LINEAR ALGEBRA II

Ch. V SCALAR PRODUCTS AND ORTHOGONALITY

§1. Scalar Products

- Let V be a vector space over a field K.
- A scalar product on V is an association which to any pair of elements v, w of V associates a scalar, denoted by $\langle v, w\rangle$, or also $v \cdot w$, satisfying: $\forall u$, v, $w \in V$ and $x \in K$,

  SP 1. $\langle v, w\rangle = \langle w, v\rangle$.
  SP 2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
  SP 3. $\langle xu, v\rangle = x\langle u, v\rangle$ and $\langle u, xv\rangle = x\langle u, v\rangle$.
- The scalar product is said to be non-degenerate if in addition it also satisfies the condition: if $v \in V$, and $\langle v, w\rangle = 0$ $\forall w \in V$, then $v = O$.

§1. Scalar Products

- The dot product in $V = K^n$ is a non-degenerate scalar product.
- $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ is a non-degenerate scalar product in the space of continuous real-valued functions on the interval [0, 1].
- v, w are said to be orthogonal or perpendicular, and write $v \perp w$, if $\langle v, w\rangle = 0$.
- Let S be a subset of V, then $S^{\perp} = \{v \in V | \langle v, s\rangle = 0 \text{ for all } s \in S\}$ is a subspace of V, called the orthogonal space of S.
- $s \in S^{\perp} \Leftrightarrow s \perp S$.
- Let U be the subspace generated by S. Then $S^{\perp} = U^{\perp}$.

§1. Scalar Products

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- $AX = O$.
- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- $W = \operatorname{span}\{A_1, \ldots, A_m\}$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^\perp = W^\perp$.

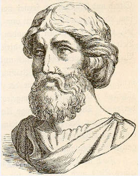
§1. Scalar Products

- Let V be a vector space over the field K, with a scalar product.
- Let $\{v_1, \dots, v_n\}$ be a basis of V. We shall say that it is an orthogonal basis, if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- We shall show later that if V is a finite dimensional vector space, with a scalar product, then there always exists an orthogonal basis.
- We shall first discuss important special cases over the real and complex numbers.

The Real Positive Definite Case

- Let V be a vector space over R, with a scalar product. We shall call this scalar product positive definite if $\langle v, v\rangle \geq 0$ for all $v \in V$ and $\langle v, v\rangle > 0$ for all $v \neq O$
- The dot product in $V = R^n$ is a positive definite scalar product.
- $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$ is a positive definite scalar product in the space of continuous real-valued functions on the interval [0, 1].
- Let V be a vector space over R, with a positive definite scalar product denoted by $\langle\ ,\ \rangle$. Let W be a subspace. Then W has a scalar product defined by the same rule defining the scalar product in V.

§1. Scalar Products



Pythagoras, 580 BC-500 BC.

- Norm. $\|v\| = \sqrt{\langle v, v\rangle}$
- $\|cv\| = |c|\|v\|$.
- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| > 0$ for all $v \neq O$
- $\mathrm{dist}(v, w) = \|v - w\|$.
- v is called a unit vector if $\|v\| = 1$. For any $0 \neq v \in V$, $v/\|v\|$ is a unit vector.
- **The Pythagoras theorem.** If $v \perp w$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.
- **The parallelogram law.** $\forall v, w$, $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.
- Let $w \in V$ and $\|w\| \neq 0$. For any v there exists a unique number c such that $v - cw \perp w$.
- $c = \frac{\langle v,w\rangle}{\langle w,w\rangle}$ $(= \langle v, w\rangle$ when $\|w\| = 1)$, the component of v along w.
- We call cw the projection of v along w.

§1. Scalar Products

- **Schwarz inequality.** $|\langle v, w\rangle| \leq \|v\| \|w\|$.
- **Triangle inequality.** $\|v + w\| \leq \|v\| + \|w\|$.

