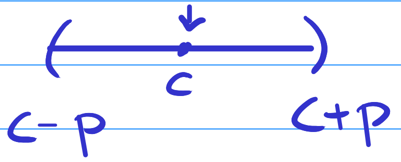
Topic 2 Limits & Continuity

Definition Let $c \in \mathbb{R}$, $f : D \longrightarrow \mathbb{R}$

where

$(c-p, c+p) \setminus \{c\} \in D \subseteq \mathbb{R}$



punctured open interval

so c does not itself need to be in D

We say $f(x)$ has limit L as x approaches c

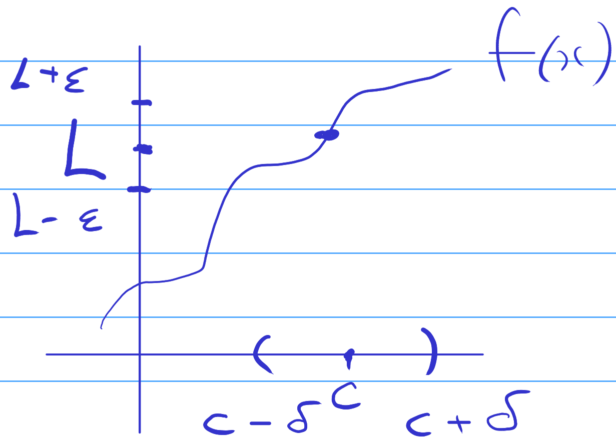
$$\boxed{\lim_{x \to c} f(x) = L}$$

if

$\forall \varepsilon > 0 \quad \exists \, \delta > 0 : 0 < |x-c| < \delta$

for all epsilon positive, exists positive delta, such that

$\Rightarrow |f(x) - L| < \varepsilon$



Def:

We say f is continuous

at $c \in \mathbb{R}$

if $f(c)$ exists

& $\lim_{x \to c} f(x)$ exists

& these are EQUAL

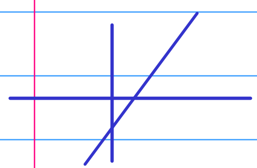
We could <u>combine</u> these two definitions

f continuous at c if

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

in other words $\lim\limits_{x \to c} f(x) = f(c)$.

<u>Examples</u> i) $f(x) = 2x - 3$



Proof of "f continuous at -1"

$c = -1$, we want to prove

$$\lim_{x \to -1} f(x) \overset{?}{=} f(-1) = -5$$

Given any $\varepsilon > 0$, we need to find some $\delta > 0$ such that

if ⊛ $|x - (-1)| < \delta$ then $|f(x) - (-5)| < \varepsilon$

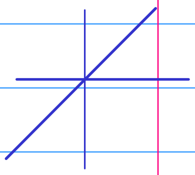
⊛ $|x + 1| < \delta \Rightarrow |2x - 3 + 5| < \varepsilon$

⊛ $|x+1| < \delta \Rightarrow |2x + 2| < \varepsilon$

Let $\delta = \varepsilon/2$. Then

$$|x - c| = |x+1| < \delta \Rightarrow |f(x) - f(c)| = |2x+2| = 2|x+1| < 2\delta = \varepsilon$$

ii) $f(x) = x$ continous everywhere
easy to prove at all $c \in \mathbb{R}$



$\forall \varepsilon > 0 \ \exists \delta > 0 : |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

Given any ε, let $\delta = \varepsilon$

If $|x-c| < \delta$ then

$$|f(x) - f(c)| = |x-c| < \delta = \varepsilon$$

iii) Hard example :

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We will not prove $f(x)$ continuous now, but a useful theorem

Pinching Theorem $\quad f, g, h : D \longrightarrow \mathbb{R}$

where D contains $(c-p, c+p) - \{c\}$

& suppose $f(x) \leq g(x) \leq h(x) \quad \forall x \in D$

Then if $\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} h(x)$

then $\lim\limits_{x \to c} g(x) = L$ also.