LINEAR ALGEBRA II

Ch. III LINEAR MAPPINGS

Ch. III Linear Mappings

- **Linear Mapping:** Let V, V' be VSs over the field K. A linear mapping
$$F : V \to V'$$
is a mapping satisfying:

  LM 1. $\forall u, v \in V$, $F(u+v) = F(u) + F(v)$.
  LM 2. $\forall c \in K$ and $v \in V$, $F(cv) = cF(v)$.

- When $V' = K$, F is called a linear functional.
- When $V = K^n$, $V' = K$, F is a linear function.
- The identity map id_V, I_V (id, I): $v \mapsto v$ is a linear mapping.
- The zero map O: $v \mapsto O$ is a linear mapping.

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- Composite mapping $G \circ F : U \ni u \mapsto G(F(u)) \in W$ of $G : V \to W$ and $F : U \to V$.
- $H \circ (G \circ F) = (H \circ G) \circ F$.
- The composite map $GF = G \circ F$ of linear maps is also a linear map.
- For linear maps, $(H + G) \circ F = H \circ F + G \circ F$; $G \circ (F + T) = G \circ F + G \circ T$; $(cG) \circ F = c(G \circ F)$.

Ch. III Linear Mappings

- A mapping $F : S \to S'$ is called injective if $x \neq y \Rightarrow F(x) \neq F(y)$ $(F(x) = F(y) \Rightarrow x = y)$.
    - A linear mapping $F : V \to V'$ is injective$\Leftrightarrow$Ker $F = \{O\}$.
    - Ker $F = \{O\}$ $\Rightarrow$If $v_1, \dots v_n$ are L.I., then $F(v_1), \dots F(v_n)$ are L.I.
    - Ker $F = \{v \in V | F(v) = O\}$ is the kernel of F, a subspace of V.
- A mapping $F : S \to S'$ is called surjective if Im $F = S'$
    - Im $F = \{F(v) | v \in V\}$ is the image of F, a subspace of V'.
- bijective=injective+surjective.

- $\dim V = \dim \operatorname{Ker} L + \dim \operatorname{Im} L$.
- Let $L : V \to W$ be a linear map. Assume that

$$\dim V = \dim W.$$

If $\operatorname{Ker} L = \{O\}$, or if $\operatorname{Im} L = W$, then L is bijective.

- We say that the mapping $F : S \to S'$ has an inverse if there exists a mapping $G : S' \to S$ such that

$$G \circ F = I_S, \text{ and } F \circ G = I_{S'}.$$

- The inverse of a linear map is a linear map.

- The mapping $F : S \to S'$ has an inverse$\Leftrightarrow$$F$ is bijective.

Ch. III Linear Mappings

- All mappings from S (a set) into V (a VS over K) is a vector space over K.

Ch. III Linear Mappings

- $\mathcal{L}(V, V')$, all linear maps from V into V' (V and V' are VSs over K) is a vector space over K.

- Let V be a finite dimensional space over K, and let $\{v_1, \ldots, v_n\}$ be a basis of V. We define a map

$$F: \ V \to K^n$$

by associating to each element $v \in V$ its coordinate vector X with respect to the basis. Thus if

$$v = x_1 v_1 + \cdots, x_n v_n,$$

with $x_i \in K$, we let

$$F(v) = (x_1, \ldots, x_n).$$

Then, F is a linear map.

- $G: K^n \to V$, $G(x_1, \ldots, x_n) = v$ a linear map.
- $GF = I$, $FG = I$.
- F is a isomorphism between V and K^n.

- Let $F : S \to K^n$ be a mapping, then

$$F(v) = (f_1(v), \ldots, f_n(v)).$$

  f_i's are coordinate (component) function(al)s of F.

- $F : V \to K^n$ (V is a VS) is a linear map $\Leftrightarrow$ f_i's are linear function(al)s.

Ch. III Linear Mappings

- Let V be the VS of functions having derivatives of all orders on the interval $0 < t < 1$, then the derivative $D = d/dt$ is a linear mapping from V into V.
- Let V be the VS of functions having derivatives of all orders, then $a_m D^m + a_{m-1} D^{m-1} + \cdots + a_1 I$ is a linear mapping from VS into V. It is also a linear operator on any one of the following finite dimensional vector spaces:
  - $P_n = \left\{ \sum_{k=0}^{n} a_k t^k | a_k \in K \right\}$.
  - $E_n = \left\{ \sum_{k=0}^{n} a_k e^{kt} | a_k \in K \right\}$.
  - $T_n = \left\{ \sum_{k=0}^{n} [a_k \cos(kt) + b_k \sin(kt)] | a_k, b_k \in K \right\}$.

Ch. III Linear Mappings

- Let A be an $m \times n$ matrix in a field K.

$$L_A : \ K^n \ni X \mapsto AX \in K^m$$

  is a linear map from K^n to K^m.

- $F : K^n \to K^r$

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_r).$$

- Operator: linear mapping $F : V \to V$ from a VS V to itself.
- $F^r = F \circ \cdots \circ F$.

- Homework:
    - P65, 14 and 15
    - Prove: $D = d/dt$ is a linear mapping from P_n to P_n.