H. A. Schwarz, 1843-1921.

- Let $v_1, \ldots, v_n$ be non-zero elements of V which are mutually perpendicular. Let c_i be the component of v along v_i. Then

$$v - c_1 v_1 - \cdots - c_n v_n \perp v_i, \forall i = 1, \ldots, n.$$

§2. Orthogonal Bases, Positive Definite Case

- $\|v - c_1 v_1 - \cdots - c_n v_n\| \le \|v - a_1 v_1 - \cdots - a_n v_n\|$.

F. W. Bessel, 1784-1846.

- **Bessel inequality.** $\sum_{i=1}^{n} c_i^2 \le \|v\|^2$.
- Let V be a vector space with a positive definite scalar product throughout this section. A basis $\{v_1, \ldots, v_n\}$ of V is said to be orthogonal if its elements are mutually perpendicular.
- Orthonormal basis.
- Gram-Schmidt orthogonalization (orthonormalization) process.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.1.** Let V be a finite dimensional vector space, with a positive definite scalar product. Let W be a subspace of V, and let $\{w_1, \ldots, w_m\}$ be an orthogonal basis of W If $W \neq V$, then there exist elements $w_{m+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_n$ is an orthogonal basis of v.

- **Corollary 2.2.** Let V be a finite dimensional vector space with a positive definite scalar product. Assume that $V \neq \{O\}$. Then V has an orthogonal basis.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.3.** Let V be a vector space over R with a positive definite scalar product, of dimension n. Let W be a subspace of V of dimension r. Let $W^\perp$ be the subspace of V consisting of all elements which are perpendicular to W. Then V is the direct sum of W and $W^\perp$, and $W^\perp$ has dimension $n - r$. In other words,

$$\dim W + \dim W^\perp = \dim V.$$

§2. Orthogonal Bases, Positive Definite Case

§2. Orthogonal Bases, Positive Definite Case

- $W^\perp$ is called the orthogonal complement of W.
- Let V be a finite dimensional vector space over R, with a positive definite scalar product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

$$v = x_1e_1 + \cdots + x_ne_n \;\text{ and }\; w = y_1e_1 + \cdots + y_ne_n.$$

Then $\langle v, w \rangle = x_1y_1 + \cdots + x_ny_n$.

§2. Orthogonal Bases, Positive Definite Case

- Homework: Ch. V, §2, 0, 2, 3, 5.

Hermitian Products on VSs Over C

Charles Hermite, 1822-1901.

- The dot product of the nonzero vector $(i) \in C^1$ with itself is -1!
- The dot product of the nonzero vector ${}^t(1, i) \in C^2$ with itself is 0!
- Dot product is not a good scalar product.

§2. Orthogonal Bases, Positive Definite Case

- Let V be a vector space over the complex numbers. A hermitian product on V is a rule which to any pair of elements v, w of V associates a complex number, denoted again by $\langle v, w\rangle$, satisfying the following properties:

  HP 1. We have $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$.

  HP 2. If u, v, w are elements of V, then

  $$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle.$$

  HP 3. If $\alpha \in C$, $u, v \in V$, then

  $$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \text{ and } \langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle.$$

- The Hermitian product is said to be positive definite, if $\langle v, v\rangle \geq 0$ for all $v \in V$, and $\langle v, v\rangle > 0$ for all $0 \neq v \in V$.

§2. Orthogonal Bases, Positive Definite Case

- Orthogonal, perpendicular, orthogonal basis, orthogonal complement can be defined same as before.
- For $X = {}^{t}(x_1, \dots, x_n)$, $Y = {}^{t}(y_1, \dots, y_n) \in C^n$, define
$$\langle X, Y \rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$
It is a positive definite Hermitian product.
- The Hermitian product of the nonzero vector $(\mathrm{i}) \in C^1$ with itself is 1!
- The Hermitian product of the nonzero vector ${}^{t}(1, \mathrm{i}) \in C^2$ with itself is 2!

