UNIVERSITY OF LEICESTER

MA1014
CALCULUS AND ANALYSIS
TUTORIAL 21

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ANNOUNCEMENTS

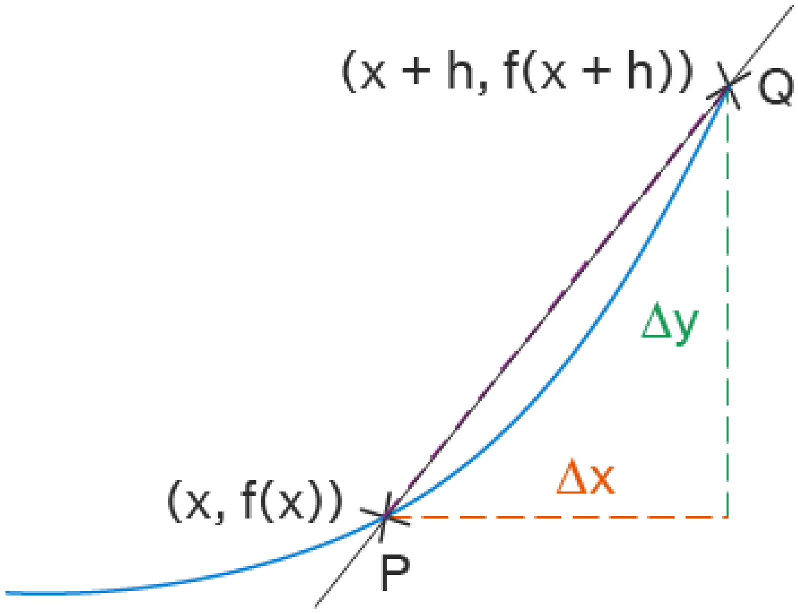
- Chapter 3 revision

THE DERIVATIVE

- Rate of Change of a function w.r.t a variable

- Definition:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



EXERCISE:
CALCULATE THE DERIVATIVES OF THE FOLLOWING FUNCTIONS FROM FIRST PRINCIPLES

a) $f(x) = \sqrt{x}$

b) $g(x) = \dfrac{1}{\sqrt{x}}$

c) $p(x) = \sin(x)$

DIFFERENTIATION RULES

- Linearity: If $h(x) = \alpha f(x) + \beta g(x)$ where $f(x)$ and $g(x)$ are differentiable on $x \subseteq \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then

$$\boxed{h'(x) = \alpha f'(x) + \beta g'(x)}$$

- Product Rule:

$$\boxed{\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)}$$

- Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

CHAIN RULE

If $F = f \circ g$ (i.e. $F(x) = f(g(x))$) where $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$ then F is differentiable at x and,

