MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part II

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Ito's lemma: core concept of stochastic calculus

Classical Ito's Lemma:

Let $f(t, x)$ be a twice partially differentiable function and $X_t = f(t, B_t)$ be a stochastic process. Then the stochastic differential $dX_t = df(t, B_t)$ is defined by the Ito formula

$$df(t, B_t) = f'_t dt + f'_{B_t} dB_t + \frac{1}{2} f''_{B_t B_t} dt$$

Ito rules: $(dB_t)^2 = dt$, $(dt)^2 = 0$, $dB_t dt = 0$

Remark

- Using the Ito's Lemma, we can write the stochastic process $X_t = f(t, B_t)$ in integral form

$$X_t = f(t, B_t) = f(0, B_0) + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where $A_u = f'_u + \frac{1}{2} f''_{B_u B_u}$ and $Y_u = f'_{B_u}$.

- And hence,

$$f(t, B_t) \sim N\left(f(0, B_0) + \int_0^t \mathbb{E}\left[f'_u + \frac{1}{2} f''_{B_u B_u} \right] du, \int_0^t \mathbb{E}\left[\left(f'_{B_u} \right)^2 \right] du \right)$$

Examples

- Find the stochastic differential $df(t, B_t)$ for the following stochastic processes
  - $f(t, B_t) = B_t^2$,
  - $f(t, B_t) = 7e^{tB_t}$.
- Find the integral forms of the stochastic processes B_t^2 and $7e^{tB_t}$.
- Compute $\mathbb{E}[B_t^2]$ using the Ito calculus.

Ito martingale

- The stochastic process $X_t = f(t, B_t)$ is a Ito martingale if

$$df(t, B_t) = Y_t dB_t$$

which has no term with dt (zero drift).

Examples

- Show that B_t is an Ito martingale.
- Show that $B_t^2 - t$ is an Ito martingale.
- Is B_t^3 an Ito martingale?

Martingale versus Ito martingale

- For any stochastic process X_t, if it is a martingale, then zero drift (Ito martingale) is equivalent to the martingale definition $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leqslant s < t$.

Conditional expectation via Ito calculus

- Conditional Ito integral:

$$\mathbb{E}\left[\int_s^t Y_u dB_u \middle| \mathcal{F}_s\right] = 0$$

- Conditional Fubini theorem:

$$\mathbb{E}\left[\int_s^t A_u du \middle| \mathcal{F}_s\right] = \int_s^t \mathbb{E}[A_u | \mathcal{F}_s] du$$

where $0 \leqslant s < t$.

Examples

- Given $t > s \geqslant 0$, compute the following conditional expectations via the Ito calculus,
  - $\mathbb{E}[t^2 B_t|\mathcal{F}_s]$.
  - $\mathbb{E}[B_t^3|\mathcal{F}_s]$.

Ito's Lemma, cont.

General Ito's Lemma:

Let $f(t,x)$ be a twice partially differentiable function and $f(t,X_t)$ be a stochastic process with $dX_t = A_t dt + Y_t dB_t$. Then, the general Ito's lemma states that

$$df(t,X_t) = f_t' dt + f_{X_t}' dX_t + \frac{1}{2} f_{X_t X_t}'' (dX_t)^2$$

where Ito rules also applied,

$$(dX_t)^2 = Y_t^2 dt \quad \text{since} \quad (dB_t)^2 = dt,\ (dt)^2 = 0,\ dB_t dt = 0$$

- For the convenience of calculation, one can also express the general Ito's lemma as follows,

$$df(t, X_t) = \left(f'_t + A_t f'_{X_t} + \frac{1}{2} Y_t^2 f''_{X_t X_t} \right) dt + Y_t f'_{X_t} dB_t$$

where A_t and Y_t are given in the SDE of dX_t.

Example

- Let X_t be defined by $dX_t = 2B_t dB_t$, find $d\left[t^3 X_t^5\right]$.

General Ito Isometry

- Given $\int_0^t Y_u dB_u$ and $\int_0^t Q_u dB_u$ are two stochastic integrals,

$$\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)\left(\int_0^t Q_u dB_u\right)\right] = \mathbb{E}\left[\int_0^t (Y_u Q_u) du\right]$$

Examples

- Let X_t be defined by $dX_t = 2B_t dB_t$ and $X_0 = 0$, find $\mathbb{E}\left[B_t X_t\right]$ and $\mathbb{E}\left[X_t^2\right]$.
- Find $\mathbb{E}[(B_t^3 - 3tB_t + 1)^2]$ by applying the Ito isometry.

Ito's Lemma versus GBM

Let $\{B_t,\ t \geqslant 0\}$ be a standard Brownian motion. Consider a continuous-time market with one risk-free bond offering a fixed interest rate of ρ and one risky asset. The asset price is modelled by a stochastic process $\{S_t,\ t \geqslant 0\}$, which is defined as a solution to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Ito's Lemma versus GBM, cont.