§2. Orthogonal Bases, Positive Definite Case

- Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$. If $f, g \in V$, we define
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\mathrm{d}t.$$
- It is a positive definite Hermitian product.
- Let $f_n(t) = e^{\mathrm{i}nt}$.
  - $\langle f_n, f_m\rangle = 0$ for $m \neq n$;
  - $\langle f_n, f_n\rangle = 2\pi$;
- The Fourier coefficients of f w.r.t. f_n is
$$\frac{\langle f, f_n\rangle}{\langle f_n, f_n\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\mathrm{i}nt}\mathrm{d}t.$$

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.4.** Let V be a finite dimensional vector space over C, with a positive definite Hermitian product. Let W be a subspace of V, and let $\{w_1, \ldots, w_m\}$ be an orthogonal basis of W If $W \neq V$, then there exist elements $w_{m+1}, \ldots, w_n$ of V such that $w_1, \ldots, w_n$ is an orthogonal basis of v.

- **Corollary 2.5.** Let V be a finite dimensional vector space over C with a positive definite scalar product. Assume that $V \neq \{O\}$. Then V has an orthogonal basis.

§2. Orthogonal Bases, Positive Definite Case

- Let V be a vector space over C, with a positive definite hermitian product.
- Norm. $\|v\| = \sqrt{\langle v, v\rangle}$
- **Schwarz inequality.** $|\langle v, w\rangle| \leq \|v\|\|w\|$.
- $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ iff $v = O$
- $\|cv\| = |c|\|v\|$ for all $c \in C$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v \in V$.

§2. Orthogonal Bases, Positive Definite Case

- The Pythagoras theorem.
- The parallelogram law.
- A unit vector, orthonormal, orthonormal basis.
- The component of v along w, the projection of v along w, the projection of v onto $\mathrm{span}\{v_1, \dots, v_n\}$.
- Bessel inequality.
- Let V be a finite dimensional vector space over C, with a positive definite Hermitian product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

  $$v = x_1 e_1 + \cdots + x_n e_n \ \text{ and } \ w = y_1 e_1 + \cdots + y_n e_n.$$

  Then $\langle v, w \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$.

§2. Orthogonal Bases, Positive Definite Case

- **Theorem 2.6. and 2.7.** Let V be a vector space over R with a positive definite scalar product, or a vector space over C with a positive definite scalar product. Assume that V has finite dimension n. Let W be a subspace of V of dimension r. Let $W^{\perp}$ be the subspace of V consisting of all elements which are perpendicular to W. Then V is the direct sum of W and $W^{\perp}$, and $W^{\perp}$ has dimension $n - r$. In other words,

$$\dim W + \dim W^{\perp} = \dim V.$$

§2. Orthogonal Bases, Positive Definite Case

- Homework: V, §2, 6, 8, 9

§3. Application to Linear Equations; The Rank

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

- $x_1 A^1 + \cdots + x_n A^n = O$.
- $AX = O$, $X \in \operatorname{Ker} A$.
- $A_1 \cdot X = 0, \ldots, A_m \cdot X = 0$.
- The solution set U of $AX = O$ is a subspace of K^n and $U = \{A_1, \ldots, A_m\}^\perp = W^\perp$. Where, $W = \operatorname{span}\{A_1, \ldots, A_m\}$.
- The row rank of A: the dimension of W.
- The column rank of A: $\dim \operatorname{span}\{A^1, \ldots, A^m\}$=$\dim \operatorname{Im} L_A$.

§3. Application to Linear Equations; The Rank

- Even if the scalar product is not positive definite, the following theorem is true (Th. 2.3, §6, Th. 6.4).
- **Theorem 3.1.** Let W be a subspace of K^n. Then

$$\dim W + \dim W^{\perp} = n.$$

- **Theorem 3.2.** Let $A = (a_{ij})$ be an $m \times n$ matrix. Then the row rank and the column rank of A are equal to the same number r . Furthermore, $n - r$ is the dimension of the space of solutions of the system of linear equations $AX = O$.
- The rank of $A \Rightarrow$ determinant
- Let S be the solution set of $AX = B$. If $S \neq \varnothing$, then, for any $X_0 \in S$, $S = X_0 + \operatorname{Ker} A$.
- $\dim\ S \triangleq \dim\ \operatorname{Ker} A$.

