Exam 2017

1 $\alpha > 0$

i)
$$L(\alpha) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n \mid \alpha = \hat{\alpha})$$
$$= \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\alpha} x_i^{(\alpha^{-1})-1} =$$
$$= \left(\frac{1}{\alpha}\right)^n \cdot \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{\alpha}-1}$$

ii) $$\ell(\alpha) = \log L(\alpha) = \log\left(\frac{1}{\alpha}\right)^n + \log\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{\alpha}-1} =$$
$$= -n\log\alpha + \left(\frac{1}{\alpha}-1\right)\cdot\log\left(\prod_{i=1}^{n} x_i\right) =$$
$$= -n\log\alpha + \left(\frac{1}{\alpha}-1\right)\cdot\sum_{i=1}^{n}\log(x_i)$$

$$\frac{d\,\ell(\alpha)}{d\alpha} = -\frac{n}{\alpha} + \left(-\frac{1}{\alpha^2}\right)\cdot\sum_{i=1}^{n}\log(x_i)$$

iii) $$\frac{d\,\ell(\alpha)}{d\alpha} = 0$$

$$\alpha^2 * \Big|\; -\frac{n}{\alpha} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\log(x_i) = 0 \qquad \alpha > 0$$
$$-\alpha n - \sum_{i=1}^{n}\log(x_i) = 0$$
$$\alpha = -\frac{1}{n}\sum_{i=1}^{n}\log(x_i) = \frac{1}{n}\sum_{i=1}^{n}(-\log x_i)$$

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n}(\log X_i)$$

$y_i = -\log X_i$ $\qquad \lambda = \frac{1}{\alpha}$ $\qquad \frac{1}{\lambda} = \alpha$

$y \sim Exp(\lambda)$ $\qquad E(y) = \frac{1}{\lambda}$

$Var(y) = \frac{1}{\lambda^2}$

iv) $E(\hat{\alpha}) = \alpha$

$$E(\hat{\alpha}) = E\Big(\frac{1}{n}\sum_{i=1}^{n}\overbrace{-\log X_i}^{"\, y_i}\Big) = \frac{1}{n}\sum_{i=1}^{n}E(y_i) =$$

$Bias(\hat{\alpha},\alpha)=0$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda} = \frac{1}{n}\cdot n\,\frac{1}{\lambda} = \frac{1}{\lambda} = \alpha \;!$$

$\hat{\alpha}$ is unbiased for α.

v) $MSE = E(\hat{\alpha}-\alpha)^2 = Var(\hat{\alpha}) + Bias^2(\hat{\alpha},\alpha)$

$Bias(\hat{\alpha},\alpha) = 0$

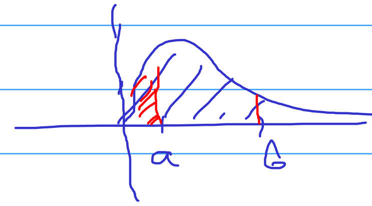
$$Var(\hat{\alpha}) = Var\Big(\frac{1}{n}\sum_{i=1}^{n}(-\log X_i)\Big) =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(\underbrace{-\log X_i}_{y_i}) =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(y_i) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{\lambda^2} = \frac{n}{\lambda^2 n^2} =$$

$$= \frac{1}{\lambda^2 n} \Rightarrow MSE(\hat{\alpha}) = \frac{\alpha^2}{n}$$

2. $P(X<\alpha) = \int_{-\infty}^{\alpha} f(x)\,dx$



$P(a<X<b) = 0{,}95$

$P(X<b) - P(X<a) = \int_{-\infty}^{b} \frac{1}{2} e^{-\frac{1}{2}x}\,dx - \int_{-\infty}^{a} \frac{1}{2} e^{-\frac{1}{2}x}\,dx =$

$x \geq 0 \quad = \int_{0}^{b} \frac{1}{2} e^{-\frac{1}{2}x}\,dx - \int_{0}^{a} \frac{1}{2} e^{-\frac{1}{2}x}\,dx =$

$= \int_{a}^{b} \frac{1}{2} e^{-\frac{1}{2}x}\,dx = -e^{-\frac{1}{2}x}\Big|_{a}^{b} =$

$= -e^{-\frac{1}{2}\log \overset{40^2}{1600}} + e^{-\frac{1}{2}\log\left(\frac{40}{39}\right)^2} =$

$= -e^{\log\sqrt{\frac{1}{(40^2)}}} + e^{\log\sqrt{\left(\frac{39}{40}\right)^2}} =$

$= -\frac{1}{40} + \frac{39}{40} = \frac{38}{40} = \frac{95}{100} = 0{,}95$

ii) $Y \sim F(\theta)$ $\qquad T \sim Exp\left(\frac{1}{2}\right)$

$T = 10(Y+\theta)$ $\qquad P(\gamma < T < \beta) = 1-\alpha$

$P(\theta_L < \theta < \theta_u) = 1-\alpha$ — ?

