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MA3077 (DLI) Operational Research

Lecture 5&6 – Feasibility and duality in linear programming

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Recap and lecture outline

Summary: so far we have learnt:

- how to solve simple (2 dimensional) linear programming problems using a graphical method
- that linear programming problems can be written in many equivalent formulations, and that one of them is know as standard form
- that some nonlinear functions such as max, min, abs, , and can be described using linear functions.
- how to solve linear programming problems in matlab.

Today: Duality theory in linear optimisation, following closely chapters 2.3 and 2.4 of the Mosek Cookbook.

Feasible set

Definition: The *feasible set* of the linear programming problem

is defined as

and is a convex polytope.

A linear programming problem is *feasible* if . Otherwise, the problem is *infeasible*.

Example of an infeasible problem

The following linear programming problem

is infeasible because multiplying the equality constraint from the left with the vector leads to

which is impossible if .

Farkas' lemma

Lemma: Let and . Then, exactly one of the following is true:

1. There exists such that .
2. There exists such that and .

Proof: Let be the closed convex cone spanned by the columns of . If , then the first alternative is true. Otherwise, standard (but not trivial) hyperplane separation theorems imply the existence of a hyperplane passing through the origin that separates and . Let be a vector normal to this hyperplane. Then, without loss of generality, and for all , which in turn implies . Finally, statements 1. and 2. cannot be true at the same time, otherwise

which is a contradiction. □

Duality – primal problem

Primal problem: We consider the following liner programming problem in standard form.

By convention, the optimal objective value is either:

- if the problem is infeasible,
- finite if the problem has an optimal solution,
- if the problem is unbounded.

Example: the problem is unbounded.

Duality – Lagrange function

Definition: To the primal problem

we associate the *Lagrangian* function defined by

The variables and are called *Lagrange multipliers* or *dual variables*.

Note that, for any feasible and any , we have

Duality – dual function

Definition: The dual function of the primal problem

is defined as

Duality – dual problem

Since for any the optimal dual objective value of the *dual problem*

is the best lower bound of the optimal objective value of the primal problem.

Remark: the optimal dual objective value is either:

- if the problem is infeasible,
- finite if the dual problem has an optimal solution,
- if the problem is unbounded.

Duality – example 1/2

Consider the primal problem

which in standard form is

Let denote the optimal objective value.

Duality – example 2/2

The Lagrangian associated to

is

and the dual problem is

- To be continued...

Duality – the dual of the dual 1/3

The dual of

is

which, we can also write as

Duality – the dual of the dual 2/3

To derive the dual of the dual problem

we introduce the Lagrangian

which, for any feasible , satisfies

Duality – the dual of the dual 3/3

Given the Lagrangian

the dual function is

and the best upper bound of is the optimal objective value of

which is the original primal problem.

Weak duality

Lemma: Let and denote the optimal objective value of

respectively. Then, .

Proof: Let and . Then,

Since this is true for any and , it follows that □

Corollary: if and are such that , then and are optimal solutions to the primal and dual problems, respectively.

Strong duality 1/3

Lemma: Let and denote the optimal objective value of

respectively. If at least one and is finite, then .

Proof: Assume that is finite and let Since is optimal, the following linear system of equations has no solutions

By Farkas' lemma, there is a vector such that

Strong duality 2/3

Lemma: Let and denote the optimal objective value of

respectively. If at least one and is finite, then .

Proof: By Farkas' lemma, there is a vector such that

Denote , with and . Then, since the primal problem is feasible, we conclude that .
Without loss of generality, we set . Then,

that is, and .

Strong duality 3/3

Lemma: Let and denote the optimal objective value of

respectively. If at least one and is finite, then .

Proof: We conclude that

The first inequality implies that , and letting in the second one implies

Hence, . Since weak duality implies , we conclude that .
The proof starting with being feasible is analogous.
□

Primal and dual Farkas' lemma

Lemma: For the primal-dual pair of linear programming problems

the following equivalences hold:

1. The primal problem is infeasible iff there is such that and .
2. The dual problem is infeasible iff there is such that and .

Summary and self-study

Summary: we have learnt

- that not all linear programming problems are feasible,
- how to derive the dual problem of a primal problem,
- *weak duality*: the optimal dual objective is a lower bound on the primal one,
- *strong duality*: if the primal or the dual are feasible, the bound is sharp.

Self-study: Write the following linear programming problem in standard form.

Then, derive its dual problem identifying clearly the Lagrangian and the dual function.