Lecture 6: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 3, 2022

Cylinders and Quadric Surfaces

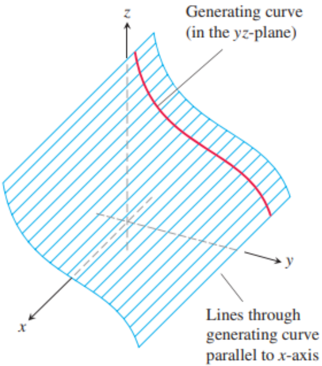
- Up to now, we have studied two special types of surfaces: **spheres and planes**.
- In today's lecture, we extend our inventory to include a variety of **cylinders and quadric surfaces**.
- Quadric surfaces are surfaces **defined by second-degree equations** in x, y, and z.
- Spheres are quadric surfaces, but there are other of equal interest.

Cylinders

• A cylinder is a surface that is **generated by moving a straight line along a given planar curve** while holding the line parallel to a given fixed line.

• The curve is called a **generating curve** for the cylinder (see Figure).

• In solid geometry, where cylinder means circular cylinder, the **generating curves are circles**, but now we allow generating curves of **any kind**.



• Any curve $g(x, z) = c$ in the xz-plane defines a cylinder parallel to the y-axis whose space equation is also $g(x, z) = c$.

• Any curve $h(y, z) = c$ defines a cylinder parallel to the x-axis whose space equation is also $h(y, z) = c$.

• The axis of a cylinder need not be parallel to a coordinate axis, however.

Quadric Surfaces

• A **quadric surface** is the graph in space of a **second-degree equation** in x, y, and z.

• We first focus on quadric surfaces given by the equation

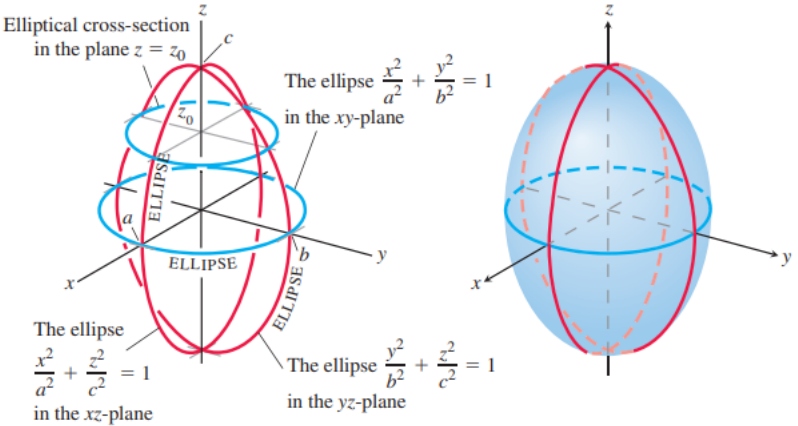
$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

• where A, B, C, D, and E are constants.

• The basic quadric surfaces are **ellipsoids, paraboloids, elliptical cones, and hyperboloids**.

• Spheres are special cases of ellipsoids.

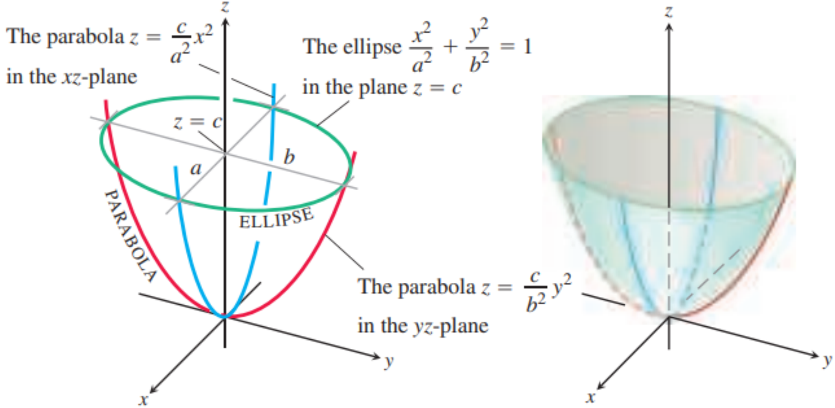
Ellipsoid



ELLIPSOID

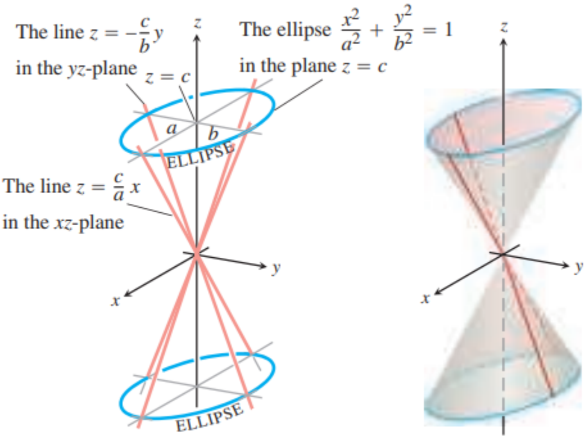
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptical Paraboloid



