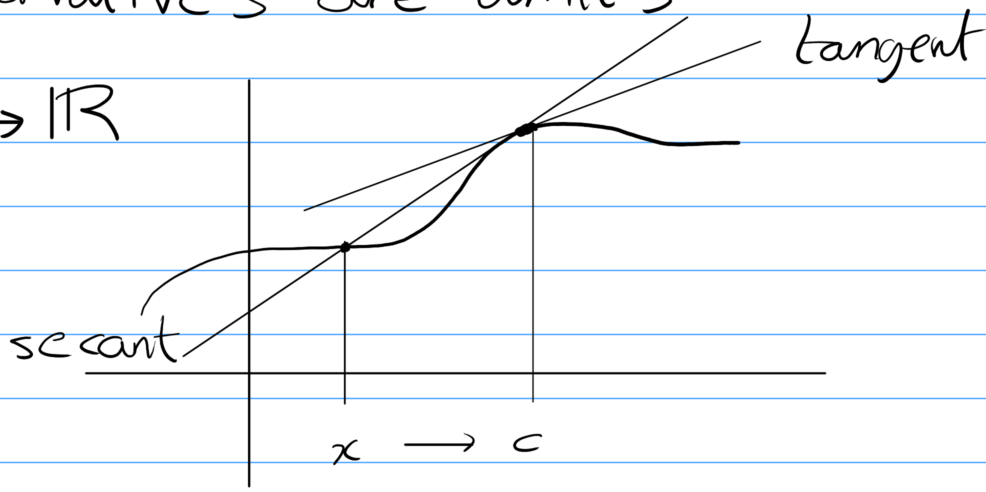
Derivatives

Idea derivatives are limits

$f : D \longrightarrow \mathbb{R}$

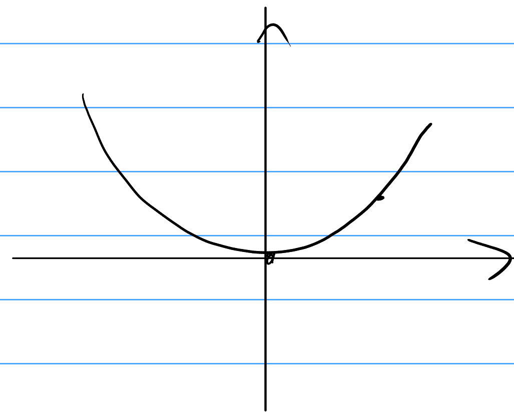
$x \in D \ni c$



secant line $\longrightarrow$ tangent

derivative : difference quotient

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



$f(x) = x^2$

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c}$$

if $x \neq c$ $\quad \dfrac{x^2 - c^2}{x - c} = \dfrac{(x + c)(x - c)}{x - c}$

$= x + c$

$f'(c) = 2c$

$f(x) = x^2 \Rightarrow f'(x) = 2x$

Other notation $y = f(x)$

Dy $\qquad \frac{dy}{dx} = f'(x)$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \left(\frac{\Delta y}{\Delta x}\right)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

"$h = x - c$"

<u>Same example as above</u>

$f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2 = (2x+h)h$$

$$= \lim_{h \to 0} (2x+h) = \underline{2x}$$

Examples using Difference Quotient def^n.

✓ i) $f(x) = x^2 \qquad f'(x) = 2x$

✓ ii) $f(x) = x^3$

✓ iii) $f(x) = x^n \qquad n = 2, 3, 4, \ldots$

iv) $f(x) = 1/x$

v) $f(x) = \sqrt{x} = x^{1/2}$

i) ✓

if $x \neq c$

ii) $$\frac{x^3 - c^3}{x - c} = \frac{\cancel{(x-c)}(x^2 + xc + c^2)}{\cancel{x-c}}$$

so $$\lim_{x \to c} \frac{x^3 - c^3}{x - c} = \lim_{x \to c} (x^2 + xc + c^2) = 3c^2$$

$$f'(c) = 3c^2 \quad f'(x) = 3x^2$$

$$(x-c)(x^2 + xc + c^2) = \begin{array}{r} x^3 + x^2c + xc^2 \\ - x^2c - xc^2 - c^3 \\ \hline x^3 \qquad\qquad - c^3 \end{array}$$

