MA3121 Complex Analysis
Lecture Notes

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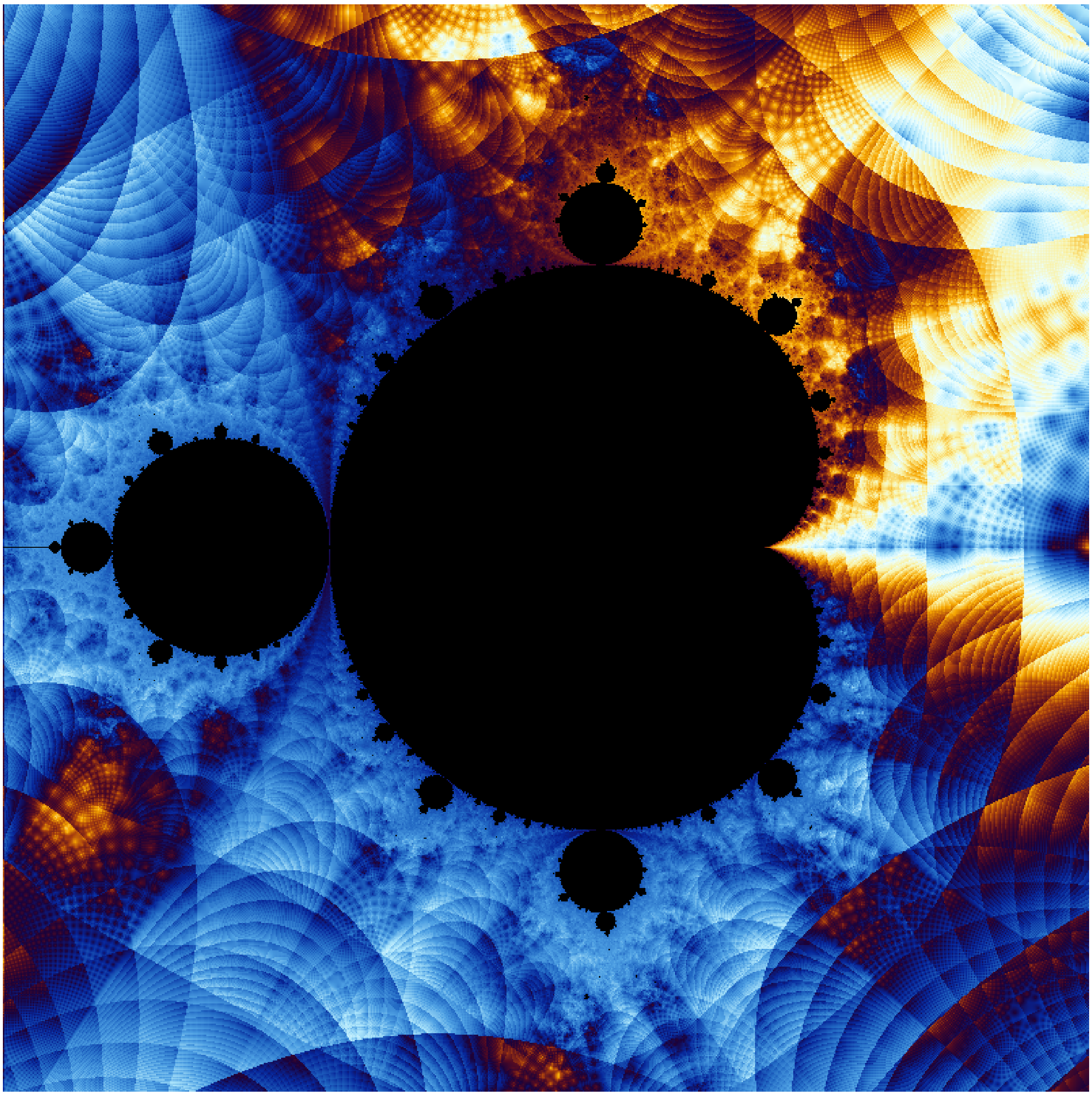


Contents

Preamble

The Notes

These notes are *not* verbatim what I will say in the lectures and what is on the slides, but the content is exactly the same. The main difference is that the notes have more complete sentences, and they may have some additional comments.

These lecture notes also give you a guide about the material of this module when reading in the book from the essential reading list:

J. M. Howie, Complex Analysis, Springer.
University of Leicester Library e-link:
`https://ebookcentral.proquest.com/lib/leicester/detail.action?docID=3075013`

If you find any errors and typos in the notes, please do let me know: k.leschke@leicester.ac.uk

How to use the notes

You can read ahead of lectures, you can use them for revision, you can use them to look up a little detail and probably in many more ways. These lecture notes also give you a guide about the material of this module when reading in the essential books. Some students like using these notes when doing exercises, because they are (automatically) searchable. It is up to you to find out how they are most useful. So my advice is to follow one of these three possibilities: take your own notes in lectures; read ahead and then listen in lectures knowing you've already seen it in the notes; listen in lectures hoping that it will all be written in the notes.

CHAPTER 1

Introduction to Complex Analysis

Complex Analysis is simultaneously a *classical* and *modern* topic in mathematics. First steps into the theory can be found in the 18th century by Euler who worked intensively with complex numbers. The basics of the modern theory were developed in the 19th century. Main contributors are Cauchy, Riemann and Weierstrass: all of them had very distinctive points of view and approaches to the theory which we will encounter during this module. Modern applications are in both pure and applied mathematics, e.g. in geometry, number theory, and complex dynamics. There are also many applications in physics, aerospace engineering, biomedical sciences, google (!), geography, biology and more.

What is Complex Analysis?

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. Surprisingly, *holomorphic functions*, that is, complex differentiable functions of a complex variable, are much “nicer” than differentiable functions in real variables!

From Cauchy’s point of view, a holomorphic function is essentially a function which is complex differentiable, with continuous derivative. He developed an integral representation of a holomorphic function and this way methods from analysis can be used: we will study the Cauchy integral formula, and its consequences, in Chapter 6, 7.

Riemann had a more geometric point of view: for him holomorphic functions were maps between areas, or more generally nowadays so–called Riemann surfaces, which are similar in their corresponding smallest parts. Riemann based his approach on physical experiences which motivated the existence of so–called conformal maps: his approach is rather intuitive and geometric. We will see aspects of this in Chapter 3.

Finally, for Weierstrass a holomorphic function are functions which are locally given by a convergent complex power series. We will see this aspect in Chapter 3 and 8.

What are the ingredients we need from previous modules?

We will need an understanding of complex numbers (real part, imaginary part, length, conjugate, polar coordinates, Euler’s formula), basic understanding of calculus (what is a differentiable function, continuous function, integrals) and topology (what is an open/closed set in the complex plane). We will go over this quickly in the next chapter but if you need more practice you can look at the material in the Appendices A, B or the book by Howie (chapter 1, 2 and 3).

CHAPTER 2

Preliminaries

In this chapter we will quickly recap some core properties of complex numbers and functions of one complex variable which are needed for the module. For more details and further examples see Appendices A and B and the book by Howie (chapter 1, 2 and 3).

1. Complex numbers

Why do we need complex numbers?

We can give two important answers:

(1) When asking to find solution of quadratic equations

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$, we obtain the general solution by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the *discriminant* $\Delta := b^2 - 4ac$ is negative, there is no real solution. However, if we postulate the existence of the number $i := \sqrt{-1}$, we get a formal solution: the solution is a complex number $x \in \mathbb{C}$. E.g., the quadratic equation

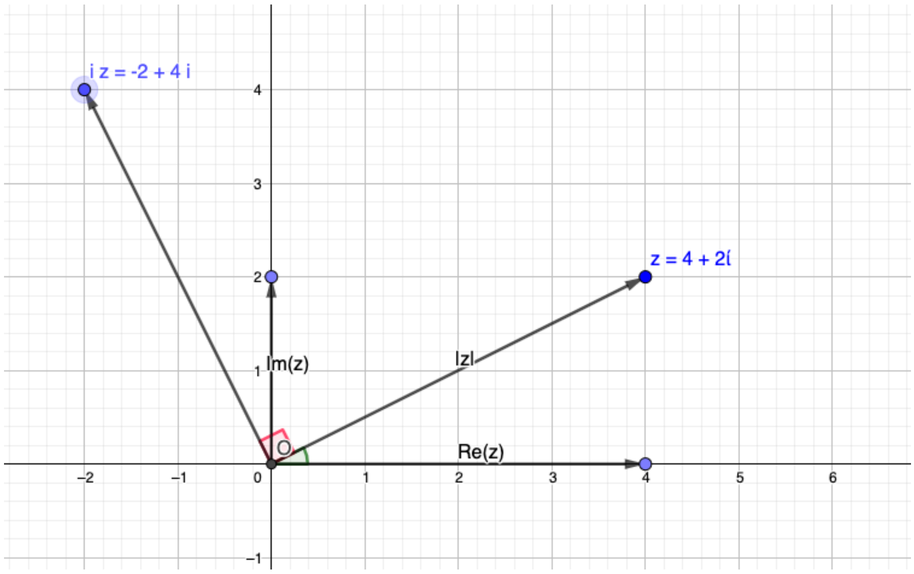
$$x^2 + 4x + 13 = 0$$

has two solutions $x = -2 \pm 3i$. More generally, the Fundamental Theorem of Algebra states that indeed any polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

with complex coefficients a_i has all its solutions in the complex numbers: complex numbers allow to solve all polynomial equations.

(2) Complex number describe geometric operations in the plane. For this we identify $\mathbb{C} = \mathbb{R}^2$ via $z = x + iy \in \mathbb{C} \longmapsto (x, y) \in \mathbb{R}^2$.



Then the map

$$z \in \mathbb{C} \longmapsto iz \in \mathbb{C}$$

is the 90 degree rotation in the plane: the vector $(x, y) \in \mathbb{R}^2$ is mapped to $(-y, x) \in \mathbb{R}^2$ since for $z = x + iy$ we have $iz = -y + ix$. We also know that the 90 degree rotation is given by the rotation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with $\theta = \frac{\pi}{2}$, that is,

$$R_{\frac{\pi}{2}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

which shows our claim. More generally, the polar representation of vectors in the plane allows to identify multiplication by complex numbers as a combination of rotations and scalings.

We now recall *key properties of complex numbers* (see also Appendix A and B):

Definition. Let $\mathbb{C} = \{z = a + ib \mid a, b \in \mathbb{R}\}$ be the set of all numbers $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 := -1$. We define:

- (A) Addition:

$$+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (a_1 + b_1 i) + (a_2 + b_2 i) := (a_1 + a_2) + (b_1 + b_2)i.$$

- (M) Multiplication:

$$\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (a_1 + b_1 i) \cdot (a_2 + b_2 i) := (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i.$$

$\mathbb{C}$ together with this addition and multiplication is called the **field of complex numbers**.
If $z = a + ib$ then the **real part** is

$$\operatorname{Re} z = a,$$

and the **imaginary part** is

$$\operatorname{Im} z = b\,.$$

The **conjugate** is given by

$$\bar{z} = a - ib\,.$$

The **length**, or **modulus**, is given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

The **distance** between two complex numbers $z, w \in \mathbb{C}$ is given by

$$d(z, w) = |z - w|\,.$$

In particular, we have

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$$

and

$$\bar{z} = z \Longleftrightarrow \operatorname{Im} z = 0\,,$$

that is, z is real, and similarly, z is purely imaginary iff

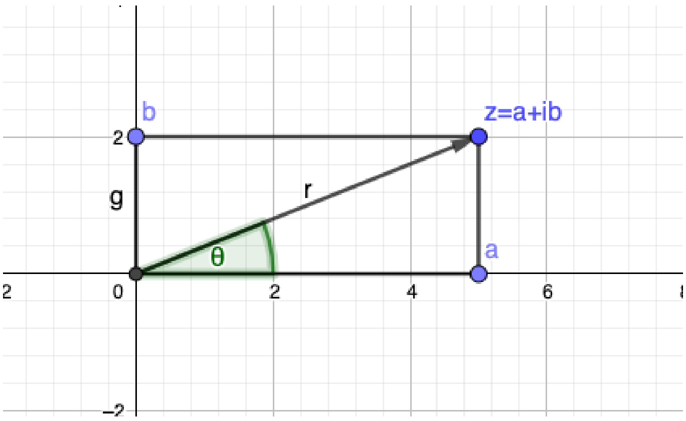
$$\bar{z} = -z \Longleftrightarrow \operatorname{Re} z = 0\,.$$

A complex number $z = a + ib$ can be written in polar coordinates as

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and the **argument** $\arg z = \theta \in (-\pi, \pi]$ is the angle with

$$\cos\theta = \frac{a}{r}, \quad \sin\theta = \frac{b}{r}\,.$$



Using the power series of cos and sin for real parameter $x \in \mathbb{R}$, we have

$$e^{ix} = \cos x + i\sin x$$

so that we see Euler's formula

$$z = re^{i\theta}\,.$$

We recall that

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

and

$$e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin(\theta)$$

so that for $z = re^{i\theta}$

$$\bar{z} = re^{-i\theta}, \qquad z^{-1} = \frac{1}{r}e^{-i\theta} = \frac{\bar{z}}{|z|^2}\,.$$

We also obtain the usual triangle inequality:

THEOREM 2.1. *Let $z, w \in \mathbb{C}$. Then we have:*

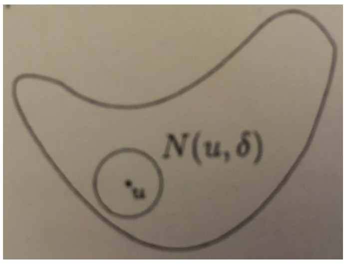
(i) $|\mathrm{Re}\,(z)| \leqslant |z|$, $|\mathrm{Im}\,(z)| \leqslant |z|$, $|\overline{z}| = |z|$.
(ii) $|zw| = |z||w|$.
(iii) $|z + w| \leqslant |z| + |w|$.
(iv) $|z - w| \geqslant ||z| - |w||$.

To discuss limits of functions in the complex plane we need to replace the notion of open/closed interval by corresponding open sets in the complex plane. We already discussed open sets in $\mathbb{R}^2$ in Vector Calculus, so we can just "translate" the notions to the complex notation by our identification $\mathbb{R}^2 = \mathbb{C}$:

Definition. Let $c \in \mathbb{C}$, $r \in \mathbb{R}$ with $r > 0$. The **neighbourhood** $N(c, r)$ with radius r around c is defined as

$$N(c, r) := \{z \in \mathbb{C} : |z - c| < r\}.$$

We will use following notations and terminolgy:
(i) $N(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ an **open neighbourhood** or **open disc**.
(ii) $\overline{N}(a, r) := \{z \in \mathbb{C} : |z - a| \leqslant r\} = \overline{N(a, r)}$ a **closed disc**.
(iii) $\kappa(a, r) := \{z \in \mathbb{C} : |z - a| = r\} = \partial N(a, r) = \partial \overline{N}(a, r)$ a **circle**.
(iv) $D'(a, r) := \{z \in \mathbb{C} : 0 < |z - a| < r\} = N(a, r)\backslash\{a\}$ a **punctured disc**.



Definition. A subset $U \subset \mathbb{C}$ is called **open** if for all $u \in U$ there exists $\delta > 0$ such that $N(u, \delta) \subset U$. A subset $D \subset \mathbb{C}$ is called **closed** if its complement $\mathbb{C}\backslash D$ in $\mathbb{C}$ is open, i.e. for all $z \notin D$ there exists $\delta > 0$ such that $N(z, \delta)$ lies wholly outside D.

2. Functions in one complex variable

Aim: We aim to study functions from the complex numbers with values in the complex numbers and analyse their fundamental properties like continuity, differentiability or integrability.

Let us look at complex valued functions $f : \mathbb{C} \to \mathbb{C}$. If $z \in \mathbb{C}$ with $z = x + iy$ and $f : \mathbb{C} \to \mathbb{C}$ is a function, then there exist real valued functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(z) = u(x, y) + iv(x, y).$$

We call

$$u = \mathrm{Re}\,(f)$$

the **real** part of f and

$$v = \mathrm{Im}\,(f)$$

the **imaginary** part of f. The **modulus function** is

$$|f| : \mathbb{C} \to \mathbb{R},\ |f|(z) := |f(z)| = \sqrt{u(x, y)^2 + v(x, y)^2}.$$

EXAMPLE 2.2. Let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then we have:

$$f(z) = (x + iy)^2 = (x^2 - y^2) + i(2xy).$$

So we have:

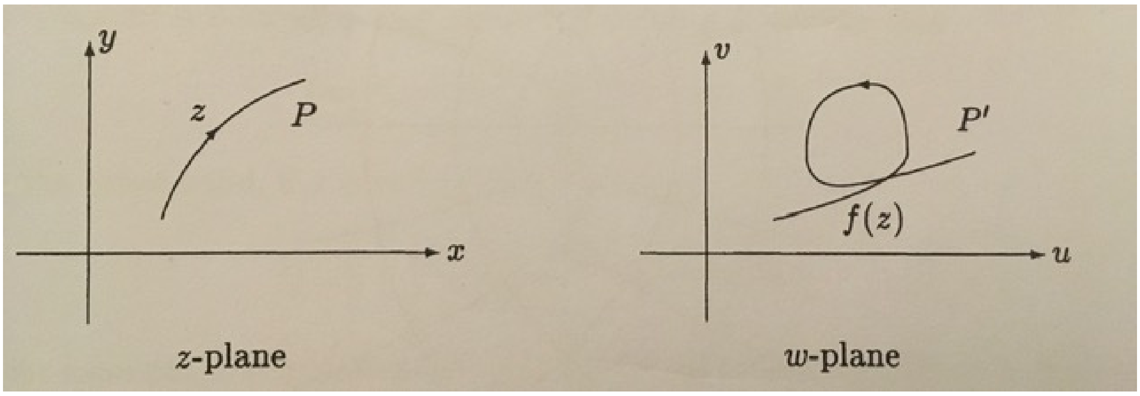
$$u : \mathbb{R}^2 \to \mathbb{R},\ u(x, y) = x^2 - y^2,\ \ v : \mathbb{R}^2 \to \mathbb{R},\ v(x, y) = 2xy.$$

Moreover,

$$|f|(z) = |f(z)| = |z|^2 = x^2 + y^2\,.$$

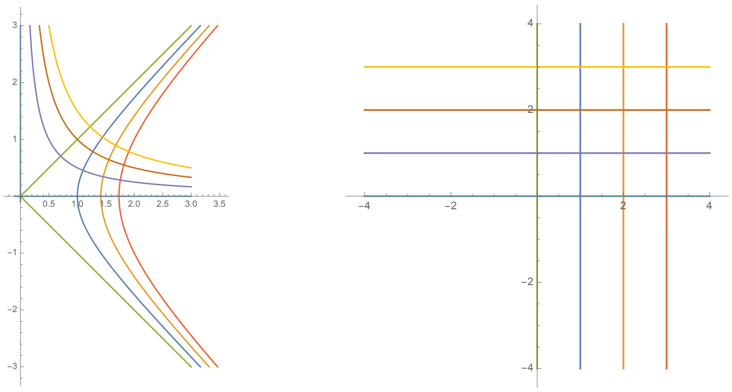
REMARK. We can't draw graphs of complex functions $f : \mathbb{C} \to \mathbb{C}$ as we do with graphs of real functions, because the graph $\{(z, f(z)) : z \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C}$ is 4-dimensional. But we can use its modulus function $|f|$ to visualise the geometry of the function f.

In order to picture a complex function $f : \mathbb{C} \to \mathbb{C}$ it is also useful to picture $z = x + iy$ and $w := f(z) = u + iv$ in two complex planes. For example, we could picture the image P' under f of a path P in the z-plane in the w-plane. As the point z moves along the path P in the z-plane, its image $f(z)$ moves along the path P' in the w-plane.



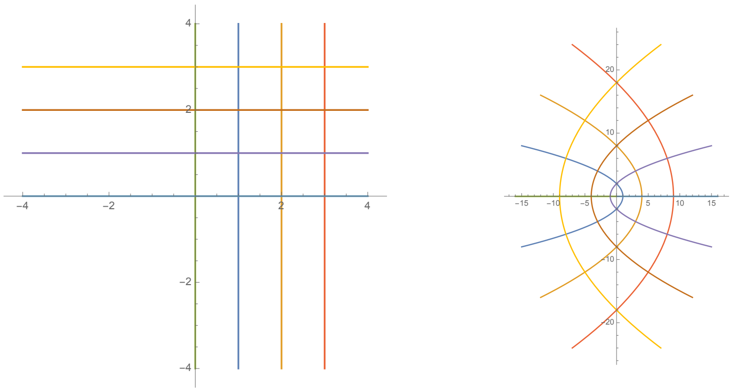
EXAMPLE 2.3. For the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$, the hyperbolic curves $x^2 - y^2 = k$ and $2xy = l$ in the z-plane transform respectively to the straight lines in the w-plane

$$u = k,\ \ v = l.$$



Similarly, one can check that the straight lines $x = k$ and $y = l$ in the z-plane transform respectively to the parabolic curves in the w-plane

$$v^2 = 4k^2(k^2 - u),\ \ v^2 = 4l^2(u + l^2).$$



There are several other methods for visualising complex functions. Let us introduce some of them. You can find more in the web.

Method I. Associate to each point $z = x + iy = re^{i\theta}$ of the complex plane $\mathbb{C}$ the real part $Re(f(z))$ and the imaginary part $Im(f(z))$ of the value $f(z)$ for the function f. As the real part and the imaginary part of $f(z)$ are both real values we get a $3D$-landscape for both parts which give a description of f and allow to see for example roots or singularities.

Here is a weblink (using GeoGebra): `https://www.geogebra.org/m/s7trZGyR`

Method II. Associate to each point $z = x + iy = re^{i\theta}$ of the complex plane $\mathbb{C}$ the absolute value or modulus $|f(z)|$. This again gives a real number and can be interpreted as the **height** of the value $f(z)$ like the height of a $3D$-landscape.

Here is a weblink (using Mathematica and Wolfram Alpha): `https://www.wolframalpha.com/input/?i=plot`

Method III. Associate to each point $z = x + iy = re^{i\theta}$ of the complex plane $\mathbb{C}$ a colour scheme with at least two parameters corresponding to the variable z and $f(z)$. For example we can use internal computer colour schemes like **HSL** given by three parameters **H**ue, **S**aturation, and **L**ightness. The best is to use hue and lightness. Hue associates a particular colour to each point in the complex plane depending on the angle θ and lightness a particular value of dark or light of the colour to each point in the complex plane depending on the value r. Saturation will be fixed at a constant value to ensure good colour saturation, normally taking the highest possible value. The result will be a $2D$-graph with colours attached to each point of the complex plane indicating the values of $f(z)$ by hue and lightness. There are also more sophisticated methods using the geometry of **conformal mappings** and visualising complex functions also as mappings from the Riemann sphere, so including the point at infinity.

Here is a weblink (using HSL): `https://jutanium.github.io/ComplexNumberGrapher/`
Here is another weblink (using HSL and conformal mappings): `http://davidbau.com/conformal/`

3. Limits

We can use the decomposition of a function in a complex variable into two real-valued functions with two real parameters, the real and imaginary part, to transfer all limit theorems we have for real-valued functions with two real parameters to the complex case. For completeness, here are the main results:

DEFINITION (Limit & continuity). Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $l, c \in \mathbb{C}$. We say that

$$\lim_{z \to c} f(z) = l$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - l| < \epsilon$$

for all z such that $0 < |z - c| < \delta$.

A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous at a point** $c \in \mathbb{C}$ if

$$\lim_{z \to c} f(z) = f(c),$$

i.e. if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(c)| < \epsilon$$

for all $z \in D'(c, \delta) = \{z \in \mathbb{C} : 0 < |z - c| < \delta\}$. The function f is **continuous** if it is continuous at *every* point $c \in \mathbb{C}$.

REMARK. The standard rules for calculus with limits hold verbatim for complex functions as they hold for real functions.

If $\lim_{z \to c} f(z) = l$ and $\lim_{z \to c} g(z) = m$, then the functions $kf(z), f(z) \pm g(z), f(z)g(z)$ and $\frac{f(z)}{g(z)}$ (if $m \neq 0$) have limits $kl, l \pm m, lm, \frac{l}{m}$ respectively.

The continuity of f and g at $c \in \mathbb{C}$ implies continuity of $kf, f \pm g, fg, \frac{f}{g}$ (unless $g(c) = 0$) at c.

Question: What about limits for $z \to \infty$? There are many paths to infinity on the complex plane!

DEFINITION (Limit). Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $L \in \mathbb{C}$. We say that

$$\lim_{z \to \infty} f(z) = L$$

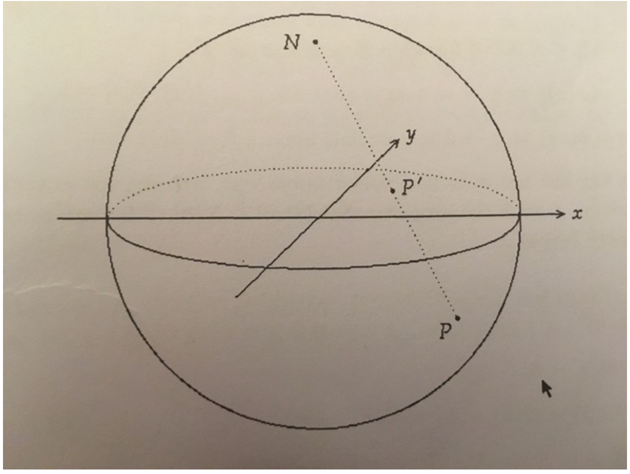
if for every $\epsilon > 0$ there exists a real number $K > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > K$.

DEFINITION (Limit). Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say that

$$f(z) \to \infty \ (z \to \infty)$$

if for every real number $E > 0$, there exists a real number $D > 0$ such that $|f(z)| > E$ whenever $|z| > D$.

REMARK. It is useful to think of ∞ as a single extra point and to extend the complex plane by adjoining this point i.e. look at $\mathbb{C} \cup \{\infty\}$. We can think of the complex plane as the equatorial plane of a sphere of radius 1 with north pole N being the point ∞.



For each point P on the sphere we let P' be the point where the line $\overline{NP}$ from the north pole N to P intersects with the equatorial plane. This gives a bijective correspondence of the points P on the sphere (except N) with the points on the equatorial x-y-plane.

Remark. We can visualise the complex numbers $\mathbb{C}$ as points on a sphere, the **Riemann sphere** and the 'missing point' $N = \infty$ is the **point at infinity**.

Remark. Very often in praxis the limit

$$\lim_{z\to\infty} f(z)$$

is the same as the limit

$$\lim_{|z|\to\infty} f(z),$$

which is easier to calculate.

CHAPTER 3

Differentiation

1. Holomorphic Functions

Question. How to define differentiability for complex functions?
Answer. Essentially as for real functions!

DEFINITION. A complex function $f : \mathbb{C} \to \mathbb{C}$ is **differentiable at a point** $c \in \mathbb{C}$ if the limit

$$f'(c) := \lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

exists. This limit is called the **derivative of f at** c and denoted by $f'(c)$.

REMARK. We often will use the equivalent formulation

$$f'(c) := \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

REMARK. Although this definition is formally identical to the one for differentiability of real functions in one variable, differentiability for complex functions requires the rate of change of f and its limit $z \to c$ to be the same for all possible directions in the complex plane $\mathbb{C}$!
On the other hand, when comparing to functions $f(x, y)$ of two real parameters to $\mathbb{R}^2$, do we obtain the same notion as in analysis/vector calculus? Recall that in this case, we also had to approach a point $c \in \mathbb{C} = \mathbb{R}^2$ from all possible paths, but the definition of the differential looked different: the function can be approximated by the linear function given by the Jacobi matrix.
More precisely, the Jacobi matrix $J_f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is given by

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

and satisfies

$$\lim_{(h_1,h_2) \longrightarrow 0} \frac{f(x + h_1, y + h_2) - f(x, y) - J_f(x, y) \cdot (h_1, h_2)}{|(h_1, h_2)|} = 0.$$

We will soon see that there is indeed a big difference between complex differentiability of a function in one complex variable and real differentiability of a function in two real variables: rather than approximating by a linear map over the reals, the functions is approximated by a linear map over the complex numbers! As a result, complex differentiable functions have much nicer properties.

