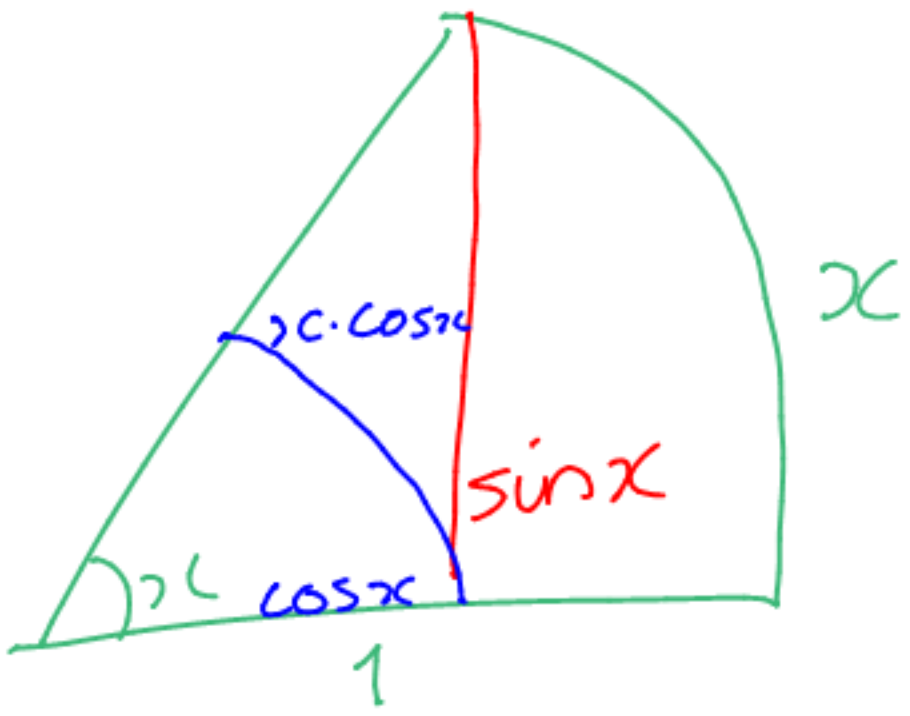


As $x \to 0$

$\cancel{-x \le}\ x\cos x \le \sin x \le x$

$\downarrow 0 \qquad \therefore \qquad \downarrow 0$

by Pinching Theorem

Similarly $\cos^2 x = 1 - \sin^2 x \to 1$

$\lim_{x \to 0} \sin(x) = 0 = \sin(0)$ so $\sin(x)$ continuous $x=0$

$\lim_{x \to 0} \cos(x) = 1 = \cos(0)$ so $\cos(x)$ cts when $x=0$

$\lim_{x \to c} \sin(x) = \sin(c)$?

$\lim_{h \to 0} \sin(x+h) = \sin(x)$?

Yes $\sin(x+h) = \sin x \cos h + \cos x \sin h$

as $h \to 0$: $\quad \downarrow \sin x \qquad \downarrow 0$

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$$\cos(x) \le \frac{\sin(x)}{x} \le 1$$

$\downarrow 1$ so $\downarrow 1$ by P.T. $\downarrow 1$

$$\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$$

$x = 2A$

$1-\cos 2A = 2\sin^2 A$

$$\frac{1-\cos(x)}{x^2} = \frac{2\sin^2 A}{4A^2}$$

$$= \frac{1}{2}\left(\frac{\sin A}{A}\right)^2 \to \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{\sin(7x)}{21x} = \frac{1}{3}$$

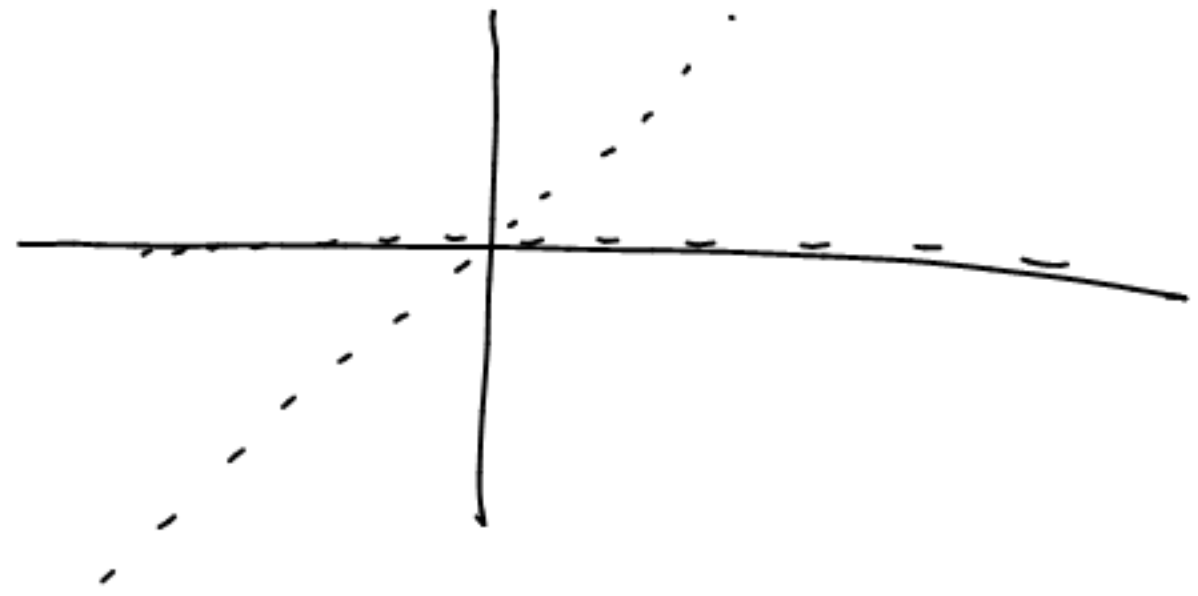
$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} x \cdot \frac{1-\cos(x)}{x^2}$$

