Lecture 31: Integrals and Vector Fields.

MA2032 Vector Calculus

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November 30, 2022

Surfaces and Area

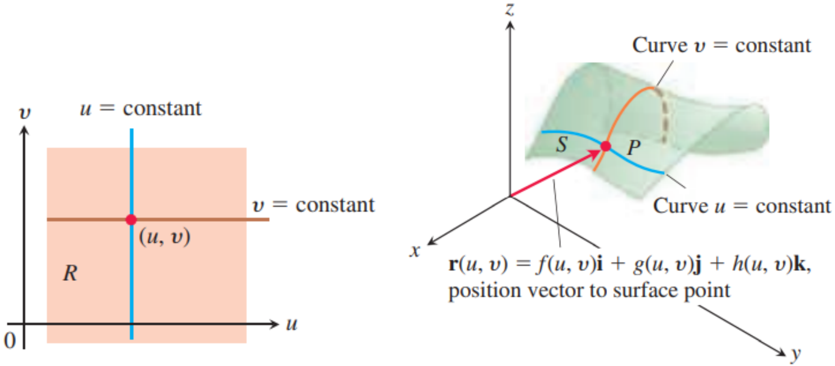
We have **defined curves** in the plane (space) in **three different ways**:

1. **Explicit** form: $y = f(x)$ $(z = f(x, y))$.
2. **Implicit** form: $F(x, y) = 0$ $(F(x, y, z) = 0)$.
3. **Parametric vector** form: $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ $(\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})$, $a \leq t \leq b$.

- There is also a **parametric form for surfaces** that gives the position of a point on the surface as a **vector function of two variables**.
- We apply the form to obtain the **area of a surface** as a double integral.

Parametrizations of Surfaces

• Suppose $\mathbf{r}(u, y) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$ (Eq. 1) is a **continuous vector function** that is defined on a region R in the uv-plane and **one-to-one** on the interior of R.



• **The range** of **r** is the **surface** S defined or traced by **r**.

Parametrizations of Surfaces

• (Eq. 1) together with the domain R constitutes a **parametrization of the surface**.

• The variables u and v are the **parameters**, and R is the **parameter domain**.

• R is rectangle defined by $a \leq u \leq b$, $c \leq y \leq d$.

• The requirement that $\mathbf{r}$ be **one-to-one** on the interior of R ensures that S **does not cross itself**.

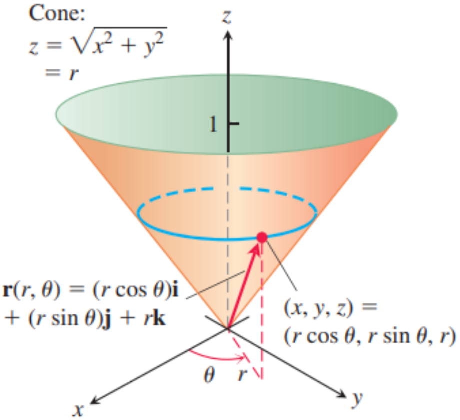
• (Eq. 1) is the vector equivalent of **three parametric equations**:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

Parametrizations of Surfaces

EXAMPLE 1 Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$



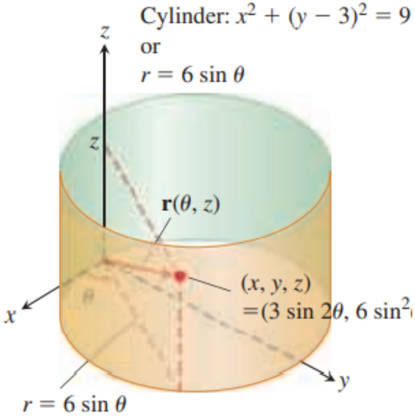
Solution Here, cylindrical coordinates provide a parametrization. A typical point (x, y, z) on the cone (Figure 16.40) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Taking $u = r$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

The parametrization is one-to-one on the interior of the domain R, though not on the boundary tip of its cone where $r = 0$. ■

