Lecture 23: Multiple Integrals.

MA2032 Vector Calculus

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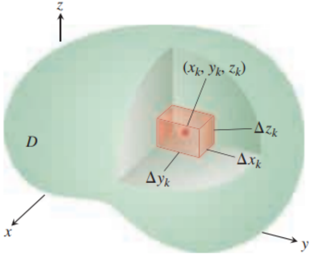
Triple Integrals in Rectangular Coordinates

• We use **triple integrals** to calculate the **volumes** of three-dimensional shapes and the **average value of a function** over a three-dimensional region.

• If $F(x, y, z)$ is a function defined on a **closed bounded region** D **in space** we partition a rectangular boxlike region containing D into **n rectangular cells** by planes parallel to the coordinate axes.



• The kth cell having **dimensions** Δx_k by Δy_k by Δz_k and **volume** $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$.

• We **choose a point** (x_k, y_k, z_k) in each cell and form the **sum**

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k) \Delta V_k.$$

Triple Integrals in Rectangular Coordinates

• It can be shown that **when F is continuous** and the bounding surface of D is formed from finitely many **smooth surfaces** joined together along finitely many **smooth curves**, then F is **integrable**.

• As $\Delta x_k \to 0, \Delta y_k \to 0, \Delta z_k \to 0$ and the number of cells n goes to ∞, the **sums S_n approach a limit**.

• We call this limit the **triple integral** of F over D and write

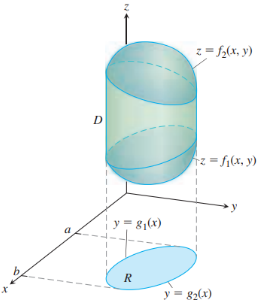
$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)dV = \iiint_D F(x,y,z)\, dx\, dy\, dz.$$

• If F is the **constant function whose value is 1**, then we therefore define the **volume of D to be the triple integral**

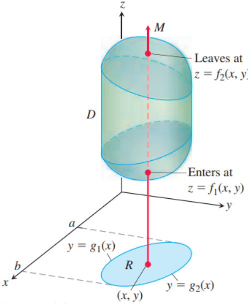
DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

- **Step 1: Sketch** the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.

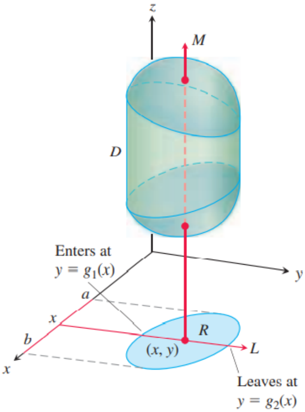


- **Step 2: Find the z-limits of integration**. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



- **Step 3: Find the y-limits of integration**. Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.

- **Step 4: Find the x-limits of integration**. Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the figure). These are the x-limits of integration.



- The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Change the order of integration

• Follow similar procedures if you **change the order of integration**.

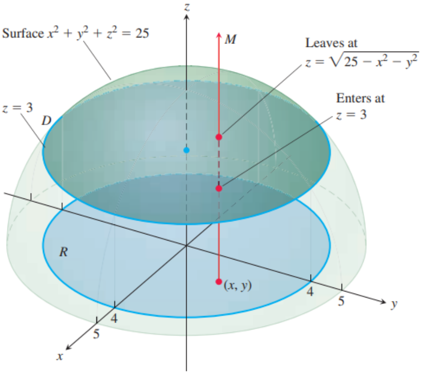
• The "shadow" of region D **lies in the plane of the last two variables** with respect to which the iterated integration takes place • The preceding

procedure applies whenever a solid region D **is bounded above and below by a surface**, and when the "shadow" region R **is bounded by a lower and upper curve**.

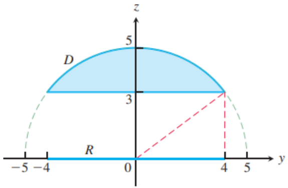
• It **does not apply** to regions with **complicated holes** through them, although sometimes such regions **can be subdivided** into simpler regions for which the procedure does apply.

Example 1

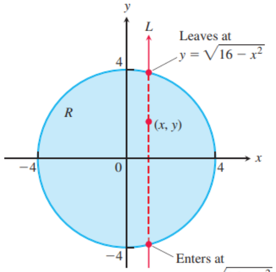
Let S be the sphere of radius 5 centered at the origin, and let D be the region under the sphere that lies above the plane $z = 3$. Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the region D.



Example 1



(a)



(b)

Solution The region under the sphere that lies above the plane $z = 3$ is enclosed by the surfaces $x^2 + y^2 + z^2 = 25$ and $z = 3$.

