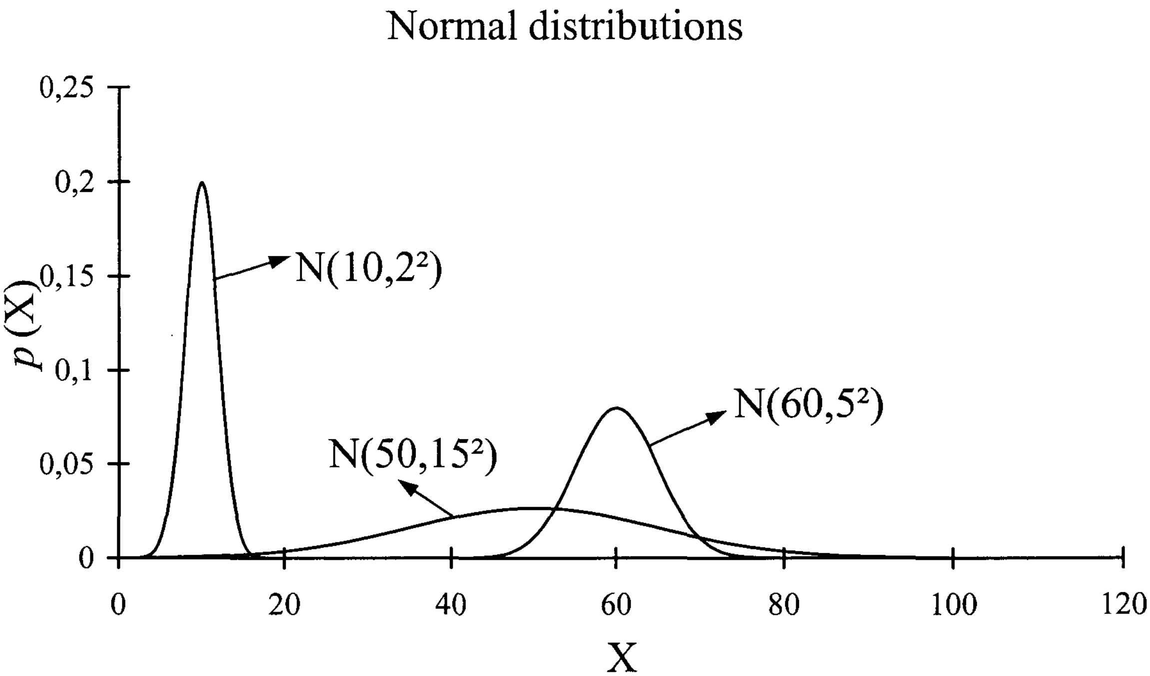
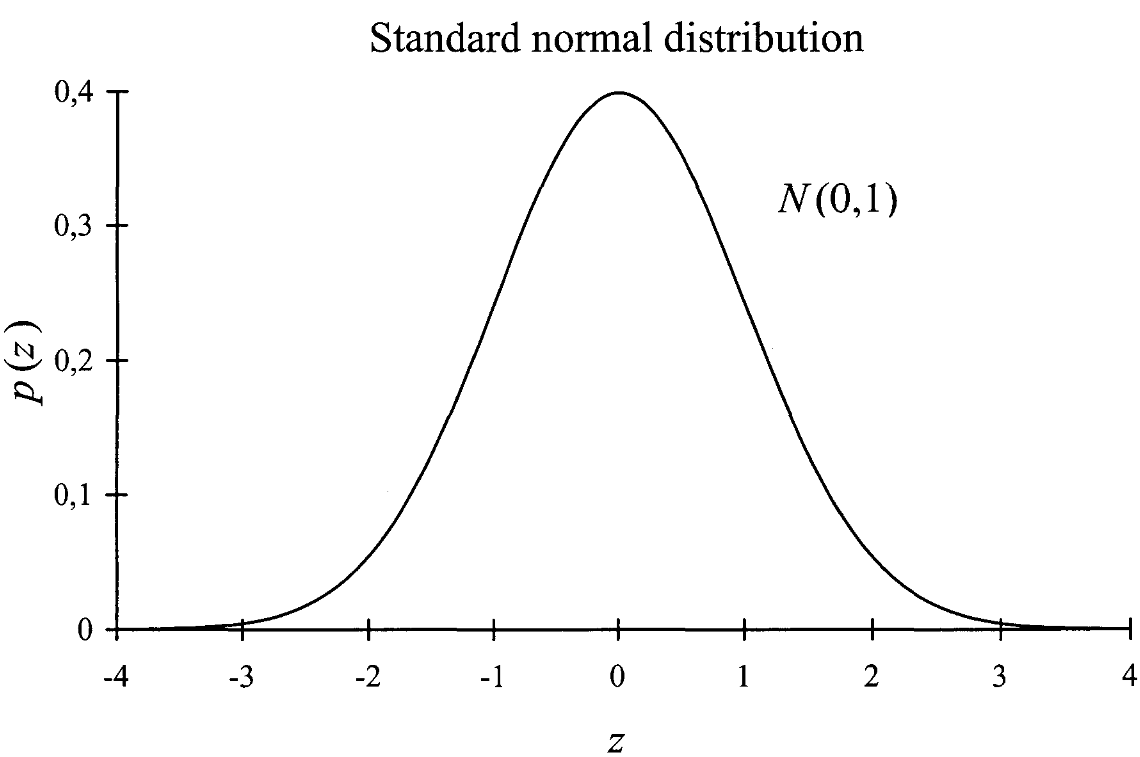
Normal distribution

- ✓ Graphical representations
- ✓ Values of the standard normal distribution function (integral), $P(z)$ (table 1)
- ✓ Percentiles of the standard normal distribution, $z(P)$ (table 2)
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation

  Calculation and moments

  Generation of pseudo random variates


Normal distributions
0,25
0,2
0,15
0,1
0,05
0
p(X)
N(10,2²)
N(60,5²)
N(50,15²)
0
20
40
60
80
100
120
X


Standard normal distribution
0,4
0,3
0,2
0,1
0
p(z)
N(0,1)
-4
-3
-2
-1
0
1
2
3
4
z

Values of the standard normal distribution function (integral), $P(z)$ (table 1)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
0.00	50000	50040	50080	50120	50160	50199	50239	50279	50319	50359
0.01	50399	50439	50479	50519	50559	50598	50638	50678	50718	50758
0.02	50798	50838	50878	50917	50957	50997	51037	51077	51117	51157
0.03	51197	51237	51276	51316	51356	51396	51436	51476	51516	51555
0.04	51595	51635	51675	51715	51755	51795	51834	51874	51914	51954
0.05	51994	52034	52074	52113	52153	52193	52233	52273	52313	52352
0.06	52392	52432	52472	52512	52551	52591	52631	52671	52711	52751
0.07	52790	52830	52870	52910	52949	52989	53029	53069	53109	53148
0.08	53188	53228	53268	53307	53347	53387	53427	53466	53506	53546
0.09	53586	53625	53665	53705	53745	53784	53824	53864	53903	53943
0.10	53983	54022	54062	54102	54142	54181	54221	54261	54300	54340
0.11	54380	54419	54459	54498	54538	54578	54617	54657	54697	54736
0.12	54776	54815	54855	54895	54934	54974	55013	55053	55093	55132
0.13	55172	55211	55251	55290	55330	55369	55409	55448	55488	55527
0.14	55567	55607	55646	55685	55725	55764	55804	55843	55883	55922
0.15	55962	56001	56041	56080	56120	56159	56198	56238	56277	56317
0.16	56356	56395	56435	56474	56513	56553	56592	56631	56671	56710
0.17	56749	56789	56828	56867	56907	56946	56985	57025	57064	57103
0.18	57142	57182	57221	57260	57299	57339	57378	57417	57456	57495
0.19	57535	57574	57613	57652	57691	57730	57769	57809	57848	57887
0.20	57926	57965	58004	58043	58082	58121	58160	58200	58239	58278
0.21	58317	58356	58395	58434	58473	58512	58551	58590	58629	58667
0.22	58706	58745	58784	58823	58862	58901	58940	58979	59018	59057
0.23	59095	59134	59173	59212	59251	59290	59328	59367	59406	59445
0.24	59483	59522	59561	59600	59638	59677	59716	59755	59793	59832
0.25	59871	59909	59948	59987	60025	60064	60102	60141	60180	60218
0.26	60257	60295	60334	60372	60411	60450	60488	60527	60565	60604
0.27	60642	60680	60719	60757	60796	60834	60873	60911	60949	60988
0.28	61026	61064	61103	61141	61179	61218	61256	61294	61333	61371
0.29	61409	61447	61486	61524	61562	61600	61638	61677	61715	61753
0.30	61791	61829	61867	61906	61944	61982	62020	62058	62096	62134
0.31	62172	62210	62248	62286	62324	62362	62400	62438	62476	62514
0.32	62552	62589	62627	62665	62703	62741	62779	62817	62854	62892
0.33	62930	62968	63006	63043	63081	63119	63156	63194	63232	63270
0.34	63307	63345	63382	63420	63458	63495	63533	63570	63608	63646
0.35	63683	63721	63758	63796	63833	63871	63908	63945	63983	64020
0.36	64058	64095	64132	64170	64207	64244	64282	64319	64356	64394
0.37	64431	64468	64505	64543	64580	64617	64654	64691	64728	64766
0.38	64803	64840	64877	64914	64951	64988	65025	65062	65099	65136
0.39	65173	65210	65247	65284	65321	65358	65395	65432	65468	65505
0.40	65542	65579	65616	65653	65689	65726	65763	65800	65836	65873
0.41	65910	65946	65983	66020	66056	66093	66129	66166	66203	66239
0.42	66276	66312	66349	66385	66422	66458	66495	66531	66567	66604
0.43	66640	66677	66713	66749	66786	66822	66858	66894	66931	66967
0.44	67003	67039	67076	67112	67148	67184	67220	67256	67292	67328
0.45	67364	67401	67437	67473	67509	67545	67581	67616	67652	67688
0.46	67724	67760	67796	67832	67868	67903	67939	67975	68011	68047
0.47	68082	68118	68154	68189	68225	68261	68296	68332	68367	68403
0.48	68439	68474	68510	68545	68581	68616	68652	68687	68723	68758
0.49	68793	68829	68864	68899	68935	68970	69005	69041	69076	69111

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+0.000	+0.001	+0.002	+0.003	+0.004	+0.005	+0.006	+0.007	+0.008	+0.009
0.50	69146	69181	69217	69252	69287	69322	69357	69392	69427	69462
0.51	69497	69532	69567	69602	69637	69672	69707	69742	69777	69812
0.52	69847	69882	69916	69951	69986	70021	70056	70090	70125	70160
0.53	70194	70229	70264	70298	70333	70368	70402	70437	70471	70506
0.54	70540	70575	70609	70644	70678	70712	70747	70781	70815	70850
0.55	70884	70918	70953	70987	71021	71055	71089	71124	71158	71192
0.56	71226	71260	71294	71328	71362	71396	71430	71464	71498	71532
0.57	71566	71600	71634	71668	71702	71735	71769	71803	71837	71871
0.58	71904	71938	71972	72005	72039	72073	72106	72140	72173	72207
0.59	72240	72274	72307	72341	72374	72408	72441	72475	72508	72541
0.60	72575	72608	72641	72675	72708	72741	72774	72807	72841	72874
0.61	72907	72940	72973	73006	73039	73072	73105	73138	73171	73204
0.62	73237	73270	73303	73336	73369	73401	73434	73467	73500	73533
0.63	73565	73598	73631	73663	73696	73729	73761	73794	73826	73859
0.64	73891	73924	73956	73989	74021	74054	74086	74118	74151	74183
0.65	74215	74248	74280	74312	74344	74377	74409	74441	74473	74505
0.66	74537	74569	74601	74633	74665	74697	74729	74761	74793	74825
0.67	74857	74889	74921	74953	74984	75016	75048	75080	75111	75143
0.68	75175	75206	75238	75270	75301	75333	75364	75396	75427	75459
0.69	75490	75522	75553	75585	75616	75647	75679	75710	75741	75772
0.70	75804	75835	75866	75897	75928	75959	75991	76022	76053	76084
0.71	76115	76146	76177	76208	76239	76270	76300	76331	76362	76393
0.72	76424	76455	76485	76516	76547	76577	76608	76639	76669	76700
0.73	76730	76761	76792	76822	76853	76883	76913	76944	76974	77005
0.74	77035	77065	77096	77126	77156	77186	77217	77247	77277	77307
0.75	77337	77367	77397	77428	77458	77488	77518	77548	77577	77607
0.76	77637	77667	77697	77727	77757	77786	77816	77846	77876	77905
0.77	77935	77965	77994	78024	78053	78083	78113	78142	78172	78201
0.78	78230	78260	78289	78319	78348	78377	78407	78436	78465	78494
0.79	78524	78553	78582	78611	78640	78669	78698	78727	78756	78785
0.80	78814	78843	78872	78901	78930	78959	78988	79017	79045	79074
0.81	79103	79132	79160	79189	79218	79246	79275	79304	79332	79361
0.82	79389	79418	79446	79475	79503	79531	79560	79588	79616	79645
0.83	79673	79701	79730	79758	79786	79814	79842	79870	79898	79927
0.84	79955	79983	80011	80039	80067	80094	80122	80150	80178	80206
0.85	80234	80262	80289	80317	80345	80372	80400	80428	80455	80483
0.86	80511	80538	80566	80593	80621	80648	80675	80703	80730	80758
0.87	80785	80812	80840	80867	80894	80921	80948	80976	81003	81030
0.88	81057	81084	81111	81138	81165	81192	81219	81246	81273	81300
0.89	81327	81354	81380	81407	81434	81461	81487	81514	81541	81567
0.90	81594	81621	81647	81674	81700	81727	81753	81780	81806	81832
0.91	81859	81885	81912	81938	81964	81990	82017	82043	82069	82095
0.92	82121	82147	82174	82200	82226	82252	82278	82304	82330	82356
0.93	82381	82407	82433	82459	82485	82511	82536	82562	82588	82613
0.94	82639	82665	82690	82716	82742	82767	82793	82818	82844	82869
0.95	82894	82920	82945	82970	82996	83021	83046	83072	83097	83122
0.96	83147	83172	83198	83223	83248	83273	83298	83323	83348	83373
0.97	83398	83423	83447	83472	83497	83522	83547	83572	83596	83621
0.98	83646	83670	83695	83720	83744	83769	83793	83818	83842	83867
0.99	83891	83916	83940	83965	83989	84013	84037	84062	84086	84110

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
1.00	84134	84159	84183	84207	84231	84255	84279	84303	84327	84351
1.01	84375	84399	84423	84447	84471	84495	84519	84542	84566	84590
1.02	84614	84637	84661	84685	84708	84732	84755	84779	84803	84826
1.03	84849	84873	84896	84920	84943	84967	84990	85013	85036	85060
1.04	85083	85106	85129	85153	85176	85199	85222	85245	85268	85291
1.05	85314	85337	85360	85383	85406	85429	85452	85474	85497	85520
1.06	85543	85566	85588	85611	85634	85656	85679	85701	85724	85747
1.07	85769	85792	85814	85836	85859	85881	85904	85926	85948	85971
1.08	85993	86015	86037	86060	86082	86104	86126	86148	86170	86192
1.09	86214	86236	86258	86280	86302	86324	86346	86368	86390	86412
1.10	86433	86455	86477	86499	86520	86542	86564	86585	86607	86628
1.11	86650	86672	86693	86715	86736	86757	86779	86800	86822	86843
1.12	86864	86886	86907	86928	86949	86971	86992	87013	87034	87055
1.13	87076	87097	87118	87139	87160	87181	87202	87223	87244	87265
1.14	87286	87307	87327	87348	87369	87390	87410	87431	87452	87472
1.15	87493	87513	87534	87554	87575	87595	87616	87636	87657	87677
1.16	87698	87718	87738	87759	87779	87799	87819	87839	87860	87880
1.17	87900	87920	87940	87960	87980	88000	88020	88040	88060	88080
1.18	88100	88120	88140	88160	88179	88199	88219	88239	88258	88278
1.19	88298	88317	88337	88357	88376	88396	88415	88435	88454	88474
1.20	88493	88512	88532	88551	88571	88590	88609	88628	88648	88667
1.21	88686	88705	88724	88744	88763	88782	88801	88820	88839	88858
1.22	88877	88896	88915	88934	88952	88971	88990	89009	89028	89046
1.23	89065	89084	89103	89121	89140	89158	89177	89196	89214	89233
1.24	89251	89270	89288	89307	89325	89343	89362	89380	89398	89417
1.25	89435	89453	89472	89490	89508	89526	89544	89562	89580	89598
1.26	89617	89635	89653	89671	89688	89706	89724	89742	89760	89778
1.27	89796	89814	89831	89849	89867	89885	89902	89920	89938	89955
1.28	89973	89990	90008	90025	90043	90060	90078	90095	90113	90130
1.29	90147	90165	90182	90199	90217	90234	90251	90268	90286	90303
1.30	90320	90337	90354	90371	90388	90405	90422	90439	90456	90473
1.31	90490	90507	90524	90541	90558	90575	90591	90608	90625	90642
1.32	90658	90675	90692	90708	90725	90741	90758	90775	90791	90808
1.33	90824	90841	90857	90873	90890	90906	90923	90939	90955	90971
1.34	90988	91004	91020	91036	91053	91069	91085	91101	91117	91133
1.35	91149	91165	91181	91197	91213	91229	91245	91261	91277	91293
1.36	91309	91324	91340	91356	91372	91387	91403	91419	91434	91450
1.37	91466	91481	91497	91512	91528	91543	91559	91574	91590	91605
1.38	91621	91636	91651	91667	91682	91697	91713	91728	91743	91758
1.39	91774	91789	91804	91819	91834	91849	91864	91879	91894	91909
1.40	91924	91939	91954	91969	91984	91999	92014	92029	92043	92058
1.41	92073	92088	92103	92117	92132	92147	92161	92176	92190	92205
1.42	92220	92234	92249	92263	92278	92292	92307	92321	92335	92350
1.43	92364	92378	92393	92407	92421	92436	92450	92464	92478	92492
1.44	92507	92521	92535	92549	92563	92577	92591	92605	92619	92633
1.45	92647	92661	92675	92689	92703	92717	92730	92744	92758	92772
1.46	92785	92799	92813	92827	92840	92854	92868	92881	92895	92908
1.47	92922	92935	92949	92962	92976	92989	93003	93016	93030	93043
1.48	93056	93070	93083	93096	93110	93123	93136	93149	93162	93176
1.49	93189	93202	93215	93228	93241	93254	93267	93280	93293	93306

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
1.50	93319	93332	93345	93358	93371	93384	93397	93409	93422	93435
1.51	93448	93461	93473	93486	93499	93511	93524	93537	93549	93562
1.52	93574	93587	93600	93612	93625	93637	93650	93662	93674	93687
1.53	93699	93712	93724	93736	93749	93761	93773	93785	93798	93810
1.54	93822	93834	93846	93858	93871	93883	93895	93907	93919	93931
1.55	93943	93955	93967	93979	93991	94003	94015	94026	94038	94050
1.56	94062	94074	94086	94097	94109	94121	94133	94144	94156	94168
1.57	94179	94191	94202	94214	94226	94237	94249	94260	94272	94283
1.58	94295	94306	94318	94329	94340	94352	94363	94374	94386	94397
1.59	94408	94420	94431	94442	94453	94464	94476	94487	94498	94509
1.60	94520	94531	94542	94553	94564	94575	94586	94597	94608	94619
1.61	94630	94641	94652	94663	94674	94684	94695	94706	94717	94728
1.62	94738	94749	94760	94771	94781	94792	94803	94813	94824	94834
1.63	94845	94855	94866	94877	94887	94898	94908	94918	94929	94939
1.64	94950	94960	94970	94981	94991	95002	95012	95022	95032	95043
1.65	95053	95063	95073	95083	95094	95104	95114	95124	95134	95144
1.66	95154	95164	95174	95184	95194	95204	95214	95224	95234	95244
1.67	95254	95264	95274	95284	95293	95303	95313	95323	95333	95342
1.68	95352	95362	95372	95381	95391	95401	95410	95420	95429	95439
1.69	95449	95458	95468	95477	95487	95496	95506	95515	95525	95534
1.70	95543	95553	95562	95572	95581	95590	95600	95609	95618	95627
1.71	95637	95646	95655	95664	95674	95683	95692	95701	95710	95719
1.72	95728	95737	95747	95756	95765	95774	95783	95792	95801	95810
1.73	95818	95827	95836	95845	95854	95863	95872	95881	95889	95898
1.74	95907	95916	95925	95933	95942	95951	95959	95968	95977	95985
1.75	95994	96003	96011	96020	96028	96037	96046	96054	96063	96071
1.76	96080	96088	96097	96105	96113	96122	96130	96139	96147	96155
1.77	96164	96172	96180	96189	96197	96205	96213	96222	96230	96238
1.78	96246	96254	96263	96271	96279	96287	96295	96303	96311	96319
1.79	96327	96335	96343	96351	96359	96367	96375	96383	96391	96399
1.80	96407	96415	96423	96431	96438	96446	96454	96462	96470	96477
1.81	96485	96493	96501	96508	96516	96524	96531	96539	96547	96554
1.82	96562	96570	96577	96585	96592	96600	96607	96615	96623	96630
1.83	96638	96645	96652	96660	96667	96675	96682	96690	96697	96704
1.84	96712	96719	96726	96734	96741	96748	96755	96763	96770	96777
1.85	96784	96792	96799	96806	96813	96820	96827	96834	96842	96849
1.86	96856	96863	96870	96877	96884	96891	96898	96905	96912	96919
1.87	96926	96933	96940	96947	96953	96960	96967	96974	96981	96988
1.88	96995	97001	97008	97015	97022	97029	97035	97042	97049	97055
1.89	97062	97069	97075	97082	97089	97095	97102	97109	97115	97122
1.90	97128	97135	97141	97148	97154	97161	97167	97174	97180	97187
1.91	97193	97200	97206	97213	97219	97225	97232	97238	97244	97251
1.92	97257	97263	97270	97276	97282	97289	97295	97301	97307	97313
1.93	97320	97326	97332	97338	97344	97350	97357	97363	97369	97375
1.94	97381	97387	97393	97399	97405	97411	97417	97423	97429	97435
1.95	97441	97447	97453	97459	97465	97471	97477	97483	97488	97494
1.96	97500	97506	97512	97518	97523	97529	97535	97541	97547	97552
1.97	97558	97564	97570	97575	97581	97587	97592	97598	97604	97609
1.98	97615	97620	97626	97632	97637	97643	97648	97654	97659	97665
1.99	97670	97676	97681	97687	97692	97698	97703	97709	97714	97720

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
2.00	97725	97730	97736	97741	97746	97752	97757	97763	97768	97773
2.01	97778	97784	97789	97794	97800	97805	97810	97815	97820	97826
2.02	97831	97836	97841	97846	97851	97857	97862	97867	97872	97877
2.03	97882	97887	97892	97897	97902	97907	97912	97917	97923	97927
2.04	97932	97937	97942	97947	97952	97957	97962	97967	97972	97977
2.05	97982	97987	97992	97996	98001	98006	98011	98016	98020	98025
2.06	98030	98035	98040	98044	98049	98054	98059	98063	98068	98073
2.07	98077	98082	98087	98091	98096	98101	98105	98110	98115	98119
2.08	98124	98128	98133	98137	98142	98147	98151	98156	98160	98165
2.09	98169	98174	98178	98183	98187	98191	98196	98200	98205	98209
2.10	98214	98218	98222	98227	98231	98235	98240	98244	98248	98253
2.11	98257	98261	98266	98270	98274	98279	98283	98287	98291	98295
2.12	98300	98304	98308	98312	98316	98321	98325	98329	98333	98337
2.13	98341	98346	98350	98354	98358	98362	98366	98370	98374	98378
2.14	98382	98386	98390	98394	98398	98402	98406	98410	98414	98418
2.15	98422	98426	98430	98434	98438	98442	98446	98450	98454	98457
2.16	98461	98465	98469	98473	98477	98481	98484	98488	98492	98496
2.17	98500	98503	98507	98511	98515	98518	98522	98526	98530	98533
2.18	98537	98541	98545	98548	98552	98556	98559	98563	98567	98570
2.19	98574	98577	98581	98585	98588	98592	98595	98599	98603	98606
2.20	98610	98613	98617	98620	98624	98627	98631	98634	98638	98641
2.21	98645	98648	98652	98655	98659	98662	98665	98669	98672	98676
2.22	98679	98682	98686	98689	98693	98696	98699	98703	98706	98709
2.23	98713	98716	98719	98723	98726	98729	98732	98736	98739	98742
2.24	98745	98749	98752	98755	98758	98762	98765	98768	98771	98774
2.25	98778	98781	98784	98787	98790	98793	98796	98800	98803	98806
2.26	98809	98812	98815	98818	98821	98824	98827	98830	98834	98837
2.27	98840	98843	98846	98849	98852	98855	98858	98861	98864	98867
2.28	98870	98873	98876	98878	98881	98884	98887	98890	98893	98896
2.29	98899	98902	98905	98908	98910	98913	98916	98919	98922	98925
2.30	98928	98930	98933	98936	98939	98942	98944	98947	98950	98953
2.31	98956	98958	98961	98964	98967	98969	98972	98975	98978	98980
2.32	98983	98986	98988	98991	98994	98996	98999	99002	99004	99007
2.33	99010	99012	99015	99018	99020	99023	99025	99028	99031	99033
2.34	99036	99038	99041	99044	99046	99049	99051	99054	99056	99059
2.35	99061	99064	99066	99069	99071	99074	99076	99079	99081	99084
2.36	99086	99089	99091	99094	99096	99098	99101	99103	99106	99108
2.37	99111	99113	99115	99118	99120	99123	99125	99127	99130	99132
2.38	99134	99137	99139	99141	99144	99146	99148	99151	99153	99155
2.39	99158	99160	99162	99164	99167	99169	99171	99174	99176	99178
2.40	99180	99182	99185	99187	99189	99191	99194	99196	99198	99200
2.41	99202	99205	99207	99209	99211	99213	99215	99218	99220	99222
2.42	99224	99226	99228	99230	99232	99235	99237	99239	99241	99243
2.43	99245	99247	99249	99251	99253	99255	99257	99260	99262	99264
2.44	99266	99268	99270	99272	99274	99276	99278	99280	99282	99284
2.45	99286	99288	99290	99292	99294	99296	99298	99299	99301	99303
2.46	99305	99307	99309	99311	99313	99315	99317	99319	99321	99323
2.47	99324	99326	99328	99330	99332	99334	99336	99338	99339	99341
2.48	99343	99345	99347	99349	99350	99352	99354	99356	99358	99359
2.49	99361	99363	99365	99367	99368	99370	99372	99374	99376	99377

