Appendix C
The function log det(X)

In this appendix we develop the matrix calculus needed to derive the gradient and Hessian of the function log $\det(X)$, and show that it is a strictly concave function.

Lemma C.1 *Let* $f : \operatorname{int}(\mathcal{S}_n^+) \mapsto \mathbf{R}$ *be given by*

$$f(X) = \log\det X,$$

Denoting

$$\nabla f(X) := \begin{bmatrix} \frac{\partial f(X)}{\partial x_{11}} & \cdots & \frac{\partial f(X)}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f(X)}{\partial x_{n1}} & \cdots & \frac{\partial f(X)}{\partial x_{nn}} \end{bmatrix},$$

one has $\nabla f(X) = X^{-1}$.

Proof:

Let $X \in \operatorname{int}(\mathcal{S}_n^+)$ be given and let $H \in \mathcal{S}_n$ be such that $X + H \in \operatorname{int}(\mathcal{S}_n^+)$. One has

$$\begin{aligned} f(X+H) - f(X) &= \log\det(X+H) - \log\det(X) \\ &= \log\det\left(X^{-1}(X+H)\right) \\ &= \log\det\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right). \end{aligned}$$

By the arithmetic-geometric inequality applied to the eigenvalues of $X^{-\frac{1}{2}}HX^{-\frac{1}{2}}$ one has

$$\begin{aligned} \log\det\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right) &\leq \log\left(\frac{1}{n}\mathbf{Tr}\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\right)^n \\ &= n\log\left(\frac{1}{n}\mathbf{Tr}\left(I + X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\right) \end{aligned}$$

$$= n\log\left(1+\frac{1}{n}\mathbf{Tr}\left(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\right).$$

Using the well-known inequality $\log(1+t)\leq t$ we arrive at

$$f(X+H)-f(X)\leq \mathbf{Tr}\left(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)=\langle X^{-1},H\rangle.$$

This shows that X^{-1} is a subgradient of f at X. Since f is assumed differentiable, the subgradient is unique and equals the gradient $\nabla f(X)$. □

The proof of the next result is trivial.

Lemma C.2 *Let $f:\mathrm{int}\,(\mathcal{S}_n^+)\mapsto\mathbf{R}$ be given by*

$$f(X)=\mathbf{Tr}\,(CX),$$

where $C\in\mathcal{S}_n$. One has $\nabla f(X)=C$.

The following result is used to derive the Hessian of the log-barrier function $f_{bar}(X)=-\log\det(X)$.

Lemma C.3 *Let $f:\mathcal{S}_n^{++}\mapsto\mathbf{R}$ be given by*

$$f(X)=\log\det X.$$

If $\nabla^2 f$ denotes the derivative of $\nabla f: X\mapsto X^{-1}$ with respect to X, then $\nabla^2 f(X)$ is the linear operator which satisfies

$$\nabla^2 f(X)H=-X^{-1}HX^{-1},\quad \forall H\in\mathcal{S}_n,$$

for a given invertible X.

Proof:

Let $\mathrm{L}(\mathcal{S}_n,\mathcal{S}_n)$ denote the space of linear operators which map $\mathcal{S}_n$ to $\mathcal{S}_n$. The Frechet derivative of ∇f is defined as the (unique) function $\nabla^2 f:\mathcal{S}_n\mapsto\mathrm{L}(\mathcal{S}_n,\mathcal{S}_n)$ such that

$$\lim_{\|H\|\to 0}\frac{\left\|\nabla f(X+H)-\nabla f(X)-\nabla^2 f(X)H\right\|}{\|H\|}=0. \tag{C.1}$$

We show that $\nabla^2 f(X)H:=-X^{-1}HX^{-1}$ satisfies (C.1). To this end, let $H\in\mathcal{S}_n$ be such that $(X+H)$ is invertible, and consider

$$\begin{aligned}
&\left\|\nabla f(X+H)-\nabla f(X)-\nabla^2 f(X)H\right\|\\
=\;&\left\|(X+H)^{-1}-X^{-1}+X^{-1}HX^{-1}\right\|\\
=\;&\left\|(X+H)^{-1}\left(I-(X+H)X^{-1}+(X+H)X^{-1}HX^{-1}\right)\right\|\\
=\;&\left\|(X+H)^{-1}\left(HX^{-1}HX^{-1}\right)\right\|\\
\leq\;&\left\|(X+H)^{-1}\right\|\,\|H\|\,\left\|X^{-1}HX^{-1}\right\|,
\end{aligned}$$

which shows that (C.1) indeed holds. □

By Lemma A.3, the Hessian of the function $f(X) = -\log\det(X)$ is a positive definite operator which implies that f is strictly convex on $\mathcal{S}_n^{++}$. We state this observation as a theorem.

Theorem C.1 *The function* $f : \mathcal{S}_n^{++} \mapsto \mathbf{R}$ *defined by*

$$f(X) = -\log\det(X)$$

is strictly convex.

An alternative proof of this theorem is given in [85] (Theorem 7.6.6).