Linear Algebra II

Ch. IV Linear Maps and Matrices

§1. The Linear Map Associated with a Matrix

- Let A be an $m \times n$ matrix in a field K.

$$L_A : K^n \ni X \mapsto AX \in K^m$$

  is a linear map from K^n to K^m.

- **Theorem 1.1.** If A, B are $m \times n$ matrices and if $L_A = L_B$, then $A = B$. In other words, if matrices A, B give rise to the same linear map, then they are equal.

§2 The Matrix Associated with a Linear Map

- **Theorem 2.1.** Let $L : K^n \to K^m$ be a linear map. Then there exists a unique matrix A such that $L = L_A$.

§2 The Matrix Associated with a Linear Map

- The identity: id_{R^n}.

- The projection: $F : R^n \to R^r$,

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_r).$$

§2 The Matrix Associated with a Linear Map

- **Theorem III, 2.1.** Let V and W be vector spaces. Let $\{v_1, \ldots, v_n\}$ be a basis of V, and let $\{w_1, \ldots, w_n\}$ be arbitrary elements of W, Then there exists a unique linear mapping $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_n) = w_n.$$

If $x_1, \ldots, x_n$ are numbers, then

$$T(x_1 v_1 + \cdots + x_n v_n) = x_1 w_1 + \cdots + x_n w_n.$$

§2 The Matrix Associated with a Linear Map

- Let $E^1, \dots, E^n$ be unit columns in R^n and $A^1, \dots, A^n$ arbitrary elements of R^m. Then the matrix associated to the unique linear mapping such that $T(E^1) = A^1, \dots, T(E^n) = A^n$ is A.

- $L_{A+B} = L_A + L_B$.

- $L_{cA} = cL_A$.

- $L_{AB} = L_A L_B = L_A \circ L_B$.

§2 The Matrix Associated with a Linear Map

- **Theorem 2.2.** Let A be an $n \times n$ matrix, and let $A^1, \ldots, A^n$ be its columns. Then A is invertible if and only if $A^1, \ldots, A^n$ are linearly independent.

§3 Bases, Matrices, and Linear Maps

- Let V and W be arbitrary finite dimensional VSs over K, $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_m\}$ be bases of V and W respectively.
- Let $F : V \to W$ be a linear map.
- $\forall v \in V$, denote by
  - $X_{\mathcal{B}}(v)$ the coordinate vector of v relative to the basis $\mathcal{B}$;
  - $X_{\mathcal{B}'}(F(v))$ the coordinate vector of $F(v)$ relative to the basis $\mathcal{B}'$
- We associate a (uniquely determined) matrix with F, depending on our choice of bases, and denoted by $M^{\mathcal{B}}_{\mathcal{B}'}(F)$, such that $\forall v \in V$

$$X_{\mathcal{B}'}(F(v)) = M^{\mathcal{B}}_{\mathcal{B}'}(F) X_{\mathcal{B}}(v).$$

- Let V be a vector space, and let $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then $\forall v \in V$

$$X_{\mathcal{B}'}(v) = M^{\mathcal{B}}_{\mathcal{B}'}(\mathrm{id}) X_{\mathcal{B}}(v).$$

§3 Bases, Matrices, and Linear Maps

- Let

$$\begin{aligned} F(v_1) &= a_{11}w_1 + \cdots + a_{m1}w_m \\ &\vdots \\ F(v_n) &= a_{1n}w_1 + \cdots + a_{mn}w_m \end{aligned}$$

  then

$$M_{\mathcal{B}'}^{\mathcal{B}}(F) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Remark.** If the order of vectors in $\mathcal{B}$ or $\mathcal{B}'$, then $M_{\mathcal{B}'}^{\mathcal{B}}(F)$ will change.

§3 Bases, Matrices, and Linear Maps

§3 Bases, Matrices, and Linear Maps

- $M_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id}) = I$.
- Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{w_1, \ldots, w_n\}$ be bases of V. If

$$\begin{aligned} w_1 &= a_{11}v_1 + \cdots + a_{n1}v_n \\ &\vdots \\ w_n &= a_{1n}v_1 + \cdots + a_{nn}v_n \end{aligned}$$

then

$$M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

§3 Bases, Matrices, and Linear Maps

- **Theorem 3.3.**
  - $M^{\mathcal{B}}_{\mathcal{B}'}(F+G) = M^{\mathcal{B}}_{\mathcal{B}'}(F) + M^{\mathcal{B}}_{\mathcal{B}'}(G)$
  - $M^{\mathcal{B}}_{\mathcal{B}'}(cF) = cM^{\mathcal{B}}_{\mathcal{B}'}(F)$
  - Let $\dim V = n$ and $\dim W = m$. The association $F \mapsto M^{\mathcal{B}}_{\mathcal{B}'}(F)$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathrm{Mat}_{m\times n}(K)$

§3 Bases, Matrices, and Linear Maps

- Let V be a vector space, $\mathcal{B}$ a bases of V and $F : V \to V$ is a linear mapping. $M_{\mathcal{B}}^{\mathcal{B}}(F)$ is called the matrix associated with F relative to $\mathcal{B}$.
- (Do this here and now) Let $P_n = \left\{ \sum_{k=0}^{n} a_k t^k | a_k \in R \right\}$. What is the matrix associate with $D = d/dt : P_n \to P_n$ relative to the basis $\{1, t, \ldots, t^n\}$?

§3 Bases, Matrices, and Linear Maps

- **Theorem 3.4.** Let V, W, U be vector spaces. Let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{B}''$ be bases for V, W, U respectively. Let $F : V \to W$ and $G : W \to U$ be linear maps. Then
$$M^{\mathcal{B}'}_{\mathcal{B}''}(G)M^{\mathcal{B}}_{\mathcal{B}'}(F) = M^{\mathcal{B}}_{\mathcal{B}''}(G \circ F)$$

- $M_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id}) = I$.
- **Corollary 3.5.** Let V be a vector spaces and $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then
$$M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id})M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}) = I = M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id})M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id}).$$
In particular, $M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id})$ is invertible.

- **Theorem 3.6.** Let V be a vector spaces and $\mathcal{B}$, $\mathcal{B}'$ be bases of V. Then there exists an invertible matrix N such that
$$M_{\mathcal{B}'}^{\mathcal{B}'}(F) = N^{-1} M_{\mathcal{B}}^{\mathcal{B}}(F) N.$$
In fact, we can take
$$N = M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}).$$
Proof. Applying Th. 3.4, we have
$$M_{\mathcal{B}'}^{\mathcal{B}'}(F) = M_{\mathcal{B}'}^{\mathcal{B}}(\mathrm{id}) M_{\mathcal{B}}^{\mathcal{B}}(F) M_{\mathcal{B}}^{\mathcal{B}'}(\mathrm{id}).$$

§3 Bases, Matrices, and Linear Maps

- Let $F : V \to V$ be a linear map. A basis $\mathcal{B}$ of V is said to diagonalize F if $M^{\mathcal{B}}_{\mathcal{B}}(F)$ is a diagonal matrix.
- If there exists such a basis which diagonalizes F, then we say that F is diagonalizable.
- If A is an $n \times n$ matrix in K, we say that A can be diagonalized (in K) if the linear map on K^n represented by A can be diagonalized.
- **Theorem 3.6.** Let V be a finite dimensional vector space over K, let $F : V \to V$ be a linear map, and let M be its associated matrix relative to a basis $\mathcal{B}$. Then F (or M) can be diagonalized (in K) if and only if there exists an invertible matrix N in K such that $N^{-1}MN$ is a diagonal matrix.
- Homework: P94, 8, 9, 10.