Last time: implicit functions & differentiation
Application to inverse functions

$$\begin{cases} y = f(x) = x^2 \\ \mathbb{R} \longrightarrow [0,\infty) \end{cases} \qquad x = g(y) = {}^{+}\sqrt{y} \quad [0,\infty) \longrightarrow [0,\infty)$$

$$* \quad \underline{f'(x) = 2x} \qquad \underline{g'(y) = \frac{1}{2}\frac{1}{\sqrt{y}} = \frac{1}{2x}}$$

<u>Theorem</u> <u>If</u> f is <u>diffentiable</u> <u>and</u> 1-1 on some open interval then so is the inverse

<u>Theorem</u> <u>If</u> f, f^{-1} are both diffentiable then $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ (assuming $f'(x) \neq 0$)

<u>Proof</u> $y = f(x) \qquad f^{-1}(y) = x$

$$f^{-1}(f(x)) = x$$

Chain $\Longrightarrow$ Rule

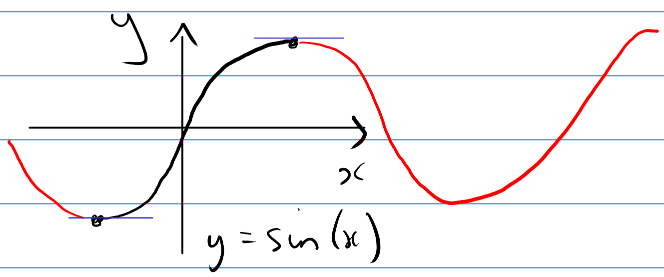
$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

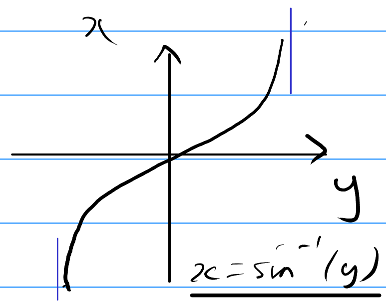
$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

List of examples

	$f(x)$	$f'(x)$	$f^{-1}(y)$	$(f^{-1})'(y)$
$\frac{1}{\cos(x)}$	$\sin(x)$	$\cos(x)$	$\sin^{-1}(y)$	$\frac{1}{\sqrt{(1-y^2)}}$
$\frac{1}{\sqrt{(1-\sin^2(x))}}$	$\cos(x)$	$-\sin(x)$	$\cos^{-1}(y)$	$\frac{-1}{\sqrt{(1-y^2)}}$
	$\tan(x)$	$\frac{1}{\cos^2(x)}$	$\tan^{-1}(y)$	$\frac{1}{1+y^2}$
$\rightarrow$	e^x	e^x	$\ln(y)$	$\frac{1}{y}$
$\frac{e^x-e^{-x}}{2}$	$\sinh(x)$	$\cosh(x)$	$\sinh^{-1}(y)$	$\frac{1}{\sqrt{(1+y^2)}}$
$\frac{e^x+e^{-x}}{2}$	$\cosh(x)$	$\sinh(x)$	$\cosh^{-1}(y)$	
$\frac{e^x-e^{-x}}{e^x+e^{-x}}$	$\tanh(x)$			

$\sin : (-\pi/2, \pi/2] \longrightarrow [-1, 1]$ one-to-one & onto



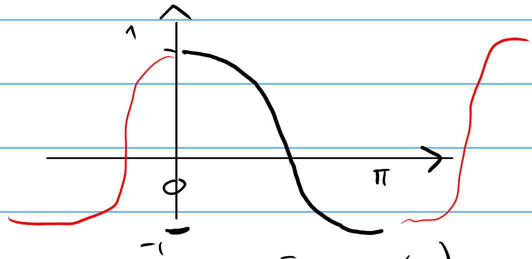


$$\frac{1}{y'} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{(1-\sin^2(x))}} = \frac{1}{\sqrt{(1-y^2)}}$$

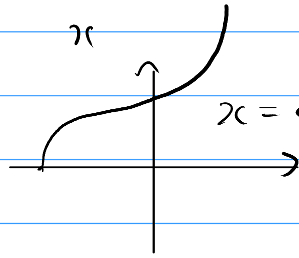
$\frac{dy}{dx}$ ✓ $\quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

$$\frac{1}{-\sin(x)} = \frac{1}{-\sqrt{1-y^2}}$$

$\cos : [0, \pi) \longrightarrow [-1,1]$ $\quad$ $\cos^{-1} : [-1,1] \to [0,\pi)$