However, many properties immediately carry over from the case of differentiable functions in one variable. We first observe the following:

THEOREM 3.1 (Rules of Differentiation).

(i) **Linearity Rule.** *If $f(z) = ku(z) + lv(z)$, with $k, l \in \mathbb{C}$ constants, then*

$$f'(z) = ku'(z) + lv'(z).$$

(ii) **Product Rule.** *If* $f(z) = u(z)v(z)$, *then*

$$f'(z) = u'(z)v(z) + u(z)v'(z).$$

(iii) **Quotient Rule.** *If* $f(z) = \frac{u(z)}{v(z)}$, *where* $v(z) \neq 0$, *then*

$$f'(z) = \frac{v(z)u'(z) - u(z)v'(z)}{(v(z))^2}.$$

PROOF. The proof is verbatim as for real functions. □

THEOREM 3.2 (Chain Rule). *If* $f(z) = u(v(z))$, *then*

$$f'(z) = u'(v(z)) \cdot v'(z).$$

PROOF. The proof is verbatim as for real functions. □

We have the following:

THEOREM 3.3.

(i) **Polynomial functions.** *Every polynomial function*

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

with $a_0, \ldots a_n \in \mathbb{C}$ *is differentiable at every point* $z \in \mathbb{C}$.

(ii) **Rational functions.** *Every rational function*

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ *and* $q(z)$ *are polynomial functions with no common factor is differentiable everywhere except at the points* $z \in \mathbb{C}$ *with* $q(z) = 0$.

We now investigate complex differentiability in terms of partial derivatives of the real and imaginary part. Let us write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ with $z = x + iy$ and $x, y \in \mathbb{R}$. Let also $c = a + ib$ with $a, b \in \mathbb{R}$.

We have with b fixed:

$$\frac{f(x + ib) - f(a + ib)}{(x + ib) - (a + ib)} = \frac{u(x, b) - u(a, b)}{x - a} + i\frac{v(x, b) - v(a, b)}{x - a}.$$

Hence the existence of the limit $f'(c)$ implies the existence of the limits

$$\lim_{x \to a} \frac{u(x, b) - u(a, b)}{x - a} \text{ and } \lim_{x \to a} \frac{v(x, b) - v(a, b)}{x - a},$$

i.e. the existence at the point (a, b) of the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

Moreover we also have:

$$f'(c) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Now let's keep a fixed:

$$\begin{aligned}\frac{f(a+iy)-f(a+ib)}{(a+iy)-(a+ib)} &= \frac{u(a,y)-u(a,b)}{i(y-b)}+\frac{i(v(a,y)-v(a,b))}{i(y-b)}\\ &= \frac{v(a,y)-v(a,b)}{y-b}-i\frac{u(a,y)-u(a,b)}{y-b}.\end{aligned}$$

Hence the differentiability of the function f at the point c implies the existence of the partial derivatives $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$.

Again we have

$$f'(c)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}.$$

Combining the two expressions for $f'(c)$ we obtain the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Theorem 3.4 (Cauchy-Riemann). *Let $f:\mathbb{C}\to\mathbb{C}$ be a complex function, which is differentiable at $c=a+ib$, and suppose that $f(x+iy)=u(x,y)+iv(x,y)$, where $x,y,u(x,y),v(x,y)$ are real. Then the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}$ all exist at the point $(a,b)\in\mathbb{R}^2$ and we have:*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Remark. *In particular, we see that a differentiable function $f:\mathbb{C}\longrightarrow\mathbb{C}$ has Jacobi matrix*

$$J_f=\begin{pmatrix}\frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x}\end{pmatrix}$$

when viewed as a real function $f(x,y)=(u(x,y),v(x,y))$ in two real parameters. Put differently, complex differentiable functions are much more restricted than real differentiable functions in two real parameters!

Example 3.5. Verify the Cauchy-Riemann equations for the function

$$f:\mathbb{C}\to\mathbb{C},\ f(z)=z^2.$$

Solution. We have:

$$u(x,y)=x^2-y^2,\ \ v(x,y)=2xy.$$

And so we get:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x,\ \ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=2y.$$

So the Cauchy-Riemann equations are satisfied.

Question. Is this necessary condition for differentiability of a complex function $f:\mathbb{C}\to\mathbb{C}$ at a point $c\in\mathbb{C}$ also sufficient?
Answer. No, not in general!

Example 3.6. Let $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ where

$$u(x,y)=\sqrt{|xy|},\ \ v(x,y)=0.$$

Show that the Cauchy-Riemann equations are satisfied at $z=0$, but that the function f is *not* differentiable at that point $z=0$.

Solution:

We have: $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ at the point $(0,0)$, because,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x\to 0} \frac{0}{x} = 0,$$

and similarly $\frac{\partial u}{\partial y} = 0$ holds.

Thus the Cauchy-Riemann equations are satisfied.

On the other hand,

$$\frac{f(z) - f(0)}{z - 0} = \frac{\sqrt{|xy|}}{x + iy} = \frac{\sqrt{|\cos\theta \sin\theta|}}{\cos\theta + i\sin\theta} = \sqrt{|\cos\theta\sin\theta|}e^{-i\theta}.$$

with $x = r\cos\theta$, $y = r\sin\theta$.
The term on the right is independent of r, so $\frac{f(z)-f(0)}{z-0}$ takes this constant value for all points on the line $x\sin\theta - y\cos\theta = 0$ arbitrarily close to 0.

For $\theta = 0$ or $\theta = \frac{\pi}{2}$ this constant value is 0, but for $\theta = \frac{\pi}{4}$ the constant value is $\frac{1-i}{2}$, so the limit does *not* exist, so f is *not* differentiable at the point $z = 0$.

So we learned that it is not enough for a function to satisfy the Cauchy–Riemann equations. However, there is a handy sufficient condition to obtain differentiability:

Theorem 3.7. *Let $D = N(c, R)$ be an open disc in $\mathbb{C}$. Let f a complex function whose domain contains D. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and suppose that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist and are continuous throughout D. Suppose that the Cauchy-Riemann equations are satisfied at the point c. Then f is differentiable at c.*

Proof. The proof is similar to the corresponding proof for differentiability of functions in two real variables. □

We also obtain the criterion to check for holomorphicity:

Theorem 3.8. *Let $D = N(c, R)$ be an open disc in $\mathbb{C}$. Let f a complex function whose domain contains D. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and suppose that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ exist and are continuous throughout D. Suppose that the Cauchy-Riemann equations are satisfied at all points in D. Then f is holomorphic in D.*

Let's return to our previous example

Example 3.9. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where

$$u(x, y) = \sqrt{|xy|},\ \ v(x, y) = 0.$$

In this example the partial derivatives are *not* continuous at the origin $(0, 0)$ so we cannot apply the above theorems:

If $x, y > 0$, then

$$\frac{\partial f}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}},\ \ \frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}}.$$

None of these has a limit as $(x, y) \to (0, 0)$.
Note however, that the continuity of the partial derivatives is not necessary for differentiability so we had to check the limit as we have done.

DEFINITION. Let U be an open set in $\mathbb{C}$. A complex function $f : \mathbb{C} \to \mathbb{C}$ is called **holomorphic in** U if it is differentiable at every point in U.
A complex function $f : \mathbb{C} \to \mathbb{C}$, which is differentiable at every point in $\mathbb{C}$ is called an **entire** function.

EXAMPLE 3.10. Every polynomial function

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_0, \dots a_n \in \mathbb{C}$ is an entire function.

EXAMPLE 3.11. The rational function

$$f(z) = \frac{z+i}{z+1}$$

is holomorphic in the open set $\mathbb{C}\backslash\{-1\}$.

EXAMPLE 3.12. Show that

$$f : \mathbb{C} \to \mathbb{C},\ f(z) = z^2$$

is an entire function.

Solution. We have:

$$u(x, y) = x^2 - y^2,\ \ v(x, y) = 2xy.$$

And so we get:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x,\ \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y.$$

So the partial derivatives are continuous and the Cauchy-Riemann equations are satisfied. We have for the derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2(x + iy) = 2z.$$

EXAMPLE 3.13. Show that the function

$$f : \mathbb{C}\backslash\{0\} \to \mathbb{C},\ f(z) = \frac{1}{z}$$

is holomorphic in its domain by using partial derivatives.

Solution. We have:

$$u(x, y) = \frac{x}{x^2 + y^2},\ \ v(x, y) = \frac{-y}{x^2 + y^2}$$

And so we get:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2},\ \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{2xy}{(x^2 + y^2)^2}.$$

The partial derivatives are continuous away from $(x, y) = (0, 0)$ and thus f is holomorphic with

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{(y^2 - x^2) + 2ixy}{(x^2 + y^2)^2}.$$

We can calculate the derivative also directly using the quotient rule:

$$f'(z) = -\frac{1}{z^2} = -\frac{\overline{z}^2}{(z\overline{z})^2} = \frac{(y^2 - x^2) + 2ixy}{(x^2 + y^2)^2}.$$

REMARK. *Note: we use results from Calculus/Vector Calculus for functions in real variables. For example, we know that any polynomial p in two variables, or any rational function $r = \frac{p}{q}$ (away from the zeros of q) in two variables, is continuous: this follows from the limit laws. Therefore, in the example above, we see that the two functions u and v are continuous away from the zeros of the polynomial $q(x, y) = (x^2 + y^2)^2$.*

EXAMPLE 3.14. Show that the function $f : \mathbb{C} \to \mathbb{C},\ f(z) = |z|^2$ is differentiable only at the point $c = 0$.

Solution. We have:

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0.$$

And so we get:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y.$$

The Cauchy-Riemann equations are only satisfied at $z = 0$, i.e. differentiability fails at all points $z \neq 0$.
To verify differentiability for $z = 0$ we have to look at the limit for $z \to 0$:

$$\frac{|z|^2 - |0|^2}{z - 0} = \frac{z\overline{z}}{z} = \overline{z} \to 0$$

as $z \to 0$, so we get:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2 - |0|^2}{z - 0} = \lim_{z \to 0} \overline{z} = 0.$$

REMARK. In Real Analysis the real functions $f(x) = x^2$ and $f(x) = |x|^2$ are identical functions, but in Complex Analysis the situation is different! The function $f(z) = z^2$ only depends on z, but $f(z) = |z|^2 = z\overline{z}$ depends on both z and $\overline{z}$ which is a new phenomena not visible for real functions.

Let us express the complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ of two real variables x and y as a function of the two complex variables z and $\overline{z}$ using:

$$x = \frac{1}{2}(z + \overline{z}), \quad y = \frac{1}{2i}(z - \overline{z}).$$

Then we obtain:

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Similarly, we can define

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

Note that the $\bar{z}$ derivative is equal to 0 (so that f is a function of z only) if and only if

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0,$$

i.e. if and only if

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0,$$

i.e. if and only if the Cauchy-Riemann equations are satisfied.
We therefore obtain:

THEOREM 3.15. *Let $f : U \longrightarrow \mathbb{C}$ be real differentiable. Then f is differentiable at $c \in U$ if and only if*

$$\frac{\partial f}{\partial \overline{z}} = 0\,.$$

In this case,

$$f'(c) = \frac{\partial f}{\partial z}(c)\,.$$

PROOF. Since f is real differentiable, it is holomorphic if and only if the Cauchy–Riemann equation hold, which we have already showed to be equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$. Moreover, the Cauchy–Riemann equation give

$$\frac{\partial f}{\partial z} = \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) = u_x + iv_x$$

so that the result follows from

$$f'(c) = u_x(c) + iv_x(c) .$$

□

REMARK. Again, we observe that the Cauchy-Riemann equations show case the difference between a true function of the complex variable z and something that is merely a complex-valued function of two variables! In the example $f(x, y) = |x+iy|^2 = x^2+y^2$ we see that f is differentiable at every point as a function in two real parameter; but it is not differentiable (holomorphic) as a function in one complex parameter!

Let us recall from Calculus and Real Analysis the following important property:

THEOREM 3.16. *Let $f : [a, b] \to \mathbb{R}$ be a real function, continuous in the interval $[a, b]$ and differentiable in the interval (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function, i.e. $f(x) = c$ for some $c \in \mathbb{R}$ and all $x \in [a, b]$.*

Question. Is there an analogue for complex functions?

THEOREM 3.17. *Let $f : U \to \mathbb{C}$ be a complex function, holomorphic in a neighbourhood $U = N(a_0, R)$. If $f'(z) = 0$ for all $z \in$ U, then f is a constant function, i.e. $f(z) = c$ for some $c \in \mathbb{C}$ and all $z \in U$.*

PROOF. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then we have:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

and hence $f'(z) = 0$ implies:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

at every point $z \in U$.
Let $p = a + bi$ and $q = c + di$ be any points in U. Then at least one of the points $r = a + di$ and $s = c + bi$ also lies in U (Excercise!).

Without loss of generality, assume $r \in U$. Then both functions $x \longmapsto u(x, d)$ and $y \longmapsto u(a, y)$ are real functions with zero derivative and therefore are constant. Thus

$$u(a, b) = u(a, d) = u(c, d)$$

and with a similar argument $v(a, b) = v(c, d)$ and hence f is constant. □

Let us recall from Calculus and Real Analysis the following important property:

THEOREM 3.18 (The Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a real function, continuous in the interval $[a, b]$ and differentiable in the interval (a, b). If $x \in (a, b)$, then there exists $\xi \in (a, b)$ such that*

$$f(x) = f(a) + (x - a)f'(\xi).$$

Moreover, if f' exists and is continuous in $[a, b]$, then

$$f(x) = f(a) + (x - a)f'(a) + (x - a)\varphi(x),$$

where $\lim_{x\to a} \varphi(x) = 0$.

Question. Is there an analogue for complex functions?

THEOREM 3.19 (Goursat's Lemma). *Let $f : U \to \mathbb{C}$ be a complex function, holomorphic in an open subset $U \subset \mathbb{C}$ and let $c \in U$. Then there exists a function ψ such that*

$$f(z) = f(c) + (z - c)f'(c) + (z - c)\psi(z, c),$$

where $\lim_{z\to c} \psi(z, c) = 0$.

PROOF. Define the function

$$\psi(z, c) := \frac{f(z) - f(c)}{z - c} - f'(c).$$

Since f is holomorphic, it follows:

$$\lim_{z\to c} \psi(z, c) = 0$$

and the result follows. □

Question. Is there a converse statement of Goursat's Lemma?

THEOREM 3.20. *Let $f : U \to \mathbb{C}$ be a complex function, defined in an open subset $U \subset \mathbb{C}$ and let $c \in U$. If there exists a complex number $\xi \in \mathbb{C}$ such that*

$$\lim_{z\to c} \frac{f(z) - f(c) - (z - c)\xi}{z - c} = 0,$$

then f is differentiable at c and $f'(c) = \xi$.

PROOF. From the limit we get immediately:

$$\lim_{z\to c} \frac{f(z) - f(c)}{z - c} - \xi = 0,$$

which shows that $f'(c)$ exists with $f'(c) = \xi$. □

THEOREM 3.21. *Let $f : U \to \mathbb{C}$ be a complex function holomorphic in a neighbourhood $U = N(a_0, R)$. If $|f|$ is constant in U, then so is f.*

PROOF. If $|f| = 0$, then also $f = 0$. Suppose that $|f(z)| = c > 0$ for all points $z \in U$. Thus with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ we have:

$$u(x, y)^2 + v(x, y)^2 = c^2$$

for all $z = x + iy \in U$. Differentiating with respect to x and y gives:

$$uu_x + vv_x = 0, \ uu_y + vv_y = 0.$$

Using the Cauchy-Riemann equations we get:

$$\begin{aligned} u^2u_x &= -uvv_x = uvu_y = -v^2v_y = -v^2u_x \\ u^2v_x &= -u^2u_y = uvv_y = uvu_x = -v^2v_x \end{aligned}$$

Using these equations and again the Cauchy-Riemann equations, we get

$$\begin{aligned} u^2u_y &= -u^2v_x = v^2v_x = -v^2u_y \\ u^2v_y &= u^2u_x = -v^2u_x = -v^2v_y \end{aligned}$$

Using $u^2 + v^2 = c^2$ we therefore have

$$c^2u_x = c^2u_y = c^2v_x = c^2v_y = 0$$

and so

$$u_x = u_y = v_x = v_y = 0$$

which implies as before that f is constant throughout the open set U. $\square$

2. Power series

Infinite series of real numbers play an important part to define functions in Real Analysis, and many results from Calculus/Vector Calculus carry over verbatim when replacing real numbers with complex numbers, including convergence tests like comparison and ratio test. For details on arguments, please see the Appendix C or the book by Howie, Chapter 4.2. Here we only collect some notations and results which will be important for us to study the exponential, trigonometric and hyperbolic functions in a complex variable. If time permits we might return to the topic at the end of the module.

DEFINITION. An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **convergent with the sum** or **converges to the sum** or **has sum** S if the sequence $(S_N)_{N=1}^{\infty}$ of partial sums $S_N := \sum_{n=1}^{N} z_n$ converges with limit S, i.e.

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} z_n = S.$$

An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **divergent** if it is not convergent.

DEFINITION. An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **absolutely convergent** if the associated infinite series of real numbers $\sum_{n=1}^{\infty} |z_n|$ is convergent.

DEFINITION. An infinite series of complex numbers of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ with $z, a, c_n \in \mathbb{C}$ is called a **power series**.

DEFINITION. The real number R is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ if the power series is convergent for all z with $|z-a| < R$, and divergent for all z with $|z-a| > R$.

The main result we need is the following:

THEOREM 3.22. *Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence $R \neq 0$ and let*

$$f(z) := \sum_{n=0}^{\infty} c_n(z-a)^n, \ \ (|z-a| < R).$$

Then the complex function f is holomorphic within the open disc $N(a, R)$, and its derivative is given as

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}.$$

Remark. The proof of the last theorem is quite technical and can be found in the book *John M. Howie, Complex Analysis, Springer*. These theorems are useful to define holomorphic functions via power series. We will meet many important examples!

Example 3.23. Find the sum of the power series

$$\sum_{n=0}^{\infty}(n+1)^2 z^n = 1^2 + 2^2 z + 3^2 z^2 + \cdots \quad (|z|<1).$$

Solution. From

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z}, \quad (|z|<1).$$

we get by differentiating term by term:

$$\sum_{n=1}^{\infty} nz^{n-1} = 1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}, \quad (|z|<1).$$

Hence:

$$\sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots = \frac{z}{(1-z)^2}, \quad (|z|<1).$$

And so by differentiation with respect to z, for all $z \in N(0,1)$ we get:

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)^2 z^n &= 1^2 + 2^2 z + 3^2 z^2 + \cdots \\ &= \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) \\ &= \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} \\ &= \frac{(1-z)[(1-z)+2z]}{(1-z)^4} \\ &= \frac{1+z}{(1-z)^3}.\end{aligned}$$

Definition. The complex **exponential function** exp is defined as the power series for all values $z \in \mathbb{C}$

$$e^z := \exp z := \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Remark. The power series used to define the exponential function is convergent for all $z \in \mathbb{C}$, which can be checked via the Ratio Test.

Remark. The exponential function $\exp z$ is holomorphic over the whole complex plane $\mathbb{C}$ and it follows for its first derivative:

$$(\exp z)' = \exp z.$$

Let $F_w(z) := \frac{\exp(z+w)}{\exp z}$ with $z, w \in \mathbb{C}$. Then by the Quotient Rule of differentiation we get:

$$F_w'(z) = \frac{(\exp z)(\exp(z+w)) - (\exp(z+w))(\exp z)}{(\exp z)^2} = 0.$$

and therefore:

$$F_w(z) = k$$

for a constant $k \in \mathbb{C}$ and all $z \in \mathbb{C}$. Since $F_w(0) = \exp w$ we see that:

$$F_w(z) = \exp w$$

for all $z \in \mathbb{C}$ and so we get the additivity property of the exponential function:

$$\exp(z + w) = (\exp z)(\exp w).$$

So we have shown:

THEOREM 3.24. *For all $z, w \in \mathbb{C}$ we have:*

$$\exp(z + w) = (\exp z)(\exp w).$$

DEFINITION (Trigonometric Functions). The complex **trigonometric functions** are defined as the following power series for all values $z \in \mathbb{C}$

$$\begin{aligned}
\cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\
\sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
\cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\
\sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots
\end{aligned}$$

REMARK. The complex trigonometric functions are holomorphic over the whole complex plane $\mathbb{C}$.

THEOREM 3.25. *For all $z \in \mathbb{C}$ we have the following identities:*

$$e^{iz} = \cos z + i \sin z, \;\; e^z = \cosh z + \sinh z.$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \;\; \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \;\; \cos^2 z + \sin^2 z = 1.$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \;\; \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

PROOF. We only have to show the relation $\cos^2 z + \sin^2 z = 1$, the other statements follow from the corresponding power series.
Using $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ we obtain

$$\cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1\,.$$

□

Using the holomorphicity and derivative of the exponential function we have:

THEOREM 3.26. *The complex trigonometric functions are holomorphic in $\mathbb{C}$ and for the first derivative we have:*

$$(\cos z)' = -\sin z, \;\; (\sin z)' = \cos z.$$

$$(\cosh z)' = \sinh z, \;\; (\sinh z)' = \cosh z.$$

Question. What can we say about special values of the complex trigonometric functions, in particular for values in the real numbers $\mathbb{R}$?

Example 3.27. As shown in Calculus and Real Analysis the trigonometric functions have the following properties for values in the real numbers $\mathbb{R}$:

$$\cos x > 0,\ x \in [0, \frac{\pi}{2}),\quad \cos(\frac{\pi}{2}) = 0.$$

$$\sin(\frac{\pi}{2}) = \pm\sqrt{1 - \cos^2(\frac{\pi}{2})} = \pm 1,$$

Since $\sin(0) = 0$ and $(\sin)'(x) = \cos x > 0$ in the interval $[0, \frac{\pi}{2})$, it follows:

$$\sin(\frac{\pi}{2}) = 1.$$

From the above theorems we also get:

$$\begin{aligned}\cos(z+w) \quad &+ \quad i\sin(z+w) = e^{i(z+w)} = e^{iz}e^{iw} \\ &= \quad (\cos z + i\sin z)(\cos w + i\sin w) \\ &= \quad (\cos z\cos w - \sin z\sin w) + i(\sin z\cos w + \cos z\sin w)\end{aligned}$$

This implies the following important formulae:

Theorem 3.28 (Addition Formulae). *For all $z, w \in \mathbb{C}$ we have:*

$$\cos(z+w) = \cos z\cos w - \sin z\sin w.$$

$$\sin(z+w) = \sin z\cos w + \cos z\sin w.$$

Example 3.29. Using the addition formulae, it follows that:

$$\cos\pi = \cos^2(\frac{\pi}{2}) - \sin^2(\frac{\pi}{2}) = -1.$$

$$\sin\pi = 2\sin(\frac{\pi}{2})\cos(\frac{\pi}{2}) = 0.$$

$$\cos 2\pi = \cos^2(\pi) - \sin^2(\pi) = 1.$$

$$\sin 2\pi = 2\sin(\pi)\cos(\pi) = 0.$$

From these special values for the trigonometric functions we get the following important property for the exponential function:

Theorem 3.30 (Periodicity Property). *For all $z \in \mathbb{C}$ we have:*

$$e^{z+2\pi i} = e^z.$$

Proof. We have: $e^{z+2\pi i} = e^z(\cos 2\pi + i\sin 2\pi) = e^z$. □

Let $z \in \mathbb{C}$ be given in standard form $z = x + iy$ with $x, y \in \mathbb{R}$. Then we have:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

So we obtain:

$$|e^z| = e^x,\ \arg e^z \equiv y \pmod{2\pi}.$$

Remark. Since $e^x \neq 0$ for all $x \in \mathbb{R}$, it follows from the above, that $e^z \neq 0$ for all $z \in \mathbb{C}$. So with

$$\exp : \mathbb{C} \to \mathbb{C},\ \exp(z) = e^z$$

being an entire function i.e. a holomorphic function on the entire complex plane, it follows that also the function

$$\exp^{-1} : \mathbb{C} \to \mathbb{C},\ \exp^{-1}(z) = e^{-z} = \frac{1}{e^z}$$

is an entire function.

3. Logarithm

In Calculus and Real Analysis we saw that the following equations for $x, y \in \mathbb{R}$, $y > 0$

$$y = e^x \Longleftrightarrow x = \log y$$

are equivalent, where $\log = \log_e$ is the **natural logarithm** to the base $e = 2.71828\ldots$. This is not longer true in Complex Analysis as the periodicity property for the complex exponential function $e^z = e^{z+2\pi i}$ implies that the complex exponential function $exp(z) = e^z$ is not longer a one-to-one function!

Let us assume

$$w = \log z \Longleftrightarrow z = e^w$$

for $z \in \mathbb{C}, z \neq 0$ and $w \in \mathbb{C},\ w = u + iv$. Then $z = e^{u+iv} = e^u e^{iv}$, in polar form and so

$$e^u = r = |z|,\ v \equiv \arg z \,(\mathrm{mod}\ 2\pi).$$

Thus $u = \log|z|$ while $v = \arg z + 2\pi$ is defined only modulo 2π.

Definition. The **principal logarithm** is defined as

$$\log z := \log|z| + i \arg z$$

for $z \in \mathbb{C}$ and $\arg z \in (-\pi, \pi]$ the **principal argument**. We also call $\log z$ the **principal value of the complex logarithm at z**.

Remark. The choice of the principal argument as lying in the interval $(-\pi, \pi]$ is a convention. We could have chosen instead also the interval $[0, 2\pi)$. Hence the choice of the principal logarithm is similarly arbitrary, but we have to make a choice to get a well-defined complex logarithm function! We also have to be careful with complex logarithms of sums and powers!

Example 3.31. We have the following special values of the complex logarithm:

$$\log(-1) = i\pi,\ \log(-i) = -i(\frac{\pi}{2}),\ \log(1 + i\sqrt{3}) = \log 2 + i(\frac{\pi}{3}).$$

Definition (Multifunction). A **multifunction** f is a rule associating each $z \in U$ in its domain $U \subset \mathbb{C}$ with a *subset* $f(z) \subset \mathbb{C}$, i.e.

$$z \longmapsto f(z) \subset \mathbb{C}.$$

The elements of the subset $f(z) \subset \mathbb{C}$ are called **values of the multifunction**.

In particular, we consider:

Definition. We define the multifunctions of the argument and complex logarithm function:

$$\mathrm{Arg}z : z \longmapsto \{\arg z + 2n\pi : n \in \mathbb{Z}\},$$
$$\mathrm{Log}z : z \longmapsto \{\log z + 2n\pi i : n \in \mathbb{Z}\}.$$

Proposition 3.32. *We have the following multiplicative properties:*

$$\mathrm{Arg}(zw) = \mathrm{Arg}z + \mathrm{Arg}w,\ \ \mathrm{Log}(zw) = \mathrm{Log}z + \mathrm{Log}w.$$

DEFINITION. We define complex powers for $c, z \in \mathbb{C}$ as

$$c^z := e^{z\,\mathrm{Log}c}.$$

In general, the complex power function $z \longmapsto c^z$ is only a multifunction.

EXAMPLE 3.33. We have:

$$\begin{aligned}(1+i)^i &= e^{i\,\mathrm{Log}(1+i)} = \{\exp[i(\log(\sqrt{2}) + (2n+\frac{1}{4})\pi i)] : n \in \mathbb{Z}\} \\ &= \{\exp[-(2n+\frac{1}{4})\pi + i\log(\sqrt{2})] : n \in \mathbb{Z}\}\end{aligned}$$

EXAMPLE 3.34. How do we interpret $e^{\log z} = z$?

