Semester 1 Tutorial Exercises

These are the problems that we will work through during the weekly tutorials. I will update them as we go along with new questions and you are encouraged to attempt them before each tutorial. They are **not** in the workbook, so they will not be used in the coursework sets. Hence, full solutions will be given! However, compared to the workbook, these will have a much smaller amount of exercises.

1 Pre-calculus

1.1. Determine the smallest[1] set each number belongs to:

(a) 5

(b) 0.334

(c) $\frac{1}{3}$

(d) $\sqrt{2}$

(e) -3

(f) π

(g) -2.75

(h) 0

(i) e^{-1}

(j) $\sqrt{-1}$

Solution:

(a) $\mathbb{N}$. See note for reasoning.

(b) $\mathbb{Q}$. $0.334 = 334/1000 \in \mathbb{Q}$ and fractions are not integers.

(c) $\mathbb{Q}$. See (b).

(d) $\mathbb{R}$. $\sqrt{2} \notin \mathbb{Q}$ i.e. irrational.

(e) $\mathbb{Z}$. $-3 \notin \mathbb{N}$.

(f) $\mathbb{R}$. See (d).

(g) $\mathbb{Q}$. $-2.75 = -11/4 \in \mathbb{Q}$.

(h) $\mathbb{N}$ or $\mathbb{N}_0$.

(i) $\mathbb{R}$. $e^{-1} = 1/e$, and $e \in \mathbb{R} \implies e^{-1} \in \mathbb{R}$.

(j) $\mathbb{C}$. $\sqrt{-1} = i = 0 + 1i$, i.e. $a = 0$ and $b = 1$.

1.2. Prove the Proposition, P : If N^2 is even $\implies$ N is even.

Solution:
The negation of P would be $\neg P$: If N^2 is even $\implies$ N is odd. If N is odd, then $\exists m \in \mathbb{Z} : N = 2m + 1$,

$$\implies N^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$$

Let $q = 2m^2 + 2m \in \mathbb{Z}$, then $N^2 = 2q + 1$ which is odd. However, by assumption N^2 is even which implies a contradiction. Therefore, by contradiction, the statement P is true.

$\square$

[1]Obviously, the word 'small' makes no sense when dealing with infinite sets! E.g. $1 \in \mathbb{R}$ but $1 \in \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, so the "smallest" set would be $\mathbb{N}$.

1.3. Prove by contradiction that $\sqrt{2}$ is irrational. Try a similar approach to prove that $\sqrt{3}$ is irrational. (*Hint: The following theorem will be useful!* **Theorem:** *If p is prime, $a \in \mathbb{Z}$ and $p|a^2$, then $p|a$.*)

Solution:
Assume that $\sqrt{2}$ is rational, so it is possible to write

$$\sqrt{2} = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}, q \neq 0.$$

Further assume that GCD$\{p, q\} = 1$ ('Greatest Common Divisor') or p and q are 'co-prime'. Hence,

$$2 = \frac{p^2}{q^2} \implies p^2 = 2q^2 \implies p^2 \text{ is even.}$$

Even is the same as saying multiple of 2 or $2|p^2$ ("2 divides p^2"). From exercise 1.2, if p^2 is even then p is even, so let $p = 2m, m \in \mathbb{Z}$. Thus,

$$(2m)^2 = q^2 \implies 4m^2 = 2q^2 \implies q^2 = 2m^2 \implies q \text{ is even.}$$

If q is even, then $2|q$ and GCD$\{p, q\} = 2 \neq 1$, a contradiction. Therefore, by contradiction, $\sqrt{2}$ is irrational.

$\square$

The proof is more or less exactly the same for $\sqrt{3}$ but the theorem in the hint is used instead of the one in exercise 1.2.

1.4. Solve the following inequalities:

(a) $x^2 - 5x + 4 < 0$

(b) $3x - 4 \geq 0$

(c) $4(x + 2) - 1 > 5 - 7(4 - x)$

(d) $2x^2 + 3x - 9 \leq 0$

Solution:

(a) There are two options: (i) immediately factorise and solve or (ii) complete the square and solve. (ii) is more direct but requires more calculation, whilst (i) requires some thinking.

  (i) $x^2 - 5x + 4 = (x - 1)(x - 4) < 0 \implies x - 1 < 0$ or $x - 4 < 0$. If $x - 1 < 0$ then $x < 1$ and $x - 4 > 0 \implies x > 4$ which is impossible. So $x > 1$ and $x < 4$ or $1 < x < 4$.

  (ii) $x^2 - 5x + 4 = (x - 5/2)^2 - \frac{25}{4} + 4 < 0$
  $\implies (x - 5/2)^2 < \frac{9}{4}$
  $\implies -\frac{3}{2} < x - \frac{5}{2} < \frac{3}{2}$
  $\implies 1 < x < 4$

(b) $3x - 4 \geq 0 \implies 3x \geq 4 \implies x \geq 4/3$

(c) $4(x + 2) - 1 > 5 - 7(4 - x) \implies 30 > 3x \implies x < 10$

(d) $2x^2 + 3x - 9 = x^2 + \frac{3x}{2} - \frac{9}{2} = (x + 3/4)^2 - \frac{9}{16} - \frac{9}{2} \leq 0$
$\implies (x + 3/4)^2 \leq 81/16$
$\implies -9/4 \leq x + 3/4 \leq 9/4$
$\implies -3 \leq x \leq 3/2$

1.5. Prove the Triangle Inequality. Provide an example where there is equality and another where there is inequality.

Solution:
Consider $(x+y)^2 = x^2 + 2xy + y^2$. Using the property of the absolute value that $xy \leq |x||y|$, then $\forall x, y \in \mathbb{R}$,

$$x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2.$$

Obviously, $|x|^2 = x^2$, so $x^2 + 2|x||y| + y^2 = |x|^2 + 2|x||y| + |y|^2$, thus

$$(x+y)^2 = x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2 = |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2,$$

or $(x+y)^2 \leq (|x| + |y|)^2$. Taking the positive root of both sides leads to the desired result.

$\square$

1.6. Fill in the following table.

Solution:
Only some are filled in, can you finish it?

Set	Upper Bound	Lower Bound	Max	Min	Sup	Inf	Is the Sup in the set?	Is the Inf in the set?
$\{x \in \mathbb{R} : 0 \leq x < 1\}$	3	-2	N/A	0	1	0	No	Yes
$\{x \in \mathbb{R} : 0 \leq x \leq 1\}$	100	-81	1	0	1	0	Yes	Yes
$\{x \in \mathbb{R} : 0 < x < 1\}$	16	-4	N/A	N/A	1	0	No	No
$\{1/n : n \in \mathbb{Z} - \{0\}\}$								
$\{1/n : n \in \mathbb{N}\}$	5	-1	1	N/A	1	0	Yes	No
$\{x \in \mathbb{R} : x < \sqrt{2}\}$								
$\{1, 4, 7, 97\}$	100	0	97	1	97	1	Yes	Yes
$\{(-1)^n \left(2 - \frac{1}{n}\right) : n \in \mathbb{N}\}$	3	-3	N/A	N/A	2	-2	No	No
$\{\ln(x) : x \in \mathbb{R}, x > 0\}$								
$\{n^{\frac{1}{n}} : n \in \mathbb{N}\}$								
$\{\arctan(x) : x \in \mathbb{R}\}$	π	-π	N/A	N/A	$\pi/2$	$-\pi/2$	No	No
$\{(-1)^n : n \in \mathbb{N}\}$								
$\{e^x : x \in \mathbb{R}\}$								

1.7. Give the Domain & Range of each function, and also classify it as either Odd, Even or Neither.

(a) $f(x) = \cos(x)$
(b) $g(x) = e^x$
(c) $h(x) = x^3$
(d) $k(x) = \sin^2(x)$
(e) $l(x) = \ln(x)$
(f) $m(x) = |x|$
(g) $p(x) = \frac{x^2+1}{3x^3+x}$
(h) $q(x) = x^5 + 4x^3 - 2x$

Solution:

(a) $\text{dom}(f) = \mathbb{R}$, $\text{range}(f) = [-1, 1]$, even as $\cos(-x) = \cos(x)$

(b) $\text{dom}(g) = \mathbb{R}$, $\text{range}(g) = \mathbb{R}^+$, neither as in general $e^{-x} \neq e^x$

(c) $\text{dom}(h) = \mathbb{R}$, $\text{range}(h) = \mathbb{R}$, odd as $h(-x) = -x^3 = -h(x)$

(d) $\text{dom}(k) = \mathbb{R}$, $\text{range}(k) = [0, 1]$, even as $\sin^2(-x) = (-\sin(x))^2 = \sin^2(x)$

(e) $\text{dom}(l) = \mathbb{R}^+$, $\text{range}(l) = \mathbb{R}$, neither as $\ln(-x)$ is not defined on the real number line.

(f) $\text{dom}(m) = \mathbb{R}$, $\text{range}(m) = [0, \infty)$, even as $|-x| = x$

(g) $\text{dom}(p) = \mathbb{R}\backslash\{0\}$, $\text{range}(p) = \mathbb{R}\backslash\{0\}$, odd as $p(-x) = -p(x)$

(h) $\text{dom}(q) = \mathbb{R}$, $\text{range}(q) = \mathbb{R}$, odd as $q(-x) = -q(x)$

1.8. If $f(x) = x^2 + 6$ and $g(x) = 2x - 1$, determine $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

$$(f \circ g)(x) = f(g(x)) = f(2x-1) = (2x-1)^2 + 6 = 4x^2 - 4x + 7$$
$$(g \circ f)(x) = g(f(x)) = g(x^2+6) = 2(x^2+6) - 1 = 2x^2 + 11$$

Note how $(f \circ g)(x) \neq (g \circ f)(x)$.

1.9. Find the inverse function of

$$f(x) = \frac{x+4}{2x-5}$$

Solution:
First, check if $f(x)$ is one-to-one. Recall that if f is one-to-one then $f(x_1) = f(x_2) \implies x_1 = x_2$. Hence,

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{x_1+4}{2x_1-5} = \frac{x_2+4}{2x_2-5} \\ &\implies (x_1+4)(2x_2-5) = (x_2+4)(2x_1-5) \\ &\implies 2x_1x_2 - 5x_1 + 8x_2 - 20 = 2x_1x_2 - 5x_2 + 8x_1 - 20 \\ &\implies 13x_2 = 13x_1 \\ &\implies x_1 = x_2 \end{aligned}$$

therefore $f(x)$ is one-to-one. Now solve $f(y) = x$,

$$\begin{aligned} f(y) = x &\implies \frac{y+4}{2y-5} = x \\ &\implies y + 4 = x(2y-5) \\ &\implies y(1-2x) + 4 = -5x \\ &\implies y = \frac{-5x-4}{1-2x} = \frac{5x+4}{2x-1} \end{aligned}$$

hence, $f^{-1}(x) = \frac{5x+4}{2x-1}$.

1.10. Determine if these functions are One-to-One. If so, find it's inverse.

(a) $f(x) = 3x - 2$

(b) $g(x) = \frac{x}{2} + 7$

(c) $h(x) = x^2$

(d) $j(x) = \sqrt[5]{2x+11}$

(e) $k(x) = \frac{1+2x}{7+x}$

(f) $l(x) = |x|$

(g) $m(x) = \cos(x)$

(h) $n(x) = (x-2)^3 + 1$

Solution:

(a) One-to-one? $f(x_1) = f(x_2) \implies 3x_1 - 2 = 3x_2 - 2 \implies x_1 = x_2$, so yes.
Inverse: $f(y) = x \implies 3y - 2 = x \implies y = \frac{x+2}{3}$. Thus, $f^{-1}(x) = \frac{x+2}{3}$.

(b) One-to-one? $g(x_1) = g(x_2) \implies \frac{x_1}{2} + 7 = \frac{x_2}{2} + 7 \implies x_1 = x_2$, so yes.
Inverse: $g(y) = x \implies \frac{y}{2} + 7 = x \implies y = 2(x-7)$. Thus, $g^{-1}(x) = 2(x-7)$.