EXAMPLE 3 Find a parametrization of the cylinder

$$x^2 + (y-3)^2 = 9, \qquad 0 \le z \le 5.$$



Solution Example 3

Solution In cylindrical coordinates, a point (x, y, z) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. For points on the cylinder $x^2 + (y-3)^2 = 9$ (Figure 16.42), the equation is the same as the polar equation for the cylinder's base in the xy-plane:

$$x^2 + (y^2 - 6y + 9) = 9$$

$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2,\ y = r\sin\theta$$

or

$$r = 6\sin\theta, \qquad 0 \le \theta \le \pi.$$

A typical point on the cylinder therefore has

$$\begin{aligned} x &= r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta \\ y &= r\sin\theta = 6\sin^2\theta \\ z &= z. \end{aligned}$$

Taking $u = \theta$ and $v = z$ in Equation (1) gives the one-to-one parametrization

$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le \pi, \quad 0 \le z \le 5.$$

Surface Area

• Our goal is to find a double integral for calculating the **area of a curved surface** S based on the parametrization

$$\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}, \quad a \leq u \leq b,\ c \leq v \leq d$$

• We need S **to be smooth** for the construction we are about to carry out.

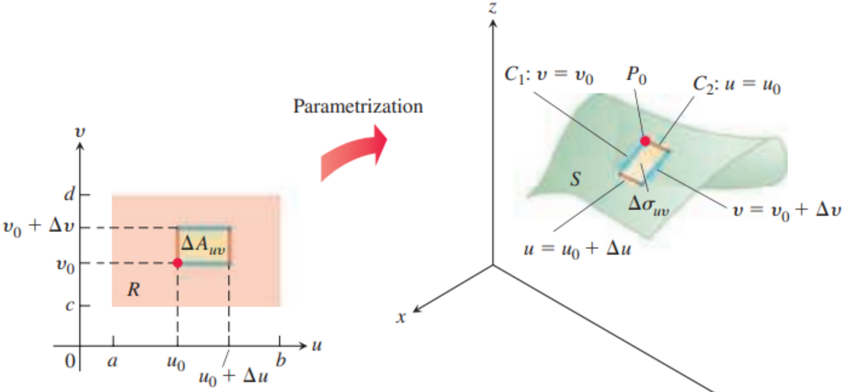
• The definition of smoothness involves the **partial derivatives** of $\mathbf{r}$ with respect to u and v:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

• The **condition** that $\mathbf{r}_u \times \mathbf{r}_v$ is **never the zero vector** in the definition of smoothness means that the two vectors r_u and r_v are nonzero and never lie along the same line, so they **always determine a plane tangent to the surface**.



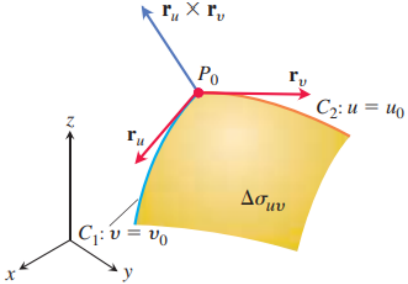
Surface Area

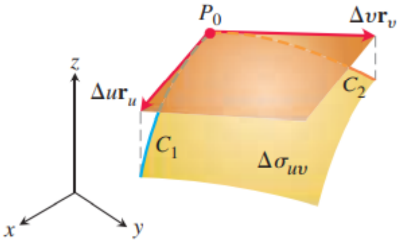
- Consider a small **rectangle** ΔA_{uv} in R

- Each side of ΔA_{uv} maps to a curve on the surface S, and together these four curves bound a **"curved patch element"** $\Delta\sigma_{uv}$.

- The **partial derivative vector** $\mathbf{r}_u(u_0, v_0)$ **is tangent** to C_1 at P_0. Likewise, $\mathbf{r}_v(u_0, v_0)$ is tangent to C_2 at P_0 .

- The **cross product** $\mathbf{r}_u \times \mathbf{r}_v$ is **normal** to the surface at P_0.





