Problem Sheet 1 Answer

Junbiao Li - 209050796

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1 Problem 1

Let x_G be the production volume of Growrite (G) in liters, and x_T be the production volume of Tomfood (T) in liters.

Each type of fertilizer requires three basic ingredients (N, P, K), and there is a limited amount of these ingredients available each day.

- For Nitrogen (N), Growrite requires 0.11 kg/L and Tomfood requires 0.08 kg/L. A total of 600 kg is available.

$$0.11 \cdot x_G + 0.08 \cdot x_T \leq 600$$

- For Phosphorus (P), Growrite requires 0.06 kg/L and Tomfood requires 0.03 kg/L. A total of 300 kg is available.

$$0.06 \cdot x_G + 0.03 \cdot x_T \leq 300$$

- For Potassium (K), Growrite requires 0.02 kg/L and Tomfood requires 0.08 kg/L. A total of 330 kg is available.

$$0.02 \cdot x_G + 0.08 \cdot x_T \leq 330$$

Biocare aims to maximize its daily income. The selling price for Growrite is £2.80/L, and for Tomfood, it's £3.00/L.

$$\max Z = 2.80 \cdot x_G + 3.00 \cdot x_T$$

Combining all the information, we have the following linear programming problem:

$$\begin{array}{rc} \max & Z = 2.80 \cdot x_G + 3.00 \cdot x_T \\ \text{s.t.} & 0.11 \cdot x_G + 0.08 \cdot x_T \leq 600 \\ & 0.06 \cdot x_G + 0.03 \cdot x_T \leq 300 \\ & 0.02 \cdot x_G + 0.08 \cdot x_T \leq 330 \\ & x_G, x_T \geq 0 \end{array}$$

2 Problem 2

After solving linear programming problem:

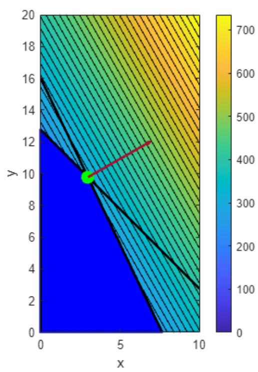


Figure 1:

It can be seen from the figure 1 that only constraint $23x + 11y \leq 176$ and constraint $4x + 4y \leq 51$ are active, since the direction of the gradient is toward the upper right corner of the image.

2.1 MATLAB Code

```
clear, close all;

% Define the coefficients for the objective function
f = [-35; -20];  % Note the negative signs for a maximization problem


% Define the inequality constraint matrix and vector
A = [23, 11; 4, 4];
b = [176; 51];

% Define the lower bound for the variables
lb = [0; 0];

% Create optimization options
options = optimoptions('linprog', 'Display', 'none');

% Solve the linear programming problem
[x, fval, exitflag, output] = linprog(f, A, b, [], [], lb, [], options);

% Convert the objective function value back to the original maximization problem
optimal_value = -fval;

% Display the results
```

```
fprintf('The optimal solution is x = %f, y = %f\n', x(1), x(2));
fprintf('The maximum value of the objective function is %f\n', optimal_value);
```

3 Problem 3

1. For the first constraint $3x - 7z \leq 176$, which is equivalent to $3x - 7z + s_1 = 176$ where, $s_1 \geq 0$.

2. For the second constraint $8z - 2y + x - 6 \geq 12$, We have $8z - 2y + x - 6 - s_2 = 12$ where, $s_2 \geq 0$.

3. The third constraint $4x + 3y = 19$ is already an equality constraint and needs no change.

4. Using new variables $x_+, x_-, y_+, y_-, z_+, z_-$ to represent the positive and negative parts of x, y, z, respectively. Specifically, $x = x_+ - x_-$, $y = y_+ - y_-$, $z = z_+ - z_-$, where $x_+, x_-, y_+, y_-, z_+, z_- \geq 0$.

