LINEAR ALGEBRA II

Ch. IV LINEAR MAPS AND MATRICES

§1. The Linear Map Associated with a Matrix

- Let A be an $m \times n$ matrix in a field K.

$$L_A : K^n \ni X \mapsto AX \in K^m$$

is a linear map from K^n to K^m.

$\forall X \in K^n$, $L_A(X) = L_B(X)$

- **Theorem 1.1.** If A, B are $m \times n$ matrices and if $L_A = L_B$, then $A = B$. In other words, if matrices A, B give rise to the same linear map, then they are equal.

Proof. If $L_A = L_B$, then $\forall X \in K^n$, $L_A(X) = L_B(X)$, $AX = BX$.
Let $E^1, \cdots, E^n$ be the standard unit vectors in K^n, $E^i = (0 \cdots, 1 \cdots 0)^T$
$AE^i = BE^i \Rightarrow (A-B)E^i = 0 \Rightarrow (A-B) \cdot I = ((A-B)E^1, \cdots, (A-B)E^n) = 0$
$\Rightarrow A - B = 0 \Rightarrow A = B$

§2 The Matrix Associated with a Linear Map

- **Theorem 2.1.** Let $L : K^n \to K^m$ be a linear map. Then there exists a unique matrix A such that $L = L_A$. $L(X) = AX$

Proof. Let $E^1, \cdots, E^n$ be standard unit vectors in K^n. $\forall X \in K^n$,

$$X = x_1 E^1 + \cdots + x_n E^n$$

$$L(X) = x_1 \underbrace{L(E^1)}_{a_1} + \cdots + x_n \underbrace{L(E^n)}_{a_n}, \qquad L(E^i) \in K^m$$

$$A = (a_1, \cdots, a_n)$$

$$L(X) = x_1 a_1 + \cdots + x_n a_n = AX, \qquad L = L_A$$

By Th.1.1, we know the matrix is unique.

§2 The Matrix Associated with a Linear Map

- The identity: id_{R^n}. $m=n$, $\underline{L=\mathrm{id}}$

  $I \cdot X = X$

  $\mathrm{id}_{R^n} \longleftrightarrow I_n$

- The projection: $F : R^n \to R^r$,

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_r).$$

$F(E^r)$

$F(E^1) = F(1, \overbrace{0 \cdots 0}^{n-1}) = {}^t(1, 0 \cdots 0)$, $F(E^2) = F(0, 1, 0 \cdots 0) = {}^t(0, 1, 0 \cdots 0)$; $\cdots$ $F(0, \cdots 0, \underset{r}{1}, 0 \cdots 0) = {}^t(0, \cdots 0, 1)$

$F(E^i) = 0 \quad (i > r)$

$A = \begin{pmatrix} 1 & 0 & & 0 & \\ 0 & 1 & \cdots & \vdots & O \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & \end{pmatrix} = (I_r, O)$

§2 The Matrix Associated with a Linear Map

- **Theorem III, 2.1.** Let V and W be vector spaces. Let $\{v_1, \ldots, v_n\}$ be a basis of V, and let $\{w_1, \ldots, w_n\}$ be arbitrary elements of W, Then there exists a unique linear mapping $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_n) = w_n.$$

If $x_1, \ldots, x_n$ are numbers, then

$$T(x_1 v_1 + \cdots + x_n v_n) = x_1 w_1 + \cdots + x_n w_n.$$

Proof. $\forall\, v \in V \qquad T(v) = ?$

$v = x_1 v_1 + \cdots + x_n v_n \qquad X = {}^t(x_1, \cdots, x_n)$

$T(v) = x_1 T(v_1) + \cdots + x_n T(v_n)$

$\quad = x_1 w_1 + \cdots + x_n w_n \in W$

§2 The Matrix Associated with a Linear Map

- Let $E^1, \ldots, E^n$ be unit columns in R^n and $A^1, \ldots, A^n$ arbitrary elements of R^m. Then the matrix associated to the unique linear mapping such that $T(E^1) = A^1, \ldots, T(E^n) = A^n$ is A.

