Composition and Inverse Mappings

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Outline

Composition of Linear Maps

Let U, V, W be sets. Let

$$F : U \to V \qquad G : V \to W$$

be mappings. Then we can form the composite mapping from U into W, denoted by $G \cdot F$. It is by definition the mapping such that

$$(G \circ F)(u) = G(F(u)) \quad \text{for all } u \text{ in } U.$$

Let A be an $m \times n$ matrix, and let B be a $q \times m$ matrix. Then we may form the product BA. Let

$$L_A : \mathbb{R}^n \to \mathbb{R}^m \text{ be the linear map such that } L_A(X) = AX$$

and let

$$L_B : \mathbb{R}^m \to \mathbb{R}^q \text{ be the linear map such that } L_B(Y) = BY.$$

Then we may form the composition linear map $L_B \circ L_A$ such that

$$(L_B \circ L_A)(X) = L_B(L_A(X)) = L_B(AX) = BAX.$$

Thus we have

$$L_B \circ L_A = L_{BA}.$$

We see that composition of linear maps corresponds to multiplication of matrices.

Let V be a vector space, and let w be an element of V. Let

$$T_w : V \to V$$

be the translation by w, that is the map such that $T_w(v) = v + w$. Then we have

$$T_{w_1}(T_{w_2}(v)) = T_{w_1}(v + w_2) = v + w_2 + w_1.$$

Thus

$$T_{w_1} \circ T_{w_2} = T_{w_1 + w_2}.$$

We can express this by saying that the composite of two translations is again a translation. Of course, the translation T_w is not a linear map if $w \neq O$ because

$$T_w(O) = O + w = w \neq O,$$

and we know that a linear map has to send O on O.

Theorem

Let U, V, W, S be sets. Let

$$F : U \to V, \quad G : V \to W, \quad H : W \to S$$

be mappings. Then

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

Theorem

Let U, V, W be vector spaces. Let

$$F : U \to V \text{ and } G : V \to W$$

be linear maps. Then the composite map $G \circ F$ is also a linear map.

Theorem

Let U, V, W be vector spaces. Let

$$F : U \to V$$

be a linear mapping, and let G, H be two linear mappings of V into W. Then

$$(G + H) \circ F = G \circ F + H \circ F.$$

If c is a number, then

$$(cG) \circ F = F \circ F + H \circ F.$$

If $T : U \to V$ is a linear mapping from U into V, then

$$G \circ (F + T) = G \circ F + G \circ T.$$

As with matrices, we see that composition and addition of linear maps behaves like multiplication and addition of numbers. However, the same warning as with matrices applies here. First, we may not have commutativity, and second we do not have "division", except as discussed in the next section for inverse, when they exist.

Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

be the linear map given by

$$F(x, y, z) = (x, y, 0)$$

and let G be the linear mapping given by

$$G(x, y, z) = (x, z, 0)$$

Then $(G \circ F)(x, y, z) = (x, 0, 0)$, but $(F \circ G)(x, y, z) = (x, z, 0)$.

Inverses

Let

$$F : V \to W$$

be a mapping (which in the application is linear). We say that F has an inverse if these exists a mapping

$$G : W \to V$$

such that

$$G \circ F = I_v \quad \text{and} \quad F \circ G = I_w.$$

By this we mean that the composite maps $G \circ F$ and $F \circ G$ are the identity mappings of V and W, respectively. If F has an inverse, we also say that F is invertible.

The inverse mapping of the translation T_u is the translation T_{-u}, because

$$T_{-u} \circ T_u(v) = T_{-u}(v + u) = v + u - u = v.$$

Thus

$$T_{-u} \circ T_u = I.$$

Similarly, $T_u \circ T_{-u} = I$.

Let A be a square $n \times n$ matrix, and let

$$L_A : \mathbb{R}^n \to \mathbb{R}^n$$

be the usual linear map such that $L_A(X) = AX$. Assume that A has an inverse matrix A^{-1}, so that $AA^{-1} = A^{-1}A = I$. Then the formula

$$L_A \circ L_B = L_{AB}$$

of the preceding sections shows that

$$L_A \circ L_{A^{-1}} = L_I = I.$$

Hence L_A has an inverse mapping, which is precisely multiplication by A^{-1}.