Letting $\mathbf{x} = [x_+, x_-, y_+, y_-, z_+, z_-, s]^T$, hence

$$\max \quad \begin{pmatrix} -30 & 30 & -21 & 21 & -18 & 18 & 0 \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z_+ \\ z_- \\ s \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} 3 & -3 & 0 & 0 & -7 & 7 & 0 \\ -1 & 1 & 2 & -2 & -8 & 8 & 0 \\ 4 & -4 & 3 & -3 & 0 & 0 & -1 \\ -4 & 4 & -3 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z_+ \\ z_- \\ s \end{pmatrix} \leq \begin{pmatrix} 176 \\ -12 \\ 19 \\ -19 \end{pmatrix}$$

$$\begin{pmatrix} x_+ \\ x_- \\ y_+ \\ y_- \\ z_+ \\ z_- \\ s \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4 Problem 4

First, we have the primal problem:

$$\begin{array}{ll} \min & \begin{pmatrix} 1 & 4 & -9 \end{pmatrix} z \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 4 & 3 & 0 \end{pmatrix} z = \begin{pmatrix} 7 \\ 2 \\ 19 \end{pmatrix} \\ & x, y, z \geq 0 \end{array}$$

To obtain its dual, we introduce the Lagrange multipliers y and construct the **Lagrangian Function**:

$$L(z, y) = \begin{pmatrix} 1 & 4 & -9 \end{pmatrix} z + y^T \left(\begin{pmatrix} 7 \\ 2 \\ 19 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 4 & 3 & 0 \end{pmatrix} z \right) - s^T z$$

Next, to obtain the **Dual Function**, we minimize the Lagrangian with respect to z:

$$\begin{aligned} g(y) &= \min_z L(z, y) \\ &= \min_z z^T \left(\begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix} + y^T \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 3 \\ -1 & 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 7 \\ 2 \\ 19 \end{pmatrix}^T y \end{aligned}$$

Based on the above dual function, we can form the **Dual Problem**:

$$\begin{array}{ll} \max & \begin{pmatrix} 7 & 2 & 19 \end{pmatrix} y \\ \text{s.t.} & \begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 3 \\ -1 & 1 & 0 \end{pmatrix} y = s \\ & s \geq 0, y \in \mathbb{R}^3 \end{array}$$

5 Problem 5

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & \|x\|_1^2 \leq 4 \end{aligned}$$

which is equivalent to:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & \|x\|_1 \leq 2 \end{aligned}$$

With auxiliary variables z_1 and z_2, transforming the problem into:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & |x_1| \leq z_1 \\ & |x_2| \leq z_2 \\ & z_1 + z_2 = 2 \end{aligned}$$

6 Problem 6

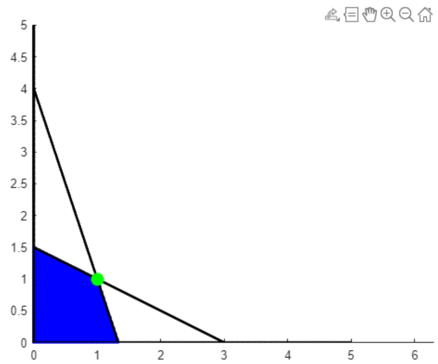
$$\begin{aligned} \min \quad & z_1 + z_2 + \cdots + z_n \\ \text{s.t.} \quad & -z \leq w \leq z \\ & Aw = y. \\ & w, z \in \mathbb{R}^n \end{aligned}$$

To make $z_1 + z_2 + \cdots + z_n$ is desired to be minimal, since A have full rank which means w is unique, we can let Let

$$\begin{aligned} 0 \leq |w| \leq z \\ 0 \leq |w_1| \leq z_1 \\ \cdots \\ 0 \leq |w_n| \leq z_n \end{aligned}$$

Therefore, when $z_1 = w_1, z_2 = w_2, \cdots, z_n = w_n \leftrightarrow z = |w|$ which satisfies $Aw = y$ Hence, that $z^* = |w^*|$.

7 Problem 7



The objective function is to maximize $c^T x$, where $c = (\cos(\alpha), \sin(\alpha))^T$. The gradient of this function is c itself. For $(1, 1)$ to be an optimal solution, the gradient c should point in a direction where the function $c^T x$ is maximized when starting from $(1, 1)$.

The relevant slopes of the boundaries at the vertex $(1, 1)$ are $-1/2$ and -3. For $(1, 1)$ to be an optimal solution, α should fall between these two angles, i.e., $\arctan(1/3) \leq \alpha \leq \arctan(2)$. Hence, for $\alpha \in [0.32, 1.11]$, the point $x = (1, 1)^T$ is an optimal solution to the given linear programming problem.