- $L_{A+B} = L_A + L_B$. $L_{A+B}(X) = (A+B)X = AX + BX = L_A(X) + L_B(X)$

- $L_{cA} = cL_A$.

- $L_{AB} = L_A L_B = L_A \circ L_B$.

§2 The Matrix Associated with a Linear Map

- **Theorem 2.2.** Let A be an $n \times n$ matrix, and let $A^1, \dots, A^n$ be its columns. Then A is invertible if and only if $A^1, \dots, A^n$ are linearly independent.

A invertible $\Leftrightarrow$ $\operatorname{Ker} A = \{0\}$

"$\Leftarrow$" $A^1, \dots, A^n$ L.I., $\Rightarrow \{A^1, \dots, A^n\}$ is a basis of K^n

$\{E^1, \dots, E^n\} \subset K^n$

$\exists!$ T s.t. $T(A^i) = E^i$, $\exists!$ $B_{n\times n}$. $BA^i = E^i$

$BA = I \Rightarrow A$ is invertible.

"$\Rightarrow$" $L_A(X) = AX = x_1 A^1 + \dots + x_n A^n = 0$ invertible

$A^{-1}AX = 0 \Rightarrow X = 0$

$\Rightarrow \operatorname{Ker} L_A = \{0\}$, $\Rightarrow A^1, \dots, A^n$ are L.I.

§3 Bases, Matrices, and Linear Maps

- Let V and W be arbitrary finite dimensional VSs over K, $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_m\}$ be bases of V and W respectively.

$X_{\mathcal{B}}(v) \to X_{\mathcal{B}'}(w)$

$y_1 w_1 + \cdots + y_m w_m = w$

- Let $F : V \to W$ be a linear map.
- $\forall v \in V$, denote by $v = x_1 v_1 + \cdots + x_n v_n$, $F(v) = x_1 F(v_1) + \cdots + x_n F(v_n)$
  - $X_{\mathcal{B}}(v)$ the coordinate vector of v relative to the basis $\mathcal{B}$;
  - $X_{\mathcal{B}'}(F(v))$ the coordinate vector of $F(v)$ relative to the basis $\mathcal{B}'$
- We associate a (uniquely determined) matrix with F, depending on our choice of bases, and denoted by $M^{\mathcal{B}}_{\mathcal{B}'}(F)$, such that $\forall v \in V$

$$X_{\mathcal{B}'}(F(v)) = M^{\mathcal{B}}_{\mathcal{B}'}(F) X_{\mathcal{B}}(v).$$

- Let V be a vector space, and let $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then $\forall v \in V$

$$X_{\mathcal{B}'}(v) = M^{\mathcal{B}}_{\mathcal{B}'}(\mathrm{id}) X_{\mathcal{B}}(v).$$

§3 Bases, Matrices, and Linear Maps

- Let

$$\begin{cases} F(v_1) = a_{11}w_1 + \cdots + a_{m1}w_m \\ \vdots \\ F(v_n) = a_{1n}w_1 + \cdots + a_{mn}w_m \end{cases}$$

then

$$A = M_{\mathcal{B}'}^{\mathcal{B}}(F) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$\forall\, v = x_1 v_1 + \cdots + x_n v_n \in V \qquad X_{\mathcal{B}'}F(v) = A X_{\mathcal{B}}(v) = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$F(v) = (A_1 X)\cdot w_1 + \cdots + (A_n X) w_m$

$F(v) = x_1 F(v_1) + \cdots + x_n F(v_n) = (a_{11}x_1 + \cdots + a_{1n}x_n) w_1 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n) w_m$