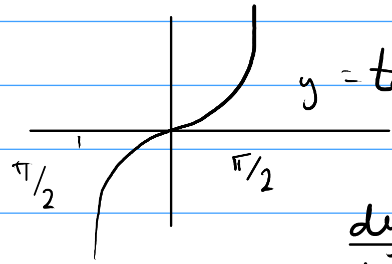


$y = \cos(x)$



$x = \cos^{-1}(y)$

tan

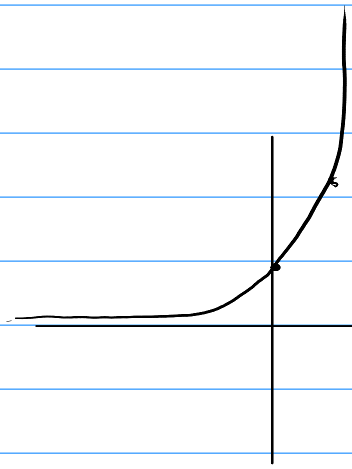


$y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ $\qquad$ $\dfrac{d}{dy}\tan^{-1}(y) = \dfrac{1}{1+y^2}$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)} = \sec^2 x$$

$$= \underline{1 + \tan^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

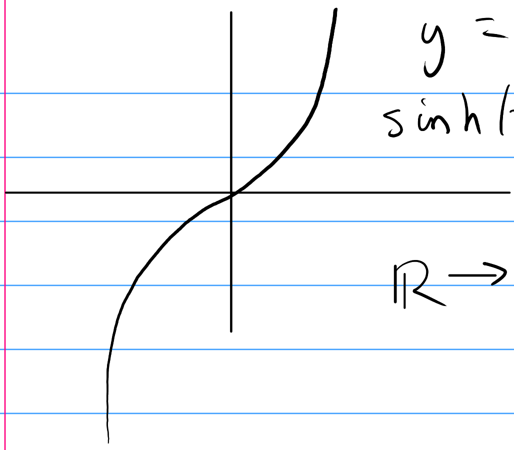
$$= 1 + y^2$$



$\dfrac{dy}{dx} = e^x = y$ $\qquad$ $\dfrac{dx}{dy} = \dfrac{1}{y}$

$y = e^x$ $\qquad$ $\dfrac{d}{dy}(\ln(y)) = \dfrac{1}{y}$

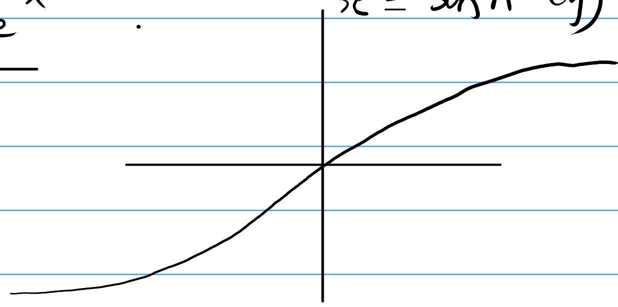
$\exp : \mathbb{R} \to (0, \infty)$ $\qquad$ $\ln : (0,\infty) \to \mathbb{R}$



$y = \sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

$\mathbb{R} \to \mathbb{R}$

$x = \sinh^{-1}(y)$



$\cos^2 + \sin^2 = 1$

$\cosh^2 - \sinh^2 = 1$

$$\frac{dx}{dy} = \frac{1}{\cosh x} = \frac{1}{\sqrt{(1+\sinh^2 x)}}$$

$$= \frac{1}{\sqrt{(1+y^2)}}$$

When <u>theory</u> / <u>exact</u> answers fail find good numerical approximations

<u>Bolzano's Theorem</u>

Problem: Solve $\underbrace{x^6 - x - 2}_{f(x)\ \text{cts}} = 0$

<u>numerically</u>

$[0,2]$ $\quad \underline{f(2) = 60} \qquad \underline{f(0) = -2}$

different signs

Problem has a solution between 0 & 2

$[0,1]$ ① $f(1)$ — $+ve$ $\Rightarrow$ between 0 & 1

$[1,2]$ $\quad f(1)$ — $-ve$ $\Rightarrow$ between 1 & 2

Repeat: find $\exists$ solution between

$[c_n, d_n]$ $f(c_n), f(d_n)$ different signs

$\Rightarrow f\left(\frac{c_n+d_n}{2}\right)$ tells us if solution

is in $\left[c_n, \frac{c_n+d_n}{2}\right]$ or $\left[\frac{c_n+d_n}{2}, d_n\right]$

$\rightsquigarrow$ converge on a solution.

<u>Next time</u> Newton's Method

<u>usually</u> faster, needs f to be differentiable

to solve $f(x) = 0$ numerically.