Lecture 30: Integrals and Vector Fields.

MA2032 Vector Calculus

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Two Forms for Green's Theorem

• A **simple closed curve C** can be traversed in **two possible directions**. (Recall that a curve is simple if it does not cross itself.)

• The curve is **traversed counterclockwise**, and said to be T**positively oriented**, if the region it encloses is always to the **left** when moving along the curve.

• If the curve is **traversed clockwise** then the enclosed region is on the **right** when moving along the curve and the curve is said to be **negatively oriented**.

• The line integral of a vector field F along C **reverses sign if we change the orientation**.

• We use the notation

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$

• for the line integral when the simple closed curve C is traversed **counterclockwise**, with its **positive orientation**.

Two Forms for Green's Theorem

- In **one form, Green's Theorem** says that the **counterclockwise circulation** of a vector field around a simple closed curve is the double integral of the **k-component of the curl** of the field over the region enclosed by the curve.

Circulation and Curl

$$\text{Circulation around } C = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds$$

$$(\text{curl}\,\mathbf{F})\cdot\mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form)
Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl}\,\mathbf{F})\cdot\mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C M\,dx + N\,dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Curl integral}} \qquad (3)$$

Two Forms for Green's Theorem

• A **second form of Green's Theorem** says that the **outward flux** of a vector field across a simple closed curve in the plane equals the **double integral of the divergence** of the field over the region enclosed by the curve.

Flux and Divergence

$$\text{Flux of } \mathbf{F} \text{ across } C = \oint_C \mathbf{F} \cdot n \, ds$$

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds}_{\text{Outward flux}} = \oint_C M \, dy - N \, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy}_{\text{Divergence integral}} \qquad (4)$$

Two Forms for Green's Theorem

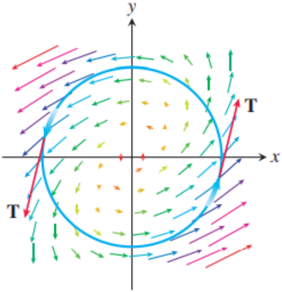
- The two forms of Green's Theorem are **equivalent**.
- Applying Equation (3) to the field $G_1 = -N_i + M_j$ gives Equation (4), and applying Equation (4) to $G_2 = N_i - M_j$ gives Equation (3).
- Both forms of Green's Theorem can be viewed as **two-dimensional generalizations** of the Fundamental Theorem of Calculus.
- The **counterclockwise circulation of F around C**, defined by the **line integral** on the left-hand side of Equation (3), is the integral of its **rate of change (circulation density)** over the region R enclosed by C, which is the double integral on the right-hand side of Equation (3).
- Likewise, the **outward flux of F across C**, defined by the **line integral** on the left-hand side of Equation (4), is the integral of its **rate of change (flux density)** over the region R enclosed by C, which is the double integral on the right-hand side of Equation (4).

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$



Solution First we evaluate the counterclockwise circulation of $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around C. On the curve C we have $x = \cos t$ and $y = \sin t$. Evaluating $\mathbf{F}(\mathbf{r}(t))$ and computing the partial derivatives of the components of $\mathbf{F}$, we have

$$M = x - y = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$
$$N = x = \cos t, \qquad dy = d(\sin t) = \cos t\, dt.$$

Therefore,

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M\, dx + N\, dy$$
$$= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t)\, dt + (\cos t)(\cos t)\, dt$$
$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt = 2\pi.$$

This gives the left side of Equation (3). Next we find the curl integral, the right side of Equation (3). Since $M = x - y$ and $N = x$, we have

$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

Two Forms for Green's Theorem

Therefore,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 - (-1))\,dx\,dy$$

$$= 2\iint_R dx\,dy = 2(\text{area inside the unit circle}) = 2\pi.$$

Thus, the right and left sides of Equation (3) both equal 2π, as asserted by the circulation-flux version of Green's Theorem.

Figure 16.35 displays the vector field and circulation around C.