$$\boxed{F'(x) = f\big(g(x)\big)' = f'\big(g(x)\big)g'(x)}$$

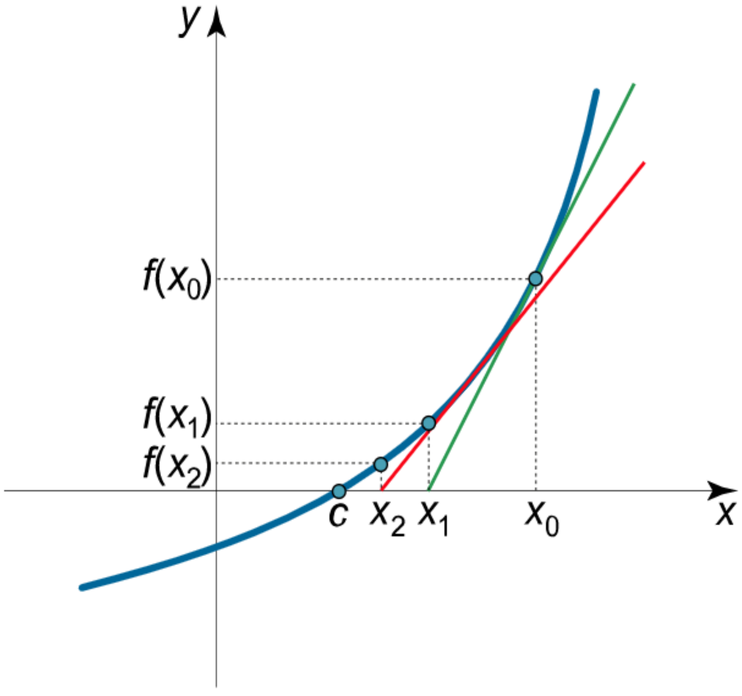
INVERSE FUNCTIONS

Let $f(x)$ be one-to-one and differentiable on $I \subseteq \mathbb{R}$. Let $a \in I$ and $f(a) = b$, if $f'(a) \neq 0$ then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'\big(f^{-1}(b)\big)}$$

NEWTON (RAPHSON) METHOD

Finds the root of a function, so it can solve non-linear equations.



1. Choose an initial guess x_0
2. Iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

EXERCISE:

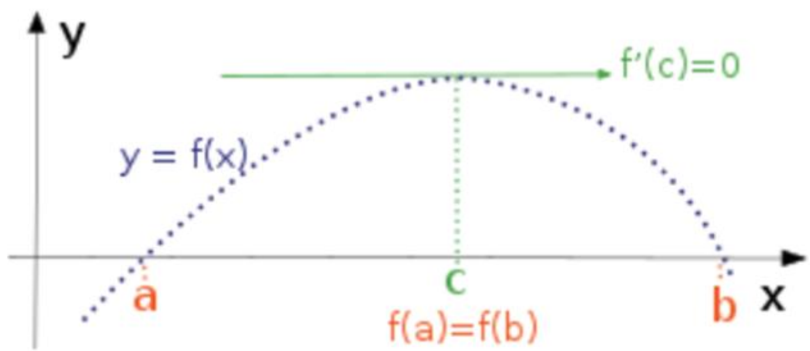
Use Newton's method to approximate:

a) $\sqrt{5}$ with $x_0 = 2$ such that $|x_n^2 - 5| < 0.01$

b) $\sqrt[3]{23}$ with $x_0 = 3$ such that $|x_n^3 - 23| < 10^{-3}$

ROLLE'S THEOREM

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose further that $f(a) = f(b)$. Then, for some $c \in (a, b), f'(c) = 0$.



MEAN VALUE THEOREM

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then for some $c \in (a, b)$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

EXERCISE:

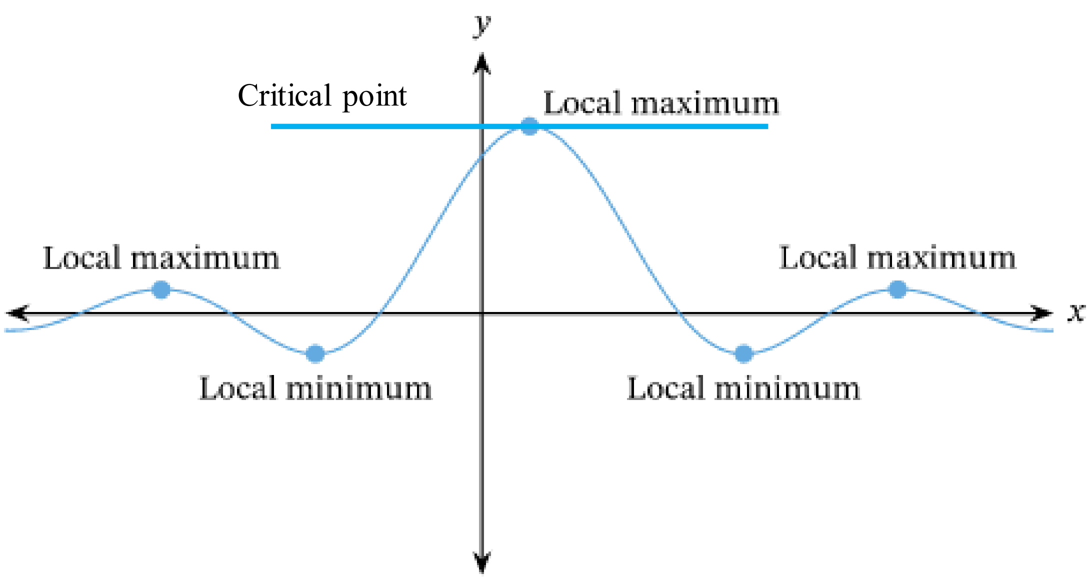
Use the Mean Value Theorem to prove that

