More Examples for Lecture 4.

MA2032 Vector Calculus

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Cross Product Calculations

Example 1

Given two vectors $u = 2i - 2j - k$ and $v = i - k$, find the length and direction (when defined) of $u \times v$ and $v \times u$.

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Solution:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}\right) \Rightarrow \text{ length } = 3 \text{ and the direction is } \tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k};$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -3\left(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}\right) \Rightarrow \text{ length } = 3 \text{ and the direction is } -\tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k}$$

Example 2

Let $u = 5i - j + k$, $v = j - 5k$, $w = -15i + 3j - 3k$. Which vectors, if any, are

(a) perpendicular?

(b) Parallel?

Give reasons for your answers.

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(b) Parallel?
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Solution:

(a) $\mathbf{u}\cdot\mathbf{v} = -6$, $\mathbf{u}\cdot\mathbf{w} = -81$, $\mathbf{v}\cdot\mathbf{w} = 18 \Rightarrow$ none are perpendicular

(b) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0} \Rightarrow \mathbf{u}$ and $\mathbf{w}$ are parallel

Area of a Triangle

Example 3

Find a concise 3×3 determinant formula that gives the area of a triangle in the xy-plane having vertices (a_1, a_2), (b_1, b_2), and (c_1, c_2).

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Find a concise 3×3 determinant formula that gives the area of a triangle in the xy-plane having vertices (a_1, a_2), (b_1, b_2), and (c_1, c_2).

Solution:

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$= \frac{1}{2}\left|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)\right| = \frac{1}{2}\left|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)\right| = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

The applicable sign ensures the area formula gives a nonnegative number.

Cross product

Example 4

By forming the cross product of two appropriate vectors, derive the trigonometric identity

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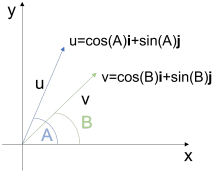
Solution:



If $\mathbf{u} = (\cos B)\mathbf{i} + (\sin B)\mathbf{j}$ and $\mathbf{v} = (\cos A)\mathbf{i} + (\sin A)\mathbf{j}$, where $A > B$, then $\mathbf{u} \times \mathbf{v} = \left[|\mathbf{u}||\mathbf{v}|\sin(A - B)\right]\mathbf{k}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos B & \sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix} = (\cos B \sin A - \sin B \cos A)\mathbf{k} \Rightarrow \sin(A - B) = \cos B \sin A - \sin B \cos A,$$ since

$|\mathbf{u}| = 1$ and $|\mathbf{v}| = 1$.

Volume of a Tetrahedron

Example 5

Determine whether the points $A(1, 1, 1)$, $B(-1, 0, 4)$, $C(0, 2, 1)$, $D(2, -2, 3)$ are coplanar.

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Solution:

$\overrightarrow{\mathrm{AB}} = u$, $\overrightarrow{\mathrm{AC}} = v$, and $\overrightarrow{\mathrm{AD}} = w$.

Let $\vec{\mathbf{u}} = \langle -2,-1,3 \rangle$, $\vec{\mathbf{v}} = \langle -1,1,0 \rangle$, and $\vec{\mathbf{w}} = \langle 1,-3,2 \rangle \Rightarrow$ volume of parallelipiped is

$$\mathrm{Vol} = \mathrm{abs}\begin{vmatrix} -2 & -1 & 3 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{vmatrix} = 0 \Rightarrow \text{points are coplanar}$$