§3 Bases, Matrices, and Linear Maps

§3 Bases, Matrices, and Linear Maps

- $M_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id}) = I$.
- Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_n\}$ be bases of V. If

$$\begin{aligned} w_1 &= a_{11}v_1 + \cdots + a_{n1}v_n \\ &\vdots \\ w_n &= a_{1n}v_1 + \cdots + a_{nn}v_n \end{aligned}$$

then

$$M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

§3 Bases, Matrices, and Linear Maps

- $M_{\mathcal{B}'}^{\mathcal{B}}(F+G) = M_{\mathcal{B}'}^{\mathcal{B}}(F) + M_{\mathcal{B}'}^{\mathcal{B}}(G)$
- $M_{\mathcal{B}'}^{\mathcal{B}}(cF) = cM_{\mathcal{B}'}^{\mathcal{B}}(F)$
- Let $\dim V = n$ and $\dim W = m$. The association $F \mapsto M_{\mathcal{B}'}^{\mathcal{B}}(F)$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathrm{Mat}_{m\times n}(K)$

Ch. III Linear Mappings

- Let V be a vector space, $\mathcal{B}$ a bases of V and $F : V \to V$ is a linear mapping. $M_{\mathcal{B}}^{\mathcal{B}}(F)$ is called the matrix associated with F relative to $\mathcal{B}$.

- Let $P_n = \left\{ \sum_{k=0}^{n} a_k t^k | a_k \in K \right\}$. What is the matrix associate with $D = d/dt : P_n \to P_n$ relative to the basis$\{1, t, \dots, t^n\}$?

$1 \to 0$

$t \to 1$

$t^2 \to 2t$

$\vdots$

$t^n \to n t^{n-1}$

If the orders of vectors in $\mathcal{B} = \{v_1 -- v_n\}$, $\mathcal{B}' = \{w_1 -- w_m\}$ change, how does the associated matrix change?

§3 Bases, Matrices, and Linear Maps

- Let

$$F(v_1) = a_{11}w_1 + \cdots + a_{m1}w_m$$
$$\vdots$$
$$F(v_n) = a_{1n}w_1 + \cdots + a_{mn}w_m$$

then

$$M_{\mathcal{B}'}^{\mathcal{B}}(F) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Remark.** If the order of vectors in $\mathcal{B}$ or $\mathcal{B}'$, then $M_{\mathcal{B}'}^{\mathcal{B}}(F)$ will change.

§3 Bases, Matrices, and Linear Maps

- **Theorem 3.3.**
  - $M^{\mathcal{B}}_{\mathcal{B}'}(F+G) = M^{\mathcal{B}}_{\mathcal{B}'}(F) + M^{\mathcal{B}}_{\mathcal{B}'}(G)$
  - $M^{\mathcal{B}}_{\mathcal{B}'}(cF) = cM^{\mathcal{B}}_{\mathcal{B}'}(F)$
  - Let $\dim V = n$ and $\dim W = m$. The association $F \mapsto M^{\mathcal{B}}_{\mathcal{B}'}(F)$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathrm{Mat}_{m\times n}(K)$

$\dim V = n$, $\dim W = m$, $\underline{\mathcal{B}}$, $\underline{\mathcal{B}'}$ $F \leftrightarrow \underline{M^{\mathcal{B}}_{\mathcal{B}'}(F)}$

$\underbrace{\mathcal{L}(V, W)}_{\dim m\cdot n}$ is isomorphic to $\underbrace{\mathrm{Mat}_{m\times n}(K)}_{\dim = m\cdot n}$

§3 Bases, Matrices, and Linear Maps

- **Theorem 3.4.** Let V, W, U be vector spaces. Let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{B}''$ be bases for V, W, U respectively. Let $F : V \to W$ and $G : W \to U$ be linear maps. Then

$$M^{\mathcal{B}'}_{\mathcal{B}''}(G)M^{\mathcal{B}}_{\mathcal{B}'}(F) = M^{\mathcal{B}}_{\mathcal{B}''}(G \circ F)$$

