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MA1014
CALCULUS AND ANALYSIS
TUTORIAL 12

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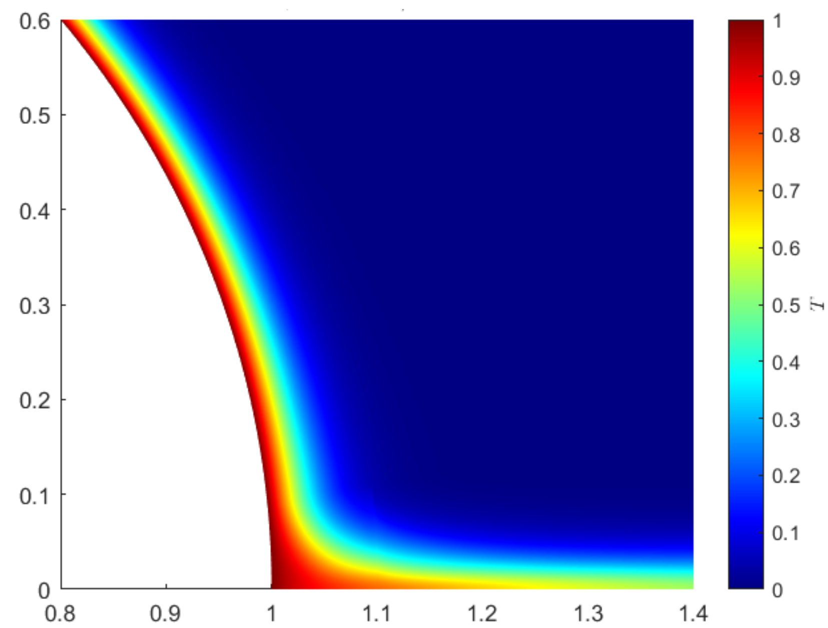
MULTIVARIATE FUNCTIONS

Functions of the form $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ i.e. $f = f(\boldsymbol{x}) = f(x_1, x_2, x_3, \dots, x_n)$

Examples:

- In 2D $f : D = (x, y) \to \mathbb{R}$ with $f = f(x, y)$
- In 3D $f : D = (x, y, z) \to \mathbb{R}$ with $f = f(x, y, z)$



Real World:

- Height of a mountain $h = h(x, y)$
- Price of an option $V = V(S, t)$
- Temperature of a room $T = T(x, y, z)$
- Wave Function in Quantum Mechanics $\Psi = \Psi(x, y, z, t)$
- Velocity of a fluid $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t) = \big(u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)\big)$

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DIFFERENTIABILITY

A function f is differentiable at $\boldsymbol{x}_0$ if the following limit exists

$$\boldsymbol{\nabla} f(\boldsymbol{x}_0) = \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{f(\boldsymbol{x}_0 + \boldsymbol{h}) - f(\boldsymbol{x}_0)}{||\boldsymbol{h}||}$$

Note: As in 1D differentiable ⇒ continuity

$\boldsymbol{\nabla} f(\boldsymbol{x}_0)$ is called the **Gradient Vector** of f at $\boldsymbol{x}_0$ and is defined as, in 2D,

$$\boldsymbol{\nabla} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Where **$\boldsymbol{\nabla}$ is called the "Differential Operator"** often called 'Nabla' or 'Del', and is defined as

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

DIRECTIONAL DERIVATIVE

The Directional Derivative gives the rate of change in the direction of a unit vector $\boldsymbol{u}$ and is

$$f_{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} f(\boldsymbol{x})$$

Example: Let $f = 2x^2y + 3y$ and $\boldsymbol{u} = (1,0)^T$. Then,

$$f_{\boldsymbol{u}}(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4xy \\ 2x^2 + 3 \end{pmatrix} = 4xy$$

EXERCISE

Let $f(x, y) = 3x^3y + xy^3$.

Calculate $\frac{\partial f}{\partial \boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} f$ at $\boldsymbol{P} = (2, -3)$, where $\boldsymbol{n} = (3, -2)$.

MAXIMA & MINIMA IN 2D

- Maximum or Minimum values of a function (can be local or global).

