LINEAR ALGEBRA II

Ch. VII SYMMETRIC, HERMITIAN, AND UNITARY OPERATORS

§1. Symmetric Operators

- Throughout this section we let V be a finite dimensional vector space over a field K. We suppose that V has a fixed non-degenerate scalar product denoted by $\langle v, w\rangle$, for $v, w \in V$.
- "The reader may take $V = K^n$ and may fix the scalar product to be the ordinary dot product

$$\langle X, Y\rangle = {}^{t}XY,$$

where X, Y are column vectors in K^n. However, in applications, it is not a good idea to fix such bases right away."
- A linear map

$$A : V \to V$$

of V into itself will also be called an (linear) operator.

§1. Symmetric Operators

- **Lemma 1.1.** Let $A : V \to V$ be an operator. Then there exists a unique operator $B : V \to V$ such that $\langle Av, w\rangle = \langle v, Bw\rangle$ for all $v, w \in V$.

§1. Symmetric Operators

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- The operator $B : V \to V$ such that $\langle Av, w\rangle = \langle v, Bw\rangle$ for all $v, w \in V$ is called the transpose of A and denoted by ${}^{t}A$.

- $B = {}^{t}A$ iff $\langle Av, w\rangle = \langle v, Bw\rangle$ for all $v, w \in V$.

- The operator A is said to be symmetric (with respect to the fixed non-degenerate scalar product $\langle\ ,\ \rangle$) if ${}^{t}A = A$.

- The operator A is symmetric iff $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$.

§1. Symmetric Operators

- Let $A : V \to W$ be a linear map. Then there exists a unique linear map (called the transpose of A and denoted by ${}^{t}A$) $B : W \to V$ such that $\langle Av, w\rangle = \langle v, Bw\rangle$ for all $v \in W$, $w \in W$.

- $M_{\mathcal{B}'}^{\mathcal{B}}({}^{t}A) = ?$.

§1. Symmetric Operators

- Let $V = K^n$ and let the scalar product be the ordinary dot product. We have
$$\langle AX, Y\rangle = {}^{t}(AX)Y = {}^{t}X\,{}^{t}AY = \langle X, {}^{t}AY\rangle,$$
where ${}^{t}A$ now means the transpose of the matrix A. Thus when we deal with the ordinary dot product of n-tuples, the transpose of the operator is represented by the transpose of the associated matrix. This is the reason why we have used the same notation in both cases.

§1. Symmetric Operators

- **Theorem 1.2.** Let V be a finite dimensional vector space over the field K, with a fixed non-degenerate scalar product $\langle v, w\rangle$. Let A, B be operators of V, and $c \in K$. Then

$$
\begin{aligned}
{}^{t}(A+B) &= {}^{t}A + {}^{t}B, \qquad & {}^{t}(AB) &= {}^{t}B\,{}^{t}A, \\
{}^{t}(cA) &= c\,{}^{t}A, \qquad & {}^{tt}A &= A.
\end{aligned}
$$

§1. Symmetric Operators

- id: $V \to V$ is symmetric.

- If $A : V \to V$ is invertible, then ${}^{\mathrm{t}}(A^{-1}) = ({}^{\mathrm{t}}A)^{-1} = {}^{\mathrm{t}}A^{-1}$ $(= A^{-\mathrm{T}})$.
  If $A : V \to V$ is invertible and symmetric, then A^{-1} is symmetric .

- If A and B are symmetric, then
  - $A \pm B$ is symmetric;
  - AB is symmetric iff $AB = BA$.

§1. Symmetric Operators

- A $n \times n$ real symmetric matrix is said to be

positive definite semi-positive	if	${}^tXAX > 0$ for all $O \neq X \in R^n$. ${}^tXAX \geq 0$ for all $X \in R^n$.

- Let V be a finite dimensional vector space over R, with a positive definite scalar product $\langle\ ,\ \rangle$. An symmetric operator A of V is said to be

positive definite semi-positive	if	$\langle Av, v\rangle > 0$ for all $O \neq v \in V$. $\langle Av, v\rangle \geq 0$ for all $v \in V$.

§1. Symmetric Operators

- Let V be a finite dimensional vector space over R, with a positive definite scalar product $\langle\ ,\ \rangle$. Suppose that $V = W + W^{\perp}$ is the direct sum of a subspace W and its orthogonal complement. Let P be the projection on W, and assume $W \neq \{O\}$. Show that P is symmetric and semipositive.

- Transpose of the infinite dimensional operator $D : f \mapsto f'$ of $C_0^{\infty}[0, 1]$ w.r.t. the scalar product $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$.

§1. Symmetric Operators

- Homework: Ch. VII, §1, 5, 7, 12.