8 Problem 8

Original problem is:

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & c - A^T y = s \\ & s \geq 0, y \in \mathbb{R}^m \end{array}$$

First we write the **Standard Form** For the constraint $c - A^T y = s$, we have

$$c - A^T y \leq s$$
$$c - A^T y \geq -s$$

And $s \geq 0$, we can rewrite as:

$$A^T y \leq c + s$$
$$-A^T y \leq s - c$$

For the purpose of writing this in standard form, we introduce the new variable. Let $y = y_1 - y_2$ The standard form is

$$\begin{array}{ll} \min & b^T (y_1 - y_2) \\ \text{s.t.} & A^T (y_1 - y_2) = c - s \\ & y_1, y_2 \geq 0 \end{array}$$

Now we are proving that this problem is infeasible if and only if there is an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$

Since this problem is infeasible, which means the feasible set $F_d = \varnothing$.

If the feasible set $F_d \neq \varnothing$, we have

$$\begin{aligned} A^T y + s &= c \\ (A^T y + s)^T x &= c^T x \\ y^T Ax + s^T x &= c^T x \\ s^T x &= c^T x \quad \text{(Since we assume that } Ax = 0) \end{aligned}$$

However, we know that $s \geq 0$ and $c^T x < 0$, which is a contradiction. Hence, the feasible set $F_d = \varnothing$. We prove that this problem is infeasible if there is an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$.

And we can prove by Farkas' lemma that this problem is infeasible then there is an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$.

Hence, we have proved that this problem is infeasible if and only if there is an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$.

9 Problem 9

optimal soln is $x = (0.00, 3.00)$, $p* = 3.00$

9.1 MATLAB Code

```
f = [1, 1];

A = [-2, -2; 12, 5];
b = [-5; 30];

lb = [0, 0];
ub = [inf, inf];

opts = optimoptions('linprog','Display','none');

intcon = [1, 2];
[x,fval]= linprog(f,A,b,[],[],lb, ub, opts);
fprintf("optimal soln is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% impose x(2) <= 2
[x,fval] = linprog(f,A,b,[],[],lb, [inf, 2], opts);
fprintf("optimal soln with y(2)<=2 is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% impose x(2) >= 3
[x,fval] = linprog(f,A,b,[],[],[0, 3],ub, opts);
fprintf("optimal soln with y(2)>=3 is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% using intlinprog
opts = optimoptions('intlinprog', 'Display', 'none');
[x, fval] = intlinprog(f, intcon, A, b, [], [], lb, ub, opts);

fprintf("optimal soln is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])
```

