Introductory Statistics

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Topic 2 - Estimators
2.2 - Properties of Estimators
Unbiasedness

- **Topic 0: Introduction**
- **Topic 1: Descriptive Statistics**
- **Topic 2: Estimators: Point estimation, Interval estimation**
  - 2.1 **Point estimation**,
  - 2.2 Properties of estimators,
  - 2.3 Interval estimation,
  - 2.4 Inference based on Normal Distribution
- Topic 3: Hypothesis Testing: Test for single population mean, test for two population mean
- Topic 4: Bayesian Estimation
- Topic 5: Goodness of Fit: The χ^2 test.

Recap

1. estimation using the method of moments:

$$\mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f_X(x;\theta)dx, \text{ if } \int_{-\infty}^{\infty} |x|^r \cdot f_X(x;\theta)dx < \infty$$

$$m_r = E\left(X^r\right) = \frac{1}{n}\sum_{i=1}^{n} X_i^r$$

$$\mu_r(\theta) = m_r$$

RECAP

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2. estimation using the method of maximum likelihood:

$$L(\theta) = L(\theta,\ x_1,\ldots,x_n) = f(x_1,\ldots,x_n,\ \theta),$$

$$l(\theta) = \ln(L(\theta,\ x_1,\ldots,x_n)) = \ln(f(x_1,\ldots,x_n,\ \theta)).$$

$$\frac{\mathrm{d}l(\theta)}{\mathrm{d}\theta} = 0.$$

EXAMPLE

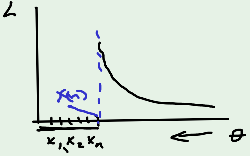
Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$, $\theta > 0$. Find the MLE of θ.

Solution

$$f_X(x) = \begin{cases} \frac{1}{\theta}, & 0 < x \le \theta \\ 0, & \text{otherwise} \end{cases} \qquad \Longrightarrow$$

$$L(x_1, \ldots, x_n, \theta) = \begin{cases} \frac{1}{\theta^n} & 0 < x_1, x_2, \ldots, x_n \le \theta \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{dL(\theta)}{d\theta} = -\frac{n}{\theta^{n+1}} = 0$$



$\theta \geq \max\{x_1, x_2, \ldots, x_n\} = x_{(n)}$

MLE: $\hat{\theta} = x_{(n)}$ $\qquad$ $\hat{\theta} = 16$

MME: $\hat{\theta} = 2\bar{X}$ $\qquad$ $\hat{\theta} = 2 \cdot \frac{30}{4} = 15$ $\qquad$ $x < \theta$

$x = \{3, 5, 6, 16\}$

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Desired properties of point estimator

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PROPERTIES OF POINT ESTIMATORS

Desired properties of point estimator

- unbiasedness,
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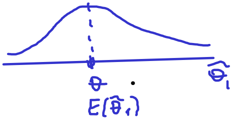

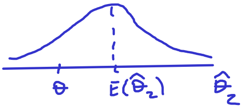

Properties of Point Estimators: Unbiasedness

Definition

A point estimator $\hat{\theta}$ is called an ***unbiased estimator*** *of the parameter θ if $E(\hat{\theta}) = \theta$ for all possible values of θ. Otherwise $\hat{\theta}$ is said to be* ***biased****.*
Furthermore, the bias of $\hat{\theta}$ is given by

$$B\left(\hat{\theta}, \theta\right) = Bias\left(\hat{\theta}, \theta\right) = E\left(\hat{\theta}\right) - \theta.$$

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where Y is the binomial random variable, $E(Y) = np$, hence,

$$\underline{E\left(\hat{p}\right)} = E\left(\frac{Y}{n}\right) = \frac{1}{n}E(Y) = \frac{1}{n}np = \underline{p}$$

PROPERTIES OF POINT ESTIMATORS: UNBIASEDNESS

THEOREM

The mean of a random sample $\bar{X}$ is an unbiased estimator of the population mean μ.

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PROOF.

Let $X_1, ..., X_n$ be random variables with mean μ. Then, the sample mean is $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$.

