Lecture 17: Partial Derivatives.

MA2032 Vector Calculus

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Tangent Planes and Differentials

• In single-variable differential calculus we saw **how the derivative defined the tangent line** to the graph of a differentiable function at a point on the graph.

• The **tangent line then provided for a linearization** of the function at the point.

• In this section, we will see analogously **how the gradient defines the tangent plane** to the level surface of a function $w = (x, y, z)$ at a point on the surface.

• The tangent plane then provides for a **linearization** of f at the point and defines the total differential of the function.

Tangent Planes and Normal Lines

• If $r(t) = x(t)i + y(t)j + z(t)k$ is a **smooth curve on the level surface** $f(x, y, z) = c$ of a differentiable function f, we found in the last lecture that

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

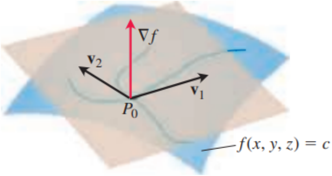
• Since f is constant along the curve **r**, the derivative on the left-hand side of the equation is 0, so the **gradient ∇f is orthogonal to the curve's velocity vector** r'.

• Now let us restrict our attention to the curves that **pass through a point** P_0.

• All the velocity vectors at P_0 are **orthogonal** to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to P_0.

• We now **define this plane**.

Tangent Planes and Normal Lines



DEFINITIONS The **tangent plane** to the level surface $f(x, y, z) = c$ of a differentiable function f at a point P_0 where the gradient is not zero is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

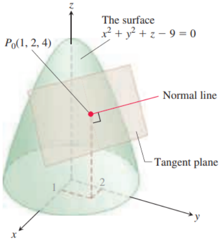
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \qquad (1)$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \qquad (2)$$

Example 1

Find the tangent plane and normal line of the level surface $f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1, 2, 4)$.



Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + (z-4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

Tangent Plane to a smooth surface

• To find an equation for the **plane tangent to a smooth surface** $z = (x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ **is equivalent to** $f(x, y) - z = 0$.

• The surface $z = f(x, y)$ is therefore the **zero level surface** of the function $F(x, y, z) = f(x, y) - z$.

• The **partial derivatives of F** are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

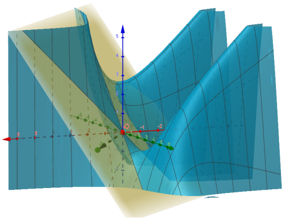
• The formula $F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$ for the plane tangent to the level surface at P_0 therefore **reduces to**

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$
The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \qquad (3)$$

Example 2

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.



Solution We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

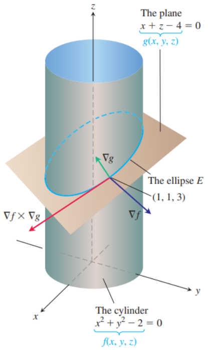
$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$

Example 3

The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ a cylinder and $g(x,y,z) = x + z - 4 = 0$ a plane meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1,1,3)$.



Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t. \qquad \blacksquare$$

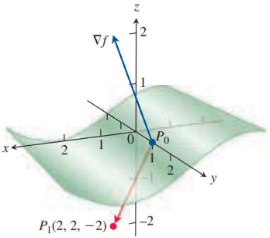
Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}} \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

Example 4

Estimate how much the value of $f(x, y, z) = y \sin x + 2yz$ will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.



Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})\Big|_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

How to Linearize a Function of Two Variables

• Functions of two variables can be quite **complicated**, and we sometimes **need to approximate** them with simpler ones that give the accuracy required for specific applications without being so difficult to work with.

• We do this in a way that is **similar** to the way we find **linear replacements** for functions of a single variable.

• Suppose the function we wish to approximate is $z = f(x, y)$ near a point (x_0, y_0) at which we know the values of f, f_x, and f_y and **at which f is differentiable**.

• If we move from (x_0, y_0) to any nearby point (x, y) by increments $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then the definition of differentiability gives the **change**

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

• where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

How to Linearize a Function of Two Variables

- If the increments Δx and Δy **are small**, the products $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ will eventually **be smaller still** and we have the **approximation**

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}.$$

- In other words, as long as Δx and Δy are small, **f will have approximately the same value as the linear function L**.

DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

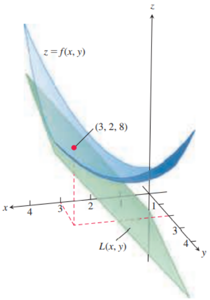
$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

- Plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0). Thus, the **linearization** of a function of two variables is a **tangent-plane approximation**.

Example 5

Find the linearization of $(x, y) = x^2 - xy + 1/2y^2 + 3$ at the point $(3, 2)$.



Solution We first evaluate f, f_x, and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)\Big|_{(3,2)} = -1,$$

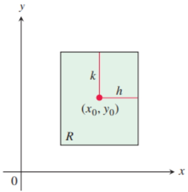
giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$ (see Figure 14.38).

The Error in the Standard Linear Approximation

- The **error** is defined by $E(x, y) = f(x, y) - L(x, y)$.



The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M\left(|x - x_0| + |y - y_0|\right)^2.$$

- To make $|E(x, y)|$ **small** for a given M, we just make $|x - x_0|$ and $|y - y_0|$ small.

Functions of More Than Two Variables

• Analogous results hold for differentiable **functions of more than two variables**.

1. The **linearization** of (x, y, z) at a point $P_0(x_0, y_0, z_0)$ is
$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$.
2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of fare continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R.
• Then the **error** $E(x, y, z) = (x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

• gives a **good approximation of the resulting change in f**.

Example 6

Find the linearization $L(x, y, z)$ of $(x, y, z) = x^2 - xy + 3\sin z$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region.

$$R: \quad |x-2| \leq 0.01, \quad |y-1| \leq 0.02, \quad |z| \leq 0.01$$

Solution Routine calculations give

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

Thus,

$$L(x, y, z) = 2 + 3(x-2) + (-2)(y-1) + 3(z-0) = 3x - 2y + 3z - 2.$$

Since

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

and $|-3\sin z| \leq 3\sin 0.01 \approx 0.03$, we may take $M = 2$ as a bound on the second partials. Hence, the error incurred by replacing f by L on R satisfies

$$|E| \leq \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016. \qquad \blacksquare$$