UNIVERSITY OF LEICESTER

MA2404
All Candidates

Semester 2 Examinations 2021-2022

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	Leicester International Institute
Module Code	MA2404
Module Title	Markov processes
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	Answer all questions
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	-
Are students permitted to bring their own Books/Statutes/Notes?	Yes
Additional Stationery	Yes

1. An insurance company receives claims of sizes $X_1, X_2, \ldots$. Hence, the total size of all claims during a week is $S = X_1 + X_2 + \cdots + X_N$, where N is the (random) number of claims during a week. The company assumes that N follows the geometric distribution with parameter p, X_i are independent from each other and from N, identically distributed, and follow the exponential distribution with parameter λ. Parameters p and λ are unknown and should be estimated from the past data.

   To protect itself from large claims, an insurance company arranged excess of loss reinsurance policy with retention level $M = 2000$, that is, if claim X_i exceeds £2000, the insurance company pays £2000, and the remaining part $X_i - 2000$ is covered by reinsurance.

   (i) [5 marks] The numbers of claims an insurance company received during the last 10 weeks were

   $$1, 3, 1, 1, 2, 4, 0, 3, 0 \text{ and } 5.$$

   Use the method of maximum likelihood to estimate the parameter p.

   (ii) [5 marks] The last 10 non-zero payments a *reinsurance* company made are

   $$200, 700, 2000, 1500, 100, 3400, 400, 600, 5000 \text{ and } 1100.$$

   Use the method of moments to estimate the parameter λ.

   (iii) [5 marks] Estimate the expectation and standard deviation of the total size S of all claims to be received by an insurance company during the next week.

   (iv) [5 marks] Let $S = I + R$, where I and R are the sums to be paid next week by insurance and reinsurance companies, respectively. Estimate the expectation of R.

   (v) [5 marks] Estimate the probability that $R = 0$.

   Total: 25 marks

2. **(i)** An insurance company receives claims that follow Burr distribution Burr$(\alpha, \lambda, \gamma)$ with density

   $$f(x) = \frac{\alpha\gamma\lambda^\alpha x^{\gamma-1}}{(\lambda + x^\gamma)^{\alpha+1}}, \quad x > 0$$

   where α, λ, γ are positive parameters. Using the limiting density ratios test, determine whether the tail of the Burr distribution becomes heavier or lighter if

   (a) **[5 marks]**. λ and γ are fixed and α increases;

   (b) **[5 marks]**. α and γ are fixed and λ increases;

   (c) **[5 marks]**. λ and α are fixed and γ increases.

   (ii) The random variables X and Y are dependent with the Clayton copula with parameter $\alpha = 1/2$.

   (a) **[5 marks]**. Calculate the coefficient of lower tail dependence of X and Y.

   (b) **[5 marks]**. Calculate the survival copula $\bar{C}(u, v)$.

   Total: 25 marks

3. Consider a no claims discount (NCD) model for car-insurance premiums. The insurance company offers discounts of 0%, 25% and 50% of the full premium $C = 1000$, determined by the following rules:

   (a) All new policyholders start at the 0% level.

   (b) If no claim is made during the current year the policyholder moves up one discount level, or remains at the 50% level.

   (c) If one or more claims are made the policyholder moves to the 0% level.

   The insurance company believes that the probability of making a claim each year depends on the current discount level and is equal to 0.3, 0.2 and 0.1 for drivers at discount levels 0%, 25% and 50%, respectively.

   (i) [5 marks] Explain why can this process be modelled as a Markov chain. Determine the state space and transition matrix.

   (ii) [5 marks] Calculate the 3-step transition matrix for this NCD system.

   (iii) [5 marks] A policyholder currently has no discount and pays the full premium. Calculate the expectation of the price of her insurance contract after 3 years.

   (iv) [5 marks] Compute the stationary distribution for this NCD system.

   (v) [5 marks] Prove that the n-step transition probabilities of this Markov chain converge to the stationary distribution.

   Total: 25 marks

4. A company provides sick pay to its employees who are unable to work. They decided to ignore the mortality rates and use the two-state, time-inhomogeneous Markov jump process with states Healthy (H) that means fit to work and Sick (S) that means unable to work. The transition rate form H to S is $\sigma(t)$, while the transition rate from S to H is $\rho(t)$.

   (i) [5 marks] Write down the generator matrix and Kolmogorov's forward equations in matrix form for this process.

   (ii) [10 marks] Given an employee is sick at the time t_1, write down an expression for the probability that he or she will stay sick continuously until time $t_2 > t_1$. Estimate this probability for $t_1 = 40$, $t_2 = 40.5$ and $\rho(t) = 100/t$.

   (iii) [10 marks] The company assumes that $\sigma(t) = at$ and would like to use linear regression with least square error to find parameter a. Data shows that $\sigma(20) \approx 0.04$, $\sigma(40) \approx 0.08$ and $\sigma(60) \approx 0.1$. Find parameter a which best approximates these data.

   Total: 25 marks