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
2.50	99379	99381	99383	99384	99386	99388	99389	99391	99393	99395
2.51	99396	99398	99400	99401	99403	99405	99407	99408	99410	99412
2.52	99413	99415	99417	99418	99420	99422	99423	99425	99426	99428
2.53	99430	99431	99433	99435	99436	99438	99439	99441	99443	99444
2.54	99446	99447	99449	99450	99452	99454	99455	99457	99458	99460
2.55	99461	99463	99464	99466	99468	99469	99471	99472	99474	99475
2.56	99477	99478	99480	99481	99483	99484	99486	99487	99489	99490
2.57	99492	99493	99494	99496	99497	99499	99500	99502	99503	99505
2.58	99506	99507	99509	99510	99512	99513	99515	99516	99517	99519
2.59	99520	99522	99523	99524	99526	99527	99528	99530	99531	99533
2.60	99534	99535	99537	99538	99539	99541	99542	99543	99545	99546
2.61	99547	99549	99550	99551	99553	99554	99555	99556	99558	99559
2.62	99560	99562	99563	99564	99565	99567	99568	99569	99571	99572
2.63	99573	99574	99576	99577	99578	99579	99581	99582	99583	99584
2.64	99585	99587	99588	99589	99590	99592	99593	99594	99595	99596
2.65	99598	99599	99600	99601	99602	99603	99605	99606	99607	99608
2.66	99609	99610	99612	99613	99614	99615	99616	99617	99618	99620
2.67	99621	99622	99623	99624	99625	99626	99627	99629	99630	99631
2.68	99632	99633	99634	99635	99636	99637	99638	99640	99641	99642
2.69	99643	99644	99645	99646	99647	99648	99649	99650	99651	99652
2.70	99653	99654	99655	99656	99657	99658	99660	99661	99662	99663
2.71	99664	99665	99666	99667	99668	99669	99670	99671	99672	99673
2.72	99674	99675	99676	99677	99678	99678	99679	99680	99681	99682
2.73	99683	99684	99685	99686	99687	99688	99689	99690	99691	99692
2.74	99693	99694	99695	99696	99697	99697	99698	99699	99700	99701
2.75	99702	99703	99704	99705	99706	99707	99707	99708	99709	99710
2.76	99711	99712	99713	99714	99715	99715	99716	99717	99718	99719
2.77	99720	99721	99721	99722	99723	99724	99725	99726	99727	99727
2.78	99728	99729	99730	99731	99732	99732	99733	99734	99735	99736
2.79	99736	99737	99738	99739	99740	99741	99741	99742	99743	99744
2.80	99744	99745	99746	99747	99748	99748	99749	99750	99751	99752
2.81	99752	99753	99754	99755	99755	99756	99757	99758	99758	99759
2.82	99760	99761	99761	99762	99763	99764	99764	99765	99766	99767
2.83	99767	99768	99769	99769	99770	99771	99772	99772	99773	99774
2.84	99774	99775	99776	99777	99777	99778	99779	99779	99780	99781
2.85	99781	99782	99783	99783	99784	99785	99785	99786	99787	99788
2.86	99788	99789	99790	99790	99791	99791	99792	99793	99793	99794
2.87	99795	99795	99796	99797	99797	99798	99799	99799	99800	99801
2.88	99801	99802	99802	99803	99804	99804	99805	99806	99806	99807
2.89	99807	99808	99809	99809	99810	99810	99811	99812	99812	99813
2.90	99813	99814	99815	99815	99816	99816	99817	99818	99818	99819
2.91	99819	99820	99820	99821	99822	99822	99823	99823	99824	99824
2.92	99825	99826	99826	99827	99827	99828	99828	99829	99829	99830
2.93	99831	99831	99832	99832	99833	99833	99834	99834	99835	99835
2.94	99836	99836	99837	99837	99838	99839	99839	99840	99840	99841
2.95	99841	99842	99842	99843	99843	99844	99844	99845	99845	99846
2.96	99846	99847	99847	99848	99848	99849	99849	99850	99850	99851
2.97	99851	99852	99852	99853	99853	99854	99854	99854	99855	99855
2.98	99856	99856	99857	99857	99858	99858	99859	99859	99860	99860
2.99	99861	99861	99861	99862	99862	99863	99863	99864	99864	99865

For values of z from 3 to 7, see the Mathematical presentation subsection.

Percentiles of the standard normal distribution, $z(P)$ (table 2)

P	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
.500	.0000	$.0^2251$	$.0^2501$	$.0^2752$	.0100	.0125	.0150	.0175	.0201	.0226
.510	.0251	.0276	.0301	.0326	.0351	.0376	.0401	.0426	.0451	.0476
.520	.0502	.0527	.0552	.0577	.0602	.0627	.0652	.0677	.0702	.0728
.530	.0753	.0778	.0803	.0828	.0853	.0878	.0904	.0929	.0954	.0979
.540	.1004	.1030	.1055	.1080	.1105	.1130	.1156	.1181	.1206	.1231
.550	.1257	.1282	.1307	.1332	.1358	.1383	.1408	.1434	.1459	.1484
.560	.1510	.1535	.1560	.1586	.1611	.1637	.1662	.1687	.1713	.1738
.570	.1764	.1789	.1815	.1840	.1866	.1891	.1917	.1942	.1968	.1993
.580	.2019	.2045	.2070	.2096	.2121	.2147	.2173	.2198	.2224	.2250
.590	.2275	.2301	.2327	.2353	.2378	.2404	.2430	.2456	.2482	.2508
.600	.2533	.2559	.2585	.2611	.2637	.2663	.2689	.2715	.2741	.2767
.610	.2793	.2819	.2845	.2871	.2898	.2924	.2950	.2976	.3002	.3029
.620	.3055	.3081	.3107	.3134	.3160	.3186	.3213	.3239	.3266	.3292
.630	.3319	.3345	.3372	.3398	.3425	.3451	.3478	.3505	.3531	.3558
.640	.3585	.3611	.3638	.3665	.3692	.3719	.3745	.3772	.3799	.3826
.650	.3853	.3880	.3907	.3934	.3961	.3989	.4016	.4043	.4070	.4097
.660	.4125	.4152	.4179	.4207	.4234	.4261	.4289	.4316	.4344	.4372
.670	.4399	.4427	.4454	.4482	.4510	.4538	.4565	.4593	.4621	.4649
.680	.4677	.4705	.4733	.4761	.4789	.4817	.4845	.4874	.4902	.4930
.690	.4959	.4987	.5015	.5044	.5072	.5101	.5129	.5158	.5187	.5215
.700	.5244	.5273	.5302	.5330	.5359	.5388	.5417	.5446	.5476	.5505
.710	.5534	.5563	.5592	.5622	.5651	.5681	.5710	.5740	.5769	.5799
.720	.5828	.5858	.5888	.5918	.5948	.5978	.6008	.6038	.6068	.6098
.730	.6128	.6158	.6189	.6219	.6250	.6280	.6311	.6341	.6372	.6403
.740	.6433	.6464	.6495	.6526	.6557	.6588	.6620	.6651	.6682	.6713
.750	.6745	.6776	.6808	.6840	.6871	.6903	.6935	.6967	.6999	.7031
.760	.7063	.7095	.7128	.7160	.7192	.7225	.7257	.7290	.7323	.7356
.770	.7388	.7421	.7454	.7488	.7521	.7554	.7588	.7621	.7655	.7688
.780	.7722	.7756	.7790	.7824	.7858	.7892	.7926	.7961	.7995	.8030
.790	.8064	.8099	.8134	.8169	.8204	.8239	.8274	.8310	.8345	.8381
.800	.8416	.8452	.8488	.8524	.8560	.8596	.8633	.8669	.8705	.8742
.810	.8779	.8816	.8853	.8890	.8927	.8965	.9002	.9040	.9078	.9116
.820	.9154	.9192	.9230	.9269	.9307	.9346	.9385	.9424	.9463	.9502
.830	.9542	.9581	.9621	.9661	.9701	.9741	.9782	.9822	.9863	.9904
.840	.9945	.9986	1.0027	1.0069	1.0110	1.0152	1.0194	1.0237	1.0279	1.0322
.850	1.0364	1.0407	1.0450	1.0494	1.0537	1.0581	1.0625	1.0669	1.0714	1.0758
.860	1.0803	1.0848	1.0893	1.0939	1.0985	1.1031	1.1077	1.1123	1.1170	1.1217
.870	1.1264	1.1311	1.1359	1.1407	1.1455	1.1503	1.1552	1.1601	1.1650	1.1700
.880	1.1750	1.1800	1.1850	1.1901	1.1952	1.2004	1.2055	1.2107	1.2160	1.2212
.890	1.2265	1.2319	1.2372	1.2426	1.2481	1.2536	1.2591	1.2646	1.2702	1.2759
.900	1.2816	1.2873	1.2930	1.2988	1.3047	1.3106	1.3165	1.3225	1.3285	1.3346
.910	1.3408	1.3469	1.3532	1.3595	1.3658	1.3722	1.3787	1.3852	1.3917	1.3984
.920	1.4051	1.4118	1.4187	1.4255	1.4325	1.4395	1.4466	1.4538	1.4611	1.4684
.930	1.4758	1.4833	1.4909	1.4985	1.5063	1.5141	1.5220	1.5301	1.5382	1.5464
.940	1.5548	1.5632	1.5718	1.5805	1.5893	1.5982	1.6072	1.6164	1.6258	1.6352
.950	1.6449	1.6546	1.6646	1.6747	1.6849	1.6954	1.7060	1.7169	1.7279	1.7392
.960	1.7507	1.7624	1.7744	1.7866	1.7991	1.8119	1.8250	1.8384	1.8522	1.8663
.970	1.8808	1.8957	1.9110	1.9268	1.9431	1.9600	1.9774	1.9954	2.0141	2.0335
.980	2.0537	2.0749	2.0969	2.1201	2.1444	2.1701	2.1973	2.2262	2.2571	2.2904
.990	2.3263	2.3656	2.4089	2.4573	2.5121	2.5758	2.6521	2.7478	2.8782	3.0902

For extreme percentiles (up to $P = 0.999999$), see the Mathematical presentation subsection.

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Reading off the tables

Table 1 gives the probability integral $P(z)$ of the standard normal distribution at z, for positive values $z = 0.000(0.001)2.999$; a hidden decimal point precedes each quantity. Such quantity $P(z)$, in a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$, denotes the probability that a random element Z lies under the indicated z value, *i.e.* $P(z) = \Pr(Z \leq z)$. For negative z values, one may use the complementary relation: $P(z) = 1-P(z)$.

Table 2 is the converse of table 1 and presents the quantile (or percentage point) z corresponding to each P value, for $P = 0.500(0.001)0.999$; when $P < 0.500$, use the relation: $z(P) = -z(1-P)$.

Illustration 1. What proportion of cases lies under $z = 1.614$, in the standard normal distribution? In table 1, taking line 1.61 and column +.004, we obtain $P = 0.94674$.

Illustration 2. What percentage of area do we find under $z = -2.037$, in the standard normal distribution? Using table 1 and relation $P(z) = 1-P(-z)$, we successively obtain $P(2.037)$= 0,97917 (line 2.03, column +.007), $1-0.97917 = 0.02083$, whence the percentage asked for is 100×0.02083, or nearly 2.

Illustration 3. Which abscissa, or z value, segregates the lower 62.5 % of the cases from the 37.5 % higher in the standard normal distribution? In table 2, integral P of 0.625 points to line 0.620 and column +.005, where value $z = 0.3186$ is given, satisfying the relation $\Pr(Z \leq 0.3186) \approx 0.625$.

Illustration 4. Which z value does divide the lower third (from the upper two thirds) in the standard normal distribution? We may approximate ⅓ with 0,333. Using table 2 and as $0.333 < 0.500$, we first obtain $1-0.333 = 0.667$, then read off $z(0.667) = 0.4316$ and, finally, with a change of sign, -0.4316. For more precision, we could also, in the second phase, interpolate between 0.666 (with $z = 0.4289$) and 0.667 (with $z = 0.4316$): for $P = ⅔$, we calculate:

$$z(2/3) = 0.4289 + \frac{2/3 - 0.666}{0.667-0.666}(0.4316-0.4289)$$

$$= 0.4307 ,$$

or $z(1/3) \approx -0.4307$, a value which is precise up to the fourth decimal digit.

Full examples

Example 1. A test of Intellectual Quotient (IQ) for children of a given age is set up by imposing a normal distribution of scores, a mean (μ) of 100 and a standard deviation (σ) of 16. Find the two IQ values that comprise approximately the central 50 % of the young population. *Solution*: The central 50 % of the area in a standard $N(0,1)$ distribution starts at integral $P = 0.25$ and ends at $P = 0.75$. Using table 2, $z(P{=}0.75) \approx 0.6745$; conversely, $z(P{=}0.25) \approx -0.6745$. The desired values are thus $(-0.6745, 0.6745)$ for the standard $N(0,1)$ distribution. These values can be converted approximately[1] into IQ scores with a $N(100,16^2)$ distribution, using QI = 100 + 16z, whence the interval is (89.208, 110.792) or, roughly, (89, 111).

Example 2. The height of people, in a given population, presents a mean of 1.655 m and a standard deviation of 0.205 m. In a representative area comprising 12 000 inhabitants, how many persons having a height of 2 m or more can one expect to find? *Solution*: In order to *predict* an approximate number, we need to stipulate a *model*; here, we favor the model of a normal distribution with the corresponding parameter values, *i.e.* $N(1.655, 0.205^2)$. Transforming height X = 2 m into a standardized z value, we get $z = (2-1.655)/0.205 \approx 1.683$. In table 1, $P(1.683) = 0.95381$, whence the proportion of cases with a height exceeding X = 2 m, or z = 1.683, approaches $1-0.95381 = 0.04619$. Multiplying this proportion by 12 000, the number of inhabitants in the designated area, we predict that there be about 554 persons of a height of 2 m or more in that area.

Example 3. A measuring device for strength in Newtons (N) allows to estimate arm flexion strength with a standard error of measurement (σ_ε) of 2.5 N. Using 5 evaluations for each arm, Robert obtains a mean strength of 93.6 N for his right arm, and of 89.8 for his left. May we assure that Robert's right arm is the stronger? *Solution*: Let us suppose that the estimates of each arm's strength fluctuate according to a normal model with means μ_j (j=1,2) and standard deviation 2.5 (= σ_ε). The difference between the two means ($\bar{x}_1 - \bar{x}_2$) is itself normally distributed, with mean $\mu_1 - \mu_2$ and standard error $\sigma_\varepsilon\sqrt{(n_1^{-1}+n_2^{-1})}$, here $2.5\times\sqrt{(5^{-1}+5^{-1})} \approx 1.581$. Assume, by hypothesis, that $\mu_1 = \mu_2$, *i.e.* both arms have equal strength. The observed difference, $\bar{x}_1 - \bar{x}_2 = 93.6-89.8 = 3.80$, standardized with:

[1] A more precise conversion would need to consider the discreteness of IQ scores (who vary by units), so that it would be generally impossible to obtain an exact interval of scores. In the same vein, the normal model, which is defined for continuous variables in the real domain, cannot be rigorously imposed to any discrete variable such as a test score.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_e \sqrt{n_1^{-1} + n_2^{-1}}} ,$$

that is, $z = (3.80 - 0)/1.581 = 2.404$, is located 2.404 units of standard error from 0. Admitting a bilateral error rate of 5 %, boundaries of statistical significance fall at the 2.5 and 97.5 percentage points, which, for the standard normal distribution in table 2, point to $z = -1.960$ and $z = 1.960$ respectively. The observed difference thus exceeds the allowed-for interval of normal variation, leading us to conclude that one arm, Robert's right arm, is truly the stronger.

Mathematical presentation

The normal law, or normal distribution, has famous origins as well as innumerable applications. It first appeared in the writings of De Moivre around 1733, and was re-discovered by Laplace and Gauss. Sir Francis Galton, in view of its quasi universality, christened it "normal", synonymously to natural, expressing order, normative: it is used as a model for the distribution of a great many measurable attributes in a population. The normal model is the foremost reference for interpreting continuous random phenomena, and it underlies an overwhelming majority of statistical techniques in estimation and hypotheses testing.

Calculation and moments

The normal law, or normal distribution, has two parameters designated by μ and σ^2, corresponding respectively to the expectation (or mean) and variance of the distributed quantity. The normal p.d.f. is:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} ,$$

where $\pi \approx 3.1416$ and $e \approx 2.7183$. As shown in the graphs, the p.d.f. is symmetrical and reaches its maximum height at $x = \mu$, μ thus being the mode, median and (arithmetic) mean of the distribution. Integration of $p(x)$ is not trivial. One usually resorts to a standardized form, $z = (x-\mu)/\sigma$, z being a *standard score*, whose density function is the so-called standard normal distribution, $N(0,1)$,

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} .$$

Maximum p.d.f., at $z = 0$, equals $p(0) \approx 0.3989$, and it decreases steadily when z goes to $+\infty$ or $-\infty$, almost vanishing (≈ 0.0044) at $z = \pm 3$.

Precise (analytic) integration of the normal p.d.f. is impossible; nevertheless authors have evolved ways and methods of calculating the normal integral, or d.f., $P(z)$: most methods use series expansions. The simplest of those is based on the expansion of e^x in a Taylor series around zero, *i.e.* $e^x = 1 + x + x^2/2! + x^3/3! +$ etc. After substitution of $x^2/2$ for x, term-by-term integration and evaluation at $x = 0$ and $x = z$, the standard normal integral is:

$$P(z) = \frac{1}{2} + \frac{z}{\sqrt{2\pi}}\left[1 - \frac{z^2}{6} + \frac{z^4}{40} - \frac{z^6}{336} + \cdots + \frac{(-1)^n z^{2n}}{2^n n!\,(2n+1)}\right],$$

the summation within brackets being pursued until the desired precision is attained.

There exist other formulae for approximating the normal d.f. $P(x)$, with varying degrees of complexity and precision. The following,

$$P(z) \approx \frac{1}{2} + \frac{z}{15.04}\left[1 + 4e^{-z^2/8} + e^{-z^2/2}\right],$$

elaborated according to Simpson's rule with one arc, is precise to 0.0002 for $|z| \le 1$, to 0.001 for $|z| \le 1.75$, and to 0.01 for $|z| \le 2.31$. For higher z values, one may use:

$$P(z) \approx 1 - \frac{e^{-z^2/2}}{\sqrt{2\pi}}\left[\frac{z^2+2}{z^3+3z}\right],$$

whose precision is nearly 0.0001 for $z \ge 2.31$ and which has the advantage of always keeping three significant digits for extreme $|z|$ values. Thus, for $z = 5$, the approximated value is 0.9999997132755, whereas the exact 14-digit integral is 0.99999971334835.

Still another approximation formula, more involved than the preceding ones but fitting for a computer program, is due to C. Hastings. Let $z \ge 0$; then,

$$P(z) \approx 1 - \frac{e^{z^2/2}}{\sqrt{2\pi}} \cdot t(b_1 + t(b_2 + t(b_3 + t(b_4 + tb_5)))),$$

where $t = 1/(1+0.2316419z)$ and $b_1 = 0.31938153$, $b_2 = -0.356563782$, $b_3 = 1.781477937$, $b_4 = -1.821255978$, $b_5 = 1.330274429$. For any (positive) z value, the precision of the calculated $P(z)$ is at least 0.000000075.

Values reported in tables 1 and 2 have been computed with great precision (12 digits or more) with the Taylor series expansion aforementioned. The two small tables below furnish some supplementary, extreme, values of the standard normal integral (note that $.9^4 6833$ should be read 0.99996833).

z	3	3.5	4	4.5	5	5.5	6	6.5	7
$P(z)$	$.9^2 8650$	$.9^3 7674$	$.9^4 6833$	$.9^5 6602$	$.9^6 7133$	$.9^7 8101$	$.9^9 0134$	$.9^{10} 5984$	$.9^{11} 8720$

P	.99	$.9^2 5$	$.9^3$	$.9^3 5$	$.9^4$	$.9^4 5$	$.9^5$	$.9^5 5$	$.9^6$
$z(P)$	2.32635	2.57583	3.09023	3.29053	3.71902	3.89059	4.26489	4.41717	4.75342

Moments. The expectation (μ) and variance (σ^2) are the two parameters of a normal distribution. The skewness index (γ_1) is zero. As for the kurtosis index (γ_2), the normal law is stipulated as a criterion, a reference shape for all other distributions, consequently this index is again zero.

For the curious reader, let us note that, for a normal $N(\mu,\sigma^2)$ distribution, the mean absolute difference, $\sum|x_i - \bar{x}|/n$, has expectation $\sigma\times\sqrt{(2/\pi)} \approx 0.79788\sigma$. Also, the mean (or expectation) of variates located in the upper 100α % of a normal population is given by $\mu + \sigma\times p(z_{1-\alpha})/\alpha$, $z_{1-\alpha}$ being the $100(1-\alpha)$ percentage point of distribution $N(0,1)$. For example, for $x = z \sim N(0,1)$, the mean of the upper 10 %, denoted $\mu_{(0.10)}$, uses $z_{[1-0.10]} = z_{[0.90]} \approx 1.2818$ (in table 2), $p(1.2816) \approx 0.17549$, and $\mu_{(0.10)} \approx 0 + 1\times 0.17549/0.10 \approx 1.7549$.

Generation of pseudo random variates

Suppose a uniform $U(0,1)$ random variate (r.v.) generator, designated UNIF (*see* the section on Random numbers for information on UNIF). A normal $N(0,1)$ r.v. is produced from two independent uniform r.v.'s using the following transformation.

Preparation : $C = 2\pi \approx 6.2831853072$

Production : **Return** $\sqrt{[-2\times\ln(\text{UNIF})]} \times \sin(C\times\text{UNIF}) \rightarrow x$.

Remarks :

1. Standard temporal cost : 4.0 × t(UNIF), *i.e.* the approximate time required to produce one normal r.v. is equivalent to 4 times t(UNIF), the time required to produce one uniform r.v..

2. This method shown above is due to Box and Muller (Devroye 1986) and has some variants. Each invocation (with the same pair of UNIF values) allows to generate a second, independent x' value, through the substitution of "cos" instead of "sin" in the conversion formula.

3. In order to produce a normal $N(\mu,\sigma)$ r.v. y, one first obtains $x \sim N(0,1)$ with the procedure outlined, then $y \leftarrow \mu + \sigma \times x$.

4. In order to produce pairs of normal $N(0,1)$ r.v.'s z_1, z_2 having mutual correlation equal to ρ, one first obtains independent r.v.'s x and x', then $z_1 \leftarrow x$ and $z_2 \leftarrow \rho \times x + \surd(1-\rho^2) \times x'$. Gentle (1998, p. 187) suggests a more elegant approach. Let $\omega = \cos^{-1}\rho$ (in radian units). Then, one first obtains $t \leftarrow \surd[-2 \times \ln(\text{UNIF})]$ and $u \leftarrow 2\pi \times \text{UNIF}$, then $z_1 \leftarrow t \times \sin(u)$, $z_2 \leftarrow t \times \sin(u-\omega)$.

Chi-square (χ^2) distribution

✓ Graphical representations

✓ Selected percentiles of Chi-square (χ^2)

✓ Reading off the table

✓ Full examples

✓ Mathematical presentation

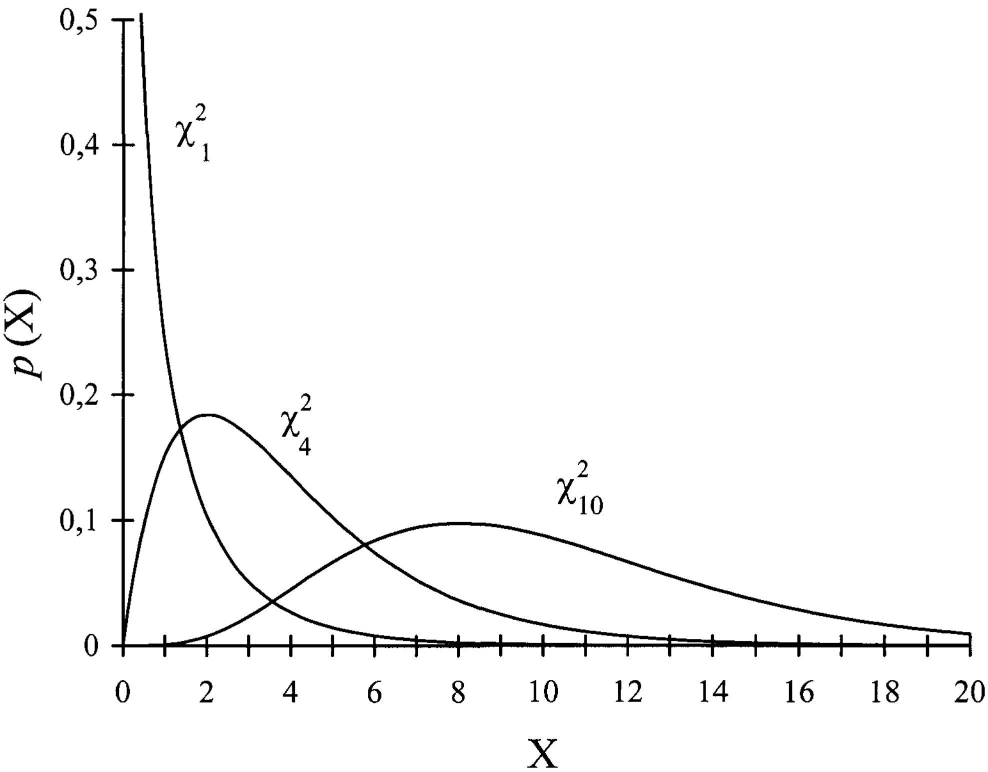
Calculation and moments

Generation of pseudo random variates

The distribution of s, the standard deviation (s.d.)

