Lecture 10: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

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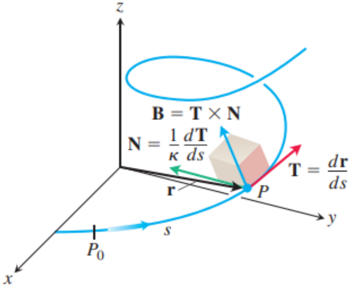
Tangential and Normal Components of Acceleration

• If you are traveling along a curve in space, the Cartesian i, j, and k coordinate system for representing the vectors describing your motion **may not be very relevant** to you.

• Instead, the vectors that represent your **forward direction** (the unit tangent vector **T**), the direction in which your **path is turning** (the unit normal vector **N**), and the tendency of your motion to **"twist" out of the plane created by these vectors** in the direction perpendicular to this plane (defined by the unit binormal vector $B = T \times N$) are likely to be **more important**.



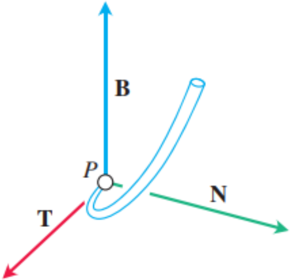
• Expressing the **acceleration vector** along the curve as a **linear combination of this TNB frame** of mutually orthogonal unit vectors traveling with the motion can reveal much about the nature of your path and your motion along it.

The TNB Frame

•The binormal vector of a curve in space is $B = T \times N$, which is **a unit vector that is orthogonal to both T and N**.

• Together T, N, and B define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space.

• It is called the Frenet (“fre-nay”) frame (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.



Tangential and Normal Components of Acceleration

• We often need to know how much of the **acceleration** acts in the direction of motion, which is the direction of the tangent vector $\mathbf{T}$.

• We can calculate this using the Chain Rule to rewrite $\mathbf{v}$ as $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$.

• Then we differentiate both ends of this string of equalities to get

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \qquad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

Tangential and Normal Components of Acceleration

DEFINITION If the acceleration vector is written as

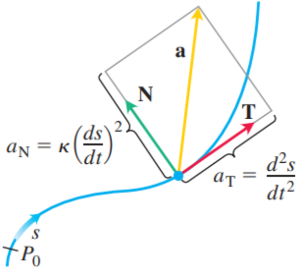
$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \tag{1}$$

then

$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{2}$$

are the **tangential** and **normal** scalar components of acceleration.

- Notice that the binormal vector **B** does not appear in Equation (1).
- No matter how the path of the moving object we are watching may appear to twist and turn in space, the acceleration **a** always lies in the plane of **T** and **N** orthogonal to **B**.
- The equation also tells us exactly how much of the acceleration takes place **tangent** to the motion (d^2s/dt^2) and how much takes place **normal** to the motion [$\kappa(ds/dt)^2$]



Tangential and Normal Components of Acceleration

• What information can we discover from **Equations (2)**?

• By definition, acceleration **a** is the **rate of change of velocity v**, and in general, both the **length** and **direction** of v change as an object moves along its path.

• The tangential component of acceleration a_T measures the **rate of change of the length** of **v** (that is, the change in the speed).

• The normal component of acceleration a_N measures the **rate of change of the direction** of **v**.

• To calculate a_N, we usually use the formula

> **Formula for Calculating the Normal Component of Acceleration**
>
> $$a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2} \tag{3}$$

• which comes from solving the equation $|a|^2 = a \cdot a = a_T^2 + a_N^2$ for a_N.

• With this formula, we can find a_N without having to calculate κ first.

Example 1

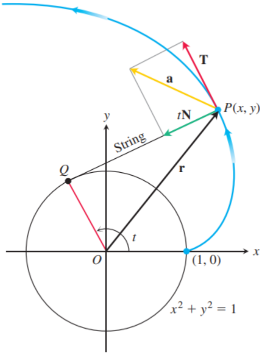
Without finding **T** and **N**, write the acceleration of the motion

$$r(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ t > 0.$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

Solution:

We use the first of Equations (2) to find a_T:



$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j}$$

$$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t \qquad t > 0$$

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1. \qquad \text{Eq. (2)}$$

Knowing a_T, we use Equation (3) to find a_N:

$$\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$$

$$|\mathbf{a}|^2 = t^2 + 1 \qquad \text{After some algebra}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2}$$

$$= \sqrt{(t^2 + 1) - (1)} = \sqrt{t^2} = t.$$

We then use Equation (1) to find **a**:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

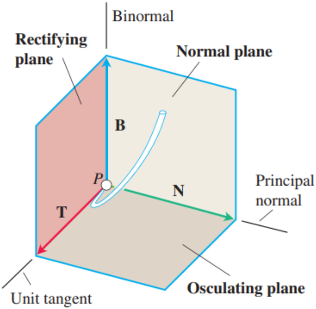
Torsion

- **Torsion measures how the curve twists**.

