# Determinants

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# Outline


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We have worked with vectors for some time, and we have often felt the need of a method to determine when vectors are linearly independent. Up to now, the only method available to us was to solve a system of linear equations by the elimination method. In this chapter, we shall exhibit a very efficient computational method to solve linear equations, and determine when vectors are linearly independent.

# Determinants of Order 2

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a $2 \times 2$ matrix. We define its determinant to be $ad - bc$. Thus the determinant is a number. We denote it by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For example, the determinant of the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

is equal to $2 \cdot 4 - 1 \cdot 1 = 7$.

The determinant can be viewed as a function of the matrix $A$. It can also be viewed as a function of its two columns. Let these be $A^1$ and $A^2$ as usual. Then we write the determinant as

$$D(A),\ Det(A), \quad \text{or} \quad D(A^1, A^2).$$

The following properties are easily verified by direct computation, which you should carry out completely.

Property 1. As a function of the column vectors, the determinant is linear. This means: suppose for instance $A^1 = C + C'$ is a sum of two columns. Then

$$D(C + C', A^2) = D(C, A^2) + D(C', A^2).$$

If $x$ is a number, then

$$D(xA^1, A^2) = xD(A^1, A^2).$$

A similar formula holds with respect to the second variable.

Property 2. If the two columns are equal, then the determinant is equal to 0.

Property 3. If $I$ is the unit matrix, $I = (E^1, E^2)$, then

$$D(I) = D(E^1, E^2) = 1.$$

Using only the above three properties we can prove others as follows.

If one adds a scalar multiple of one column to the other, then the value of the determinant does not change.

In other words, let $x$ be a number. Then

$$D(A^1 + xA^2, A^2) = D(A^1, A^2).$$

If the two columns are interchanged, then the determinant changes by a sign.

In other words, we have $D(A^2, A^1) = -D(A^1, A^2)$.

The determinant of $A$ is equal to the determinant of its transpose, i.e. $D(A) = D(A^t)$.
The vectors $A^1, A^2$ are linearly dependent if and only if the determinant is 0.

## Theorem

*Let $\phi$ be a function of two vector variables $A^1, A^2 \in \mathbb{R}^2$ such that:*
*$\phi$ is bilinear, that is $\phi$ is linear in each variable.*
*$\phi(A^1, A^1) = 0$ for all $A^1 \in \mathbb{R}^2$.*
*$\phi(E^1, E^2) = 1$ if $E^1, E^2$ are the standard unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.*

*Then $\phi(A^1, A^2)$ is the determinant.*

## Proof.

Write

$$A^1 = aE^1 + cE^2 \quad \text{and} \quad A^2 = bE^1 + dE^2.$$

Then

$$\begin{aligned} \phi(A^1, A^2) &= \phi(aE^1 + cE^2, bE^1 + dE^2) \\ &= ad - bc. \end{aligned}$$

$\square$

# $3 \times 3$ and $n \times n$ Determinants

Let

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a $3 \times 3$ matrix. We define its determinant according to the formula known as the expansion by a row, say the first row. That is, we define

$$\begin{aligned} Det(A) &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \end{aligned}$$

We may describe this sum as follows. Let $A_{ij}$ be the matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column. Then the sum expressing $Det(A)$ can be written

$$a_{11}Det(A_{11}) - a_{12}Det(A_{12}) + a_{13}Det(A_{13}).$$

In other words, each term consists of the product of an element of the first row and the determinant of the $2 \times 2$ matrix obtained by deleting the first row and the $j$-th column, and putting the appropriate sign to this term as shown.

Furthermore, there is no particular reason why we selected the expansion according to the first row. We can also use the second row, and write a similar sum, namely:

$$\begin{aligned} Det(A) &= -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ &= -a_{21} Det(A_{21}) + a_{22} Det(A_{22}) - a_{23} Det(A_{23}). \end{aligned}$$

Again, each term is the product of $a_{2j}$, the determinant of the $2 \times 2$ matrix obtained by deleting the second row and $j$-th column, and putting the appropriate sign in front of each term. This sign is determined according to the pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

One can see directly that the determinant can be expanded according to any row by multiplying out all the terms, and expanding the $2 \times 2$ determinants, thus obtaining the determinant as an alternating sum of six terms:

$$\begin{aligned} Det(A) = & a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{21}a_{23}a_{31} \\ & + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

We can also expand according to columns following the same principle. For instance, the expansion according to the first column:

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

yields precisely the same six terms.

In the case of $3 \times 3$ determinants, we therefore have the following result.