Three normal approximations to Chi-square

Chi-square (χ^2) distributions



Selected percentiles of Chi-square (χ^2)

$\nu \setminus P$	.001	.005	.010	.025	.050	.250	.500	.750	.950	.975	.990	.995	.999	ν
1	$.0^516$	$.0^439$	$.0^316$	$.0^398$	$.0^239$	.10	.45	1.32	3.84	5.02	6.63	7.88	10.83	1
2	$.0^220$	.010	.020	.051	.10	.58	1.39	2.77	5.99	7.38	9.21	10.60	13.82	2
3	.024	.072	.11	.22	.35	1.21	2.37	4.11	7.81	9.35	11.34	12.84	16.27	3
4	.091	.21	.30	.48	.71	1.92	3.36	5.39	9.49	11.14	13.28	14.86	18.47	4
5	.21	.41	.55	.83	1.15	2.67	4.35	6.63	11.07	12.83	15.09	16.75	20.52	5
6	.38	.68	.87	1.24	1.64	3.45	5.35	7.84	12.59	14.45	16.81	18.55	22.46	6
7	.60	.99	1.24	1.69	2.17	4.25	6.35	9.04	14.07	16.01	18.48	20.28	24.32	7
8	.86	1.34	1.65	2.18	2.73	5.07	7.34	10.22	15.51	17.53	20.09	21.96	26.12	8
9	1.15	1.73	2.09	2.70	3.33	5.90	8.34	11.39	16.92	19.02	21.67	23.59	27.88	9
10	1.48	2.16	2.56	3.25	3.94	6.74	9.34	12.55	18.31	20.48	23.21	25.19	29.59	10
11	1.83	2.60	3.05	3.82	4.57	7.58	10.34	13.70	19.68	21.92	24.73	26.76	31.26	11
12	2.21	3.07	3.57	4.40	5.23	8.44	11.34	14.85	21.03	23.34	26.22	28.30	32.91	12
13	2.62	3.57	4.11	5.01	5.89	9.30	12.34	15.98	22.36	24.74	27.69	29.82	34.53	13
14	3.04	4.07	4.66	5.63	6.57	10.17	13.34	17.12	23.68	26.12	29.14	31.32	36.12	14
15	3.48	4.60	5.23	6.26	7.26	11.04	14.34	18.25	25.00	27.49	30.58	32.80	37.70	15
16	3.94	5.14	5.81	6.91	7.96	11.91	15.34	19.37	26.30	28.85	32.00	34.27	39.25	16
17	4.42	5.70	6.41	7.56	8.67	12.79	16.34	20.49	27.59	30.19	33.41	35.72	40.79	17
18	4.90	6.26	7.01	8.23	9.39	13.68	17.34	21.60	28.87	31.53	34.81	37.16	42.31	18
19	5.41	6.84	7.63	8.91	10.12	14.56	18.34	22.72	30.14	32.85	36.19	38.58	43.82	19
20	5.92	7.43	8.26	9.59	10.85	15.45	19.34	23.83	31.41	34.17	37.57	40.00	45.31	20
21	6.45	8.03	8.90	10.28	11.59	16.34	20.34	24.93	32.67	35.48	38.93	41.40	46.80	21
22	6.98	8.64	9.54	10.98	12.34	17.24	21.34	26.04	33.92	36.78	40.29	42.80	48.27	22
23	7.53	9.26	10.20	11.69	13.09	18.14	22.34	27.14	35.17	38.08	41.64	44.18	49.73	23
24	8.08	9.89	10.86	12.40	13.85	19.04	23.34	28.24	36.42	39.36	42.98	45.56	51.18	24
25	8.65	10.52	11.52	13.12	14.61	19.94	24.34	29.34	37.65	40.65	44.31	46.93	52.62	25
26	9.22	11.16	12.20	13.84	15.38	20.84	25.34	30.43	38.89	41.92	45.64	48.29	54.05	26
27	9.80	11.81	12.88	14.57	16.15	21.75	26.34	31.53	40.11	43.19	46.96	49.64	55.48	27
28	10.39	12.46	13.56	15.31	16.93	22.66	27.34	32.62	41.34	44.46	48.28	50.99	56.89	28
29	10.99	13.12	14.26	16.05	17.71	23.57	28.34	33.71	42.56	45.72	49.59	52.34	58.30	29
30	11.59	13.79	14.95	16.79	18.49	24.48	29.34	34.80	43.77	46.98	50.89	53.67	59.70	30
31	12.20	14.46	15.66	17.54	19.28	25.39	30.34	35.89	44.99	48.23	52.19	55.00	61.10	31
32	12.81	15.13	16.36	18.29	20.07	26.30	31.34	36.97	46.19	49.48	53.49	56.33	62.49	32
33	13.43	15.82	17.07	19.05	20.87	27.22	32.34	38.06	47.40	50.73	54.78	57.65	63.87	33
34	14.06	16.50	17.79	19.81	21.66	28.14	33.34	39.14	48.60	51.97	56.06	58.96	65.25	34
35	14.69	17.19	18.51	20.57	22.47	29.05	34.34	40.22	49.80	53.20	57.34	60.27	66.62	35
36	15.32	17.89	19.23	21.34	23.27	29.97	35.34	41.30	51.00	54.44	58.62	61.58	67.99	36
37	15.97	18.59	19.96	22.11	24.07	30.89	36.34	42.38	52.19	55.67	59.89	62.88	69.35	37
38	16.61	19.29	20.69	22.88	24.88	31.81	37.34	43.46	53.38	56.90	61.16	64.18	70.70	38
39	17.26	20.00	21.43	23.65	25.70	32.74	38.34	44.54	54.57	58.12	62.43	65.48	72.05	39
40	17.92	20.71	22.16	24.43	26.51	33.66	39.34	45.62	55.76	59.34	63.69	66.77	73.40	40
41	18.58	21.42	22.91	25.21	27.33	34.58	40.34	46.69	56.94	60.56	64.95	68.05	74.74	41
42	19.24	22.14	23.65	26.00	28.14	35.51	41.34	47.77	58.12	61.78	66.21	69.34	76.08	42
43	19.91	22.86	24.40	26.79	28.96	36.44	42.34	48.84	59.30	62.99	67.46	70.62	77.42	43
44	20.58	23.58	25.15	27.57	29.79	37.36	43.34	49.91	60.48	64.20	68.71	71.89	78.75	44
45	21.25	24.31	25.90	28.37	30.61	38.29	44.34	50.98	61.66	65.41	69.96	73.17	80.08	45
46	21.93	25.04	26.66	29.16	31.44	39.22	45.34	52.06	62.83	66.62	71.20	74.44	81.40	46
47	22.61	25.77	27.42	29.96	32.27	40.15	46.34	53.13	64.00	67.82	72.44	75.70	82.72	47
48	23.29	26.51	28.18	30.75	33.10	41.08	47.34	54.20	65.17	69.02	73.68	76.97	84.04	48
49	23.98	27.25	28.94	31.55	33.93	42.01	48.33	55.27	66.34	70.22	74.92	78.23	85.35	49
50	24.67	27.99	29.71	32.36	34.76	42.94	49.33	56.33	67.50	71.42	76.15	79.49	86.66	50

Selected percentiles of Chi-square (χ^2) (cont.)

ν \ P	.001	.005	.010	.025	.050	.250	.500	.750	.950	.975	.990	.995	.999	ν
51	25.37	28.73	30.48	33.16	35.60	43.87	50.33	57.40	68.67	72.62	77.39	80.75	87.97	51
52	26.07	29.48	31.25	33.97	36.44	44.81	51.33	58.47	69.83	73.81	78.62	82.00	89.27	52
53	26.76	30.23	32.02	34.78	37.28	45.74	52.33	59.53	70.99	75.00	79.84	83.25	90.57	53
54	27.47	30.98	32.79	35.59	38.12	46.68	53.33	60.60	72.15	76.19	81.07	84.50	91.87	54
55	28.17	31.73	33.57	36.40	38.96	47.61	54.33	61.66	73.31	77.38	82.29	85.75	93.17	55
56	28.88	32.49	34.35	37.21	39.80	48.55	55.33	62.73	74.47	78.57	83.51	86.99	94.46	56
57	29.59	33.25	35.13	38.03	40.65	49.48	56.33	63.79	75.62	79.75	84.73	88.24	95.75	57
58	30.30	34.01	35.91	38.84	41.49	50.42	57.33	64.86	76.78	80.94	85.95	89.48	97.04	58
59	31.02	34.77	36.70	39.66	42.34	51.36	58.33	65.92	77.93	82.12	87.17	90.72	98.32	59
60	31.74	35.53	37.48	40.48	43.19	52.29	59.33	66.98	79.08	83.30	88.38	91.95	99.61	60
61	32.46	36.30	38.27	41.30	44.04	53.23	60.33	68.04	80.23	84.48	89.59	93.19	100.89	61
62	33.18	37.07	39.06	42.13	44.89	54.17	61.33	69.10	81.38	85.65	90.80	94.42	102.17	62
63	33.91	37.84	39.86	42.95	45.74	55.11	62.33	70.16	82.53	86.83	92.01	95.65	103.44	63
64	34.63	38.61	40.65	43.78	46.59	56.05	63.33	71.23	83.68	88.00	93.22	96.88	104.72	64
65	35.36	39.38	41.44	44.60	47.45	56.99	64.33	72.28	84.82	89.18	94.42	98.11	105.99	65
66	36.09	40.16	42.24	45.43	48.31	57.93	65.33	73.34	85.96	90.35	95.63	99.33	107.26	66
67	36.83	40.94	43.04	46.26	49.16	58.87	66.33	74.40	87.11	91.52	96.83	100.55	108.53	67
68	37.56	41.71	43.84	47.09	50.02	59.81	67.33	75.46	88.25	92.69	98.03	101.78	109.79	68
69	38.30	42.49	44.64	47.92	50.88	60.76	68.33	76.52	89.39	93.86	99.23	103.00	111.06	69
70	39.04	43.28	45.44	48.76	51.74	61.70	69.33	77.58	90.53	95.02	100.43	104.21	112.32	70
71	39.78	44.06	46.25	49.59	52.60	62.64	70.33	78.63	91.67	96.19	101.62	105.43	113.58	71
72	40.52	44.84	47.05	50.43	53.46	63.58	71.33	79.69	92.81	97.35	102.82	106.65	114.84	72
73	41.26	45.63	47.86	51.26	54.33	64.53	72.33	80.75	93.95	98.52	104.01	107.86	116.09	73
74	42.01	46.42	48.67	52.10	55.19	65.47	73.33	81.80	95.08	99.68	105.20	109.07	117.35	74
75	42.76	47.21	49.48	52.94	56.05	66.42	74.33	82.86	96.22	100.84	106.39	110.29	118.60	75
76	43.51	48.00	50.29	53.78	56.92	67.36	75.33	83.91	97.35	102.00	107.58	111.50	119.85	76
77	44.26	48.79	51.10	54.62	57.79	68.31	76.33	84.97	98.48	103.16	108.77	112.70	121.10	77
78	45.01	49.58	51.91	55.47	58.65	69.25	77.33	86.02	99.62	104.32	109.96	113.91	122.35	78
79	45.76	50.38	52.72	56.31	59.52	70.20	78.33	87.08	100.75	105.47	111.14	115.12	123.59	79
80	46.52	51.17	53.54	57.15	60.39	71.14	79.33	88.13	101.88	106.63	112.33	116.32	124.84	80
81	47.28	51.97	54.36	58.00	61.26	72.09	80.33	89.18	103.01	107.78	113.51	117.52	126.08	81
82	48.04	52.77	55.17	58.84	62.13	73.04	81.33	90.24	104.14	108.94	114.69	118.73	127.32	82
83	48.80	53.57	55.99	59.69	63.00	73.99	82.33	91.29	105.27	110.09	115.88	119.93	128.56	83
84	49.56	54.37	56.81	60.54	63.88	74.93	83.33	92.34	106.39	111.24	117.06	121.13	129.80	84
85	50.32	55.17	57.63	61.39	64.75	75.88	84.33	93.39	107.52	112.39	118.24	122.32	131.04	85
86	51.08	55.97	58.46	62.24	65.62	76.83	85.33	94.45	108.65	113.54	119.41	123.52	132.28	86
87	51.85	56.78	59.28	63.09	66.50	77.78	86.33	95.50	109.77	114.69	120.59	124.72	133.51	87
88	52.62	57.58	60.10	63.94	67.37	78.73	87.33	96.55	110.90	115.84	121.77	125.91	134.75	88
89	53.39	58.39	60.93	64.79	68.25	79.68	88.33	97.60	112.02	116.99	122.94	127.11	135.98	89
90	54.16	59.20	61.75	65.65	69.13	80.62	89.33	98.65	113.15	118.14	124.12	128.30	137.21	90
91	54.93	60.00	62.58	66.50	70.00	81.57	90.33	99.70	114.27	119.28	125.29	129.49	138.44	91
92	55.70	60.81	63.41	67.36	70.88	82.52	91.33	100.75	115.39	120.43	126.46	130.68	139.67	92
93	56.47	61.63	64.24	68.21	71.76	83.47	92.33	101.80	116.51	121.57	127.63	131.87	140.89	93
94	57.25	62.44	65.07	69.07	72.64	84.42	93.33	102.85	117.63	122.72	128.80	133.06	142.12	94
95	58.02	63.25	65.90	69.92	73.52	85.38	94.33	103.90	118.75	123.86	129.97	134.25	143.34	95
96	58.80	64.06	66.73	70.78	74.40	86.33	95.33	104.95	119.87	125.00	131.14	135.43	144.57	96
97	59.58	64.88	67.56	71.64	75.28	87.28	96.33	106.00	120.99	126.14	132.31	136.62	145.79	97
98	60.36	65.69	68.40	72.50	76.16	88.23	97.33	107.05	122.11	127.28	133.48	137.80	147.01	98
99	61.14	66.51	69.23	73.36	77.05	89.18	98.33	108.09	123.23	128.42	134.64	138.99	148.23	99
100	61.92	67.33	70.06	74.22	77.93	90.13	99.33	109.14	124.34	129.56	135.81	140.17	149.45	100
z	-3.090	-2.576	-2.326	-1.960	-1.645	-.674	.000	.674	1.645	1.960	2.326	2.576	3.090	z

For degrees of freedom (ν) beyond 100, percentiles of χ^2 may be approximated with: $\chi^2_{\nu[P]} = ½\left(z_{[P]} + \sqrt{2\nu - 1}\right)^2$, utilizing the normal percentiles $z_{[P]}$ at the foot of the table.

Reading off the table

The table furnishes a set of percentage points of the Chi-square (χ^2_ν) distribution for degrees of freedom (ν) from 1 to 100. For larger ν, the approximation formula printed at the foot of the table is recommended.

Illustration 1. What is the value of $\chi^2_{6[.95]}$, *i.e.* the 95th percentage point of Chi-square with $\nu = 6$? Looking up line 6 (= ν) in the table under column 0.95, we read off 12.59, hence $\chi^2_{6[.95]} = 12.59$. In the same way, we obtain $\chi^2_{13[.99]} = 27.69$ and $\chi^2_{20[.975]} = 34.17$.

Illustration 2. Find $\chi^2_{110[.95]}$. As $\nu = 110 > 100$, it is necessary to calculate some estimate of the required percentage point. Using the recommended formula, with $z_{[.95]} = 1.6449$ as indicated, we calculate $\chi^2_{110[.95]} \approx ½[1.6449 + \sqrt{(2\times110-1)}]^2 \approx 135.20$. The exact value (when available) is 135.48.

Full examples

Example 1. In a sample containing 50 observations, we obtain $s^2 = 16.43$ as an estimate of variance. What are the limits within which sould lie the true variance σ^2, using a confidence coefficient of 95 %? *Solution*: We must suppose that the individual observations (X_i) obey the normal law, with (unknown) mean μ and variance σ^2. Under that assumption, the sample variance s^2 is distributed as Chi-square with $n-1$ *df*, specifically $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$. Using the appropriate percentage points of χ^2 and inverting this formula, we obtain the interval:

$$\left\{\frac{(n-1)s^2}{\chi^2_{n-1[(1+c)/2]}} \; ; \; \frac{(n-1)s^2}{\chi^2_{n-1[(1-c)/2]}}\right\}$$

which comprises σ^2 with probability c. For our data, $n = 50$, $c = 0.95$, $\chi^2_{49[.975]} = 70.22$ and $\chi^2_{49[.025]} = 31.55$; the left boundary is thus $(49\times16.43)/70.22 \approx 11.465$, the right one, $(49\times16.43)/31.55 \approx 25.517$. We may state that $\Pr\{\, 11.465 < \sigma^2 < 25.517 \,\} = 0.95$ or, by taking square roots, $\Pr\{\, 3.386 < \sigma < 5.051 \,\} = 0.95$.

Example 2. In an opinion poll bearing on social and moral issues, 200 people must declare their views as "Against", "Uncertain" or "In favor" relatively to the death penalty. Here are the obtained frequencies of opinion, divided between the two genders:

Gender \ Option	Against	Uncertain	In favor
Men	29	17	51
Women	52	8	43

Can we suppose that, in the entire population, men and women share the same views? *Solution*: The statistical analysis of frequency (or contingency) tables is perhaps the foremost application of Chi-square. Here, the (null) hypothesis according to which the answers are scattered irrespective of gender, *i.e.* the *independence hypothesis*, allows to determine the theoretical frequencies ($ft_{i,j}$), with the multiplicative formula:

$$ft_{i,j} = \mathrm{N}(p_{\mathrm{L}_i} \times p_{\mathrm{C}_j}) \; ;$$

the quantities shown are estimated from the proportions in each line (p_{L_i}) and each colum (p_{C_j}). Other equivalent formulae are possible. The independence hypothesis will be discarded at significance level α if the test statistic $\mathrm{X}^2 = \Sigma_{i,j}[(f_{i,j} - ft_{i,j})^2/ft_{i,j}]$ exceeds $\chi^2_{\nu[1-\alpha]}$, with ν = (nbr. of lines$-$1)$\times$(nbr. of columns$-$1). The table below summarizes the calculations. Note that quantity $ft_{i,j}$ is printed in italics at the lower right corner of each cell, and individual X^2 components, $(f_{i,j} - ft_{i,j})^2/ft_{i,j}$, at the upper left corner.

	Column 1	Column 2	Column 3	Total
Line 1	2.693 29 *39.285*	1.960 17 *12.125*	0.642 51 *45.590*	97 $p_{\mathrm{L}1} = 0.485$
Line 2	2.536 52 *41.715*	1.846 8 *12.875*	0.605 43 *48.410*	103 $p_{\mathrm{L}2} = 0.515$
Total	81 $p_{\mathrm{C}1} = 0.405$	25 $p_{\mathrm{C}2} = 0.125$	94 $p_{\mathrm{C}3} = 0.470$	N = 200

Adding all six components $(f_{i,j}-ft_{i,j})^2/ft_{i,j}$, we get $X^2 = 2.693 + 1.960 + 0.642 + 2.536 + 1.846 + 0.605 = 10.282$. The appropriate tabular value significant at 5 % and with $df = (2-1)\times(3-1) = 2$, is $\chi^2_{2[.95]} = 5.99$. As the obtained value (10.282) exceeds the critical value, we may conclude that there is some dependence (or interaction, indeed correlation in a broad sense) between lines and columns, that the frequency profiles vary from one line to the other; in other words, the respondant's gender seems to bias his or her opinion on the death penalty.

Mathematical presentation

A Chi-square variate with ν degrees of freedom is equivalent to the sum of ν independent, squared, standard normal variates, $\sum_{i=1}^{\nu} z_i^2$ and it is denoted χ^2_ν. As an example, the variance (s^2) from a sample of normally distributed observations is distributed as χ^2, the parameter ν being referred to as the *degrees of freedom* (*df*) of the calculated variance. Symbolically, we write:

$$\frac{\nu \cdot s^2}{\sigma^2} \sim \chi^2_\nu \ .$$

In the case of the statistic s^2 based upon n observations from a $N(\mu,\sigma^2)$ distribution, where $s^2 = \sum(x_i-\bar{x})^2/(n-1)$, the *df* are equal to $\nu = n-1$. The Chi-square distribution is also used for the analysis of frequency (or contingency) tables and as an approximation to the distribution of many complex statistics.

Calculation and moments

The Chi-square distribution, a particular case of the *Gamma* distribution (*see* Mathematical complements), has p.d.f.:

$$p_{\chi^2}(x) = [2^{\nu/2}\Gamma(\nu/2)]^{-1} x^{(\nu-2)/2} e^{-x/2} \quad \{ x \geq 0 \} \ ,$$

where $\Gamma(x)$ is the *Gamma* function and $e \approx 2.7183$. Integration of the χ^2 density depends on whether ν is even or odd. Integrating by parts, we obtain for even ν :

$$P_{\chi^2}(x) = \Pr(X \leq x) = 1 - e^{-y}\left[1 + y + \frac{y^2}{2!} + \cdots + \frac{y^{\nu/2-1}}{(\nu/2-1)!}\right] ,$$

and for odd ν:

$$P_{\chi^2}(x) = \Pr(X \le x) = 2\Phi(\sqrt{x}) - 1 - e^{-y}\sqrt{y}\left[\frac{1}{\Gamma(3/2)} + \frac{y}{\Gamma(5/2)} + \cdots + \frac{y^{(\nu-3)/2}}{\Gamma(\nu/2)}\right] ;$$

in each expression, $y = x/2$. When $\nu = 1$, $\chi^2 = z^2$ by definition, therefore $P_{\chi^2}(x) = 2\Phi(\sqrt{x}) - 1$, $\Phi()$ designating the normal d.f.. For $\nu = 2$, the χ^2 variable is the same as a r.v. from the (standard) exponential distribution and $P_{\chi^2}(x) = 1 - \exp(-x/2)$; centiles ($C_P$) of this χ^2_2 distribution may be obtained by inversion, *i.e.* $C_P = \chi^2_{2[P]} = -2\ln(1-P)$.

Moments. The expectation, variance and moments for skewness and kurtosis of a χ^2 variable with degrees of freedom ν are:

$$E(x) = \mu = \nu \; ; \quad \text{var}(x) = \sigma^2 = 2\nu \; ; \quad \gamma_1 = \sqrt{(8/\nu)} \; ; \quad \gamma_2 = 12/\nu \; .$$

The distribution is positively skew, the more large and right-shifted as ν grows and approaching a normal form. The mode is seated at $\nu-2$ (for $\nu \ge 2$), and the median is approximately equal to $\nu - \frac{2}{3} + \frac{1}{9\nu}$.

Some authors "standardize" the χ^2 variable by dividing it by its parameter ν, *i.e.* $x' = x/\nu$: in that case, $\mu(x') = 1$ and $\text{var}(x') = 2/\nu$. This form facilitates somewhat interpolation of χ^2 for untabled values of ν ; note in that context that $\chi^2/\nu \to 1$ when $\nu \to \infty$.

Three normal approximations to χ^2

The p.d.f. and d.f. of χ^2 can be approximated by the normal distribution through diverse transformations. The simplest one is trivial and uses only the first two moments, *i.e.* $z = (X-\nu)/\sqrt{(2\nu)}, X \sim \chi^2$, and is globally not to be recommended except for large ν such as $\nu > 500$.

Fisher proposes another approximation which compensates for the skewness of X. It reads like:

$$\sqrt{(2X)} - \sqrt{(2\nu - 1)} \sim N(0,1) \; ;$$

it is the method whose inversion formula is proposed at the foot of the table for extending it to larger ν.

The third method, attributed to Wilson and Hilferty, is quite accurate. Defining $A = 2/(9\nu)$, it may be written:

$$[\, \sqrt[3]{(X/\nu)} - 1 + A \,] \, / \, \sqrt{A} \sim N(0,1) \; .$$

Its inversion, for the determination of percentage points, is:

$$\chi^2_{\nu[P]} = \nu[z_P\sqrt{A} + 1 - A]^3 \; .$$

With the help of a pocket calculator or of a short computer program, this last method can make up for most current applications of χ^2, even when $\nu < 100$.

The distribution of s, the standard deviation (s.d.)

Just as the χ^2 law governs the distribution of variances (s^2) originating from samples of n normal data, with $\nu = n-1$ *df*, the χ ("Chi") law, more precisely $\chi/\sqrt{\nu}$, represents the sampling distribution of s.d.'s (s). Its p.d.f. is:

$$p_\chi(x) = 2(\nu/2)^{\nu/2}[\Gamma(\nu/2)]^{-1}x^{\nu-1}e^{-\nu x^2/2} \quad \{ x > 0 \} .$$

The χ variable being the positive square root of χ^2, its centiles or percentage points may be obtained from it in that way. Thus, centile C_P of the distribution of a s/σ with ν degrees of freedom is given by $\sqrt{[\chi^2_{\nu[P]}/\nu]}$.

Moments. The first two moments of $\chi/\sqrt{\nu}$ are:

$$E(x) = \mu = \sqrt{(2/\nu)}\Gamma[(\nu+1)/2]/\Gamma(\nu/2) \approx 1 - 1/(4\nu)$$

$$\mathrm{var}(x) = \sigma^2 = 1 - \mu^2 \approx (4\nu-1)/(8\nu^2) .$$

The s.d. s being distributed as $\sigma\chi/\sqrt{\nu}$, the expectation above shows that $E(s) < \sigma$, *i.e.* that the sample s.d. underestimates the parameter σ, notwithstanding the fact that $E(s^2) = \sigma^2$. Lastly, the mode of $\chi/\sqrt{\nu}$ equals $\sqrt{(1-1/\nu)}$ and the median is approximated by $1-\frac{1}{3\nu}$.

Generation of pseudo random variates

The schema of a program below allows the production of r.v.'s from χ^2_ν, the Chi-square distribution with ν ($\nu > 2$) *df*, and it requires a function (designated UNIF) which generates serially r.v.'s from the standard uniform $U(0,1)$ distribution. Particular cases, especially those with $\nu = 1$ and 2, are covered in Remark 3.

