Topic 1. Precalculus

1.1. Using only the field axioms prove:

(a) For all real numbers x and y, we have $x + y = 0$ if and only if $x = -y$.

(b) For all real numbers y, we have $y + y = y \Leftrightarrow y = 0$.

(c) If $x \in \mathbb{R}$ then $0x = x0 = 0$.

(d) If $x, y \in \mathbb{R}$ then xy is zero if and only if at least one of x, y is zero.

(e) If $x \in \mathbb{R}$ then $(-1)x = -x$.

(f) If $x, y, z \in \mathbb{R}$ and $x \neq 0$ then $xy = xz \;\Rightarrow\; y = z$

(g) If $x, y, z \in \mathbb{R}$ and $x + y = x + z$ then $y = z$

(h) If $k \in \mathbb{R}$ then $k = 0$ if and only if $k + x = x$ for all $x \in \mathbb{R}$.

(i) If $k \in \mathbb{R}$ then $k = 1$ if and only if $kx = x$ for all $x \in \mathbb{R}$.

1.2. Prove the following for all real numbers a, b, c:

(a) If $c > 0$ then $c^{-1} > 0$.

(b) If $ab > 0$ then $a < b \;\Rightarrow\; \dfrac{1}{b} < \dfrac{1}{a}$

(c) If $a + b > 0$ then $a^2 < b^2 \;\Leftrightarrow\; a < b$

(d) $0 \leq a \leq b \;\Rightarrow\; 0 \leq \sqrt{a} \leq \sqrt{b}$.

(e) $0 \leq a \leq b \;\Rightarrow\; \dfrac{a}{1+a} \leq \dfrac{b}{1+b}$.

(f) $a \leq b \;\Rightarrow\; a \leq \frac{1}{2}(a + b) \leq b$.

(g) If $0 \leq a \leq b$ then $a \leq \sqrt{ab} \leq \frac{1}{2}(a + b) \leq b$.

1.3. Prove the following for all real numbers a, b:

(a) $\big|\,|a| - |b|\,\big| \leq |a - b|$

(b) $|a + b| \geq \Big||a| - |b|\Big|$

(c) $|a - b| \leq |a| + |b|$

(d) $|a + b| = |a| + |b| \;\Leftrightarrow\; ab \geq 0$.

1.4. State whether the following sets are bounded above and/or bounded below.

If bounded above, give an upper bound; if bounded below, give a lower bound.

(a) $S = \{1, 3, 5, 7, \ldots\}$

(b) $S = \{x : x \leq 1\}$

(c) $S = \{x : |x + 2| < 3\}$

(d) $S = \{(-1/n)^n : n \in \mathbb{N}\}$

1.5. Find the least upper bound (if it exists) and the greatest lower bound (if it exists).

(a) $(0, \infty)$

(b) $\{x : x^2 < 4\}$

(c) $\{x : \ln x < 1\}$

(d) $\{x : x^2 + x - 1 < 0\}$

1.6. Prove that the least upper bound of a set of negative numbers cannot be positive.

1.7. Solve the following inequalities. Express the solution as an interval or as the union of intervals. Mark the solution on a number line.

(a) $3x + 5 < \frac{1}{2}(4 - x)$

(b) $x^2 - x - 6 \geq 0$

(c) $x(x^2 - 3x + 2) \leq 0$

(d) $\frac{x+1}{(x+2)(x-2)} > 0$

(e) $\frac{x^2-4x+4}{x^2-2x-3} \leq 0$

(f) $|x - 2| < 1$

(g) $|3x - 2| \geq 4$

(h) $\left|\frac{2}{x+4}\right| > 2.$

1.8. Give the domain and range of the following functions.

(a) $f_1(x) = 4 - x^2$

(b) $f_2(x) = 3x - 2$

(c) $f_3(x) = \sqrt{x - 4}$

(d) $f_4(x) = \frac{1}{2}\sqrt{1 - 4x^2}$

(e) $f_5(x) = |\sin x|$

(f) $f_6(x) = \sin^2 x + \cos^2 x$

(g) $f_7(x) = 2\cos(3x)$.

(h) $f_8(x) = 1 + \tan^2(x)$.

1.9. Show the following functions are one–to–one. Find the inverse function f^{-1} of each f. Give the domain and the range of each function f, and the domain and the range of f^{-1}.

(a) $f(x) = \frac{x-1}{x-2}$

(b) $f(x) = x^2 + 1,\ x > 0$

(c) $f(x) = \dfrac{x}{2x - 4} + \dfrac{1}{2}$

(d) $f(x) = \dfrac{3x - 5}{x - 2}$

1.10. State whether the following functions are odd, even, or neither.

(a) $f(x) = \sin(3x)$

(b) $f(x) = 1 + \cos(2x)$

1.11. Assume that f and g are one–to–one. Show that $f \circ g$ is one–to–one.

1.12. Prove that if $ad \neq bc$ then $f(x) = \dfrac{ax + b}{cx + d}$ is a one–to–one function. What about if $ad = bc$? Find the inverse function $f^{-1}(x)$. What are the domains and ranges of these functions?

1.13. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function then it is not one–to–one.

1.14. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is any function, then $g(x) = \frac{1}{2}(f(x) + f(-x))$ is an even function and $h(x) = \frac{1}{2}(f(x) - f(-x))$ is an odd function.

1.15. Prove that any function can be expressed as the sum of an even and an odd function. Is this expression unique?

1.16. Prove by induction, for all positive integers n, that

(a) $1 + 2n \leq 3^n$

(b) $n! \geq 2^{n-1}$

(c) $\displaystyle\sum_{\ell=1}^{n} 2\ell + 1 \;=\; n(n + 2)$

Topic 2. Limits and continuity

2.1. Which of the following says that $f(x)$ has limit L as x approaches c?

(a) For any positive real number δ there exists a positive real number ϵ such that the distance from $f(x)$ to L is at most ϵ whenever x is not equal to but is within δ of c.

