

## Problem Sheet 1 - Submission Deadline: 16 October at 6pm (GMT) on Blackboard

- 1) **[5pts]** Formulate the following problem as a linear programming problem. Write the decision variables, constraints and objective function.

Biocare makes liquid plant food for fruit and vegetables. It makes two types: Growrite (G), a high nitrogen fertiliser for green vegetables, and Tomfood (T), with a high potassium content for tomatoes, cucumbers and so on. Both types need the same basic ingredients - Ammonium Nitrate for Nitrogen (N), Phosphorus Pentoxide for Phosphorus (P), and Potassium Dioxide for Potassium (K) - but in different amounts. One litre of G requires 0.11kg of N, 0.06kg P and 0.02kg of K. One litre of T requires 0.08kg of N, 0.03kg P and 0.08kg of K. There are available each day 600kg of N, 300kg of P and 330kg of K. The selling prices per litre are £2.80 for Growrite and £3.00 for Tomfood. At these prices, Biocare can sell all it produces. Biocare wishes to maximise its daily income. How should it do so?

- 2) Consider the following linear programming problem.

$$\begin{array}{ll} \max & 35x + 20y \\ \text{s. t.} & 23x + 11y \leq 176 \\ & 4x + 4y \leq 51 \\ & x, y \geq 0 \end{array}$$

- a) **[4 pts]** Draw (by hand or using some software) the feasible set. Then determine graphically an optimal solution and indicate which constraints are active.  
b) **[4 pts]** Determine an approximate optimal solution using Matlab's function linprog. You can either write the commands by hand or include your m-file in your submission.
- 3) **[5 pts]** Write the following linear programming problem in standard form.

$$\begin{array}{ll} \min & 30x + 21y + 18z \\ \text{s. t.} & 3x - 7z \leq 176 \\ & 8z - 2y + x - 6 \geq 12 \\ & 4x + 3y = 19 \\ & x, y, z \in \mathbb{R} \end{array}$$

- 4) **[5 pts]** Derive the dual problem of the following linear programming problem, identifying clearly the Lagrangian and the dual function.

$$\begin{array}{ll} \min & x + 4y - 9z \\ \text{s. t.} & x - z = 7 \\ & x + y + z = 2 \\ & 4x + 3y = 19 \\ & x, y, z \geq 0 \end{array}$$

5) [5 Marks] Formulate the following optimization problem as a linear programming problem.

$$\max x_1 + x_2 \text{ s. t. } \|x\|_1^2 \leq 4, x \in \mathbb{R}^2$$

6) [5 Marks] Let  $m, n \in \mathbb{N}$  be such that  $m < n$ . Let  $A \in \mathbb{R}^{m,n}$  have full rank, and let  $y \in \mathbb{R}^m$  be in the column space of  $A$ . Let  $x^* := (w^*, z^*) \in \mathbb{R}^{2n}$  be the optimal solution to the linear programming problem

$$\begin{aligned} \min \quad & z_1 + z_2 + \cdots + z_n \\ \text{s. t.} \quad & -z \leq w \leq z \\ & Aw = y \\ & w, z \in \mathbb{R}^n \end{aligned}$$

Prove that  $z^* = |w^*|$ .

7) [5 pts] Let  $c = (\cos(\alpha), \sin(\alpha))^T$ . For which values of  $\alpha \in [0, 2\pi)$  is  $x = (1, 1)^T$  an optimal solution to the following linear programming problem?

$$\begin{aligned} \max \quad & c^T x \\ \text{s. t.} \quad & x + 2y \leq 3 \\ & 3x + y \leq 4 \\ & x, y \geq 0 \end{aligned}$$

8) [6 Marks] Write the following linear programming problem in standard form. Then, show that this problem is infeasible if and only if there is an  $x \geq 0$  such that  $Ax = 0$  and  $c^T x < 0$ .

$$\begin{aligned} \max \quad & b^T y \\ \text{s. t.} \quad & c - A^T y = s \\ & s \geq 0, y \in \mathbb{R}^m \end{aligned}$$

9) [6 Marks] Solve the following problem by use of the branch-and-bound method.

$$\begin{aligned} \min \quad & x + y \\ \text{s. t.} \quad & 2x + 2y \geq 5 \\ & 12x + 5y \leq 30 \\ & x, y \geq 0, \quad x, y \in \mathbb{N} \end{aligned}$$