Lecture 12: Partial Derivatives.

MA2032 Vector Calculus

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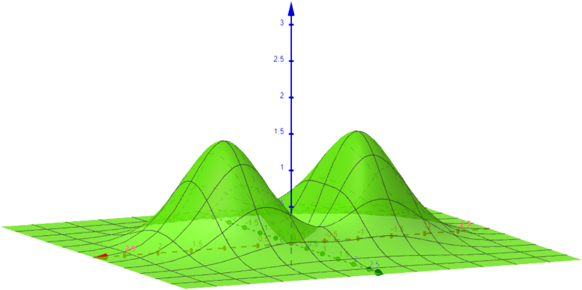
Functions of Several Variables

• We extend the ideas of single-variable differential calculus to **functions of several variables**.

• Their **derivatives are more varied and interesting** because of the different ways the variables can interact.

• The **applications** of these derivatives are also **more varied** than for single-variable calculus, and we will see that the same is true for integrals involving several variables.



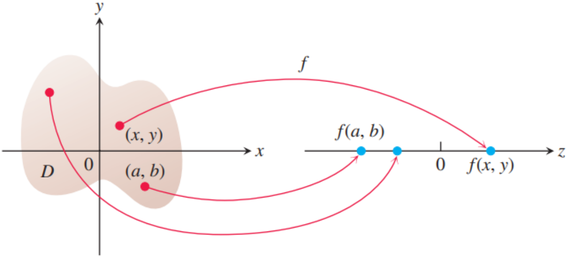
Functions of Several Variables

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

• If f is a function of **two independent variables**, we usually call the independent variables **x and y** and the **dependent variable z**, and we picture the **domain of f** as a **region in the xy-plane**.



• In **defining a function** of more than one variable, we follow the usual practice of **excluding** inputs that lead to **complex numbers or division by zero**.

• The **domain** of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly.

• The **range** consists of the set of **output values for the dependent variable**.

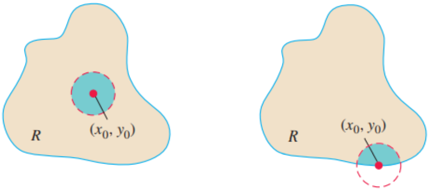
Example 1:

Function	**Domain**	**Range**
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

Functions of Two Variables

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



(a) Interior point (b) Boundary point

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Graphs, Level Curves, and Contours

• There are **two standard ways** to picture the values of a function f(x, y).

• One is to draw and **label curves** in the domain on which f has a constant value.

• The other is to **sketch the surface** z = f(x, y) in space.

> **DEFINITIONS** The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $\boldsymbol{z = f(x, y)}$.

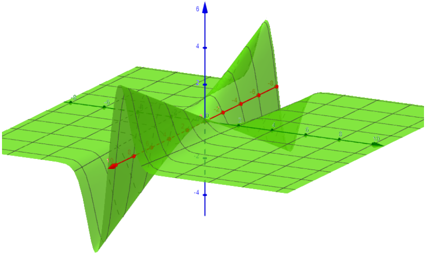
• The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$.
• It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f.

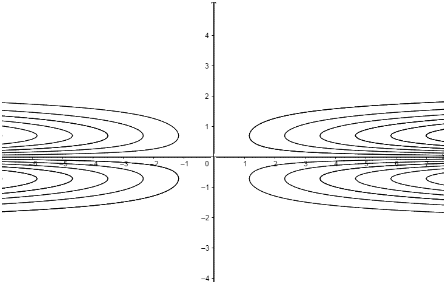
Example 2

Graph a) $f(x, y) = xye^{-y^2}$ and b) $f(x, y) = \sin x + 2\sin y$ and plot some of the level curves.

