Lecture 15: Partial Derivatives.

MA2032 Vector Calculus

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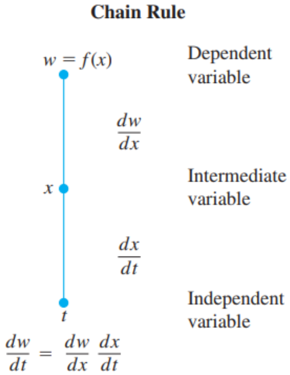
The Chain Rule

• The **Chain Rule** for functions of a **single variable** says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and dw/dt can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$



• We display the Chain Rule in a **"dependency diagram"** in the margin. Such diagrams capture which variables depend on which.

• For functions of several variables the **Chain Rule has more than one form**, which depends on how many independent and intermediate variables are involved.

The Chain Rule. Functions of Two Variables

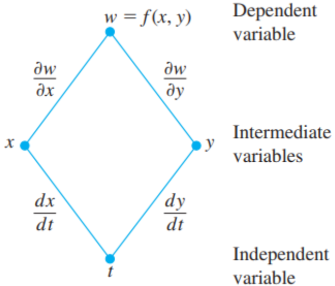
THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables
If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

The Chain Rule

Example 1

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}(2\cos 2t) = \cos 2t.$$

In either case, at the given value of t,

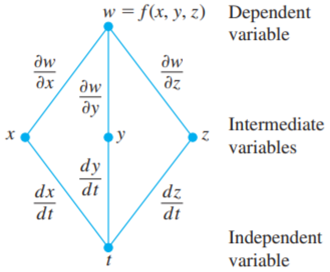
$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\frac{\pi}{2}\right) = \cos\pi = -1. \quad \blacksquare$$

The Chain Rule. Functions of Three Variable

THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables
If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 2

Find dw/dt if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. In this example the values of $w(t)$ are changing along the path of a helix as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for intermediate variables.

so

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

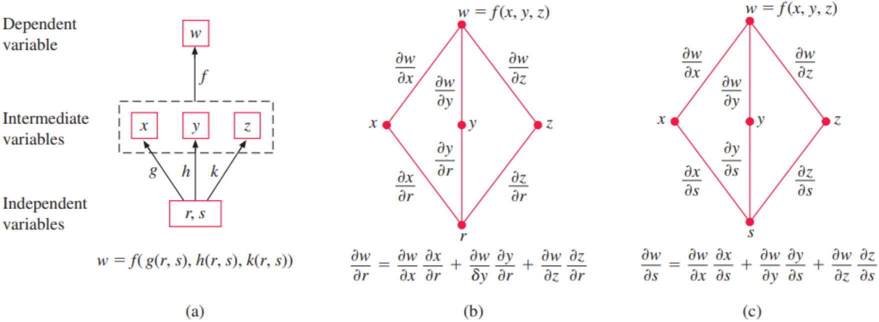
The Chain Rule. Functions Defined on Surfaces

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The Chain Rule. Functions of Three Variable



Example 3

Express $\partial w/dr$ and $\partial w/ds$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \qquad \text{Substitute for intermediate variable } z. \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.
\end{aligned}$$

Implicit Differentiation Revisited

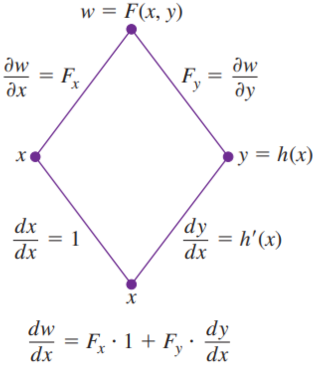
- Suppose that

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

- Since $w = F(x, y) = 0$, the derivative dw/dx must be zero.
- Computing the derivative from the Chain Rule, we find

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$



- If $F_y = dw/dy \neq 0$, we can solve this equation for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Implicit Differentiation Revisited

THEOREM 8—A Formula for Implicit Differentiation
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

EXAMPLE 5 Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than a single-variable calculation using implicit differentiation. ■

Implicit Differentiation Revisited

- The result in Theorem 8 is **easily extended** to three variables.

- Suppose that the equation $F(x, y, z) = 0$ defines the **variable z implicitly** as a function $z = f(x, y)$.

- Then for all (x, y) in the domain of f, we have $F(x, y, f(x, y)) = 0$.

- Assuming that **F and f are differentiable** functions, we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to the independent variable x:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x}, \end{aligned}$$

y is constant when differentiating with respect to x.

- A similar calculation for differentiating with respect to the independent variable y is performed.

Implicit Differentiation Revisited

• If the partial derivatives F_x, F_y, and F_z are **continuous** throughout an open region R in space containing the point (x_0, y_0, z_0), and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ **defines z implicitly** as a differentiable function of x and y near (x_0, y_0, z_0), and the **partial derivatives of z** are given by

$$\frac{dz}{dx} = -\frac{F_x}{F_z} \text{ and } \frac{dz}{dy} = -\frac{F_y}{F_z}$$

Implicit Differentiation Revisited

Example 4

Find dz/dx and dz/dy at $(0,0,0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Solution Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines z as a differentiable function of x and y near the point $(0, 0, 0)$. From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

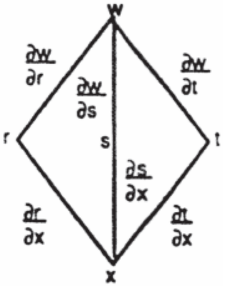
$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1. \qquad \blacksquare$$

Example 5

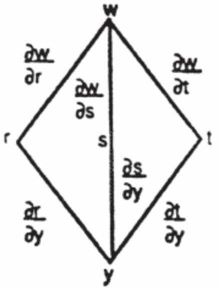
Draw a dependency diagram and write a Chain Rule formula for each derivative
$\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$.

Solution:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$$



$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$



Example 6

Assume that $z = f(x,y)^2$, $x = g(t)$, $y = h(t)$, $f_x(1,0) = -1$, $f_y(1,0) = 1$, and f(1,0)=2.
If $g(3) = 1$, $h(3) = 0$, $g'(3) = -3$, and $h'(3) = 4$, find $\frac{dz}{dt}\,|_{t=3}$

Solution:

$$\tfrac{dz}{dt} = 2f(x,y)\left[f_x(x,y)g'(t) + f_y(x,y)\cdot h'(t)\right] \Rightarrow \tfrac{dz}{dt}\Big|_{t=3} = 2f(1,0)\left[f_x(1,0)g'(3) + f_y(1,0)h'(3)\right]$$

$$= 2(2)\left[(-1)(-3) + (1)(4)\right] = 28$$