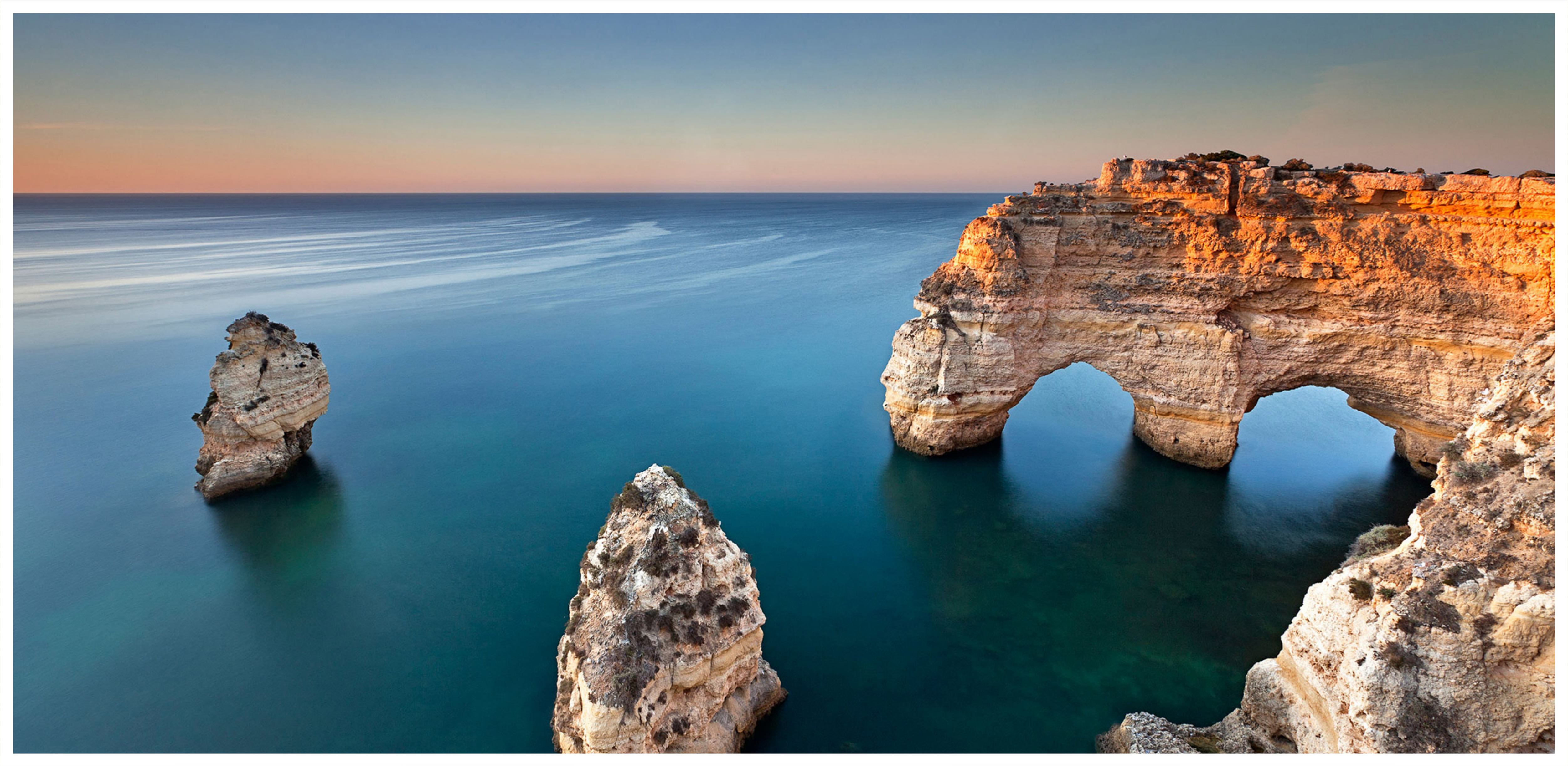


LINEAR ALGEBRA II

Bo Yu（于波）Dalian University of Technology

线性代数II（B.YU）

Ch. X Triangulation of Matrices and Linear Maps

§1. EXISTENCE OF TRIANGULATION

- Let V be a finite dimensional vector space over the field K, and assume $n = \dim V \geq 1$. $A : V \to V$ be a linear map.

