Solutions for Tutorial Problem Sheet 1, September 29.

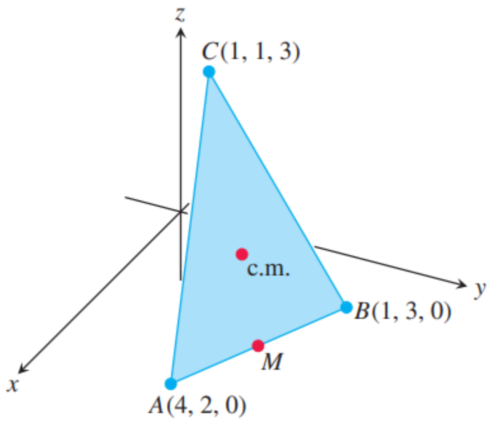
(Vectors and the Geometry of Space)

Problem 1. Suppose that A, B, and C are the corner points of the thin triangular plate of constant density shown here.

a) Find the vector from C to the midpoint M of side AB.

b) Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect. This point is the plate's center of mass. (See the figure.)



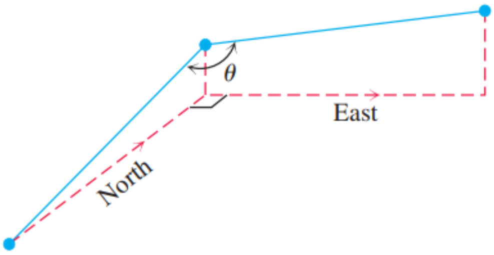
Solution:

(a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2}-1\right)\mathbf{i} + \left(\frac{5}{2}-1\right)\mathbf{j} + (0-3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$

(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass

Problem 2. A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east, as it shown in Figure.



Solution:

$\mathbf{u} = 10\mathbf{i} + 2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{j} + \mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\sqrt{101}}\right) \approx 1.55 \text{ rad} \approx 88.88°$.

Problem 3. Given three points in the $\mathbb{N}^3$ space $P(1,-1,2), Q(2,0,-1), R(0,2,1)$

a) Find the area of the triangle determined by the points P, Q, and R.

b) Find a unit vector perpendicular to plane PQR.

Solution:

(a) $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \Rightarrow$ Area $= \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{64+16+16} = 2\sqrt{6}$

(b) $\mathbf{u} = \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

Problem 4. For what value or values of a will the vectors $u = 2i + 4j - 5k$ and $v = -4i - 8j + ak$ be parallel?

Solution:

$$\mathbf{u} \text{ and } \mathbf{v} \text{ are parallel when } \mathbf{u}\times\mathbf{v} = \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a-40)\mathbf{i} + (20-2a)\mathbf{j} + (0)\mathbf{k} = \mathbf{0}$$

$$\Rightarrow 4a - 40 = 0 \text{ and } 20 - 2a \Rightarrow a = 10$$

Problem 5. Express the velocity vector $v = (e^t \cos t - e^t \sin t)i + (e^t \sin t + e^t \cos t)j$ when $t = \ln 2$ in terms of their lengths and directions.

Solution:

$$t = \ln 2 \Rightarrow \mathbf{v} = \left(e^{\ln 2}\cos(\ln 2) - e^{\ln 2}\sin(\ln 2)\right)\mathbf{i} + \left(e^{\ln 2}\sin(\ln 2) + e^{\ln 2}\cos(\ln 2)\right)\mathbf{j}$$

$$= \left(2\cos(\ln 2) - 2\sin(\ln 2)\right)\mathbf{i} + \left(2\sin(\ln 2) + 2\cos(\ln 2)\right)\mathbf{j} = 2\left[\left(\cos(\ln 2) - (\sin(\ln 2)\right)\mathbf{i} + \left(\sin(\ln 2) + \cos(\ln 2)\right)\mathbf{j}\right]$$

$$\text{length} = \left|2\left[\left(\cos(\ln 2) - \sin(\ln 2)\right)\mathbf{i} + \left(\sin(\ln 2) + \cos(\ln 2)\right)\mathbf{j}\right]\right| = 2\sqrt{\left(\cos(\ln 2) - \sin(\ln 2)\right)^2 + \left(\cos(\ln 2) + \sin(\ln 2)\right)^2}$$

$$= 2\sqrt{2\cos^2(\ln 2) + 2\sin^2(\ln 2)} = 2\sqrt{2}; \quad 2\left[\left(\cos(\ln 2) - \sin(\ln 2)\right)\mathbf{i} + \left(\sin(\ln 2) + \cos(\ln 2)\right)\mathbf{j}\right]$$

$$= 2\sqrt{2}\left(\frac{(\cos(\ln 2) - \sin(\ln 2))\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\mathbf{j}}{\sqrt{2}}\right) \Rightarrow \text{direction } \frac{(\cos(\ln 2) - \sin(\ln 2))}{\sqrt{2}}\mathbf{i} + \frac{(\sin(\ln 2) + \cos(\ln 2))}{\sqrt{2}}\mathbf{j}$$