Introductory Statistics

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Topic 0 - Introduction. Part 2

- Topic 0: Introduction
- Topic 1: Descriptive Statistics
- Topic 2: Estimators: Point estimation, Interval estimation
- Topic 3: Hypothesis Testing
- Topic 4: Goodness of Fit
- Topic 5: Bayesian Estimation.

Introduction

- What is statistics?
- Examples
- How do we use statistics?
- Population vs Sample
- Random variable, Probability Distributions
- Moments and Moment Generating Functions

DISCRETE PROBABILITY DISTRIBUTIONS

- A random variable X is said to be *uniformly distributed* over the numbers $1, 2, 3, \ldots, n$ if
$$P(X = i) = \frac{1}{n}, \text{ for } i = 1, 2, \ldots, n.$$
The *uniform probability function* is defined as:
$$p(i) = P(X = i) = \frac{1}{n}, \text{ for } i = 1, 2, \ldots, n.$$
The *cumulative uniform distribution function* is:
$$F(x) = P(X \leq x) = \sum_{i \leq x} p(i) = \sum_{i \leq x} \frac{1}{n} = \frac{x}{n},$$
for $0 \leq x \leq n$.

Discrete Probability Distributions

- *Bernoulli experiment and trials*
  - $S = \{'\text{success}', '\text{failure}'\}$, i.e. $X \in \{1, 0\}$;
  - *Bernoulli trials* are the fixed sequence of n identical repetitions of the same Bernoulli experiment;
  - for each trial probability of success is p, probability of failure is $q = 1 - p$, these probabilities are constant for every trial;
  - the outcome of each experiment is independent on previous experiments and does not influence any subsequent outcomes.

Discrete Probability Distributions

- *Bernoulli experiment and trials*
- a *binomial random variable* is a discrete random variable that describes the number of successes ($X = 1$) in a sequence of n Bernoulli trials. The *binomial probability function* is defined as:

$$p(k) = P(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}, \; k = 0, 1, 2, 3..., n.$$

Cumulative binomial distribution function is:

$$F(x) = P(X \le x) = \sum_{k \le x} p(k) = \sum_{k \le x} \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k},$$

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- Hypergeometric random variable

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  The corresponding probability distribution functions and the cumulative distribution functions are:
  - for X representing number of 'successes'

$$p(k) = P(X = k) = (1-p)^{k-1}p, \text{ for } k = 1, 2, 3, ...$$

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  - for Y representing number of 'failures'

$$p(k) = P(Y = k) = (1-p)^k p, \text{ for } k = 0, 1, 2, 3, ...$$

$$F(y) = P(Y \leq y) = \sum_{k \leq y} (1-p)^k p = 1 - (1-p)^{k+1}$$

DISCRETE PROBABILITY DISTRIBUTIONS

- A *Poisson random variable* describes the random events occurring in continuous fixed units of time and space.
  - k is the number of times an event occurs in an interval and $k = 0, 1, 2, ...$
  - The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
  - The average rate at which events occur is independent of any occurrences.
  - Two events cannot occur at exactly the same instant; instead, at each very small sub-interval, either exactly one event occurs, or no event occurs.

DISCRETE PROBABILITY DISTRIBUTIONS

- A *Poisson random variable* describes the random events occurring in continuous fixed units of time and space.
  A discrete random variable X has a *Poisson distribution* with parameter $\lambda > 0$ if it's probability mass function can be described by:

$$p(k) = P(X = k) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

  The cumulative distribution function is:

$$F(x) = P(X \leq x) = \sum_{k \leq x} p(k) = \sum_{k \leq x} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \cdot \sum_{k \leq x} \frac{\lambda^k}{k!},$$

  for $0 \leq x < \infty$.

Continuous Probability Distributions

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$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function is:

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

for $0 \leq x < \infty$.

Continuous Probability Distributions

- A random variable X is called *Gaussian* or *normal* if its probability density function $f(x)$ is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where $-\infty < x < \infty$ and μ and σ are two parameters, such that $-\infty < \mu < \infty$, $\sigma > 0$.
The cumulative distribution function is:

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

for $-\infty < x < \infty$.