(b) There exists a positive real number ϵ such that for all positive real numbers δ the distance from $f(x)$ to L is at most ϵ whenever x is not equal to but is within δ of c.

(c) There exists a positive real number δ such that for all positive real numbers ϵ the distance from $f(x)$ to L is at most ϵ whenever $x \neq c$ is not equal to but is within δ of c.

(d) For any positive real number ϵ there exists a positive real number δ such that the distance from $f(x)$ to L is at most ϵ whenever x is within δ of c.

(e) For any positive real number ϵ there exists a positive real number δ such that the distance from $f(x)$ to L is at most ϵ whenever x is not equal to but is within δ of c.

(f) For any positive real number δ there exists a positive real number ϵ such that the distance from $f(x)$ to L is at most ϵ whenever x is within δ of c.

(g) There exists a positive real number ϵ such that for all positive real numbers δ the distance from $f(x)$ to L is at most ϵ whenever x is within δ of c.

(h) There exists a positive real number δ such that for all positive real numbers ϵ the distance from $f(x)$ to L is at most ϵ whenever x is within δ of c.

2.2. The definition of "$f(x)$ has limit L as $x \to c$" is:

(a) $\forall\, \delta > 0,\ \exists\, \varepsilon > 0$, such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(b) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$, such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(c) $\exists\, \delta > 0,\ \forall\, \varepsilon > 0$, such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(d) $\exists\, \varepsilon > 0,\ \forall\, \delta > 0$, such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(e) $\forall\, \delta > 0,\ \exists\, \varepsilon > 0$, such that $|x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(f) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$, such that $|x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(g) $\exists\, \delta > 0,\ \forall\, \varepsilon > 0$, such that $|x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

(h) $\exists\, \varepsilon > 0,\ \forall\, \delta > 0$, such that $|x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

2.3. Which of the following says that $f(x)$ is continuous at $x = c$?

(a) For any positive real number δ there exists a positive real number ϵ such that the distance from $f(x)$ to $f(c)$ is at most ϵ whenever x is within δ of c.

(b) There exists a positive real number ϵ such that for all positive real numbers δ the distance from $f(x)$ to $f(c)$ is at most ϵ whenever x is within δ of c.

(c) For any positive real number ϵ there exists a positive real number δ such that the distance from $f(x)$ to $f(c)$ is at most ϵ whenever x is within δ of c.

(d) There exists a positive real number δ such that for all positive real numbers ϵ the distance from $f(x)$ to $f(c)$ is at most ϵ whenever x is within δ of c.

2.4. Let $f(x) = 2x + 4$ and let $c = 2$.

What is the largest value of A that ensures that if $|x - c| < A$ then $|f(x) - f(c)| < 1$?

What is the largest value of A that ensures that if $|x - c| < A$ then $|f(x) - f(c)| < \frac{1}{10}$?

Prove that $f(x)$ is continuous at $x = c$.

2.5. What is the largest value of A that ensures that if $0 < |x - 1| < A$ then $|\sqrt{x} - 1| < \frac{1}{2}$?

2.6. What is the largest value of δ that ensures that if $|x - 8| < \delta$ then $|\sqrt[3]{x} - \sqrt[3]{8}| < 1$?

2.7. Prove, using only the ε-δ definition of limit, that

(a) $f(x) = \sqrt{x-1}$ is continous at $x = 5$.

(b) $f(x) = |x|$ is continuous on all of $\mathbb{R}$.

[Prove it is continous at $x = c$ for any $c > 0$, for any $c < 0$, and for $c = 0$].

(c) $f(x) = \sqrt{x}$ is continuous at $x = c$ for any $c > 0$.

2.8. Prove, using the ε-δ definition of '*continuity from the right*' that $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

2.9. Suppose f is continuous at a point c. Show there exist $B > 0$ and $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x)| < B$. [That is, f is bounded in some open interval around c].

2.10. Prove that if $|f(x)| \to 0$ as $x \to c$ then $f(x) \to 0$ as $x \to c$.

Show that if $|g(x)| \to 1$ as $x \to c$ then it is not necessarily true that $g(x) \to 1$ as $x \to c$.

2.11. Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous from the right at $x = a$.

Using only the definition of continuity from the right, prove that

(a) there exists $\xi \in (a, b)$ such that f is bounded on $[a, \xi)$.

(b) if $f(a) < 0$, then there exists $\xi \in (a, b)$ such that $f(x) < 0$ for all $x \in [a, \xi)$.

(c) if $f(a) > 0$, then there exists $\xi \in (a, b)$ such that $f(x) > 0$ for all $x \in [a, \xi)$.

2.12. Use the Pinching theorem:

(a) to find: $\displaystyle\lim_{x\to 1}(x-1)\cos\left(\frac{\pi}{x-1}\right)$. (b) to find: $\displaystyle\lim_{x\to 0} x\sin(1/x)$

(c) to prove: if $\exists B \in \mathbb{R}$ such that $\left|\frac{f(x)}{x}\right| \leq B$, $\forall x \neq 0$, then $\displaystyle\lim_{x\to 0} f(x) = 0$.

2.13. Find the following limits (ϵ, δ proof not required!)

(a) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ (b) $\displaystyle\lim_{x\to 4}\frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}$ (c) $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

(d) $\displaystyle\lim_{x\to 9}\frac{\frac{1}{\sqrt{x}}-\frac{1}{3}}{x-9}$ (e) $\displaystyle\lim_{x\to c}\frac{x^{\frac{1}{2}}-c^{\frac{1}{2}}}{x-c}$ (f) $\displaystyle\lim_{x\to c}\frac{x^{-\frac{1}{2}}-c^{-\frac{1}{2}}}{x-c}$

(g) $\displaystyle\lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h}$ (h) $\displaystyle\lim_{h\to 0}\frac{\cos(c+h)-\cos(c)}{h}$

2.14. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(c+h) = L$.