We look at the associated multifunction:

$$\begin{aligned}e^{\mathrm{Log} z} &= \{e^{\log|z| + i\arg z + 2n\pi i} : n \in \mathbb{Z}\} = \{|z|e^{i\arg z}e^{2n\pi i} : n \in \mathbb{Z}\} \\ &= \{ze^{2n\pi i} : n \in \mathbb{Z}\} = \{z\}.\end{aligned}$$

EXAMPLE 3.35. How do we interpret $\log(e^z) = z$?
We look at the associated multifunction $\mathrm{Log}(e^z)$.

$$w \in \mathrm{Log}(e^z) \iff e^w = e^z \iff w \in \{z + 2n\pi i : n \in \mathbb{Z}\} = \{x + (y + 2n\pi)i : n \in \mathbb{Z}\}.$$

This set *includes* z, but we can't be sure that using the principal logarithm gives us z, e.g. if $z = 5i\pi/2$ then $\log(e^z) = i\pi/2 \neq z.$, so the statement is not true in general!

EXAMPLE 3.36. How do we interpret $(e^z)^w = e^{zw}$?
We look at the associated multifunction:

$$(e^z)^w = e^{w\,\mathrm{Log}(e^z)} = \{e^{w(z+2n\pi i)} : n \in \mathbb{Z}\} = \{e^{zw}e^{2n\pi i w} : n \in \mathbb{Z}\}.$$

All we can say is that e^{zw} is a value of the multifunction $(e^z)^w$ being a complex power function and so only defined via an associated multifunction.

REMARK. In practice it is usually easy to decide if a formula is true for a complex function or only for a multifunction. If $a \in \mathbb{R}$ with $a > 0$, then we regard $z \longmapsto a^z$ as a complex function and not as a multifunction. We have:

$$a^z = e^{z\log a},$$

where log has its usual meaning as a real function from Calculus and Real Analysis.

Question. Does the formula for the differentiation of the real logarithm function extends to the complex plane?

THEOREM 3.37. *We have for the first derivative of the principal logarithm function:*

$$(\log z)' = \frac{1}{z}.$$

PROOF. Let $z = x + iy = re^{i\theta}$. Then the values of the multifunction $\mathrm{Log}z$ are given as:

$$\mathrm{Log}z = \{\log|z| + i\arg z + 2n\pi i : n \in \mathbb{Z}\} = \{\log r + i(\theta + 2n\pi) : n \in \mathbb{Z}\}.$$

If we choose any of these values we get:

$$\log z = \frac{1}{2}\log(x^2+y^2) + i(\tan^{-1}(\frac{y}{x}) + 2n\pi + C),$$

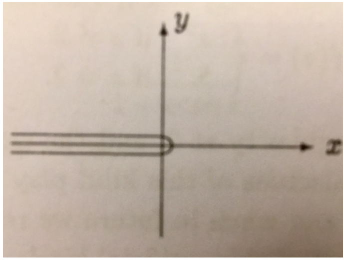
where C is a constant with $C = 0$ or $C = \pm\pi$. We need to take $\pm\pi$ as $-\frac{\pi}{2} < \tan^{-1}(\frac{y}{x}) < \frac{\pi}{2}$, as e.g. $\arg(-1-i) = -3\frac{\pi}{4} = \tan^{-1} 1 - \pi$.
We calculate the partial derivatives of the real and imaginary parts with respect to the real variable x and get:

$$(\log z)' = \frac{x}{x^2+y^2} + i\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{-y}{x^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z},$$

and so the formula for the first derivative of the principal logarithm extends to the complex plane. □

REMARK. We can define many different multifunctions, however we will mostly use and study $\mathrm{Log}z$, $\mathrm{Arg}z$ and the n-th root $z^{\frac{1}{n}}$.

We can think of the complex plane $\mathbb{C}$ as having a **cut** along the negative x-axis, preventing $\arg z$ from leaving the interval $(-\pi, \pi]$. If z moves around any closed path wholly contained in $\mathbb{C}\backslash(-\infty, 0]$, the logarithm value $\log z$ changes continuously and returns to its original value.



As with the definition of the principal logarithm $\log z$ and principal argument $\arg z$, it is a choice where we make the cut in the complex plane. The cut needs to contain the origin 0 and should go off to infinity, we say that 0 and ∞ are the **branch points**.

EXAMPLE 3.38. How do we define the n-th root multifunction $z^{\frac{1}{n}}$ for $n \in \mathbb{N}$?

If $z = re^{i\theta}$, then as a multifunction we set:

$$z^{\frac{1}{n}} = \{r^{\frac{1}{n}}e^{i\frac{(\theta+2k\pi)}{n}} : k = 0, 1, \ldots, n-1\}.$$

We make the convention to define $r^{\frac{1}{n}}e^{i\frac{\theta}{n}}$ to be the principal value of $z^{\frac{1}{n}}$ i.e. the **principal n-th root** and make a cut along $[0, \infty)$.

If z moves around any closed path wholly contained in $\mathbb{C}\backslash[0, \infty)$, then the value $z^{\frac{1}{n}}$ changes continuously and returns to its original value. The origin 0 and ∞ are the branch points.

REMARK. For more complicated multifunctions it can be more difficult to find the right cut and branch points in order to define a principal value.

4. Singularities

We will now look at the cases when the holomorphic function $f : D'(a, r) \longrightarrow \mathbb{C}$ is defined away from an isolated point a (typically, we would think of a function with $\lim_{z\longrightarrow a} f(z) = \infty$) or is not continuous at a.

DEFINITION. Let $f : U \to \mathbb{C}$ be a complex function whose domain contains the neighbourhood $N(c, r)$. If the limit

$$\lim_{z\to c} f(z)$$

exists, but is not equal to the value $f(c)$, we say that f has a **removable singularity at the point** c.

REMARK. If a complex function $f : U \to \mathbb{C}$ has a removable singularity at the point c we can simply redefine the value $f(c)$ as the limit, i.e.

$$f(c) := \lim_{z\to c} f(z).$$

EXAMPLE 3.39. Let $f : \mathbb{C} \to \mathbb{C}$ be the complex function given by:

$$f(z) = \begin{cases} z^2 & \text{if } z \neq 2 \\ 5 & \text{if } z = 2 \end{cases}$$

f has a removable singularity at $c = 2$, as $\lim_{z\to 2} f(z) = 4 \neq 5$. The singularity disappears if we redefine $f(2) = 4$.

DEFINITION (Singularity). For a complex function $f : U \to \mathbb{C}$, a point $c \in \mathbb{C}$ such that $\lim_{z\to c} f(z)$ is not finite, is called a **singularity**. If there exists and integer $n \geqslant 1$ such that $\lim_{z\to c}(z-c)^n f(z)$ exists and is finite, we say that the singularity is a **pole**. The **order** of the pole is the least value of $n \geqslant 1$ for which $\lim_{z\to c}(z-c)^n f(z)$ exists and is finite. Poles of order $1, 2$ and 3 are called (respectively) **simple**, **double** and **triple poles**.

DEFINITION. If a complex function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open subset $H \subset \mathbb{C}$ except possible for poles, we say that f is **meromorphic in** H.

EXAMPLE 3.40. The complex function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z}$ is meromorphic in all of $H = \mathbb{C}$, as f has a simple pole ($n = 1$) at the point $c = 0$, because the limit exists

$$\lim_{z\to c}(z-c)^n f(z) = \lim_{z\to 0} z\frac{1}{z} = \lim_{z\to 0} 1 = 1.$$

and is finite.

EXAMPLE 3.41. Show that the complex function $f : \mathbb{C}\backslash\{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}$, $f(z) = \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$ with simple poles at $c = n\pi$ with $n \in \mathbb{Z}$.

Solution. We have (Exercise!)

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

and we know that $\cosh y \geqslant 1$ for all $y \in \mathbb{R}$ and $\sinh y = 0 \iff y = 0$.
Hence:

$$\operatorname{Re}(\sin(x + iy)) = 0 \iff \sin x = 0 \iff x = n\pi, n \in \mathbb{Z}.$$

Also as $\cos n\pi = \pm 1$ we have:

$$\operatorname{Im}(\sin(x + iy)) = 0 \iff y = 0.$$

Thus the singularities of the function $f(z) = \frac{1}{\sin z}$ occur exactly at the points $c = n\pi$ for $n \in \mathbb{Z}$.
We also have for all $n \in \mathbb{Z}$ (Exercise):

$$\sin z = (-1)^n \sin(z - n\pi).$$

Hence finally we get for each $n \in \mathbb{Z}$:

$$\lim_{z\to n\pi} \frac{z - n\pi}{\sin z} = \lim_{z\to n\pi} \frac{(-1)^n(z - n\pi)}{\sin(z - n\pi)} = (-1)^n.$$

DEFINITION. If $p : \mathbb{C} \to \mathbb{C}$ and $q : \mathbb{C} \to \mathbb{C}$ are complex polynomial functions, we say that the complex function $r : U \to \mathbb{C}$, $r(z) := \frac{p(z)}{q(z)}$ defined on the domain

$$U = \{z \in \mathbb{C} : q(z) \neq 0\}$$

is a **rational function**.

EXAMPLE 3.42. If the polynomial p and q have no common factors, then the rational function $r : U \to \mathbb{C}$, $r(z) := \frac{p(z)}{q(z)}$ is a meromorphic function with poles at the roots $z \in \mathbb{C}$ with $q(z) = 0$. The rational function

$$r : \mathbb{C}\backslash\{0, 1\} \to \mathbb{C},\ r(z) = \frac{z+1}{z(z-1)^2}$$

has a simple pole at $c = 0$ and a double pole at $c = 1$.

EXAMPLE 3.43. The complex function

$$f : \mathbb{C}\backslash\{0\} \to \mathbb{C},\ f(z) = e^{\frac{1}{z}}$$

has a singularity at the point $c = 0$, but this is *not* a pole, since for all $n \geqslant 1$ we have:

$$z^n e^{\frac{1}{z}} = z^n \left(1 + \frac{1}{z} + \cdots + \frac{1}{(n+1)!z^{n+1}} + \cdots\right).$$

But the limit $\lim_{z\to 0} z^n e^{\frac{1}{z}}$ does not exist as a finite number.

In such a case, we call $c = 0$ an **isolated essential singularity**.

EXAMPLE 3.44. The complex function given by $z \longmapsto f(z) = \tan(\frac{1}{z})$ has a sequence of singularities $\{c = \frac{2}{n\pi} : n \in \mathbb{N}\}$, which are poles with limit 0.

In such a case we call $c = 0$ a **non-isolated essential singularity**.

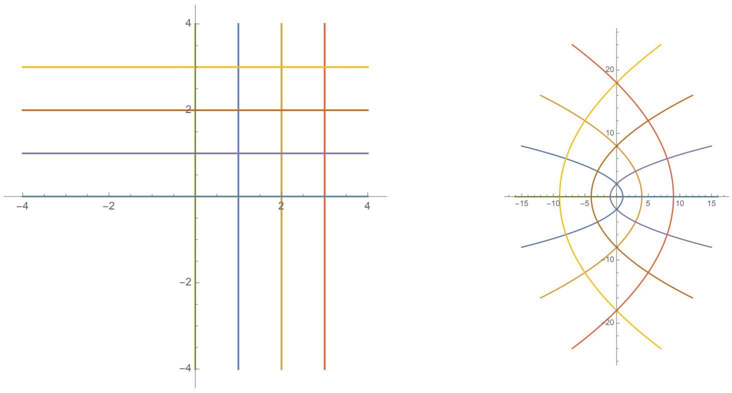
CHAPTER 4

Conformal Mappings

In this chapter we will discuss special holomorphic maps and their transformations.
Maps which preserve angles have lots of important application in Physics, Engineering, Medicine, Computer Graphics etc. We give a short introduction, for more examples see Howie, Chapter 11.

1. Conformal Mappings

In our Example 2.3



we can visually see that the lines $x = k, y = l$ have for the function $f(z) = z^2$, $z \neq 0$, parameter lines that are mutually perpendicular. We will show that this is behaviour is typically for holomorphic functions whenever $f'(z) \neq 0$.

Question: What is meant by the angle between curves?

Consider a parametrised differentiable curve $c : I \subset \mathbb{R} \longrightarrow \mathbb{C}$. Then the tangent line of c at the point t_0 is given by

$$T_c(t_0) = \{c(t_0) + tc'(t_0) \mid t \in I\}$$

provided that $c'(t_0) \neq 0$. If $c'(t_0) = 0$ then there is no well-defined tangent at the point $c(t_0)$. Now, we can define the angle between to intersecting curves as the angle between the tangent lines:

Definition. Let c_1, c_2 be two curves and assume $c_1(0) = c_2(0)$ and $c_1'(0) \neq 0, c_2'(0) \neq 0$.

Then the **angle** between c_1 and c_2 at 0 is given by

$$\arg(c_2'(0) - c_1'(0)) \, .$$

Remark. This is a reasonable definition: for $i = 1, 2$, $\arg(c_i'(0))$ is the angle made by the non-zero vector $c_i'(0)$ with the positive x-axis, and the angle between $c_1'(0)$ and $c_2'(0)$ is the difference between the two arguments.

We know show that angles are preserved under holomorphic functions, by taking into account the potential ambiguity of the argument function.

Theorem 4.1 (Angle preserving). *Let $f : U \longrightarrow \mathbb{C}$ be holomorphic. Suppose that $c_1(t), c_2(t)$ are curves in U that intersect at $t = 0$ in the point $c = c_1(0) = c_2(0)$. Furthermore, suppose that $f'(c) \neq 0$ as well as $c_i'(0) \neq 0, i = 1, 2$.*

Then the angle α between c_1 and c_2 is the same as the angle β between the curves $f(c_1)$ and $f(c_2)$ up to multiples of 2π that is,

$$\alpha = \beta (\mathrm{mod}\ 2\pi)\,.$$

Proof. For every pair of complex numbers $c, d \in \mathbb{C}\backslash\{0\}$ we have

$$\arg(\frac{c}{d}) = (\arg c - \arg d)(\mathrm{mod}\ 2\pi)\,.$$

The angle β is given by

$$\beta = (\arg(f \circ c_1)'(0) - \arg(f \circ c_2)'(0))(\mathrm{mod}\ 2\pi) = \arg\left(\frac{(f \circ c_1)'(0)}{(f \circ c_2)'(0)}\right)(\mathrm{mod}\ 2\pi)\,.$$

By the chain rule,

$$\frac{(f \circ c_1)'(0)}{(f \circ c_2)'(0)} = \frac{f'(c_1(0))c_1'(0)}{f'(c_2(0))c_2'(0)} = \frac{f'(c)c_1'(0)}{f'(c)c_2'(0)} = \frac{c_1'(0)}{c_2'(0)}$$

Hence,

$$\begin{aligned}\beta &= \arg\left(\frac{(f \circ c_1)'(0)}{(f \circ c_2)'(0)}\right)(\mathrm{mod}\ 2\pi) = \arg\left(\frac{c_1'(0)}{c_2'(0)}\right)(\mathrm{mod}\ 2\pi)\\ &= \arg(c_1)'(0) - \arg(c_2'(0))(\mathrm{mod}\ 2\pi) = \alpha(\mathrm{mod}\ 2\pi)\,.\end{aligned}$$

□

Remark. *The trivial observation that*

$$\frac{|f(z) - f(c)|}{|z - c|} \longrightarrow |f'(c)| \text{ as } z \longrightarrow c$$

has a geometric interpretation, that the ***local magnification*** *of the mapping f at the point c is $|f'(c)|$.*

Remark. *The condition $f'(c) \neq 0$ is essential: for example, the function $f(z) = z^2$ has $f'(0) = 0$ and the positive x–axis maps to itself whereas the line $\theta = \frac{\pi}{4}$ maps to the positive y–axis. The angle between the lines doubles.*

Definition. A complex function f is **conformal** in an open set U if it is holomorphic in U and if $f'(c) \neq 0$ for all $c \in U$.

Remark. By our previous discussion, conformal mappings preserve angles.

Example 4.2. The function $f(z) = z^2$ is conformal on the open set $U = \mathbb{C}\backslash\{0\}$.

A rather beautiful result is the Riemann Mapping Theorem but the proof of it is beyond the scope of this module (see e.g. Beardon, A primer on Riemann surfaces).

Theorem 4.3 (Riemann Mapping Theorem). *Let c be a closed, piecewise smooth curve with no self–intersections. Then there exists a one-to-one conformal mapping f from the interior $I(\gamma)$ onto $N(0, 1)$, so that $f^{-1} : N(0, 1) \longrightarrow I(\gamma)$ is also conformal.*

In practice, it is hard to find such a conformal map and there is no practical general method to find one. However, there are certain conformal mappings which are helpful in some situations: the Möbius transformations.

2. Möbius transformation

To define Möbius transformations, we recall that we can consider $\mathbb{C} \cup \infty = \mathbb{C}^*$ by adding a point at ∞ to the complex plane.

DEFINITION. A **Möbius transformation**, also called a **bilinear transformation**, is a map $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by

$$f(z) = \frac{az+b}{cz+d} \qquad \text{with } ad - bc \neq 0, a, b, c, d \in \mathbb{C}\,.$$

REMARK. *The condition $ad - bc \neq 0$ is necessary for the transformation to be of interest: if $a = d = b = c = 0$ the formula is meaningless, otherwise $ad - bc$ gives $\frac{a}{c} = \frac{b}{d} = k$ (say), and the transformation reduces to $z \longmapsto kz$.*

It is clear that a Möbius transformation is holomorphic on $\mathbb{C}\backslash\{-\frac{d}{c}\}$ and has a simple pole $z = -\frac{d}{c}$. It's derivative is

$$f'(z) = \frac{ad - bc}{(cz+d)^2}$$

which doesn't vanish on the domain $\mathbb{C}\backslash\{-\frac{d}{c}\}$. Put differently, a Möbius transformation is a conformal mapping.

THEOREM 4.4. *The inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation.*

PROOF. As is easily verified, the Möbius transformation

$$w \longmapsto \frac{dw - b}{-cw + a}$$

is the inverse of $z \longmapsto \frac{az+b}{cz+d}$.
We compute for two Möbius transformations

$$f_i(z) = \frac{a_i z + b_i}{c_i z + d_i}$$

that

$$(f_1 \circ f_2)(z) = \frac{Az + B}{Cz + D}$$

with

$$A = a_1 a_2 + b_1 c_2, B = a_1 b_2 + b_1 d_2, C = c_1 a_2 + d_1 c_2, D = c_1 b_2 + d_1 d_2\,.$$

A routine calculation shows

$$AD - BC = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0$$

and $f_1 \circ f_2$ is a Möbius transformation. □

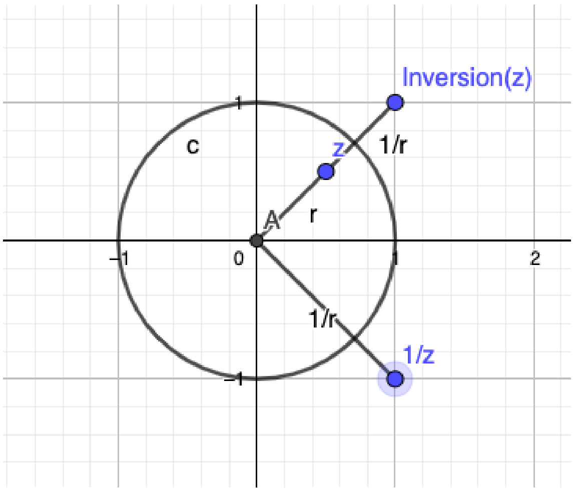
Amongst all Möbius transformations, there are some which play a special role:

(M1) $z \longmapsto az$

(M2) $z \longmapsto z + b$

(M3) $z \longmapsto \frac{1}{z}$.

Writing $a = re^{i\theta}$ in polar coordinates, we see that (M1) is a scaling by the factor r, followed by a rotation by the angle θ. The transformation in (M2) is a translation. The transformation in (M3) is an **inversion** followed by a reflection. An inversion is a reflection across a circle. In our case, the circle is the unit circle.



Theorem 4.5. *Every Möbius transformation*

$$f(z) = \frac{az+b}{cz+d}$$

is a composition of transformations of type (M1), (M2), (M3).

Proof. If $c = 0$ then $ad - bc \neq 0$ implies that $a, d \neq 0$ and $f(z) = \frac{a}{d}z + \frac{b}{d}$: we can first do a scaling (M1)

$$g_1(z) = \frac{a}{d}z$$

by $\frac{a}{d}$ and follow by a translation (M2)

$$g_2(z) = z + \frac{b}{d}$$

to obtain

$$f = g_2 \circ g_1 .$$

In the case, when $c \neq 0$ we first consider the denominator:

$$cz + d = g_2 \circ g_1(z)$$

where

$$g_1(z) = cz, \qquad g_2(z) = z + d\,,$$

and use (M3) to get

$$g_3 \circ g_2 \circ g_1(z) = \frac{1}{cz+d}$$

where $g_3(z) = \frac{1}{z}$. Now using $c \neq 0$ we can consider

$$g_4(z) = \frac{1}{c}(bc - ad)z \quad \text{and} \quad g_5(z) = z + \frac{a}{c}$$

to obtain

$$g_5 \circ g_4(z) = \frac{1}{c}((bc - cd)z + a)$$

and finally

$$f = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1$$

□

It is clear that transformations of type (M1) and (M2) preserve shapes, and in particular, they transform circles to circles. The transformation (M3) also transforms circles to circles. Let us first consider the simple case that the circle has center $m = 0$. Then for

$$z \in \kappa(m, r) = \{z \in \mathbb{C} \mid z = re^{i\theta}\}$$

we have

$$f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$

so that $f(z) \in \kappa(0, \frac{1}{r})$.
In the case of a circle with center $m \neq 0$ and radius r the situation is slightly more complicated: the image under $\frac{1}{z}$ is either a circle or a straight line. It is custom to call in this setting a line an (generalised) circle. We think of the line as a circle with infinite radius.

THEOREM 4.6. *A Möbius transformation transforms circles into circles.*

PROOF. Consider $z \in \kappa(m, r)$, that is,

$$(z - m)(\bar{z} - \bar{m}) = r^2 ,$$

and thus

$$|z|^2 + |m|^2 - z\bar{m} - m\bar{z} = r^2 .$$

Putting $w = f(z) = \frac{1}{z}$ this implies

$$\frac{1}{|w|^2} + |m|^2 - \frac{\bar{m}}{w} - \frac{m}{\bar{w}} = r^2 . \tag{1}$$

We can reformulate this as

$$|w|^2 - \frac{\bar{m}\bar{w}}{|m|^2 - r^2} - \frac{mw}{|m|^2 - r^2} = -\frac{1}{|m|^2 - r^2} ,$$

provided $|m| \neq |r|$. In this case, the circle $\kappa(m, r)$ does not go through the origin. We can now rewrite the above as

$$(w - \frac{\bar{m}}{|m|^2 - r^2})(\bar{w} - \frac{m}{|m|^2 - r^2}) = \frac{r^2}{(|m|^2 - r^2)^2} .$$

This shows that the image $w = f(z)$ lies on the circle with centre $\frac{\bar{m}}{|m|^2 - r^2}$ and radius $\frac{r}{||m|^2 - r^2|}$.
If the circle $\kappa(m, r)$ goes through the origin, then (1) shows that

$$1 - \bar{m}\bar{w} - mw = 0 ,$$

which gives

$$\operatorname{Re}(w)\operatorname{Re}(m) - \operatorname{Im}(w)\operatorname{Im}(m) = \frac{1}{2}$$

which is the equation of a line.

Thus, we have demonstrated that the function $f(z) = \frac{1}{z}$ maps (standard) circles to circles or lines. When considering the inverse function, we see that it therefore maps lines to circles, concluding the proof since $f^{-1} = f$. □

EXAMPLE 4.7. Find a Möbius transformation mapping $-1, 0, 1$ to $-1, i, 1$.
Solution: We want $f(z) = \frac{az+b}{cz+d}$ with $f(-1) = -1$, $f(0) = i$, $f(1) = 1$. This gives 3 equations for the unknowns:

$$f(-1) = -1 = \frac{-a + b}{-c + d} \Rightarrow -a + b = c - d$$

$$f(0) = i = \frac{b}{c} \Rightarrow b = ic$$

$$f(1) = 1 = \frac{a + b}{c + d} \Rightarrow a + b = c + d$$

Combining the equations we get

$$f(z) = \frac{z+i}{iz+1}$$

We easily can verify that f maps $-1, 0, 1$ to $-1, i, 1$.

This example is a special case of the following theorem which is an important feature of Möbius transformations:

Theorem 4.8. *Let (z_1, z_2, z_3) and (w_1, w_2, w_3) be triples of distinct points. Then there is a unique Möbius tranformation f with $f(z_i) = w_i$, $i = 1, 2, 3$.*

Remark. *For practical purposes it is normally easier to find the Möbius transformation as in the previous example rather than computing the inverse and composition.*

Proof. The Möbius transformation

$$g(z) = \frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}$$

satisfies

$$g(z_1) = 0, g(z_2) = 1, g(z_3) = \infty\,.$$

Similarly,

$$h(w) = \frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}$$

satisfies

$$h(w_1) = 0, h(w_2) = 1, h(w_3) = \infty\,.$$

Therefore, $h^{-1} \circ g$ maps z_1, z_2, z_3 to w_1, w_2, w_3.
We leave the uniqueness as an exercise. □

Example 4.9. Since circles are uniquely defined by 3 points, we can find for any two circles a Möbius transformation transforming one into the other.

Example 4.10. Find a conformal mapping between the real axis $L = \{z \in \mathbb{C} \mid \operatorname{Im} z = 0\}$ and the circle $S = \kappa(0, 1)$.

Solution: Choose $-1, 0, 1$ on L and $-1, i, 1$ on the circle, and compute the corresponding Möbius transformation as

$$F(z) = \frac{z+i}{iz+1}\,.$$

Since a Möbius transformation is conformal, F is a conformal mapping.
Note that the solution is not unique: we could have chosen different initial points.

Example 4.11. Find a conformal mapping which maps the lower-half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$ to the disc $N(0, 1) = \{z \in \mathbb{C} \mid |z| < 1\}$.

Solution: For $\operatorname{Im} z < 0$ we know that z is closer to $-i$ then i, therefore

$$|z+i| < |z-i|\,.$$

Therefore, for $F(z) = \frac{z+i}{iz+1}$ we have

$$|F(z)| = \left|\frac{z+i}{z-i}\right| < 1\,.$$

CHAPTER 5

Complex Integration

In this chapter we will discuss path integration of a complex function. For this we will need some notions you might have seen in Calculus, Vector Calculus and Topology.

1. Compactness

In this section we will discuss that a compact set in case of the complex plane is the same as a bounded and closed set. Recall:

DEFINITION. A subset $S \subset \mathbb{C}$ is **bounded** if there exists a real number $K > 0$ such that for all $z \in S$ we have:

$$|z| \leqslant K,$$

i.e. the set S lies inside the closed disc $S \subset \overline{N}(0, K)$.

Moreover, we recall:

DEFINITION (Coverings). An **open covering** of a set $S \subset \mathbb{C}$ is a possibly infinite collection

$$\mathscr{C} := \{V_i : i \in I\}$$

of open sets V_i such that

$$S \subset \bigcup_{i \in I} V_i.$$

If the index set I is a finite set, we call the covering **finite**.

A **subcovering** of an open covering $\mathscr{C} = \{V_i : i \in I\}$ of the set $S \subset \mathbb{C}$ is a selection $\mathscr{S}$ of the open sets V_i

$$\mathscr{S} = \{V_i : i \in J\}$$

where $J \subset I$ and

$$S \subset \bigcup_{i \in J} V_i.$$

It is called a **finite** subcovering if J is a finite subset of I.

DEFINITION (Compactness). A subset $S \subset \mathbb{C}$ such that every open covering has a finite subcovering is called **compact**.

THEOREM 5.1 (Heine-Borel Theorem). *A subset $S \subset \mathbb{C}$ is compact, i.e. every open covering of S contains a finite subcovering of S if and only if S is closed and bounded.*

REMARK. Both 'closed' and 'bounded' are required in the Heine-Borel Theorem!

EXAMPLE 5.2. Let $S = N(0,1)$ be the bounded open disc, then

$$\mathscr{C} = \{N(0, 1 - \frac{1}{n}) : n \in \mathbb{N}\}$$

is an open covering of S, but no finite collection of the open sets in $\mathscr{C}$ will suffice to cover S.

EXAMPLE 5.3. If S is the closed, but unbounded set

$$S = \{z \in \mathbb{C} : Re(z) \geqslant 0, Im(z) \geqslant 0\} \subset \mathbb{C}$$

i.e. the first quadrant of the complex plane, then

$$\mathscr{C} = \{N(0,n) : n \in \mathbb{N}\}$$

is an open covering of S, but no finite subcovering of $\mathscr{C}$ exists.