- A fan of A: a sequence of subspaces $\{V_1, \ldots, V_n\}$, such that
  ① $V_i \subset V_{i+1}$;
  ② $\dim V_i = i$;
  ③ Each V_i is A-invariant

- A fan basis of V w.r.t. A: a basis $\{v_1, \ldots, v_n\}$of V such that $\{V_1, \ldots, V_n\}$ is a fan of A, where $V_i = \mathrm{span}\{v_1, \ldots, v_i\}$;

§1. EXISTENCE OF TRIANGULATION

- **Theorem 1.1.** *Let* $\{v_1, \ldots, v_n\}$ *be a fan basis for* A. *Then the matrix associated with* A *relative to this basis is an upper triangular matrix.*
-

§1. EXISTENCE OF TRIANGULATION

- If A is a triangular matrix, then $\{e^1, \ldots, e^n\}$ is a fan basis of K^n for A. Thus the converse of Theorem 1.1 is also true.

- A linear map A is triangulable: there exists a basis for V for which the associated matrix of A is triangular.
- An $n \times n$ matrix A over K is triangulable over K if it is triangulable as an operator of K^n.
- A matrix A over K is triangulable iff there exists a non-singular matrix B in K such that $B^{-1}AB$ is an upper triangular matrix.

§1. EXISTENCE OF TRIANGULATION

- **Theorem 1.2.** *Let V be a finite dimensional vector space over the complex numbers, and assume that* $\dim V \geqq 1$. *Let* $A: V \to V$ *be a linear map. Then there exists a fan of A in V.*
-

§1. EXISTENCE OF TRIANGULATION

§1. EXISTENCE OF TRIANGULATION

- **Corollary 1.3.** *Let V be a finite dimensional vector space over the complex numbers, and assume that* $\dim V \geqq 1$. *Let $A: V \to V$ be a linear map. Then there exists a basis of V such that the matrix of A with respect to this basis is a triangular matrix.*

- **Corollary 1.4.** *Let M be a matrix of complex numbers. There exists a non-singular matrix B such that $B^{-1}MB$ is a triangular matrix.*

§1. EXISTENCE OF TRIANGULATION

- **Theorem (Schur decomposition).** For any $n \times n$ complex matrix A, there exists a unitary matrix U, such that $U^*AU = R$ is an upper triangular matrix.

§1. EXISTENCE OF TRIANGULATION

- **Theorem (real Schur decomposition).** For any $n \times n$ real matrix A, there exists a real unitary matrix U, such that $U^{\mathrm{T}}AU = R$ is a block-wise upper triangular real matrix with diagonal blocks of order 1 or 2.

§1. EXISTENCE OF TRIANGULATION

§1. EXISTENCE OF TRIANGULATION

- Let $A = (a_{ij})$ be an $n \times n$ complex matrix. If the sum of the elements of each column is 1 then A is called a **Markov matrix**. In symbols, for each j we have

$$\sum_i a_{ij} = 1.$$

We leave the following properties as exercises.

Property 1. Prove that if A, B are Markov matrices, then so is AB. In particular, if A is a Markov matrix, then A^k is a Markov matrix for every positive integer k.

- *Property* 2. Prove that if A, B are Markov matrices such that $|a_{ij}| \leqq 1$ and $|b_{ij}| \leqq 1$ for all i, j and if $AB = C = (c_{ij})$, then $|c_{ij}| \leqq 1$ for all i, j.

- $A = \begin{pmatrix} 0.5 & 1 & 1 & -1 \\ 0.5 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 1 \\ -0.5 & 0 & 0 & 0 \end{pmatrix}$? $\quad (A^2)_{11} = 1.75$

§1. EXISTENCE OF TRIANGULATION

- **Theorem 1.5.** *Let A be a Markov matrix such that $|a_{ij}| \leqq 1$ for all i, j. Then every eigenvalue of A has absolute value $\leqq 1$.*

§2. THEOREM OF HAMILTON-CAYLEY

- **Theorem 2.1.** *Let V be a finite dimensional vector space over the complex numbers, of dimension $\geqq 1$, and let $A: V \to V$ be a linear map. Let P be its characteristic polynomial. Then $P(A) = O$.*

§2. THEOREM OF HAMILTON-CAYLEY

§2. THEOREM OF HAMILTON-CAYLEY

- **Corollary 2.2.** *Let A be an $n \times n$ matrix of complex numbers, and let P be its characteristic polynomial. Then $P(A) = O$.*