Now we compute the two sides of Equation (4) in the flux-divergence form of Green's Theorem, starting with the outward flux:

$$\oint_C M\,dy - N\,dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi.$$

Next we compute the divergence integral:

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1 + 0)\,dx\,dy = \iint_R dx\,dy = \pi.$$

Hence the right and left sides of Equation (4) both equal π, as asserted by the flux-divergence version of Green's Theorem. ∎

Using Green's Theorem to Evaluate Line Integrals

EXAMPLE 4 Evaluate the line integral

$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's Theorem to change the line integral into a double integral over the square, where C is the square's boundary and R is its interior.

1. *With the Tangential Form* Equation (3): Taking $M = -y^2$ and $N = xy$ gives the result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (y - (-2y))\,dx\,dy$$

$$= \int_0^1 \int_0^1 3y\,dx\,dy = \int_0^1 \left[3xy\right]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}.$$

2. *With the Normal Form* Equation (4): Taking $M = xy$, $N = y^2$, gives the same result:

$$\oint_C xy\,dy - y^2\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (y + 2y)\,dx\,dy = \frac{3}{2}.$$

Using Green's Theorem to Evaluate Line Integrals

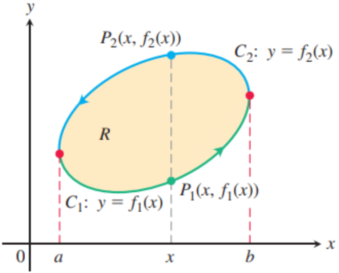
EXAMPLE 5 Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = 2e^{xy}$, $N = y^3$, C the square, and R the square's interior, we have

$$\begin{aligned}
\text{Flux} &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \qquad \text{Green's Theorem, Eq. (4)} \\
&= \int_{-1}^{1}\int_{-1}^{1} (2ye^{xy} + 3y^2)\,dx\,dy = \int_{-1}^{1} \left[2e^{xy} + 3xy^2\right]_{x=-1}^{x=1} dy \\
&= \int_{-1}^{1} (2e^{y} + 6y^2 - 2e^{-y})\,dy = \left[2e^{y} + 2y^3 + 2e^{-y}\right]_{-1}^{1} = 4. \qquad \blacksquare
\end{aligned}$$

Proof of Green's Theorem for Special Regions

- Let C be a **smooth simple closed curve** in the xy-plane with the **property** that lines parallel to the axes cut it at no more than two points.
- Let R be the region enclosed by C and suppose that M, N, and their **first partial derivatives** are **continuous** at every point of some open region containing C and R.



- We want to prove the **circulation-curl form** of Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

- Figure shows C made up of **two directed parts**

$$C_1:\quad y = f_1(x),\quad a \le x \le b,\qquad C_2:\quad y = f_2(x),\quad b \ge x \ge a.$$

Proof of Green's Theorem for Special Regions

- For any x between a and b, we can **integrate** $\partial M/\partial y$ **with respect to** y from $y = f_1(x)$ to $y = f_2(x)$ and obtain

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

- We can then **integrate** this **with respect to** x from a to b:

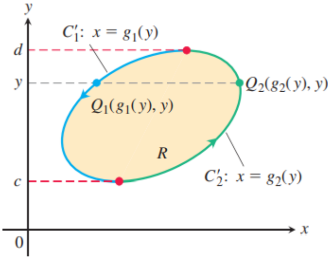
$$\begin{aligned}\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy\, dx &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx \\ &= -\int_b^a M(x, f_2(x))\, dx - \int_a^b M(x, f_1(x))\, dx \\ &= -\int_{C_2} M\, dx - \int_{C_1} M\, dx \\ &= -\oint_C M\, dx.\end{aligned}$$

Proof of Green's Theorem for Special Regions

- Therefore, **reversing the order of the equations**, we have

$$\oint_C M\,dx = \iint_R \left(-\frac{\partial M}{\partial y}\right) dx\,dy.$$

- These Equation is **half the result** we need. We derive the other half by **integrating** $\partial N/\partial x$ **first with respect to** x and then **with respect to** y, as suggested by Figure.
- The curve C of Figure decomposed into the **two directed parts**



$$C_1'\colon x = g_1(y), d \geq y \geq c \text{ and } C_2'\colon x = g_2(y), c \leq y \leq d.$$

- The result of this double integration is

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x} dx\,dy.$$