# Lecture 34: Infinite Sequences and Series.

MA2032 Vector Calculus


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# The Integral Test

• The most basic **question** we can ask about a series **is whether it converges**.

• We begin to study this question, starting with **series that have nonnegative terms**.

• **Such a series converges if its sequence of partial sums is bounded.**

• If we establish that a given series does converge, **we generally do not have a formula available for its sum**.

• So to get an **estimate for the sum** of a convergent series, we investigate the **error** involved when using a **partial sum to approximate the total sum**.

**Corollary of Theorem 6**

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

**DEFINITION** A sequence $\{a_n\}$ is **bounded from above** if there exists a number $M$ such that $a_n \leq M$ for all $n$. The number $M$ is an **upper bound** for $\{a_n\}$. If $M$ is an upper bound for $\{a_n\}$ but no number less than $M$ is an upper bound for $\{a_n\}$, then $M$ is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number $m$ such that $a_n \geq m$ for all $n$. The number $m$ is a **lower bound** for $\{a_n\}$. If $m$ is a lower bound for $\{a_n\}$ but no number greater than $m$ is a lower bound for $\{a_n\}$, then $m$ is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

# The Integral Test

**THEOREM 9—The Integral Test**
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

# The Integral Test

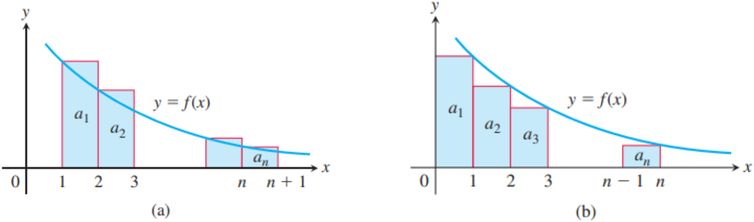
**Proof** We establish the test for the case $N = 1$. The proof for general $N$ is similar.

We start with the assumption that $f$ is a decreasing function with $f(n) = a_n$ for every $n$. This leads us to observe that the rectangles in Figure 10.12a, which have areas $a_1, a_2, \dots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n.$$

In Figure 10.12b the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle of area $a_1$, we see that

$$a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx.$$



# The Integral Test

If we include $a_1$, we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Combining these results gives

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

These inequalities hold for each $n$, and continue to hold as $n \to \infty$.

If $\int_1^\infty f(x)\,dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^\infty f(x)\,dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are either both finite or both infinite. ■

**EXAMPLE 5** Determine the convergence or divergence of the series.

**(a)** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

**Solutions**

**(a)** We apply the Integral Test and find that

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^{\infty} \frac{du}{e^u} \qquad u = x^2,\ du = 2x\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-u}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2e^b} + \frac{1}{2e}\right) = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

# Solution. Example

**(b)** Again applying the Integral Test,

$$\int_1^\infty \frac{dx}{2^{\ln x}} = \int_0^\infty \frac{e^u\,du}{2^u} \qquad u = \ln x,\ x = e^u,\ dx = e^u\,du$$

$$= \int_0^\infty \left(\frac{e}{2}\right)^u du$$

$$= \lim_{b\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)} \left(\left(\frac{e}{2}\right)^b - 1\right) = \infty. \qquad (e/2) > 1$$

The improper integral diverges, so the series diverges also.

# Error Estimation

• For some convergent series, such as the geometric series, we can actually find the **total sum of the series**.

• That is, we can find the **limiting value** $S$ **of the sequence of partial sums**.

• For **most convergent series**, however, **we cannot easily find the total sum**.

• Nevertheless, we can **estimate the sum** by adding the first $n$ terms to get $s_n$, but we need to know **how far off $s_n$ is from the total sum $S$**.

• An approximation to a function or to a number is more useful when it is accompanied by a **bound on the size of the worst possible error** that could occur.

