Odinary Differential Equations

. Explain the difference between classes of differential equations.

. Analyse initial value problems in order to determine whether or not they have unique solutions.

. State, explain, and prove basic existence and uniqueness theorems.

. Use and apply methods for finding general solutions of ordinary differential equations.

. Apply and write programs for finding numerical solutions of ordinary differential equations.

Odinary Differential Equations

Chapter 1. First order equations: some integrable cases

Chapter 2. Theory of first order differential equations

Chapter 3. First order systems.

Chapter 4. linear differential equations of order n

Chapter 5. Stability and asymptotic behavior

Chapter 1 First Order Equations: Some Integrable Cases

1. Ordinary differential equation and solution

Implicit first order differential equation

$$F(x, y, y') = 0. \qquad (1)$$

Explicit first order differential equation

$$y' = f(x, y).$$

A function $y(x): J \to R$ *is called a solution to the differential equation* (1)

If y *is differential in* J *and* (1) *holds. Where* J *is an interval.*

General solutions ⟹ *Families of integral curves*

Special solutions ⟹ *Families of integral curves*

2.Equations with separated variables. $\boxed{\boldsymbol{y' = f(x)g(y)}}$ $\longleftrightarrow$ $\frac{dy}{dx} = f(x)g(y)$,

$Case\ 1.\ g(y) \neq 0, integration\ to\ the\ \ equation \int \frac{dy}{g(y)} = \int f(x)dx$,

general solutions can be obtained by solving for $y = y(x, C)$.

This is accomplished under the general hypothesis:

(H) $f(x) is\ continuous\ in\ J_x = (a, b); g(y) is\ continuous\ in\ J_y = (\alpha, \beta)$.

$Case\ 2.\ If\ \ g(y_0) = 0, y_0 \in (\alpha, \beta), then\ one\ solution\ can\ be\ given: y(x) = y_0$.

Examples. *Determine all of the solutions to the following differential equations.*

1. $\dfrac{dy}{dx} = \dfrac{y}{x}$. 2. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2}$, *with initial condition* $y(0) = 0$.

Exercise.

1. $x(y^2 - 1)dx + y(x^2 - 1)dy = 0$.

3.***Homogeneous differential equation.*** $\boxed{\boldsymbol{y'(x) = g\left(\frac{y}{x}\right)}}$

$Using\ the\ ansatz\ u = \dfrac{y(x)}{x}\ (x \neq 0) and\ calculating\ the\ derivative, one\ obtains$

$the\ relation\ y' = u + xu' = g(u), and\ thus\ a\ differential\ equation\ for\ u(x) with$

$separated\ variables, u' = \dfrac{g(u) - u}{x}.$

$If\ \ g(u) - u \neq 0, then\ general\ solutions\ can\ be\ given\ x = Ce^{\phi\left(\frac{y}{x}\right)},$

$where\ \phi(u) = \displaystyle\int \frac{du}{g(u) - u}.$

$If\ \ g(u_0) - u_0 = 0\big(in\ the\ case\ where\ u_0 \in J_y\big), then\ one\ solution\ is\ y = u_0 x.$

$$\boldsymbol{y'(x) = g\left(\frac{y}{x}\right)}$$

$If\ g(u) - u \neq 0, then\ general\ solutions\ can\ be\ given\ x = Ce^{\phi\left(\frac{y}{x}\right)},$

$If\ g(u_0) - u_0 = 0\left(in\ the\ case\ where\ u_0 \in J_y\right), then\ one\ solution\ is\ y = u_0 x.$

$Example.$

1. The initial value problem $y' = \frac{y}{x} - \frac{x^2}{y^2}, y(1) = 1.$

$$2. x^2 \frac{dy}{dx} = xy - y^2.$$

$$\boxed{y' = f\left(\frac{ax + by + c}{\alpha x + \beta y + \gamma}\right)}$$

$In\ the\ case\ where\ the\ determinant\ \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0, that\ is\ , where\ a = \lambda\alpha\ and\ b = \lambda\beta,$

$the\ equation\ can\ be\ reduced\ to\ one\ of\ the\ types\ we\ have\ already\ considered.$

$If\ this\ determinant\ is\ not\ zero, then\ the\ linear\ system\ of\ equations$

$\begin{cases} ax + by + c = 0, \\ \alpha x + \beta y + \gamma = 0, \end{cases}$ $has\ a\ unique\ solution\ (x_0, y_0). If\ a\ new\ system\ coordinates\ (\xi, \eta)$

$is\ introduced\ by\ translating\ the\ origin\ to\ the\ point\ (x_0, y_0),$

$\xi = x - x_0, \eta = y - y_0$, *in the new coordinate system a solution curve* $y(x)$ *is described*

$\eta = y(\xi + x_0) - y_0$.

The differential equation in the (ξ, η) *coordinate system*

$$\frac{dy}{dx} = \frac{dy}{d\eta}\frac{d\eta}{d\xi}\frac{d\xi}{dx} = \frac{d\eta}{d\xi} = y'(\xi + x_0) = f\left(\frac{a(\xi + x_0) + b(\eta + y_0) + c}{\alpha(\xi + x_0) + \beta(\eta + y_0) + \gamma}\right) = f\left(\frac{a\xi + b\eta}{\alpha\xi + \beta\eta}\right),$$

is just the special case $c = \gamma = 0$ *of the original equation.*

How to proceed.

1. *Determine the point* (x_0, y_0) *that satisfies* $\begin{cases} ax + by + c = 0, \\ \alpha x + \beta y + \gamma = 0, \end{cases}$.
2. *Solve the differential equation with* $c = \gamma = 0$ *using techniques form above.*
3. *A solution* $\eta(\xi)$ *of this equation generates a solution to the original equation using the substitution* $\xi = x - x_0, \eta = y - y_0$, *that is,* $y(x) = y_0 + \eta(x - x_0)$.

Example.

1. $\frac{dy}{dx} = \frac{x-y+1}{x+y-3}$.

2. $(2x + y + 1)dx - (4x + 2y - 3)dy = 0$.

Exercises.

1. $(y^2 - 2xy)dx + x^2 dy = 0$.

2. $(2x^2 + 3y^2 - 7)xdx - (3x^2 + 2y^2 - 8)ydy = 0$.

4. The linear differential equation. Related equations

$A\ first\ order\ linear\ diffeential\ equation\ is\ an\ equation\ of\ the\ form$

$$\boxed{y' + p(x)y = f(x);} \qquad (1)$$

$we\ assume\ that\ the\ two\ given\ functions\ p(x)\ and\ f(x) are\ continuous\ on\ an\ interval\ J.$

$If\ \boldsymbol{f(x) \equiv 0}, then\ equation\ (1) is\ called\ \boldsymbol{homogeneous}, \boldsymbol{otherwise\ nonhomogeneous}.$

$The\ homogeneous\ equation.\ y' + p(x)y = 0 \qquad (2)$

$This\ is\ an\ equation\ with\ separated\ variables. We\ obtain\ the\ family\ of\ solutions$

$$\boxed{y = Ce^{-\int p(x)dx}.} \qquad (3)$$

The nonhomogeneous equation.

$$\boxed{y' + p(x)y = f(x).} \qquad (4)$$

Solutions to the nonhomogeneous equation can be obtained with the help of **the method of variation of constants**. In the method, the constant C is replaced by a function C(x). The Calculation of an appropriate choice of C(x) gives a solution of the nonhomogeneous equation. Indeed, the ansatz

$$y(x) = C(x)e^{-\int p(x)dx}$$

Leads to $C'(x)e^{-\int p(x)dx} = f(x), or\ equivalently, C(x) = \int f(x)e^{\int p(x)dx}dx + C.$

Example.

1. $y' = \frac{y}{x} + x^2.$

Exercise.

1. $y' - \cot x\, y = 2x\sin x.$

$Example. \ y' = \frac{y}{x} + x^2.$

$Solution.$ $Solve\ the\ realted\ homogeneous\ ODE\ y' = \frac{y}{2x}, we\ have\ y = Cx.$

$Setting\ \ y = c(x)x\ is\ the\ solution\ of\ the\ original\ ODE, c(x) = \frac{x^2}{2} + c\ can\ be\ obtained.$

$y = \left(\frac{x^2}{2} + c\right)x\ is\ the\ general\ solution\ .$

Remark. $If\ y, y_1\ are\ two\ solutions\ to\ the\ nonhomogeneous\ equation, then\ z = y - y_1$ $is\ a\ solution\ of\ the\ homogeneous\ equation.$ $Thus\ all\ solutions\ y(x)\ of\ the\ nonhomogeneous\ equation\ can\ be\ written\ in\ the\ form$

$$y(x) = y_1(x) + z(x) \qquad (5).$$

$where\ y_1(x)\ is\ a\ fixed\ solution\ of\ the\ nonhomogeneous\ equation\ and\ z(x)\ runs\ through\ all\ solutions\ of\ the\ homogenesou\ equation.$

$For\ initial\ problem \begin{cases} y' + p(x)y = f(x) \\ y(x_0) = y_0 \end{cases},\ the\ ansatz\ y(x) = C(x)e^{-\int_{x_0}^{x} p(\tau)d\tau}\ leads\ to$

$C(x) = \int_{x_0}^{x} f(s)e^{\int_{x_0}^{s} p(\tau)d\tau} ds + C.\ Then\ we\ substitude\ C(x)\ and\ initial\ conditon\ into$

$y(x) = C(x)e^{-\int_{x_0}^{x} p(\tau)d\tau}.\ It\ follows\ that$

$$y = y_0 e^{-\int_{x_0}^{x} p(\tau)d\tau} + \int_{x_0}^{x} f(s)e^{\int_{x_0}^{s} p(\tau)d\tau} ds.$$

Example.

$$y = y_0 e^{-\int_{x_0}^{x} p(\tau)d\tau} + \int_{x_0}^{x} f(s) e^{\int_{x_0}^{s} p(\tau)d\tau} ds.$$

$y' + y\sin x = sin^3\, x$.

$Hence\ z(x) = Ce^{\cos x}\ is\ the\ general\ solution\ of\ the\ homogeneous\ equation\ and$

$$y_1(x) = \int_0^x \sin^3 t e^{\cos x - \cos t} dt = \sin^2 x - 2\cos x - 2 + 4e^{\cos x - 1}\ \ is\ a\ solution\ to\ the$$

$nonhomogeneous\ equation. Then\ the\ general\ solution\ of\ the\ nonhomogeneous\ equation$

$is\ given\ by\ y(x) = \sin^2 x - 2\cos x - 2 + Ce^{\cos x}.$

Bernoulli's equation. $\boxed{\boldsymbol{y' + p(x)y + f(x)y^{\alpha} = 0, \alpha \neq 0, 1}}$.

This differential equation can be transformed into a linear differential equation. Let us assume that the functions g, h *are continuous in* J *and that* $y > 0$. *If the equation is multiplied by* $(1-\alpha)y^{-\alpha}$ *and the relation* $(1-\alpha)y^{-\alpha}y' = (y^{1-\alpha})'$ *is used, then one obtains a linear differential equation,*

$z' + (1-\alpha)g(x)z + (1-\alpha)h(x) = 0$, *where the function* $z = y^{1-\alpha}$.

Example. $y' = \dfrac{y}{2x} + \dfrac{x^2}{2y}$.

Exercise. $y' = y + xy^5$.

$Example.\, y' = \dfrac{y}{2x} + \dfrac{x^2}{2y}.$

$Solution.\, set\ y^2 = z, the\ ODE\ transforms\ into\ z' - \dfrac{z}{x} = x^2.$

$use\ the\ method\ of\ variation\ of\ constants\ \ to\ solve\ the\ nonhomogeneous\ ODE,$

$we\ have\ z = \left(\dfrac{x^2}{2} + c\right)x, so\ the\ general\ solution\ of\ the\ original\ ODE\ is$

$$y = \pm\sqrt{\left(\frac{x^2}{2} + c\right)x}.$$

5. Exact differential equations. $\boxed{M(x,y)dx + N(x,y)dy = 0}$. $\quad (1)$

$A\ differential\ equation\ of\ the\ form\ (1) is\ called\ an\ exact\ equation.\ If\ there\ exists\ a$

$function\ U(x,y) \in C^1\ such\ that\ U_x(x,y) = M(x,y), U_y(x,y) = (x,y).$

$The\ function\ U\ is\ called\ a\ potential\ function.$

$\boldsymbol{Example}.$

$xdx + ydy = 0\ is\ an\ exact\ equation, and\ U(x,y) = \frac{1}{2}(x^2 + y^2) is\ a\ potential\ function.$

Theorem on potential functions.

If $M(x,y), N(x,y)$ *are continuously differentiable*

in the domain $D: |x-x_0| \le a, |y-y_0| \le b,$ *then there exists a potential function*

$U(x,y)$ *satisfying* $U_x(x,y) = M(x,y), U_y(x,y) = N(x,y)$ *if and only if* $M_y \equiv N_x$ *in* D.

The potential function is obtained as a line integral

$$U(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C.$$

$Example.\,(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0.$

$M(x, y)$ and $N(x, y)$ are continuous differentiable on the xOy − plane. Let's set $x_0 = 0,$

$y_0 = 0.$ Then potential functions are given by $\displaystyle\int_0^x (3x^2 + 6xy^2)dx + \int_0^y 4y^3 dy = C\,.$

$$Example.\begin{cases} xydx + \frac{1}{2}(x^2 + y)dy = 0 \\ y(0) = 2 \end{cases}.$$

$$\boxed{U(x, y) = \int_{x_0}^{x} M(x, y)dx + \int_{y_0}^{y} N(x_0, y)dy = C.}$$

$Is - ydx + xdy = 0\ an\ exact\ differential\ equation?$

$$Is\ \frac{(-ydx + xdy)1}{x^2} = 0\ \ an\ exact\ differential\ equation?$$

2. *Integrating Factors*.

The differential equation $ydx + 2xdy = 0$ *is not exact. However, it can easily be made an exact differential equation (in the domain* $x > 0$*) by multiplying the equation* $\frac{1}{\sqrt{x}}$. *The resulting differential equation* $\frac{y}{\sqrt{x}}dx + 2\sqrt{x}dy = 0$ *is exact, and a potential function is given by* $F(x, y) = 2y\sqrt{x}\ (x > 0)$.

Definition. *If the funtions* $M(x, y), N(x, y)$ *are continuous in* D, *then a continuous function* $\mu(x, y) \neq 0$ *defined in* D *is called an integrating factor or Euler multiplier for the differential equation* (1) *if the differential equation*

$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$ *is exact.*

If $\mu(x, y)$ *is a integrating factor,* $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ *is necessary.*

Integrating factor depending on only one variable.

(1)An *integrating factor can be found that* ***depends only on x*** *if* and only if

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \text{ depends only on } x. \text{ Thus the integrating factor is given by}$$

$$\mu(x)=e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx}.$$

(2)An *integrating factor can be found that* ***depends only on y*** *if* and only if

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \text{ depends only on } y. \text{ Thus the integrating factor is given by}$$

$$\mu(y)=e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dy}.$$

Example.

1. $-ydx + xdy = 0$

2. $(3x + 6xy + 3y^2)dx + (2x^2 + 3xy)dy = 0.$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ depends only on } x \Longrightarrow \quad \mu(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}.$$

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ depends only on } y \Longrightarrow \quad \mu(y) = e^{-\int \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dy}.$$

Exercises.

1. $e^{-y}dx - (2y + xe^{-y})dy = 0$

2. $(x^2 + y^2 + x)dx + xydy = 0.$

$\boldsymbol{Implicit\ first\ order\ differential\ equation}$ $\boxed{\boldsymbol{F(x,y,y')=0}}$

Throughout this section, we assume that the Function $F(x,y,p)$ is continuous in a domain.

Case 1. *If we can get the case of explicit differential equations* $y' = f_i(x,y)(i = 1,2,\dots,n)$, *integrable methods can be used to solve these explicit differential equation.*

Example. $y'^2 - (x+y)y' + xy = 0.$

From the equation, we obtain $(y'-x)(y'-y) = 0.$

From the equations $y' = x$ *and* $y' = y$, *we can get the solutions* $y = \dfrac{1}{2}x^2 + C$ *and* $y = Ce^x$.

Case 2. *Parametric Representation.*

In this section, we will discuss two kinds of implicit differential equations will could be solved by ansatze.