THEOREM 5.4. *Let $S \subset \mathbb{C}$ be a closed bounded subset and $f : U \to \mathbb{C}$ a complex continuous function with domain containing $S \subset U$. Then f is bounded on S, i.e. the set*

$$T := \{|f(z)| : s \in S\} \subset \mathbb{C}$$

is bounded. Moreover, if $M := \sup_S |f(z)| = \sup T$ is the supremum of T, then there exists $z_0 \in S$ such that $|f(z_0)| = M$.

PROOF. Let $\varepsilon = 1$. By continuity of f for each $c \in S$ there exists $\delta = \delta(c)$ such that

$$|f(z) - f(c)| < \varepsilon = 1.$$

for all $z \in N(c, \delta(c))$. The sets $N(c, \delta(c))$ give an open covering of $S \subset \mathbb{C}$.

By the Heine-Borel Theorem there exists then a finite subcovering of S

$$\{N(c_i, \delta(c_i) : i = 1, 2, \ldots m\}.$$

Let $z \in S$ and

$$K := \max\{|f(c_1)| + 1, |f(c_2)| + 1, \ldots, |f(c_m)| + 1\}.$$

Then $z \in N(c_i, \delta(c_i))$ for at least one $i \in \{1, 2, \ldots, m\}$, hence

$$|f(z)| - |f(c_i)| \leqslant |f(z) - f(c_i)| < 1$$

Hence:

$$|f(z)| < |f(c_i)| + 1 \leqslant K.$$

Let $M := \sup_S |f(z)|$ and suppose by contradiction that $|f(z)| < M$ for all $z \in S$. Then the function

$$g : S \to \mathbb{R}, \; g(z) := \frac{1}{(M - |f(z)|)}$$

being continuous is bounded on S.

But for all $K > 0$ there exists $z \in S$ with $M - |f(z)| < \frac{1}{K}$ (otherwise a smaller upper bound would be possible). Hence:

$$|g(z)| = g(z) > K$$

and so g is not bounded. This contradiction gives therefore $|f(z_0)| = M$ for a $z_0 \in S$. $\square$

COROLLARY 5.5. *Let $S \subset \mathbb{C}$ be a closed bounded subset and $f : U \to \mathbb{C}$ a complex continuous function with domain containing $S \subset U$ with $f(z) \neq 0$ for all $z \in S$. Then*

$$\inf\{|f(z)| : z \in S\} > 0.$$

Proof. From the hypothesis it follows that the function $\frac{1}{f}$ is also continuous for all $z \in S$. So by the last theorem there exists a real number $M > 0$ such that

$$\sup\{|\frac{1}{f(z)}| : z \in S\} = M.$$

From this it follows that

$$\inf\{|f(z)| : z \in S\} = \frac{1}{M} > 0.$$

□

2. Parametric curves

We recall the notion of a parametrised curve from Vector Calculus, translated from the $\mathbb{R}^2$ notation to the complex plane:

Definition (Parametric representation of curves). A **(parametrised) curve** or **path** is a complex continuous function

$$\gamma : [a, b] \to \mathbb{C},\ \gamma(t) := \alpha(t) + i\beta(t)\,.$$

The image of γ is the **complex curve**

$$\gamma^* = \{\gamma(t) : t \in [a, b]\}.$$

The **orientation** of the curve γ^* is given by the direction of travel of a point on the curve as the parameter t increases from a to b.

If $\gamma(a) = \gamma(b)$ we say that γ^* is a **closed** curve.

If $a \leqslant t < t' \leqslant b$ and $|t' - t| < b - a$ implies that $\gamma(t) \neq \gamma(t')$ we say that γ^* is a **simple** curve. Visually a simple curve has no self-crossings.

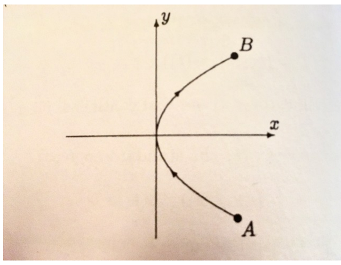
Remark. The parametric description has advantages over describing a curve as a **graph** of a continuous function $f : [a, b] \to \mathbb{R}$,

$$\mathscr{C} = \{(x, f(x)) : x \in [a, b]\}$$

as it allows the curve to become vertical or having self-crossings.

Example 5.6. Let $\gamma(t) = e^{it}$ with $t \in [0, 2\pi]$, then γ^* is a simple closed curve, namely a circle around the origin $(0, 0) \in \mathbb{R}^2$ of radius 1 beginning and ending at the point $(1, 0) \in \mathbb{R}^2$. The orientation of γ^* is anti-clockwise.

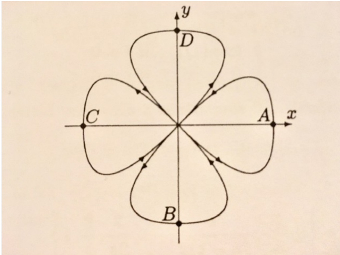
Example 5.7. Let $\gamma(t) = t^2 + it$ with $t \in [-1, 1]$, then the curve γ^* is a parabola. It is a simple, but not closed curve. It begins at the point $A = (1, -1)$ and ends at the point $B = (1, 1)$.



Example 5.8. Let $\gamma(t) = \cos(2t)e^{it}$ with $t \in [0, 2\pi]$, then the curve γ^* is a closed curve, but not simple, since

$$\gamma(\frac{\pi}{4}) = \gamma(\frac{3\pi}{4}) = \gamma(\frac{5\pi}{4}) = \gamma(\frac{7\pi}{4}) = (0, 0).$$

As t increases from 0 to 2π the point $\gamma(t)$ follows a smooth path from A to O to B to O to C to O to D to O and back to A.



Consider a curve

$$\gamma^* = \{\gamma(t) : t \in [a, b]\},$$

and let

$$D := \{a = t_0, t_1, \cdots, t_n = b\}$$

be a partition of the interval $[a, b]$ with $t_0 < t_1 < \ldots < t_n$. Each $t_i \in D$ corresponds to a point $P_i = \gamma(t_i)$ on the curve γ^* and we can estimate the length of the curve between the point $A = P_0$ and $B = P_n$ as

$$\mathscr{L}(\gamma^*, D) := |P_0P_1| + |P_1P_2| + \cdots |P_{n-1}P_n|.$$

and using vectors we get:

$$\mathscr{L}(\gamma^*, D) = \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \, .$$

If we refine the partition D of the curve γ^* by adding additional points then $\mathscr{L}(\gamma^*, D)$ increases: if Q is a point between P_{i-1} and P_i, then, by the triangle equality $|P_{i-1}Q| + |QP_i| \geqslant |P_{i-1}P_i|$.

DEFINITION. Let $\mathscr{D}$ be the set of all partitions of the interval $[a, b]$. If the set

$$\{\mathscr{L}(\gamma^*, D) : D \in \mathscr{D}\}$$

is bounded above, we say that the curve γ^* is **rectifiable** and define its **length** $\Lambda(\gamma^*)$ by

$$\Lambda(\gamma^*) := \sup\{\mathscr{L}(\gamma^*, D) : D \in \mathscr{D}\}$$

EXAMPLE 5.9. The curve $\gamma^* := \{t + ir_2(t) : t \in [0, 1]\}$ is *not* rectifiable, where

$$r_2(t) = \begin{cases} t\,\sin(\frac{1}{t}) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

THEOREM 5.10. *Let $\gamma^* = \{\gamma(t) : t \in [a, b]\}$ be a complex curve and suppose that the functions $\operatorname{Re}\gamma, \operatorname{Im}\gamma : [a, b] \to \mathbb{R}$ are differentiable and the first derivatives are continuous on $[a, b]$. Then γ^* is rectifiable and its length $\Lambda(\gamma^*)$ is given by*

$$\Lambda(\gamma^*) = \int_a^b |\gamma'(t)| dt.$$

EXAMPLE 5.11. Determine the length $\Lambda(\gamma^*)$ of the curve γ^* given by the complex function

$$\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = re^{it},$$

where $r \in \mathbb{R}$ with $r > 0$.
Solution. We have for the continuous function

$$\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = re^{it}$$

the following first derivative

$$\gamma'(t) = \frac{d}{dt} re^{it} = ire^{it}$$

and so $|\gamma'(t)| = |ire^{it}| = |i||re^{it}| = r$. So for the length of the curve γ^* which is in fact a circle of radius r, we get:

$$\Lambda(\gamma^*) = \int_0^{2\pi} |\gamma'(t)|dt = \int_0^{2\pi} rdt = 2\pi r.$$

EXAMPLE 5.12. Determine the length $\Lambda(\gamma^*)$ of the curve γ^* given by the complex function

$$\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = t - ie^{-it}.$$

Solution. Since

$$\gamma'(t) = 1 - e^{-it} = (1 - \cos t) + i \sin t,$$

we get:

$$|\gamma'(t)|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t = 4\sin^2(\frac{t}{2}).$$

As the function $\sin(\frac{t}{2}) \geqslant 0$ for all $t \in [0, 2\pi]$ we get:

$$|\gamma'(t)| = 2\sin(\frac{t}{2}).$$

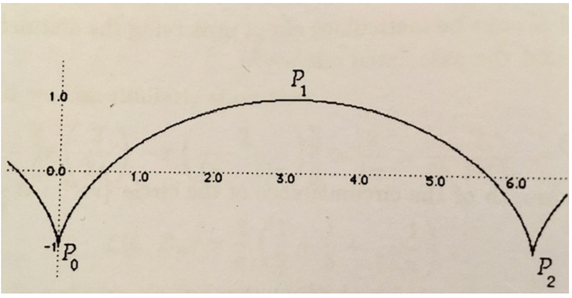
and finally for the length of the curve γ^*

$$\Lambda(\gamma^*) = \int_0^{2\pi} 2\sin(\frac{t}{2})dt = \left[-4\cos(\frac{t}{2})\right]_0^{2\pi} = 8.$$

REMARK. The curve γ^* given by the complex function

$$\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = t - ie^{-it}.$$

is called a **cycloid** and looks like this in the complex plane:



Geometrically, it is the path of a point on the circumference of a wheel of radius 1 rolling along the line $y = -1$ and making one complete rotation. The points P_0, P_1, P_2 correspond respectively to the values $0, \pi, 2\pi$ of t. It shows up in many applications in geometry, physics and engineering.

3. Integration

Question. How do we define the integral of a complex function along a curve in the complex plane?

Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth complex function, i.e. its first derivative γ' is continuous on $[a, b]$. Now let $f : U \to \mathbb{C}$ be *suitable* complex function, whose domain contains the curve γ^* associated to the function γ, i.e. $\gamma(t) \subset U$ for all $t \in [a, b]$. We define:

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

In general, if we define $g : [a, b] \to \mathbb{C}$ by

$$g : [a, b] \to \mathbb{C},\ g(t) := f(\gamma(t))\gamma'(t),$$

then $g(t) = u(t) + iv(t)$, where $u, v : [a, b] \to \mathbb{R}$ are continuous real functions and we define

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

THEOREM 5.13 (Rules for integration). *Let $a < b < c$ be real numbers, $f, g : \mathbb{R} \to \mathbb{C}$ complex functions and $k \in \mathbb{C}$ a constant. Then we have:*

$$\int_a^b (f \pm g)\,dt = \int_a^b f\,dt \pm \int_a^b g\,dt,$$

$$\int_a^b k \cdot f\,dt = k\int_a^b f\,dt,$$

$$\int_a^b f\,dt + \int_b^c f\,dt = \int_a^c f\,dt.$$

REMARK. We also define for $a > b$ real numbers:

$$\int_a^a f\,dt = 0, \qquad \int_b^a f\,dt = -\int_a^b f\,dt.$$

A function $f : U \to \mathbb{C}$ is *suitable* if and only if the integral

$$\int_\gamma f(z)dz$$

is defined, i.e. if and only if the function $f(\gamma(t))\gamma'(t)$ is an **integrable** function. We know from Calculus and Real Analysis that in particular any continuous function is integrable.

DEFINITION. The integral

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

is called the **integral of f along** γ. If γ gives rise to a closed curve γ^* we call it the **integral round** γ.

THEOREM 5.14. *Let $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) := e^{it}$, so that the associated curve γ^* is the unit circle C with centre 0 and radius $r = 1$. Let $n \in \mathbb{Z}$ be an integer. Then:*

$$\int_\gamma z^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. By the definition of the integral along the curve we have

$$\int_\gamma z^n dz = \int_0^{2\pi} e^{nit} i e^{it} dt = i\int_0^{2\pi} e^{(n+1)it} dt.$$

If $n = -1$, this gives:

$$\int_\gamma z^{-1} dz = i\int_0^{2\pi} e^{(-1+1)it} dt = i\int_0^{2\pi} dt = 2\pi i.$$

Otherwise, if $n \neq -1$, this gives:

$$\begin{aligned}\int_\gamma z^n dz &= i\int_0^{2\pi} [\cos(n+1)t + i\sin(n+1)t]dt \\ &= i\left(\left[\frac{\sin(n+1)t}{n+1} - i\frac{\cos(n+1)t}{n+1}\right]_0^{2\pi}\right) = 0.\end{aligned}$$

□

REMARK. Although we write the integral as $\int_\gamma f(z)dz$, the integral does *not* depend on the particular (smooth increasing) parametrisation of the contour curve γ^*. Thus, we could in the theorem above use a different parametrisation of the unit circle by using the function

$$\delta : [0,\pi] \to \mathbb{C},\ \delta(t) := e^{2it}.$$

The value of the integral $\int_\delta f(z)dz$ would be the same.

REMARK. In the last Theorem we have used the parametrisation $\gamma(t) = e^{it}$ so the closed curve was traversed in positive (anti-clockwise) orientation. We could also use the clockwise orientation of the parametrisation $\gamma(t) = e^{-it}$. If $n \neq -1$ there will be no difference in the calculation of the intergral (Check!), but if $n = -1$ we get instead:

$$\int_\gamma z^{-1}dz = \int_0^{2\pi} e^{it}(-ie^{-it})dt = -i\int_0^{2\pi} dt = -2\pi i.$$

DEFINITION. If $\gamma : [a,b] \to \mathbb{C}$ is a complex curve, let $\overleftarrow{\gamma}$ be the same path, but with opposite orientation given by:

$$\overleftarrow{\gamma} : [a,b] \to \mathbb{C},\ \overleftarrow{\gamma}(t) := \gamma(a+b-t).$$

THEOREM 5.15. *Let $\gamma : [a,b] \to \mathbb{C}$ be a complex curve, and $f(z)$ a suitable complex function, then we have:*

$$\int_{\overleftarrow{\gamma}} f(z)dz = -\int_\gamma f(z)dz.$$

PROOF. We have:

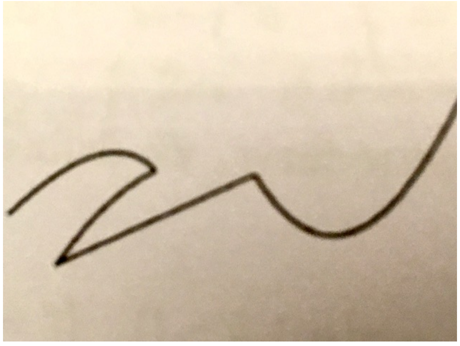
$$\begin{aligned}\int_{\overleftarrow{\gamma}} f(z)dz &= \int_a^b f(\overleftarrow{\gamma}(t))(\overleftarrow{\gamma})'(t)dt \\ &= \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))dt \\ &= \int_b^a f(\gamma(u))(\gamma'(u))(du) \text{ where } u := a+b-t \\ &= -\int_a^b f(\gamma(u))\gamma'(u)du = -\int_\gamma f(z)dz.\end{aligned}$$

□

REMARK. The requirement that in the definition of the integral

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

that the curve γ is smooth is often too restrictive, so we like to extend the definition to so-called piecewise smooth curves. Geometrically, these curves consists of finitely many smooth segments, but are not everywhere smooth.



DEFINITION (Integration over piecewise smooth curve). A complex curve $\gamma : [a, b] \to \mathbb{C}$ is **piecewise smooth** if there are real numbers

$$a = c_0 < c_1 < \cdots c_m = b$$

and smooth functions

$$\gamma_i : [c_{i-1}, c_i] \to \mathbb{C} \ (i = 1, 2, \ldots m)$$

such that

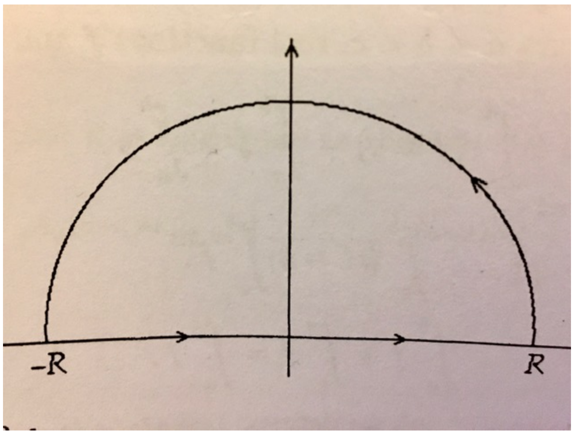
$$\gamma_i(c_i) = \gamma_{i+1}(c_i) \ (i = 1, 2, \ldots m - 1).$$

If $f : U \to \mathbb{C}$ is a complex function such that the image of γ lies in U, then we define:

$$\int_\gamma f(z)dz := \sum_{i=1}^{m} \left(\int_{\gamma_i} f(z)dz \right).$$

EXAMPLE 5.16. Let us look at a particular curve γ. Let $\gamma = \sigma(0, R) \subset \mathbb{C}$ be the closed semicircle shown below. Determine the integral

$$\int_\gamma z^2 dz.$$



Solution. The curve γ is given by two pieces, both being smooth, namely the straight line segment γ_1 from the point $(-R, 0)$ to the point $(R, 0)$ and the semicircular arc γ_2 in the positive (i.e. anti-clockwise) orientation from the $(R, 0)$ back to the point $(-R, 0)$.

We parametrise γ_1 in the complex plane by:

$$\gamma_1(t) = t + i \cdot 0, \ (t \in [-R, R]).$$

Then we get:

$$\int_{\gamma_1} z^2 dz = \int_{-R}^{R} t^2 dt = \left[\frac{1}{3} t^3 \right]_{-R}^{R} = \frac{2}{3} R^3.$$

We parametrise γ_2 in the complex plane by:

$$\gamma_2(t) = Re^{it}. \ (t \in [0, \pi]).$$

Then we get:

$$\int_{\gamma_2} z^2 dz = \int_0^\pi R^2 e^{2it} \cdot Rie^{it} dt = \left[\frac{1}{3} R^3 e^{3it}\right]_0^\pi = -\frac{2}{3} R^3.$$

And so finally:

$$\int_\gamma z^2 dz = \int_{\gamma_1} z^2 dz + \int_{\gamma_2} z^2 dz = 0.$$

Theorem 5.17. *Let $f : [a, b] \to \mathbb{C}$ be a continuous complex function, and let*

$$F(x) := \int_a^x f(t) dt, \ (x \in [a, b])$$

Then $F'(x) = f(x)$ for all $x \in [a, b]$. If $\Theta : [a, b] \to \mathbb{C}$ is any complex function such that $\Theta' = f$, then

$$\int_a^b f(t) dt = \Theta(b) - \Theta(a).$$

Proof. Let $g := \mathrm{Re}\,(f), h := \mathrm{Im}\,(f)$. Then

$$\begin{aligned} F(x) &= \int_a^x (g(t) + ih(t)) dt = \int_a^x g(t) dt + i \int_a^x h(t) dt \\ &= G(x) + iH(x), \end{aligned}$$

for some real functions G, H. By the Fundamental Theorem of Calculus we get:

$$F'(x) = G'(x) + iH'(x) = g(x) + ih(x) = f(x),$$

for all $x \in [a, b]$. Suppose now that $\Theta' = f$. Then with $\Theta = \Phi + i\Psi$ we have:

$$\Phi' = G' = g, \ \Psi' = H' = h.$$

With some constants $C, C' \in \mathbb{R}$ we get for all $x \in [a, b]$ then

$$G(x) = \Phi(x) + C, \ H(x) = \Psi(x) + C'.$$

Setting $x = a$ gives $C = -\Phi(a)$ and $C' = -\Psi(a)$ and hence:

$$\begin{aligned} \int_a^b f(t) dt &= G(b) + iH(b) = (\Phi(b) - \Phi(a)) + i(\Psi(b) - \Psi(a)) \\ &= (\Phi(b) + i\Psi(b)) - (\Phi(a) + i\Psi(a)) = \Theta(b) - \Theta(a), \end{aligned}$$

which proves the theorem. $\square$

Theorem 5.18. *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve. Let $F : U \to \mathbb{C}$ be a complex function with domain U an open set containing γ^*. Suppose F' exists and is continuous at each point of γ^*. Then*

$$\int_\gamma F'(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, then

$$\int_\gamma F'(z) dz = 0.$$

Proof. Suppose first that γ is smooth, then $F \circ \gamma$ is differentiable on $[a, b]$. Since $(F \circ \gamma)'(t) = F'(\gamma(t)\gamma'(t)$ the previous Theorem gives:

$$\int_\gamma F'(z) dz = \int_a^b (F \circ \gamma)'(t) dt = F(\gamma(b)) - F(\gamma(a)).$$

Now suppose γ is piecewise smooth, i.e. there are real numbers

$$a = c_0 < c_1 < \cdots c_m = b$$

and smooth functions $\gamma_i : [c_{i-1}, c_i] \to \mathbb{C}$ $(i = 1, 2, \ldots m)$ such that $\gamma_i(c_i) = \gamma_{i+1}(c_i)$ $(i = 1, 2, \ldots m-1)$. Then we get by using the previous formula for the smooth parts:

$$\begin{aligned}
\int_\gamma F'(z)dz &= \sum_{i=1}^m \left(\int_{\gamma_i} F'(z)dz \right) \\
&= [F(\gamma_1(c_1)) - F(\gamma_1(a))] + [F(\gamma_2(c_2)) - F(\gamma_2(c_1))] + \cdots \\
&\quad \cdots + [F(\gamma_m(b)) - F(\gamma_m(c_{m-1}))] \\
&= F(\gamma_m(b)) - F(\gamma_1(a)) \\
&= F(\gamma(b)) - F(\gamma(a))
\end{aligned}$$

If γ^* is closed, then $\gamma(b) = \gamma(a)$ and hence:

$$\int_\gamma F'(z)dz = 0,$$

which proves the theorem. $\square$

Corollary 5.19. *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth and closed curve. Then we have:*

$$\int_\gamma 1dz = \int_\gamma zdz = 0.$$

Proof. In the previous Theorem take first $F(z) := z$ and then $F(z) := \frac{z^2}{2}$ which gives the result. $\square$

Example 5.20. Let γ^* be the upper half of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

with positive (i.e. counterclockwise) orientation. Determine the integral

$$\int_\gamma \cos(z)dz.$$

Solution. The ellipse intersects the x-axis at the points $(a, 0)$ and $(-a, 0)$. By the previous Theorem we don't need to find an explicit parametrisation of the curve γ, hence:

$$\int_\gamma \cos(z)dz = [\sin(z)]_a^{-a} = -2\sin(a).$$

Remark. A closed piecewise smooth curve γ can be quite complicated and having many self-crossings. But as we have just seen for suitable functions F the integral of F' round γ has the value 0. We are mostly interested in simple curves without crossings.

Theorem 5.21. *Let $g : [a, b] \to \mathbb{C}$ be a continuous complex function. Then we have:*

$$\left| \int_a^b g(t)dt \right| \leqslant \int_a^b |g(t)|dt.$$

Proof. From Calculus and Real Analysis we know that such an inequality holds for continuous real functions, i.e. for functions $g : [a, b] \to \mathbb{R}$ we have:

$$\left| \int_a^b g(t)dt \right| \leqslant \int_a^b |g(t)|dt.$$

To show it also holds for complex functions g we observe first, that for some θ using polar form:

$$\left|\int_a^b g(t)dt\right| = e^{i\theta}\int_a^b g(t)dt = \int_a^b e^{i\theta}g(t)dt = \int_a^b \mathrm{Re}\,(e^{i\theta}g(t))dt + i\int_a^b \mathrm{Im}\,(e^{i\theta}g(t))dt.$$

Since the imaginary part of the left-hand side of the equation is zero, we get:

$$\left|\int_a^b g(t)dt\right| = \int_a^b \mathrm{Re}\,(e^{i\theta}g(t))dt \leqslant \int_a^b |e^{i\theta}g(t)|dt = \int_a^b |g(t)|dt.$$

This proves the theorem. □

THEOREM 5.22 ("Maximum-Length" Theorem). *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve, and let f be a continuous complex function such that $|f(z)| \leqslant M$ for all points z of the curve γ^* and a real constant $M \in \mathbb{R}$. Then*

$$\left|\int_\gamma f(z)dz\right| \leqslant M \cdot l,$$

where $l := \int_a^b |\gamma'(t)|dt$ is the length of the curve γ^*.

PROOF. We have by the last theorem:

$$\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leqslant M\int_a^b |\gamma'(t)|dt = M \cdot l.$$

This proves the theorem. □

EXAMPLE 5.23. Let f be the complex function given by $f(z) := \frac{1}{z^3+1}$ and let γ^* be the contour with $\gamma(t) := Re^{it}, t \in [0,\pi], R \in \mathbb{R}$. Show that:

$$\lim_{R\to\infty}\left|\int_\gamma f(z)dz\right| = 0.$$

Solution. For each point on the contour γ^* we have:

$$|f(z)| = \left|\frac{1}{R^3e^{3it}+1}\right| = \frac{1}{|R^3e^{3it}+1|}.$$

Furthermore:

$$|R^3e^{3it}+1| \geqslant |R^3e^{3it}| - 1 = R^3 - 1.$$

and the length of γ^* is $\Lambda(\gamma^*) = R\pi$. Hence:

$$0 \leqslant \left|\int_\gamma f(z)dz\right| \leqslant \frac{R\pi}{R^3-1} \to 0\,(R\to\infty).$$

EXAMPLE 5.24. Let f be the complex function given by $f(z) := \frac{1}{z+\frac{1}{2}}$ and let γ be the contour with $\gamma(t) := e^{it}, t \in [0,\pi]$. Show that:

$$\left|\int_\gamma f(z)dz\right| \leqslant 2\pi.$$

Solution: Here we get:

$$|f(z)| = \frac{1}{|z+\frac{1}{2}|} \leqslant \frac{1}{|z|-\frac{1}{2}} \leqslant 2$$

for all points z on the contour γ^* and the length of γ^* is $\Lambda(\gamma^*) = \pi$. So the desired inequality follows immediately:

$$\left|\int_\gamma f(z)dz\right| \leqslant \int_\gamma |f(z)|dz \leqslant 2\pi.$$

DEFINITION. A **contour** γ in the complex plane $\mathbb{C}$ is a piecewise smooth, simple, closed curve. If not specified otherwise we will always assume that contours are traversed with positive (i.e. anti-clockwise or counterclockwise) orientation.

REMARK. Let γ be a contour. After the Jordan Curve Theorem, the complement $\mathbb{C}\backslash\gamma^*$ of the curve γ^* is the disjoint union of two open sets, the **interior** $I(\gamma)$ and the **exterior** $E(\gamma)$ of γ^*, i.e.

$$\mathbb{C}\backslash\gamma^* = I(\gamma) \sqcup E(\gamma)$$

By a **convex** contour we mean a contour γ such that for all $a, b \in I(\gamma)$ the straight line segment $L(a, b)$, which connects the points a and b lies wholly in $I(\gamma)$, i.e. $L(a, b) \subset I(\gamma)$.

