LINEAR ALGEBRA II

Ch. V SCALAR PRODUCTS AND ORTHOGONALITY

§1. Scalar Products

inner product

- Let V be a vector space over a field K.
- A scalar product on V is an association which to any pair of elements v, w of V associates a scalar, denoted by $\langle v, w\rangle$, or also $v \cdot w$, satisfying: $\forall u$, $v, w \in V$ and $x \in K$,

  $\langle \cdot \rangle : V \times V \to K$

  SP 1. $\langle v, w\rangle = \langle w, v\rangle$. 对称

  $\langle w, u+v\rangle = \langle w, u\rangle + \langle w, v\rangle$

  SP 2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.

  $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

  SP 3. $\langle xu, v\rangle = x\langle u, v\rangle$ and $\langle u, xv\rangle = x\langle u, v\rangle$.

  bilinear 双线性

- The scalar product is said to be non-degenerate if in addition it also satisfies the condition: if $v \in V$, and $\langle v, w\rangle \underset{=0}{} \forall w \in V$, then $v = O$.

§1. Scalar Products

- The dot product in $V = K^n$ is a non-degenerate scalar product.
- $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ is a non-degenerate scalar product in the space of continuous real-valued functions on the interval [0, 1].
- v, w are said to be orthogonal or perpendicular, and write $v \perp w$, if $\langle v, w\rangle = 0$.
- Let S be a subset of V, then $S^{\perp} = \{v \in V | \langle v, s\rangle = 0 \text{ for all } s \in S\}$ is a subspace of V, called the orthogonal space of S.
- $s \in S^{\perp} \Leftrightarrow s \perp S$.
- Let U be the subspace generated by S. Then $S^{\perp} = U^{\perp}$.

$U = \mathrm{span}(S)$

$\forall v \in U^{\perp}, \forall u \in S \subset U$

$\langle v, u\rangle = 0$

$\forall v \in S^{\perp} \quad \forall u \in U,$

$\exists u_1, \cdots, u_n \in S, a_1, \cdots, a_n \in K$, s.t $u = a_1u_1 + \cdots + a_nu_n$

$\langle v, u\rangle = a_1\langle v, u_1\rangle + \cdots + a_n\langle v, u_n\rangle = 0, \quad v \in U^{\perp}$

$\langle v, a_1u_1 + \cdots + a_nu_n\rangle = \langle v, a_1u_1\rangle + \cdots + \langle v, a_nu_n\rangle$

§1. Scalar Products

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- $AX = O$.
- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- $W = \text{span}\{A_1, \ldots, A_m\}$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^{\perp} = W^{\perp}$.

$\dim U = \dim W^{\perp}$

§1. Scalar Products

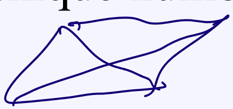
- Let V be a vector space over the field K, with a scalar product.
- Let $\{v_1, \ldots, v_n\}$ be a basis of V. We shall say that it is an orthogonal basis, if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- We shall show later that if V is a finite dimensional vector space, with a scalar product, then there always exists an orthogonal basis.
- We shall first discuss important special cases over the real and complex numbers.

§1. Scalar Products

- The real positive definite case
- Let V be a vector space over R, with a scalar product. We shall call this scalar product positive definite if $\langle v, v\rangle \geq 0$ for all $v \in V$ and $\langle v, v\rangle > 0$ for all $v \neq O$
- The dot product in $V = R^n$ is a positive definite scalar product.
- $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ is a positive definite scalar product in the space of continuous real-valued functions on the interval [0, 1].
- Let V be a vector space over R, with a positive definite scalar product denoted by $\langle\ ,\ \rangle$. Let W be a subspace. Then W has a scalar product defined by the same rule defining the scalar product in V.

§1. Scalar Products

- Norm. $\|v\| = \sqrt{\langle v, v\rangle}$ 范数. 模
- $\|cv\| = |c|\|v\|$. 正齐性 homogeneous
- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| > 0$ for all $v \neq O$ 正定性.
- $\operatorname{dist}(v, w) = \|v - w\|$.
- v is called a unit vector if $\|v\| = 1$. For any $0 \neq v \in V$, $v/\|v\|$ is a unit vector.
- **The Pythagoras theorem.** If $v \perp w$, then $\|v \pm w\|^2 = \|v\|^2 + \|w\|^2$.
- **The parallelogram law.** $\forall v, w$, $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.
- Let $w \in V$ and $\|w\| \neq 0$. For any v there exists a unique number c such that $v - cw \perp w$.