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CONTINUOUS PROBABILITY DISTRIBUTIONS

- The **exponential distribution** is the probability distribution of the time between events in a *Poisson point process*, i.e., a process in which events occur continuously and independently at a constant average rate.
  The probability density function of an exponentially distributed random variable X ($x \in [0, \infty)$) can be parametrised in two ways:

by *a rate parameter* $\lambda > 0$	by *a scale parameter* $\beta > 0, \beta = 1/\lambda$
the pdf has the form: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0, & x < 0 \end{cases}$;	the pdf has the form: $f(x) = \begin{cases} \frac{e^{-x/\beta}}{\beta} & x \geq 0 \\ 0, & x < 0 \end{cases}$;
the cumulative distribution function is: $F(x) = \begin{cases} (1 - e^{-\lambda x}) & x \geq 0 \\ 0, & x < 0 \end{cases}$	the cumulative distribution function is: $F(x) = \begin{cases} (1 - e^{-x/\beta}) & x \geq 0 \\ 0, & x < 0 \end{cases}$

Probability Distributions in R

- Discrete probability distributions

Distribution	pdf	cdf	inverse cdf	random deviates
Binomial	dbinom	pbinom	qbinom	rbinom
Geometric	dgeom	pgeom	qgeom	rgeom
Hypergeometric	dhyper	phyper	qhyper	rhyper
Poisson	dpois	ppois	qpois	rpois

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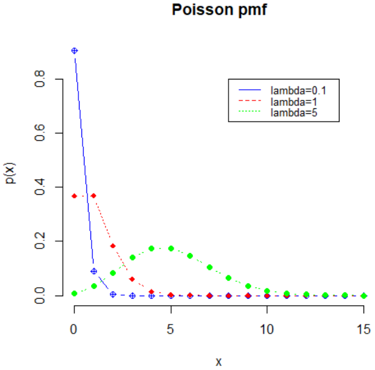
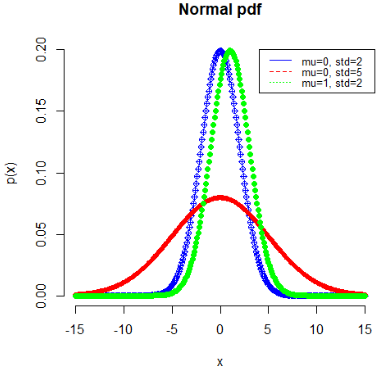
- Continuous probability distributions

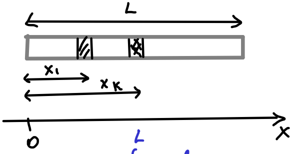
Distribution	pdf	cdf	inverse cdf	random deviates
Normal	dbinom	pbinom	qbinom	rbinom
Exponential	dexp	pexp	qexp	rexp
Uniform	dunif	punif	qunif	runif

Probability Distributions in R





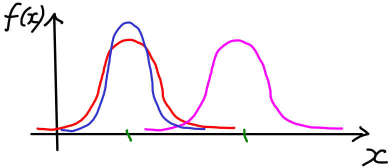
Moments



$$x_{cm} = \frac{\int_0^L x \cdot dm}{\int_0^L dm}$$

$$\rho = \frac{dm}{dx} \text{ - density}$$

$$x_{cm} = \frac{\int_0^L x \rho dx}{\int_0^L \rho dx} \quad \int_0^L \rho dx = 1$$



$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$\mu' = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

Moments and Moment Generating Functions

Definition

Let X be any random variable with pdf $f_X(x)$. For any positive integer r:

- ***The r-th moment of X about the origin**, μ_r, is given by*

$$\mu_r = E[X^r] = \int_{-\infty}^{\infty} w^r \cdot f_X(w)dw,$$

provided $\int_{-\infty}^{\infty} |w|^r \cdot f_W(w)dw < \infty$.
(When $r = 1$, the subscript is usually omitted, i.e. $\mu_1 = \mu$.)

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- ***The r-th moment of X about the mean,** μ'_r, is given by*

$$\mu'_r = E\left((X-\mu)^r\right) = \int_{-\infty}^{\infty} (w-\mu)^r \cdot f_X(w)dw,$$

  provided fitness conditions of the part 1 hold.
  (When r=2, $\mu'_2 = \sigma^2$ is called the `variance`, and σ is called the `standard deviation`.)

MOMENTS AND MOMENT GENERATING FUNCTIONS

DEFINITION (CONTINUED...)

- ***The** $r-$**th standardized moment**, $\tilde{\mu}_r$, is a moment that is normalized, typically by the standard deviation raised to the power of r, σ^r.*

$$\tilde{\mu}_r = \frac{E\left((W-\mu)^r\right)}{\sigma^r}.$$

Moments and Moment Generating Functions

$E[X]$ - the *mean* - the first moment about origin:

$$\mu = E[X] = \int_{-\infty}^{\infty} w \cdot f_X(w)dw$$

$$\mu = E[X] = \sum_{\text{all } x} x \cdot p_X(x)$$

$E[(X-\mu)^2]$ - the *variance* - the second moment about the mean;

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (w-\mu)^2 \cdot f_X(w)dw$$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 \cdot p_X(x)$$

$E[(X-\mu)^3]/(E[(X-\mu)^2])^{3/2}$ - the *skewness* - the third standardized moment;
$E[(X-\mu)^4]/(E[(X-\mu)^2])^2$ - the *kurtosis* - the forth standardized moment.

