog

(see OR08_branch_and_bound.m)

```
f = ones(1,7);
intcon = 1:7;
v = [1 0 0 1 1 1 1];
A = -toeplitz(v([1, 7:-1:2]), v);  % create a circulant matrix
b = -[45,50,61,49,60,50,26];
lb = zeros(size(f));
ub = 35*ones(1,7);
[x,fval] = intlinprog(f,intcon,A,b,[],[],lb,ub);
```

Matlab's solution is . Another solution is

Mixed-integer optimisation problems are difficult

More precisely, they are *NP hard*, meaning that we can verify that a solution is correct in polynomial time, and we do not know whether we can solve them with a deterministic algorithm in polynomial time (however, if you find out how to do it, you will become very rich and famous).

The beautiful duality theory we developed for linear programming problems does not apply.

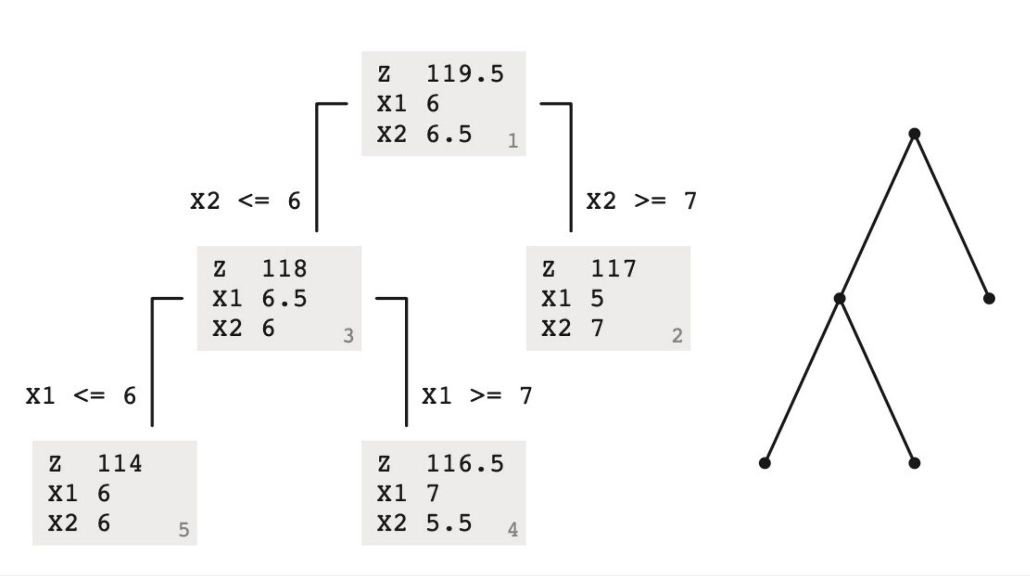
In many practical cases, an exact solution cannot be found in reasonable time.

To find out what Matlab's `intlinprog` does, see this documentation page.

The branch and bound method

(see OR08_branch_and_bound.m)

Consider the mixed-integer linear programming problem



Branch and bounding the A&E unit problem

Question: What happens if we solve the relaxed version of the A&E problem?

Answer: We get the same solution!!! But why? This was a bit of a coincidence. Indeed, the relaxed problems with `b = -[46,50,61,49,60,50,26];` gives a non-integer (relaxed) solution.

However, sometimes one always gets an integer solution to the relaxed problem. This is known as integer solution property.

Definition: A matrix is called totally unimodular if every square non-singular submatrix of A has determinant .

Theorem: Consider the integer programming problem

If the matrix is totally unimodular and is integer valued, then every extreme point of this integer programming problem is integer valued.

Remark: verifying that a matrix is totally unimodular is not always easy, but there are some tricks.

Example with integer solution property

Transportation problem: you employ two artisans who produce chairs and you have three customers who need chairs. Artisan 1 produced 10 chairs and artisan 2 produced 20 chairs. Customer 1 needs 6 chairs, customer 2 needs 8 chairs, and customer 3 needs 16 chairs. The cost to ship a chair from each artisan to any of the three customers is summarized in the following table:

	Customer 1	Customer 2	Customer 3
Artisan 1	£14	£11	£12
Artisan 2	£10	£12	£18

Question: To whom should artisans 1 and 2 ship their chairs to minimize the shipping costs for your company?

Transportation problem - model

Let denote how many chairs artisan sends to customer . Then, what we want to solve is

subject to

and

Transportation problem – totally unimodular

The constraints

can be written as

The matrix is totally unimodular!

Summary and self-study

Summary: today we have learnt

- how to model some mixed-integer linear programming problems,
- how to solve them in Matlab using `intlinprog`,
- and what the branch and bound technique is.

Self-study: Formulate the following problem as mixed-integer linear programming problem.

Scenario: A colleague of mine convenes the module MA3513 Industrial Mathematics Project. In this module, students work in groups for one year on a project set by an industrial partner.
This summer my colleague collected 7 potential projects and now he must allocate students to projects. This year there are 32 students and he would like to let them choose what project they'd like to work on. To this end, he asked them to rank their top-three preferred projects.

Question: How should he allocate students to projects to maximise their happiness while at the same time ensuring that no project has fewer than 4 or more than 5 students?