Solutions for Preparation Sheet

Problem 1. Vectors and the Geometry of Space. Vector-Valued Functions and Motion in Space.

a) [**5 marks**] Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.

Solution:

$x = 2t,\ y = -t,\ z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3}$

$\Rightarrow \left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ is the point of intersection

b) [**6 marks**] Find a vector of magnitude 2 parallel to the line of intersection of the planes $x + 2y + z - 1 = 0$ and $x - y + 2z + 7 = 0$.

Solution:

A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{25+1+9} = \sqrt{35}$

$\Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ is the desired vector.

c) [**6 marks**] Suppose $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$. Show that the angle between $\mathbf{r}$ and $\mathbf{a}$ never changes. What is the angle?

Solution:

$$\mathbf{r} = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j} \Rightarrow \mathbf{v} = \left(e^t \cos t - e^t \sin t\right)\mathbf{i} + \left(e^t \sin t + e^t \cos t\right)\mathbf{j}$$

$$\Rightarrow \mathbf{a} = \left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\mathbf{i} + \left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)\mathbf{j} = \left(-2e^t \sin t\right)\mathbf{i} + \left(2e^t \cos t\right)\mathbf{j}.$$

Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}||\mathbf{a}|}\right) = \cos^{-1}\left(\frac{-2e^{2t}\sin t\cos t + 2e^{2t}\sin t\cos t}{\sqrt{\left(e^t\cos t\right)^2 + \left(e^t \sin t\right)^2}\sqrt{\left(-2e^t\sin t\right)^2 + \left(2e^t\cos t\right)^2}}\right)$

$= \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2}$ for all t

d) **[8 marks]** Find equations for the osculating, normal, and rectifying planes of the curve $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ at the point $(1,1,1)$.

Solution:

$\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2+9t^4} \Rightarrow |\mathbf{v}(1)| = \sqrt{14} \Rightarrow \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i}+\frac{2}{\sqrt{14}}\mathbf{j}+\frac{3}{\sqrt{14}}\mathbf{k}$, which is normal to the normal plane $\Rightarrow \frac{1}{\sqrt{14}}(x-1)+\frac{2}{\sqrt{14}}(y-1)+\frac{3}{\sqrt{14}}(z-1)=0$ or $x+2y+3z=6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j}+6t\mathbf{k}$

$\Rightarrow \mathbf{a}(1) = 2\mathbf{j}+6\mathbf{k} \Rightarrow \mathbf{v}(1)\times\mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i}-6\mathbf{j}+2\mathbf{k} \Rightarrow |\mathbf{v}(1)\times\mathbf{a}(1)| = \sqrt{76} \Rightarrow \kappa(1) = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} = \frac{\sqrt{19}}{7\sqrt{14}}$;

$\frac{ds}{dt} = |\mathbf{v}(t)| \Rightarrow \left.\frac{d^2s}{dt^2}\right|_{t=1} = \left.\frac{1}{2}\left(1+4t^2+9t^4\right)^{-1/2}\left(8t+36t^3\right)\right|_{t=1} = \frac{22}{\sqrt{14}}$, so $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T}+\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$

$\Rightarrow 2\mathbf{j}+6\mathbf{k} = \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}}\right)+\frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2\mathbf{N} \Rightarrow \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\mathbf{i}-\frac{8}{7}\mathbf{j}+\frac{9}{7}\mathbf{k}\right)$

$\Rightarrow -\frac{11}{7}(x-1)-\frac{8}{7}(y-1)+\frac{9}{7}(z-1)=0$ or or $11x+8y-9z=10$ is an equation of the rectifying plane. Finally,

$\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i}-3\mathbf{j}+\mathbf{k}) \Rightarrow 3(x-1)-3(y-1)+(z-1)=0$ or $3x-3y+z=1$ is an equation of the osculating plane.

Total: 25 mark

Problem 2. Partial Derivatives.

a) [**7 marks**] What is the largest value that the directional derivative of $f(x,y,z) = xyz$ can have at the point $(1,1,1)$?

Solution:

$f(x,y,z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1,1,1)$ we get $\nabla f = \mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow$ the maximum value of $D_{\mathbf{u}}f\big|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

b) [**10 marks**] Find the extreme values of $f(x,y,z) = x(y+z)$ on the curve of intersection of the right circular cylinder $x^2+y^2=1$ and the hyperbolic cylinder $xz=1$.

