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MA3077 (DLI) Operational Research

Lecture 12 &13– Maximal flows and minimal cuts

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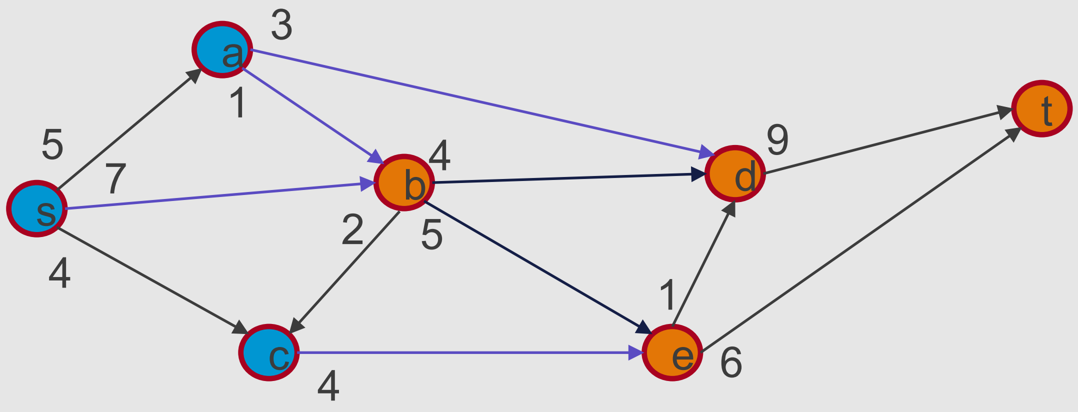
Recap and plan of the day

Summary: in the previous lectures we learnt:

- about graphs and networks
- how to determine a minimal spanning tree,
- how to determine a shortest path tree.

Today: Oriented networks, maximal flows, and minimal cuts, following loosely Ch. 10 of the book by Hillier and Lieberman.

Directed networks - cuts



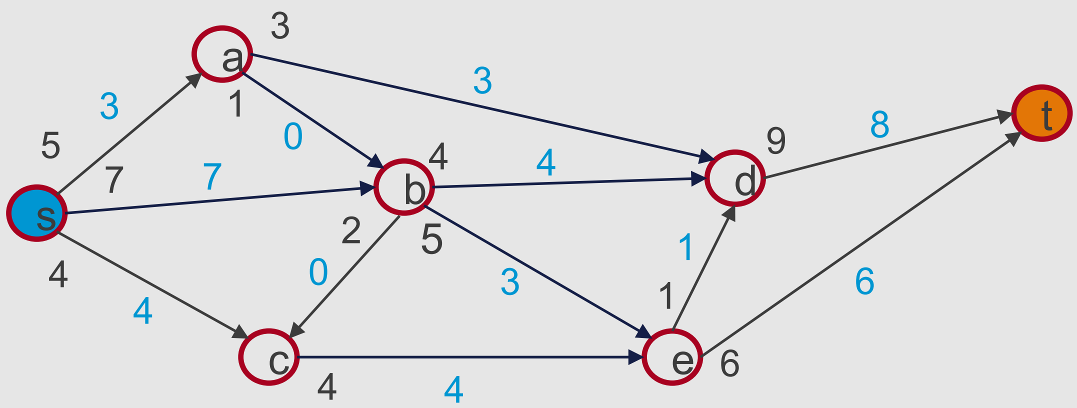
Definition:

- A *network* is a graph () together with a function . In this lecture, the number denotes the *capacity* of the edge
- A network is called *directed* if its edges are directed, that is, if the edges are defined by ordered pairs of vertices.
- A *source* is a node with no incoming edges, whereas a *sink* is a node with no outgoing edges.
- An S-T *cut* of a directed network with a source and a sink is a partition of such that
- The *cut-set* of a cut of a directed network is
- The *capacity* of an s-t cut is the sum of the capacities of the edges in its cut-set:

Example: K={S,A,C}, ={B,E,D,T} is an S-T cut, is the cut-set, and .

Challenge: can you find an S-T cut with minimal capacity?

Directed networks - flows



Definition:

Let be a directed network with source and sink . A *flow* is a function that satisfies

- the *capacity constraint*: for every
- the *conservation of flows*: for every ,

The value of a flow is defined by

Example: the function is a flow and

Challenge: can you find a flow with maximal value?