2.15. Decide whether or not the following limits exist, and evaluate them if they do.

(a) $\lim_{x\to 1} \dfrac{x}{x+1}$ (b) $\lim_{x\to 0} \dfrac{x^2(1+x)}{2x}$ (c) $\lim_{x\to 0} \dfrac{x(1+x)}{2x^2}$ (d) $\lim_{x\to 4} \dfrac{x}{\sqrt{x}+1}$

(e) $\lim_{x\to 1} \dfrac{x^4-1}{x-1}$ (f) $\lim_{x\to -1} \dfrac{1-x}{x+1}$ (g) $\lim_{x\to 0} \dfrac{x}{|x|}$ (h) $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$

2.16. Let f be a function for which we know only that $0 < |x-3| < 1 \ \Rightarrow\ |f(x)-5| < 0.1$. Which of the following statements are necessarily true?

(a) If $|x-3| < 1$, then $|f(x)-5| < 0.1$.

(b) If $|x-2.5| < 0.3$, then $|f(x)-5| < 0.1$.

(c) $\lim_{x\to 3} f(x) = 5$

(d) If $0 < |x-3| < 2$, then $|f(x)-5| < 0.1$.

(e) If $0 < |x-3| < 0.5$, then $|f(x)-5| < 0.1$.

(f) If $0 < |x-3| < \frac{1}{4}$, then $|f(x)-5| < \frac{1}{4}(0.1)$.

(g) If $0 < |x-3| < 1$, then $|f(x)-5| < 0.2$.

(h) If $0 < |x-3| < 1$, then $|f(x)-4.95| < 0.05$.

(i) If $\lim_{x\to 3} f(x) = L$, then $4.9 \le L \le 5.1$.

2.17. Prove that if $\lim_{x\to 0} f(x)/x$ exists, then $\lim_{x\to 0} f(x) = 0$.

2.18. Find the values of a, b and c which make the function f and g, defined below, continuous.

$$f(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax+b & \text{for } -1 \le x < 1 \\ x^2+2 & \text{for } x \ge 1. \end{cases} \qquad g(x) = \begin{cases} \sin(2x)/x & x \neq 0 \\ c & x = 0 \end{cases}$$

2.19. If $f : [0,1] \to [0,1]$ is continuous, show $\exists c \in [0,1]$ with $f(c) = c$.

2.20. Suppose $f : (a,b) \to \mathbb{R}$ is continuous and one–to–one.

Show $f(x)$ is either strictly increasing on (a,b) or strictly decreasing on (a,b).

2.21. Consider two continuous functions

$$f, g : [a,b] \to \mathbb{R} \quad \text{such that } f(a) < g(a) \text{ and } g(b) < f(b).$$

Show that there exists $c \in (a,b)$ with $f(c) = g(c)$.

2.22. Prove that $\sin(x) = 1 - x$ has a solution between $x = 0$ and $x = \pi/2$.

2.23. Prove that if $\lim_{x\to c} f(x) = b$, and $g(y)$ is continuous at $y = b$, then $\lim_{x\to c} g(f(x)) = g(b)$.

2.24. Give an example to show that the following is **not** true:

if $\lim_{x\to c} f(x) = b$, and $\lim_{y\to b} g(y) = a$, then $\lim_{x\to c} g(f(x)) = a$.

Topic 3. Differentiation

3.1. Use exercise 2.13 to calculate the derivatives of the functions:

(a) $f(x) = \sqrt{x}$ (b) $f(x) = 1/\sqrt{x}$ (c) $f(x) = \sin(x)$ (d) $f(x) = \cos(x)$.

3.2. Using only the definition of the derivative as the limit of a different quotient, determine $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right)$ when n is a positive integer.

3.3. Using only the product rule and the fact that $\frac{dx}{dx} = 1$, prove by induction the formula for the derivative of x^n when n is a positive integer.

3.4. Consider the function: $f(x) = |x|x^3 = \begin{cases} x^4 & (x \geq 0) \\ -x^4 & (x < 0) \end{cases}$

(a) State whether f is an odd function, an even function, or neither.

(b) Prove that f is one-to-one. Find the inverse function f^{-1}.

(c) Calculate $f'(x)$ for all $x \in \mathbb{R}$.

(d) State whether $f'(x)$ is an odd or even function, and whether or not it is invertible.

3.5. Prove the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^3/6 & x \geq 0 \\ -x^3/6 & x < 0 \end{cases}$ is differentiable everywhere and that its derivative $g(x) = \dfrac{df}{dx}$ is also differentiable everywhere.

Is the function $h(x) = \dfrac{dg}{dx} = \dfrac{d^2 f}{dx^2}$ continuous at $x = 0$? Is it differentiable there?

3.6. Use the mean value theorem (MVT) to prove that

(a) $e^x \geq 1 + x$ for all $x \in \mathbb{R}$ (b) $\dfrac{x}{1+x} < \ln(1+x) < x$ for all $x > 0$.

3.7. Assume f and g are differentiable on the interval $(-c, c)$ and $f(0) = g(0)$.

(a) Show that if $f'(x) > g'(x)$ for all $x \in (0, c)$, then $f(x) > g(x)$ for all $x \in (0, c)$.

(b) Show that if $f'(x) > g'(x)$ for all $x \in (-c, 0)$, then $f(x) < g(x)$ for all $x \in (-c, 0)$.

