Lecture V Partitioned Matrices

Lei Du
dulei@dlut.edu.cn

School of Mathematical Sciences
Dalian University of Technology

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Introduction

Partitioned matrices, also called block matrices, play an important role in matrix and tensor calculus. They are usually employed for the computation of matrix products, especially for Khatri-Rao and Kronecker products. Partitioned matrices are also underlying the definition of certain structured matrices such as Hamiltonian, Hadamard, Fourier, Toeplitz, and Hankel matrices, and subsequently block-Toeplitz and block-Hankel matrices.

This chapter has several objectives:

- to define the notions of submatrices and partitioned matrices
- to describe examples of partitioned matrices for the computation of matrix products
- to present a few special cases such as block-diagonal matrices, Jordan forms, block-triangular matrices, block-Toeplitz and Hankel matrices
- to define block operations such as transposition, trace, addition and multiplication, as well as the determinants, and the ranks of certain partitioned matrices
- to introduce elementary operations and associated matrices, used for block triangularization, block-diagonalization, block-factorization, block- inversion, and generalized inversion of 2×2 block matrices
- to use inversion formulae of block matrices to deduce several fundamental results such as the matrix inversion lemma, the inversion of a partitioned Gram matrix, and recursive inversion with respect to the order of a square partitioned matrix
- matrix representations of a linear map and a bilinear/sesquilinear form, Quadratic forms and Hermitian forms
- eigenvalue and eigenvector, and generalized eigenvalue problem
- to provide an example of application of the recursive inversion formula of a 2×2 block matrix, for demonstrating the Levinson algorithm which is an algorithm widely used in signal processing for the estimation of the parameters of an autoregressive (AR) model,

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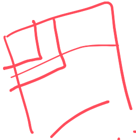
Submatrices

A submatrix $\mathbf{B}\left(m_i, n_j\right)$ of a matrix $\mathbf{A}(m, n)$, with $m_i \leq m$ and $n_j \leq n$, is a matrix whose elements are positioned at the intersections of the m_i rows and n_j columns of A defined by the sets of indices:

$$\alpha_{m_i} = \{i_k, k \in \langle m_i \rangle\} \subseteq \langle m \rangle, \beta_{n_j} = \{j_l, l \in \langle n_j \rangle\} \subseteq \langle n \rangle.$$

Thus, the element $a_{i_k j_l}$ of $\mathbf{A}$ is given by $a_{i_k j_l} = \left(\mathbf{e}_{i_k}^{(m)}\right)^T \mathbf{A}\mathbf{e}_{j_l}^{(n)}$. Subsequently, by defining the row and column selection matrices:



$$\mathbf{M} = \left[\mathbf{e}_{i_1}^{(m)}, \cdots, \mathbf{e}_{i_{m_i}}^{(m)}\right] \text{ and } \mathbf{N} = \left[\mathbf{e}_{j_1}^{(n)}, \cdots, \mathbf{e}_{j_{n_j}}^{(n)}\right],$$

we can write $\mathbf{B}\left(m_i, n_j\right)$ as:

A(1:3, 2:4)

$$\mathbf{B}\left(m_i, n_j\right) = \mathbf{M}^T\mathbf{A}\mathbf{N} = \begin{bmatrix} a_{i_1, j_1} & \cdots & a_{i_1, j_{n_j}} \\ \vdots & & \vdots \\ a_{i_{m_i}, j_1} & \cdots & a_{i_{m_i}, j_{n_j}} \end{bmatrix}.$$

In the case of a square matrix $\mathbf{A}$ of order n, a principal submatrix of order r is a submatrix $\mathbf{B}(r, r)$ whose elements are positioned at the intersections of the same set of r rows and r columns, that is, defined by the same set of indices $\alpha_r = \{i_k, k \in \langle r \rangle\} \subseteq \langle n \rangle$. A principal submatrix of order r contains r elements of the main diagonal of $\mathbf{A}$. There are $C_n = \frac{n!}{r!(n-r)!}$ principal submatrices of order r.

Partitioned matrices

Let $\{\alpha_{m_1}, \cdots, \alpha_{m_R}\}$ and $\{\beta_{n_1}, \cdots, \beta_{ns}\}$ be partitions of the sets $\{1, \cdots, m\}$ and $\{1, \cdots, n\}$, respectively, with $m_r \in \langle m \rangle$ and $n_s \in \langle n \rangle$, such that $\sum_{r=1}^{R} m_r = m$ and $\sum_{s=1}^{S} n_s = n$. It is said that matrices $\mathbf{A}_{rs}$ of dimensions $(m_{r+} n_s)$ form a partition of the matrix $\mathbf{A} \in \mathbb{K}^{m \times n}$ into (R, S) blocks, or that $\mathbf{A}$ is partitioned into (R, S) blocks, if $\mathbf{A}$ can be written as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1S} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{R1} & \mathbf{A}_{R2} & \cdots & \mathbf{A}_{RS} \end{bmatrix} = [\mathbf{A}_{rs}], r \in \langle R \rangle, s \in \langle S \rangle. \tag{1}$$

Such a partitioning with blocks of different dimensions is said to be unbalanced.

Partitioned matrices

The submatrix $\mathbf{A}_{rs}$ can be expressed as:

$$\mathbf{A}_{rs} = \begin{bmatrix} a_{m_1+\cdots+m_{r-1}+1, n_1+\cdots+n_{x-1}+1} & \cdots & a_{m_1+\cdots+m_{r-1}+1, n_1+\cdots+n_{x-1}+n_x} \\ \vdots & \ddots & \vdots \\ a_{m_1+\cdots+m_{r-1}+m_r, n_1+\cdots+n_{s-1}+1} & \cdots & a_{m_1+\cdots+m_{n-1}+m_{r,n_1+\cdots+n_{s-1}+n_s}} \end{bmatrix}$$
$$\in \mathbb{K}^{m_r \times n_s}.$$

All submatrices of the same row-block (r) contain the same number (m_r) of rows. Similarly, all submatrices of the same column-block (s) contain the same number (n_s) of columns, that is:

$$\begin{bmatrix} \mathbf{A}_{r1} & \mathbf{A}_{r2} & \cdots & \mathbf{A}_{rS} \end{bmatrix} \in \mathbb{K}^{m_r \times n}, \begin{bmatrix} \mathbf{A}_{1s} \\ \mathbf{A}_{2s} \\ \vdots \\ \mathbf{A}_{Rs} \end{bmatrix} \in \mathbb{K}^{m \times n_s}.$$

It is then said that the submatrices $\mathbf{A}_{rs}$ are of compatible dimensions.

Partitioned matrices

In the particular case where $n = 1$, the partitioned matrix (1) becomes a blockcolumn vector:

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_R \end{bmatrix} \in \mathbb{K}^{m\times 1}, \mathbf{a}_r \in \mathbb{K}^{m_r\times 1}, r \in \langle R\rangle.$$

Similarly, when $m = 1$, the partitioned matrix (1) becomes a block-row vector:

$$\mathbf{a}^T = \left[\mathbf{a}_1^T\, \mathbf{a}_2^T\, \cdots\, \mathbf{a}_S^T\right] \in \mathbf{K}^{1\times n}, \mathbf{a}_s \in \mathbb{K}^{n_s\times 1}, s \in \langle S\rangle.$$

If all the blocks $\mathbf{A}_{rs}$ have the same dimensions $P \times Q$, that is, when $m_r = P, \forall r \in \langle R\rangle$, and $n_s = Q, \forall s \in \langle S\rangle$, then the space of partitioned matrices into (R, S) blocks, with entries in the space $\mathbb{K}^{P\times Q}$ (also written $\mathcal{M}_{P\times Q}(\mathbb{K})$), will be denoted $\mathcal{M}_{R\times S}(\mathcal{M}_{P\times Q}(\mathbb{K}))$. The partitioning is then said to be balanced.

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Matrix products

Given two rectangular matrices $\mathbf{A} \in \mathbb{K}^{I \times J}$ and $\mathbf{B} \in \mathbb{K}^{J \times K}$, the product $\mathbf{C} = \mathbf{AB} \in \mathbb{K}^{I \times K}$ can be written in terms of matrices partitioned into column blocks or row blocks:

$$\mathbf{AB} = [\mathbf{AB}_{.1}, \mathbf{AB}_{.2}, \cdots, \mathbf{AB}_{.K}] = \begin{bmatrix} \mathbf{A}_{1}\mathbf{B} \\ \mathbf{A}_{2.}\mathbf{B} \\ \vdots \\ \mathbf{A}_{I}\mathbf{B} \end{bmatrix}.$$

Two matrix products play an important role in matrix calculation. These are the Kronecker and Khatri-Rao products.

Vector Kronecker product

Let $\mathbf{u} \in \mathbb{K}^I$ and $\mathbf{v} \in \mathbb{K}^J$. Their Kronecker product is defined as:

$$\mathbf{x} = \mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} u_1 \mathbf{v} \\ \vdots \\ u_I \mathbf{v} \end{bmatrix} \in \mathbb{K}^{IJ}$$
$$= \left[u_1 v_1, u_1 v_2, \cdots, u_1 v_J, u_2 v_1, \cdots, u_I v_J\right]^T.$$

This is a vector partitioned into I blocks of dimension J. The element $u_i v_j$ is positioned at position $j + (i-1)J$.