a) $\frac{x}{1+x} < \ln(1+x) < x$ for all $x > 0$

b) $\sin(x) < x$ if x is positive

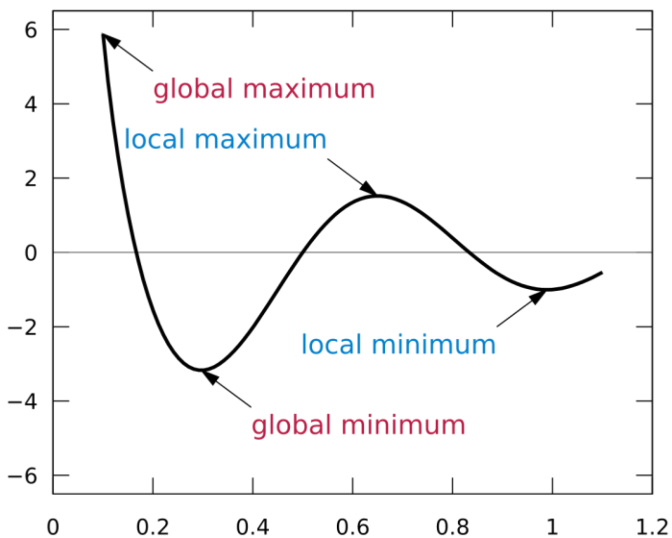
LOCAL EXTREMA

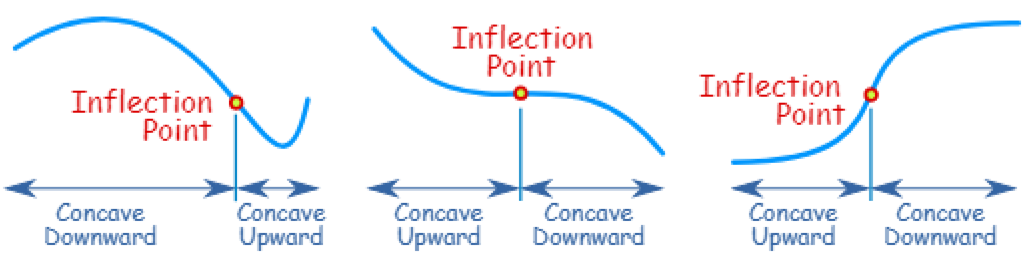
- If $f(c) \leq f(x)$ $\forall x$ in the neighbourhood of c, this is called a **local minimum**.
- If $f(c) \geq f(x)$ $\forall x$ in the neighbourhood of c, this is called a **local maximum**.
- **Critical point**: A point where $f'(x) = 0$ i.e. the tangent line is horizontal/constant or $f'(x)$ does not exist.



GLOBAL EXTREMA

- If $f(c) \leq f(x)\ \forall x \in \mathrm{dom}(f)$ this is called the **global minimum**
- If $f(c) \geq f(x)\ \forall x \in \mathrm{dom}(f)$ this is called the **global maximum**
- If at $(c, f(c))$ the concavity changes, then this is called an **Inflection point**.