Preparation: Let $n \equiv \nu$ (the degrees of freedom)

$C_1 = 1 + \sqrt{(2/e)} \approx 1{,}8577638850$; $\quad C_2 = \sqrt{(n/2)}$
$C_3 = (3n^2-2) / [3n(n-2)]$; $\quad C_4 = 4 / (n-2)$
$C_5 = n-2$

Production : **Repeat**

  Repeat $t \leftarrow$ UNIF ; $u \leftarrow t + (1 - C_1 \times \text{UNIF})/C_2$

  Until $0 < u \leq 1$;

 $w \leftarrow C_3 \times t \,/\, u$

 Until $C_4 \times (u-1) + w + 1/w < 2$ **or** $C_4 \times \ln(u) - \ln(w) + w < 1$;

 Return $C_5 \times w \rightarrow x$.

Remarks :

1. Standard temporal cost : 7.8 à 8.7 × t(UNIF)

2. This algorithm, known under the codename "GMK2" (Cheng et Feast 1979, *in* Fishman 1996), performs equally well for any value ν (= n). It uses up from 3 to 3.5 uniform r.v.'s per call.

3. There are many other methods, the following being noteworthy. Considering that "$x_{(2)} \leftarrow -2 \times \ln(\text{UNIF})$" produces a χ^2_2 r.v. and capitalizing on the additive property of χ^2, we can produce, for instance, a χ^2_8 r.v. with "$x_{(8)} \leftarrow -2 \times \ln(\text{UNIF} \times \text{UNIF} \times \text{UNIF} \times \text{UNIF})$". Also from the definition, "$x_{(1)} \leftarrow y^2$" furnishes one r.v. from χ^2_1 using y, a standard $N(0,1)$ normal r.v.. Lastly and for example, we may fabricate a χ^2_5 r.v. through "$x_{(5)} \leftarrow -2 \times \ln(\text{UNIF} \times \text{UNIF}) + y^2$" , once more using $y \sim N(0,1)$.

Student's *t* distribution

✓ Graphical representations

✓ Selected percentiles of Student's *t* distribution (table 1)

✓ Critical values of *t* according to Dunn-Šidák's criterion (table 2)

✓ Minimum sample size (n) such that a given correlation coefficient r be significant at various thresholds (table 3)

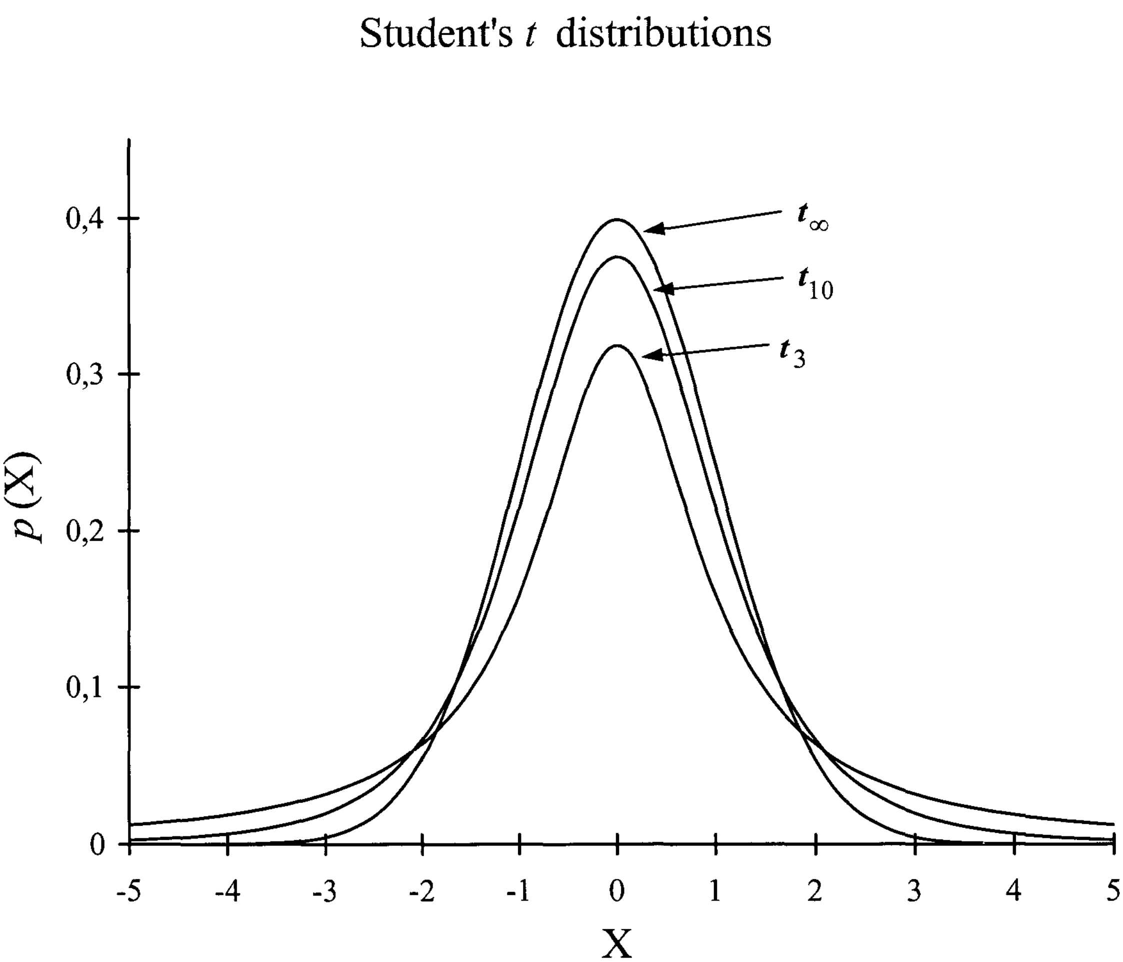
✓ Reading off the table

✓ Full examples

✓ Mathematical presentation

Calculation and moments

Generation of pseudo random variates



Student's t distributions
t_∞
t_{10}
t_3
p (X)
0,4
0,3
0,2
0,1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
X

Selected percentiles of Student's t distribution (table 1)

$\nu \setminus P$	.750	.950	.975	.990	.995	$\nu \setminus P$	.750	.950	.975	.990	.995
1	1.000	6.314	12.71	31.82	63.66	51	.679	1.675	2.008	2.402	2.676
2	.816	2.920	4.303	6.965	9.925	52	.679	1.675	2.007	2.400	2.674
3	.765	2.353	3.182	4.541	5.841	53	.679	1.674	2.006	2.399	2.672
4	.741	2.132	2.776	3.747	4.604	54	.679	1.674	2.005	2.397	2.670
5	.727	2.015	2.571	3.365	4.032	55	.679	1.673	2.004	2.396	2.668
6	.718	1.943	2.447	3.143	3.707	56	.679	1.673	2.003	2.395	2.667
7	.711	1.895	2.365	2.998	3.499	57	.679	1.672	2.002	2.394	2.665
8	.706	1.860	2.306	2.896	3.355	58	.679	1.672	2.002	2.392	2.663
9	.703	1.833	2.262	2.821	3.250	59	.679	1.671	2.001	2.391	2.662
10	.700	1.812	2.228	2.764	3.169	60	.679	1.671	2.000	2.390	2.660
11	.697	1.796	2.201	2.718	3.106	61	.679	1.670	2.000	2.389	2.659
12	.695	1.782	2.179	2.681	3.055	62	.678	1.670	1.999	2.388	2.657
13	.694	1.771	2.160	2.650	3.012	63	.678	1.669	1.998	2.387	2.656
14	.692	1.761	2.145	2.624	2.977	64	.678	1.669	1.998	2.386	2.655
15	.691	1.753	2.131	2.602	2.947	65	.678	1.669	1.997	2.385	2.654
16	.690	1.746	2.120	2.583	2.921	66	.678	1.668	1.997	2.384	2.652
17	.689	1.740	2.110	2.567	2.898	67	.678	1.668	1.996	2.383	2.651
18	.688	1.734	2.101	2.552	2.878	68	.678	1.668	1.995	2.382	2.650
19	.688	1.729	2.093	2.539	2.861	69	.678	1.667	1.995	2.382	2.649
20	.687	1.725	2.086	2.528	2.845	70	.678	1.667	1.994	2.381	2.648
21	.686	1.721	2.080	2.518	2.831	71	.678	1.667	1.994	2.380	2.647
22	.686	1.717	2.074	2.508	2.819	72	.678	1.666	1.993	2.379	2.646
23	.685	1.714	2.069	2.500	2.807	73	.678	1.666	1.993	2.379	2.645
24	.685	1.711	2.064	2.492	2.797	74	.678	1.666	1.993	2.378	2.644
25	.684	1.708	2.060	2.485	2.787	75	.678	1.665	1.992	2.377	2.643
26	.684	1.706	2.056	2.479	2.779	76	.678	1.665	1.992	2.376	2.642
27	.684	1.703	2.052	2.473	2.771	77	.678	1.665	1.991	2.376	2.641
28	.683	1.701	2.048	2.467	2.763	78	.678	1.665	1.991	2.375	2.640
29	.683	1.699	2.045	2.462	2.756	79	.678	1.664	1.990	2.374	2.640
30	.683	1.697	2.042	2.457	2.750	80	.678	1.664	1.990	2.374	2.639
31	.682	1.696	2.040	2.453	2.744	81	.678	1.664	1.990	2.373	2.638
32	.682	1.694	2.037	2.449	2.738	82	.677	1.664	1.989	2.373	2.637
33	.682	1.692	2.035	2.445	2.733	83	.677	1.663	1.989	2.372	2.636
34	.682	1.691	2.032	2.441	2.728	84	.677	1.663	1.989	2.372	2.636
35	.682	1.690	2.030	2.438	2.724	85	.677	1.663	1.988	2.371	2.635
36	.681	1.688	2.028	2.434	2.719	86	.677	1.663	1.988	2.370	2.634
37	.681	1.687	2.026	2.431	2.715	87	.677	1.663	1.988	2.370	2.634
38	.681	1.686	2.024	2.429	2.712	88	.677	1.662	1.987	2.369	2.633
39	.681	1.685	2.023	2.426	2.708	89	.677	1.662	1.987	2.369	2.632
40	.681	1.684	2.021	2.423	2.704	90	.677	1.662	1.987	2.368	2.632
41	.681	1.683	2.020	2.421	2.701	91	.677	1.662	1.986	2.368	2.631
42	.680	1.682	2.018	2.418	2.698	92	.677	1.662	1.986	2.368	2.630
43	.680	1.681	2.017	2.416	2.695	93	.677	1.661	1.986	2.367	2.630
44	.680	1.680	2.015	2.414	2.692	94	.677	1.661	1.986	2.367	2.629
45	.680	1.679	2.014	2.412	2.690	95	.677	1.661	1.985	2.366	2.629
46	.680	1.679	2.013	2.410	2.687	96	.677	1.661	1.985	2.366	2.628
47	.680	1.678	2.012	2.408	2.685	97	.677	1.661	1.985	2.365	2.627
48	.680	1.677	2.011	2.407	2.682	98	.677	1.661	1.984	2.365	2.627
49	.680	1.677	2.010	2.405	2.680	99	.677	1.660	1.984	2.365	2.626
50	.679	1.676	2.009	2.403	2.678	100	.677	1.660	1.984	2.364	2.626
						500	.675	1.648	1.965	2.334	2.586
						∞	.674	1.645	1.960	2.326	2.576

Critical values of *t* according to Dunn-Šidák's criterion for one-tailed 5 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	4.273	5.292	6.144	6.892	7.566	8.185	8.760	9.300	9.810	10.29	10.76	11.20	11.63	12.04
3	3.166	3.716	4.146	4.506	4.819	5.097	5.349	5.580	5.793	5.993	6.180	6.357	6.525	6.685
4	2.764	3.169	3.474	3.723	3.935	4.121	4.286	4.436	4.574	4.701	4.819	4.929	5.034	5.132
5	2.560	2.897	3.146	3.346	3.514	3.660	3.788	3.904	4.009	4.106	4.195	4.278	4.357	4.430
6	2.438	2.736	2.954	3.127	3.270	3.394	3.503	3.600	3.688	3.769	3.843	3.912	3.976	4.037
7	2.356	2.630	2.828	2.984	3.112	3.223	3.319	3.405	3.482	3.553	3.618	3.678	3.735	3.787
8	2.298	2.555	2.739	2.883	3.002	3.103	3.191	3.269	3.340	3.404	3.463	3.517	3.568	3.615
9	2.254	2.499	2.673	2.809	2.920	3.015	3.097	3.170	3.235	3.295	3.349	3.400	3.447	3.490
10	2.221	2.456	2.623	2.752	2.858	2.947	3.025	3.094	3.156	3.212	3.263	3.310	3.354	3.395
11	2.194	2.422	2.582	2.707	2.808	2.894	2.968	3.034	3.093	3.146	3.195	3.240	3.282	3.320
12	2.172	2.394	2.550	2.670	2.768	2.851	2.922	2.986	3.042	3.093	3.140	3.183	3.223	3.260
13	2.153	2.370	2.523	2.640	2.735	2.815	2.885	2.946	3.000	3.050	3.095	3.136	3.175	3.211
14	2.138	2.351	2.500	2.614	2.707	2.785	2.853	2.912	2.965	3.013	3.057	3.097	3.134	3.169
15	2.125	2.334	2.480	2.592	2.683	2.760	2.826	2.884	2.935	2.982	3.025	3.064	3.100	3.134
16	2.113	2.320	2.463	2.573	2.663	2.738	2.802	2.859	2.910	2.955	2.997	3.035	3.071	3.104
17	2.103	2.307	2.449	2.557	2.645	2.718	2.782	2.838	2.887	2.932	2.973	3.011	3.045	3.077
18	2.094	2.296	2.436	2.543	2.629	2.702	2.764	2.819	2.868	2.912	2.952	2.989	3.023	3.054
19	2.087	2.286	2.424	2.530	2.615	2.687	2.748	2.802	2.850	2.894	2.933	2.969	3.003	3.034
20	2.080	2.277	2.414	2.518	2.603	2.673	2.734	2.788	2.835	2.878	2.917	2.952	2.985	3.016
22	2.068	2.262	2.397	2.499	2.582	2.651	2.710	2.762	2.809	2.850	2.888	2.923	2.955	2.985
24	2.058	2.250	2.382	2.483	2.564	2.632	2.690	2.742	2.787	2.828	2.865	2.899	2.931	2.960
26	2.049	2.239	2.370	2.470	2.550	2.617	2.674	2.724	2.769	2.809	2.846	2.879	2.910	2.939
28	2.042	2.231	2.360	2.458	2.537	2.603	2.660	2.710	2.754	2.793	2.829	2.862	2.893	2.921
30	2.036	2.223	2.351	2.448	2.527	2.592	2.648	2.697	2.741	2.780	2.815	2.848	2.878	2.905
32	2.031	2.216	2.343	2.440	2.517	2.582	2.638	2.686	2.729	2.768	2.803	2.835	2.865	2.892
34	2.026	2.210	2.337	2.432	2.509	2.573	2.628	2.676	2.719	2.757	2.792	2.824	2.853	2.880
36	2.022	2.205	2.331	2.426	2.502	2.566	2.620	2.668	2.710	2.748	2.783	2.814	2.843	2.870
38	2.018	2.201	2.325	2.420	2.496	2.559	2.613	2.660	2.702	2.740	2.774	2.806	2.834	2.861
40	2.015	2.197	2.321	2.415	2.490	2.553	2.607	2.654	2.695	2.733	2.767	2.798	2.826	2.853
42	2.012	2.193	2.316	2.410	2.485	2.547	2.601	2.648	2.689	2.726	2.760	2.791	2.819	2.846
44	2.010	2.190	2.313	2.406	2.480	2.542	2.596	2.642	2.683	2.720	2.754	2.784	2.813	2.839
46	2.007	2.186	2.309	2.402	2.476	2.538	2.591	2.637	2.678	2.715	2.748	2.779	2.807	2.833
48	2.005	2.184	2.306	2.398	2.472	2.534	2.587	2.633	2.673	2.710	2.743	2.773	2.801	2.827
50	2.003	2.181	2.303	2.395	2.469	2.530	2.583	2.628	2.669	2.705	2.738	2.769	2.796	2.822
55	1.998	2.176	2.296	2.388	2.461	2.522	2.574	2.619	2.659	2.696	2.728	2.758	2.786	2.811
60	1.995	2.171	2.291	2.382	2.455	2.515	2.567	2.612	2.652	2.687	2.720	2.749	2.777	2.802
65	1.991	2.167	2.287	2.377	2.449	2.509	2.561	2.605	2.645	2.681	2.713	2.742	2.769	2.794
70	1.989	2.164	2.283	2.373	2.445	2.504	2.555	2.600	2.639	2.675	2.707	2.736	2.763	2.788
75	1.986	2.161	2.279	2.369	2.441	2.500	2.551	2.595	2.634	2.670	2.701	2.731	2.757	2.782
80	1.984	2.158	2.277	2.366	2.437	2.496	2.547	2.591	2.630	2.665	2.697	2.726	2.753	2.777
85	1.983	2.156	2.274	2.363	2.434	2.493	2.544	2.588	2.626	2.661	2.693	2.722	2.748	2.773
90	1.981	2.154	2.272	2.360	2.431	2.490	2.541	2.584	2.623	2.658	2.689	2.718	2.745	2.769
95	1.980	2.152	2.270	2.358	2.429	2.488	2.538	2.581	2.620	2.655	2.686	2.715	2.741	2.766
100	1.978	2.151	2.268	2.356	2.427	2.485	2.535	2.579	2.618	2.652	2.683	2.712	2.738	2.763
125	1.974	2.145	2.261	2.349	2.419	2.477	2.526	2.569	2.607	2.642	2.673	2.701	2.727	2.751
150	1.970	2.141	2.257	2.344	2.413	2.471	2.520	2.563	2.601	2.635	2.665	2.694	2.719	2.743
175	1.968	2.138	2.253	2.340	2.409	2.467	2.516	2.558	2.596	2.630	2.660	2.688	2.714	2.738
200	1.966	2.136	2.251	2.337	2.406	2.464	2.512	2.555	2.592	2.626	2.657	2.684	2.710	2.734
250	1.964	2.133	2.247	2.334	2.402	2.459	2.508	2.550	2.588	2.621	2.651	2.679	2.704	2.728
300	1.962	2.131	2.245	2.331	2.400	2.456	2.505	2.547	2.584	2.618	2.648	2.675	2.701	2.724
350	1.961	2.130	2.244	2.329	2.398	2.454	2.503	2.545	2.582	2.615	2.645	2.673	2.698	2.721
400	1.960	2.129	2.242	2.328	2.396	2.453	2.501	2.543	2.580	2.613	2.643	2.671	2.696	2.719
450	1.960	2.128	2.241	2.327	2.395	2.452	2.500	2.542	2.579	2.612	2.642	2.669	2.694	2.718
500	1.959	2.127	2.241	2.326	2.394	2.451	2.499	2.541	2.578	2.611	2.641	2.668	2.693	2.716
1000	1.957	2.124	2.237	2.322	2.390	2.446	2.494	2.536	2.573	2.606	2.636	2.663	2.688	2.711
∞	1.955	2.121	2.234	2.319	2.386	2.442	2.490	2.531	2.568	2.601	2.630	2.657	2.682	2.705

Critical values of t according to Dunn-Šidák's criterion for two-tailed 5% tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	6.164	7.582	8.774	9.823	10.77	11.64	12.45	13.21	13.93	14.61	15.26	15.89	16.49	17.07
3	4.156	4.826	5.355	5.799	6.185	6.529	6.842	7.128	7.394	7.642	7.876	8.096	8.306	8.505
4	3.481	3.941	4.290	4.577	4.822	5.036	5.228	5.402	5.562	5.710	5.848	5.977	6.099	6.214
5	3.152	3.518	3.791	4.012	4.197	4.358	4.501	4.630	4.747	4.855	4.955	5.049	5.136	5.219
6	2.959	3.274	3.505	3.690	3.845	3.978	4.095	4.200	4.296	4.383	4.464	4.539	4.609	4.675
7	2.832	3.115	3.321	3.484	3.620	3.736	3.838	3.929	4.011	4.086	4.156	4.220	4.280	4.336
8	2.743	3.005	3.193	3.342	3.464	3.569	3.661	3.743	3.816	3.883	3.945	4.002	4.055	4.105
9	2.677	2.923	3.099	3.237	3.351	3.448	3.532	3.607	3.675	3.736	3.793	3.845	3.893	3.939
10	2.626	2.860	3.027	3.157	3.264	3.355	3.434	3.505	3.568	3.625	3.677	3.726	3.771	3.813
11	2.586	2.811	2.970	3.094	3.196	3.283	3.358	3.424	3.484	3.538	3.587	3.633	3.675	3.715
12	2.553	2.770	2.924	3.044	3.141	3.224	3.296	3.359	3.416	3.468	3.515	3.558	3.598	3.636
13	2.526	2.737	2.886	3.002	3.096	3.176	3.245	3.306	3.361	3.410	3.455	3.497	3.535	3.571
14	2.503	2.709	2.854	2.967	3.058	3.135	3.202	3.261	3.314	3.362	3.406	3.446	3.483	3.518
15	2.483	2.685	2.827	2.937	3.026	3.101	3.166	3.224	3.275	3.321	3.364	3.402	3.439	3.472
16	2.467	2.665	2.804	2.911	2.998	3.072	3.135	3.191	3.241	3.286	3.327	3.365	3.400	3.433
17	2.452	2.647	2.783	2.889	2.974	3.046	3.108	3.163	3.212	3.256	3.296	3.333	3.367	3.399
18	2.439	2.631	2.766	2.869	2.953	3.024	3.085	3.138	3.186	3.229	3.269	3.305	3.338	3.370
19	2.427	2.617	2.750	2.852	2.934	3.004	3.064	3.116	3.163	3.206	3.245	3.280	3.313	3.343
20	2.417	2.605	2.736	2.836	2.918	2.986	3.045	3.097	3.143	3.185	3.223	3.258	3.290	3.320
22	2.400	2.584	2.712	2.810	2.889	2.956	3.014	3.064	3.109	3.150	3.187	3.220	3.252	3.281
24	2.385	2.566	2.692	2.788	2.866	2.931	2.988	3.037	3.081	3.121	3.157	3.190	3.220	3.249
26	2.373	2.551	2.675	2.770	2.847	2.911	2.966	3.014	3.058	3.096	3.132	3.164	3.194	3.222
28	2.363	2.539	2.661	2.755	2.830	2.893	2.948	2.995	3.038	3.076	3.111	3.142	3.172	3.199
30	2.354	2.528	2.649	2.742	2.816	2.878	2.932	2.979	3.021	3.058	3.092	3.124	3.153	3.180
32	2.346	2.519	2.639	2.730	2.804	2.865	2.918	2.965	3.006	3.043	3.077	3.108	3.136	3.163
34	2.340	2.511	2.630	2.720	2.793	2.854	2.906	2.952	2.993	3.030	3.063	3.094	3.122	3.148
36	2.334	2.504	2.622	2.711	2.784	2.844	2.896	2.941	2.982	3.018	3.051	3.081	3.109	3.135
38	2.328	2.498	2.614	2.703	2.775	2.835	2.887	2.932	2.972	3.008	3.040	3.070	3.098	3.123
40	2.323	2.492	2.608	2.696	2.768	2.827	2.878	2.923	2.963	2.998	3.031	3.060	3.088	3.113
42	2.319	2.487	2.602	2.690	2.761	2.820	2.871	2.915	2.954	2.990	3.022	3.051	3.079	3.104
44	2.315	2.482	2.597	2.684	2.755	2.813	2.864	2.908	2.947	2.982	3.014	3.043	3.070	3.095
46	2.312	2.478	2.592	2.679	2.749	2.807	2.858	2.901	2.940	2.975	3.007	3.036	3.063	3.088
48	2.309	2.474	2.588	2.674	2.744	2.802	2.852	2.895	2.934	2.969	3.001	3.029	3.056	3.081
50	2.306	2.470	2.584	2.670	2.739	2.797	2.847	2.890	2.929	2.963	2.995	3.023	3.050	3.074
55	2.299	2.463	2.575	2.660	2.729	2.786	2.835	2.878	2.916	2.951	2.982	3.010	3.036	3.060
60	2.294	2.456	2.568	2.653	2.721	2.777	2.826	2.869	2.906	2.940	2.971	2.999	3.025	3.049
65	2.289	2.451	2.562	2.646	2.714	2.770	2.818	2.860	2.898	2.931	2.962	2.990	3.016	3.039
70	2.285	2.446	2.557	2.640	2.707	2.764	2.812	2.853	2.891	2.924	2.954	2.982	3.008	3.031
75	2.282	2.442	2.552	2.635	2.702	2.758	2.806	2.847	2.884	2.918	2.948	2.975	3.001	3.024
80	2.279	2.439	2.548	2.631	2.698	2.753	2.801	2.842	2.879	2.912	2.942	2.969	2.995	3.018
85	2.277	2.436	2.545	2.627	2.694	2.749	2.796	2.838	2.874	2.907	2.937	2.964	2.989	3.012
90	2.274	2.433	2.542	2.624	2.690	2.745	2.792	2.833	2.870	2.903	2.932	2.959	2.984	3.008
95	2.272	2.431	2.539	2.621	2.687	2.742	2.789	2.830	2.866	2.899	2.928	2.955	2.980	3.003
100	2.271	2.428	2.537	2.618	2.684	2.739	2.786	2.827	2.863	2.895	2.925	2.952	2.976	2.999
125	2.264	2.420	2.527	2.608	2.673	2.727	2.774	2.814	2.850	2.882	2.911	2.938	2.962	2.985
150	2.259	2.415	2.521	2.602	2.666	2.720	2.766	2.806	2.841	2.873	2.902	2.928	2.953	2.975
175	2.256	2.411	2.517	2.597	2.661	2.715	2.760	2.800	2.835	2.867	2.896	2.922	2.946	2.968
200	2.253	2.408	2.514	2.593	2.657	2.711	2.756	2.796	2.831	2.862	2.891	2.917	2.941	2.963
250	2.250	2.404	2.509	2.588	2.652	2.705	2.750	2.790	2.825	2.856	2.884	2.910	2.934	2.956
300	2.248	2.401	2.506	2.585	2.649	2.701	2.746	2.786	2.820	2.852	2.880	2.906	2.929	2.951
350	2.246	2.399	2.504	2.583	2.646	2.699	2.744	2.783	2.817	2.849	2.877	2.902	2.926	2.948
400	2.245	2.398	2.502	2.581	2.644	2.697	2.741	2.781	2.815	2.846	2.874	2.900	2.924	2.945
450	2.244	2.397	2.501	2.580	2.643	2.695	2.740	2.779	2.813	2.844	2.872	2.898	2.922	2.943
500	2.243	2.396	2.500	2.579	2.641	2.694	2.739	2.778	2.812	2.843	2.871	2.897	2.920	2.942
1000	2.240	2.392	2.495	2.574	2.636	2.688	2.733	2.772	2.806	2.837	2.864	2.890	2.913	2.935
∞	2.236	2.388	2.491	2.569	2.631	2.683	2.727	2.766	2.800	2.830	2.858	2.883	2.906	2.928