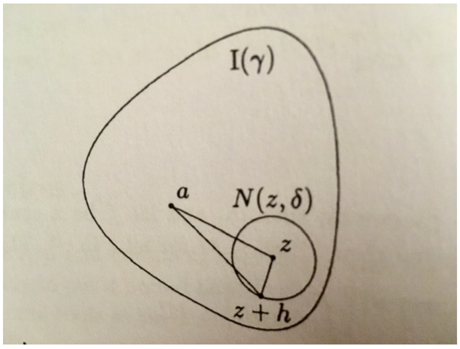
THEOREM 5.25. *Let f be a continuous complex function and γ be a convex contour and assume that $\int_\tau f(z)dz = 0$ for every triangular contour τ within the interior $I(\gamma)$. Then there exists a complex function F, holomorphic in $I(\gamma)$ such that*

$$F'(z) = f(z)$$

for all points $z \in I(\gamma)$.

PROOF. Let $z \in I(\gamma)$. Since $I(\gamma)$ is open, there exists $\delta > 0$ such that $N(z, \delta) \subset I(\gamma)$. If we choose $h \in \mathbb{C}$ such that $|h| < \delta$, then always $z + h \in I(\gamma)$. Let $a \in I(\gamma)$ an arbitrary fixed point. Then by the convexity of the open set $I(\gamma)$, the entire triangle $\Delta(z, a, z + h)$ with vertices $z, a, z + h$ lies wholly in $I(\gamma)$ i.e.

$$\Delta(z, a, z + h) \subset I(\gamma).$$



The hypothesis of the theorem implies:

$$\int_{[a,z]} f(w)dw + \int_{[z,z+h]} f(w)dw - \int_{[a,z+h]} f(w)dw = 0.$$

Let us define the complex function F by:

$$F(z) := \int_{[a,z]} f(w)dw.$$

Then we have:

$$F(z + h) - F(z) = \int_{[z,z+h]} f(w)dw.$$

For any constant $k \in \mathbb{C}$ we have also:

$$\int_{[z,z+h]} k\, dw = k \cdot (z + h - z) = k \cdot h.$$

and so in particular we get:

$$\int_{[z,z+h]} f(z)dw = hf(z).$$

As the complex function f is continuous it is bounded in $I(\gamma)$ and so by the last theorem:

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \frac{1}{|h|} \left| \int_{[z,z+h]} (f(w) - f(z))dw \right| \\ &\leqslant \frac{1}{|h|} \cdot |h| \sup_{w \in [z,z+h]} |f(w) - f(z)|. \end{aligned}$$

Therefore we get as f is continuous:

$$\lim_{h \to 0} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = 0,$$

which shows that F' exists with $F' = f$ and so F is holomorphic in $I(\gamma)$. □

Corollary 5.26. *Let f be a continuous complex function and γ a convex contour and assume $\int_\tau f(z)dz = 0$ for every triangular contour τ within the interior $I(\gamma)$, then*

$$\int_\gamma f(z)dz = 0.$$

Proof. This follows from the previous theorem, because a contour is a closed curve. □

CHAPTER 6

Cauchy's Theorem

Our main goal is to prove the following fundamental theorem due to **Augustin-Louis Cauchy (1789-1857)**.

THEOREM 6.1 (Cauchy's Theorem). *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth function and γ^* the associated contour in the complex plane. Furthermore, let $f : U \to \mathbb{C}$ be a complex function holomorphic in an open domain U containing $I(\gamma) \cup \gamma^*$, i.e. $I(\gamma) \cup \gamma^* \subset U$. Then we have:*

$$\int_\gamma f(z)dz = 0.$$

REMARK. We will only prove this result in the case that $I(\gamma) \cup \gamma^* \subset U$ is *convex* or *polygonal*. The proof for the general case can be found for example in the book by John M. Howie: Complex Analysis.

1. Cauchy's Theorem

We will first prove the theorem in the case that the contour has triangular shape.

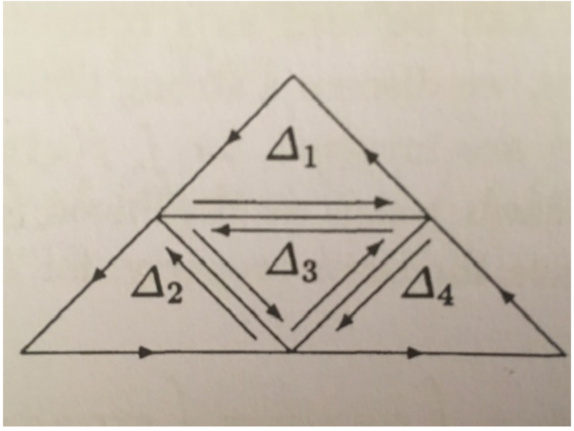
THEOREM 6.2 (Cauchy's Theorem for triangular contours). *Let $T \subset \mathbb{C}$ be a triangular contour in the complex plane and let $f : U \to \mathbb{C}$ be a complex function holomorphic in an open domain U containing $I(T) \cup T$, i.e. $I(T) \cup T \subset U$. Then we have:*

$$\int_T f(z)dz = 0.$$

PROOF BY CONTRADICTION. Let T be a triangle in the complex plane $\mathbb{C}$ whose longest side is of length l. Suppose we have for a real constant $h \in \mathbb{R}$:

$$\left| \int_T f(z)dz \right| = h > 0.$$

We divide the triangle T into four congruent subtriangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ as below:



For $i = 1, 2, 3, 4$ let $U_i = \partial\Delta_i$ be the boundary of the triangle Δ_i with orientation as shown above. Then we have the partition:

$$\int_T f(z)dz = \sum_{i=1}^{4} \int_{U_i} f(z)dz.$$

This partition holds because in the sum on the right hand side each of the internal lines is traversed once with positive and once with negative orientation, so we are only left with the outer contour which is T.

Since

$$h = \left|\int_T f(z)dz\right| \leqslant \sum_{i=1}^{4} \left|\int_{U_i} f(z)dz\right|,$$

we must have:

$$\left|\int_{U_i} f(z)dz\right| \geqslant \frac{h}{4}$$

for *at least* one of the triangular contours $U_i = \partial\Delta_i$.
Let's choose one of these triangles and rename it as Δ_1 with triangular contour $T_1 = \partial\Delta_1$. $T_1 =$ has then a longest side of length $\frac{l}{2}$ and we have:

$$\left|\int_{T_1} f(z)dz\right| \geqslant \frac{h}{4}.$$

We can iterate this process by subdividing now the triangle with boundary T_1 and choose a T_2 at the end with longest side of length $\frac{l}{4}$ such that:

$$\left|\int_{T_2} f(z)dz\right| \geqslant \frac{h}{16}.$$

So continuing this process in the same fashion we obtain for each $n \in \mathbb{N}, n \geqslant 1$ a triangular contour T_n with longest side of length $\frac{l}{2^n}$ such that:

$$\left|\int_{T_n} f(z)dz\right| \geqslant \frac{h}{4^n}.$$

For each $n \in \mathbb{N}, n \geqslant 1$ we select a point α_n inside T_n and obtain a Cauchy sequence $(\alpha_n) \subset \mathbb{C}$ with a limit $\alpha \in \mathbb{C}$ lying inside *every* triangular contour T_n.
Let $\varepsilon > 0$ be given. Then by Goursat's Lemma there exists a $\delta > 0$ such that:

$$|f(z) - f(\alpha) - (z - \alpha)f'(\alpha)| < \varepsilon|z - \alpha| \tag{2}$$

for all $z \in N(\alpha, \delta)$. Now choose n so that $T_n \subset N(\alpha, \delta)$ lies inside the open disc around the point α with radius δ.
We also have as $f(\alpha) \in \mathbb{C}$ is a constant:

$$\int_{T_n} f(\alpha)dz = 0 \quad \text{and} \quad \int_{T_n} (z - \alpha)f'(\alpha)dz = 0,$$

therefore we get:

$$\int_{T_n} f(z)dz = \int_{T_n} [f(z) - f(\alpha) - (z-\alpha)f'(\alpha)]dz.$$

The perimeter of the triangular contour T_n is at most $\frac{3l}{2^n}$ and the maximum value for of $|z-\alpha|$ for z and α inside or on T_n is $\frac{l}{2^n}$, as $\frac{l}{2^n}$ is the length of the longest side.

Hence from (2) we get the following estimation:

$$\left|\int_{T_n} f(z)dz\right| \leqslant \varepsilon \frac{3l}{2^n}\frac{l}{2^n} = \frac{3l^2\varepsilon}{4^n}.$$

But we have already seen that the left hand side is greater or equal $\frac{h}{4^n}$, therefore we must have:

$$\frac{3l^2\varepsilon}{4^n} \geqslant \frac{h}{4^n},$$

which implies:

$$h \leqslant 3l^2\varepsilon.$$

As ε can be chosen arbitrarily small, this gives a contradiction, and so an $h > 0$ as assumed to exist at the start of the proof can't exist and so it follows:

$$\int_T f(z)dz = 0,$$

which proves the theorem. □

COROLLARY 6.3 (Cauchy's Theorem for convex contours). *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth function determining a convex contour γ^* in the complex plane. Furthermore, let $f : U \to \mathbb{C}$ be a complex function holomorphic in an open domain U containing $I(\gamma) \cup \gamma^*$, i.e. $I(\gamma) \cup \gamma^* \subset U$. Then we have:*

$$\int_\gamma f(z)dz = 0.$$

PROOF. From the previous theorem and as γ^* is a convex contour, there exists a function F such that $F' = f$ by Theorem 5.25 and therefore we get by the complex analog of the Fundamental Theorem of Calculus:

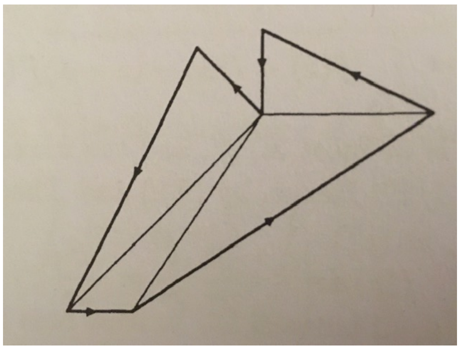
$$\int_\gamma f(z)dz = 0,$$

which gives the statement. □

COROLLARY 6.4. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth function determining a polygonal contour γ^* in the complex plane. Furthermore, let $f : U \to \mathbb{C}$ be a complex function holomorphic in an open domain U containing $I(\gamma) \cup \gamma^*$, i.e. $I(\gamma) \cup \gamma^* \subset U$. Then we have:*

$$\int_\gamma f(z)dz = 0.$$

PROOF. The polygonal contour γ^* can be subdivided into finitely many triangles $\Delta_1, \Delta_2, \dots, \Delta_n$ with triangular contours $T_1, T_2, \dots T_n$ and $T_i = \partial\Delta_i$ for all $i = 1, 2, \dots, n$.



Then by Cauchy's Theorem for triangular contours we get:

$$\int_\gamma f(z)dz = \sum_{i=1}^{n} \int_{T_i} f(z)dz = 0,$$

which gives the statement. □

2. Deformation

We can use Cauchy's Theorem to obtain results for integrals by "deforming" the curve.

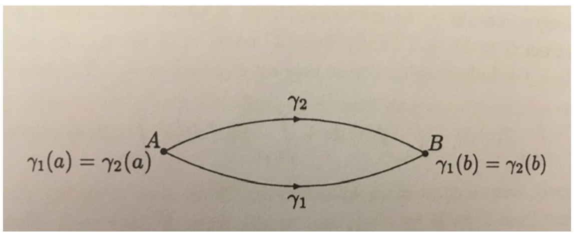
THEOREM 6.5. *Let $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$ be two piecewise smooth curves such that*

$$\gamma_1(a) = \gamma_2(a),\ \gamma_1(b) = \gamma_2(b),\ \gamma_1(t_1) \neq \gamma_2(t_2)\ (t_1, t_2 \in (a, b)).$$

If $f : U \to \mathbb{C}$ is a complex function, which is holomorphic throughout an open set $U \subset \mathbb{C}$ containing γ_1^*, γ_2^* and the region between, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

PROOF. We have the following picture:



Let σ^* be the contour given by the simple closed curve traversing from point A to point B via γ_1 and then from point B back to point A via $-\gamma_2$. Since f is holomorphic in the open domain U with $I(\sigma) \cup \sigma^* \subset U$, we get from Cauchy's Theorem

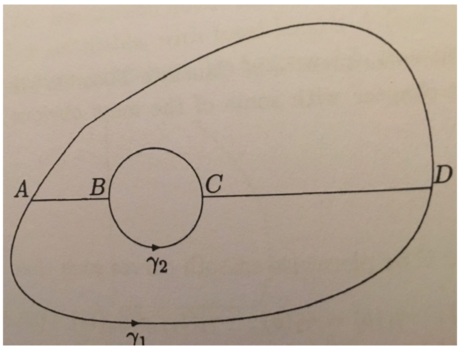
$$0 = \int_\sigma f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz,$$

which proves the statement. □

THEOREM 6.6 (The Deformation Theorem). *Let γ_1, γ_2 be contours, with $\gamma_2 \subset I(\gamma_1)$ and suppose that $f : U \to \mathbb{C}$ is a complex function, which is holomorphic throughout an open set $U \subset \mathbb{C}$ containing the region between γ_1 and γ_2. Then we have:*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

PROOF. We have the following picture:



Join the two contours γ_1, γ_2 by straight lines AB and CD. Denote the lower section of γ_1 from A to D by γ_{1l} and the upper section of γ_1 from D to A by γ_{1u}. Similarly, denote the lower section of γ_2 from B to C by γ_{2l} and the upper section of γ_2 from C to B by γ_{2u}.

Form a contour σ_1 by traversing from A to B, then from B to C by $-\gamma_{2u}$, then from C to D, and finally from D back to A by γ_{1u}. By hypothesis, the complex function f is holomorphic inside and on σ_1 and so after Cauchy's Theorem:

$$\int_{\sigma_1} f(z)dz = 0,$$

which means:

$$\int_{AB} f(z)dz - \int_{\gamma_{2u}} f(z)dz + \int_{CD} f(z)dz + \int_{\gamma_{1u}} f(z)dz = 0\,. \tag{3}$$

Similarly, form a contour σ_2 by traversing from A to D by γ_{1l}, then from D to C, then from C to B via $-\gamma_{2l}$, and finally from B back to A . By hypothesis, the complex function f is holomorphic inside and on σ_2 and so after Cauchy's Theorem:

$$\int_{\sigma_2} f(z)dz = 0,$$

which means:

$$\int_{\gamma_{1l}} f(z)dz - \int_{CD} f(z)dz - \int_{\gamma_{2l}} f(z)dz - \int_{AB} f(z)dz = 0\,. \tag{4}$$

Adding (3) and (4) gives:

$$\left(\int_{\gamma_{1u}} f(z)dz + \int_{\gamma_{1l}} f(z)dz\right) - \left(\int_{\gamma_{2u}} f(z)dz + \int_{\gamma_{2l}} f(z)dz\right) = 0,$$

hence:

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

which proves the statement. □

EXAMPLE 6.7. We have seen already, that if σ is a contour given by the unit circle $\kappa(0, 1) \subset \mathbb{C}$ with centre the origin 0, then

$$\int_\sigma \frac{1}{z} = 2\pi i.$$

More generally, if $\kappa(a,r) \subset \mathbb{C}$ is the circle of radius $r > 0$ around the point $a \in \mathbb{C}$, then writing $z = a + re^{i\theta}$ we get:

$$\int_{\kappa(a,r)} \frac{1}{z-a} dz = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = 2\pi i.$$

Remark. These contour integrals are important ingredients for determining values of holomorphic functions on points which lie inside a particular contour. It will turn out that the value of the functions depends on the above fundamental contour integrals.

CHAPTER 7

Applications of Cauchy's Theorem

1. Cauchy's Integral Formula

Cauchy's Integral Formula indicates that integration round a contour depends critically on the singularities of the integrand within the contour:

THEOREM 7.1 (Cauchy's Integral Formula). *Let γ be a contour and let $f : U \to \mathbb{C}$ b a complex function holomorphic in an open domain containing $I(\gamma) \cup \gamma^*$, i.e. $I(\gamma) \cup \gamma^* \subset U$. Then for every point $a \in I(\gamma)$, we have:*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

PROOF. Let $a \in I(\gamma)$. By the differentiability of f at a we have by Goursat's Lemma:

$$f(z) = f(a) + (z-a)f'(a) + \psi(z,a)(z-a),$$

where $\lim_{z\to a} \psi(z,a) = 0$, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\psi(z,a)| < \varepsilon$ for all $z \in N(a,\delta)$.

Let $\kappa = \kappa(a,r)$ the circle with centre a and radius r, where r is chosen so that:

(i) the disc $\sigma = N(a,r)$ lies wholly inside γ, i.e. $N(a,r) \subset I(\gamma) \cup \gamma^*$

(ii) $r < \delta$.

Since $\frac{f(z)}{z-a}$ is holomorphic in the region between κ and σ, the Deformation Theorem gives:

$$\begin{aligned} \int_\gamma \frac{f(z)}{z-a} dz &= \int_\kappa \frac{f(z)}{z-a} dz \\ &= f(a)\int_\kappa \frac{1}{z-a} dz + f'(a)\int_\kappa 1 dz + \int_\kappa \psi(z,a) dz \\ &= 2\pi i\, f(a) + \int_\kappa \psi(a,z) dz \end{aligned}$$

Hence

$$\left| \int_\gamma \frac{f(z)}{z-a} dz - 2\pi i\, f(a) \right| = \left| \int_\kappa \psi(z,a) dz \right| < 2\pi r \varepsilon.$$

As this holds for *every* $\varepsilon > 0$, it follows that:

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz,$$

which proves the theorem. □

REMARK. Dividing $f(z)$ by $z - a$ introduces a singularity (unless $f(a) = 0$). Cauchy's Integral Formula indicates that integration round a contour depends critically on the singularities of the integrand within the contour.

Example 7.2. Evaluate the integral

$$\int_{\kappa(0,2)} \frac{\sin z}{z^2+1} dz.$$

Solution. Since

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right),$$

we may deduce from Cauchy's Integral Formula

$$\begin{aligned}\int_{\kappa(0,2)} \frac{\sin(z)}{z^2+1} dz &= \frac{1}{2i}\int_{\kappa(0,2)} \frac{\sin(z)}{z-i} - \frac{1}{2i}\int_{\kappa(0,2)} \frac{\sin(z)}{z+i} \\ &= \pi(\sin(i) - \sin(-i)) = 2\pi\sin(i) = \frac{\pi}{i}(e^{-1}-e) = \pi i(e-e^{-1}).\end{aligned}$$

If we could be sure that the procedure of differentiating under the integral sign is valid, we could deduce from Cauchy's Integral Formula that

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{d}{da}\left(\frac{f(z)}{z-a}\right) dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2} dz.$$

Indeed, this statement is true, even more generally:

Theorem 7.3. *Let γ be a contour and let $f : U \to \mathbb{C}$ b a complex function holomorphic in an open domain containing $I(\gamma) \cup \gamma^*$, i.e. $I(\gamma) \cup \gamma^* \subset U$. Then f has an n-th derivative $f^{(n)}$ for all $n \geqslant 1$ and for every point $a \in I(\gamma)$, we have:*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz.$$

Remark. We will not prove these results here as we will focus more on applications and examples of these results. Complete proofs can be found again in the book: John M. Howie, Complex Analysis, Springer.

Let us emphasise that this theorem gives a very surprising statement

Corollary 7.4. *Any differentiable function f has higher derivatives of every order.*

Example 7.5. These results are surprising, because that last statement says that any differentiable function f necessarily has higher derivatives of every order, which is not true for real functions studied in Calculus and Real Analysis, e. g. the real function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is differentiable at 0, but f' is not even continuous, so higher derivatives don't exist!

Remark. We can use the last theorem also to calculate integrals by writing instead the formula equivalently in the following form:

$$\int_\gamma \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a).$$

Example 7.6. Evaluate the integral:

$$\int_{\kappa(0,1)} \frac{e^{\sin z}}{z^3} dz.$$

Solution. The value of the integral is given as where $f(z) = e^{\sin z}$, $n = 3$ and $a = 0$:

$$\int_{\kappa(0,1)} \frac{e^{\sin z}}{z^3} dz = \frac{1}{2!} 2\pi i\, f''(0).$$

We have:

$$f'(z) = e^{\sin z} \cos z,\ f''(z) = e^{\sin z}(\cos^2 z - \sin z).$$

Thus $f''(0) = 1$ and we get finally:

$$\int_{\kappa(0,1)} \frac{e^{\sin z}}{z^3} dz = \pi i.$$

The following theorem is a converse to Cauchy's Theorem:

THEOREM 7.7 (Morera's Theorem). *Let $D \subset \mathbb{C}$ be an open set and $f : D \to \mathbb{C}$ be a continuous complex function. If*

$$\int_\gamma f(z) dz = 0$$

for every contour $\gamma \subset D$, then f is holomorphic in D.

PROOF. Let $a \in D$ and $r \in \mathbb{R}$, $r > 0$ such that $N(a, r) \subset D$. Within this convex open set every contour γ and in particular every triangular contour γ is such that

$$\int_\gamma f(z) dz = 0.$$

Therefore there exists a complex function F, holomorphic in $N(a, r)$ such that:

$$F'(z) = f(z)$$

for all $z \in N(a, r)$. By the previous theorem F has derivatives of all orders within $N(a, r)$ and so also $f'(a)$ exists. Since $a \in D$ was chosen arbitrarily, it follows that f is a holomorphic function in D. □

2. Liouville's Theorem

Question. Do there exist bounded entire functions, i.e. bounded functions which are holomorphic throughout the whole complex plane?

THEOREM 7.8 (Liouville's Theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded entire function. Then f is a constant function.*

PROOF. Suppose that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$ and a real number $M \in \mathbb{R}$, $M > 0$. Let $a \in \mathbb{C}$ be any point in the complex plane and let γ_R be the circular contour given by

$$\gamma_R := \kappa(a, R) = \{z \in \mathbb{C} : |z - a| = R\}.$$

Then by Theorem 7.3 and the maximum-length theorem Theorem 5.22 we get:

$$|f'(a)| = \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-a)^2} dz \right| \leqslant \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}.$$

But this holds for all values of $R > 0$, so $f'(a) = 0$. Since $f'(a) = 0$ for any point $a \in \mathbb{C}$ it follows that f is a constant function. □

Liouville's Theorem is important in many areas in mathematics, for example, it can be used to prove the Fundamental Theorem of Algebra, see Appendix D for more applications.

THEOREM 7.9. *Let $p(z)$ be a polynomial of degree $n \geqslant 1$ with coefficients in $\mathbb{C}$, i.e. $p \in \mathbb{C}[z]$ with*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$$

with $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.

PROOF. Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then both $p(z)$ and $\frac{1}{p(z)}$ give entire functions. Also

$$\lim_{|z| \to \infty} |p(z)| = \infty.$$

Hence there exists a real number $R \in \mathbb{R}, R > 0$ such that $|\frac{1}{p(z)}| \leqslant 1$ whenever $|z| > R$. Since $\frac{1}{p(z)}$ is continuous by assumption, it is also bounded on the closed bounded set $S := \{z \in \mathbb{C} : |z| \leqslant R\}$ by Theorem 5.4. Thus, $\frac{1}{p(z)}$ is an entire function. Hence by Liouville's Theorem, the function $\frac{1}{p(z)}$ must be constant, which gives a contradiction: the polynomial $p(z)$ must have at least one root, i.e. there exists $a \in \mathbb{C}$ with $p(a) = 0$. □

THEOREM 7.10 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial of degree $n \geqslant 1$ with coefficients in $\mathbb{C}$, i.e. $p \in \mathbb{C}[z]$ with*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$$

with $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then there exist complex numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that

$$p(z) = \beta(z - \alpha_1)(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n).$$

PROOF (BY INDUCTION ON THE DEGREE $n \geqslant 1$). The result is clear for degree $n = 1$, as $p(z)$ is then just a linear function. Suppose that the result is true for all polynomials of degree $n - 1$ and let $p(z)$ be a polynomial of degree n. By the previous theorem, there exists then an $\alpha_1 \in \mathbb{C}$ such that $p(\alpha_1) = 0$.

Hence we have:

$$p(z) = (z - \alpha_1) \cdot q(z),$$

where $q(z)$ is a polynomial of degree $n - 1$. By induction hypothesis there exists $\beta, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that:

$$q(z) = \beta(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n).$$

Hence we obtain

$$p(z) = \beta(z - \alpha_1)(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n),$$

as required. □

REMARK. The Fundamental Theorem of Algebra was first proved by C. F. Gauss, who also provided many different proofs for it. Nowadays there are many different proofs known, for example using topology or complex analysis or Galois theory. In particular, the proof presented here using Liouville's Theorem is particularly short. There are more algebraic proofs of it, but all need some fundamental properties of the field $\mathbb{R}$ of real numbers, namely the important Intermediate Value Theorem from Calculus and Real Analysis.

3. Taylor series

In real analysis, there are distinctions between functions which are differentiable, infinitely differentiable and analytic (having a Taylor series expansion). This is in contrast to the behaviour of complex differentiable functions. We already have seen that a holomorphic function is infinitely often differentiable. In fact, it also has a Taylor series expansion:

Theorem 7.11. *Let $c \in \mathbb{C}$ and suppose that the function f is holomorphic in some neighbourhood $N(c,R)$ of c. Then, within $N(c,R)$*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

where, for $n \in \mathbb{N}$,

$$a_n = \frac{f^{(n)}(c)}{n!}\,.$$

Proof. It is helpful first to record the sum of the following finite geometric series:

$$\begin{aligned} \sum_{k=0}^{n} \frac{h^k}{(z-c)^{k+1}} &= \frac{1}{z-c} + \frac{h}{(z-c)^2} + \ldots + \frac{h^n}{(z-c)^{n+1}} \\ &= \frac{1}{z-c-h} - \frac{h^{n+1}}{(z-c)^{n+1}(z-c-h)}\,. \end{aligned} \tag{5}$$

Let $0 < R_1 < R_2 < R$. Then f is holomorphic throughout the closed disc $\bar{N}(c,R_2)$. Let $C = \kappa(c,R_2)$ and $c+h \in \bar{N}(c,R_1)$. Then, by Cauchy's Integral Formula, Theorem 7.3 and (5) we have

$$\begin{aligned} f(c+h) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-c-h}dz = \frac{1}{2\pi i}\left[\sum_{k=0}^{n} h^k \int_C \frac{f(z)}{(z-c)^{k+1}}dz + h^{n+1}\int_C \frac{f(z)}{(z-c)^{n+1}(z-c-h)}dz\right] \\ &= \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(c) + E_n \end{aligned}$$

where

$$E_n = \frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)}{(z-c)^{n+1}(z-c-h)}dz\,.$$

We now show that $E_n \longrightarrow 0$ as $n \longrightarrow \infty$. By our assumptions, we know that the image of f is bounded, that is, there exists $M > 0$ such that $|f(z)| \leqslant M$ for all z on the circle C. For all z on C we also have

$$|z-c-h| \geqslant |z-c| - |h| \geqslant R_2 - R_1\,,$$

since $|z-c| = R_2$, and $|h| \leqslant R_2$. Hence we can estimate by a previous theorem:

$$|E_n| \leqslant \frac{|h|^{n+1}}{2\pi} 2\pi R_2 \frac{M}{R_2^{n+1}(R_2-R_1)} = \frac{M|h|}{R_2-R_1}\left(\frac{|h|}{R_2}\right)^n$$

Since $\frac{|h|}{R_2} < 1$ we conclude that $\lim_{n\longrightarrow\infty} E_n = 0$. Therefore, we have

$$f(c+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(c)\,.$$

Substituting $z = c+h$ gives the result. □

Remark. *The Taylor series of a function f is unique for given $c \in \mathbb{C}$. If $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ then by Theorem 3.22*

$$f^{(n)}(z) = n!\,a_n + \text{ positive powers of } (z-c).$$

Therefore,

$$a_n = \frac{f^{(n)}(c)}{n!}\,.$$

Thus, if we find, by whatever method, a power series for a function, the series we find must be the Taylor series.

DEFINITION. The series
$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n\,.$$
is called the **Taylor series** of f **centred** on c.

However, the Taylor series is not unique when we allow c to change:

EXAMPLE 7.12. Find the Taylor series of $f(z) = \frac{1}{(1+z)^2}$ centered at $c = 0$.

