Lecture 26: Integrals and Vector Fields.

MA2032 Vector Calculus

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Integrals and Vector Fields. Overview

• **We extend the theory of integration** to general curves and surfaces in space.

• The resulting **line and surface integrals** give powerful mathematical tools for science and engineering.

• **Line integrals are used** to find the **work done by a force** in moving an object along a path, and to find the **mass of a curved wire** with variable density.

• **Surface integrals** are used to find the **rate of flow** of a fluid across a surface and to describe the **interactions of electric and magnetic forces**.

• We present the **fundamental theorems of vector integral calculus**, and discuss their mathematical consequences and physical applications.

• The theorems of vector calculus are then shown to be **generalized versions of the Fundamental Theorem of Calculus**.

Line Integrals of Scalar Functions

• We need a **more general notion of integral** than was defined in Integral Calculus.

• We need **to integrate over a curve C rather than over an interval** $[a, b]$.

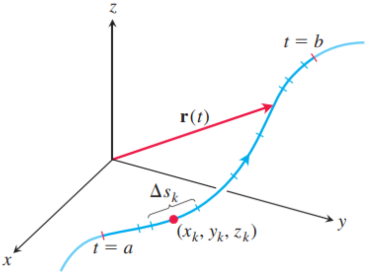
• These more general integrals are called **line integrals**.

• Suppose that $f(x, y, z)$ is a **real-valued function** we wish to integrate over the curve $C \in D_f$ **parametrized by** $r(t) = g(t)i + h(t)j + k(t)k$, $a \leq t \leq b$.

• The **values of f along C are given by the composite function** $f(g(t), h(t), k(t))$.

• We are going to **integrate this composition** with respect **to arc length** $s(t)$ **from** $t = a$ **to** $t = b$.

• We first **partition** C into a finite number n of **subarcs** with length Δs_k.



Line Integrals of Scalar Functions

- In each subarc we choose a point (x_k, y_k, z_k) and form the **sum**

$$S_n = \sum_{k=1}^{n} \underbrace{f(x_k, y_k, z_k)}_{\substack{\text{value of } f \text{ at a point} \\ \text{on the subarc}}} \underbrace{\Delta s_k}_{\substack{\text{length of a small} \\ \text{subarc of the curve}}},$$

- which is similar to a **Riemann sum**.
- If f is **continuous** and the functions g, h, and k **have continuous first derivatives**, then **these sums approach a limit as** n **increases** and the lengths Δs_k approach zero.
- This leads to the following **definition**

DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \leq t \leq b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k, \tag{1}$$

provided this limit exists.

Line Integrals of Scalar Functions

• If the **curve** C **is smooth** for $a \leq t \leq b$ (so $v = dr/dt$ is continuous and never 0) and the function f is **continuous on** C, then the limit in Equation (1) can be shown to **exist**.

• We can then apply the Fundamental Theorem of Calculus **to differentiate the arc length equation**,

$$s(t) = \int_a^t |\mathbf{v}(\tau)| \, d\tau,$$

• to express ds in Equation (1) as $ds = |v(t)|dt$ and evaluate the integral of f over C as

$$\int_C f(x, y, z) \, ds = \int_a^b f(g(t), h(t), k(t)) \, |\mathbf{v}(t)| \, dt.$$

• The formula evaluates the line integral correctly **no matter what smooth parametrization for** C is used.

• Note that the parameter **t defines a direction** along the path.

How to Evaluate a Line Integral

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

2. Evaluate the integral as

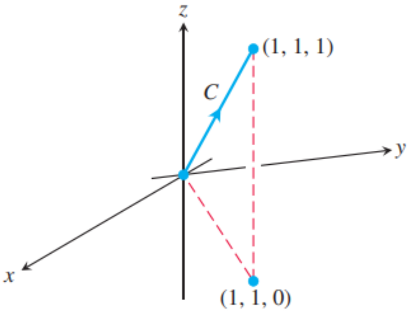
$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt.$$

• If f has the constant value 1, then the integral of f over C gives the length of C from $t = a$ to $t = b$.

• We also write $f(\mathbf{r}(t))$ for the evaluation $f(g(t), h(t), k(t))$ along the curve $\mathbf{r}$.

Example 1

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$.



Example 1

Solution Since any choice of parametrization will give the same answer, we choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$$

The components have continuous first derivatives and $|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of f over C is

$$\int_C f(x, y, z)\, ds = \int_0^1 f(t, t, t)\sqrt{3}\, dt \qquad \text{Eq. (2), } ds = |\mathbf{v}(t)|\, dt = \sqrt{3}\, dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt$$

$$= \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = \sqrt{3}\Big[t^2 - t^3\Big]_0^1 = 0. \qquad \blacksquare$$

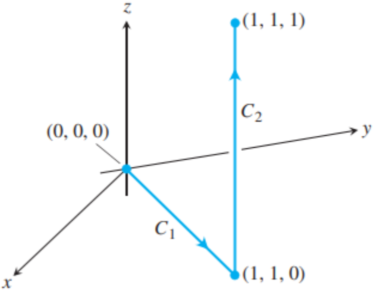
• Line integrals have the **useful property** that if a piecewise smooth curve C is **made by joining a finite number of smooth curves** $C_1, C_2, ..., C_n$ end to end, then the integral of a function over C is the sum of the integrals over the curves that make it up:

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds.$$

• The **value of the line integral along a path** joining two points **can change if you change the path** between them.