Critical values of *t* according to Dunn-Šidák's criterion for one-tailed 1 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	9.912	12.17	14.06	15.73	17.24	18.63	19.92	21.13	22.28	23.37	24.41	25.41	26.37	27.29
3	5.836	6.733	7.444	8.041	8.563	9.028	9.451	9.839	10.20	10.54	10.85	11.15	11.44	11.71
4	4.601	5.162	5.592	5.945	6.247	6.513	6.750	6.966	7.165	7.348	7.520	7.680	7.832	7.975
5	4.030	4.452	4.769	5.026	5.242	5.431	5.599	5.750	5.887	6.014	6.132	6.242	6.345	6.443
6	3.705	4.055	4.313	4.520	4.694	4.844	4.976	5.095	5.203	5.302	5.393	5.478	5.558	5.633
7	3.498	3.803	4.026	4.204	4.352	4.479	4.591	4.691	4.781	4.864	4.940	5.011	5.077	5.139
8	3.354	3.630	3.830	3.988	4.119	4.232	4.330	4.418	4.497	4.570	4.636	4.698	4.755	4.809
9	3.248	3.503	3.687	3.832	3.951	4.054	4.143	4.222	4.294	4.359	4.419	4.474	4.526	4.574
10	3.168	3.407	3.579	3.714	3.825	3.919	4.002	4.075	4.141	4.201	4.256	4.306	4.354	4.398
11	3.104	3.332	3.494	3.621	3.726	3.815	3.892	3.960	4.022	4.078	4.129	4.177	4.220	4.262
12	3.053	3.271	3.426	3.547	3.647	3.731	3.804	3.869	3.927	3.980	4.028	4.073	4.114	4.153
13	3.011	3.221	3.371	3.487	3.582	3.662	3.732	3.794	3.850	3.900	3.946	3.988	4.028	4.064
14	2.976	3.179	3.324	3.436	3.527	3.605	3.672	3.732	3.785	3.833	3.877	3.918	3.956	3.991
15	2.945	3.144	3.284	3.393	3.482	3.557	3.622	3.679	3.731	3.777	3.820	3.859	3.895	3.929
16	2.920	3.113	3.250	3.356	3.442	3.515	3.578	3.634	3.684	3.729	3.770	3.808	3.843	3.876
17	2.897	3.087	3.221	3.324	3.408	3.479	3.541	3.595	3.644	3.688	3.728	3.764	3.799	3.830
18	2.877	3.064	3.195	3.296	3.378	3.448	3.508	3.561	3.608	3.651	3.690	3.726	3.759	3.790
19	2.860	3.043	3.172	3.271	3.352	3.420	3.479	3.531	3.577	3.619	3.657	3.693	3.725	3.755
20	2.844	3.025	3.152	3.249	3.329	3.396	3.454	3.504	3.550	3.591	3.628	3.663	3.695	3.724
22	2.818	2.994	3.117	3.212	3.289	3.354	3.410	3.459	3.503	3.543	3.579	3.612	3.643	3.671
24	2.796	2.968	3.089	3.182	3.257	3.320	3.374	3.422	3.465	3.503	3.539	3.571	3.601	3.628
26	2.778	2.947	3.065	3.156	3.230	3.291	3.345	3.391	3.433	3.471	3.505	3.537	3.566	3.593
28	2.762	2.929	3.045	3.135	3.207	3.267	3.320	3.366	3.406	3.443	3.477	3.508	3.536	3.562
30	2.749	2.914	3.028	3.116	3.187	3.247	3.298	3.343	3.383	3.420	3.453	3.483	3.511	3.537
32	2.737	2.900	3.013	3.100	3.170	3.229	3.280	3.324	3.364	3.399	3.432	3.461	3.489	3.514
34	2.727	2.889	3.000	3.086	3.155	3.213	3.263	3.307	3.346	3.381	3.413	3.443	3.470	3.495
36	2.718	2.878	2.989	3.074	3.142	3.200	3.249	3.292	3.331	3.366	3.397	3.426	3.453	3.478
38	2.711	2.869	2.979	3.063	3.131	3.188	3.236	3.279	3.317	3.352	3.383	3.412	3.438	3.463
40	2.703	2.861	2.970	3.053	3.120	3.177	3.225	3.267	3.305	3.339	3.370	3.399	3.425	3.449
42	2.697	2.853	2.962	3.044	3.111	3.167	3.215	3.257	3.294	3.328	3.359	3.387	3.413	3.437
44	2.691	2.847	2.954	3.036	3.102	3.158	3.206	3.247	3.284	3.318	3.348	3.376	3.402	3.426
46	2.686	2.840	2.947	3.029	3.095	3.150	3.197	3.239	3.276	3.309	3.339	3.366	3.392	3.416
48	2.681	2.835	2.941	3.022	3.088	3.142	3.190	3.231	3.267	3.300	3.330	3.358	3.383	3.406
50	2.677	2.830	2.936	3.016	3.081	3.136	3.183	3.223	3.260	3.293	3.322	3.350	3.375	3.398
55	2.667	2.819	2.923	3.003	3.067	3.121	3.167	3.208	3.244	3.276	3.305	3.332	3.357	3.380
60	2.659	2.809	2.913	2.992	3.056	3.109	3.155	3.195	3.230	3.262	3.291	3.318	3.342	3.365
65	2.653	2.802	2.905	2.983	3.046	3.099	3.144	3.184	3.219	3.251	3.279	3.306	3.330	3.352
70	2.647	2.795	2.897	2.975	3.038	3.090	3.135	3.174	3.209	3.241	3.269	3.295	3.319	3.342
75	2.642	2.789	2.891	2.968	3.031	3.083	3.127	3.166	3.201	3.232	3.260	3.286	3.310	3.332
80	2.638	2.785	2.886	2.963	3.024	3.076	3.121	3.159	3.194	3.225	3.253	3.279	3.302	3.324
85	2.634	2.780	2.881	2.957	3.019	3.070	3.115	3.153	3.187	3.218	3.246	3.272	3.295	3.317
90	2.631	2.776	2.877	2.953	3.014	3.065	3.109	3.148	3.182	3.212	3.240	3.266	3.289	3.311
95	2.628	2.773	2.873	2.949	3.010	3.061	3.105	3.143	3.177	3.207	3.235	3.260	3.284	3.305
100	2.625	2.770	2.869	2.945	3.006	3.057	3.100	3.138	3.172	3.203	3.230	3.255	3.279	3.300
125	2.615	2.758	2.856	2.931	2.991	3.041	3.084	3.122	3.155	3.185	3.212	3.237	3.260	3.281
150	2.608	2.750	2.848	2.922	2.982	3.031	3.074	3.111	3.144	3.174	3.201	3.225	3.248	3.269
175	2.603	2.745	2.842	2.915	2.975	3.024	3.066	3.103	3.136	3.165	3.192	3.217	3.239	3.260
200	2.600	2.741	2.837	2.911	2.970	3.019	3.061	3.097	3.130	3.159	3.186	3.210	3.233	3.253
250	2.595	2.735	2.831	2.904	2.962	3.011	3.053	3.089	3.122	3.151	3.177	3.201	3.223	3.244
300	2.591	2.731	2.827	2.899	2.958	3.006	3.048	3.084	3.116	3.145	3.171	3.195	3.217	3.238
350	2.589	2.728	2.824	2.896	2.954	3.003	3.044	3.080	3.112	3.141	3.167	3.191	3.213	3.233
400	2.587	2.726	2.821	2.894	2.952	3.000	3.041	3.077	3.109	3.138	3.164	3.188	3.210	3.230
450	2.586	2.725	2.820	2.892	2.950	2.998	3.039	3.075	3.107	3.136	3.162	3.185	3.207	3.228
500	2.585	2.723	2.818	2.890	2.948	2.996	3.037	3.073	3.105	3.134	3.160	3.184	3.205	3.226
1000	2.580	2.718	2.812	2.884	2.941	2.989	3.030	3.065	3.097	3.125	3.151	3.175	3.196	3.216
∞	2.575	2.712	2.806	2.877	2.934	2.981	3.022	3.057	3.089	3.117	3.143	3.166	3.187	3.207

Critical values of *t* according to Dunn-Šidák's criterion for two-tailed 1 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	14.07	17.25	19.92	22.28	24.41	26.37	28.20	29.91	31.53	33.07	34.54	35.95	37.31	38.62
3	7.447	8.565	9.453	10.20	10.85	11.44	11.97	12.45	12.90	13.33	13.72	14.10	14.46	14.80
4	5.594	6.248	6.751	7.166	7.520	7.832	8.112	8.367	8.600	8.817	9.019	9.208	9.387	9.556
5	4.771	5.243	5.599	5.888	6.133	6.346	6.535	6.706	6.862	7.006	7.139	7.264	7.381	7.491
6	4.315	4.695	4.977	5.203	5.394	5.559	5.704	5.835	5.954	6.063	6.164	6.258	6.345	6.428
7	4.027	4.353	4.591	4.782	4.941	5.078	5.198	5.306	5.404	5.493	5.576	5.652	5.724	5.791
8	3.831	4.120	4.331	4.498	4.637	4.756	4.860	4.953	5.038	5.115	5.185	5.251	5.312	5.370
9	3.688	3.952	4.143	4.294	4.419	4.526	4.619	4.703	4.778	4.846	4.909	4.967	5.021	5.072
10	3.580	3.825	4.002	4.141	4.256	4.354	4.439	4.515	4.584	4.646	4.703	4.756	4.806	4.852
11	3.495	3.726	3.892	4.022	4.129	4.221	4.300	4.371	4.434	4.492	4.545	4.594	4.639	4.682
12	3.427	3.647	3.804	3.927	4.029	4.114	4.189	4.256	4.315	4.369	4.419	4.465	4.507	4.547
13	3.371	3.582	3.733	3.850	3.946	4.028	4.099	4.162	4.218	4.270	4.317	4.360	4.400	4.438
14	3.324	3.528	3.673	3.785	3.878	3.956	4.024	4.084	4.138	4.187	4.232	4.273	4.311	4.347
15	3.285	3.482	3.622	3.731	3.820	3.895	3.961	4.019	4.071	4.118	4.160	4.200	4.237	4.271
16	3.251	3.443	3.579	3.684	3.771	3.844	3.907	3.963	4.013	4.058	4.100	4.138	4.173	4.206
17	3.221	3.409	3.541	3.644	3.728	3.799	3.860	3.914	3.963	4.007	4.047	4.084	4.118	4.150
18	3.195	3.379	3.508	3.609	3.691	3.760	3.820	3.872	3.920	3.962	4.001	4.037	4.071	4.102
19	3.173	3.353	3.479	3.578	3.658	3.725	3.784	3.835	3.881	3.923	3.961	3.996	4.029	4.059
20	3.152	3.329	3.454	3.550	3.629	3.695	3.752	3.802	3.848	3.888	3.926	3.960	3.992	4.021
22	3.118	3.289	3.410	3.503	3.579	3.643	3.698	3.747	3.790	3.830	3.865	3.898	3.929	3.957
24	3.089	3.257	3.375	3.465	3.539	3.601	3.654	3.702	3.744	3.782	3.816	3.848	3.878	3.905
26	3.066	3.230	3.345	3.433	3.505	3.566	3.618	3.664	3.705	3.742	3.776	3.807	3.835	3.862
28	3.046	3.207	3.320	3.407	3.477	3.536	3.587	3.632	3.672	3.708	3.741	3.771	3.799	3.825
30	3.029	3.188	3.298	3.384	3.453	3.511	3.561	3.605	3.644	3.680	3.712	3.742	3.769	3.794
32	3.014	3.171	3.280	3.364	3.432	3.489	3.538	3.582	3.620	3.655	3.687	3.716	3.743	3.768
34	3.001	3.156	3.264	3.346	3.414	3.470	3.518	3.561	3.599	3.633	3.665	3.693	3.720	3.744
36	2.990	3.143	3.249	3.331	3.397	3.453	3.501	3.543	3.581	3.614	3.645	3.673	3.699	3.724
38	2.979	3.131	3.237	3.318	3.383	3.438	3.485	3.527	3.564	3.597	3.628	3.656	3.681	3.705
40	2.970	3.121	3.225	3.305	3.370	3.425	3.472	3.513	3.549	3.582	3.612	3.640	3.665	3.689
42	2.962	3.111	3.215	3.295	3.359	3.413	3.459	3.500	3.536	3.569	3.599	3.626	3.651	3.674
44	2.955	3.103	3.206	3.285	3.348	3.402	3.448	3.488	3.524	3.557	3.586	3.613	3.638	3.661
46	2.948	3.095	3.197	3.276	3.339	3.392	3.438	3.478	3.513	3.546	3.575	3.601	3.626	3.649
48	2.942	3.088	3.190	3.268	3.330	3.383	3.428	3.468	3.504	3.535	3.564	3.591	3.615	3.638
50	2.936	3.082	3.183	3.260	3.322	3.375	3.420	3.459	3.495	3.526	3.555	3.581	3.606	3.628
55	2.924	3.068	3.167	3.244	3.305	3.357	3.401	3.440	3.475	3.506	3.534	3.560	3.584	3.606
60	2.914	3.056	3.155	3.230	3.291	3.342	3.386	3.425	3.459	3.489	3.517	3.543	3.567	3.589
65	2.905	3.046	3.144	3.219	3.279	3.330	3.373	3.411	3.445	3.476	3.503	3.529	3.552	3.574
70	2.898	3.038	3.135	3.209	3.269	3.319	3.362	3.400	3.434	3.464	3.491	3.516	3.539	3.561
75	2.892	3.031	3.128	3.201	3.261	3.310	3.353	3.390	3.424	3.454	3.481	3.506	3.528	3.550
80	2.886	3.025	3.121	3.194	3.253	3.302	3.345	3.382	3.415	3.445	3.472	3.496	3.519	3.540
85	2.881	3.019	3.115	3.188	3.246	3.295	3.338	3.374	3.407	3.437	3.464	3.488	3.511	3.532
90	2.877	3.014	3.110	3.182	3.240	3.289	3.331	3.368	3.401	3.430	3.457	3.481	3.503	3.524
95	2.873	3.010	3.105	3.177	3.235	3.284	3.326	3.362	3.395	3.424	3.450	3.475	3.497	3.518
100	2.870	3.006	3.101	3.172	3.230	3.279	3.320	3.357	3.389	3.418	3.445	3.469	3.491	3.512
125	2.857	2.992	3.085	3.155	3.213	3.260	3.301	3.337	3.369	3.397	3.423	3.447	3.469	3.489
150	2.848	2.982	3.074	3.144	3.201	3.248	3.288	3.324	3.355	3.383	3.409	3.433	3.454	3.474
175	2.842	2.975	3.067	3.136	3.192	3.239	3.279	3.314	3.346	3.374	3.399	3.422	3.444	3.464
200	2.838	2.970	3.061	3.130	3.186	3.233	3.273	3.307	3.339	3.366	3.392	3.415	3.436	3.456
250	2.831	2.963	3.053	3.122	3.177	3.224	3.263	3.298	3.329	3.356	3.381	3.404	3.425	3.445
300	2.827	2.958	3.048	3.116	3.171	3.217	3.257	3.291	3.322	3.349	3.374	3.397	3.418	3.438
350	2.824	2.954	3.044	3.112	3.167	3.213	3.252	3.287	3.317	3.345	3.369	3.392	3.413	3.432
400	2.822	2.952	3.041	3.110	3.164	3.210	3.249	3.283	3.314	3.341	3.366	3.388	3.409	3.429
450	2.820	2.950	3.039	3.107	3.162	3.207	3.247	3.281	3.311	3.338	3.363	3.385	3.406	3.426
500	2.819	2.948	3.038	3.105	3.160	3.205	3.244	3.279	3.309	3.336	3.361	3.383	3.404	3.423
000	2.812	2.941	3.030	3.097	3.151	3.196	3.235	3.269	3.299	3.326	3.350	3.373	3.393	3.412
∞	2.806	2.934	3.022	3.089	3.143	3.188	3.226	3.260	3.289	3.316	3.340	3.362	3.383	3.402

Minimum sample size (n) such that a given correlation coefficient de correlation r be significant (*i.e.* significantly different from zero) at various thresholds (table 3)

r	P = 0.95	0.975	0.99	0.995	r	P = 0.95	0.975	0.99	0.995
		minimum n					minimum n		
.99	3	4	4	5	.49	13	17	23	27
.98	4	4	5	5	.48	13	18	24	28
.97	4	4	5	5	.47	14	18	25	30
.96	4	4	5	5	.46	14	19	26	31
.95	4	5	5	6	.45	15	20	27	32
.94	4	5	5	6	.44	16	21	28	34
.93	4	5	6	6	.43	16	22	29	35
.92	4	5	6	6	.42	17	23	31	37
.91	4	5	6	7	.41	18	24	32	39
.90	5	5	6	7	.40	19	25	34	41
.89	5	5	6	7	.39	19	26	36	43
.88	5	5	7	7	.38	20	28	38	46
.87	5	6	7	8	.37	21	29	40	48
.86	5	6	7	8	.36	22	31	42	51
.85	5	6	7	8	.35	24	32	44	54
.84	5	6	7	8	.34	25	34	47	57
.83	5	6	8	9	.33	26	36	50	61
.82	5	6	8	9	.32	28	39	53	64
.81	5	7	8	9	.31	30	41	57	69
.80	6	7	8	9	.30	32	44	60	73
.79	6	7	8	10	.29	34	47	65	79
.78	6	7	9	10	.28	36	50	69	84
.77	6	7	9	10	.27	39	54	74	91
.76	6	7	9	11	.26	42	58	80	98
.75	6	8	9	11	.25	45	63	87	106
.74	6	8	10	11	.24	49	68	94	115
.73	6	8	10	12	.23	53	74	103	125
.72	7	8	10	12	.22	58	80	112	137
.71	7	8	11	12	.21	63	88	123	150
.70	7	9	11	13	.20	69	97	136	166
.69	7	9	11	13	.19	77	107	150	183
.68	7	9	12	14	.18	85	120	167	204
.67	7	9	12	14	.17	95	134	188	229
.66	8	10	12	15	.16	107	151	212	259
.65	8	10	13	15	.15	122	172	241	295
.64	8	10	13	16	.14	140	197	276	338
.63	8	11	14	16	.13	162	228	321	392
.62	9	11	14	17	.12	190	268	376	460
.61	9	11	15	17	.11	225	319	448	548
.60	9	12	15	18	.10	272	385	541	663
.59	9	12	16	18	.09	336	475	668	819
.58	10	12	16	19	.08	424	601	846	1036
.57	10	13	17	20	.07	554	785	1105	1354
.56	10	13	17	21	.06	753	1068	1504	1843
.55	10	14	18	21	.05	1084	1538	2165	2654
.54	11	14	19	22	.04	1693	2402	3383	4146
.53	11	15	19	23	.03	3008	4269	6014	7372
.52	12	15	20	24	.02	6766	9605	13530	16587
.51	12	16	21	25	.01	27057	38416	54119	66349
.50	12	16	22	26					

Reading off the tables

Table 1 furnishes values of Student's t statistic corresponding to probability integrals $P = 0.75$, 0.95, 0.975, 0.99 and 0.995 for $df\,(\nu)$ from 1 to 99, plus some others. For $P < 0.50$, one may use relation: $t_{[P]} = -t_{[1-P]}$ due to the symmetry of the distribution about zero.

Table 2 also gives t values for $P = 0.95$, 0.975, 0.99 and 0.995 following Dunn-Šidák protection criterion: the other parameters are degrees of freedom (ν, or df) and number of planned comparisons (nc).

Table 3 is a significance table for the correlation coefficient r. For each significance level α ($P = 1-\alpha = 0.95, 0.975, 0.99, 0.995$), it indicates the minimum sample size n for which any r value departs significantly from zero. This table is based on a logical inversion of the t test appropriate for this situation.

Illustration 1. Find $t_{14[.95]}$. In table 1, at line $\nu = 14$ and under $P = 0.95$, we read $t = 1.761$. Likewise, for $t_{9[.05]}$, we use $-t_{9[.95]} = -1.833$.

Illutration 2. With 72 df, what are the (two) critical values of Student's t for a two-tailed 1 % test? The total extreme probability of 1 %, assigned equally to the right and left, leads to probability grades 0.005 (on the left) and 0.995 (on the right). In table 1, under $\nu = 72$ and $P = 0.995$, we obtain 2.646, so the searched-for values are -2.646 and 2.646.

Illustration 3. Find $t_{DS}(20[.95];4)$, *i.e.* the appropriate t value under the Dunn-Šidák criterion, for $\nu = 20$ df, a one-tailed 5 % threshold and $nc = 4$ comparisons. In table 2, on the page prepared for one-tailed 5 % tests, line $\nu = 20$ and column $nc = 4$ point to $t_{DS} = 2.414$.

Illustration 4. Find $n(r = \pm 0.30)$ for $P = 0.975$ (or, equivalently, for a two-tailed 5 % test). In table 3, at line $r = 0.30$ and under $P = 0.975$, we read $n = 44$. Hence, any coefficient $r \geq 0.30$ or $r \leq -0.30$ differs significantly from zero (*i.e.* is significant) at the 5 % (bilateral) level if $n \geq 44$.

Full examples

Example 1. With a sample of 40 children in first grade of primary school, we measured an average weight ($\bar{x}$) of 19.89 kg, with a s.d. of 4.74. Using a confidence coefficient

$c = 0.95$, find the interval inside which the true mean weight (μ_X) of children in this category should lie. *Solution*: The confidence interval for a mean, or average, is given by:

$$\Pr\{\, \bar{x}+t_{n-1[(1-c)/2]}s/\sqrt{n} < \mu_X < \bar{x}+t_{n-1[(1+c)/2]}s/\sqrt{n} \,\} = c \ .$$

Here, $c = 0.95$, $n = 40$, $\bar{x} = 19.89$, $s = 4.74$. In table 1, we find $t_{39[.975]} = 2.023$, hence $t_{39[.025]} = -2.023$ by symmetry. These data allow us to calculate and state that:

$$\Pr\{\ 18{,}37 < \mu_X < 21{,}41\ \} \approx 0.95.$$

Example 2. Thirty-eight patients suffering a phobia to dogs were randomly assigned either to therapy 1 or 2. After four individual sessions of one hour, a questionnaire measuring the respondent's "bravery toward dogs" was administered to each patient. A summary of results follows:

Therapy 1	$n_1 = 20$	$\bar{x}_1 = 3.49$	$s_1 = 0.96$;
Therapy 2	$n_2 = 18$	$\bar{x}_2 = 2.04$	$s_2 = 0.93$.

May we conclude that one therapy is more effective than the other? *Solution*: This simple research paradigm and the associated t-test procedure are among the most currently encountered in statistical practice. Under appropriate conditions (*i.e.* independent samples from a unique normally distributed population, homogeneity of variance estimates), the t test for the difference of means between two samples is defined by:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} ,$$

and is distributed as Student's t with $n_1 + n_2 - 2$ *df*. Selecting a 5 % significance level with a two-tailed test (whence $P = 0.025$ and 0.975) and with $df = 20 + 18 - 2 = 36$, table 1 furnishes $t = \pm 2.028$. Calculating the above formula with our data, we get $t \approx 1.45 / \sqrt{(0.8948 \times 0.10556)} \approx 1.45 / 0.30733 \approx 4.718$. As the calculated t test exceeds the critical value, we reject the hypothesis of the equality of population means. Therefore, therapy 1 appears more effective, at least in the short term, than therapy 2.

Example 3. A researcher in experimental biology allocates 50 laboratory rats into 5 groups of 10, in a random manner. In his research protocol, he compares three experimental diets: E1, E2 and E3, and two control diets, C1 and C2. The body weight — the dependent variable in this study — is measured in grams (g) at the end of the experiment. Here is a summary of results:

Diet group	n	$\bar{x}$	s
E1	10	352.7	32.14
E2	10	318.2	29.56
E3	10	336.4	33.37
C1	10	308.5	28.61
C2	10	273.4	30.19

The researcher wishes to determine, using a 1 % significance threshold, whether each experimental diet has benefitted the rats' weight in contrast with each control diet. *Solution*: The researcher's questioning design corresponds to a set of 6 planned comparisons (and 6 one-tailed tests), each of type H_1: $\mu(E_j) > \mu(C_r)$, $j = 1,2,3$ and $r = 1,2$; the criterion and method of Dunn-Šidák are appropriate here. The t test for this situation is:

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} .$$

The (error) variance estimate, $\hat{\sigma}^2$, is given here by the Within-group mean square (MS_{Within}) of a one-way analysis of variance, here $MS_{Within} = 950.06$ (calculted as a weighted average of each group's s^2, with weights equal to $n_j - 1$), with $\nu = 5\times9 = 45$ *df.* Table 2 furnishes, for one-tailed tests, $\alpha = 0.01$ and $nc = 6$ comparisons, the two values $t_{DS}(44[0.99];6) = 3.102$ and $t_{DS}(46[0.99];6) = 3.095$, whence, by interpolation for $\nu = 45$, $t(45) \approx 3.099$. To be significant, the t test must exceed 3.099; therefore, each difference $\bar{x}(E_j) - \bar{x}(C_r)$ must exceed $2.099\times\sqrt{[950.06\times(1/10+1/10)]} \approx 42.72$. Computations show that only differences (E1 − C1), (E1 − C2) and (E3 − C2) are significant at the prescribed level.

