Lecture 3: Vectors and the Geometry of Space.

MA2032 Vector Calculus

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The Dot Product

Dot products are also called **inner or scalar products** because the product results in a scalar, not a vector.
The Dot Product is a key part of the calculation of the **angle between two vectors** from their components.

Theorem: Angle Between Two Vectors

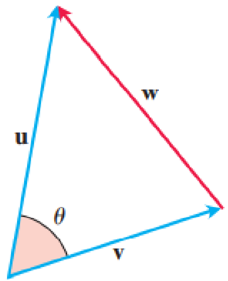
The angle θ between two nonzero vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{||u||\ ||v||}\right) = \cos^{-1}\left(\frac{u \cdot v}{||u||\ ||v||}\right).$$

Definition

The **dot product** $u \cdot v$ of vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = ||u||\ ||v|| \cos\theta.$$



Applying the law of cosines to the triangle in Figure, we find that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$
$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

Because w = u - v, the component form of w is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$|\mathbf{u}|^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$$
$$|\mathbf{v}|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2$$
$$|\mathbf{w}|^2 = \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}\right)^2$$
$$= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$
$$= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \quad \Rightarrow$$

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Proof of Theorem: Angle Between Two Vectors

Therefore,

$$
\begin{aligned}
2|\mathbf{u}||\mathbf{v}|\cos\theta &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3) \\
|\mathbf{u}||\mathbf{v}|\cos\theta &= u_1v_1 + u_2v_2 + u_3v_3 \\
\cos\theta &= \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}.
\end{aligned}
$$

Since $0 \leqslant \theta < \pi$, we have

$$
\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right).
$$

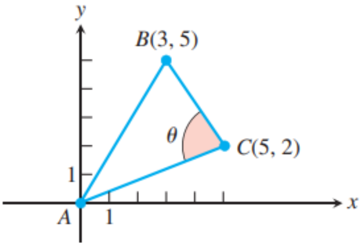
Example

Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ as showed on Figure.

Solution:

The angle θ is the angle between the vectors $\overrightarrow{\mathrm{CA}}$ and $\overrightarrow{\mathrm{CB}}$. The component forms of these two vectors are

$$\overrightarrow{CA} = \langle -5, -2 \rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3 \rangle.$$



First we calculate the dot product and magnitudes of these two vectors.

$$\begin{aligned} \overrightarrow{CA} \cdot \overrightarrow{CB} &= (-5)(-2) + (-2)(3) = 4 \\ |\overrightarrow{CA}| &= \sqrt{(-5)^2 + (-2)^2} = \sqrt{29} \\ |\overrightarrow{CB}| &= \sqrt{(-2)^2 + (3)^2} = \sqrt{13} \end{aligned}$$

Then applying the angle formula, we have $\approx 78.1^\circ$ or 1.36 radians:

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$

Orthogonal Vectors

Two nonzero vectors u and v are**perpendicular (orthogonal)** if the angle between them is $\pi/2$.

For such vectors, we have $u \cdot v = 0$ because $\cos(\pi/2) = 0$.

The **converse is also true**: If u and v are nonzero vectors with $u \cdot v = ||u|| \, ||v|| \cos\theta = 0$, then $cos\theta = 0$ and $\theta = \cos^{-1} = \pi/2$.

The following definition also allows for one or both of the vectors to be the zero vector.

Definition

Vectors u and v are orthogonal if $u \cdot v = 0$.

Example

To determine if two vectors are orthogonal, calculate their dot product.

Solution:

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

(c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) = 0. \end{aligned}$$

Dot Product Properties

Properties of the Dot Product

If u, v, and w are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

1. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$

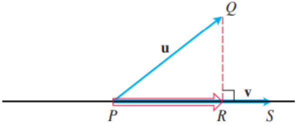
3.
$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\
&= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3 \\
&= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

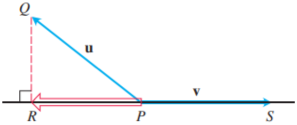
We now return to the problem of **projecting one vector onto another**.

The vector projection of $u = \overrightarrow{\mathrm{PQ}}$ onto a nonzero vector $v = \overrightarrow{\mathrm{PS}}$, see Figure, is the vector $\overrightarrow{\mathrm{PR}} = proj_v u$ determined by **dropping a perpendicular** from Q to the line PS.

If the angle θ between u and v is **acute**, $proj_v u$ has length $|u| \cos\theta$ called **scalar component** of u in the direction of v and direction $v/|v|$.

If u is **obtuse**, cos < 0 and $proj_v u$ has length $-|u| \cos\theta$ and direction $-v/|v|$. In both cases





$$
\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \left(|\mathbf{u}| \cos\theta\right) \frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right) \frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right) \mathbf{v}.
\end{aligned}
$$

$$|\mathbf{u}| \cos\theta = \frac{|\mathbf{u}||\mathbf{v}| \cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

Example

Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v.

Solution:

Using the previously given Equations we find:

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\
&= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}.
\end{aligned}$$

$$\begin{aligned}
|\mathbf{u}| \cos\theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k} \right) \\
&= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.
\end{aligned}$$

Application: Work

Definition

The work done by a constant force F acting through a displacement $D = \overrightarrow{\mathrm{PQ}}$ is

$$W = (\text{scalar component of F in the direction of D})(\text{length of D}) = (|F|\cos\theta)|D| = F \cdot D$$

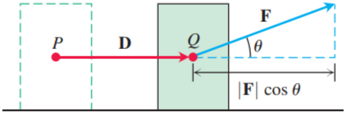


Figure: The work done by a constant force F during a displacement D is the dot product $F \cdot D$.