Lecture 9: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

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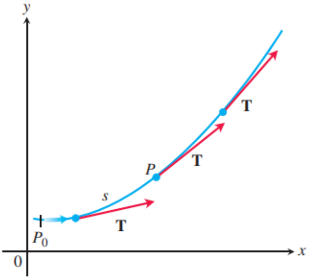
Curvature and Normal Vectors of a Curve

- We study how a curve **turns or bends**.
- To gain perspective, we look first at curves in the coordinate plane, then we consider curves in space.
- As a particle moves along a smooth curve in the plane, $\mathbf{T} = d\mathbf{r}/ds$ turns as the curve bends.
- Since $\mathbf{T}$ is a unit vector, its length remains constant and **only its direction changes** as the particle moves along the curve.
- The **rate at which $\mathbf{T}$ turns per unit of length** along the curve is called the **curvature** κ.



DEFINITION If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Curvature

• If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large.

• If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller.

• We can calculate the curvature as

$$\begin{aligned}\kappa = \left|\frac{d\mathbf{T}}{ds}\right| &= \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| && \text{Chain Rule}\\ &= \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right| \\ &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|. && \frac{ds}{dt} = |\mathbf{v}|\end{aligned}$$

Formula for Calculating Curvature

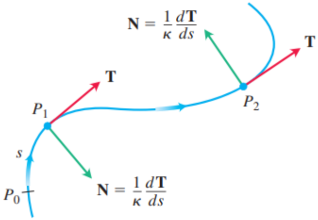
If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|, \tag{1}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Normal Vectors of a Curve

- Among the vectors **orthogonal to the unit tangent vector T**, there is one of particular significance because it points in the **direction in which the curve is turning**.
- Since $\mathbf{T}$ has constant length (because its length is always 1), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$.
- Therefore, if we divide $d\mathbf{T}/ds$ by its length κ, we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$.



DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

- The principal normal vector **N** will point toward the concave side of the curve.
- Following formula enables us to find **N** without having to find **k** and **s** first.

Formula for Calculating N

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \tag{2}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Example 1

Find **T** and **N** for the circular motion

$$r(t) = (\cos 2t)i + (\sin 2t)j.$$

Solution:

We first find **T**, then its derivative:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}$$
$$|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$
$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}.$$

$$\frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$
$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

and

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}.$$

Notice that $T \cdot N = 0$, verifying that **N** is orthogonal to **T**. Notice too, that for the circular motion here, **N** points from $r(t)$ toward the circle's center at the origin.

Circle of Curvature for Plane Curves

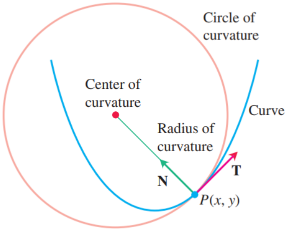
The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is **tangent to the curve** at P (has the same tangent line the curve has)
2. has the **same curvature** the curve has at P
3. has **center that lies toward the concave or inner side** of the curve.

• The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is $\rho = \frac{1}{\kappa}$.

• To find ρ, we find κ and take the reciprocal.

• The **center of curvature** of the curve at P is the center of the circle of curvature.



Example 2

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution:

We parametrize the parabola using the parameter $t = x$ $r(t) = ti + t^2 j$.
First we find the curvature of the parabola at the origin

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{1 + 4t^2}$$

so that

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}.$$

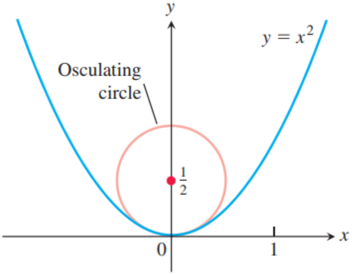
From this we find

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + \left[2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}\right]\mathbf{j}.$$

Solution for Example 2 continuation:

• At the origin, $t = 0$, so the curvature is

$$\kappa(0) = \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right|$$
$$= \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}|$$
$$= (1)\sqrt{0^2 + 2^2} = 2.$$



• Therefore, the radius of curvature is $1/k = 1/2$. At the origin we have $t = 0$ and $\mathbf{T} = i$, so $\mathbf{N} = j$. Thus the center of the circle is $(0, 1/2)$. The equation of the osculating circle is $(x - 0)^2 + (y - 1/2)^2 = (1/2)^2$.

