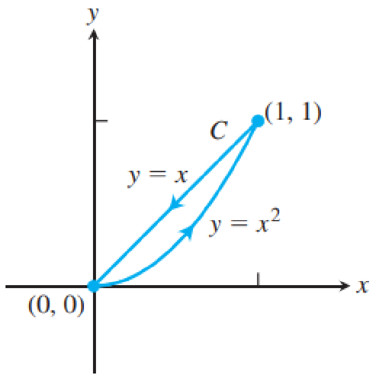
Solutions for Tutorial Problem Sheet 9, November 24.

(Integrals and Vector Fields)

Problem 1. Evaluate $\int_C (x + \sqrt{y})\, ds$ where C is given in the accompanying figure.



Solution:

$C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2};\ C_2 : \mathbf{r}(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1$

$\Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2} \Rightarrow \int_C \left(x + \sqrt{y}\right) ds = \int_{C_1} \left(x + \sqrt{y}\right) ds + \int_{C_2} \left(x + \sqrt{y}\right) ds$

$= \int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{1+4t^2}\ dt + \int_0^1 \left((1-t) + \sqrt{1-t}\right)\sqrt{2}dt = \int_0^1 2t\sqrt{1+4t^2}\,dt + \int_0^1 \left(1 - t + \sqrt{1-t}\right)\sqrt{2}dt$

$= \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 + \sqrt{2}\left[t - \frac{1}{2}t^2 - \frac{2}{3}(1-t)^{3/2}\right]_0^1 = \frac{5\sqrt{5}-1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5}+7\sqrt{2}-1}{6}$

Problem 2. Integrate $f(x,y) = x^2 - y$ over the curve $C:\ x^2 + y^2 = 4$ in the first quadrant from $(0,2)$ to $(\sqrt{2}, \sqrt{2})$.

Solution:

$$\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j},\ 0 \le t \le \tfrac{\pi}{4} \Rightarrow \tfrac{d\mathbf{r}}{dt} = (2\cos t)\mathbf{i} + (-2\sin t)\mathbf{j} \Rightarrow \left|\tfrac{d\mathbf{r}}{dt}\right| = 2;\ f(x,y) = f(2\sin t, 2\cos t)$$
$$= 4\sin^2 t - 2\cos t \Rightarrow \int_C f\, ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\, dt = \left[4t - 2\sin 2t - 4\sin t\right]_0^{\pi/4} = \pi - 2\left(1+\sqrt{2}\right)$$

Problem 3. Along the curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$, evaluate each of the following integrals:
a) $\int_C (x+y-z)\, dx$; b) $\int_C (x+y-z)\, dy$; c) $\int_C (x+y-z)\, dz$.

Solution:

$\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}, 0 \le t \le 1 \Rightarrow dx = dt, dy = 0, dz = 2t\, dt$

(a) $\int_C (x+y-z)\, dx = \int_0^1 \left(t-1-t^2\right) dt = \left[\frac{1}{2}t^2 - t - \frac{1}{3}t^3\right]_0^1 = -\frac{5}{6}$

(b) $\int_C (x+y-z)\, dy = \int_0^1 \left(t-1-t^2\right) \cdot 0 = 0$

(c) $\int_C (x+y-z)\, dz = \int_0^1 \left(t-1-t^2\right) 2t\, dt = \int_0^1 \left(2t^2 - 2t - 2t^3\right) dt = \left[\frac{2}{3}t^3 - t^2 - \frac{1}{2}t^4\right]_0^1 = -\frac{5}{6}$

Problem 4. Find the flow of the velocity field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ along each of the following paths from $(1,0)$ to $(-1,0)$ in the xy-plane.

a) The upper half of the circle $x^2+y^2=1$.

b) The line segment from $(1,0)$ to $(-1,0)$.

c) The line segment from $(1,0)$ to $(0,-1)$ followed by the line segment from $(0,-1)$ to $(-1,0)$.

Solution:

(a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and $\mathbf{F} = (\cos t + \sin t)\mathbf{i} - \left(\cos^2 t + \sin^2 t\right)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin t\cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ $= \int_0^{\pi}\left(-\sin t\cos t - \sin^2 t - \cos t\right)dt = \left[-\frac{1}{2}\sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t\right]_0^{\pi} = -\frac{\pi}{2}$

(b) $\mathbf{r} = (1-2t)\mathbf{i},\ 0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\mathbf{j} \Rightarrow$ $\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 4t-2 \Rightarrow \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 (4t-2)\,dt = \left[2t^2-2t\right]_0^1 = 0$

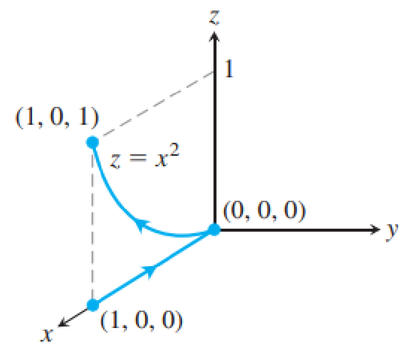
(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - \left(1-2t+2t^2\right)\mathbf{j}$ $\Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}_1}{dt} = (2t-1) + \left(1-2t+2t^2\right) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1}\mathbf{F}\cdot\frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2\,dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - \left(t^2+t^2-2t+1\right)\mathbf{j}$ $= -\mathbf{i} - \left(2t^2-2t+1\right)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}_2}{dt} = 1 - \left(2t^2-2t+1\right) = 2t-2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2}\mathbf{F}\cdot\frac{d\mathbf{r}_2}{dt} = \int_0^1\left(2t-2t^2\right)dt$ $= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

Problem 5. Find the work done by $\mathbf{F} = (x^2+y)\mathbf{i} + (y^2+x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from $(1,0,0)$ to $(1,0,1)$.

a) The line segment $x = 1, \;\; y = 0, \;\; 0 \le z \le 1$.

b) The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, \;\; 0 \le t \le 2\pi$.

c) The x-axis from $(1,0,0)$ to $(0,0,0)$ followed by the parabola $z = x^2, \;\; y = 0$ from $(0,0,0)$ to $(1,0,1)$.



Solution:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \; \frac{\partial M}{\partial y} = 0 = \frac{\partial P}{\partial x}, \; \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative } \Rightarrow \text{ there exists an } f \text{ so that } \mathbf{F} = \nabla f;$$

$$\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x,y,z) = \tfrac{1}{3}x^3 + xy + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y,z) = \tfrac{1}{3}y^3 + h(z)$$

$$\Rightarrow f(x,y,z) = \tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C$$

$$\Rightarrow f(x,y,z) = \tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + ze^z - e^z\right)$$

(a) $\text{work} = \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\tfrac{1}{3} + 0 + 0 + e - e\right) - \left(\tfrac{1}{3} + 0 + 0 - 1\right) = 1$

(b) $\text{work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) $\text{work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\tfrac{1}{3}x^3 + xy + \tfrac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

<u>Note</u>: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1,0,0)$ to $(1,0,1)$.