$P(a < T < b) = 0{,}95$

from q. i) from q. i)

$$P(a < 10(y+\theta) < b) = 0{,}95$$

$$P\left(\frac{a-10y}{10} < \theta < \frac{b-10y}{10}\right) = 0{,}95 \qquad y = 5$$

$$P\left(\frac{\log\left(\frac{40}{39}\right)^2 - 50}{10} < \theta < \frac{\log 40^2 - 50}{10}\right) = 0{,}95$$

$$P(-4{,}99 < \theta < -4{,}26) = 0{,}95$$

$$\theta \in (-4{,}99 \, , -4{,}26)$$

iii)

$$Z \sim Exp\left(\frac{1}{5}\right)$$

$$z_1, \dots, z_n \qquad n = 100$$

$$[0,2) \quad RFreq = \int_0^2 \frac{1}{5} \cdot e^{-\frac{1}{5}x} dx = -e^{-\frac{1}{5}x}\Big|_0^2 =$$

$$= -e^{-\frac{2}{5}} + 1 = \underline{0{,}33}$$

$$nE_1 = 100 \cdot 0{,}33 = 33.0$$

$$[2,4) \Rightarrow \quad nE_2 = \left(\int_2^4 \frac{1}{5} e^{-\frac{1}{5}x} dx\right) \cdot 100 = \underline{22.1}$$

$$[4,6) \Rightarrow \quad nE_3 = \left(\int_4^6 \frac{1}{5} e^{-\frac{1}{5}x} dx\right) \cdot 100 = \underline{14.8}$$

$$[6,8) \Rightarrow \quad nE_4 = \left(\int_6^8 \frac{1}{5} e^{-\frac{1}{5}x} dx\right) \cdot 100 = \underline{9.93}$$

$$[8,\infty) \Rightarrow \quad nE_5 = \left(\int_8^\infty \frac{1}{5} e^{-\frac{1}{5}x} dx\right) \cdot 100 = \underline{20{,}2}$$

iv)

$$G = \sum_{i=1}^{k} \frac{(\text{observed}_i - \text{expected}_i)^2}{\text{expected}_i} =$$

$$= \sum_{i=1}^{k} \frac{(X_i - nE_i)^2}{nE_i} \sim \chi^2_{k-1-t}$$

k = number of intervals

t = $1 - 1 -$ estimated param.

$t = 0$, $k = 5$

$$G \sim \chi^2_{5-1-0} = \chi^2_4$$

H_0: $Y \sim Exp(\frac{1}{5})$

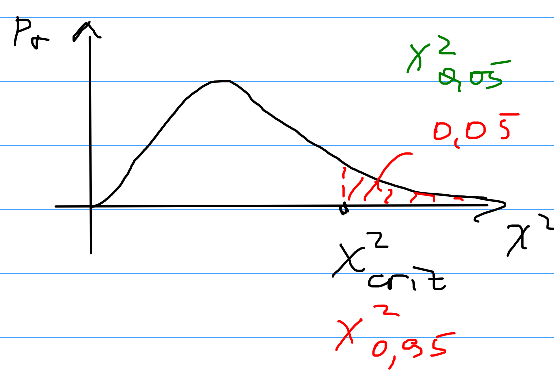
H_a: Y does not follow $Exp(\frac{1}{5})$

$$g_{obs} = \frac{(30 - 33,0)^2}{33,0} + \frac{(20 - 22,1)^2}{22,1} \dots + \frac{(21 - 20,2)^2}{20,2} =$$

$$= 1,31 \qquad \chi^2_{0,95,4}$$

$$g_{obs} \; ? \; \chi^2_{crit} = \chi^2_{0,05,4} = 9,49$$

$g_{obs} < \chi^2_{crit} \Rightarrow$ H_0 is not rejected



$Y \sim Exp(\frac{1}{5})$

3.

Rand. Var	Observ.
B_i	b_i
A_i	a_i
D_i	d_i

i) No

$$D \sim N(\delta, \sigma^2)$$

$\bar{D}$ – sample mean

S – sample standard deviation

ii) i person $\quad b_i > a_i \quad b_i - a_i > 0$

$$\underline{\underline{E(D) = \delta > 0}} \, !$$

iii) $$T = \frac{\bar{D} - \delta}{S/\sqrt{15}} \sim t_{n-1} = t_{14} \qquad \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

iv) $H_0: \delta = 0$

$H_a: \delta > 0$

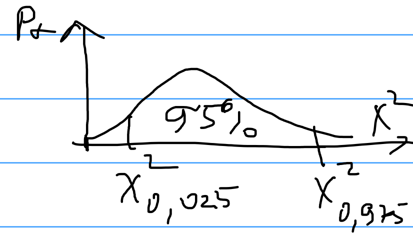
TS: $$t_{obs} = \frac{0{,}98 - 0}{\sqrt{0{,}24/15}} = 7{,}75$$

RR: $P(T > t_{0{,}95,14}) = P(T > 1{,}761) = 0{,}05$

$t_{obs} > t_{crit} \Rightarrow$ reject H_0

conclusion

v) $T = \frac{14 S^2}{\sigma^2} \sim \chi^2_{14}$ – pivot RV



$P(a < T < b) = 0,95$

$a = \chi^2_{0,025,14} = 5,63$

$b = \chi^2_{0,975,14} = 26,12$

$$P\left(5,63 < \frac{14 \cdot 0,24}{\sigma^2} < 26,12\right) = 0,95$$

$$P\left(\frac{14 \cdot 0,24}{26,12} < \sigma^2 < \frac{14 \cdot 0,24}{5,63}\right) = 0,95$$

95% CI for σ^2 is $(0,13 \,,\, 0,60)$.