To find the limits of integration, we first sketch the region, as shown in Figure 15.31. The "shadow region" R in the xy-plane is a circle of some radius centered at the origin. By considering a side view of the region D, we can determine that the radius of this circle is 4; see Figure 15.32a.

If we fix a point (x, y) in R and draw a vertical line M above (x, y), then we see that this line enters the region D at the height $z = 3$ and leaves the region at the height $z = \sqrt{25 - x^2 - y^2}$; see Figure 15.31. This gives us the z-limits of integration.

Example 1

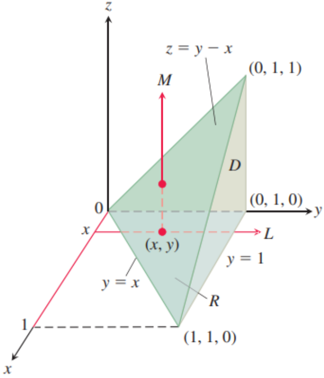
To find the y-limits of integration, we consider a line L that lies in the region R, passes through the point (x, y), and is parallel to the y-axis. For clarity we have separately pictured the region R and the line L in Figure 15.32b. The line L enters R when $y = -\sqrt{16 - x^2}$ and exits when $y = \sqrt{16 - x^2}$. This gives us the y-limits of integration.

Finally, as L sweeps across R from left to right, the value of x varies from $x = -4$ to $x = 4$. This gives us the x-limits of integration. Therefore, the triple integral of F over the region D is given by

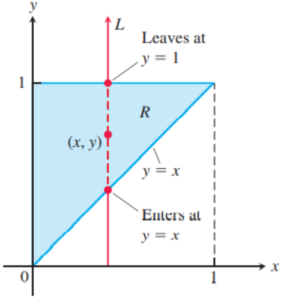
$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} F(x, y, z)\, dz\, dy\, dx. \qquad \blacksquare$$

Example 2

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D whose vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dz\, dy\, dx$.



(a)



(b)

Example 2

Solution The region D and its "shadow" R in the xy-plane are shown in Figure 15.33a. The "side" face of D is parallel to the xz-plane, the "back" face lies in the yz-plane, and the "top" face is contained in the plane $z = y - x$.

To find the z-limits of integration, fix a point (x, y) in the shadow region R, and consider the vertical line M that passes through (x, y) and is parallel to the z-axis. This line enters D at the height $z = 0$, and it exits at height $z = y - x$.

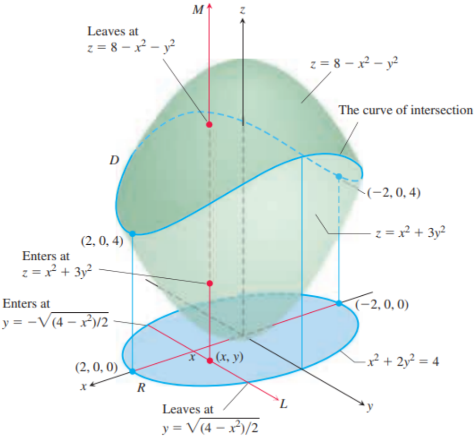
To find the y-limits of integration we again fix a point (x, y) in R, but now we consider a line L that lies in R, passes through (x, y), and is parallel to the y-axis. This line is shown in Figure 15.33a and also in the face-on view of R that is pictured in Figure 15.33b. The line L enters R when $y = x$ and exits when $y = 1$.

Finally, as L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. Therefore, the triple integral of F over the region D is given by

$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx. \qquad \blacksquare$$

Example 3

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Example 3

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.35) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Example 3

Next we find the y-limits of integration. The line L through (x, y) that lies parallel to the y-axis enters the region R when $y = -\sqrt{(4 - x^2)/2}$ and leaves when $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$V = \iiint_D dz\, dy\, dx \qquad \text{Integrand is 1 when computing volume.}$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \qquad \text{Substitute limits of integration.}$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \qquad \text{Integrate over } z \text{ and evaluate.}$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \qquad \text{Integrate over } y.$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx \qquad \text{Evaluate.}$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2 \sin u$$

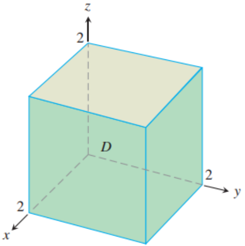
Average Value of a Function in Space

- The **average value of a function** F **over a region** D in space is defined by the formula

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV.$$

Example 4

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.



Example 4

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.36). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2\left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2\left[y^2z\right]_{y=0}^{y=2} dz = \int_0^2 4z\,dz = \left[2z^2\right]_0^2 = 8.$$

With these values, Equation (2) gives

$$\begin{array}{l}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint\limits_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\,dy\,dz$, but any of the other five possible orders would have done as well. ■