Solutions for Tutorial Problem Sheet 6, November 3.

(Partial Derivatives.)

Problem 1. Find equations for the

(a) tangent plane and

(b) normal line at the point $P_0(0,0,1)$ on the given surface defined by $ye^x - ze^{y^2} = z$.

Solution:

(a) $ye^x - ze^{y^2} - z = 0 \Rightarrow \nabla f = \left(ye^x\right)\vec{\mathbf{i}} + \left(e^x - 2yze^{y^2}\right)\vec{\mathbf{j}} + \left(-e^{y^2} - 1\right)\vec{\mathbf{k}} \Rightarrow \nabla f(0,0,1) = (0)\vec{\mathbf{i}} + (1)\vec{\mathbf{j}} + (-2)\vec{\mathbf{k}}$

$\Rightarrow$ Tangent plane: $y - 2(z-1) = 0$

(b) Normal line: $x = 0,\ y = t,\ z = 1 - 2t$

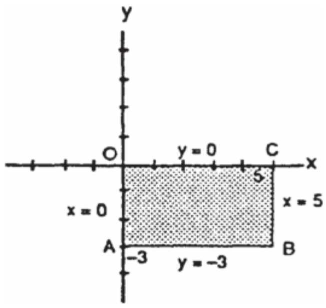
Problem 2. Find the linearization $L(x, y)$ of the function $f(x, y) = x^2 - 3xy + 5$ at $P_0(2, 1)$. Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approxeq L(x, y)$ over the rectangle $R : |x - 2| \le 0.1, \quad |y - 1| \le 0.1$.

Solution:

$f(2,1) = 3$, $f_x(x, y) = 2x - 3y \Rightarrow f_x(2,1) = 1$, $f_y(x, y) = -3x \Rightarrow f_y(2,1) = -6 \Rightarrow L(x, y) = 3 + 1(x-2) - 6(y-1)$ $= 7 + x - 6y$; $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 0$, $f_{xy}(x, y) = -3 \Rightarrow M = 3$; thus $|E(x, y)| \le \left(\frac{1}{2}\right)(3)\left(|x-2| + |y-1|\right)^2$ $\le \left(\frac{3}{2}\right)(0.1 + 0.1)^2 = 0.06$

Problem 3. Find the absolute maxima and minima of the function $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5, \quad -3 \le y \le 0$.

Solution:



(i) On $OC, T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \le x \le 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7, T(0, 0) = 2,$ and $T(5, 0) = -3$

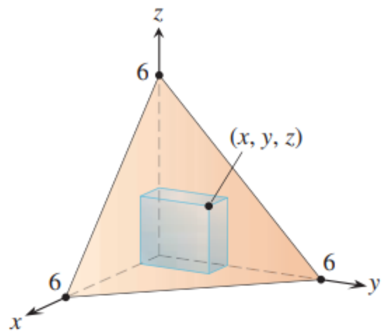
(ii) On $CB, T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \le y \le 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{37}{4}$ and $T(5, -3) = -9$

(iii) On $AB, T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{37}{4}$ and $T(0, -3) = 11$

(iv) On $AO, T(x, y) = T(0, y) = y^2 + 2$ on $-3 \le y \le 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.

Problem 4. A rectangular box is inscribed in the region in the first octant bounded above by the plane with x-intercept 6, y-intercept 6, and z-intercept 6.



a) Find an equation for the plane.

b) Find the dimensions of the box of maximum volume.

Solution:

(a) plane: $x+y+z=6$

(b) Minimize volume $V(x,y,z)=xyz$; $z=6-x-y \Rightarrow V(x,y)=xy(6-x-y)=6xy-x^2y-xy^2 \Rightarrow$ $V_x(x,y)=6y-2xy-y^2=y(6-2x-y)=0$ and $V_y(x,y)=6x-x^2-2xy=x(6-x-2y)=0 \Rightarrow$ critical point is $(2,2)$; $V_{xx}(2,2)=-4$, $V_{yy}(2,2)=-4$, $V_{xy}(2,2)=-2 \Rightarrow V_{xx}V_{yy}-\left(V_{xy}\right)^2=12>0$ and $V_{xx}<0 \Rightarrow$ local maximum of $V(2,2,2)=8$

Problem 5. Use the method of Lagrange multipliers to find
a) Minimum on a hyperbola: The minimum value of $x+y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$.
b) Maximum on a line: The maximum value of xy, subject to the constraint $x + y = 16$.
Comment on the geometry of each solution.

Solution:

(a) $\nabla f = \mathbf{i}+\mathbf{j}$ and $\nabla g = y\mathbf{i}+x\mathbf{j}$ so that $\nabla f = \lambda\nabla g \Rightarrow \mathbf{i}+\mathbf{j} = \lambda(y\mathbf{i}+x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4, 4)$. Now, $xy = 16, x > 0, y > 0$ is a branch of a hyperbola in the first quadrant with the x- and y-axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.

(b) $\nabla f = y\mathbf{i}+x\mathbf{j}$ and $\nabla g = \mathbf{i}+\mathbf{j}$ so that *s* $\nabla f = \lambda\nabla g \Rightarrow y\mathbf{i}+x\mathbf{j} = \lambda(\mathbf{i}+\mathbf{j}) \Rightarrow y = \lambda = x \Rightarrow y + y = 16 \Rightarrow y = 8$ $\Rightarrow x = 8 \Rightarrow f(8, 8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and

$y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x- and y-axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.