3.8. Show that, if the derivative $f'(x)$ of a function exists, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

3.9. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersects the graph of f at a point $(a, f(a))$, where $0 < a < 1$.

Show that there exists $c \in [0, 1]$ such that $f''(c) = 0$.

3.10. Prove that there exists no differentiable function $f : [0, 2] \to \mathbb{R}$ satisfying $f(0) = 1$, $f(2) = 6$ and $f'(x) \leq 2$ for all $x \in [0, 2]$.

3.11. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and is differentiable on (a, b). Prove that

(a) if $y = f(x)$ and $y = x$ intersect at least twice, then $f'(c) = 1$ for some $c \in (a, b)$.

(b) if f has a local maximum or a local minimum at $x = c \in (a, b)$, then $f'(c) = 0$.
[Hint: the proof uses the same idea as the proof of Rolle's Theorem]

(c) if $f' : (a, b) \to \mathbb{R}$ is continuous and never zero then f is either strictly increasing on all of $[a, b]$ or strictly decreasing on all of $[a, b]$.

(d) $f' : (a, b) \to \mathbb{R}$ is always zero $\Leftrightarrow$ f is constant.

3.12. Suppose f and g are two functions whose product $f \cdot g$ is differentiable at $x = c$.

If f is differentiable at $x = c$ does it follow that g is? [Give a proof, or a counterexample]

3.13. Use the Mean Value Theorem to prove that $\sin(x) < x$ if x is positive.

Use the Cauchy Mean Value Theorem to prove that $\cos(x) > 1 - \frac{x^2}{2}$ if $x \neq 0$.

3.14. Prove that if $\lim\limits_{x \to c} f'(x)$ exists then f is differentiable at $x = c$.

Give an example to show the converse is not true.

3.15. If f is twice differentiable, prove that $f''(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

3.16. Use logarithms to calculate (a) $\dfrac{d}{dx}\left(x^x\right)$, (b) $\dfrac{d}{dx}\left(x^{(x^x)}\right)$, (c) $\dfrac{d}{dx}\left((x^x)^x\right)$

3.17. Find $\lim\limits_{x \to 0} \dfrac{\sin(x) - x}{x^3}$, $\lim\limits_{x \to 0} \dfrac{e^x - (1 + x + \frac{x^2}{2})}{x^3}$, $\lim\limits_{x \to 0^+} \dfrac{\ln(1+x)}{x}$, $\lim\limits_{z \to \infty} \left(1 + \dfrac{1}{z}\right)^z$.

3.18. For each of the following functions find all local maxima and minima and the intervals where the function is increasing, decreasing, 'concave up' and 'concave down':

(a) $f(x) = x^3 - 9x$ (b) $f(x) = 3x^4 + 4x^2 + 1$ (c) $f(x) = 2x^3 + 3x^2 + 1$

3.19. Find the global extreme values of the functions

(a) $f : [0, 3] \to \mathbb{R}$, $f(x) = x^2 - 4x + 1$ (b) $f : [-2, 0] \to \mathbb{R}$, $f(x) = 2x^2 + 5x - 1$

3.20. Find the Taylor polynomial centred at $x = c$ of degree n, and the Lagrangian remainder:

(a) $f(x) = \sqrt{x}$, $c = 4$, $n = 3$ (b) $f(x) = \cos(x)$, $c = \frac{\pi}{3}$, $n = 4$

(c) $f(x) = \sin(x)$, $c = \frac{\pi}{4}$, $n = 4$ (d) $f(x) = \ln(x)$, $c = 1$, $n = 5$

(e) $f(x) = \tan^{-1}(x)$, $c = 1$, $n = 3$ (f) $f(x) = \cos(\pi x)$, $c = \frac{1}{2}$, $n = 4$

3.21. Use Newton's method to approximate $\sqrt{5}$.

Start the iteration with $x_0 = 2$ and give an approximation x_n with $|x_n^2 - 5| < 0.01$.

3.22. Use Newton's method to approximate $\sqrt[3]{23}$.

Start the iteration with $x_0 = 3$ and give an approximation x_n with $|x_n^3 - 23| < 0.001$.

3.23. Show that $f(x) = x^4 - 7x^2 - 8x - 3$ has exactly one critical point $x = c$ with $c \in (2, 3)$.

Use Newton's method to find an approximation x_n to c with $|f'(x_n)| < 0.01$.

3.24. Find the global maximum and minimum values of the function $f : [-2, 6] \to \mathbb{R}$ defined by $f(x) = 3x^3 - 9x + 10$.

3.25. Find the global maximum and minimum values of $z = f(x, y) = 2x^2 + y + xy + 4$ on the edges of the triangle with vertices $(-1, -1)$, $(3, 3)$, $(-5, -1)$.

3.26. Find the global maximum and the global minimum of $z = f(x, y) = 4y - \frac{2}{3}y^3 - 4x^2y$ when the point (x, y) is restricted to the curve C given by the ellipse

$$\{(x, y) \in \mathbb{R}^2 : x^2 + (\tfrac{y}{2})^2 = 1\}.$$

3.27. Consider a metal triangular plate with vertices $V_1 = (0, -4)$, $V_2 = (6, -4)$, $V_3 = (0, 8)$. What are the global maximum and minimum temperatures on the edge of the plate, if the temperature at a point on the plate is given by $T(x, y) = x^2 + xy + 2y^2 - 3x + 2y$?

3.28. In exercise 3.27, what are the maximum and minimum temperatures on the whole plate (the interior as well as the boundary)?

3.29. Find the global maximum and minimum values of $z = f(x, y) = 2x^2 - y^2 + 6y$ on

(a) the disc of radius 4, centre $(0, 0)$ (b) the square of side length 2, centre $(0, 0)$

In each case find the global extreme values on the boundary (including, in case (b), the vertices) and the global extreme values on the interior (including the boundary).

