# Lecture 33: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

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# Sequences. Representing Sequences

• **Sequences are fundamental** to the study of **infinite series** and to many aspects of mathematics.

• A sequence is a **list of numbers**

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

• **in a given order**. **Order is important!**

• Each of $a_1, a_2, a_3$ and so on **represents a number**. These are the **terms of the sequence**.

• The integer $n$ is called the **index** of $a_n$, and **indicates where $a_n$ occurs in the list**.

• An **infinite sequence of numbers is a function of index** $n$ whose domain is the set of positive integers.

# Sequences. Representing Sequences

- **For example**, the function associated with the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

- sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on.

- The **general behavior** of this sequence is described by the **formula** $a_n = 2n$.

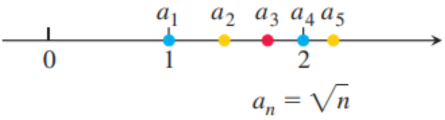
- Sequences can be **described by writing rules** that specify their terms, such as $a_n = \sqrt{n}$ or **by listing terms**

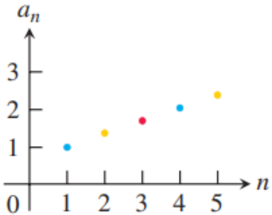
$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

- We also sometimes write a sequence using its rule, as with

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

- Figure shows two ways to **represent sequences graphically**.



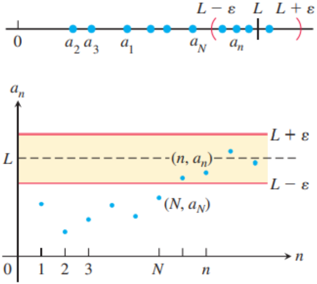



**DEFINITIONS** The sequence $\{a_n\}$ **converges** to the number $L$ if for every positive number $\varepsilon$ there corresponds an integer $N$ such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number $L$ exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to $L$, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call $L$ the **limit** of the sequence (Figure 10.2).



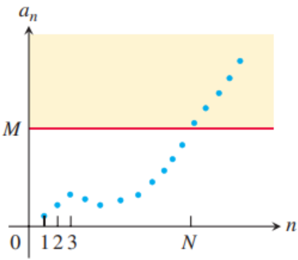
# Convergence and Divergence

**DEFINITION** The sequence $\{a_n\}$ **diverges to infinity** if for every number $M$ there is an integer $N$ such that for all $n$ larger than $N$, $a_n > M$. If this condition holds we write

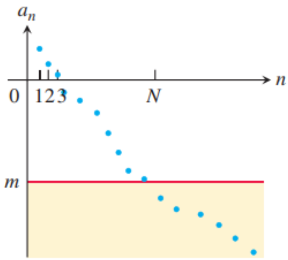
$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly, if for every number $m$ there is an integer $N$ such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$



(a)



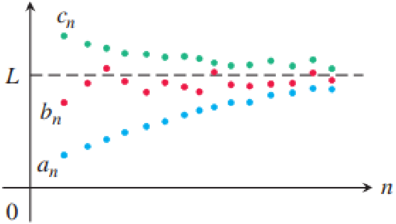
(b)

# Calculating Limits of Sequences

**THEOREM 1** Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $A$ and $B$ be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number $k$)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

**THEOREM 2—The Sandwich Theorem for Sequences**
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all $n$ beyond some index $N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.



**EXAMPLE** Since $1/n \to 0$, we know that

**(a)** $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

**(b)** $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

**(c)** $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

**(d)** If $|a_n| \to 0$, then $a_n \to 0$ because $-|a_n| \le a_n \le |a_n|$.

# Using L'Hôpital's Rule

**THEOREM 3—The Continuous Function Theorem for Sequences**
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if $f$ is a function that is continuous at $L$ and defined at all $a_n$, then $f(a_n) \to f(L)$.

**THEOREM 4** Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n\to\infty} a_n = L \qquad \text{whenever} \qquad \lim_{x\to\infty} f(x) = L.$$

**EXAMPLE** Does the sequence whose $n$th term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

**Solution** The limit leads to the indeterminate form $1^\infty$. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of $a_n$:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$
\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) && \infty \cdot 0 \text{ form} \\
&= \lim_{n\to\infty} \frac{\ln\left(\dfrac{n+1}{n-1}\right)}{1/n} && \frac{0}{0} \text{ form} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} && \text{L'Hôpital's Rule: differentiate numerator and denominator.} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2. && \text{Simplify and evaluate.}
\end{aligned}
$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 3 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to $e^2$.

**THEOREM 5** The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), $x$ remains fixed as $n \to \infty$.

# Infinite Series

**DEFINITIONS** Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number $a_n$ is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number $s_n$ being the ***n*th partial sum**. If the sequence of partial sums converges to a limit $L$, we say that the series **converges** and that its **sum** is $L$. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

**Geometric series** are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which $a$ and $r$ are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** $r$ can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2,\, a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3,\, a = 1$$

# Geometric Series

If $r = 1$, the $n$th partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of $a$. If $r = -1$, the series diverges because the $n$th partial sums alternate between $a$ and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$\begin{aligned}
s_n &= a + ar + ar^2 + \cdots + ar^{n-1} && \text{Write the } n\text{th partial sum.}\\
rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n && \text{Multiply } s_n \text{ by } r.\\
s_n - rs_n &= a - ar^n && \text{Subtract } rs_n \text{ from } s_n. \text{ Most of the terms on the right cancel.}\\
s_n(1 - r) &= a(1 - r^n) && \text{Factor.}\\
s_n &= \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1). && \text{We can solve for } s_n \text{ if } r \neq 1.
\end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 10.1), so $s_n \to a/(1 - r)$ in this case. On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

# Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

**EXAMPLE** The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

**EXAMPLE** The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

# The nth-Term Test for a Divergent Series

• One reason that a series may fail to converge is that its terms don't become small.

We now show that $\lim_{n\to\infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let $S$ represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the $n$th partial sum. When $n$ is large, both $s_n$ and $s_{n-1}$ are close to $S$, so their difference, $a_n$, is close to zero. More formally,

$$a_n = s_n - s_{n-1} \quad \rightarrow \quad S - S = 0.$$

Difference Rule
for sequences

This establishes the following theorem.

**THEOREM 7** If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 7 leads to a test for detecting the kind of divergence

**The *n*th-Term Test for Divergence**

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

# Combining Series

**THEOREM 8** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number $k$).

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

# Combining Series

**EXAMPLE** Find the sums of the following series.

**(a)** 
$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$ Difference Rule

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)}$$ Geometric series with $a = 1$ and $r = 1/2, 1/6$

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

**(b)** 
$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$$ Constant Multiple Rule

$$= 4\left(\frac{1}{1-(1/2)}\right)$$ Geometric series with $a = 1, r = 1/2$

$$= 8$$