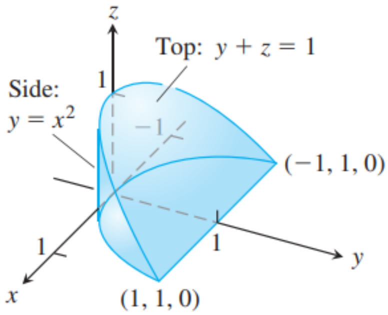
Solutions for Tutorial Problem Sheet 8, November 17.

(Multiple Integrals)

Problem 1. Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx.$$



Rewrite the integral as an equivalent iterated integral in the order
a) $dy\, dz\, dx$; b) $dy\, dx\, dz$; c) $dx\, dy\, dz$; d) $dx\, dz\, dy$; e) $dz\, dx\, dy$.

Solution:

(a) $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\, dz\, dx$

(b) $\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\, dx\, dz$

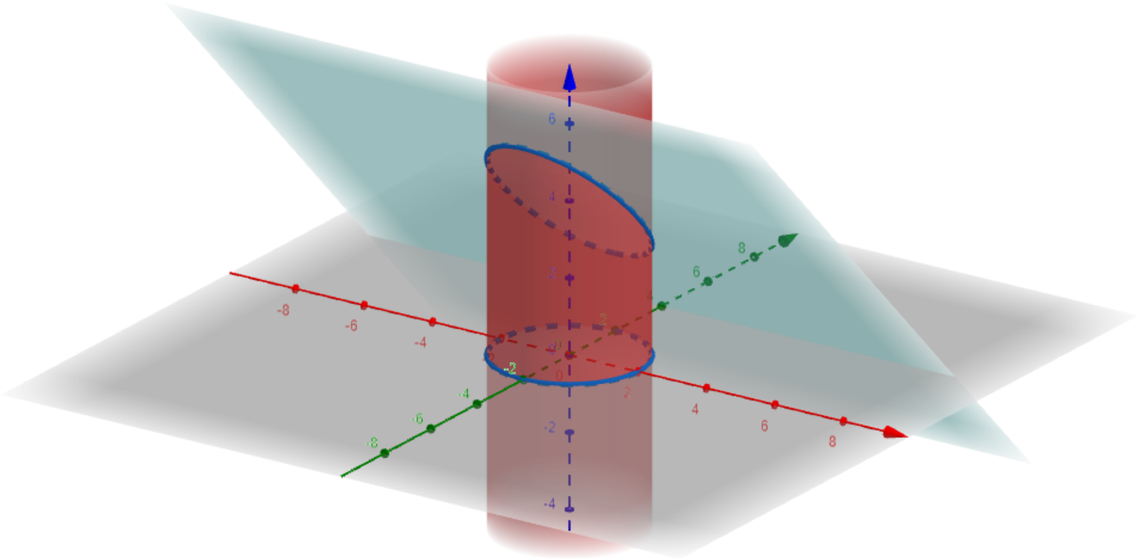
(c) $\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$

(d) $\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\, dz\, dy$

(e) $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\, dx\, dy$

Problem 2. Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

Solution:



$$V = \int_0^{2\pi} \int_0^2 \int_0^{4-r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left(4r - r^2 \sin\theta\right) dr\, d\theta = 8\int_0^{2\pi} \left(1 - \frac{\sin\theta}{3}\right) d\theta = 16\pi$$

Problem 3. Let D be the region in xyz-space defined by the inequalities $1 \le x \le 2$, $0 \le xy \le 2$, $0 \le z \le 1$. Evaluate

$$\iiint_D (x^2y + 3xyz)dx\,dy\,dz$$

by applying the transformation $u = x$, $y = xy$, $w = 3z$ and integrating over an appropriate region G in uyw-space.

Solution:

$$u = x,\ v = xy, \text{ and } w = 3z \Rightarrow x = u,\ y = \tfrac{v}{u}, \text{ and } z = \tfrac{1}{3}w \Rightarrow J(u, v, w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u};$$

$$\iiint\limits_D \left(x^2y + 3xyz\right) dx\,dy\,dz = \iiint\limits_G \left[u^2\left(\tfrac{v}{u}\right) + 3u\left(\tfrac{v}{u}\right)\left(\tfrac{w}{3}\right)\right] |J(u, v, w)|\,du\,dv\,dw = \tfrac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \tfrac{vw}{u}\right) du\,dv\,dw$$

$$= \tfrac{1}{3}\int_0^3\int_0^2 (v + vw \ln 2)\,dv\,dw = \tfrac{1}{3}\int_0^3 (1 + w\ln 2)\left[\tfrac{v^2}{2}\right]_0^2 dw = \tfrac{2}{3}\int_0^3 (1 + w\ln 2)\,dw = \tfrac{2}{3}\left[w + \tfrac{w^2}{2}\ln 2\right]_0^3$$

$$= \tfrac{2}{3}\left(3 + \tfrac{9}{2}\ln 2\right) = 2 + 3\ln 2 = 2 + \ln 8$$