- These occur at either Stationary points ($\boldsymbol{\nabla} f = \mathbf{0}$) or Boundary points (if the domain is confined).

- Note that the converse is not true, i.e. if $\boldsymbol{\nabla} f = \mathbf{0}$ at a point (x_0, y_0) then this **does not** imply that the function takes a max/min at (x_0, y_0). See Saddle points.

EXERCISE

a. Find the (x, y) co-ordinates of the stationary points (there may only be one) of the function

$$f(x, y) = \frac{xy}{1 + x^2 + y^2} \text{ for } (x, y) \in \mathbb{R}^2.$$

b. Investigate the points to decide if they are maxima or minima by looking at the directional derivatives of f near to the stationary points.

THE SECOND DERIVATIVE TEST

Consider the surface $f(x, y)$. Assuming it has at least one stationary point (x_0, y_0), i.e. $\boldsymbol{\nabla} f|_{x_0,y_0} = \mathbf{0}$.

Define the Hessian matrix as

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Then if:

- $\det\left(H|_{x_0,y_0}\right) > 0$ & $f_{xx}|_{x_0,y_0} > 0$, then (x_0, y_0) is a local min
- $\det\left(H|_{x_0,y_0}\right) > 0$ & $f_{xx}|_{x_0,y_0} < 0$, then (x_0, y_0) is a local max
- $\det\left(H|_{x_0,y_0}\right) < 0$, then (x_0, y_0) is a saddle
- $\det\left(H|_{x_0,y_0}\right) = 0$, then the test is inconclusive

EXERCISE:

Find and classify the stationary points of

$$f(x, y) = \frac{x^3}{3} + 5x^2y + 24y^2x + 63y$$

Hints:

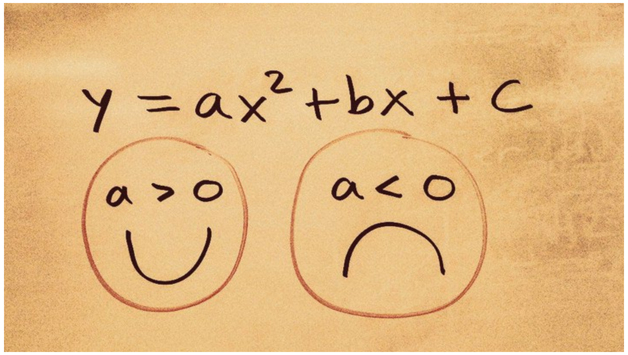
- Find and solve $f_x = 0$ and $f_y = 0$
- Determine the Hessian matrix: $H_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$
- What's $\det(H)$?

EXERCISE

Let $f(x, y) = -5x^2 - xy - y^2 - 4x - 4y$.

Find and classify the stationary points.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

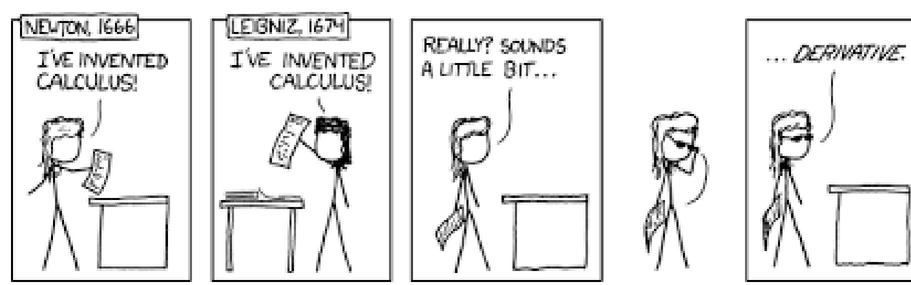


$$\int_a^b f(x)dx = F(b) - F(a)$$

ANY QUESTIONS?

$$m\frac{d^2x}{dt^2} = -kx$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$



EXTRA TIME

Consider the Heat equation of an Incompressible fluid of velocity $\boldsymbol{u}$ and temperature T:

$$\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})T = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})T$$

a. Identify where and state the Directional Derivative in the above equation.

b. Why do you think it appears here?