UNIVERSITY OF LEICESTER

Lecture 24: Multiple Integrals.

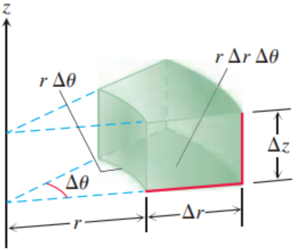
MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 14, 2022

Integration in Cylindrical Coordinates



• We partition the region of integration D into n small **cylindrical wedges**.
• The kth cylindrical wedge having **dimensions** Δr_k by $\Delta\theta_k$ by Δz_k.
• We calculated in polar coordinates that **area of wedges' base** is $\Delta A_k = r_k \Delta r_k \Delta\theta_k$.
• So the volume of wedge is $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta\theta_k$

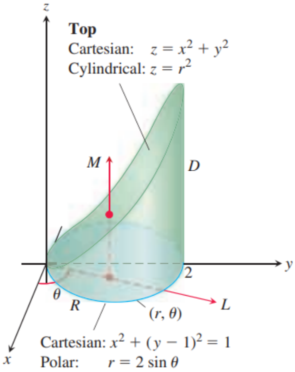
• We define the **triple integral as a limit of Riemann sums** using these wedges as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\Delta V_k = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

• In most situations we will need to **restrict** θ to $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

Example 1

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



Example 1

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

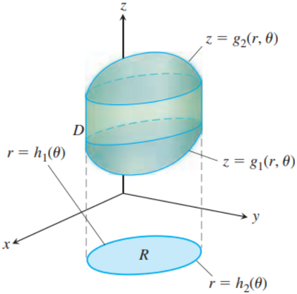
The region is sketched in Figure 15.49.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

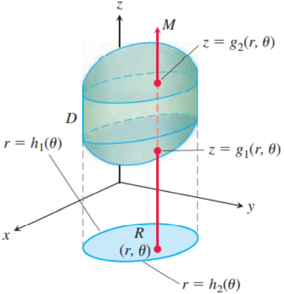
Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \qquad \blacksquare$$



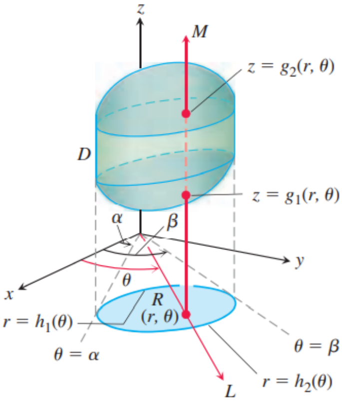
- **Step 1: Sketch** Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



- **Step 2: Find the z-limits of integration**. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.

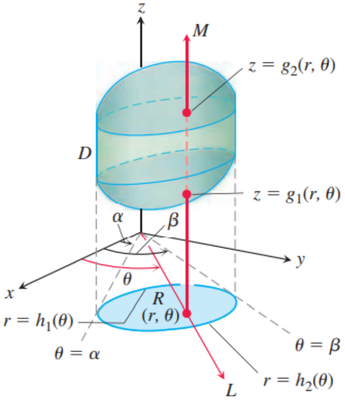
Whiteboard

$$\iiint\limits_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$



- **Step 3: Find the r-limits of integration**. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.

- **Step 4: Find the θ-limits of integration**. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.

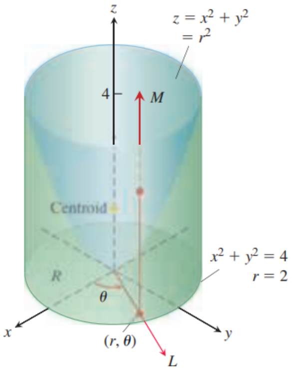


- The integral is

$$\iiint_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

Example 2

Find the centroid ($\sigma = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.



Example 2

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.50). Its base R is the disk $0 \leq r \leq 2$ in the xy-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

Example 2

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^5}{2} dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[z\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

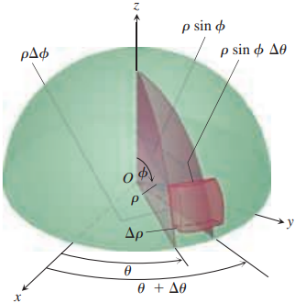
and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the z-axis, outside the solid.

Triple Integrals in Spherical Coordinates

• We partition the region of integration D into n small **spherical wedges**.

• The kth spherical wedge having **dimensions** $\Delta\rho_k$ by $\Delta\phi_k$ by $\Delta\theta_k$.

• It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$, a point chosen inside the wedge.



• We define the **triple integral as a limit of Riemann sums** using these wedges as

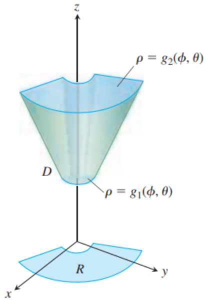
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\Delta V_k = \iiint_D f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

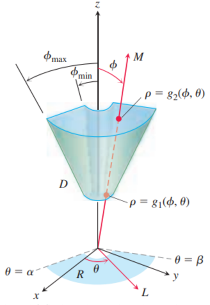
• In most situations we will need to **restrict** θ to $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

Finding Limits of Integration in the Order $d\rho\ d\phi\ d\theta$

- **Step 1: Sketch** Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.

- **Step 2: Find the ρ-limits of integration**. Draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As r increases, M enters D at $r = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the -limits of integration.





Finding Limits of Integration in the Order $d\rho\ d\phi\ d\theta$

- **Step 3: Find the ϕ-limits of integration**. For any given θ, the angle ϕ that M makes with the z-axis runs from $f = f_{min}$ to $f = f_{max}$. These are the ϕ-limits of integration.

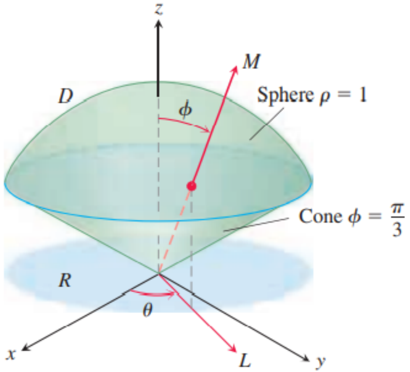
- **Step 4: Find the θ-limits of integration**. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration.

- The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Example 3

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.



Example 3

Solution The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure 15.56).

The ρ-limits of integration. We draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ-limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

The θ-limits of integration. The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3} \right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} \cos\phi \right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3} \right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

Example 4

A solid of constant density $\sigma = 1$ occupies the region D in Example 3. Find the solid's moment of inertia about the z-axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint_D (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint_D (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \iiint_D \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta.$$

For the region D in Example 5, this becomes

$$I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi\, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3} (1 - \cos^2\phi) \sin\phi\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi} \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi} \frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}. \quad \blacksquare$$

Triple Integrals in Spherical Coordinates

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$