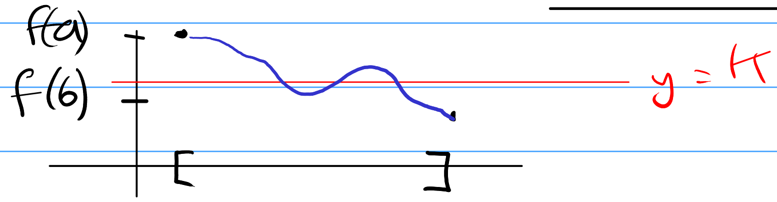
Lecture 11

26 Oct 2021

Applications of Continuity

① Bolzano's Theorem & the Intermediate Value Theorem (IVT)

Thm (IVT) If $f:[a,b] \longrightarrow \mathbb{R}$ is continuous & $K \in \mathbb{R}$ strictly between $f(a)$ and $f(b)$. Then $\exists\, c \in (a,b)$ such that $f(c) = K$



Thm (Bolzano) Special case for $k = 0$

If $f:[a,b] \to \mathbb{R}$ continuous
& $f(a)$ & $f(b)$ have different signs
(i.e. $f(a)f(b) < 0$)
Then $\exists\, c \in (a,b)$ such that $f(c) = 0$

IVT $\Longleftrightarrow$ Bolzano

Let $g(x) = f(x) - K$ $\quad f,g:[a,b] \to \mathbb{R}$

f cts $\Leftrightarrow$ g cts

$f(a) < K < f(b) \Leftrightarrow g(a) < 0 < g(b)$

or $f(b) < K < f(a) \Leftrightarrow g(b) < 0 < g(a)$

$\exists c \in (a,b)\ f(c) = K \Leftrightarrow \exists c \in (a,b)\ g(c) = 0$

Before we prove Bolzano's Theorem ...

Proof of Bolzano's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ cts
& $f(a) < 0$ $f(b) > 0$
($f(a) > 0$ & $f(b) < 0$ similar proof)

Let $\underline{c}$ = L.U.B. (S) where

$S = \{ x \in (a, b) \mid f$ is negative on all of $[a, x) \}$

(Aim: prove $\underline{f(c) = 0}$!)

① $f(c) < 0$ <u>impossible</u> as the Lemma would say f is negative on all of some interval $(c-\delta, c+\delta)$
so negative on

$$[a, c-\delta/2] \cup [c-\delta/2, c+\delta)$$

as c is <u>Least</u> upper bound

$$= [a, c+\delta)$$

$c+\delta \in S$!

② $f(c) > 0$ impossible: Lemma would say f positive on $(c-\delta, c+\delta)$
But f is negative on $[a, c-\delta/2)$

So $\exists c$ with $f(c) = 0$

Bounded Functions

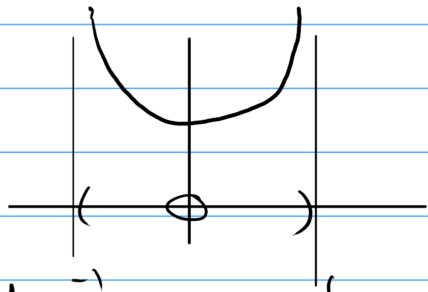
function f whose image is a bounded subset of $\mathbb{R}$

$\exists B$ such that $|f(x)| < B$

I interval , $f : I \to \mathbb{R}$ cts

$(0, \infty)$ $(-1, 1)$ $[-1, 1]$

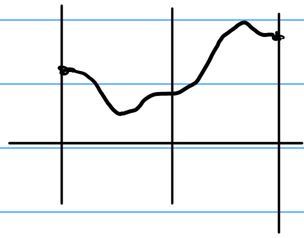
not bounded on open interval



$f(x) = 1/(1-x^2)$

unbounded on open interval

BOUNDED



Lemma $f : [a,b] \to \mathbb{R}$

1. cts from above at $x = a$
2. cts from below at $x = b$
3. cts at all $x = c \in (a,b)$

Then f is a bounded function.

Proof (take $\varepsilon = 1$ in definition of continuity)

1. $\exists \delta > 0 : x \in [a, a+\delta) \Rightarrow f(x) \in (f(a)-1, f(a)+1)$
2. $\exists \delta > 0 : x \in (b-\delta, b] \Rightarrow f(x) \in (f(b)-1, f(b)+1)$
3. $\exists \delta > 0 : x \in (c-\delta, c+\delta) \Rightarrow f(x) \in (f(c)-1, f(c)+1)$

$$\textcircled{1} \qquad\qquad\qquad\qquad \textcircled{3}$$

f bounded on $[a, a+\delta/2]$, $[b+\delta/2, b]$, $[c-\delta/2, c+\frac{\delta}{2}]$

Let $S = \{ x \in [a,b] \mid f \text{ is bounded on } [a,x] \}$

① says $S \neq \emptyset$, $a+\delta/2 \in S$

③ says $\underline{c = LUB(S)} < b$ is $\underline{\text{impossible}}$

it would mean bounded on

$$\underbrace{[a, c-\delta/2]}_{\text{by def}^n \text{ of } S} \cup \underbrace{[c-\delta/2, c+\delta/2]}_{③ \Rightarrow \text{bounded}}$$

$$= [a, \underline{c}+\delta/2]$$

$c+\delta/2 \in S$

So $\underline{c=b}$ & $\underline{f \text{ is bounded on}}$

$$[a, b-\delta/2] \cup \underbrace{[b-\delta/2, b]}_{②} = \underline{[a,b]}$$