- $c = \frac{\langle v,w\rangle}{\langle w,w\rangle}$ $(= \langle v, w\rangle$ when $\|w\| = 1)$, the component of v along w.
- We call cw the projection of v along w.

§1. Scalar Products

- **Schwarz inequality.** $|\langle v, w\rangle| \leq \|v\|\,\|w\|$.
- **Triangle inequality.** $\|v + w\| \leq \|v\| + \|w\|$.
- Let $v_1, \ldots, v_n$ be non-zero elements of V which are mutually perpendicular. Let c_i be the component of v along v_i. Then

$$v - c_1 v_1 - \cdots - c_n v_n \perp v_i, \forall i = 1, \ldots, n.$$

- $\|v - c_1 v_1 - \cdots - c_n v_n\| \leq \|v - a_1 v_1 - \cdots - a_n v_n\|, \forall a_i \in R$.
- **Bessel inequality.** $\sum_{i=1}^{n} c_i^2 \leq \|v\|^2$.

$v = v - c_1 v_1 - \cdots - c_n v_n + c_1 v_1 + \cdots + c_n v_n$

$(a_1 - c_1)v_1 + \cdots$

$(a_n - c_n)v_n$

$\langle v, v_i\rangle - c_1\langle v_1, v_i\rangle - \cdots - c_n\langle v_n, v_i\rangle$

$= \langle v, v_i\rangle - c_i\langle v_i, v_i\rangle = 0$

$c_i = \dfrac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle}$

§2. Orthogonal Bases, Positive Definite Case

- Let V be a vector space with a positive definite scalar product throughout this section. A basis $\{v_1, \ldots, v_n\}$ of V is said to be orthogonal if its elements are mutually perpendicular.
- Orthonormal basis. $\|v_i\| = 1$ 标准正交基。正规正交基。
- Gram-Schmidt orthogonalization (orthonormalization) process.
- **Theorem 2.1.** Let V be a finite dimensional vector space, with a positive definite scalar product. Let W be a subspace of V, and let $\{w_1, \ldots, w_m\}$ be an orthogonal basis of W If $W \neq V$, then there exist elements $w_{m+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_n$ is an orthogonal basis of v.
- **Corollary 2.2.** Let V be a finite dimensional vector space with a positive definite scalar product. Assume that $V \neq \{O\}$. Then V has an orthogonal basis.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.3.** Let V be a vector space over R with a positive definite scalar product, of dimension n. Let W be a subspace of V of dimension r. Let $W^{\perp}$ be the subspace of V consisting of all elements which are perpendicular to W. Then V is the direct sum of W and $W^{\perp}$, and $W^{\perp}$ has dimension $n - r$. In other words,

$$\dim W + \dim W^{\perp} = \dim V.$$

§2. Orthogonal Bases, Positive Definite Case

§2. Orthogonal Bases, Positive Definite Case

- $W^{\perp}$ is called the orthogonal complement of W.
- Let V be a finite dimensional vector space over R, with a positive definite scalar product. Let $\{e_l, \ldots, e_n\}$ be an orthonormal basis.

  $$v = x_1e_1 + \cdots + x_ne_n \text{ and } w = y_1e_1 + \cdots + y_ne_n.$$

  Then $\langle v, w\rangle = x_1y_1 + \cdots + x_ny_n$.

- Homework: Ch. V, §2, 0, 2, 3, 5.

$$\begin{pmatrix} 1 \\ \cdot \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad 2$$

$$\begin{pmatrix} 1 \\ - \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ - \\ 1 \end{pmatrix} = 2$$

Linear Algebra II

Ch. V Scalar Products and Orthogonality

§1. Scalar Products

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  SP 1. $\langle v, w\rangle = \langle w, v\rangle$.
  SP 2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
  SP 3. $\langle xu, v\rangle = x\langle u, v\rangle$ and $\langle u, xv\rangle = x\langle u, v\rangle$.
- The scalar product is said to be non-degenerate if in addition it also satisfies the condition: if $v \in V$, and $\langle v, w\rangle = 0$ $\forall w \in V$, then $v = O$.

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- Let S be a subset of V, then $S^{\perp} = \{v \in V | \langle v, s\rangle = 0 \text{ for all } s \in S\}$ is a subspace of V, called the orthogonal space of S.
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- Let U be the subspace generated by S. Then $S^{\perp} = U^{\perp}$.

