Problem Sheet 5 for the Tutorial, October 27.

(Partial Derivatives.)

Problem 1. Find partial derivatives f_x, f_y, and f_z of the functions:

a) $f(x, y, z) = x - \sqrt{y^2 + z^2}$.
b) $f(x, y, z) = yz \ln(xy)$

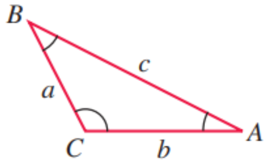
Solution:

$$f_x = 1, f_y = -\frac{y}{\sqrt{y^2+z^2}}, \quad f_z = -\frac{z}{\sqrt{y^2+z^2}}$$

$$f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}, \quad f_y = z\ln(xy) + yz \cdot \frac{\partial}{\partial y}\ln(xy) = z\ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z\ln(xy) + z,$$

$$f_z = y\ln(xy) + yz \cdot \frac{\partial}{\partial z}\ln(xy) = y\ln(xy)$$

Problem 2. Given a triangle ABC as shown in Figure below



a) Express **A** implicitly as a function of **a, b**, and **c** and calculate $\partial A/\partial a$ and $\partial A/\partial b$.

b) Express **a** implicitly as a function of **A, b**, and **B** and calculate $\partial a/\partial A$ and $\partial a/\partial B$.

Solution:

$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow 2a = (2bc\sin A)\frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc\sin A};\ \text{also}\ \ 0 = 2b - 2c\cos A + (2bc\sin A)\,\frac{\partial A}{\partial b}$$
$$\Rightarrow 2c\cos A - 2b = (2bc\sin A)\,\frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A)\,\frac{\partial a}{\partial A} - a\cos A}{\sin^2 A} = 0 \Rightarrow (\sin A)\frac{\partial a}{\partial x} - a\cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a\cos A}{\sin A};\ \text{also}\ \left(\frac{1}{\sin A}\right)\frac{\partial a}{\partial B} = b(-\csc B\cot B)$$
$$\Rightarrow \frac{\partial a}{\partial B} = -b\csc B\cot B\sin A$$

Problem 3. Assuming that the following equations define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

a) $xe^y + \sin xy + y - \ln 2 = 0, \quad (0, \ln 2),$

b) $(x^3 - y^4)^6 + \ln(x^2 + y) = 1, \quad (-1, 0).$

Solution:

Let $F(x, y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x, y) = e^y + y\cos xy$ and $F_y(x, y) = xe^y + x\sin xy + 1$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\sin xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$$

Let $F(x, y) = \left(x^3 - y^4\right)^6 + \ln\left(x^2 + y\right) \Rightarrow F_x(x, y) = 18x^2\left(x^3 - y^4\right)^5 + \frac{2x}{x^2+y}$ and

$$F_y(x, y) = -24y^3\left(x^3 - y^4\right)^5 + \frac{1}{x^2+y} \Rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-18x^2\left(x^3-y^4\right)^5 + \frac{2x}{x^2+y}}{-24y^3\left(x^3-y^4\right)^5 + \frac{1}{x^2+y}} \Rightarrow \frac{dy}{dx}(-1, 0) = 20$$

Problem 4. Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

Solution:

(a) $\frac{\partial T}{\partial x} = 8x - 4y$ and $\frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (8x-4y)(-\sin t) + (8y-4x)(\cos t)$

$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) = 4\sin^2 t - 4\cos^2 t \Rightarrow \frac{d^2T}{dt^2} = 16\sin t\cos t;$

$\frac{dT}{dt} = 0 \Rightarrow 4\sin^2 t - 4\cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t$ or $\sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \Rightarrow T$ has a minimum at $(x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow T$ has a maximum at $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow T$ has a minimum at $(x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \Rightarrow T$ has a maximum at $(x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$T = 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y$, and $\frac{\partial T}{\partial y} = 8y - 4x$ so the extreme values occur at the four points found in part (a): $T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 6$, the maximum and $T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 2$, the minimum

Problem 5. Find the directions in which the function

$$f(x, y, z) = \ln xy + \ln yz + \ln xz,$$

increase and decrease most rapidly at $P_0(1, 1, 1)$. Then find the derivatives of the function in these directions.

Solution:

$\nabla f = \left(\frac{1}{x}+\frac{1}{x}\right)\mathbf{i}+\left(\frac{1}{y}+\frac{1}{y}\right)\mathbf{j}+\left(\frac{1}{z}+\frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$