Previous lectures:

Real numbers $\mathbb{R}$ satisfy the field axioms
and the ordered field axioms

What about the rational numbers $\mathbb{Q}$

$\mathbb{Q}$ is also an ordered field

$\mathbb{R}$ is complete, but $\mathbb{Q}$ is not.

satisfies the Least Upper Bound axiom

Bound of a set $S \subseteq \mathbb{R}$
not empty $S \neq \emptyset$

Def An upper bound for S is any M
which no elements of S are
larger than.

$$\forall x \in S, \quad x \leq M$$

Similar definition of lower bound
m for S : $\forall x \in S, \quad x \geq m$

Example

closed interval $[0,1] \subseteq \mathbb{R}$ Upper bound $M = 1000$

S $\quad \forall\, 0 \le x \le 1,\ x \le 1000$

So we could find "better" upper bound, $M = 2$, or "best" $M = 1$

A <u>Least</u> upper bound for S is an upper bound which no other upper bounds are smaller than.

<u>M is a L.U.B. for S if</u>

U.B. ① $x \le M \quad \forall x \in S$

L. ② if $x \le \overline{M} \quad \forall x \in S$

then $M \le \overline{M}$

Similar definition of greatest lower bound

$m = GLB(S)$ if

① $x \ge m \quad \forall x \in S$

② if $x \ge \overline{m} \quad \forall x \in S$

then $\overline{m} \le m$

Examples

open interval $(0, \infty)$ Not bounded above
($\nexists$ any upper bound)

Bounded below:

$-2, -1, -\frac{1}{2}, 0$ lower bounds

$\uparrow$ GLB

half-open $[1, \sqrt{2}) = \{x \in \mathbb{R} : 1 \leq x < \sqrt{2}\}$

<u>Bounded</u> (above & below) $= \{x \in \mathbb{R} : 1 \leq x^2 < 2\}$
$x > 0$

Least Upper Bound Axiom

Any non-empty set of real numbers

$S \subseteq \mathbb{R}$, $S \neq \emptyset$

$S \subseteq \mathbb{R}$, $S \neq \emptyset$

Then if S is bounded above
it has a least upper bound

$\mathbb{R}$ satisfies this axiom
Any bounded-above nonempty $S \subseteq \mathbb{R}$
has a LUB

$\{x \in \mathbb{R} : x > 0, x^2 < 2\}$

GLB $= 0$, LUB $= \sqrt{2}$

$\sqrt{2}$ is not a rational number

$\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$

has a GLB $0 \in \mathbb{Q}$

but has <u>no</u> LUB $m \in \mathbb{Q}$

(LUB $\sqrt{2} \in \mathbb{R}$)