Sequences $(a_n)_{n \in \mathbb{N}}$

Limit: $\lim_{n \to \infty} a_n = L$

$\forall \varepsilon > 0 \; \exists N : n > N \Rightarrow |a_n - L| < \varepsilon$

We defined subsequence as a selection of some terms from a sequence.

Idea: subsequences might have better properties than the original sequence

Example: $a_n = (-1)^n$

divergent $1, -1, 1, -1, 1, \ldots$

convergent $1, \quad 1 \quad, 1 \quad 1 \ldots$

Theorem

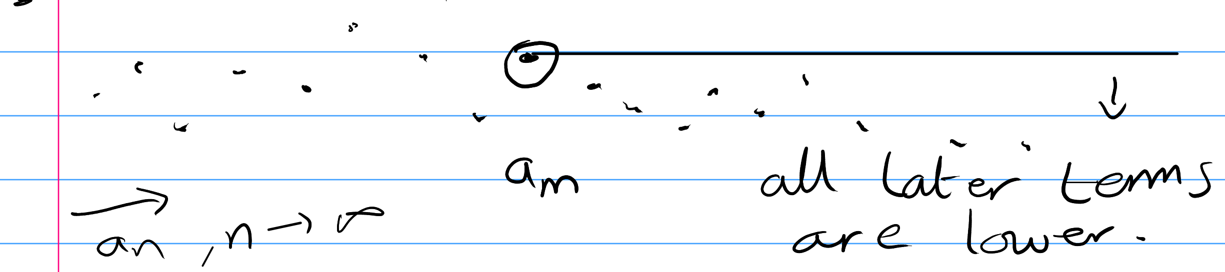
Every sequence (a_n) has a monotonic subsequence.

Intuitively: try and find an increasing subsequence, or a decreasing one.

Proof Suppose $(a_n)_{n \in \mathbb{N}}$ is a sequence, and we make a definition:

m is a <u>peak</u> of the sequence if

$$a_n \leq a_m \quad \text{for all } n \geq m$$



Observation:

$(a_n)_{n \in \mathbb{N}}$ increasing $\Rightarrow$ no peaks

$(a_n)_{n \in \mathbb{N}}$ decreasing $\Rightarrow$ all m are peaks

Let's consider two possibilities

Either

(1) The sequence has infinitely many peaks

(2) It only has a finite number

In case (1) we will prove there exists a subsequence which is monotonic <u>decreasing</u>

& in case (2) there is a subsequence which is monotonic increasing.

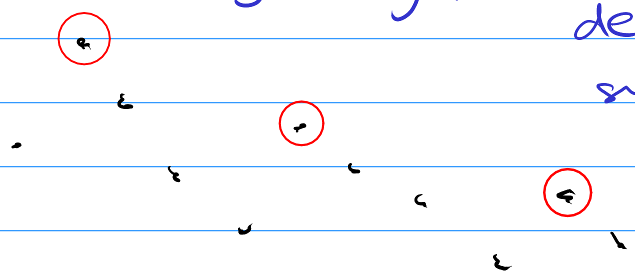
<u>Case (1)</u> Let $m_1 < m_2 < m_3 < \ldots\ldots$ be <u>infinitely</u> many peak for our sequence $(a_n)_{n \in \mathbb{N}}$

$$m_k < m_{k+1} \Rightarrow a_{m_{k+1}} \leq a_{m_k}$$

peak

So $(a_{m_k})_{k=1}^{\infty}$ is a decreasing subsequence

Picture ∞ly many peaks $\Rightarrow$ monotonic decreasing subsequence



<u>Case (2)</u> Let $M_0 >$ all peaks

That is, M_0 is bigger than the last of the finitely many peaks

So $a_{m_0}, a_{m_0+1}, a_{m_0+2}, \ldots\ldots$ has <u>no</u> peaks

M_0 is not a peak so $\exists M_1 > M_0$ with $a_{M_1} > a_{M_0}$

M_1 not a peak $\Rightarrow \exists M_2 > M_1$ with $a_{M_2} > a_{M_1}$

and we form an increasing subsequence

$$a_{M_0} < a_{M_1} < a_{M_2} < \dots < a_{M_k} < a_{M_{k+1}}$$

M_k not a peak, so there is a later term $a_{M_{k+1}}$ in (a_n) with $a_{M_{k+1}} > a_{M_k}$

Consequences of the fact that every sequence has a monotonic subsequence

Theorem Every bounded sequence has a convergent subsequence

Proof If $(a_n)_{n\in\mathbb{N}}$ is bounded

$$|a_n| < B \quad \forall n$$

By the previous theorem, $(a_n)_{n\in\mathbb{N}}$

has a <u>monotonic</u> subsequence
$(a_{M_k})_{k \in \mathbb{N}}$

The subsequence is still <u>bounded</u>

$$|a_{M_k}| < B \quad \forall k$$

Therefore it converges.

[Reminder: a monotonic decreasing sequence, bounded below, converges to its greatest lower bound
& a monotonic increasing sequence that is bounded above converges to its least upper bound]

Convergent sequence $\Rightarrow$ Cauchy sequence
$\Leftarrow$

→ Any bounded sequence has a convergent subsequence

→ Any sequence has a monotonic subsequence

First thing to prove is

① Any Cauchy sequence is bounded

$(a_n)_{n\in\mathbb{N}}$ is Cauchy if

$$\forall \varepsilon>0 \ \exists N : m,n>N \Rightarrow |a_m-a_n|<\varepsilon$$

Proof $(a_n)_{n\in\mathbb{N}}$ Cauchy $\Rightarrow$ if we choose $\varepsilon=1$ we get

$$\exists N : m,n>N \Rightarrow |a_m-a_n|<1$$

$$\Rightarrow |a_{N+1}-a_n|<1 \quad \forall n>N$$

i.e. $a_{N+1}-1 < a_n < a_{N+1}+1$

Let $B=\max\{|a_0|,|a_1|,\dots,|a_N|,|a_{N+1}|,$
$|a_{N+1}-1|,|a_{N+1}+1|\}$

Then $|a_n|\leq B$ for all $n\in\mathbb{N}$

Put all results together

Cauchy $\Rightarrow$ bounded, has a convergent subsequence

Cauchy ⇒ Convergent

Theorem Any Cauchy sequence has a limit

Proof We have a convergent subsequence

$$a_{M_k} \longrightarrow L \quad \text{as } k \to \infty$$

Aim of the proof is to show

$$a_n \longrightarrow L \quad \text{as } n \to \infty$$

A Cauchy sequence converges to the same limit as its convergent subsequences

(Finish next time!)