# Lecture 32: Integrals and Vector Fields.

MA2032 Vector Calculus

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# Surface Integrals

• **Surface integral** is the two-dimensional **extension** of the line integral concept used to integrate over a one-dimensional curve.

• Like line integrals, surface integrals arise in **two forms**: The first occurs when we **integrate a scalar function** over a surface.

• The second form involves **surface integrals of vector fields**.

• Suppose that the function $G(x, y, z)$ gives the **mass density** (mass per unit area) at each point on a surface $S$.

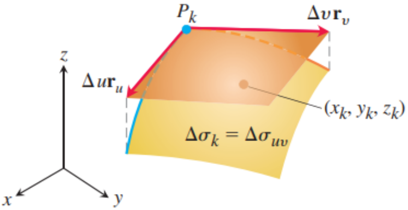
• Then we can calculate the **total mass** of $S$ as an integral in the following way.

• Assume, that the **surface** $S$ **is defined parametrically** on a region $R$ in the $uv$-plane.

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

• The surface $S$ is divided into $n$ **curved surface elements** of area

$$\Delta\sigma_{uv} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$




• To form a **Riemann sum** over $S$, we choose a point $(x_k, y_k, z_k)$ in the $k$-th patch, multiply the value of the function $G$ at that point by the area $\Delta\sigma_k$, and add together the products:

$$\sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k .$$

• Then we take the limit as the number $n$ of surface patches increases, their areas shrink to zero.

• **This limit**, whenever it exists, **defines the surface integral** of $G$ over the surface $S$ as

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n \to \infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k .$$

## Formulas for a Surface Integral of a Scalar Function

**1.** For a smooth surface $S$ defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on $S$, the surface integral of $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

## Formulas for a Surface Integral of a Scalar Function

**2.** For a surface $S$ given **implicitly** by $F(x, y, z) = c$, where $F$ is a continuously differentiable function, with $S$ lying above its closed and bounded shadow region $R$ in the coordinate plane beneath it, the surface integral of the continuous function $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \qquad (3)$$

**3.** For a surface $S$ given **explicitly** as the graph of $z = f(x, y)$, where $f$ is a continuously differentiable function over a region $R$ in the $xy$-plane, the surface integral of the continuous function $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy. \qquad (4)$$

**EXAMPLE 1** Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$.

**Solution** Using Equation (2) and the calculations from Example 4 in Section 16.5, we have $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$ and

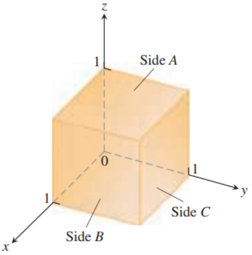
$$\begin{aligned}
\iint_S x^2\, d\sigma &= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(\sqrt{2}r)\, dr\, d\theta \qquad x = r\cos\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, dr\, d\theta \\
&= \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}.
\end{aligned}$$

# Surface Integrals

• When $S$ is **partitioned** by smooth curves into a finite number of smooth patches with **non-overlapping interiors** (i.e., if $S$ is piecewise smooth), then the integral over $S$ is the sum of the integrals over the patches:

$$\iint_S G\, d\sigma = \iint_{S_1} G\, d\sigma + \iint_{S_2} G\, d\sigma + \cdots + \iint_{S_n} G\, d\sigma.$$

**EXAMPLE 2** Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$



**Solution** We integrate $xyz$ over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \iint_{\text{Side } A} xyz\, d\sigma + \iint_{\text{Side } B} xyz\, d\sigma + \iint_{\text{Side } C} xyz\, d\sigma.$$

Side $A$ is the surface $f(x, y, z) = z = 1$ over the square region $R_{xy}$: $0 \le x \le 1$, $0 \le y \le 1$, in the $xy$-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{1}{1} dx\, dy = dx\, dy \qquad \text{Eq. (3)}$$

$$xyz = xy(1) = xy$$

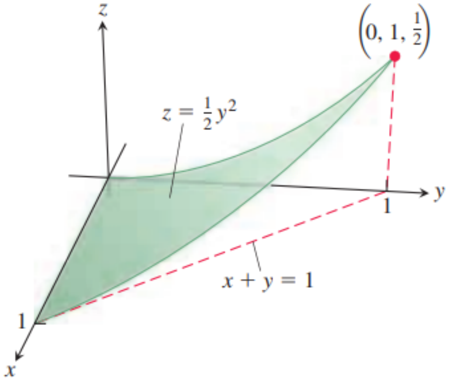
and

$$\iint_{\text{Side } A} xyz\, d\sigma = \iint_{R_{xy}} xy\, dx\, dy = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2} dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of $xyz$ over sides $B$ and $C$ are also $1/4$. Hence,