SECOND DERIVATIVE TEST

Suppose $f'(c) = 0$ and $f''(c)$ exists.

i. If $f''(c) < 0$ then f has a local maximum at c
ii. If $f''(c) > 0$ then f has a local minimum at c
iii. If $f''(c) = 0$ then f has an inflection point at c

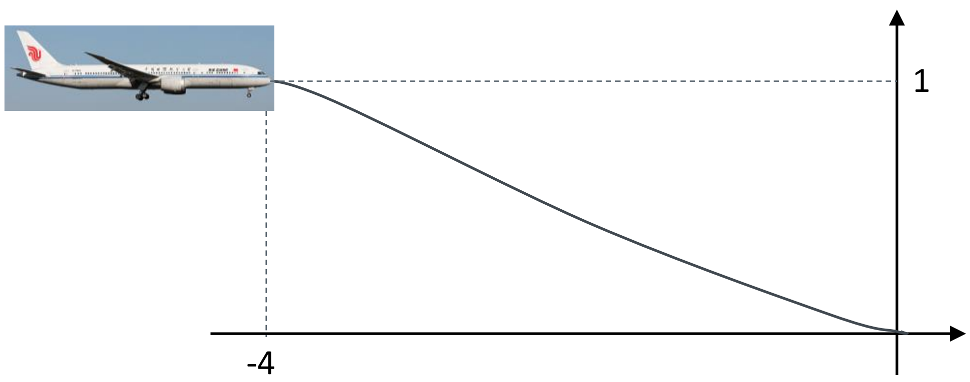
EXERCISE

A small aircraft starts its descent from an altitude of 1km, 4km west of the runway.

a) Find the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where $x \in [-4,0]$ that describes the smooth glide path for the landing.

b) The function $f(x)$ models the glide path of the plane. When would the plane be descending at the greatest rate.

c) If $x(t) = -4 + t^2 : t \in [0,2]$, calculate $f(t)$ and hence determine the velocity of the plane. When does it travel at its fastest?



L'HÔPITAL'S RULE

If the functions $f(x)$ and $g(x)$ are differentiable on an interval $I = (a, b)\backslash\{c\} : c \in (a, b)$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty$$

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

(If the second limit/RHS exists!)

EXERCISE:
DETERMINE THE FOLLOWING LIMITS

a) $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

b) $\lim_{x\to 0} \frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}$

c) $\lim_{x\to 0^+} \frac{\ln(1+x)}{x}$

TAYLOR'S THEOREM

Suppose $f(x)$ is n-times differentiable over $[a, x]$. Then,

$$\boxed{f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^n}{n!} f^{(n)}(c)}$$

for some $c \in (a, x)$.

EXERCISE:
FIND THE TAYLOR POLYNOMIAL AT $x = c$ OF DEGREE n, AND LAGRANGIAN REMAINDER

a) $f(x) = \sqrt{x}, c = 4, n = 3$

b) $g(x) = \sin(x), c = \frac{\pi}{4}, n = 4$

c) $h(x) = \tan^{-1}(x), c = 1, n = 3$

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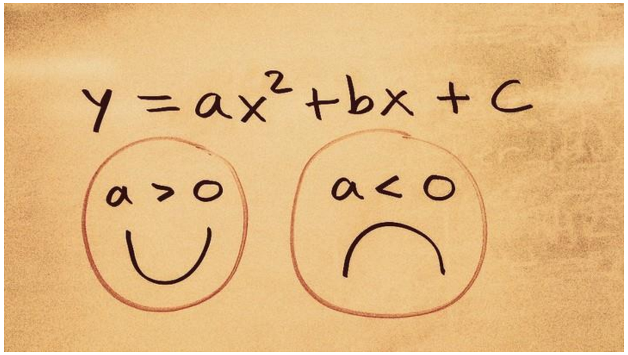

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

ANY QUESTIONS?

$$m\frac{d^2x}{dt^2} = -kx$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$$