# §7. Quadratic Forms

- A scalar product on a vector space $V$ is also called a symmetric bilinear form.

$$g : V \times V \to K,$$
$$g(v, w) = \langle v, w \rangle.$$

- By the quadratic form determined by g, we shall mean the function

$$f : V \to K$$

such that

$$f(v) = g(v, v) = \langle v, v \rangle.$$

# §7. Quadratic Forms

- For $V = K^n, f(X) = X \cdot X = x_1^2 + \cdots + x_n^2$ is the quadratic form determined by the ordinary dot product.
- If $A \in \mathrm{Mat}_{n\times n}(K)$ is symmetric, then

$$g_A(X, Y) = {}^{\mathrm{t}}XAY = \sum_{i,j=1}^{n} a_{ij}x_iy_j.$$

is a symmetric bilinear form (scalar product) and

$$f_A(X) = {}^{\mathrm{t}}XAX = \sum_{i,j=1}^{n} a_{ij}x_ix_j.$$

is the quadratic form determined by $g_A$.

- If $A = \mathrm{diag}(a_1, \ldots, a_n)$, then

$$f_A(X) = a_1x_1^2 + \cdots + a_nx_n^2.$$

## §7. Quadratic Forms

- The scalar product $g$ can be uniquely determined by the quadratic form $f$.

$$g(v,w) = \frac{1}{4}[f(v+w, v+w) - f(v-w, v-w)],$$

i.e.

$$\langle v,w\rangle = \frac{1}{4}[\langle v+w, v+w\rangle - \langle v-w, v-w\rangle],$$

or

$$g(v,w) = \frac{1}{2}[f(v+w, v+w) - f(v,v) - f(w,w)],$$

i.e.

$$\langle v,w\rangle = \frac{1}{2}[\langle v+w, v+w\rangle - \langle v,v\rangle - \langle w,w\rangle].$$

# §7. Quadratic Forms

- Given the quadratic form

$$f(x, y) = 2x^2 + 3xy + y^2,$$

find the matrix $A$ of its symmetric bilinear form $g$.

$$(x, y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2$$

$$a = 2, \quad b = \frac{3}{2}, \quad c = 1$$

$$\begin{pmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$$

# §7. Quadratic Forms

- Let $f : R^n \to R$ be a function which has partial derivatives of order 1 and 2, and such that the partial derivatives are continuous functions. Assume that

$$f(tX) = t^2 f(X), \forall X \in R^n.$$

homogeneous

Then $f$ is a quadratic form, that is there exists a symmetric matrix $A = (a_{ij})$ such that

$$f(X) = \sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

$$\underline{f(tx_1, \cdots, tx_n)}$$

$$\| $$

$$f(tX) = t^2 f(X)$$

$$\sum_{i=1}^{n} f_i'(tX)\cdot x_i = 2t f(X)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}''(tX)\cdot x_i x_j = 2 f(X)$$

Let $t=0$, we have $f(X) = \frac{1}{2} \sum_{i,j=1}^{n} f_{ij}''(0) x_i x_j = X^T A X$

$$\boxed{\begin{array}{l} A = (a_{ij})_{n\times n} \\ a_{ij} = \frac{1}{2} f_{ij}''(0) \end{array}}$$

# §7. Quadratic Forms

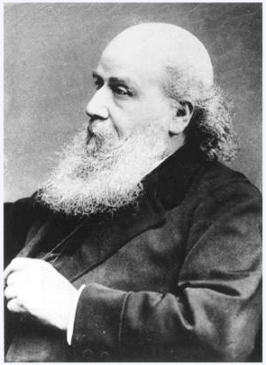
- Homework: Ch. V, §7, 1, 2, 3(c).

# §8. Sylvester's Theorem

- Let $V$ be a finite dimensional vector space over $R$, with a positive definite scalar product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

$$v = x_1e_1 + \cdots + x_ne_n \text{ and } w = y_1e_1 + \cdots + y_ne_n.$$

Then $\langle v, w\rangle = x_1y_1 + \cdots + x_ny_n$.



James Sylvester, 1814-1897.

- $\langle v, v\rangle = x_1^2 + \cdots + x_n^2$.
- Let $V$ be a finite dimensional vector space over $C$, with a positive definite Hermitian product. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis.

$$v = x_1e_1 + \cdots + x_ne_n \text{ and } w = y_1e_1 + \cdots + y_ne_n.$$

Then $\langle v, w\rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$.

- $\langle v, v\rangle = |x_1|^2 + \cdots + |x_n|^2$.

# §8. Sylvester's Theorem

- Let $V = R^2$, and let the symmetric bilinear form be represented by the matrix

$$A = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}.$$

Then the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

forms an orthogonal basis for the form, and

$$\langle v_1, v_1 \rangle = -1, \quad \langle v_2, v_2 \rangle = 0.$$

- $\forall v = x_1 v_1 + x_2 v_2$, $\langle v, v \rangle = -x_1^2 + 0 \cdot x_2^2$. $= \langle v_1, v_1 \rangle x_1^2 + \langle v_2, v_2 \rangle x_2^2$
- General situation?

# §8. Sylvester's Theorem

- Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis of $V$ and

$$c_i = \langle v_i, v_i \rangle.$$

After renumbering the elements of our basis if necessary, we may assume that $\{v_1, \ldots, v_n\}$ are so ordered that:

$$\begin{aligned} c_1, \ldots, c_r &> 0, \\ c_{r+1}, \ldots, c_s &< 0, \\ c_{s+1}, \ldots, c_n &= 0. \end{aligned}$$

- $\forall v = x_1 v_1 + \cdots + x_n v_n$,

$$f(X) = \langle v, v \rangle = c_1 x_1^2 + \cdots + c_r x_r^2 + c_{r+1} x_{r+1}^2 + \cdots + c_s x_s^2.$$

- $r$ and $s$ do not depend on the orthogonal basis.

