Yesterday Topic 4 : Sequences

$a : \mathbb{N} \longrightarrow \mathbb{R}$ $\quad a_0, a_1, a_2, a_3, \ldots$

$(a_n)_{n \in \mathbb{N}}$

Bounded $\quad \exists B \in \mathbb{R} : |a_n| \leq B \ \forall n$

Monotonic Increasing $\quad a_{n+1} \geq a_n \ \forall n$

— " — Decreasing $\quad a_{n+1} \leq a_n \ \forall n$

Limit of a sequence $(a_n)_{n \in \mathbb{Z}}$ is L

$\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $n > N \Rightarrow$

$|a_n - L| < \varepsilon$

We say $a_n \longrightarrow L$ as $n \to \infty$

a_n converges

Theorem Convergent $\Rightarrow$ Bounded

✓ Theorem Monotonic increasing, bounded above

$\Longrightarrow$ convergent

Similar Monotonic decreasing, bounded below

$\Rightarrow$ convergent.

Today 17/11/2021

Pf If $(a_n)_{n \in \mathbb{Z}}$ bounded above

then let $L = \text{L.U.B.} \{a_n : n \in \mathbb{N}\}$

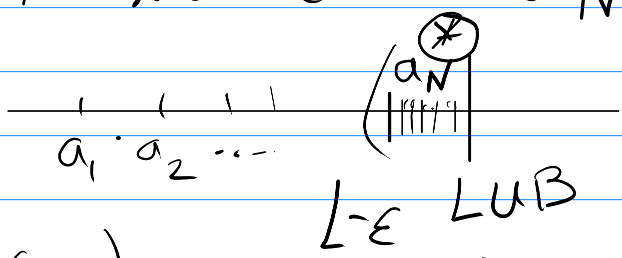
Need to prove $a_n \longrightarrow L$ as $n \to \infty$.

Suppose we are given any $\varepsilon > 0$

L is the Least upper bound

$\Rightarrow L - \varepsilon$ is not an upper bound

$\Rightarrow \exists N \in \mathbb{N}$ such that $a_N > L - \varepsilon$



If we know (a_n) is monotonic increasing

$a_{n+1} \geq a_n \; \forall n$, so if

$n > N$ we have $a_n \geq a_N$

(*) so $a_n > L - \varepsilon \quad \forall n > N$

$$|a_n - L| < \varepsilon \quad \forall n > N$$

& a_n converges to L.

Basic Laws for Limits of Sequences

Suppose $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$ are both convergent, with limits L M

<u>Then</u>

$(a_n+b_n)_{n\in\mathbb{N}}$ converges to $L+M$

$(ka_n)_{n\in\mathbb{N}}$ converges to kL

$(a_nb_n)_{n\in\mathbb{N}}$ converges to LM

$(a_n/b_n)_{n\in\mathbb{N}}$ (if b_n is never zero) converges to L/M (if $M\neq 0$)

converge
$$\begin{cases} a_n = \dfrac{1+n}{n} = 1+\dfrac{1}{n} \longrightarrow 1 = L & \text{as } n\to\infty \\ b_n = \dfrac{1}{n} \longrightarrow \underline{0 = M} & \text{as } n\to\infty \end{cases}$$

diverges $\dfrac{a_n}{b_n} = 1+n$ unbounded so not convergent

Proofs of limit laws are similar to those in Topic 2 "δ" "N"

Pinching Theorem

Suppose that $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ are 3 sequences such that

1. $a_n \leq b_n \leq c_n \quad \forall n \geq \underline{K}$
2. $(a_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ <u>convergent</u>
3. their limits are <u>equal</u>.

Then $(b_n)_{n\in\mathbb{N}}$ also converges to this limit.

<u>Proof</u> Suppose $a_n \to L$, $c_n \to L$ as $n \to \infty$

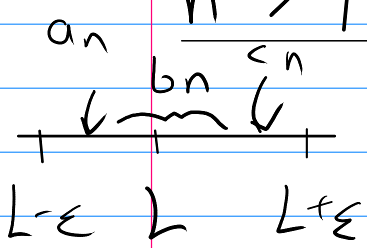
Given <u>any $\varepsilon > 0$</u>, so we know

$$\exists N_a \in \mathbb{N} : n > N_a \Rightarrow |a_n - L| < \varepsilon$$

$$N_c \in \mathbb{N} : n > N_c \Rightarrow |c_n - L| < \varepsilon$$

If $\underline{N} = \max(N_a, N_c, K)$

$$\underline{n > N} \Rightarrow L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$



$$\Rightarrow |b_n - L| < \varepsilon$$

i.e. $b_n \to L$ as $n \to \infty$

We know convergent sequences are bounded, but the converse in general the converse is false.

<u>Example</u> $a_n = (-1)^n$

$1, -1, 1, -1, 1, -1, \ldots$ bounded

<u>not</u> convergent

But it <u>does</u> have convergent <u>subsequences</u>

$1, 1, 1, 1, \ldots \; (a_{2n})$

still bounded but now also convergent

Similarly $a_{2n+1} = -1 \quad \forall n$

<u>Definition</u> A subsequence of $(a_n)_{n \in \mathbb{N}}$ is a sequence $(b_m)_{m \in \mathbb{N}}$ defined by. $b_m = a_{n_m} \quad \forall m$

where $n_0 < n_1 < n_2 < \ldots$

is a <u>strictly</u> increasing sequence of natural numbers

Theorem (Bolzano Weierstrass theorem)

Every bounded sequence has at least one convergent subsequence

(Proof next video!)

Definition (Cauchy sequences)

A sequence $(a_n)_{n \in \mathbb{N}}$ has the Cauchy property if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$n, m > N \Rightarrow |a_n - a_m| < \varepsilon$$

Compare with definition of "converges to L"

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

Strategy: to prove

Cauchy $\iff$ convergent

One direction is easy:

If (a_n) is convergent then it is Cauchy

<u>Pf</u> If $a_n \to L$ as $n \to \infty$

then given $\underline{\varepsilon > 0}$ we know

$\exists \, \underline{N \in \mathbb{N}} : n > N \Rightarrow |a_n - L| < \varepsilon/2$

So $\underline{n, m > N} \Rightarrow$

$$|a_n - L| < \frac{\varepsilon}{2} \quad \& \quad |a_m - L| < \varepsilon/2$$

& $\underline{|a_n - a_m|} \leq |a_n - L| + |a_m - L|$

by the triangle inequality

$$\underline{<} \; \varepsilon/2 + \varepsilon/2$$

$$\underline{= \varepsilon}$$

So we have shown $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence