# MA1014
# CALCULUS AND ANALYSIS
# TUTORIAL 19

Dr. Andrew Tonks: apt12@le.ac.uk

Ben Smith: bjs30@le.ac.uk

# ANNOUNCEMENTS

- Chapter 1 revision

# ORDER & INEQUALITIES

The real numbers $\mathbb{R}$ are an ordered field, i.e. $\exists$ a relation $<$, that $\forall a, b, c \in \mathbb{R}$:

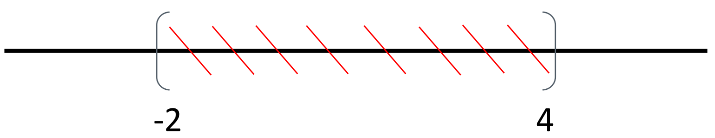
- Total Order: Either $a < b, b < a$ or $b = a$
- Transitivity: If $a < b$ and $b < c \ \Rightarrow a < c$
- Compatibility: If $a < b \ \Rightarrow a + c < b + c$. If $a < b$ and $c > 0$ then $ac < bc$

Example:

If $x, y > 0$, then Compatibility $\Rightarrow x + y > y$ and $xy > 0$ (i.e. $a = 0, b = x, c = y$)

Also we can denote intervals with these inequalities.

Example: $-2 < x < 4$



# EXERCISE:
# SOLVE THE FOLLOWING INEQUALITIES

a) $3x + 5 < \frac{4-x}{2}$

c) $\frac{x^2-4x+4}{x^2-2x-3} \leq 0$

b) $x(x^2 - 3x + 2) \leq 0$

d) $|3x - 2| \geq 4$

# BOUNDS

Let $S \subset F$, then

- An **Upper Bound** of $S$ is $M \in F : \forall s \in S,\ s \leq M$.
- A **Lower Bound** of $S$ is $m \in F : \forall s \in S,\ m \leq s$.
- The **Supremum** of $S$ or $\sup(S)$ is the smallest/least upper bound
- The **Infimum** of $S$ or $\mathrm{Inf}(S)$ is the largest/greatest lower bound

# TRIANGLE INEQUALITIES

Triangle Inequality: $|x+y| \le |x| + |y|$

Reverse Tringle Inequality: $\big||x| - |y|\big| \le |x-y|$

Exercise: Prove the following

a) $|a-b| \le |a| + |b|$

b) $\big||a| - |b|\big| \le |a+b|$

# FUNCTIONS

A function takes an input, say $x$, does something to it ($f: x \to \mathbb{R}$), and gives an output $f(x)$.

More precisely, a function maps a **Domain** ($D = \mathrm{dom}(f) \subseteq \mathbb{R}$) to a **Range** ($\mathrm{range}(f) = \{f(x) : x \in D\}$)

**Example:** $f(x) = \sin(x)$, $\mathrm{dom}(f) = (-\infty, \infty)$ or $\mathbb{R}$ and $\mathrm{range}(f) = [-1,1]$

Further, it is possible to classify some functions as either **Odd** or **Even**:

- If $f(-x) = -f(x)$, then $f$ is said to be **Odd**
- If $f(-x) = f(x)$, then $f$ is said to be **Even**

# COMPOSITIONS OF FUNCTIONS

If $f: D_f \to \mathbb{R}, g: D_g \to \mathbb{R}:\ \text{range}(g) \subseteq D_f$ then the Composition, $f \circ g$, on $D_g$ can be defined as

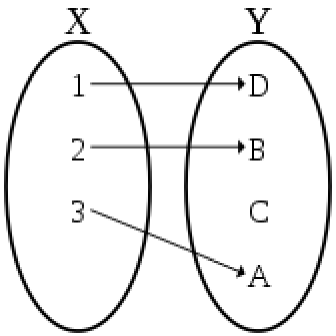
$$\boxed{(f \circ g)(x) = f\big(g(x)\big), \qquad x \in D_g}$$

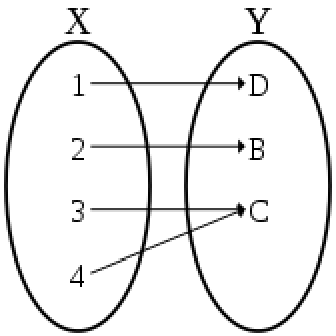
Note: Compositions are not generally commutative (i.e. $f \circ g \neq g \circ f$)

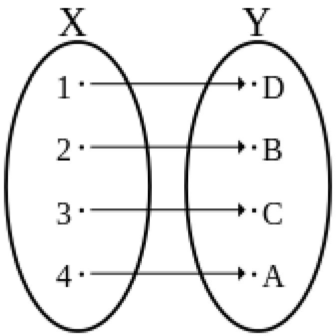
# ONE-TO-ONE, ONTO & BIJECTIVITY

- A function, $f$, is **One-to-One** (Injective) if $\forall x_1, x_2 \in D_f, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- A function, $f$, is **Onto** (Surjective) if $\forall y \in \text{range}(f), \exists x \in D_f : f(x) = y$
- A function is **Bijective** if it is **both** One-to-One and Onto