# Error Estimation

**Bounds for the Remainder in the Integral Test**

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where $f$ is a continuous positive decreasing function of $x$ for all $x \geq n$, and that $\Sigma a_n$ converges to $S$. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \tag{1}$$

If we add the partial sum $s_n$ to each side of the inequalities in (1), we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq s_n + \int_{n}^{\infty} f(x)\,dx \tag{2}$$

since $s_n + R_n = S$. The inequalities in (2) are useful for estimating the error in approximating the sum of a series known to converge by the Integral Test. The error can be no larger than the length of the interval containing $S$, with endpoints given by (2).

# Error Estimation

**EXAMPLE 6** Estimate the sum of the series $\Sigma(1/n^2)$ using the inequalities in (2) and $n = 10$.

**Solution** We have that

$$\int_n^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

Using this result with the inequalities in (2), we get

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

Taking $s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977$, these last inequalities give

$$1.64068 \le S \le 1.64977.$$

If we approximate the sum $S$ by the midpoint of this interval, we find that

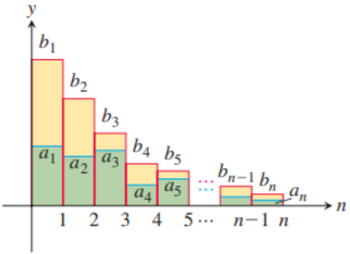
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452.$$

The error in this approximation is then less than half the length of the interval, so the error is less than 0.005. Using a trigonometric *Fourier series* (studied in advanced calculus), it can be shown that $S$ is equal to $\pi^2/6 \approx 1.64493$. ■

• We can **test the convergence** of many more series **by comparing their terms** to those of a series whose **convergence is already known**.

**THEOREM 10—Direct Comparison Test**
Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all $n$. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.



**Proof** The series $\sum a_n$ and $\sum b_n$ have nonnegative terms. The Corollary of Theorem 6 stated in Section 10.3 tells us that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number $M$. The partial sums $\sum_{n=1}^{N} a_n$ are all bounded from above by $M = \sum b_n$, since

$$s_N = a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N \leq \sum_{n=1}^{\infty} b_n = M.$$

Since the partial sums of $\sum a_n$ are bounded from above, the Corollary of Theorem 6 implies that $\sum a_n$ converges. We conclude that when $\sum b_n$ converges, then so does $\sum a_n$. Figure 10.12 illustrates this result, with each term of each series interpreted as the area of a rectangle.

In Part (2), where we assume that $\sum a_n$ diverges, the partial sums of $\sum_{n=1}^{\infty} b_n$ are not bounded from above. If they were, the partial sums for $\sum a_n$ would also be bounded from above, since

$$a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N,$$

and this would mean that $\sum a_n$ converges. We conclude that if $\sum a_n$ diverges, then so does $\sum b_n$. ∎

# The Limit Comparison Test

**THEOREM 11—Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ an integer).

**1.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

**2.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

**3.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

**Proof** We will prove Part 1. Parts 2 and 3 are left as Exercises 57a and b.

Since $c/2 > 0$, there exists an integer $N$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \qquad \text{whenever} \qquad n > N.$$

Limit definition with $\varepsilon = c/2, L = c$, and $a_n$ replaced by $a_n/b_n$

Thus, for $n > N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right)b_n < a_n < \left(\frac{3c}{2}\right)b_n.$$

If $\sum b_n$ converges, then $\sum(3c/2)b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum(c/2)b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test. ■

# The Limit Comparison Test

**EXAMPLE 3** Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

**Solution** Because $\ln n$ grows more slowly than $n^c$ for any positive constant $c$ (Section 10.1, Exercise 115), we can compare the series to a convergent $p$-series. To get the $p$-series, we see that

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for $n$ sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \\
&= \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} \qquad \text{l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.
\end{aligned}$$

Since $\sum b_n = \sum(1/n^{5/4})$ is a $p$-series with $p > 1$, it converges. Therefore $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■