(c) One-to-one? No as $(\pm x)^2 = x^2$.

(d) One-to-one?
$j(x_1) = j(x_2) \implies \sqrt[5]{2x_1 + 11} = \sqrt[5]{2x_2 + 11} \implies 2x_1 + 11 = 2x_2 + 11 \implies x_1 = x_2$, so yes.
Inverse: $j(y) = x \implies \sqrt[5]{2y + 11} = x \implies 2y + 11 = x^5 \implies y = \frac{x^5 - 11}{2}$.
Thus $j^{-1}(x) = \frac{x^5 - 11}{2}$.

(e) One-to-One?

$$\begin{aligned} k(x_1) = k(x_2) &\implies \frac{1 + 2x_1}{7 + x_1} = \frac{1 + 2x_2}{7 + x_2} \\ &\implies (1 + 2x_1)(7 + x_2) = (1 + 2x_2)(7 + x_1) \\ &\implies 7 + 2x_1x_2 + x_2 + 14x_1 = 7 + x_1 + 14x_2 + 2x_1x_2 \\ &\implies 13x_1 = 13x_2 \\ &\implies x_1 = x_2, \text{ so yes.} \end{aligned}$$

Inverse:

$$\begin{aligned} k(y) = x &\implies \frac{1 + 2y}{7 + y} = x \\ &\implies 1 + 2y = x(7 + y) \\ &\implies 1 + y(2 - x) = 7x \\ &\implies y = \frac{7x - 1}{2 - x} \end{aligned}$$

Thus $k^{-1}(x) = \frac{7x - 1}{2 - x}$.

(f) One-to-one? No as $|\pm x| = |x|$.

(g) One-to-one? No as $\cos\left(\frac{n\pi}{2}\right) = 0 \ \forall n \in \mathbb{Z}$.

(h) One-to-one? There are two ways to show it is, the normal way i.e. $n(x_1) = n(x_2) \implies x_1 = x_2$ is possible but the calculations are long. If you want to try this way and need help, contact me. Otherwise a much simpler way is a proof by contradiction.

Assume $n(x_1) = n(x_2) \implies x_1 \neq x_2$, i.e. $x_1 > x_2$ or $x_1 < x_2$. First consider if $x_1 > x_2$,

$$\begin{aligned} x_1 > x_2 &\implies x_1 - 2 > x_2 - 2 \\ &\implies (x_1 - 2)^3 > (x_2 - 2)^3 \text{ as } x^3 \text{ is an increasing function} \\ &\implies (x_1 - 2)^3 + 1 > (x_2 - 2)^3 + 1 \\ &\implies n(x_1) > n(x_2) \end{aligned}$$

i.e. a contradiction, which implies $x_1 \leq x_2$. A similar argument can be made for $x_1 < x_2$ (make sure you do this!), thus by contradiction $n(x_1) = n(x_2) \implies x_1 = x_2$ and $n(x)$ is one-to-one.

$\square$

Inverse:

$$\begin{aligned} n(y) = x &\implies (y - 2)^3 + 1 = x \\ &\implies (y - 2)^3 = x - 1 \\ &\implies y - 2 = \sqrt[3]{x - 1} \\ &\implies y = 2 + \sqrt[3]{x - 1} \end{aligned}$$

Thus, $n^{-1}(x) = 2 + \sqrt[3]{x - 1}$.

1.11. Prove by induction that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(6i+3) = 3n(n+2)$$

Solution:

1. Prove Base case: $n = 1$

$$\implies \sum_{i=1}^{1}(6i+3) = 6+3 = 9 = 3(1)(1+2) = 3(3)\checkmark$$

2. Assume $n = k$ is true, i.e.

$$\sum_{i=1}^{k}(6i+3) = 3(k)(k+2).$$

3. Prove $n = k+1$ is true, i.e

$$\sum_{i=1}^{k+1}(6i+3) = 3(k+1)(k+3).$$

Extracting the $k+1$ term of the sum,

$$\begin{aligned}\sum_{i=1}^{k+1}(6i+3) &= \sum_{i=1}^{k}(6i+3) + 6(k+1) + 3 \\ &= 3k(k+2) + 6k + 9 \\ &= 3k^2 + 12k + 9 \\ &= 3(k^2 + 4k + 3) \\ \implies \sum_{i=1}^{k+1}(6i+3) &= 3(k+1)(k+3)\end{aligned}$$

Therefore, by the principle of induction, $\sum_{i=1}^{n}(6i+3) = 3n(n+2)$.

$\square$

1.12. Prove by induction that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Solution:

1. Prove Base case: $n = 1$

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1\checkmark$$

2. Assume $n = k$ is true, i.e.

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}.$$

3. Prove $n = k+1$ is true, i.e.
$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}.$$
Extracting the $k+1$ term of the sum,
$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2}{4}\big(k^2+4(k+1)\big) \\ &= \frac{(k+1)^2}{4}(k^2+4k+4) \\ \implies \sum_{i=1}^{k+1} i^3 &= \frac{(k+1)^2(k+2)^2}{4}\end{aligned}$$
Hence, by the principle of induction, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

□

1.13. Prove by induction that $\forall n \geq 2, x_j \in \mathbb{R}, j \in \mathbb{N}$,
$$\left|\sum_{j=1}^{n} x_j\right| \leq \sum_{j=1}^{n} |x_j|$$
(*Hint:* $|x+y| \leq |x| + |y|$)

Solution:

1. Prove Base case: $n = 2$
$$\left|\sum_{j=1}^{2} x_j\right| = |x_1 + x_2| \leq |x_1| + |x_2| = \sum_{j=1}^{2} |x_j| \checkmark$$
as $|x_1+x_2| \leq |x_1| + |x_2|$ by the triangle inequality.
2. Assume $n = k$ is true, i.e.
$$\left|\sum_{j=1}^{k} x_j\right| \leq \sum_{j=1}^{k} |x_j|.$$
3. Prove $n = k+1$ is true, i.e.
$$\left|\sum_{j=1}^{k+1} x_j\right| \leq \sum_{j=1}^{k+1} |x_j|.$$
Expanding the left hand sum,
$$\left|\sum_{j=1}^{k+1} x_j\right| = \left|\sum_{j=1}^{k} x_j + x_{k+1}\right| \leq \left|\sum_{j=1}^{k} x_j\right| + |x_{k+1}| \leq \sum_{j=1}^{k} |x_j| + |x_{k+1}| = \sum_{j=1}^{k+1} |x_j|$$
again using the triangle inequality and the assumption. Thus, by the principle of induction, $\left|\sum_{j=1}^{n} x_j\right| \leq \sum_{j=1}^{n} |x_j|$.

□

1.14. Prove by induction that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i^m = \frac{n^{m+1}}{m+1} + q(n)$$

where $q(n)$ is a polynomial of degree m.

Solution:

1. Prove Base case: $n = 1$

$$\sum_{i=1}^{1} i^m = 1^m = 1 = \frac{1^{m+1}}{m+1} + q(1) \implies q(1) = 1 - \frac{1}{m+1} \checkmark$$

2. Assume $n = k$ is true, i.e.

$$\sum_{i=1}^{k} i^m = \frac{k^{m+1}}{m+1} + q(k).$$

3. Prove $n = k+1$ is true, i.e.

$$\sum_{i=1}^{k+1} i^m = \frac{(k+1)^{m+1}}{m+1} + q(k+1).$$

Extracting the $k+1$ term,

$$\sum_{i=1}^{k+1} i^m = \sum_{i=1}^{k} i^m + (k+1)^m = \frac{k^{m+1}}{m+1} + q(k) + (k+1)^m = \frac{k^{m+1}}{m+1} + \sum_{i=0}^{m} \binom{m}{i} k^i + q(k)$$

using the assumption and Binomial Theorem.

$$\frac{k^{m+1}}{m+1} + \sum_{i=0}^{m} \binom{m}{i} k^i + q(k) = \frac{1}{m+1} \left[k^{m+1} + m \sum_{i=0}^{m} \binom{m}{i} k^i + \sum_{i=0}^{m} \binom{m}{i} k^i \right] + q(k).$$

Recall the identity,

$$\binom{m}{i} = \binom{m+1}{i} - \binom{m}{i-1},$$

then,

$$\begin{aligned}\frac{1}{m+1} &\left[k^{m+1} + m \sum_{i=0}^{m} \binom{m}{i} k^i + \sum_{i=0}^{m} \binom{m+1}{i} k^i - \sum_{i=1}^{m} \binom{m}{i-1} k^i \right] + q(k) \\ &= \frac{1}{m+1} \left[(k+1)^{m+1} + m(k+1)^m - \sum_{i=1}^{m} \binom{m}{i-1} k^i \right] + q(k)\end{aligned}$$

because

$$\sum_{i=0}^{m+1} \binom{m+1}{i} k^i = (k+1)^{m+1} \quad \text{as} \quad \binom{m+1}{m+1} = 1$$

i.e. absorbing the k^{m+1} term into the sum. Further, defining

$$q(k+1) = \frac{1}{m+1} \left(m(k+1)^m - \sum_{i=1}^{m} \binom{m}{i-1} k^i \right) + q(k)$$

which is a polynomial of order of at most m. Therefore, by the principle of induction, $\sum_{i=1}^{n} i^m = \frac{n^{m+1}}{m+1} + q(n)$.

$\square$

1.15. Prove that the minimum number of moves to solve the Hanoi Tower with n rings is $N_n = 2^n - 1$.

2 Limits and Continuity

2.1. Let $f(x) = x^2 - 4x + 5$. Show that $\lim_{x \to 2} f(x) = 1$ by using an $\epsilon - \delta$ argument.

Solution:

1. Find δ: Let $|x - 2| < \delta$ (i.e. $c = 2$ and $L = 1$), then

$$|f(x) - 1| = |x^2 - 4x + 5 - 1| = |x^2 - 4x + 4| = |x - 2|^2 < \delta^2,$$

so choose $\delta < \sqrt{\epsilon}$.

2. Show δ works: $\forall \epsilon > 0$ and let $\delta < \sqrt{\epsilon}$, then if $|x - 2| < \delta$,

$$|f(x) - 1| = |x - 2|^2 < \delta^2 < \left(\sqrt{\epsilon}\right)^2 = \epsilon,$$

therefore $\lim_{x \to 2} f(x) = 1$.

2.2. Let $f(x) = -2x - 5$.

(a) Find $A > 0$ such that if $0 < |x + 2| < A$ then $|f(x) + 1| < 1/200$.

(b) Given $\epsilon > 0$ find $\delta > 0$ so that if $0 < |x + 2| < \delta$ then $|f(x) + 1| < \epsilon$.

(c) Hence, prove $\lim_{x \to -2} f(x) = -1$.

Solution:

(a) If $0 < |x + 2| < A$, then

$$|f(x) + 1| = |-2x - 5 + 1| = |-2x - 4| = |(-2)(x + 2)| = 2|x + 2| < 2A < \frac{1}{200} \implies A < \frac{1}{400}$$

(b) If $0 < |x + 2| < \delta$, then $|f(x) + 1| = 2|x + 2| < 2\delta < \epsilon$, so choose $\delta < \epsilon/2$.

(c) $\forall \epsilon > 0$ and letting $\delta < \epsilon/2$, then if $|x - (-2)| = |x + 2| < \delta$,

$$|f(x) - (-1)| = |-2x - 5 + 1| = 2|x + 2| < 2\delta < 2\left(\frac{\epsilon}{2}\right) = \epsilon,$$

therefore $\lim_{x \to -2} f(x) = -1$.

