Linear Algebra II

Ch. VI Determinants

- $\mathrm{Det}(A) = |A| = D(A) = D(A^1, \ldots, A^n)$.
  1. $D(A^1, \ldots, C + C', \ldots, A^n) = D(A^1, \ldots, C, \ldots, A^n) + D(A^1, \ldots, C', \ldots, A^n)$;
  2. $D(A^1, \ldots, tC, \ldots, A^n) = tD(A^1, \ldots, C, \ldots, A^n)$;
  3. $D(A^1, \ldots, A^j, A^j, \ldots, A^n) = 0$;
  4. $D(A^1, \ldots, A^i, \ldots, A^j, \ldots, A^n) = -D(A^1, \ldots, A^j, \ldots, A^i, \ldots, A^n)$;
  5. $D(A^1, \ldots, A^j, \ldots, A^j, \ldots, A^n) = 0$;
  6. $D(A^1, \ldots, A^i, \ldots, A^j + tA^i, \ldots, A^n) = D(A^1, \ldots, A^i, \ldots, A^j, \ldots, A^n)$;
  7. $D(A^{\mathrm{T}}) = D(A)$;

- Definition of the determinant:
  - By induction, expansion according to the 1st row:

$$D(A) = (-1)^{1+1} a_{11} D(A_{11}) + \cdots + (-1)^{1+n} a_{1n} D(A_{1n}).$$

  - By permutation, the number of inversions, expansion formula.

$$D(A) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}.$$

  - By Property 1-3 with $D(I) = 1$.

Ch. 6. Determinants

- $D(A) = (-1)^{i+1}a_{i1}D(A_{i1}) + \cdots + (-1)^{i+n}a_{in}D(A_{in})$.
- $D(A) = (-1)^{1+j}a_{1j}D(A_{1j}) + \cdots + (-1)^{n+j}a_{nj}D(A_{nj})$.
- Laplace Theorem.

Ch. 6. Determinants

- $D(AB) = D(A)D(B)$;
- $D(A^k) = D(A)^k$;
- $D(A) \neq 0 \Leftrightarrow A$ is invertible (non-singular) $\Leftrightarrow$ rank $A = n \Leftrightarrow$ Columns (rows) of A are L. I.
- $D(A^{-1}) = D(A)^{-1}$.
- Cramer's rule: $x_i = \dfrac{D(A^1, \dots, B, \dots, A^n)}{D(A)}$.
- $B = (b_{ij})$, where,

  $$b_{ij} = \frac{D(A^1, \dots, E^j, \dots, A^n)}{D(A)}.$$

  is the inverse of A.

- $\tilde{A} = (\tilde{a}_{ij})$, where,
$$\tilde{a}_{ij} = D(A^1, \ldots, E^j, \ldots, A^n).$$
is the called the adjugate matrix of A. It satisfies
$$A\tilde{A} = \tilde{A}A = D(A)I.$$

- rank $\tilde{A}$ =?.

Ch. 6. Determinants

- The column rank and row rank of a matrix A.
- The rank of a set of vectors $\{v_1, \dots, v_n\}$.
- The rank of a linear map.
- Rank r of A = the highest order of non-zero minors (subdeterminant, determinant of submatrix) of A.$\Leftrightarrow$ There exists a nonzero minor of order r, and all minors of order $r+1$ are 0.
- rank A =colum rank of A=row rank of A.

Ch. 6. Determinants

- column (row) operation;
- column (row) equivalent;
- Triangulation by column (row) operations.
- rank $A_{m\times n}$=r $\Leftrightarrow$ There exist non-singular matrices $P_{m\times m}$ and $Q_{n\times n}$, s.t.

$$PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.$$

Ch. 6. Determinants

- $\begin{vmatrix} A & C \\ O & B \end{vmatrix} = |A| \cdot |B|, \quad \begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A| \cdot |B|, \quad \begin{vmatrix} A & O \\ O & B \end{vmatrix} = |A| \cdot |B|.$

Ch. 6. Determinants

- partitioned column (row) operations;

Ch. 6. Determinants

Ch. 6. Determinants

- $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A + B| \cdot |A - B|.$

- $\begin{vmatrix} I_n & B \\ A & I_m \end{vmatrix} = |I_m - AB| = |I_n - BA|.$

Ch. 6. Determinants

- compute inverse by (partitioned) row operations;

Ch. 6. Determinants

- $\begin{pmatrix} A & O \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}.$

- $\operatorname{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} \geq \operatorname{rank}(A)+\operatorname{rank}(B)$.

- $\operatorname{rank}\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \operatorname{rank}(A)+\operatorname{rank}(B)$.

- $\mathrm{rank}(A_{m\times k})+\mathrm{rank}(B_{k\times n})-k \leq \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

- rank(A, B) $\leq$rank(A)+rank(B).

- rank$(A + B)$ $\leq$rank(A)+rank(B).

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- $$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}$$