Lecture 38: Infinite Sequences and Series.

MA2032 Vector Calculus

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Fourier Series

• We can express a given function as an **infinite sum of sines and/or cosines**.

• These trigonometric series are called **Fourier series**;

• they are somewhat analogous to Taylor series in that both types of series provide a means of **expressing quite complicated functions in terms of certain familiar elementary functions**.

Fourier Series

• We begin with a series of the form Eq. (1)

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

• On the set of points where the **series (1) converges**, it defines a function f, whose value at each point is the sum of the series for that value of x.

• In this case the series (1) is said to be the **Fourier series for** f.

• Our **immediate goals** are to determine **what functions** can be represented by a Fourier series and to **find the coefficients** a_0**,** a_m **and** b_m.

• Fourier series are useful in **partial differential equations**, in the analysis of **mechanical or electrical systems** acted on by periodic external forces, and etc.

Periodicity of the Sine and Cosine Functions

- To discuss Fourier series, it is necessary to develop **certain properties of the trigonometric functions** $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer.
- The **first property** is their **periodic character**.
- A function f is said to be **periodic with period** $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if Eq. (2)

$$f(x + T) = f(x)$$

- for every value of x.

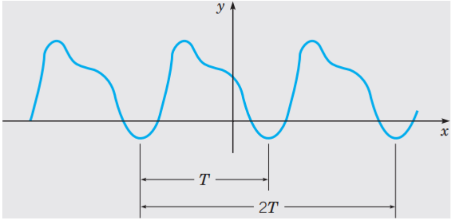

FIGURE A periodic function of period T.

Periodicity of the Sine and Cosine Functions

• If T is a period of f, then $2T$ is also a period, and so indeed is **any multiple of T**.

• The smallest value of T for which Eq. (2) holds is called the **fundamental period of f**.

• If f and g are any two periodic functions with common period T, then any **linear combination** $c_1 f + c_2 g$ is **also periodic** with period T.

• To prove this statement, let $F(x) = c_1 f(x) + c_2 g(x)$; then, for any x,

$$F(x+T) = c_1 f(x+T) + c_2 g(x+T) = c_1 f(x) + c_2 g(x) = F(x).$$

• Moreover, it can be shown that the **sum of any finite number**, or even the sum of a convergent infinite series, of functions of period T is also periodic with period T.

In a similar way, you can show that the **product $f\, g$ is periodic** with period T.

Orthogonality of the Sine and Cosine Functions

• In particular, the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, 3, \ldots$, are periodic with **fundamental period** $T = 2L/m$ and **common period** $2L$.

• To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, we **generalize the concept of orthogonality** of vectors.

• The **standard inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \leq x \leq \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx.$$

• The functions u and v are said to be **orthogonal** on $\alpha \leq x \leq \beta$ if their inner **product is zero**, that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0.$$

Orthogonality of the Sine and Cosine Functions

• A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n; \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0, \qquad \text{all } m, n;$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

Orthogonality of the Sine and Cosine Functions

• These results can be obtained by **direct integration**. For example, to derive last Equation

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right]dx$$

$$= \frac{1}{2}\frac{L}{\pi}\left\{\frac{\sin[(m-n)\pi x/L]}{m-n} - \frac{\sin[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L}$$

$$= 0$$

• as long as $m+n$ and $m-n$ are not zero.

• Since m and n are positive, $m+n \neq 0$.

• On the other hand, if $mn = 0$, then $m = n$, and the integral must be **evaluated in different way**.