3.30. Find the global maximum and minimum values of $z = f(x, y) = xe^x(y^2 - 25)$,

$$\text{on } C = \{(x, y) : x^2 + y^2 = 25\}, \quad \text{and on } D = \{(x, y) : x^2 + y^2 \leq 25\}$$

(the domain D is the disc of radius 5, centre $(0, 0)$, and the circle C is its boundary).

3.31. Consider the rectangle in $\mathbb{R}^2$ defined by the vertices $(\pm 2, \pm\pi)$. Find the global maximum and minimum values of $z = f(x, y) = 2x^2y - 2\sin y$ on the rectangle (the interior, as well as the edges and vertices).

3.32. For all (x, y) in the real plane $\mathbb{R}^2$, consider the function

$$f(x, y) = \tfrac{2}{3}x^3 + 4xy^2 - 4x.$$

Find the global maximum and global minimum values of the function $z = f(x, y)$ when restricted to each of the following domains in $\mathbb{R}^2$:

$$C = \{(x, y) : \tfrac{x^2}{4} + y^2 = 1\} \quad \text{and} \quad D = \{(x, y) : \tfrac{x^2}{4} + y^2 \leq 1\}.$$

3.33. For all (x, y) in the real plane, consider the function $f(x, y) = x^2y - \cos(\pi y)$.

Find the global maximum value of f when restricted to the square with vertices $(\pm 1, \pm 1)$, and when restricted to the boundary of this square.

Topic 4. Sequences

4.1. Consider the sequence $(a_n)_{n\geq 1}$ with $a_n = \dfrac{n+(-1)^n}{n}$. Is it

(a) bounded? (If so give an upper and a lower bound).

(b) monotonic? (Justify your answer).

(c) convergent? (If it is, give the limit).

4.2. State the Pinching theorem for limits of sequences. Determine whether or not the following sequences converge as $n \to \infty$, and give the values of the limits if they do.

(a) $\dfrac{1}{n}\sin(n))$ (b) $\dfrac{n\cos(1+n^2)}{1+n^2}$

4.3. Find $\lim\limits_{n\to\infty}(a_n + b_n)$ if $a_n = \dfrac{n+1000000}{n^2}$ and $b_n = \dfrac{\cos^2(3n^2-4)}{n}$.

4.4. Determine whether the following sequences converge, and find their limits if they do.

(You should justify your answers, but no need for an ε-N proof)

(a) $(\sin(\pi n))_{n\geq 0}$ (b) $(\sin(3n))_{n\geq 0}$ (c) $\left(\frac{2}{n+1}\sin(3n)\right)_{n\geq 0}$ (d) $(\tan(n\pi))_{n\geq 0}$

(e) $(\tan(3n))_{n\geq 0}$ (f) $\left(\frac{2}{n+1}\tan(3n)\right)_{n\geq 0}$ (g) $(\cos(n\pi))_{n\geq 0}$ (h) $\left(\frac{\cos n^2}{1+n}\right)_{n\geq 0}$

4.5. Prove directly from the ϵ–N definition that, as $n \to \infty$,

(a) $\dfrac{n+2}{n+1} \to 1$ (b) $\dfrac{n}{1+n^2} \to 0$

4.6. If $a_n \to L_a$ and $b_n \to L_b$, prove: $a_n + b_n \to L_a + L_b$, $a_n - b_n \to L_a - L_b$, $a_n b_n \to L_a L_b$.

4.7. Prove that if $(a_n)_{n\geq 0}$ converges to L then $(|a_n|)_{n\geq 0}$ converges to $|L|$.

4.8. Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_n = \sqrt{n^2+1} - n$.

(a) Prove that $0 < a_n < \frac{1}{2n}$ for all $n \geq 1$.

(b) Find the limit of the sequence $(a_n)_{n\geq 1}$, justifying your answer.

4.9. (a) Give the definition of a *Cauchy sequence*.

(b) Prove that every Cauchy sequence is bounded, but the converse is false.

(c) If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences, prove $(a_n - b_n)_{n\geq 1}$ and $(a_n b_n)_{n\geq 1}$ are.

4.10. Let $r \neq 0$ and let $(x)_{n\geq 1}$ be a sequence satisfying $|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|$, $\forall n \geq 1$.

Prove $|x_m - x_n| \leq \left|\dfrac{r^m - r^n}{r-1}(x_1 - x_0)\right|$ for all m, n and that $(x_n)_{n\geq 0}$ is Cauchy if $|r| < 1$.

4.11. Newton's method for finding an approximate solution of $x^2 - 2 = 0$ gives a sequence of rational numbers defined by $a_0 = 2$ and $a_{n+1} = a_n - \dfrac{a_n^2 - 2}{2a_n} = \dfrac{1}{a_n} + \dfrac{a_n}{2}$ $(n \geq 0)$.

Prove by induction that the sequence is monotonic decreasing and bounded below.

Find the limit of the sequence (a_n).

4.12. Prove that if (a_n) and (b_n) are both convergent sequences, then so is the product sequence (c_n) defined by $c_n = a_n \cdot b_n$ for all $n \geq 0$.

Show that if (a_n) does not converge but (b_n) does, then (c_n) might or might not converge.

Show that if neither (a_n) nor (b_n) converge, then (c_n) might or might not converge.

4.13. Give an example of a sequence that has no convergent subsequence.

4.14. Consider subsequences (b_n) and (c_n) of any sequence (a_n), where $b_n = a_{2n}$ and $c_n = a_{2n+1}$.

Prove (a_n) is convergent $\Leftrightarrow$ (b_n) and (c_n) are convergent and have the same limit.

Topic 5. Integration

5.1. Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ that is not integrable.

Give an example of a function such that $|f|$ and f^2 are integrable but f is not.