Maximal flows and minimal cuts 1/2

Proposition: Let be a directed network with source and sink . For any flow and any s-t cut , it holds .

Proof: The conservation of flows implies that, for any

and the definition of sink implies that . Therefore,

Maximal flows and minimal cuts 2/2

Proposition: Let be a directed network with source and sink . For any flow f and any S-T cut K, it holds .

Proof: Therefore,

□

Maximal flows and minimal cuts - corollary

Proposition: Let be a directed network with source and sink . For any flow and any s-t cut , it holds .

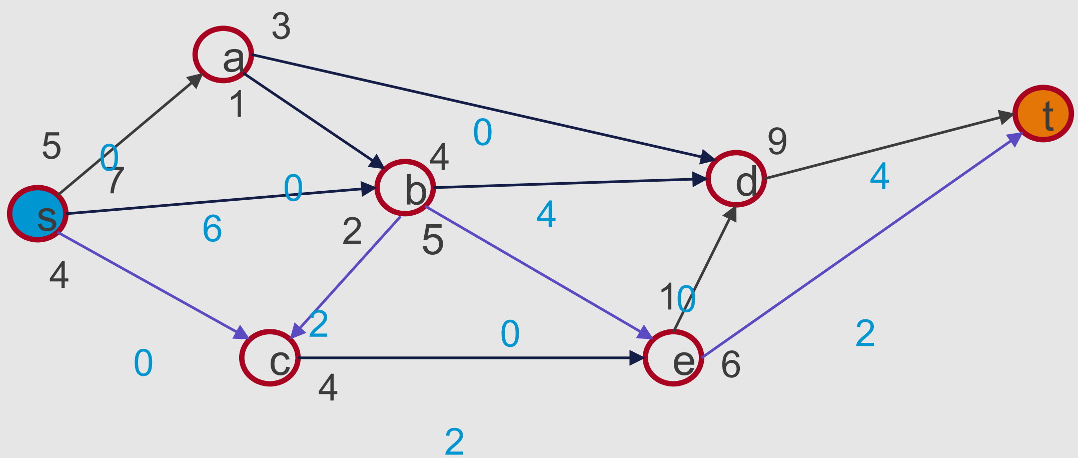
Corollary: If , then is a maximal flow and a minimal cut.

Proof: Let be a maximal flow and a minimal cut. Then

□

- Next lecture

Unsaturated paths



Definition:

Let be a directed network with source and sink , and let be a flow.

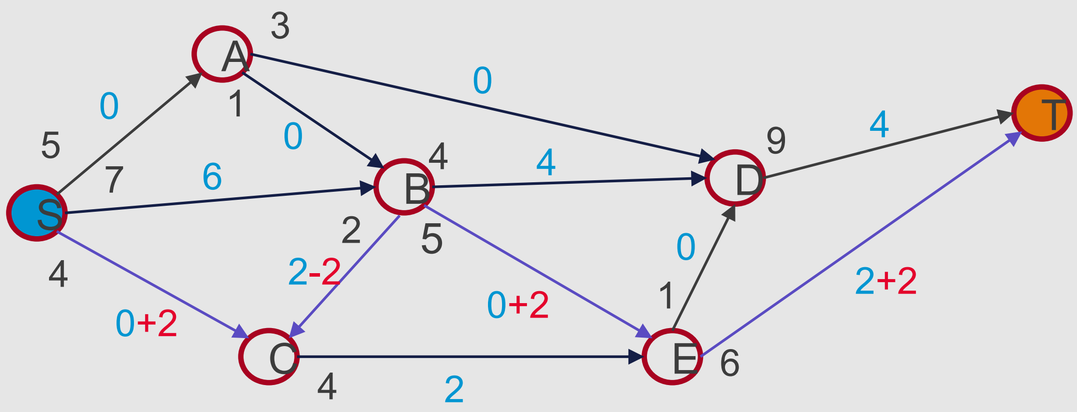
A *path* in is a finite sequence of unique nodes such that either or for every .

The capacity of a path in is defined by

A path with is called *-unsaturated* . An -unsaturated path that connects to is called *-augmenting*.

Example: The capacity of the path ={s,c,b,e,t} is . Therefore, is -augmenting.

Augmented flows



Idea: Let be a directed network with source and sink , let be a flow, and an -augmenting path. Let be defined as follows

Then, is a flow and .

