Lecture 29: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

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Green's Theorem in the Plane

• If F is a **conservative field**, then we know $F = \nabla f$ for a differentiable function f, and we can calculate the **line integral** of F over any path C joining point A to point B as $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

• Today we derive a **method for computing a work or flux integral** over a closed curve C in the plane when the **field F is not conservative**.

• This method comes from **Green's Theorem**, which allows us to **convert the line integral into a double integral** over the region enclosed by C.

• The discussion is given in terms of **velocity fields of fluid flows** (a fluid is a liquid or a gas) because they are easy to visualize.

• However, Green's Theorem **applies to any vector field**, provided the assumptions of the theorem are satisfied.

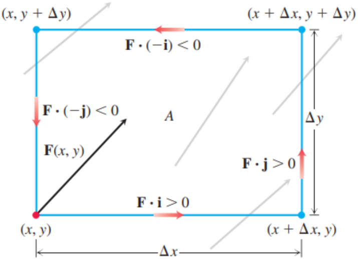
• We introduce two new ideas for Green's Theorem: **circulation density** around an axis perpendicular to the plane and **divergence** (or flux density).

Spin Around an Axis: The k-Component of Curl

• Suppose that $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is the **velocity field** of a fluid flowing in the plane.

• The first partial derivatives of M and N are **continuous** at each point of a region R

• Let A be a **small rectangle** lies entirely in R.



• The first idea we use to convey **Green's Theorem** quantifies **the rate at which** a floating paddle wheel, with axis perpendicular to the plane, **spins at a point** in a fluid flowing in a plane region.

• Physicists sometimes refer to this as the **circulation density of a vector field F** at a point.

• To approximate the **rate of circulation** at the point (x, y), we calculate the (approximate) **flow rates along each edge** in the directions of the red arrows, sum these rates, and then divide the sum by the area of A.

$$\begin{array}{ll}
\text{Top:} & \mathbf{F}(x, y + \Delta y)\cdot(-\mathbf{i})\Delta x = -M(x, y + \Delta y)\,\Delta x \\
\text{Bottom:} & \mathbf{F}(x, y)\cdot\mathbf{i}\,\Delta x = M(x, y)\Delta x \\
\text{Right:} & \mathbf{F}(x + \Delta x, y)\cdot\mathbf{j}\,\Delta Y = N(x + \Delta x, y)\Delta y \\
\text{Left:} & \mathbf{F}(x, y)\cdot(-\mathbf{j})\Delta y = -N(x, y)\Delta y
\end{array}$$

$$\text{Top and bottom:} \quad -(M(x, y + \Delta y) - M(x, y))\,\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\text{Right and left:} \quad (N(x + \Delta x, y) - N(x, y))\,\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y.$$

• **Taking the limit** as $\Delta x \to 0$ and $\Delta y \to 0$ gives the **rate of the circulation per unit area**.

$$\text{Circulation rate around rectangle} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\, \Delta y.$$

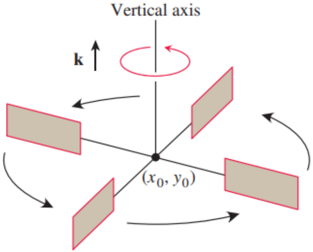
$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Green's Theorem in the Plane

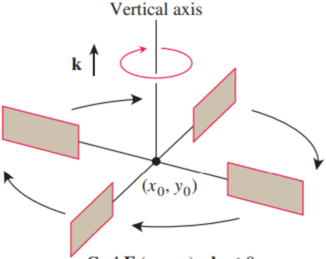
DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{1}$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl}\,\mathbf{F}) \cdot \mathbf{k}$.



Curl $\mathbf{F}\,(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation



Curl $\mathbf{F}\,(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

EXAMPLE 1 The following vector fields represent the velocity of a gas flowing in the xy-plane. Find the circulation density of each vector field and interpret its physical meaning. Figure 16.31 displays the vector fields.

(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$ c a constant

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

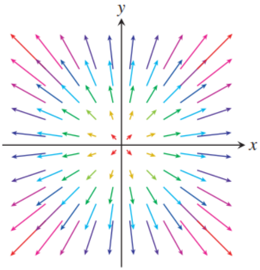
(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

Solution Example 1

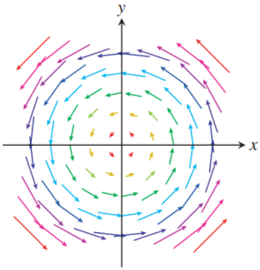
Solution

(a) *Uniform expansion:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial}{\partial x}(cy) - \dfrac{\partial}{\partial y}(cx) = 0$. The gas is not circulating at very small scales.