- **Proof of Theorem 3.2.**

§4. Bilinear Maps and Matrices

- Let U, V, W be vector spaces over K, and let

$$g : U \times V \to W$$

be a map. We say that g is bilinear if for each fixed $u \in U$ the map

$$v \mapsto g(u, v)$$

is linear, and for each fixed $v \in U$ the map

$$u \mapsto g(u, v)$$

is linear.

§4. Bilinear Maps and Matrices

- Let A be $m \times n$ matrix, $A = (a_{ij})$. We can define a map

$$g_A : K^m \times K^n \to K$$

by letting

$$g_A(X, Y) = {}^t XAY = (x_1, \ldots, x_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j.$$

It is a bilinear map.

- Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}.$$

Then

$$g_A(X, Y) = {}^t XAY = x_1 y_1 + 2x_1 y_2 + 3x_2 y_1 - x_2 y_2$$

§4. Bilinear Maps and Matrices

- **Theorem 4.1.** Given a bilinear map $g : K^m \times K^n \to K$, there exists a unique matrix A such that $g = g_A$, i.e. such that

  $$g(X, Y) = {}^t XAY.$$

  The set of bilinear maps of $K^m \times K^n$ into K is a vector space, denoted by $\mathrm{Bil}(K^m \times K^n, K)$, and the association

  $$A \mapsto g_A$$

  gives an isomorphism between $\mathrm{Mat}_{m\times n}(K)$ and $\mathrm{Bil}(K^m \times K^n, K)$.

§4. Bilinear Maps and Matrices

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- If A is an $n \times n$ symmetric matrix in K, then

$$g_A(X, Y) = g_A(Y, X), \forall X, Y \in K^n,$$

and g_A is a scalar product.

- If A is an $n \times n$ matrix in K such that

$$g_A(X, Y) = g_A(Y, X), \forall X, Y \in K^n,$$

then A is symmetric.

§4. Bilinear Maps and Matrices

- Homework: Ch. V, §4, 1, 3, 5(d), 6.

§5. General Orthogonal Bases

- Let V be a finite dimensional vector space over the field K, with a scalar product which need not be positive definite.
  - R^2, $\langle X, Y \rangle = x_1y_1 - x_2y_2$.
  - R^4, $\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$.
- The scalar product is then said to be null (a null scalar product) and V is called a null space, if $\langle u, u \rangle = 0$ for every $u \in V$.
  - $\langle v, w \rangle = 0$ for all $v,\ w \in V$.
  - Any basis of V is then an orthogonal basis by definition.

§5. General Orthogonal Bases

- **Theorem 5.1.** Let V be a finite dimensional vector space over the field K, and assume that V has a scalar product. If $V \neq \{O\}$, then V has an orthogonal basis.

§5. General Orthogonal Bases

§5. General Orthogonal Bases

- For R^2 with the scalar product $\langle X, Y\rangle = x_1y_1 - x_2y_2$,
  - $(1, 0)$ and $(0, 1)$ form an orthogonal basis;
  - $(1, 2)$ and $(2, 1)$ form an orthogonal basis.
- For X, $Y \in R^3$, define their scalar product to be

$$\langle X, Y\rangle = x_1y_1 - x_2y_2 - x_3y_3.$$

Let

- $U = \text{span}\{A, B\}$, where $A = (1, 2, 1)$, $B = (1, 1, 1)$.
- $V = \text{span}\{C, D\}$, where $C = (\sqrt{2}, 1, 1)$, $D = (1, 1, 1)$.
- $W = \text{span}\{G, H\}$, where $G = (1, 1, 0)$, $H = (1, 0, 1)$.

Find orthogonal bases of U, V and W with respect to this product.

§5. General Orthogonal Bases

§5. General Orthogonal Bases

- Homework: Ch. V, §5, 1(b), 2.