I) $\{S_t,\ t \geqslant 0\}$ is a geometric Brownian motion with mean parameter $a = \mu - \frac{\sigma^2}{2}$ and volatility parameter σ.

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

II) Moreover, $e^{-\rho t} S_t$ is a martingale if and only if $\mu = \rho$.

III) The discounted financial derivative $e^{-\rho t} g(t, S_t)$ is a Martingale iff the Black-Scholes equation holds:

$$g'_t + \mu S_t g'_{S_t} + \frac{1}{2}\sigma^2 {S_t}^2 g''_{S_t S_t} = \rho g$$

GBM, cont.

- S_t follows a log-normal distribution, such that

$$\log\left(\frac{S_t}{S_0}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

- GBM has the following expected value and variance:
  - $\mathbb{E}[S_t] = S_0 e^{\mu t}$
  - $Var[S_t] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Conditional expectation of GBM

Let $S_t = S_0 e^{at+\sigma B_t}$ be a GBM, then we have

$$\begin{aligned}\mathbb{E}[f(S_T)|\mathcal{F}_t] &= \mathbb{E}\left[f\left(S_t \cdot \frac{S_T}{S_t}\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\Big|\mathcal{F}_t\right]\end{aligned}$$

where $\{\mathcal{F}_t, 0 \leq t < T\}$ is a filtration of S_t.

Specifically, when $t = 0$,

$$\mathbb{E}[f(S_T)|\mathcal{F}_0] = \mathbb{E}[f(S_T)]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(S_t)|\mathcal{F}_t] = f(S_t)$,
- $\mathbb{E}[S_T|\mathcal{F}_t] = S_t e^{\mu(T-t)}$
- $\mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle| \mathcal{F}_t\right] = e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}$, where k is a constant.
- $S_t \perp e^{a(T-t)+\sigma(B_T-B_t)}$ since $B_t \perp B_T - B_t$.

for all $T > t \geqslant 0$.

Examples

- Find $\mathbb{E}[6S_T^3|\mathcal{F}_t]$ with $S_t = S_0 e^{2t+B_t}$.
- Find $\mathbb{E}[\log(S_T)|\mathcal{F}_t]$ with $S_t = S_0 e^{at+\sigma B_t}$.

Multivariate Ito's Lemma

Multivariate Ito's Lemma:

Let $f\left(x_1, \cdots, x_k\right)$ be a twice partially differentiable function and $f\left(X_t^{(1)}, \cdots, X_t^{(k)}\right)$ be a stochastic process with $dX_t^{(i)} = A_t^{(i)} dt + Y_t^{(i)} dB_t,\ i = 1, \cdots, k$. Then, the multivariate Ito's lemma states that

$$df\left(X_t^{(1)}, \cdots, X_t^{(k)}\right) = \sum_{i=1}^{k} f'_{X_t^{(i)}} dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^{k} f''_{X_t^{(i)} X_t^{(i)}} \left(dX_t^{(i)}\right)^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f''_{X_t^{(i)} X_t^{(j)}} dX_t^{(i)} dX_t^{(j)}$$

where $\left(dX_t^{(i)}\right)^2 = \left(Y_t^{(i)}\right)^2 dt$ and $dX_t^{(i)} dX_t^{(j)} = Y_t^{(i)} Y_t^{(j)} dt$.

Bivariate Ito's Lemma

In particular, when $k = 2$, $dX_t = A_t dt + Y_t dB_t$ and $dZ_t = G_t dt + Q_t dB_t$. Then, the bivariate Ito formula is

$$
\begin{aligned}
df(X_t, Z_t) = f'_{X_t} dX_t + f'_{Z_t} dZ_t + \frac{1}{2} f''_{X_t X_t} (dX_t)^2 \\
+ \frac{1}{2} f''_{Z_t Z_t} (dZ_t)^2 + f''_{X_t Z_t} dX_t dZ_t
\end{aligned}
$$

And in the integral form,

$$
\begin{aligned}
f(X_t, Z_t) = f(X_0, Z_0) + \int_0^t f'_{X_s} dX_s + \int_0^t f'_{Z_s} dZ_s \\
+ \frac{1}{2} \int_0^t f''_{X_s X_s} (dX_s)^2 + \frac{1}{2} \int_0^t f''_{Z_s Z_s} (dZ_s)^2 + \int_0^t f''_{X_s Z_s} dX_s dZ_s
\end{aligned}
$$

Examples

- Find $d[M_t S_t]$ through the bivariate Ito formula.
- If $dM_t = B_t^4 dB_t$ and $dS_t = B_t^2 dB_t$, $M_0 = S_0 = 0$, find $\mathbb{E}[M_t S_t]$.
- If $dM_t = (1 + t^2 B_t) dB_t$ and $M_0 = 0$, let $Y_t = tB_t M_t$. Is Y_t a martingale?