MA3071 – DLI
Financial Mathematics – Section 1
Introduction to options

Dr. Ting Wei

University of Leicester

Options - definitions

- An **option** gives the buyer of the option (or the person granted the option) the right, but not the obligation, to buy or sell a specified asset (the underlying asset) on a predetermined future date (the expiry/maturity date) for a predetermined price (the strike price, usually denoted by K).
  - A **call option** grants the right, but not the obligation, to **purchase** an underlying asset on a specified maturity date in the future for a predetermined strike price.
  - A **put option** grants the right, but not the obligation, to **sell** an underlying asset on a specified maturity date in the future for a predetermined strike price.
- A **European option** can only be exercised at maturity, while an **American option** can be exercised at any time before its maturity.

Example

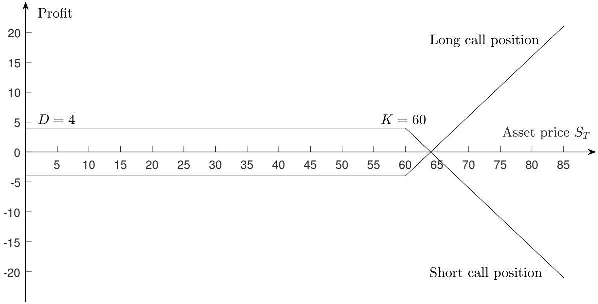
Let S_t denote the price of an underlying asset at time t, T denote the maturity, K be the strike price, and D be the option premium at time T.

- European call option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?
- European put option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?

Long and short positions

- A **long position** on an option is when the option has been purchased, while a **short position** is when the option is sold.
- Therefore, a long position on a European option gives the holder (i.e. the buyer, or owner of the option) the right but not the obligation to exercise the option. The holder of the short position (i.e. the seller, or writer of the option) will be obliged to sell, or buy, the underlying asset for the agreed price, if the option is exercised.

European call option profit at maturity

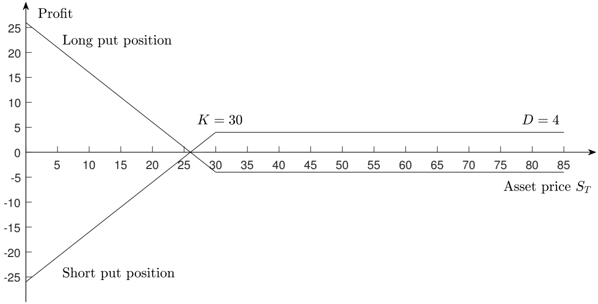


The profit (denoted by P) at maturity T for the holder of a call option is calculated using the formula:

$$P = (S_T - K)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(S_T - K)_+$ is called **call option claim/payoff**.

European put option profit at maturity



The profit P at maturity T for the holder of a put option is calculated using the formula:

$$P = (K - S_T)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(K - S_T)_+$ is called **put option claim/payoff**.

Some discussions

- The loss for a short position and the profit for a long position in a put option are both limited to the difference between the strike price and the premium (i.e. $K - D$).
- In contrast, with a call option, the loss for a short position and the profit for a long position can be positively infinite as the price of the underlying asset theoretically has no upper limit. Therefore, call options are more risky for short positions but can be more profitable for long positions.
- Asset price S_T is a random variable, thus the option profit P is also a random variable.
- The profit for the writer of a European option is always the opposite of the profit for the holder. Thus, only when $\mathbb{E}[P] = 0$ would it be considered fair for both the writer and the holder.

Question

- If you want to buy a European option that expires at T, how much do you need to pay at time 0?

Basic concept of option prices

- Under the assumption that $\mathbb{E}[P] = 0$, the price of a European option at time t (denoted as V_t) is the present value of the expected payoff at time T, that is
  - Call option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$

  where ρ represents the risk-free interest rate for the period $[0, 1]$, and $(1+\rho)^{-(T-t)}$ is the discount factor for the period $[t, T]$.

- In the case of continuous compounding, discount factor is modified as $e^{-\rho(T-t)}$, and
  - Call option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$

- The option premium at T is calculated by $D = V_t(1+\rho)^{T-t}$ or $D = V_t e^{\rho(T-t)}$.

Option pricing models

- Option pricing is one of main tasks of this module.
- We will talk about the following option pricing models in this course:
  - Binomial tree models.
  - Black–Scholes models.
  - Monte-Carlo methods.
- We focus more on the option pricing of European style options.

The concept of arbitrage

- We will encounter the concept of arbitrage when discussing those pricing models. An arbitrage opportunity arises when you can execute a series of transactions that result in a profit without incurring any risk. More formally, arbitrage occurs when:
  - The initial net investment is zero.
  - The probability of making a loss is zero.
  - The probability of making a profit is strictly greater than zero.
- Clearly, if such an opportunity was available then investors would trade as much as they could to take advantage of this "free lunch".

Example

In a financial market, bank A offers a one-year fixed-term bond with an interest rate of ρ_1, and bank B offers another one-year fixed-term bond with an interest rate of ρ_2. Assuming short sales are allowed in this market, are there any arbitrage opportunities when $\rho_1 \neq \rho_2$?

No-arbitrage principle in option pricing

- Since the value of options depends on the value of some other financial variables (price of underlying assets S_t, strike price K, maturity T, etc.), their price will be a function of those variables. If the option price is not consistent with the underlying, then arbitrage would be possible.

- This would result in a loss to one of the parties to the transaction as others traded to take advantage of the arbitrage opportunity. Everyone is therefore keen to avoid this outcome.

- Even if it were to happen, market activity would move prices such that they would fall back in line with the no arbitrage equivalent.

- It ensures that the calculated prices for the relevant options are consistent with the related market variables.

Factors affecting option prices

- The price of the underlying asset (S_t, usually referred to as the spot value) compared to the strike price (K).
- The volatility of the underlying (usually meaning the annualised standard deviation, σ), as a measure of how uncertain the value in future is.
- Interest rates (ρ), since we are estimating present values.
- The term to expiry (T).

Relationship between these key factors and option prices

- A longer term to maturity means a higher option value. The increased uncertainty regarding values further into the future implies a greater potential benefit for the option holder.
- A higher level of volatility also leads to a higher option value (this is the most significant variable because it has the greatest individual impact on the option value). This is due to the greater uncertainty.

Relationship between these key factors and option prices

- Higher interest rates lead to higher call option prices and lower put option prices. As interest rates rise in the market, the expected return demanded by investors in stocks tends to increase. Conversely, the present value of future cash flows generated by option contracts decreases. The overall impact of these two factors is an increase in call option value and a decrease in put option value.
- For fixed levels of the above variables, a higher spot price of the underlying asset increases the option value for call options due to their higher intrinsic value, while it decreases the option value for put options.