Example 2

Figure shows another path from the origin to $(1, 1, 1)$, formed from two line segments C_1 and C_2. Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.



Example 2

Solution We choose the simplest parametrizations for C_1 and C_2 we can find, calculating the lengths of the velocity vectors as we go along:

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

With these parametrizations we find that

$$\int_{C_1 \cup C_2} f(x, y, z)\, ds = \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds \qquad \text{Eq. (3)}$$

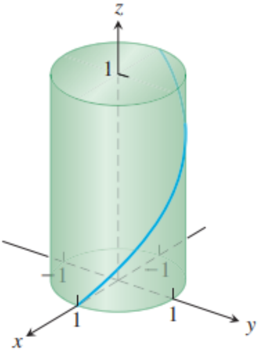
$$= \int_0^1 f(t, t, 0)\sqrt{2}\, dt + \int_0^1 f(1, 1, t)(1)\, dt \qquad \text{Eq. (2)}$$

$$= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\, dt + \int_0^1 (1 - 3 + t)(1)\, dt$$

$$= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}.$$

Example 3

Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi$.



Example 3

Solution For the helix (Figure 16.4) we find $\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$ and $|\mathbf{v}(\mathrm{t})| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$. Evaluating the function f at the point $\mathbf{r}(t)$, we obtain

$$f(\mathbf{r}(t)) = f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t} = \sin 2t + \sqrt{t}.$$

The line integral is given by

$$\begin{aligned}
\int_C f(x, y, z)\, ds &= \int_0^{\pi} \left(\sin 2t + \sqrt{t}\right)\sqrt{2}\, dt \\
&= \sqrt{2}\left[-\frac{1}{2}\cos 2t + \frac{2}{3}t^{3/2}\right]_0^{\pi} \\
&= \frac{2\sqrt{2}}{3}\pi^{3/2} \approx 5.25.
\end{aligned}$$

Mass and Moment Calculations

• We treat **coil springs** (Left Figure) and **wires** (Right Figure) as **masses distributed along smooth curves in space**.

Mass and Moment Calculations

- The distribution is described by a **continuous density function** $\delta(x, y, z)$ representing **mass per unit length**.

- When a curve C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$ then x, y, and z are functions of the parameter t, the **density is the function** $\delta(x(t), y(t), z(t))$, and the **arc length differential** is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

- The spring's or wire's **mass, center of mass, and moments** are then calculated using the formulas in Table on next slide, with the integrations in terms of the parameter t over the interval $[a, b]$.
- For example, the formula for mass becomes

$$M = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Mass and Moment Calculations

TABLE 16.1 Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $M = \int_C \delta \, ds$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x \, \delta \, ds, \qquad M_{xz} = \int_C y \, \delta \, ds, \qquad M_{xy} = \int_C z \, \delta \, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

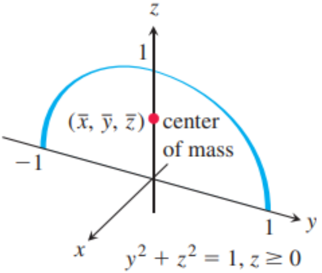
$$I_x = \int_C (y^2 + z^2)\, \delta \, ds, \qquad I_y = \int_C (x^2 + z^2)\, \delta \, ds, \qquad I_z = \int_C (x^2 + y^2)\, \delta \, ds,$$

$$I_L = \int_C r^2 \delta \, ds \qquad r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L$$

Mass and Moment Calculations

Example 4

A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1,\ z \geq 0$, in the yz-plane. Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.



Example 4

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the yz-plane with its mass distributed symmetrically about the z-axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \qquad 0 \le t \le \pi.$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1,$$

so $ds = |\mathbf{v}|\, dt = dt$.

The formulas in Table 16.1 then give

$$M = \int_C \delta\, ds = \int_C (2 - z)\, ds = \int_0^\pi (2 - \sin t)\, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\delta\, ds = \int_C z(2 - z)\, ds = \int_0^\pi (\sin t)(2 - \sin t)\, dt$$

$$= \int_0^\pi (2\sin t - \sin^2 t)\, dt = \frac{8 - \pi}{2} \qquad \text{Routine integration}$$

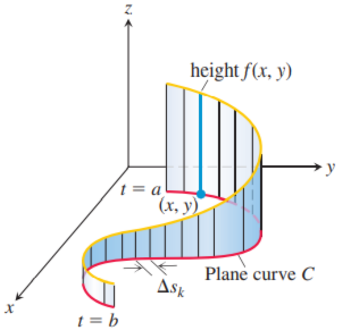
$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

With $\bar{z}$ to the nearest hundredth, the center of mass is (0, 0, 0.57).

Line Integrals in the Plane

- Line integrals **for curves in the plane** have a natural geometric interpretation.

- If C is a smooth curve in the xy-plane parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leq t \leq b$, we generate a **cylindrical surface** by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis.
- The cylinder cuts through the surface $z = (x, y)$, forming a **"curved wall"**.



- At any point (x, y) along the curve, the height of the wall is $f(x, y)$.

$$\int_C f\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta s_k,$$

- The line integral $\int_C f\ ds$ is the area of the wall shown in the figure.