§6. The Dual Space and Scalar Products

- Let V be a vector space over the field K. The set of all linear maps of V into K (functionals) is called the dual space, and will be denoted by V^*.
- $V^* = \mathcal{L}(V, K)$.
- Suppose that V is of finite dimension n, then V is isomorphic to K^n.
- Let $V = K^n$. $\forall \varphi \in V^*$, there exists a unique element $A \in K^n$, such that

$$\varphi(X) = A \cdot X \quad \text{for all} \quad X \in K^n.$$

  Thus $\varphi = L_A$.
- The association

$$A \mapsto L_A$$

  is a linear map, and hence an isomorphism between K^n and V^*.

§6. The Dual Space and Scalar Products

- **Theorem 6.1.** Let V be a vector space of finite dimension. Then $\dim V^* = \dim V$.

- Let $V = K^n$. The coordinate functions:

$$\varphi_i(X) = x_i,\ X = (x_1, \ldots, x_n) = x_1 E^1 + \cdots + x_n E^n.$$

$\forall \varphi \in V^*$,

$$\varphi(X) = x_1\varphi(E^1) + \cdots + x_n\varphi(E^n) = \varphi(E^1)\varphi_1(X) + \cdots + \varphi(E^n)\varphi_n(X).$$

$\Rightarrow \{\varphi_1, \ldots, \varphi_n\}$ is a basis of V^*.

§6. The Dual Space and Scalar Products

- Let V be a VS of dimension n. Let $\{v_1, \dots, v_n\}$ be a basis of V. For each i, let

$$\varphi_i : V \to K$$

be the functional such that

$$\varphi_i(v_i) = 1 \text{ and } \varphi_i(v_j) = 0 \ (i \neq j).$$

Then $\forall v = x_1 v_1 + \cdots + x_n v_n \in V$,

$$\varphi_i(v) = x_i.$$

$\forall \varphi \in V^*$,

$$\varphi(v) = x_1\varphi(v_1) + \cdots + x_n\varphi(v_n) = \varphi(v_1)\varphi_1(v) + \cdots + \varphi(v_n)\varphi_n(v).$$

$\Rightarrow \{\varphi_1, \dots, \varphi_n\}$ is a basis of V^*, called the dual basis of $\{v_1, \dots, v_n\}$.

§6. The Dual Space and Scalar Products

- Let V be a VS over K, with a scalar product. Let v_0 be an element of V. The map

$$v \mapsto \langle v, v_0 \rangle = \langle v_0, v \rangle \ (K = C, \text{ Hermitian product?}), \ v \in V,$$

is a functional. ($\langle v_0, v \rangle$ is not linear, and is anti-linear!)

- Let V be the vector space of continuous real-valued functions on the interval $[0, 1]$. If f_0 is a fixed element of V, we can define a functional $L : V \to R$ by the formula

$$f \mapsto \int_0^1 f_0(t) f(t) \mathrm{d}t.$$

  - $f_0(t) = 1$, $L(f) = \int_0^1 f(t) \mathrm{d}t$.
  - $f_0(t) = \delta(t)$, $\delta(f) = f(0)$, called the Dirac functional.

§6. The Dual Space and Scalar Products

- Let V be a VS over K, with a scalar product. To each element $v \in V$ we can associate a functional L_v in the dual space, namely the map such that
$$L_v(w) = \langle v, w \rangle$$
for all $w \in V$.
- The map
$$v \mapsto L_v$$
is a linear map of V into the dual space V^*.

- **Theorem 6.2.** Let V be a finite dimensional vector space over K with a non-degenerate scalar product. Then the map

$$v \mapsto L_v$$

is an isomorphism of V with the dual space V^*.

- We say that the vector v represents the functional L with respect to the non-degenerate scalar product.

§6. The Dual Space and Scalar Products

- Let $V = K^n$ with the dot product. $\forall \varphi \in V^*$, there exists a unique element $A \in K^n$, such that
$$\varphi(X) = A \cdot X \quad \text{for all} \quad X \in K^n.$$
This allows us to represent the functional φ by the vector A.