§2. Hermitian Operators

- Throughout this section we let V be a finite dimensional vector space over C. We suppose that V has a fixed positive definite hermitian product (Hermitian form) denoted by $\langle v, w\rangle$, for $v, w \in V$.
- The reader may take $V = C^n$ and may fix the hermitian product to be the standard product
$$\langle X, Y\rangle = {}^{t}X\bar{Y},$$
where X, Y are column vectors in C^n.
- Let $A : V \to V$ be an operator.
- $L_w : v \mapsto \langle Av, w\rangle$ is a (complex) functional on V.

§2. Hermitian Operators

- **Theorem 2.1.** Let V be a finite dimensional vector space over C with a positive definite Hermitian form $\langle\ ,\ \rangle$. Given a functional L on V, there exists a unique $w' \in V$ such that $L(v) = \langle v, w' \rangle$ for all $v \in V$.
- The association $w \mapsto L_w$ is not an isomorphism.
- **Lemma 2.2.** Given an operator $A : V \to V$, there exists a unique operator $A^* : V \to V$ such that for all $v, w \in V$ we have
$$\langle Av, w \rangle = \langle v, A^* w \rangle.$$

§2. Hermitian Operators

- A^* is called the adjoint of A.
- Let $V = C^n$ and let the form be the standard form given by

$$(X, Y) \mapsto {}^{t}X\bar{Y} = \langle X, Y \rangle,$$

for X, Y column vectors in C^n. Then for any matrix A representing a linear map of V into itself, we have

$${}^{t}X\overline{(A^*Y)} = \langle X, A^*Y \rangle = \langle AX, Y \rangle = {}^{t}(AX)\bar{Y} = {}^{t}X\,{}^{t}A\bar{Y} = {}^{t}X\,\overline{({}^{t}\bar{A}Y)}.$$

This means that

$$\boxed{A^* = {}^{t}\bar{A}.}$$

- An operator A is called hermitian (or self-adjoint) if $A^* = A$. This means that for all $v, w \in V$ we have

$$\langle Av, w \rangle = \langle v, Aw \rangle.$$

§2. Hermitian Operators

- A complex matrix A is called hermitian if $A = A^* \triangleq {}^t\bar{A}$, or equivalently, ${}^tA = \bar{A}$.
- If A is a hermitian matrix, then we can define on C^n a hermitian product by the rule

$$(X, Y) \mapsto {}^t(AX)\bar{Y}.$$

§2. Hermitian Operators

- **Theorem 2.3.** Let V be a finite dimensional vector space over the field C, with a fixed positive definite hermitian form $\langle v, w\rangle$. Let A, B be operators of V, and $\alpha \in C$. Then

$$(A+B)^* = A^* + B^*, \qquad (AB)^* = B^*A^*,$$
$$(\alpha A)^* = \bar{\alpha}A^*, \qquad A^{**} = A.$$

- **Polarization identity:**

$$\langle A(v+w), v+w\rangle - \langle A(v-w), v-w\rangle = 2[\langle Aw, v\rangle + \langle Av, w\rangle]$$

or

$$\langle A(v+w), v+w\rangle - \langle Av, v\rangle - \langle Aw, w\rangle = \langle Aw, v\rangle + \langle Av, w\rangle$$

- $$4\langle Av, w\rangle = \\ \langle A(v+w), v+w\rangle - \langle A(v-w), v-w\rangle + \mathrm{i}\langle A(v+\mathrm{i}w), v+\mathrm{i}w\rangle - \mathrm{i}\langle A(v-\mathrm{i}w), v-\mathrm{i}w\rangle$$

§2. Hermitian Operators

- **Theorem 2.4.** Let V be a finite dimensional vector space over C, with a fixed positive definite hermitian form $\langle v, w\rangle$. Let A be an operator such that $\langle Av, v\rangle = 0$ for all $v \in V$. Then $A = O$.

§2. Hermitian Operators

- **Theorem 2.5.** Let V be a finite dimensional vector space over C, with a fixed positive definite hermitian form $\langle v, w\rangle$. Let A be an operator. Then A is hermitian if and only if $\langle Av, v\rangle$ is real for all $v \in V$.

§2. Hermitian Operators

- If $A : V \to V$ is invertible, then $(A^{-1})^* = (A^*)^{-1} = A^{-*}$.
  If $A : V \to V$ is invertible and hermitian, then A^{-1} is hermitian.

- If A and B are hermitian matrices, then
  - ${}^{t}A$ and $\bar{A}$ are hermitian;
  - $A \pm B$ is hermitian;
  - AB is hermitian iff $AB = BA$.

§2. Hermitian Operators

- Skew-symmetric matrix and operator.

- Skew-hermitian matrix and operator.

§2. Hermitian Operators

- A $n \times n$ hermitian matrix is said to be

positive definite semi-positive	if	$X^*AX > 0$ for all $O \neq X \in C^n$. $X^*AX \geq 0$ for all $X \in C^n$.