Example: The value of the flow obtained by augmenting with ={s,c,b,e,t} is = 6+0+2 =   = 8.

Maximal flows and minimal cuts revisited

Theorem: Let be a directed network with source and sink .

1. A flow is maximal iff there are no augmenting paths.
2. A flow is maximal iff its value is equal to the capacity of a minimal cut.

Proof: If is a flow and is an augmenting path, then the flow defined in the previous slide satisfies , and is not maximal. Otherwise, assume there are no augmenting paths, and let

Then, the defines an s-t cut. Its definition implies that for any , and that for any edge with and . Therefore,

Hence, the previous corollary implies that is maximal, and the cut defined by is minimal. □

Maximal flow via linear programming

Let be a directed network with source and sink with many edges. Let We can compute the maximal flow by solving

Minimal cut via linear programming

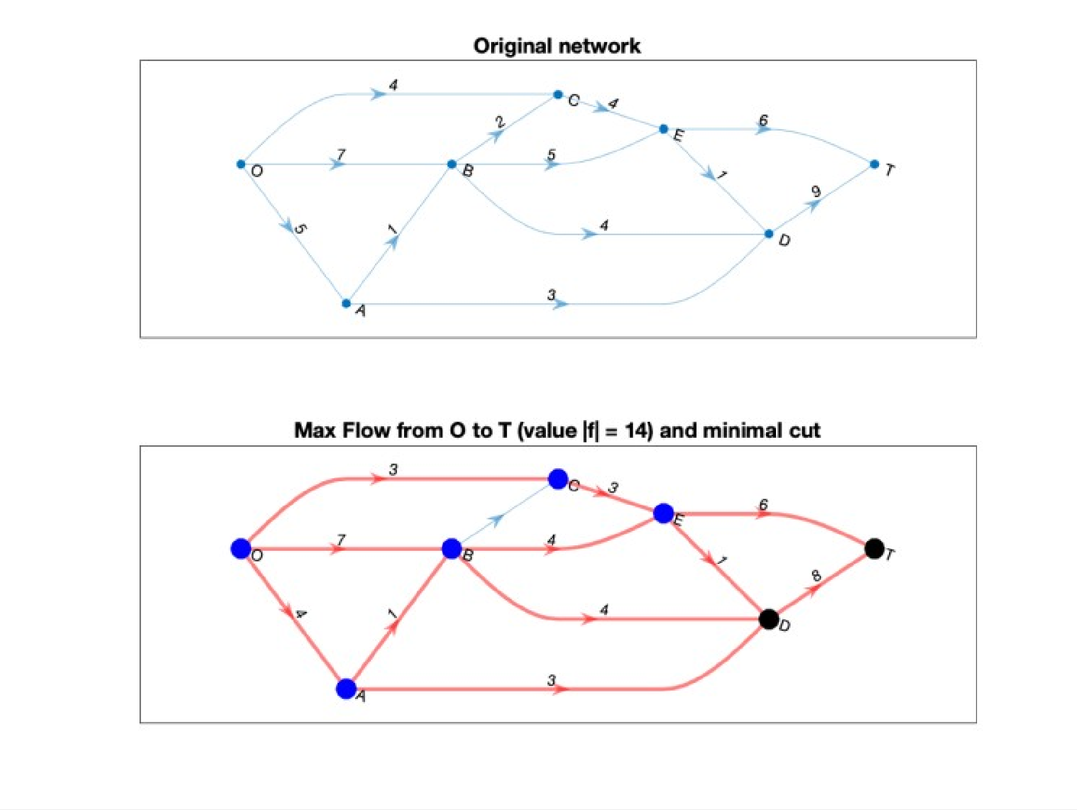
We obtained as the dual to the maximal flow problem. Let and

Interpretation: let identify a minimal cut

- otherwise
- , otherwise
- implies that if and
- implies that if
- implies that if

Maximal flows and minimal cuts in Matlab

(see OR13_cutsandflows.m)



Summary and self-study

Summary: In this lecture and previous lecture we have learnt:

- about directed networks,
- that to a maximal flow corresponds a minimal cut,
- that these are linear programming problems,
- and that you can increase a flow with -augmenting paths.

Self-study: Consider the directed network with source and sink on the right. Let be defined by

and otherwise. Verify that f is a flow. Then, identify an f-augmenting path and compute its capacity.