UNIVERSITY OF LEICESTER

MA1014
CALCULUS AND ANALYSIS
TUTORIAL 22

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ANNOUNCEMENTS

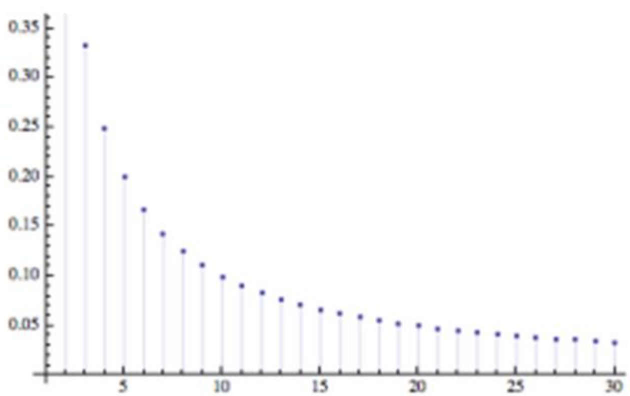
- Chapter 4 revision

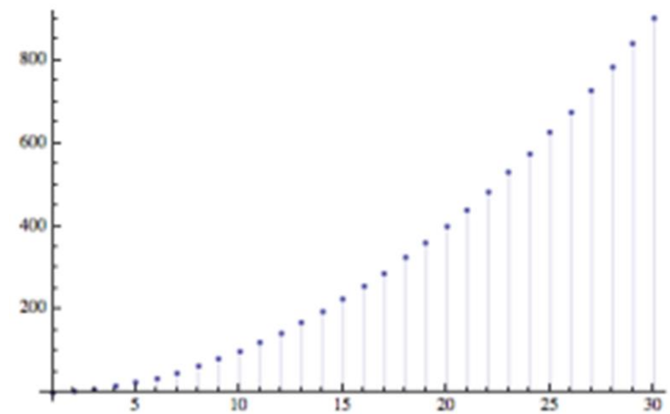
SEQUENCES

- A sequence is a function, $a : \mathbb{N} \to \mathbb{R}$ with $a_n = a(n),\ n \in \mathbb{N}$
  e.g. $a_n = n^2 \Rightarrow a_1 = 1, a_2 = 4, a_3 = 9, \ldots$

- Monotonic: If $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$ (increasing) or $a_n \geq a_{n+1}$ (decreasing)

- Bounded: If $\exists m, M \in \mathbb{R} : m \leq a_n \leq M\ \forall n \in \mathbb{N}$





LIMITS OF SEQUENCES

If

$$\boxed{\forall \varepsilon > 0, \exists K \in \mathbb{N} : |a_n - L| < \varepsilon \ \forall n \geq K}$$

then $\lim_{n\to\infty} a_n = L$ (convergent)

EXERCISE

Consider $(a_n)_{n\geq 1} = \frac{n+(-1)^n}{n}$. Is it

a) Bounded? (If so, give both an upper and lower bound)

b) Monotonic? (Justify your answer)

c) Convergent? (If so give a limit and prove it using an ε-K argument)

THEOREMS FOR SEQUENCES

- **Every Convergent sequence is bounded**

  e.g. $a_n = n$ is not bounded, so it diverges

- Monotone Convergence Theorem:
  - If a sequence is **bounded and monotonic** then it is convergent (to it's supremum or infimum)
  - E.g. $0 < a_n = \frac{1}{n} < 2$ and $a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}$ so it converges to 0

LIMIT LAWS

If a_n and b_n are **convergent** sequences, then

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \pm \left(\lim\limits_{n\to\infty} b_n\right)$

- $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$

- $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided that $b_n \neq 0\ \forall n$ and $\lim\limits_{n\to\infty} b_n \neq 0$

PINCHING THEOREM (FOR SEQUENCES)

Let a_n, b_n, c_n be sequences such that $\exists N \in \mathbb{N}$ with

$$a_n \leq b_n \leq c_n \quad \forall n \geq N$$

If a_n and c_n **are convergent** with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then b_n is convergent with

$$\lim_{n\to\infty} b_n = L$$

EXERCISE:
DETERMINE IF THE FOLLOWING LIMITS CONVERGE.
IF THEY DO, FIND THE LIMIT.

a) $$\lim_{n\to\infty} \frac{\sin(n)}{n}$$

b) $$\lim_{n\to\infty} \frac{n\cos(1+n^2)}{1+n^2}$$

c) $$\lim_{n\to\infty} \left(\frac{n+10^6}{n^2} + \frac{\cos^2(3n^2-4)}{n}\right)$$

d) $$\lim_{n\to\infty} \tan(n\pi)$$

e) $$\lim_{n\to\infty} \frac{2}{n+1}\tan(3n)$$

f) $$\lim_{n\to\infty} \frac{\cos(n^2)}{1+n}$$

SUBSEQUENCES

If $(a_n)_{n\geq 1}$ is a sequence, then a subsequence of a_n is a sequence

$$(b_k)_{k\geq 1} : \forall k \in \mathbb{N}, \exists n_k \in \mathbb{N} : n_1 < n_2 < \cdots < n_k \text{ with } b_k = a_{n_k}$$

e.g. $a_n = n^2 : n_k = 2k, b_k = a_{n_k} = a_{2k} = (2k)^2 = 4k^2$

<u>Bolzano-Weierstrass Theorem:</u>

If $(a_n)_{n\geq 1}$ is a bounded sequence, then there exists a convergent subsequence $\left(a_{n_k}\right)_{k\geq 1}$

CAUCHY SEQUENCES

- A sequence $(a_n)_{n\geq 1}$ is Cauchy if

$$\boxed{\forall \varepsilon > 0, \exists N \in \mathbb{N} : \ \forall n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon}$$

- Every Cauchy sequence is bounded
- A sequence is convergent iff it is Cauchy

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EXERCISE:
PROVE THE FOLLOWING

a) Prove every Cauchy sequence is bounded, but the converse is false

b) If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences, prove $(a_n - b_n)_{n\geq 1}$ and $(a_n b_n)_{n\geq 1}$ are Cauchy.

c) Consider subsequences $(b_k)_{k\geq 1}$ and $(c_k)_{k\geq 1}$ of any sequence $(a_n)_{n\geq 1}$ where $b_k = a_{2k}$ and $c_k = a_{2k+1}$. Then

$$a_n \text{ is convergent } \Leftrightarrow b_n \,\&\, c_n \text{ converge to the same limit}$$

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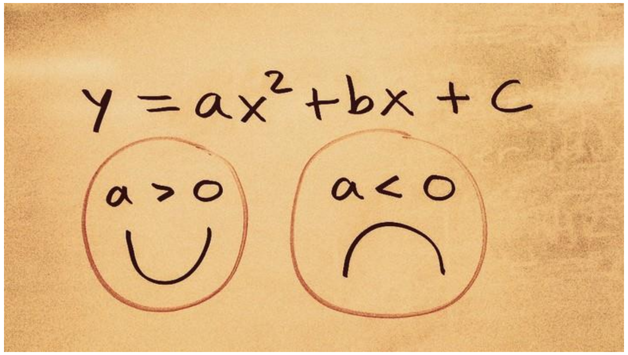

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

ANY QUESTIONS?

$$m\frac{d^2x}{dt^2} = -kx$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$