Linear Algebra II, III

Ch. II Linear Mappings

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- **Linear Mapping:** Let V, V' be VSs over the field K. A linear mapping
$$F : V \to V'$$
is a mapping satisfying:

  LM 1. $\forall u, v \in V$, $F(u+v) = F(u) + F(v)$.
  LM 2. $\forall c \in K$ and $v \in V$, $F(cv) = cF(v)$.

- When $V' = K$, F is called a linear functional.
- When $V = K^n$, $V' = K$, F is a linear function.
- The identity map id_V, I_V (id, I): $v \mapsto v$ is a linear mapping.
- The zero map O: $v \mapsto O$ is a linear mapping.

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- Composite mapping $G \circ F : U \ni u \mapsto G(F(u)) \in W$ of $G : V \to W$ and $F : U \to V$.
- $H \circ (G \circ F) = (H \circ G) \circ F$. $= H(G(F(u)))$
- The composite map $GF = G \circ F$ of linear maps is also a linear map.
- For linear maps, $(H + G) \circ F = H \circ F + G \circ F$; $G \circ (F + T) = G \circ F + G \circ T$; $(cG) \circ F = c(G \circ F)$.

Ch. II Linear Mappings

- A mapping $F : S \to S'$ is called injective if $x \neq y \Rightarrow F(x) \neq F(y)$ $(F(x) = F(y) \Rightarrow x = y)$.
  - A linear mapping $F : V \to V'$ is injective$\Leftrightarrow$Ker $F = \{O\}$.

"$\Leftarrow$" $F(x)=F(y)$, $F(x)-F(y)=0=F(x-y) \Rightarrow x-y=0 \Rightarrow x=y$

"$\Rightarrow$" $F(x)=0$, $F(0)=0$, $F(x)=F(0) \Rightarrow x=0 \Rightarrow \mathrm{Ker}\,F=\{0\}$.

  - Ker $F = \{O\} \Rightarrow$If $v_1, \ldots v_n$ are L.I., then $F(v_1), \ldots F(v_n)$ are L.I.

$0 = c_1F(v_1)+\cdots+c_nF(v_n) = F(c_1v_1+\cdots+c_nv_n)$

  - Ker $F = \{v \in V | F(v) = O\}$ is the kernel of F, a subspace of V.
- A mapping $F : S \to S'$ is called surjective if Im $F = S'$

$\forall y \in S'$, $\exists x \in S$ s.t. $F(x)=y$

  - Im $F = \{F(v) | v \in V\}$ is the image of F, a subspace of V'.
- bijective=injective+surjective.

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- $\dim V = \dim \operatorname{Ker} L + \dim \operatorname{Im} L$.

$L : V \to V'$ linear mapping

$\dim V$ finite

- Let $L : V \to W$ be a linear map. Assume that

$$\dim V = \dim W.$$

If $\operatorname{Ker} L = \{O\}$, or if $\operatorname{Im} L = W$, then L is bijective.

Proof.. $\ker L = \{0\}$. $\underbrace{\dim V}_{\underbrace{\dim W}_{0}} = \underbrace{\dim \ker L}_{0} + \underbrace{\dim \operatorname{Im} L}_{\dim W}$

$\operatorname{Im} L \subset W$ subspace

$\operatorname{Im} L = W$

$\operatorname{Im} L = W$

$\Rightarrow \dim \ker L = 0 \Rightarrow \ker L = \{0\}$

- We say that the mapping $F : S \to S'$ has an inverse if there exists a mapping $G : S' \to S$ such that

$$G \circ F = I_S, \text{ and } F \circ G = I_{S'}.$$

- The inverse of a linear map is a linear map.

- The mapping $F : S \to S'$ has an inverse$\Leftrightarrow F$ is bijective.

Proof. "$\Leftarrow$" $G: S' \to S$

$\forall y \in S'$, F is bijective, so it is surjective, $\exists x \in S$, s.t. $F(x)=y$.

F is also injective, x is unique.

Let : $G(y) = x$

$$\begin{cases} G \circ F(x) = \underline{G(F(x))} = x \\ F \circ \underbrace{G(y)}_{=x} = F(G(y)) = y \quad \because F(x) \end{cases}$$

"z"

$\forall x \in S$, $\overset{y=}{F(x)} \in S'$ y

$\exists! z \in S$, s.t. $F(z) = F(x)$

$z = x$ (F is injective)

$\Rightarrow G$ is a inverse of F

"$\Rightarrow$" If F has a inverse G

① $F(x) = F(y) \Rightarrow x = G(F(x)) = G(F(y)) = y \Rightarrow x = y \Rightarrow F$ is injective

② $\forall y \in S'$, let $x = G(y) \in S$, $F(x) = F(G(y)) = y \Rightarrow F$ is surjective

Ch. II Linear Mappings

$S \to S'$ vector

- All mappings from S (a set) into V (a VS over K) is a linear space over K.