Solution:

$\nabla f = (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda\nabla g + \mu\nabla h \Rightarrow (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$ $= \lambda(2x\mathbf{i}+2y\mathbf{j}) + \mu(z\mathbf{i}+x\mathbf{k}) \Rightarrow y+z = 2\lambda x + \mu z$, $x = 2\lambda y$, $x = \mu x \Rightarrow x=0$ or $\mu = 1$.

CASE1: $x=0$ which is impossible since $xz=1$.

CASE 2: $\mu = 1 \Rightarrow y+z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y=0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz=1$ we obtain the points $(1,0,1)$ and $(-1,0,-1)$. If $4\lambda^2=1$, then $\lambda = \pm\frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y=-x$ so $x^2+y^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$. For $\lambda = \frac{1}{2}$, $y = x$ $\Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.

Evaluations give $f(1,0,1) = 1$, $f(-1,0,-1) = 1$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{3}{2}$, and $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$, and the absolute minimum is $\frac{1}{2}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

c) [**8 marks**] Find the points on the surface $(y+z)^2 + (z-x)^2 = 16$ where the normal line is parallel to the yz-plane.

Solution:

$(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y+z = \pm 4$. Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t\pm 4, t)$, t a real number.

Total: 25 mark

Problem 3. Multiple Integrals. Integrals and Vector Fields

a) [**7 marks**] Find the area of the "triangular" region in the xy-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.

Solution:

$$A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx\, dy = \int_1^4 \left(\sqrt{y} - 2 + y\right) dy = \tfrac{37}{6}$$

b) [**10 marks**] Convert

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 \;\; dz\, r\, dr\, d\theta, \qquad r \geq 0$$

to (a) rectangular coordinates with the order of integration $dz\, dx\, dy$ and

(b) spherical coordinates.

Then (c) evaluate one of the integrals.

Solution:

(a) $\displaystyle \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\, dz\, dx\, dy$

(b) $\displaystyle \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 3\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(c) $\displaystyle \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi} \int_0^{\sqrt{2}} \left[r\left(4-r^2\right)^{1/2} - r^2 \right] dr\, d\theta = 3\int_0^{2\pi} \left[-\tfrac{1}{3}\left(4-r^2\right)^{3/2} - \tfrac{r^3}{3} \right]_0^{\sqrt{2}} d\theta$

$$= \int_0^{2\pi} \left(-2^{3/2} - 2^{3/2} + 4^{3/2}\right) d\theta = \left(8 - 4\sqrt{2}\right) \int_0^{2\pi} d\theta = 2\pi\left(8 - 4\sqrt{2}\right)$$

c) [**8 marks**] Use Green's Theorem to find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of D: the entire surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.

Solution:

$$\tfrac{\partial}{\partial x}(xz) = z, \tfrac{\partial}{\partial z}(yz) = z, \tfrac{\partial}{\partial z}(1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow \text{Flux} = \iiint_D 2z\, r\, dr\, d\theta\, dz$$

$$\int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25-r^2}} 2z\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^4 r\left(16 - r^2\right) dr\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$$

Total: 25 mark

Problem 4. Infinite Sequences and Series. Fourier series.

a) **[15 marks]** Given

$$\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\,3^n}$$

(i) find the series' radius and interval of convergence. Then identify the values of x for which the series converges

(ii) absolutely and

(iii) conditionally.

Solution:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x+4)^n}\right|<1 \Rightarrow \frac{|x+4|}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)<1 \Rightarrow \frac{|x+4|}{3}<1 \Rightarrow |x+4|<3 \Rightarrow -3<x+4<3$$

$\Rightarrow -7<x<-1$; at $x=-7$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x=-1$ we have $\sum_{n=1}^{\infty}\frac{3^n}{n3^n}=\sum_{n=1}^{\infty}\frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7\le x<-1$

(b) the interval of absolute convergence is $-7<x<-1$

(c) the series converges conditionally at $x=-7$

b) **[10 marks]** The series

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$$

is the value of the Maclaurin series at $x=0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

Solution:

The given series has the form $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots=\sin x$, where $x=\pi$; the sum is $\sin\pi=0$

Total: 25 mark