Solution:
$$a_n = \frac{f^{(n)}(0)}{n!} = (-1)^n(n+1)$$
so that for $|z| < 1$
$$f(z) = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n\,.$$

EXAMPLE 7.13. Find the Taylor series of $f(z) = \frac{1}{(1+z)^2}$ centered at $c \neq -1$.

Solution: First
$$\frac{1}{(1+z)^2} = \frac{1}{((c+1)+(z-c))^2} = \frac{1}{(c+1)^2}\frac{1}{(1+\frac{z-c}{c+1})^2}\,.$$
Substituting $w = \frac{z-c}{c+1}$ and applying the previous example we obtain
$$\frac{1}{(1+z)^2} = \frac{1}{(c+1)^2}\sum_{n=0}^{\infty}(-1)^n(n+1)w^n = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(c+1)^{n+2}}(z-c)^n\,.$$

EXAMPLE 7.14. Show that
$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots, \qquad (|z| < 1)\,.$$

Solution: Within the neighbourhood $N(0,1)$, $1+z$ stays clear of the cut locus of the logarithm. Therefore, f is well defined for $|z| < 1$ and we have for $n > 1$
$$f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{(1+z)^n}$$
so that
$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}, \qquad |z| < 1\,.$$

CHAPTER 8

Laurent Series and the Residue Theorem

1. Laurent Series

In Chapter 3, Section 4, we already briefly looked at functions with isolated singularities. It is clear that at a singularity c, the function cannot have a Taylor series. Instead it has a **Laurent Series**, a generalised version of a Taylor Series in which there are negative as well as positive powers of $(z-c)$.

THEOREM 8.1 (Laurent expansion). *Let f be holomorphic in the punctured disc $D'(c,R)$, $R>0$. Then there exist $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$, such that for all $z \in D'(c,R)$,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n .$$

If $0 < r < R$, then

$$a_n = \frac{1}{2\pi i} \int_{\kappa(c,r)} \frac{f(w)}{(w-c)^{n+1}} dw . \tag{6}$$

PROOF. Due to time constraints, I will not give the proof here. The main ingredients for the proof are the Deformation Theorem, and the Cauchy Integral formula. The proof can be found in the book by Howie. □

DEFINITION. The series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n .$$

is called the **Laurent series** of f in the punctured disc $D'(c,R)$. The sum

$$g(z) = \sum_{n=-\infty}^{-1} a_n (z-c)^n .$$

is called the **principal part** of f at c.

As for Taylor series, there is a uniqueness theorem: if there is a power expansion (with possibly negative powers) then the coeffients are the Laurent coefficients.

THEOREM 8.2 (Laurent expansion). *Let f be holomorphic in the punctured disc $D'(c,R)$, $R>0$. If for $z \in D'(c,R)$,*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-c)^n .$$

then for $n \in \mathbb{Z}$

$$b_n = \frac{1}{2\pi i} \int_{\kappa(c,r)} \frac{f(w)}{(w-c)^{n+1}} dw .$$

Corollary 8.3. *If f has a Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n\,.$$

in the punctured disc $D'(c,R)$, then

$$\int_{\kappa(c,r)} f(z)dz = 2\pi i a_{-1}\,.$$

for $r < R$.

Proof. Simply put $n = 1$ in the formula (6). □

This rather innocent looking result has far-reaching consequences, as we shall see shortly: the most important coefficient of the principal part is the coefficient a_{-1}.

Definition. The coefficient a_{-1} is called the **residue** of f at c, and we will denote it by

$$\operatorname{res}(f,c) = a_{-1}\,.$$

Remark. *With this notation the previous corollary reads as: If f has a Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n\,.$$

in the punctured disc $D'(c,R)$, then

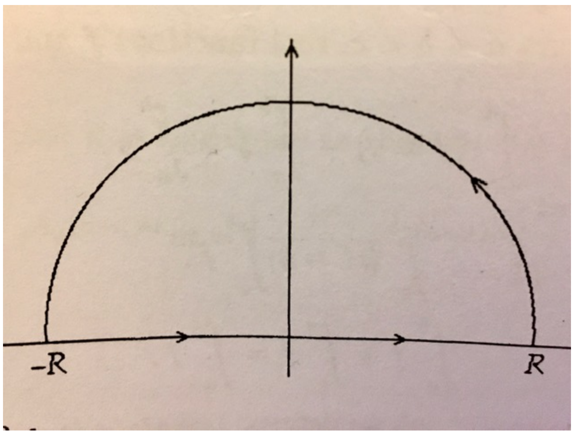
$$\int_{\kappa(c,r)} f(z)dz = 2\pi i \operatorname{res}(f,c)\,.$$

for $r < R$.

Example 8.4. Determine

$$\int_\gamma \frac{1}{z^2+1}dz\,,$$

where γ is the semi-circle $[-R,R] \cup \{z \mid |z| = R, \operatorname{Im} z \geqslant 0\}$, transversed in the positive direction, with $R > 1$.



Solution: The integrand $f(z) = \frac{1}{1+z^2}$ has a singularity at i, and otherwise is holomorphic in $I(\gamma) \cup \gamma^*$. We can compute the integral by using the residue of f at i: since f is holomorphic between the circle $\kappa(i,r)$ (for $0 < r < \min\{|R-1|, 1\}$) and γ we can use the Deformation Theorem to compute

$$\int_\gamma f(z)dz = \int_{\kappa(i,r)} f(z)dz = 2\pi i \operatorname{res}(f,i)\,.$$

Using the geometric series, the Laurent expansion at i is given by

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\cdot\frac{1}{2i(1-(-1)\frac{(z-i)}{2i})}\\ &= \frac{1}{2i(z-i)}\left(1-\frac{z-i}{2i}+\frac{(z-i)^2}{(2i)^2}-\frac{(z-i)^3}{(2i)^3}+\dots\right)\,.\end{aligned}$$

Therefore, the function $f(z) = \frac{1}{z^2+1}$ has a simple pole at i and the residue is $\operatorname{res}(f,i) = \frac{1}{2i}$. Hence

$$\int_\gamma \frac{1}{z^2+1}dz = 2\pi i\frac{1}{2i} = \pi\,.$$

Remark. *For a function f that is holomorphic in an opend domain containing the disc $D(c,R)$, we have a Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n\,,$$

where

$$a_n = \frac{f^{(n)}(c)}{n!} = \frac{1}{2\pi i}\int_{\kappa(c,R)}\frac{f(w)}{(w-c)^{n+1}}dw\,.$$

This is also the Laurent expansion: if f is holomorphic, then the negative coefficients in the Laurent expansion are all 0.

Example 8.5. Calculate the first few terms of the Laurent series for $f(z) = \frac{1}{\sin z}$ at 0 and the residue at 0.

Solution: The function has a singularity at $z = 0$ but is otherwise holomorphic in the neighbourhood $N(0,\pi)$. We know that for $z \longrightarrow 0$

$$\sin z = z - \frac{z^3}{6} + O(z^5)\,.$$

Hence, for z near 0, we have with the geometric series that

$$\frac{1}{\sin z} = \frac{1}{z}\left(1-(\frac{z^2}{6}+O(z^4))\right)^{-1} = \frac{1}{z}\left(1+\frac{z^2}{6}+O(z^4)\right) = \frac{1}{z}+\frac{z}{6}+O(z^3)$$

If we need more terms, it is in principle easy to compute them. The residue is $\operatorname{res}(f,0) = 1$.

2. Classification of singularities

We encountered singularities before, but the Laurent series now helps us to understand them better.

Definition. Let c be a singularity of f, and let $f(z)$ have a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n\,.$$

Then

- If $a_n \neq 0$ for infinitely many $n < 0$, then c is an **essential singularity**.
- If c is not an essential singularity, we say that n is the order of f at c,

  $$\operatorname{ord}(f,c) = n$$

  if n is the least integer (positive or negative) such that $a_n \neq 0$.

- If $\text{ord}(f,c) \geqslant 0$, that is, $a_n = 0$ for all $n < 0$ then f has a **removable singularity**.
- If $\text{ord}(f,c) < 0$ then c is a **pole** of $\text{ord}(f,c) = n$.

REMARK.

- *If f has a removable singularity, then f becomes differentiable at c if we define $f(c) = a_0$. Thus, whether $f(c)$ was undefined, or had another value than a_0, then we can remove the singularity by redefining f at the point c. Note that f has at worst a removable singularity if $\lim_{z\longrightarrow c} f(z)$ is finite.*
- *If f has not an essential singularity at c, then the order $\text{ord}(f,c)$ is the unique integer n such that*
$$f(z) = (z-c)^n g(z)$$
with g differentiable and $g(c) \neq 0$.
- *If f has an essential singularity, then $\lim_{z\longrightarrow c} f(z)$ does not exist. Indeed, a theorem due to Casorati and Weierstrass, see below, says that in this case within any arbitrary small neighbourhood, $f(z)$ gets arbitrarily close to* **any** *complex number whatever!*

PROPOSITION 8.6. *Let f, g have finite order at c. Then*
$$\text{ord}(fg,c) = \text{ord}(f,c) + \text{ord}(g,c)$$
$$\text{ord}(\frac{1}{f},c) = -\text{ord}(f,c)$$
Moreover, if $\text{ord}(f,c) < \text{ord}(g,c)$ then $\text{ord}(f+g,c) = \text{ord}(f,c)$.

PROOF. The proof is straightforward and is left to the reader. □

THEOREM 8.7 (The Casorati–Weierstrass Theorem). *Let f have an essential singularity at c, and let d be an arbitrary complex numbers. Then, for all $\delta > 0$ and for all $\epsilon > 0$ there exists $z \in D'(c,\delta)$ such that*
$$|f(z) - d| < \epsilon\,.$$

PROOF. Suppose for a contradiction that for some $d \in \mathbb{C}$ there exists $\epsilon > 0$ and $\delta > 0$ such that
$$|f(z) - d| \geqslant \epsilon$$
for all $z \in D'(c,\delta)$. Let $g(z) = \frac{1}{f(z)-d}$. Then for all $z \in D'(c,\delta)$ we have
$$|g(z)| < \frac{1}{\epsilon}\,.$$
Therefore, g has at worst a removable singularity. Since g is not identically zeros, $\text{ord}(g,c) = k \geqslant 0$, and so, using the previous proposition, we have
$$\text{ord}(f,c) = \text{ord}(f-d,c) = -k\,.$$
This contradicts the assumption that f has an essential singularity at c. □

EXAMPLE 8.8. Show that $f(z) = \cos(\frac{1}{z})$ has an essential singularity at 0.

Solution: Since $\cos(\frac{1}{z})$ has Laurent expansion
$$\cos(\frac{1}{z}) = \sum_{n=-\infty}^{0} (-1)^{-n} \frac{z^{2n}}{(-2n)!} = 1 - \frac{1}{2!} z^{-2} + \frac{1}{4!} z^{-4} - \dots$$
the function f has an essential singularity at $z = 0$. The residue is $\text{res}(f,0) = 0$.

EXAMPLE 8.9. Show that $\frac{1}{\sin z}$ has a simple pole at $z = 0$.

Solution: We already showed that

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + O(z^3)\,.$$

Therefore, the least n with non-vanishing a_n is $n = -1$, so that the function has a simple pole at $z = 0$.

3. The Residue Theorem

We have already seen in Corollary 8.3 that if we integrate a meromorphic function on a contour such that there is only a single pole in the interior, then we obtain the residue up to $2\pi i$. More generally, we have the following:

THEOREM 8.10 (Residue Theorem). *Let γ be a contour, and let f be a function holomorphic in an open domain U containing $I(\gamma) \cup \gamma^*$, except for finally many poles at $c_1, \ldots, c_m \in I(\gamma)$. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} \operatorname{res}(f, c_k)\,.$$

PROOF. For $k = 1, 2, \ldots, m$, denote by f_k the principal part of f at c_k. Furthermore, let N_k the order of the pole c_k then the Laurent series of f at c_k is

$$f(z) = \sum_{n=-N_k}^{\infty} a_n^{(k)}(z - c_k)^n\,.$$

Then

$$f_k(z) = \sum_{n=-N_k}^{-1} a_n^{(k)}(z - c_k)^n\,.$$

Note that f_k has exactly one singularity, a pole of order N_k at c_k. Notice also that

$$f(z) - f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)}(z - c_k)^n$$

is holomorphic in some neighbourhood of c_k. Let now $g = f - (f_1 + f_2 + \ldots + f_m)$. When writing

$$g = (f - f_k) - \sum_{j \neq k} f_j$$

we see that g is holomorphic at c_k since both $f - f_k$ and all $f_j, j \neq k$ are holomorphic at c_k. Since all potential singularities of g are given by the c_k's, we conclude that g is holomorphic in U. Therefore, by Cauchy's Theorem

$$\int_\gamma g(z)dz = 0\,.$$

Therefore,

$$\int_\gamma f(z)dz = \sum_{k=1}^{m} \int_\gamma f_k(z)dz$$

and Corollary 8.3

$$\int_\gamma f_k(z)dz = 2\pi i a_{-1}^{(k)} = 2\pi i \operatorname{res}(f, c_k)\,.$$

□

Therefore, the key to integration around a contour is the calculation of residues, and it is important to be able to calculate those without computing the entire Laurent series. Simple poles are the easiest:

THEOREM 8.11. *Let f have a simple pole at c. Then*

$$\operatorname{res}(f,c) = \lim_{z\longrightarrow c}(z-c)f(z)\,.$$

Thus, if

$$f(z) = \frac{a}{z-c} + O(1)$$

then $\operatorname{res}(f,c) = a$.

PROOF. Suppose that f has a simple pole at c, so that the Laurent series is

$$f(z) = a_{-1}(z-c)^{-1} + \sum_{n=0}^{\infty} a_n(z-c)^n\,.$$

Then

$$\lim_{z\longrightarrow c}(z-c)f(z) = \lim_{z\longrightarrow c} a_{-1} + \sum_{n=0}^{\infty} a_n(z-c)^{n+1} = a_{-1} = \operatorname{res}(f,c)\,.$$

□

EXAMPLE 8.12. Evaluate

$$\int_\gamma \frac{\sin(\pi z)}{z^2+1}dz$$

where γ is any contour such that $i, -i \in I(\gamma)$.

Solution: The integrand $f(z) = \frac{\sin(\pi z)}{z^2+1}$ has simple poles at $\pm i$. Recalling that $\sin(iz) = i\sinh z$ we obtain form the previous theorem that

$$\begin{aligned}\operatorname{res}(f,i) &= \lim_{z\longrightarrow i}(z-i)\frac{\sin(\pi z)}{z^2+1} = \lim_{z\longrightarrow i}\frac{\sin(\pi z)}{z+i} = \frac{\sinh\pi}{2}\\ \operatorname{res}(f,-i) &= \lim_{z\longrightarrow -i}\frac{\sin(\pi z)}{z-i} = \frac{\sinh\pi}{2}\,.\end{aligned}$$

Hence by the Residue Theorem

$$\int_\gamma \frac{\sin(\pi z)}{z^2+1}dz = 2\pi i(\operatorname{res}(f,i) + \operatorname{res}(f,-i)) = 2i\pi\sinh\pi\,.$$

EXAMPLE 8.13. Evaluate

$$\int_\gamma \frac{1}{z^2+1}dz$$

where γ is the circle of radius 2, parametrised anticlockwise.

Solution: The poles of $f(z) = \frac{1}{z^2+1}$ are at $z = \pm i$. We have by partial fraction decomposition

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

so that

$$\operatorname{res}(f,i) = -\frac{i}{2}, \quad \operatorname{res}(f,-i) = \frac{i}{2}$$

and

$$\int_\gamma \frac{1}{z^2+1}dz = \pi i(-i+i) = 0\,.$$

In this example, the integrand $f(z)$ can be written as quotient of g/h where g, h are holomorphic and $h(c) = 0$. A technique applying to this situation is worth recording:

THEOREM 8.14. *Let $f(z) = \frac{g(z)}{h(z)}$ where both g and h are holomorphic in a neighbourhood of c, and additionally, $h(c) = 0$ but $h'(c) \neq 0$. Then*

$$\operatorname{res}(f, c) = \frac{g(c)}{h'(c)}\,.$$

PROOF. By limit laws we have

$$\operatorname{res}(f, c) = \lim_{z \longrightarrow c} g(z)\frac{z-c}{h(z)} = g(c) \lim_{z \longrightarrow c} \frac{z-c}{h(z) - h(c)} = \frac{g(c)}{h'(c)}\,.$$

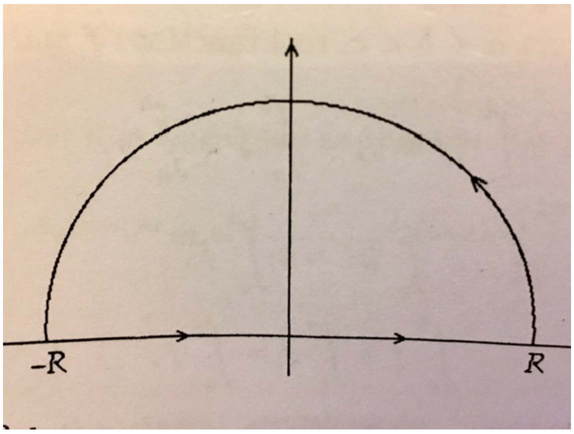
□

In the previous example this observation makes little difference, but it can help in other cases.

EXAMPLE 8.15. Evaluate

$$\int_\gamma \frac{1}{z^4+1} dz\,,$$

where γ is the semicircle $[-R, R] \cup \{z : |z| = R, \operatorname{Im} z > 0\}$, parametrised anticlockwise, $R > 1$.



Solution: The zeroes of $f(z) = \frac{1}{z^4+1}$ are at $z_k = e^{\frac{i\pi}{4} + i\frac{(k-1)\pi}{2}}$, $k = 1, 2, 3, 4$, but $z_3, z_4 \in \mathbb{C}\backslash(I(\gamma) \cup \gamma^*)$. We compute the residues at $z_1, z_2 \in I(\gamma) \cup \gamma^*$ by the previous theorem:

$$\operatorname{res}(f, e^{\frac{i\pi}{4}}) = -\frac{1+i}{4\sqrt{2}}, \qquad \operatorname{res}(f, e^{\frac{3i\pi}{4}}) = \frac{1-i}{4\sqrt{2}}\,.$$

Therefore,

$$\int_\gamma \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}\,.$$

(Note again that we used the deformation theorem!).

Similar to the previous theorem, one can obtain residues of higher order poles of f by evaluating higher order derivatives of $g = (z-c)^m f$, but in this case often it is easier to compute the Laurent expansion directly by different tools.

THEOREM 8.16. *Suppose that f has a pole of order m at c, so that*

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-c)^n \,,$$

and $a_{-m} \neq 0$. Then

$$\operatorname{res}(f,c) = \frac{1}{(m-1)!} g^{(m-1)}(c)\,,$$

where $g(z) = (z-c)^m f(z)$.

PROOF. By definition we have

$$g(z) = a_{-m} + a_{-m+1}(z-c) + \ldots + a_{-1}(z-c)^{m-1} + \ldots$$

Differentiating $m-1$ times, we deduce that

$$g^{(m-1)}(z) = (m-1)!\, a_{-1} + \text{ positive powers of } (z-c)\,.$$

Therefore,

$$g^{(m-1)}(c) = (m-1)! \operatorname{res}(f,c)\,.$$

□

EXAMPLE 8.17. Evaluate

$$\int_\gamma \frac{1}{(z^2+1)^2} dz$$

where γ is as before the semicircle $[-R,R] \cup \{z : |z| = R, \operatorname{Im} z > 0\}$, parametrised anticlockwise, $R > 1$.

Solution: The integrand $f(z) = \frac{1}{(z^2+1)^2}$ has a double pole at i in $I(\gamma)$. Moreover,

$$g(z) = (z-i)^2 f(z) = \frac{1}{(z+i)^2}\,.$$

We have $g'(z) = -\frac{2}{(z+i)^3}$ and so

$$\operatorname{res}(f,i) = g'(i) = \frac{1}{4i}\,.$$

Hence

$$\int_\gamma \frac{1}{(z^2+1)^2} dz = \frac{\pi}{2}\,.$$

EXAMPLE 8.18. Find the residue of $f(z) = \frac{1}{z^2 \sin z}$ at the triple pole $z = 0$.

Solution: Recall that we derived before that $\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + O(z^3)$. Therefore,

$$f(z) = \frac{1}{z^2}\left(\frac{1}{z} + \frac{z}{6} + O(z^3)\right) = \frac{1}{z^3} + \frac{1}{6z} + O(z)$$

so that the residue of f at 0 is $\operatorname{res}(f,0) = \frac{1}{z}$.

The alternative method, which involves calculating $\lim_{z->0} g''(z)$ for $g = z^3 f(z) = \frac{z}{\sin z}$, is much harder.

In real analysis, we can consider $\frac{f'}{f} = (\log f)'$ for real-valued positive functions. In Complex Analysis we can use the Residue Theorem to evaluate the integral around a contour by using the residues at the singularities.

THEOREM 8.19. *Let γ be a contour, $f : U \longrightarrow \mathbb{C}$ meromorphic, $I(\gamma) \cup \gamma^* \subset U$. Suppose that $Q = \{q \in I(\gamma) \mid \operatorname{ord}(f,q) \neq 0\}$ is finite. Then*

$$\frac{1}{2\pi} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{q \in Q} \operatorname{ord}(f,q)\,.$$

Proof. The function $\frac{f'}{f}$ is differentiable in $I(\gamma)\backslash Q$. Suppose that $m = \text{ord}(f, q)$ for $q \in Q$. Then $f(z) = (z-q)^m g(z)$ for holomorphic g with $g(z) \neq 0$. Then

$$f'(z) = m(z-q)^{m-1}g(z) + (z-q)^m g'(z)\,,$$

so that

$$\frac{f'(z)}{f(z)} = \frac{m}{z-q} + \frac{g'(z)}{g(z)}\,.$$

Since $\frac{g'}{g}$ is holomorphic at q we deduce that $\text{res}(\frac{f'}{f}, q) = m = \text{ord}(f, q)$. The result now follows from the Residue Theorem. □

Example 8.20. Let $f(z) = \frac{(z-1)(z-2)^3}{(z-3)^2(z-4)^2(z-5)^2}$, $\gamma = \kappa(0, 6)$. Compute

$$\int_\gamma \frac{f'(z)}{f(z)} dz$$

Solution:

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i((1+3) - (2+2+2)) = -4\pi i\,.$$

One can compute

$$\frac{f'(z)}{f(z)} = \frac{-2z^4 + 13z^3 - 2z^2 - 97z + 112}{(z-5)(z-4)(z-3)(z-2)(z-1)} = -\frac{2}{z-4} - \frac{2}{z-3} + \frac{3}{z-2} + \frac{1}{z-1} - \frac{2}{z-5}\,.$$

4. Further results

There are lots of applications of Complex Analysis in various areas of mathematics, physics, etc. Some notable results/methods are listed here if you are interested to look into further material.

A. Real integrals. It is often useful to integrate functions in two real variables by transferring them to complex integrals. For example, if an integral is of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$$

one can view $\cos\theta = \text{Re}\,(e^{i\theta}), \sin\theta = \text{Im}\,(e^{i\theta})$. Writing $z = e^{i\theta}$ we thus have $\cos\theta = \frac{1}{2}(z + \frac{1}{z}), \sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, and the integral can be expressed as

$$\int_{\kappa(0,1)} g(z) dz\,.$$

Example 8.21. Evaluate $\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta$, $0 < b < a$.

Solution:

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta = -2i \int_{\kappa(0,1)} \frac{1}{bz^2 + 2az + b} dz$$

Denoting the zeroes of $bz^2 + 2az + b = 0$ by α, β we see that one of the singularities does not lie within the contour $\kappa(0, 1)$. For the other one can compute the residue to be $\frac{1}{2\sqrt{a^2 - b^2}}$ so that

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}\,.$$

B. Rouche's Theorem. Rouche's Theorem is a consequence of the Residue Theorem and gives under suitable assumptions the integral of $\frac{f' + g'}{f + g}$ as the integral of $\frac{f}{g}$.
It gives an immediate proof of the Fundamental Theorem of Algebra.

C. The Open Mapping Theorem. Every non-constant holomorphic map maps an open set U in its domain to an open set $f(U)$.

A consequence of the Open Mapping Theorem is the Maximum Modulus Theorem: If f is holomorphic, non-constant, on a domain U and γ a contour with $I(\gamma) \cup \gamma^* \subset U$ and $M = \sup\{|f(z)| : z \in I(\gamma) \cup \gamma^*\}$, then $|f(z)| < M$ for all $z \in I(\gamma)$.

Additionally, the Inverse Function Theorem holds: A holomorphic function with $f'(c) \neq 0$ for some c, is bijective in a neighbourhood of c and its inverse function has derivative $\frac{1}{f'(f^{-1}(z))}$.

D. Riemann's Zeta Function. Riemann's Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad \operatorname{Re} s > 1$$

plays an important role in number theory, as it is linked to the set of all prime numbers.
The Riemann Hypothesis states that all zeros of ζ are in the strip $\{s : 0 \leqslant \operatorname{Re} s \leqslant 1\}$. A proof of this conjecture would have wide-reaching consequences in number theory.

E. Julia sets and Fractals. Julia sets are given by an iteration of holomorphic functions: these iterations play an important role in dynamical systems. The Mandelbrot set (you can see an image of that set on the front page of these lecture notes!) is one of the most famous associated fractal. It is the set of complex numbers for which the function $f_c(z) = z^2 + c$ does not diverge to infinity when iterated at $z = 0$, that is, all $c \in \mathbb{C}$ such that $f_c(0), f_c(f_c(0)), \ldots$ remains bounded.

APPENDIX A

The Field $\mathbb{C}$ of Complex Numbers

The following is from the Lecture Notes from Frank Neumann. There is some overlap with the previously stated material but in some cases more technical details and examples are provided. Please compare also to the book by Howie, chapter 1-3.

Question: Why do we need complex numbers?

They appear naturally when solving general **quadratic equations**

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{Q}$ and $a \neq 0$.

We can further assume that $a, b, c \in \mathbb{Z}$, otherwise we just multiply through with all denominators to get an equivalent equation with integer coefficients.

The general solution is given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $\Delta := b^2 - 4ac$ be the **discriminant**. If $\Delta = r^2$, then the equation has solutions in the field $\mathbb{Q}$ of rational numbers, in particular if $\Delta = 0$ we only have one solution $x = -\frac{b}{2a}$.

If $\Delta > 0$ then we obtain two solutions in the field $\mathbb{R}$ of real numbers.
If $\Delta < 0$ then there is no solution in $\mathbb{R}$.

Question: Can we extend the field $\mathbb{R}$ to a new field of numbers, such that there exist solutions when $\Delta < 0$?

Can modify the general solution:

$$x = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} = \frac{-b \pm \sqrt{(-1)}\sqrt{4ac - b^2}}{2a},$$

where $4ac - b^2 > 0$. So if we postulate the existence of the number $i := \sqrt{-1}$, we get a formal solution, but not in $\mathbb{R}$.

Example A.1. Solve the quadratic equation

$$x^2 + 4x + 13 = 0.$$

Then $x = -2 \pm 3i$ are the two solutions. Check by inserting into the equation and use that $i^2 = -1$.

Definition. Let $\mathbb{C}$ be the set of all numbers $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 := -1$. We define:

- ◇ (A) Addition:

$$+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (a_1 + b_1 i) + (a_2 + b_2 i) := (a_1 + a_2) + (b_1 + b_2)i.$$

- ◇ (M) Multiplication:

$$\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ (a_1 + b_1 i) \cdot (a_2 + b_2 i) := (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i.$$

$\mathbb{C}$ is called the set of **complex numbers**.

REMARK. It can be shown by checking the field axioms from algebra that $\mathbb{C}$ is indeed a field, the **field $\mathbb{C}$ of complex numbers** and $\mathbb{R}$ is a subfield of $\mathbb{C}$, i.e. $\mathbb{R} \subset \mathbb{C}$. Every $r \in \mathbb{R}$ can be written as $r = r + 0i$.