$\sum\limits_{i=1}^{s} c_i x_i^2$

# §8. Sylvester's Theorem

- If $\{v_1, \ldots, v_n\}$ is orthonormal,

$$\langle v_i, v_i \rangle = \begin{cases} 1, & i = 1, \ldots, r, \\ -1, & i = r+1, \ldots, s, \\ 0, & i = s+1, \ldots, n. \end{cases}$$

  then

$$f(X) = \langle v, v \rangle = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2.$$

- $r$ and $s$ do not depend on the orthonormal basis.
- Normalization:

$$v_i' = \begin{cases} v_i/\sqrt{c_i}, & i = 1, \ldots, r, \\ v_i/\sqrt{-c_i}, & i = r+1, \ldots, s, \\ v_i, & i = s+1, \ldots, n. \end{cases}$$

# §8. Sylvester's Theorem

not necessarily non-degenerate

- **Theorem 8.1.** Let $V$ be a finite dimensional vector space over $R$, with a scalar product. Assume dim $V > 0$. Let $V_0$ be the subspace of $V$ consisting of all vectors $v \in V$ such that $\langle v, w\rangle = 0$ for all $w \in V$. Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis for $V$. Then the number of integers $i$ such that $\langle v_i, v_i\rangle = 0$ is equal to the dimension of $V_0$.

$V^\perp$

Proof. Suppose $v_1, \ldots, v_n$ is ordered such that

$\langle v_1, v_1\rangle \neq 0, \ldots, \langle v_s, v_s\rangle \neq 0, \quad \langle v_i, v_i\rangle = 0 \ (i > s)$

$\forall\, v \in V_0 \subset V$, it can be written as

$v = x_1 v_1 + \cdots + x_s v_s + x_{s+1} v_{s+1} + \cdots + x_n v_n$

$(j \leq s)$

$0 = \langle v, v_j\rangle = x_j \underbrace{\langle v_j, v_j\rangle}_{\neq 0}$

$x_j = 0$, $\quad v = x_{s+1} v_{s+1} + \cdots + x_n v_n$. $\{v_{s+1}, \ldots, v_n\}$ is an orthogonal basis of $V_0$, $\dim V_0 = n - s$

# §8. Sylvester's Theorem

零化指标. 零指标,

- The dimension $n - s$ of $V_0$ in Th. 8.1 is called the index of nullity of the form.
- The form is non-degenerate if and only if its index of nullity is 0.

**Theorem 8.2 (Sylvester's theorem).** Let $V$ be a finite dimensional vector space over $R$, with a scalar product. There exists an integer $r \geq 0$ having the following property. If $\{v_1, \ldots, v_n\}$ is an orthogonal basis of $V$, then there are precisely $r$ integers $i$ such that $\langle v_i, v_i \rangle > 0$.

- The integer $r$ of Sylvester's theorem is called the index of positivity of the scalar product.

  正指标.

- The integer $s - r$ is the number of integers $i$ such that $\langle v_i, v_i \rangle < 0$. It does not depend on the orthonormal basis and is called the index of negativity of the scalar product.

  负指标.

- **Proof of Theorem 8.2.**

Let $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_n\}$ be orthogonal bases of $V$, in which the vectors are ordered such that

$$\langle v_i, v_i\rangle \begin{cases} >0, & 1\le i\le r, \\ <0, & r< i\le s, \\ =0, & s<i\le n. \end{cases} \qquad \langle w_i, w_i\rangle \begin{cases} >0, & 1\le i\le r' \\ <0, & r'< i\le s \\ =0, & s<i\le n \end{cases}$$

$$\Rightarrow \boxed{v_1, \dots, v_r, w_{r'+1}, \dots, w_n \text{ are L. I.}} \Rightarrow \boxed{r \le r'}$$

If $\quad x_1 v_1 + \dots + x_r v_r + y_{r'+1} w_{r'+1} + \dots + y_n w_n = 0$

$$\Rightarrow \quad x_1 v_1 + \dots + x_r v_r = -(y_{r'+1} w_{r'+1} + \dots + y_n w_n)$$

Taking the scalar product of each side with itself. $\Rightarrow$

$$0 \le c_1 x_1^2 + \dots + c_r x_r^2 = d_{r'+1} y_{r'+1}^2 + \dots + d_n y_n^2 \le 0,$$

$c_i = \langle v_i, v_i\rangle > 0 \ (1\le i\le r)$, $\quad d_i = \langle w_i, w_i\rangle \le 0$.

# §8. Sylvester's Theorem

$c_1 x_1^2 + \cdots + c_r x_r^2 = 0 \Rightarrow x_1 = \cdots = x_r = 0$

$\Rightarrow y_{r'+1} w_{r'+1} + \cdots + y_n w_n = 0 \Rightarrow y_{r'+1} = \cdots = y_n = 0$

$\Rightarrow v_1, \ldots, v_r, w_{r'+1}, \ldots, w_n$ are L.I.

$\Rightarrow r + n - r' \leq n \Rightarrow r \leq r'$

Similarly, we can also prove that: $r \geq r'$

$\Rightarrow r = r'$

# §8. Sylvester's Theorem

- Index of positivity of the form represented by $A$ = number of positive eigenvalues of $A$.

  Index of negativity of the form represented by $A$ = number of negative eigenvalues of $A$.

  Index of nullity of the form represented by $A$ = number of zero eigenvalues of $A$.

- Homework: Ch. V, §8, 1, 2.