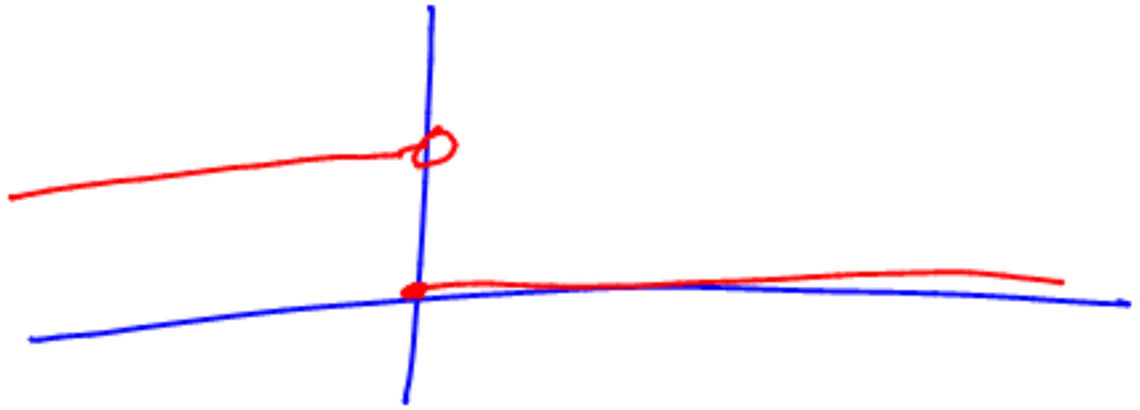
Continuous functions

Non-examples

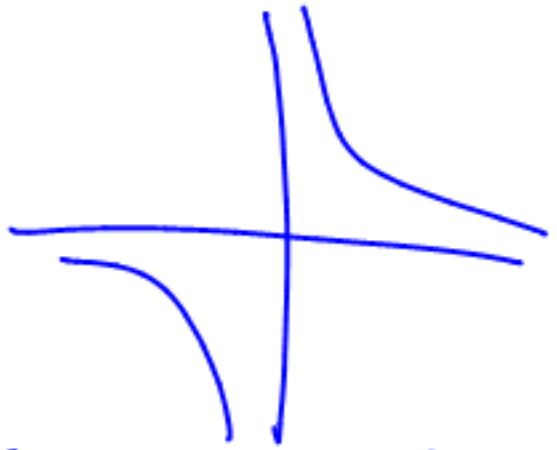
a) $f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$



not continuous at origin

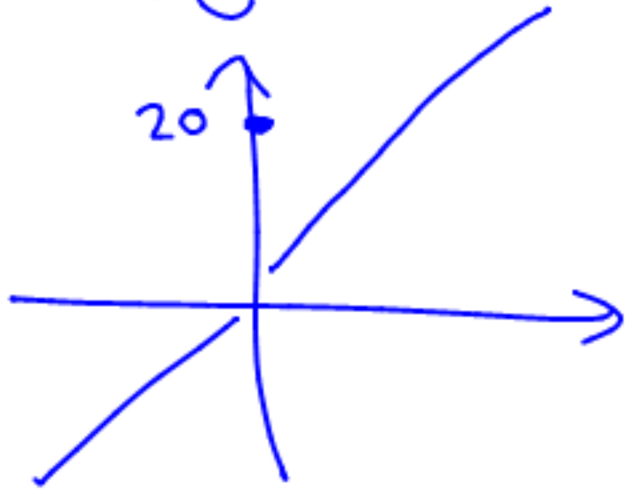
$\lim_{x \to 0^-} f(x) = 1$ $\quad \lim_{x \to 0^+} f(x) = 0$ so f has no limit.

b) $f(x) = \frac{1}{x}$



Neither $\lim_{x \to 0^-} f(x)$ nor $\lim_{x \to 0^+} f(x)$ exist.