$I.\ F(x,y') = 0 \quad (F(y,y') = 0).$

$II.\ y = f(x,y') \quad \big(x = f(y,y')\big).$

$\boldsymbol{I.\ F(x,y')=0.}$ **(1)**

$We\ can\ use\ the\ parametric\ representation\ as \begin{cases} x=\phi(t) \\ y'=\psi(t) \end{cases}\ to\ represent\ curve\ of$

$F(x,y')=0, where\ t\ is\ the\ parameter, it\ follows\ that\ F\big(\phi(t),\psi(t)\big)=0.$

$Consider\ \phi(t)\ and\ \psi(t)\ are\ continuous\ differentiable\ in\ an\ interval\ with\ \ the\ property$

$dy=y'dx, then\ the\ equation\ can\ be\ represented\ by\ dy=\psi(t)\phi'(t)dt, and\ moreover,$

$it\ is\ given\ that\ y=\int\psi(t)\phi'(t)dt+C.$

$Then \begin{cases} x=\phi(t) \\ y=\int\psi(t)\phi'(t)dt+C \end{cases}\ satisfies\ the\ equation\ (1).$

$We\ can\ discuss\ eqaution\ F(y,y')=0\ as\ described\ above.$

$Examples.$

1. $x\sqrt{1+y'^2} = y'.$

2. $y - y'^5 - y'^3 - y' - 5 = 0.$

$\boldsymbol{II}.\ \boldsymbol{y = f(x, y')} \quad \boldsymbol{\left(x = f(y, y')\right)}$.

The ansatze that used here all have the property that they lead to solution curves with a special parametric representation in which $p = y'$ *is the parameter, and the parametric representation is* $\begin{cases} x = x \\ y' = p \\ y = f(x,p) \end{cases}$. *Consider the property* $dy = y'dx$, *we have*

$f_x(x,p)dx + f_p(x,p)dp = pdx$, *and* $p(x) = p(x, C)$ *can be derived by solving this equation.*

The substitution $p(x)$ *into equation* $y = f(x, y')$ *gives the general solution*

$y = f\left(x, p(x, C)\right)$.

We can discuss eqaution $x = f(y, y')$ *as described above.*

$Examples.$

$1. \, y = y'^2 - xy' + \frac{1}{2}x^2.$

$2. \, Clairaut's \; equation$

$y = xy' + \phi(y'), where \; \phi \; is \; twice \; continuous \; differentiable \; and \; \phi'' \neq 0.$

$Some\ special\ nth\ order\ ODEs$

$Case\ 1.\, F\left(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}\right) = 0$

$Set\ y^{(k)} = z, then\ the\ ODE\ transforms\ into\ F\left(x, z, z', \dots, z^{(n-k)}\right) = 0$

$Example.$

$$y^{(5)} - \frac{1}{x} y^{(4)} = 0.$$

$Some\ special\ nth\ order\ ODEs$

$Case\ 2.\ F\left(y, y', \ldots, y^{(n)}\right) = 0$

$Set\ y' = p, then\ the\ ODE\ transforms\ into\ \tilde{F}\left(y, p, p', \ldots, p^{(n-1)}\right) = 0 \Longrightarrow p = p(y).$

$Example.$

$$y'' + y = 0.\ (y'' = \frac{d^2y}{dx^2} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy})$$

$Exercises\ 1. Determine\ all\ the\ solutions\ to\ the\ following\ differential\ equations.$

(1) $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$

(2) $x^2 \dfrac{dy}{dx} = xy - y^2.$

(3) $y' = \frac{y+1}{x+2} - \exp\left(\frac{y+1}{x+2}\right).$

(4) $y' - y \cot x = 2x \sin x\,.$

(5) $(y^2 e^{xy} + 3x^2 y)dx + (x^3 + (1+xy)e^{xy})dy = 0\ is\ exact\ in\ R^2.$

Chapter 2. ***Theory of first order differential equations***

Line element.

We consider the explicit first order differential equation $y' = f(x, y)(1)$.

$f(x, y)$ *is assumed to be defined as a real* − *valued fucntion on a set D in the xy* − *plane. If* $y(x)$ *is an integral curve of equation*(1)*that passes through a point* (x_0, y_0), *then the differential equation specifies the slop of the curve at that point*: $y'(x_0) = f(x_0, y_0)$. *The unit line section which the center is* (x_0, y_0) *can be obtained by taking the slope as* $f(x_0, y_0)$, *and the line section is* ***line element*** *of* (x_0, y_0).

Example. Discuss the line element field of $y' = \frac{y}{x}$ *and* $y' = -\frac{x}{y}$.

Remark. A solution $y(x)$ of equation (1)"*fits*" its line element field. The slope at each point on the solution curve agrees with the slope of the line element at that point.

Euler's method

In this section, we assume that $f(x, y)$ is continuous and bounded in $[a, b]$.

In order to obtain the approximate solution of the initial problem $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

in $[x_0, b]$, we should take the Euler's method as follows.

Step 1. Divide the $[x_0, b]$ into n equal parts, and $x_k = x_0 + kh, k = 0,1, \dots, n,$ and

$$h = \frac{b - x_0}{n}, x_n = b.$$

Step 2. We take the function $y = y_0 + f(x_0, y_0)(x - x_0)$ as the approximate solution in $[x_0, x_1]$.

$Step\ 3.\ We\ get\ y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)\ as\ the\ approximation\ of\ y(x_1).$

$Step\ 4.\ We\ take\ the\ function\ y = y_1 + f(x_1, y_1)(x - x_1)\ as\ the\ approximate\ solution$

$in\ [x_1, x_2], and\ we\ can\ get\ y_2 \approx y(x_2)\ from\ the\ function.$

$Step\ 5.\ And\ so\ on.\ y_k = y_{k-1} + f(x_{k-1}, y_{k-1})h, k = 1,2, \dots, n.$

$So\ the\ line\ segments\ we\ obtained\ in\ [x_0, b]\ are\ the\ approximation\ of\ y(x).$

$Exercise 1.$

1. $Solve\ the\ approximate\ solution\ at\ x = 1.4\ for\ the\ initial\ problem$

$$\begin{cases} y' = x^2 + y^2 \\ y(1) = 1 \end{cases}\ by\ Euler's\ method, and\ take\ the\ step\ as\ h = 0.1.$$

2. $Solve\ the\ approximate\ solution\ in\ [0,2\pi]\ for\ the\ initial\ problem$

$$\begin{cases} y' = \cos x \\ y(0) = 2 \end{cases}\ by\ Euler's\ method, and\ display\ the\ fitting\ curve.$$

Lipschitz condition. *Function* $f(x)$ *satisfies a Lipschitz condition in* D $\left(\begin{array}{c}\text{with Lipschitz}\\ \text{constant } N\end{array}\right)$

if $|f(x)-f(y)| \le N|x-y|$ *for* $x, y \in D$.

It's easy to check that such an function is uniformly continuous in D.

We consider the following initial value problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ (1).

Existence and uniqueness theorem. Let $f(x,y)$ is continuous in the strip $R: x_0 - a \le x \le x_0 + a, y_0 - b \le y \le y_0 + b$ and satisfy the Lipschitz condition with respect to y in $R: |f(x,y) - f(x,\bar{y})| \le N|y - \bar{y}|$. Then the initial value problem (1) has exactly one solution $y = \phi(x)$ in an interval $x_0 - h_0 \le x \le x_0 + h_0$, where

$$h_0 = \min\left(a, \frac{b}{M}\right), M = \max|f(x,y)|, (x,y) \in R.$$

Remark 1. $If\ there\ exist\ the\ partial\ derivative\ of\ f(x,y)\ with\ respect\ to\ y\ in\ R,$

$and\ f_y'(x,y)\ is\ bounded\ which\ \left|f_y'(x,y)\right| \le N.$

$Acoording\ to\ Lagrange\ Mean\ Value\ Theorem, we\ have$

$|f(x,y)-f(x,\bar{y})| = \left|f_y'(x,\xi)\right||y-\bar{y}| \le N|y-\bar{y}|, where\ y<\xi<\bar{y}.$

Remark 2. $The\ initial\ value\ problem\ (1) is\ equivalent\ to\ the\ integral\ equation$

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi,y)d\xi \quad (2).$$

The proof of existence.

1. ***Construct Picard's iterative sequence.***

Finding any $y = \phi_0(x)$ *which satisfies* $\phi_0(x_0) = y_0$, *and* $y_0 - b \le \phi_0(x) \le y_0 + b$

$\forall\, x_0 - h_0 \le x \le x_0 + h_0$. *We take* $\phi_0(x) = y_0$. y_0 *is substituted into equation* (2), *thus*

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(\xi, y_0)\,d\xi\,.$$

$y = \phi_1(x)$ *is substituted into equation* (2), *we have*

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(\xi, \phi_1(\xi))\,d\xi$$

. *And so on , then*

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f\big(\xi, \phi_{n-1}(\xi)\big)\,d\xi\,. \quad (3)$$

Exercise. Prove $|\phi_n(x) - y_0| \le b, n = 1,2, \dots$.

Proving $|\phi_n(x) - y_0| \leq b\ in\ [x_0 - h_0, x_0 + h_0]\ by\ mathematical\ induction.$

$Obviously, |\phi_0(x) - y_0| \leq b.$ Assume $that\ |\phi_{n-1}(x) - y_0| \leq b, then\ we\ have$

$$|\phi_n(x) - y_0| \leq \left| \int_{x_0}^{x} f\big(\xi, \phi_{n-1}(\xi)\big) d\xi \right| \leq M|x - x_0| \leq Mh_0 \leq b.$$

2. *Prove the convergence of approximate sequence* $\{\boldsymbol{\phi_n(x)}\}$.

Consider the functional series

$$\phi_0(x) + [\phi_1(x) - \phi_0(x)] + \cdots + [\phi_n(x) - \phi_{n-1}(x)] + \cdots (4).$$

$$|\phi_1(x) - \phi_0(x)| \le \left| \int_{x_0}^{x} |f(\xi, y_0)| d\xi \right| \le M|x - x_0|,$$

According to Lipschitz conditon, we have

$$|\phi_2(x) - \phi_1(x)| \le \left| \int_{x_0}^{x} \left| f\big(\xi, \phi_1(\xi)\big) - f\big(\xi, \phi_0(\xi)\big) \right| d\xi \right| \le N \left| \int_{x_0}^{x} |\phi_1(\xi) - \phi_0(\xi)| d\xi \right|$$

$$\le MN \left| \int_{x_0}^{x} |\xi - x_0| d\xi \right| \le MN \frac{|x - x_0|^2}{2!},$$

$Assume\ that\ |\phi_n(x)-\phi_{n-1}(x)| \le MN^{n-1}\frac{|x-x_0|^n}{n!}, then\ we\ have$

$$|\phi_{n+1}(x)-\phi_n(x)| \le \left|\int_{x_0}^{x}\left|f\big(\xi,\phi_n(\xi)\big)-f\big(\xi,\phi_{n-1}(\xi)\big)\right|d\xi\right| \le N\left|\int_{x_0}^{x}|\phi_n(\xi)-\phi_{n-1}(\xi)|d\xi\right|$$

$$\le MN^n\left|\int_{x_0}^{x}\frac{|\xi-x_0|^n}{n!}d\xi\right| \le MN^n\frac{|x-x_0|^{n+1}}{(n+1)!}\ .$$

The $positive\ series\ Mh_0+MN\frac{h_0^2}{2}+\cdots+MN^{n-1}\frac{h_0^n}{n!}+\cdots is\ convergence.$

$Note\ that\ |x-x_0| \le h_0, so\ the\ \boldsymbol{functional\ series\ (4)'s\ uniform\ convergence}\ can\ be$

$obtained\ by\ Weierstrass\ discriminance.$

$\mathbf{3.}$ $\boldsymbol{Let\ \phi(x) = \lim_{n\to\infty} \phi_n(x)\ , prove\ \phi(x)\ is\ a\ solution\ of\ equation\ (2).}$

$$\left|\int_{x_0}^{x} f\big(\xi,\phi_n(\xi)\big)d\xi - \int_{x_0}^{x} f\big(\xi,\phi(\xi)\big)d\xi\right| \le \left|\int_{x_0}^{x} \left|f\big(\xi,\phi_n(\xi)\big) - f\big(\xi,\phi(\xi)\big)\right|d\xi\right|$$

$$\le N\left|\int_{x_0}^{x} |\phi_n(\xi)-\phi(\xi)|d\xi\right| \le Nh_0 \max|\phi_n(x)-\phi(x)|\ \ for\ x\in[x_0-h_0,x_0+h_0].$$

$Because\ of\ the\ uniform\ convergence\ of\ functional\ series\ \{\phi_n(x)\}, so\ \forall\epsilon, \exists n_0 \in N^+,$

$st. \forall n \ge n_0\ , |\phi_n(x)-\phi(x)| < \epsilon\ for\ x\in[x_0-h_0,x_0+h_0].$

$$That\ means\ \left|\int_{x_0}^{x} f\big(\xi,\phi_n(\xi)\big)d\xi - \int_{x_0}^{x} f\big(\xi,\phi(\xi)\big)d\xi\right| \le Nh_0\epsilon, so$$

$$\lim_{n\to\infty}\int_{x_0}^{x} f\big(\xi,\phi_n(\xi)\big)d\xi \;=\int_{x_0}^{x} f\big(\xi,\phi(\xi)\big)d\xi$$

$\lim_{n\to\infty}\phi_n(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f\big(\xi,\phi_{n-1}(\xi)\big)d\xi$ *is obtained by taking limit of equation* (3) .

That is $\phi(x) = y_0 + \int_{x_0}^{x} f\big(\xi,\phi(\xi)\big)d\xi$ *, thus the* $\phi(x)$ *is a solution of equation* (2).

1. *The initial value problem* (1)*is equivalent to the integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(\xi, y)d\xi \quad (2).$$

2. *Construct Picard's iterative sequence.* $\phi_n(x) = y_0 + \int_{x_0}^{x} f\big(\xi, \phi_{n-1}(\xi)\big)d\xi \,.(3)$

3. *Prove the convergence of approximate sequence* $\{\phi_n(x)\}$.

4. *Let* $\phi(x) = \lim_{n\to\infty} \phi_n(x)$, *prove* $\phi(x)$ *is a solution of equation* (2).

The proof of uniqueness.

Bellman Lemma.

Let $y(x)$ *is a nonnegative and continuous in* $[a,b]$. *If we can find* $\delta \geq 0$ *and* $k \geq 0$ *so that* $y(x) \leq \delta + k\left|\int_{x_0}^{x} y(t)\,dt\right|$ *for* $a \leq x_0 \leq b$, *where* $x \in [a,b]$. *Then*

$y(x) \leq \delta e^{k|x-x_0|}, x \in [a,b].$

The proof of uniqueness.

Bellman Lemma.

Let $y(x)$ *is nonnegative and continuous in* $[a,b]$*. If we can find* $\delta \geq 0$ *and* $k \geq 0$ *so*

$$\text{that } y(x) \leq \delta + k\left|\int_{x_0}^{x} y(t)\,dt\right| \text{ for } a \leq x_0 \leq b, \text{where } x \in [a,b]. \text{Then}$$

$y(x) \leq \delta e^{k|x-x_0|}, x \in [a,b].$

Let $y_1(x)$ *and* $y_2(x\ )$ *are two equations of equation* (2)*, the following estimation can be obtained by Lipschitze condition.*

$$|y_1 - y_2| \leq \left|\int_{x_0}^{x} \left|f\big(t, y_1(t)\big) - f\big(t, y_2(t)\big)\right| dt\right| \leq N\left|\int_{x_0}^{x} |y_1(t) - y_2(t)| dt\right|.$$

According to Bellman lem*ma, we have* $y(x) = 0.$

$Example$ 1.

$Show\ the\ solution\ of\ the\ equation \dfrac{dy}{dx} = \begin{cases} 0, & y = 0 \\ y\ln|y|, & y \neq 0 \end{cases}$ $which\ pass\ through\ a\ point$ $(x_0, y_0) \in R^2\ is\ unique.$

$Lipschitz\ condition\ is\ not\ the\ necessary\ condition\ for\ existence\ and\ uniqueness$ $of\ solution\ for\ the\ initial\ problem.$

Example 2.

Discuss the uniqueness of the solution of $\dfrac{dy}{dx} = 3y^{\frac{2}{3}}$.