Matrix Kronecker product

Given $\mathbf{A} \in \mathbb{K}^{I \times J}$ and $\mathbf{B} \in \mathbb{K}^{M \times N}$, the Kronecker product to the right of $\mathbf{A}$ by $\mathbf{B}$ is the matrix $\mathbf{C} \in \mathbb{K}^{IM \times JN}$ defined as :

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1J}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2J}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{I1}\mathbf{B} & a_{I2}\mathbf{B} & \cdots & a_{IJ}\mathbf{B} \end{bmatrix} = [a_{ij}\mathbf{B}] \, .$$

This is a matrix partitioned into (I, J) blocks, the block (i, j) being the matrix $a_{ij}\mathbf{B} \in \mathbb{K}^{M \times N}$. The element $a_{ij}b_{mn}$ is positioned at position $((i-1)M + m, (j-1)N + n)$ in $\mathbf{A} \otimes \mathbf{B}$.

Matrix Kronecker product

The j th column-block of $\mathbf{A} \otimes \mathbf{B}$ is given by:

$$\mathbf{A}_{.j} \otimes \mathbf{B} = \begin{bmatrix} a_{1j}\mathbf{B} \\ \vdots \\ a_{Ij}\mathbf{B} \end{bmatrix} = \left[\mathbf{A}_{.j} \otimes \mathbf{B}_{.1} \mathbf{A}_{.j} \otimes \mathbf{B}_{.2} \cdots \mathbf{A}_{.j} \otimes \mathbf{B}_{.N}\right], j \in \langle J \rangle$$

Subsequently, the columns of $\mathbf{A} \otimes \mathbf{B}$ are composed of all the Kronecker products of a column of $\mathbf{A}$ with a column of $\mathbf{B}$, the columns being taken in lexicographical order. Similarly, $\mathbf{A} \otimes \mathbf{B}$ can be decomposed into I row-blocks $\mathbf{A}_{i.} \otimes \mathbf{B}$, with $i \in \langle I \rangle$, the IM rows being composed of all the Kronecker products of a row of $\mathbf{A}$ with a row of $\mathbf{B}$. Therefore, $\mathbf{A} \otimes \mathbf{B}$ can be broken into blocks such that:

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= [\mathbf{A}_{.1} \otimes \mathbf{B} \cdots \mathbf{A}_{.J} \otimes \mathbf{B}] \\ &= [\mathbf{A}_{.1} \otimes \mathbf{B}_{.1} \cdots \mathbf{A}_{.1} \otimes \mathbf{B}_{.N} \cdots \mathbf{A}_{.J} \otimes \mathbf{B}_{.1} \cdots \mathbf{A}_{.J} \otimes \mathbf{B}_{.N}] \\ &= \begin{bmatrix} \mathbf{A}_{1.} \otimes \mathbf{B} \\ \vdots \\ \mathbf{A}_{I.} \otimes \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{1.} \otimes \mathbf{B}_{1.} \\ \vdots \\ \mathbf{A}_{1.} \otimes \mathbf{B}_{M.} \\ \vdots \\ \mathbf{A}_{I.} \otimes \mathbf{B}_{1.} \\ \vdots \\ \mathbf{A}_{I.} \otimes \mathbf{B}_{M.} \end{bmatrix}. \end{aligned}$$

Matrix Kronecker product

The Kronecker product can be used to write the matrix $\mathbf{A}$ partitioned into (R, S) blocks, defined in (1), as follows:

$$\mathbf{A} = \sum_{r=1}^{R} \sum_{s=1}^{S} \mathbf{E}_{rs}^{(R \times S)} \otimes \mathbf{A}_{rs}$$

where $\mathbf{E}_{rs}^{(R \times S)}$, for $r \in \langle R \rangle$ and $s \in \langle S \rangle$, are the matrices of the canonical basis of the space $\mathbb{K}^{R \times S}$, that is, with the (r, s) th element equal to 1 and all others equal to zero.

例

For $R = 2$ and $S = 3$, we have:

$$\mathbf{A} = \sum_{r=1}^{2} \sum_{s=1}^{3} \mathbf{E}_{rs}^{(2 \times 3)} \otimes \mathbf{A}_{rs} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \end{bmatrix}.$$

Khatri-Rao product

Given $\mathbf{A} \in \mathbb{K}^{I \times J}$ and $\mathbf{B} \in \mathbb{K}^{K \times J}$ having the same number of columns, the Khatri-Rao product of $\mathbf{A}$ with $\mathbf{B}$, denoted by $\mathbf{A} \diamond \mathbf{B} \in \mathbb{K}^{IK \times J}$, is defined as:

$$\mathbf{A} \diamond \mathbf{B} = [\mathbf{A}_{.1} \otimes \mathbf{B}_{.1}, \mathbf{A}_{.2} \otimes \mathbf{B}_{.2}, \cdots, \mathbf{A}_{.J} \otimes \mathbf{B}_{.J}]$$

This is a matrix that is partitioned into J column-blocks, the j th block being equal to the Kronecker product of the j th column of $\mathbf{A}$ with the j th column of $\mathbf{B}$. It is said that $\mathbf{A} \diamond \mathbf{B}$ is a columnwise Kronecker product of $\mathbf{A}$ and $\mathbf{B}$.

Khatri-Rao product

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定理

The Khatri-Rao product can also be written as a matrix partitioned into I row-blocks:

$$\mathbf{A} \diamond \mathbf{B} = \begin{bmatrix} \mathbf{B}\mathbf{D}_1(\mathbf{A}) \\ \mathbf{B}\mathbf{D}_2(\mathbf{A}) \\ \vdots \\ \mathbf{B}\mathbf{D}_I(\mathbf{A}) \end{bmatrix}$$

where $\mathbf{D}_i(\mathbf{A}) = \operatorname{diag}(a_{i1}, a_{i2}, \cdots, a_{iJ})$ refers to the diagonal matrix whose diagonal elements are the elements of the ith row of $\mathbf{A}$.

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Block-diagonal matrices

A square matrix $\mathbf{A} \in \mathbb{K}^{n \times n}$ partitioned into (R, R) blocks, of diagonal blocks $\mathbf{A}_{rr} \in \mathbb{K}^{n_r \times n_r}$, with $r \in \langle R \rangle$ and $\sum_{r=1}^{R} n_r = n$, whose off-diagonal blocks are zero, is called a block-diagonal matrix and can be written as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0}_{n_1 \times n_2} & \cdots & \mathbf{0}_{n_1 \times n_R} \\ \mathbf{0}_{n_2 \times n_1} & \mathbf{A}_{22} & \cdots & \mathbf{0}_{n_2 \times n_R} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_R \times n_1} & \mathbf{0}_{n_R \times n_2} & \cdots & \boxed{\mathbf{A}_{RR}} \end{bmatrix}$$

blk diag

It is also written $\mathrm{diag}(\mathbf{A}_{11}, \mathbf{A}_{22}, \cdots, \mathbf{A}_{RR})$ or simply$\mathrm{diag}\,(\mathbf{A}_{rr})$ with the number R of diagonal blocks implied.

Signature matrices

The signature matrix of a symmetric matrix $\mathbf{A}$, of full rank, is a diagonal matrix whose diagonal elements are equal to 1 or -1:

$$\mathbf{S} = \mathrm{diag}(\underbrace{1, \cdots, 1}_{p \text{ terms}}, \underbrace{-1, \cdots, -1}_{q \text{ terms}}) \text{ with } p \geq 0, q \geq 0$$

p and q correspond to the numbers of positive and negative eigenvalues of $\mathbf{A}$, respectively. A signature matrix is thus a block-diagonal matrix consisting of two diagonal blocks $\mathbf{I}_p$ and $-\mathbf{I}_q$:

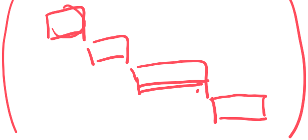
$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0}_{p \times q} \\ \mathbf{0}_{q \times p} & -\mathbf{I}_q \end{bmatrix}.$$

Direct sum

Given $\mathbf{A} \in \mathbb{K}^{I\times J}$ and $\mathbf{B} \in \mathbb{K}^{K\times L}$, the direct sum of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \oplus \mathbf{B}$, is the block-diagonal matrix $\begin{bmatrix} \boxed{\mathbf{A}} & \cancel{\mathbf{0}_{I\times L}} \\ \cancel{\mathbf{0}_{K\times J}} & \boxed{\mathbf{B}} \end{bmatrix} \in \mathbb{K}^{(I+K)\times(J+L)}$. In the case of P matrices $\mathbf{A}^{(p)} \in \mathbb{K}^{I_p\times J_p}$, we have:

$$\bigoplus_{p=1} \mathbf{A}_p = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_P \in \mathbb{K}^{\sum_{p=1}^{P} I_p \times \sum_{p=1}^{P} J_p}$$

$$= \operatorname{diag}(\mathbf{A}_1, \cdots, \mathbf{A}_P) = \operatorname{diag}(\mathbf{A}_p).$$



Jordan forms

A non-diagonalizable matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ can be transformed into a Jordan form:

$A^{100} \to P B^{100} P^{-1}$

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & & & \mathbf{0} \\ & \mathbf{B}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{B}_p \end{bmatrix}, \tag{2}$$

① 求特征值

$(\lambda_i \underset{n\times n}{I} - A)x = 0$

$$\mathbf{B}_i = \begin{bmatrix} \lambda_i & 1 & & \mathbf{0} \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda_i \end{bmatrix} \in \mathbb{K}^{n_i \times n_i},\ i \in \langle p \rangle,$$

where $\{\lambda_1, \cdots, \lambda_p\}$ are the eigenvalues of $\mathbf{A}$, and n_i is the multiplicity order of λ_i. Block-diagonal decomposition (2) into Jordan blocks $\mathbf{B}_i$, is called the Jordan form of A. This decomposition is little used in practice because its numerical determination may be unstable.

