Application of differentiation 15/11

→ finding local extreme values

→ finding global extreme values

f, g are continuous on $[a,b]$
& differentiable on (a,b)

Thm $f'(x) = g'(x) \quad \forall x \in (a,b)$

$\iff f(x) - g(x)$ is constant
$= C \quad \forall x \in [a,b]$

Proof First simplify: consider

$h(x) = f(x) - g(x)$ still continuous and differentiable

$h'(x) = f'(x) - g'(x)$

So we have to show

$h'(x) = 0 \ \forall x \iff h(x)$ is constant

$\Leftarrow$) If $h(x) = c \ \forall x$, $h'(x) = 0$

$\Rightarrow$) Use the mean value theorem

for $h : [c,d] \longrightarrow \mathbb{R}$

for $a \leq c < d \leq b$

$z \in$

$$\exists z : \frac{h(d) - h(c)}{d - c} = h'(z) = 0 \Rightarrow h(c) = h(d)$$

So derivative always 0 $\Rightarrow$

$\forall c < d$, $h(c) = h(d)$

so h is constant

Theorem $h'(x) > 0$ for all $x \in (a,b)$
$\Rightarrow h$ is strictly increasing

Same proof: if we choose any $c < d$ between a & b
the MVT says $\exists z$ between c & d

$$\frac{h(d) - h(c)}{d - c} = f'(z) > 0$$

$\Rightarrow h(d) - h(c) > 0$ whenever $c < d$.
$\Rightarrow h$ strictly increasing.

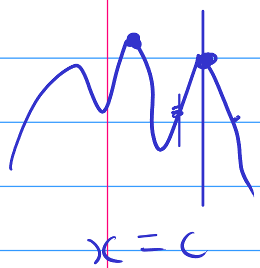
Theorem $h'(x) < 0 \Rightarrow$ strictly decreasing.

Finding max & min values

$f : [a,b] \to \mathbb{R}$ cts

$f' : (a,b) \to \mathbb{R}$ exists

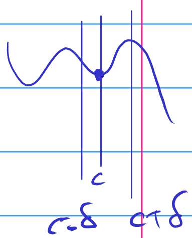
f has a <u>local</u> maximum at $x = c$



$\underline{\exists\, \delta > 0} : \underline{c - \delta < x \leq c} \Rightarrow \underline{f(x) \leq f(c)}$

$\& \quad \underline{c \leq x < c + \delta} \Rightarrow \underline{f(x) \leq f(c)}$

f has a local minimum at $x = c$

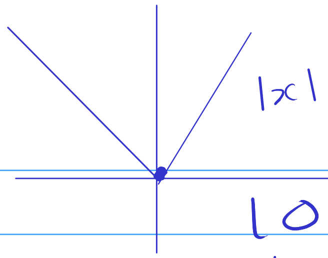


$\exists\, \delta > 0$ such that if $x \in (c - \delta, c + \delta)$ then $f(x) \geq f(c)$

$$\left\{\begin{array}{ll} \underline{\text{local max}} \ \nearrow\searrow \Rightarrow & f'(x) \geq 0 \ \forall x \in (c-\delta, c) \\ & f'(x) \leq 0 \ \forall x \in (c, c+\delta) \\ \underline{\text{local min}} \ \searrow\nearrow \Rightarrow & f'(x) \leq 0 \ \forall x \in (c-\delta, c) \\ & f'(x) \geq 0 \ \forall x \in (c, c+\delta) \end{array}\right.$$

$$\left.\begin{array}{l} \underline{\text{Local extreme value}} \text{ at } x = c \\ \& \text{ if } f'(c) \text{ exists} \end{array}\right\} \Rightarrow f'(c) = 0$$

$f(x) = |x|$



local min
but not differentiable

To disguish local max from local min

1) Look at $f(x)$ $x < c$ $x > c$

max: $f(x) \leq f(c)$ min: $f(x) \geq f(c)$

2) look at $f'(x)$ $x < c$, $x > 0$

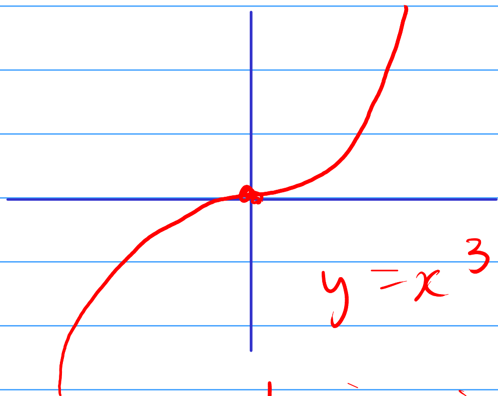
max: f' from +ve to −ve min: f' changes from −ve to +ve

3) If $f''(c)$

max: $f''(c) < 0$ min: $f''(c) > 0$

Careful: not every point $x = c$ with $f'(c) = 0$ is a local max or min.

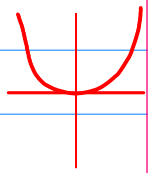
$f(x) = x^3$



$f'(x) = 3x^2$

$f'(0) = 0$

monotonic <u>increasing</u>



$f(x) = x^4$ $f'(0) = 0$ $f''(0) = 0$

$f'''(0) = 0$

Global Extreme Values

E.V.T. $f:[a,b] \longrightarrow \mathbb{R}$ cts

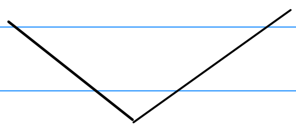
$\Rightarrow \exists\, c, d$

$$\underset{\text{min}}{f(c)} \le f(x) \le \underset{\text{max}}{f(d)}$$

Global extreme value could be

$f(c)$?

1) $x = a$
2) $x = b$
3) at $x = c$ with f not differentiable at $x = c$ — critical
4) at $x = c$ with $f'(c) = 0$



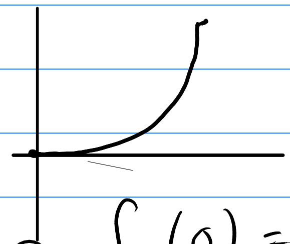
$f(x) = |x|$

$x = 0$ global minimum

(case 3)

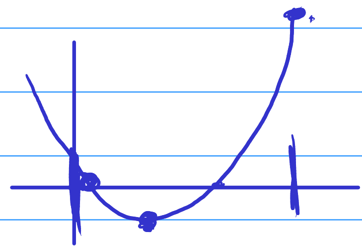
$f:[0,1] \to \mathbb{R}$

$f(x) = x^2$



$x = 0$, $f(0) = 0$, global min

$x = 1$, $f(1) = 1$, global max



$f:[0,2] \to \mathbb{R}$ $f(x) = x^2 - x$

$\cancel{f(0) = 0}$ $f(2) = 2$

$f'(x) = 2x - 1$ $f(\frac{1}{2}) = -1/4$