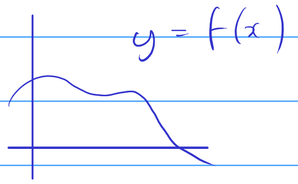
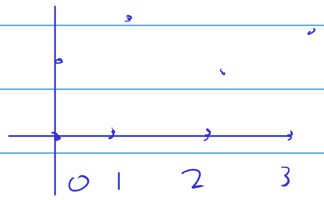
Topic 4 Sequences 16/11/2021

Functions ✓ Sequences $(a_n)_{n \in \mathbb{N}}$





$f : \mathbb{R} \to \mathbb{R}$ $a : \mathbb{N} \longrightarrow \mathbb{R}$

$f : D \to \mathbb{R}$ $a_0, a_1, a_2, a_3, \ldots \in \mathbb{R}$

domain

examples $f_0 = 0, \; f_1 = 1,$

inductively $f_{n+1} = f_n + f_{n-1}$

$0, 1, 1, 2, 3, 5, 8, \ldots$

Behaviour : 1) Unbounded

2) Bounded

$a_n = (-1)^n$ $\underline{1, -1, 1, -1, 1, -1, \ldots}$

$-1 \leq a_n \leq 1 \quad \forall n$

3) Monotonic increasing

$n < m \Rightarrow a_n < a_m$

$$\left[\; x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \; \right]$$

4) or monotonic decreasing

Limits

For $f : \mathbb{R} \to \mathbb{R}$ we have defined $\lim_{x \to c} f(x)$

We can define

$$\lim_{x \to \infty} f(x) = L \text{ if and only if}$$

$\forall \varepsilon > 0 \ \exists N$ such that if $x > N$ then $|f(x) - L| < \varepsilon$

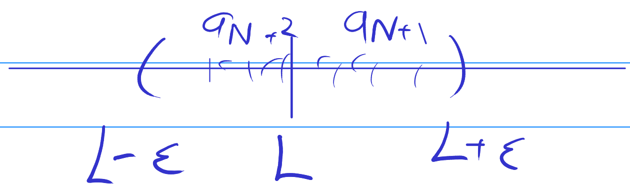
For sequences, we say

$(a_n)_{n \in \mathbb{N}}$ converges to L

$a_n \to L$ as $n \to \infty$, or

$\lim_{n \to \infty} a_n = L$ means:

$$\forall \varepsilon > 0 \ \exists N : n > N \Rightarrow |a_n - L| < \varepsilon$$



Sequences can be $\begin{cases} \text{bounded} \\ \text{unbounded} \end{cases}$ $\begin{cases} \text{monotonic} \\ \text{not} \end{cases}$ $\begin{cases} \text{convergent} \\ \text{divergent} \end{cases}$

Theorem If a limit of a sequence exists then it is unique

$a_n \to L$ & $a_n \to M \Rightarrow L = M$

Proof By contradiction, if $L \neq M$

let $\varepsilon = |L - M|/2 > 0$

$\exists N \quad |a_n - L| < \varepsilon \quad$ if $n > N$

$\exists N' \quad |a_n - M| < \varepsilon \quad$ if $n > N'$

Triangle inequality

if $n > N, N' \quad |L - M| \leq |a_n - L| + |a_n - M|$

$$< 2\varepsilon$$

contradiction $\#$ $\qquad = |L - M|/2$

Examples $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \to 0$

$$a_n = \frac{1}{n+1}, \quad n \in \mathbb{N}$$

Does (a_n) converge?

Yes $a_n \to 0$ ✓

Proof Given any $\varepsilon > 0$, choose $N \in \mathbb{N}$ larger than $1/\varepsilon$. Then $\frac{1}{N} < \varepsilon$ & $n > N \Rightarrow$
$|a_n - 0| < 1/N < \varepsilon$

Example: $\frac{3}{2}, 1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \frac{8}{12}, \frac{9}{14}$

$$a_n = \frac{n+3}{2n+2}$$

Monotonic decreasing: $\forall n \in \mathbb{N}$

$$a_{n+1} < a_n \iff \frac{n+4}{2n+4} < \frac{n+3}{2n+2}$$

$$\iff (n+4)(2n+2) < (n+3)(2n+4)$$

$$\iff \cancel{2n^2 + 10n} + 8 < \cancel{2n^2 + 10n} + 12$$

$$\iff 8 < 12$$

Convergent?

$$a_{1000000} = \frac{1000003}{2000002} \sim \frac{1}{2}$$

Proof $\lim_{n \to \infty} \frac{n+3}{2n+2} = \frac{1}{2}$ $\quad \frac{1}{2} = \frac{n+1}{2n+2}$

Control $\left| a_n - \frac{1}{2} \right| = \left| \frac{n+3}{2n+2} - \frac{1}{2} \right|$

$$= \left| \frac{(n+3)-(n+1)}{2n+2} \right| = \frac{2}{2n+2} = \frac{1}{n+1}$$

Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$

$n > N \Rightarrow \left| a_n - \frac{1}{2} \right| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon$

so a_n converges to $1/2$

We just saw two examples of monotonic bounded sequences

Theorem If a sequence $(a_n)_{n \in \mathbb{N}}$ is

monotonic increasing and bounded above

or monotonic decreasing and bounded below

Then a_n converges to

$LUB\{a_n : n \in \mathbb{N}\}$ or $GLB\{a_n : n \in \mathbb{N}\}$

Theorem If a sequence $(a_n)_{n \in \mathbb{N}}$ is convergent then it is bounded

Proof Given $\varepsilon = 1$ in $\lim_{n\to\infty} a_n = L$

$\exists N$: if $n > N$ then $|a_n - L| < 1$

$| \; a_{N+1}, a_{N+2}, a_{N+3}, \ldots \in (L-1, L+1)$

So $|a_n| \leq \max(|L+1|, |L-1|)$
if $n > N$

& $|a_n| \leq \max\{|a_0|, |a_1|, \ldots, |a_N|, |L+1|, |L-1|\}$

$\forall n \in \mathbb{N}$

& a_n is bounded.

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bounded monotonic $\Rightarrow$ convergent (Pf)
convergent $\Rightarrow$ bounded