Lecture 8: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

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Integrals of Vector Functions

• Today we investigate **integrals of vector functions** and their **application to motion** along a path in space or in the plane.

• A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval I if $d\mathbf{R}/dt = \mathbf{r}$ at each point of I.

• If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on I, it can be shown, working one component at a time, that every antiderivative of $\mathbf{r}$ on I has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$.

• The set of all antiderivatives of $\mathbf{r}$ on I is the **indefinite integral** of $\mathbf{r}$ on I.

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

Example 1

- To integrate a vector function, we integrate each of its components.

Example 1:

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k} \qquad (1)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \qquad (2)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad C = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

- **Definite integrals** of vector functions are best defined in terms of **components**.
- The definition is consistent with how we compute limits and derivatives of vector functions.

> **DEFINITION** If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is
>
> $$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

• The **Fundamental Theorem of Calculus** for continuous vector functions says that

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

• where **R** is any antiderivative of **r**, so that $\mathbf{R}'(t) = \mathbf{r}(t)$.
• Notice that an **antiderivative** of a vector function is also a **vector function**, whereas a **definite integral** of a vector function is a **single constant vector**.

Example 2:

• As in Example 1, we integrate each component:

$$\begin{aligned}\int_0^\pi ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt &= \left(\int_0^\pi \cos t\,dt\right)\mathbf{i} + \left(\int_0^\pi dt\right)\mathbf{j} - \left(\int_0^\pi 2t\,dt\right)\mathbf{k} \\ &= \Big[\sin t\Big]_0^\pi \mathbf{i} + \Big[t\Big]_0^\pi \mathbf{j} - \Big[t^2\Big]_0^\pi \mathbf{k} \\ &= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k} \\ &= \pi\mathbf{j} - \pi^2\mathbf{k}\end{aligned}$$

Arc Length Along a Space Curve

• We study the mathematical **features of a curve's shape** that describe the **sharpness** of its **turning** and its **twisting**.

• One of the **features** of smooth space and plane curves is that they have a **measurable length**. This enables us to locate points along these curves by giving their **directed distance** s along the curve from some base point.

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (1)$$

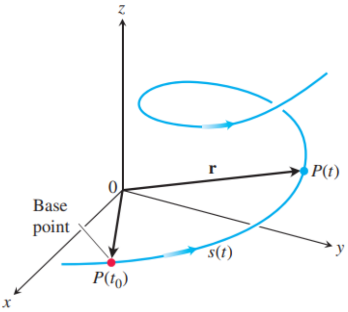
Arc Length Formula: $L = \int_b^a |v| dt$.

Arc Length in Space

- If we choose a **base point** $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a **"directed distance"**

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

measured along C from the base point.



- If $t > t_0$, $s(t)$ is the **distance** along the curve from $P(t_0)$ to $P(t)$.
- If $t < t_0$, $s(t)$ is the **negative of the distance**.
- We call s an **arc length parameter** for the curve.
- The parameter's value increases in the direction of increasing t.

Unit Tangent Vector

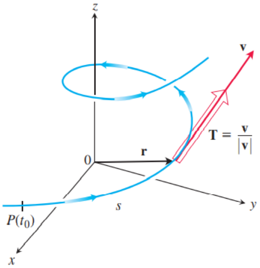
• We already know the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and that the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the (smooth) curve, called the **unit tangent vector**.

• The unit tangent vector $\mathbf{T}$ is a differentiable function of t whenever $\mathbf{v}$ is a differentiable function of t.

• $\mathbf{v}$ is one of the vectors in a traveling reference frame that is used to **describe the motion** of objects traveling **in three dimensions**.