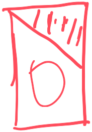
Block-triangular matrices

When $\mathbf{A}_{rs} = \mathbf{0}_{m_r, n_s}$ for $s < r$ in (1), that is:

$$\mathbf{A} = \begin{bmatrix} \boxed{\mathbf{A}_{11}} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1S} \\ \mathbf{0} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2S} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{RS} \end{bmatrix},$$

A_{SS}

0 0 - - - 0



$\mathbf{A}$ is said to be an upper block-triangular matrix.

lower

Block-triangular matrices

When $\mathbf{A}_{rs} = \mathbf{0}_{m_r,n_s}$ for $s < r$ in (1), that is:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1S} \\ \mathbf{0} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{RS} \end{bmatrix},$$

$\mathbf{A}$ is said to be an upper block-triangular matrix. Similarly, when $\mathbf{A}_{rs} = \mathbf{0}_{m_r,n_s}$ for $s > r$, that is:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{R1} & \mathbf{A}_{R2} & \cdots & \mathbf{A}_{RS} \end{bmatrix},$$

then $\mathbf{A}$ is called a lower block-triangular matrix.

Block Toeplitz and Hankel matrices

An $I \times J$ block-Toeplitz matrix is an $IM \times JN$ matrix partitioned in the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{F}_0 & \mathbf{F}_{-1} & \mathbf{F}_{-2} & \cdots & \mathbf{F}_{1-J} \\ \mathbf{F}_1 & \mathbf{F}_0 & \mathbf{F}_{-1} & \cdots & \mathbf{F}_{2-J} \\ \mathbf{F}_2 & \mathbf{F}_1 & \mathbf{F}_0 & \cdots & \mathbf{F}_{3-J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{I-1} & \mathbf{F}_{I-2} & \mathbf{F}_{I-3} & \cdots & \mathbf{F}_0 \end{bmatrix}$$

where $\mathbf{F}_t \in \mathbb{C}^{M \times N}$, with $1 - J \leq t \leq I - 1$. When $M = N = 1$, we have a standard $I \times J$ Toeplitz matrix.

Block Toeplitz and Hankel matrices

An $I \times J$ block-Toeplitz matrix is an $IM \times JN$ matrix partitioned in the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{F}_0 & \mathbf{F}_{-1} & \mathbf{F}_{-2} & \cdots & \mathbf{F}_{1-J} \\ \mathbf{F}_1 & \mathbf{F}_0 & \mathbf{F}_{-1} & \cdots & \mathbf{F}_{2-J} \\ \mathbf{F}_2 & \mathbf{F}_1 & \mathbf{F}_0 & \cdots & \mathbf{F}_{3-J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{I-1} & \mathbf{F}_{I-2} & \mathbf{F}_{I-3} & \cdots & \mathbf{F}_0 \end{bmatrix}$$

where $\mathbf{F}_t \in \mathbb{C}^{M \times N}$, with $1 - J \leq t \leq I - 1$. When $M = N = 1$, we have a standard $I \times J$ Toeplitz matrix. An $IJ \times IJ$ block-Hankel matrix is of the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{F}_0 & \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_I \\ \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 & \cdots & \mathbf{F}_{I+1} \\ \mathbf{F}_2 & \mathbf{F}_3 & \mathbf{F}_4 & \cdots & \mathbf{F}_{I+2} \\ \vdots & \vdots & \vdots & . & \vdots \\ \mathbf{F}_I & \mathbf{F}_{I+1} & \mathbf{F}_{I+2} & \cdots & \mathbf{F}_{2I} \end{bmatrix}$$

where $\mathbf{F}_t$ is a $J \times J$ matrix for $t = 0, 1, \cdots, 2I$. As a Hankel matrix $\mathbf{A} = [a_{ij}] = [a_{i+j}]$, with $0 \leq i, j \leq I$, is determined by its first column and last row, a blockHankel matrix is such that $\mathbf{A} = [\mathbf{A}_{ij}] = [\mathbf{F}_{i+j}]$ with $0 \leq i, j \leq I$. When each block $\mathbf{F}_t$ is a Hankel matrix, then $\mathbf{A}$ is a block-Hankel matrix with Hankel blocks.

Transposition and conjugate transposition

The transposition (or conjugate transposition) of a matrix $\mathbf{A}$ partitioned into (R, S) blocks $\mathbf{A}_{rs}, r \in \langle R \rangle, s \in \langle S \rangle$, is obtained by transposing (or transconjugating) the blocks, followed by a blockwise transposition.

定理

For a matrix partitioned into $(2,2)$ blocks with square blocks of same dimensions:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

$M^T = M$

$M^H = \begin{bmatrix} A^H & C^H \\ B^H & D^H \end{bmatrix}$

we have:

$$\mathbf{M} \text{ symmetric } \Leftrightarrow \mathbf{A}^T = \mathbf{A}, \mathbf{C} = \mathbf{B}^T, \mathbf{D}^T = \mathbf{D},$$

$$\mathbf{M} \text{ Hermitian } \Leftrightarrow \mathbf{A}^H = \mathbf{A}, \mathbf{C} = \mathbf{B}^H, \mathbf{D}^H = \mathbf{D},$$

that is, the diagonal blocks must be symmetric/Hermitian and the off-diagonal blocks transposed/conjugate transposed with respect to one another.

Trace

The trace of a partitioned matrix $\mathbf{A} = [\mathbf{A}_{rs}]$, with $r, s \in \langle R \rangle$, of dimensions (n, n) :

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1R} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2R} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{R1} & \mathbf{A}_{R2} & \cdots & \mathbf{A}_{RR} \end{bmatrix},$$

with $\dim(\mathbf{A}_{rr}) = (n_r, n_r)$ and $\sum_{r=1}^{R} n_r = n$, is given by:
$\mathrm{tr}(\mathbf{A}) = \sum_{r=1}^{R} \mathrm{tr}(\mathbf{A}_{rr})$.

Vectorization

Let us consider a balanced partitioning of $\mathbf{A}$ into (R, S) blocks of dimensions $P \times Q$. The partitioned matrix $\mathbf{A}$ can be vectorized column-blockwise (or rowblockwise), that is, by vectorizing each column (or row) of blocks, and then stacking the resulting vectors. The corresponding vectorization operators are denoted as $\text{vec}_c(.)$ and $\text{vec}_r(.)$, respectively.

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例

For $R = S = 2$, we have:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \Rightarrow \text{vec}_c(\mathbf{A}) = \begin{bmatrix} \text{vec}\,(\mathbf{A}_{11}) \\ \text{vec}\,(\mathbf{A}_{21}) \\ \text{vec}\,(\mathbf{A}_{12}) \\ \text{vec}\,(\mathbf{A}_{22}) \end{bmatrix}, \text{vec}_r(\mathbf{A}) = \begin{bmatrix} \text{vec}\,(\mathbf{A}_{11}) \\ \text{vec}\,(\mathbf{A}_{12}) \\ \text{vec}\,(\mathbf{A}_{21}) \\ \text{vec}\,(\mathbf{A}_{22}) \end{bmatrix}.$$

Blockwise addition

Let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{I \times J}$ be two matrices partitioned into blocks having the same dimensions $\mathbf{A}_{rs}, \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $\sum_{r=1}^{R} I_r = I, \sum_{s=1}^{S} J_s = J$. Their sum is a partitioned matrix $\mathbf{C} = \mathbf{A} + \mathbf{B} = [\mathbf{A}_{rs} + \mathbf{B}_{rs}]$, with $\mathbf{C}_{rs} = \mathbf{A}_{rs} + \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$.

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$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix},$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} \\ \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} \end{bmatrix}$$

Blockwise multiplication

Let $\mathbf{A} \in \mathbb{K}^{I \times J}$ and $\mathbf{B} \in \mathbb{K}^{J \times L}$ be two matrices partitioned into blocks $\mathbf{A}_{rs} \in \mathbb{K}^{I_r \times J_s}$ and $\mathbf{B}_{sn} \in \mathbb{K}^{J_s \times L_n}$, with $\sum_{r=1}^{R} I_r = I, \sum_{s=1}^{S} J_s = J$, and $\sum_{n=1}^{N} L_n = L$. The product $\mathbf{C} = \mathbf{AB}$ is a matrix that is partitioned into blocks $\mathbf{C}_{rn} = \sum_{s=1}^{S} \mathbf{A}_{rs}\mathbf{B}_{sn} \in \mathbb{K}^{I_r \times L_n}$.

