MA1014
All candidates

UNIVERSITY OF LEICESTER

Semester 2 Examinations 2019

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
Department	Mathematics
Module Code	MA1014
Module Title	Calculus and Analysis
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	5
Instructions to Candidates	Answer all questions. All marks gained will be counted. All questions carry equal weight.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

1. (a) **[6 marks]** Prove the *Pinching Theorem*:

   Suppose $f, g, h : (c-p, c+p) \to \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all x.

   If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ then $\lim_{x \to c} g(x) = L$ also.

   (b) **[7 marks]** Use the Pinching Theorem to:

   i. prove that $h \sin(\frac{1}{h}) \to 0$ as $h \to 0$,

   ii. find $\lim_{x \to 3} \; 2x + (x-3)^2 \sin\left(\dfrac{1}{x-3}\right)$.

   (c) **[2 marks]** Hence state the value of c that makes the following function continuous:

$$f(x) = \begin{cases} 2x + (x-3)^2 \sin(\frac{1}{x-3}) & (x \neq 3) \\ c & (x = 3) \end{cases}$$

   (d) **[5 marks]** Find $f'(3)$, using the difference-quotient definition of the derivative.

2. Consider the integral $I_n = \displaystyle\int_0^x t^n e^{-kt} dt$ where n is a non-negative integer and $k > 0$.

   (a) **[6 marks]** Find $I_0 = \displaystyle\int_0^x e^{-kt} dt$ and deduce the value of the improper integral $\displaystyle\int_0^\infty e^{-kt} dt$.

   (b) **[8 marks]** By integrating I_n by parts, prove the following relation holds (if $n \geq 1$):

$$kI_n = -x^n e^{-kx} + nI_{n-1}.$$

   Deduce a similar relation between the corresponding improper integrals

$$\int_0^\infty t^n e^{-kt} dt \quad \text{and} \quad \int_0^\infty t^{n-1} e^{-kt} dt.$$

   [Hint: since $k > 0$ you can assume that $x^n e^{-kx} \to 0$ as $x \to \infty$.]

   (c) **[6 marks]** Prove by induction that, for all $n \geq 0$,

$$\int_0^\infty t^n e^{-kt} dt = \frac{n!}{k^{n+1}}.$$

3. (a) **[8 marks]** Prove *Abel's theorem*:

If an infinite series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for some $x = x_0$ then it is absolutely convergent for all x satisfying $|x| < |x_0|$.

(b) **[12 marks]** Consider the geometric series $S(x) = \sum_{n=0}^{\infty} (-x^2)^n$.

i. State the radius of convergence r of $S(x)$, and the value of $S(x)$ if $-r < x < r$.
ii. By integration, find a power series representation $T(x)$ of $\tan^{-1}(x)$.
iii. Investigate the convergence behaviour of $T(x)$ when $x = \pm r$.
iv. Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}.$$

4. Consider the function of two variables $f(x,y) = \begin{cases} \dfrac{2x^3 - y^3}{2x^2 + 4y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

(a) **[5 marks]** Prove f is continuous at the origin.
(b) **[5 marks]** Calculate the partial derivative $f_x(x,y)$ for $(x,y) \neq (0,0)$.
(c) **[5 marks]** Show that the function $f_x(x,y)$ is not continuous at $(0,0)$.
(d) **[5 marks]** Show the directional derivative $f_{\underline{\hat{u}}}(0,0)$ at the origin, in the direction of the unit vector $\hat{u} = (p,q)$, equals $f(p,q)$.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = e^{-\frac{x^3}{3} + x - y^2}$.

(a) **[5 marks]** Find the derivative ∇f and the critical points of $f(x,y)$.
(b) **[5 marks]** Classify these critical points. You may assume that the Hessian matrix is

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} ((x^2-1)^2 - 2x)\, f(x,y) & 2y(x^2-1)\, f(x,y) \\ 2y(x^2-1)\, f(x,y) & (4y^2-2)\, f(x,y) \end{pmatrix}.$$

(c) **[8 marks]** Find the extreme values of the functions $g, h, k : [0,7] \to \mathbb{R}$ given by

$$g(x) = f(x,0) = e^{-\frac{x^3}{3} + x}, \qquad h(y) = f(0,y) = e^{-y^2}, \qquad k(x) = f(x, 7-x) = e^{-\frac{x^3}{3} - x^2 + 15x - 49}.$$

(d) **[2 marks]** Hence write down the extreme values of $z = f(x,y)$ when restricted to the closed domain given by the triangle ABC with $A = (0,0)$, $B = (7,0)$, $C = (0,7)$.