ELLIPTICAL PARABOLOID $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$

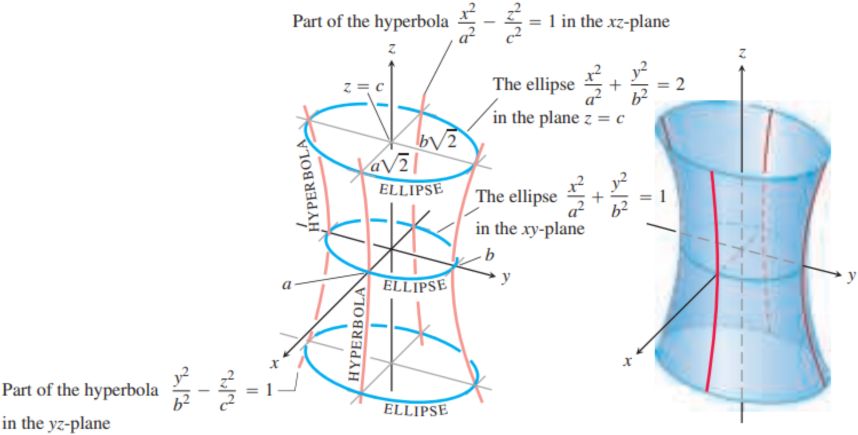
Elliptical Cone



ELLIPTICAL CONE

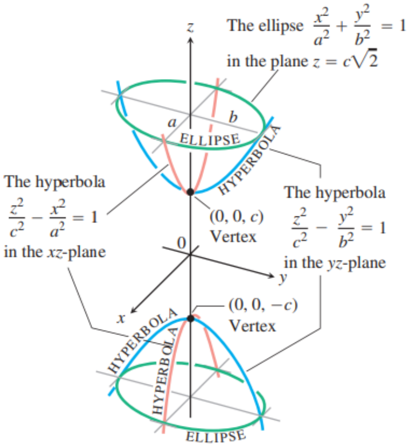
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Hyperboloid of one Sheet



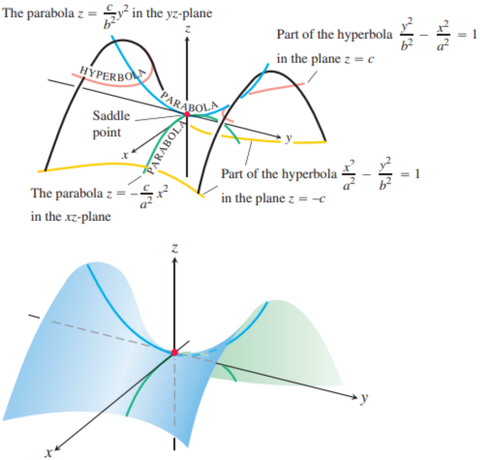
HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Hyperboloid of two sheets



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Hyperbolic Paraboloid



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

General Quadric Surfaces

• The quadric surfaces we have considered have **symmetries** relative to the x-, y-, or z-axes.

• The general equation of second degree in three variables x, y, z is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0,$$

• where A, B, C, D, E, F, G, H, I, and J are constants.

• This equation leads to surfaces similar to those presented above, but in general these surfaces might be **translated and rotated** relative to the x-, y-, and z-axes.

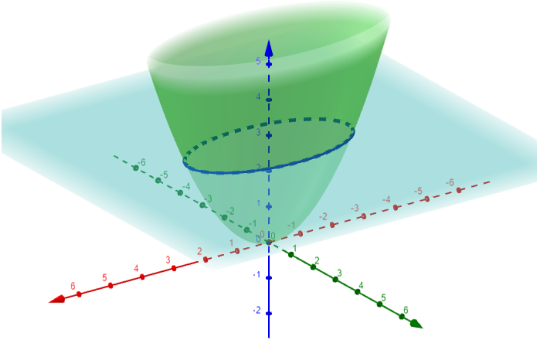
• Terms of the type Gx, Hy, or Iz in the above formula **lead to translations**, which can be seen by a process of completing the square.

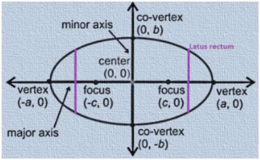
Example 1

Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude.