$$(x-c)(x^2 + c^2) = \begin{array}{r} x^3 - c^3 \\ + xc^2 - cx^2 \\ \hline \end{array}$$

$$\neq x^3 - c^3$$

$$y = x^3, \quad \frac{dy}{dx} = 3x^2$$

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iii) $$(x-c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \ldots + c^{n-2}x + c^{n-1})$$

$$= x^n - c^n$$

All other terms appear twice with opposite signs, & cancel

$$\lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} \left(\sum_{i=0}^{n-1} c^i x^{n-1-i} \right) = nc^{n-1}$$

$y = x^n$ $\qquad \boxed{\dfrac{dy}{dx} = n x^{n-1}}$

Alternative formula $\qquad$ Binomial Theorem

If $h \neq 0$, $\dfrac{(x+h)^n - x^n}{h} = \dfrac{\cancel{x^n} + nhx^{n-1} + \binom{n}{2}h^2 x^{n-2} + \dots + \cancel{-x^n}}{h}$

$$= nx^{n-1} + \binom{n}{2} h x^{n-2} + \dots \dots$$

higher powers of h

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = n x^{n-1}$$

iv) $f(x) = \dfrac{1}{x}$, $x, c > 0$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c}$$

$$= \lim_{x \to c} \frac{\frac{c - x}{xc}}{x - c} = \lim_{x \to c} \frac{-1}{xc}$$

$$= -1/c^2 = -c^{-2}$$

$y = x^{-1}$, $\dfrac{dy}{dx} = -x^{-2}$

v) $f(x) = \sqrt{x} = x^{1/2}$, $x, c \geq 0$

If $x \neq c$

$$\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\cancel{\sqrt{x}-\sqrt{c}}}{\cancel{(\sqrt{x}-\sqrt{c})}(\sqrt{x}+\sqrt{c})}$$

$$f'(c) = \lim_{x \to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2}\frac{1}{\sqrt{c}} = \frac{1}{2}c^{-1/2}$$

$$y = x^{1/2} \quad , \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

-1 $\quad f(x) = 7 \qquad f'(x) = 0$

0 $\quad f(x) = x \left(\frac{x-c}{x-c}\right) \qquad f'(x) = 1$

Basic Laws for derivatives

f is differentiable at $x = c$

if $\displaystyle\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exist

Theorem Suppose $f(x)$, $g(x)$ are differentiable at $x = c$

Then

i) $(f+g)'(c) = f'(c) + g'(c)$

ii) $(k \cdot f)'(c) = k \cdot f'(c)$

iii) $(f \cdot g)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$

(induction $\Rightarrow \frac{d}{dx}(x^n) = n x^{n-1}$)

Proof of (iii) Use limit laws

$f(x) \cdot g(x) - f(c) \cdot g(c)$

$= f(x) \cdot g(x) \underline{- f(c) \cdot g(x) + f(c) \cdot g(x)} - f(c) \cdot g(c)$

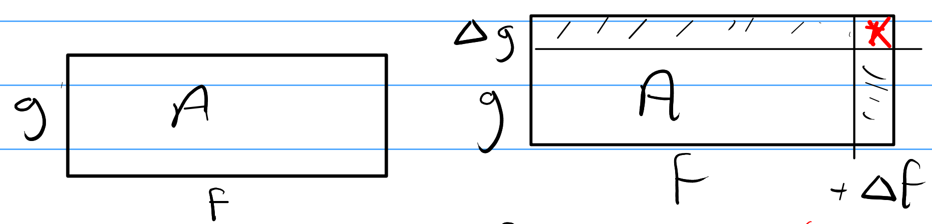
$$\frac{f(x)g(x) - f(c)g(x)}{x - c}$$

$(f \cdot g)' \neq f'g'$

$$= \frac{f(x)g(x) - f(c)g(x)}{x - c} + \frac{f(c)g(x) - f(c)g(c)}{x - c}$$

As $x \to c$ $\to$ $\underline{f'(c) \cdot g(c) + f(c)\,g'(c)}$

$f \times g$



$\Delta A \simeq \Delta f \times g + f \times \Delta g \ (+\Delta f \Delta g)$