5.2. [Harder] Prove that if $f : [a, b] \to \mathbb{R}$ is integrable then so are $|f|$ and f^2.

5.3. Calculate $U_f(P)$ and $L_f(P)$ in the following two situations:

$$f : [0, 1] \to \mathbb{R},\ f(x) = 2x,\ P = \{0, \tfrac{1}{4}, \tfrac{1}{2}, 1\};$$

$$f : [-1, 0] \to \mathbb{R},\ f(x) = x^2,\ P = \{-1, -\tfrac{1}{2}, -\tfrac{1}{4}, 0\}\,.$$

5.4. Consider the two functions $f : [0, 1] \to \mathbb{R}$ below, and the partition P_n of $[0, 1]$ into n intervals of equal widths, $\Delta x_i = \frac{1}{n}$.

Find $U_f(P_n)$ and $L_f(P_n)$ for each function and use your answer to calculate $\int_0^1 f(x)dx$.

$$\text{a)}\ f(x) = x + 3; \qquad \text{b)}\ f(x) = -3x\,.$$

5.5. Find partitions P_n such that $U_f(P_n) - L_f(P_n) \to 0$ which prove integrability of

$$\text{a)}\ f(x) = x \text{ on } [0, 1], \qquad \text{b)}\ f(x) = x^2 \text{ on } [0, 1], \qquad \text{c)}\ f(x) = 1/x \text{ on } [1, 2],$$

Calculate the value of the definite integral, that is, $\lim_{n\to\infty} U_f(P_n)$, in each case.

5.6. Consider the function $f : [0, 1] \to \mathbb{R}$, with $f(x) = n^4$ if $x = \frac{1}{n}$ for some positive integer n, and $f(x) = 0$ otherwise. TRUE or FALSE: the function f is integrable on $[0, 1]$.

Justify (give a proof) your answer, and if ‘TRUE’ write down the value of the integral.

5.7. Consider integrable functions $f, g : [a, b] \to \mathbb{R}$ that satisfy $\int_a^b g(x)dx < \int_a^b f(x)dx$.

Which of the following statements necessarily hold for all partitions P of $[a, b]$?

$$L_g(P) < L_f(P); \qquad L_g(P) < U_f(P); \qquad U_g(P) < U_f(P);$$

$$L_g(P) < \int_a^b f(x)dx; \qquad U_g(P) < \int_a^b f(x)dx.$$

5.8. Prove that if a function is integrable on an interval $[a, b]$ and on an interval $[b, c]$ then it is integrable on $[a, c]$.

5.9. Assume that $f : [a, b] \to \mathbb{R}$ is non-increasing: $f(x_1) \leq f(x_2)$ if $x_1 \geq x_2$. Explain why f is bounded. Prove that f is integrable.

5.10. [Hard] Prove the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, x = 1, \text{ or } x \text{ irrational} \\ \dfrac{1}{n} & \text{if } 0 < x < 1 \text{ and } x = \dfrac{m}{n} \text{ in lowest terms} \end{cases}$$

is integrable, even though it is discontinuous at every rational number.

5.11. Write down two integrable functions $g, h : [0, 1] \to [0, 1]$ whose compostite $g \circ h : [0, 1] \to [0, 1]$ is not integrable.

5.12. Write down "a formula" for a function $F(x)$ such that $F'(x) = \sin(x^2)$ and $F(1) = 2$.

5.13. Let $F(x) = \displaystyle\int_0^x \frac{t-1}{1+t^2} dt$, $x \in \mathbb{R}$. Find the critical points of F. Determine, at each critical point, whether F has a local maximum, local minimum or neither.

5.14. Prove that if $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $\displaystyle\int_0^1 f(x)dx = 0$, then $\exists c \in (0, 1)$ with $f(c) = 0$.

5.15. Prove that there exists an integrable, but not continuous, function $f : [0, 1] \to \mathbb{R}$ such that $\displaystyle\int_0^1 f(x)dx = 0$ but $f(c) \neq 0 \ \forall c \in [0, 1]$.

5.16. Use an appropriate substitution to evaluate $\displaystyle\int_2^x \frac{dt}{t \ln(t)}$.

5.17. Use an appropriate substitution to evaluate $\displaystyle\int_2^x \frac{dt}{t\,(\ln(t))^2}$.

5.18. Use an appropriate substitution to evaluate $\displaystyle\int_2^x \frac{dt}{t\,(\ln(t))^p}$ for any $p > 1$.

5.19. Let f be continuous on $[-a, 0]$. Prove that $\displaystyle\int_{-a}^0 f(x)dx = \int_0^a f(-x)dx$.

5.20. Prove by induction that for all $r > -1$ and for all integers $n \geq 0$,

$$\int_0^1 y^n (1-y)^r dy = \frac{n!}{(r+1)(r+2)\dots(r+n+1)} .$$

5.21. Assume that $f : [a, b] \to \mathbb{R}$ has continuous second derivative. Use integration by parts and the Fundamental Theorems of Calculus to prove the identity

$$f(b) = f(a) + f'(a)(b-a) - \int_a^b f''(x)(x-b)dx .$$

5.22. Assume that f has continuous third derivative. Use integration by parts and the Fundamental Theorems of Calculus to prove the identity

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{f'''(x)}{2}(x-b)^2 dx\,.$$

5.23. The previous two questions are 'integral formulas' for the remainder between f and its Taylor polynomials P_1 and P_2, respectively.

For *any* $n \geq 1$ guess an integral formula for the remainder between f and its Taylor polynomial P_n stating the required conditions on f. Prove by induction that your general formula holds true for all positive integers n.