The standard form of the equation of an ellipse with center $(0,0)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a,0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$

Ellipse Equation

Area of ellipse $=\pi ab$

Perimeter of ellipse $=2\pi\sqrt{\frac{(a^2+b^2)}{2}}$

The standard form of the equation of an ellipse with center $(h,\ k)$ is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h\pm a,k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h,k\pm b)$
- the coordinates of the foci are $(h\pm c,k)$, where $c^2=a^2-b^2$

We calculate the volume by the slicing method, taking slices parallel to the xy-plane. For fixed z, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)}+\frac{y^2}{\left(\frac{zb^2}{c}\right)}=1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right)=\frac{\pi abz}{c}$ (see Exercise 45a). Hence the volume is given by $V=\int_0^h\frac{\pi abz}{c}\,dz=\left[\frac{\pi abz^2}{2c}\right]_0^h=\frac{\pi abh^2}{2c}$. Now the area of the elliptic base when $z=h$ is $A=\frac{\pi abh}{c}$, as determined previously. Thus, $V=\frac{\pi abh^2}{2c}=\frac{1}{2}\left(\frac{\pi abh}{c}\right)h=\frac{1}{2}$ (base)(altitude), as claimed.

Examples of Cylinders and Quadric Surfaces

Example 2

a) Express the area A of the cross-section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

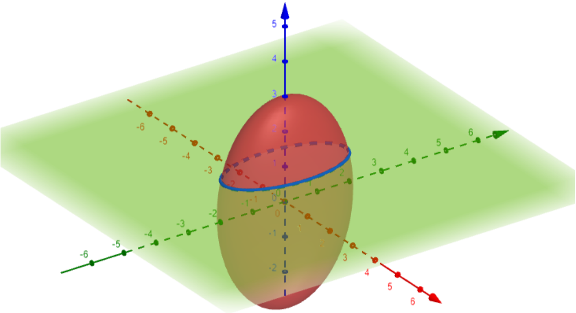
by the plane $z = c$ as a function of c.

b) Use slices perpendicular to the z-axis to find the volume of the ellipsoid in part (a).

c) Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Does your formula give the volume of a sphere of radius a if a = b = c?



Solution for Example 2

(a) If $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$ and $z=c$, then $x^2+\frac{y^2}{4}=\frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)}+\frac{y^2}{\left[\frac{4\left(9-c^2\right)}{9}\right]}=1 \Rightarrow A=ab\pi$

$=\pi\left(\frac{\sqrt{9-c^2}}{3}\right)\left(\frac{2\sqrt{9-c^2}}{3}\right)=\frac{2\pi\left(9-c^2\right)}{9}$

(b) From part (a), each slice has the area $\frac{2\pi\left(9-z^2\right)}{9}$, where $-3 \le z \le 3$. Thus $V=2\int_0^3 \frac{2\pi}{9}\left(9-z^2\right)dz$

$=\frac{4\pi}{9}\int_0^3\left(9-z^2\right)dz=\frac{4\pi}{9}\left[9z-\frac{z^3}{3}\right]_0^3=\frac{4\pi}{9}(27-9)=8\pi$

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{x^2}{\left[\frac{a^2\left(c^2-z^2\right)}{c^2}\right]}+\frac{y^2}{\left[\frac{b^2\left(c^2-z^2\right)}{c^2}\right]}=1 \Rightarrow A=\pi\left(\frac{a\sqrt{c^2-z^2}}{c}\right)\left(\frac{b\sqrt{c^2-z^2}}{c}\right)$

$\Rightarrow V=2\int_0^c \frac{\pi ab}{c^2}\left(c^2-z^2\right)dz=\frac{2\pi ab}{c^2}\left[c^2z-\frac{z^3}{3}\right]_0^c=\frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right)=\frac{4\pi abc}{3}$. Note that if $r=a=b=c$,

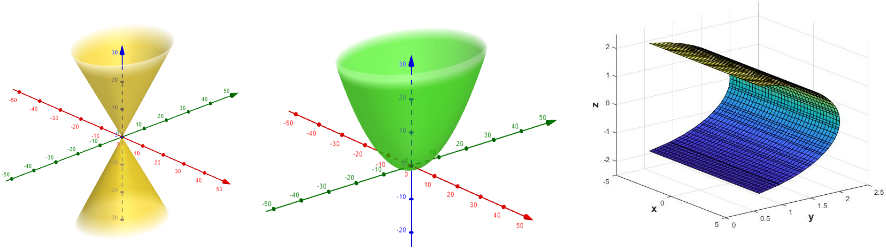
then $V=\frac{4\pi r^3}{3}$, which is the volume of a sphere.

Example 3

Use a Geogebra to plot the surfaces. Identify the type of quadric surface from your graph.

a) $5x^2 = z^2 - 3y^2$, b) $\frac{y^2}{16} = 1 - \frac{x^2}{9} + z$, c) $y - \sqrt{4 - z^2} = 0$.

Solution



a) Elliptical cone, b) elliptical paraboloid, c) half of cylinder.