MA2252 Introduction to Computing

Lecture 11
Solving System of Linear Equations
Part 1

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Learning outcomes

At the end of lecture, students will be able to

- understand basic theory of system of linear equations
- use MATLAB to find solutions to the system

A system of linear equations is represented as $m \times n$

$m \neq n$

$$
\left.\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_2
\end{array}\right\} \qquad (1)
$$

m equations

$x_1, x_2, \ldots, x_n \rightarrow n$ variables

The aim of this lecture is to find solution to the above system.

Consider first this simple equation:

$A_{1\times 1}$ special case of $Ax = b$

$$ax = b, \quad a, b \in \mathbb{R}. \tag{2}$$

Find a solution in the following cases.

$\frac{a \times b}{a} = b$

1. $a \neq 0$ $\quad x = b/a$
2. $a = 0$ and $b \neq 0$ $\quad \underbrace{0 \times x}_{=0} = \underbrace{b}_{\neq 0}$
3. $a = 0$ and $b = 0$ $\quad 0 \times x = 0$ $\quad x \in \mathbb{R}$ $\quad 0 \times 1 = 0,\ 0 \times 2 = 0,\ 0 \times 3 = 0$

Introduction (contd.)

The solutions in the three cases:

1. x=b/a (unique solution)
2. $x = \emptyset$ (no solution) empty set
3. $x \in \mathbb{R}$ (infinitely many solutions)

Now consider this matrix equation: $x = 0$

$$Ax = b \qquad x \;\&\; x^* \qquad x = x^* \quad \text{exercise} \tag{3}$$

where A is a n x n matrix. Find a solution in the following cases.

square matrix

1. $|A| \neq 0$ $\quad x = A^{-1}b$ unique solution; A is non-singular
2. $|A| = 0$ and $b \neq \underline{0}$ $\quad$ zero vector; no solution or infinitely many solutions
3. $|A| = 0$ and $b = \underline{0}$ $\quad$ infinitely many sol^n

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

$x = A^{-1}b$

rank(A) < n when A is singular.

The solutions in the three cases:

$Ax = \underline{0}$

$x = \underline{0}$ $|A| = 0$

1. $x = A^{-1}b$ (unique solution)
2. $x = \emptyset$ (no solution) $b = \underline{0}$
3. $x = N(A)$ (infinitely many solutions)

Here, $N(A)$ means nullspace of matrix A.

A^{-1} exists

$x = A^{-1}b$
$= 0$

Question: What is the solution to (3) when $|A| \neq 0$ and $b = 0$?

$Ax = b$

$x = A\backslash b$ solves the system (1) of linear equations $Ax = b$.

back slash

forward slash

Example: Solve the system of equations:

$$[A\ b] = \begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{aligned} 2x + y &= 4 \\ x - y &= -1 \end{aligned} \tag{4}$$

Here, $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

$2 \times 1 + 2 = 4$

$1 - 2 = -1$

So, $x = A\backslash b$ gives $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$x = A^{-1} b$

inv(A) * b

Question: Can you find this solution using MATLAB's inv() function?

inv(A) $= A^{-1}$

Backslash operator (contd.)

Demo

Backslash operator (contd.)

$Ax = b$

$ax = b$

- If A is a square matrix then $A \backslash b$ and $inv(A) * b$ are equivalent.
- For scalars a and b, $a \backslash b$ solves the equation $ax = b$. So, $a \backslash b$ and b/a are equivalent.

Let us return back to system of equations (1).

For solving this system, $x = A\backslash b$ gives unexpected results when

1. the system has no solution. → no need to find inv(A)*b using MATLAB
2. the system has infinitely many solutions. In this case, a particular solution may be found using $x = pinv(A) * B$. Here, $pinv(A)$ computes the 'pseudo-inverse' of A.