Question: Do we need to extend the field $\mathbb{C}$ further?
No, it is not necessary as we have the following theorem which we will prove later:

THEOREM A.2 (Fundamental Theorem of Algebra). *For all $n \geqslant 1$ every polynomial equation*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 = 0$$

with $z \in \mathbb{C}$ and coefficients $a_0, a_1, \ldots a_n \in \mathbb{C}$ with $a_n \neq 0$ has all its solutions within $\mathbb{C}$.

We can visualise any complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$ on the Euclidean plane.
Real numbers x appear as points $(x, 0)$ on the x-axis.
Pure imaginary numbers yi appear as points $(0, y)$ on the y-axis.

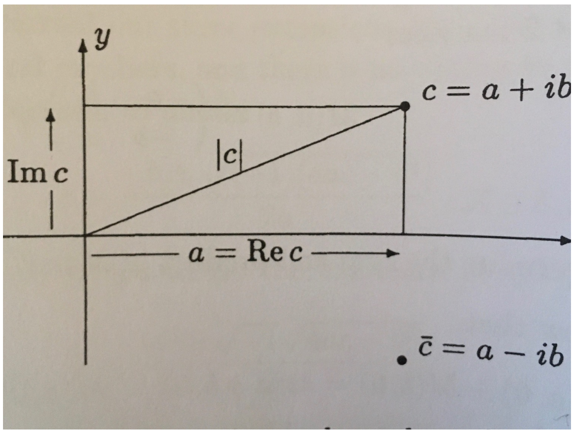
The x-axis is also referred to as the **real axis**.
The y-axis is also referred to as the **imaginary axis**.

DEFINITION. If $z = x + iy$ with $x, y \in \mathbb{R}$, then x is the **real part** of z, written $x = Re(z)$.
If $z = x + iy$ with $x, y \in \mathbb{R}$, then y is the **imaginary part** of z, written $y = Im(z)$.
The complex number $\overline{z} := x - iy$ is called the **conjugate** of z.

PROPOSITION A.3. *For all $z, w \in \mathbb{C}$ we have:*

(1) $\overline{\overline{z}} = z$, $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$.
(2) $z + \overline{z} = 2\,Re(z)$, $z - \overline{z} = 2i\,Im(z)$.
(3) $\overline{z} = z$ *if and only if z is real and* $\overline{z} = -z$ *if and only if z is pure imaginary.*

We have the following geometric representation of a complex number $c = a + bi$.



REMARK. The product $c\overline{c} = a^2 + b^2$ and $|c| := \sqrt{c\overline{c}} = \sqrt{a^2 + b^2}$ is the **modulus** of c and measures the distance of the point $(a, b) \in \mathbb{R}^2$ from the origin. If $c \in \mathbb{R}$ then $|c|$ is just the absolute value of c.

THEOREM A.4. *Let $z, w \in \mathbb{C}$. Then we have:*

(i) $|Re(z)| \leqslant |z|$, $|Im(z)| \leqslant |z|$, $|\overline{z}| = |z|$.
(ii) $|zw| = |z||w|$.
(iii) $|z + w| \leqslant |z| + |w|$.
(iv) $|z - w| \geqslant ||z| - |w||$.

Proof: (i) follows immediately from the definitions. Check!
(ii) We have:

$$|zw|^2 = (zw)(\overline{zw}) = (z\overline{z})(w\overline{w}) = (|z||w|)^2.$$

and the result follows by taking square roots. (iii) We have:

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$$

Now also

$$z\overline{w} + w\overline{z} = z\overline{w} + \overline{z\overline{w}} = 2\,Re(z\overline{w}) \leqslant 2|z\overline{w}| = 2|z||\overline{w}| = 2|z||w|,$$

and so

$$|z+w|^2 \leqslant |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2.$$

The result follows by taking square roots.

(iv) We observe that
$|z| = |(z-w)+w| \leqslant |z-w| + |w|$ implies $|z-w| \geqslant |z| - |w|$.
Similarly,
$|w| = |z-(z-w)| \leqslant |z| + |z-w|$ implies $|z-w| \geqslant |w| - |z|$.

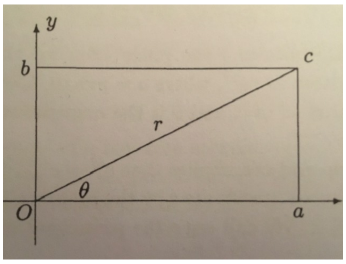
Since for any $x \in \mathbb{R}$ we have $|x| = \max\{x, -x\}$ these together imply now:

$$|z-w| \geqslant ||z| - |w||.$$

This finishes the proof.

$\square$

We will also consider the **Argand diagram**:



We identify a complex number $c = a + bi$ with the point $(a,b) \in \mathbb{R}^2$. The point $c = (a,b)$ lies on the circle around the origin 0 given by the equation $x^2 + y^2 = r^2$ where $r = |c| = \sqrt{a^2+b^2}$.
If $c \neq 0$, there is a unique $\theta \in (-\pi, \pi]$ such that

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

So we can write the point $c = (a,b)$ in **polar coordinates** as

$$c = r(\cos\theta + i\sin\theta),$$

where $r(\cos\theta + i\sin\theta)$ is called the **polar form** of c.
Using some trigonometry we get:

$$\begin{aligned} & (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ = \; & (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ = \; & \cos(\theta+\phi) + i\sin(\theta+\phi). \end{aligned}$$

By extending series definition of exponential function to complex number values we get for any $\theta \in \mathbb{R}$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots \\ &= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots) + i(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Using this expansion we get:

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}.$$

We also get:

$$e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin(\theta)$$

which implies:

PROPOSITION A.5 (Euler's formulae).

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

So we can write the polar form for a complex number $c = a + bi$ as

$$c = a + bi = re^{i\theta}$$

where $a = r\cos\theta$, $b = r\sin\theta$.

We call $r = |c|$ the **modulus** of c and $\theta =: \arg c$ the **argument** of c.

The polar form of the conjugate of c is $\bar{c} = re^{-i\theta}$.

The periodicity of the sine and cosine function give for every $n \in \mathbb{Z}$: $e^{i\theta} = e^{i(\theta+2n\pi)}$

We call $\arg c$ the **principal argument** for $c = e^{i\theta}$ with $-\pi < \theta \leqslant \pi$.

Multiplication of complex numbers can now be easily described in polar form:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1+\theta_2)}.$$

Because $|c_1 c_2| = |c_1||c_2|$ for any $c_1, c_2 \in \mathbb{C}$ we have:

$$\arg(c_1 c_2) \equiv \arg c_1 + \arg c_2 \pmod{2\pi},$$

i.e. $\arg(c_1 c_2) = \arg c_1 + \arg c_2 + 2k\pi$ for $k \in \mathbb{Z}$.

More generally we have by induction:

PROPOSITION A.6. *Let $c, c_1, c_2, \dots c_n \in \mathbb{C}$ and $n \in \mathbb{N}$ Then we have:*

(i) $|c_1 c_2 \dots c_n| = |c_1||c_2| \dots |c_n|$.
(ii) $\arg c_1 c_2 \dots c_n \equiv \arg c_1 + \arg c_2 + \cdots \arg c_n \pmod{2\pi}$.
(iii) $|c^n| = |c|^n$ *and* $\arg c^n \equiv n \arg c \pmod{2\pi}$.

EXAMPLE A.7. Determine the modulus and argument of c^5 where $c = 1 + i\sqrt{3}$.
We have $|c| = 2$ and $\arg c = \theta$, where $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$, i.e. $\arg c = \pi/3$. So $|c^5| = 2^5 = 32$ and $\arg(c^5) \equiv 5\pi/3 \equiv -\pi/3$.
The standard form of

$$c^5 = 32(\cos(-\pi/3) + i\sin(-\pi/3)) = 16(1 - i\sqrt{3}).$$

How to determine the inverse or reciprocal of a given $c \in \mathbb{C}$?
If $c = re^{i\theta}$, then

$$\frac{1}{c} = \frac{1}{r}e^{-i\theta}.$$

If $c = a + bi$, then

$$\frac{1}{c} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

EXAMPLE A.8. Express

$$\frac{3+7i}{2+5i}$$

in standard form as $a + bi$.

$$\frac{3+7i}{2+5i} = \frac{(3+7i)(2-5i)}{(2+5i)(2-5i)} = \frac{1}{29}(41 - i).$$

EXAMPLE A.9. Express
$$\frac{3+2i}{1+i}$$
in standard form as $a+bi$.
$$\frac{3+2i}{1+i} = \frac{(3+2i)(1-i)}{(1+i)(1-i)} = \frac{1}{2}(5-i).$$

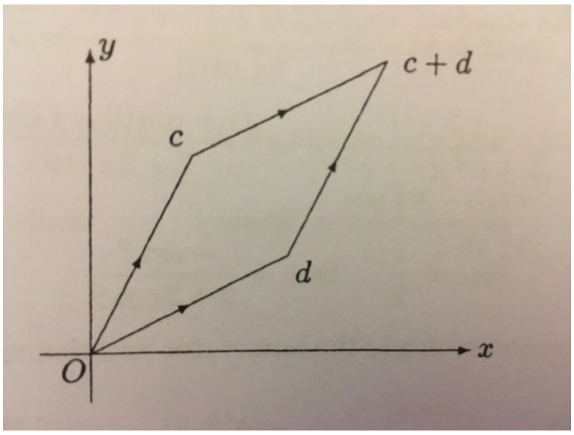
PROPOSITION A.10. *Every complex number $z \in \mathbb{C}$ has a square root $\sqrt{z} \in \mathbb{C}$.*

Proof:
For $z = re^{i\theta}$ we have: $\sqrt{r}e^{i(\theta/2)}\sqrt{r}e^{i(\theta/2)} = re^{i\theta} = z$, so $\sqrt{z} = \sqrt{r}e^{i(\theta/2)}$. $\square$

Every quadratic equation
$$az^2 + bz + c = 0$$
with $a, b, c \in \mathbb{C}$ and $a \neq 0$ has a solution in $z \in \mathbb{C}$, namely
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
We can simply do the same procedure of 'completing the square' in $\mathbb{C}$ as we can do in $\mathbb{R}$, but now we always get a solution.

EXAMPLE A.11. Solve the quadratic equation
$$z^2 + 2iz + (2-4i) = 0.$$
By the formula the solution is given as:
$$z = \frac{1}{2}(-2i \pm \sqrt{(-2i)^2 - 4(2-4i)}) = \frac{1}{2}(-2i \pm \sqrt{-12+16i} = -i \pm \sqrt{-3+4i}.$$
But we also have
$$(1+2i)^2 = -3+4i.$$
So we can write for the solution:
$$z = -i \pm (1+2i),$$
i.e. $z = 1+i$ or $z = -1-3i$.

The addition in $\mathbb{C}$ can be described geometrically as vector addition in $\mathbb{R}^2$:



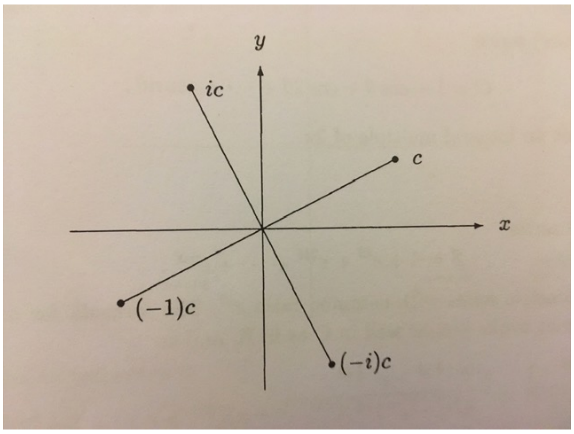
The multiplication in $\mathbb{C}$ can also be described geometrically in $\mathbb{R}^2$ using the polar form:

If we multiply $c \in \mathbb{C}$ by $re^{i\theta}$ we multiply $|c|$ by a factor of r and we add θ to $\arg c$.

If $r = 1$, multiplication by $e^{i\theta}$ is given as rotation by angle θ.

EXAMPLE A.12. We have:
(i) multiplication by $-1 = e^{i\pi}$ is rotation by π
(ii) multiplication by $i = e^{i\pi/2}$ is rotation by $\pi/2$

(iii) multiplication by $-i = e^{-i\pi/2}$ is rotation by $-\pi/2$



Example A.13. Find the real and imaginary parts of $c = 1/(1+e^{i\theta})$.

Solution: We have:

$$c = \frac{1+e^{-i\theta}}{(1+e^{i\theta})(1+e^{-i\theta})} = \frac{(1+\cos\theta) - i\sin\theta}{2+2\cos\theta}$$

so get
$Re(c) = \frac{1}{2}$ and $Im(c) = -\frac{\sin\theta}{(2+2\cos\theta)}$.
Could instead also multiply through with $e^{-i(\theta/2)}$:

$$c = \frac{e^{-i(\theta/2)}}{e^{-i(\theta/2)} + e^{i(\theta/2)}} = \frac{\cos(\theta/2) - i\sin(\theta/2)}{2\cos(\theta/2)}$$

hence
$Re(c) = \frac{1}{2}$ and $Im(c) = -\frac{1}{2}\tan(\theta/2)$.

Example A.14. Sum the (finite) series:

$$C = 1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta),$$

where $\theta \neq 2k\pi$ for any $k \in \mathbb{Z}$.

Solution: Consider the geometric series

$$Z = 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta}.$$

We get using Euler's formula:

$$Z = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} = \frac{e^{i(n+1/2)\theta} - e^{-1/2\, i\theta}}{e^{1/2\, i\theta} - e^{-1/2\, i\theta}} = \frac{e^{i(n+1/2)\theta} - e^{-1/2\, i\theta}}{2i\sin(1/2\,\theta)}.$$

As $1/i = -i$ we get:

$$Z = \frac{-i(\cos((n+1/2)\theta) + i\sin((n+1/2)\theta)) + i(\cos(1/2\,\theta) - i\sin(1/2\,\theta))}{2sin(1/2\,\theta)}$$

therefore

$$Z = \frac{(\sin((n+1/2)\theta) + \sin(1/2\,\theta)) + i(\cos(1/2\,\theta) - \cos((n+1/2)\theta))}{2sin(1/2\,\theta)}.$$

Equating the real parts gives ($\theta \neq 2k\pi$):

$$C = \frac{\sin((n+1/2)\theta) + \sin(1/2\,\theta)}{2sin(1/2\,\theta)}.$$

Equating the imaginary parts gives as bonus:

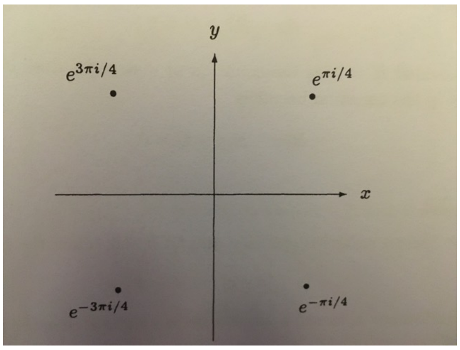
$$\sin(\theta) + \sin(2\theta) + \ldots \sin(n\theta) = \frac{\cos(1/2\,\theta) - \cos((n+1/2)\,\theta)}{2sin(1/2\,\theta)}.$$

Example A.15. Find all roots of the equation

$$z^4 + 1 = 0.$$

Factorise this polynomial in the field $\mathbb{C}$ and in the field $\mathbb{R}$.

Solution: We have:
$z^4 = -1 = e^{i\pi}$ if and only if $z = e^{\pm\pi i/4}$ or $z = e^{\pm 3\pi i/4}$.
The roots all lie in the unit circle, and are equally spaced.



The factorisation in $\mathbb{C}$ is then given by

$$z^4 + 1 = (z - e^{\pi i/4})(z - e^{-\pi i/4})(z - e^{3\pi i/4})(z - e^{-3\pi i/4}).$$

The factorisation in $\mathbb{R}$ is obtained by combining conjugate factors:

$$z^4 + 1 = (z^2 - 2z\cos(\pi/4) + 1)(z^2 - 2z\cos(3\pi/4) + 1)$$

And so finally:

$$z^4 + 1 = (z^2 - z\sqrt{2} + 1)(z^2 + z\sqrt{2} + 1).$$

Remark. We can study geometrical objects in the Euclidean plane $\mathbb{R}^2$ by using complex numbers as operations of complex numbers are given by transformations of the plane.

Example A.16. The circle C with centre $c \in \mathbb{C}$ and radius $r = |c| \geqslant 0$:

$$C = \{z \in \mathbb{C} : |z - c| = r\}.$$

If $z = x + iy$ and $c = a + bi$, then we have:
$|z - c| = r$ if and only if $|z - c|^2 = r^2$ if and only if $(x - a)^2 + (y - b)^2 = r^2$.

The general equation of a circle as a quadratic polynomial in two variables

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

can be rewritten then as (with $h = g - if$).

$$z\overline{z} + hz + \overline{hz} + c = 0.$$

Let us look at the general equation:

$$Az\overline{z} + Bz + \overline{Bz} + C = 0,$$

where $A, C \in \mathbb{R}$, $A \neq 0$ and $B \in \mathbb{C}$. Then the set

$$C = \{z \in \mathbb{C} : Az\overline{z} + Bz + \overline{Bz} + C = 0\}$$

is:

(C1) a circle with centre $-\overline{B}/A$ and radius R where $R^2 = (B\overline{B} - AC)/A^2$ if $B\overline{B} - AC \geqslant 0$.
(C2) the empty set if $B\overline{B} - AC < 0$.

If $A = 0$ we get:

$$Bz + \overline{Bz} + C = 0,$$

which is the equation of a straight line. If $B = B_1 + iB_2$ and $z = x + iy$, then the equation becomes:

$$B_1x - B_2y + C = 0.$$

APPENDIX B

Functions in one complex variable

Aim. We want to develop calculus and analysis in the field $\mathbb{C}$ of complex numbers!

Need to define limits, sequences, continuous functions, derivations...

DEFINITION. The **distance** between two complex numbers $a, b \in \mathbb{C}$ is given as:

$$|a - b| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

where $a = a_1 + ia_2$ and $b = b_1 + ib_2$.

Question. Do we have an order relation $\leqslant$ in $\mathbb{C}$ similar as in $\mathbb{R}$?

Answer. No. It is not possible, as we would need to have $a^2 > 0$ for all $a \in \mathbb{C}$ with $a \neq 0$, but we have: $i^2 = -1 < 0$. So complex numbers can't be ordered!

DEFINITION. A sequence (c_n) of complex numbers is **convergent with limit** γ if for every $\epsilon > 0$ there exists a natural number $N = N(\epsilon) \in \mathbb{N}$ such that

$$|c_n - \gamma| < \epsilon$$

for all natural numbers $n > N$.

DEFINITION. A sequence (c_n) of complex numbers is called a **Cauchy sequence** if for every $\epsilon > 0$ there exists a natural number $N = N(\epsilon) \in \mathbb{N}$ such

$$|c_m - c_n| < \epsilon$$

for all natural numbers $m, n > N$.

THEOREM B.1 (Completeness Property of $\mathbb{C}$). *If (c_n) is a Cauchy sequence in $\mathbb{C}$, then (c_n) is a convergent sequence.*

Proof. We know the statement is true in $\mathbb{R}$. Write $c_n = a_n + ib_n$ with $a_n, b_n \in \mathbb{R}$. Assume (c_n) is a Cauchy sequence, i.e. for all $\epsilon > 0$ there exists N such that

$$|c_m - c_n| < \epsilon$$

for all natural numbers $m, n > N$. Then

$$\begin{aligned} |c_m - c_n| &= |(a_m - a_n) + i(b_m - b_n)| \\ &= ((a_m - a_n)^2 + (b_m - b_n)^2)^{1/2} \\ &\geqslant \max\{|a_m - a_n|, |b_m - b_n|\}. \end{aligned}$$

$(a_n), (b_n)$ are both Cauchy sequences in $\mathbb{R}$, so are convergent with limits α, β respectively, i.e. there exists N_1 such that $|a_n - \alpha| < \epsilon/2$ for all $n > N_1$ and an N_2 such that $|b_n - \beta| < \epsilon/2$ for all $n > N_2$. Hence for all $n > \max\{N_1, N_2\}$

$$|c_n - (\alpha + i\beta)| = |(a_n - \alpha) + i(b_n - \beta)| \leqslant |a_n - \alpha| + |b_n - \beta| < \epsilon,$$

i.e. $\lim_{n\to\infty} c_n = \gamma := \alpha + i\beta$, so (c_n) is convergent with limit γ. □

THEOREM B.2 (General Principle of Convergence). *Let $\sum_{n=1}^{\infty} c_n$ be a series of complex numbers. If, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\left| \sum_{r=n+1}^{m} c_r \right| < \epsilon$$

for all $m > n > N$, then the series $\sum_{n=1}^{\infty} c_n$ is convergent.

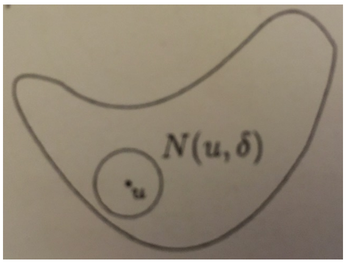
Proof. This follows from the previous theorem by applying it to the sequence of partial sums (C_n) with $C_n := \sum_{k=1}^{n} c_k$. $\square$

Remark. Any constructions and definitions involving the modulus $|a|$ or distance $|a-b|$ can be extended from the field $\mathbb{R}$ of real numbers $\mathbb{R}$ to the field $\mathbb{C}$ of complex numbers. But the geometry of $\mathbb{C}$ is the geometry of the plane, while the geometry of $\mathbb{R}$ is the geometry of the line. Instead of dealing with intervals like $[a,b] \subset \mathbb{R}$ we will have to deal with subsets in the plane.

Definition. Let $c \in \mathbb{C}$, $r \in \mathbb{R}$ with $r > 0$. The **neighbourhood** $N(c,r)$ with radius r around c is defined as

$$N(c,r) := \{z \in \mathbb{C} : |z-c| < r\}.$$

Definition. A subset $U \subset \mathbb{C}$ is called **open** if for all $u \in U$ there exists $\delta > 0$ such that $N(u,\delta) \subset U$. A subset $D \subset \mathbb{C}$ is called **closed** if its complement $\mathbb{C} \backslash D$ in $\mathbb{C}$ is open, i.e. for all $z \notin D$ there exists $\delta > 0$ such that $N(z,\delta)$ lies wholly outside D.



Example B.3. (i) The set $\mathbb{C}$ and the empty set $\varnothing$ are both open and closed.
(ii) Any subset of $\mathbb{C}$ with finitely many elements is closed.
(iii) For given $c \in \mathbb{C}$, $r \in \mathbb{R}$, the sets $N(c,r) := \{z \in \mathbb{C} : |z-c| < r\}$ and $\{z \in \mathbb{C} : |z-c| > r\}$ are open.
(iv) For given $c \in \mathbb{C}$, $r \in \mathbb{R}$, the sets $\overline{N}(c,r) := \{z \in \mathbb{C} : |z-c| \leqslant r\}$ and $\{z \in \mathbb{C} : |z-c| \geqslant r\}$ are closed.

Theorem B.4. *Let S be a non-empty subset of $\mathbb{C}$. Then we have:*
(i) $I(S)$ is open and $I(S) = S$ if and only if S is open.
(ii) $\overline{S}$ is closed and $S = \overline{S}$ if and only if S is closed.
(iii) $\mathbb{C}$ and $\varnothing$ are the only subsets of $\mathbb{C}$ which are both open and closed.

Definition. We will use following notations and terminolgy:
(i) $N(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ an **open neighbourhood** or **open disc**.
(ii) $\overline{N}(a,r) := \{z \in \mathbb{C} : |z-a| \leqslant r\} = \overline{N(a,r)}$ a **closed disc**.
(iii) $\kappa(a,r) := \{z \in \mathbb{C} : |z-a| = r\} = \partial N(a,r) = \partial\overline{N}(a,r)$ a **circle**.
(iv) $D'(a,r) := \{z \in \mathbb{C} : 0 < |z-a| < r\} = N(a,r)\backslash\{a\}$ a **punctured disc**.

Aim. We aim to study functions from the complex numbers with values in the complex numbers and analyse their fundamental properties like continuity, differentiability or integrability.

Let us look at complex valued functions $f : \mathbb{C} \to \mathbb{C}$. If $z \in \mathbb{C}$ with $z = x + iy$ and $f : \mathbb{C} \to \mathbb{C}$ is a function, then there exist real valued functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(z) = u(x,y) + iv(x,y).$$

We call $u = Re(f)$ the **real** part of f and $v = Im(f)$ the **imaginary** part of f. The **modulus function** is $|f| : \mathbb{C} \to \mathbb{R}$, $|f|(z) := |f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}$.

Example B.5. Let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then we have:

$$f(z) = (x+iy)^2 = (x^2 - y^2) + i(2xy).$$

So we have:

$$u : \mathbb{R}^2 \to \mathbb{R},\ u(x,y) = x^2 - y^2,\ \ v : \mathbb{R}^2 \to \mathbb{R},\ v(x,y) = 2xy.$$

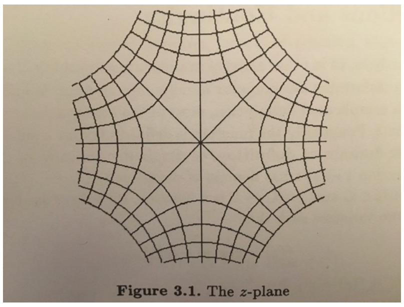


Figure 3.1. The z-plane

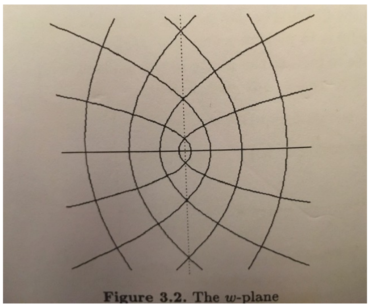


Figure 3.2. The w-plane

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. The **modulus function** is given by:

$$|f| : \mathbb{C} \to \mathbb{R}, \; |f|(z) := |f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}.$$

Example B.6. Let $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2$. Then we have:

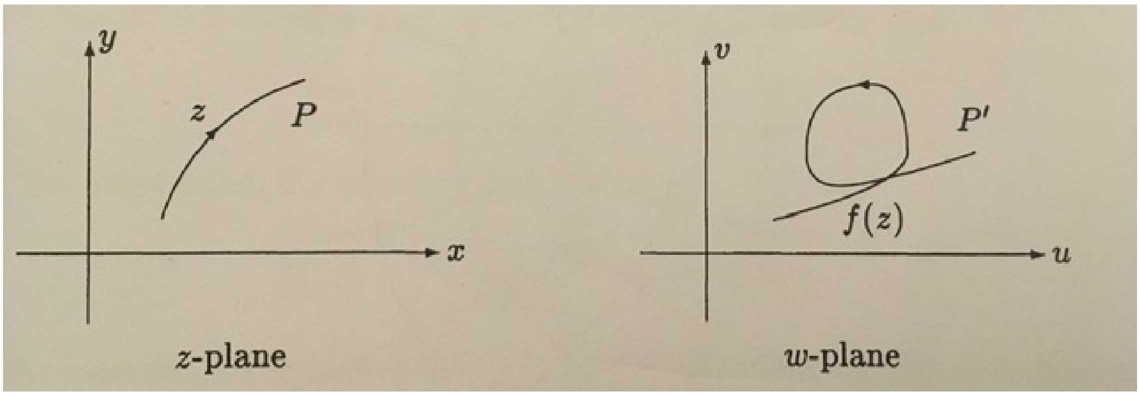
$$|f|(z) = |f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2} = \sqrt{(x^2 - y^2)^2 + (2xy)^2}.$$

So we get:

$$|f|(z) = |f(z)| = x^2 + y^2.$$

Remark. We can't draw graphs of complex functions $f : \mathbb{C} \to \mathbb{C}$ as we do with graphs of real functions, because the graph $\{(z, f(z)) : z \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C}$ is 4-dimensional. But we can use its modulus function $|f|$ to visualise the geometry of the function f.

In order to picture a complex function $f : \mathbb{C} \to \mathbb{C}$ it is also useful to picture $z = x + iy$ and $w := f(z) = u + iv$ in two complex planes. For example, we could picture the image P' under f of a path P in the z-plane in the w-plane. As the point z moves along the path P in the z-plane, its image $f(z)$ moves along the path P' in the w-plane.