Surface Area

• We next **approximate** the surface patch element $\Delta\sigma_{uv}$ **by the parallelogram** on the tangent plane whose area is

$$\left|\Delta u\,\mathbf{r}_u \times \Delta v\,\mathbf{r}_v\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u\,\Delta v.$$

• A **partition** of the surface S into surface patch elements **approximates** the area of S by the sum

$$\sum_n \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u\,\Delta v.$$

Surface Area

• This sum **in limit** sense $(\Delta u \to 0,\ \Delta v \to 0,\ n \to \infty)$ approaches the double integral

DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv. \tag{4}$$

Surface Area

EXAMPLE 4 Find the surface area of the cone in Example 1

Solution In Example 1, we found the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

To apply Equation (4), we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}.$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The area of the cone is

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta \qquad \text{Eq. (4) with } u = r, v = \theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2} \text{ square units.}$$

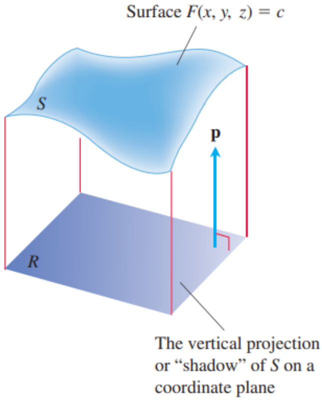
Implicit Surfaces

• Surfaces are often presented as **level sets of a function**, described by an equation such as $F(x, y, z) = c$, for some constant c.
• It may be **difficult to find explicit formulas** for the functions f, g, and h that describe the surface in the form
$\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$.

• We choose $\mathbf{p}$ to be a **unit vector** (it can be $\mathbf{k}$) **normal** to the plane region R ($\in xy$-plane).
• By assumption, we then have

$$\nabla F \cdot \mathbf{p} = \nabla F \cdot \mathbf{k} = F_z \neq 0 \text{ on } S.$$

• The Implicit Function Theorem implies that S is then the **graph of a differentiable function** $z = h(x, y)$.
• Defining $u = x$ and $v = y$, then $z = h(u, v)$ and $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + h(u, v)\,\mathbf{k}$ gives a **parametrization of the surface** S.



- We use previous results to **find the area** of S:

Calculating the partial derivatives of $\mathbf{r}$, we find

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

Applying the Chain Rule for implicit differentiation (see Equation (2) in Section 14.4) to $F(x, y, z) = c$, where $x = u$, $y = v$, and $z = h(u, v)$, we obtain the partial derivatives

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}. \qquad F_z \neq 0$$

Substitution of these derivatives into the derivatives of $\mathbf{r}$ gives

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}.$$

From a routine calculation of the cross product we find

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -F_x/F_z \\ 0 & 1 & -F_y/F_z \end{vmatrix} \quad \begin{array}{c} \text{cross product of} \\ \mathbf{r}_u \\ \mathbf{r}_v \end{array}$$

$$= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$$

$$= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}}$$

$$= \frac{\nabla F}{\nabla F \cdot \mathbf{p}}. \qquad \mathbf{p} = \mathbf{k}$$

Therefore, the surface area differential is given by

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dx\, dy. \qquad u = x \text{ and } v = y$$

We obtain similar calculations if instead the vector $\mathbf{p} = \mathbf{j}$ is normal to the xz-plane when $F_y \neq 0$ on S, or if $\mathbf{p} = \mathbf{i}$ is normal to the yz-plane when $F_x \neq 0$ on S. Combining these results with Equation (4) then gives the following general formula.

Implicit Surfaces

Formula for the Surface Area of an Implicit Surface

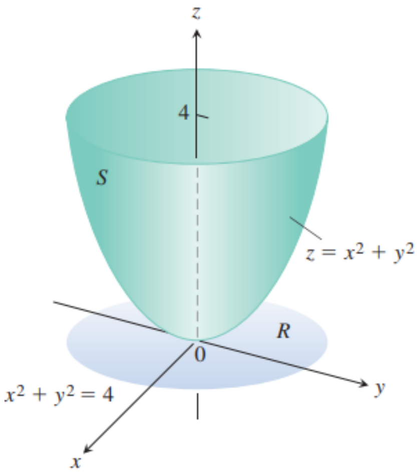
The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Implicit Surfaces

EXAMPLE 7 Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.



Solution Example 7

Solution We sketch the surface S and the region R below it in the xy-plane (Figure 16.48). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal to the plane of R, we can take $\mathbf{p} = \mathbf{k}$.

At any point (x, y, z) on the surface, we have

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \mathbf{p}| &= |\nabla F \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

In the region R, $dA = dx\,dy$. Therefore,

Solution Example 7

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA \qquad \text{Eq. (7)}$$

$$= \iint_{x^2+y^2 \le 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \qquad \text{Polar coordinates}$$

$$= \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$