MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part I

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Background

- The option pricing models that we will encounter in future sections are based on a form of modelling known as stochastic calculus. This is the approach used to model financial variables in **continuous time**.
- This can result, under certain conditions, in closed-form solutions that are therefore very **computationally efficient**.

Brownian motion

- A continuous time process that has proved useful for financial modelling purposes is Brownian motion. This was first developed as an idea in the physical sciences, with botanist Robert Brown noting the erratic motion of pollen particles in water in 1827.
- Norbert Wiener developed a mathematically rigorous construction of the stochastic process in a more abstract, general sense. You will sometimes therefore see it called the Wiener process.
- One way to think about this process is the continuous time equivalent of a binomial tree (i.e. a discrete time process, where at each time point the value can increase or decrease with equal probability).

Standard Brownian motion

$\{B_t,\ t \geqslant 0\}$ is called a standard Brownian motion (SBM), if

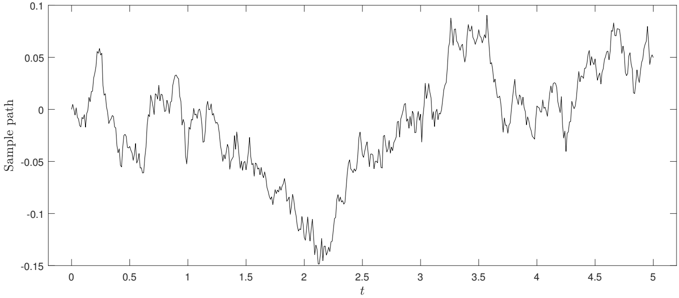
(1) $B_0 = 0$;

(2) $B_t - B_s \backsim N(0, t-s)$ for all $t > s$;

(3) B_t has independent increments such that $B_{t_1},\ B_{t_2} - B_{t_1},\ \cdots,\ B_{t_k} - B_{t_{k-1}}$ are independent for all $t_1 < t_2 < \cdots < t_{k-1} < t_k$;

Sample path of SBM

- An example of a sample path of an SBM is shown in the graph below,



- Notice how jagged the process is. In fact, it is possible to show that the sample path is nowhere differentiable: the limit for a mathematical derivative does not exist because the variable is not bounded, even on a short interval, so no unique tangent can be found.

Properties of SBM

- B_t has stationary increments: the distribution of $(B_t - B_s)$ depends only on $t - s$, where $t > s$.
- $\{B_t,\ t \geqslant 0\}$ is a Markov process.
- $\{B_t,\ t \geqslant 0\}$ returns infinitely often to any level, including 0.
- B_t has continuous sample paths: $t \to B_t$.

Properties of SBM, cont.

- $\mathbb{E}[B_t] = 0$.
- $\mathrm{Var}(B_t) = t$.
- $Cov(aB_t, bB_s) = ab\mathbb{E}(B_t B_s) = ab \min(t, s)$, where a and b are constants.
- $B_t - B_s$ has the same distribution as B_{t-s}, $t > s$. But they are different,
  - $B_t - B_s$ is the difference between two different random variables.
  - B_{t-s} is a single random variable.

Conditional expectation of SBM

The conditional expectation encountered in this course is no longer conditioning on a single random variable, but many (including a filtration, in the abstract sense).

Let $X_t = f(t, B_t)$ be a stochastic process, then we have

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[f(t, B_t)|\mathcal{F}_s] = \mathbb{E}[f(t, B_s + (B_t - B_s))|\mathcal{F}_s]$$

where $\{\mathcal{F}_s, 0 \leq s < t\}$ is a filtration of X_t.

Specifically, when $s = 0$,

$$\mathbb{E}[X_t|\mathcal{F}_0] = \mathbb{E}[X_t]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(s, B_s)|\mathcal{F}_s] = f(s, B_s)$,
- $\mathbb{E}[(B_t - B_s)^{2m+1}|\mathcal{F}_s] = 0,\ m = 0, 1, \ldots$
- $\mathbb{E}[(B_t - B_s)^{2m}|\mathcal{F}_s] = (t-s)^m(2m-1)!!,\ m = 0, 1, \ldots,$ where $(2m-1)!! = 1 \times 3 \times 5 \times (2m-1)$.
- If $f(t, B_t) = g(t)h(B_t)$, $\mathbb{E}[g(t)h(B_t)|\mathcal{F}_s] = g(t)\mathbb{E}[h(B_t)|\mathcal{F}_s]$
- $\mathbb{E}[g(t)|\mathcal{F}_s] = g(t)$ and $\mathrm{Var}[g(t)] = 0$.

for all $t > s \geqslant 0$.

Examples

- Show that B_t is a martingale.
- Show that $B_t^2 - t$ is a martingale.
- Is B_t^3 a martingale?
- Find $\mathbb{E}[t^2 B_t | \mathcal{F}_s]$, $t > s$.

