$f(x)$ is continuous at $x=c$ if $\forall \varepsilon>0\ \exists \delta>0$ s.t. if $0<|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

Some time we are interested in case where $f(x)$ is not defined

If $f(c)$ does not excist, we could still consider

$$\lim_{x\to c} f(x) = L$$

Limit is L : $\forall \varepsilon>0\ \exists \delta>0$ s.t. if $0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$

Worked example: $f(x)=\sqrt{x}$ has limit 2 as $x\to 4$.

$\varepsilon=1$

How could we guarantee $2-1<\sqrt{x}<2+1$? $\quad |\sqrt{x}-2|<1$

Easy: we just need $1<x<9$ $\quad \Uparrow$

So we see that if $4-3<x<4+3<9$ $\quad |x-4|<3$

then $2-1<\sqrt{x}<2+1$

$\delta=3$

What about $\varepsilon=0.01$? or $\varepsilon=0.0001$?

Can we guarantee $1.99 < \sqrt{x} < 2.01$? As before: $3.9601 < x < 4.0401$

So if $3.9601 = 4 - 0.0399 < x < 4 + 0.0399 < 4.0401$ it is ok. So if $\varepsilon = 0.01$ we can take $\delta = 0.0399$ & $|x-4| < 0.0399 \Rightarrow |\sqrt{x} - 2| < 0.01$

General Proof Given any $\varepsilon > 0$, how can we guarantee

$$|\sqrt{x} - 2| < \varepsilon \iff 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon \underset{[\text{if } \varepsilon < 2]}{\iff} 4 - 4\varepsilon + \varepsilon^2 < x < 4 + 4\varepsilon + \varepsilon^2$$

So take $\delta = 4\varepsilon - \varepsilon^2$, so $4 - \delta < x < 4 + \delta$

$$\Rightarrow 4 - 4\varepsilon + \varepsilon^2 < x < 4 + 4\varepsilon - \varepsilon^2 < 4 + 4\varepsilon + \varepsilon^2$$

$$\Rightarrow 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon$$

That is, $\forall \varepsilon > 0$ $(\& \ \varepsilon \not\geq 2)$, we take $\delta = 4\varepsilon - \varepsilon^2$ & then $|x - 4| < \delta \Rightarrow |\sqrt{x} - 2| < \varepsilon$

[If $\varepsilon \geq 2$, take $\delta = 4$ & check $|x-4| < 4 \Rightarrow |\sqrt{x} - 2| < 2$]

Given $\varepsilon > 0$, how to find $\delta > 0$ s.t. $|x-4| < \delta \Rightarrow |\sqrt{x}-2| < \varepsilon$

We could use $2|\sqrt{x}-2| \leq |x-4| = |(\sqrt{x}-2)\underbrace{(\sqrt{x}+2)}_{\text{at least } 2}|$

So if we choose $\delta = 2\varepsilon$ it works:

$$|x-4| < \delta \Rightarrow 2|\sqrt{x}-2| \leq |x-4| < \delta = 2\varepsilon$$

$$\Rightarrow |\sqrt{x}-2| < \varepsilon$$

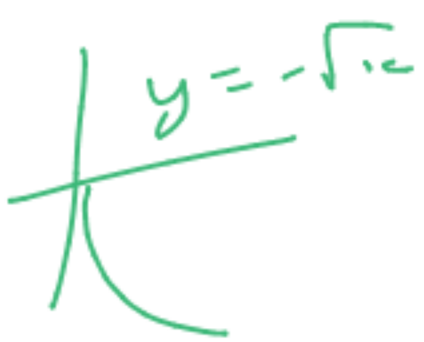
Now we know $\lim_{x\to 4} \sqrt{x} = 2 = \sqrt{4}$

So $\sqrt{x}$ is continuous at $x=4$.

We could repeat the proof(s) replacing 4, 2 by c, $\sqrt{c}$.

to prove $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ so $\sqrt{x}$ is continuous

What about the origin?



New definition Left & right LIMITS.

$$\lim_{x\to c^+} f(x) = L : \forall \varepsilon>0\ \exists \delta>0 : 0<x-c<\delta \Rightarrow |f(x)-L|<\varepsilon$$

$$\lim_{x\to c^-} f(x) = L : \forall \varepsilon>0\ \exists \delta>0 : 0<c-x<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Note: $\lim_{x\to c} f(x) = L \Longleftrightarrow$ both left & right limits are L

f continuous from left or right at $x=c$ if left or right limit $= f(c)$.

E.g. $\sqrt{x}$ is continuous from right at $x=0$.

Proof?.