Example 4. The (linear) correlation between the height of 20 women and that of their consort has been calculated at $r = 0.54$. With this value, can we infer that there exists some relation between the statures of partners in human couples in the whole population, a non-zero correlation (using a 5 %, two-tailed test)? *Solution*: In table 3, under column $P = 0.975$, we find $n(0.54) = 14$, this being the minimum sample size for our correlation to be significant. As our sample is of size 20, we conclude that the obtained correlation is statistically significant, hence that the members of human couples tend moderately to be akin in stature.

Mathematical presentation

Student's t variable is basically the quotient of a random normal $N(\mu,\sigma^2)$ variable divided by an estimate s of σ having ν *df*, that estimate itself emanating from the same statistical population[1]. As the following equality shows, the t formula compensates for our ignorance of the parametric value of σ^2 :

$$t_\nu = \frac{x-\mu}{s_\nu} = \frac{(x-\mu)/\sigma}{\sqrt{\frac{\nu \cdot s_\nu^2}{\sigma^2}/\nu}} = \frac{z}{\sqrt{\chi_\nu^2/\nu}} .$$

We see, in addition, that the t variable can be manufactured, or generated, by dividing a standard normal r.v. (z) with the square root of a χ_ν^2 r.v., itself divided by its parameter ν.

The most popular applications of Student's t distribution concern the sampling distribution of the mean $\bar{x}$, the distribution of a difference $(\bar{x}_1 - \bar{x}_2)$ of two independent or two paired means, and the significance of a correlation coefficient. The distribution was discovered by W. S. Gosset (alias *Student*), and R. A. Fisher established it mathematically. Fisher also established the kinship of Student's t with correlation coefficient r in a bivariate normal population where $\rho = 0$, and with the χ^2 and F distributions. Fisher promoted the use of t tests and other small-sample tests in fundamental and applied research.

Calculation and moments

The p.d.f. of t having ν *df* is:

$$p(t_\nu) = K_\nu(1+t^2/\nu)^{-(\nu+1)/2}$$

where:

$$K_\nu = \Gamma[(\nu+1)/2]/\{\sqrt{(\nu\pi)}\Gamma(\nu/2)\} .$$

[1] This last clause is not mandatory. It is stated here as it evokes the habitual context in which most t tests are applied, a context wherein some sample mean is divided by the square root of the same sample's variance estimate. The reader is reminded that, in samples from a normal distribution, the sample estimates $\bar{X}$ and s^2 are independently distributed.

The distribution has one mode, is symmetrical about zero and is somewhat wider than the normal density, tending progressively to it as $\nu \to \infty$. $\Gamma(x)$ indicates the *Gamma* function.

Integration of the above p.d.f. is not simple, that being the case for most statistical distributions. The binomial expansion of $(1 + t^2/\nu)^{-r}$, followed by term-by-term integration on t, results in the converging power series:

$$P(t_\nu) = \tfrac{1}{2} + \mathrm{K}_\nu t \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(2i+1)} \left(\frac{\nu+1}{2}\right)^{(i)} \left(\frac{t^2}{\nu}\right)^i ,$$

where $x^{(i)}$ denotes the ascending factorial function. Gosset, using the change of variables $\theta = \tan^{-1}(t/\sqrt{\nu})$, developped the following series, one for even ν:

$$P(t_\nu) = \frac{1}{2} + \frac{\sin\theta}{2}\left\{1 + \sum_{i=1}^{\frac{1}{2}(\nu-2)} (\cos\theta)^{2i} \prod_{j=1}^{i} \frac{2j-1}{2j}\right\},$$

the other for odd ν:

$$P(t_\nu) = \frac{1}{2} + \frac{1}{\pi}\left\{\theta + \sin\theta\cos\theta\left[1 + \sum_{i=1}^{\frac{1}{2}(\nu-3)} (\cos\theta)^{2i} \prod_{j=1}^{i} \frac{2j}{2j+1}\right]\right\};$$

both series become cluttered for large ν values. For such cases, the preceding binomial expansion seems preferable, or else, the approximation proposed by Fisher:

$$P(t_\nu) \approx \Phi(t) + \varphi(t)\cdot t\,[\mathrm{A}/(4\nu) + \mathrm{B}/(96\nu^2) + \mathrm{C}/(384\nu^3) + \mathrm{D}/(92160\nu^4) + \cdots],$$

where Φ and φ are respectively the d.f. and the p.d.f. of the standard normal distribution, and $\mathrm{A} = t^2+1$, $\mathrm{B} = 3t^6 - 7t^4 - 5t^2 - 3$, $\mathrm{C} = t^{10} - 11t^8 + 14t^6 + 6t^4 - 3t^2 - 15$, $\mathrm{D} = 15t^{14} - 375t^{12} + 2225t^{10} - 2141t^8 - 939t^6 - 213t^4 + 915t^2 + 945$. Many other computational algorithms are available.

The t variable with $\nu = 1$ has p.d.f. $1/[\pi(1+t^2)]$ and coincides with a standard Cauchy variable. This variable, which also corresponds to the quotient of two standard normal r.v.'s ($= z_1/z_2$), has no expectation, no variance, nor moments of any order. The d.f. is simply $P(t_1) = \tfrac{1}{2} + (\tan^{-1}t)/\pi$, whence, by inversion, we may obtain the percentage point $t_1[P] = \tan\{\pi(P-\tfrac{1}{2})\}$.

For $\nu = 2$, through Gosset's trigonometric series, we have $P(t_2) = \tfrac{1}{2} + \tfrac{1}{2}t/\sqrt{(t^2+2)}$, and $t_2[P] = (2P-1)/\sqrt{[2P(1-P)]}$.

Note, finally, the kinship of t_ν and $F_{1,\nu}$, from which we have, for instance, the correspondance:

$$t_{\nu[P]} = +\sqrt{F_{1,\nu[2P-1]}} \ \{ P > ½ \} \ ,$$

and also $t_\nu \to z \sim N(0,1)$ as $\nu \to \infty$, reflecting the trend of t toward the normal law. These diverse relations may help to interpolate in a table of percentage points of t, when necessary.

Moments. Thanks to the symmetry of its p.d.f. centered at zero, the expectation, median and mode of t_ν are all zero, as is the skewness index, γ_1. The variance (σ^2) equals $\nu/(\nu-2)$, and the kurtosis index (γ_2) is $6/(\nu-4)$, reflecting a leptokurtic, somewhat narrow, center. Note that the moment of order r ($r = 1$ for μ, 2 for σ^2, etc.) is defined only if $\nu > r$.

With only the first two moments of t_ν, we can approximate a t centile using the rough equivalence: $t_{\nu[P]} \approx z_{[P]}\sigma_\nu$, $z_{[P]}$ being the standard normal P-centile, and $\sigma_\nu = \sqrt{[\nu/(\nu-2)]}$. The literature offers a better approximation:

$$t_{\nu[P]} \approx z_{[P]}\left(1 + \frac{z_{[P]}^2 + 1}{4\nu - 4}\right),$$

which renders centile values with a ±0.01 precision as soon as $\nu \geq 7$ for $P \leq 0.975$, and $\nu \geq 21$ for $P \leq 0.995$.

The Dunn-Šidák criterion (see table 2)

The t distribution is also useful in the realm of analysis of variance (ANOVA), for doing multiple comparisons of means associated with a factor or dimension of ANOVA. In some cases, a few comparisons of type "Mean *vs.* Mean" are planned (before obtaining the data or independently of it). A standard statistical practice consists in establishing a global error rate α for all nc comparisons or tests, and in performing each test using an effective $\alpha_C = \alpha/nc$ significance level: this is the so-called Bonferroni criterion. Now, according to a theorem by Z. Šidák:

$$\alpha_{(\text{global})} \leq 1 - (1-\alpha_C)^{nc} ,$$

whether the nc comparisons be statistically independent or not. O. J. Dunn then proposes, for each comparison, to calculate a t test and apply, for the significance criterion, an individual significance level α_C obtained with:

$$\alpha_C = 1 - \sqrt[nc]{(1-\alpha)},$$

this being slightly more powerful than Bonferroni's (α/nc) value. Table 2 was built to meet the requirements of such a procedure, with parameters ν, α and nc: the values given are simply $t_{\nu[1-\alpha_c]}$ or $t_{\nu[1-\alpha_c/2]}$, for one-tailed or two-tailed tests respectively.

The significance or r (see table 3)

The distribution of the correlation coefficient r for samples from a bivariate normal population having correlation $\rho = 0$ has been discovered by Fisher, who linked it with Student's t through the transformation:

$$\frac{r}{\sqrt{\dfrac{1-r^2}{n-2}}} \rightarrow t_{n-2}$$

This direct correspondance between Student's t and correlation coefficient r allowed us to set up an exhaustive significance table of r, in table 3. The table indicates the sample size n (= ν+2) sufficient for a given r value or a larger one to be significant (*i.e.* significantly different from zero) at the prescribed level. Note the approximate relation: $n_{[P]} \approx t^2_{n[P]}/r^2$; linear interpolation on r and $1/\sqrt{n}$ can also be used to determine the sufficient size n for intermediate r values.

Generation of pseudo random variates

The following program outline permits the generation of r.v.'s from Student's $t(\nu)$ distribution using a function "UNIF" that produces serially independent uniform $U(0,1)$ r.v.'s. Some particular cases are given in Remark 2.

Preparation : Let $n \equiv \nu$ (the degrees of freedom, *df*)

$\mathrm{C} = -2/n$

Production : **Repeat** $t \leftarrow 2\times\mathrm{UNIF}-1$; $u \leftarrow 2\times\mathrm{UNIF}-1$; $r \leftarrow t^2 + u^2$
Until $r < 1$;
Return $t \times \sqrt{[n \times (r^{\mathrm{C}}-1)/r]} \rightarrow x$.

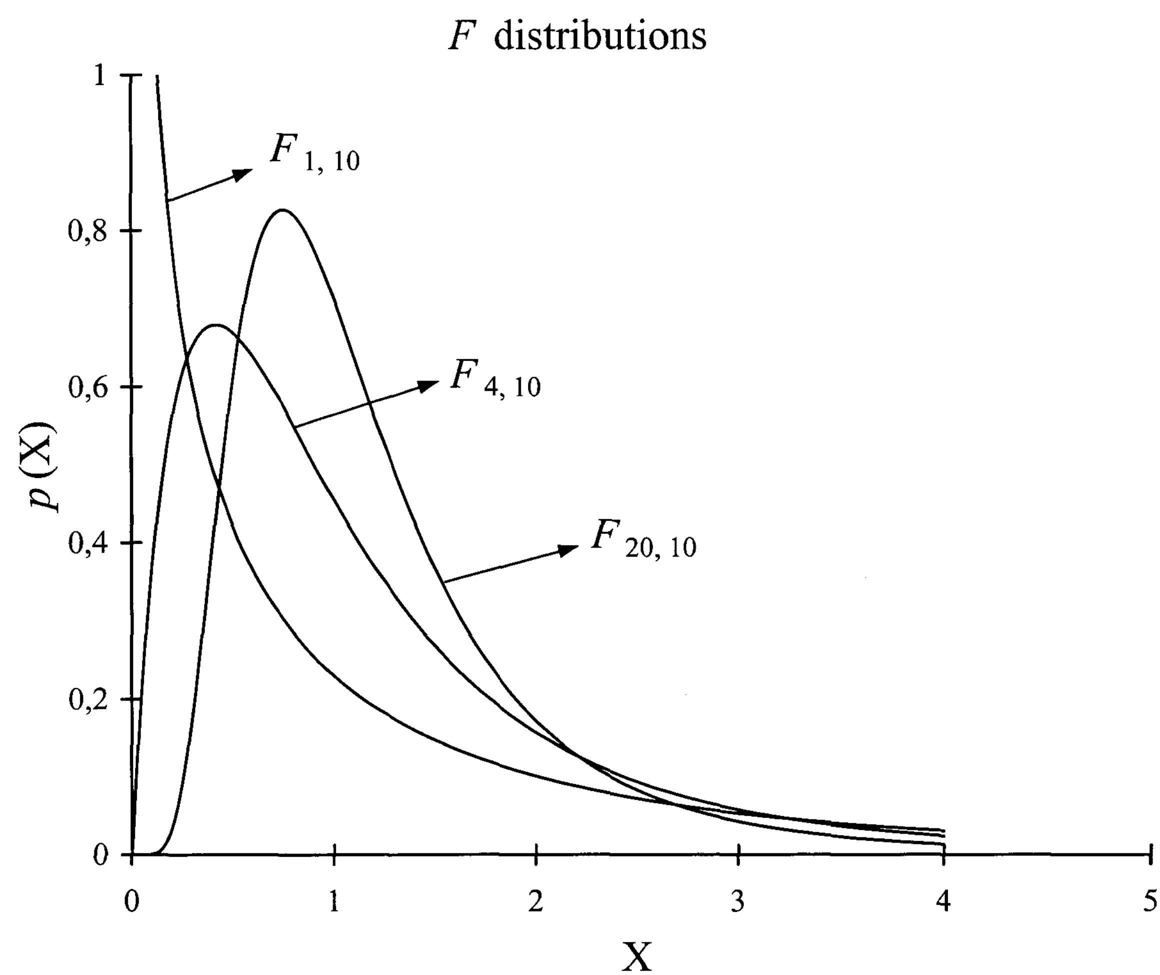
Remarks :

1. Standard temporal cost : 9.9 × t(UNIF)

2. The above method is attributed to R. W. Bailey (1994, in Gentle 1998) and is valid for any ν (= n) value. The generation of one x variate requires an expected number of $8/\pi \approx 2.55$ calls to the UNIF generator. When $\nu = 1$ or 2, one may also exploit the simple inversion formulae given earlier, while substituting $u \sim$ UNIF instead of the required P value.

F distribution

- ✓ Graphical representations
- ✓ Percentiles 95, 97.5, 99 and 99.5 of the *F* distribution
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation
  - Calculation and moments
  - Relationship between *F* and binomial distributions
  - Generation of pseudo random variates



F distributions
$F_{1, 10}$
$F_{4, 10}$
$F_{20, 10}$
p(X)
X
0
0,2
0,4
0,6
0,8
1
0
1
2
3
4
5

Percentile 95 of the *F* distribution

$\nu_2 \backslash \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.0	243.9
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40	19.41
3	10.13	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.763	8.745
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.936	5.912
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.704	4.678
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	4.027	4.000
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.603	3.575
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.313	3.284
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.102	3.073
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.943	2.913
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.818	2.788
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.717	2.687
13	4.667	3.806	3.411	3.179	3.026	2.915	2.832	2.767	2.714	2.671	2.635	2.604
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.565	2.534
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.507	2.475
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.456	2.425
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.413	2.381
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.374	2.342
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.340	2.308
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.310	2.278
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.259	2.226
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.216	2.183
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.265	2.220	2.181	2.148
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190	2.151	2.118
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.126	2.092
32	4.149	3.295	2.901	2.669	2.512	2.399	2.313	2.244	2.189	2.142	2.103	2.070
34	4.130	3.276	2.883	2.650	2.494	2.380	2.294	2.225	2.170	2.123	2.084	2.050
36	4.113	3.259	2.866	2.634	2.477	2.364	2.277	2.209	2.153	2.106	2.067	2.033
38	4.098	3.245	2.852	2.619	2.463	2.349	2.262	2.194	2.138	2.091	2.051	2.017
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	2.038	2.003
42	4.073	3.220	2.827	2.594	2.438	2.324	2.237	2.168	2.112	2.065	2.025	1.991
44	4.062	3.209	2.816	2.584	2.427	2.313	2.226	2.157	2.101	2.054	2.014	1.980
46	4.052	3.200	2.807	2.574	2.417	2.304	2.216	2.147	2.091	2.044	2.004	1.970
48	4.043	3.191	2.798	2.565	2.409	2.295	2.207	2.138	2.082	2.035	1.995	1.960
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026	1.986	1.952
55	4.016	3.165	2.773	2.540	2.383	2.269	2.181	2.112	2.055	2.008	1.968	1.933
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.952	1.917
65	3.989	3.138	2.746	2.513	2.356	2.242	2.154	2.084	2.027	1.980	1.939	1.904
70	3.978	3.128	2.736	2.503	2.346	2.231	2.143	2.074	2.017	1.969	1.928	1.893
75	3.968	3.119	2.727	2.494	2.337	2.222	2.134	2.064	2.007	1.959	1.919	1.884
80	3.960	3.111	2.719	2.486	2.329	2.214	2.126	2.056	1.999	1.951	1.910	1.875
85	3.953	3.104	2.712	2.479	2.322	2.207	2.119	2.049	1.992	1.944	1.903	1.868
90	3.947	3.098	2.706	2.473	2.316	2.201	2.113	2.043	1.986	1.938	1.897	1.861
95	3.941	3.092	2.700	2.468	2.310	2.196	2.107	2.037	1.980	1.932	1.891	1.856
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927	1.886	1.850
125	3.917	3.069	2.677	2.444	2.287	2.172	2.084	2.013	1.956	1.907	1.866	1.830
150	3.904	3.056	2.665	2.432	2.274	2.160	2.071	2.001	1.943	1.894	1.853	1.817
175	3.895	3.048	2.656	2.423	2.266	2.151	2.062	1.992	1.934	1.885	1.844	1.808
200	3.888	3.041	2.650	2.417	2.259	2.144	2.056	1.985	1.927	1.878	1.837	1.801
250	3.879	3.032	2.641	2.408	2.250	2.135	2.046	1.976	1.917	1.869	1.827	1.791
500	3.860	3.014	2.623	2.390	2.232	2.117	2.028	1.957	1.899	1.850	1.808	1.772
∞	3.841	2.996	2.605	2.372	2.214	2.099	2.010	1.938	1.880	1.831	1.789	1.752

Percentile 95 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	244.7	245.4	245.9	246.5	247.3	248.0	249.3	250.1	251.1	251.8	253.0	254.3
2	19.42	19.42	19.43	19.43	19.44	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	8.729	8.715	8.703	8.694	8.675	8.660	8.634	8.617	8.594	8.581	8.554	8.526
4	5.891	5.873	5.858	5.844	5.821	5.803	5.769	5.746	5.717	5.699	5.664	5.628
5	4.655	4.636	4.619	4.604	4.579	4.558	4.521	4.496	4.464	4.444	4.405	4.365
6	3.976	3.956	3.938	3.922	3.896	3.874	3.835	3.808	3.774	3.754	3.712	3.669
7	3.550	3.529	3.511	3.494	3.467	3.445	3.404	3.376	3.340	3.319	3.275	3.230
8	3.259	3.237	3.218	3.202	3.173	3.150	3.108	3.079	3.043	3.021	2.975	2.928
9	3.048	3.025	3.006	2.989	2.960	2.936	2.893	2.864	2.826	2.803	2.756	2.707
10	2.887	2.865	2.845	2.828	2.798	2.774	2.730	2.700	2.661	2.637	2.588	2.538
11	2.761	2.739	2.719	2.701	2.671	2.646	2.601	2.570	2.531	2.507	2.457	2.404
12	2.660	2.637	2.617	2.599	2.568	2.544	2.498	2.466	2.426	2.401	2.350	2.296
13	2.577	2.554	2.533	2.515	2.484	2.459	2.412	2.380	2.339	2.314	2.261	2.206
14	2.507	2.484	2.463	2.445	2.413	2.388	2.341	2.308	2.266	2.241	2.187	2.131
15	2.448	2.424	2.403	2.385	2.353	2.328	2.280	2.247	2.204	2.178	2.123	2.066
16	2.397	2.373	2.352	2.333	2.302	2.276	2.227	2.194	2.151	2.124	2.068	2.010
17	2.353	2.329	2.308	2.289	2.257	2.230	2.181	2.148	2.104	2.077	2.020	1.960
18	2.314	2.290	2.269	2.250	2.217	2.191	2.141	2.107	2.063	2.035	1.978	1.917
19	2.280	2.256	2.234	2.215	2.182	2.155	2.106	2.071	2.026	1.999	1.940	1.878
20	2.250	2.225	2.203	2.184	2.151	2.124	2.074	2.039	1.994	1.966	1.907	1.843
22	2.198	2.173	2.151	2.131	2.098	2.071	2.020	1.984	1.938	1.909	1.849	1.783
24	2.155	2.130	2.108	2.088	2.054	2.027	1.975	1.939	1.892	1.863	1.800	1.733
26	2.119	2.094	2.072	2.052	2.018	1.990	1.938	1.901	1.853	1.823	1.760	1.691
28	2.089	2.064	2.041	2.021	1.987	1.959	1.906	1.869	1.820	1.790	1.725	1.654
30	2.063	2.037	2.015	1.995	1.960	1.932	1.878	1.841	1.792	1.761	1.695	1.622
32	2.040	2.015	1.992	1.972	1.937	1.908	1.854	1.817	1.767	1.736	1.669	1.594
34	2.021	1.995	1.972	1.952	1.917	1.888	1.833	1.795	1.745	1.713	1.645	1.569
36	2.003	1.977	1.954	1.934	1.899	1.870	1.815	1.776	1.726	1.694	1.625	1.547
38	1.988	1.962	1.939	1.918	1.883	1.853	1.798	1.760	1.708	1.676	1.606	1.527
40	1.974	1.948	1.924	1.904	1.868	1.839	1.783	1.744	1.693	1.660	1.589	1.509
42	1.961	1.935	1.912	1.891	1.855	1.826	1.770	1.731	1.679	1.646	1.574	1.492
44	1.950	1.924	1.900	1.879	1.844	1.814	1.758	1.718	1.666	1.633	1.560	1.477
46	1.940	1.913	1.890	1.869	1.833	1.803	1.747	1.707	1.654	1.621	1.547	1.463
48	1.930	1.904	1.880	1.859	1.823	1.793	1.737	1.697	1.644	1.610	1.536	1.450
50	1.921	1.895	1.871	1.850	1.814	1.784	1.727	1.687	1.634	1.599	1.525	1.438
55	1.903	1.876	1.852	1.831	1.795	1.764	1.707	1.666	1.612	1.577	1.501	1.412
60	1.887	1.860	1.836	1.815	1.778	1.748	1.690	1.649	1.594	1.559	1.481	1.389
65	1.874	1.847	1.823	1.802	1.765	1.734	1.676	1.635	1.579	1.543	1.464	1.370
70	1.863	1.836	1.812	1.790	1.753	1.722	1.664	1.622	1.566	1.530	1.450	1.353
75	1.853	1.826	1.802	1.780	1.743	1.712	1.653	1.611	1.555	1.518	1.437	1.338
80	1.845	1.817	1.793	1.772	1.734	1.703	1.644	1.602	1.545	1.508	1.426	1.325
85	1.837	1.810	1.786	1.764	1.726	1.695	1.636	1.593	1.536	1.499	1.416	1.313
90	1.830	1.803	1.779	1.757	1.720	1.688	1.629	1.586	1.528	1.491	1.407	1.302
95	1.825	1.797	1.773	1.751	1.713	1.682	1.622	1.579	1.521	1.484	1.399	1.292
100	1.819	1.792	1.768	1.746	1.708	1.676	1.616	1.573	1.515	1.477	1.392	1.283
125	1.799	1.772	1.747	1.725	1.687	1.655	1.594	1.551	1.491	1.452	1.364	1.248
150	1.786	1.758	1.734	1.711	1.673	1.641	1.580	1.535	1.475	1.436	1.345	1.223
175	1.776	1.749	1.724	1.702	1.663	1.631	1.569	1.525	1.464	1.424	1.331	1.204
200	1.769	1.742	1.717	1.694	1.656	1.623	1.561	1.516	1.455	1.415	1.321	1.189
250	1.759	1.732	1.707	1.684	1.645	1.613	1.550	1.505	1.443	1.402	1.306	1.166
500	1.740	1.712	1.686	1.664	1.625	1.592	1.528	1.482	1.419	1.376	1.275	1.113
∞	1.720	1.692	1.666	1.644	1.604	1.571	1.506	1.459	1.394	1.350	1.243	1.000

