Sequence $(a_n)_{n\in\mathbb{N}}$ bounded and monotonic $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$ then $\begin{matrix} a_n \to LUB \\ a_n \to GLB \end{matrix}$

Cauchy sequences : $(a_n)_{n\in\mathbb{N}}$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : n_1, n_2 > N \Rightarrow |a_{n_1} - a_{n_2}| < \varepsilon$

Cauchy sequences are bounded

Convergent sequences are Cauchy

Aim: All Cauchy sequences converge

Peak points Lemma : <u>every</u> sequence has a <u>monontonic</u> subsequence

<u>Theorem</u> Every <u>bounded</u> sequence $(a_n)_{n\in\mathbb{Z}}$ has a subsequence $(a_{m_k})_{k\in\mathbb{Z}}$ which converges to a limit L.

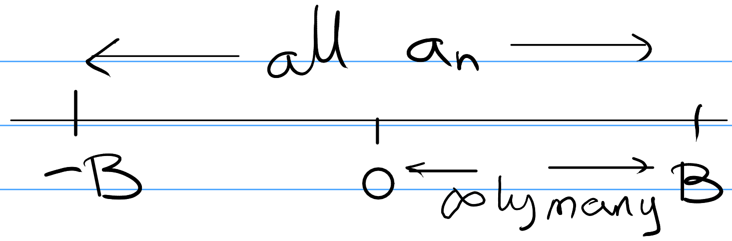
<u>Proof</u> First proof : $(a_n)_{n\in\mathbb{Z}}$ has a monotonic sequence $(a_{m_k})_{k\in\mathbb{Z}}$ which is bounded as $(a_n)_{n\in\mathbb{Z}}$ is. So it converges.

Second Proof by bisection

$\boxed{a_{m_0} = a_0}$ and $[x_0, y_0] = [-B, B]$

where $|a_n| < B$ (sequence bounded)

Inductively

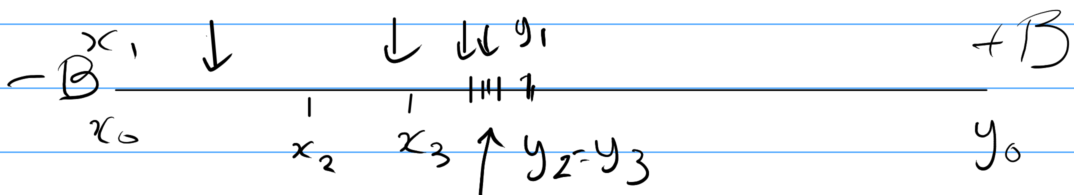


Choose $[x_1, y_1]$ to be either $[-B, 0]$ or $[0, B]$ so it contains a_n for infinitely many n.

Suppose $[x_{k-1}, y_{k-1}]$ with $a_n \in [x_{k-1}, y_{k-1}]$ for infinitely many n & choose $[x_k, y_k]$ to be either

$$\left[x_{k-1}, \tfrac{1}{2}(x_{k-1} + y_{k-1})\right] \text{ or } \left[\tfrac{1}{2}(x_{k-1}, y_{k-1}), y_{k-1}\right]$$

so it contains infinitely many terms of the sequence.



infinitely many $a_n \in [x_k, y_k]$

Choose a subsequence

$a_{m_0}, a_{m_1}, a_{m_2}, a_{m_3}, \ldots, a_{m_k} \in [x_k, y_k]$

We have $(x_k)_{k\in\mathbb{N}}$ is monotonic increasing

$(y_k)_{k\in\mathbb{N}}$ is monotonic decreasing

since $x_k \leq x_{k+1} < y_{k+1} \leq y_k$

& both sequences are bounded

$$y_k - x_k = \frac{2B}{2^k} \longrightarrow 0$$

x_k, y_k converge

$$\lim y_k - \lim x_k = \lim y_k - x_k = 0$$

So $\lim y_k = \lim x_k$

Pinching Theorem: $x_k \leq a_{m_k} \leq y_k$

$$\begin{array}{ccccc} x_k & \leq & a_{m_k} & \leq & y_k \\ \downarrow & & \downarrow & & \downarrow \\ L & \therefore & L & & L \end{array}$$

so our subsequence converges.

Back to Cauchy Sequence

$(a_n)_{n\in\mathbb{N}}$ Cauchy $\Rightarrow$ bounded

$\Rightarrow$ has a convergent subsequence

$a_{m_k} \longrightarrow L$ as $k\to\infty$

Aim prove that $a_n \longrightarrow L$

(so $\underline{\text{Cauchy} \Leftrightarrow \text{convergent}}$)

Proof Given $\varepsilon > 0$

(a_n) **Cauchy**: $\exists N : n_1, n_2 > N \Rightarrow |a_{n_1} - a_{n_2}| < \frac{\varepsilon}{2}$

(a_{m_k}) **convergent** $\exists K > N : k > K \Rightarrow |a_{m_k} - L| < \frac{\varepsilon}{2}$

$$n > N \Rightarrow |a_n - L| = |a_n - a_{m_k} + a_{m_k} - L| \quad < \varepsilon$$

Triangle inequality $\leq \underbrace{|a_n - a_{m_k}|}_{< \varepsilon/2} + \underbrace{|a_{m_k} - L|}_{< \varepsilon/2}$

by Cauchy property

$m_1 \geq 1,\ m_2 \geq 2$

$m_k \geq k \geq N$

Reached the end of Topic 4

Next week: concentrate on example, revision

Tell me which topics are useful!