Lecture 18: Partial Derivatives.

MA2032 Vector Calculus

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Extreme Values and Saddle Points

- Continuous functions of two variables assume **extreme values** on closed, bounded domains.

- We see in this section that we can narrow the search for these extreme values by **examining the functions' first partial derivatives**.
- A function of two variables can assume extreme values only at **boundary points** of the domain or at **interior domain points** where both first partial derivatives are zero or where one or both of the first partial derivatives fail to exist.
- However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value.

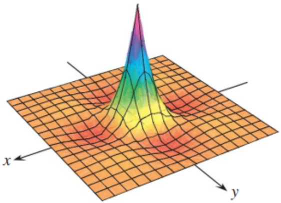
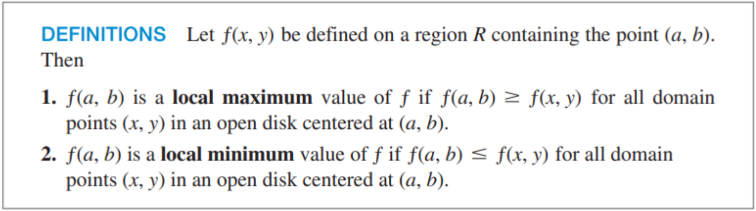


Figure: $z = (\cos x)(\cos y)e - \sqrt{x^2 + y^2}$ with max=1 and a min=-0.067 on $|x| \leq 3\pi/2, |y| \leq 3\pi/2$.

- The surface that is the graph of the function might be shaped like a **saddle** right above (a, b) and cross its tangent plane there.

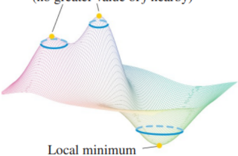
Derivative Tests for Local Extreme Values

• For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a **horizontal tangent plane**.
• At such points, we then look for **local maxima, local minima, and saddle points**.



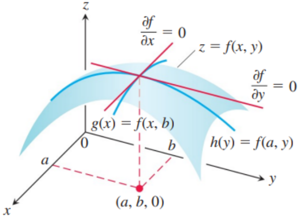
DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).



Derivative Tests for Local Extreme Values

THEOREM 10—First Derivative Test for Local Extreme Values
If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

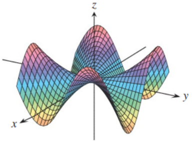


DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Extreme Values and Saddle Points

• As with differentiable functions of a single variable, **not every critical point gives rise to a local extremum**.
• A differentiable function of a single variable might have a **point of inflection**.
• A differentiable function of two variables might have a **saddle point**, with the graph of f **crossing the tangent plane defined there**.

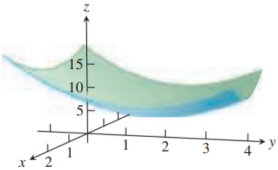
DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.45).



$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$

Example 1

Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.



Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

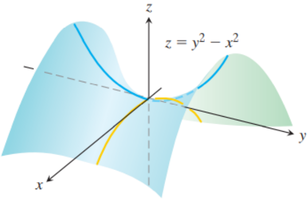
$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum (Figure 14.46).

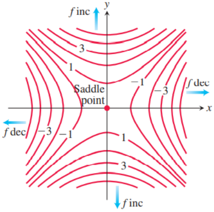
Example 2

Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$. Along the positive x-axis, however, f has the value $f(x, 0) = -x^2 < 0$; along the positive y-axis, f has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the xy-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (Figure 14.47a). Figure 14.47b displays the level curves (they are hyperbolas) of f, and shows the function decreasing and increasing in an alternating fashion among the four groupings of hyperbolas. ∎



(a)



(b)

Second Derivative Test for Local Extreme Values

- That $f_x = f_y = 0$ at an interior point (a, b) of R **does not guarantee** f has a local extreme value there.
- If f and its **first and second partial derivatives are continuous on R**, however, we may be able to learn more from the following theorem

THEOREM 11—Second Derivative Test for Local Extreme Values
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

- The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the discriminant or **Hessian of f.**
- It is sometimes easier to remember it in **determinant form**,

$$f_{xx}f_{yy} - f_{xy}{}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Example 3

Find the local extreme values of the function
$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Solution The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ∎

Example 4

Find the local extreme values of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

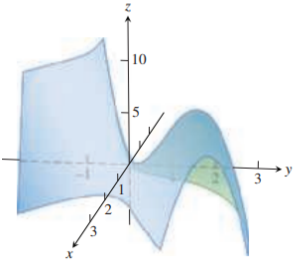
To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point $(2, 2)$ we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48.



Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the **absolute extrema** of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. List the **interior points of R** where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.

2. List the **boundary points of R** where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.

3. Look through the lists for the maximum and minimum values of f. These will be the **absolute maximum and minimum values of f on R**.

Example 5

Find the absolute maximum and minimum values of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0$, and $y = 9 - x$.

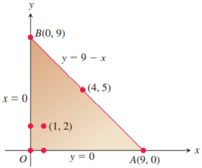
Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary (Figure 14.50a).

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of f there is

$$f(1, 2) = 7.$$



(a)

Example 5

(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, $y = 0$. The function

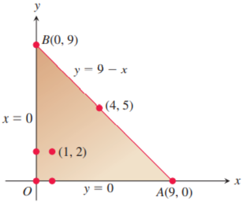
$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values (as we know from Chapter 4) may occur at the endpoints

$$\begin{aligned} x &= 0 \quad \text{where} \quad f(0, 0) = 2 \\ x &= 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61 \end{aligned}$$

or at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$



(a)

Example 5

ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of y defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$

iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of the line segment AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

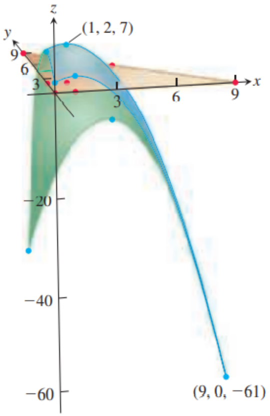
$$x = 4.$$

At this value of x,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

Example 5

Summary We list all the function value candidates: 7, 2, −61, 3, −43, 6, −11. The maximum is 7, which f assumes at (1, 2). The minimum is −61, which f assumes at (9, 0). See Figure 14.50b. ∎



(b)

Summary of Max-Min Tests

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist)

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**