Example B.7. For the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$, the hyperbolic curves $x^2 - y^2 = k$ and $2xy = l$ in the z-plane transform respectively to the straight lines in the w-plane

$$u = k, \;\; v = l.$$

Similarly, one can check that the straight lines $x = k$ and $y = l$ in the z-plane transform respectively to the parabolic curves in the w-plane

$$u^2 = 4k^2(k^2 - u), \;\; v^2 = 4l^2(u + l^2).$$

(see Fig. 3.1 and Fig. 3.2 above)

Definition (Limit). Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $l, c \in \mathbb{C}$. We say that

$$\lim_{z \to c} f(z) = l$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - l| < \epsilon$$

for all z such that $0 < |z - c| < \delta$.

Remark. Formally limits for complex functions are defined as for real functions, but now the set $0 < |z - c| < \delta$ is not a punctured interval, but a punctured disc! So while for real numbers on the line $\mathbb{R}$ we have only two directions to approach a point $c \in \mathbb{R}$, for complex numbers on the plane $\mathbb{C}$ there are infinitely many paths to approach a point $c \in \mathbb{C}$, so the limit should exist for every possible path!

Example B.8. Let $f(z = x + iy) = u(x, y) + iv(x, y)$ with $v(x, y) = 0$ for all $x, y \in \mathbb{R}$ and

$$u(x, y) = \frac{xy}{x^2 + y^2}, \ ((x, y) \neq (0, 0)).$$

Show that $\lim_{x\to 0} f(x + i0)$ and $\lim_{y\to 0} f(0 + iy)$ both exist, but $\lim_{z\to 0} f(z)$ does not exist.

Solution. Since $u(x, 0) = 0$ for all $x \in \mathbb{R}$ and $u(0, y) = 0$ for all $y \in \mathbb{R}$ we have:

$$\lim_{x\to 0} f(x + i0) = \lim_{y\to 0} f(0 + iy) = 0.$$

If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, then $|f(z)| = \left|\frac{r^2\cos\theta\sin\theta}{r^2}\right| = |\cos\theta\sin\theta|$. So, if e.g. $\theta = \pi/4$, then $|f(z)| = 1/2$ for all r, so the limit $\lim_{z\to 0} f(z)$ can't exist, because inside $N(0, \epsilon)$ there is a $z = (\epsilon/2) + i0$ with $f(z) = 0$ and $z = (\epsilon/2)e^{i\pi/4}$ with $f(z) = 1/2$.

Definition (Continuity). A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous at a point** $c \in \mathbb{C}$ if

$$\lim_{z\to c} f(z) = f(c),$$

i.e. if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(c)| < \epsilon$$

for all $z \in D'(c, \delta) = \{z \in \mathbb{C} : 0 < |z - c| < \delta\}$. The function f is **continuous** if it is continuous at *every* point $c \in \mathbb{C}$.

Example B.9. Show that the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = |z|^2$ is a continuous function, i.e. continuous at every point $c \in \mathbb{C}$.

Solution. Let $\epsilon > 0$ and $c \in \mathbb{C}$. Then

$$|f(z) - f(c)| = ||z|^2 - |c|^2| = ||z| - |c||(|z| + |c|) \leqslant |z - c|(|z| + |c|)$$

Let $\delta \leqslant 1$. Then $0 < |z - c| < 1$ implies $|z| - |c| < 1$ and so $|z| < |c| + 1$.
Hence

$$|f(z) - f(c)| \leqslant (2|c| + 1)|z - c|.$$

Hence, if $\delta = \min\{1, \epsilon/(2|c| + 1)\}$, then if $z \in D'(c, \delta)$ we have $|f(z) - f(c)| < \epsilon$. So f is continuous at c.

Remark. The standard rules for calculus with limits hold verbatim for complex functions as they hold for real functions.
If $\lim_{z\to c} f(z) = l$ and $\lim_{z\to c} g(z) = m$, then the functions $kf(z), f(z) \pm g(z), f(z)g(z)$ and $f(z)/g(z)$ (if $m \neq 0$) have limits $kl, l \pm m, lm, l/m$ respectively.
The continuity of f and g at $c \in \mathbb{C}$ implies continuity of $kf, f \pm g, fg, f/g$ (unless $g(c) = 0$) at c.

Question: What about limits for $z \to \infty$?
There are many paths to infinity on the complex plane!

Definition (Limit). Let $f : \mathbb{C} \to \mathbb{C}$ be a function and $L \in \mathbb{C}$. We say that

$$\lim_{z\to\infty} f(z) = L$$

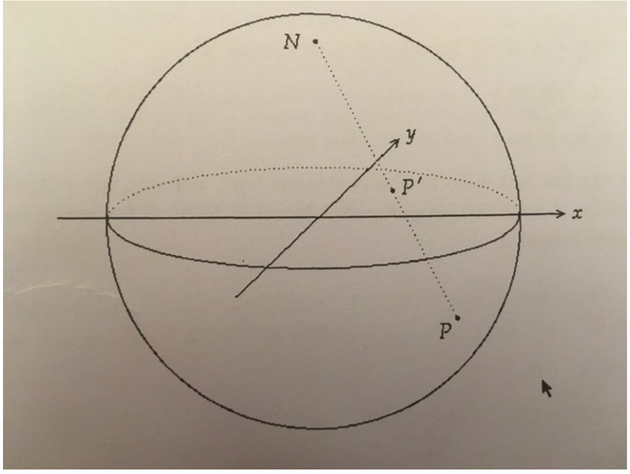
if for every $\epsilon > 0$ there exists a real number $K > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > K$.

Definition (Limit). Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say that

$$f(z) \to \infty \ (z \to \infty)$$

if for every real number $E > 0$, there exists a real number $D > 0$ such that $f(z) > E$ whenever $|z| > D$.

Remark. It is useful to think of ∞ as a single extra point and to extend the complex plane by adjoining this point i.e. look at $\mathbb{C} \cup \{\infty\}$. We can think of the complex plane as the equatorial plane of a sphere of radius 1 with north pole N being the point ∞.



For each point P on the sphere we let P' be the point where the line $\overline{NP}$ from the north pole N to P intersects with the equatorial plane. This gives a bijective correspondence of the points P on the sphere (except N) with the points on the equatorial x-y-plane.

Remark. We can visualise the complex numbers $\mathbb{C}$ as points on a sphere, the **Riemann sphere** and the 'missing point' $N = \infty$ is the **point at infinity**.

Remark. Very often in praxis the limit

$$\lim_{z \to \infty} f(z)$$

is the same as the limit

$$\lim_{|z| \to \infty} f(z),$$

which is easier to calculate.

Question: How can we quantify some values as 'small' in comparison to other values?
To say that a quantity q is small compared to a quantity Q is to say that in the limit the ratio q/Q is zero.

Definition. Let $f, \phi : \mathbb{C} \to \mathbb{C}$ be complex functions, then:

(i) $f(z) = O(\phi(z))$ as $z \to \infty$ means that there is a real number $K > 0$ such that $|f(z)| \leqslant K|\phi(z)|$ for all sufficiently large $|z|$.
(ii) $f(z) = O(\phi(z))$ as $z \to 0$ means that there is a real number $K > 0$ such that $|f(z)| \leqslant K|\phi(z)|$ for all sufficiently small $|z|$.
(iii) $f(z) = o(\phi(z))$ as $z \to \infty$ means that $\lim_{|z| \to \infty} f(z)/\phi(z) = 0$.
(iv) $f(z) = o(\phi(z))$ as $z \to 0$ means that $\lim_{z \to 0} f(z)/\phi(z) = 0$.

Remark. We will also write $O(\phi)$ for any function $f : \mathbb{C} \to \mathbb{C}$ with the property that $|f(z)| \leqslant K|\phi(z)|$ for sufficiently small (or sufficiently large) z.

Theorem B.10. *For $z \to 0$ we have:*

(i) $O(z) + O(z) = O(z)$ *and* $o(z) + o(z) = o(z)$.
(ii) For all $K \in \mathbb{C}$ *with* $K \neq 0$*:* $KO(z) = O(z)$ *and* $Ko(z) = o(z)$.
(iii) $O(z)O(z) = O(z^2)$ *and* $O(z)O(z) = o(z)$.

Proof. (i) Let $f_1(z) = O(z), f_2(z) = O(z)$. Thus there exist real numbers $K_1, K_2 > 0$ such that for sufficiently small z:

$$|f_1(z)| \leqslant K_1|z|,\ |f_2(z)| \leqslant K_2|z|.$$

Hence:

$$|f_1(z) + f_2(z)| \leqslant |f_1(z)| + |f_2(z)| \leqslant (K_1 + K_2)|z|$$

and so $f_1(z) + f_2(z) = O(z)$.
Let $f_1(z) = o(z), f_2(z) = o(z)$. Then as $z \to 0$ we have: $f_1(z)/z \to 0$ and $f_2(z)/z \to 0$. It follows:

$$\frac{f_1(z) + f_2(z)}{z} = \frac{f_1(z)}{z} + \frac{f_2(z)}{z} \longrightarrow 0$$

which implies $f_1(z) + f_2(z) = o(z)$.

(ii) Let $f(z) = O(z)$. Thus there exists a real number $M > 0$ such that $|f(z)| \leqslant M|z|$ for sufficiently small z. Hence:

$$|Kf(z)| = |K||f(z)| \leqslant M|K||z|,$$

so $Kf(z) = O(z)$.
Let $f(z) = o(z)$. Thus $f(z)/z \to 0$ for $z \to 0$. Hence: $Kf(z)/z \to 0$ and so $Kf(z) = o(z)$.

(iii) Let $f_1(z) = O(z), f_2(z) = O(z)$. Thus there exist real numbers $K_1, K_2 > 0$ such that for sufficiently small z:

$$|f_1(z)| \leqslant K_1|z|,\ |f_2(z)| \leqslant K_2|z|.$$

So $|f_1(z)f_2(z)| \leqslant K_1K_2|z^2|$ which gives $f_1(z)f_2(z) = O(z^2)$. We also have:

$$\left|\frac{f_1(z)f_2(z)}{z}\right| \leqslant K_1K_2|z| \longrightarrow 0,$$

which implies $f_1(z)f_2(z) = o(z)$. □

Remark. $f(z) = o(\phi(z))$ implies $f(z) = O(\phi(z))$, but the converse is *not* true. We have, e.g. $1 + z = O(1)$ as $z \to 0$, but it is not true that $1 + z = o(1)$ as $z \to 0$.

Example B.11. Show that, as $z \to 0$, we have:

$$\frac{1}{(1+z)^2} = 1 - 2z + O(z^2).$$

Solution. We have for all $|z| \leqslant \frac{1}{2}$:

$$\begin{aligned}
\left|\frac{1}{(1+z)^2} - (1-2z)\right| &= \left|\frac{1-(1-2z)(1+2z+z^2)}{(1+z)^2}\right| \\
&= \frac{|3z^2+2z^3|}{|1+z|^2} \leqslant \frac{3|z|^2+2|z|^3}{(1-|z|^2)} \\
&\leqslant \frac{4|z|^2}{1/4} = 16|z|^2,
\end{aligned}$$

since $|z| \leqslant \frac{1}{2}$ implies $(1-|z|^2) \geqslant \frac{1}{4}$ and $3|z|^2 + 2|z|^3 \leqslant 4|z|^2$. Hence:

$$\frac{1}{(1+z)^2} - (1-2z) = O(z^2).$$

APPENDIX C

Power Series

Question. Infinite series of real numbers play an important part to define functions in Real Analysis. Can we study them also in Complex Analysis?

DEFINITION. An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **convergent with the sum** or **converges to the sum** or **has sum** S if the sequence $(S_N)_{N=1}^{\infty}$ of partial sums $S_N := \sum_{n=1}^{N} z_n$ converges with limit S, i.e.

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} z_n = S.$$

An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **divergent** if it is not convergent.

PROPOSITION C.1. *If the infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers is convergent, then*

$$\lim_{n\to\infty} z_n = 0.$$

REMARK. Most of the essential definitions and results are easily extended from infinite series of real numbers to infinite series of complex numbers. Check your Calculus and Real Analysis lectures!

DEFINITION. An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is **absolutely convergent** if the associated infinite series of real numbers $\sum_{n=1}^{\infty} |z_n|$ is convergent.

PROPOSITION C.2. *If the infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is absolutely convergent, then it is itself convergent.*

Proof. The proof follows verbatim the argument as in the case of infinite series of real numbers. $\square$.

REMARK. Since $\sum_{n=1}^{\infty} |z_n|$ is an infinite series of real numbers, we can easily extend the standard tests for convergence like the Comparison Test and the Ratio Test.

THEOREM C.3 (Comparison Test). *Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers $z_n \in \mathbb{C}$.*

(i) If the infinite series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $|z_n| \leqslant |a_n|$ for all sufficiently large n, then also $\sum_{n=1}^{\infty} z_n$ is absolute convergent.
(ii) If the infinite series $\sum_{n=1}^{\infty} a_n$ is divergent and $|z_n| \geqslant |a_n|$ for all sufficiently large n, then also $\sum_{n=1}^{\infty} z_n$ is divergent.

THEOREM C.4 (Ratio Test). *Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers $z_n \in \mathbb{C}$.*

(i) If $\lim_{n\to\infty} |z_{n+1}/z_n| = l < 1$, then $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.
(ii) If $\lim_{n\to\infty} |z_{n+1}/z_n| = l > 1$, then $\sum_{n=1}^{\infty} z_n$ is divergent.

DEFINITION. An infinite series of complex numbers of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ with $z, a, c_n \in \mathbb{C}$ is called a **power series**.

THEOREM C.5. *If the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ with $z, a, c_n \in \mathbb{C}$ converges for $z - a = d$, then it converges (absolutely) for all $z \in \mathbb{C}$ with $|z-a| < |d|$.*

Proof. Since $\lim_{n\to\infty} c_n d^n = 0$, there exist a real $K > 0$ with $|c_n d^n| \leqslant K$ for all n. Let z be such that $|z-a| < |d|$. Then the geometric series $\sum_{n=0}^{\infty} (|z-a|/|d|)^n$ converges and for all n we have:

$$|c_n(z-a)^n| = |c_n d^n| \left|\frac{z-a}{d}\right|^n \leqslant K \left(\frac{|z-a|}{|d|}\right)^n.$$

So by the Comparison Test, the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges (absolutely). $\square$

Theorem C.6. *A power series* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *satisfies exactly one of the following three conditions:*

(i) *the series converges for all* $z \in \mathbb{C}$,
(ii) *the series converges only for* $z = a$,
(iii) *there exists a real number* $R > 0$ *such that the series converges for all* z *with* $|z-a| < R$ *and diverges for all* z *with* $|z-a| > R$.

Proof. Let $\mathscr{D}$ be the set of all z such that $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges and let $\mathscr{M} = \{|z-a| : z \in \mathscr{D}\}$. Suppose $\mathscr{M}$ is unbounded. Then for every $z \in \mathbb{C}$ there exists $d \in \mathscr{D}$ with $|d| > |z-a|$ and so $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges (absolutely). This gives (i).
Suppose $\mathscr{M}$ is bounded and let $R := \sup \mathscr{M}$. If $R = 0$, then $\mathscr{D} = \{a\}$. This gives (ii).
Now suppose $R > 0$, and let z be such that $|z-a| < R$. Then there exist d with

$$|z-a| < |d| < R$$

and the series $\sum_{n=0}^{\infty} c_n d^n$ converges and so $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges (absolutely).
Now let z be such that $|z-a| > R$ and suppose $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges. Then $z - a \in \mathscr{D}$ and we get a contradiction as $R = \sup \mathscr{M}$ is an upper bound of $\mathscr{M}$. This gives (iii). □

Definition. The real number R is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$. In case (i) we write: $R = \infty$, in case (ii) we write $R = 0$ and in case (iii) we write $R > 0$.
The circle $\{z \in \mathbb{C} : |z-a| = R\}$ is called the **circle of convergence** of the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$.

Theorem C.7. *Let* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *be a power series with radius of convergence* R. *Then:*

(i) If

$$\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lambda,$$

then $\lambda = R$.

(ii) If

$$\lim_{n\to\infty} |c_n|^{-1/n} = \lambda,$$

then $\lambda = R$.

Proof. The proof works just verbatim as for real power series. □

Theorem C.8. *The power series* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *and its derivative* $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ *have the same radius of convergence* R.

Theorem C.9. *Let* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *be a power series with radius of convergence* $R \neq 0$ *and let*

$$f(z) := \sum_{n=0}^{\infty} c_n(z-a)^n, \ (|z| < R).$$

Then the complex function f *is holomorphic within the open disc* $N(a, R)$, *and its derivative is given as*

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}.$$

Remark. The proofs of the last two theorems are quite technical and can be found in the book *John M. Howie, Complex Analysis, Springer.* These theorems are useful to define holomorphic functions via power series. We will meet many important examples!

Example C.10. Find the sum of the power series

$$\sum_{n=0}^{\infty} (n+1)^2 z^n = 1^2 + 2^2 z + 3^2 z^2 + \cdots \ (|z| < 1).$$

Solution. From

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z}, \ (|z| < 1).$$

we get by differentiating term by term:

$$\sum_{n=1}^{\infty} nz^{n-1} = 1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}, \ (|z| < 1).$$

Hence:

$$\sum_{n=1}^{\infty} nz^{n} = z + 2z^2 + 3z^3 + \cdots = \frac{z}{(1-z)^2}, \ (|z| < 1).$$

And so by differentiation with respect to z, for all $z \in N(0,1)$ we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1)^2 z^n &= 1^2 + 2^2 z + 3^2 z^2 + \cdots \\
&= \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) \\
&= \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} \\
&= \frac{(1-z)[(1-z) + 2z]}{(1-z)^4} \\
&= \frac{1+z}{(1-z)^3}.
\end{aligned}$$

Definition. The complex **exponential function** exp is defined as the power series for all values $z \in \mathbb{C}$

$$e^z := \exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

Remark. The power series used to define the exponential function is convergent for all $z \in \mathbb{C}$, which can be checked via the Ratio Test. (Check it!)

Remark. The exponential function $\exp z$ is holomorphic over the whole complex plane $\mathbb{C}$ and it follows for its first derivative:

$$(\exp z)' = \exp z.$$

Let $F_w(z) := \exp(z+w)/\exp z$ with $z, w \in \mathbb{C}$. Then by the Quotient Rule of differentiation we get:

$$F_w'(z) = \frac{(\exp z)(\exp(z+w)) - (\exp(z+w))(\exp z)}{(\exp z)^2} = 0.$$

and therefore:

$$F_w(z) = k$$

for a constant $k \in \mathbb{C}$ and all $z \in \mathbb{C}$. Since $F_w(0) = \exp w$ we see that:

$$F_w(z) = \exp w$$

for all $z \in \mathbb{C}$ and so we get the additivity property of the exponential function:

$$\exp(z+w) = (\exp z)(\exp w).$$

So we have shown:

Proposition C.11. *For all $z, w \in \mathbb{C}$ we have:*

$$\exp(z+w) = (\exp z)(\exp w).$$

DEFINITION (Trigonometric Functions). The complex **trigonometric functions** are defined as the following power series for all values $z \in \mathbb{C}$

$$\begin{aligned}
\cos z &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\
\sin z &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
\cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\
\sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots
\end{aligned}$$

REMARK. The complex trigonometric functions are holomorphic over the whole complex plane $\mathbb{C}$.

THEOREM C.12. *For all $z \in \mathbb{C}$ we have the following identities:*

$$e^{iz} = \cos z + i \sin z,\ e^z = \cosh z + \sinh z.$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}),\ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}),\ \cos^2 z + \sin^2 z = 1.$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z}),\ \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

THEOREM C.13. *The complex trigonometric functions are holomorphic in $\mathbb{C}$ and for the first derivative we have:*

$$(\cos z)' = -\sin z,\ (\sin z)' = \cos z.$$

$$(\cosh z)' = \sinh z,\ (\sinh z)' = \cosh z.$$

Question. What can we say about special values of the complex trigonometric functions, in particular for values in the real numbers $\mathbb{R}$?

EXAMPLE C.14. As shown in Calculus and Real Analysis the trigonometric functions have the following properties for values in the real numbers $\mathbb{R}$:

$$\cos x > 0,\ x \in [0, \pi/2),\ \ \cos(\pi/2) = 0.$$

$$\sin(\pi/2) = \pm\sqrt{1 - \cos^2(\pi/2)} = \pm 1,$$

Since $\sin(0) = 0$ and $(\sin)'(x) = \cos x > 0$ in the interval $[0, \pi/2)$, it follows:

$$\sin(\pi/2) = 1.$$

From the above theorems we also get:

$$\begin{aligned}
\cos(z + w) \quad &+ \quad i \sin(z + w) = e^{i(z+w)} = e^{iz} e^{iw} \\
&= \quad (\cos z + i \sin z)(\cos w + i \sin w) \\
&= \quad (\cos z \cos w - \sin z \sin w) + i(\sin z \cos w + \cos z \sin w)
\end{aligned}$$

This implies the following important formulae:

THEOREM C.15 (Addition Formulae). *For all $z, w \in \mathbb{C}$ we have:*

$$\cos(z + w) = \cos z \cos w - \sin z \sin w.$$

$$\sin(z + w) = \sin z \cos w + \cos z \sin w.$$

EXAMPLE C.16. Using the addition formulae, it follows that:

$$\cos \pi = \cos^2(\pi/2) - \sin^2(\pi/2) = -1.$$

$$\sin \pi = 2 \sin(\pi/2) \cos(\pi/2) = 0.$$

$$\cos 2\pi = \cos^2(\pi) - \sin^2(\pi) = 1.$$

$$\sin 2\pi = 2 \sin(\pi) \cos(\pi) = 0.$$

From these special values for the trigonometric functions we get the following important property for the exponential function:

Theorem C.17 (Periodicity Property). *For all $z \in \mathbb{C}$ we have:*

$$e^{z+2\pi i} = e^z.$$

Proof. We have: $e^{z+2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z$. $\square$
Let $z \in \mathbb{C}$ be given in standard form $z = x + iy$ with $x, y \in \mathbb{R}$. Then we have:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

So we obtain:

$$|e^z| = e^x, \ \arg e^z \equiv y \,(\mathrm{mod}\ 2\pi).$$

Remark. Since $e^x \neq 0$ for all $x \in \mathbb{R}$, it follows from the above, that $e^z \neq 0$ for all $z \in \mathbb{C}$. So with

$$\exp : \mathbb{C} \to \mathbb{C}, \exp(z) = e^z$$

being an entire function i.e. a holomorphic function on the entire complex plane, it follows that also the function

$$\exp^{-1} : \mathbb{C} \to \mathbb{C}, \exp^{-1}(z) = e^{-z} = \frac{1}{e^z}$$

is an entire function.

APPENDIX D

The Fundamental Theorem of Algebra

THEOREM D.1. *Let $p(z)$ be a polynomial of degree $n \geqslant 1$ with coefficients in $\mathbb{C}$, i.e. $p \in \mathbb{C}[z]$ with*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$$

with $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.

Proof. Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then both $p(z)$ and $1/p(z)$ give entire functions. Also

$$\lim_{|z| \to \infty} |p(z)| = \infty.$$

Hence there exists a real number $R \in \mathbb{R}, R > 0$ such that $|1/p(z)| \leqslant 1$ whenever $|z| > R$. Therefore the function $1/p(z)$ is also bounded on the closed bounded set $S := \{z \in \mathbb{C} : |z| \leqslant R\}$. Hence by Liouville's Theorem, the function $1/p(z)$ being a bounded entire function must be constant, which gives a contradiction, therefore the polynomial $p(z)$ must have at least one root, i.e. there exists $a \in \mathbb{C}$ with $p(a) = 0$. □

THEOREM D.2 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial of degree $n \geqslant 1$ with coefficients in $\mathbb{C}$, i.e. $p \in \mathbb{C}[z]$ with*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$$

with $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then there exist complex numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that

$$p(z) = \beta(z - \alpha_1)(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n).$$

Proof. (by induction on the degree $n \geqslant 1$)
The result is clear for degree $n = 1$, as $p(z)$ is then just a linear function. Suppose that the result is true for all polynomials of degree $n-1$ and let $p(z)$ be a polynomial of degree n. By the previous theorem, there exists then an $\alpha_1 \in \mathbb{C}$ such that $p(\alpha_1) = 0$.
Hence we have:

$$p(z) = (z - \alpha_1) \cdot q(z),$$

where $q(z)$ is a polynomial of degree $n - 1$.
By induction hypothesis there exists $\beta, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that:

$$q(z) = \beta(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n).$$

Hence we obtain

$$p(z) = \beta(z - \alpha_1)(z - \alpha_2) \cdot \ldots \cdot (z - \alpha_n),$$

as required. □

REMARK. The Fundamental Theorem of Algebra was first proved by C. F. Gauss, who also provided many different proofs for it. Nowadays there are many different proofs known, for example using topology or complex analysis or Galois theory. In particular, the proof presented here using Liouville's Theorem is particularly short. There are more algebraic proofs of it, but all need some fundamental properties of the field $\mathbb{R}$ of real numbers, namely the important Intermediate Value Theorem from Calculus and Real Analysis.

Question. What can we say about factorisation of real polynomials instead of complex polynomials?

Theorem D.3. *Let $p(x)$ be a polynomial of degree $n \geqslant 1$ with coefficients in $\mathbb{R}$, i.e. $p \in \mathbb{R}[x]$ with*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

with $a_0, a_1, \ldots, a_n \in \mathbb{R}, a_n \neq 0$. Then $p(x)$ factorises into linear and quadratic factors, i.e. for some $k, l \geqslant 0$ such that $k + 2l = n$, there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k; \beta_1, \beta_2, \ldots, \beta_l, \gamma_1, \gamma_2, \ldots, \gamma_l$ such that

$$p(x) = a_n(x - \alpha_1)\ldots(x - \alpha_k)(x^2 + \beta_1 x + \gamma_1)\ldots(x^2 + \beta_l x + \gamma_l).$$

Proof. By the Fundamental Theorem of Algebra the polynomial $p(x)$ seen as a polynomial in $\mathbb{C}[x]$ factorises over $\mathbb{C}$ as

$$p(x) = a_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n)$$

with $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{C}$.
Suppose the roots of the polynomial $p(x)$ are ordered, so that $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, $(k \geqslant 0)$ are real, and $\alpha_{k+1}, \ldots \alpha_n \in \mathbb{C}\backslash\mathbb{R}$ are not real.
These last roots always occur in complex conjugate pairs, i.e. of the form $x - \mu, x - \bar{\mu}$, and therefore $l = n - k$ is even.
Any of these complex conjugate factors combine to give a real quadratic factor of the form

$$(x - \mu)(x - \bar{\mu}) = x^2 - 2Re(\mu)x + |\mu|^2.$$

Hence the polynomial $p(x)$ has the described factorisation into linear and quadratic factors over the field $\mathbb{R}$ of real numbers. □

Corollary D.4. *Any real polynomial $p(x)$ of odd degree must have at least one real root.*

Proof. If the degree of the real polynomial $p(x)$ is odd, then in the proof of the last theorem we get that $k = n - l$ is also odd as l is even, so is at least 1 i.e. a real root exist. □

Example D.5. We have the following factorisation:

$$x^6 + 1 = (x - e^{\pi i/6})(x - e^{-\pi i/6})(x - e^{3\pi i/6})(x - e^{-3\pi i/6})(x - e^{5\pi i/6})(x - e^{-5\pi i/6}),$$

so we get:

$$\begin{aligned} x^6 + 1 &= (x^2 - 2x\cos(\pi/6) + 1)(x^2 + 1)(x^2 - 2x\cos(5\pi/6) + 1) \\ &= (x^2 + 1)(x^2 - x\sqrt{3} + 1)(x^2 + x\sqrt{3} + 1) \end{aligned}$$

So here we have $k = 0$ and only quadratic factors.

Example D.6. We have the following factorisations:

$$x^4 - 3x^3 + 4x^2 - 6x + 4 = (x - 1)(x - 2)(x^2 + 2).$$

$$x^4 + 3x^3 - 3x^2 - 7x + 6 = (x - 1)(x - 1)(x + 2)(x + 3), \text{ so } l = 0.$$

