<u>Lemma</u> If $k(x)$ has limit M as $x \to c$, & $h(x)$ has limit 0 as $x \to c$. then $h(x)k(x)$ has limit 0 as $x \to c$

(special case of limits preserving products)

<u>Proof</u>: Take $\varepsilon = 1$ in the definition of $\lim_{x \to c} k(x) = M$

so $\exists\ \delta > 0$: $0 < |x - c| < \delta \implies |k(x) - M| < 1$

$\implies M - 1 < k(x) < M + 1$

The product $h(x)k(x)$ satisfies

$(M-1)h(x) < h(x)k(x) < (M+1)h(x)$ when $0 < |x-c| < \delta$

Now just use <u>Pinching Theorem</u>. We know $(M-1)h(x) \to 0$ as $x \to c$

$(M+1)h(x) \to 0$

which says $h(x)k(x) \to 0$ as $x \to c$.

<u>Theorem</u> If $\lim_{x \to c} f_1(x) = L_1$, $\lim_{x \to c} f_2(x) = L_2$

then $\lim (f_1(x) f_2(x)) = L_1 L_2$

Algebraic trick: $\left| f_1(x) f_2(x) - L_1 L_2 \right| = \left| f_1(x)\left(f_2(x) - L_2\right) + \left(f_1(x) - L_1\right) L_2 \right|$

By $\triangle$ inequality

$$|f_1(x)f_2(x) - L_1L_2| \leq \underbrace{|\underbrace{f_1(x)}_{\text{limit } L_1}\underbrace{(f_2(x) - L_2)}_{\text{limit } 0}|}_{\text{limit } 0 \text{ by Lemma}} + \underbrace{|(f_1(x) - L_1)L_2|}_{\substack{\text{limit } 0 \times \text{constant} \\ \text{so also limit } 0}}$$

So we have

$$\underline{0} \leq |f_1(x)f_2(x) - L_1L_2| \leq \text{something with limit } 0.$$

Pinching theorem: $|f_1(x)f_2(x) - L_1L_2|$ also has limit 0 as $x \to c$.

So $f_1(x)f_2(x)$ has limit L_1L_2 as $x \to c$.

<u>Examples</u>

a) $\sin(x)$ is continuous at $x=0$

b) $\frac{1}{x}$ is continuous at all points $\neq 0$.

