UNIVERSITY OF LEICESTER

MA7404
All Candidates

Semester 1 Examinations 2022

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	Computing and Mathematical Sciences
Module Code	MA7404
Module Title	Markov processes
Exam Duration	Two hours + 45 minutes upload time
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	Answer all questions
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	Formulae and tables for actuarial examinations
Are students permitted to bring their own Books/Statutes/Notes?	Yes
Additional Stationery	Yes

1. An insurance company receives two types of claims, which we call type A and type B claims. The type A and type B claims arrive according to independent Poisson processes with parameters μ_A and μ_B claims per week, respectively. Parameters μ_A and μ_B are unknown and should be estimated from the past data. The claim sizes are independent. The sizes of type A claims follow gamma distribution with parameters $\alpha_A = 3$, $\lambda_A = 0.01$, while the sizes of type B claims follow Pareto distribution with parameters $\alpha_B = 4$, $\lambda_B = 600$.

   (i) [5 marks] The numbers of type A claims during the last 7 weeks were

   $$10, 12, 15, 6, 8, 13 \text{ and } 6.$$

   Use the method of maximum likelihood to estimate parameter μ_A.

   (ii) [5 marks] The numbers of type B claims during the last 7 weeks were

   $$2, 3, 1, 2, 0, 1 \text{ and } 2.$$

   Use the method of percentile at level $\alpha = e^{-2} \approx 0.135$ to estimate parameter μ_B.

   For the questions below, use the values of μ_A and μ_B found in parts (i) and (ii), or use $\mu_A = 5$ and $\mu_B = 1$ if you cannot solve (i) or (ii).

   (iii) [5 marks] Estimate the mean and standard deviation of the total size S_3 of all claims to be received in the next 3 weeks.

   (iv) [5 marks] Estimate the probability that, during the next 2 weeks, the company will receive exactly 7 claims in total.

   (v) [5 marks] Estimate the probability that the next three claims the company will receive will happen to be the type A claims.

   Total: 25 marks

2. A company models their total expenses in 2022 as a random variable X that follows Weibull distribution with parameters $c = 0.002$ and $\gamma = 1/2$. They also model their total expenses in 2023 as Weibull distribution with parameters $c = 0.01$ and $\gamma = 1/3$. They consider three possible copulas to model the dependence of X and Y:

   (1) The independence copula $C(u,v) = uv$;

   (2) The co-monotonic copula $C(u,v) = \min(u,v)$, and

   (3) The Clayton copula with parameter $\alpha = 1$.

   (i) [5 marks] Using the limiting density ratios test, determine which random variable (X or Y) has distribution with a heavier tail.

   (ii) For each of the three models for copula listed above:

   (a) **[10 marks]**. Compute the survival copula $\bar{C}(u,v)$.

   (b) **[10 marks]**. Estimate the probability that both X and Y exceed a million pounds.

   Total: 25 marks

3. A credit rating agency assigns two possible ratings for a bond B: I - investment grade, and J - junk grade. The rating is reviewed once a year. From the past data, the agency noticed the following

   - Among the bonds with investment grades, 70% stay at the investment grade next year, 25% move to the junk grade, and only 5% default within a year.
   - Among the bonds that just moved from the investment grade to junk grade, 30% return to the investment grade next year, 60% stay at the junk grade, and 10% default.
   - Among the bonds that are in the junk grade for at least two consecutive years, 10% move to the investment grade next year, 70% stays at the junk grade, and 20% default.

   (i) [5 marks] Based on these data, can the credit rating dynamic be modelled as a Markov chain with states I - investment grade, J - junk grade, and D - default? If not, suggest an alternative state space.

   (ii) [5 marks] Model the process as a Markov chain with state space described in part (i). Write down the one-step transition matrix. Draw the transition graph.

   (iii) [5 marks] Estimate the probability that the bond currently at investment grade will not default within the next 2 years but then will default during the third year.

   (iv) [5 marks] Find the stationary distribution of this Markov chain.

   (v) (a) **[3 marks]** Is this Markov chain finite, irreducible, and aperiodic?

   (b) **[2 marks]** Does it converge to its stationary distribution?

   Total: 25 marks

4. Assume that the marriage rate for an unmarried person aged t is $\alpha(t)$, the divorce rate for a married person aged t is $d(t)$, and the mortality rate for a person aged t is $\mu(t)$, independently of the martial status.

   (i) [5 marks] Build the marriage model that consists on three states: N - not married, M - married, and D - dead. Write down the generator matrix and the transition graph. You may assume that every married couple has the same age.

   (ii) [10 marks] Assume that $\mu(t) = at + b$ for some constants a and b. Assume that we have data points $\mu(20) = 0.002$, $\mu(40) = 0.006$, $\mu(60) = 0.02$, and $\mu(80) = 0.08$. Use simple linear regression to find constants a and b such that $\mu(t)$ best approximates the data.

   (iii) [10 marks] Using $\mu(t)$ from part (ii) (or use $\mu(t) = 0.001t - 0.03$ for $t \geq 30$ if you cannot solve (ii)) and $d(t) = 0.02$, find the probability that a person aged 40, currently married, will stay in the same marriage more than 10 years but less than 20 years.

   Total: 25 marks