2.3. Let $h(x) = x^2 - 2x + 3$. Show that

$$\lim_{x \to -1} h(x) = 6.$$

Solution:
Given $\epsilon > 0$, if $|x - (-1)| = |x + 1| < \delta$, then

$$|h(x) - 6| = |x^2 - 2x + 3 - 6| = |x^2 - 2x - 3| = |x - 3||x + 1|,$$

As $|x + 1| < \delta$, then

$$\begin{aligned} |x+1| < \delta &\implies -\delta < x + 1 < \delta \\ &\implies -4 - \delta < x - 3 < \delta - 4 \\ &\implies |x - 3| < \max\{|-\delta - 4| = |\delta + 4|, |\delta - 4|\} = |\delta + 4| = \delta + 4 \end{aligned}$$

as $\delta > 0$. Assume $\delta < 1$ then $|x - 3| < \delta + 4 < 5$, thus $|x - 3||x + 1| < 5\delta < \epsilon$, so choose $\delta < \epsilon/5$.
Hence, $\forall \epsilon > 0$, let $\delta < \min\{1, \epsilon/5\}$, then if $|x + 1| < \delta$

$$|f(x) - 6| = |x - 3||x + 1| < 5\delta < 5\left(\frac{\epsilon}{5}\right) = \epsilon,$$

therefore, $\lim\limits_{x \to -1} h(x) = 6$.

2.4. Let

$$g(x) = \begin{cases} x^2, & x < 1 \\ 2 - x, & x \geq 1 \end{cases}$$

Show that $\lim\limits_{x \to 1} g(x) = 1$.

Solution:
First prove the one sided limit $\lim\limits_{x \to 1^+} g(x) = 1$, so $g(x) = 2 - x$, thus given $\epsilon > 0$ let $|x - 1| < \delta$, then

$$|g(x) - 1| = |2 - x - 1| = |1 - x| = |x - 1| < \delta < \epsilon,$$

so let $\delta < \epsilon$ and $\lim\limits_{x \to 1^+} g(x) = 1$.
The other one sided limit $\lim\limits_{x \to 1^-} g(x) = 1$, so $g(x) = x^2$, hence given $\epsilon > 0$, let $|x - 1| < \delta$, then

$$|g(x) - 1| = |x^2 - 1| = |x + 1||x - 1|.$$

As $|x - 1| < \delta$,

$$\begin{aligned} |x - 1| < \delta &\implies -\delta < x - 1 < \delta \\ &\implies 2 - \delta < x + 1 < \delta + 2 \\ &\implies |x + 1| < \max\{|2 - \delta|, |2 + \delta|\} = |\delta + 2| = \delta + 2 \end{aligned}$$

as $\delta > 0$. Assume $\delta < 1$, then $|x + 1| < \delta + 2 < 3$, so $|x + 1||x - 1| < 3\delta < \epsilon$, so choose $\delta < \epsilon/3$.
Therefore, $\forall \epsilon > 0$, let $\delta < \min\{1, \epsilon/3\}$, then if $|x - 1| < \delta$

$$|g(x) - 1| = |x + 1||x - 1| < 3\delta < 3\left(\frac{\epsilon}{3}\right) = \epsilon,$$

hence, $\lim\limits_{x \to 1^-} g(x) = 1$. As $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^-} g(x) = 1$, then $\lim\limits_{x \to 1} g(x) = 1$.

2.5. Prove that $\lim_{x\to 0} x^2 = 0$

Solution:
Given $\epsilon > 0$, let $|x - 0| = |x| < \delta$, then

$$|x^2 - 0| = |x|^2 < \delta^2 < \epsilon$$

so let $\delta < \sqrt{\epsilon}$. Hence, $\forall \epsilon > 0$, let $\delta < \sqrt{\epsilon}$, if $|x| < \delta$ then

$$|x^2 - 0| = |x|^2 < \delta^2 < \left(\sqrt{\epsilon}\right)^2 = \epsilon,$$

therefore $\lim_{x\to 0} x^2 = 0$.

2.6. Use the Pinching Theorem to determine

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$$

Solution:
Recall that $\forall y \in \mathbb{R}$, $\cos(y) \in [-1, 1]$ or $-1 \leq \cos(y) \leq 1$, i.e.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1, \forall x \neq 0 \implies -x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2, \forall x \neq 0.$$

As we know $\lim_{x\to 0} x^2 = 0$, hence by the Pinching Theorem $\lim_{x\to 0} x^2 \cos(1/x) = 0$.

2.7. Prove that $\lim_{x\to 1} x - 1 = 0$. Thus, determine

$$\lim_{x\to 1} (x-1)\cos\left(\frac{\pi}{x-1}\right)$$

2.8. Determine the following limit,

$$\lim_{x\to 0^-} \frac{x + |x|(1+x)}{x} \sin\left(\frac{1}{x}\right)$$

Solution:
As the limit approaches 0 from below then $x < 0$, hence $|x| = -x > 0$,

$$\implies \frac{x + |x|(1+x)}{x} = \frac{x - x(1+x)}{x} = -\frac{x^2}{x} = -x,$$

therefore,

$$\lim_{x\to 0^-} \frac{x + |x|(1+x)}{x} \sin\left(\frac{1}{x}\right) = \lim_{x\to 0^-} -x \sin\left(\frac{1}{x}\right).$$

Recall, $-1 \leq \sin(y) \leq 1, \forall y \in \mathbb{R}$, then

$$\implies -1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \forall x \neq 0$$

$$\implies x \leq -x \sin\left(\frac{1}{x}\right) \leq -x, \forall x \neq 0.$$

As $\lim_{x\to 0} x = 0$, then by the Pinching Theorem $\lim_{x\to 0^-} \frac{x+|x|(1+x)}{x} \sin\left(\frac{1}{x}\right) = 0$.

2.9. Given that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, determine $\lim_{x\to 0} \frac{1-\cos(x)}{x}$. Hence find,

$$\lim_{x\to 0} \frac{\sqrt{1+\sin(x)} - \sqrt{1-\sin(x)}}{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{x(1+\cos(x))} &= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))} \\
&= \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \sin(x)\right)\left(\lim_{x\to 0} \frac{1}{1+\cos(x)}\right) \\
&= (1)(0)\left(\frac{1}{2}\right) = 0
\end{aligned}$$

using the multiplicative limit law. Therefore,

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\sin(x)} - \sqrt{1-\sin(x)}}{x} &= \lim_{x\to 0} \frac{\left(\sqrt{1+\sin(x)} - \sqrt{1-\sin(x)}\right)\sqrt{1+\sin(x)}}{x\sqrt{1+\sin(x)}} \\
&= \lim_{x\to 0} \frac{1+\sin(x) - \sqrt{1-\sin^2(x)}}{x\sqrt{1+\sin(x)}} \\
&= \lim_{x\to 0} \frac{1+\sin(x)-\cos(x)}{x\sqrt{1+\sin(x)}} \\
&= \left(\lim_{x\to 0} \frac{1+\sin(x)-\cos(x)}{x}\right)\left(\lim_{x\to 0} \frac{1}{\sqrt{1+\sin(x)}}\right) \\
&= \left(\lim_{x\to 0} \frac{1-\cos(x)}{x} + \lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{1}{\sqrt{1+\sin(x)}}\right) \\
&= (0+1)(1) = 1
\end{aligned}$$

using the additive and multiplicative limit laws. Hence,

$$\lim_{x\to 0} \frac{\sqrt{1+\sin(x)} - \sqrt{1-\sin(x)}}{x} = 1.$$

2.10. Let $f(x) = x^2 - 4x + 5$. Show that $\lim_{x\to 2} f(x) = 1$ by using $\epsilon - \delta$ argument. Hence, conclude that $f(x)$ is continuous at $x = 2$.

Solution:
See exercise 2.1. As $f(2) = 2^2 - 4(2) + 5 = 1$, and $\lim_{x\to 2} f(x) = f(2) = 1$, then $f(x)$ is continuous at $x = 2$.

2.11. Let

$$g(x) = \begin{cases} x^2, & x < 1 \\ 2 - x, & x \geq 1 \end{cases}$$

Show that $g(x)$ is continuous at $x = 1$ using an $\epsilon - \delta$ argument.

Solution:
See exercise 2.4 but use $L = g(1) = 1$. As $\lim_{x \to 1} g(x) = g(1) = 1$, then $g(x)$ is continuous at $x = 1$.

2.12. Prove that $f(x) = x^3 - 3x^2 + 12x - 25$ has at least one real root.

Solution:
As $f(-10) = -1445$ and $f(10) = 795$, so if $x \in [-10, 10]$ then,

$$-1445 = f(-10) \leq f(x) \leq f(10) = 795.$$

Hence, by the Intermediate Value Theorem then $\exists c \in (-10, 10)$ such that $f(c) = 0$, i.e. $\exists$ at least one real root in $x \in (-10, 10)$.

3 Derivatives

3.1. Let $f(x) = \frac{1}{x}$ such that $x > 0$, determine $\frac{df}{dx}$.

Solution:

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{1}{x+h} - \frac{1}{x}\right) &= \lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{x-(x-h)}{x(x+h)}\right) \\ &= \lim_{h\to 0} -\frac{1}{x(x+h)} \\ f'(x) &= -\frac{1}{x^2} \end{aligned}$$

3.2. Consider $f(x) = x^3 + x^2$. Determine $f'(x)$ using first principles.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 + (x+h)^2 - (x^3 + x^2)}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) + (x^2 + 2xh + h^2) - x^3 - x^2}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 2x + h \\ &= 3x^2 + 2x \end{aligned}$$

3.3. Let

$$g(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

Show that $g(x)$ is continuous at $x = 0$. Is $g(x)$ differentiable at $x = 0$?

Solution:
Recall for continuity that $\lim\limits_{x\to 0} g(x) = g(0) = 0$. First look at left limit,

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} -\sqrt{-x} = 0,$$

and the right limit,

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} \sqrt{x} = 0.$$

As $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^+} g(x) = 0$, then $\lim\limits_{x\to 0} g(x) = g(0) = 0$ and $g(x)$ is continuous at $x = 0$.
Now, if $g(x)$ is differentiable at $x = 0$ then the following limit exists,

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h}.$$

Thus, consider the limit,

$$\lim_{h\to 0^+} \frac{g(h) - g(0)}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}}$$

which does not exist. Similarly,

$$\lim_{h\to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-} \frac{-\sqrt{-h}}{h} = \lim_{h\to 0^-} \frac{1}{\sqrt{-h}}$$

which also does not exist, hence $g(x)$ is not differentiable at $x = 0$.

3.4. From first principles, if $n \in \mathbb{Z}$ show that

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

.

3.5. Determine the derivatives of the following functions:

(a) $f(x) = x(2x+5)^3$

(b) $g(x) = \sin(x)\cos(x)$

(c) $h(x) = x\tan(x)$

(d) $p(x) = \sin(x)\sqrt{x^2+7}$

(e) $q(x) = \frac{e^{3x}}{\cos(4x)}$

(f) $r(x) = \sin\left(\cos\left(1+x^3\right)\right)$

(g) $s(x) = \sqrt{\frac{x-3}{x^2+2}}$

(h) $y(x) = (\sin(x)+1)^x$

Solution:

(a) Using the product rule and chain rule,

$$f'(x) = [x]'(2x+5)^3 + x\left[(2x+5)^3\right]' = (2x+5)^3 + x(3)(2x+5)^2(2) = (8x+5)(2x+5)^2$$

(b)

$$g'(x) = [\sin(x)]'\cos(x) + \sin(x)[\cos(x)]' = \cos^2(x) - \sin^2(x) = \cos(2x)$$

(c)

$$\begin{aligned} h'(x) = [x]'\tan(x) + x[\tan(x)]' &= \tan(x) + x\left[\frac{\sin(x)}{\cos(x)}\right]' \\ &= \tan(x) + x\left(\frac{\cos(x)}{\cos(x)} + \sin(x)\frac{(-1)(-\sin(x))}{\cos^2(x)}\right) \\ &= \tan(x) + x\left(\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\right) \\ &= \tan(x) + x\sec^2(x) \end{aligned}$$

(d)

$$\begin{aligned} p'(x) = [\sin(x)]'\sqrt{x^2+7} + \sin(x)\left[\sqrt{x^2+7}\right]' &= \cos(x)\sqrt{x^2+7} + \sin(x)\left(\frac{1}{2}\right)(x^2+7)^{-\frac{1}{2}}(2x) \\ &= \frac{(x^2+7)\cos(x) + x\sin(x)}{\sqrt{x^2+7}} \end{aligned}$$

