UNIVERSITY OF LEICESTER

MA1012
2016-17 and 2017-18 candidates only

January Examinations 2018

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
Department	Mathematics
Module Code	MA1012
Module Title	Calculus and Analysis I
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	Answer all questions. All marks gained will be counted. All questions carry equal weight.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

1. (a) Let c be a real number, and let f be a real function defined on some open interval around c (except, possibly, at c itself.)

   i. **[6 marks]** Give the ε, δ definition of the mathematical statement '$\lim_{x \to c} f(x) = L$'.

   ii. **[7 marks]** Use the ε, δ definition to prove $\lim_{x \to 1} \sqrt{x} = 1$.

   (b) i. **[9 marks]** Prove the Pinching Theorem for limits of real functions.

   ii. **[3 marks]** Use the Pinching Theorem to find

$$\lim_{x \to 1} (x-1) \cos\left(\frac{\pi}{x-1}\right).$$

[25 marks]

2. (a) **[7 marks]** Consider two continuous functions

$$f, g : [a,b] \to \mathbb{R} \quad \text{such that } f(a) < g(a) \text{ and } g(b) < f(b).$$

   Show that there exists $c \in (a,b)$ with $f(c) = g(c)$.

   Hint: consider the function $h(x) = f(x) - g(x)$.

   Bolzano's theorem may be assumed, as long as it is stated precisely.

   (b) **[11 marks]** State and prove the Extreme Value Theorem.

   You may assume that any continuous function $f : [a,b] \to \mathbb{R}$ is bounded.

   (c) **[7 marks]** Suppose $h : [a,b] \to \mathbb{R}$ is a differentiable function that intersects the line $y = x$ at least twice. Prove that $h'(x) = 1$ for some $c \in (a,b)$

   Rolle's Theorem may be assumed, as long as it is stated precisely.

[25 marks]

3. (a) **[9 marks]**

   i. Define what it means for a function f to be *differentiable* at a point $x = c$.

   ii. Using this definition, differentiate $f(x) = x^n/n$ when n is a positive integer.

   (b) **[16 marks]** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) \;=\; \begin{cases} x^3/6 & x \geq 0 \\ -x^3/6 & x < 0 \end{cases}$$

   i. Prove that f is differentiable everywhere and that its derivative $g(x) = \dfrac{df}{dx}$ is also differentiable everywhere.

   ii. Prove that the function $g(x) = \dfrac{df}{dx}$ is one-to-one and find the inverse function.

   iii. Sketch the function $h(x) = \dfrac{dg}{dx} = \dfrac{d^2 f}{dx^2}$. Is it continuous at $x = 0$? Is it differentiable there?

   iv. State which of f, g, h are even, which are odd, and which are neither.

[25 marks]

4. (a) **[8 marks]** Prove that $n! \geq 2^{n-1}$ by induction for all positive integers n.

   (b) i. **[3 marks]** State the Cauchy (or Extended) Mean Value Theorem (CMVT).

   ii. **[6 marks]** Use the CMVT to prove the following for all $0 < x < 2\pi$:

   - $\sin(x) < x$
   - $1 - \cos(x) < \frac{1}{2}x^2$

   (c) i. **[5 marks]** Find the Taylor polynomial $P_3(x)$ of degree 3, centred at the point $c = 1$, for the function $f(x) = x^{-1}$.

   ii. **[3 marks]** Find also the Lagrangian remainder term.

[25 marks]