**Theorem**

*The determinant satisfies the rule for expansion according to rows and columns, and* $Det(A) = Det(A^t)$. *In other words, the determinant of a matrix is equal to the determinant of its transpose.*

Next, let $A = (a_{ij})$ be an arbitrary $n \times n$ matrix. Let $A_{ij}$ be the $(n-1) \times (n-1)$ matrix obtained by deleting the $i$-th row and $j$-th column from $A$. We give an expression for the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices. Let $i$ be an integer, $1 \leq i \leq n$. We define

$$D(A) = (-1)^{i+1} a_{i1} Det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} Det(A_{in}).$$

This sum can be descried in words. For each element of the $i$-th row, we have a contribution of one term in the sum. This term is equal to $+$ or $-$ the product of this element, times the determinant of the matrix obtained from $A$ by deleting the $i$-th row and the corresponding column. The sign $+$ or $-$ is determined according to the chess-board pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \end{pmatrix}.$$

This sum is called the expansion of the determinant according to the $i$-th row.

In particular, the determinant satisfies the rule of expansion according to the $j$-th column, for any $j$. Thus we have the expansion formula:

$$D(A) = (-1)^{1+j} a_{1j} Det(A_{1j}) + \cdots + (-1)^{n+j} a_{nj} Det(A_{nj}).$$

In practice, the computation of a determinant is always done by using an expansion according to some row or column.

**Theorem**

*The determinant satisfies the following properties:*
*1. As a function of each column vector, the determinant is linear.*
*2. If two columns are equal, i.e. if $A^j = A^k$, with $j \neq k$, then the determinant $D(A)$ is equal to $0$.*
*3. If $I$ is the unit matrix, then $D(I) = 1$.*

A function of several variables which is linear in each variable, i.e. which satisfies the first property of determinants, is called multilinear. A function which satisfies the second property is called alternating.

To compute determinants efficiently, we need additional properties.
4. Let $j, k$ be integers with $1 \leq j \leq n$ and $1 \leq k \leq n$, and $j \neq k$. If the $j$-th column and $k$-th column are interchanged, then the determinant changes by a sign.
5. If one adds a scalar multiple of one column to another then the value of the determinant does not change.
Since the determinant of a matrix is equal to the determinant of its transpose, that is $Det(A) = Det(A^t)$, we obtain the following general fact: All the properties stated above for rows or column operations are valid for both row and column operations.

Compute the determinant

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 5 \\ -1 & 4 & 2 \end{vmatrix}.$$

Compute the determinant

$$\begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 1 & 5 & 2 \\ 1 & -1 & 2 & 3 \\ 4 & 1 & -3 & 7 \end{vmatrix}.$$

# The Rank of a Matrix and Subdeterminants

In this section we give a test for linear independence by using determinants.

**Theorem**

*Let $A^1, ..., A^n$ be column vectors of dimension $n$. They are linearly dependent if and only if*

$$D(A^1, ..., A^n) = 0.$$

## Proof

Suppose $A^1, ..., A^n$ are linearly dependent, so there exists a relation

$$x_1 A^1 + \cdots + x_n A^n = O$$

with numbers $x_1, ..., x_n$ not all 0. Say $x_j \neq 0$. Subtracting and dividing by $x_j$, we can find numbers $c_k$ with $k \neq j$ such that

$$A^j = \sum_{k \neq j} c_k A^k$$

Thus

$$D(A) = D(A^1, ..., \sum_{k \neq j} c_k A^k, ..., A^n)$$
$$\sum_{k \neq j} c_k D(A^1, ..., A^k, ..., A^n)$$

where $A^k$ occurs in the $j$-th place. But $A^k$ also occurs in the $k$-th place and $k \neq j$. Hence the determinant is equal to 0. This concludes the proof of the first part.

## Proof

As to converse, we recall that a matrix is row equivalent to an echelon matrix. Suppose that $A^1, ..., A^n$ are linearly independent. Then the matrix

$$A = (A^1, ..., A^n)$$

is row equivalent to a triangular matrix. Indeed, it is row equivalent to a matrix $B$ in echelon form

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$$

and the operations of row equivalence do not change the property of rows or columns being linearly independent. Hence all the diagonal elements $b_{11}, ..., b_{nn}$ are $\neq 0$. The determinant of this matrix is the product

$$b_{11} \cdots b_{nn} \neq 0$$

## Proof.

by the rule of expansion according to columns. Under the operations of row equivalence, the property of the determinant being $\neq 0$ does not change, because row equivalences involve multiplying a row by a nonzero scalar which multiplies the determinant by this scalar; or interchanging rows, which multiplies the determinant by $-1$; or adding a multiple of one row to another, which does not change the value of the determinant. Since $Det(B) \neq 0$ it follows that $Det(A) \neq 0$. This concludes the proof. $\square$

## Corollary

If $A^1, ..., A^n$ are column vectors of $\mathbb{R}^n$ such that $D(A^1, ..., A^n) \neq 0$, and if $B$ is a column vector, then there exist numbers $x_1, ..., x_n$ such that

$$x_1 A^1 + \cdots + x_n A^n = B.$$

These numbers are uniquely determined by $B$.

