Lecture 14: Partial Derivatives.

MA2032 Vector Calculus

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Partial Derivatives

• When we **hold all** but one of the independent variables of a function **constant** and **differentiate with respect to that one** variable, we get a **"partial" derivative**.

• Today, I am going to show how partial derivatives are **defined and interpreted geometrically**, and how to calculate them by applying the rules for differentiating functions of a single variable.

• The idea of **differentiability** for functions of several variables requires more than the existence of the partial derivatives because a point can be **approached from many different directions**.

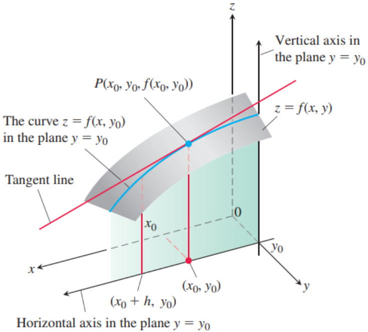
Partial Derivatives of a Function of Two Variables

• If (x_0, y_0) is a **point in the domain of a function** $f(x, y)$, the **vertical plane** $y = y_0$ will **cut the surface** $z = f(x, y)$ in the curve $z = f(x, y_0)$.

• This **curve is the graph of the function** $z = f(x, y_0)$ in the plane $y = y_0$.

The horizontal coordinate in this plane is x; the vertical coordinate is z.

• The **y-value is held constant at** y_0, so y is not a variable.



• We define the **partial derivative of f with respect to x** at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

• In the definition, h represents a real number, positive or negative.

Partial Derivatives of a Function of Two Variables

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

• The **slope of the curve** $z = f(x, y_0)$ at the point $\mathrm{P}(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the **value of the partial derivative** of f with respect to x at (x_0, y_0).

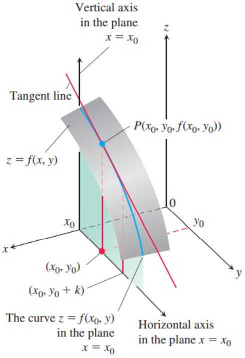
• The **tangent line to the curve** at P is the line in the plane $y = y_0$ that passes through P with this slope.

• The partial derivative $\frac{\partial f}{\partial x}$ at (x_0, y_0) gives the **rate of change of f with respect to x** when y is held fixed at the value y_0.

DEFINITION The **partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0)** is

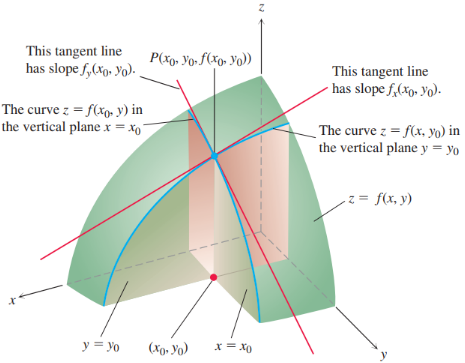
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.



- The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x.

Partial Derivatives of a Function of Two Variables



• Notice that we now have **two tangent lines** associated with the surface $z = f(x, y)$ at the point $\mathrm{P}(x_0, y_0, f(x_0, y_0))$.

• Is the plane **they determine tangent to the surface at P**?

• We will see that **it is for the differentiable functions** and we will learn how to find the tangent plane.

First we have to better understand partial derivatives.

Calculations

The definitions of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ give us **two different ways of differentiating f at a point**:

1) **with respect to x** in the usual way while treating y as a constant and

2) **with respect to y** in the usual way while treating x as a constant.

Example 1

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$.

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

Example 2

Find $\frac{\partial f}{\partial y}$ as a function if $f(x,y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y) \\ &= (y \cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$

Example 3

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as a function if $f(x,y) = \frac{2y}{y+\cos x}$.