A'

$Ax = b$

$\cancel{A^TA}$ $\cancel{A}$

$Ax = b \Rightarrow x = A^{-1}b$

$A^TAx = A^Tb$

$(A^TA)^{-1}A^TAx = (A^TA)^{-1}A^T * b$

Backslash operator (contd.)

Exercise

When A^{-1} exists
then $A^{-1} = (A^TA)^{-1}A^T$

$$\underset{\| \atop x}{I}x = \underbrace{(A^TA)^{-1}A^T}_{\text{pseudo inverse}} b$$

Demo

Rank of a matrix

To find out if the system (1) has a unique or infinitely many solutions, we need to understand 'rank' of a matrix.

Definition

Rank of a matrix A is defined as the maximum number of linearly independent rows/columns of A.

v_1 & v_2

$\lambda_1 v_1 + \lambda_2 v_2 = 0$ if & only if $\lambda_1 = \lambda_2 = 0$

Question: Find out the rank of these matrices.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

MATLAB's rank() function finds the rank of a matrix.

⇒ for big matrices, it is very useful.

Rank of a matrix (contd.)

Demo

→ rank test

Rank can be used to determine if the system (1) has no solution, unique solution or infinitely many solutions.

n = no. of variables of A → RHS $\neq 0$

n = no. of columns of

$[A \; b]$ → augmented matrix $A \mid b$

- Non-homogeneous equations ($Ax = b$, $b \neq 0$)
  1. $rank(A) < rank([A \quad b]) \implies$ No solution
  2. $rank(A) = rank([A \quad b]) = n \implies$ Unique solution
  3. $rank(A) = rank([A \quad b]) = k < n \implies$ Infinitely many solutions

$$\left[\, A \;\middle|\; b \,\right]$$

- Homogeneous equations ($Ax = 0$) $n \leq m$
  1. $rank(A) = n \implies$ Unique solution (the trivial solution)
  2. $rank(A) = k < n \implies$ Infinitely many solutions

$x = 0$

sol^n always exists because 0 is always a sol^n

Finding solutions

For non-homogeneous equations:

- First, check the existence of solution using Rank method.
- If the solution exists and is unique, find the solution using $x = A\backslash b$.
- If there are infinitely many solutions, first find the particular solution (say x^*) using $x^* = pinv(A) * b$. The general solution is given by $x = x^* + N(A)$ where $N(A)$ is the nullspace of A.

$Ax^* = b$ $\quad A(x^* + X) = b$ $\quad AX = 0$

$A(x^* + N(A)) = b$

Rank-nullity theorem

$\text{rank}(A) + \underbrace{\text{nullity of } A}_{p} = n$

Follow these steps to find the nullspace $N(A)$:

- Use MATLAB's null(A) function to create a matrix containing orthonormal basis of N as column vectors.

no. of columns

- Let $P =$ null(A) and $p =$nullity of A. Then $p = n - rank(A)$, (**Why?**)

size(P, 2)

no. of columns ~~vectors~~ in null(A) or dimension of basis of A.

- The nullspace of A is then given by

size(P, 1)

$$N(A) = c_1 * P(:, 1) + c_2 * P(:, 2) + \cdots + c_p P(:, p)$$

where the constants $c_1, c_2, \cdots, c_p \in \mathbb{R}$.

Example: Solve the system of equations:

$$\begin{aligned} x + y + z + w &= 6 \\ x + 2y - 3z - w &= -4 \\ y - 4z - 2w &= -10 \\ 2x + 3y - 2z &= 2 \end{aligned} \tag{5}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 \\ 0 & 1 & -4 & -2 \\ 2 & 3 & -2 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ -4 \\ -10 \\ 2 \end{bmatrix}$$

Finding solutions (contd.)

Demo

Finding solutions (contd.)

For homogeneous equations:

- Find $rank(A)$. If $rank(A) = n$, then the trivial solution $x = \underline{0}$ is the only solution.
- Otherwise, the solution is $x = N(A)$.

End of Lecture 11

Please provide your feedback here