Lecture 37: Infinite Sequences and Series.

MA2032 Vector Calculus

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Power Series and Convergence

• Now that we can test many infinite series of numbers for convergence, we can study **sums that look like "infinite polynomials."**

• We call these sums **power series** because they are defined as infinite series of powers of some variable, in our case x.

• Like polynomials, power series **can be added, subtracted, multiplied, differentiated, and integrated** to give new power series.

• With power series we can **extend the methods of calculus** to a vast array of functions, making the techniques of calculus applicable in an even wider setting.

Power Series and Convergence

• We begin with the **formal definition**, which specifies the **notation and terminology** used for power series.

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots . \qquad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

• We will see that a **power series defines a function $f(x)$** on a certain interval where it converges.

• Moreover, this function will be shown **to be continuous and differentiable** over the interior of that interval.

Power Series and Convergence

EXAMPLE 1 Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

- We think of the partial sums of the series on the right as **polynomials** $P_n(x)$ that **approximate the function** on the left.

• For **values of x near zero**, we need take only a **few terms of the series to get a good approximation**.

• As we **move toward $x = 1$**, or -1, we must **take more terms**.

• The function $f(x) = 1/(1-x)$ is **not continuous** on intervals containing $x = 1$, where it has a vertical asymptote.

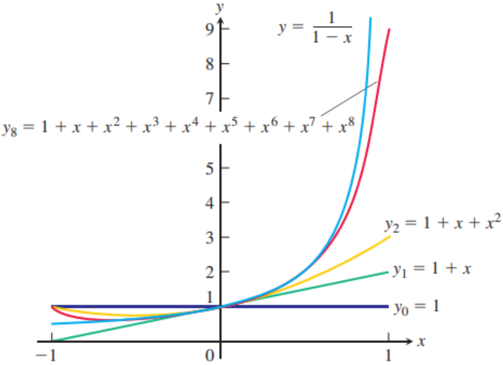


FIGURE The graphs of $f(x) = 1/(1-x)$ in Example 1 and four of its polynomial approximations.

• The **approximations do not apply** when $x \geq 1$.

- The following example illustrates **how we test a power series for convergence** by using the Ratio Test to see where it converges and diverges.

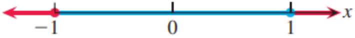
EXAMPLE 3 For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$

By the Ratio Test, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series, which diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



• We will see that **this series converges to the function** $\ln(1+x)$ on the interval $(-1, 1]$

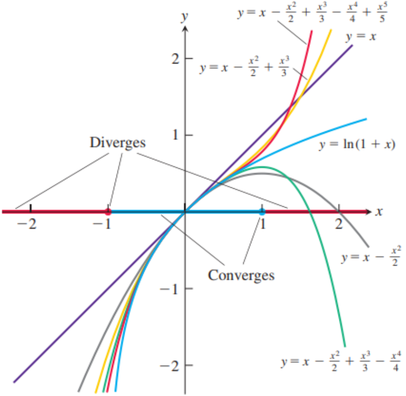


FIGURE The power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ converges on the interval $(-1, 1]$.

Power Series and Convergence

• The next result shows that if a power series converges at more than one value, then it **converges over an entire interval of values**.

• The interval might be **finite or infinite** and contain one, both, or none of its endpoints.

• We will see that **each endpoint** of a finite interval **must be tested independently** for convergence or divergence.

THEOREM 18—The Convergence Theorem for Power Series
If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

The Radius of Convergence of a Power Series

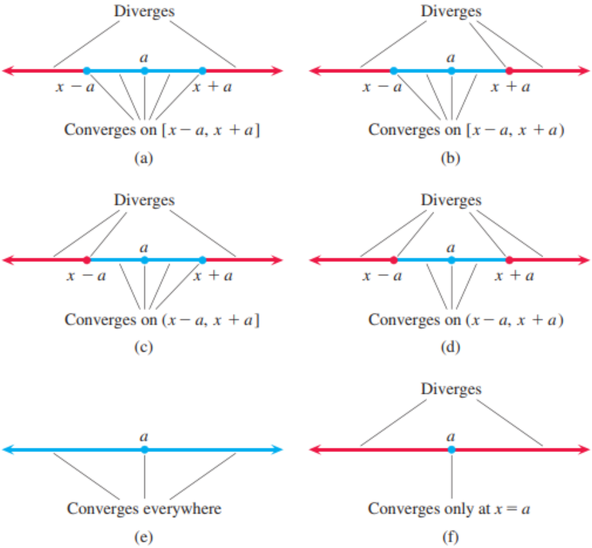


FIGURE The six possibilities for an interval of convergence.

Power Series and Convergence

Corollary to Theorem 18
The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x $(R=\infty)$.
3. The series converges at $x=a$ and diverges elsewhere $(R=0)$.

- R is the **radius of convergence** of the power series, and the interval of radius R centered at $x=a$ is the **interval of convergence**.
- The interval of convergence may be open, closed, or half-open.
- At points x with $|x-a|<R$, the series **converges absolutely**.
- If the series converges for all values of x, then R **is infinite**. If it converges only at $x=a$, then $R=0$.

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely,

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If R is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

THEOREM 19—Series Multiplication for Power Series
If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)| < R$.

Operations on Power Series

THEOREM 21—Term-by-Term Differentiation

If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

EXAMPLE Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Solution We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

Operations on Power Series

THEOREM 22—Term-by-Term Integration
Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Operations on Power Series

EXAMPLE Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1. \qquad \text{Theorem 21}$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1}x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1}x, \qquad -1 < x < 1. \tag{6}$$

It can be shown that the series also converges to $\tan^{-1}x$ at the endpoints $x = \pm 1$, but we omit the proof. ■

EXAMPLE The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 22}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem. A proof of this is outlined in Exercise .