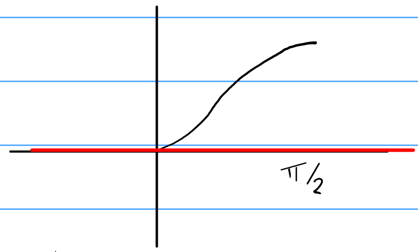
1. Tangent and normal lines ✓
2. Implicit differentiation
3. Higher Derivatives

Tangent has gradient = derivative

$f(x) = x \sin x$

$f'(x) = 1 \sin x + x \cos x$

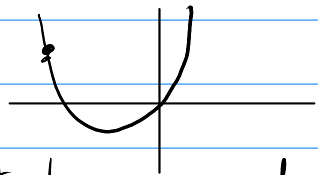


What is the tangent line at the origin

$f'(0) = 1 . \sin 0 + 0 \cos 0 = 0$

$y = 0$

$f(x) = x^2 + 2x$



What is the tangent line at $x = -3$

$f'(x) = 2x + 2 \qquad f'(-3) = -6 + 2 = -4$

$y = -4x + c$

$x = -3, \ y = f(-3) = 12 + c$

$9 - 6 = 12 + c \Rightarrow c = -9$

$y = -4x - 9$

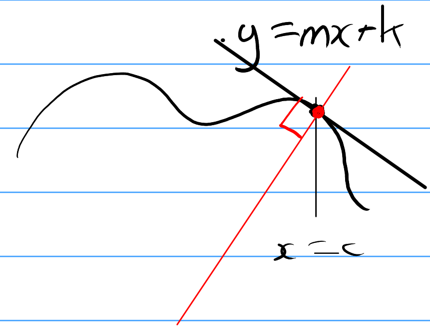
General formula

Tangent line to the curve $y = f(x)$

at some point $x = c$

is $y = \underbrace{f'(c)}_{\text{gradient } m}\, x + \left(f(c) - f'(c)\,c\right)$ ✓

Goes through $\underline{(c, f(c))}$ ✓

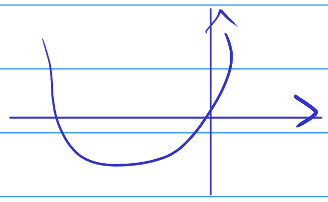


Normal line

perpendicular

gradient $-\frac{1}{m}$

$f(x) = x^2 + 2x$



at $x = -3$

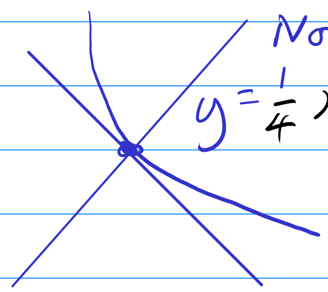
Normal line has gradient $-\frac{1}{-4} = \frac{1}{4}$

$y = \frac{1}{4}x + \underbrace{3 + \frac{3}{4}}_{15/4}$

$x = -3,\ y = 3$

Normal

$y = \frac{1}{4}x + \frac{15}{4}$



$y = -4x - 9$

tangent

Chain Rule

Implicit definition of y

$y = f(x)$ with $y^5 x + x^3 + \sin(x) = 0$

$y' = f'(x)$

dy/dx ?

$$\underbrace{\frac{dy^5}{dx} \cdot x + y^5 \frac{dx}{dx}}_{\text{product rule}} + 3x^2 + \cos(x) = 0$$

$$5y^4 \frac{dy}{dx} x + y^5 \boxed{\cancel{\frac{dy}{dx}}}^{\,1} + 3x^2 + \cos(x) = 0$$

$$5y^4 x \frac{dy}{dx} + y^5 + 3x^2 + \cos(x) = 0$$

Corrected 5 Nov

$$\frac{dy}{dx} = -\frac{y^5 + 3x^2 + \cos x}{5y^4 x}$$

Higher Derivatives

$f'(x)$

$f(x) = y = x^3 + 7x^2$ $\qquad \frac{dy}{dx} = 3x^2 + 14x$

$$\frac{d}{dx}\left(\frac{d}{dx} y\right) = \frac{d^2 y}{dx^2} = f''(x) = 6x + 14$$

$$\frac{d^3 y}{dx^3} = f'''(x) = 6$$

$$\frac{d^4 y}{dx^4} = f''''(x) = f^{(4)}(x) = 0$$

$f(x) = \sin x \quad f'(x) = \cos x$

$f''(x) = -\sin x \quad f'''(x) = -\cos x$

$f''''(x) = \sin x = f(x) \quad = f^{(100)}(x)$

n larger than degree of polynomial $f(x)$

$$f^{(n)}(x) = 0$$

$f(x) = x^5 \quad 5x^4, \ 20x^3, \ 60x^2, \ 120x,$

$120, \ \underline{\underline{0}}$

$u(x)\, v(x) \qquad (u \cdot v)' = uv' + u'v$

$$(u \cdot v)'' = u'v' + u''v + uv'' + u'v'$$

$$= u''v + 2u'v' + uv''$$

$$(u \cdot v)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$u = x^5 \quad v = \cos x \quad \dfrac{d^n}{dx^n}(u \cdot v)$

$n = 1, 2, 3, 4, 5$

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ 1 & 4 & & 6 & & 4 & 1 \end{array}$$

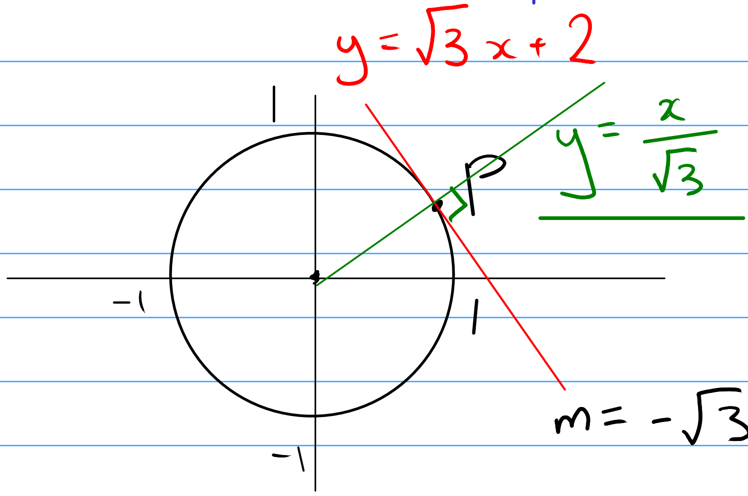
$\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Binomial coeffs in Pascal's triangle

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

Example

What are the equations of the tangent and normal lines to the circle of radius 1 at the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



Solution

$$x^2 + y^2 = 1$$

$$\rightarrow 2x + 2y\frac{dy}{dx} = 0$$

so $\frac{dy}{dx} = -\frac{x}{y} = -\sqrt{3}$

Tangent $y = -\sqrt{3}x + c_1$

Normal $y = \frac{1}{\sqrt{3}}x + c_2$

$$\frac{1}{2} = -\sqrt{3}\,\frac{\sqrt{3}}{2} + c_1 \Leftrightarrow \frac{1}{2} = -\frac{3}{2} + c_1 \Leftrightarrow c_1 = 2$$

$$\frac{1}{2} = \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{2} + c_2 \Leftrightarrow c_2 = 0$$