5.24. Calculate the following improper integrals, or prove they are divergent

(a) $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}}$ (b) $\displaystyle\int_{27}^\infty \frac{dx}{\sqrt[3]{x}}$ (c) $\displaystyle\int_0^2 \frac{dx}{x^3}$ (d) $\displaystyle\int_{\frac{1}{2}}^\infty \frac{dx}{x^3}$

5.25. Determine for which values of p each of the following integrals converge, and find their values when they do.

(a) $\displaystyle\int_0^1 \frac{dx}{x^p}$ (b) $\displaystyle\int_1^\infty \frac{dx}{x^p}$ (c) $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ (d) $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

Topic 6. Differential equations

6.1. Solve the Initial Value Problem $y'' + y = x, \quad y(0) = y'(0) = 2.$

6.2. find the general solutions to the differential equations

(a) $y' = 6xe^{x+4y}$ (b) $y'' - 6y' + 9y = 0$ (c) $y' = y^4\sin(2x-1)$

6.3. Solve the initial value problem $x^2y' + xy = x^2 + 2$, $y(1) = 4.5$.

6.4. Find the general solution to the inhomogeneous second order linear ordinary differential equation

$$y'' - 2y' - 3y = 3x^2 + 4x - 5$$

Find also the solution to the IVP with $y(0) = 9$ and $y'(0) = -4$.

6.5. Find the general solution to the following differential equation and determine all possible behaviors of the solution as $x \to \infty$

$$2y' - y = 4\sin(3x)$$

6.6. Solve the following IVP and find the domain of validity of the solution

$$y' = \frac{xy^3}{\sqrt{1+x^2}}, \qquad y(0) = -1.$$

6.7. Solve the following IVP

$$y'' - 8y' + 17y = 85 \qquad y(0) = 1, \quad y'(0) = -1.$$

6.8. Use the method of Separable Variables to solve the following Initial Value Problems, and determine the intervals of validity of your solutions.

(a) $y' = 6xy^2$, $y(1) = 1/25$,

(b) $y' = e^{-y}(2x - 4)$, $y(5) = 0$

(c) $y\,y' = x^2y^2 + x^2 + y^2 + 1$, $y(0) = 1$

6.9. Use the Integrating Factor method to solve

(a) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$, $y(\pi/4) = 3\sqrt{2}$, where $0 \leq x < \pi/2$.

(b) $xy' - 2y = x^5 \sin(2x) - x^3 + 4x^4$, $y(\pi) = 3\pi^4/2$.

6.10. Find and solve the characteristic polynomials associated to each of the following differential equations. Give the general solutions of the differential equations and also the solution which satisfies the given initial value problem

(a) $3y'' + 2y' - 8y = 0$, $y(0) = -6$, $y'(0) = -18$.

(b) $4y'' - 5y' = 0$, $y(-2) = 0$, $y'(-2) = 7$.

(c) $4y'' + 24y' + 37y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$.

(d) $y'' + 14y' + 49y = 0$, $y(-4) = -1$, $y'(-4) = -5$.

(e) $y''' - 5y'' - 22y' + 56y = 0$, $y(0) = 1$, $y'(0) = -2$, $y''(0) = -4$.

(f) $y'' - 4y' - 12y = 3e^{5x}$, $y(0) = \frac{18}{7}$, $y'(0) = -\frac{1}{7}$.

Topic 7. Infinite series

7.1. Prove, without using the integral test or the p-test, that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Using the comparison theorem, deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \geq 2$.

7.2. Prove, without using the integral test or the p-test, that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Using the comparison theorem, deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $0 < p \leq 1$.

7.3. Prove by induction the formula for the sum of a geometric progression: $\displaystyle\sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r}$.

Deduce that the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$ diverges if $|r| \geq 1$, but converges to $\frac{1}{1-r}$ if $|r| < 1$.

What are the partial sums s_n in the case $r = -1$?

7.4. Write down a formula for the partial sums s_n of the series

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}.$$

7.5. Use the formula for the sum of a geometric series to determine the values of

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}, \qquad \sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$$

7.6. Express $\frac{1}{k^2-1}$ in terms of partial fractions and hence prove that

$$\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2n+2}.$$

Hence determine if the following series converges or diverges, and if it converges determine its sum.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

7.7. Using a similar technique to the previous question, investigate the following series

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2+4n+3}.$$

7.8. Determine if the following series converges or diverges, and if it converges determine its sum

$$\sum_{n=0}^{\infty} \frac{4n^2-n^3}{10+2n^3}.$$

Hint: look at the behaviour of a single term a_n in the series, as $n \to \infty$.

7.9. Determine for which values of the parameter p the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$ is convergent, and for which values it is absolutely convergent, justifying your statements.

7.10. For which values of the parameter $c \in \mathbb{R}\backslash\{0\}$ is the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(5c)^n}$ absolutely convergent?

7.11. Determine the radius of convergence and the interval of convergence for the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{4^n}(x+3)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$ (e) $\displaystyle\sum_{n=1}^{\infty} n!(2x+1)^n$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{n \cdot 3^n}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\left(4-\frac{1}{n}\right)^n}$ (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n^n}$

(j) $\displaystyle\sum_{n=0}^{\infty} (-1)^n\, n\, 2^n\, x^n$ (k) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{n+1}}{n \cdot 3^n}$ (l) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{n!}(x-10)^n$

In each case state the convergence behaviour in the interval of convergence and investigate the convergence at the endpoints of the interval.

7.12. Write the following as power series, then find the interval of convergence of the series.

(a) $\dfrac{1}{1-x}$ (b) $\dfrac{1}{1+x^3}$ (c) $\dfrac{2x^2}{1+x^3}$

(d) $\dfrac{x}{5-x}$ (e) $\dfrac{1}{(1-x)^2}$ (f) $\ln(5-x)$

7.13. Find a Taylor series, about $x=0$, for the function $F(x)$ defined by

$$F(x) = \int_0^x \frac{\sin t\,dt}{t}$$

7.14. Find the Taylor series centred at $x=0$ for the function $\sin x$, and prove that this power series is defined on the whole real line (i.e., the radius of convergence is infinite). By differentiation, deduce the series representation of $\cos x$. Were is this defined?