§6. The Dual Space and Scalar Products

- **Theorem 6.3.** Let V be a vector space of dimension n. Let W be a subspace of V and let

$$W^{\perp} = \{\varphi \in V^{*} \text{ such that } \varphi(W) = 0\}.$$

Then

$$\dim W + \dim W^{\perp} = n.$$

§6. The Dual Space and Scalar Products

§6. The Dual Space and Scalar Products

- Two possible orthogonal complements of W:
  - $\operatorname{perp}_V(W) = \{v \in V \text{ such that } \langle v, w\rangle = 0 \text{ for all } w \in W\}.$
  - $\operatorname{perp}_{V^*}(W) = \{\varphi \in V^* \text{ such that } \varphi(W) = 0\}.$
- The map: $v \mapsto L_v$ of Th. 6.2 gives an isomorphism
$$\operatorname{perp}_V(W) \to \operatorname{perp}_{V^*}(W).$$
- The following theorem is a corollary of Th. 6.3.
- **Theorem 6.4.** Let V be a finite dimensional vector space with a non-degenerate scalar product. Let W be a subspace. Let $W^\perp$ be the subspace of V consisting of all elements $v \in V$ such that $\langle v, w\rangle = 0$ for all $w \in W$. Then
$$\dim W + \dim W^\perp = n.$$
- Th. 3.1 is a corollary of Th. 6.4: $V = K^n$ with the dot product.

§6. The Dual Space and Scalar Products

- Homework: Ch. V, §6, 3, 4, 7.

§7. Quadratic Forms

- A scalar product on a vector space V is also called a symmetric bilinear form.

$$g : V \times V \to K,$$
$$g(v, w) = \langle v, w\rangle.$$

- By the quadratic form determined by g, we shall mean the function

$$f : V \to K$$

such that

$$f(v) = g(v, v) = \langle v, v\rangle.$$

§7. Quadratic Forms

- For $V = K^n, f(X) = X \cdot X = x_1^2 + \cdots + x_n^2$ is the quadratic form determined by the ordinary dot product.
- If $A \in \mathrm{Mat}_{n\times n}(K)$ is symmetric, then

$$g_A(X,Y) = {}^{\mathrm{t}} XAY = \sum_{i,j=1}^{n} a_{ij}x_i y_j.$$

is a symmetric bilinear form (scalar product) and

$$f_A(X) = {}^{\mathrm{t}} XAX = \sum_{i,j=1}^{n} a_{ij}x_i x_j.$$

is the quadratic form determined by g_A.

- If $A = \mathrm{diag}(a_1, \ldots, a_n)$, then

$$f_A(X) = a_1 x_1^2 + \cdots + a_n x_n^2.$$

§7. Quadratic Forms

- The scalar product g can be uniquely determined by the quadratic form f.

$$g(v,w) = \frac{1}{4}[f(v+w,v+w) - f(v-w,v-w)],$$

i.e.

$$\langle v,w\rangle = \frac{1}{4}[\langle v+w,v+w\rangle - \langle v-w,v-w\rangle],$$

or

$$g(v,w) = \frac{1}{2}[f(v+w,v+w) - f(v,v) - f(w,w)],$$

i.e.

$$\langle v,w\rangle = \frac{1}{2}[\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle].$$

§7. Quadratic Forms

- Given the quadratic form

$$f(x,y) = 2x^2 + 3xy + y^2,$$

find the matrix A of its symmetric bilinear form g.

§7. Quadratic Forms

- Let $f : R^n \to R$ be a function which has partial derivatives of order 1 and 2, and such that the partial derivatives are continuous functions. Assume that
$$f(tX) = t^2 f(X), \forall X \in R^n.$$
Then f is a quadratic form, that is there exists a symmetric matrix $A = (a_{ij})$ such that
$$f(X) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

§7. Quadratic Forms

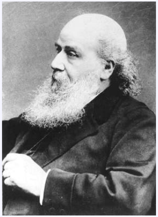
- Homework: Ch. V, §7, 1, 2, 3(c).

§8. Sylvester's Theorem

- Let V be a finite dimensional vector space over R, with a positive definite scalar product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

$$v = x_1 e_1 + \cdots + x_n e_n \text{ and } w = y_1 e_1 + \cdots + y_n e_n.$$

Then $\langle v, w \rangle = x_1 y_1 + \cdots + x_n y_n$.



