MA2252 Introduction to Computing

Lecture 18

Numerical Integration

Sharad Kumar Keshari

School of Computing and Mathematical Sciences

University of Leicester

Learning outcomes

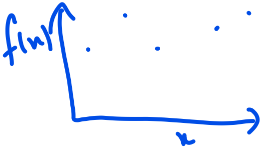
At the end of lecture, students will be able to

- apply numerical methods to evaluate integrals
- understand geometrical interpretation of these methods
- implement these methods in MATLAB
- use MATLAB built-in integration functions

Introduction

Why study numerical integration?

- The anti-derivatives of many functions cannot be represented in terms of elementary functions. **Examples:** $\frac{\sin x}{x}$, e^{-x^2} and $\frac{1}{\ln x}$
- Analytical form of the integrand function(say $f(x)$) may be unknown. **Example:** The values of $f(x)$ are only known at a set of data points x_i.



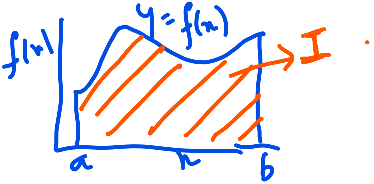
Problem statement

Consider a function $f(x)$ defined over a interval $[a, b]$. We want to evaluate

$$I = \int_a^b f(x)dx. \tag{1}$$

definite integral

This integral can be geometrically seen as area under the curve $y = f(x)$ for $x \in [a, b]$.



Problem statement (contd.)

Steps to evaluate (1) numerically:

- Create a numerical grid $x_i \quad (i = 0, 1, 2, \cdots, n)$ such that $x_0 = a, x_n = b$ and $x_{i+1} - x_i = h$(say).
- Using some appropriate method, calculate the area A_i under $f(x)$ for each sub-interval $[x_i, x_{i+1}] \quad (i = 0, 1, 2, \cdots, n-1)$.
- Compute the sum of the areas A_i over the interval $[a, b]$ i.e.

$$I \approx \sum_{i=0}^{n-1} A_i \tag{2}$$

sum the areas of the Ai strips

Numerical integration methods

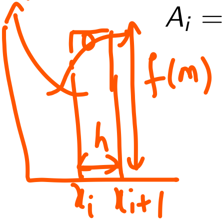
- Midpoint rule
- Trapezoidal rule
- Simpson's rule

Midpoint rule

Steps:

- The value of function in a subinterval $[x_i, x_{i+1}]$ is interpolated by a constant function with the value $f(\frac{x_i + x_{i+1}}{2})$.
- The area A_i is calculated by area of rectangle under the constant function.

$A_i = f(m) \times h$



$$A_i = h * f(\frac{x_i + x_{i+1}}{2}) \quad (3)$$

$I \approx \sum_{i=0}^{n-1} A_i$

$m = \frac{x_i + x_{i+1}}{2}$

Midpoint rule (contd.)

Example: Write a script file which uses Midpoint rule to approximate $\int_0^{\pi} \sin x \, dx$.

$$\int_0^{\pi} \sin x \, dx = 2$$

Midpoint rule (contd.)

Demo

Trapezoidal rule

trapezoid = trapezium

Steps:

- Here, the function in the subinterval $[x_i, x_{i+1}]$ is approximated using a straight line joining points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ (linear interpolation).
- The area A_i is calculated by the area of trapezium formed under this straight line.

$$A_i = \frac{1}{2}(f(x_i) + f(x_{i+1}))h \tag{4}$$

Trapezoidal rule (contd.)

Example: Write a script file which uses Trapezoidal rule to approximate $\int_0^\pi \sin x \, dx$.

Trapezoidal rule (contd.)

Demo

Simpson's rule

Steps:

- Here, the function $f(x)$ is approximated on two subintervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ taken together. The interpolating function is a quadratic passing through points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$.

- The area B_i over interval $[x_{i-1}, x_{i+1}]$ is derived as

$$B_i = \frac{h}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})) \tag{5}$$

- The integral I is given by

$$I \approx \sum_{i=1, i=odd}^{n-1} B_i \tag{6}$$

weights

(6) can also be expressed in the form:

$$I \approx \frac{h}{3}\left[f(x_0) + 4\left(\sum_{i=1, i=odd}^{n-1} f(x_i) \right) + 2\left(\sum_{i=2, i=even}^{n-2} f(x_i) \right) + f(x_n) \right] \quad (7)$$

Note: Since B_i is calculated for two consecutive subintervals taken together, Simpson's rule requires even number of subintervals i.e. n should be even.

Example: Write a script file which uses Simpson's rule to approximate $\int_0^{\pi} \sin x \, dx$.

Demo

MATLAB's built-in integration functions

Two useful functions are trapz() and integral().

- trapz(x,f) takes of numerical grid x and function f as vector arguments and computes the value of integral I using trapezoidal rule.
- integral(fun,xmin,xmax) integrates the function fun from lower limit xmin to upper limit xmax.

MATLAB's built-in integration functions (contd.)

Write a script file using trapz() and integral() functions to approximate $\int_0^{\pi} \sin x \, dx$.

MATLAB's built-in integration functions (contd.)

Demo

End of Lecture 18

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