UNIVERSITY OF LEICESTER

MA1013
All candidates

May Examinations 2018

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
Department	Mathematics
Module Code	MA1013
Module Title	Calculus and Analysis II
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	Answer all questions. All marks gained will be counted. All questions carry equal weight.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

1. (a) Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_n = \sqrt{n^2+1} - n$.
   i. Prove that $0 < a_n < \frac{1}{2n}$ for all $n \geq 1$.
   ii. Find the limit of the sequence $(a_n)_{n\geq 1}$, justifying your answer.

[6 marks]

(b) i. Give the definition of a *Cauchy sequence*.
   ii. Prove that every Cauchy sequence is bounded.
   iii. Prove that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences then so is $(a_n b_n)_{n\geq 1}$.

[12 marks]

(c) Use an integrating factor to solve the Initial Value Problem

$$y' - y/x = x, \qquad y(1) = 2 \qquad\qquad (x > 0).$$

[7 marks]

2. (a) Consider the power series $S(x) = \sum_{n=0}^{\infty} (\sin n)(x/2)^n$.
   i. State the vanishing test for convergence of series.
   ii. Prove that $S(x)$ diverges if $x = \pm 2$.
   iii. State the comparison test for convergence of series.
   iv. Prove that $S(x)$ converges absolutely if $|x| < 2$, justifying carefully your answer.
   v. Write down the radius of convergence of $S(x)$.

[12 marks]

(b) Consider the infinite series $\sum_{n=0}^{\infty} a_n$ with $a_n = \ln \dfrac{n+2}{n+1}$.
   i. Calculate the partial sum s_k.
   ii. Determine if the infinite series is convergent or divergent.
   iii. State the integral test for convergence of series.
   iv. Prove that $f(x) = \ln \dfrac{x+2}{x+1}$ is a positive, monotonic decreasing function for $x > 0$.
   v. Determine if the improper integral $\displaystyle\int_1^{\infty} \ln \frac{x+2}{x+1}\, dx$ converges or diverges.

[13 marks]

3. Consider the function of two variables $f(x,y) = \begin{cases} \dfrac{x^3+y^5}{x^2+y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

(a) Prove f is continuous at the origin. [4 marks]

(b) Using only the definition of partial derivatives, prove $f_x(0,0) = f_y(0,0) = 1$. [4 marks]

(c) Calculate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$, for $(x,y) \neq (0,0)$. [4 marks]

(d) By considering the limits $\lim\limits_{y\to 0} f_x(0,y)$ and $\lim\limits_{x\to 0} f_y(x,0)$, show that the functions f_x and f_y are both discontinuous at $(0,0)$. [4 marks]

(e) Find the equation of the tangent plane at the point $(x,y) = (-1,-1)$ to the surface $z = f(x,y)$. [4 marks]

(f) i. Define the *directional derivative* of a function of two variables f at a point $\underline{x} = (x,y)$ in the direction of a unit vector $\underline{\hat{u}} = (p,q)$.

ii. For the function f above, show the directional derivative $f_{\underline{\hat{u}}}(0,0)$ at the origin equals p if $p \neq 0$, or q if $p = 0$.

[5 marks]

4. (a) Define the notion of a *critical point* of a function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$. [3 marks]

(b) State the Extreme Value Theorem for functions $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$. [4 marks]

(c) Find the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = \frac{1}{6}x^2 + xy + y^3$. [5 marks]

(d) Use the Hessian to classify these critical points. [5 marks]

(e) Find the global maximum and the global minimum values attained by the function f when restricted to the elliptical region

$$D = \left\{ (x,y) : \frac{1}{6}x^2 + xy + 3y^2 \leq 6 \right\}.$$

[8 marks]