Lemma $|x| = \sqrt{x^2}$

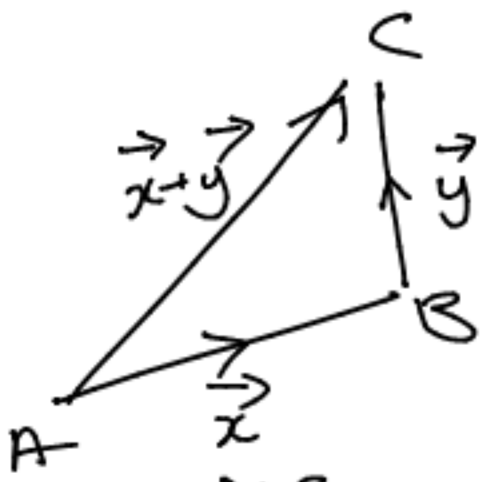
$|0| = \sqrt{0^2}$

$x>0$ : $|x| = x = \sqrt{x^2}$

$x<0$ : $|x| = -x = \sqrt{(x^2)}$

Proof of $\triangle$ inequality:

$|x+y| \leq |x| + |y|$




Lemma says we have to prove

$$\sqrt{(x+y)^2} \leq \sqrt{x^2} + \sqrt{y^2}$$

$$\iff (x+y)^2 \leq \left(\sqrt{x^2} + \sqrt{y^2}\right)^2$$

$$\iff \cancel{x^2} + 2xy + \cancel{y^2} \leq \cancel{x^2} + 2\sqrt{x^2}\sqrt{y^2} + \cancel{y^2}$$

$$\iff \cancel{2}xy \leq \cancel{2}|x|\cdot|y|$$

($x, y$ same sign : $=$
different signs : $<$ )

# Mathematical Induction

$\mathbb{N} = \{0, 1, 2, \dots\}$

* $P(n) \equiv$ " $1+3+5+\dots+(2n-1) = n^2$ "

Method of Proof

$P(0)$ $0 = 0^2 = 0$ Base case is true

$P(1)$ $1 = 1^2 = 1$

Inductive Step $P(n) \Rightarrow P(n+1)$

So $P(n)$ $\forall n$.

* if $1+3+5+\dots+(2n-1) = n^2$

then $\underbrace{1+3+5+\dots+(2n-1)}_{n^2} + (2(n+1)-1) =$

$+2(n+1)-1 = (n+1)^2$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

$F_0 = 0, \quad F_1 = 1$

$F_{n+1} = F_n + F_{n-1}$

**Proposition**: $F_n^2 + F_{n+1}^2 = F_{2n+1}$

$n=3 \qquad 2^2 + 3^2 = 13$

$n=0 \qquad 0^2 + 1^2 = 1$

I don't know any proof $P(n) \Rightarrow P(n+1)$

**Lemma**: (*) $F_n F_m + F_{n+1} F_{m+1} = F_{n+m+1} \quad \forall m, n \in \mathbb{N}$

(Put $m = n \Rightarrow$ Proposition)

Let $P(n)$ be "(*) $\forall m$"

Base case $P(0)$

$F_0 F_m + F_1 F_{m+1} = F_{m+1} \quad \forall m$

$\Leftrightarrow 0 \times F_m + 1 \times F_{m+1} = F_{m+1}$ true $\forall m$.

Suppose we knew $P(n), P(n-1)$

$\forall m \quad F_n F_m + F_{n+1} F_{m+1} = F_{n+m+1} \quad P(n)$

$\forall m \quad F_{n-1} F_m + F_n F_{m+1} = F_{n+m} \quad P(n-1)$

$\forall m \quad F_{n+1} F_m + F_{n+2} F_{m+1} = F_{n+m+2} \quad P(n+1)$

$(F_n + F_{n-1}) F_m$

So $P(n-1), P(n) \Rightarrow P(n+1)$

& $P(0), P(1)$ are true.

So $P(n) \; \forall n$

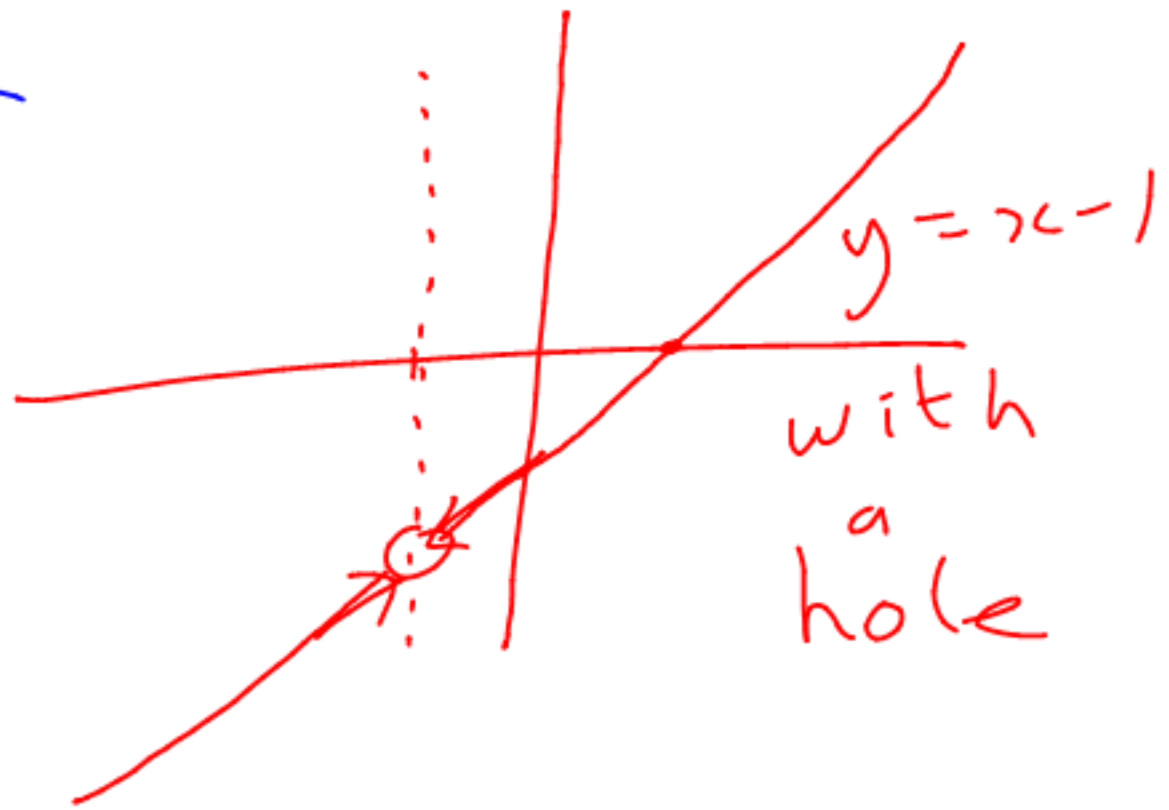
# Chapter 2 Limits

"$\forall \varepsilon > 0 \ \exists \delta > 0 , |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$"

Example $f(x) = \dfrac{x^2-1}{x+1}$

$$\frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$$

$x \neq -1$ : $f(x) = x-1$




$$\lim_{x \to -1} f(x) = -2$$

"$\lim_{x \to c} f(x) = L$"

" lim