UNIVERSITY OF LEICESTER

MA1014
CALCULUS AND ANALYSIS
TUTORIAL 17

Dr. Andrew Tonks: apt12@le.ac.uk

Ben Smith: bjs30@le.ac.uk

THEOREMS FOR SEQUENCES

- **Every Convergent sequence is bounded**

  e.g. $a_n = n$ is not bounded, so it diverges

- Monotone Convergence Theorem:
  - If a sequence is **bounded and monotonic** then it is convergent (to it's supremum or infimum)
  - E.g. $0 < a_n = \frac{1}{n} < 2$ and $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ so it converges to 0

LIMIT LAWS

If a_n and b_n are **convergent** sequences, then

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \pm \left(\lim\limits_{n\to\infty} b_n\right)$

- $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$

- $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided that $b_n \neq 0\ \forall n$ and $\lim\limits_{n\to\infty} b_n \neq 0$

PINCHING THEOREM (FOR SEQUENCES)

Let a_n, b_n, c_n be sequences such that $\exists N \in \mathbb{N}$ with

$$a_n \leq b_n \leq c_n \quad \forall n \geq N$$

If a_n and c_n **are convergent** with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ then b_n is convergent with

$$\lim_{n\to\infty} b_n = L$$

RATIO TEST (FOR SEQUENCES)

If $(a_n)_{n\geq 1}$ is a sequence such that $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $(a_n)_{n\geq 1}$ is convergent and $\lim\limits_{n\to\infty} a_n = 0$

E.g. $(a_n)_{n\geq 1} = \frac{2^n}{3^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2^{n+1}}{3^{n+1}}\right)}{\left(\frac{2^n}{3^n}\right)}\right| = \lim_{n\to\infty}\left|\frac{(2^{n+1})(3^n)}{(2^n)(3^{n+1})}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\right| = \frac{2}{3} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

EXERCISE: DETERMINE IF THE FOLLOWING LIMITS CONVERGE, CALCULATE THE LIMITS IF THEY DO

(Justify your answers, but , $\varepsilon - K$ arguments are not required)

a) $(a_n)_{n\geq 1} = \cos(n\pi)$

b) $(b_n)_{n\geq 1} = \sin(n\pi)$

c) $(c_n)_{n\geq 1} = \left(\frac{\cos(n)}{1+n}\right)$

d) $(d_n)_{n\geq 1} = \frac{1+\sin(n)}{n}$

e) $(e_n)_{n\geq 1} = \frac{n^2}{n!}$

f) $(f_n)_{n\geq 2} = \frac{n-1}{n} - \frac{n}{n-1}$

g) $(g_n)_{n\geq 1} = \left(1 + \frac{1}{n}\right)^n$

h) $(h_n)_{n\geq 1} = ne^{-\frac{n}{2}}$

EXERCISE: COMPOUND INTEREST

Consider the sequence,

$$A_n = P\left(1 + \frac{r}{12}\right)^n$$

where P is the principal sum, r is the annual interest rate and A_n is the account balance after n months.

a) Calculate the first ten terms of A_n if $P = £10{,}000$ and $r = 0.055$

b) Does A_n converge? Explain your answer

EXTRA TIME:

1) Prove the ratio test for sequences
(Hint: let $\varepsilon = 1 - r : r \in (L, 1)$)

2) Prove the following limits using the $\varepsilon - K$ definition

i. $$\lim_{n\to\infty} \frac{\cos(n)}{1+n} = 0$$

ii. $$\lim_{n\to\infty} \frac{1+\sin(n)}{n} = 0$$

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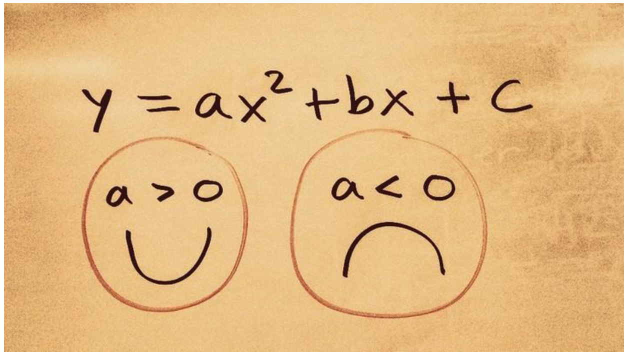

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

ANY QUESTIONS?

$$m\frac{d^2x}{dt^2} = -kx$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$