Lecture 25: Multiple Integrals.

MA2032 Vector Calculus

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Substitutions in Multiple Integrals

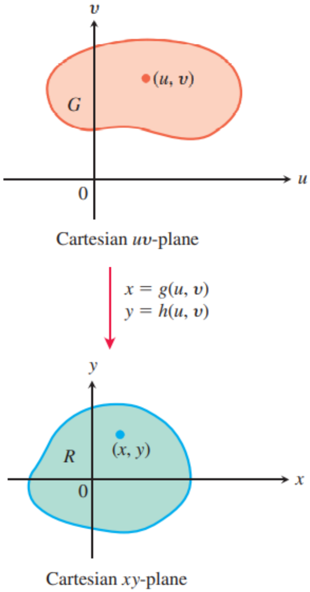
• Today we introduce the ideas involved in **coordinate transformations** to evaluate multiple integrals by **substitution**.

• The method **replaces complicated integrals** by ones that are **easier to evaluate**.

• Substitutions accomplish this by **simplifying the integrand, the limits of integration, or both**.

• The **polar coordinate substitution is a special case** of a more general substitution method for double integrals.

• Suppose that a region G in the uv-plane is **transformed** into the region R in the xy-plane by equations of the form $x = g(u, v), y = h(u, v)$



Substitutions in Double Integrals

- We assume the transformation is **one-to-one** on the interior of G.
- We call R the **image** of G under the transformation, and G the **preimage** of R.
- Any function $f(x, y)$ defined on R **can be thought** of as a function $f(g(u, \nu), h(u, \nu))$ defined on G as well.
- **How is the integral** of $f(x, y)$ over R **related** to the integral of $f(g(u, y), h(u, y))$ over G?
- To gain some **insight into the question**, we look at the **single variable case**: substitution method for single integrals.

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du. \qquad x = g(u),\ dx = g'(u)\,du$$

- To propose an analogue for substitution in a double integral $\iint_R f(x, y)dx\,dy$, we need a **derivative factor** like $g'(u)$ as a **multiplier that transforms the area element** $du\,d\nu$ in the region G to its corresponding area element $dx\,dy$ in the region R.

Substitutions in Double Integrals

- We denote this factor by J:

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \qquad (1)$$

- The Jacobian measures **how much the transformation is expanding or contracting the area** around the point (u, ν).

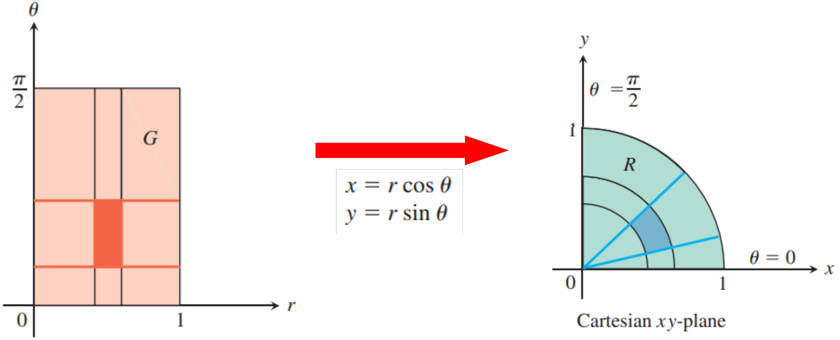
Substitutions in Double Integrals

THEOREM 3—Substitution for Double Integrals
Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \tag{2}$$

Example 1

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta,\ y = r\sin\theta$, and write the Cartesian integral $\iint_R f(x,y)dx\,dy$ as a polar integral.



Cartesian $r\theta$-plane

Cartesian xy-plane

Example 1

Solution Figure 15.58 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1$, $0 \le \theta \le \pi/2$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy-plane.

For polar coordinates, we have r and θ in place of u and v. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \ge 0$ when integrating in polar coordinates, $|J(r,\theta)| = |r| = r$, so that Equation (2) gives

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \tag{3}$$

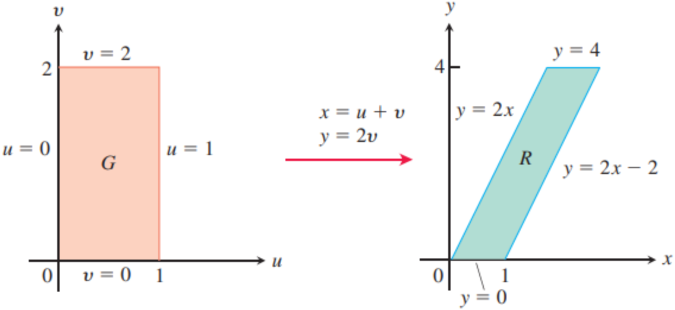
This is the same formula we derived independently using a geometric argument for polar area in Section 15.4. ■

Example 2

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy$$

by applying the transformation $u = \frac{2x-y}{2}$, $\nu = \frac{y}{2}$ and integrating over an appropriate region in the $u\nu$-plane.



Example 2

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Figure 15.59).