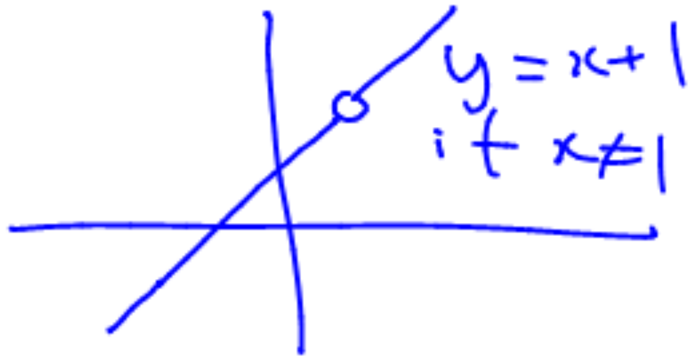
c) $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 20 & \text{if } x = 0 \end{cases}$



$\lim_{x \to 0} f(x) = 0 \neq f(0)$

d) $f(x) = \frac{x^2 - 1}{x - 1}$

$f(x) = \begin{cases} x+1 & x \neq \\ \text{undefined} & x \neq \end{cases}$



$\lim_{x \to 1} f(x$

$= \lim_{x \to 1} x+1$

$= 2$

Basic Limit Theorems

Thm 1 If a limit exists it is unique

If $\lim_{x \to c} f(x) = L_1$ & $\lim_{x \to c} f(x) = M = L_2$ Then $L_1 = M = L_2$

$\forall \varepsilon > 0 : \exists \delta_1, \delta_2$ such that $|f(x) - L_k| < \varepsilon$ when $|x - c| < \delta_k$

$k = 1$ or $k = 2$

If $|x - c| < \min(\delta_1, \delta_2)$ then $|f(x) - L_1|, |f(x) - L_2| < \varepsilon$

& $|L_1 - L_2| = |(f(x) - L_2) - (f(x) - L_1)| \leq |f(x) - L_2| + |-(f(x) - L_1)| < 2\varepsilon$

$|L_1 - L_2|$ smaller than any +ve number $\Rightarrow L_1 - L_2 = 0$.

Thm 2. If $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, then we have

i) $\lim_{x \to c} (f(x) \pm g(x)) = L \pm M$ ii) $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$

iii) If $M \neq 0$ then $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$, so using (ii) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

Pf of (i) Given: $\forall \varepsilon > 0 \exists \delta_f > 0$ s.t. $|f(x) - L| < \varepsilon/2$ whenever $|x - c| < \delta_f$

$\exists \delta_g > 0$ s.t. $|g(x) - M| < \varepsilon/2$ whenever $|x - c| < \delta_g$