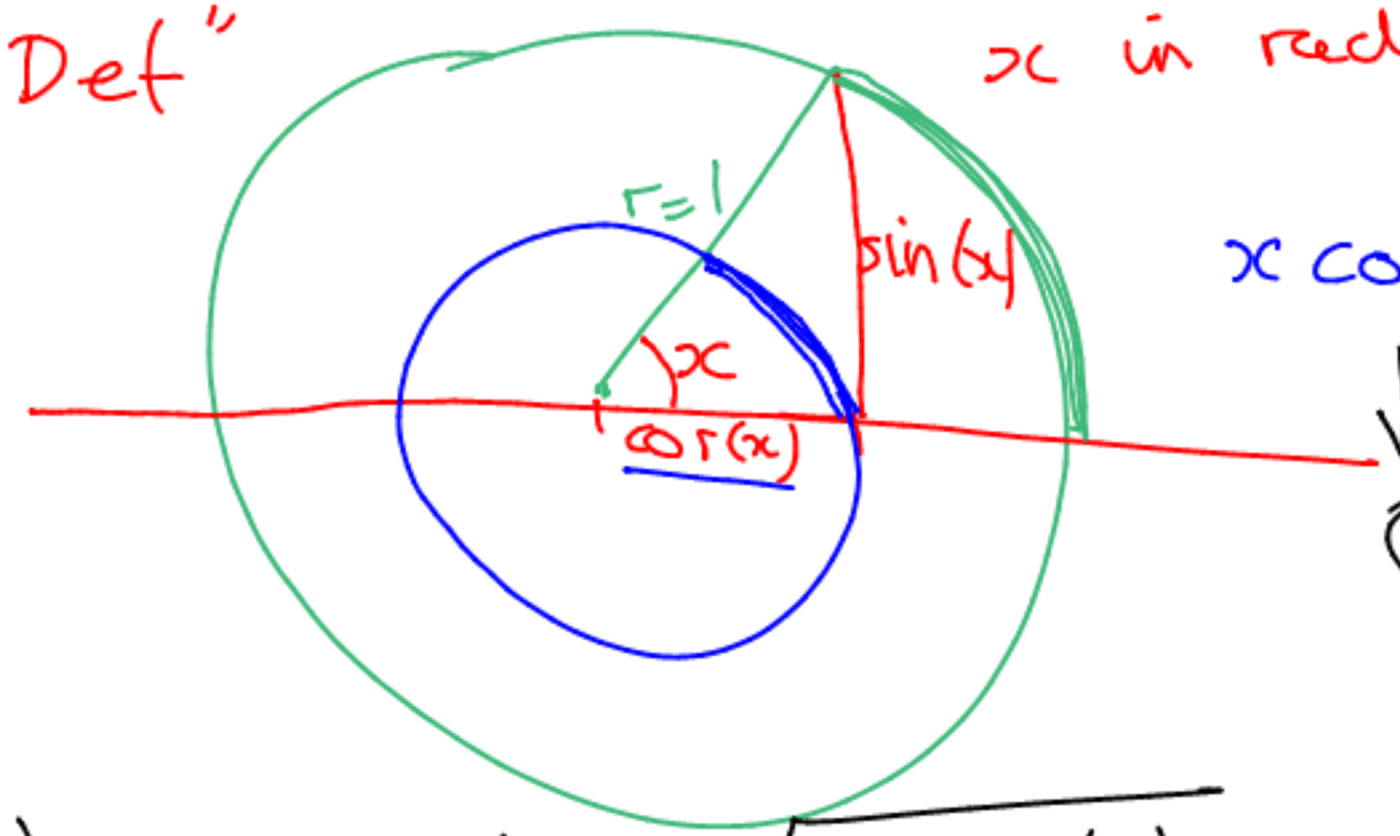
c) $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ continuous.

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

d) $\cos(x)$ cts

e) $f(x) = \begin{cases} \frac{1-\cos(x)}{x^2} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$

a) "Def" x in radians.



$$x\cos(x) \le \sin(x) \le x$$

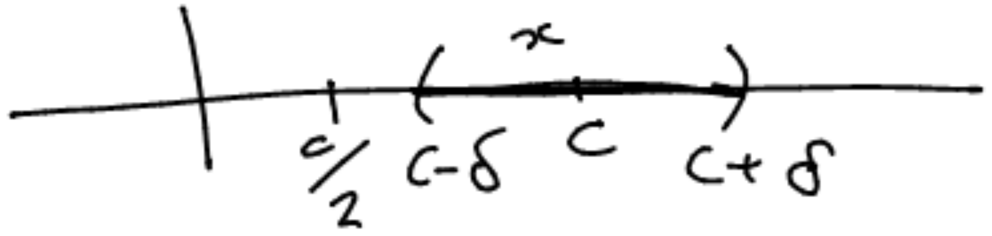
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \quad \text{as } x \to 0$$

$$0 \qquad \text{so} \qquad 0 \qquad\qquad 0$$

as $x \to 0$ by Pinching Theorem

d) $\cos(x) = \sqrt{1-\sin^2(x)} \to \sqrt{1-0^2} = \underline{\underline{1}}$

b) $\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| = \frac{|x-c|}{|x|\,|c|}$



$$\left|\frac{a}{bc}\right| = \frac{|-a|}{|b|\,|c|}$$

If we take $\delta < \frac{|c|}{2}$, $|x| > \frac{|c|}{2}$, $\frac{1}{|x|} < \frac{2}{|c|}$

so $\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}|x-c|$

Now given any $\varepsilon > 0$, take $\delta < \frac{|c|}{2}$ and $\delta < \frac{c^2}{2}\varepsilon$

So $0 < |x-c| < \delta \implies \left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}|x-c| < \frac{2}{c^2}\delta < \varepsilon.$