Theorem

Let $F : U \to V$ be a linear map, and assume that this map has an inverse mapping $G : V \to U$

Proof.

Let $v_1, v_2 \in V$. We must first show that

$$G(v_1 + v_2) = G(v_1) + G(v_2).$$

Let $u_1 = G(v_1)$ and $u_2 = G(v_2)$. By definition, this means that

$$F(u_1) = v_1 \text{ and } F(u_2) = v_2.$$

Since F is linear, we find that

$$F(u_1 + u_2) = F(u_1) + F(u_2) = v_1 + v_2.$$

By definition of the inverse map, this means that $G(v_1 + v_2) = u_1 + u_2$, thus proving what we wanted. Similarly, we can prove $G(cv) = cG(v)$. □

Let $L : V \to V$ be a linear map such that $L^2 = O$. Then $I + L$ is invertible, becasue

$$(I + L)(I - L) = I^2 - L^2 = I,$$

and similarly on the other side, $(I - L)(I + L) = I$. Thus we have

$$(I + L)^{-1} = I - L.$$

Let

$$F : V \to W$$

be a mapping. We say that F is injective (in older terminology, one-to-one) if given elements v_1, v_2 in V such that $v_1 \neq v_2$ then $F(v_1) \neq F(v_2)$. Suppose that F is a linear map whose kernel is not $\{O\}$. Then there is an element $v \neq O$ in the kernel, and we have

$$F(O) = F(v) = O.$$

Hence F is not injective.

Theorem

A linear map $F : V \to W$ is injective if and only if its kernel is $\{O\}$.

Proof.

We have already proved one implication. Conversely, assume that the kernel is $\{O\}$. We must prove that F is injective. Let $v_1 \neq v_2$ be distinct elements of V. We must show that $F(v_1) \neq F(v_2)$. But

$$F(v_1) - F(v_2) = F(v_1 - v_2) \quad \text{because } F \text{ is linear.}$$

Since the kernel of F is $\{O\}$, and $v_1 - v_2 \neq O$, it follows that $F(v_1 - v_2) \neq O$. Hence $F(v_1) - F(v_2) \neq O$, so $F(v_1) \neq F(v_2)$. This proves the theorem. □

Let $F : V \to W$ be a mapping. If the image of F is all of W then we say that F is surjective. The two notions for a mapping to be injective or surjective combine to give a basic criterion for F to have an inverse.

Theorem

A mapping $F : V \to W$ has an inverse if and only if it is both injective and surjective.

Proof

Suppose F is both injective and surjective. Given an element w in W, there exists an element v in V such that $F(v) = w$ (because F is surjective). There is only one such element v (because F is injective). Thus we may define

$$G(w) = \text{unique element } v \text{ such that } F(v) = w.$$

By the way we have defined G, it is then clear that

$$G(F(v)) = v \quad \text{and} \quad F(G(w)) = w.$$

Thus G is the inverse mapping of F.

Proof.

Conversely, suppose F has an inverse mapping G. Let v_1, v_2 be elements of V such that $F(v_1) = F(v_2)$. Applying G yields

$$v_1 = G \circ F(v_1) = G \circ F(v_2) = v_2,$$

so F is injective. Secondly, let w be an element of W. The equation

$$w = F(G(w))$$

shows that $w = F(v)$ for some v, namely $v = G(w)$, so F is surjective. This proves the theorem. □

Theorem

Let $F : V \to W$ be a linear map. Assume that

$$dim\ V = dim\ W.$$

(i) If Ker $F = \{O\}$ then F is invertible.
(ii) If F is surjective, then F is invertible.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map such that

$$F(x, y) = (3x - y, 4x + 2y).$$

F has an inverse.
A linear map $F : U \to V$ which has an inverse $G : V \to U$ (we also say invertible) is called an isomorphism.
A square matrix A is invertible if and only if its columns $A^1, ..., A^n$ are linearly independent.