Exercise 2.

1. *Discuss the region which satisfy the existence and uniqueness of solution for the following initial problems.*

$(1)y' = x^2 + y^2$; $\quad (2)y' = x^{-\frac{1}{3}}$; $\quad (3)y' = \sqrt{|y|}$.

2. *Get all solutions of the initial problem* $\begin{cases} y' = \frac{3}{2}y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$.

Exercise 2.

3. *Get the approximate solutions* $\phi_0(x), \phi_1(x), \phi_2(x)$ *for the initial problem* $\begin{cases} y' = x - y^2 \\ y(0) = 0 \end{cases}$ *by Picard's iterative method.*

3. *Prove* $|\phi_n(x) - \phi(x)| \le \dfrac{MN^n}{(1+n)!}|x - x_0|^{n+1}$ *in existence and uniqueness theorem.*

Exercise 2.

4\. *The solution of the initial problem* $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ *is unique in* $R: a \le x \le b$, $|y| < +\infty$. *Show* $y_1(x) < y_2(x)$ *in* $x_0 \le x \le b$ *for any two solutions* $y_1(x), y_2(x)$ *which satisfy* $y_1(x_0) < y_2(x_0)$.

The extension of solutions.

Local Lipschitz condition. *The function* $f(x, y)$ *is said to satisfy a local Lipschitz condition with respect to* y *in* $D \subset R^2$ *if for every* $(x_0, y_0) \in D$ *there exists a neighbor* – *hood* $U = U(x_0, y_0)$ *and an* $L = L(x_0, y_0)$ *such that in* $U \cap D$ *the function* f *satisfies the Lipschitz condition* $|f(x, y) - f(x, \bar{y})| \leq L|y - \bar{y}|$.

Theorem on local solvability. *If* D *is open and* $f \in C(D)$ *satisfies a local Lipschitz condition in* D, *then the initial value problem* (1) *is locally uniquely solvable for* $(x_0, y_0) \in D$; *i.e., there is a neighborhood* I *of* x_0 *such that exactly one solution exists in* I.

Theorem on the extension of solutions. Let $f \in C(D)$ satisfy a local Lipschitz conditon with respect to y in D. Then for every $(x_0, y_0) \in D$ the initial value problem $y' = f(x,y), y(x_0) = y_0$ has a solution ϕ that can be extended to the left and to the right comes arbitrarily close to the boundary of D.

The Peano existence theorem. *If* $f(x,y)$ *is continuous in a domain* D *and* (ξ,η) *is a point in* D, *then at least one solution of the differential equation* $y' = f(x,y)$ *goes through* (ξ,η). *Every solution can be extended to the left and to the right up to the boundary of* D.

Example 1.

Discuss the existence of the solutions of $y' = y^2, y(1) = 1$ *and* $y' = y^2, y(3) = -1$.

Example 2.

Discuss the existence of the solution of $y' = -\frac{1}{x^2}\cos\frac{1}{x}$.

$Exercise\ 3.$

$1. Let\ f(x,y) \in C(R^2)\ and\ f'(x,y) \in C(R^2). Prove\ the\ solution\ of\ y' = (y^2 - a^2)f(x,y),$

$y(x_0) = y_0\ exists\ in\ (-\infty, +\infty)\ for\ arbitrary\ x_0\ and\ |y_0| < a.$

$2. f(y)\ is\ continuous\ and\ differentiable\ in\ (-\infty, +\infty), and\ yf(y) < 0 (y \neq 0).$

$Show\ the\ initial\ problem\ y' = f(y), y(x_0) = y_0\ has\ a\ solution\ in\ [x_0, \infty].$

$Assume\ that\ y(x)\ is\ a\ solution, show\ \lim_{x \to +\infty} y(x) = 0.$

$3. f(y)\ is\ continuous\ and\ differentiable\ in\ (-\infty, +\infty), and\ yf(y) < 0 (y \neq 0).$

$Show\ the\ equation\ y' = x^2 f(\sin y)\ has\ a\ solution\ y = y(x) in\ (-\infty, +\infty), and\ if\ y(x)$

$is\ not\ a\ constant, then\ it\ is\ a\ monotonic\ increasing\ function.$

Comparison theorem.

Here we consider two initial value problems:

$y' = f(x,y), y(x_0) = y_0 \quad (1),$

$y' = F(x,y), y(x_0) = y_0 \quad (2).$

Let $f(x,y), F(x,y) \in C(D)$ *satisfy Lipschitz condition with respect to* y *in* D. $y = \phi(x)$ *and* $\mathrm{y} = \Phi(x)$ *are the solutions of equation* (1) *and equation*(2) *respectively*.

If $f(x,y) < F(x,y)$ *in* D. *Then* $\phi(x) < \Phi(x)$ *for* $x > x_0$, $\phi(x) > \Phi(x)$ *for* $x < x_0$.

Upper solutions, Lower solutions.

Let $f(x,y)$ *be defined in* $D, D \subset R^2$ *arbitary. The function* $v(x)$ *is called a lower solution (or subsolution) and* $w(x)$ *is called an upper solution (or supersolution) of the initial value problem* $y' = f(x,y)$ *in* $J = [x_0, x_0 + h_0], y(x_0) = y_0$,

if it is differentiable in J *and*

$v' < f(x,v)$ *in* J, $\quad v(x_0) \le y_0$, $\quad$ *lower solution*,

$w' > f(x,w)$ *in* J, $\quad w(x_0) \ge y_0$, $\quad$ *upper solution*.

Naturally, $v(x) < y(x) < w(x)$ *in* $J_0 : x_0 < x \le x_0 + h_0$.

$Example$.

$We\ consider\ the\ equation\ \ y' = x^2 + y^2, y(0) = 1.$

Singular solution. *The integral curve is called a singular integral curve, if the differential equation has no unique solution for every point in the integral curve*.

Example.

Determine whether there exist singular solutions for the follwing differential equations.

1. $y' = x^2 + y^2$. *The problem has no singular solution*.

2. $y' = 3y^{\frac{2}{3}}$. $y = 0$ *is a singular solution*.

3. $y' = \sqrt{y - x} + 2$. *The problem has no singular solution*.

4. $y' = \sqrt{|y|}$;

5. $y' = \sqrt{y - x}$.

Continuous dependence of solutions on initial value.

If D is open and $f \in C(D)$ satisfies Lipschitz condition in D. $y = \phi(x, x_0^*, y_0^*)$ is the solution of the initial value problem $\begin{cases} y' = f(x,y) \\ y(x_0^*) = y_0^* \end{cases}$, *and* $\left(x, \phi(x, x_0^*, y_0^*)\right) \in D$ *for* $a \leq x \leq b$. *Then for every* $\epsilon > 0$, *there exist* $\delta > 0$, *such that* $|\phi(x, x_0, y_0) - \phi(x, x_0^*, y_0^*)| < \epsilon$ *for every* (x_0, y_0) *which satisfies* $|x - x_0^*| \leq \delta, |y - y_0^*| \leq \delta$, *where* $y = \phi(x, x_0, y_0)$ *is the solution of of the initial value problem* $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ *in* $[a, b]$.

$\boldsymbol{Chapter\ 3. First\ Order\ Systems}.$

$A\ first\ order\ system\ of\ differential\ equations\ (in\ explicit\ form)\ of\ \ the\ form$

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \qquad\quad \vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases}. \ (1)$$

A $n - order\ differential\ equation\ y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)\ can\ be\ described\ as$

$$\begin{cases} y' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-2}' = y_{n-1} \\ y_{n-1}' = f(x, y, y_1, \dots, y_{n-1}) \end{cases}.$$

$We\ denote\ column\ vectors\ with\ boldface\ letters, as\ shown\ in\ the\ follwing$:

$$\boldsymbol{Y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \qquad \boldsymbol{F}(x, \boldsymbol{Y}) = \begin{pmatrix} f_1(x, y_1, \dots, y_n) \\ \vdots \\ f_n\ (x, y_1, \dots, y_n) \end{pmatrix}.$$

$Derivatives\ and\ integrals\ of\ a\ vector\ function\ Y(x)\ are\ also\ defined\ component$ wise:

$$\boldsymbol{Y}'(x) = \begin{pmatrix} y'_1(x) \\ \vdots \\ y'_n(x) \end{pmatrix}, \qquad \int_{x_0}^{x} \boldsymbol{F}(x)dx = \begin{pmatrix} \int_{x_0}^{x} f_1(x)dx \\ \vdots \\ \int_{x_0}^{x} f_n(x)dx \end{pmatrix}.$$

$Written\ in\ vector\ notation, system\ (1)\ reads$

$$\boldsymbol{Y}' = \boldsymbol{F}(x, \boldsymbol{Y})$$

The initial condition of system (1) $y_1(x_0) = y_{10}, \dots, y_n(x_0) = y_{n0}$ *can be written as*

$$\boldsymbol{Y}(x_0) = \boldsymbol{Y}_0, where\ \boldsymbol{Y}_0 = \begin{pmatrix} y_{10} \\ \vdots \\ y_{n0} \end{pmatrix}.$$

Initial value problem. $\begin{cases} \boldsymbol{Y}' = \boldsymbol{F}(x, \boldsymbol{Y}) \\ \boldsymbol{Y}(x_0) = \boldsymbol{Y}_0 \end{cases}.$

Norm. *A real value function* $||\cdot||$ *defined for* $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ *and* $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

is called a norm if it has the properties

$\left||Y|\right| > 0$ *for* $Y \neq O$ *definiteness,*

$\left||\alpha Y|\right| = |\alpha| \cdot \left||Y|\right|$ *for every cosntant* α, *homogeneity,*

$\left||Y + X|\right| \leq \left||X|\right| + \left||Y|\right|$, *triangle inequality*

$We\ define\ \left\|Y\right\| = \sum_{i=1}^{n} |y_i|, \left\|A\right\| = \sum_{k,j=1}^{n} |a_{kj}|.\ Based\ on\ the\ definition, we\ have$

$\left\| \int_{x_0}^{x} F(x)dx \right\| \leq | \int_{x_0}^{x} ||F(x)||dx |.$

$\forall x \in [a,b], a\ sequence\ \{Y_n\}\ converges\ in\ the\ norm\ to\ Y\ if\ \lim_{n\to\infty} \left\|Y_n(x) - Y(x)\right\| = 0.$

Lipschitz condition. *A vector function* $F(x, Y)$ *satisfies a Lipschitz condition with respect to* Y *in* D *(with Lipschitz constant* L*) if*

$$|F(x,Y) - F(x,\bar{Y})| \leq L|Y - \bar{Y}| \ \forall\ (x,Y),(x,\bar{Y}) \in D.$$

Local Lipschitz condition. *A function* f *is said to satisfy in* D *a local Lipschitz condition with respect to* Y *if for every point* $(x, Y) \in D$, *there exists a neihborhood* $U: |x - \bar{x}| < \delta, |y - \bar{y}| < \delta\ (\delta > 0)$ *such that* F *satisfies a Lipschitz condition in* $D \cap U$.

Lemma. *If* $F \in C(D)$ *satisfies in* D *a local Lipschitz condition in* Y, *then* F *satisfies a Lipschitz condition in* Y *on compact subsets of* D.

Existence and uniqueness theorem. *Let* $F(x,Y)$ *be continuous and satisfy the Lipschitz condition in* $J \times R^n, J = [\xi, \xi + a]$. *Then there is exactly one solution to the initial value problem* $Y' = F(x,Y), Y(\xi) = \eta$. *The solution exists in of* J.

Let $F(x,Y)$ *be continuous in a domain* $D \subset R^{n+1}$ *and satisfy a local Lipschitz condition with respect to* y *in* D. *If* $(\xi,\eta) \in D$, *then the initial value problem* $Y' = F(x,Y), Y(\xi) = \eta$ *has exactly one solution. Then solution can be extended to the left and right up to the boundary of* D.

Chapter $\mathbf{3}$. ***First Order Systems***.

A first order system of differential equations (in explicit form) of the form

$$\begin{cases} y_1' = f_1(x, y_1, \dots, y_n) \\ \qquad\vdots \\ y_n' = f_n(x, y_1, \dots, y_n) \end{cases}. \quad (1)$$

A $n-$ *order differential equation* $y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)$ *can be described as*

$$\begin{cases} y' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-2}' = y_{n-1} \\ y_{n-1}' = f(x, y, y_1, \dots, y_{n-1}) \end{cases}.$$

$We\ denote\ column\ vectors\ with\ boldface\ letters, as\ shown\ in\ the\ follwing$:

$$\boldsymbol{Y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \qquad \boldsymbol{F}(x, \boldsymbol{Y}) = \begin{pmatrix} f_1(x, y_1, \dots, y_n) \\ \vdots \\ f_n\ (x, y_1, \dots, y_n) \end{pmatrix}.$$

$Derivatives\ and\ integrals\ of\ a\ vector\ function\ Y(x)\ are\ also\ defined\ component$ wise:

$$\boldsymbol{Y}'(x) = \begin{pmatrix} y'_1(x) \\ \vdots \\ y'_n(x) \end{pmatrix}, \qquad \int_{x_0}^{x} \boldsymbol{F}(x) dx = \begin{pmatrix} \int_{x_0}^{x} f_1(x) dx \\ \vdots \\ \int_{x_0}^{x} f_n(x) dx \end{pmatrix}.$$

$Written\ in\ vector\ notation, system\ (1)\ reads$

$$\boldsymbol{Y}' = \boldsymbol{F}(x, \boldsymbol{Y})$$

The initial condition of system (1) $y_1(x_0) = y_{10}, \dots, y_n(x_0) = y_{n0}$ *can be written as*

$$\boldsymbol{Y}(x_0) = \boldsymbol{Y}_0, \text{where } \boldsymbol{Y}_0 = \begin{pmatrix} y_{10} \\ \vdots \\ y_{n0} \end{pmatrix}.$$

Initial value problem. $\begin{cases} \boldsymbol{Y}' = \boldsymbol{F}(x, \boldsymbol{Y}) \\ \boldsymbol{Y}(x_0) = \boldsymbol{Y}_0 \end{cases}.$

Norm. *A real value function* $||\cdot||$ *defined for* $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ *and* $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

is called a norm if it has the properties

$\left||Y|\right| > 0$ *for* $Y \neq O$ *definiteness,*

$\left||\alpha Y|\right| = |\alpha| \cdot \left||Y|\right|$ *for every cosntant* α, *homogeneity,*

$\left||Y + X|\right| \leq \left||X|\right| + \left||Y|\right|$, *triangle inequality*

$We\ define\ \left\|Y\right\| = \sum_{i=1}^{n} |y_i|\,, \left\|A\right\| = \sum_{k,j=1}^{n} |a_{kj}|\,.\,Based\ on\ the\ definition, we\ have$

$\left\|\int_{x_0}^{x} F(x)dx\right\| \leq |\int_{x_0}^{x} ||F(x)||dx\,|.$

$\forall x \in [a,b], a\ sequence\ \{Y_n\}\ converges\ in\ the\ norm\ to\ Y\ if\ \lim_{n\to\infty}\left\|Y_n(x) - Y(x)\right\| = 0\,.$

Lipschitz condition. A vector function $F(x,Y)$ satisfies a Lipschitz condition with respect to Y in D (with Lipschitz constant L) if

$$||F(x,Y)-F(x,\bar{Y})|| \le L||Y-\bar{Y}|| \ \forall\ (x,Y),(x,\bar{Y}) \in D.$$

Local Lipschitz condition. A function F is said to satisfy in D a local Lipschitz condition with respect to Y if for every point $(x,Y) \in D$, there exists a neihborhood $U: |x-\bar{x}| < \delta, ||Y-\bar{Y}|| < \delta\ (\delta > 0)$ such that F satisfies a Lipschitz condition in $D \cap U$.

Lemma. If $F \in C(D)$ satisfies in D a local Lipschitz condition in Y, then F satisfies a Lipschitz condition in Y on compact subsets of D.

Existence and uniqueness theorem. *Let* $F(x, Y)$ *be continuous and satisfy the Lipschitz condition in* $R: |x - x_0| \le a, \left\| Y - Y_0 \right\| \le b$. *Then there is exactly one solution to the initial value problem* $Y' = F(x, Y), Y(x_0) = Y_0$ *in* $|x - x_0| \le h_0$, *where* $h_0 = \min\left(a, \frac{b}{M}\right)$, $M = \max ||F(x, Y)||$.