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \int_{-L}^{L} \left(\sin\frac{m\pi x}{L}\right)^2 dx$$

$$= \frac{1}{2}\int_{-L}^{L} \left[1 - \cos\frac{2m\pi x}{L}\right] dx$$

$$= \frac{1}{2}\left\{x - \frac{\sin(2m\pi x/L)}{2m\pi/L}\right\}\Bigg|_{-L}^{L}$$

$$= L.$$

The Euler–Fourier Formulas. Now let us suppose that a series of the form (1) converges, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{9}$$

The coefficients a_m and b_m can be related to $f(x)$ as a consequence of the orthogonality conditions (6), (7), and (8). First multiply Eq. (9) by $\cos(n\pi x/L)$, where n is a *fixed* positive integer ($n > 0$), and integrate with respect to x from $-L$ to L. Assuming that the integration can be legitimately carried out term by term,[3] we obtain

$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx. \tag{10}$$

The Euler–Fourier Formulas

Keeping in mind that n is fixed whereas m ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only nonzero term on the right side of Eq. (10) is the one for which $m = n$ in the first summation. Hence

$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = La_n, \qquad n = 1, 2, \ldots. \tag{11}$$

To determine a_0, we can integrate Eq. (9) from $-L$ to L, obtaining

$$\begin{aligned} \int_{-L}^{L} f(x)\, dx &= \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx \\ &= La_0, \end{aligned} \tag{12}$$

The Euler–Fourier Formulas

since each integral involving a trigonometric function is zero. Thus

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots. \tag{13}$$

By writing the constant term in Eq. (9) as $a_0/2$, it is possible to compute all the a_n from Eq. (13). Otherwise, a separate formula would have to be used for a_0.

A similar expression for b_n may be obtained by multiplying Eq. (9) by $\sin(n\pi x/L)$, integrating termwise from $-L$ to L, and using the orthogonality relations (7) and (8); thus

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{14}$$

Equations (13) and (14) are known as the Euler–Fourier formulas for the coefficients in a Fourier series. Hence, if the series (9) converges to $f(x)$, and if the series can be integrated term by term, then the coefficients *must be given* by Eqs. (13) and (14).

Fourier series. Example

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad (15)$$
$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus in this case $L = 2$, and the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{2} + b_m \sin \frac{m\pi x}{2} \right), \qquad (16)$$

where the coefficients are computed from Eqs. (13) and (14) with $L = 2$. Substituting for $f(x)$ in Eq. (13) with $m = 0$, we have

$$a_0 = \frac{1}{2} \int_{-2}^{0} (-x)\, dx + \frac{1}{2} \int_{0}^{2} x\, dx = 1 + 1 = 2. \qquad (17)$$

Fourier series. Example

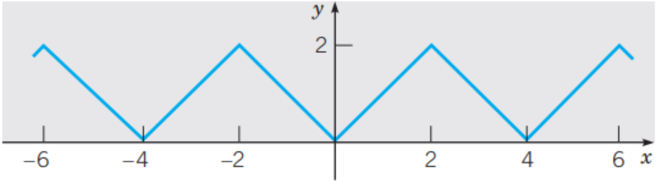

FIGURE The triangular wave in Example 1.

For $m > 0$, Eq. (13) yields

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{m\pi x}{2}\,dx.$$

These integrals can be evaluated through integration by parts, with the result that

Fourier series. Example

$$
\begin{aligned}
a_m &= \frac{1}{2}\left[-\frac{2}{m\pi}x\sin\frac{m\pi x}{2} - \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{-2}^{0} \\
&\quad + \frac{1}{2}\left[\frac{2}{m\pi}x\sin\frac{m\pi x}{2} + \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{0}^{2} \\
&= \frac{1}{2}\left[-\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2\cos m\pi + \left(\frac{2}{m\pi}\right)^2\cos m\pi - \left(\frac{2}{m\pi}\right)^2\right] \\
&= \frac{4}{(m\pi)^2}(\cos m\pi - 1), \qquad m = 1, 2, \ldots \\
&= \begin{cases} -8/(m\pi)^2, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \qquad (18)
\end{aligned}
$$

Finally, from Eq. (14) it follows in a similar way that

$$b_m = 0, \qquad m = 1, 2, \ldots. \tag{19}$$

By substituting the coefficients from Eqs. (17), (18), and (19) in the series (16), we obtain the Fourier series for f:

$$\begin{aligned} f(x) &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\cos(m\pi x/2)}{m^2} \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}. \end{aligned} \tag{20}$$