(e)

$$q'(x) = \frac{[e^{3x}]'}{\cos(4x)} + e^{3x}\left[\frac{1}{\cos(4x)}\right]' = \frac{3e^{3x}}{\cos(4x)} + e^{3x}\frac{(-1)(-\sin(4x))(4)}{\cos^2(4x)} = e^{3x}\sec(4x)(3+4\tan(x))$$

(f)

$$r'(x) = \cos(\cos(1+x^3)(-\sin(1+x^3))(3x^2) = -3x^2\sin(1+x^3)\cos(\cos(1+x^3))$$

(g)

$$s'(x) = \frac{1}{2}\sqrt{\frac{x^2+2}{x-3}}\left(\frac{x-3}{x^2+2}\right)' = \frac{1}{2}\sqrt{\frac{x^2+2}{x-3}}\left(\frac{1}{x^2+2} - \frac{(x-3)(2x)}{(x^2+2)^2}\right) = \frac{1}{2}\sqrt{\frac{x^2+2}{x-3}}\left(\frac{2+6x-x^2}{(x^2+2)^2}\right)$$
$$= \frac{2+6x-x^2}{2\sqrt{(x-3)(x^2+2)^3}}$$

(h)

$$\begin{aligned} y'(x) = [\exp\{\ln(\sin(x)+1)^x\}]' &= [\exp\{x\ln(\sin(x)+1)\}]' \\ &= \exp\{x\ln(\sin(x)+1)\}\,[x\ln(\sin(x)+1)]' \\ &= \exp\{x\ln(\sin(x)+1)\}\left(\ln(\sin(x)+1) + \frac{x\cos(x)}{\sin(x)+1}\right) \\ &= \ln(\sin(x)+1)^x\left(\ln(\sin(x)+1) + \frac{x\cos(x)}{\sin(x)+1}\right) \end{aligned}$$

3.6. Determine the derivatives of the following functions:

(a) $f(x) = \cot^{-1}(x)$ (b) $g(x) = \sec^{-1}(x)$ (c) $h(x) = \mathrm{cosec}^{-1}(x)$

Solution:
Note: If $y = f(x)$ and $g(y) = f^{-1}(y) = x$, then $f'(x)g'(y) = 1$.

(a) Let $y(x) = \cot(x)$, so $g(y) = \cot^{-1}(y)$. Hence,

$$\begin{aligned} \frac{d}{dx}\cot(x) = \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = -\frac{\sin(x)}{\sin(x)} - \frac{\cos^2(x)}{\sin^2(x)} &= -\frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)} = -\mathrm{cosec}^2(x) \\ \implies \left(\frac{d}{dx}\cot(x)\right)\left(\frac{d}{dy}\cot^{-1}(y)\right) &= 1 \\ \implies \frac{d}{dy}\cot^{-1}(y) &= -\sin^2(x). \end{aligned}$$

As $y = \cot(x)$, then $\sin(x) = 1/\sqrt{y^2+1}$, thus

$$\frac{d}{dy}\cot^{-1}(y) = -\frac{1}{y^2+1} \implies \frac{df(x)}{dx} = -\frac{1}{1+x^2}.$$

(b) Let $y(x) = \sec(x)$, so $G(y) = \sec^{-1}(y)$. Hence,

$$\begin{aligned} \frac{d}{dx}\sec(x) = \frac{d}{dx}\frac{1}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)} &= \tan(x)\sec(x) \\ \implies \left(\frac{d}{dx}\sec(x)\right)\left(\frac{d}{dy}\sec^{-1}(y)\right) &= 1 \\ \implies \frac{d}{dy}\sec^{-1}(y) &= \cos(x)\cot(x). \end{aligned}$$

As $y = \sec(x)$, then $\cos(x) = 1/y$ and $\cot(x) = 1/\sqrt{y^2-1}$, thus

$$\frac{d}{dy}\sec^{-1}(y) = \frac{1}{y\sqrt{y^2-1}} \implies \frac{dg(x)}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

(c) Let $y(x) = \text{cosec}(x)$, so $g(y) = \text{cosec}^{-1}(y)$. Hence,

$$\begin{aligned}
\frac{d}{dx}\text{cosec}(x) = \frac{d}{dx}\frac{1}{\sin(x)} &= -\frac{\cos(x)}{\sin^2(x)} = -\cot(x)\text{cosec}(x) \\
\implies \left(\frac{d}{dx}\text{cosec}(x)\right)&\left(\frac{d}{dy}\text{cosec}^{-1}(y)\right) = 1 \\
\implies \frac{d}{dy}&\text{cosec}^{-1}(y) = -\tan(x)\sin(x).
\end{aligned}$$

As $y = \text{cosec}(x)$, then $\sin(x) = 1/y$ and $\tan(x) = 1/\sqrt{y^2-1}$, thus

$$\frac{d}{dy}\text{cosec}^{-1}(y) = -\frac{1}{y\sqrt{y^2-1}} \implies \frac{dh(x)}{dx} = -\frac{1}{x\sqrt{x^2-1}}.$$

3.7. Find an approximation of the solution to $x^5 = 3$ using Newton's method with $x_0 = 4$.

Solution:
Recall that Newton's method is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which solves equations of the form $f(x) = 0$. Hence, let $f(x) = x^5 - 3$ so $f'(x) = 5x^4$, so Newton's method becomes,

$$x_{n+1} = x_n - \frac{x_n^5 - 3}{5x_n^4}.$$

Therefore, we have,

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^5-3}{5x_0^4} = 4 - \frac{4^5-3}{5(4)^4} \approx 3.2023 \\
x_2 &= x_1 - \frac{x_1^5-3}{5x_1^4} \approx 2.5676 \\
x_3 &= x_2 - \frac{x_2^5-3}{5x_2^4} \approx 2.0679 \\
x_4 &\approx 1.6871, x_5 \approx 1.4237, x_6 \approx 1.2850, x_7 \approx 1.2481, x_8 \approx 1.2457
\end{aligned}$$

Therefore solution is 1.25 to 2 decimal places.

3.8. Find an approximation to the solution of $f(x) = x^3 - \cos(x) = 0$ to 4 decimal places.

Solution:
Using Newton's method, $x_{n+1} = x_n - f(x_n)/f'(x_n)$, and $f'(x) = 3x^2 + \sin(x)$, so

$$x_{n+1} = x_n - \frac{x_n^3 - \cos(x_n)}{3x_n^2 + \sin(x_n)}.$$

Therefore, using $x_0 = 1$ we have,

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^3 - \cos(x_0)}{3x_0^2 + \sin(x_0)} = 1 - \frac{1^3 - \cos(1)}{3(1)^2 + \sin(1)} \approx 0.88033 \\
x_2 &= x_1 - \frac{x_1^3 - \cos(x_1)}{3x_1^2 + \sin(x_1)} \approx 0.86568, x_3 \approx 0.86547, x_4 \approx 0.86547
\end{aligned}$$

Hence, solution is 0.8655 to 4 decimal places.

3.9. Use the Newton method on the function $y(x) = 2x^3 - 6x^2 + 6x - 1$ with $x_0 = 1$. What happens? Why? Try again using $x_0 = 0.5$.

Solution:
As $y'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x-1)^2$, then Newton's method becomes

$$x_{n+1} = x_n - \frac{2x_n^3 - 6x_n^2 + 6x_n - 1}{6(x_n - 1)^2},$$

and using $x_0 = 1$ then results in '1/0' and diverges. This is because $f'(1) = 0$, i.e. a horizontal tangent line, which cannot direct the solution towards the root, hence divergence.
Using $x_0 = 0.5$ then yields,

$$\begin{aligned} x_1 &= x_0 - \frac{2x_0^3 - 6x_0^2 + 6x_0 - 1}{6(x_0 - 1)^2} = 0.5 - \frac{2(0.5)^3 - 6(0.5)^2 + 6(0.5) - 1}{6(0.5-1)^2} = 0 \\ x_2 &= 1/6, x_3 = 46/225 \approx 0.2044, x_4 \approx 0.2063, x_5 \approx 0.2063 \end{aligned}$$

So to 3 decimal places $x = 0.206$.

3.10. Prove that $f(x) = x^3 - 3x^2 + 12x - 25$ has exactly one real root.

Solution:
As $f(-10) = -1445$ and $f(10) = 795$, so if $x \in [-10, 10]$ then, $f(-10) \leq f(x) \leq f(10)$. Hence, by the Intermediate Value Theorem then $\exists x_1 \in (-10, 10)$ such that $f(x_1) = 0$. Assume that $\exists x_2 : f(x_2) = 0$, thus by Rolle's Theorem, $\exists c \in (x_1, x_2) : f'(c) = 0$. However, $f'(x) = 3x^2 - 6x + 12 = 3(x^2 - 2x + 4) = 3(x-1)^2 + 9 > 0$ for all x which is a contradiction. Thus, by proof by contradiction, there can only be one real root.

3.11. Implement/program the Newton method in order to solve equations of the form $f(x) = 0$. Hence, find the root of the function $f(x) = x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$.

3.12. Find and classify the critical points of $f(x) = -3x^5 + 5x^3$.

Solution:
Recall a critical point satisfies $f'(x) = 0$, thus need to solve

$$f'(x) = -15x^4 + 15x^2 = x^2(15 - 15x^2) = 0 \implies x^2 = 0 \ \& \ 15 - 15x^2 = 0 \implies x = 0 \ \& \ x = \pm 1.$$

Hence, the critical points are $(0, 0)$, $(-1, -2)$ and $(1, 2)$. In order to classify them, use the second derivative test, so $f''(x) = 2x(15 - 15x^2) + x^2(-30x) = 30x(1 - 2x^2)$. Therefore,

$$\begin{aligned} (0,0) : f''(0) = 0 &\implies \text{Inflection Point} \\ (-1,-2) : f''(-1) = 30 > 0 &\implies \text{Local Minimum} \\ (1,2) : f''(1) = -30 < 0 &\implies \text{Local Maximum} \end{aligned}$$

3.13. Find and classify the critical points of $f(x) = -\frac{1}{8}(x+2)^2(x-4)^2$. Find all points of inflection, and hence sketch a graph of $f(x)$.

Solution:
First solve

$$f'(x) = -\frac{1}{8}[2(x+2)(x-4)^2 + 2(x+2)^2(x-4)] = -\frac{(x+2)(x-4)}{4}[2x-2] = -\frac{(x+2)(x-4)(x-1)}{2} = 0,$$

which has solutions $x = -2$, $x = 4$, $x = 1$. Thus, the critical points are $(-2, 0)$, $(1, -81/8)$ and $(4, 0)$. Also,

$$f''(x) = -\frac{(x-4)(x-1) + (x+2)(x-1) + (x+2)(x-4)}{2} = -\frac{3(x^2 - 2x - 2)}{2}.$$

Thus,

$$\begin{aligned}(-2,0) : f''(-2) = -9 < 0 &\implies \text{Local Maximum}\\ (1,-81/8) : f''(1) = 9/2 > 0 &\implies \text{Local Minimum}\\ (4,0) : f''(4) = -9 < 0 &\implies \text{Local Maximum}\end{aligned}$$

Recall for points of inflection that $f''(x) = 0$, so need to solve

$$f''(x) = -\frac{3(x^2-2x-2)}{2} = 0 \implies x^2 - 2x - 2 = 0 \implies x = \frac{2 \pm \sqrt{4-(4)(-2)}}{2} = 1 \pm \sqrt{3}.$$

Hence, inflection points are $(1 \pm \sqrt{3}, -9/2)$. Your sketch should look like Figure 1.

