Yesterday : product rule

$$\frac{f(x)g(x) - f(c)g(c)}{x-c} = \frac{f(x)g(x) - f(c)g(x)}{x-c} + \frac{f(c)g(x) - f(c)g(c)}{x-c}$$

as $x \to c$ $\qquad f'(c)\,g(c) + f(c)\,g'(c)$ *

$$(f.g)' = f'g + fg'$$

Problem here : assumed $g(x) \longrightarrow g(c)$

Theorem Any differentiable function is continuous

Pf : We know $\displaystyle\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$ exists

So
$$\lim_{x\to c} g(x) = \lim_{x\to c}\big(g(c) + g'(c).(x-c)\big)$$
$$= g(c)$$

Chain Rule : Derivative of composition of differentiable functions

$$\mathbb{R} \xrightarrow{\sin} \mathbb{R} \xrightarrow{\exp} \mathbb{R} \qquad e^{\sin(x)}$$

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R} \qquad f\circ g$$

$$x \qquad g(x) \qquad f(g(x)) = (f\circ g)(x)$$

$$\qquad y \qquad z$$

Theorem If f and g are differentiable functions

g is differentiable at x $\qquad g'(x)$

f ——— " ——— $g(x)$ $\qquad f'(g(x))$ exists

$y = g(x) \quad z = f(y) = f(g(x))$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \qquad *$$

$$\boxed{\text{ie. } (f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

$f \circ g$ is differentiable

Pf Define $H(y) = \begin{cases} f'(g(x)) & y = g(x) \\ \dfrac{f(y) - f(g(x))}{y - g(x)} & y \neq g(x) \end{cases}$

Calculate $\underline{f'(g(x))}$ ..

$$(f \circ g)'(x) = \lim_{t \to x} \frac{f(g(t)) - f(g(x))}{t - x}$$

$$= \lim_{t \to x} H(g(t)) \cdot \frac{g(t) - g(x)}{t - x}$$

$$= f'(g(x)) \cdot g'(x) \qquad \square$$

Examples

$f(x) = \sin(x)$ $\quad f'(x) =$

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$\lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cos x \right)$$

$$\frac{\cos h - 1}{h} = \underbrace{h}_{\substack{\text{limit} \\ 0}} \cdot \underbrace{\frac{\cos h - 1}{h^2}}_{\text{limit } -\frac{1}{2}} \longrightarrow 0$$

$$\frac{\sin h}{h} \longrightarrow 1 \text{ as } h \to 0$$

So $\sin'(x) = \sin x \cdot 0 + 1 \cdot \cos x$

$$\underline{\sin' = \cos}$$

Similarly $\cos'(x) = -\sin(x)$

Derivative of $\underline{\sin^2(x)}$ $\qquad \frac{d}{dy}(y^2) = 2y$

is $\quad 2\sin(x) \cdot \cos(x)$ $\qquad y = \sin x$

$\qquad \frac{dz}{dy} \cdot \frac{dy}{dx}$ $\qquad z = y^2$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = (f \circ g)'(x)$$

$$g(x) = \sin(x)$$

$$f(y) = 1/y$$

$$-\frac{1}{\sin^2(x)} \cdot \cos(x)$$

$$\frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = -\frac{1}{\cos^2(x)} \cdot (-\sin(x))$$

by chain rule

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\left(\sin x \cdot \frac{1}{\cos x}\right)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right)$$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-1}{g^2(x)} \cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - f(x)\frac{g'(x)}{g^2(x)} = \boxed{\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}}$$

Proved quotient rule from product & chain rules

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos^2 x + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2 x} = \underline{\sec^2 x}$$

$$\left((\sin(x))^2\right)'$$

$$= \left(\sin^2(x)\right)' = 2\sin x \cdot \cos x$$

$$\left(\sin\left(x^2\right)\right)' = \cos\left(x^2\right) \cdot 2x$$

<u>Next week</u> $\sin^{-1}(x) = \arcsin(x)$

(derivatives of inverse functions.) $\neq \frac{1}{\sin(x)}$

$\sin(x)\cos^3(x)$?

$x / 1+x^2$?