MOMENTS AND MOMENT GENERATING FUNCTIONS

EXAMPLE

To find out the prevalence of smallpox vaccine use, a researcher inquired into the number of times a randomly selected 200 people aged 16 and over in an African village had been vaccinated. He obtained the following figures:

N	0	1	2	3	4	5
proportion	17/200	30/100	58/200	50/200	38/200	7/200

Assume that these proportions continue to hold exhaustively for the population of that village.

Questions:

a) What is the expected number of times those people in the village had been vaccinated?

b) What is the standard deviation?

EXAMPLE (SOLUTION)

The random variable X represent the number of times a person has been vaccinated, then X has distribution p(x) as in the table.

x	0	1	2	3	4	5
p(x)	17/200	30/200	58/200	50/200	38/200	7/200

Then

$$E[X] = \sum xp(x) = 0 \cdot \frac{17}{200} + 1 \cdot \frac{30}{200} + 2 \cdot \frac{58}{200} \dots + 5 \cdot \frac{7}{200} = \frac{483}{200} = 2.415$$

The variance

$$\sigma^2 = Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2 =$$

$$= 0 \cdot \frac{17}{200} + 1^2 \cdot \frac{30}{200} + 2^2 \cdot \frac{58}{200} + \dots + 5^2 \cdot \frac{7}{200} - (2.415)^2 = 1.643 \text{ (3 d.p.)}$$

Example

Let X be a random variable with pdf:

$$f(x) = \begin{cases} \frac{3}{64}y^2(4-y) & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases} .$$

Question:

Find the expected value and variance of X.

In Problem Sheet 1

MOMENTS AND MOMENT GENERATING FUNCTIONS

EXAMPLE

Find the expectation and the variance for a random variable X:

- with Poisson distribution, $Poi(\lambda)$:

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \text{ for } k = 0, 1, 2, \ldots$$

$$E(k) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \cdot \lambda \overset{t=k-1}{=} e^{-\lambda} \lambda \cdot \underbrace{\sum_{t=0}^{\infty} \frac{\lambda^t}{(t-1)!}}_{= e^{\lambda}}$$

$$\Rightarrow \underline{\underline{E(k) = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda = \lambda}}$$

- with normal distribution, $N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

was discussed in Probability module

Definition

For a random variable X, suppose that there is a positive number h such that for $-h < t < h$ the mathematical expectation $E(e^{tX})$ exists. The ***moment-generating function*** *(mgf) of the random variable X is defined by*

$$M_X = E(e^{tX}) = \sum_{-\infty}^{\infty} e^{tk} p_X(k), \quad \textit{if } X \textit{ is discrete}$$

$$M_X = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \quad \textit{if } X \textit{ is continuous.}$$

MOMENT-GENERATING FUNCTIONS

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + ... + \frac{(tX)^n}{n!} + ...$$

The moment generating function

$$M_X = E(e^{tX}) = 1 + tE(X) + \frac{(t)^2}{2!}E(X^2) + ... + \frac{(t)^n}{n!}E(X^n) + ...$$

Moment-Generating functions

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$$\frac{dM_X}{dt} = M_X'(t) = E(X) + tE(X^2) + ... + \frac{(t)^{n-1}}{(n-1)!}E(X^n) + ...$$

Evaluating for $t = 0$

$$M_X'(0) = E(X), \qquad M_X''(0) = E(X^2), \qquad ... M_X^{(n)}(0) = E(X^n), \; n \geq 1$$

MOMENT-GENERATING FUNCTIONS

THEOREM

If $M_X(t)$ exists, then for any positive integer r,

$$\left.\frac{d^r M_X}{dt^r}\right|_{t=0} = M_X^{(r)}(0) = \mu_r$$

Moment-Generating functions

Theorem

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Properties of the moment-generating function:

1. The moment-generating function of X is unique in the sense that, if two random variables X and Y have the same mgf ($M_X(t) = M_Y(t)$, for t in an interval containing 0), then X and Y have the same distribution.

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2. If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$. That is, the mgf of the sum of two independent random variables is the product of the mgfs of the individual random variables. The result can be extended to n random variables.

Moment-generating functions

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3. Let $Y = aX + b$. Then $M_Y(t) = e^{bt}M_X(at)$.

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2. Difference between Population and Sampling, between Parameter and Statistic;
3. Different ways of sampling(simple random sampling, cluster sampling, and stratified sampling);
4. Find some examples of descriptive and inferential statistics.