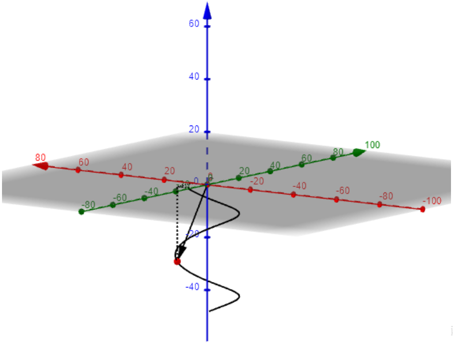
Arc Length in Space

Example 2

Find the point on the curve

$$r(t) = (12\sin t)i - (12\cos t)j + 5tk$$

at a distance 13π units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.



Solution for Example 2

Solution:

Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and

$-13\pi = \int_0^{t_0} \sqrt{144\cos^2 t + 144\sin^2 t + 25}\, dt = \int_0^{t_0} 13\, dt = 13t_0 \Rightarrow t_0 = -\pi,$ and the point is

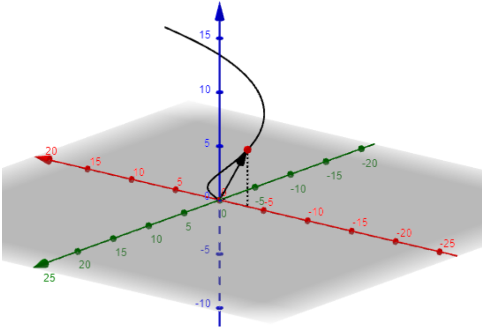
$P(-\pi) = \left(12\sin(-\pi), -12\cos(-\pi), -5\pi\right) = (0, 12, -5\pi)$

Example 3

Find the curve's

$$r(t) = (t\cos t)i + (t\sin t)j + 2\sqrt{2}/3t^{3/2}k, \quad 0 \le t \le \pi$$

unit tangent vector. Also, find the length of the indicated portion of the curve.



Solution:

$$\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \tfrac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}t^{1/2}\right)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2t}\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1, \text{ if } t \geq 0;$$

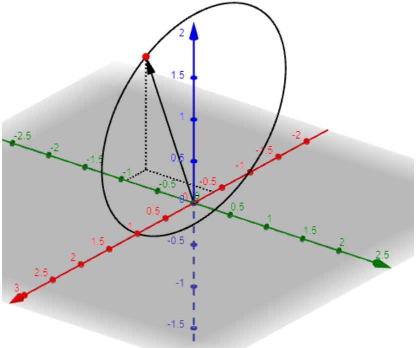
$$\mathbf{T} = \tfrac{\mathbf{v}}{|\mathbf{v}|} = \left(\tfrac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\tfrac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\tfrac{\sqrt{2}t^{1/2}}{t+1}\right)\mathbf{k} \text{ and Length } = \int_0^{\pi}(t+1)\,dt = \left[\tfrac{t^2}{2} + t\right]_0^{\pi} = \tfrac{\pi^2}{2} + \pi$$

Example 4

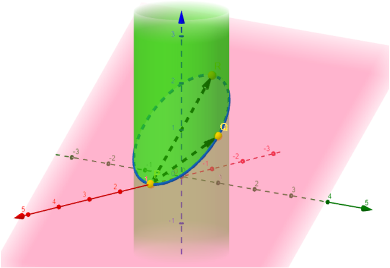
Show that the curve

$$r(t) = (\cos t)i + (\sin t)j + (1 - \cos t)k, \quad 0 \leq t \leq 2\pi,$$

is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.



Solution for Example 4



$\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1-\cos t)\mathbf{k}, 0 \le t \le 2\pi \Rightarrow x = \cos t,\ y = \sin t,\ z = 1-\cos t$

$\Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1,$ a right circular cylinder with the z-axis as the axis and radius $= 1$.

Therefore $P(\cos t, \sin t, 1-\cos t)$ lies on the cylinder $x^2 + y^2 = 1; t = 0 \Rightarrow P(1, 0, 0)$ is on the curve;

$t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and

$\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} = 2\mathbf{i} + 2\mathbf{k}$ is a vector normal to the plane of P, Q, and R. Then

the plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.