Lecture 36: Infinite Sequences and Series.

MA2032 Vector Calculus

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Alternating Series and Conditional Convergence

- A series in which the terms are alternately positive and negative is an **alternating series**.

- Here are three **examples**:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \tag{3}$$

- We see from these examples that the *n*-**th term** of an alternating series **is of the form**

$$a_n = (-1)^{n+1} u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

- where $u_n = |a_n|$ is a positive number.

Alternating Series and Conditional Convergence

• Series (1), called the **alternating harmonic series**, **converges**, as we will see in a moment.

• Series (2), a **geometric series** with ratio $r = -1/2$, **converges** to $-2/[1 + (1/2)] = -4/3$.

• Series (3) **diverges** because the n-th term does not approach zero.

• We prove the convergence of the alternating harmonic series by applying the **Alternating Series Test**.

• This test is for convergence of an alternating series and **cannot be used to conclude** that such a **series diverges**.

THEOREM 15—The Alternating Series Test

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Proof We look at the case where $u_1, u_2, u_3, u_4, \ldots$ is nonincreasing, so that $N = 1$. If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L. \qquad \text{Theorem 6} \tag{4}$$

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m\to\infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L. \tag{5}$$

Combining the results of Equations (4) and (5) gives $\lim_{n\to\infty} s_n = L$

■

• **Rather than directly verifying** the definition $u_n \geq u_{n+1}$, a **second way** to show that the sequence $\{u_n\}$ **is nonincreasing** is to **define a differentiable function** $f(x)$ satisfying $f(n) = u_n$.

• That is, the values of f match the values of the sequence at every positive integer n.

• If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ **is nonincreasing** for $x \geq N$.

• It follows that $(n) \geq (n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

EXAMPLE We show that the sequence $u_n = 10n/(n^2 + 16)$ is eventually nonincreasing. Define $f(x) = 10x/(x^2 + 16)$. Then from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \qquad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$. That is, the sequence $\{u_n\}$ is nonincreasing for $n \geq 4$. ■

• A graphical interpretation of the partial sums **shows how an alternating series converges** to its limit L when the three conditions of Theorem 15 are satisfied with $N = 1$.

• The **limit** L **lies between any two successive sums** s_n **and** s_{n+1} and hence differs from s_n by an amount less than u_{n+1}.

• Because $|L - s_n| \leq u_{n+1}$ for $n \geq N$, we can make **useful estimates of the sums of convergent alternating series**.

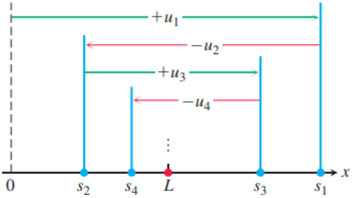

FIGURE 10.15 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

THEOREM 16—The Alternating Series Estimation Theorem
If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

EXAMPLE We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is $s_8 = 0.6640625$ and the sum of the first nine terms is $s_9 = 0.66796875$. The sum of the geometric series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3},$$

and we note that $0.6640625 < (2/3) < 0.66796875$. The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and is less than $(1/256) = 0.00390625$. ■

Conditional Convergence

• If we **replace all the negative terms** in the alternating series in Example (slide 10), **changing them to positive terms** instead, we obtain the geometric series $\sum 1/2^n$.

• The **original series and the new series of absolute values both converge** (although to different sums).

• For an **absolutely convergent series**, changing infinitely many of the negative terms in the series to positive values **does not change its property of still being a convergent series**.

• **Other convergent series may behave differently**.

Conditional Convergence

• The **convergent alternating harmonic series** has infinitely many negative terms, but if we change its negative terms to positive values, **the resulting series is the divergent harmonic series**.

• So the presence of **infinitely many negative terms is essential to the convergence** of the alternating harmonic series.

• The following **terminology distinguishes these two types of convergent series**.

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent.**

Conditional Convergence

EXAMPLE 4 If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely as an ordinary p-series. If $0 < p \leq 1$, the series converges conditionally by the alternating series test. For instance,

$$\text{Absolute convergence } \left(p = 3/2\right): \quad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

$$\text{Conditional convergence } \left(p = 1/2\right): \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \quad \blacksquare$$

Rearranging Series

• We can always **rearrange the terms of a finite collection of numbers** without changing their sum.

• The same result **is true for an infinite series that is absolutely convergent**.

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

• On the other hand, **if we rearrange the terms of a conditionally convergent series, we can get different results**.

Conditional Convergence

• Next Example shows that **we cannot rearrange the terms of a conditionally convergent series** and expect the new series to be the same as the original one.

• When we use a **conditionally convergent series**, the **terms must be added together in the order they are given** to obtain a correct result.

• **In contrast, Theorem 17** guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.

Conditional Convergence

EXAMPLE We know that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges to some number L. Moreover, by Theorem 16, L lies between the successive partial sums $s_2 = 1/2$ and $s_3 = 5/6$, so $L \neq 0$. If we multiply the series by 2 we obtain

$$2L = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right)$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots.$$

Now we change the order of this last sum by grouping each pair of terms with the same odd denominator, but leaving the negative terms with the even denominators as they are placed (so the denominators are the positive integers in their natural order). This rearrangement gives

$$(2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\left(\frac{2}{7}-\frac{1}{7}\right)-\frac{1}{8}+\cdots$$
$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\cdots\right)$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=L.$$

So by rearranging the terms of the conditionally convergent series $\sum_{n=1}^{\infty} 2(-1)^{n+1}/n$, the series becomes $\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which is the alternating harmonic series itself. If the two series are the same, it would imply that $2L = L$, which is clearly false since $L \neq 0$. ■

Summary of Tests to Determine Convergence or Divergence

1. **The nth-Term Test for Divergence:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.