DEFINITION Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \tag{4}$$

- The three planes determined by **T, N**, and **B** are named and shown in Figure.
- The curvature $\kappa = |dT/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path.
- Similarly, the torsion $\tau = -(dB/ds) \cdot N$ is the rate at which the osculating plane turns about T as P moves along the curve.



Formulas for Computing Curvature and Torsion

- We now give **easy-to-use formulas for computing the curvature and torsion** of a smooth curve.
- From Equations (1) and (2), we have

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right] && \mathbf{v} = dr/dt = (ds/dt)\mathbf{T} \\
&= \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T} \times \mathbf{N}) \\
&= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}. && \mathbf{T} \times \mathbf{T} = 0 \text{ and } \mathbf{T} \times \mathbf{N} = \mathbf{B}
\end{aligned}$$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa|\mathbf{v}|^3. \qquad \frac{ds}{dt} = |\mathbf{v}| \quad \text{and} \quad |\mathbf{B}| = 1$$

Solving for κ gives the following formula.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \tag{5}$$

Formulas for Computing Curvature and Torsion

Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}) \tag{6}$$

• This formula calculates the torsion directly from the derivatives of the component functions $x = (t), y = g(t), z = h(t)$ that make up **r**.

• The determinant's first row comes from **v**, the second row comes from **a**, and the third row comes from a da/dt.

• This formula for torsion is traditionally written using Newton's dot notation for derivatives.

Formulas for Computing Curvature and Torsion

Example 2

Use Equations (5) and (6) to find the curvature κ and torsion τ for the helix

$$r(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Solution:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$\mathbf{a} = -(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (ab\sin t)\mathbf{i} - (ab\cos t)\mathbf{j} + a^2\mathbf{k}$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2+b^2)^{3/2}} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}.$$

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a\sin t)\mathbf{i} - (a\cos t)\mathbf{j}.$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2}$$

Value of $|\mathbf{v}\times\mathbf{a}|$ from Eq. (7)

$$= \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2+b^2)}$$

$$= \frac{b}{a^2+b^2}.$$

- From this last equation we see that the torsion of a helix about a circular cylinder is constant.
- In fact, constant curvature and constant torsion characterize the helix among all curves in space.

Formulas for Computing Curvature and Torsion

Computation Formulas for Curves in Space

Unit tangent vector:
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Principal unit normal vector:
$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

Binormal vector:
$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Curvature:
$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

Torsion:
$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

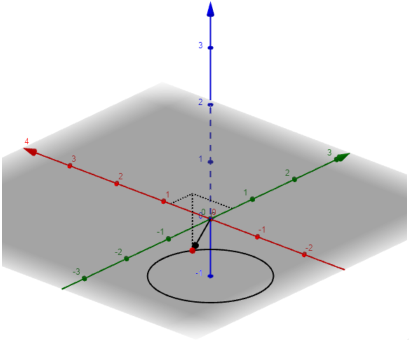
Tangential and normal scalar components of acceleration:
$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$$
$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$
$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}^2}$$

Example 3

Given

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k},$$

Find **r, T, N**, and **B** at the $t = \pi/4$. Then find equations for the osculating, normal, and rectifying planes at that value of $t = \pi/4$.



Solution:

$\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}-\mathbf{k}\Rightarrow \mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}\Rightarrow |\mathbf{v}|=\sqrt{(-\sin t)^2+(\cos t)^2}=1$

$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}\Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j};\ \frac{d\mathbf{T}}{dt}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|\sqrt{(-\cos t)^2+(-\sin t)^2}=1\Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}\Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j};$

$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 0\\ -\cos t & -\sin t & 0\end{vmatrix}=\mathbf{k}\Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right)=\mathbf{k}$, the normal to the osculating plane; $\mathbf{r}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}-\mathbf{k}$

$\Rightarrow P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-1\right)$ lies on the osculating plane $\Rightarrow 0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+(z-(-1))=0\Rightarrow z=-1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0$

$\Rightarrow -\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0\Rightarrow -x+y=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane

$\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0\Rightarrow -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y=-1\Rightarrow x+y=\sqrt{2}$ is the rectifying plane.