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例

For $R = S = N = 2$, we have:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

$$\mathbf{A}\mathbf{B} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}.$$

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例

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$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}.$$

例

In the case of the product of two partitioned matrices built by adding a row and a column, respectively, we have:

$$\begin{bmatrix} \mathbf{X}^T \\ \mathbf{x}^T \end{bmatrix} [\mathbf{Y} \ \mathbf{y}] = \begin{bmatrix} \mathbf{X}^T\mathbf{Y} & \mathbf{X}^T\mathbf{y} \\ \mathbf{x}^T\mathbf{Y} & \mathbf{x}^T\mathbf{y} \end{bmatrix}.$$

$Yx^T + yx^T =$

Hadamard product of partitioned matrices

It should be remembered first that the Hadamard product of two matrices $\mathbf{A} \in \mathbb{K}^{I \times J}$ and $\mathbf{B} \in \mathbb{K}^{I \times J}$, of the same dimensions, gives a matrix $\mathbf{C} \in \mathbb{K}^{I \times J}$ defined as:

$$\mathbf{C} = \mathbf{A} \odot \mathbf{B} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1J}b_{1J} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2J}b_{2J} \\ \vdots & \vdots & \vdots & \\ a_{I1}b_{I1} & a_{I2}b_{I2} & \cdots & a_{IJ}b_{IJ} \end{bmatrix}$$

that is, $c_{ij} = a_{ij}b_{ij}$, and thus, $\mathbf{C} = [a_{ij}b_{ij}]$.

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Given $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{I \times J}$, partitioned into (R, S) blocks $\mathbf{A}_{rs}, \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $I = \sum_{r=1}^{R} I_r$ and $J = \sum_{s=1}^{S} J_s$, then their Hadamard product $\mathbf{A} \odot \mathbf{B}$ is a partitioned matrix into (R, S) blocks $\mathbf{C}_{rs} = \mathbf{A}_{rs} \odot \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $r \in \langle R \rangle, s \in \langle S \rangle$.

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Given $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{I \times J}$, partitioned into (R, S) blocks $\mathbf{A}_{rs}, \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $I = \sum_{r=1}^{R} I_r$ and $J = \sum_{s=1}^{S} J_s$, then their Hadamard product $\mathbf{A} \odot \mathbf{B}$ is a partitioned matrix into (R, S) blocks $\mathbf{C}_{rs} = \mathbf{A}_{rs} \odot \mathbf{B}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $r \in \langle R \rangle, s \in \langle S \rangle$.

If all the blocks have the same dimensions $P \times Q$, that is, $I_r = P, \forall r \in \langle R \rangle$, and $J_s = Q, \forall s \in \langle S \rangle$, then $\mathbf{A}$ and $\mathbf{B}$, and consequently $\mathbf{A} \odot \mathbf{B}$ belong to the space denoted by $\mathcal{M}_{R \times S}(\mathcal{M}_{P \times Q}(\mathbb{K}))$.

Kronecker product of partitioned matrices

Given a matrix $\mathbf{A} \in \mathbb{K}^{I \times J}$ partitioned into (R, S) blocks $\mathbf{A}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $\sum_{r=1}^{R} I_r = I$ and $\sum_{s=1}^{S} J_s = J$, and a matrix $\mathbf{B} \in \mathbb{K}^{M \times N}$, then their Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is a matrix partitioned into (R, S) blocks $\mathbf{A}_{rs} \otimes \mathbf{B} \in \mathbb{K}^{I_r M \times J_s N}$.

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例

For $R = 2, S = 3$, we have:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \end{bmatrix} \Rightarrow \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{A}_{11} \otimes \mathbf{B} & \mathbf{A}_{12} \otimes \mathbf{B} & \mathbf{A}_{13} \otimes \mathbf{B} \\ \mathbf{A}_{21} \otimes \mathbf{B} & \mathbf{A}_{22} \otimes \mathbf{B} & \mathbf{A}_{23} \otimes \mathbf{B} \end{bmatrix}.$$

Kronecker product of partitioned matrices

More generally, in the case of two matrices partitioned into blocks $\mathbf{A}_{rs} \in \mathbb{K}^{I_r \times J_s}$, with $(r \in \langle R \rangle, s \in \langle S \rangle)$, and $\mathbf{B}_{mn} \in \mathbb{K}^{K_m \times L_n}$, with $(m \in \langle M \rangle, n \in \langle N \rangle)$, the block Kronecker product, called the Tracy-Singh product, is defined as:

$A_{12} \otimes (B_{mn})$

$$
\mathbf{A} \otimes_b \mathbf{B} = \begin{bmatrix}
\mathbf{A}_{11} \otimes \mathbf{B}_{11} & \cdots & \mathbf{A}_{11} \otimes \mathbf{B}_{1N} & \cdots & \mathbf{A}_{1S} \otimes \mathbf{B}_{11} & \cdots & \mathbf{A}_{1S} \otimes \mathbf{B}_{1N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{A}_{11} \otimes \mathbf{B}_{M1} & \cdots & \mathbf{A}_{11} \otimes \mathbf{B}_{MN} & \cdots & \mathbf{A}_{1S} \otimes \mathbf{B}_{M1} & \cdots & \mathbf{A}_{1S} \otimes \mathbf{B}_{MN} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{A}_{R1} \otimes \mathbf{B}_{11} & \cdots & \mathbf{A}_{R1} \otimes \mathbf{B}_{1N} & \cdots & \mathbf{A}_{RS} \otimes \mathbf{B}_{11} & \cdots & \mathbf{A}_{RS} \otimes \mathbf{B}_{1N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{A}_{R1} \otimes \mathbf{B}_{M1} & \cdots & \mathbf{A}_{R1} \otimes \mathbf{B}_{MN} & \cdots & \mathbf{A}_{RS} \otimes \mathbf{B}_{M1} & \cdots & \mathbf{A}_{RS} \otimes \mathbf{B}_{MN}
\end{bmatrix}
$$

of dimensions $\left(\sum_{r=1}^{R}\sum_{m=1}^{M} I_r K_m, \sum_{s=1}^{S}\sum_{n=1}^{N} J_s L_n\right)$. Note that if $R = S = M = N = 1$, this block Kronecker product becomes the classical Kronecker product, with $\mathbf{A} = \mathbf{A}_{11}$ and $\mathbf{B} = \mathbf{B}_{11}$.

Kronecker product of partitioned matrices

Another Kronecker product of partitioned matrices, called the strong Kronecker product and denoted by $|\otimes|$, was introduced for generating orthogonal matrices from Hadamard matrices. This Kronecker product is also used to represent tensor train decompositions. Given two matrices partitioned into blocks $\mathbf{A}_{rs} \in \mathbb{K}^{I \times J}$ and $\mathbf{B}_{sn} \in \mathbb{K}^{K \times L}$, with $r \in \langle R \rangle, s \in \langle S \rangle$, and $n \in \langle N \rangle$, the strong Kronecker product $\mathbf{A}| \otimes |\mathbf{B}$ is defined as the matrix partitioned into (R, N) blocks $\mathbf{C}_{rn} \in \mathbb{K}^{IK \times JL}$, with $r \in \langle R \rangle$ and $n \in \langle N \rangle$, such as:

$$\mathbf{C}_{rn} = \sum_{s=1}^{S} \mathbf{A}_{rs} \otimes \mathbf{B}_{sn}.$$

This operation, which is completely determined by the parameters (R, S, N), preserves the orthogonality.

- The matrix $\mathbf{1}_{RP\times SQ}$, whose all entries are equal to 1, is the identity element for $\odot$ in the space $\mathcal{M}_{R\times S}(\mathcal{M}_{P\times Q}(\mathbb{K}))$:

$$\mathbf{A}\odot\mathbf{1}_{RP\times SQ}=\mathbf{1}_{RP\times SQ}\odot\mathbf{A}=\mathbf{A},\forall\mathbf{A}\in\mathcal{M}_{R\times S}(\mathcal{M}_{P\times Q}(\mathbb{K}))$$

  Note that, for the Kronecker product, there is no identity element $\mathbf{E}$ such that $\mathbf{A}\otimes\mathbf{E}=\mathbf{E}\otimes\mathbf{A}=\mathbf{A},\forall\mathbf{A}$.

- Commutativity of $\odot$:

$$\mathbf{A}\odot\mathbf{B}=\mathbf{B}\odot\mathbf{A}$$

- Associativity of $\odot$ and $\otimes$:

$$\mathbf{A}\odot(\mathbf{B}\odot\mathbf{C})=(\mathbf{A}\odot\mathbf{B})\odot\mathbf{C}$$
$$\mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C})=(\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C}$$

- Distributivity of $\odot$ and $\otimes$ over the addition:

$$(\mathbf{A}+\mathbf{B})\odot\mathbf{C} = (\mathbf{A}\odot\mathbf{C}) + (\mathbf{B}\odot\mathbf{C})$$
$$\mathbf{A}\odot(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\odot\mathbf{B}) + (\mathbf{A}\odot\mathbf{C})$$
$$(\mathbf{A}+\mathbf{B})\otimes\mathbf{C} = (\mathbf{A}\otimes\mathbf{C}) + (\mathbf{B}\otimes\mathbf{C})$$
$$\mathbf{A}\otimes(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\otimes\mathbf{B}) + (\mathbf{A}\otimes\mathbf{C})$$

- Distributivity of $\odot$ and $\otimes$ over the scalar multiplication. For any $\lambda \in \mathbb{K}$:

$$\lambda(\mathbf{A}\odot\mathbf{B}) = (\lambda\mathbf{A})\odot\mathbf{B} = \mathbf{A}\odot(\lambda\mathbf{B})$$
$$\lambda(\mathbf{A}\otimes\mathbf{B}) = (\lambda\mathbf{A})\otimes\mathbf{B} = \mathbf{A}\otimes(\lambda\mathbf{B})$$

Elementary operations and elementary matrices

The elementary operations consist of:

- interchanging the ith and jth rows (columns):

$$\mathbf{A}_{i.} \leftrightarrow \mathbf{A}_{j.} \; (\mathbf{A}_{.i} \leftrightarrow \mathbf{A}_{.j})$$

- multiplying the elements of the ith row (column) by a scalar $k \neq 0$:

$$k\mathbf{A}_{i.} \rightarrow \mathbf{A}_{i.} \; (k\mathbf{A}_{.i} \rightarrow \mathbf{A}_{.i})$$

- adding to the elements of the ith row (column), the corresponding elements of the jth row (column) multiplied by k :

$$\mathbf{A}_{i.} + k\mathbf{A}_{j.} \rightarrow \mathbf{A}_{i.} \; (\mathbf{A}_{.i} + k\mathbf{A}_{.j} \rightarrow \mathbf{A}_{.i})$$

The three corresponding transformations, respectively, denoted by $\mathbf{P}_{ij}, \mathbf{P}_i(k)$, and $\mathbf{P}_{ij}(k)$ can be represented using the so-called elementary matrices.