**Example:** Let $f: X \to Y$, classify the diagrams below.



# INVERSE FUNCTIONS

If $f(x)$ is a one-to-one function, there exists a unique inverse function which we denote by

$$f^{-1} : \operatorname{ran}(f) \rightarrow \operatorname{dom}(f)$$

such that

$$f^{-1} \circ f(x) = x \ \forall x \in \operatorname{dom}(f)$$

and

$$f \circ f^{-1}(y) = y \ \forall y \in \operatorname{ran}(f)$$

EXERCISE:
SHOW THE FOLLOWING FUNCTIONS ARE ONE-TO-ONE. FIND THE INVERSE FUNCTION AND GIVE BOTH THE DOMAIN AND RANGE OF BOTH FUNCTIONS.

a) $f(x) = \frac{x-1}{x-2}$

b) $g(x) = \frac{x}{2x-4} + \frac{1}{2}$

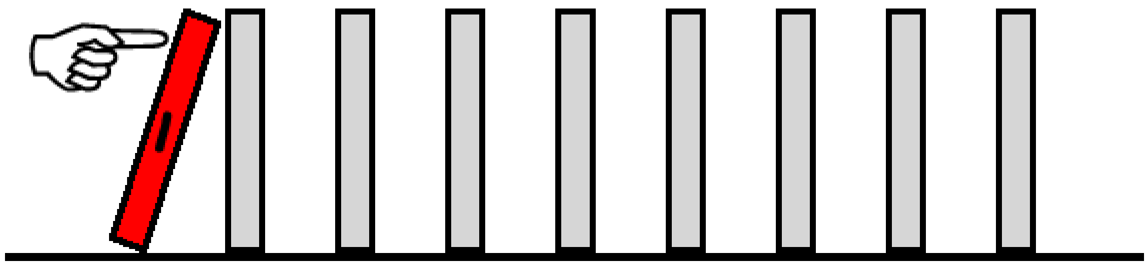
c) $p(x) = x^2 + 1, x > 0$

d) $q(x) = \frac{3x-5}{x-2}$

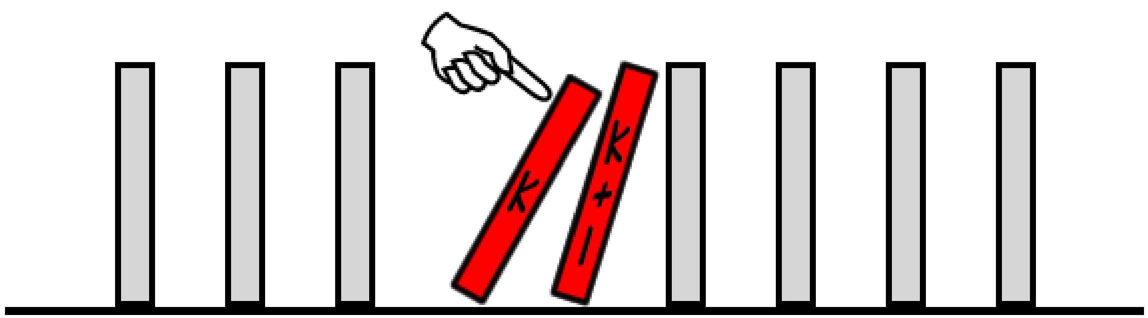
# PROOF BY INDUCTION

Powerful way to prove iterative statements $P(n), n \in \mathbb{N}$.

1. First step: Prove Base Case (smallest possible $n$)




2. Assumptive step: Assume $n = k \in \mathbb{N}$ is true
3. Inductive step: Show $n = k$ is true $\Rightarrow n = k + 1$ is true



# EXERCISE:

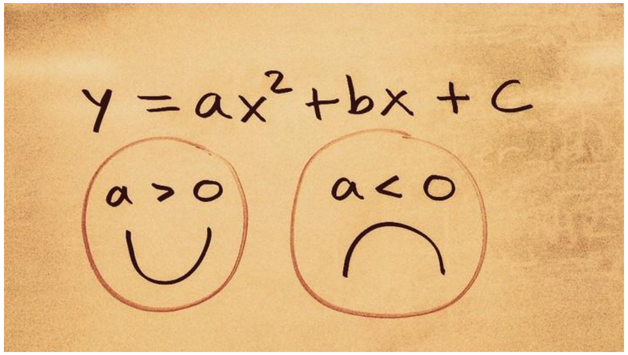
Prove by Induction:

a) $1 + 2n \leq 3^n$

b) $n! \geq 2^{n-1}$

c) $\sum_{l=1}^{n} 2l + 1 = n(n+2)$

UNIVERSITY OF LEICESTER




$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

## ANY QUESTIONS?

$$m\frac{d^2x}{dt^2} = -kx$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$