Solution:

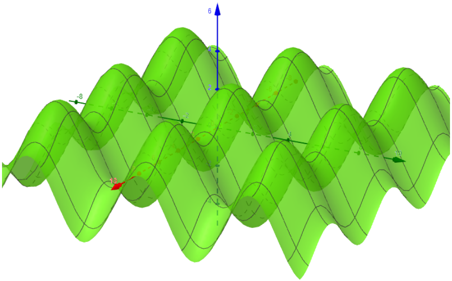
a)

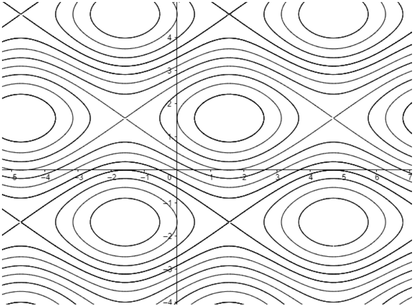




Solution Example 2:

b)





Functions of Three Variables

DEFINITION The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

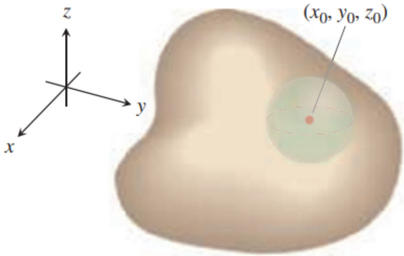
• Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in a **four-dimensional space**, we cannot sketch them effectively in our three-dimensional frame of reference.

• We can see how the function behaves, however, by looking at its **three-dimensional level surfaces**.

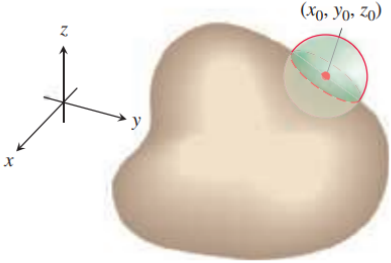
Functions of Three Variables

DEFINITIONS A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.9a). A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R (Figure 14.9b). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.



(a) Interior point



(b) Boundary point

Example 3

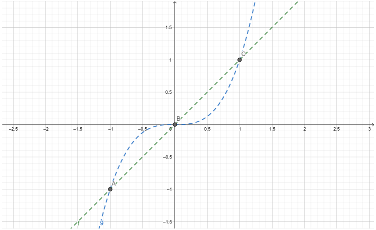
Find and sketch the domain for each function:

a) $f(x,y) = \frac{(x-1)(y+2)}{(y-x)(y-x^3)}$, b) $f(x,y) = cos^{-1}(y - x^2)$,

c) $f(x,y) = \sqrt{(x^2-4)(y^2-9)}$.

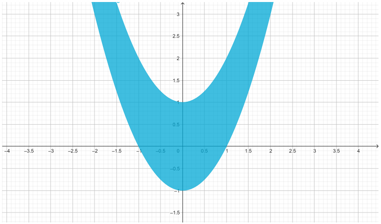
Solution:

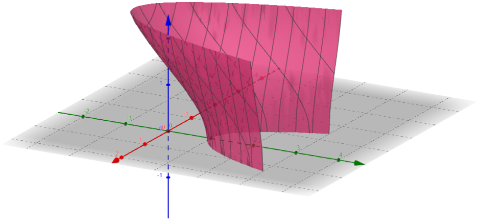
a) Domain: all points (x, y) not lying on the graph of $y = x$, $y = x^3$.



Solution Example 3:

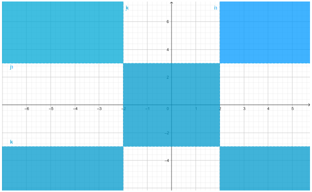
b) Domain: As $\cos^{-1}(y - x^2)$ is defined on the $[-1, 1]$, those all points (x, y) satisfying $x^2 - 1 \leq y \leq x^2 + 1$.

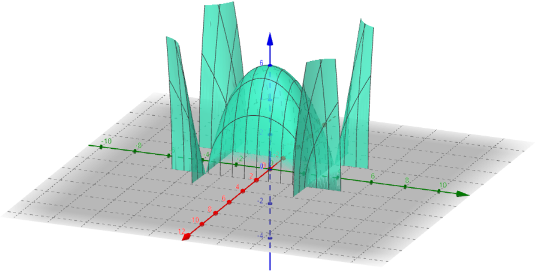




Solution Example 3:

c) Domain: all points (x, y) x y satisfying $(x+2)(x-2)(y-3)(y+3) \geq 0$.