Designating by $(p_1, \cdots, p_n)$ a permutation of $(1, \cdots, n)$, these elementary matrices are such that:

$$
\begin{aligned}
\mathbf{P}_{ij} &= [\mathbf{e}_{p_1} \cdots \mathbf{e}_{p_n}] \ \text{ with } \left[\begin{array}{c} \mathbf{e}_{p_i} = \mathbf{e}_j \text{ and } \mathbf{e}_{p_j} = \mathbf{e}_i \\ \mathbf{e}_{p_k} = \mathbf{e}_k \text{ if } k \neq i \text{ and } j \end{array} \right] \\
&= \mathbf{I} - (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \\
\mathbf{P}_i(k) &= [\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}, k\mathbf{e}_i, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_n] \\
\mathbf{P}_{ij}(k) &= \left[\mathbf{e}_1, \cdots, \mathbf{e}_{j-1}, \mathbf{e}'_j, \mathbf{e}_{j+1}, \cdots, \mathbf{e}_n\right] \qquad \text{for rows} \\
&= \mathbf{I} + k\mathbf{e}_i\mathbf{e}_j^T \\
\mathbf{P}_{ij}(k) &= \left[\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}, \mathbf{e}'_i, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_n\right] \qquad \text{for columns} \\
&= \mathbf{I} + k\mathbf{e}_j\mathbf{e}_i^T
\end{aligned}
$$

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

where $\mathbf{e}'_j\,(\mathbf{e}'_i)$ is a vector consisting of 0s except its ith (jth) component equal to k and its jth (ith) component equal to 1.

例

Consider $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. For $\mathbf{P}_{12} = \mathbf{I}_3 - (\mathbf{e}_2 - \mathbf{e}_1)(\mathbf{e}_2 - \mathbf{e}_1)^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

we have:

$$\mathbf{P}_{12}\mathbf{A} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}\mathbf{P}_{12} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}.$$

and

$$\mathbf{P}_{12}(k) = \mathbf{I}_3 + k\mathbf{e}_1\mathbf{e}_2^T = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{P}_{12}(k)\mathbf{A} = \begin{bmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{P}_{12}(k) = \mathbf{I}_3 + k\mathbf{e}_2\mathbf{e}_1^T = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{A}\mathbf{P}_{12}(k) = \begin{bmatrix} a_{11} + ka_{12} & a_{12} & a_{13} \\ a_{21} + ka_{22} & a_{22} & a_{23} \\ a_{31} + ka_{32} & a_{32} & a_{33} \end{bmatrix}.$$

Elementary operations and elementary matrices

Similarly, we can define elementary operations involving row-blocks or column-blocks of a partitioned matrix, to:

- interchange the i th and j th row-blocks (column-blocks);
- multiply the ith row-block (column-block) on the left-hand side (right-hand side) by a non-singular matrix;
- add the ith row-block (column-block) to the jth row-block (column-block) multiplied on the left-hand side (right-hand side) by a non-singular matrix.

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- add the ith row-block (column-block) to the jth row-block (column-block) multiplied on the left-hand side (right-hand side) by a non-singular matrix.

For example, consider a matrix partitioned into $(2,2)$ blocks, with square diagonal blocks:

$$\mathbf{M} = \begin{bmatrix} \overset{n\times n}{\mathbf{A}} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}_{m\times n} \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$$

$\mathbf{A}, \mathbf{E} \in \mathbb{C}^{n\times n}, \mathbf{D}, \mathbf{F} \in \mathbb{C}^{m\times m}, \mathbf{C}, \mathbf{G} \in \mathbb{C}^{m\times n}, \mathbf{B}, \mathbf{H} \in \mathbb{C}^{n\times m}$, with non-singular $\mathbf{E}$ and $\mathbf{F}$.

Elementary operations and elementary matrices

For the first type of elementary operation, we have:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{A} & \mathbf{B} \end{bmatrix}, \quad \mathbf{M} \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{D} & \mathbf{C} \end{bmatrix}.$$

Elementary operations and elementary matrices

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For the second type of elementary operation:

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathrm{M} = \begin{bmatrix} \mathbf{EA} & \mathbf{EB} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \mathrm{M} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{AEB} & B \\ \mathbf{CE} & D \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \mathrm{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{FC} & \mathbf{FD} \end{bmatrix}, \mathrm{M} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{BF} \\ \mathbf{C} & \mathbf{DF} \end{bmatrix}.$$

For the third type of elementary operation:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{H} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{A} + \mathbf{HC} & \mathbf{B} + \mathbf{HD} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{G} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{BG} & \mathbf{B} \\ \mathbf{C} + \mathbf{DG} & \mathbf{D} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{G} & \mathbf{I}_m \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} + \mathbf{GA} & \mathbf{D} + \mathbf{GB} \end{bmatrix}$$

$$\mathbf{M} \begin{bmatrix} \mathbf{I}_n & \mathbf{H} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} + \mathbf{AH} \\ \mathbf{C} & \mathbf{D} + \mathbf{CH} \end{bmatrix}.$$

Inversion of partitioned matrices

This section is devoted to the inversion of 2×2 block matrices. We must note that for a 2×2 block matrix $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, whose blocks $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ have the dimensions $m \times n, m \times p, q \times n$ and $q \times p$, respectively, with $m + q = n + p$, its inverse $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$ is such that the blocks $\mathbf{E}, \mathbf{F}, \mathbf{G}$, and $\mathbf{H}$ must be of dimensions $n \times m$ $n \times q, p \times m$ and $p \times q$, respectively, in order to satisfy $\mathbf{M}\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m \times q} \\ \mathbf{0}_{q \times m} & \mathbf{I}_q \end{bmatrix}$ and $\mathbf{M}^{-1}\mathbf{M} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times p} \\ \mathbf{0}_{p \times n} & \mathbf{I}_p \end{bmatrix}$. So, we can conclude that the blocks of $\mathbf{M}^{-1}$ have the same dimensions as those of $\mathbf{M}^T$, and consequently, the partition of $\mathbf{M}^{-1}$ is transposed of that of $\mathbf{M}$.

Inversion of block-diagonal matrices

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} I & -A^{-1}B \\ & I \end{pmatrix} \begin{pmatrix} A^{-1} & \\ & D^{-1} \end{pmatrix}$$

$$M = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix}$$

$$= \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

Assuming that $\mathbf{A}$ and $\mathbf{D}$ are non-singular, we have:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \quad \Rightarrow \quad \mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix}.$$

For block upper and lower triangular matrices, with non-singular square diagonal blocks, we have:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{D}^{-1} \end{bmatrix}. \tag{4}$$

When $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{D} = \mathbf{I}_m$, that is for unit block-triangular matrices, formulae (3) and (4) become:

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{B} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_m & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_n \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ -\mathbf{C} & \mathbf{I}_n \end{bmatrix}.$$

Inversion of block-triangular matrices

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{-1} = \left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} C & 0 \\ A & B \end{pmatrix}\right]^{-1} = \begin{pmatrix} C & 0 \\ A & B \end{pmatrix}^{-1}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1}$$

Similarly, with non-singular square off-diagonal blocks, we have:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \mathbf{0} & \mathbf{C}^{-1} \\ \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{A}\mathbf{C}^{-1} \end{bmatrix}},$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{C}^{-1}\mathbf{D}\mathbf{B}^{-1} & \mathbf{C}^{-1} \\ \mathbf{B}^{-1} & \mathbf{0} \end{bmatrix},$$

with the following particular cases:

$$\begin{bmatrix} \mathbf{A} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ \mathbf{I}_m & -\mathbf{A} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ \mathbf{I}_n & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{D} & \mathbf{I}_n \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix}.$$

When the partitioned matrix has no special structure, its inverse and its determinant are determined from block-triangular factorization, this factorization being itself obtained from block-diagonalization.