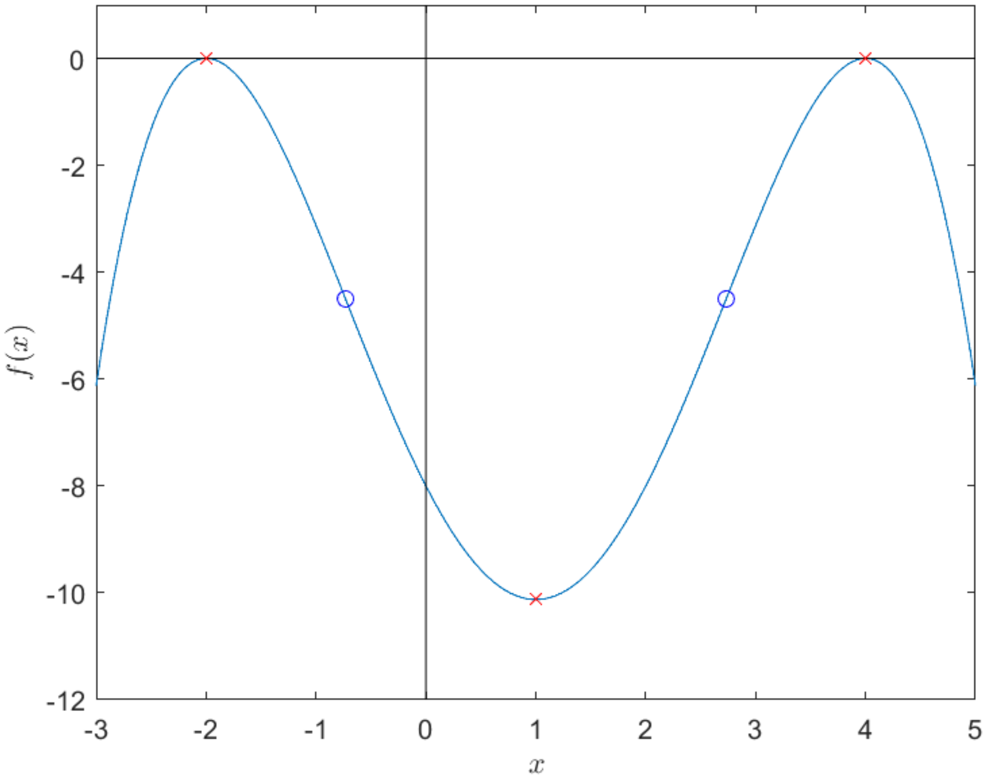

Figure 1: $f(x) = -\frac{1}{8}(x+2)^2(x-4)^2$ for $x \in [-3, 5]$. Red crosses are critical points and blue circles are inflection points.

3.14. The deflection D of a beam of length L is $D = 2x^4 - 5Lx^3 + 3L^2x^2$, where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

Solution:
Maximum points occur at critical points or at boundaries, so solve $D'(x) = 0$,

$$\begin{aligned}D'(x) &= 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0\\ &\implies x = 0 \ \& \ 8x^2 - 15Lx + 6L^2 = 0\\ &\implies x = \frac{15L \pm \sqrt{225L^2 - (4)(8)6L^2}}{16} = \frac{15 \pm \sqrt{33}}{16}L.\end{aligned}$$

Therefore, critical points at $x = 0$ and $x = (15 \pm \sqrt{33})L/16$. As $D''(x) = 24x^2 - 30Lx + 6L^2$, then to find the maximum use second derivative test,

$$\begin{aligned}
D''(0) = 6L^2 > 0 &\implies \text{Local Minimum}\\
D''((15-\sqrt{33})L/16) \approx -3.32L^2 < 0 &\implies \text{Local Maximum}\\
D''((15+\sqrt{33})L/16) \approx 7.45L^2 > 0 &\implies \text{Local Minimum}
\end{aligned}$$

As there is only one maximum point with $D((15-\sqrt{33})L/16) \approx 0.26L^4$, so only need to check boundary points: $D(0) = D(L) = 0$. Hence maximum deflection is at $x \sim 0.578L$ with $D \sim 0.26L^4$.

3.15. A small aircraft starts its descent from an altitude of 1km, 4km west of the runway.

(a) Find the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where $x \in [-4, 0]$ that describes the smooth glide path for the landing.

(b) The function $f(x)$ models the glide path of the plane. When would the plane be descending at the greatest rate.

(c) If $x(t) = -4 + t^2 : t \in [0, 2]$, calculate $f(t)$ and hence determine the velocity of the plane. When does it travel at its fastest?

Solution:

(a) At endpoints we have $f(0) = 0 \implies f(0) = d = 0$ and $f(-4) = -64a + 16b - 4c = 1$; also assume at endpoints the trajectory is flat i.e. $f'(x) = 0$, thus

$$f'(x) = 3ax^2 + 2bx + c = 0 \implies f'(0) = 0 \;\&\; f'(-4) = 48a - 8b = 0.$$

This gives two equations for unknowns:

$$\begin{cases} -64a + 16b = 1 \\ 48a - 8b = 0 \end{cases} \implies a = \frac{1}{32} \;\&\; b = \frac{3}{16}$$

Hence,

$$f(x) = \frac{x^3}{32} + \frac{3x^2}{16} = \frac{x^2(x+6)}{32}.$$

(b) The plane would be descending greatest at an inflection point, $f''(x) = 0$. Therefore,

$$f''(x) = 6ax + 2b = \frac{3x}{16} + \frac{3}{8} = 0 \implies x = -2,$$

i.e. 2 miles west of the runway.

(c) As $f(t) = f(x(t))$, then

$$f(t) = f(x(t)) = f(t^2 - 4) = \frac{(t^2-4)^2((t^2-4)+6)}{32} = \frac{(t^2-4)^2(t^2+2)}{32} = \frac{t^6 - 6t^4 + 32}{32}.$$

The velocity is the rate of change of position, i.e. $v = \dot{f} = df/dt$,

$$v = \dot{f} = \frac{6t^5 - 24t^3}{32} = \frac{3t^3(t^2-4)}{16}.$$

As $v(-4) = v(0) = 0$, then the plane will travel at it's fastest when $\dot{v} = 0$, thus

$$\dot{v} = \ddot{f} = \frac{3(3t^2)(t^2-4)}{16} + \frac{3t^3(2t)}{16} = \frac{3t^2\left(3(t^2-4) + 2t^2\right)}{16} = \frac{3t^2(5t^2-12)}{16} = 0$$

$$\implies t^2(5t^2 - 12) = 0 \implies t = 0 \;\&\; 5t^2 - 12 = 0 \implies t = \sqrt{\frac{12}{5}} \sim 1.549$$

Even though this is a minimum (you should check this), the maximum speed is $|v| \sim 1.115$ at $t = \sqrt{12/5}$.

3.16. Determine $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$.

Solution:
As $\lim_{x\to 0}(e^{2x}-1) = 0$ and $\lim_{x\to 0} x = 0$, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2$$

3.17. Determine the following limits:

(a) $\lim_{x\to 0} f(x) = \begin{cases} x^2-1, & x>0 \\ -\cos(x), & x\le 0 \end{cases}$

(b) $\lim_{x\to 2} g(x) = \begin{cases} 2\sin(2-x)-4, & x\le 2 \\ -2x, & x>2 \end{cases}$

(c) $\lim_{x\to 0} \dfrac{\cos(x)\sin(x)}{x}$

(d) $\lim_{x\to 0} \dfrac{e^x-1}{\sin(2x)}$

Solution:

(a) Consider, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-1 = -1$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -\cos(x) = -1$. As $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = -1$, then $\lim_{x\to 0} f(x) = -1$.

(b) Consider, $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} -2x = -4$ and $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} 2\sin(2-x)-4 = -4$. As $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^-} g(x) = -4$, then $\lim_{x\to 2} g(x) = -4$.

(c) As $\lim_{x\to 0}\cos(x)\sin(x) = \lim_{x\to 0} x = 0$ then use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\cos(x)\sin(x)}{x = \lim_{x\to 0}} \frac{-\sin^2(x)+\cos^2(x)}{1} = 1.$$

(d) As $\lim_{x\to 0} e^x - 1 = \lim_{x\to 0}\sin(2x) = 0$ then use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{e^x-1}{\sin(2x)} = \lim_{x\to 0} \frac{e^x}{2\cos(2x)} = \frac{1}{2}.$$

3.18. Evaluate

(a) $\lim_{x\to -\infty} \dfrac{x^2}{e^{-x}}$

(b) $\lim_{x\to 0^+} (\sin(x))^x$

(c) $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(x)}$

Solution:

(a) Taking the limit straight away yields,

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \frac{\lim_{x\to -\infty} x^2}{\lim_{x\to -\infty} e^{-x}} = \frac{`\infty\text{'}}{`\infty\text{'}}$$

so using L'Hôpital's rule:

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}} = \lim_{x\to -\infty} 2e^x = 0.$$

The second equality is obtained from L'Hôpital's rule again as this gives the indeterminate '$-\infty$'/'$-\infty$'.

(b) Assume that the limit exists and $\lim_{x\to 0^+} (\sin(x))^x = y \in \mathbb{R}/\{0\}$, so it's possible to do the following,

$$\begin{aligned}
y = \lim_{x\to 0^+} (\sin(x))^x \implies \ln(y) &= \ln\left[\lim_{x\to 0^+} (\sin(x))^x\right] \\
&= \lim_{x\to 0^+} \ln\left[(\sin(x))^x\right], \text{ as the natural logarithm is continuous,} \\
&= \lim_{x\to 0^+} x \ln\left[\sin(x)\right] \\
&= \lim_{x\to 0^+} \frac{\ln\left[\sin(x)\right]}{(1/x)} \left(= \frac{`-\infty\text{'}}{`\infty\text{'}}\right), \text{ so use L'Hôpital's rule,} \\
&= \lim_{x\to 0^+} \frac{(\cos(x)/\sin(x))}{-(1/x^2)} \\
&= -\lim_{x\to 0^+} \frac{x^2}{\tan(x)} \left(= \frac{`0\text{'}}{`0\text{'}}\right), \text{ so use L'Hôpital's rule again,} \\
&= -\lim_{x\to 0^+} \frac{2x}{\sec^2(x)} \\
\implies \ln(y) &= -\lim_{x\to 0^+} 2x\cos^2(x) = 0.
\end{aligned}$$

Hence, exponentiating both sides then gives

$$y = \lim_{x\to 0^+} (\sin(x))^x = e^0 = 1.$$

(c) As $\lim_{x\to 0} x = \lim_{x\to 0} \tan^{-1}(x) = 0$, then use L'Hôpital's rule,

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(2x)} = \lim_{x\to 0} \frac{1}{\left[\tan^{-1}(2x)\right]'} = \lim_{x\to 0} \frac{1}{\left(\frac{2}{1+4x^2}\right)} = \lim_{x\to 0} \frac{1+4x^2}{2} = \frac{1}{2}.$$

The second equality is obtained via noting that $[\tan(x)/2]' = \sec^2(x)/2$ so $[\tan^{-1}(2y)]' = 2\cos^2(x) = 2/(1+4y^2)$ [see exercise 3.6 with $y(x) = \tan(x)/2$].

3.19. Find the Taylor series of $\cos(x)$ at $x = 0$.

Solution:
Recall the Taylor series of a function around a point $x = a = 0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

First, determine the derivative terms,

$$\begin{aligned}
f^{(0)}(x) &= \cos(x) \implies f^{(0)}(0) = \cos(0) = 1 \\
f^{(1)}(x) &= -\sin(x) \implies f^{(1)}(0) = -\sin(0) = 0 \\
f^{(2)}(x) &= -\cos(x) \implies f^{(2)}(0) = -\cos(0) = -1 \\
f^{(3)}(x) &= \sin(x) \implies f^{(3)}(0) = \sin(0) = 0 \\
f^{(4)}(x) &= \cos(x) \implies f^{(4)}(0) = \cos(0) = 1
\end{aligned}$$

which suggests that for odd n, $f^{(n)}(0) = 0$, and for even $n = 2m, m \in \mathbb{N}_0$ that $f^{(2m)}(0) = (-1)^m$. Thus,

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6).$$

3.20. Determine the Taylor series at $x = 0$ of the following functions:

(a) $f(x) = \cos(ax)$ (b) $g(x) = \sin(bx)$ (c) $h(x) = e^{cx}$

where $a, b, c \in \mathbb{R}$. Thus show that $e^{inx} = \cos(nx) + i\sin(nx)$ with $i = \sqrt{-1} \in \mathbb{C}$.