- Let V be a finite dimensional vector space over C, with a positive definite hermitian product $\langle\ ,\ \rangle$. An hermitian operator A of V is said to be

positive definite semi-positive	if	$\langle Av, v\rangle > 0$ for all $O \neq v \in V$. $\langle Av, v\rangle \geq 0$ for all $v \in V$.

§2. Hermitian Operators

- Homework: Ch. VII, §2, 2, 3, 8

§3. Unitary Operators

- Let V be a finite dimensional vector space over R, with a positive definite scalar product $\langle\ ,\ \rangle$. An operator A of V is said to be real unitary (orthogonal) if

$$\langle Av, Aw \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

- **Theorem 3.1.** Let V be a finite dimensional vector space over R with a positive definite scalar product, and A be an operator of V. The following conditions on A are equivalent:
(1) A is unitary.
(2) A preserves the norm of vectors, i.e. for every $v \in V$, we have

$$\|Av\| = \|v\|.$$

(3) For every unit vector $v \in V$, the vector Av is also a unit vector.

§3. Unitary Operators

§3. Unitary Operators

- **Theorem 3.2.** Let V be a finite dimensional vector space over R with a positive definite scalar product. An operator A of V is unitary iff

$$ {}^{t}AA = I. $$

- A unitary map is invertible.

- A real matrix A is said to be unitary (or orthogonal) if ${}^{t}AA = I$ or equivalently, ${}^{t}A = A^{-1}$.

§3. Unitary Operators

- The only unitary maps of the plane R^2 into itself are the maps whose matrices are of the type
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

§3. Unitary Operators

- Let A be a real unitary operator.
  (a) Show that ${}^{t}A$ is unitary.
  (b) Show that A^{-1} exists and is unitary.
  (c) If B is real unitary, show that AB is unitary, and that $B^{-1}AB$ is unitary.

§3. Unitary Operators

The complex case.

- Let V be a finite dimensional vector space over C, with a positive definite hermitian product $\langle\ ,\ \rangle$. An operator A of V is said to be real unitary (orthogonal) if
$$\langle Av, Aw\rangle = \langle v, w\rangle$$
for all $v, w \in V$.

- **Theorem 3.1'.** Let V be a finite dimensional vector space over C with a positive definite hermitian product, and A be an operator of V. The following conditions on A are equivalent:
(1) A is unitary.
(2) A preserves the norm of vectors, i.e. for every $v \in V$, we have
$$\|Av\| = \|v\|.$$
(3) For every unit vector $v \in V$, the vector Av is also a unit vector.

§3. Unitary Operators

- **Theorem 3.3.** Let V be a finite dimensional vector space over R with a positive definite hermitian product. An operator A of V is unitary iff

$$A^*A = I.$$

- A unitary map is invertible.

- A complex matrix A is said to be unitary (or orthogonal) if $A^*A = {}^t\bar{A}A = I$ or equivalently, $A^* = {}^t\bar{A} = A^{-1}$.

§3. Unitary Operators

- **Theorem 3.4.** Let V be a finite dimensional vector space over R with a positive definite scalar product, or over C with a positive definite hermitian product. Let A be an operator of V and $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V.
  (a) If A is unitary then $\{Av_1, \ldots, Av_n\}$ is an orthonormal basis.
  (b) Let $\{w_1, \ldots, w_n\}$ be another orthonormal basis. Suppose that $Av_i = w_i$ for $i = 1, \ldots, n$. Then A is unitary.

§3. Unitary Operators

- Let A and B be $n \times n$ matrices satisfying $AB = I$. Show that A, B are nonsingular and $B = A^{-1}$, $A = B^{-1}$.
- Let V, W be vector spaces over K and $\dim V = \dim W$. Let $F : V \to W$ and $G : W \to V$ be linear maps satisfying $F \circ G = \mathrm{id}$. Show that F, G are invertible and $G = F^{-1}$, $F = G^{-1}$.
- Let V, W be finite dimensional vector spaces over K with bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. Show that a linear map $F : V \to W$ is invertible iff $M^{\mathcal{B}}_{\mathcal{B}'}(F)$ is nonsingular.

§3. Unitary Operators

- Let V be a finite dimensional vector space over C with a non-degenerate hermitian product $\langle\ ,\ \rangle$. Let A be an operator of V and $\mathcal{B}$ be a basis of V.
  - Show that $M_{\mathcal{B}}^{\mathcal{B}}(A^*) = \left(M_{\mathcal{B}}^{\mathcal{B}}(A)\right)^*$.
  - Is the statement "A is hermitian iff $M_A = M_{\mathcal{B}}^{\mathcal{B}}(A)$ is hermitian" right? Is it right when $\mathcal{B}$ is an orthogonal (orthonormal) basis?
  - Is the statement "A is unitary iff $M_A = M_{\mathcal{B}}^{\mathcal{B}}(A)$ is unitary" right? Is it right when $\mathcal{B}$ is an orthogonal (orthonormal) basis?

§3. Unitary Operators

- Homework: Ch. VII, §3, 5, 13, 14.