Lecture 28: Integrals and Vector Fields.

MA2032 Vector Calculus

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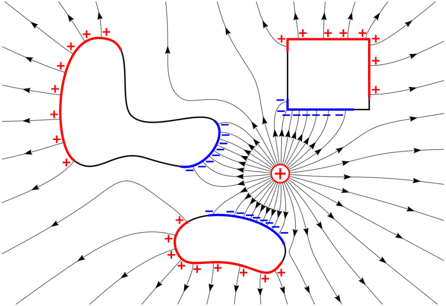
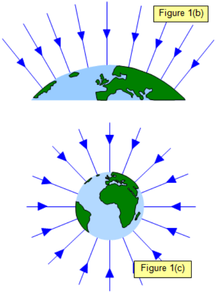
Gravitational Field. Electric Field

• A **gravitational field** G is a vector field that represents the **effect of gravity** at a point in space due to the **presence of a massive object**. The **gravitational force** on a body of mass m placed in the field is given by $F = mG$.





• Similarly, an **electric field** E is a vector field in space that represents the **effect of electric forces** on a **charged particle** placed within it.

• The **force on a body of charge** q placed in the field is given by $F = qE$.

• In gravitational and electric fields, the amount of **work it takes to move a mass or charge** from one point to another **depends on the initial and final positions** of the object — **not on which path is taken** between these positions.

• We study vector fields with this **independence-of-path property** and the calculation of **work integrals** associated with them.

• If A and B are two points in an open region D in space, the **line integral** of F along C from A to B for a field **F** defined on D **usually depends on the path C** taken, as we saw in the previous lectures.

• For some **special fields**, however, the **integral's value is the same for all paths from A to B**.

Path Independence

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

- The **word conservative** comes **from physics**, where it refers to fields in which the **principle of conservation of energy holds**.

- Under reasonable differentiability conditions that we will specify, we will show that a field **F is conservative if and only if it is the gradient field of a scalar function** f — that is, if and only if $F = \nabla f$ for some f.

- The function f then has a **special name**.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.

Assumptions on Curves, Vector Fields, and Domains

• Once we have found a **potential function** f for a field $\mathbf{F}$, we **can evaluate all the line integrals** in the domain of **F over any path between** A **and** B by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

• The curves we consider are **piecewise smooth**.

• We consider vector fields **F** whose components have **continuous first partial derivatives**.

• The domains D we consider are **connected**: this means that any two points in D can be joined by a smooth curve that lies in the region. Being in **"one piece"**

• Some results require D to be **simply connected**, which means that every loop in D can be contracted to a point in D without ever leaving D. **Not having any "loop-catching holes."**

• Like the Fundamental Theorem of Calculus, Theorem 1 gives a **direct way to evaluate line integrals**

THEOREM 1—Fundamental Theorem of Line Integrals
Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Proof of Theorem 1 Suppose that A and B are two points in the region D and that C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \leq t \leq b$, is a smooth curve in D joining A to B. In Section 14.5 we found that the derivative of a scalar function f along a path C is the dot product $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$, so we have

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \\
&= \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt && \text{Eq. (2) of Section 16.2 for computing } d\mathbf{r} \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt && \text{Eq. (7) of Section 14.5 giving derivative along a path} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) && \text{Fundamental Theorem of Calculus} \\
&= f(B) - f(A). && \mathbf{r}(a) = A,\ \mathbf{r}(b) = B \qquad \blacksquare
\end{aligned}
$$

Conservative Fields

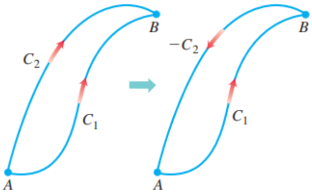
THEOREM 2—Conservative Fields are Gradient Fields
Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

THEOREM 3—Loop Property of Conservative Fields
The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

- The following diagram **summarizes the results of Theorems 2 and 3**.

$$\mathbf{F} = \nabla f \text{ on D} \quad \overset{\text{Theorem 2}}{\Longleftrightarrow} \quad \mathbf{F} \text{ conservative on D} \quad \overset{\text{Theorem 3}}{\Longleftrightarrow} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any loop in D}$$

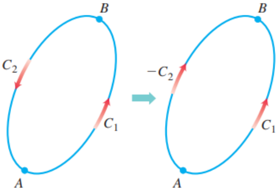


Proof that Part 1 ⇒ Part 2 We want to show that for any two points A and B in D, the integral of $\mathbf{F}\cdot d\mathbf{r}$ has the same value over any two paths C_1 and C_2 from A to B. We reverse the direction on C_2 to make a path $-C_2$ from B to A (Figure 16.27). Together, C_1 and $-C_2$ make a closed loop C, and by assumption,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C} \mathbf{F}\cdot d\mathbf{r} = 0.$$

Thus, the integrals over C_1 and C_2 give the same value. Note that the definition of $\mathbf{F}\cdot d\mathbf{r}$ shows that changing the direction along a curve reverses the sign of the line integral.

Proof of Theorem 3



Proof that Part 2 ⇒ Part 1 We want to show that the integral of $\mathbf{F} \cdot d\mathbf{r}$ is zero over any closed loop C. We pick two points A and B on C and use them to break C into two pieces: C_1 from A to B followed by C_2 from B back to A (Figure 16.28). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} - \int_A^B \mathbf{F} \cdot d\mathbf{r} = 0. \qquad \blacksquare$$

EXAMPLE 2 Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

in moving an object along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Solution With $f(x, y, z) = xyz$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \text{ and path independence} \\
&= f(B) - f(A) && \text{Theorem 1} \\
&= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} \\
&= (1)(6)(-4) - (-1)(3)(9) \\
&= -24 + 27 = 3.
\end{aligned}$$

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Conservative Fields

Two questions arise:

1. **How do we know** whether a given vector field **F** is **conservative**?

2. If F is in fact conservative, **how do we find** a **potential function** f (so that $\mathbf{F} = \nabla f$)?

Finding Potentials for Conservative Fields

Component Test for Conservative Fields
Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

• We can view the component test as saying that on a simply connected region, the **vector**

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

• **is zero if and only if F is conservative**.
• This interesting vector is called the **curl of F**

Finding Potentials for Conservative Fields

- Once we know that **F** is conservative, we often want to **find a potential function** for **F**.
- This requires **solving the equation** $\nabla f = F$ or

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

- for f.
- We accomplish this by **integrating the three equations**

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P,$$

EXAMPLE 3 Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

Solution The natural domain of $\mathbf{F}$ is all of space, which is open and simply connected. We apply the test in Equations (2) to

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

The partial derivatives are continuous, so these equalities tell us that $\mathbf{F}$ is conservative, so there is a function f with $\nabla f = \mathbf{F}$ (Theorem 2).

We find f by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z. \tag{3}$$

We integrate the first equation with respect to x, holding y and z fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

Finding Potentials for Conservative Fields

We write the constant of integration as a function of y and z because its value may depend on y and z, though not on x. We then calculate $\partial f/\partial y$ from this equation and match it with the expression for $\partial f/\partial y$ in Equations (3). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g/\partial y = 0$. Therefore, g is a function of z alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f/\partial z$ from this equation and match it to the formula for $\partial f/\partial z$ in Equations (3). This gives

$$xy + \frac{dh}{dz} = xy + z, \qquad \text{or} \qquad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We found infinitely many potential functions of **F**, one for each value of C. ■