Solution:

(a) Determine the derivative terms,

$$\begin{aligned}
f^{(0)}(x) &= \cos(ax) \implies f^{(0)}(0) = \cos(0) = 1 \\
f^{(1)}(x) &= -a\sin(ax) \implies f^{(1)}(0) = -a\sin(0) = 0 \\
f^{(2)}(x) &= -a^2\cos(ax) \implies f^{(2)}(0) = -a^2\cos(0) = -a^2 \\
f^{(3)}(x) &= a^3\sin(ax) \implies f^{(3)}(0) = a^3\sin(0) = 0 \\
f^{(4)}(x) &= a^4\cos(ax) \implies f^{(4)}(0) = a^4\cos(0) = a^4
\end{aligned}$$

which suggests that for odd n, $f^{(n)}(0) = 0$, and for even $n = 2m, m \in \mathbb{N}_0$ that $f^{(2m)}(0) = (-1)^m a^{2m}$. Thus,

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m} x^{2m}}{(2m)!}.$$

(b) Determine the derivative terms,

$$\begin{aligned}
g^{(0)}(x) &= \sin(bx) \implies g^{(0)}(0) = \sin(0) = 0 \\
g^{(1)}(x) &= b\cos(bx) \implies g^{(1)}(0) = b\cos(0) = b \\
g^{(2)}(x) &= -b^2\sin(bx) \implies g^{(2)}(0) = -b^2\sin(0) = 0 \\
g^{(3)}(x) &= -b^3\cos(bx) \implies g^{(3)}(0) = -b^3\cos(0) = -b^3 \\
g^{(4)}(x) &= b^4\sin(bx) \implies g^{(4)}(0) = b^4\sin(0) = 0
\end{aligned}$$

which suggests that for even n, $g^{(n)}(0) = 0$, and for odd $n = 2k+1, k \in \mathbb{N}_0$ that $g^{(2k+1)}(0) = (-1)^k a^{2k+1}$. Thus,

$$g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k+1} x^{2k+1}}{(2k+1)!}.$$

(c) Determine the derivative terms,

$$\begin{aligned}
h^{(0)}(x) &= e^{cx} \implies h^{(0)}(0) = e^0 = 1 \\
h^{(1)}(x) &= ce^{cx} \implies h^{(1)}(0) = ce^0 = c \\
h^{(2)}(x) &= c^2e^{cx} \implies h^{(2)}(0) = c^2e^0 = c^2 \\
h^{(3)}(x) &= c^3e^{cx} \implies h^{(3)}(0) = c^3e^0 = c^3
\end{aligned}$$

which suggests that $h^{(n)}(0) = c^n$. Thus,

$$h(x) = \sum_{n=0}^{\infty} \frac{c^n x^n}{n!}.$$

Now consider,

$$e^{inx} = \sum_{k=0}^{\infty} \frac{(in)^k x^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k n^k x^k}{k!},$$

and note that $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i$, suggesting that if $k = 2m$ (i.e. even) then $i^{2m} = (-1)^m$ and if $k = 2m+1$ (i.e. odd) then $i^{2m+1} = (-1)^m i$. As k is composed of either odd or even numbers, then $k = \{2m : m \in \mathbb{N}_0\} \cup \{2m+1 : m \in \mathbb{N}_0\}$, hence

$$\begin{aligned}
e^{inx} = \sum_{k=0}^{\infty} \frac{i^k n^k x^k}{k!} &= \sum_{m=0}^{\infty} \left(\frac{i^{2m} n^{2m} x^{2m}}{(2m)!} + \frac{i^{2m+1} n^{2m+1} x^{2m+1}}{(2m+1)!} \right) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m n^{2m} x^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(-1)^m i n^{2m+1} x^{2m+1}}{(2m+1)!} \\
&= \cos(nx) + i \sum_{m=0}^{\infty} \frac{(-1)^m n^{2m+1} x^{2m+1}}{(2m+1)!} \\
\implies e^{inx} &= \cos(nx) + i \sin(nx)
\end{aligned}$$

3.21. Albert is very clever and has figured out that the Energy of a particle is

$$E = \sqrt{m^2 c^4 + p^2 c^2}$$

where p is the particle's momentum, v is the particle's velocity, m is the particle's mass and $c \sim 3 \times 10^5$ km/s is the speed of light.

(a) Find the first two terms of the Taylor Series of $\dfrac{1}{\sqrt{1-x^2}}$ around $x = 0$.

(b) What is E if $p = \dfrac{mv}{\sqrt{1-(v/c)^2}}$?

(c) Assume that $v \ll c$ and help Albert find a nice expression for E. What is E if $v = 0$?

Solution:

(a) If $f(x) = 1/\sqrt{1-x^2} = (1-x^2)^{-1/2}$, then

$$\begin{aligned}
f^{(0)}(x) &= (1-x^2)^{-\frac{1}{2}} \implies f^{(0)}(0) = 1 \\
f^{(1)}(x) &= x(1-x^2)^{-\frac{3}{2}} \implies f^{(1)}(0) = 0 \\
f^{(2)}(x) &= (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}} \implies f^{(2)}(0) = 1
\end{aligned}$$

Therefore,

$$f(x) = \frac{1}{\sqrt{1-x^2}} \approx \sum_{n=0}^{2} \frac{f^{(n)}(0) x^n}{n!} = 1 + \frac{x^2}{2}$$

(b)

$$E = \sqrt{m^2c^4 + \left(\frac{mv}{\sqrt{1-(v/c)^2}}\right)^2 c^2} = \sqrt{m^2c^4 + \frac{m^2v^2c^2}{1-(v/c)^2}}$$

$$= \sqrt{\frac{m^2c^4\left(1-(v/c)^2\right) + m^2v^2c^2}{1-(v/c)^2}}$$

$$= \sqrt{\frac{m^2c^4 - m^2v^2c^2 + m^2v^2c^2}{1-(v/c)^2}}$$

$$= \sqrt{\frac{m^2c^4}{1-(v/c)^2}}$$

$$\implies E = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$

(c) If $v \ll c$ then $v/c \ll 1$, or $v/c \sim 0$, thus let $v/c = x$, so

$$E = \frac{mc^2}{\sqrt{1-x^2}} \approx mc^2\left(1 + \frac{x^2}{2}\right)$$

by using the Taylor series around $x = 0$ from part (a). Hence,

$$E \approx mc^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right) = mc^2 + \frac{mv^2}{2}.$$

If $v = 0$, then $E = mc^2$.

4 Sequences

4.1. Prove that
$$\lim_{n\to\infty}\frac{2n+4}{n}=2$$

Solution:
Given $\epsilon>0$, find $K\in\mathbb{N}$ such that $\left|\frac{2n+4}{n}-2\right|<\epsilon, \forall n\geq K$,
$$\implies \left|\frac{2n+4}{n}-2\right|=\left|\frac{4}{n}\right|=\frac{4}{n}<\epsilon \implies \frac{4}{K}<\epsilon \implies K>\frac{4}{\epsilon}$$

Therefore,
$$\forall\epsilon>0, \exists K\in\mathbb{N}: K>4/\epsilon \implies \forall n\geq K \text{ then } \left|\frac{2n+4}{n}-2\right|=\frac{4}{n}\leq\frac{4}{K}<\epsilon,$$
thus $\lim\limits_{n\to\infty}\frac{2n+4}{n}=2$.

4.2. Let $a_n=\dfrac{3n-8}{4n+1}$

(a) Determine $K\in\mathbb{N}$ such that $|a_n-3/4|<0.01$ for all $n\geq K$

(b) Given $\epsilon>0$, determine $K\in\mathbb{N}$ such that $|a_n-3/4|<\epsilon$ for all $n\geq K$

(c) Prove that $\lim\limits_{n\to\infty}a_n=3/4$

Solution:

(a) If $|a_n-3/4|<0.01$ then,
$$\begin{aligned}\left|\frac{3n-8}{4n+1}-\frac{3}{4}\right|=\left|\frac{4(3n-8)-3(4n+1)}{4(4n+1)}\right|=\left|-\frac{35}{4(4n+1)}\right|=\frac{35}{4(4n+1)}<\frac{35}{4(4n)}=\frac{35}{16n}<0.01\\ \implies \frac{35}{16K}<0.01\\ \implies K>\frac{35}{16(0.01)}=218.75\end{aligned}$$

Therefore, choose $K=219$.

(b) Given $\epsilon>0$ and $|a_n-3/4|<\epsilon$, then
$$\begin{aligned}\left|\frac{3n-8}{4n+1}-\frac{3}{4}\right|=\left|\frac{4(3n-8)-3(4n+1)}{4(4n+1)}\right|=\left|-\frac{35}{4(4n+1)}\right|=\frac{35}{4(4n+1)}<\frac{35}{4(4n)}=\frac{35}{16n}<\epsilon\\ \implies \frac{35}{16K}<\epsilon\\ \implies K>\frac{35}{16\epsilon}\end{aligned}$$

Therefore, choose $K>35/(16\epsilon)$.

(c) $\forall \epsilon > 0, \exists K \in \mathbb{N} : K > 35/(16\epsilon)$

$$\implies \forall n \geq K, \left|\frac{3n-8}{4n+1} - \frac{3}{4}\right| = \frac{35}{4(4n+1)} < \frac{35}{16n} \leq \frac{35}{16K} < \epsilon,$$

hence $\lim_{n\to\infty} a_n = 3/4$.

4.3. If $a_n = (4n+1)/n$, calculate a_{10}, a_{100} and a_{1000} and make a guess for the limit L as $n \to \infty$. Prove that a_n tends to this limit.

Solution:
By calculating a_{10}, a_{100} and a_{1000},

$$a_{10} = \frac{4(10)+1}{10} = \frac{41}{10} = 4.1$$
$$a_{100} = \frac{4(100)+1}{100} = \frac{401}{100} = 4.01$$
$$a_{1000} = \frac{4(1000)+1}{1000} = \frac{4001}{1000} = 4.001$$

then a reasonable guess for L would be 4.
If given $\epsilon > 0$ and let $|a_n - 4| < \epsilon$, then

$$\implies \left|\frac{4n+1}{n} - 4\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon \implies K > \frac{1}{\epsilon}.$$

Hence,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : K > 1/\epsilon \implies \left|\frac{4n+1}{n} - 4\right| = \frac{1}{n} \leq \frac{1}{K} < \epsilon,$$

so $\lim_{n\to\infty} a_n = 4$.

4.4. Prove the following limits:

(a) $a_n = \cos(an)/n \to 0$ as $n \to \infty$ where $a \in \mathbb{R}$

(b) $b_n = \dfrac{n^2+1}{n^2-1} \to 1$ as $n \to \infty$

(c) $c_n = \left(\sqrt{n+1} - \sqrt{n}\right) \to 0$ as $n \to \infty$

Solution:

(a) Given $\epsilon > 0$,

$$\left|\frac{\cos(an)}{n} - 0\right| = \left|\frac{\cos(an)}{n}\right| \leq \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon \implies K > \frac{1}{\epsilon}.$$

Thus,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : K > 1/\epsilon \implies \left|\frac{\cos(an)}{n} - 0\right| \leq \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{K} < \epsilon,$$

so $\lim_{n\to\infty} a_n = 0$.

(b) Given $\epsilon > 0$,

$$\left|\frac{n^2+1}{n^2-1} - 1\right| = \left|\frac{2}{n^2-1}\right| = \frac{2}{n^2-1} < \epsilon \implies \frac{2}{K^2-1} \implies K > \sqrt{1+\frac{2}{\epsilon}},$$

therefore,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : K > \sqrt{1+\frac{2}{\epsilon}} \implies \left|\frac{n^2+1}{n^2-1} - 1\right| = \frac{2}{n^2-1} \leq \frac{2}{K^2-1} < \epsilon,$$

so $\lim_{n\to\infty} b_n = 1$.