Let $F(x, Y)$ *be continuous in a domain* $D \subset R^{n+1}$ *and satisfy a local Lipschitz condition with respect to* y *in* D. *If* $(\xi, \eta) \in D$, *then the initial value problem* $Y' = F(x, Y), Y(\xi) = \eta$ *has exactly one solution. Then solution can be extended to the left and right up to the boundary of* D.

Peano existence theorem. *If* $F(x, Y)$ *is continuous in the domain* D *and* $(\xi, \eta) \in D$, *then the initial value problem* (1) *has at least one solution. Every solution can be extended to the left and right up to the boundary of* D.

Homogeneous linear systems

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + f_1(x) \\ \vdots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + f_n(x) \end{cases}$$ *(1)is a first order linear systems.*

$$Set\ A(x) = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix}\ and\ \ F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x\ ) \end{pmatrix}.$$

$Written\ in\ vector\ notation, system\ (1)\ reads\ Y' = A(x)Y + F(x), if\ F(x) \equiv \mathbf{0}, then$

$Y' = A(x)Y\ is\ called\ homogeneous, otherwise, it\ is\ called\ inhomogeneous.$

$\boldsymbol{Theorem}. If\ A(x)\ and\ F(x) are\ continuous\ in\ [a, b], for\ every\ x_0 \in [a, b], Y_0 \in R^n\ or\ C^n,$

$there\ exists\ exactly\ one\ solution\ for\ the\ initial\ problem Y' = A(x)Y + F(x), Y(x_0) = Y_0$

$in\ [a, b]\ .$

A set of solutions $Y_1, \dots, Y_k$ is called **linearly dependent** if there exist constants $c_1, \dots, c_k$ with $|c_1| + \cdots + |c_k| > 0$ such that $c_1 Y_1 + \cdots + c_k Y_k = 0$.

The k solutions are said to be **linearly independent** if they are not linearly dependent

Proposition of homogeneous linear systems .

(a) $Y \equiv \mathbf{0}$ in J is a solution of the **homogeneous linear systems**.

(b) There exist n linearly independent solutions $Y_1, \dots, Y_n$. Every such set of n linearly independent solutions is called a **fundamental system of solutions**. If $Y_1, \dots, Y_n$ is a fundamental system, then every solution y can be written in a unique way as a linear combination $Y = C_1 Y_1 + \cdots + C_n Y_n$.

(c) A system of n solutions $Y_1, \dots, Y_n$ can be assembled into an $n \times n$ solution matrix $\Phi(x) = (Y_1, \dots, Y_n)$. If n solutions $Y_1, \dots, Y_n$ are linearly independent, then $\Phi(x)$ is a **fundamental matrix**.

$Example.$ Show vector functions $Y_1(x) = \begin{pmatrix} \cos^2 x \\ 1 \\ x \end{pmatrix}$ and $Y_2(x) = \begin{pmatrix} \sin^2 x - 1 \\ -1 \\ -x \end{pmatrix}$ are linearly dependent in (a, b).

$Example.$ Show vector functions $Y_1(x) = \begin{pmatrix} e^{3x} \\ e^{3x} \\ e^{3x} \end{pmatrix}$ and $Y_2(x) = \begin{pmatrix} e^{6x} \\ -2e^{6x} \\ e^{6x} \end{pmatrix}$ are linearly independent in $(-\infty, +\infty)$.

$Excersise.$ Show vector functions $Y_1(x) = \begin{pmatrix} e^{-2x} \\ 0 \\ -e^{-2x} \end{pmatrix}$ and $Y_2(x) = \begin{pmatrix} 0 \\ e^{-2x} \\ -e^{-2x} \end{pmatrix}$ are linearly independent in $(-\infty, +\infty)$.

The Wronskian. *If* $\Phi(x) = (Y_1, \dots, Y_n)$ *is a solution matrix of* $Y' = A(x)Y$, *then its determinant* $W(x) = |\Phi(x)|$ *is called the Wronskian determinant.*

Theorem. *If* $Y_1, \dots, Y_n$ *are linearly dependent in* J, *then the Wronskian* $W(x) \equiv 0$.

Proof. $Y_1, \dots, Y_n$ *are linearly dependent in* J, *so there exists* $C = (c_1, \dots, c_n)^T$ *which satisfies* $|c_1| + \cdots + |c_n| > 0$ *st.* $\Phi(x)C = 0$ *for every* $x \in J$.

That means the homegeneous linear equations $\Phi(x)C = 0$ *has non* $-$ *zero solutions for every* $x \in J$, *so* $w(x) = |\Phi(x)| = 0$ *for every* $x \in J$.

$\boldsymbol{Theorem}. If\, Y_1, \ldots, Y_n$ *are linearly dependent in* J, *then the Wronskian* $W(x) \equiv 0$.

$\boldsymbol{Theorem}. If\ Y_1, \ldots, Y_n$ *is a fundamental system of equation* $Y' = A(x)Y$, *then the Wronskian* $W(x) \neq 0$ *in* J.

Proof. *If there exists* $x_0 \in J$ *st.* $W(x_0) = 0$, *the linear equations* $\Phi(x_0)C = 0$ *has non − zero solutions. That means* $\exists\ C = (c_1, \ldots, c_n) \neq 0$ *st.* $c_1 Y_1(x_0) + \cdots + c_n Y_n(x_0) = 0$, *and* $Y(x) = c_1 Y_1(x) + \cdots + c_n Y_n(x)$ *is a solution of the initial problem* $Y' = A(x)Y, Y(x_0) = 0$. *Obviously,* $Y(x) = 0$ *is a solution of such a initial problem.*

According to the existence and uniqueness theorem, the initial problem has exactly one solution in J. *So* $W(x) \neq 0$ *in* J.

$\boldsymbol{Corollary}$. *The Wronskian is either* $= 0$ *or* $\neq 0$ *in* J. *The nonvanishing of the Wronskian is necessary and sufficient condition for* $\Phi(x)$ *to be a fundamental matrix.*

Theorem. $If Y_1, \dots, Y_n$ *are linearly dependent in* J*, then the Wronskian* $W(x) \equiv 0$.

Theorem. *If* $Y_1, \dots, Y_n$ *is a fundamental system of equation* $Y' = A(x)Y$*, then the Wronskian* $W(x) \neq 0$ *in* J.

Theorem. *There exists a fundamental system of solutios for eqaution* $Y' = A(x)Y$.

Proof. *According to the existence and uniqueness theorem, the initial problem*

$$Y' = A(x)Y, Y_i(x_0) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to i \ (i = 1, \dots, n) \text{ has exactly one solution. Note that}$$

$W(x_0) = |Y_1(x_0), \dots, Y_n(x_0)| = |E| = 1$, *so* $Y_1(x), \dots, Y_n(x)$ *is a fundamental system of the linear homogeneous differential system* $Y' = A(x)Y$.

$\boldsymbol{Theorem}.$ *If* $Y_1, \ldots, Y_n$ *is a fundamental system of the linear homogeneous differential system* $Y' = A(x)Y$, *then every solution* Y *can be written in a unique way as a linear combination* $Y = C_1Y_1 + \cdots + C_nY_n$.

Proof. $\forall\, c_1, \ldots, c_n$, *set* $Y(x) = c_1Y_1(x) + \cdots + c_nY_n(x)$, *then we have*

$Y'(x) = [c_1Y_1(x) + \cdots + c_nY_n(x)]' = A(x)Y(x)$, *so* $Y(x)$ *is a solution of* $Y' = A(x)Y$.

If $Y(x)$ *is a solution of the initial problem* $Y' = A(x)Y, Y(x_0) = Y_0$, *then there exists*

excatly one $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq 0$ *st.* $\Phi(x_0)C = Y_0$ *because of* $W(x_0) = |\Phi_0(x_0)| \neq 0$.

According to the existence and uniqueness theorem, $Y(x) = \Phi(x)C$ *is the unique solution of such initial problem.*

$\boldsymbol{Theorem}.\, If\ Y_1, \dots, Y_n\ is\ a\ fundamental\ system\ of\ equation\ Y' = A(x)Y, then\ every$

$solution\, Y\ can\ be\ written\ in\ a\ unique\ way\ as\ a\ linear\ combination Y = C_1Y_1 + \cdots + C_nY_n.$

$\boldsymbol{Theorem}.\, If\ A(x) is\ real - valued\ and\ continuous\ in\ [a, b], then\ the\ set\ of\ real$

$solutions\, Y(x)\ of\ the\ homogeneous\ equation\ \ Y' = A(x)Y\ forms\ a\, n - dimensional$

$linear\ space.$

$Exercise.$

1. $If\ \Phi(x)\ and\ \Psi(x) are\ two\ fundamental\ matrixes\ of\ Y' = A(x)Y.$

$Show\ there\ exists\ a\ nonsigular\ matrix\ B\ such\ that\ \Phi(x) = \Psi(x)B.$

Theorem. *If* $A(x)$ *is continuous in* J, *then the Weonskian* $W(x) = W(x_0)e^{\int_{x_0}^{x}[trA(t)]dt}$,

where $trA(t) = a_{11}(t) + \cdots + a_{nn}(t)$. *This formula is called* ***Liouville formula***.

***Inhomogeneous Systems*.**

$Y' = A(x)Y + F(x) \quad (1).$

Theorem. *Let* $\tilde{Y}(x)$ *be a fixed solution of the inhomogeneous equation* (1). *If* $Y_0(x)$ *is an arbitrary solution of the homogeneous equaiton, then* $Y(x) = \tilde{Y}(x) + Y_0(x)$ *is a solution of the inhomogeneous equation, and all solutions of the inhomogeneous equation are obtained in this way.*

Remark. *The general solution of the inhomogeneous equation is given by*

$Y(x) = C_1Y_1(x) + \cdots + C_nY_n(x) + \tilde{Y}(x)$, *where* $Y_1(x), \dots, Y_n(x)$ *is a fundamental system of the related homogeneous equation and* $C_1, \dots, C_n$ *are arbitrary constants.*

Method of Variation of constants.

$$Y' = A(x)Y. \qquad (1)$$

$$Y' = A(x)Y + F(x). \qquad (2)$$

$Y(x) = \Phi(x)C$ *is the general solution of the homogeneous systems* (1).

In the method of variation of constants the constants $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ *are replaced by functions of* $C(x) = \begin{pmatrix} C_1(x) \\ \vdots \\ C_n(x) \end{pmatrix}$.

Substituting $\tilde{Y}(x) = \Phi(x)C(x)$ *into* (2) *gives* $\widetilde{Y}(x) = \int_{x_0}^{x} \Phi(x)\,\Phi^{-1}(t)F(t)dt$.

The general solution of the inhomogeneous systems (2) *is*

$$Y(x) = \Phi(x)C + \int_{x_0}^{x} \Phi(x)\,\Phi^{-1}(t)F(t)dt.$$

Methods for solving linear differential systems with constant coefficients

$\boldsymbol{Linear\ Transformations}.$

$We\ consider\ the\ homogeneous\ system\ Y' = AY.(1)$

$If\ C\ is\ a\ nonsingular\ constant\ matrix, then\ the\ mapping\ Y = CZ, Z = C^{-1}Y(\det C \neq 0)$

$transforms\ a\ solution\ of\ (1)\ into\ a\ solution\ Z(t)\ of\ the\ system\ Z' = C^{-1}ACZ\ (2).$

Theorem. $Suppose\ that\ A\ has\ n\ different\ eigenvalues, then\ it\ has\ n\ linearly$

$independent\ eigenvectors\ C_1, \ldots C_n. If\ one\ sets\ C = (C_1, \ldots C_n), then\ AC = (\lambda_1 C_1, \ldots, \lambda_n C_n)$

$= CD, where\ D = diag(\lambda_1, \ldots, \lambda_n). Thus\ for\ this\ choice\ of\ C\ st. C^{-1}AC = D\ and\ (2) reads$

$simply\ z_1' = \lambda_1 z_1, \ldots, z_n' = \lambda_n z_n. It's\ easy\ to\ find\ a\ fundamental\ system\ of\ solutions\ for$

$$this\ system, namely\ Z(t) = (z_1, \ldots, z_n) = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}.$$

$Going\ back\ to\ Y = CZ, we\ obtain\ the\ fundamental\ system\ of$

$Y_i = CZ_i = e^{\lambda_i x} C_i, i = 1, \dots, n.$

$Example. Determin\ the\ general\ solution\ of\ the\ system$ $\begin{cases} \dfrac{dx}{dt} = 3x - y + z \\ \dfrac{dy}{dt} = -x + 5y - z. \\ \dfrac{dz}{dt} = x - y + 3z \end{cases}$

$Solution.\, A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}.\, From\, \det(A - \lambda E) = 0, it\ follows\ that\ \lambda_1 = 2, \lambda_2 = 3, \lambda_3$
$= 6. The\ corresponding\ eigevectors\ are\ T_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

$So\ the\ general\ solution\ is\ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{6t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

$Example. Solve\ the\ nonhomogeneous\ linear\ differential\ system \begin{cases} x' = 2x + 3y + 5t \\ y' = 3x + 2y + 8e^t \end{cases}.$

$Solution. The\ general\ solution\ of\ the\ related\ homogeneous\ system\ is$

$C_1 e^{5t}\binom{1}{1} + C_2 e^{-t}\binom{1}{-1}.$

$Set\ the\ special\ solution\ of\ the\ nonhomogeneous\ system\ is$

$C_1(t) e^{5t}\binom{1}{1} + C_2(t) e^{-t}\binom{1}{-1}.$

$We\ have\ C_1'(t) e^{5t}\binom{1}{1} + C_2'(t) e^{-t}\binom{1}{-1} = \binom{5t}{8e^t}.$

$$Moreover, \begin{cases} C_1(t) = \left(-\frac{1}{2}t - \frac{1}{10}\right)e^{-5t} - e^\wedge - 4t \\ C_2(t) = \left(\frac{5}{2}t - \frac{5}{2}\right)e^{t} - 2e^{2t} \end{cases}.$$

$Exercise.$ $Determine\ the\ general\ solution\ of\ the\ systems$

$$(a).\ \begin{cases} 2x' - 5y' = 4y - x, \\ 3x' - 4y' = 2x - y. \end{cases}$$

$$(b).\ \begin{cases} x' = y + 2e^t, \\ y' = x + t^2. \end{cases}$$

Remark. A *is a real coefficient matrix, so the complex eigenvalues of the matrix come in pairs, and the correspongding eigenvectors are conjugate too.*

Remark. *If* $Y(x) = U(x) + iV(x)$ *is a solution of* $Y' = A(x)Y$, *then* $U(x)$ *and* $V(x)$ *are solutions of the homogeneous system.*

Theorem. *If* $\lambda = \mu + iv (v \neq 0)$ *is a complex eigenvalue of the real matrix* A *and* $c = a + ib$ *is a corresponding eigenvector, then the complex solution* $Y = ce^{\lambda x}$ *products two real solutions*:

$$Y_1(x) = Re\, Y = e^{\mu x}(a \cos vx - b \sin vx),$$

$$Y_2(x) = Im\, Y = e^{\mu x}(a \sin vx + b \cos vx).$$

$Example. Determin\ the\ general\ solution\ of\ the\ system$ $\begin{cases} \dfrac{dx}{dt} = x - y - z \\ \dfrac{dy}{dt} = x + y \\ \dfrac{dz}{dt} = 3x + z \end{cases}.$

$Solution.\ A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.\ From\ \det(A - \lambda E) = 0, it\ follows\ that\ \lambda_1 = 1, \lambda_2 = \lambda_3 = 1 \pm$
$2i. The\ corresponding\ eigevectors\ are\ T_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, T_2 = \begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix}, T_3 = \begin{pmatrix} -2i \\ 1 \\ 3 \end{pmatrix}.$

$So\ the\ general\ solution\ is$ $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_2 e^t \begin{pmatrix} -2\sin 2t \\ \cos 2t \\ 3\cos 2t \end{pmatrix} + C_3 e^t \begin{pmatrix} 2\cos 2t \\ \sin 2t \\ 3\sin 2t \end{pmatrix}.$

$\boldsymbol{Jordan\ normal\ form\ of\ a\ matrix}.$ $The\ matrix\ theory\ says\ for\ every\ real\ or\ complex\ matrix\ A\ there\ exists\ a\ nonsingular\ matrix\ C(in\ general, C\ will\ be\ complex), such\ that$

