Solutions for Tutorial Problem Sheet 12, December 15.

(Infinite Sequences and Series)

Problem 1.

a) Use the Ratio Test to determine if the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2(n+2)!}{n!\,3^{2n}}$ converges absolutely or diverges.

Solution:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^2((n+1)+2)!}{(n+1)!3^{2(n+1)}}}{(-1)^n\frac{n^2(n+2)!}{n!3^{2n}}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^2(n+3)(n+2)!}{(n+1)\cdot n!3^{2n}\cdot 3^2}\cdot\frac{n!3^{2n}}{n^2(n+2)!}\right) = \lim_{n\to\infty}\left(\frac{n^3+5n^2+7n+3}{9n^3+9n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3n^2+15n+7}{27n^2+18n}\right) = \lim_{n\to\infty}\left(\frac{6n+15}{54n+18}\right) = \lim_{n\to\infty}\left(\frac{6}{54}\right) = \frac{1}{9} < 1 \Rightarrow \sum_{n=1}^{\infty}(-1)^n\frac{n^2(n+2)!}{n!3^{2n}} \text{ converges}$$

b) Use the Root Test to determine if the series $\sum_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^n$ converges absolutely or diverges.

Solution:

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{4n+3}{3n-5}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{4n+3}{3n-5}\right) = \lim_{n\to\infty}\left(\frac{4}{3}\right) = \frac{4}{3} > 1 \Rightarrow \sum_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^n \text{ diverges}$$

c) Use any method to determine if the series $\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$ converges or diverges. Give reasons for your answer.

Solution:

$$\text{converges by the Ratio Test: } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10}\left(\frac{1}{10}\right) = \frac{1}{10} < 1$$

Problem 2. Determine if the series converges absolutely, converges, or diverges? Give reasons for your answers.

a) $\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{3+n}{5+n}$,

Solution:

diverges by the nth-Term Test since $\lim_{n\to\infty}\frac{3+n}{5+n}=1\neq 0$

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,(n!)^2}{(2n)!}$.

Solution:

converges absolutely by the Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{((n+1)!)^2}{((2n+2)!)}\cdot\frac{(2n)!}{(n!)^2}=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}=\frac{1}{4}<1$

Problem 3. Use any method to determine whether the series converges or diverges. Give reasons for your answer.

a) $\sum_{n=0}^{\infty}(-1)^n \frac{(n+2)!}{(2n)!}$,

Solution:

converges by the Ratio Test: $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{(n+3)!}{(2(n+1))!}\cdot\frac{(2n)!}{(n+2)!} = \lim\limits_{n\to\infty}\frac{n+3}{(2n+2)(2n+1)} = 0 < 1$

b) $\sum_{n=2}^{\infty}\frac{3}{10+n^{4/3}}$.

Solution:

converges by the Direct Comparison Test: $\frac{3}{10+n^{4/3}} < \frac{3}{n^{4/3}}$, which is the nth term of a convergent p-series

Problem 4. Given a series $\sum_{n=0}^{\infty} \frac{(-1)^n\, x^n}{n!}$
a) find the series' radius and interval of convergence.
For what values of x does the series converge b) absolutely, c) conditionally?

Solution:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|<1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{1}{n+1}\right)<1 \text{ for all } x$$

(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

Problem 5.

a) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{8}{4^{n+2}} x^n$.

b) Represent the power series in part a) as a power series about $x = 3$ and identify the interval of convergence of the new series.

Solution:

We can write the given series as $\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$ which shows that the interval of convergence is $-4 < x < 4$.

The function represented by the series in (a) is $\frac{2}{4-x}$ for $-4 < x < 4$. If we rewrite this function as $\frac{2}{1-(x-3)}$ we can represent it by the geometric series $\sum_{n=0}^{\infty} 2(x-3)^n$ which will converge only for $|x-3| < 1$ or $2 < x < 4$.