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

**EXAMPLE 4** Evaluate $\iint_S \sqrt{x(1+2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region $R$: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the $xy$-plane



# Solution Example 4

**Solution** The function $G$ on the surface $S$ is given by

$$G(x, y, z) = \sqrt{x(1 + 2z)} = \sqrt{x}\sqrt{1 + y^2}.$$

With $z = f(x, y) = y^2/2$, we use Equation (4) to evaluate the surface integral:

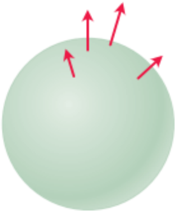
$$d\sigma = \sqrt{f_x{}^2 + f_y{}^2 + 1}\,dx\,dy = \sqrt{0 + y^2 + 1}\,dx\,dy$$

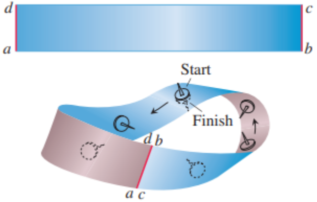
and

$$\begin{aligned}
\iint_S G(x, y, z)\,d\sigma &= \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\,dx\,dy \\
&= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1 + y^2)\,dy\,dx \\
&= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right]dx && \text{Integrate and evaluate.} \\
&= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right)dx && \text{Routine algebra} \\
&= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1 \\
&= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30.
\end{aligned}$$

# Orientation of a Surface

- To specify an **orientation on a surface** in space $S$, we specify a **normal vector** at each point on the surface.
- A parametrization of a surface $\mathbf{r}(u, v)$ gives a **vector $\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface**, and so gives an orientation wherever the parametrization applies.
- A **second choice of orientation** is found by taking $-(\mathbf{r}_u \times \mathbf{r}_v)$, giving a vector that points to the **opposite side of the surface** at each point.
- In essence, an orientation is a way of consistently choosing **one of the two sides of a surface**.
- **Not all surfaces have orientations**.





# Surface Integrals of Vector Fields

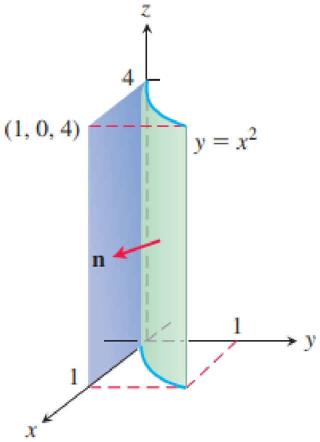
• We defined the **line integral** of a vector field along a path $C$ as $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where $\mathbf{T}$ is the unit **tangent vector** to the path pointing in the **forward oriented direction**.

• We have a **similar definition for surface integrals**.

> **DEFINITION** Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface $S$ having a chosen field of normal unit vectors $\mathbf{n}$ orienting $S$. Then the **surface integral of F over** $S$ is
>
> $$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \tag{5}$$
>
> This integral is also called the **flux** of the vector field $\mathbf{F}$ across $S$.

• If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, then the flux of $\mathbf{F}$ across $S$ is the **net rate at which fluid is crossing** $S$ **per unit time** in the chosen positive direction $\mathbf{n}$ defined by the orientation of $S$.

**EXAMPLE 5** Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in



# Solution. Example 5

**Solution** On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \le x \le 1$, $0 \le z \le 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The unit normal vectors pointing outward from the surface as indicated in Figure 16.54 are

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{4x^2 + 1}}((x^2z)(2x) + (x)(-1) + (-z^2)(0)) = \frac{2x^3z - x}{\sqrt{4x^2 + 1}}.$$

# Solution. Example 5

The flux of $\mathbf{F}$ outward through the surface is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\left|\mathbf{r}_x \times \mathbf{r}_z\right|\,dx\,dz \qquad d\sigma = \left|\mathbf{r}_x \times \mathbf{r}_z\right|\,dx\,dz$$

$$= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\sqrt{4x^2+1}\,dx\,dz$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2\right]_{x=0}^{x=1} dz$$

$$= \int_0^4 \frac{1}{2}(z-1)\,dz = \frac{1}{4}(z-1)^2\Big]_0^4$$

$$= \frac{1}{4}(9) - \frac{1}{4}(1) = 2.$$