$B = C^{-1}AC\ has\ the\ so-called\ Jordan\ normal\ form\ B = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_i & 0 \\ 0 & 0 & J_k \end{bmatrix}$, where the Jordan

$block\ J_i\ is\ a\ square\ matrix\ of\ the\ form\ J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}\ with\ r_i\ rows\ and\ columns; Out$

$of\ the\ Jordan\ blocks, B\ consists\ entirely\ of\ zeros. Here\ r_1 + \cdots + r_k = n, and$

$\det(A - \lambda E) = P_n(\lambda) = (-1)^n(\lambda - \lambda_1)^{r_1} \dots (\lambda - \lambda_k)^{r_k}.$

$Note\ that\ the\ main\ diagonal\ of\ B\ consists\ of\ eigenvalues\ of\ A\ and\ that\ each\ block\ is\ made\ up\ of\ one\ and\ the\ same\ eigenvalue.$

$The\ system\ corresponding\ to\ a\ Jordan\ block\ J\ with\ r\ rows\ and\ diagonal\ element\ \lambda\ is$

$given\ by\ X_r' = JX_r\ or\ \begin{cases} x_1' = \lambda x_1 + x_2 \\ x_2' = \lambda x_2 + x_3 \\ \quad\vdots \\ x_r' = \lambda x_r \end{cases}\ can\ be\ easily\ solved\ (one\ begins\ with\ the\ last$

$equation).For\ example, if\ J = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, then\ the\ corresponding\ system$

$Z' = JZ\ can\ be\ rewritten\ as$

$$\begin{cases} z_1' = \lambda_1 z_1 + z_2 \\ z_2' = \lambda_1 z_2 + z_3 \\ z_3' = \lambda_1 z_3 \end{cases} \text{ and } \begin{cases} z_4' = \lambda_2 z_4 + z_5 \\ z_5' = \lambda_2 z_5 \end{cases}.$$ *We can obtain solutions of such two systems*

$$\begin{cases} z_1 = (\frac{C_3}{2!}x^2 + C_2 x + C_1)e^{\lambda_1 x} \\ z_2 = (C_3 x + C_2)e^{\lambda_1 x} \\ z_3 = C_3 e^{\lambda_1 x} \end{cases} \text{ and } \begin{cases} z_4 = (C_5 x + C_4)e^{\lambda_2 x} \\ z_5 = C_5 e^{\lambda_2 x} \end{cases}.$$

Set $C_1 = 1, C_2 = C_3 = C_4 = C_5 = 0; C_2 = 1, C_1 = C_3 = C_4 = C_5 = 0;$ $C_3 = 1, C_1 = C_2 = C_4 = C_5 = 0;$ $C_4 = 1, C_1 = C_2 = C_3 = C_5 = 0;$ $C_5 = 1, C_1 = C_2 = C_3 = C_4 = 0,$ *we get the fundamental system of solutions.*

Summary. *For every* $k-fold$ *eigenvalue* λ *there exist* k *linearly independent solutions* $Y_1 = \boldsymbol{p}_0(x)e^{\lambda x}, \dots, Y_k = \boldsymbol{p}_{k-1}(x)e^{\lambda x}$ *in which every component of* $\boldsymbol{p}_m(x) = \left(p_1^m(x), \dots, p_n^m(x)\right)^T \ (m = 0,1,\dots,k-1)$ *is a polynomial of degree* $\leq m$.

Example. $\begin{cases} x' = x - y \\ y' = 4x - 3y \end{cases}.$

From $A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$ *follows that* $\lambda_1 = \lambda_2 = -1$.

The corresponding solution is $\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *A second, linearly independent solution*

can be obtained using the ansatz $\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\begin{pmatrix} a + bt \\ c + dt \end{pmatrix}$.

From $\begin{pmatrix} x' \\ y' \end{pmatrix} = e^{-t}\begin{pmatrix} b - a - bt \\ d - c - dt \end{pmatrix} = Ae^{-t}\begin{pmatrix} a + bt \\ c + dt \end{pmatrix}$, *we have* $\begin{pmatrix} x \\ y \end{pmatrix} = e^{-t}\begin{pmatrix} t \\ -1 + 2t \end{pmatrix}$.

Theorem. *For every $k-fold$ eigenvalue λ there exist k linearly independent solutions* $Y=(R_0+R_1x+\cdots+R_{k-1}x^{k-1})e^{\lambda x}$ *in which* $R_0,\ldots,R_{k-1}$ *are solved by*

$$\begin{cases} (A-\lambda E)R_0 = R_1 \\ (A-\lambda E)R_1 = 2R_2 \\ \ldots \\ \ldots \\ (A-\lambda E)R_{k-2} = (k-1)R_{k-1} \\ (A-\lambda E)^k R_0 = O \end{cases}.$$

Example. Determine the general solution of the system $\begin{cases} y_1' = y_2 + y_3, \\ y_2' = y_1 + y_3, \\ y_3' = y_1 + y_2. \end{cases}$

Solution. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. *From* $\det(A - \lambda E) = 0$, *it follows that* $\lambda_1 = 2, \lambda_2 = \lambda_3 = -1$.

The corresponding solution about λ_1 *is* $Y_1 = e^{2x}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = -1$, *we have* $\begin{cases} (A + E)R_0 = R_1 \\ (A + E)^2 R_0 = O. \end{cases}$

From $(A + E)^2 R_0 = O$, *we obtain linearly independent vectors are* $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$,
the corresponding R_1 *are zeros. Then we have two lineary independent solutions*
$Y_2 = e^{-x}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $Y_3 = e^{-x}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

$$Exercise.\,Determine\ the\ general\ solution\ of\ the\ system\ \begin{cases} y_1' = 3y_1 + y_2 - y_3, \\ y_2' = -y_1 + 2y_2 + y_3, \\ y_3' = y_1 + y_2 + y_3. \end{cases}$$

$*$ ***Matrix functions***.

Power series of matrices.

If B *is an* $n \times n$ *matrix and* $p(s)$ *is the polynomial* $p(s) = c_0 + c_1 s + \cdots + c_k s^k$,

then $p(B)$ *is defined to be the matrix* $p(B) = c_0 E + c_1 B + \cdots + c_k B^k$.

For $B = At \left(b_{ij} = a_{ij} t\right), p(At) = c_0 E + c_1 At + \cdots + c_k A^k t^k$.

$$For\ C = \sum_{k=0}^{\infty} C_k\ ,$$

convergence *is defined as usual*: $S_p = C_0 + \cdots + C_p \to C$ *as* $p \to \infty$, *i.e.*, $\left|\left|S_p - C\right|\right| \to 0$.

The matrix series is ***absolutely convergent*** *if the real series* $\sum ||C_k||$ *converges*.

The exponential matrix functions.

If A is an $n \times n$ matrix, the series

$$e^{A} = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \cdots + \frac{A^n}{n!} + \cdots$$

converges absolutely for all A.

A simple calculation shows that $\frac{d}{dx} e^{Ax} = Ae^{Ax}$, *so* e^{Ax} *is a fundamental matrix for the linear system* $Y' = AY$.

Standard fundamental matrix: e^{Ax} *is a fundamental matrix for the IVP* $\begin{cases} Y' = AY \\ Y(0) = E \end{cases}$.

$Example.\ If\ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, we\ have\ the\ series$

$$e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \cdots + \frac{A^n}{n!} + \cdots$$

$$= \begin{bmatrix} 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots & 0 \\ 0 & 1 + 2 + \frac{1}{2!}2^2 + \cdots + \frac{1}{n!}2^n + \cdots \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}.$$

$converges\ absolutely\ for\ all\ A.$

$e^{Ax} = \begin{bmatrix} e^x & 0 \\ 0 & e^{2x} \end{bmatrix}$ $is\ the\ standard\ fundamental\ matrix\ of\ Y' = AY,$

$then\ e^A = \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix}.$

*Exercise*1.

1. *Determine the general solution of the autonomous differential system*

$$\begin{cases}\dfrac{dx}{dt}=p(t)x+q(t)y\\ \dfrac{dy}{dt}=q(t)x+p(t)y\end{cases},\text{where } p(t) \text{ and } q(t) \text{ are continuous.}$$

2. *If* $A_1(t)$ *and* $A_2(t)$ *are continuous in* (a,b), *and* $\Phi(t)$ *is a fundamental matrix of the differential system* $\dfrac{dX}{dt}=A_1(t)\,X$ *and* $\dfrac{dX}{dt}=A_2(t)X$. *Show* $A_1(t)\equiv A_2(t)$.

3. *Consider the linear homogeneous differential system* $\dfrac{dY}{dx}=A(x)Y$, *where* $A(x)$ *is a* $T-$*periodic continuous function.*

(a) *If* $\Phi(x)$ *is a fundamental matrix of such a system, then show* $\Phi(x+T)$ *is a fundamental matrix.*

(b) *Show there exists a nonsingular matrix* B *such taht* $\Phi(x+T)=\Phi(x)B$.

Exercise2.

1. *Determine the general solution of the following differential systems*

(a) $\begin{cases} \dfrac{dy}{dx} = 5y + 4z \\ \dfrac{dz}{dx} = 4y + 5z \end{cases}$. (b) $\begin{cases} 2x' - 5y' = 4y - x \\ 3x' - 4y' = 2x - y \end{cases}$. (c) $\begin{cases} x' = y + 2e^t \\ y' = x + t^2 \end{cases}$.

2. *If* $A(x)$ *and* $F(x)$ *are continuous.* $Y_1(x), \dots Y_{n+1}(x)$ *are solutions of the linear nonhomogeneous system* $\dfrac{dY}{dx} = A(x)Y + F(x)$ *and they are linearly independent.*
Show the general solution of such nonhomogeneous system is
$Y(x) = a_1Y_1(x) + \cdots + a_{n+1}Y_{n+1}(x)$, *where* $a_1, \dots, a_{n+1}$ *are some constants which satisty* $a_1 + \cdots + a_{n+1} = 1$.

3. *Show* $\forall\, t\ Be^{At} = e^{At}B$ *if and only if* $AB = BA$.

$\boldsymbol{Chapter}$ 4. Linear differential equations of order n

$A\ linear\ differential\ equation\ of\ order\ n$

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x) \quad (1)$$

$which\ has\ the\ initial\ condition\ y(x_0) = y_0, y'(x_0) = y_0', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \quad (2)$

$$is\ equivalent\ to\ the\ system \begin{cases} \dfrac{dy}{dx} = y_1 \\ \dfrac{dy_1}{dx} = y_2 \\ \cdots \\ \dfrac{dy_{n-2}}{dx} = y_{n-1} \\ \dfrac{dy_{n-1}}{dx} = -p_1(x)y_{n-1} - \cdots - p_{n-1}(x)y_1 - p_n(x)y + f(x) \end{cases}.$$

$And\ this\ can\ be\ written\ in\ the\ form \dfrac{dY}{dx} = A(x)Y + F(x), Y(x_0) = Y_0, where$

$$A(x)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n(x) & \cdots & \dots & & -p_1(x) \end{bmatrix}, F(x)=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}, Y=\begin{pmatrix} y \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

The initial condition can be proposed as $Y(x_0)=Y_0$, *where* $Y_0=\begin{pmatrix} y(x_0) \\ y_1(x_0) \\ \vdots \\ y_{n-1}(x_0) \end{pmatrix}=\begin{pmatrix} y_0 \\ {y_0}' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}.$

Existence and uniqueness theorem. *If the coefficients* $p_k(x)(k=1,2,\dots,n)$ *and* $f(x)$ *are continuous in an interval* J *and if* $x_0 \in J$, *then the initial value problem*

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y=f(x),$$

$y(x_0)=y_0, y'(x_0)=y_0', \dots, y^{(n-1)}(x_0)=y_0^{(n-1)}$ *has exactly one solution in* J.

$\boldsymbol{The\ homogeneous\ differential\ equations\ of\ order\ n\ Ly = 0}$.

$The\ nth - order\ homogeneous\ differential\ equation\ can\ be\ written\ as\ \dfrac{dY}{dx} = A(x)Y,$

$so\ the\ solutions\ of\ the\ differential\ equation\ form\ an\ n - dimensional\ vector\ space.$

$\boldsymbol{Propositions}$. $(a)\ A\ set\ of\ functions\ \phi_1(x), \dots, \phi_n(x)\ is\ called\ linearly\ dependent\ if$

$there\ exist\ constants\ c_1, \dots, c_n\ with\ |c_1| + \cdots + |c_n| > 0\ st.\ c_1\phi_1(x) + \cdots c_n\phi_n(x) = 0\ in\ J.$

$(b)\ If\ \phi_k(x)\ k = 1, \dots n\ are\ n\ solutions\ of$

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0, then Y_k = \left(\phi_k(x), \phi_k'(x) \dots, \phi_k^{(n-1)}(x)\right)^T$$

$k = 1, \dots, n\ are\ n\ solutions\ of\ the\ corresponding\ system\ \dfrac{dY}{dx} = A(x)Y. Thus\ the$

$$wronskian\ of\ the\ n\ solutions\ is\ the\ determinant\ W(x) = \begin{vmatrix} \phi_1 & \dots & \phi_n \\ \phi_1{}' & \dots & \phi_n{}' \\ \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix}.$$

(c) A set of functions $\phi_1(x), \ldots, \phi_n(x)$ is linearly dependent in J if and only if

$$\begin{pmatrix} \phi_1(x) \\ \phi_1{}'(x) \\ \vdots \\ \phi_1^{(n-1)}(x) \end{pmatrix}, \ldots, \begin{pmatrix} \phi_n(x) \\ \phi_n'(x) \\ \vdots \\ \phi_n^{(n-1)}(x) \end{pmatrix} \text{ is linearly dependent in } J.$$

(d) A set of solutions $\phi_1(x), \ldots, \phi_n(x)$ of the homogeneous differential equation

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$ is linearly independent(dependent)

in J if and only if there exists $x_0 \in J$ such that $W(x_0) \neq 0(W(x_0) = 0)$.

(e) *There exist* n *linearly independent solutions of* $\phi_1(x), \ldots, \phi_n(x)$ *for the equation. Every such set of* n *linearly independent solutions is called a fundamental system of solutions. If* $\phi_1(x), \ldots, \phi_n(x)$ *is a fundamental system, then every solution* y *can be written in a unique way as a linear combination*

$$y = C_1\phi_1(x) + \cdots + C_n\phi_n(x).$$

(f) *If* $\phi_1(x), \ldots, \phi_n(x)$ *are* n *solutions of the homogeneous differential equation, then the Wronskian determinant* $W(x) = W(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}$ *for every* $x_0 \in J$.

The inhomogeneous differential equations of order n $Ly = f(x)$.

Theorem. *Every solution* $y(x)$ *of the inhomogeneous differential equation* $Ly = f(x)$ *can be written in the form of* $y(x) = y^*(x) + \phi(x)$, *where* $y^*(x)$ *is a particular solution of the inhomogeneous differential equation and* $\phi(x)$ *is the general solution to the homogeneous differential equation.*

Method of variation of constants.

Let $y(x) = c_1(x)\phi_1(x) + \cdots + c_n(x)\phi_n(x)$, *where* $\phi_1(x), \ldots, \phi_n(x)$ *is a fundamental system of the homogeneous differential equation and* $c_1(x), \ldots, c_n(x)$ *are funcitons that are yet to be determined.*

We refer to the result in chapter 3, $\Phi(x)C'(x) = F(x)$, *where*

$$C(x) = \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix}, \Phi(x) = \begin{bmatrix} \phi_1 & \ldots & \phi_n \\ \phi_1{}' & \ldots & \phi_n{}' \\ \vdots & \ldots & \vdots \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{bmatrix}, F(x) = \begin{pmatrix} 0 \\ \vdots \\ f\ (x) \end{pmatrix}.$$

Due to $|\Phi(x)| \neq 0$, *the unique* $C(x)$ *can be obtained.*

$\textbf{\textit{Example}}.\ y_1 = \cos x$ *and* $y_2 = \sin x$ *are two solutions of* $y'' + y = 0$.

The corresponding Wronskian determinant is $\begin{vmatrix} y_1, & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = 1 \neq 0$ *in* $(-\infty, +\infty)$.

