Solutions for Tutorial Problem Sheet 10, December 1.

(Integrals and Vector Fields)

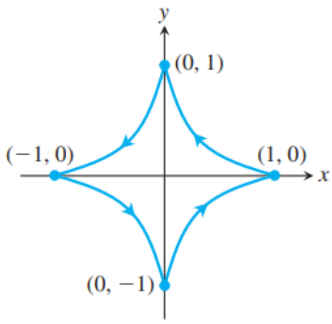
Problem 1. Evaluate the line integral $\int_C 2x\cos y\,dx - x^2\sin y\,dy$ along the following paths C in the xy-plane.

a) The parabola $y = (x-1)^2$ from $(1,0)$ to $(0,1)$.

b) The line segment from $(-1,\pi)$ to $(1,0)$.

c) The x-axis from $(-1,0)$ to $(1,0)$.

d) The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + \sin^3 t)\mathbf{j},\ 0 \le t \le 2\pi$, counterclockwise from $(1,0)$ back to $(1,0)$.



Solution:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2x\sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative } \Rightarrow \text{ there exists an } f \text{ so that } \mathbf{F} = \nabla f;$$

$$\frac{\partial f}{\partial x} = 2x\cos y \Rightarrow f(x,y,z) = x^2\cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -x^2\sin y + \frac{\partial g}{\partial y} = -x^2\sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$$

$$\Rightarrow f(x,y,z) = x^2\cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2\cos y + C \Rightarrow \mathbf{F} = \nabla\left(x^2\cos y\right)$$

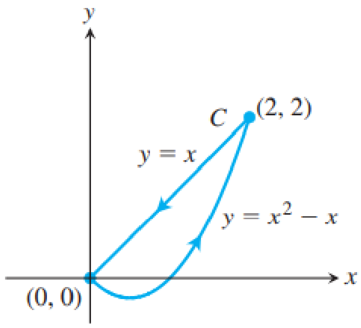
(a) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

Problem 2. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F} = x^3y^2\ \mathbf{i} + \frac{1}{2}x^4y\ \mathbf{j}$ and curve C shown on the following Figure



Solution:

$$M = x^3y^2,\ N = \tfrac{1}{2}x^4y \Rightarrow \tfrac{\partial M}{\partial x} = 3x^2y^2,\ \tfrac{\partial M}{\partial y} = 2x^3y,\ \tfrac{\partial N}{\partial x} = 2x^3y,\ \tfrac{\partial N}{\partial y} = \tfrac{1}{2}x^4 \Rightarrow \text{Flux} = \iint\limits_R \left(3x^2y^2 + \tfrac{1}{2}x^4\right) dy\, dx$$

$$= \int_0^2 \int_{x^2-x}^{x} \left(3x^2y^2 + \tfrac{1}{2}x^4\right) dy\, dx = \int_0^2 \left(3x^5 - \tfrac{7}{2}x^6 + 3x^7 - x^8\right) dx = \tfrac{64}{9};\ \text{Circ} = \iint\limits_R \left(2x^3y - 2x^3y\right) dy\, dx = 0$$

Problem 3. Evaluate the integral

$$\oint_C 4x^3y\,dx + x^4\,dy$$

for any closed path C.

Solution:

The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_C 4x^3y\,dx + x^4 dy = \iint\limits_R \left[\frac{\partial}{\partial x}\left(x^4\right) - \frac{\partial}{\partial y}\left(4x^3y\right)\right] dx\,dy$

$= \iint\limits_R \underbrace{\left(4x^3 - 4x^3\right)}_{0} dx\,dy = 0.$

Problem 4. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same. They should have the same values, however.)

Plane inside cylinder: The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$.

Solution:

Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= \left(r\sin^2\theta + r\cos^2\theta\right)\mathbf{i} + \left(r\sin\theta\cos\theta - r\sin\theta\cos\theta\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} = r\mathbf{i} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\pi\sqrt{2}$$