Block-triangularization and Schur complements

Assuming that $\mathbf{D}$ is non-singular, and applying the elementary transformation, the partitioned matrix $\mathbf{M}$, can be transformed into a lower block-triangular form. Indeed, by choosing $\mathbf{H} = -\mathbf{B}\mathbf{D}^{-1}$, we obtain:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{I}_n & -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{X}_{\mathbf{D}} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix},$$

$$\mathbf{X}_{\mathbf{D}} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C},$$

where $\mathbf{X}_{\mathbf{D}}$, also denoted by $(\mathbf{M}/\mathbf{D})$, is called the Schur complement of $\mathbf{D}$ in $\mathbf{M}$. Similarly, assuming that $\mathbf{A}$ is non-singular and choosing $\mathbf{H} = -\mathbf{A}^{-1}\mathbf{B}$, we have:

$$\mathbf{M} \begin{bmatrix} \mathbf{I}_n - \mathbf{A}^{-1}\mathbf{B} & \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{X}_{\mathbf{A}} \end{bmatrix},$$

$$\mathbf{X}_{\mathbf{A}} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B},$$

where $\mathbf{X}_{\mathbf{A}}$, also denoted by $(\mathbf{M}/\mathbf{A})$, is the Schur complement of $\mathbf{A}$ in $\mathbf{M}$. Similarly, elementary transformations as follows can be used to transform the partitioned matrix $\mathbf{M}$ into an upper block triangular form:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{X}_{\mathbf{A}} \end{bmatrix}, \quad \begin{bmatrix} I & -BA^{-1} \\ & I \end{bmatrix} = \begin{pmatrix} A & 0 \\ 0 & X_A \end{pmatrix}$$

$$\mathbf{M} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{X}_{\mathbf{D}} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}.$$

Block-diagonalization and block-factorization

Assuming that $\mathbf{A}$ is invertible, it is possible to put $\mathbf{M}$ in a block-diagonal form:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_\mathbf{A} \end{bmatrix}. \tag{5}$$

-1 $\quad -1$ $\quad \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$

Using the inversion formulae, the partitioned matrix $\mathbf{M}$ can be written in the following block-factorized form:

$\cancel{\begin{bmatrix} I & A^{-1}B \\ & I \end{bmatrix}} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cancel{\begin{bmatrix} I & \\ CA^{-1} & I \end{bmatrix}} = \begin{bmatrix} A^{-1} & \\ & X_A^{-1} \end{bmatrix} \quad \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}. \tag{6}$$

Similarly, assuming that $\mathbf{D}$ is invertible, a second block-diagonal form is obtained:

$$\begin{bmatrix} \mathbf{I}_n & -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{X}_\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \tag{7}$$

from which the following block-factorized form is deduced:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{C} & \mathbf{I}_m \end{bmatrix}. \tag{8}$$

Block-inversion and partitioned inverse

The block-factorized form (6) gives the so-called Banachiewicz-Schur form, with $\mathbf{X_A} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$:

$$\begin{bmatrix} \overset{n\times n}{\mathbf{A}} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_n & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X_A}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{X_A}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{X_A}^{-1} \\ -\mathbf{X_A}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{X_A}^{-1} \end{bmatrix}. \qquad (9)$$

This inversion formula is valid if $\mathbf{A}$ and $\mathbf{X_A}$ are invertible.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\overset{n\times n}{A} x = 0$

$\overset{m\times m}{X_A} y = 0$

Block-inversion and partitioned inverse

The block-factorized form (6) gives the so-called Banachiewicz-Schur form, with $\mathbf{X_A} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$:

$(A+UV^T)^{-1}$ $U, V \in \mathbb{R}^{n \times k}$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_n & -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X_A}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{X_A}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{X_A}^{-1} \\ -\mathbf{X_A}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{X_A}^{-1} \end{bmatrix}. \quad (9)$$

$(A+UV^T)^{-1}$

$(I+UV^T)^{-1}$

This inversion formula is valid if $\mathbf{A}$ and $\mathbf{X_A}$ are invertible. Similarly, the factorized form (8) leads to:

Sherman-Morrison-Woodbury

$$\begin{bmatrix} \mathbf{A} & \overset{U}{\mathbf{B}} \\ \underset{V^T}{\mathbf{C}} & \underset{I}{\mathbf{D}} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{X_D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{X_D}^{-1} & -\mathbf{X_D}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{X_D}^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{X_D}^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}. \quad (10)$$

This inversion formula is valid if $\mathbf{D}$ and $\mathbf{X_D} = \boxed{\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}}$ are invertible.

Other formulae for the partitioned 2×2 inverse

From (9) and (10), the following other forms of $\mathbf{M}^{-1}$ can be deduced:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{A}^{-1}\mathbf{B} \\ \mathbf{I}_m \end{bmatrix} \mathbf{X}_{\mathbf{A}}^{-1} \left[-\mathbf{C}\mathbf{A}^{-1} \mathbf{I}_m \right]$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix} + \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{D}^{-1}\mathbf{C} \end{bmatrix} \mathbf{X}_{\mathbf{D}}^{-1} \left[\mathbf{I}_n - \mathbf{B}\mathbf{D}^{-1} \right].$$

Combining formulae (9) and (10), we can also rewrite $\mathbf{M}^{-1}$ as:

$$\begin{aligned} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} &= \begin{bmatrix} \mathbf{X}_{\mathbf{D}}^{-1} & -\mathbf{X}_{\mathbf{D}}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{X}_{\mathbf{A}}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{X}_{\mathbf{A}}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{X}_{\mathbf{D}}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_{\mathbf{A}}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{X}_{\mathbf{D}}^{-1} & \mathbf{X}_{\mathbf{A}}^{-1} \end{bmatrix}, \end{aligned} \tag{11}$$

which gives the following block-factorizations:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{X}_{\mathbf{D}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{\mathbf{A}}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & -\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_m \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{I}_n & -\mathbf{A}^{-1}\mathbf{B} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathbf{D}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{\mathbf{A}}^{-1} \end{bmatrix}.$$

Other formulae for the partitioned 2×2 inverse

By taking the first row-block of (9) and the second row-block of (10) , we get:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{X}_{\mathbf{A}}^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_{\mathbf{A}}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{X}_{\mathbf{D}}^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{X}_{\mathbf{D}}^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}.$$

Consider the following system of linear equations:

$$\left\{\begin{array}{ll} \mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{x}_2 & = \mathbf{y}_1 \\ \mathbf{C}\mathbf{x}_1 + \mathbf{D}\mathbf{x}_2 & = \mathbf{y}_2 \end{array}\right.,$$

or equivalently,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}.$$

Applying the inversion formula (11) gives us the following solution:

$$\mathbf{x}_1 = \mathbf{X}_{\mathbf{D}}^{-1}\left(\mathbf{y}_1 - \mathbf{B}\mathbf{D}^{-1}\mathbf{y}_2\right),$$
$$\mathbf{x}_2 = \mathbf{X}_{\mathbf{A}}^{-1}\left(\mathbf{y}_2 - \mathbf{C}\mathbf{A}^{-1}\mathbf{y}_1\right),$$

where $\mathbf{X}_{\mathbf{A}}$ and $\mathbf{X}_{\mathbf{D}}$ are the Schur complements.

Inversion of a partitioned Gram matrix

Given a matrix partitioned into two column blocks $\mathbf{A} = [\mathbf{A}_1 \; \mathbf{A}_2]$, with $\mathbf{A}_1 \in \mathbb{R}^{n\times m}$ and $\mathbf{A}_2 \in \mathbb{R}^{n\times p}$, its Gram matrix $\mathbf{A}^T\mathbf{A}$ is partitioned as :

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{A}_1^T\mathbf{A}_1 & \mathbf{A}_1^T\mathbf{A}_2 \\ \mathbf{A}_2^T\mathbf{A}_1 & \mathbf{A}_2^T\mathbf{A}_2 \end{bmatrix}$$

Inversion of this type of matrix is employed for solving linear prediction problems or for estimating linear regression models. The application of inversion formula (11), with $\mathbf{A} = \mathbf{A}_1^T\mathbf{A}_1, \mathbf{B} = \mathbf{A}_1^T\mathbf{A}_2$, $\mathbf{C} = \mathbf{A}_2^T\mathbf{A}_1$, $\mathbf{D} = \mathbf{A}_2^T\mathbf{A}_2$, gives:

$$[\mathbf{A}^T\mathbf{A}]^{-1} = \begin{bmatrix} (\mathbf{A}_1^T\mathbf{P}_2^\perp\mathbf{A}_1)^{-1} & -(\mathbf{A}_1^T\mathbf{P}_2^\perp\mathbf{A}_1)^{-1}\mathbf{A}_1^T\mathbf{A}_2(\mathbf{A}_2^T\mathbf{A}_2)^{-1} \\ -(\mathbf{A}_2^T\mathbf{P}_1^\perp\mathbf{A}_2)^{-1}\mathbf{A}_2^T\mathbf{A}_1(\mathbf{A}_1^T\mathbf{A}_1)^{-1} & (\mathbf{A}_2^T\mathbf{P}_1^\perp\mathbf{A}_2)^{-1} \end{bmatrix}$$

where $\mathbf{P}_1^\perp$ and $\mathbf{P}_2^\perp$ are the orthogonal complements of orthogonal projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ on column spaces $C(\mathbf{A}_1)$ and $C(\mathbf{A}_2)$, respectively, that is:

$$\mathbf{P}_1^\perp = \mathbf{I}_n - \mathbf{P}_1 = \mathbf{I}_n - \mathbf{A}_1\left(\mathbf{A}_1^T\mathbf{A}_1\right)^{-1}\mathbf{A}_1^T,$$
$$\mathbf{P}_2^\perp = \mathbf{I}_n - \mathbf{P}_2 = \mathbf{I}_n - \mathbf{A}_2\left(\mathbf{A}_2^T\mathbf{A}_2\right)^{-1}\mathbf{A}_2^T.$$