The general solution is $y = C_1 \cos x + C_2 \sin x$, *where* C_1 *and* C_2 *are arbitrary constants.*

$\textbf{\textit{Example}}.$ *Determine the general solution of* $y'' + y = \dfrac{1}{\cos x}$.

Solution. The general solution of the related homogeneous differential equation is

$y = c_1 \cos x + c_2 \sin x$.

Set $y_1 = c_1(x) \cos x + c_2(x) \sin x$ *is a special solution of the inhomogeneous differenti al equation. So* $c_1'(x)$ *and* $c_2'(x)$ *satisfy the following system*

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} c_1{}'(x) \\ c_2{}'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{\cos x} \end{pmatrix}$$

, then we have $c_1'(x) = -\dfrac{\sin x}{\cos x}$, $c_2'(x) = 1$.

By integrating, we obtain $c_1(x) = \ln|\cos x|$, $c_2(x) = x$.

Thus the general solution is $y = c_1 \cos x + c_2 \sin x + \ln|\cos x| \cos x + x \sin x$.

Linear equations of order n with constant coefficients.

$$Now\ let\ Ly = \sum_{i=0}^{n} a_i y^{(i)}(x) = 0, a_i\ are\ constants, a_n = 1\ (1).$$

$$The\ characteristic\ polynomial\ is\ P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i.$$

Theorem. *If* λ *is a zero of the characteristic polynomial of multiplicity* k*, then there are* k *solutions of the differential equation* (1) $e^{\lambda x}, xe^{\lambda x}, \dots, x^{k-1}e^{\lambda x}$ *that correspond to* λ*. In this manner, one obtains* n *linearly independent solutions from the* n *zeros of the characteristic polynomial* $P(\lambda)$ *(each counted according to its multiplicity), that is a fundamental system.*

$$Ly=\sum_{i=0}^{n} a_i y^{(i)}(x)=0, a_i \text{ are constants}, a_n=1 \ (1).$$

Remark. *If* a_i *are real and there exist complex zeros, then this fundamental system contains complex solutions. A real fundamental system can be obtained by splitting the k solutions corresponding to a complex zero* $\lambda=\mu+i\nu$ $(\nu\neq 0)$ *into real and imaginary parts,* $x^i e^{\mu x}\cos\nu x$, $x^i e^{\mu x}\sin\nu x$ $(i=0,1,\dots,k-1)$ *(and discarding the solutions corresponding to* $\bar{\lambda}$.

Example. *Determine the general solution of* $y'' - 5y' = 0$.

Solution. *The characteristic equation is* $\lambda^2 - 5\lambda = 0$, $\lambda_1 = 0, \lambda_2 = 5$ *are the roots*.

The general solution of such equation is $y = c_1 + c_2 e^{5x}$.

Example. *Determine the general solution of* $y''' - 3y'' + 3y' - y = 0$.

Solution. *The characteristic polynomial is* $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$.

The characteristic roots are $\lambda_{1,2,3} = 1$, *so the fundamental system is* e^x, xe^x, x^2e^x.

Then the general solution of such equation is $y = e^x(c_1 + c_2x + c_3x^2)$.

Example. *Determine thegeneral solution of* $y'''' - 4y''' + 5y'' - 4y' + 4y = 0$.

Solution. The characteristic equation is $\lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4\lambda = (\lambda - 2)^2(\lambda^2 + 1)$.

So the characteristic roots are $\lambda_{1,2} = 2, \lambda_3 = i, \lambda_4 = -i$.

Then the fundamental system is $e^{2x}, xe^{2x}, \cos x, \sin x$.

The general solution is $y = e^{2x}(c_1 + c_2x) + c_3 \cos x + c_4 \sin x$.

$Exercise.$

1. $Determine\ the\ solution\ of\ the\ initial\ problem$ $\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 1, y'(0) = 2 \end{cases}.$

2. $Determine\ the\ general\ solution\ of\ y''' - 3y'' + 9y' + 13y = 0.$

3. $Dtermine\ the\ general\ solution\ of\ y'' + 4y' + 4y = 0.$

4. $Discuss\ how\ to\ choose\ \lambda\ so\ that\ the\ initial\ problem$ $\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$ $has\ non$

$-\ zero\ solution.$

Nonhomogeneous linear equations of order n

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x), (1)$

Case 1. $\boldsymbol{f(x) = e^{ax}P_m(x)}$, ***where*** $\boldsymbol{P_m(x) = p_0x^m + p_1x^{m-1} + \cdots + p_{m-1}x + p_m}$.

$\boldsymbol{Remark}$.

1. $If\ a\ is\ not\ a\ characristic\ root, we\ can\ set$

$$y(x) = Q_m(x)e^{ax}$$

$as\ the\ special\ solution\ of\ the\ eqution\ (1).$

2. $If\ a\ is\ a\ k-fold\ characristic\ root, we\ can\ set$

$$y(x) = x^kQ_m(x)e^{ax}$$

$as\ the\ special\ solution\ of\ the\ eqution\ (1).$

$Where\ Q_m(x) = q_0x^m + q_1x^{m-1} + \cdots + q_{m-1}x + q_m.$

Example.

1. *Determine the general solution of* $y'' - 3y' = e^{5x}$.

We can set $y = Ae^{5x}$ *as the special solution of such a equation*.

2. *Determine the general solution of* $y'' - y = \frac{1}{2}e^{x}$.

We can set $y = Axe^{x}$ *as the special solution of such a equation*.

3. *Determine the general solution of* $y'' - 5y' + 6y = 6x^2 - 10x + 2$.

We can set $y = Ax^2 + Bx + C$ *as the special solution of such a equation*.

4. *Determine the general solution of* $y'' - 5y' = -5x^2 + 2x$.

We can set $y = x(Ax^2 + Bx + C)$ *as the special solution of such a equation*.

Exercise.

1. *Determine the general solution of* $y'' - 4y' + 4y = 2e^{2x}$.

2. *Determine the general solution of* $y'' - 2y' + 4y = (x + 2)e^{3x}$.

$Chapter$ 4. Linear differential equations of order n

$A\ linear\ differential\ equation\ of\ order\ n$

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y=f(x)\ \ (1)$$

$which\ has\ the\ initial\ condition\ y(x_0)=y_0, y'(x_0)=y_0', \ldots, y^{(n-1)}(x_0)=y_0^{(n-1)}\ \ (2)$

$$is\ equivalent\ to\ the\ system \begin{cases} \dfrac{dy}{dx}=y_1 \\ \dfrac{dy_1}{dx}=y_2 \\ \cdots \\ \dfrac{dy_{n-2}}{dx}=y_{n-1} \\ \dfrac{dy_{n-1}}{dx}=-p_1(x)y_{n-1}-\cdots-p_{n-1}(x)y_1-p_n(x)y+f(x) \end{cases}.$$

$And\ this\ can\ be\ written\ in\ the\ form \dfrac{dY}{dx}=A(x)Y+F(x), Y(x_0)=Y_0, where$

$$A(x)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n(x) & \cdots & \dots & & -p_1(x) \end{bmatrix}, F(x)=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}, Y=\begin{pmatrix} y \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

The initial condition can be proposed as $Y(x_0)=Y_0$, *where* $Y_0=\begin{pmatrix} y(x_0) \\ y_1(x_0) \\ \vdots \\ y_{n-1}(x_0) \end{pmatrix}=\begin{pmatrix} y_0 \\ {y_0}' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}.$

Existence and uniqueness theorem. *If the coefficients* $p_k(x)(k=1,2,\dots,n)$ *and* $f(x)$ *are continuous in an interval* J *and if* $x_0 \in J$, *then the initial value problem*

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_{n-1}(x)y'+p_n(x)y=f(x),$$

$y(x_0)=y_0, y'(x_0)=y_0', \dots, y^{(n-1)}(x_0)=y_0^{(n-1)}$ *has exactly one solution in* J.

The homogeneous differential equations of order n $\boldsymbol{Ly = 0}$.

The nth − order homogeneous differential equation can be written as $\dfrac{dY}{dx} = A(x)Y$,

so the solutions of the differential equation form an n − dimensional vector space.

Propositions. *(a) A set of functions* $\phi_1(x), \dots, \phi_n(x)$ *is called linearly dependent if there exist constants* $c_1, \dots, c_n$ *with* $|c_1| + \cdots + |c_n| > 0$ *st.* $c_1\phi_1(x) + \cdots c_n\phi_n(x) = 0$ *in J.*

(b) If $\phi_k(x)$ $k = 1, \dots n$ *are n solutions of*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0, \text{then} Y_k = \left(\phi_k(x), \phi_k'(x) \dots, \phi_k^{(n-1)}(x)\right)^T$$

$k = 1, \dots, n$ *are n solutions of the corresponding system* $\dfrac{dY}{dx} = A(x)Y$. *Thus the*

wronskian of the n solutions is the determinant $W(x) = \begin{vmatrix} \phi_1 & \dots & \phi_n \\ \phi_1' & \dots & \phi_n' \\ \vdots & \dots & \vdots \\ \phi_1^{(n-1)} & \dots & \phi_n^{(n-1)} \end{vmatrix}$.

(c) $A\ set\ of\ functions\ \phi_1(x), \ldots, \phi_n(x) is\ linearly\ dependent\ in\ J\ if\ and\ only\ if$

$$\begin{pmatrix} \phi_1(x) \\ \phi_1{}'(x) \\ \vdots \\ \phi_1^{(n-1)}(x) \end{pmatrix}, \ldots, \begin{pmatrix} \phi_n(x) \\ \phi_n'(x) \\ \vdots \\ \phi_n^{(n-1)}(x) \end{pmatrix} is\ linearly\ dependent\ in\ J.$$

(d) $A\ set\ of\ solutions\ \phi_1(x), \ldots, \phi_n(x)\ of\ the\ homogeneous\ differential\ equation$

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0\ is\ linearly\ independent(dependent)$

$in\ J\ if\ and\ only\ if\ there\ exists\ x_0 \in J\ such\ that\ W(x_0) \neq 0(W(x_0) = 0).$

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(f) If $\phi_1(x), \dots, \phi_n(x)$ are n solutions of the homogeneous differential equation, then the Wronskian determinant $W(x) = W(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}$ for every $x_0 \in J$.

The inhomogeneous differential equations of order n $Ly = f(x)$.

Theorem. *Every solution* $y(x)$ *of the inhomogeneous differential equation* $Ly = f(x)$ *can be written in the form of* $y(x) = y^*(x) + \phi(x)$, *where* $y^*(x)$ *is a particular solution of the inhomogeneous differential equation and* $\phi(x)$ *is the general solution to the homogeneous differential equation.*

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Let $y(x) = c_1(x)\phi_1(x) + \cdots + c_n(x)\phi_n(x)$, *where* $\phi_1(x), \dots, \phi_n(x)$ *is a fundamental system of the homogeneous differential equation and* $c_1(x), \dots, c_n(x)$ *are funcitons that are yet to be determined*.

We refer to the result in chapter 3, $\Phi(x)C'(x) = F(x)$, *where*

$$C(x) = \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix}, \Phi(x) = \begin{bmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1{}' & \cdots & \phi_n{}' \\ \vdots & \cdots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{bmatrix}, F(x) = \begin{pmatrix} 0 \\ \vdots \\ f\ (x) \end{pmatrix}.$$

Due to $|\Phi(x)| \neq 0$, *the unique* $C(x)$ *can be obtained*.

$\boldsymbol{Example.}\ y_1 = \cos x$ *and* $y_2 = \sin x$ *are two solutions of* $y'' + y = 0$.

The corresponding Wronskian determinant is $\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = 1 \neq 0$ *in* $(-\infty, +\infty)$.

The general solution is $y = C_1 \cos x + C_2 \sin x$, *where* C_1 *and* C_2 *are arbitrary constants.*

$\boldsymbol{Example.}$ *Determine the general solution of* $y'' + y = \dfrac{1}{\cos x}$.

Solution. The general solution of the related homogeneous differential equation is

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Set $y_1 = c_1(x)\cos x + c_2(x)\sin x$ *is a special solution of the inhomogeneous differenti al equation. So* $c_1'(x)$ *and* $c_2'(x)$ *satisfy the following system*

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} c_1{}'(x) \\ c_2{}'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \hline \cos x \end{pmatrix},$$ *then we have* $c_1'(x) = -\dfrac{\sin x}{\cos x}, c_2'(x) = 1$.

By integrating, we obtain $c_1(x) = \ln|\cos x|, c_2(x) = x$.

Thus the general solution is $y = c_1 \cos x + c_2 \sin x + \ln|\cos x| \cos x + x \sin x$.

Linear equations of order n with constant coefficients.

$$Now\ let\ Ly = \sum_{i=0}^{n} a_i y^{(i)}(x) = 0, a_i\ are\ constants, a_n = 1\ (1).$$

$$The\ characteristic\ polynomial\ is\ P(\lambda) = \sum_{i=0}^{n} a_i \lambda^i.$$

Theorem. *If* λ *is a zero of the characteristic polynomial of multiplicity* k, *then there are* k *solutions of the differential equation (1)* $e^{\lambda x}, xe^{\lambda x}, \ldots, x^{k-1}e^{\lambda x}$ *that correspond to* λ. *In this manner, one obtains* n *linearly independent solutions from the* n *zeros of the characteristic polynomial* $P(\lambda)$ *(each counted according to its multiplicity), that is a fundamental system.*

$$Ly = \sum_{i=0}^{n} a_i y^{(i)}(x) = 0, a_i \; are \; constants, a_n = 1 \; (1).$$

$\boldsymbol{Remark}. If \; a_i \; are \; real \; and \; there \; exist \; complex \; zeros, then \; this \; fundamental \; system$ $contains \; complex \; solutions.$ $A \; real \; fundamental \; system \; can \; be \; obtained \; by \; splitting$ $the \; k \; solutions \; corresponding \; to \; a \; complex \; zero \; \lambda = \mu + iv \; (\nu \neq 0) \; into \; real \; and \; ima$ $ginary \; parts, x^i e^{\mu x} \cos \nu x, \qquad x^i e^{\mu x} \sin \nu x \quad (i = 0,1, \dots, k-1)(\; and \; discarding \; the$ $solutions \; corresponding \; to \; \bar{\lambda}.$

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Solution. *The characteristic equation is* $\lambda^2 - 5\lambda = 0$, $\quad \lambda_1 = 0, \lambda_2 = 5$ *are the roots*.

The general solution of such equation is $y = c_1 + c_2 e^{5x}$.

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The characteristic roots are $\lambda_{1,2,3} = 1$, *so the fundamental system is* e^x, xe^x, x^2e^x.

Then the general solution of such equation is $y = e^x(c_1 + c_2x + c_3x^2)$.

$\boldsymbol{Example}.\ Determine\ the general\ solution\ of\ y'''' - 4y''' + 5y'' - 4y' + 4y = 0.$

$Solution.\ The\ characteristic\ equation\ is\ \lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4\lambda = (\lambda - 2)^2(\lambda^2 + 1).$

$So\ the\ characteristic\ roots\ are\ \lambda_{1,2} = 2, \lambda_3 = i, \lambda_4 = -i.$

$Then\ the\ fundamental\ system\ is\ e^{2x}, xe^{2x}, \cos x, \sin x.$

$The\ general\ solution\ is\ y = e^{2x}(c_1 + c_2 x) + c_3 \cos x + c_4 \sin x.$

$Exercise.$

1. $Determine\ the\ solution\ of\ the\ initial\ problem\ \begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 1, y'(0) = 2 \end{cases}.$

2. $Determine\ the\ general\ solution\ of\ y''' - 3y'' + 9y' + 13y = 0.$

3. $Dtermine\ the\ general\ solution\ of\ y'' + 4y' + 4y = 0.$

4. $Discuss\ how\ to\ choose\ \lambda\ so\ that\ the\ initial\ problem \begin{cases} y'' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}\ has\ non$

$-\ zero\ solution.$

Nonhomogeneous linear equations of order n

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x), (1)$

Case 1. $\boldsymbol{f(x) = e^{ax}P_m(x)}$, ***where*** $\boldsymbol{P_m(x) = p_0x^m + p_1x^{m-1} + \cdots + p_{m-1}x + p_m}$.

$\boldsymbol{Remark}$.

1. $If\ a\ is\ not\ a\ characristic\ root, we\ can\ set$

$$y(x) = Q_m(x)e^{ax}$$

$as\ the\ special\ solution\ of\ the\ eqution$ (1).