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- $AX = O$.
- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- $W = \text{span}\{A_1, \ldots, A_m\}$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^\perp = W^\perp$.

§1. Scalar Products

- Let V be a vector space over the field K, with a scalar product.
- Let $\{v_1, \ldots, v_n\}$ be a basis of V. We shall say that it is an orthogonal basis, if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
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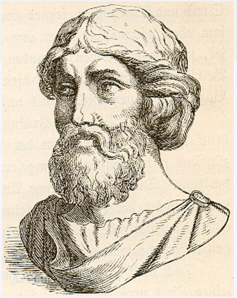
§1. Scalar Products

The Real Positive Definite Case

- Let V be a vector space over R, with a scalar product. We shall call this scalar product positive definite if $\langle v, v\rangle \geq 0$ for all $v \in V$ and $\langle v, v\rangle > 0$ for all $v \neq O$
- The dot product in $V = R^n$ is a positive definite scalar product.
- $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ is a positive definite scalar product in the space of continuous real-valued functions on the interval [0, 1].
- Let V be a vector space over R, with a positive definite scalar product denoted by $\langle\ ,\ \rangle$. Let W be a subspace. Then W has a scalar product defined by the same rule defining the scalar product in V.

§1. Scalar Products



Pythagoras, 580 BC-500 BC.

- Norm. $\|v\| = \sqrt{\langle v, v\rangle}$
- $\|cv\| = |c|\|v\|$.
- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| > 0$ for all $v \neq O$
- $\mathrm{dist}(v, w) = \|v - w\|$.
- v is called a unit vector if $\|v\| = 1$. For any $0 \neq v \in V$, $v/\|v\|$ is a unit vector.
- **The Pythagoras theorem.** If $v \perp w$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.
- **The parallelogram law.** $\forall v, w$, $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.
- Let $w \in V$ and $\|w\| \neq 0$. For any v there exists a unique number c such that $v - cw \perp w$.
- $c = \frac{\langle v,w\rangle}{\langle w,w\rangle}$ $(= \langle v, w\rangle$ when $\|w\| = 1)$, the component of v along w.
- We call cw the projection of v along w.

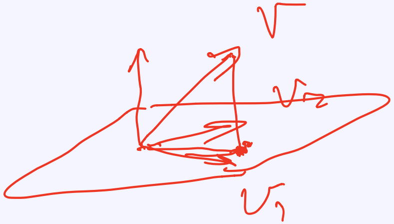
§1. Scalar Products

H. A. Schwarz, 1843-1921.

- **Schwarz inequality.** $|\langle v, w\rangle| \leq \|v\| \|w\|$.
- **Triangle inequality.** $\|v + w\| \leq \|v\| + \|w\|$.

- Let $v_1, \ldots, v_n$ be non-zero elements of V which are mutually perpendicular. Let c_i be the component of v along v_i. Then

$$v - c_1 v_1 - \cdots - c_n v_n \perp v_i, \forall i = 1, \ldots, n.$$



Span{v1, ... vn}

§2. Orthogonal Bases, Positive Definite Case

- $\|v - c_1 v_1 - \cdots - c_n v_n\| \leq \|v - a_1 v_1 - \cdots - a_n v_n\|$.

F. W. Bessel, 1784-1846.

- **Bessel inequality.** $\sum_{i=1}^{n} c_i^2 \leq \|v\|^2$.
- Let V be a vector space with a positive definite scalar product throughout this section. A basis $\{v_1, \ldots, v_n\}$ of V is said to be orthogonal if its elements are mutually perpendicular.
- Orthonormal basis.
- Gram-Schmidt orthogonalization (orthonormalization) process.

$v_1, \ldots, v_n$

$v_1', \ldots, v_n'$

$v_1' = v_1$

$v_2' = v_2 - a_{21} v_1'$

$v_{r+1}' = v_{r+1} - a_{r+1,r} v_r' - \cdots - a_{r+1,1} v_1'$

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.1.** Let V be a finite dimensional vector space, with a positive definite scalar product. Let W be a subspace of V, and let $\{w_1, \ldots, w_m\}$ be an orthogonal basis of W If $W \neq V$, then there exist elements $w_{m+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_n$ is an orthogonal basis of v.