Appendix

A MATLAB Code

```
%%Problem 2
clear, close all;

% Define the coefficients for the objective function
f = [-35; -20];  % Note the negative signs for a maximization problem

% Define the inequality constraint matrix and vector
A = [23, 11; 4, 4];
b = [176; 51];

% Define the lower bound for the variables
lb = [0; 0];

% Create optimization options
options = optimoptions('linprog', 'Display', 'none');

% Solve the linear programming problem
[x, fval, exitflag, output] = linprog(f, A, b, [], [], lb, [], options);

% Convert the objective function value back to the original maximization problem
optimal_value = -fval;

% Display the results
fprintf('The optimal solution is x = %f, y = %f\n', x(1), x(2));
fprintf('The maximum value of the objective function is %f\n', optimal_value);

% Create meshgrid for plotting
[X, Y] = meshgrid(linspace(0, 10, 401), linspace(0, 20, 401));

% Define and plot objective function
f = @(x,y) 35*x + 20*y;
contourf(X, Y, f(X,Y), 50); colorbar;
xlim([0, 10]), ylim([0, 20]), axis equal, hold on;

% Determine and plot feasible region using logical indexing
idx1 = 23*X + 11*Y <= 176;  % Constraint 1
idx2 = 4*X + 4*Y <= 51;     % Constraint 2
idx3 = X >= 0;              % Non-negative constraint for x
idx4 = Y >= 0;              % Non-negative constraint for y

% Combine all constraints
idx = idx1 & idx2 & idx3 & idx4;

% Extract coordinates of feasible region
Xf = X(idx); Yf = Y(idx);

% Plot feasible region
plot(Xf, Yf, '.b')

% Plot boundary of feasible region
t = linspace(0, 10, 400);
lw = 'linewidth';
```

```
plot(t, (176 - 23*t)/11, '-k', lw, 2)  % Constraint boundary 1
plot(t, (51 - 4*t)/4, '-k', lw, 2)     % Constraint boundary 2
plot(zeros(size(t)), t, '-k', lw, 2)   % Non-negative constraint for x
plot(t, zeros(size(t)), '-k', lw, 2)   % Non-negative constraint for y

% Mark the optimal solution
opt_x = x(1);
opt_y = x(2);
plot(opt_x, opt_y, 'go', 'MarkerSize', 10, 'MarkerFaceColor', 'g');

% Plot normalized gradient
grad_f = [35; 20];
grad_f = 5*grad_f / norm(grad_f);
quiver(opt_x, opt_y, grad_f(1), grad_f(2), 'linewidth', 2)
hold off

%%Problem 7
clear, close all;

% Define the coefficients for the objective function
f = [-35; -20];  % Note the negative signs for a maximization problem

% Define the inequality constraint matrix and vector
A = [23, 11; 4, 4];
b = [176; 51];

% Define the lower bound for the variables
lb = [0; 0];

% Create optimization options
options = optimoptions('linprog', 'Display', 'none');

% Solve the linear programming problem
[x, fval, exitflag, output] = linprog(f, A, b, [], [], lb, [], options);

% Convert the objective function value back to the original maximization problem
optimal_value = -fval;

% Display the results
fprintf('The optimal solution is x = %f, y = %f\n', x(1), x(2));
fprintf('The maximum value of the objective function is %f\n', optimal_value);

% Create meshgrid for plotting
[X, Y] = meshgrid(linspace(0, 10, 1001), linspace(0, 10, 1001));

% Define and plot objective function
%f = @(x,y) 35*x + 20*y;
%contourf(X, Y, f(X,Y), 50); colorbar;
xlim([0, 5]), ylim([0, 5]), axis equal, hold on;

% Determine and plot feasible region using logical indexing
idx1 = X + 2*Y <= 3;  % Constraint 1
idx2 = 3*X + Y <= 4;      % Constraint 2
idx3 = X >= 0;            % Non-negative constraint for x
idx4 = Y >= 0;            % Non-negative constraint for y

% Combine all constraints
```

```
idx = idx1 & idx2 & idx3 & idx4;

% Extract coordinates of feasible region
Xf = X(idx); Yf = Y(idx);

% Plot feasible region
plot(Xf, Yf, 'b.')

% Plot boundary of feasible region
t = linspace(0, 5, 400);
lw = 'linewidth';
plot(t, 3/2-t/2, '--k', lw, 2)  % Constraint boundary 1
plot(t, -3*t+4, '--k', lw, 2)      % Constraint boundary 2
plot(zeros(size(t)), t, '--k', lw, 2)   % Non-negative constraint for x
plot(t, zeros(size(t)), '--k', lw, 2)   % Non-negative constraint for y

% Mark the optimal solution
opt_x = 1;
opt_y = 1;
plot(opt_x, opt_y, 'go', 'MarkerSize', 10, 'MarkerFaceColor', 'g');

% Plot normalized gradient
%grad_f = [35; 20];
%grad_f = 5*grad_f / norm(grad_f);
%quiver(opt_x, opt_y, grad_f(1), grad_f(2), 'linewidth', 2)

%%Problem 9
f = [1, 1];

A = [-2, -2; 12, 5];
b = [-5; 30];

lb = [0, 0];
ub = [inf, inf];

opts = optimoptions('linprog','Display','none');

intcon = [1, 2];
[x,fval]= linprog(f,A,b,[],[],lb, ub, opts);
fprintf("optimal soln is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% impose x(2) <= 2
[x,fval] = linprog(f,A,b,[],[],lb, [inf, 2], opts);
fprintf("optimal soln with y(2)<=2 is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% impose x(2) >= 3
[x,fval] = linprog(f,A,b,[],[],[0, 3],ub, opts);
fprintf("optimal soln with y(2)>=3 is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])

%% using intlinprog
opts = optimoptions('intlinprog', 'Display', 'none');
[x, fval] = intlinprog(f, intcon, A, b, [], [], lb, ub, opts);

fprintf("optimal soln is x = (%1.2f, %1.2f), p* = %1.2f\n", [x; fval])
```