2. $If\ a\ is\ a\ k-fold\ characristic\ root, we\ can\ set$

$$y(x) = x^k Q_m(x)e^{ax}$$

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$Where\ Q_m(x) = q_0x^m + q_1x^{m-1} + \cdots + q_{m-1}x + q_m$.

$Example.$

$1.$ $Determine\ the\ general\ solution\ of\ y'' - 3y' = e^{5x}.$

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$2.$ $Determine\ the\ general\ solution\ of\ y'' - y = \frac{1}{2}e^{x}.$

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$3.$ $Determine\ the\ general\ solution\ of\ y'' - 5y' + 6y = 6x^2 - 10x + 2.$

$We\ can\ set\ y = Ax^2 + Bx + C\ as\ the\ special\ solution\ of\ such\ a\ equation.$

$4.$ $Determine\ the\ general\ solution\ of\ y'' - 5y' = -5x^2 + 2x.$

$We\ can\ set\ y = x(Ax^2 + Bx + C)\ as\ the\ special\ solution\ of\ such\ a\ equation.$

Nonhomogeneous linear equations of order n

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x), (1)$

Case 2. $\boldsymbol{f(x) = e^{\alpha x}[P^1{}_m(x)cos\beta x, +P_m^2(x)sin\ \beta x]}$ ***where*** $\boldsymbol{P_m(x) = p_0x^m + \cdots + p_m}$.

$\boldsymbol{Remark}$.

1\. $If\ \alpha + i\beta\ is\ not\ a\ characristic\ root, we\ can\ set$

$$y(x) = e^{\alpha x}[Q_m^{(1)}(x)\cos\beta x + Q_m^{(2)}(x)\sin\beta x]$$

$as\ the\ special\ solution\ of\ the\ eqution\ (1).$

2\. $If \alpha + i\beta\ is\ a\ k - fold\ characristic\ root, we\ can\ set$

$$y(x) = x^k e^{\alpha x}[Q_m^{(1)}(x)\cos\beta x + Q_m^{(2)}(x)\sin\beta x$$

$as\ the\ special\ solution\ of\ the\ eqution\ (1).$

$Where\ Q_m(x) = q_0x^m + q_1x^{m-1} + \cdots + q_{m-1}x + q_m.$

Example.

1. *Determine the general solution of* $y'' + y' - 2y = e^x(\cos x - 7\sin x)$.

We can set $y = e^x(A\cos x + B\sin x)$ *as the special solution of such a equation.*

2. *Determine the general solution of* $y'' + y = 2\sin x$.

We can set $y = x(A\cos x + B\sin x)$ *as the special solution of such a equation.*

$Example.$

1. $Determine\ the\ general\ solution\ of\ y'' - 6y' + 5y = -3e^{x} + 5x^{2}.$

$Solution.\ The\ characteristic\ euqation\ to\ the\ related\ homogeneous\ equation\ is\ \lambda^{2}$

$- 6\lambda + 5 = 0.$

$Then\ the\ general\ solution\ of\ the\ homogeneous\ equation\ is\ y = C_{1}e^{x} + C_{2}e^{5x}.$

$We\ set\ y_{1} = Axe^{x}\ as\ the\ special\ solution\ to\ the\ equation\ y'' - 6y' + 5y = -3e^{x}.$

$Set\ y_{2} = Bx^{2} + Cx + D\ as\ the\ special\ solution\ to\ the\ equation\ y'' - 6y' + 5y = 5x^{2}.$

$Thus\ , y = y_{1} + y_{2}\ is\ a\ special\ solution\ to\ the\ original\ nonhomogeneous\ equation.$

Exercise.

1. *Determine the general solution of* $y'' - 4y' + 4y = 2e^{2x}$.

2. *Determine the general solution of* $y'' - 2y' + 4y = (x + 2)e^{3x}$.

3. *Determine the general solution of* $y'' + 9y = 18\cos 3x - 30\sin 3x$.

$\boldsymbol{Chapter\ 5.\ Stability\ and\ Asymptotic\ Behavior}$

$\boldsymbol{Stability, Asymptotic\ Stability}.\ Let\ x = \phi(t, t_0, x_1) be\ a\ solution\ of\ the\ differential$

$$system \frac{dx}{dt} = f(t, x)\ for\ t_0 \leq t < \infty\ with\ initial\ condition\ x(t_0) = x_1. We\ assume\ that$$

$f(t, x)\ is\ defined\ and\ continuous\ at\ least\ in\ S_\alpha: -\infty \leq t < \infty, x \in D \subseteq R^n\ and\ satisfies$

$Lipschitz\ condition.$

$The\ solution\ x = \phi(t, t_0, x_1) is\ said\ to\ be\ stable\ (in\ the\ sense\ of\ Lyapunov)\ if\ the$

$following\ statement\ is\ ture:$

$For\ every\ \epsilon > 0, there\ exists\ \delta > 0\ such\ that\ every\ solution\ x = x(t, t_0, x_0)\ with$

$||x_0 - x_1|| < \delta\ exists\ for\ all\ t \geq t_0\ and\ satisfies\ the\ inequality$

$$||x(t, t_0, x_0) - \phi(t, t_0, x_1)|| < \epsilon\ for\ t_0 \leq t < \infty.$$

A solution $x = \phi(t, t_0, x_1)$ is called asymptotically stable if it is stable and if there exists $\delta_1 > 0$ such that every solution $x = x(t, t_0, x_0)$ with $||x_0 - x_1|| < \delta_1$ satisfies

$$\lim_{t\to\infty} ||x(t, t_0, x_0) - \phi(t, t_0, x_1)|| = 0 .$$

A solution $x(t)$ is called unstable if it is not stable.

Example. Let $y(t)$ be the solution of $y' = y, y(0) = \eta$ and $z(t)$ be a solution with the initial value $z(0) = \eta + \epsilon$. Then $z(t) - y(t) = \epsilon e^t$, i.e., the difference between two solutions to the same differential equation with different initial conditions tends to ∞ like e^t.

On the other hand, if y and z are two solutions of the differential eqution $y' = -y$ with initial values η and $\eta + \epsilon$, then the difference is given by $z(t) - y(t) = \epsilon e^{-t}$, and hence converges to 0 as $t \to \infty$.

$Set\ x(t) = \phi(t, t_0, x_0), \phi(t) = \phi(t, t_0, x_1), set\ y = x(t) - \phi(t), then\ \frac{dy}{dt} = \frac{dx(t)}{dt} - \frac{d\phi(t)}{dt} =$

$f\big(t, x(t)\big) - f\big(t, \phi(t)\big) = f(t, \phi(t) + y) - f\big(t, \phi(t)\big) = F(t, y), obviously, F(t, 0) = 0.$

$Then\ we\ translate\ the\ stability\ of\ \phi(t) for\ \frac{dx}{dt} = f(t, x)\ into\ the\ sability\ of\ y = 0\ of\ \frac{dy}{dt}$

$= F(t, y).$

$Example.\ Determine\ the\ stability\ of\ the\ zero\ solution\ of\ \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}.$

Solution. set $t_0 = 0$, then for every $t \geq 0$, the solution which satisfies $x(0) = x_0, y(0) = y_0$ is $\begin{cases} x(t) = x_0 \cos t + y_0 \sin t \\ y(t) = -x_0 \sin t + y_0 \cos t \end{cases}$. For every $\epsilon > 0$, set $\delta = \epsilon$, if $(x_0^2 + y_0^2)^{\frac{1}{2}} < \delta$, then we have $[x(t)^2 + y(t)^2]^{\frac{1}{2}} = (x_0^2 + y_0^2)^{\frac{1}{2}} < \delta = \epsilon$.

So the zero solution of such a system is stable.

However, $\lim_{t\to\infty} [x(t)^2 + y(t)^2]^{\frac{1}{2}} = (x_0^2 + y_0^2)^{\frac{1}{2}} \neq 0$.

So zero solution is not asymptotically stable.

$\boldsymbol{The\ method\ of\ Lyapunov}. We\ consider\ real\ antonomous\ systems \dfrac{dx}{dt} = F(x)\ \ (1),$
$where\ F(x) = (F_1(x), \dots, F_n(x))^T is\ continuous\ and\ locally\ Lipschitz\ continuous\ in\ G$
$= \left\{x \in R^n \middle| \|x\| \le K\right\}, and\ F(0) = 0.$

$\boldsymbol{Definition}. \boldsymbol{A\ Lyapunov\ function}\ for\ (1)\ is\ a\ \ real - valued\ function\ V \in C^1(G)\ that$

$satisfies\ the\ relations\ V(0) = 0, V(x) > 0\ for\ x \neq 0\ and \dfrac{dV}{dt} \le 0, where\ \dfrac{dV}{dt}$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i(x) .$$

$\boldsymbol{Stability\ Theorem}\ (\boldsymbol{Lyapunov}). Let\ F \in C(D) with\ F(0) = 0\ and\ let\ there\ exist\ a$

$Lyapunov\ function\ V\ \ for \dfrac{dx}{dt} = F(x). Then$

$(a) \dfrac{dV}{dt} \le 0\ in\ G \Rightarrow the\ zero\ solution\ of\ \ (1)\ is\ stable\ .$

$(b) \frac{dV}{dt} < 0\ in\ G \backslash \{0\} \Rightarrow the\ zero\ solution\ of\ \ (1)\ is\ asymptotically\ stable\ .$

$Example.$ $Determine\ the\ stability\ of\ the\ zero\ solution\ of$ $\begin{cases} x' = -y + x(x^2 + y^2 - 1) \\ y' = x + y(x^2 + y^2 - 1) \end{cases}$ $(2).$

$Proof.$

$We\ consider\ V(x, y) = \dfrac{1}{2}(x^2 + y^2)\ as\ a\ Lyapunov\ function\ in\ D = \{(x, y) | x^2 + y^2 < 1\},$

$it\ has\ \dfrac{dV}{dt} = xx' + yy' = (x^2 + y^2)(x^2 + y^2 - 1) < 0\ in D \backslash \{0\}.$

$So\ the\ zero\ solution\ of\ (2)\ is\ asymptotically\ stable.$

Exercise.

1. *Determine the stability of the zero solution of* $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y \end{cases}.$

2. *Discuss the stability of zero solution of the autonomous system* $\begin{cases} x_1' = Ax_1 - x_1x_2^2 \\ x_2' = Ax_2 + x_1^2x_2 \end{cases}.$

3. *Consider the linear system with constant coefficients* $X' = Ax$.

Show the zero solution of such system is asymptotically stable if all the eigenvalues of A are real and less than 0.

Exercise.

Solve the following differential equations (1-8).

1. $\frac{dy}{dx} = \frac{y^2-1}{2}$

Solution. Obviously, $y = \pm 1$ *are constant solutions to this equation.*

If $y \neq \pm 1$, *the general integral form of the equation is* $\int \frac{2dy}{y^2-1} = x + C$.

The general solution of the equation is $y = \frac{1+ce^x}{1-ce^x}$.

2. $x^2 y' = xy - y^2$.

Solution. The equation can be written $\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$, *and for* $y = ux$ *the differential equation is* $x\frac{du}{dx} = -u^2$.

Clearly, $u = 0$ *is a solution, so* $y = 0$ *is a solution of the original differential equation.*

If $u \neq 0$, *we obtain* $u = \frac{1}{\ln|x|+C}$. *The functions* $y = \frac{x}{\ln|x|+C}$ *are the solutions of the original differential equation.*

3. $x(y' - y) = e^x$.

Solution. $y = Ce^x$ *is the general solution of the homogeneous differential equation* $x\frac{dy}{dx} = xy$.

Let $y = C(x)e^x$ *be a solution of the nonhomogeneous differential equation.*

After a simple calculation, one obtains $C'(x) = \frac{1}{x} \Rightarrow C(x) = ln\,|x| + C$.

Therefore, the general solution of the nonhomogeneous differential equation is $y(x) =$

$(ln|x| + C)e^x$.

4. $y' - y = xy^5$.

Solution. Obviously $y = 0$ *is a particular solution of the equation.*

If $y \neq 0$, *the equation can be transformed into* $\frac{1}{y^5}y' = \frac{1}{y^4} + x$ *by multiplying* $\frac{1}{y^5}$.

Then the function $z = \frac{1}{y^4}$ *satisfies a linear differential equation* $-\frac{1}{4}\frac{dz}{dx} = z + x$. *The general*

solution of the corresponding homogeneous differential equation is $\bar{z} = Ce^{-4x}$, *and* $z^* =$

$\int_0^x -4t \cdot e^{4t} dt \cdot e^{-4x} = -x + \frac{1}{4}$ *is a solution of the nonhomogeneous differential equation.*

It follows that the general solution of the nonhomogeneous equation is

$$z = Ce^{-4x} - x + \frac{1}{4}.$$

Therefore, the solutions of the original differential equation read

$$\frac{1}{y^4} = Ce^{-4x} - x + \frac{1}{4}.$$

5. $y' = \frac{x-y+2}{x+y^2+4}$.

Solution. The differential equation $(x-y+2)dx-(x+y^2+4)dy=0$ *is exact.*

A potential function is given by

$$F(x,y)=\int_0^x (x-y+2)dx+\int_0^y -(y^2+4)dy=\frac{x^2}{2}-xy+2x-\frac{y^3}{3}-4y.$$

Therefore, the solutions of the exact equation are given by

$$F(x,y)=\frac{x^2}{2}-xy+2x-\frac{y^3}{3}-4y=C.$$

6. $(x^2+y^2+x)dx+xydy=0.$

Solution. Set $M=x^2+y^2+x, N=xy$. *The differential equation is not exact.*

However, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1}{x}$ *, and hence* $\mu(x)=x$ *is an integrating factor.*

An exact differential equation can be obtained by multiplying the original differential equation by the integrating factor.

A potential function can be determined by

$$F(x,y)=\int_0^x x(x^2+y^2+x)dx+\int_0^y 0dy=\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}.$$

Thus, the solutions are given by

$$F(x,y)=\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=C.$$

7. $y = y'^2 - xy' + \frac{1}{2}x^2.$

Solution. Set $y' = p$, *the equation* $y = p^2 - xp + \frac{1}{2}x^2$ *implies* $(-p + x)dx + (2p - x)dp = pdx$ *with the property* $dy = y'dx$. *Then, we have* $(2p - x)\left(\frac{dp}{dx} - 1\right) = 0.$

From $2p - x = 0$, *we obtain* $p = \frac{x}{2}$, *hence,* $y = \frac{x^2}{4} + C$.

Substituting $y = \frac{x^2}{4} + C$ *in* $y = p^2 - xp + \frac{1}{2}x^2$ *gives* $C = 0$, *thus* $y = \frac{x^2}{4}$ *is a particular solution to the original differential equation.*

From $\frac{dp}{dx} - 1 = 0$, *we have* $p = x + C$, *it follows that* $y = \frac{1}{2}x^2 + Cx + D$.

Substituting $y = \frac{1}{2}x^2 + Cx + D$ *in* $y = p^2 - xp + \frac{1}{2}x^2$ *gives* $D = C^2$, *thus* $y = \frac{1}{2}x^2 + Cx + C^2$ *is the general solution of the original equation.*

8. $x\sqrt{1+y'^2} = y'$.

Solution. Set $\begin{cases} x = \sin t \\ y' = \tan t \end{cases}$, $dy = \tan t \cdot \cos t \; dt = \sin t \; dt$ *is obtained by*

$dy = y'\,dx$. *Then we have the parametric solution is* $\begin{cases} x = \sin t \\ y = -\cos t + C \end{cases}$

That implies $y = C \pm \sqrt{1-x^2}$.