Iterative inversion of a partitioned square matrix

$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$a_{11} = l_{11} \times u_{11}$

Consider the square matrix $\mathbf{M}_n$ of order n, partitioned into the following form:

$$\mathbf{M}_n = \begin{bmatrix} \mathbf{M}_{n-1} & \mathbf{c}_n \\ \mathbf{r}_n^T & \sigma_n \end{bmatrix}$$

where $\mathbf{M}_{n-1}$ is a square matrix of order $n-1$, and $\mathbf{c}_n, \mathbf{r}_n \in \mathbb{K}^{n-1}$. Assuming $\mathbf{M}_{n-1}$ and $\mathbf{M}_n$ are invertible, the application of the inversion formula (9) allows to perform the calculation of the inverse $\mathbf{M}_n^{-1}$ recursively with respect to order n, that is, in terms of $\mathbf{M}_{n-1}^{-1}$:

$$\mathbf{M}_n^{-1} = \begin{bmatrix} \mathbf{M}_{n-1}^{-1} + k_n \mathbf{M}_{n-1}^{-1}\mathbf{c}_n\mathbf{r}_n^T\mathbf{M}_{n-1}^{-1} & -k_n\mathbf{M}_{n-1}^{-1}\mathbf{c}_n \\ -k_n\mathbf{r}_n^T\mathbf{M}_{n-1}^{-1} & k_n \end{bmatrix},$$

where $k_n = (\sigma_n - \mathbf{r}_n^T\mathbf{M}_{n-1}^{-1}\mathbf{c}_n)^{-1}$ This recursive inversion formula will be used to demonstrate the Levinson-Durbin algorithm.

A^{-1} 或 $A=LU$

$\boxed{(A-BD^{-1}C)}\,x=b$

定理

Let $\mathbf{A} \in \mathbb{K}^{n\times n}, \mathbf{B} \in \mathbb{K}^{n\times m}, \mathbf{C} \in \mathbb{K}^{m\times n}$, *and* $\mathbf{D} \in \mathbb{K}^{m\times m}$. *By identifying the blocks* $(1,1)$ *of the right-hand sides of (9) and (10), it can be deduced that:*

$$\begin{aligned}\underbrace{\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}}_{n\times n} &= \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\mathbf{X}_{\mathbf{A}}^{-1}\mathbf{C}\mathbf{A}^{-1}\\ &= \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\underbrace{\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}}_{m\times m}\mathbf{C}\mathbf{A}^{-1}\end{aligned} \tag{12}$$

This formula is known as the matrix inversion lemma. It is also called the Sherman-Morrison-Woodbury formula.

It should be noted that $\mathbf{X}_\mathbf{D} = \mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is defined if $\mathbf{D}$ is invertible, and its inverse can be calculated using (12) if $\mathbf{A}$ and $\mathbf{X}_\mathbf{A} = \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ are invertible.

Applications of the matrix inversion lemma

For different choices of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, the inversion lemma provides the following identities:

- For $\mathbf{D} = -\mathbf{I}_m$, the matrix inversion lemma (12) gives:

$$[\mathbf{A} + \mathbf{B}\mathbf{C}]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left[\mathbf{I}_m + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right]^{-1}\mathbf{C}\mathbf{A}^{-1}$$

- From this identity, it can be deduced that for $\mathbf{A} = \mathbf{I}_n$ and $\mathbf{C} = \mathbf{B}^H$

$$\left[\mathbf{I}_n + \mathbf{B}\mathbf{B}^H\right]^{-1} = \mathbf{I}_n - \mathbf{B}\left[\mathbf{I}_m + \mathbf{B}^H\mathbf{B}\right]^{-1}\mathbf{B}^H.$$

- For $m = n$, $\mathbf{D} = -\Delta^{-1}$ and $\mathbf{B} = \mathbf{C} = \mathbf{I}_n$

$$[\mathbf{A} + \Delta]^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}[\mathbf{A}^{-1} + \Delta^{-1}]^{-1}\mathbf{A}^{-1}.$$

- For $m = 1$, $\mathbf{D} = -1/\alpha$, $\mathbf{B} = \mathbf{u} \in \mathbb{K}^n$, and $\mathbf{C}^T = \mathbf{v} \in \mathbb{K}^n$, assuming that $\alpha^{-1} + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$, we get

$$\left[\mathbf{A} + \alpha\mathbf{u}\mathbf{v}^T\right]^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{\alpha^{-1} + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}$$

Generalized inverses of 2×2 block matrices

Consider the extension of the Banachiewicz-Schur form (9) to the case of singular or rectangular matrices partitioned into 2×2 blocks, more specifically with singular or rectangular submatrices, written as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \in \mathbb{K}^{(m+q)\times(n+p)}, \tag{13}$$

with $\mathbf{A} \in \mathbb{K}^{m\times n}, \mathbf{B} \in \mathbb{K}^{m\times p}, \mathbf{C} \in \mathbf{K}^{q\times n}$, and $\mathbf{D} \in \mathbf{K}^{q\times p}$. Unlike the case of a partitioned matrix $\mathbf{M}$ with square diagonal blocks, previously addressed, we now consider rectangular diagonal blocks, needing to define a generalized inverse for $\mathbf{M}$.

Generalized inverses of 2×2 block matrices

The notion of generalized inverse was introduced by Moore (1935), in a book published after his death. It was Penrose (1955) who demonstrated the uniqueness of the Moore generalized inverse, which explains the name given to the Moore-Penrose pseudo-inverse. This pseudo-inverse that generalizes the inverse of a regular square matrix to the case of rectangular matrices plays a very important role for solving systems of linear equations using the method of least squares.

Generalized inverses of 2×2 block matrices

The notion of generalized inverse was introduced by Moore (1935), in a book published after his death. It was Penrose (1955) who demonstrated the uniqueness of the Moore generalized inverse, which explains the name given to the Moore-Penrose pseudo-inverse. This pseudo-inverse that generalizes the inverse of a regular square matrix to the case of rectangular matrices plays a very important role for solving systems of linear equations using the method of least squares. Different types of generalized inverse $\mathbf{A}^{\#} \in \mathbb{K}^{n \times m}$ of a matrix $\mathbf{A} \in \mathbb{K}^{m \times n}$ can be defined according to the equations that are satisfied among:

1. $\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}$
2. $\mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}$
3. $\left(\mathbf{A}\mathbf{A}^{\#}\right)^{H} = \mathbf{A}\mathbf{A}^{\#}$
4. $\left(\mathbf{A}^{\#}\mathbf{A}\right)^{H} = \mathbf{A}^{\#}\mathbf{A}$.

Note that (3) and (4) means that $\mathbf{A}\mathbf{A}^{\#}$ and $\mathbf{A}^{\#}\mathbf{A}$ are Hermitian, respectively, or symmetric in the real case.

Generalized inverses of 2×2 block matrices

Any inverse only satisfying conditions $\{c_1\}$, or $\{c_1, c_2\}$, or $\{c_1, c_2, c_3\}$, with $c_1, c_2, c_3 \in \{(1), (2), (3), (4)\}$, is denoted $\mathbf{A}^{\{c_1\}}, \mathbf{A}^{\{c_1,c_2\}}$, and $\mathbf{A}^{\{c_1,c_2,c_3\}}$, respectively.

- The properties of this type of inverse were studied by Ben-Israel and Greville (2001), where the inverses $\mathbf{A}^{\{1\}}, \mathbf{A}^{\{2\}}, \mathbf{A}^{\{1,2\}}$, and $\mathbf{A}^{(1,2,3)}$ are often called inner inverse, outer inverse, reflexive generalized inverse (or semi-inverse), and weak generalized inverse (or least-squares reflexive generalized inverse), respectively.
- For any matrix $\mathbf{A}$, there exists a unique matrix $\mathbf{A}^{\{1,2,3,4\}}$. This matrix corresponds to the Moore-Penrose pseudo-inverse of $\mathbf{A}$ and is often denoted by $\mathbf{A}^{\dagger}$.
- When $\mathbf{M}$ and $\mathbf{A}$ in (13) are singular, the Banachiewicz-Schur formula (9) can be extended by replacing the inverses of $\mathbf{A}$ and $\mathbf{X}_{\mathbf{A}}$ by generalized inverses $\mathbf{A}^{\#}$ and $\mathbf{X}_{\mathbf{A}}^{\#}$.