Proof. $\forall\, v \in V$, $X_{\mathcal{B}''}(G\circ F(v)) = M^{\mathcal{B}}_{\mathcal{B}''}(G\circ F)\, X_{\mathcal{B}}(v)$

$X_{\mathcal{B}'}(F(v)) = M^{\mathcal{B}}_{\mathcal{B}'}(F) \cdot X_{\mathcal{B}}(v)$

$X_{\mathcal{B}''}(G\circ F(v)) = X_{\mathcal{B}''}\left(G(F(v))\right) = M^{\mathcal{B}'}_{\mathcal{B}''}(G) \cdot X_{\mathcal{B}'}(F(v))$

$= M^{\mathcal{B}'}_{\mathcal{B}''}(G) \cdot M^{\mathcal{B}}_{\mathcal{B}'}(F)\, X_{\mathcal{B}}(v)$

$\parallel$

$M^{\mathcal{B}}_{\mathcal{B}''}(G\circ F)$

- $M_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id}) = I$.
- **Corollary 3.5.** Let V be a vector spaces and $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then

$$M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id})M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}) = I = M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id})M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id}).$$

In particular, $M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id})$ is invertible.

Proof. Applying Th. 3.4 to $F = \mathrm{id} : \underset{\mathcal{B}'}{V} \to \underset{\mathcal{B}}{V}$ and

$G = \mathrm{id} : \underset{\mathcal{B}'}{V} \to \underset{\mathcal{B}}{V}$, we have

$$M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id})\, M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}) = M_{\mathcal{B}'}^{\mathcal{B}'}(\mathrm{id}) = I$$

The second equality can be proven similarly.

- **Theorem 3.6.** Let V be a vector spaces and $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then there exists an invertible matrix N such that

$$M^{\mathcal{B}'}_{\mathcal{B}'}(F) = N^{-1} M^{\mathcal{B}}_{\mathcal{B}}(F) N.$$

相似! Similar

In fact, we can take

$$N = M^{\mathcal{B}'}_{\mathcal{B}}(\mathrm{id}).$$

Proof. Applying Th. 3.4, we have

$$M^{\mathcal{B}'}_{\mathcal{B}'}(F) = \overbrace{M^{\mathcal{B}}_{\mathcal{B}'}(\mathrm{id})}^{N^{-1}} M^{\mathcal{B}}_{\mathcal{B}}(F) \underbrace{M^{\mathcal{B}'}_{\mathcal{B}}(\mathrm{id})}_{N}.$$

$M^{\mathcal{B}'}_{\mathcal{B}}(F)$

$M^{\mathcal{B}'}_{\mathcal{B}'}(\mathrm{id} \circ F)$

§3 Bases, Matrices, and Linear Maps

- Let $F : V \to V$ be a linear map. A basis $\mathcal{B}$ of V is said to diagonalize F if $M_{\mathcal{B}}^{\mathcal{B}}(F)$ is a diagonal matrix.

  对角化

- If there exists such a basis which diagonalizes F, then we say that F is diagonalizable. 是可对角化的

  $L_A : K^n \to K^n$ $L_A(X) = AX$

- If A is an $n \times n$ matrix in K, we say that A can be diagonalized (in K) if the linear map on K^n represented by A can be diagonalized.

- **Theorem 3.6.** Let V be a finite dimensional vector space over K, let $F : V \to V$ be a linear map, and let M be its associated matrix relative to a basis $\mathcal{B}$. Then F (or M) can be diagonalized (in K) if and only if there exists an invertible matrix N in K such that $N^{-1}MN$ is a diagonal matrix.

- Homework: P94, 8, 9, 10.

F is diagonalizable $\Leftrightarrow \exists\, B$, s.t $M_B^B(F)$ is a diagonal matrix $\Leftrightarrow M = M_{B'}^{B'}(F)$, $\exists\, N$ invertibl s.t. $\underbrace{M_B^B(F)}_{\text{diagonal}} = N^{-1} M N$ diagonal.