$$= 0 \cdot \frac{1}{2}$$

$$= 0$$

$$=$$

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \text{ irrational} \end{cases}$$



$f(x)$ is continuous at $x=0$

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

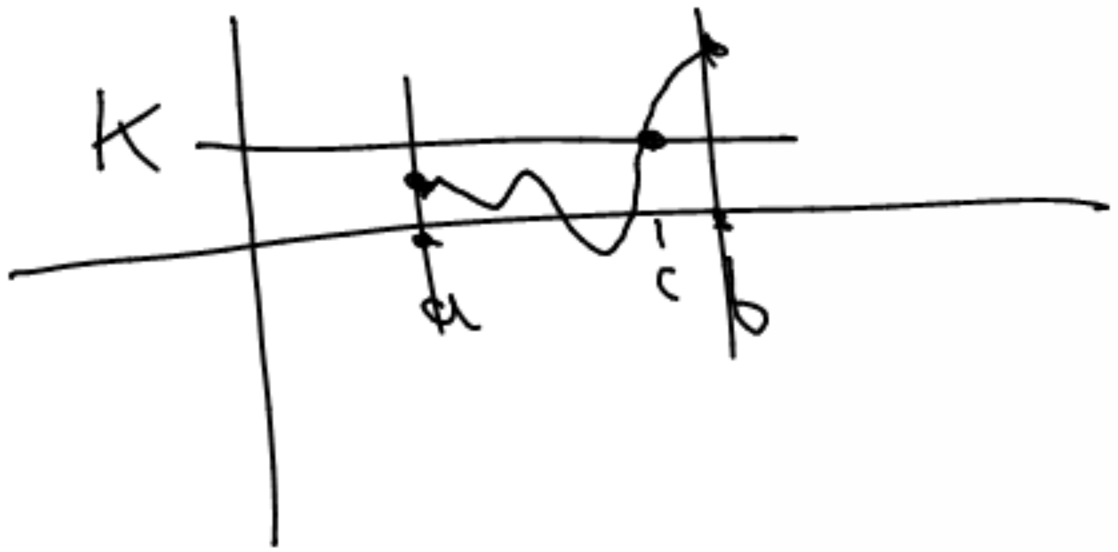
$$-x \le f(x) \le x \quad \forall x$$

$$\downarrow \qquad \therefore \overset{\downarrow}{0} \qquad \downarrow$$

Bolzano's Theorem & Intermediate Value Theorem

If $K \in \mathbb{R}$, $f(a) < K < f(b)$ or $f(b) < K < f(a)$

$\exists c \in (a,b) : f(c) = K$



for continuous function $f : [a,b] \to \mathbb{R}$

Bolzano: If $f(a) \cdot f(b) < 0$
<u>then</u> $f(c) = 0$ for some $c \in (a,b)$

$f(a), f(b)$ different signs, f continuous
$\Rightarrow \exists c, \quad a < c < b, \quad f(c) = 0$

Proofs Next Tuesday!

Applications:

Bisection Method

$f(x)$ real, continuous

Solve $f(x) = 0$

Guess a_1, b_1

with $f(a_1) < 0$ $f(b_1) > 0$

Bolzano: $\exists$ solution between a_1 & b_1

n Look at $x = \frac{1}{2}(a_n + b_n)$

$\rightarrow f(x) > 0$ $\begin{cases} a_{n+1} = a_n \\ b_{n+1} = x \end{cases}$

$\rightarrow f(x) < 0$ $\begin{cases} a_{n+1} = x \\ b_{n+1} = b_n \end{cases}$

(Note true for $\mathbb{R}$ not for $\mathbb{Q}$)

$f(x) = x^2 - 2$

$f(0) = -2$ $f(2) = 2$

$\exists c : c^2 - 2 = 0$

$(c = \sqrt{2})$