• The osculating circle is a better approximation to the parabola at the origin than is the tangent line approximation $y = 0$.

Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $r(t)$ as a function of some parameter t, and if s is the arc length parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$. The **curvature** in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

just as for plane curves. The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$, and we define the **principal unit normal** to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

Example 3

Find the curvature for the helix

$$r(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Find N for the helix and describe how the vector is pointing.

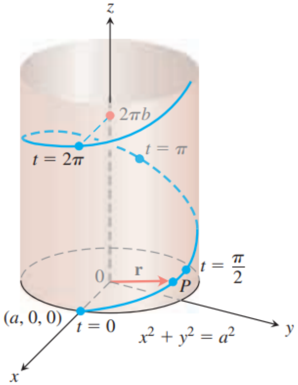
Solution:

We calculate T from the velocity vector v then κ using definition:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}}\left[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}\right].$$



Solution for Example 3 continuation:

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left|\frac{1}{\sqrt{a^2+b^2}}\left[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}\right]\right| \\
&= \frac{a}{a^2+b^2}\left|-(\cos t)\mathbf{i} - (\sin t)\mathbf{j}\right| \\
&= \frac{a}{a^2+b^2}\sqrt{(\cos t)^2 + (\sin t)^2} = \frac{a}{a^2+b^2}.
\end{aligned}$$

• From this equation, we see that increasing b for a fixed a decreases the curvature.

• Decreasing a for a fixed b eventually decreases the curvature as well.

• If $b = 0$, the helix reduces to a circle of radius a and its curvature reduces to $1/a$, as it should.

• If $a = 0$, the helix becomes the z-axis, and its curvature reduces to 0, again as it should.

Solution for Example 3 continuation:

We have

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

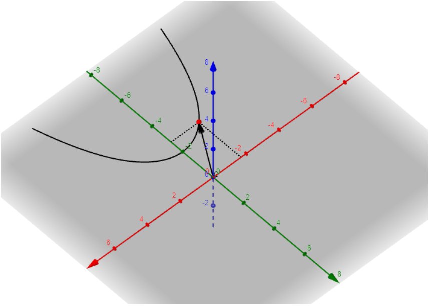
$$= -\frac{\sqrt{a^2+b^2}}{a}\cdot\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]$$

$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

Thus, **N** is parallel to the *xy*-plane and always points toward the z-axis.

Example 4 :

Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = (2\ln t)\mathbf{i} - \left[t + (1/t)\right]\mathbf{j}$, $e^{-2} \leq t \leq e^{2}$, at the point $(0, -2)$, where $t = 1$.



Solution for Example 4:

$$\mathbf{r} = (2\ln t)\mathbf{i} - \left(t + \tfrac{1}{t}\right)\mathbf{j} \Rightarrow \mathbf{v} = \left(\tfrac{2}{t}\right)\mathbf{i} - \left(1 - \tfrac{1}{t^2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\frac{4}{t^2} + \left(1 - \frac{1}{t^2}\right)^2} = \frac{t^2+1}{t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{t^2+1}\mathbf{i} - \frac{t^2-1}{t^2+1}\mathbf{j};$$

$$\frac{d\mathbf{T}}{dt} = \frac{-2\left(t^2-1\right)}{\left(t^2+1\right)^2}\mathbf{i} - \frac{4t}{\left(t^2+1\right)^2}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4\left(t^2-1\right)^2 + 16t^2}{\left(t^2+1\right)^4}} = \frac{2}{t^2+1}. \text{ Thus } \kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{t^2}{t^2+1} \cdot \frac{2}{t^2+1} = \frac{2t^2}{\left(t^2+1\right)^2}$$

$\Rightarrow \kappa(1) = \frac{2}{2^2} = \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$. The circle of curvature is tangent to the curve at $P(0, -2) \Rightarrow$ circle has same tangent as the curve $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the y-axis. If $t \neq 1 (t > 0)$, then $(t-1)^2 > 0 \Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$ since $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -\left(t + \frac{1}{t}\right) < -2 \Rightarrow y < -2$ on both sides of $(0, -2) \Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0, -4)$

$\Rightarrow x^2 + (y+4)^2 = 4$ is an equation of the circle of curvature