1 Introduction

2 Submatrices

3 Matrix products and partitioned matrices

4 Special cases of partitioned matrices

5 Determinants of partitioned matrices

Determinant of block-diagonal matrices

Assuming that $\mathbf{A}$ and $\mathbf{D}$ are square matrices, we have:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \Rightarrow \det(\mathbf{M}) = \det(\mathbf{A})\det(\mathbf{D}).$$

Determinant of block-triangular matrices

For upper and lower block-triangular matrices, with square diagonal blocks, we have:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \Rightarrow \det(\mathbf{M}) = \det(\mathbf{A})\det(\mathbf{D}).$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \Rightarrow \det(\mathbf{M}) = \det(\mathbf{A})\det(\mathbf{D}).$$

For unit block-triangular matrices $(\mathbf{A} = \mathbf{I}_n, \mathbf{D} = \mathbf{I}_m)$, we have $\det(\mathbf{M}) = 1$. More generally, we have:

$$\det \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1R} \\ & \mathbf{A}_{22} & & \vdots \\ & 0 & \ddots & \vdots \\ & & & \mathbf{A}_{RR} \end{bmatrix} = \prod_{r=1}^{R} \det\left(\mathbf{A}_{rr}\right).$$

From relations (5) and (7), the following expressions can be deduced:

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{A})\det(\mathbf{X_A}) = \det(\mathbf{A})\det\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) \quad (14)$$

$$= \det(\mathbf{D})\det(\mathbf{X_D}) = \det(\mathbf{D})\det\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right). \quad (15)$$

Determinants of specific partitioned matrices

$\left|\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\right| = 1\times4 - 3\times2$

定理

When $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ *are square matrices of order n, we have:*

2×2

- *If* $\mathbf{A}$ *and* $\mathbf{C}$ *commute:* $\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det(\mathbf{AD} - \mathbf{CB})$.
- *If* $\mathbf{B}$ *and* $\mathbf{D}$ *commute:* $\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det(\mathbf{DA} - \mathbf{BC})$.
- *If* $\mathbf{A}$ *and* $\mathbf{B}$ *commute:* $\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det(\mathbf{DA} - \mathbf{CB})$.
- *If* $\mathbf{C}$ *and* $\mathbf{D}$ *commute:* $\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det(\mathbf{AD} - \mathbf{BC})$.

Determinants of specific partitioned matrices

定理

The application of the previous formulae yields:

$$\det\begin{bmatrix} \mathbf{I}_n & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{D}-\mathbf{CB}); \qquad \det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{I}_n \end{bmatrix} = \det(\mathbf{A}-\mathbf{BC}).$$

$$\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{A} & \mathbf{D} \end{bmatrix} = \det(\mathbf{A})\det(\mathbf{D}-\mathbf{B}); \det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} \end{bmatrix} = \det(\mathbf{B})\det(\mathbf{A}-\mathbf{C}).$$

$$\det\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \det\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = (-1)^n\det(\mathbf{B})\det(\mathbf{C}).$$

$$\begin{pmatrix} A & 0 \\ A & B \end{pmatrix}$$

Determinants of specific partitioned matrices

Defining:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \text{ with } \mathbf{B}_{11} \in \mathbb{K}^{n\times n}, \mathbf{B}_{22} \in \mathbb{K}^{m\times m} \mathbf{A} \in \mathbb{K}^{n\times n}, \mathbf{C} \in \mathbb{K}^{m\times m},$$

and using the property $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, as well as determinant formulae for block diagonal matrices and block-triangular matrices, the other following determinants can be deduced:

$$\det \begin{bmatrix} \mathbf{AB}_{11} & \mathbf{AB}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \det \left(\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{B} \right) = \det(\mathbf{A})\det(\mathbf{B}),$$

and

$$\det \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} + \mathbf{CB}_{11} & \mathbf{B}_{22} + \mathbf{CB}_{12} \end{bmatrix} = \det \left(\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_m \end{bmatrix} \mathbf{B} \right) = \det(\mathbf{B}).$$

Eigenvalues of CB and BC

定理

For $\mathbf{B} \in \mathbb{K}^{n\times m}$ *and* $\mathbf{C} \in \mathbb{K}^{m\times n}$, *we have:*

$$\lambda^n \det\left(\lambda \mathbf{I}_m - \mathbf{CB}\right) = \lambda^m \det\left(\lambda \mathbf{I}_n - \mathbf{BC}\right).$$

For $\lambda = 1$, *we obtain:*

$$\det\left(\mathbf{I}_m - \mathbf{CB}\right) = \det\left(\mathbf{I}_n - \mathbf{BC}\right).$$

Eigenvalues of CB and BC

定理

For $\mathbf{B} \in \mathbb{K}^{n \times m}$ *and* $\mathbf{C} \in \mathbb{K}^{m \times n}$*, we have:*

$$\lambda^n \det(\lambda \mathbf{I}_m - \mathbf{CB}) = \lambda^m \det(\lambda \mathbf{I}_n - \mathbf{BC}).$$

For $\lambda = 1$*, we obtain:*

$$\det(\mathbf{I}_m - \mathbf{CB}) = \det(\mathbf{I}_n - \mathbf{BC}).$$

证明.

By formulae (14) and (15), with $\mathbf{A} = \lambda \mathbf{I}_n$ and $\mathbf{D} = \mathbf{I}_m$, then:

$$\begin{aligned} \det \begin{bmatrix} \lambda \mathbf{I}_n & \mathbf{B} \\ \mathbf{C} & \mathbf{I}_m \end{bmatrix} &= \det(\lambda \mathbf{I}_n) \det\left(\mathbf{I}_m - \lambda^{-1} \mathbf{CB}\right) = \lambda^{n-m} \det(\lambda \mathbf{I}_m - \mathbf{CB}) \\ &= \det(\lambda \mathbf{I}_n - \mathbf{BC}) \end{aligned}$$

from which the identity (13) is deduced. $\square$

Rank of partitioned matrices

Let the partitioned matrix be:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

Obviously, we have:

$$\begin{aligned} \mathrm{r}(\mathbf{M}) &\leq \mathrm{r}([\mathbf{A} \quad \mathbf{B}]) + \mathrm{r}([\mathbf{C} \quad \mathbf{D}]) \\ &\leq \mathrm{r}\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}\right) + r\left(\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}\right). \end{aligned}$$

In general, for a partitioned matrix $\mathbf{A} = [\mathbf{A}_{ij}]$, we have $\mathbf{r}(\mathbf{A}_{ij}) \leq \mathbf{r}(\mathbf{A})$
Based on block-factorization formulae (5) and (7), the following relations can be deduced, for $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{D} \in \mathbb{C}^{m\times m}$:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \Rightarrow \quad \begin{pmatrix} A & 0 \\ 0 & \boxed{X_A} \end{pmatrix}$$

$$\begin{aligned} \mathrm{r}(\mathbf{M}) = \mathrm{r}(\mathbf{A}) + \mathrm{r}(\mathbf{X_A}) &= \mathrm{r}(\mathbf{A}) + \mathrm{r}\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) \\ = \mathrm{r}(\mathbf{D}) + \mathrm{r}(\mathbf{X_D}) &= \mathrm{r}(\mathbf{D}) + \mathrm{r}\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right). \end{aligned}$$

$Tx = b \quad T \in \mathbb{R}^{n\times n} \quad O(2n^3) \quad 10^6 \quad 10^{12}$

$O(4n^2) \quad 10^8$

Autoregressive (AR) processes or models are widely used in signal processing for the representation and classification of random signals. The autocorrelation function of such a process satisfies a system of linear equations in AR coefficients, called Yule–Walker equations. These equations form a Toeplitz system whose inversion can be achieved in a numerically efficient way, by means of the Levinson algorithm, also known as the Levinson-Durbin algorithm, which is an algorithm recursive with respect to the order of the model and plays a fundamental role in signal processing.

Yule–Walker equations

LU分解

$Tx = b$ T 正定 $-b$ $\frac{2}{3}n^3$

$T = LU$ ← 上三角

L: 单位下三角

$$\begin{pmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ t_1 & \ddots & & \vdots \\ \vdots & & \ddots & t_1 \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = - \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-1} \\ t_n \end{pmatrix}$$

t_n 任意

$\frac{1}{3}n^3$

$T = LL^T$ Cholesky 分解

L: 下三角

Yule-Walker

假设 $T_k x^{(k)} = -b^{(k)}$

T 的 K 阶顺序主子阵

$$T_k = \begin{pmatrix} t_0 & \cdots & t_{k-1} \\ t_1 & \ddots & \vdots \\ \vdots & & \\ t_{k-1} & \cdots & t_0 \end{pmatrix}$$

$$b^{(k)} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

$$E_k = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

$T_{k+1} x^{(k+1)} = -b^{(k+1)}$

$$T_{k+1} = \begin{pmatrix} T_k & \begin{matrix} t_k \\ t_{k-1} \\ \vdots \end{matrix} \\ t_k \cdots & t_0 \end{pmatrix} = \begin{pmatrix} T_k & E_k b^{(k)} \\ (E_k b^{(k)})^T & t_0 \end{pmatrix}, \quad b^{(k+1)} = \begin{pmatrix} b^{(k)} \\ t_{k+1} \end{pmatrix}$$

Levinson-Durbin algorithm

$$x^{(k+1)} = \begin{pmatrix} z_k \\ \alpha \end{pmatrix} \qquad \begin{pmatrix} T_k & E_k b^{(k)} \\ (E_k b^{(k)})^T & t_0 \end{pmatrix} \begin{pmatrix} z_k \\ \alpha \end{pmatrix} = -\begin{pmatrix} b^{(k)} \\ t_{k+1} \end{pmatrix}$$

$$\begin{cases} T_k z_k + \alpha E_k b^{(k)} = -b^{(k)} & (1) \\ (E_k b^{(k)})^T z_k + t_0 \alpha = -t_{k+1} & (2) \end{cases} \qquad T_k E_k = E_k T_k$$

$$T_k z_k = -b^{(k)} - \alpha E_k b^{(k)} \qquad \boxed{T_k^{-1} E_k = E_k T_k^{-1}}$$

$$z_k = -T_k^{-1} b^{(k)} - \alpha T_k^{-1} E_k b^{(k)} \quad (3) \qquad \sum_{k=1}^{n} 2k \qquad n(n+1)$$

$$= x^{(k)} + \alpha E_k x^{(k)}$$