9. Let $f(x,y)$ *be continuous and satisfy the Lipschitz condition with respect to* y *in* D*:*

$|x-x_0| \le a, |y-y_0| \le b\ (a,b>0)$ *(with Lipschitz condition* N*). If the sequence of Picard*

iterations $\{\phi_n(x)\}$ *of the problem* $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ *converges to* $\phi(x)$*, show* $|\phi_n(x) - \phi(x)| \le$

$\frac{MN^n}{(n+1)!}|x-x_0|^{n+1}$ *for* $x \in D$*, where* $M = max|f(x,y)|$ *in* D*.*

Proof. Since $\phi_n(x) = y_0 + \int_{x_0}^{x} f\big(\xi, \phi_{n-1}(\xi)\big)d\xi$, $\phi_0(x) = y_0$*, and* $\phi(x) = y_0 +$

$\int_{x_0}^{x} f\big(\xi, \phi(\xi)\big)d\xi$*, we have*

$$|\phi_0(x) - \phi(x)| = \left|\int_{x_0}^{x} f\big(\xi, \phi(\xi)\big)d\xi\right| \le M|x-x_0|.$$

Use the Lipschitz condition property of $f(x,y)$*, we have*

$$|\phi_1(x)-\phi(x)|=\left|\int_{x_0}^{x}\left[f\big(\xi,\phi_0(\xi)\big)-f\big(\xi,\phi(\xi)\big)\right]d\xi\right|\le N\left|\int_{x_0}^{x}|\phi_0(\xi)-\phi(\xi)|d\xi\right|$$

$$\le MN\left|\int_{x_0}^{x}|\xi-x_0|d\xi\right|=MN\frac{|x-x_0|^2}{2}.$$

Suppose $|\phi_{n-1}(x)-\phi(x)|\le\frac{MN^{n-1}}{n!}|x-x_0|^n$*, then we have*

$$|\phi_n(x)-\phi(x)|=\left|\int_{x_0}^{x}\left[f\big(\xi,\phi_{n-1}(\xi)\big)-f\big(\xi,\phi(\xi)\big)\right]d\xi\right|\le N\left|\int_{x_0}^{x}|\phi_{n-1}(\xi)-\phi(\xi)|d\xi\right|\le$$

$$\frac{MN^n}{n!}\left|\int_{x_0}^{x}|\xi-x_0|^n d\xi\right|=\frac{MN^n}{(n+1)!}|x-x_0|^{n+1}.$$

So we prove the estimate by mathematical induction.

10. Let $f(y)$ be continuous and differentiable in $(-\infty,+\infty)$, and $yf(y) < 0 (y \neq 0)$. Show every solution of the differential equation $y' = x^2 f(\sin y)$ exists in $(-\infty,+\infty)$, and $y(x)$ is a strictly monotonic function if it's a non-constant solution of the differential equation.

Proof. Since $yf(y) < 0$ in $(-\infty,+\infty)\backslash\{0\}$, we have $\begin{cases} f(y) < 0, y > 0 \\ f(y) > 0, y < 0 \end{cases}$.

Moreover, $f(0) = 0$ is obtained because of $f(y)$ is continuous in $(-\infty,+\infty)$.

It is easy to check that the straight lines $y = n\pi, n = 0, \pm 1, \pm 2, \dots$ are solutions of $y' = x^2 f(\sin y)$. If $y = y(x)$ is an arbitrary non-constant solution, then $m\pi < y(x) < (m+1)\pi$, where m is an integer. Moreover, $y(x)$ can be extended to infinity by the existence and uniqueness theorem and extension theorem, so $y(x)$ exists in $(-\infty,+\infty)$.

In addition, the function $\sin y(x) > 0 (< 0)$ in $(-\infty,+\infty)$ if $m\pi < y(x) < (m+1)\pi$, that implies $y(x)' = x^2 f(\sin y(x)) < 0 (> 0)$ in $(-\infty,+\infty)$. Hence, $y(x)$ is a strictly monotonic function.

11. Consider the initial problem $\begin{cases} \frac{dy}{dx} = y^2 - x^2 \\ y(0) = 1 \end{cases}.$

(a) Use the sequence of Euler lines to calculate the value of the approximate solution at $x = 0.2$*. Here we take the step as* $h = 0.1$*.*

Solution. Here $y_{n+1} = y_n + (y_n^2 - x_n^2) \times h$*, then we have*

$x = 0.1 \Rightarrow y = 1 + (1^2 - 0^2) \times 0.1 = 1.1,$

$x = 0.2 \Rightarrow y = 1.1 + (1.1^2 - 0.1^2) \times 0.1 = 1.22.$

11. Consider the initial problem $\begin{cases} \dfrac{dy}{dx} = y^2 - x^2 \\ y(0) = 1 \end{cases}$.

(b) Consider the sequence of Picard iterations $\{\phi_n(x)\}$, *and calculate* $\phi_2(x)$.

Solution. We construct the sequence of Picard iterations:

$\phi_0(x) = 1,$

$$\phi_1(x) = 1 + \int_0^x [\phi_0^2(\xi) - \xi^2]d\xi = 1 + \int_0^x (1 - \xi^2)d\xi = 1 + x - \frac{1}{3}x^3,$$

$$\phi_2(x) = 1 + \int_0^x [\phi_1^2(\xi) - \xi^2]d\xi = 1 + x + x^2 - \frac{1}{6}x^4 - \frac{2}{15}x^5 + \frac{1}{63}x^7,$$

$$\vdots$$

$\phi_n(x) = \phi_0(x) + \int_0^x [\phi_{n-1}^2(\xi) - \xi^2]d\xi.$

12. Determine whether there exist singular solutions for the following ODEs.

$$y' = \sqrt{y - x}$$

Solution. Set $f(x, y) = \sqrt{y - x}$, *then* $f(x, y)$ *and* $f_y{}' = \frac{1}{2}\frac{1}{\sqrt{y-x}}$ *are continuous for* $y > x$.

By the existence and uniqueness theorem, we just need to consider the point set on $y = x$.

However, $y = x$ *is not a solution to the equation.*

So the equation has no singular solutions.

13. Determine whether there exist singular solutions for the following ODEs.

$y' = 3y^{2/3}$.

Solution. Set $f(x, y) = 3y^{2/3}$, *then* $f(x, y)$ *and* $f_y{}' = 2y^{-1/3}$ *are continuous for* $y \neq 0$.

By the existence and uniqueness theorem, we just need to consider the point set on $y = 0$.

Obviously, $y = 0$ *is a solution to the equation,* $y = (x + C)^3$ *is the general solution of the equation.*

Then $y = (x - x_0)^3$ *and* $y = 0$ *are two solutions which pass through an arbitrary point* $(x_0, 0)$ *on the X-axis.*

So $y = 0$ *is a singular solution to the equation.*

Determine the general real solution of the following differential systems (14-17).

14. $\begin{cases} \dfrac{dy}{dx} = 5y + 4z \\ \dfrac{dz}{dx} = 4y + 5z \end{cases}.$

Solution. From the system, we have $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 9$.

The corresponding eigenvectors are $\boldsymbol{T_1} = (1, -1)^T, \boldsymbol{T_2} = (1,1)^T$.

Then solutions $y_1 = e^x \cdot \boldsymbol{T_1}, y_2 = e^{9x} \cdot \boldsymbol{T_2}$ constitute a fundamental system. For this choice

of fundamental system, the general solution of this system is $\begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = C_1 \begin{pmatrix} e^x \\ -e^x \end{pmatrix} + C_2 \begin{pmatrix} e^{9x} \\ e^{9x} \end{pmatrix}$.

15. $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 3y - 2x \end{cases}.$

Solution. From the system, we have $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = 2 + i, \lambda_2 = 2 - i$.

The corresponding eigenvectors are $\boldsymbol{T_1} = (1,1+i)^T, \boldsymbol{T_2} = (1,1-i)^T$.

Then real solutions $y_1 = e^{2t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}, y_2 = e^{2t}\begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}$

constitute a fundamental system.

For this choice of fundamental system, the general solution of this system is

$$y(x) = C_1 e^{2t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}.$$

16. $$\begin{cases} \dfrac{dy_1}{dx} = 3y_1 + y_2 - y_3 \\ \dfrac{dy_2}{dx} = -y_1 + 2y_2 + y_3 \\ \dfrac{dy_3}{dx} = y_1 + y_2 + y_3 \end{cases}$$

Solution. From the system, we have $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

Set $Y(x) = (R_0 + R_1 x + R_2 x^2)e^{2x}$ *as the solution to the equation, where*

$$\begin{cases} (A - 2E)^3 R_0 = 0 \\ (A - 2E)R_0 = R_1 \\ (A - 2E)R_1 = 2R_2 \end{cases}$$

By a simple calculation, we have $\begin{cases} R_0 = (1,0,0)^T \\ R_1 = (1,-1,1)^T \\ R_2 = \left(-\frac{1}{2},0,-\frac{1}{2}\right)^T \end{cases}, \begin{cases} R_0 = (0,1,0)^T \\ R_1 = (1,0,1)^T, \\ R_2 = (0,0,0)^T \end{cases} \begin{cases} R_0 = (0,0,1)^T \\ R_1 = (-1,1,-1)^T \\ R_2 = \left(\frac{1}{2},\frac{0,1}{2}\right)^T \end{cases}.$

Then the corresponding fundamental system is Y_1, Y_2, Y_3.

The general solution of this system is $Y = C_1Y_1 + C_2Y_2 + C_3Y_3$.

17. $\begin{cases} \dfrac{dx}{dt} = 2x + 3y + 5t \\ \dfrac{dy}{dt} = 3x + 2y + 8e^t \end{cases}.$

Solution. From the related homogeneous system, we have $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = 5, \lambda_2 = -1$.

The corresponding eigenvectors are $\boldsymbol{T_1} = (1,1)^T, \boldsymbol{T_2} = (1,-1)^T$.

Then $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ *constitute a fundamental system.*

Using the method of variation of constants, we set $\begin{pmatrix} x \\ y \end{pmatrix} = C_1(t) \begin{pmatrix} e^{5t} \\ e^{5t} \end{pmatrix} + C_2(t) \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ *as a solution of the nonhomogeneous equation. We obtain*

$$\begin{cases} C_1'(t)e^{5t} + C_2'(t)e^{-t} = 5t \\ -C_1'(t)e^{5t} - C_2{}'(t)e^{-t} = 8e^t \end{cases} \Longrightarrow \begin{cases} C_1(t) = \\ C_2(t) = \end{cases}.$$

The general solution to the homogeneous equation is $\begin{cases} x(t) = \\ y(t) = \end{cases}$

18. If $\boldsymbol{A}(x)_{n\times n}$ is continuous in J, and the set of n solutions $\boldsymbol{Y_1}(x),\ldots,\boldsymbol{Y_n}(x)$ is a fundamental system of solutions to the differential system $\frac{d\,\boldsymbol{Y}}{dx} = A(x)\boldsymbol{Y}$.

Show the general solution of the differential system could be proposed as

$$\boldsymbol{Y}(x) = C_1\boldsymbol{Y_1}(x) + C_2\boldsymbol{Y_2}(x) + \ldots + C_n\boldsymbol{Y_n}(x),$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Proof. For arbitrary constants $C_1, \dots, C_n$*, we have* $\frac{d\boldsymbol{Y}}{dx} = \frac{d[C_1\boldsymbol{Y_1}(x)+\cdots+C_n\boldsymbol{Y_n}(x)]}{dx} = \boldsymbol{A}(x)\boldsymbol{Y}(x)$*, that implies* $\mathbf{Y}(x)$ *is a solution of the differential system.*

Set $\boldsymbol{\Phi}(x) = \left(\boldsymbol{Y_1}(x), \dots, \boldsymbol{Y_n}(x)\right)$*, then* $\boldsymbol{\Phi}(x)$ *is a nonsingular matrix in* J*. If* $\boldsymbol{Z}(x)$ *is an arbitrary solution of the differential system and goes through* $(x_0, \boldsymbol{Y_0})$*, then there exists exactly one vector* $\boldsymbol{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$ *such that* $\boldsymbol{\Phi}(x_0)\boldsymbol{C} = \boldsymbol{Y_0}$ *because of* $|\boldsymbol{\Phi}(x_0)| \neq 0$*. According to the existence and uniqueness theorem,* $\boldsymbol{Z}(x) = \boldsymbol{\Phi}(x)\boldsymbol{C}$ *is the unique solution of the differential system that goes through* $(x_0, \boldsymbol{Y_0})$*. Since this argument can be applied for every solution of the differential system, it follows that every solution can be written in a unique way as a linear combination*

$\boldsymbol{Y}(x) = C_1\boldsymbol{Y_1}(x) + C_2\boldsymbol{Y_2}(x) + \dots + C_n\boldsymbol{Y_n}(x)$.

19. A set of $n+1$ *solutions* $\boldsymbol{Y_1}(x), \ldots, \boldsymbol{Y_{n+1}}(x)$ *to the linear differential system* $\frac{d\boldsymbol{Y}}{dx} = \boldsymbol{A}(x)\boldsymbol{Y} + \boldsymbol{F}(x)$ *is linearly independent, where* $\boldsymbol{A}(x)_{n\times n}$ *and* $\boldsymbol{F}(x)$ *are continuous in* J.

Show the general solution of the differential system is $\boldsymbol{Y}(x) = \sum_{i=1}^{n+1} a_i \boldsymbol{Y_i}(x)$, *where* $a_1, \ldots, a_{n+1}$ *are arbitrary constants which satisfy* $\sum_{i=1}^{n+1} a_i = 1$.

Proof. Set $\boldsymbol{Y}_{\boldsymbol{i}}^*(x) = \boldsymbol{Y}_{\boldsymbol{i}}(x) - \boldsymbol{Y}_{\boldsymbol{n+1}}(x), i = 1, \dots, n,$ *and hence* $\boldsymbol{Y}_{\boldsymbol{i}}^*(x)(i = 1, \dots, n)$ *are n solutions of the corresponding homogeneous differential system.*

A simple calculation shows that

$\sum_{i=1}^{n} c_i \boldsymbol{Y}_{\boldsymbol{i}}^*(x) = \sum_{i=1}^{n} c_i \boldsymbol{Y}_{\boldsymbol{i}}(x) - \sum_{i=1}^{n} c_i \boldsymbol{Y}_{\boldsymbol{n+1}}(x)$, *hence the set of n solutions* $\boldsymbol{Y}_{\boldsymbol{1}}^*(x), \dots, \boldsymbol{Y}_{\boldsymbol{n}}^*(x)$ *is a fundamental system of solutions to the homogeneous differential system because of* $\boldsymbol{Y}_{\boldsymbol{1}}(x), \dots, \boldsymbol{Y}_{\boldsymbol{n+1}}(x)$ *are linearly independent.*

Therefore, the general solution of $\frac{d\boldsymbol{Y}}{dx} = \boldsymbol{A}(x)\boldsymbol{Y} + \boldsymbol{F}(x)$ *is*

$\boldsymbol{Y}(x) = \sum_{i=1}^{n} a_i \boldsymbol{Y}_{\boldsymbol{i}}^*(x) + \boldsymbol{Y}_{\boldsymbol{n+1}}(x) = \sum_{i=1}^{n} a_i \boldsymbol{Y}_{\boldsymbol{i}}(x) + [(1 - \sum_{i=1}^{n} a_i)\boldsymbol{Y}_{\boldsymbol{n+1}}(x)]$, *where* $a_i (i = 1, \dots, n)$ *are arbitrary constants.*

20. Solve the following high order differential equations

(a) $y''' - 3y'' + 3y' - y = 0.$

Solution. The characteristic equation $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$ *has three roots*

$\lambda_{1,2,3} = 1.$

A fundamental system of solutions is given by e^x, xe^x, x^2e^x.

The general solution of the differential equation is

$y = C_1e^x + C_2xe^x + C_3x^2e^x$.

(b) $y'' - 5y' = -5x^2 + 2x.$

Solution. The characteristic roots are $\lambda_1 = 0, \lambda_2 = 5.$

A fundamental system of solutions to the related homogeneous equation is given by $1, e^{5x}$.

We can set $y = x(Ax^2 + Bx + C)$ *as the special solution to the nonhomogeneous equation.*

Substituting the special solution into the equation, we obtain

$$A = \frac{1}{3}, B = 0, C = 0.$$

**C discriminant method*

$\Phi(x,y,C)=0$ *is the general integral of the differential equation* $y'=f(x,y)$.

$$\mathrm{T}:\begin{cases}x=\phi(C)\\ y=\psi(C)\end{cases}\ satisfies\ \begin{cases}\Phi(x,y,C)=0\\ \Phi'_c(x,y,C)=0\end{cases}\ and\ \begin{cases}\phi'^2(C)+\psi'^2(C)\neq 0\\ \Phi'^2_x(\phi(C),\psi(C),C)+\Phi'^2_y(\phi(C),\psi(C),C)\neq 0\end{cases}$$

T *is the envelop to the integral curves*.

Example. Determine the sigular solutions to the following equations.

1. $y'=3y^{\frac{2}{3}}$.

2. $y'=\sqrt{1-y^2}$.