(b) *Rotation:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial}{\partial x}(cx) - \dfrac{\partial}{\partial y}(-cy) = 2c$. The constant circulation density indicates rotation around every point. If $c > 0$, the rotation is counterclockwise; if $c < 0$, the rotation is clockwise.



(a)



(b)

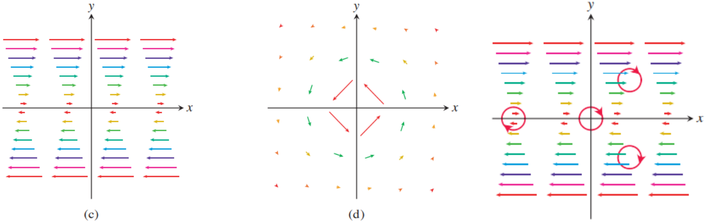
Solution Example 1

(c) *Shear:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = -\frac{\partial}{\partial y}(y) = -1$. The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average rotational effect of the fluid flow is to push fluid clockwise around each of the small circles shown in Figure 16.32.

(d) *Whirlpool:*

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

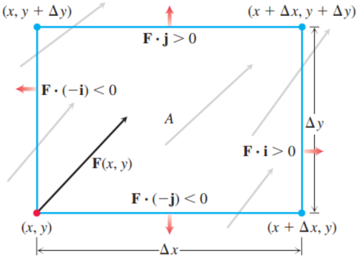
The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined. ■



Divergence

• Consider again the **velocity field** $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in a domain containing the **rectangle A**, as shown in Figure.

• As before, we assume the field components **do not change sign** throughout a small region containing the rectangle A.



• Our interest now is to determine **the rate at which the fluid leaves A** by flowing across its boundary.

• To approximate the flow rate at the point (x, y), we calculate the (approximate) **flow rates across each edge** in the directions of the red arrows, **sum these rates**, and then **divide the sum by the area of A**.

Divergence

Fluid Flow Rates:

	Top:	$\mathbf{F}(x, y+\Delta y) \cdot \mathbf{j}\,\Delta x = N(x, y+\Delta y)\,\Delta x$
	Bottom:	$\mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta x = -N(x, y)\,\Delta x$
	Right:	$\mathbf{F}(x+\Delta x, y) \cdot \mathbf{i}\,\Delta y = M(x+\Delta x, y)\,\Delta y$
	Left:	$\mathbf{F}(x, y) \cdot (-\mathbf{i})\Delta y = -M(x, y)\,\Delta y$

Summing opposite pairs gives

$$\text{Top and bottom:} \qquad (N(x, y+\Delta y) - N(x, y))\,\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

$$\text{Right and left:} \qquad (M(x+\Delta x, y) - M(x, y))\,\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$$

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y.$$

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

• **Taking the limit** as $\Delta x \to 0$ and $\Delta y \to 0$ gives the **flow rate per unit area**.

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

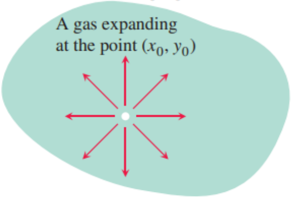
Divergence

• A gas is **compressible**, unlike a liquid, and the **divergence of its velocity field** measures to what **extent it is expanding or compressing** at each point.

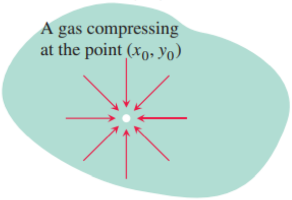
• Intuitively, if a gas is **expanding** at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_0, y_0), the divergence of F at (x_0, y_0) would be **positive**.

• If the gas were **compressing** instead of expanding, the divergence would be **negative**.

Source: div $\mathbf{F}\,(x_0, y_0) > 0$



Sink: div $\mathbf{F}\,(x_0, y_0) < 0$



EXAMPLE 2 Find the divergence, and interpret what it means, for each vector field in Example 1 representing the velocity of a gas flowing in the xy-plane.

Solution

(a) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

(b) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(-cy) + \dfrac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

(c) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

(d) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{-y}{x^2 + y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2 + y^2}\right) = \dfrac{2xy}{(x^2 + y^2)^2} - \dfrac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field. ■