$F: S \to V$, $G: S \to V$, $c \in K$

$(F+G)(x) = F(x) + G(x)$, $(c \cdot F)(x) = c \cdot F(x)$

- $\mathcal{L}(V, V')$, all linear maps from V into V' (V and V' are VSs over K) is a linear space over K.

vector

$F: V \to V'$ linear mapping

G :

$c \in K$, $a \in K$

$(F+G)(u+v) = F(u+v) + G(u+v)$
$= F(u) + F(v) + G(u) + G(v)$
$= [F(u) + G(u)] + [F(v) + G(v)]$
$= (F+G)(u) + (F+G)(v)$

$(F+G)(cu) = F(cu) + G(cu)$
$= c[F(u) + G(u)]$
$= c(F+G)(u)$

$\Rightarrow F+G: V \to V'$ is a linear mapping

$(cF)(u+v) = cF(u+v) = c(F(u) + F(v)) = cF(u) + cF(v) = (cF)(u) + (cF)(v)$

$(cF)(au) = cF(au) = c \cdot a F(u) = a \cdot cF(u) = a(cF)(u) \Rightarrow cF$ is a linear mapping from V to V'.

- Let V be a finite dimensional space over K, and let $\{v_1, \ldots, v_n\}$ be a basis of V. We define a map

$$F: \ V \to K^n$$

by associating to each element $v \in V$ its coordinate vector X with respect to the basis. Thus if

$$v = x_1 v_1 + \cdots, x_n v_n,$$

with $x_i \in K$, we let

$$F(v) = (x_1, \ldots, x_n).$$

Then, F is a linear map.

- $G : K^n \to V$, $G(x_1, \ldots, x_n) = v$ a linear map.
- $GF = I$, $FG = I$.

  $V \to V$ $K^n \to K^n$

- Isomorphism. $V \to V'$ $\exists$ inversable $F: V \to V'$ V and V' are isomorphic. F is called a isomorphism between V and V'

Ch. II Linear Mappings

- Let $F : S \to K^n$ be a mapping, then $\forall v \in S \quad F(v) \in K^n$

$$F(v) = (f_1(v), \ldots, f_n(v)).$$

  f_i's are coordinate (component) function(al)s of F.

- $F : V \to K^n$ (V is a VS) is a linear map$\Leftrightarrow$ f_i's are linear functions. al

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- Let V be the VS of functions having derivatives of all orders on the interval $0 < t < 1$, then the derivative $D = d/dt$ is a linear mapping from V into V.

$Df(t) = \frac{df(t)}{dt}$ $\quad D(f+g) = Df + Dg$

- Let V be the VS of functions having derivatives of all orders, then $a_m D^m + a_{m-1} D^{m-1} + \cdots + a_1 I$ is a linear mapping from VS into V.

$D(cf) = c\,D(f)$

  - $P_n = \left\{ \sum_{k=0}^{n} a_k t^k \mid a_k \in K \right\}$
  - $E_n = \left\{ \sum_{k=0}^{n} a_k e^{kt} \mid a_k \in K \right\}$
  - $T_n = \left\{ \sum_{k=0}^{n} [a_k \cos(kt) + b_k \sin(kt)] \mid a_k, b_k \in K \right\}$

$D^m f = \frac{d^m f(t)}{dt^m}$

$$(a_m D^m + a_{m-1} D^{m-1} + \cdots + a_0 I) f(t) = a_m \frac{d^m f(t)}{dt^m} + a_{m-1} \frac{d^{m-1} f(t)}{dt^{m-1}} + \cdots + a_0 f(t)$$

- Let A be an $m \times n$ matrix in a field K.

$$L_A : K^n \ni X \mapsto AX \in K^m$$

  is a linear map from K^n to K^m.

- $F : K^n \to K^r$

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_r).$$

- Operator: linear mapping $F : V \to V$ from a VS V to itself.

- $F^r = \underbrace{F \circ \cdots \circ F}_{r \text{ times}}.$

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Prove: D is a linear mapping from P_n to P_n