Take $0 < \delta < \min(\delta_f, \delta_g)$

Then $|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

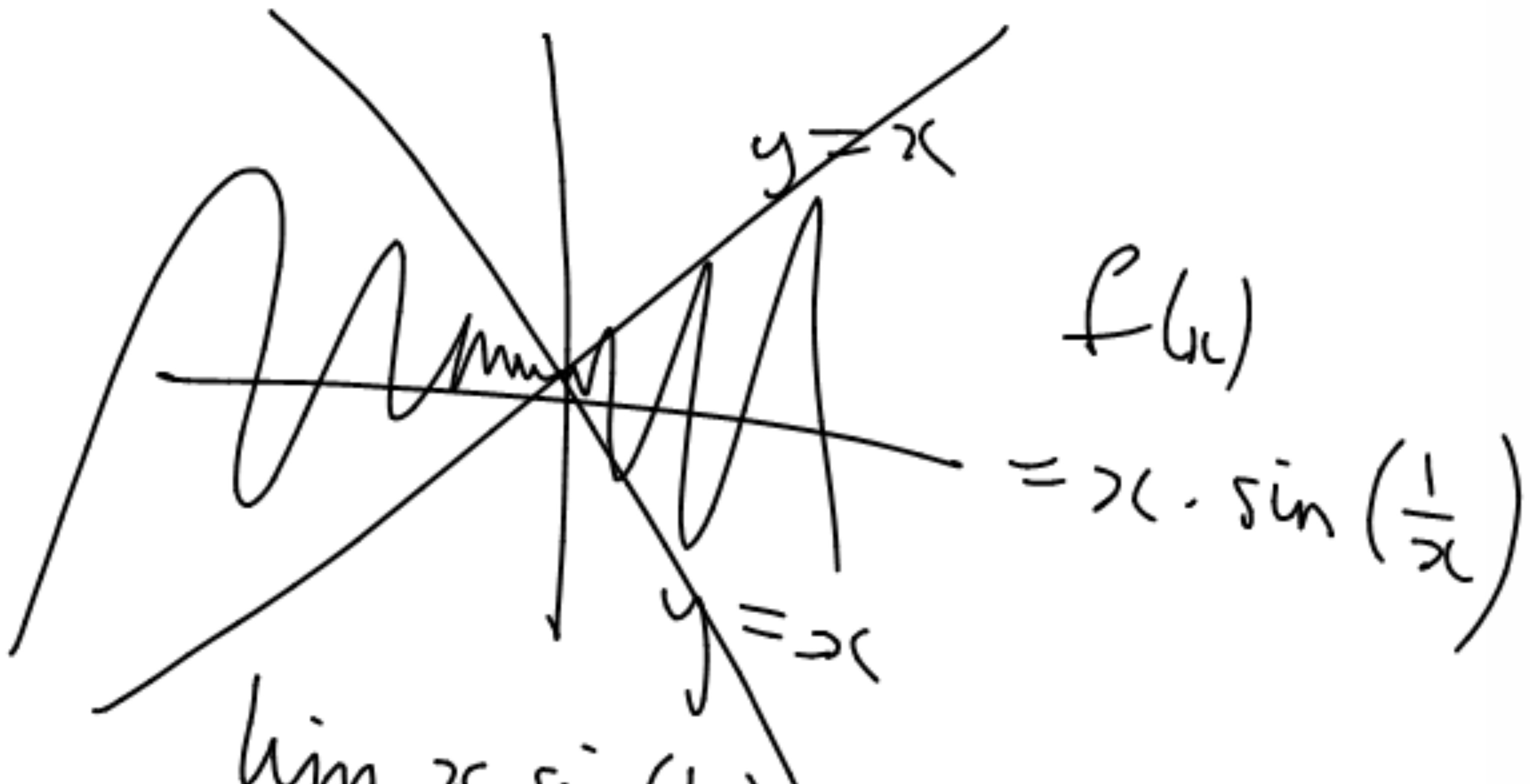
whenever $|x - c| < \delta$

PINCHING THEOREM

If $f(x) \le g(x) \le h(x)$

& $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} (h(x)) = L$

Then $\lim\limits_{x\to c} g(x) = L$



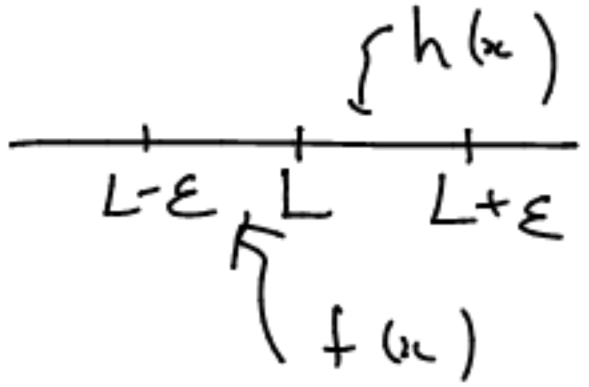
$\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Pf $\forall \varepsilon > 0 \ \exists \delta_f, \delta_h : |x-c| < \delta_f \Rightarrow$

$|f(x)-L| < \varepsilon$

$|h(x)-L| < \varepsilon$

Let $\delta = \min(\delta_f, \delta_h)$



This says

$\underline{L-\varepsilon < f(x)} \le g(x) \le \underline{h(x) < L+\varepsilon}$

So $|g(x)-L| < \varepsilon$.