7.15. Using the series for $\cos(x)$ you obtained in the previous question, write down the first 4 non-zero terms of the Taylor series, about $x=0$, of:

(a) $\cos(x^4)$ (b) $\cos(2x)$ (c) $\cos^2(x)$ (d) $\sin^2(x)$

7.16. Find the first 3 non-zero terms of the Taylor series, about $x=0$, for

$$e^{x^4}, \qquad e^{2x}, \qquad e^{e^x}.$$

7.17. The power series form of the binomial theorem says that if $|z|<1$ then

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \frac{n(n-1)(n-2)(n-3)}{4!}z^4 + \dots$$

Use this to find the first 4 non-zero terms of a power series centred at $x=0$ for the function $f(x) = (1-x^2)^{-\frac{1}{2}}$. Deduce the first 4 terms of the Maclaurin series for the arcsin function.

Topic 8. Calculus of several variables

8.1. Prove that $\dfrac{x^2y^2}{x^2+y^4}$ approaches a limit as (x,y) approaches the origin.

8.2. Prove that $\dfrac{xy^2}{x^2+y^4}$ does not have a limit as (x,y) approaches the origin.

8.3. Prove that $\dfrac{(x-1)^2\ln(x)}{(x-1)^2+y^2}$ does approach a limit as $(x,y)\to(1,0)$.

8.4. Consider the function of two variables $f(x,y) = \begin{cases} \dfrac{x^3+y^5}{x^2+y^4} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0). \end{cases}$

(a) Prove f is continuous at the origin.

(b) Using only the definition of partial derivatives, prove $f_x(0,0) = f_y(0,0) = 1$.

(c) Calculate the partial derivatives $f_x(x,y)$ and $f_y(x,y)$, for $(x,y) \neq (0,0)$.

(d) Show that the functions f_x and f_y are both discontinuous at $(0,0)$.

(e) Find the equation of the tangent plane at $(x,y) = (-1,-1)$ to the surface $z = f(x,y)$.

(f) i. Define the *directional derivative* of a function of two variables f at a point $\underline{x} = (x,y)$ in the direction of a unit vector $\underline{\hat{u}} = (p,q)$.

ii. For the function f above, show the directional derivative $f_{\underline{\hat{u}}}(0,0)$ at the origin equals p if $p \neq 0$, or q if $p = 0$.

8.5. Consider the two functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases} \qquad g(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

(a) Prove that *one* of these functions is continuous at the origin.
Prove that the other function is *not* continuous at the origin.

(b) Write down the partial derivative $f_y(x,y)$ when $(x,y) \neq (0,0)$.
Find $f_y(0,0)$ by evaluating the limit of a difference quotient.
Prove that f_y is not continuous at the origin by considering the curve $y = x^3$.

8.6. Consider the function of two variables defined by $f(x,y) = \begin{cases} \dfrac{3xy^2}{2x^2+2y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

(a) Determine whether or not f is continuous at the origin.

(b) Compute the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.

(c) Use the definition of partial derivative to find f_x and f_y at the origin, and prove they are not continuous there.

(d) Define the *directional derivative* $f_{\hat{u}}(\underline{x})$ of f in the direction of the unit vector $\hat{u} = (p,q)$.
Calculate $f_{\hat{u}}(0,0)$ if $p \neq 0$.

8.7. Consider the function of two variables $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+3y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

(a) Prove that f is not continuous at the origin.

(b) Compute the first partial derivatives f_x and f_y of f, at any $(x,y) \in \mathbb{R}^2$.
Prove that they are not continuous at the origin.

(c) The *directional derivative* of f at $\underline{x}$ in the direction of the unit vector $\hat{u} = (p,q)$ is

$$f_{\hat{u}}(\underline{x}) = \lim_{h \to 0} \frac{f(\underline{x} + h\hat{u}) - f(\underline{x})}{h},$$

Prove that for the function f above the directional derivative at the origin in the direction $\hat{u} = (p, q)$ does not exist unless p or q is zero.

8.8. Consider the function of two variables $f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$

(a) Prove f is continuous at the origin.

(b) i. Compute the first partial derivatives $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.
ii. Prove, directly from the definition, that $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$.

(c) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1, \frac{1}{2})$.

(d) Prove that neither f_x nor f_y are continuous at the origin.

(e) The *directional derivative* of f at $\underline{x}$ in the direction of the unit vector $\underline{\hat{u}} = (p, q)$ is

$$f_{\underline{\hat{u}}}(\underline{x}) = \lim_{h \to 0} \frac{f(\underline{x} + h\underline{\hat{u}}) - f(\underline{x})}{h}.$$

Prove that for the function f above the directional derivative at the origin in the direction $\underline{\hat{u}} = (p, q)$ is given by $f_{(p,q)}(0, 0) = p^3$.

8.9. (a) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ \dfrac{x^3 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \end{cases}$

is continuous at the origin.

(b) By using the definition of partial derivative, show that the function f from part (a) has first partial derivatives f_x and f_y at the origin.

(c) Compute the first partial derivatives of the function f from part (a) for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(d) By using the definition of partial derivative, show that the function f from part (a) has second partial derivatives f_{xy} and f_{yx} at the origin.

(e) Compute the mixed second partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(f) Compute the limit at the origin of the mixed second partial derivatives $f_{xy}(x, y)$ along the line $y = 0$.

(g) State the equality of the mixed partial derivatives theorem. In view of this theorem, comment on your findings from parts (d) and (f).