(c) Given $\epsilon > 0$,

$$\begin{aligned}\left|\sqrt{n+1} - \sqrt{n}\right| = \left|\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \epsilon \\ \implies \frac{1}{2\sqrt{K}} < \epsilon \\ \implies K > \frac{1}{4\epsilon^2}\end{aligned}$$

Thus,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : K > \frac{1}{4\epsilon^2} \implies \left|\sqrt{n+1} - \sqrt{n} - 0\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{K}} < \epsilon,$$

so $\lim_{n\to\infty} c_n = 0$.

4.5. Determine if the following limits converge, calculate the limits if they do.

(i) $(a_n)_{n\geq 1} = \cos(n\pi)$ (iii) $(c_n)_{n\geq 2} = \frac{n-1}{n} - \frac{n}{n-1}$ (v) $(e_n)_{n\geq 1} = \frac{\cos(n)}{1+n}$ (vii) $(g_n)_{n\geq 1} = \left(1+\frac{1}{n}\right)^n$

(ii) $(b_n)_{n\geq 1} = \sin(n\pi)$ (iv) $(d_n)_{n\geq 1} = \frac{1+\sin(n)}{n}$ (vi) $(f_n)_{n\geq 1} = \frac{n^2}{n!}$ (viii) $(h_n)_{n\geq 1} = ne^{-\frac{n}{2}}$

Solution:

(i) $a_n = \cos(n\pi) = (-1)^n$ which is a divergent sequence, so a_n does not converge.

(ii) $b_n = \sin(n\pi) = 0$ for all n, so b_n converges to 0.

(iii) Consider,

$$\lim_{n\to\infty} \frac{n-1}{n} - \frac{n}{n-1} = \lim_{n\to\infty} \frac{(n-1)^2 - n^2}{n(n-1)} = \lim_{n\to\infty} \frac{1-2n}{n(n-1)} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right) - \left(\frac{2}{n}\right)}{1 - \left(\frac{1}{n}\right)} = \frac{\lim_{n\to\infty} \left(\frac{1}{n^2} - \frac{2}{n}\right)}{\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)} = 0,$$

by using the limit laws as both these limits exist and $1 - 1/n \neq 0$ hence $\lim_{n\to\infty} c_n = 0$.

(iv) As $-1 \leq \sin(n) \leq 1 \implies 0 \leq \frac{1+\sin(n)}{n} \leq \frac{2}{n}$, and as $\lim_{n\to\infty} \frac{2}{n} = 0$ then by the pinching theorem $\lim_{n\to\infty} d_n = 0$.

(v) As $-1 \leq \cos(n) \leq 1 \implies -\frac{1}{n+1} \leq \frac{\cos(n)}{n+1} \leq \frac{1}{n+1}$, and as $\lim_{n\to\infty} \frac{1}{n+1} = 0$ then by the pinching theorem $\lim_{n\to\infty} e_n = 0$.

(vi) There are a couple of ways to do this:

(1)

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{n!} = \lim_{n\to\infty} \frac{n}{(n-1)!} = \lim_{n\to\infty} \frac{n}{(n-2)!(n-1)} &= \lim_{n\to\infty} \left(\frac{1}{(n-2)!} \frac{n}{n-1} \right) \\
&= \left(\lim_{n\to\infty} \frac{1}{(n-2)!} \right) \left(\lim_{n\to\infty} \frac{n}{n-1} \right) \\
&= \left(\lim_{n\to\infty} \frac{1}{(n-2)!} \right) \left(\lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} \right) \\
&= \left(\lim_{n\to\infty} \frac{1}{(n-2)!} \right) \left(\frac{1}{\lim\limits_{n\to\infty} \left(1-\frac{1}{n}\right)} \right) \\
&= (0)(1) \\
\implies \lim_{n\to\infty} \frac{n^2}{n!} &= 0
\end{aligned}$$

(2) Using the Ratio Test for sequences,

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{(n+1)^2}{(n+1)!}\right)}{\left(\frac{n^2}{n!}\right)} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 n!}{n^2 (n+1)!} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2(n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| \\
&= \lim_{n\to\infty} \frac{n+1}{n^2} \\
&= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n} \right) + \lim_{n\to\infty} \left(\frac{1}{n^2} \right) \\
&= 0 + 0 \\
\implies \lim_{n\to\infty} \left| \frac{f_{n+1}}{f_n} \right| &= 0 < 1
\end{aligned}$$

Therefore, by the Ratio Test, $\lim\limits_{n\to\infty} f_n = 0$.

(vii) Obviously, by definition, $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \sim 2.718$, but why do we know it converges? First, consider

$$\begin{aligned}
\frac{g_{n+1}}{g_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} &= \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^n \\
&= \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(\frac{n^2+2n+1-1}{n^2+2n+1}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(1-\frac{1}{(n+1)^2}\right)^{n+1} \left(\frac{n+1}{n}\right)
\end{aligned}$$

Using Bernoulli's inequality: $(1+x)^r \geq 1+rx$ for all $r \in \mathbb{N}_0$ if $x > -1$, then

$$\frac{g_{n+1}}{g_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{n+1}{n}\right) \geq \left(1 - \frac{n+1}{(n+1)^2}\right)\left(\frac{n+1}{n}\right) = \frac{n+1}{n} - \frac{1}{n} = \frac{n}{n} = 1$$

$$\implies \frac{g_{n+1}}{g_n} \geq 1$$

$$\implies g_{n+1} \geq g_n \text{ i.e. monotonic increasing.}$$

Also,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\left(\frac{1}{n}\right)^k &= \sum_{k=0}^{n} \frac{n(n-1)...(n-k+1)}{k!}\left(\frac{1}{n}\right)^k \\
&= \sum_{k=0}^{n} \frac{1}{k!n^k}\prod_{m=0}^{k-1}(n-m) \\
&= \sum_{k=0}^{n} \frac{1}{k!}\prod_{m=0}^{k-1}\frac{n-m}{n} \\
&= \sum_{k=0}^{n} \frac{1}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right) < \sum_{k=0}^{n}\frac{1}{k!}
\end{aligned}$$

as $\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right) < 1$ for all $m < k-1 < n$. Hence,

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!}\prod_{m=0}^{k-1}\left(1-\frac{m}{n}\right) < \sum_{k=0}^{n}\frac{1}{k!} &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... + \frac{1}{n!} \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{(3)(2)} + \frac{1}{(4)(3)(2)} + ... + \frac{1}{n(n-1)...(2)} \\
&< 2 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n-1}} = 2 + \sum_{k=1}^{n-1}\frac{1}{2^k}
\end{aligned}$$

Consider,

$$s_n = \sum_{k=1}^{n}\frac{1}{2^k} \implies 2s_n = \sum_{k=1}^{n}\frac{1}{2^{k-1}} = 1 + \sum_{k=1}^{n-1}\frac{1}{2^k} = 1 + s_{n-1} = 1 + \left(s_n - \frac{1}{2^n}\right)$$

$$\implies 2s_n = 1 + s_n - \frac{1}{2^n}$$

$$\implies s_n = 1 - \frac{1}{2^n}$$

therefore,

$$\left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n}\frac{1}{k!} < 2 + \sum_{k=1}^{n-1}\frac{1}{2^k} = 2 + \left(1 - \frac{1}{2^{n-1}}\right) = 3 - \frac{1}{2^{n-1}} < 3 \implies \left(1+\frac{1}{n}\right)^n < 3$$

i.e. bounded from above by 3. Therefore, by the Monotone Convergence Theorem g_n must converge to it's supremum.

(viii) Again, there are a couple of ways to do this,

(1) It can be shown that h_n is bounded below with an infimum of 0 (why?) and that it decreases monotonically (show this via induction), thus by the Monotone Convergence Theorem, it converges to 0.

(2) Consider

$$\lim_{n\to\infty}\left|\frac{h_{n+1}}{h_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)e^{-\frac{n+1}{2}}}{ne^{-\frac{n}{2}}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| e^{\frac{n}{2}}e^{-\frac{n+1}{2}} = \lim_{n\to\infty}\frac{n+1}{n}e^{\frac{n}{2}-\frac{n+1}{2}}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{2}}$$

$$= e^{-\frac{1}{2}}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$\implies \lim_{n\to\infty}\left|\frac{h_{n+1}}{h_n}\right| = e^{-\frac{1}{2}} < 1$$

hence by the Ratio Test, h_n converges with $\lim_{n\to\infty} h_n = 0$.

4.6. Consider the sequence

$$A_n = P\left(1+\frac{r}{12}\right)^n$$

where A_n is the balance of a savings account after n months with a principal sum, P, and annual interest rate r.

(a) Calculate the first ten terms of A_n if $P = £10,000$ and $r = 0.055$

(b) Does A_n converge? Explain your answer

Solution:

(a)

$A_1 = £10000\left(1+\frac{0.055}{12}\right)^1 \approx £10045.83$, $A_2 \approx £10091.88$, $A_3 \approx £10138.13$, $A_4 \approx £10184.60$, $A_5 \approx £10231.28$, $A_6 \approx £10278.17$, $A_7 \approx £10325.28$, $A_8 \approx £10372.60$, $A_9 \approx £10420.14$, $A_{10} \approx £10467.90$

(b) Let $\beta = 1 + r/12 > 1$ if $r > 0$ so $A_n = P\beta^n \implies \lim_{n\to\infty} A_n = P\lim_{n\to\infty}\beta^n$.
A sequence is bounded if $\exists M \in \mathbb{R}, N \in \mathbb{N} \implies \forall n \geq N, |a_n| < M$, thus for an unbounded sequence, $\forall M > 0, \exists N \in \mathbb{N} \implies \forall n \geq N, |A_n| > M$. Consider,

$$|A_n| = |P\beta^n| = P\beta^n > M \implies \beta^n > \frac{M}{P}$$

$$\implies n\ln(\beta) > \ln\left(\frac{M}{P}\right)$$

$$\implies n > \frac{\ln(M/P)}{\ln(\beta)}$$

hence $\forall M > 0, \exists N \in \mathbb{N} : N > \ln(M/P)/\ln(\beta) \implies \forall n \geq N, |A_n| > M$ i.e. unbounded. As all convergent sequences are bounded then A_n must diverge.

4.7. Prove the following:

(i) $\forall \epsilon > 0, \exists K_1, K_2 \in \mathbb{N}:$

(a) $\forall n \geq K_1 \implies \left|\frac{\cos(n)}{1+n}\right| < \epsilon$ (b) $\forall n \geq K_2 \implies \left|\frac{1+\sin(n)}{n}\right| < \epsilon$

(ii) Consider $(b_n)_{n\geq 1}$, $K > 0$ and $\lim_{n\to\infty} a_n = 0$. If $a_n > 0, \forall n \in \mathbb{N}$, and $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|b_n - L| \leq K a_n$, then $\lim_{n\to\infty} b_n = L$.

