Solutions for Tutorial Problem Sheet 3, October 13.

(Vector-Valued Functions and Motion in Space.)

Problem 1. Find the length of the curve $\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k}$ from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

Solution:

$$\mathbf{r} = \left(\sqrt{2}t\right)\mathbf{i} + \left(\sqrt{2}t\right)\mathbf{j} + \left(1-t^2\right)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + (-2t)^2} = \sqrt{4+4t^2}$$

$$= 2\sqrt{1+t^2} \Rightarrow \text{ Length } = \int_0^1 2\sqrt{1+t^2}\, dt = \left[2\left(\tfrac{t}{2}\sqrt{1+t^2} + \tfrac{1}{2}\ln\left(t + \sqrt{1+t^2}\right)\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$$

Problem 2. Find $\mathbf{T}$, $\mathbf{N}$, and κ for the space curves defined by position vector $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 < t < \pi/2$.

Solution:

$$\mathbf{r} = \left(\cos^3 t\right)\mathbf{i} + \left(\sin^3 t\right)\mathbf{j},\ 0 < t < \tfrac{\pi}{2} \Rightarrow \mathbf{v} = \left(-3\cos^2 t \sin t\right)\mathbf{i} + \left(3\sin^2 t \cos t\right)\mathbf{j}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{\left(-3\cos^2 t\sin t\right)^2 + \left(3\sin^2 t\cos t\right)^2} = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = 3\cos t\sin t,\ \text{ since } 0 < t < \tfrac{\pi}{2}$$

$$\Rightarrow \mathbf{T} = \tfrac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \tfrac{d\mathbf{T}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\tfrac{d\mathbf{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j};\ \ \kappa = \tfrac{1}{|\mathbf{v}|}\cdot\left|\tfrac{d\mathbf{T}}{dt}\right| = \tfrac{1}{3\cos t\sin t}\cdot 1 = \tfrac{1}{3\cos t\sin t}.$$

Problem 3. Show that a moving particle will move in a straight line if the normal component of its acceleration is zero. **Solution:**

$a_N = 0 \Rightarrow \kappa|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

Problem 4. Find parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k}$ at $t = 0$. **Solution:**

$\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \Rightarrow \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$; $\mathbf{r}(0) = \mathbf{i} \Rightarrow (1, 0, 0)$ is on the line $\Rightarrow x = 1 + t,\ y = t,$ and $z = -t$ are parametric equations of the line

Problem 5. Evaluate the integrals:

$$\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\, dt$$

$$\int_0^1 [te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}]\, dt$$

Solution:

$$\int_0^1\left[t^3\mathbf{i}+7\mathbf{j}+(t+1)\mathbf{k}\right]dt=\left[\tfrac{t^4}{4}\right]_0^1\mathbf{i}+[7t]_0^1\mathbf{j}+\left[\tfrac{t^2}{2}+t\right]_0^1\mathbf{k}=\tfrac{1}{4}\mathbf{i}+7\mathbf{j}+\tfrac{3}{2}\mathbf{k}$$

$$\int_0^1\left(te^{t^2}\mathbf{i}+e^{-t}\mathbf{j}+\mathbf{k}\right)dt=\left[\tfrac{1}{2}e^{t^2}\right]_0^1\mathbf{i}-\left[e^{-t}\right]_0^1\mathbf{j}+[t]_0^1\mathbf{k}=\tfrac{e-1}{2}\mathbf{i}+\tfrac{e-1}{e}\mathbf{i}+\mathbf{k}$$