MA3077 (DLI) Operational Research

Lecture 14 – Game theory

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Recap and lecture outline

Summary: In previous lectures we learnt:

- about graphs and networks
- how to determine a minimal spanning tree,
- how to determine the shortest path tree
- that maximal flows and minimal cuts are inherently related to each other.

Today: A first glimpes into game theory (following closely Ch. 15 of the book by Hillier and Lieberman).

Two-person, zero-sum games

Definitions:

- Consider a game involving two players.
- The game is called *zero-sum* if one player wins what the other player loses, so that the sum of their net winnings is zero.
- A *strategy* is a predetermined rule that specifies how a player responds to each possible circumstance at each stage of the game.

Two-person, zero-sum games - example

Example: Playing a simplified version of *odds and evens.*

- Each player shows simultaneously one finger or two fingers.
- If the sum of the fingers is even, player A wins the bet and player B pays the bet. Otherwise, player B wins and player A pays.

We can summarize this game in the so-called *payoff table,* which shows the gains of player 1. Negative numbers denote the losses of player 1 (= gains of player 2).

Odds and evens		**Player 2**	
	Strategy	**1**	**2**
Player 1	**1**	1	-1
	2	-1	1

Game theory objectives and assumptions

Goal: The primary objective of game theory is the development of rational criteria for selecting a strategy (possibly the best one).

Key assumptions:

1. Both players are rational.
2. Both players choose their strategies solely to promote their own welfare (no compassion for the opponent , i.e., the game is adversarial, there is no co-operation).

Example 1

Consider the following scenario. Again, the game is assumed to be zero-sum and the table shows the gains of player 1. Negative numbers denote the losses of player 1 (= gains of player 2).

Abstract game		Player 2		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Example 1 – dominated strategies 1/6

Consider the following scenario.

Abstract game		**Player 2**		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Definition: A strategy is *dominated* by a second strategy if the latter is at least as good regardless of what the opponent chooses.

Example 1 – dominated strategies 2/6

Consider the following scenario.

Abstract game		**Player 2**		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Player 2 has not dominated strategies, but player 1 does, because strategy 1 dominates strategy 3. Since the players are rational, we can remove strategy 3 of player 1.

Example 1 – dominated strategies 3/6

Consider the following scenario.

Abstract game		**Player 2**		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Now player 2 has a dominated strategy, because both strategies 1 and 2 lead to lower losses than strategy 3. Hence, we should remove the latter.

Example 1 – dominated strategies 4/6

Consider the following scenario.

Abstract game		Player 2		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Now, player 1's strategy 1 dominates their strategy 2.

Example 1 – dominated strategies 5/6

Consider the following scenario.

Abstract game		Player 2		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Now, player 2's strategy 1 dominates their strategy 2.

Example 1 – dominated strategies 6/6

Consider the following scenario.

Abstract game		Player 2		
	Strategy	**1**	**2**	**3**
Player 1	**1**	1	2	4
	2	1	0	5
	3	0	1	-1

Question: Which strategy should each player select?

Each player should pick their own strategy 1. Unfortunately to player 2, this will always lead to a win for player 1.

Game values and fair games

Definition: The payoff to player 1 when both players play optimally is called the *value of the game.*

Definition: If the value of the game is zero, the game is called *fair*.

Example 2

Consider the following scenario.

Abstract game		Player 2		
	Strategy	**1**	**2**	**3**
Player 1	**1**	-3	-2	6
	2	2	0	2
	3	5	-2	-4

Question: Which strategy should each player select?

This game has no dominated strategies.

Example 2 – minimax and maximin 1/2

Consider the following scenario.

Abstract game		Player 2			Min
	Strategy	**1**	**2**	**3**	
Player 1	**1**	-3	-2	6	-3
	2	2	0	2	**0**
	3	5	-2	-4	-4
Max		5	**0**	6	

Question: Which strategy should each player select?

Idea: Each player minimises their maximum losses. Player 1 determines their *maximin* payoff, whereas player 2 determines their *minimax* loss.

Example 2 - minimax and maximin 2/2

Consider the following scenario.

Abstract game		**Player 2**			**Min**
	Strategy	**1**	**2**	**3**	
Player 1	**1**	-3	-2	6	-3
	2	2	0	2	**0**
	3	5	-2	-4	-4
Max		5	**0**	6	

Question: Which strategy should each player select?

In this case, if both player pick strategy two, neither improves upon their best guarantee and both force the opponent into the same position.

Example 2 – stable solution

Consider the following scenario.

Abstract game		Player 2			Min
	Strategy	**1**	**2**	**3**	
Player 1	**1**	-3	-2	6	-3
	2	2	0	2	**0**
	3	5	-2	-4	-4
Max		5	**0**	6	

Question: Which strategy should each player select?

In this case, if both player pick strategy two, neither improves upon their best guarantee (not even a-posteriori) and both force the opponent into the same position. This is a *stable solution* (aka *equilibrium solution*).

Example 2 – saddle point

Consider the following scenario.

Abstract game		Player 2			Min
	Strategy	**1**	**2**	**3**	
Player 1	**1**	-3	-2	6	-3
	2	2	0	2	**0**
	3	5	-2	-4	-4
Max		5	**0**	6	

Question: Which strategy should each player select?

Note: The maximin and the minimax values are the same, because this entry is both the minimum in its rows and the maximum in its columns. This is called a *saddle point*.

Summary and self-study

Summary: today we have considered two-person zero-sum games and learnt

- the basic principles of game theory,
- pay-off matrix reduction by dominated strategies,
- what minimax and maximin strategies are
- and that saddle-points characterise stable solution.

Self-study: apply reduction by dominance to the following two-person zero-sum game and then, if necessary, apply the minimax/maximin criterion to determine optimal strategies. Is the solution stable?

Abstract game		Player 2			
	Strategy	**1**	**2**	**3**	**4**
Player 1	**1**	2	-2	-5	1
	2	4	-2	-3	-3
	3	0	-1	2	3
	4	3	-3	-3	-4