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n\mu = \mu.$$

Hence, $\bar{X}$ is an unbiased estimator of μ. □

Properties of Point Estimators: Unbiasedness

Example

Recall, that $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$
Find $E(\hat{\sigma}^2)$:
Solution:

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Solution:

$-\mu + \mu$

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$= \frac{1}{n}\left(E\left[\sum_{i=1}^{n}(X_i-\mu)^2\right] - 2E\left[\underbrace{\sum_{i=1}^{n}(X_i-\mu)}_{*}(\bar{X}-\mu)\right] + E\left[\sum_{i=1}^{n}(\bar{X}-\mu)^2\right]\right)$
$= \frac{1}{n}\left(\sum_{i=1}^{n}E\left[(X_i-\mu)^2\right] - 2E\left[\underbrace{n(\bar{X}-\mu)}_{*}(\bar{X}-\mu)\right] + \sum_{i=1}^{n}E\left[(\bar{X}-\mu)^2\right]\right)$

$* \ \Sigma(x_i - \mu) =$
$\Sigma x_i - \Sigma\mu$
$\frac{1}{n}\Sigma x_i = \bar{x}$
$\Sigma\mu = n\mu$
$\frac{n}{n}(\Sigma(x_i-\mu)) =$
$n(\bar{x}-\mu)$

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$$= \frac{1}{n}\left(\sum_{i=1}^n \underbrace{E\left[(X_i - \mu)^2\right]}_{} - 2E\left[n\underbrace{(\bar{X} - \mu)(\bar{X} - \mu)}_{E(\bar{X}-\mu)^2}\right] + \underbrace{\sum_{i=1}^n E\left[(\bar{X} - \mu)^2\right]}_{nVar(\bar{X})}\right)$$

$Var(\bar{X})$

$$= \frac{1}{n}\left(\sum_{i=1}^n Var(X_i) - 2nVar(\bar{X}) + nVar(\bar{X})\right)$$

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$= \frac{1}{n}\left(\sum_{i=1}^n E\left[(X_i - \mu)^2\right] - 2E\left[n(\bar{X} - \mu)(\bar{X} - \mu)\right] + \sum_{i=1}^n E\left[(\bar{X} - \mu)^2\right]\right)$

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$= \frac{1}{n}\left(\sum_{i=1}^n Var(X_i) - nVar(\bar{X})\right) = \frac{1}{n}\left(n\sigma^2 - n\sigma^2/n\right)$

$Var(\bar{X}) = \frac{\sigma^2}{n}$

PROPERTIES OF POINT ESTIMATORS: UNBIASEDNESS

EXAMPLE

Recall, that $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ ← is biased

Find $E(\hat{\sigma}^2)$:

Solution:

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$$E(\hat{\sigma}^2) = E\left[\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2\right] = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

PROPERTIES OF POINT ESTIMATORS: UNBIASEDNESS

THEOREM

Let $X_1, \dots, X_n$ be random sample drawn from an infinite population with variance $\sigma^2 < \infty$. If $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the variance of the random sample, then S^2 is an unbiased estimator for σ^2.

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PROOF.

$$E\left(S^2\right) = \frac{1}{n-1} E\left[\sum_{i=1}^n (X_i - \bar{X})^2\right] = \frac{1}{n-1} E\left[\sum_{i=1}^n \left((X_i \overset{-\mu}{-} \mu) \overset{+\mu}{-} (\bar{X} - \mu)\right)^2\right]$$

...

$$= \frac{1}{n-1} \sum_{i=1}^n \underbrace{E\left[(X_i - \mu)^2\right]}_{Var(X_i)} - n\left(\underbrace{E\left[\bar{X} - \mu\right]}_{Var(\bar{X})}\right)^2 = \frac{1}{n-1}\overset{n\sigma^2 - \sigma^2}{\left(\sum_{i=1}^n \sigma^2 - n\frac{\sigma^2}{n}\right)} = \sigma^2.$$

Hence, S^2 is an unbiased estimator for σ^2 □

PROPERTIES OF POINT ESTIMATORS: UNBIASEDNESS

EXAMPLE

Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of θ.
Show that

$$\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2, \quad 0 \le a \le 1$$

is an unbiased estimator of θ. Note that $\hat{\theta}_3$ is a convex combination of $\hat{\theta}_1$ and $\hat{\theta}_2$.

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Solution.

We are given that $E(\hat{\theta}_1) = \theta$ and $E(\hat{\theta}_2) = \theta$. Therefore,

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$$= a\theta + (1-a)\theta = \theta.$$

Hence, $\hat{\theta}_3$ is unbiased.