Geometric Brownian Motion (preview)

- One of the shortcomings of standard Brownian motion, from the point of view of financial modelling, is that it allows negative values. It is not helpful when modelling equity prices, since these cannot be worth less than nothing.
- This can be addressed by assuming that the log of a stochastic process follows standard Brownian motion. The process itself will then be log-normally distributed.

Geometric Brownian Motion (preview)

- The asset price at time t can be modelled by

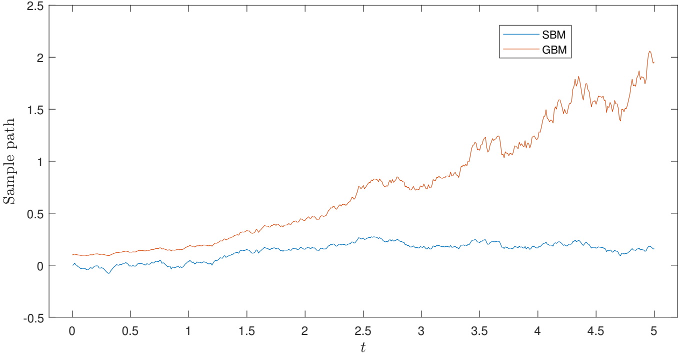
$$S_t = S_0 e^{at + \sigma B_t}$$

where S_0 is the price at time 0, a is a constant and σ is the standard deviation/volatility of the return.

- This model is referred to as geometric Brownian motion (GBM) and is central to the development of many models in mathematical finance.

Sample path of GBM

- An example sample path, with a comparison to standard Brownian motion is shown below,



Stochastic calculus

- Since the sample paths of Brownian motion are not differentiable, that might suggest that the powerful applications that calculus can bring are not available.
- However, it is in fact possible to develop a definition of integrals of the form $\int_0^t Y_u dB_u$ for suitable random integrands Y_u (i.e. those that are adapted to the filtration of the Brownian motion).
- These stochastic integrals are also referred to as "Ito integrals", after the Japanese mathematician Kiyoshi Ito who first developed the idea in the 1940s.

Stochastic integrals

- The basic principle is that, while Brownian motion is "too erratic" to allow for a sensible definition of a mathematical derivative, there is enough regularity that it is possible to define a sequence of sums that will converge to a finite value under certain conditions. These can be considered an integral, in the same way as a standard integral is the limit of a sequence of Riemann sums.
- These stochastic integrals are themselves random variables, since B_t is random. We will therefore be interested in their distribution, expectation and variance.

Properties of stochastic integrals

(1) $\left(\int_0^t Y_u dB_u, t \geqslant 0\right)$ is a martingale.

(2) $\mathbb{E}[\int_0^t Y_u dB_u] = 0$.

(3) Ito isometry: $\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$.

(4) $\int_0^t Y_u dB_u$ follows a normal distribution with mean 0 and variance $\mathrm{Var}(\int_0^t Y_u dB_u) = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$, that is

$$\int_0^t Y_u dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t Y_u^2 du\right]\right)$$

Example

- State the distribution of the Ito integral $\int_0^t \sqrt{u} dB_u$.

Ito process

- The Ito process X_t is a stochastic process, which can be defined by the stochastic differential equation (SDE)

$$dX_t = A_t dt + Y_t dB_t$$

where A_t and Y_t are two expressions in terms of B_t and t.

- It is common to write the Ito process X_t in integral form

$$X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where X_0 is a constant and the initial value of X_t.

Ito process, cont.

- Note that $\int_0^t A_u du$ is deterministic and is called a deterministic integral. Therefore, the Ito process X_t has a deterministic mean and a random component, such that

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t A_u du\right] + \mathbb{E}\left[\int_0^t Y_u dB_u\right] \\
&= X_0 + \mathbb{E}\left[\int_0^t A_u du\right] \\
\mathrm{Var}[X_t] &= \mathrm{Var}[X_0] + \mathrm{Var}\left[\int_0^t A_u du\right] + \mathrm{Var}\left[\int_0^t Y_u dB_u\right] \\
&= \mathbb{E}\left[\int_0^t Y_u^2 du\right]
\end{aligned}$$

since the deterministic integral $\int_0^t A_u du$ is independent of the stochastic integral $\int_0^t Y_u dB_u$.

Properties of deterministic integral

(1) Fubini theorem: $\mathbb{E}\left[\int_0^t A_u du\right] = \int_0^t \mathbb{E}[A_u]du$

(2) $\mathrm{Var}\left[\int_0^t A_u du\right] = 0$

(3) If $X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$, then

$$X_t \sim N\left(X_0 + \int_0^t \mathbb{E}\left[A_u\right] du, \int_0^t \mathbb{E}\left[Y_u^2\right] du\right)$$

Example

- If $dX_t = -2tdt + 5\sqrt{t}dB_t$, state the distribution of the stochastic integral $\int_0^t \sqrt{u}dX_u$.