Last time Bolzano, IVT

A continuous function on a closed interval $\left\{ \begin{array}{l} \text{is bounded} \\ \text{has bounded range} \end{array} \right.$ $\exists B$ $|f(x)| < B$ $\forall x \in$ domain $[a,b]$

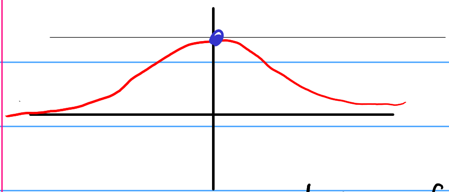
$$f : [a,b] \longrightarrow \mathbb{R}$$

range(f) bounded $\subseteq \mathbb{R}$

Examples

$f : \mathbb{R} \to \mathbb{R}$

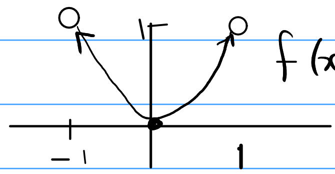
$f(x) = \dfrac{1}{1+x^2}$



$f(x) > 0$, $\lim_{x \to \pm\infty} f(x) = 0$ not attained

Bounded above by $1 = f(0)$

$f : (-1,1) \to \mathbb{R}$



$f(x) = x^2$

$f(0) = 0$

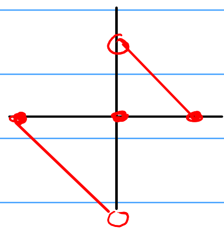
$\lim_{x \to 1^-} f(x) = 1$ not attained

$\lim_{x \to -1^+} f(x) = 1$

not cts

$f : [-1,1] \to \mathbb{R}$

$f(x) = \begin{cases} 1-x & x > 0 \\ 0 & 0 \\ -1-x & x < 0 \end{cases}$



Bounds ± 1 not attained

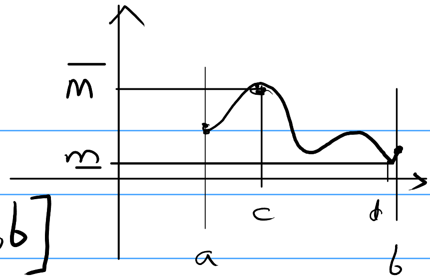
Extreme Value Theorem (EVT)

If $f : [a,b] \to \mathbb{R}$ is continuous then it attains its bounds

More explicitly: Let $\underline{m} = GLB(\text{range}(f))$ & let $\overline{M} = LUB(\text{range}(f))$ then $\exists c, d \in [a,b]$

$\underline{m} \leq f(x) \leq \overline{M}$ $\forall x \in [a,b]$

$$f(c) = \underline{m} \leq f(x) \leq \overline{M} = f(d)$$



Proof by contradiction

Suppose $\underline{f(x) \neq \underline{m}}$ $\forall x \in [a,b]$

$$d(x) = f(x) - \underline{m} > 0, \quad \underset{\text{cts}}{g(x)} = \frac{1}{d(x)} > 0$$

$$f, \overset{\text{cts}}{d}, g : [a,b] \to \mathbb{R}$$

g bounded as it is continuous on $[a,b]$

$$0 < g(x) < B \quad \forall x \in [a,b]$$

$$d(x) > \frac{1}{B} \quad \forall x \in [a,b]$$

$$f(x) - \underline{m} > \frac{1}{B}, \quad f(x) > \underline{m} + \frac{1}{B}$$

<u>Contradiction</u> Greater lower bound than $\underline{m}$

So if $\underline{m} = GLB(\text{range}(f))$

$\exists\, x = c \in [a,b]$ such that $f(c) = \underline{m}$

Leave the existence of d s.t. $f(d) = LUB$ $\overline{m}$ proof for the students.