Solutions for Tutorial Problem Sheet 7, November 10.

(Partial Derivatives. Multiple Integrals)

Problem 1. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Solution:

$f(x, y) = \cos x \cos y \Rightarrow f_x = -\sin x \cos y$, $f_y = -\cos x \sin y$, $f_{xx} = -\cos x \cos y$, $f_{xy} = \sin x \sin y$, $f_{yy} = -\cos x \cos y \Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$ $= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot (-1) + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 - \frac{x^2}{2} - \frac{y^2}{2}$, quadratic approximation. Since all partial derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal to 1 $\Rightarrow E(x, y) \leq \frac{1}{6}[(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3] \leq 0.00134$.

Problem 2. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle R : $0 \leq x \leq 1, \quad 0 \leq y \leq 2$.

Solution:

$$V = \iint_R f(x, y)\, dA = \int_0^1 \int_0^2 \left(4 - y^2\right) dy\, dx = \int_0^1 \left[4y - \frac{1}{3}y^3\right]_0^2 dx = \int_0^1 \left(\frac{16}{3}\right) dx = \left[\frac{16}{3}x\right]_0^1 = \frac{16}{3}$$

Problem 3. Sketch the region of integration, reverse the order of integration, and evaluate the integral

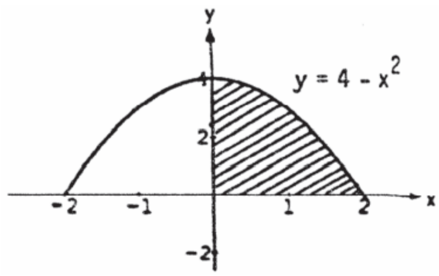
$$\text{a)} \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy\, dx,$$

$$\text{b)} \int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} dy\, dx.$$

Solution:

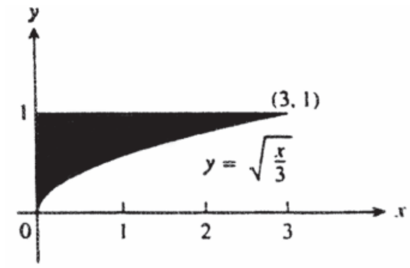
$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)} \right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{4} \right]_0^4 = \frac{e^8 - 1}{4}$$



$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy = \left[e^{y^3} \right]_0^1 = e - 1$$



Problem 4. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

Solution:

$$V = \int_{-4}^{1}\int_{3x}^{4-x^2}(x+4)\,dy\,dx = \int_{-4}^{1}\left[xy+4y\right]_{3x}^{4-x^2}\,dx = \int_{-4}^{1}\left[x\left(4-x^2\right)+4\left(4-x^2\right)-3x^2-12x\right]dx$$
$$= \int_{-4}^{1}\left(-x^3-7x^2-8x+16\right)dx = \left[-\tfrac{1}{4}x^4-\tfrac{7}{3}x^3-4x^2+16x\right]_{-4}^{1} = \left(-\tfrac{1}{4}-\tfrac{7}{3}+12\right)-\left(\tfrac{64}{3}-64\right) = \tfrac{157}{3}-\tfrac{1}{4} = \tfrac{625}{12}$$

Problem 5. Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral

$$\text{a)}\ \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{(1+x^2+y^2)^2}\,dy\,dx,$$

$$\text{b)}\ \int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}}\frac{1}{(x^2+y^2)^2}\,dy\,dx.$$

Solution:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{\left(1+x^2+y^2\right)^2}\,dy\,dx = 4\int_{0}^{\pi/2}\int_{0}^{1}\frac{2r}{\left(1+r^2\right)^2}\,dr\,d\theta = 4\int_{0}^{\pi/2}\left[-\frac{1}{1+r^2}\right]_{0}^{1}d\theta = 2\int_{0}^{\pi/2}d\theta = \pi$$

$$\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}}\frac{1}{\left(x^2+y^2\right)^2}\,dy\,dx = \int_{0}^{\pi/4}\int_{\sec\theta}^{2\cos\theta}\frac{1}{r^4}\,r\,dr\,d\theta = \int_{0}^{\pi/4}\left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta}d\theta = \int_{0}^{\pi/4}\left(\tfrac{1}{2}\cos^2\theta-\tfrac{1}{8}\sec^2\theta\right)d\theta$$
$$= \left[\tfrac{1}{4}\theta+\tfrac{1}{8}\sin 2\theta-\tfrac{1}{8}\tan\theta\right]_{0}^{\pi/4} = \tfrac{\pi}{16}$$