Chapter 1: Pre-Calculus

(i) Solve the inequality $|3x - 2| \geq 2$.

(ii) Prove that $||a| - |b|| \leq |a + b|$ for all $a, b \in \mathbb{R}$.

(iii) If $f(x) = x^2 + 6$ and $g(x) = 2x - 1$, determine $(f \circ g)(x)$ and $(g \circ f)(x)$.

(iv) Show that $f(x) = \dfrac{x+4}{2x-5}$ is one-to-one, and hence, find the inverse function $f^{-1}(x)$ stating it's domain and range.

Chapter 2: Limits

(i) Using an ϵ-δ argument, show that $g(x) = \begin{cases} x^2, & x < 1, \\ 2 - x, & x \geq 1, \end{cases}$ is continuous at $x = 1$.

(ii) Use the Pinching Theorem to show that $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right) = 0$.

(iii) Consider two continuous functions

$$f, g : [a, b] \to \mathbb{R} \text{ such that } f(a) < g(a) \text{ and } g(b) < f(b).$$

Show that there exists $c \in (a, b)$ with $f(c) = g(c)$.

Chapter 3: Differentiation

(i) From first principles and using Induction, if $n \in \mathbb{N}$ show that $\dfrac{d(x^n)}{dx} = nx^{n-1}$.
You may assume the Product Rule.

(ii) Determine $\dfrac{d}{dx} \tan^{-1}(x)$ and hence calculate $\lim\limits_{x \to 0} \dfrac{x}{\tan^{-1}(x)}$.

(iii) Using Newton's method, approximate $\sqrt{5}$ using $x_0 = 2$ until $|x_n^2 - 5| < 10^{-2}$.

(iv) Find the Taylor polynomial at $x = 1$ of degree 3 of the function $\tan^{-1}(x)$, also stating it's Lagrangian Remainder.

Chapter 4: Sequences

(i) If $a_n = (7n + 1)/n$, calculate a_{10}, a_{100} and a_{1000} and make a guess for the limit L as $n \to \infty$. Prove that a_n tends to this limit.

(ii) Determine if $\lim\limits_{n \to \infty} \left(\dfrac{n + 10^6}{n^2} + \dfrac{\cos^2(3n^2 - 4)}{n} \right)$ exists and if so, find it's limit.

(iii) Consider the sequence $(a_n)_{n \geq 1} = (-1)^n$. Show that there exists a convergent subsequence b_k and give an example of such a subsequence.

(iv) Prove that if $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are Cauchy sequences, then $(a_n b_n)_{n \geq 1}$ is also Cauchy.

Chapter 5: Integration

(i) Using n partitions of equal width, determine $U_f(n)$ and $L_f(n)$ of the integral $4\int_0^x t^2\,dt$. What are the corresponding errors? Why is $y=(2t)^2$ integrable?

(ii) Show that the function $g(x) = \int_0^{1/x} \frac{dt}{1+t^2} + \int_0^x \frac{dt}{1+t^2}$ is constant for $x>0$.

(iii) Determine $\int \frac{dx}{\sqrt{1-x^2}}$.

(iv) Show that $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$. Hence calculate $\int e^t\sqrt{1-e^{2t}}\,dt$.

Chapter 6: Differential Equations

(i) Solve $\frac{dy}{dx} = \frac{1+y^2}{9+6x}$ such that $y(1)=0$.

(ii) Find the general solution to $\frac{dy}{dx} - \tan(x)y = x^4$.

(iii) Solve $y'' + 6y' + 9y = 0$ such that $y(0)=0$ and $y(1)=1$.

Chapter 7: Infinite Series

(i) Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $s_{2n} - s_n \geq \frac{1}{2}$. Does $s_n = \sum_{k=1}^{n}\frac{1}{k}$ converge? Explain your answer.

(ii) Write down a formula for the partial sums s_k of the series

$$\sum_{n=1}^{\infty} \frac{1}{(-7)^{n-1}}.$$

Hence determine if the series converges, and if so determine its sum.

(iii) Which values of $c \in \mathbb{R}\backslash\{0\}$ make the series $\sum_{n=1}^{\infty} \frac{n}{(5c)^n}$ absolutely convergent?

(iv) Does $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ converge? Justify your answer.

Chapter 8: Multivariate Differentiation

(i) Show that the function $g(x,y) = \begin{cases} \frac{xy^2}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is continuous at the origin.

(ii) Let $f(x,y) = 3xy^2 + 2x^2y^2$. Find the tangent plane to the surface f at $\boldsymbol{P} = (-2,1)$.

(iii) Let $f(x,y) = 3x^3y + xy^3$. Calculate $\frac{\partial f}{\partial \boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \nabla f$ at $\boldsymbol{P} = (2,-3)$ where $\boldsymbol{n} = (3,-2)$.

(iv) Find and classify the stationary points of $f(x,y) = \frac{x^3}{3} + 5x^2y + 24y^2x + 63y$.