Percentile 97.5 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	973.0	976.7
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.41
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.37	14.34
4	12.22	10.65	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.794	8.751
5	10.01	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.568	6.525
6	8.813	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.410	5.366
7	8.073	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.709	4.666
8	7.571	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.243	4.200
9	7.209	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.912	3.868
10	6.937	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.665	3.621
11	6.724	5.256	4.630	4.275	4.044	3.881	3.759	3.664	3.588	3.526	3.474	3.430
12	6.554	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.321	3.277
13	6.414	4.965	4.347	3.996	3.767	3.604	3.483	3.388	3.312	3.250	3.197	3.153
14	6.298	4.857	4.242	3.892	3.663	3.501	3.380	3.285	3.209	3.147	3.095	3.050
15	6.200	4.765	4.153	3.804	3.577	3.415	3.293	3.199	3.123	3.060	3.008	2.963
16	6.115	4.687	4.077	3.729	3.502	3.341	3.219	3.125	3.049	2.986	2.934	2.889
17	6.042	4.619	4.011	3.665	3.438	3.277	3.156	3.061	2.985	2.922	2.870	2.825
18	5.978	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.814	2.769
19	5.922	4.508	3.903	3.559	3.333	3.172	3.051	2.956	2.880	2.817	2.765	2.720
20	5.871	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.721	2.676
22	5.786	4.383	3.783	3.440	3.215	3.055	2.934	2.839	2.763	2.700	2.647	2.602
24	5.717	4.319	3.721	3.379	3.155	2.995	2.874	2.779	2.703	2.640	2.586	2.541
26	5.659	4.265	3.670	3.329	3.105	2.945	2.824	2.729	2.653	2.590	2.536	2.491
28	5.610	4.221	3.626	3.286	3.063	2.903	2.782	2.687	2.611	2.547	2.494	2.448
30	5.568	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.458	2.412
32	5.531	4.149	3.557	3.218	2.995	2.836	2.715	2.620	2.543	2.480	2.426	2.381
34	5.499	4.120	3.529	3.191	2.968	2.808	2.688	2.593	2.516	2.453	2.399	2.353
36	5.471	4.094	3.505	3.167	2.944	2.785	2.664	2.569	2.492	2.429	2.375	2.329
38	5.446	4.071	3.483	3.145	2.923	2.763	2.643	2.548	2.471	2.407	2.353	2.307
40	5.424	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.334	2.288
42	5.404	4.033	3.446	3.109	2.887	2.727	2.607	2.512	2.435	2.371	2.317	2.271
44	5.386	4.016	3.430	3.093	2.871	2.712	2.591	2.496	2.419	2.355	2.302	2.255
46	5.369	4.001	3.415	3.079	2.857	2.698	2.577	2.482	2.405	2.341	2.287	2.241
48	5.354	3.987	3.402	3.066	2.844	2.685	2.565	2.470	2.393	2.329	2.274	2.228
50	5.340	3.975	3.390	3.054	2.833	2.674	2.553	2.458	2.381	2.317	2.263	2.216
55	5.310	3.948	3.364	3.029	2.807	2.648	2.528	2.433	2.355	2.291	2.237	2.190
60	5.286	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.216	2.169
65	5.265	3.906	3.324	2.990	2.769	2.610	2.489	2.394	2.317	2.252	2.198	2.151
70	5.247	3.890	3.309	2.975	2.754	2.595	2.474	2.379	2.302	2.237	2.183	2.136
75	5.232	3.876	3.296	2.962	2.741	2.582	2.461	2.366	2.289	2.224	2.170	2.123
80	5.218	3.864	3.284	2.950	2.730	2.571	2.450	2.355	2.277	2.213	2.158	2.111
85	5.207	3.854	3.274	2.940	2.720	2.561	2.440	2.345	2.268	2.203	2.148	2.101
90	5.196	3.844	3.265	2.932	2.711	2.552	2.432	2.336	2.259	2.194	2.140	2.092
95	5.187	3.836	3.257	2.924	2.703	2.544	2.424	2.328	2.251	2.186	2.132	2.084
100	5.179	3.828	3.250	2.917	2.696	2.537	2.417	2.321	2.244	2.179	2.124	2.077
125	5.147	3.800	3.222	2.890	2.670	2.511	2.390	2.295	2.217	2.153	2.098	2.050
150	5.126	3.781	3.204	2.872	2.652	2.494	2.373	2.278	2.200	2.135	2.080	2.032
175	5.111	3.768	3.192	2.860	2.640	2.481	2.361	2.265	2.187	2.122	2.067	2.020
200	5.100	3.758	3.182	2.850	2.630	2.472	2.351	2.256	2.178	2.113	2.058	2.010
250	5.085	3.744	3.169	2.837	2.618	2.459	2.338	2.243	2.165	2.100	2.045	1.997
500	5.054	3.716	3.142	2.811	2.592	2.434	2.313	2.217	2.139	2.074	2.019	1.971
∞	5.024	3.689	3.116	2.786	2.567	2.408	2.288	2.192	2.114	2.048	1.993	1.945

Percentile 97.5 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	979.8	982.5	984.9	986.9	990.4	993.1	998.1	1001	1006	1008	1013	1018
2	39.42	39.43	39.43	39.44	39.44	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	14.30	14.28	14.25	14.23	14.20	14.17	14.12	14.08	14.04	14.01	13.96	13.90
4	8.715	8.684	8.657	8.633	8.592	8.560	8.501	8.461	8.411	8.381	8.319	8.257
5	6.488	6.456	6.428	6.403	6.362	6.329	6.268	6.227	6.175	6.144	6.080	6.015
6	5.329	5.297	5.269	5.244	5.202	5.168	5.107	5.065	5.012	4.980	4.915	4.849
7	4.628	4.596	4.568	4.543	4.501	4.467	4.405	4.362	4.309	4.276	4.210	4.142
8	4.162	4.130	4.101	4.076	4.034	3.999	3.937	3.894	3.840	3.807	3.739	3.670
9	3.831	3.798	3.769	3.744	3.701	3.667	3.604	3.560	3.505	3.472	3.403	3.333
10	3.583	3.550	3.522	3.496	3.453	3.419	3.355	3.311	3.255	3.221	3.152	3.080
11	3.392	3.359	3.330	3.304	3.261	3.226	3.162	3.118	3.061	3.027	2.956	2.883
12	3.239	3.206	3.177	3.152	3.108	3.073	3.008	2.963	2.906	2.871	2.800	2.725
13	3.115	3.082	3.053	3.027	2.983	2.948	2.882	2.837	2.780	2.744	2.671	2.595
14	3.012	2.979	2.949	2.923	2.879	2.844	2.778	2.732	2.674	2.638	2.565	2.487
15	2.925	2.891	2.862	2.836	2.792	2.756	2.689	2.644	2.585	2.549	2.474	2.395
16	2.851	2.817	2.788	2.761	2.717	2.681	2.614	2.568	2.509	2.472	2.396	2.316
17	2.786	2.753	2.723	2.697	2.652	2.616	2.548	2.502	2.442	2.405	2.329	2.247
18	2.730	2.696	2.667	2.640	2.596	2.559	2.491	2.445	2.384	2.347	2.269	2.187
19	2.681	2.647	2.617	2.591	2.546	2.509	2.441	2.394	2.333	2.295	2.217	2.133
20	2.637	2.603	2.573	2.547	2.501	2.464	2.396	2.349	2.287	2.249	2.170	2.085
22	2.563	2.528	2.498	2.472	2.426	2.389	2.320	2.272	2.210	2.171	2.090	2.003
24	2.502	2.468	2.437	2.411	2.365	2.327	2.257	2.209	2.146	2.107	2.024	1.935
26	2.451	2.417	2.387	2.360	2.314	2.276	2.205	2.157	2.093	2.053	1.969	1.878
28	2.409	2.374	2.344	2.317	2.270	2.232	2.161	2.112	2.048	2.007	1.922	1.829
30	2.372	2.338	2.307	2.280	2.233	2.195	2.124	2.074	2.009	1.968	1.882	1.787
32	2.341	2.306	2.275	2.248	2.201	2.163	2.091	2.041	1.975	1.934	1.846	1.750
34	2.313	2.278	2.248	2.220	2.173	2.135	2.062	2.012	1.946	1.904	1.815	1.717
36	2.289	2.254	2.223	2.196	2.148	2.110	2.037	1.986	1.919	1.877	1.787	1.687
38	2.267	2.232	2.201	2.174	2.126	2.088	2.015	1.963	1.896	1.854	1.763	1.661
40	2.248	2.213	2.182	2.154	2.107	2.068	1.994	1.943	1.875	1.832	1.741	1.637
42	2.231	2.196	2.164	2.137	2.089	2.050	1.976	1.924	1.856	1.813	1.720	1.615
44	2.215	2.180	2.149	2.121	2.073	2.034	1.960	1.908	1.839	1.796	1.702	1.596
46	2.201	2.165	2.134	2.106	2.058	2.019	1.945	1.893	1.824	1.780	1.685	1.577
48	2.188	2.152	2.121	2.093	2.045	2.006	1.931	1.879	1.809	1.765	1.670	1.561
50	2.176	2.140	2.109	2.081	2.033	1.993	1.919	1.866	1.796	1.752	1.656	1.545
55	2.150	2.114	2.083	2.055	2.006	1.967	1.891	1.838	1.768	1.723	1.625	1.511
60	2.129	2.093	2.061	2.033	1.985	1.944	1.869	1.815	1.744	1.699	1.599	1.482
65	2.111	2.075	2.043	2.015	1.966	1.926	1.850	1.796	1.724	1.678	1.577	1.457
70	2.095	2.059	2.028	1.999	1.950	1.910	1.833	1.779	1.707	1.660	1.558	1.436
75	2.082	2.046	2.014	1.986	1.937	1.896	1.819	1.765	1.692	1.645	1.542	1.417
80	2.071	2.035	2.003	1.974	1.925	1.884	1.807	1.752	1.679	1.632	1.527	1.400
85	2.060	2.024	1.992	1.964	1.915	1.874	1.796	1.741	1.668	1.620	1.514	1.385
90	2.051	2.015	1.983	1.955	1.905	1.864	1.787	1.731	1.657	1.610	1.503	1.371
95	2.043	2.007	1.975	1.946	1.897	1.856	1.778	1.723	1.648	1.600	1.493	1.359
100	2.036	2.000	1.968	1.939	1.890	1.849	1.770	1.715	1.640	1.592	1.483	1.347
125	2.009	1.973	1.940	1.911	1.862	1.820	1.741	1.685	1.609	1.559	1.448	1.303
150	1.991	1.955	1.922	1.893	1.843	1.801	1.722	1.665	1.588	1.538	1.423	1.271
175	1.978	1.942	1.909	1.880	1.830	1.788	1.708	1.651	1.573	1.522	1.406	1.248
200	1.969	1.932	1.900	1.870	1.820	1.778	1.698	1.640	1.562	1.511	1.393	1.229
250	1.955	1.919	1.886	1.857	1.806	1.764	1.683	1.625	1.546	1.495	1.374	1.201
500	1.929	1.892	1.859	1.830	1.779	1.736	1.655	1.596	1.515	1.462	1.336	1.137
∞	1.903	1.866	1.833	1.803	1.751	1.708	1.626	1.566	1.484	1.428	1.296	1.000

Percentile 99 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	4052	5000	5403	5625	5764	5859	5928	5981	6022	6056	6083	6106
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.41	99.42
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.13	27.05
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.45	14.37
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.963	9.888
6	13.75	10.92	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.790	7.718
7	12.25	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.538	6.469
8	11.26	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.734	5.667
9	10.56	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	5.178	5.111
10	10.04	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.772	4.706
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.462	4.397
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.220	4.155
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100	4.025	3.960
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939	3.864	3.800
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.730	3.666
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691	3.616	3.553
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.519	3.455
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.434	3.371
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434	3.360	3.297
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.294	3.231
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	3.184	3.121
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	3.094	3.032
26	7.721	5.526	4.637	4.140	3.818	3.591	3.421	3.288	3.182	3.094	3.020	2.958
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.120	3.032	2.959	2.896
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.906	2.843
32	7.499	5.336	4.459	3.969	3.652	3.427	3.258	3.127	3.021	2.934	2.860	2.798
34	7.444	5.289	4.416	3.927	3.611	3.386	3.218	3.087	2.981	2.894	2.821	2.758
36	7.396	5.248	4.377	3.890	3.574	3.351	3.183	3.052	2.946	2.859	2.786	2.723
38	7.353	5.211	4.343	3.858	3.542	3.319	3.152	3.021	2.915	2.828	2.755	2.692
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.727	2.665
42	7.280	5.149	4.285	3.802	3.488	3.266	3.099	2.968	2.863	2.776	2.703	2.640
44	7.248	5.123	4.261	3.778	3.465	3.243	3.076	2.946	2.841	2.754	2.680	2.618
46	7.220	5.099	4.238	3.757	3.444	3.222	3.056	2.925	2.820	2.733	2.660	2.598
48	7.194	5.077	4.218	3.737	3.425	3.204	3.037	2.907	2.802	2.715	2.642	2.579
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698	2.625	2.562
55	7.119	5.013	4.159	3.681	3.370	3.149	2.983	2.853	2.748	2.662	2.589	2.526
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.559	2.496
65	7.041	4.947	4.098	3.622	3.313	3.093	2.928	2.798	2.693	2.607	2.534	2.471
70	7.011	4.922	4.074	3.600	3.291	3.071	2.906	2.777	2.672	2.585	2.512	2.450
75	6.985	4.900	4.054	3.580	3.272	3.052	2.887	2.758	2.653	2.567	2.494	2.431
80	6.963	4.881	4.036	3.563	3.255	3.036	2.871	2.742	2.637	2.551	2.478	2.415
85	6.943	4.864	4.021	3.548	3.241	3.022	2.857	2.728	2.623	2.537	2.464	2.401
90	6.925	4.849	4.007	3.535	3.228	3.009	2.845	2.715	2.611	2.524	2.451	2.389
95	6.910	4.836	3.995	3.523	3.216	2.998	2.833	2.704	2.600	2.513	2.440	2.378
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503	2.430	2.368
125	6.842	4.779	3.942	3.473	3.167	2.950	2.785	2.657	2.552	2.466	2.393	2.330
150	6.807	4.749	3.915	3.447	3.142	2.924	2.761	2.632	2.528	2.441	2.368	2.305
175	6.782	4.728	3.895	3.428	3.123	2.906	2.743	2.614	2.510	2.424	2.351	2.288
200	6.763	4.713	3.881	3.414	3.110	2.893	2.730	2.601	2.497	2.411	2.337	2.275
250	6.737	4.691	3.861	3.395	3.091	2.875	2.711	2.583	2.479	2.393	2.319	2.257
500	6.686	4.648	3.821	3.357	3.054	2.838	2.675	2.547	2.443	2.356	2.283	2.220
∞	6.635	4.605	3.782	3.319	3.017	2.802	2.639	2.511	2.407	2.321	2.248	2.185

Percentile 99 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	6126	6143	6157	6170	6192	6209	6240	6261	6287	6303	6334	6366
2	99.42	99.43	99.43	99.44	99.44	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	26.98	26.92	26.87	26.83	26.75	26.69	26.58	26.50	26.41	26.35	26.24	26.13
4	14.31	14.25	14.20	14.15	14.08	14.02	13.91	13.84	13.75	13.69	13.58	13.46
5	9.825	9.770	9.722	9.680	9.610	9.553	9.449	9.379	9.291	9.238	9.130	9.020
6	7.657	7.605	7.559	7.519	7.451	7.396	7.296	7.229	7.143	7.091	6.987	6.880
7	6.410	6.359	6.314	6.275	6.209	6.155	6.058	5.992	5.908	5.858	5.755	5.650
8	5.609	5.559	5.515	5.477	5.412	5.359	5.263	5.198	5.116	5.065	4.963	4.859
9	5.055	5.005	4.962	4.924	4.860	4.808	4.713	4.649	4.567	4.517	4.415	4.311
10	4.650	4.601	4.558	4.520	4.457	4.405	4.311	4.247	4.165	4.115	4.014	3.909
11	4.342	4.293	4.251	4.213	4.150	4.099	4.005	3.941	3.860	3.810	3.708	3.602
12	4.100	4.052	4.010	3.972	3.909	3.858	3.765	3.701	3.619	3.569	3.467	3.361
13	3.905	3.857	3.815	3.778	3.716	3.665	3.571	3.507	3.425	3.375	3.272	3.165
14	3.745	3.698	3.656	3.619	3.556	3.505	3.412	3.348	3.266	3.215	3.112	3.004
15	3.612	3.564	3.522	3.485	3.423	3.372	3.278	3.214	3.132	3.081	2.977	2.868
16	3.498	3.451	3.409	3.372	3.310	3.259	3.165	3.101	3.018	2.967	2.863	2.753
17	3.401	3.353	3.312	3.275	3.212	3.162	3.068	3.003	2.920	2.869	2.764	2.653
18	3.316	3.269	3.227	3.190	3.128	3.077	2.983	2.919	2.835	2.784	2.678	2.566
19	3.242	3.195	3.153	3.117	3.054	3.003	2.909	2.844	2.761	2.709	2.602	2.489
20	3.177	3.130	3.088	3.051	2.989	2.938	2.843	2.778	2.695	2.643	2.535	2.421
22	3.067	3.019	2.978	2.941	2.879	2.827	2.733	2.667	2.583	2.531	2.422	2.305
24	2.977	2.930	2.889	2.852	2.789	2.738	2.643	2.577	2.492	2.440	2.329	2.211
26	2.904	2.857	2.815	2.778	2.715	2.664	2.569	2.503	2.417	2.364	2.252	2.131
28	2.842	2.795	2.753	2.716	2.653	2.602	2.506	2.440	2.353	2.300	2.187	2.064
30	2.789	2.742	2.700	2.663	2.600	2.549	2.453	2.386	2.299	2.245	2.131	2.006
32	2.744	2.696	2.655	2.618	2.555	2.503	2.406	2.340	2.252	2.198	2.082	1.956
34	2.704	2.657	2.615	2.578	2.515	2.463	2.366	2.299	2.211	2.156	2.040	1.911
36	2.669	2.622	2.580	2.543	2.480	2.428	2.331	2.263	2.175	2.120	2.002	1.872
38	2.638	2.591	2.549	2.512	2.449	2.397	2.299	2.232	2.143	2.087	1.968	1.837
40	2.611	2.563	2.522	2.484	2.421	2.369	2.271	2.203	2.114	2.058	1.938	1.805
42	2.586	2.539	2.497	2.460	2.396	2.344	2.246	2.178	2.088	2.032	1.911	1.776
44	2.564	2.516	2.475	2.437	2.374	2.321	2.223	2.155	2.065	2.008	1.887	1.750
46	2.544	2.496	2.454	2.417	2.353	2.301	2.203	2.134	2.044	1.987	1.864	1.726
48	2.525	2.478	2.436	2.399	2.335	2.282	2.184	2.115	2.024	1.967	1.844	1.703
50	2.508	2.461	2.419	2.382	2.318	2.265	2.167	2.098	2.007	1.949	1.825	1.683
55	2.472	2.424	2.382	2.345	2.281	2.228	2.129	2.060	1.968	1.910	1.784	1.638
60	2.442	2.394	2.352	2.315	2.251	2.198	2.098	2.028	1.936	1.877	1.749	1.601
65	2.417	2.369	2.327	2.289	2.225	2.172	2.072	2.002	1.909	1.850	1.720	1.568
70	2.395	2.348	2.306	2.268	2.204	2.150	2.050	1.980	1.886	1.826	1.695	1.541
75	2.377	2.329	2.287	2.249	2.185	2.132	2.031	1.960	1.866	1.806	1.674	1.516
80	2.361	2.313	2.271	2.233	2.169	2.115	2.015	1.944	1.849	1.788	1.655	1.495
85	2.347	2.299	2.257	2.219	2.154	2.101	2.000	1.929	1.834	1.773	1.638	1.475
90	2.334	2.286	2.244	2.206	2.142	2.088	1.987	1.916	1.820	1.759	1.623	1.458
95	2.323	2.275	2.233	2.195	2.130	2.077	1.976	1.904	1.808	1.746	1.610	1.442
100	2.313	2.265	2.223	2.185	2.120	2.067	1.965	1.893	1.797	1.735	1.598	1.428
125	2.276	2.228	2.185	2.147	2.082	2.028	1.926	1.853	1.756	1.693	1.551	1.371
150	2.251	2.203	2.160	2.122	2.057	2.003	1.900	1.827	1.729	1.665	1.520	1.331
175	2.233	2.185	2.143	2.105	2.039	1.985	1.882	1.808	1.709	1.645	1.498	1.302
200	2.220	2.172	2.129	2.091	2.026	1.971	1.868	1.794	1.694	1.629	1.481	1.279
250	2.202	2.154	2.111	2.073	2.007	1.953	1.849	1.774	1.674	1.608	1.457	1.244
500	2.166	2.118	2.075	2.036	1.970	1.915	1.810	1.735	1.633	1.566	1.408	1.164
∞	2.130	2.082	2.039	2.000	1.934	1.878	1.773	1.696	1.592	1.523	1.358	1.000

Percentile 99.5 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24225	24334	24426
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.52	43.39
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.82	20.70
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.49	13.38
6	18.64	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.13	10.03
7	16.24	12.40	10.88	10.05	9.522	9.155	8.885	8.678	8.514	8.380	8.270	8.176
8	14.69	11.04	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	7.104	7.015
9	13.61	10.11	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.314	6.227
10	12.83	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.746	5.661
11	12.23	8.912	7.600	6.881	6.422	6.102	5.865	5.682	5.537	5.418	5.320	5.236
12	11.75	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.988	4.906
13	11.37	8.186	6.926	6.233	5.791	5.482	5.253	5.076	4.935	4.820	4.724	4.643
14	11.06	7.922	6.680	5.998	5.562	5.257	5.031	4.857	4.717	4.603	4.508	4.428
15	10.80	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.329	4.250
16	10.58	7.514	6.303	5.638	5.212	4.913	4.692	4.521	4.384	4.272	4.179	4.099
17	10.38	7.354	6.156	5.497	5.075	4.779	4.559	4.389	4.254	4.142	4.050	3.971
18	10.22	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.938	3.860
19	10.07	7.093	5.916	5.268	4.853	4.561	4.345	4.177	4.043	3.933	3.841	3.763
20	9.944	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.756	3.678
22	9.727	6.806	5.652	5.017	4.609	4.322	4.109	3.944	3.812	3.703	3.612	3.535
24	9.551	6.661	5.519	4.890	4.486	4.202	3.991	3.826	3.695	3.587	3.497	3.420
26	9.406	6.541	5.409	4.785	4.384	4.103	3.893	3.730	3.599	3.492	3.402	3.325
28	9.284	6.440	5.317	4.698	4.300	4.020	3.811	3.649	3.519	3.412	3.322	3.246
30	9.180	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.255	3.179
32	9.090	6.281	5.171	4.559	4.166	3.889	3.682	3.521	3.392	3.286	3.197	3.121
34	9.012	6.217	5.113	4.504	4.112	3.836	3.630	3.470	3.341	3.235	3.146	3.071
36	8.943	6.161	5.062	4.455	4.065	3.790	3.585	3.425	3.296	3.191	3.102	3.027
38	8.882	6.111	5.016	4.412	4.023	3.749	3.545	3.385	3.257	3.152	3.063	2.988
40	8.828	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	3.028	2.953
42	8.779	6.027	4.940	4.339	3.953	3.680	3.477	3.318	3.191	3.086	2.997	2.922
44	8.735	5.991	4.907	4.308	3.923	3.651	3.448	3.290	3.162	3.057	2.969	2.894
46	8.695	5.958	4.877	4.280	3.896	3.625	3.422	3.264	3.137	3.032	2.944	2.869
48	8.659	5.929	4.850	4.255	3.871	3.601	3.398	3.240	3.113	3.009	2.921	2.846
50	8.626	5.902	4.826	4.232	3.849	3.579	3.376	3.219	3.092	2.988	2.900	2.825
55	8.554	5.843	4.773	4.181	3.800	3.531	3.330	3.173	3.046	2.942	2.854	2.779
60	8.494	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.817	2.742
65	8.445	5.755	4.692	4.105	3.726	3.459	3.259	3.103	2.977	2.873	2.785	2.711
70	8.402	5.720	4.661	4.076	3.698	3.431	3.232	3.075	2.950	2.846	2.759	2.684
75	8.366	5.691	4.635	4.050	3.674	3.407	3.208	3.052	2.927	2.823	2.736	2.661
80	8.335	5.665	4.612	4.029	3.652	3.387	3.188	3.032	2.907	2.803	2.716	2.641
85	8.307	5.643	4.591	4.009	3.634	3.368	3.170	3.014	2.889	2.786	2.698	2.624
90	8.282	5.623	4.573	3.992	3.617	3.352	3.154	2.999	2.873	2.770	2.683	2.608
95	8.260	5.605	4.556	3.977	3.603	3.338	3.140	2.985	2.860	2.756	2.669	2.595
100	8.241	5.589	4.542	3.963	3.589	3.325	3.127	2.972	2.847	2.744	2.657	2.583
125	8.166	5.530	4.488	3.912	3.540	3.277	3.079	2.925	2.801	2.698	2.611	2.536
150	8.118	5.490	4.453	3.878	3.508	3.245	3.048	2.894	2.770	2.667	2.580	2.506
175	8.083	5.462	4.427	3.855	3.485	3.223	3.026	2.872	2.748	2.645	2.559	2.484
200	8.057	5.441	4.408	3.837	3.468	3.206	3.010	2.856	2.732	2.629	2.543	2.468
250	8.022	5.412	4.382	3.812	3.443	3.182	2.987	2.833	2.710	2.607	2.520	2.446
500	7.948	5.355	4.330	3.764	3.397	3.137	2.941	2.789	2.665	2.563	2.476	2.402
∞	7.880	5.299	4.279	3.715	3.350	3.091	2.897	2.744	2.621	2.519	2.432	2.358