To apply Equation (2), we need to find the corresponding uv-region G and the Jacobian of the transformation. To find them, we first solve Equations (4) for x and y in terms of u and v. From those equations it is easy to find algebraically that

$$x = u + v, \qquad y = 2v. \tag{5}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R (Figure 15.59)

xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

Example 2

From Equations (5) the Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v) \\[2ex] \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (2):

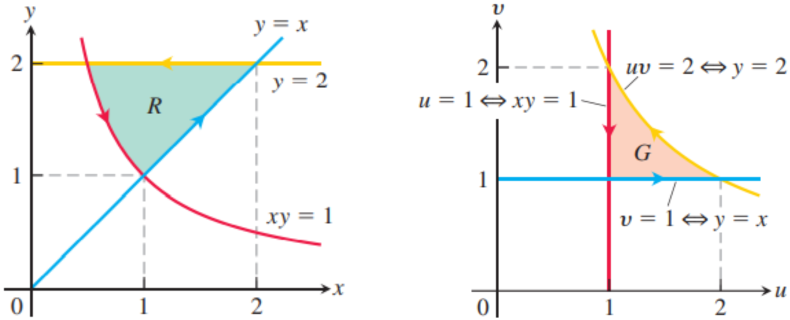
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\, |J(u, v)|\, du\, dv$$

$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[u^2 \Big]_0^1 dv = \int_0^2 dv = 2. \quad \blacksquare$$

Example 3

Evaluate the integral

$$\int_1^2 \int_{x=1/y}^{x=y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx\, dy.$$



Example 3. Whiteboard

Example 3

Solution The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$. Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

Example 3

If G is the region of integration in the uv-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G v e^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration R of the original integral in the xy-plane is shown in Figure 15.61. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for R is the vertical line segment $u = 1$, $2 \geq v \geq 1$, in G (see Figure 15.62). Likewise, the right-hand boundary $y = x$ of R maps to the horizontal line segment $v = 1$, $1 \leq u \leq 2$, in G. Finally, the horizontal top boundary $y = 2$ of R

Example 3

maps to $uv = 2$, $1 \le v \le 2$, in G. As we move counterclockwise around the boundary of the region R, we also move counterclockwise around the boundary of G, as shown in Figure 15.62. Knowing the region of integration G in the uv-plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du.$$

Note the order of integration.

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned}
\int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \Big[vue^u \Big]_{v=1}^{v=2/u} du \\
&= 2\int_1^2 (2e^u - ue^u)\, du \\
&= 2\int_1^2 (2-u)e^u\, du \\
&= 2\Big[(2-u)e^u + e^u \Big]_{u=1}^{u=2} \\
&= 2(e^2 - (e+e)) = 2e(e-2).
\end{aligned}$$

Integrate by parts.

Substitutions in Triple Integrals

• The **cylindrical and spherical coordinate substitutions** are special cases of a substitution method that pictures changes of variables in **triple integrals** as transformations of three-dimensional regions.

• The method is like the method for double integrals except that now **we work in three dimensions** instead of two.

• Suppose that a region G in uvw-space **is transformed** one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

• Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

• defined on G.

Substitutions in Triple Integrals

• If g, h, and k have **continuous first partial derivatives**, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, y, w)$ over G by the equation

$$\iiint\limits_D F(x, y, z)\, dx\, dy\, dz = \iiint\limits_G H(u, v, w)\,|J(u, v, w)|\, du\, dv\, dw.$$

• The factor $J(u, \nu, w)$, whose absolute value appears in this equation, is

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

the **Jacobian determinant**

• This determinant measures how much the **volume** near a point in G is being **expanded or contracted** by the transformation from (u, ν, w) to (x, y, z) coordinates.

Substitutions in Triple Integrals. Cylindrical coordinates

For cylindrical coordinates, r, θ, and z take the place of u, v, and w. The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

(Figure 15.64). The Jacobian of the transformation is

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

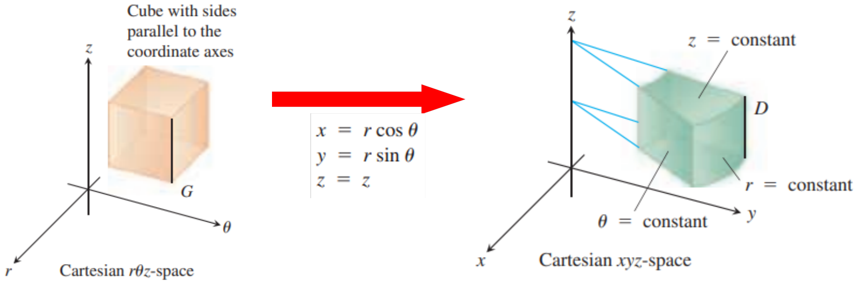
$$= r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(r,\theta,z)\,|r|\,dr\,d\theta\,dz.$$

We can drop the absolute value signs because $r \geq 0$.

Substitutions in Triple Integrals. Cylindrical coordinates



For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is given by

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

(Figure 15.65). The Jacobian of the transformation (see Exercise 23) is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin \phi.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(\rho, \phi, \theta)\, \left|\rho^2 \sin \phi\right|\, d\rho\, d\phi\, d\theta.$$

Substitutions in Triple Integrals. Spherical coordinates