LINEAR ALGEBRA II

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线性代数II（B.YU）

Ch. IX Polynomials and Matrices

§1. POLYNOMIALS

- By a polynomial over K, we shall mean a formal expression

  $$f(t) = a_n t^n + \cdots + a_0$$

- Coefficients
- Degree
- Degree of zero polynomial
- The leading coefficient
- The constant term
- $K[t]$

§1. POLYNOMIALS

- **Theorem 1.1.** *Let f, g be polynomials with coefficients in K. Then*

$$\deg(fg) = \deg f + \deg g.$$

- **Theorem 1.2.** *Let f be a polynomial with complex coefficients, of degree $\geqq 1$. Then f has a root in $\mathbf{C}$.*

- **Theorem 1.3.** *Let f be a polynomial with complex coefficients, leading coefficient 1, and $\deg f = n \geqq 1$. Then there exist complex numbers $\alpha_1, \ldots, \alpha_n$ such that*

$$f(t) = (t - \alpha_1) \cdots (t - \alpha_n).$$

The numbers $\alpha_1, \ldots, \alpha_n$ are uniquely determined up to a permutation. Every root α of f is equal to some α_i, and conversely.

- $f(t) = (t - \alpha_1)^{m_1} \cdots (t - \alpha_r)^{m_r}$, multiplicity

§1. POLYNOMIALS

§2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

an operator of V

- Let A be a square matrix with coefficients in K. Let $f \in K[t]$, and write

$$f(t) = a_n t^n + \cdots + a_0$$

with $a_i \in K$. We define

$$f(A) = a_n A^n + \cdots + a_0 I.$$

- **Example 1.** *Let* $f(t) = 3t^2 - 2t + 5$. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. Then

$$f(A) = 3\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^2 - \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}.$$

§2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

an operator

- **Theorem 2.1.** *Let $f,\ g \in K[t]$. Let A be a square matrix with coefficients in K. Then*

$$(f+g)(A) = f(A) + g(A),$$

$$(fg)(A) = f(A)g(A).$$

If $c \in K$, then $(cf)(A) = cf(A)$.

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§2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

- **Example 2.** Let $f(t) = (t-1)(t+3) = t^2 + 2t - 3$. Then

$$f(A) = A^2 + 2A - 3I = (A - I)(A + 3I).$$

- **Example 3.** Let $\alpha_1, \ldots, \alpha_n$ be numbers. Let

$$f(t) = (t - \alpha_1) \cdots (t - \alpha_n).$$

Then

$$f(A) = (A - \alpha_1 I) \cdots (A - \alpha_n I).$$

§2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

operator of n-dimensional Vector space V

- **Theorem 2.2.** *Let A be an $n \times n$ matrix in a field K. Then there exists a non-zero polynomial $f \in K[t]$ such that $f(A) = O$.*
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- Homework:
  - P236: 3, 5,