Percentile 99.5 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	24505	24572	24630	24682	24767	24836	24960	25044	25148	25211	25337	25465
2	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	43.27	43.17	43.08	43.01	42.88	42.78	42.59	42.47	42.31	42.21	42.02	41.83
4	20.60	20.51	20.44	20.37	20.26	20.17	20.00	19.89	19.75	19.67	19.50	19.32
5	13.29	13.21	13.15	13.09	12.98	12.90	12.76	12.66	12.53	12.45	12.30	12.14
6	9.950	9.877	9.814	9.758	9.664	9.589	9.451	9.358	9.241	9.170	9.026	8.879
7	8.097	8.028	7.968	7.915	7.826	7.754	7.623	7.534	7.422	7.354	7.217	7.076
8	6.938	6.872	6.814	6.763	6.678	6.608	6.482	6.396	6.288	6.222	6.088	5.951
9	6.153	6.089	6.032	5.983	5.899	5.832	5.708	5.625	5.519	5.454	5.322	5.188
10	5.589	5.526	5.471	5.422	5.340	5.274	5.153	5.071	4.966	4.902	4.772	4.639
11	5.165	5.103	5.049	5.001	4.921	4.855	4.736	4.654	4.551	4.488	4.359	4.226
12	4.836	4.775	4.721	4.674	4.595	4.530	4.412	4.331	4.228	4.165	4.037	3.904
13	4.573	4.513	4.460	4.413	4.334	4.270	4.153	4.073	3.970	3.908	3.780	3.647
14	4.359	4.299	4.247	4.200	4.122	4.059	3.942	3.862	3.760	3.698	3.569	3.436
15	4.181	4.122	4.070	4.024	3.946	3.883	3.766	3.687	3.585	3.523	3.394	3.260
16	4.031	3.972	3.920	3.875	3.797	3.734	3.618	3.539	3.437	3.375	3.246	3.112
17	3.903	3.844	3.793	3.747	3.670	3.607	3.492	3.412	3.311	3.248	3.119	2.984
18	3.793	3.734	3.683	3.637	3.560	3.498	3.382	3.303	3.201	3.139	3.009	2.873
19	3.696	3.638	3.587	3.541	3.465	3.402	3.287	3.208	3.106	3.043	2.913	2.776
20	3.611	3.553	3.502	3.457	3.380	3.318	3.203	3.123	3.022	2.959	2.828	2.690
22	3.469	3.411	3.360	3.315	3.239	3.176	3.061	2.982	2.880	2.817	2.685	2.545
24	3.354	3.296	3.246	3.201	3.125	3.062	2.947	2.868	2.765	2.702	2.569	2.428
26	3.259	3.202	3.151	3.107	3.031	2.969	2.853	2.774	2.671	2.607	2.473	2.330
28	3.180	3.123	3.073	3.028	2.952	2.890	2.775	2.695	2.592	2.527	2.392	2.247
30	3.113	3.056	3.006	2.961	2.885	2.823	2.708	2.628	2.524	2.459	2.323	2.176
32	3.056	2.998	2.948	2.904	2.828	2.766	2.650	2.570	2.466	2.401	2.264	2.114
34	3.005	2.948	2.898	2.854	2.778	2.716	2.600	2.520	2.415	2.350	2.212	2.060
36	2.961	2.905	2.854	2.810	2.734	2.672	2.556	2.475	2.371	2.305	2.166	2.013
38	2.923	2.866	2.816	2.771	2.695	2.633	2.517	2.436	2.331	2.265	2.125	1.970
40	2.888	2.831	2.781	2.737	2.661	2.598	2.482	2.401	2.296	2.230	2.088	1.932
42	2.857	2.800	2.750	2.706	2.630	2.567	2.451	2.370	2.264	2.198	2.056	1.897
44	2.829	2.772	2.722	2.678	2.602	2.540	2.423	2.342	2.236	2.169	2.026	1.866
46	2.804	2.747	2.697	2.653	2.577	2.514	2.398	2.316	2.210	2.143	1.999	1.837
48	2.781	2.724	2.674	2.630	2.554	2.491	2.375	2.293	2.186	2.119	1.974	1.811
50	2.760	2.703	2.653	2.609	2.533	2.470	2.353	2.272	2.164	2.097	1.951	1.786
55	2.714	2.658	2.608	2.563	2.487	2.425	2.308	2.226	2.118	2.049	1.902	1.733
60	2.677	2.620	2.570	2.526	2.450	2.387	2.270	2.187	2.079	2.010	1.861	1.689
65	2.646	2.589	2.539	2.495	2.419	2.356	2.238	2.155	2.046	1.977	1.826	1.650
70	2.619	2.563	2.513	2.468	2.392	2.329	2.211	2.128	2.019	1.949	1.796	1.619
75	2.597	2.540	2.490	2.445	2.369	2.306	2.188	2.105	1.995	1.925	1.771	1.590
80	2.577	2.520	2.470	2.425	2.349	2.286	2.168	2.084	1.974	1.903	1.749	1.564
85	2.559	2.503	2.453	2.408	2.332	2.269	2.150	2.067	1.956	1.885	1.729	1.541
90	2.544	2.487	2.437	2.393	2.316	2.253	2.134	2.051	1.939	1.868	1.711	1.521
95	2.530	2.474	2.424	2.379	2.303	2.239	2.120	2.037	1.925	1.853	1.695	1.503
100	2.518	2.461	2.411	2.367	2.290	2.227	2.108	2.024	1.912	1.840	1.681	1.486
125	2.472	2.415	2.365	2.320	2.244	2.180	2.061	1.976	1.863	1.790	1.627	1.420
150	2.441	2.385	2.335	2.290	2.213	2.150	2.030	1.944	1.830	1.756	1.590	1.375
175	2.420	2.363	2.313	2.268	2.191	2.128	2.007	1.922	1.807	1.733	1.564	1.340
200	2.404	2.347	2.297	2.252	2.175	2.112	1.991	1.905	1.790	1.715	1.544	1.314
250	2.381	2.325	2.275	2.230	2.153	2.089	1.968	1.882	1.765	1.690	1.516	1.275
500	2.337	2.281	2.230	2.185	2.108	2.044	1.922	1.835	1.717	1.640	1.460	1.184
∞	2.294	2.237	2.187	2.142	2.064	2.000	1.877	1.789	1.669	1.590	1.402	1.000

Reading off the table

The preceding table gives the critical values of $F(\nu_1,\nu_2)$ at percentage points $P = 0.95$, 0.975, 0.99 and 0.995, for pairs of degrees of freedom (ν_1,ν_2), ν_1 usually referring to the numerator, ν_2 to the denominator of F.

Illustration 1. Find $F_{3,8[.95]}$, *i.e.* the 95th percentage point of F with 3 and 8 *df.* Taking the first page for percentile 95, at the junction of column $\nu_1 = 3$ and line $\nu_2 = 8$, we read off 4.066. In the same manner, we find $F_{10,12[.99]} = 4.296$ and $F_{20,5[.975]} = 6.329$.

Illustration 2. Determine $F_{12,3[.05]}$. The table furnishes only percentage points over the 50th. For lower percentage points, we must resort to the inversion formula for F, *i.e.*

$$F_{\nu_1,\nu_2[P]} = 1 \,/\, F_{\nu_2,\nu_1[1-P]} \;.$$

Finding first $F_{3,12[.95]} = 3.490$ (note the interversion of the two *df*'s and the transposition of percentiles $.05 \rightarrow .95$), we obtain $F_{12,3[.05]} = 1 \,/\, F_{3,12[.95]} = 1 \,/\, 3.490 \approx 0.287$.

Illustration 3. Determine $F_{4,160[.99]}$. At page 1 for percentile 99, we find only $F_{4,150} = 3.447$ and $F_{4,175} = 3.428$; consequently, we must interpolate. Linear interpolation seems adequate for estimating F for $\nu_2 = 160$ between values at $\nu_2 = 150$ and 175. Simplifying the notation, we calculate:

$$F_{\nu} \approx F_{\nu'} + (F_{\nu''} - F_{\nu'}) \times (\nu - \nu')/(\nu'' - \nu') \;.$$

Here, $F_{160} \approx F_{150} + (F_{175} - F_{150}) \times (160-150)/(175-150) = 3.439$, a value precise to 3 decimal places.

Illustration 4. Determine if a calculated F value corresponding to $F_{22,140}$ is significant at the 5 % (one-tailed) significance level, *i.e.* at $P = 0.95$. The two pages for percentage point 95 show neither $\nu_1 = 22$, nor $\nu_2 = 140$. However, many manners of solution are possible:

a) For $P = .95$, the table furnishes $F_{20,125} = 1.655$ and $F_{25,150} = 1.580$. Therefore, if our test value equals, say, 2.19, we can declare it significant (because it goes beyond the more demanding $F_{20,125}$), whereas if it equals 1.50, it is certainly not significant.

b) If it is required, we can determine approximately $F_{22,140}$ via a double interpolation scheme: first, on ν_1 for each value of ν_2, then on ν_2. Harmonic interpolation is generally superior; we must resort to it anyway with our table if $\nu_1 > 100$ or $\nu_2 > 500$. However, if the interpolation gap is narrow, simple linear interpolation may suffice. For this case, on page 2 for $P = .95$, we find

$F_{20,125} = 1.655$; $F_{20,150} = 1.641$; $F_{25,125} = 1.594$; $F_{25,150} = 1.580$. Using harmonic interpolation, let us first calculate $F_{22,125} \approx 1.6272$ $[= 1.655 + (1.594 - 1.655) \times (22^{-1} - 20^{-1})/(25^{-1} - 20^{-1})]$ and $F_{22,150} \approx 1.6133$; then, combining these two interpolated values, one more harmonic interpolation renders $F_{22,140} \approx 1.618$. Linear interpolation would have produced 1.622, the exact value being 1.61885.

c) Lastly, there exists a remarkable normal approximation to F (traced back to Wilson and Hilferty), whose formula is:

$$z = \frac{(1-\mathrm{B})F^{1/3} - (1-\mathrm{A})}{\sqrt{\mathrm{B}F^{2/3} + \mathrm{A}}} ,$$

where $\mathrm{A} = 2/(9\nu_1)$ and $\mathrm{B} = 2/(9\nu_2)$. The obtained z value is then compared to percentile $z_{[P]}$ in the appropriate table of the standard $N(0,1)$ distribution. For example, in that table, we may read $z_{[.95]} = 1.6449$, whence any value $z \geq 1.6449$ would be declared significant at 5 %. The interpolated value $F_{22,140[.95]} \approx 1.618$, with $\mathrm{A} = 2/(9\times 22)$ and $B = 2/(9\times 140)$, $z \approx 1.644$, would (correctly) be deemed non-significant, and $F = 1.619$, giving $z = 1.646$, would be declared significant.

Full examples

Example 1. A researcher wants to compare the effects of three alimentary diets by evaluating weight loss (in kg). He recruits 18 subjects, randomly assigning 6 of them to each group. Here are his summary results.

Diet	n	$\bar{x}$	s^2
1	6	4.40	1.300
2	6	6.00	1.333
3	6	4.00	1.200

Using the 0.05 significance level, can he confirm that the diets have truly different effects? *Solution*: This example illustrates the method of "analysis of variance", or ANOVA, conceived by R. A. Fisher and for which he also derived the so-called F distribution. In the research design presented here, the null hypothesis at stake, $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$, postulates that all data come from a common normal population (or, equivalently, from normal populations with identical parameter values). Under that hypothesis, the quotient $F = \mathrm{MS}_{\text{Between-groups}}/\mathrm{MS}_{\text{Within-groups}}$

is distributed as $F_{k-1,N-k}$, $N = \sum n_j = 18$; hypothesis H_0 can be rejected, or discredited, if $F > F_{k-1,N-k[1-\alpha]} = F_{2,15[.95]} = 3.682$. Formulae for calculating the two MS's are:

$$MS_{Between} = n_h s^2(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$$

$$MS_{Within} = \sum(n_j - 1)s_j^2 \;/\; (N - k) \;.$$

Quantity n_h in $MS_{Between}$ represents the *harmonic mean* of n_j's, obtained with $n_h = k/(1/n_1 + 1/n_2 + \cdots + 1/n_k)$; when all n_j are equal to n, then $n_h = n$.

Calculations for the above data are: $n_h = 6$; $s^2(\bar{x}_1, \bar{x}_2, \ldots) = s^2(4.40;\ 6.00;\ 4.00) \approx 1.120$; $MS_{Between} = 6 \times 1.120 = 6.720$; $MS_{Within} = [5 \times 1.3 + 5 \times 1.333 + 5 \times 1.2]/[18 - 3] \approx 1.278$ and, lastly, $F = MS_{Between}/MS_{Within} = 6.720/1.278 \approx 5.258$. This value exceeds $F_{2,15[.95]} = 3.682$, and the researcher may reject H_0 and state that the various diets entail different amounts of weight loss.

There exist a very great number of research designs and a corresponding variety of procedures for ANOVA (factorial designs, with or without repeated measures, with nested factors, by latin squares, with confounded interactions, etc.). Also, different calculating formulae are possible [see B. J. Winer, D. R. Brown, & K. M. Michels, "Statistical principles in experimental design" (3rd ed.), 1991, New York, McGraw-Hill; R. E. Kirk, "Experimental design: Procedures for the behavioral sciences" (2nd ed.), 1994, Belmont (CA), Brooks/Cole].

Example 2. In order to investigate a theory on the relationship between gender and abilities, a university psychologist puts forward the idea that mechanical ability is higher in boys than in girls: a test of ability is suitably chosen. With random samples composed of 25 girls and 31 boys, all aged 12 years, she measures each subject and obtains, in brief:

Girls :	$n = 25$	$\bar{x} = 18.56$	$s^2 = 86.816$;
Boys :	$n = 31$	$\bar{x} = 26.43$	$s^2 = 19.431$.

Now, beyond the fact that the boys' mean ability is higher as predicted, our researcher notes that the scores' variance for girls is higher, as if ability levels were sprinkled in a more heterogeneous manner among girls. Is there a difference in scatter of ability between boys and girls? Recall that, anyhow, the t test appropriate for deciding on the difference between means $(\bar{x}_1 - \bar{x}_2)$ subsumes the condition called "homogeneity of variance", which brings us back to a comparison of variances. *Solution*: The null hypothesis to be tested is that s_1^2 and s_2^2 are two sample estimates of the same parametric variance σ^2. Hence, according to hypothesis H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the quotient s_1^2/s_2^2 is distributed as $F(\nu_1, \nu_2)$, $\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$. The appropriate test, the test of homogeneity for two variance estimates, calls for a bilateral criterion; the resulting F quotient will be deemed significant if $F \leq F_{\nu_1,\nu_2[1-\alpha/2]}$ or $F \geq F_{\nu_1,\nu_2[\alpha/2]}$. Let us here choose $\alpha = 0{,}10$ and put the greater variance in the numerator (to simplify the location of the critical value). We obtain 86.816/19.431 $\approx$ 4.468, vs. $F_{24,30[.95]} \approx 1.887$ [approximated through harmonic interpolation, between $F_{20,30} = 1.932$ and $F_{25,30} = 1.878$]. The girls' subpopulation thus appears more heterogeneous than the boys' as regards to mechanical ability: there are very talented girls and other much less talented, the range of variation being broader for girls than for boys. On the other hand, for the purpose of testing her first hypothesis (on $\bar{x}_1 - \bar{x}_2$), the researcher should resort to a different test procedure other than the usual t test, for instance to Welch's t^* procedure (see Winer 1991).

To test the homogeneity hypothesis for $k \geq 2$ variance estimates, other techniques are available, in particular Hartley's $F_{\max}$ test and Cochran's C (see appropriate sections of this book).

Mathematical presentation

The quotient of two variances originating from two independant normal samples is distributed according to the F distribution or, more precisely :

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{\nu_1,\nu_2} ,$$

where ν_1 and ν_2 are the df's associated with variances s_1^2 and s_2^2 respectively, and σ_1^2 and σ_2^2 are the (unknown) true variances of the corresponding normal populations. In fact, the F r.v. corresponds to the quotient of two independent χ^2 (Chi-square) r.v.'s, each one divided by its own df :

$$F_{\nu_1,\nu_2} = \frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2} .$$

When, by hypothesis, we postulate that $\sigma_1^2 = \sigma_2^2$, the quotient s_1^2/s_2^2 allows us to test the homogeneity of variance condition as stipulated for ANOVA. In accordance with its definition, the F variable may be *inverted*, concurrently with a swap of the two df's. Through that inversion, one obtains the lower percentage points of F (e.g. $P = 0.05$) from the higher ones (e.g. $P = 0.95$), using:

$$F_{\nu_1,\nu_2[P]} = 1 \,/\, F_{\nu_2,\nu_1[1-P]} ,$$

where the quantity within brackets designates the centile rank (reduced to range 0-1) of the given *F* value.

Calculation and moments

The p.d.f. of the *F* distribution, for given ν_1 and ν_2, is :

$$p(F) = K_{\nu_1,\nu_2} F^{\nu_1/2-1}\left(1+\frac{\nu_1}{\nu_2}F\right)^{-\frac{1}{2}(\nu_1+\nu_2)} ,$$

where:

$$K_{\nu_1,\nu_2} = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B(\nu_1/2,\nu_2/2)} = \frac{(\nu_1/\nu_2)^{\nu_1/2}\,\Gamma([\nu_1+\nu_2]/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} .$$

According to the values of ν_1 and ν_2, this function can be integrated either analytically (by integration by parts) or numerically (with the generalized Simpson's rule, for instance). We thus obtain the d.f. $P(F) = \int_0^F p(x)\,dx = \Pr\{x \le F \mid \nu_1,\nu_2\}$, enabling us to determine percentage points or critical values of *F*. In the formulae for constant $K(\nu_1,\nu_2)$, $B(a,b)$ denotes the *Beta* function and $\Gamma(x)$, the *Gamma* function.

Apart from the possible swapping of *df*'s ν_1 and ν_2, a few special cases of *F* permit an easy computing of the d.f. and critical values. In order to find $F(\nu_1,\nu_2[P])$, we can use:

$$\nu_1=1,\ \nu_2=1 \ :\ \tan^2(\tfrac{1}{2}\pi P)$$

$$\nu_1=1,\ \nu_2\ge 1 \ :\ t^2_{\nu_2[(1+P)/2]}$$

$$\nu_1,\ \ \nu_2=\infty \ :\ \chi^2_{\nu_1[P]}/\nu_1$$

$$\nu_1=2,\ \nu_2\ge 1 \ :\ \frac{\nu_2}{2}\left[\frac{1}{\sqrt[\nu_2]{(1-P)^2}} - 1\right] .$$

In these expressions, t_ν is a Student's *t* variable with ν *df.* Kendall and Stuart (1977) note one more interesting case,

$$\frac{\sqrt{\nu}}{2}\left[\sqrt{F_{\nu,\nu[P]}} - \sqrt{1/F_{\nu,\nu[P]}}\right] \sim t_{\nu[(1+P)/2]} .$$

Finally, except for the relation of F's d.f. with the binomial distribution (see below), exact integration of the F distribution is laborious[1], and we resort readily to numerical methods.

Relationship between *F* and binomial distributions

There exists a working relationship between F's d.f. and the binomial sum of probabilities, through an underlying *Beta* distribution. The respective parameters are, for F : the *df* ν_1, ν_2, and for the binomial : n, x, π ; the binomial distribution pertains to the probability of obtaining x (= 0, 1, ···, n) "successes" out of n independent "trials", the probability of a success being π at each trial (see section on binomial distribution). We illustrate the F - binomial relationship with three categories of applications.

– Case 1 : Is the number of "successes" (x) significant?

Let us suppose a binomial process for $n = 10$ trials and yielding $x = 7$ successes. The hypothesized probability of success is $\pi = 0.3$. Does the observed rate of success of 7/10 contradict the hypothetical $\pi = 0.3$, at the significance level $\alpha = 0.01$?

To answer this question, we may convert the binomial data (n, x, p) into F data (ν_1, ν_2, F), with:

$$2(n-x+1) \to \nu_1\,;\; 2x \to \nu_2\,;\; \nu_2(1-\pi)/(\nu_1\pi) \to F,$$

obtaining $F = 4.0833$, $\nu_1 = 8$ and $\nu_2 = 14$. The critical value $F_{8,14[.99]}$ is 4.140; thus, our F = 4.0833 is not significant at the 0.01 level, and the proposed probability $\pi = 0.3$ remains plausible.

Note that the computed binomial sum for our example is $\Pr\{x \geq 7\,|\,10;\, 0.3\} \approx 0.010592$, a non-significant result. In (binomial) table 3a, we find π^*, the maximum value of π such that $\Pr\{r \geq x\,|\,n,\pi\} \leq \alpha$. Here, for $n = 10$, $x = 7$ and $\alpha = 0.01$, table 3a gives $\pi^* = 0.2971$, our $\pi = 0.3$ being too high to obtain significance.

– Case 2 : What is the maximum value π^* such that a number x of successes be sgnificant?

With n trials and x observed successes, the success rate (x/n) will be judged significant only if the individual probability of success (π) is small. In fact, π must lie in the interval (0, π^*), within which $\Pr\{r \geq x\,|\,n,\pi\} \leq \alpha$; value $\pi = \pi^*$ brings about equality with α. Take an example with n = 12, $x = 3$ and $\alpha = 0.05$. Under what value π^* of π does the success rate of 3 out of 12 trials will be found significant at 5 %?

[1] For example, for $\nu_1 = \nu_2 = 3$, $P(F) = \Pr(x \leq F) = K_{3,3}[(F+1)^2\tan^{-1}\sqrt{F} + \sqrt{F}(F-1)]\,/\,(F+1)^2$, where $K_{3,3} \approx$ 2.54648.

We can exploit again the $F(\nu_1,\nu_2)$ distribution, by determining as above ν_1 = $2(12-3+1) = 20$ and $\nu_2 = 2\times3 = 6$. Percentile 95 of $F_{20,6}$ is about 3.874, and $\pi^* = \nu_2/(\nu_2+\nu_1F)$ ≈ 0.071874, as can be found also in (binomial) table 3a. Thus, for any probability $\pi \leq 0.07187$, the occurence of 3 or more successes in 12 trials is a rare, unusual event, according to the 5 % significance level.

– Case 3 : Is a given *F* value significant?

The binomial-*F* relationship can also be serve to determine the significance of an observed *F* value, with the restriction that both *F*'s ν_1 and ν_2 be even. For instance, let $\nu_1 = 6$, $\nu_2 = 4$ and $F = 6.30$. At the (one-tailed) upper 5 % significance level, does this *F* value reach significance?

In order to answer that question, we first compute:

$$n = \tfrac{1}{2}[\nu_1 + \nu_2 - 2]\,;\; x = \nu_2/2\,;\; \pi = \nu_2/[\nu_2 + \nu_1 F]\;,$$

obtaining $n = 4$, $x = 2$, $\pi \approx 0.0957$. Then, by direct computation of the binomial sum, $\Pr\{r\geq 2\,|\,4,0.0957\} \approx 0.0482$, a significant result at $\alpha = 0.05$. Besides, in (binomial) table 3a with $(n,x,\alpha) = (4,2,0.05)$, we read an upper limit $\pi^* = 0.0976$; our $\pi = 0.0957$ falls under this upper limit and, therefore, carries significance.

Normal approximation to the F distribution. Any function for approximating the *F* distribution must take into account its usually marked positive skewness. Based upon the very good Wilson-Hilferty normal approximation to χ^2, the following:

$$z = \frac{(1-\mathrm{B})F^{1/3} - (1-\mathrm{A})}{\sqrt{\mathrm{B}F^{2/3} + \mathrm{A}}}\;,$$

with $\mathrm{A} = 2/(9\nu_1)$ and $\mathrm{B} = 2/(9\nu_2)$, is approximately distributed as a standard $N(0,1)$ r.v..

Moments of F. The moments and shape coefficients of the *F* distribution depend upon its two parameters, ν_1 and ν_2. They are given below, simpler expressions being obtained when both parameters are equal ($\nu_1 = \nu_2$).

$$\mathrm{E}(F) = \mu = \frac{\nu_2}{\nu_2 - 2}\;;$$

$$\mathrm{var}(F) = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{4\nu(\nu-1)}{(\nu-2)^2(\nu-4)} \qquad (\text{when } \nu_1 = \nu_2 = \nu)\;;$$

$$\gamma_1 = \frac{2\nu_1+\nu_2-2}{\nu_2-6}\cdot\sqrt{\frac{8(\nu_2-4)}{\nu_1(\nu_1+\nu_2-2)}}$$

$$= \frac{2(3\nu-2)}{\nu-6}\cdot\sqrt{\frac{\nu-4}{\nu(\nu-1)}}\ ; \qquad \text{(when } \nu_1=\nu_2=\nu\text{)}\ ;$$

$$\gamma_2 = \frac{12\left[(\nu_2-2)^2(\nu_2-4)+\nu_1(\nu_1+\nu_2-2)(5\nu_2-22)\right]}{\nu_1(\nu_2-6)(\nu_2-8)(\nu_1+\nu_2-2)}$$

$$= \frac{6(11\nu^3-62\nu^2+64\nu-16)}{\nu(\nu-1)(\nu-6)(\nu-8)}\ . \qquad \text{(when } \nu_1=\nu_2=\nu\text{)}\ .$$

These moments are determinate when ν_2 (or ν) is sufficient; for instance, var(F) exists if $\nu_2 \geq 5$.

Furthermore, the mode of F, Mo(F), equals $\nu_2(\nu_1-2)/[\nu_1(\nu_2+2)]$; it is used notably in the numerical integration of the p.d.f. and it represents also a lower limit for the median $F_{[0.5]}$. Note that the median satisfies inequalities: Mo(F) < $F_{[0.5]}$ < $\mu(F)$. Moreover, $F_{[0.5]} < 1$ if $\nu_1 < \nu_2$, $F_{[0.5]} = 1$ if $\nu_1 = \nu_2$, and $F_{[0.5]} > 1$ if $\nu_1 > \nu_2$.

For the inquisitive reader, the quotient of two *correlated* variances is not distributed as F. If both variances originate from a normal bivariate population with parametric correlation ρ, the (parametric) correlation coefficient between sample variances, *i.e.* $\rho(s_1^2,s_2^2)$, equals ρ^2. Also, postulating $E(s_1^2) = E(s_2^2)$ as for F, then the quotient s_1^2/s_2^2 has expectation $(\nu-2\rho^2)/(\nu-2)$ and variance $4[\nu(\nu-1)-\rho^2(5\nu-8)](1-\rho^2)/[(\nu-2)^2(\nu-4)]$, where ν is the common parameter of s_1^2 and s_2^2. The p.d.f. is also known. More information, some of it about the comparative validity of tests on the equality of two correlated variances (*see* Supplementary examples, n° 8), is to be found in Laurencelle (2000).

Generation of pseudo random variates

The schema of a program below allows the production of r.v.'s from the $F(\nu_1,\nu_2)$ distribution; it requires a function (designated UNIF) which generates serially r.v.'s from the standard uniform $U(0,1)$ distribution.

Preparation : Let ν_1 and ν_2 (the degrees of freedom)

$a = \nu_1/2$; $b = \nu_2/2$; $s = a + b$;
If min(a,b) ≤ 1 **then** $\beta \leftarrow 1$ / min(a,b)
else $\beta \leftarrow \sqrt{[(s-2)/(2\times a\times b-s)]}$;
$g \leftarrow a + 1/\beta$; C = ln(4) $\approx$ 1,3862943611 .

Production : **Repeat** $t \leftarrow$ UNIF ; $u \leftarrow \beta \times \ln[t/(1-t)]$; $w \leftarrow a \times \exp(u)$
Until $s \times \ln[s/(b+w)] + g \times u - C \geq \ln(t^2 \times \text{UNIF})$;
Return $w / a \rightarrow x$.

Remarks :

1. Standard temporal cost : 8.7 à 16.0 × t(UNIF), depending on values of ν_1 et ν_2.

2. This method is due to R. C. H. Cheng (1978, *in* Bratley, Fox and Schrage 1987) and it is suitable for any (ν_1, ν_2) combination. There are adaptations of this method, and other methods, that are more efficient for particular values of (ν_1,ν_2), like those in which $\max(\nu_1, \nu_2) < 2$ or $\nu_1 = \nu_2$ (Devroye 1986). The inversion formulae given earlier (for cases where $\nu_1 = \nu_2 = 1$ or $\nu_1 = 2$, $\nu_2 \geq 2$) allow the generation of r.v.'s from the *F* distribution through the simple replacement of *P* by one r.v. from the standard $U(0,1)$ distribution, *i.e.* $u \leftarrow$ UNIF.

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