Lecture 16: Partial Derivatives.

MA2032 Vector Calculus

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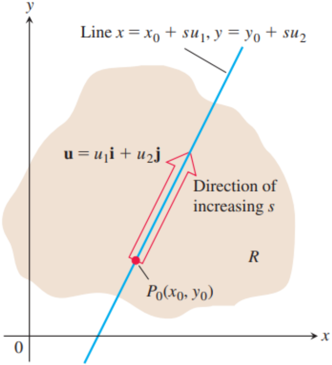
- Suppose that the function $f(x, y)$ is defined throughout a region **R** in the xy-plane, that $P_0(x_0, y_0)$ is a point in **R**, and that $u = u_1 i + u_2 j$ is a unit vector.
- Then the equations $x = x_0 + su_1$, $y = y_0 + su_2$ parametrize the line through P_0 parallel to **u**.
- If the parameter **s** measures arc length from P_0 in the direction of **u**, we find the **rate of change of f at P_0 in the direction of u by calculating df/ds at P_0.**



DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

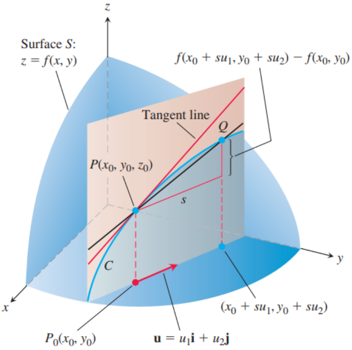
Interpretation of the Directional Derivative

• The equation $z = f(x, y)$ represents a surface S in space.

• If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S.

• The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to u intersects S in a curve C.

• The rate of change of f in the direction of u is the slope of the tangent to C at P in the right-handed system formed by the vectors u and k.



• The partial derivatives $f_x(x_0, y_0)$ and $_y(x_0, y_0)$ are the **directional derivatives** of f at P_0 in the **i** and **j** directions.

Calculation and Gradients

- We now develop an **efficient formula** to calculate the directional derivative for a differentiable function f.
- By the Chain Rule we find

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \left.\frac{\partial f}{\partial x}\right|_{P_0}\frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0}\frac{dy}{ds} && \text{Chain Rule for differentiable } f\\
&= \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 && \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2\\
&= \underbrace{\left[\left.\frac{\partial f}{\partial x}\right|_{P_0}\mathbf{i} + \left.\frac{\partial f}{\partial y}\right|_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}.
\end{aligned}$$

DEFINITION The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f|_{P_0} \qquad \text{or} \qquad \nabla f(x_0, y_0).$$

Directional Derivative

THEOREM 9—The Directional Derivative Is a Dot Product
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 with the vector $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

Properties of the Directional Derivative $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

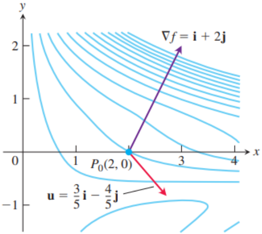
$$D_{\mathbf{u}} f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example 1

Find the derivative of $f(x, y) = xe^y + cos(xy)$ at the point $(2, 0)$ in the direction of $v = 3i - 4j$.



Solution Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))\Big|_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))\Big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.29). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned} D_{\mathbf{u}}f|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4) with the } D_{\mathbf{u}}f|_{P_0} \text{ notation} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1. \end{aligned}$$

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Example 2

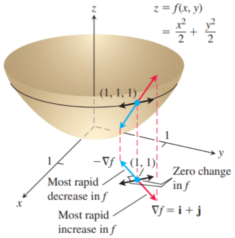
Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$
(a) increases most rapidly at the point (1, 1), and
(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$\nabla f|_{(1,1)} = (x\mathbf{i} + y\mathbf{j})\Big|_{(1,1)} = \mathbf{i} + \mathbf{j}.$$



Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

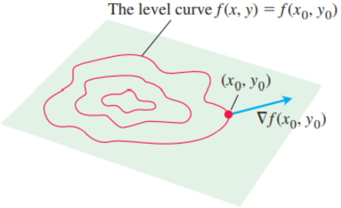
(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

Gradients and Tangents to Level Curves

• If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $r = g(t)i + h(t)j$ (making the curve part of a level curve of f), then $f(g(t), h(t)) = c$.

• Differentiating both sides of this equation with respect to t leads to the equation



$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$

> At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.31).

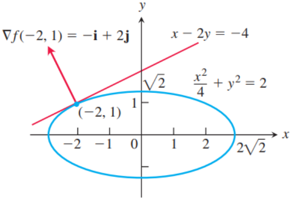
- This observation also enables us to find **equations for tangent lines to level curves**.
- They are the **lines normal to the gradients**.
- The line through a point $P_0(x_0, y_0)$ normal to a nonzero vector $N = Ai + Bj$ has the equation $A(x - x_0) + B(y - y_0) = 0$
- If N is the gradient $\nabla f|_{(x_0,y_0)} = f_x(x_0, y_0)i + f_y(x_0, y_0)j$, and this gradient is not the zero vector, then this equation gives the following formula.

Tangent Line to a Level Curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \tag{6}$$

Example 3

Find an equation for the tangent to the ellipse $x^2/4 + y^2 = 2$ at the point $(-2, 1)$.



Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Big|_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad \text{Eq. (6)}$$
$$x - 2y = -4. \qquad \text{Simplify.}$$

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

Functions of Three Variables

- For a differentiable function $f(x, y, z)$ and a unit vector $u = u_1 i + u_2 j + u_3 k$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

- The **directional derivative** can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

- so the properties listed earlier for functions of two variables extend to three variables.
- At any given point, f **increases most rapidly** in the direction of ∇f and **decreases most rapidly** in the direction of $-\nabla f$.
- In any **direction orthogonal** to ∇f, the derivative is zero.

Example 4

(a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $v = 2i - 3j + 6k$.
(b) In what directions does f change most rapidly at P_0 , and what are the rates of change in these directions?.

Solution

(a) The direction of **v** is obtained by dividing **v** by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

Example 4. Solution

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3. \qquad \blacksquare$$

The Chain Rule for Paths

• If $r(t) = x(t)i + y(t)j + z(t)k$ is a smooth path C, and $w = f(r(t))$ is a scalar function evaluated along C, then according to the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

• The partial derivatives on the right-hand side of the above equation are **evaluated along the curve** $r(t)$, and the derivatives of the intermediate variables are **evaluated at t**.

• If we express this equation using **vector notation**, we have

The Derivative Along a Path

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \tag{7}$$

• What Equation (7) says is that the derivative of the **composite function f(r(t))** is the “derivative” (gradient) of the **outside function f** “times” (dot product) the derivative of the **inside function r**.