MA2252 Introduction to Computing

Lecture 15
Taylor series

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Learning outcomes

At the end of lecture, students will be able to

- understand Taylor polynomials
- understand Taylor series
- find Taylor series approximations and estimate error

Introduction

Many functions in Mathematics can be approximated by polynomials upto desired accuracy.

To perform such approximations, we need to understand **Taylor polynomials** of a function.

A Taylor polynomial of a function $f(x)$ centered at $x = a$ is a polynomial approximation of $f(x)$.

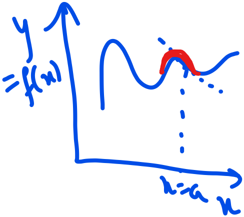


Figure: Brook Taylor

Taylor polynomials (contd.)

Question: Find a quadratic polynomial to approximate $f(x)$ near $x = a$.

$$p_2(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$$

$$= f(a) + f'(a)(x-a) + f''(a)(x-a)^2$$

Taylor polynomials (contd.)

A *nth* order Taylor Polynomial is given as

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n. \quad (1)$$

Do it as an exercise

- $p_0(x) = f(a)$ **(constant function)**
- $p_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$ **(linear approximation)**
- $p_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$

(quadratic approximation)

at $x = 0$ $(a=0)$

Example: The Taylor polynomials of $f(x) = \cos x$ upto degree 6:

$\cancel{p_0 \neq 0}$

$p_0(x) = f(0) = \cos 0 = 1$

$p_1(x) = f(0) + \frac{f'(0)}{1!}(x-0)$

$= 1 + f'(0)x$

$f'(x) = -\sin x$

$f'(0) = 0$

- $n = 0: \quad p_0(x) = 1$
- $n = 2: \quad p_2(x) = 1 - \frac{x^2}{2}$
- $n = 4: \quad p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
- $n = 6: \quad p_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$

$p_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1$

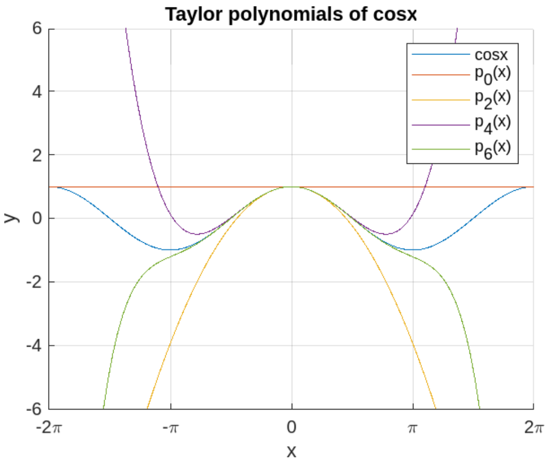
Note: It is possible that a *nth* order Taylor polynomial is not of degree n.

$= 1 - \frac{x^2}{2}$

Example: Write a script file to plot the Taylor polynomials of $\cos x$ upto degree 6.

Demo

Taylor polynomials (contd.)



Taylor series

A Taylor series is an infinite series expansion of a function at a given point in its domain.

Taylor series of a real-valued function $f(x)$ at $x = a$ is defined as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \tag{2}$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \tag{3}$$

Taylor series (contd.)

special case of Taylor series

When $a = 0$, we obtain a **Maclaurin series**.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \tag{4}$$

Example: Find the Taylor series of $\cos x$ at $x = 0$.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f''''(0)x^4}{4!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Taylor's theorem

Theorem

Suppose a function $f(x)$ has $n+1$ continuous derivatives in an open interval I containing $x = a$ then $\forall n$ and $\forall x \in I$,

$$f(x) = \underbrace{f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f''(n)}{2!}(x-a)^n}_{n\text{th order Taylor polynomial}}+R_n(x) \tag{5}$$

where

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1} \tag{6}$$

$R_n(x) \to 0$ as $n \to \infty$

for some c between a and x.

Here, $R_n(x)$ is called the Taylor remainder or error term.

We don't know the exact value of $R_n(x)$ since exact value of c is unknown. However, an upper bound on the error term can still be found.

If $|f^{n+1}(x)| \leq M$ for all t between a and x, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{7}$$

Lagrange upper bound on remainder

$$\cos x = \underbrace{1 - \frac{x^2}{2!}}_{P_2(x)} + \underbrace{\frac{x^4}{4!} + \cdots}_{R_2(x)}$$

2nd order Taylor polynomial near $x = 0$

$1 - \frac{(x-0)^2}{2}$

Example: Find the maximum error if $p_2(x) = 1 - \frac{x^2}{2}$ is used to estimate the value of $cos(x)$ at $x = 0.3$. Verify that the error estimate in MATLAB is less than the maximum error.

$$R_2(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$$= \frac{f^{3}(c)}{3!} x^3 = \frac{\sin c \cdot x^3}{6}$$

$-1 \leq \sin c \leq 1$

$|\sin c| \leq 1$

$M = 1$

$|R_2(x)| = \frac{|\sin c| \, |x|^3}{6}$

$\leq \frac{|x|^3}{6} = \frac{(0.3)^3}{6}$

$= \frac{0.027}{6}$

$= 0.0045$

max error

Demo

$0 < c < 0.3$

$\sin c < \sin(0.3)$

$\sin(0.3)$

Applications of Taylor series

Some applications:

- Small-angle approximations e.g. $\sin\theta \approx \theta$ for small values of θ.
- Finding non-elementary definite integrals e.g. $\int_1^2 \frac{sinx}{x} dx$
- Deriving formulas for numerical differentiation and integration (we'll study later in this course)

End of Lecture 15

Please provide your feedback here