In terms of linear equations, this corollary shows:
If a system of $n$ linear equations in $n$ unknowns has a matrix of coefficients whose determinant is not 0, then this system has a unique solution.

Let

$$A = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

This is a $3 \times 4$ matrix. Its rank is at most 3. If we can find three linearly independent columns, then we know that its rank is exactly 3. But the determinant

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 4$$

is not equal to 0. Hence rank $A = 3$.

It may be that in a $3 \times 4$ matrix, some determinant of a $3 \times 3$ submatrix is 0, but the $3 \times 4$ matrix has rank 3. For instance, let

$$B = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 1 \end{pmatrix}.$$

The determinant of the first three columns

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & -1 \\ 4 & 3 & 1 \end{vmatrix}$$

is equal to 0. But the determinant

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix}$$

is not zero so that again the rank of $B$ is equal to 3.

If the rank of a $3 \times 4$ matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{pmatrix}$$

is 2 or less, then determinant of every $3 \times 3$ submatrix must be 0, otherwise we would argue as above to get three linearly independent columns. We note that there are four such subdeterminants, obtained by eliminating successively any one of the four columns. Conversely, if every such subdeterminant of every $3 \times 3$ submatrix is equal to 0, then it is easy to see that the rank is at most 2. Because if the rank were equal to 3, then there would be three linearly independent columns, and their determinant would not be 0. Thus we can compute such subdeterminants to get an estimate on the rank, and then use trial and error, and some judgement, to get the exact rank.

Let

$$C = \begin{pmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 7 \end{pmatrix}.$$

If we compute every $3 \times 3$ subdeterminant, we shall find 0. Hence the rank of $C$ is at most equal to 2. However, the first two rows are linearly independent, for instance because the determinant

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

is not equal to 0. Hence the rank is equal to 2.

Of course, if we notice that the last row of $C$ is equal to the sum of the first two, then we see at once that the rank is $\leq 2$.

# Cramer's Rule

The properties of determinants can be used to prove a well-known rule used in solving linear equations.

## Theorem

*Let $A^1, ..., A^n$ be column vectors such that*

$$D(A^1, ..., A^n) \neq 0.$$

*Let $B$ be a column vector. If $x_1, ..., x_n$ are numbers such that*

$$x_1 A^1 + \cdots + x_n A^n = B,$$

*then for each $j = 1, ..., n$ we have*

$$x_j = \frac{D(A^1, ..., B, ..., A^n)}{D(A^1, ..., A^n)}$$

*where $B$ occurs in the $j$-th column instead of $A^j$.*

Solve the system of linear equations:

$$\begin{array}{rcrcrcl} 3x & + & 2y & + & 4z & = & 1 \\ 2x & - & y & + & z & = & 0 \\ x & + & 2y & + & 3z & = & 1. \end{array}$$

# Inverse of a Matrix

Let $A$ be an $n \times n$ matrix. If $B$ is a matrix such that $AB = I$ and $BA = I$ ($I =$ unit $n \times n$ matrix), then we called $B$ an inverse of $A$, and we write $B = A^{-1}$. If there exists an inverse of $A$, then it is unique. Indeed, let $C$ be an inverse of $A$. Then $CA = I$. Multiplying by $B$ on the right, we obtain $CAB = B$. But $CAB = C(AB) = CI = C$, Hence $C = B$.

**Theorem**

*Let $A = (a_{ij})$ be an $n \times n$ matrix, and assume that $D(A) \neq 0$. Then $A$ is invertible. Let $E^j$ be the $j$-th column unit vector, and let*

$$b_{ij} = \frac{D(A^1, \ldots, E^j, \ldots, A^n)}{D(A)},$$

*where $E^j$ occurs in the $i$-th place. Then the matrix $B = (b_{ij})$ is an inverse for $A$.*

Let $A_{ij}$ be the matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column. Then

$$b_{ij} = \frac{(-1)^{i+j} Det(A_{ji})}{Det(A)}$$

(note the reversal of indices !) and thus we have the formula

$$A^{-1} = \text{transpose of } \left( \frac{(-1)^{i+j} Det(A_{ji})}{Det(A)} \right).$$

A square matrix whose determinant is $\neq 0$, or equivalently which admits an inverse, is called non-singular.

Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix}.$$

**Theorem**

*For any two $n \times n$ matrices $A, B$ the determinant of the product is equal to the product of the determinants, that is*

$$Det(AB) = Det(A)Det(B).$$

Then as a special case, we find that for an invertible matrix $A$,

$$Det(A^{-1}) = Det(A)^{-1}.$$