(iii) (Ratio Test for sequences): If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\lim_{n\to\infty} a_n = 0$.
(*Hint: Let* $\epsilon = r - L : r \in (L, 1)$)

Solution:

(i) (a) If $\left|\frac{\cos(n)}{1+n}\right| < \epsilon$, then

$$\left|\frac{\cos(n)}{1+n}\right| \leq \left|\frac{1}{1+n}\right| = \frac{1}{1+n} < \frac{1}{n} < \epsilon \implies K_1 > \frac{1}{\epsilon}$$

Therefore, $\forall \epsilon > 0, \exists K_1 \in \mathbb{N} : K_1 > 1/\epsilon \implies \forall n \geq K_1, \left|\frac{\cos(n)}{1+n}\right| < 1/n \leq 1/K_1 < \epsilon.$

□

(b) If $\left|\frac{1+\sin(n)}{n}\right| < \epsilon$, then

$$\left|\frac{1+\sin(n)}{n}\right| \leq \left|\frac{2}{n}\right| = \frac{2}{n} < \epsilon \implies K_2 > \frac{2}{\epsilon}$$

Therefore, $\forall \epsilon > 0, \exists K_2 \in \mathbb{N} : K_2 > 1/\epsilon \implies \forall n \geq K_2, \left|\frac{1+\sin(n)}{n}\right| < 2/n \leq 2/K_2 < \epsilon.$

□

(ii) As $\lim_{n\to\infty} a_n = 0$, then

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N \implies |a_n - 0| = |a_n| = a_n < \epsilon,$$

hence let $\epsilon_0 = \epsilon/K$, therefore $\exists N_2 \in \mathbb{N} : \forall n \geq N_2 \implies a_n < \epsilon_0 = \epsilon/K$, which implies that $K a_n < \epsilon$ for all $n \geq N_2$. Thus,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : N = \max\{N_1, N_2\} \implies \forall n \geq N, |b_n - L| \leq K a_n < \epsilon \implies \lim_{n\to\infty} b_n = 0.$$

□

(iii) As $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then $\forall\epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N \implies \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon$. Let $r \in (L, 1)$, so for $\epsilon = r - L > 0$,

$$\begin{aligned}
\exists N : \forall n \geq N &\implies \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon = r - L \\
&\implies -(r-L) < \left|\frac{a_{n+1}}{a_n}\right| - L < r - L \\
&\implies \left|\frac{a_{n+1}}{a_n}\right| < r \\
&\implies |a_{n+1}| < |a_n| r \quad \forall n \geq N \\
&\implies |a_{n+1}| < |a_n| r < |a_{n-1}| r^2 < ... < |a_{N-1}| r^{n-N} < |a_N| r^{n-N+1}
\end{aligned}$$

Letting $K = |a_N|/r^N$, then $|a_N| r^{n-N+1} = K r^{n+1}$, thus $\forall n \geq N, |a_n| r < K r^{n+1}$ so $|a_n| < K r^n$. As $\lim_{n\to\infty} r^n = 0$ as $r \in (0,1)$, then by the theorem in part (ii), $\lim_{n\to\infty} a_n = 0$.

$\square$

4.8. Consider the sequence $(a_n)_{n\geq 1} = (-1)^n$.

(a) Is a_n bounded? Is it monotonic? Explain your answers.

(b) Show that a_n diverges.

(c) Does there exist a convergent subsequence $(b_k)_{k\geq 1} = a_{n_k}$? If so, explain why and give an example of such a subsequence.

(d) Show that your chosen b_k is a Cauchy sequence. Why is a_n not a Cauchy sequence?

Solution:

(a) If n is even then $a_n = 1$, and if n is odd then $a_n = -1$, hence a_n is bounded. Consider $a_1 = -1 > a_2 = 1$ but $a_3 = -1 < a_2$, thus a_n is not monotonic.

(b) Let $\epsilon = 1/2$, then if a_n is convergent then $\exists N \in \mathbb{N} : \forall n \geq N \implies |a_n - L| < 1/2$. Since $\forall n \geq N$, $|a_n - L| < 1/2$ so if n is odd then,

$$|a_n - L| = |-1 - L| < 1/2 \implies -1/2 < -1 - L < 1/2 \implies -3/2 < L < -1/2$$

And if n is even then,

$$|a_n - L| = |1 - L| < 1/2 \implies -1/2 < 1 - L < 1/2 \implies 1/2 < L < 3/2$$

However this suggests that $L \in (1/2, 3/2)$ and $L \in (-3/2, -1/2)$ i.e. a contradiction, therefore a_n diverges.

(c) Yes as a_n is bounded then by the Bolzano-Weierstrass Theorem there exists a convergent subsequence. For example, $n_k = 2k$ (i.e. even terms), so $b_k = (-1)^{2k} = 1$.

(d) Recall the definition of a Cauchy sequence:

$$\forall\epsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N \implies |a_n - a_m| < \epsilon$$

If $b_k = (-1)^{2k} = 1$,

$$\forall\epsilon > 0, \exists N \in \mathbb{N} : N \geq 1 \implies \forall n, m \geq N, |b_n - b_m| = |1 - 1| = 0 < \epsilon$$

Therefore b_k is a Cauchy sequence. a_n is not a Cauchy sequence because it diverges and all Cauchy sequences converge.

4.9. Consider $x^3 = 3$.

(a) Write down Newton's method to solve this equation.

(b) By choosing $x_0 = 3$, prove by induction that the resulting sequence is monotonic decreasing and bounded below.

(c) Hence, why does the sequence converge, and find the limit.

Solution:

(a) Let $f(x) = x^3 - 3 = 0$, then

$$x_{n+1} = x_n - \frac{x_n^3 - 3}{3x_n^2} = \frac{3x_n^3 - x_n^3 + 3}{3x_n^2} = \frac{2x_n}{3} + \frac{1}{x_n^2}$$

(b) As x_{n+1} are additions of previous terms then a reasonable lower bound would be 0, hence want to prove: $0 < x_{n+1} < x_n$. Using induction then,

(1) Prove Base case: $n = 0$

$$x_1 = \frac{2x_0}{3} + \frac{3}{x_0^2} = \frac{2(3)}{3} + \frac{3}{3^2} = 2 + \frac{1}{9} < 3 = x_0 > 0\checkmark$$

(2) Assume $n = k$ is true, i.e.

$$0 < x_{k+1} < x_k$$

(3) Prove $n = k+1$ is true, i.e. $0 < x_{k+2} < x_{k+1}$

$$x_{k+2} = \frac{2x_{k+1}}{3} + \frac{3}{x_{k+1}^2} < \frac{2x_k}{3} + \frac{3}{x_{k+1}^2} < x_{k+1},$$

as $x_{k+1} = 2x_k/3 + 1/x_k^2 > 2x_k/3 + 1/x_{k+1}^2$. As $x_{k+1} > 0$ then $x_{k+2} > 0$, hence $0 < x_{k+2} < x_{k+1}$ i.e. x_{n+1} is bounded below and monotonic decreases.

$\square$

(c) As x_{n+1} is bounded below and monotonically decreases, then by the Monotone Convergence theorem the sequence converges to it's infimum which is $\sqrt[3]{3}$.

4.10. Let $(a_n)_{n\geq 1}$ be a sequence, and $(a_{n_k})_{k\geq 1}$ be a subsequence. Prove by induction that $n_k \geq k, \forall k \geq 1$.

Solution:

(1) Prove Base case: $k = 1$

$$n_1 \in \mathbb{N} \implies n_1 \geq 1 = k\checkmark$$

(2) Assume $k = \gamma$ is true, i.e.

$$n_\gamma \geq \gamma$$

(3) Prove $k = \gamma + 1$ is true, i.e. $n_{\gamma+1} \geq \gamma + 1$

$$n_{\gamma+1} \geq n_\gamma + 1 \geq \gamma + 1 \implies n_{\gamma+1} \geq \gamma + 1$$

Therefore by induction $n_k \geq k$ for all $k \in \mathbb{N}$.

$\square$

4.11. Show that all bounded divergent sequences possess at least two convergent subsequences.

Solution:
As a_n is bounded then by the Bolzano-Weierstrass Theorem there is a convergent subsequence $b_k = a_{n_k}$ with $\lim_{n\to\infty} b_k = b$.
As a_n is divergent then

$$\exists\epsilon > 0 : \forall N \in \mathbb{N}, \exists n > N : |a_n - b| \geq \epsilon \implies \exists c_m = a_{n_m} : m > N \text{ with } |c_m - b| \geq \epsilon$$

i.e. c_m is a subsequence of a_n that does not converge to b.
As c_m is bounded (as a_n is bounded) then by the Bolzano-Weierstrass Theorem there exists a convergent subsequence c_{m_j} with $\lim_{j\to\infty} c_{m_j} \neq b$.
Therefore, there exists at least two convergent subsequences.

□

5 Workbook

5.1. Solve the following inequalities.

(a) $3x + 5 < \dfrac{4-x}{2}$

(b) $x(x^2 - 3x + 2) \leq 0$

(c) $\dfrac{x^2 - 4x + 4}{x^2 - 2x - 3} \leq 0$

(d) $|3x - 2| \geq 4$

5.2. Prove the following:

(a) $|a - b| \leq |a| + |b|$

(b) $||a| - |b|| \leq |a + b|$

5.3. Show the following functions are one-to-one. Find the inverse function and give both the domain and range of both functions.

(a) $f(x) = \dfrac{x-1}{x-2}$

(b) $g(x) = \dfrac{x}{2x-4} - \dfrac{1}{2}$

(c) $p(x) = x^2 + 1$, $x > 0$

(d) $q(x) = \dfrac{3x-5}{x-2}$

5.4. Prove by induction:

(a) $1 + 2n \leq 3^n$

(b) $n! \geq 2^{n-1}$

(c) $\sum_{l=1}^{n} 2l + 1 = n(n+2)$

5.5. Using the ϵ-δ definition of the limit, show the following,

(a) $f(x) = \sqrt{x-1}$ is continuous at $x = 5$.

(b) $f(x) = |x|$ is continuous on all of $\mathbb{R}$.

(c) $f(x) = \sqrt{x}$ is continuous at $x = c > 0$.

5.6. Find the following limits (ϵ-δ proof not required).

(a) $\displaystyle\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

(b) $\displaystyle\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

(c) $\displaystyle\lim_{x \to 9} \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9}$

(d) $\displaystyle\lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c}$

(e) $\displaystyle\lim_{x \to c} \frac{x^{-\frac{1}{2}} - c^{-\frac{1}{2}}}{x - c}$

(f) $\displaystyle\lim_{h \to 0} \frac{\sin(c+h) - \sin(c)}{h}$

5.7. Prove that $\sin(x) = 1 - x$ has a solution between $x = 0$ and $x = \pi/2$.

5.8. Calculate the derivatives of the following functions from first principles.

(a) $f(x) = \sqrt{x}$

(b) $g(x) = x^{-\frac{1}{2}}$

(c) $p(x) = \sin(x)$

5.9. Use Newton's method to approximate:

(a) $\sqrt{5}$ using $x_0 = 2$ such that $|x_n^2 - 5| < 0.01$

(b) $\sqrt[3]{23}$ using $x_0 = 3$ such that $|x_n^3 - 23| < 10^{-3}$

5.10. Use the Mean Value Theorem to prove that

(a) $\dfrac{x}{1+x} < \ln(1+x) < x$ for all $x > 0$

(b) $\sin(x) < x$ if x is positive

5.11. Determine the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - (1 + x + x^2/2)}{x^3}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{\ln(1+x)}{x}$

5.12. Find the Taylor polynomial at $x = c$ of degree n, and Lagrangian Remainder:

(a) $f(x) = \sqrt{x}$, $c = 4$, $n = 3$

(b) $g(x) = \sin(x)$, $c = \pi/4$, $n = 4$

(c) $h(x) = \tan^{-1}(x)$, $c = 1$, $n = 3$

5.13. Consider $(a_n)_{n\geq 1} = \dfrac{n + (-1)^n}{n}$. Is it

(a) Bounded? (If so, give both an upper and lower bound)

(b) Monotonic? (Justify your answer)

(c) Convergent? (If so give a limit and prove it using an ϵ-K argument)

5.14. Determine if the following limits converge. If they do, find the limit.

(a) $\displaystyle\lim_{n\to\infty} \frac{\sin(n)}{n}$

(b) $\displaystyle\lim_{n\to\infty} \tan(n\pi)$

(c) $\displaystyle\lim_{n\to\infty} \frac{n\cos(1+n^2)}{1+n^2}$

(d) $\displaystyle\lim_{n\to\infty} \frac{2}{n+1} \tan(3n)$

(e) $\displaystyle\lim_{n\to\infty} \frac{\cos(n^2)}{1+n}$

(f) $\displaystyle\lim_{n\to\infty} \left(\frac{n + 10^6}{n^2} + \frac{\cos^2(3n^2 - 4)}{n} \right)$

5.15. Prove the following:

(a) Prove every Cauchy sequence is bounded, but the converse is false.

(b) If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences, then $(a_n - b_n)_{n\geq 1}$ and $(a_n b_n)_{n\geq 1}$ are Cauchy.

(c) Consider subsequences $(b_k)_{k\geq 1}$ and $(c_k)_{k\geq 1}$ of any sequence $(a_n)_{n\geq 1}$ where $b_k = a_{2k}$ and $c_k = a_{2k+1}$. Then

$$a_n \text{ is convergent } \iff b_k \,\&\, c_k \text{ converge to the same limit}$$