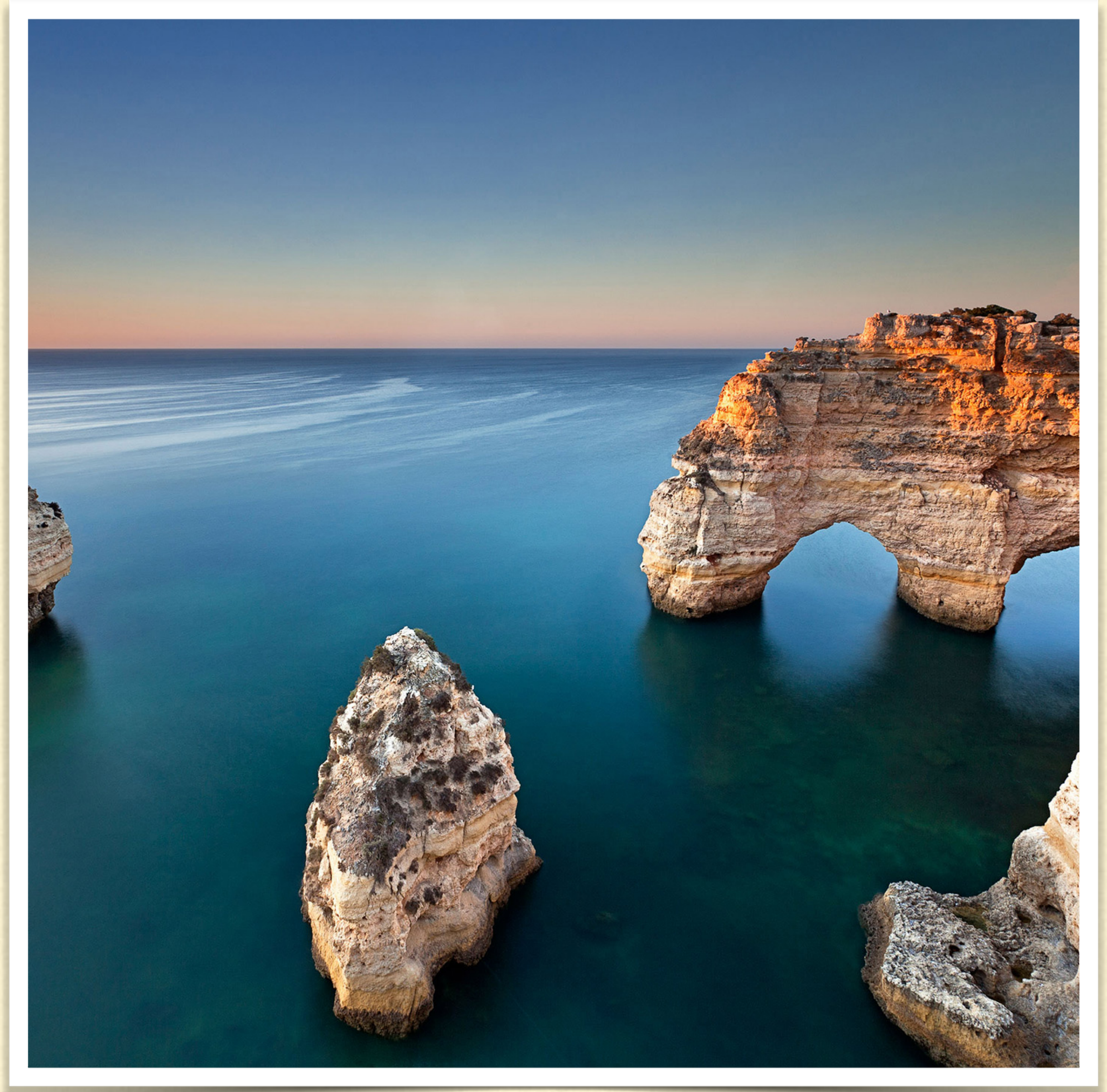
- **Corollary 2.3.** *Let V be a finite dimensional vector space over the field K, and let $A: V \to V$ be a linear map. Let P be the characteristic polynomial of A. Then $P(A) = O$.*

§3. DIAGONALIZATION OF UNITARY MAPS

- **Theorem 3.1.** *Let V be a finite dimensional vector space over the complex numbers, and let* $\dim V \geqq 1$. *Assume given a positive definite hermitian product on V. Let $A: V \to V$ be a unitary map. Then there exists an orthogonal basis of V consisting of eigenvectors of A.*
- New proof.

§3. DIAGONALIZATION OF UNITARY MAPS

- **Corollary 3.2.** *Let A be a complex unitary matrix. Then there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.*



- Homework:
  - P240: 2, 3, 6.
- Problem solving and discussion:
  - P240: 5, 7