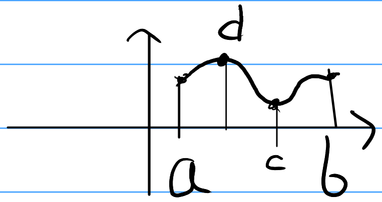
Applications of derivatives

10 Nov 2021

If $f:[a,b] \longrightarrow \mathbb{R}$ cts & differentiable on (a,b)

then

Rolle's Theorem If $f(a) = f(b)$ then $\exists c \in (a,b)$ such that $f'(c) = 0$



i.e. the function has a turning point.

Proof ① If f is constant

$f(x) = f(a) = f(b) \quad \forall x$

$\Rightarrow \quad f'(c) = 0 \quad \forall c$

② If f is not constant

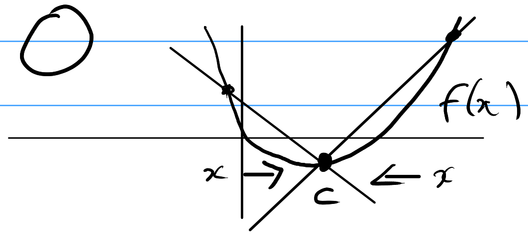
Extreme Value Theorem (f continuous)

f attains its min $f(c)$ & max $f(d)$

We need to prove $f'(c) = 0$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

≤ 0



≥ 0

so $f'(c) = 0$

Mean Value Theorem :

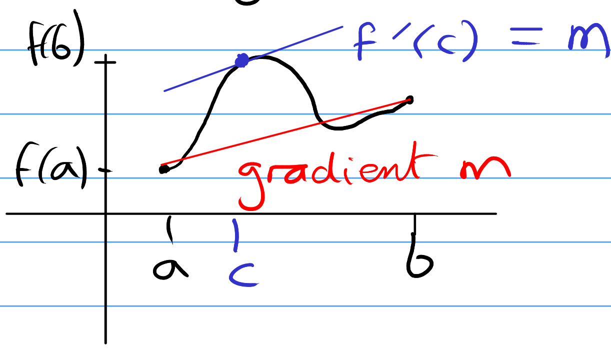
similar to Rolle's Theorem

Suppose $f : [a,b] \to \mathbb{R}$ cts
& differentiable of (a,b)

We do not assume $f(a) = f(b)$, (case $m=0$)

but we let $m = \dfrac{f(b) - f(a)}{b - a}$

$\left\{ \begin{array}{l} \text{mean slope} \\ \text{average gradient} \end{array} \right.$



MVT $\exists c \in (a,b)$ such that

$$f'(c) = m = \frac{f(b) - f(a)}{b - a}$$

Proof Rolle $\Leftrightarrow$ MVT

Given f , let $g(x) = f(x) - mx$

$g(b) = f(b) - mb$ f continuous $\Rightarrow$ g cts

$g(a) = f(a) - ma$ f differentiable $\Rightarrow$ so is g

$$g(b) - g(a) = f(b) - f(a) - m(b-a)$$

$$= \cancel{f(b) - f(a)} - \frac{\cancel{f(b) - f(a)}}{\cancel{b-a}} \cancel{(b-a)}$$

so $g(a) = g(b)$

Applying Rolle's Theorem to g

$$\exists c \in (a,b) \quad g'(c) = 0$$

$$\underline{g'(c)} = f'(c) - m = \underline{0}$$

& we have $f'(c) = m$

Even more "general" version:

// <u>Cauchy Mean Value Theorem</u>

Consider now $f, g : [a,b] \longrightarrow \mathbb{R}$

continuous & differentiable on (a,b)

Then $\exists c \in (a,b)$ such that

$$f'(c)\left(\underline{g(b) - g(a)}\right)$$
$$= \underline{g'(c)}\left(f(b) - f(a)\right)$$

Special case: $g(a) = g(b)$

Rolle $\Rightarrow \exists c : g'(c) = 0$ ✓

If $g(a) \neq g(b)$ we can write

$\exists c : \underline{f'(c) = m\, g'(c)}$ where $m = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

Why is this true?

Consider $k(x) = f(x) - m\,g(x)$

f, g cts & diff^{ble} $\Rightarrow k$ is too

$$k(b) = f(b) - m \cdot g(b)$$
$$k(a) = f(a) - m \cdot g(a)$$

$$k(b) - k(a) = \cancel{f(b) - f(a)} - \cancel{\frac{f(b)-f(a)}{g(b)-g(a)}} \cancel{(g(b) - g(a))}$$

so $k(b) = k(a)$

& we can apply Rolle's Theorem :

$$\exists\, \underline{c \in (a,b)} : k'(c) = 0$$

But $k'(x) = f'(x) - m\,g'(x)$

so $\quad f'(c) = m\,g'(c)$ ✓

L'Hôpital's rule

Two versions

① f, g cts on $[a,b]$, diff on (a,b)

Assume f', g' are continuous on (a,b)

If $\underline{f(c), g(c) = 0}$, $\underline{g'(c) \neq 0}$

then $\frac{f(x)}{g(x)}, \frac{f'(x)}{g'(x)} \longrightarrow \frac{f'(c)}{g'(c)}$ as $\underline{x \to c}$

$\frac{0}{0}$?

Proof

$$\frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{\frac{f(x) - \cancel{f(c)}}{\cancel{x - c}}}{\frac{g(x) - \cancel{g(c)}}{\cancel{x - c}}} = \lim_{x \to c} \frac{f(x)}{g(x)}$$

② $f(x), g(x) \longrightarrow 0$ as $x \to c^+$

If $\dfrac{f'(x)}{g'(x)} \longrightarrow L$ as $x \to c^+$

then $\dfrac{f(x)}{g(x)} \longrightarrow L$ as $x \to c^+$

+ Proof ?

Example

$$\lim_{x \to 0} \frac{\sin(x)}{x} \overset{①}{=} \lim_{x \to 0} \frac{\cos(x)}{1}$$

$$= \frac{\cos(0)}{1} = \underline{\underline{1}}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \overset{②}{=} \lim_{x \to 0} \frac{\sin(x)}{2x} = \underline{\underline{\frac{1}{2}}}$$

Rolle $\Leftrightarrow$ MVT $\Leftrightarrow$ Cauchy MVT

+ Hôpital + Ex