James Sylvester, 1814-1897.

- $\langle v, v \rangle = x_1^2 + \cdots + x_n^2$.
- Let V be a finite dimensional vector space over C, with a positive definite Hermitian product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

$$v = x_1 e_1 + \cdots + x_n e_n \text{ and } w = y_1 e_1 + \cdots + y_n e_n.$$

Then $\langle v, w \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$.

- $\langle v, v \rangle = |x_1|^2 + \cdots + |x_n|^2$.

§8. Sylvester's Theorem

- Let $V = R^2$, and let the symmetric bilinear form be represented by the matrix

$$A = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}.$$

Then the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

forms an orthogonal basis for the form, and

$$\langle v_1, v_1 \rangle = -1, \quad \langle v_2, v_2 \rangle = 0.$$

- $\forall v = x_1 v_1 + x_2 v_2$, $\langle v, v \rangle = -x_1^2 + 0 \cdot x_2^2$.
- General situation?

§8. Sylvester's Theorem

- Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis of V and

$$c_i = \langle v_i, v_i \rangle.$$

After renumbering the elements of our basis if necessary, we may assume that $\{v_1, \ldots, v_n\}$ are so ordered that:

$$\begin{aligned} c_1, \ldots, c_r &> 0, \\ c_{r+1}, \ldots, c_s &< 0, \\ c_{s+1}, \ldots, c_n &= 0. \end{aligned}$$

- $\forall v = x_1 v_1 + \cdots + x_n v_n$,

$$f(X) = \langle v, v \rangle = c_1 x_1^2 + \cdots + c_r x_r^2 + c_{r+1} x_{r+1}^2 + \cdots + c_s x_s^2.$$

- r and s do not depend on the orthogonal basis.

§8. Sylvester's Theorem

- If $\{v_1, \ldots, v_n\}$ is orthonormal,

$$\langle v_i, v_i \rangle = \begin{cases} 1, & i = 1, \ldots, r, \\ -1, & i = r+1, \ldots, s, \\ 0, & i = s+1, \ldots, n. \end{cases}$$

  then

$$f(X) = \langle v, v \rangle = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2.$$

- r and s do not depend on the orthonormal basis.
- Normalization:

$$v_i' = \begin{cases} v_i/\sqrt{c_i}, & i = 1, \ldots, r, \\ v_i/\sqrt{-c_i}, & i = r+1, \ldots, s, \\ v_i, & i = s+1, \ldots, n. \end{cases}$$

§8. Sylvester's Theorem

- **Theorem 8.1.** Let V be a finite dimensional vector space over R, with a scalar product. Assume $\dim V > 0$. Let V_0 be the subspace of V consisting of all vectors $v \in V$ such that $\langle v, w \rangle = 0$ for all $w \in V$. Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis for V. Then the number of integers i such that $\langle v_i, v_i \rangle = 0$ is equal to the dimension of V_0.

§8. Sylvester's Theorem

- The dimension $n - s$ of V_0 in Th. 8.1 is called the index of nullity of the form.
- The form is non-degenerate if and only if its index of nullity is 0.

Theorem 8.2 (Sylvester's theorem). Let V be a finite dimensional vector space over R, with a scalar product. There exists an integer $r > 0$ having the following property. If $\{v_1, \ldots, v_n\}$ is an orthogonal basis of V, then there are precisely r integers i such that $\langle v_i, v_i \rangle > 0$.

- The integer r of Sylvester's theorem is called the index of positivity of the scalar product.
- The integer $s - r$ is the number of integers i such that $\langle v_i, v_i \rangle < 0$. It does not depend on the orthonormal basis and is called the index of negativity of the scalar product.

- **Proof of Theorem 8.2.**

§8. Sylvester's Theorem

§8. Sylvester's Theorem

- Index of positivity of the form represented by A=number of positive eigenvalues of A.
  Index of negativity of the form represented by A=number of negative eigenvalues of A. Index of nullity of the form represented by A=number of zero eigenvalues of A.
- Homework: Ch. V, §8, 1, 2.