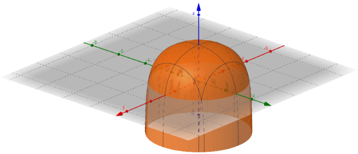
Function of two variables

Example 4

Given function $f(x, y) = \ln(9 - x^2 - y^2)$,
a) find the function's domain, b) find the function's range, c) describe the function's level curves, d) find the boundary of the function's domain, e) determine if the domain is an open region, a closed region, or neither, and (f) decide if the domain is bounded or unbounded.

Solution:

(a) Domain: all points (x, y) inside the circle $x^2 + y^2 = 9$
(b) Range: $z < \ln 9$
(c) Circles centered at the origin with radii $r < 9$
(d) Boundary: the circle $x^2 + y^2 = 9$
(e) open
(f) bounded



Level Surface of the Function

Example 4

Find an equation for the level surface of the function $f(x, y, z) = \sqrt{x - y} - \ln z$ through the point $(3, -1, 1)$.

Solution:

$f(x, y, z) = \sqrt{x - y} - \ln z$ at $(3, -1, 1) \Rightarrow w = \sqrt{x - y} - \ln z$; at $(3, -1, 1) \Rightarrow w = \sqrt{3 - (-1)} - \ln 1 = 2$
$\Rightarrow \sqrt{x - y} - \ln z = 2$

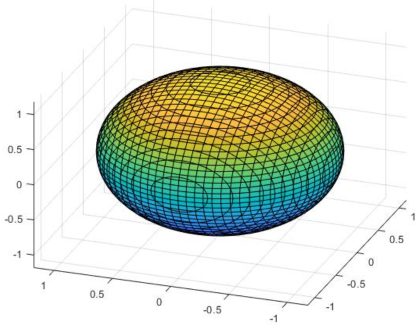
Level Surface of the Function

Example 5

Sketch a typical level surface for the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

Solution:

For example, $\ln(x^2 + y^2 + z^2) = 0$

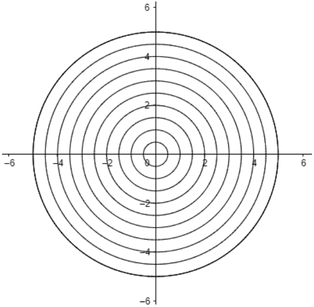


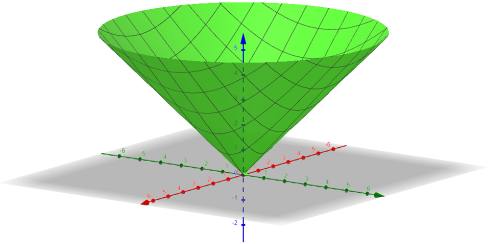
Example 6

Display the values of the functions in two ways:
(a) by sketching the surface $z = f_n(x, y)$ and (b) by drawing an assortment of level curves in the function's domain.
$f_1(x, y) = \sqrt{x^2 + y^2}$, $\quad f_2(x, y) = 1 - |x| - |y|$, $\quad f_3(x, y) = x^2 - y$.

Solution:

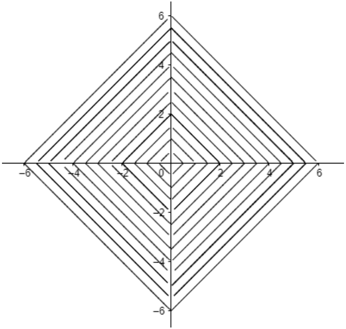
Function f_1

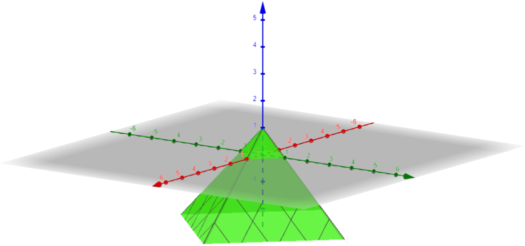




Solution for Example 6:

Function f_2





Solution:

Function f_2