- **Corollary 2.2.** Let V be a finite dimensional vector space with a positive definite scalar product. Assume that $V \neq \{O\}$. Then V has an orthogonal basis.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.3.** Let V be a vector space over R with a positive definite scalar product, of dimension n. Let W be a subspace of V of dimension r. Let $W^\perp$ be the subspace of V consisting of all elements which are perpendicular to W. Then V is the direct sum of W and $W^\perp$, and $W^\perp$ has dimension $n - r$. In other words,

$$\dim W + \dim W^\perp = \dim V.$$

Proof. If $W=\{0\}$, then $W^\perp = V$. If $W=V$, then $W^\perp=\{0\}$

$\Rightarrow$ The results hold. If $W \neq V$, and $W \neq \{0\}$, let $\{w_1, \cdots, w_r\}$ be a orthonormal basis of W. By Th 2.1, $\exists$ $\underline{u_{r+1}, \cdots, u_n}$, s.t. $\{w_1, \cdots, w_r, u_{r+1}, \cdots, u_n\}$ is an orthonormal basis of V.

Let $u \in W^\perp$. Then $\exists$ $x_1, \cdots, x_n \in R$, s.t.

$$u = x_1 w_1 + \cdots + x_r w_r + x_{r+1} u_{r+1} + \cdots + x_n u_n$$

$0 = \langle u, w_i \rangle = x_i \langle w_i, w_i \rangle = x_i$

$u = x_{r+1} u_{r+1} + \cdots + x_n u_n$

§2. Orthogonal Bases, Positive Definite Case

Let $u = x_{r+1} u_{r+1} + \cdots + x_n u_n$

$\langle u, w_i \rangle = 0 \Rightarrow u \in W^\perp$

$u_{r+1}, \cdots, u_n$ are perpendicular to each other,

Ther form an orthonormal basis of $\underline{W}^\perp$

$\Rightarrow \dim W^\perp = n - r.$ and $W + W^\perp = V.$

$\Updownarrow$

$\dim W + \dim W^\perp = \dim V$

§2. Orthogonal Bases, Positive Definite Case

- $W^{\perp}$ is called the orthogonal complement of W.
- Let V be a finite dimensional vector space over R, with a positive definite scalar product. Let $\{e_l, \ldots, e_n\}$ be an orthonormal basis.

$$v = x_1e_1 + \cdots + x_ne_n \text{ and } w = y_1e_1 + \cdots + y_ne_n.$$

Then $\langle v, w\rangle = x_1y_1 + \cdots + x_ny_n$.

§2. Orthogonal Bases, Positive Definite Case

- Homework: Ch. V, §2, 0, 2, 3, 5.

Hermitian Products on VSs Over C

埃尔米特

Hermit

Charles Hermite, 1822-1901.

- The dot product of the nonzero vector $(\mathrm{i}) \in C^1$ with itself is -1!
- The dot product of the nonzero vector ${}^t(1, \mathrm{i}) \in C^2$ with itself is 0!
- Dot product is not a good scalar product.

§2. Orthogonal Bases, Positive Definite Case

- Let V be a vector space over the complex numbers. A hermitian product on V is a rule which to any pair of elements v, w of V associates a complex number, denoted again by $\langle v, w\rangle$, satisfying the following properties:

  HP 1. We have $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$.

  HP 2. If u, v, w are elements of V, then

  $$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle.$$

  HP 3. If $\alpha \in C$, $u, v \in V$, then

  $$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \text{ and } \langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle.$$

- The Hermitian product is said to be positive definite, if $\langle v, v\rangle \geq 0$ for all $v \in V$, and $\langle v, v\rangle > 0$ for all $0 \neq v \in V$.

§2. Orthogonal Bases, Positive Definite Case

- Orthogonal [illegible]), perpendicular, orthogonal [illegible] basis, orthogonal complement can be defined same as before.
- For $X = {}^t(x_1, \ldots, x_n)$, $Y = {}^t(y_1, \ldots, y_n) \in C^n$, define

$$\langle X, Y \rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$

  It is a positive definite Hermitian product.
- The Hermitian product of the nonzero vector $(\mathrm{i}) \in C^1$ with itself is 1!
- The Hermitian product of the nonzero vector ${}^t(1, \mathrm{i}) \in C^2$ with itself is 2!

§2. Orthogonal Bases, Positive Definite Case

- Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$. If $f, g \in V$, we define
$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t.$$
- It is a positive definite Hermitian product.
- Let $f_n(t) = e^{int}$.
  - $\langle f_n, f_m \rangle = 0$ for $m \neq n$;
  - $\langle f_n, f_m \rangle = 2\pi$;
- The Fourier coefficients of f w.r.t. f_n is
$$\frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t.$$

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.4.** Let V be a finite dimensional vector space over C, with a positive definite Hermitian product. Let W be a subspace of V, and let $\{w_1, \ldots, w_m\}$ be an orthogonal basis of W If $W \neq V$, then there exist elements $w_{m+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_n$ is an orthogonal basis of v.

- **Corollary 2.5.** Let V be a finite dimensional vector space over C with a positive definite scalar product. Assume that $V \neq \{O\}$. Then V has an orthogonal basis.

§2. Orthogonal Bases, Positive Definite Case

- Let V be a vector space over C, with a positive definite hermitian product.
- Norm. $\|v\| = \sqrt{\langle v, v\rangle}$
- **Schwarz inequality.** $|\langle v, w\rangle| \leq \|v\| \|w\|$.
- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ iff $v = O$
- $\|cv\| = |c| \|v\|$ for all $c \in C$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v \in V$.

$\langle v, w\rangle \leq |\langle v, w\rangle| \leq \|v\| \cdot \|w\|$

$\|v+w\|^2 = \langle v+w, v+w\rangle = \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle$

$\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$ real

$= (\|v\| + \|w\|)^2$

$\leq |\langle v, w\rangle + \langle w, v\rangle|$

$\leq |\langle v, w\rangle| + |\langle w, v\rangle|$

§2. Orthogonal Bases, Positive Definite Case

- The Pythagoras theorem.
- The parallelogram law.
- A unit vector, orthonormal, orthonormal basis.
- The component of v along w, the projection of v along w, the projection of v onto span$\{v_1, \ldots, v_n\}$. $\{v_1, \ldots, v_n\}$ is a orthonormal basis
- Bessel inequality.
- Let V be a finite dimensional vector space over C, with a positive definite Hermitian product. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis.

$$v = x_1 e_1 + \cdots + x_n e_n \text{ and } w = y_1 e_1 + \cdots + y_n e_n.$$

Then $\langle v, w \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.6. and 2.7.** Let V be a vector space over R with a positive definite scalar product, or a vector space over C with a positive definite scalar product. Assume that V has finite dimension n. Let W be a subspace of V of dimension r. Let $W^{\perp}$ be the subspace of V consisting of all elements which are perpendicular to W. Then V is the direct sum of W and $W^{\perp}$, and $W^{\perp}$ has dimension $n-r$. In other words,

$$\dim W + \dim W^{\perp} = \dim V.$$

§2. Orthogonal Bases, Positive Definite Case

- Homework: V, §2, 6, 8, 9

§3. Application to Linear Equations; The Rank

- A system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\
&\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

(xA)

- $x_1A^1 + \cdots + x_nA^n = O$.
- $AX = O$, $X \in \operatorname{Ker} A$.
- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^\perp = W^\perp$. Where, $W = \operatorname{span}\{A_1, \ldots, A_m\}$.

§3. Application to Linear Equations; The Rank

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- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^{\perp} = W^{\perp}$. Where, $W = \operatorname{span}\{A_1, \ldots, A_m\}$.
- The row rank of A: the dimension of W.
- The column rank of A: $\dim \operatorname{span}\{A^1, \ldots, A^m\}$=$\dim \operatorname{Im} L_A$.

§3. Application to Linear Equations; The Rank

- Even if the scalar product is not positive definite, the following theorem (Th. 2.3, for a VS over R with a positive definite scalar product) is true (§6, Th. 6.4).

- **Theorem 3.1.** Let W be a subspace of K^n. Then

$$\dim W + \dim W^{\perp} = n.$$

- **Theorem 3.2.** Let $A = (a_{ij})$ be an $m \times n$ matrix. Then the row rank and the column rank of A are equal to the same number r . Furthermore, $n - r$ is the dimension of the space of solutions of the system of linear equations $AX = O$.

- The rank of $A \Rightarrow$ determinant

$\dim \ker L + \dim \operatorname{Im} L = \dim V$

§3. Application to Linear Equations; The Rank

Proof Consider the linear map $L : K^n \to K^n$

$$L(X) = AX = x_1 A^1 + \cdots + x_n A^n$$

$$\mathrm{Im}\, L = \mathrm{span}\{A^1, \dots, A^n\}$$

①
$$\underbrace{\dim \operatorname{Ker} L}_{\dim(\text{space of solutions})} + \underbrace{\dim \operatorname{Im} L}_{\text{column rank}} = n$$

②
$$\underbrace{\dim W}_{\text{row rank}} + \underbrace{\dim W^{\perp}}_{\dim(\text{space of solutions})} = n$$