Solution We treat f as a quotient. With y held constant, we use the quotient rule to get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}.$$

With x held constant and again applying the quotient rule, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}.$$

Calculations

• **Implicit differentiation** works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates

Example 4

Find $\frac{\partial z}{\partial x}$ assuming that the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

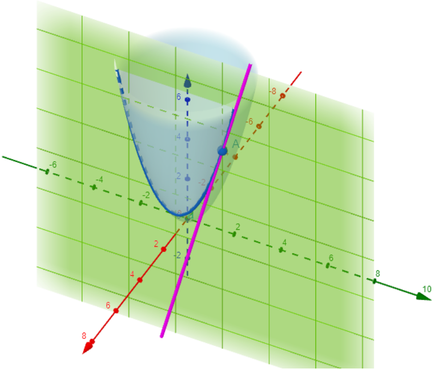
$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \qquad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}.$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

Example 5

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.



Calculations

Solution The parabola lies in a plane parallel to the yz-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at $(1, 2)$:

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

■

• The definitions of the partial derivatives of functions of more than two independent variables are **similar to the definitions for functions of two variables**.

• They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example 6

If x, y, and z are independent variables and $f(x, y, z) = x \sin(y + 3z)$, find $\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x \sin(y + 3z)\right] = x \frac{\partial}{\partial z} \sin(y + 3z) \qquad x \text{ held constant}$$

$$= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) \qquad \text{Chain rule}$$

$$= 3x \cos(y + 3z). \qquad y \text{ held constant}$$

• When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives.

• These derivatives are usually denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$

Example 7

If $f(x, y) = x \cos y + ye^x$, find all second-order derivatives.

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

The Mixed Derivative Theorem

THEOREM 2—The Mixed Derivative Theorem
If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example 8

Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$.

Solution The symbol $\partial^2 w/\partial x\, \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y\, \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x\, \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0).

Partial Derivatives of Still Higher Order

• Although we will **deal mostly with first- and second-order partial derivatives**, because these appear the most frequently in applications

• There is **no theoretical limit** to how many times we can differentiate a function as long as the **derivatives involved exist**.

• Thus, we get third- and fourth-order derivatives denoted by symbols like $\frac{\partial^3 f}{\partial x \partial y^2} = f_{xyy}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{xxyy}$ and so on.

• As with second-order derivatives, the **order of differentiation is immaterial** as long as all the derivatives through the order in question are **continuous**.

Partial Derivatives of Still Higher Order

Example 9

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution: We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

Partial Derivatives and Continuity. Differentiability

• A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point **without the function being continuous there**.

• This is **different from functions of a single variable**, where the existence of a derivative implies continuity.

• The **concept of differentiability** for functions of several variables is **more complicated** than for single-variable functions because a point in the domain can be approached **along more than one path**.

• For the existence of differentiability, a **property is needed** to ensure that **no abrupt change occurs** in the function resulting from small changes in the independent variables **along any path** approaching (x_0, y_0).

• If the partial derivatives of $f(x, y)$ exist and are continuous throughout a **disk centered at** (x_0, y_0), however, then f is continuous at (x_0, y_0).

Differentiability

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

How to set the function PATH to a point?

• Two-Path Test for **Nonexistence of a Limit**.

• There is **infinite number of different path** that led to a point $(x_0, y_0) \in \mathbb{R}^2$

Steps to choose a Path:

• First, Try two **basic paths** $x = 0$ and $y = 0$.

• If it doesn't work, chose others $x = g(y)$ or $y = h(x)$:

a) **Be certain** that point (x_0, y_0) is actually on your path,

b) Try to substitute so **degrees** of the numerator and denominator **are equal**.

How to set the function PATH to a point?

$$\lim_{(x,y)\to(1,0)} \frac{xe^y-1}{xe^y-1+y} \qquad \frac{1\cdot e^0-1}{1\cdot e^0-1+0}=\frac{0}{0} \text{ ind.}$$

- $(1,0) \notin x=0$
- Path : $y=0$ $\quad \lim_{x\to 1}\frac{x-1}{x-1}=1$
- Path : $x=1$ $\quad \lim_{y\to 0}\frac{e^y-1}{e^y-1+y} \overset{\frac{0}{0}\text{ ind}}{=} \lim_{y\to 0}\frac{e^y}{e^y+1}=\frac{1}{2}$

L'Hopital's Rule

or

- Path : $y=\ln x$ $\quad \lim_{x\to 1}\frac{x\cdot e^{\ln x}-1}{x\cdot e^{\ln x}-1+y} = \lim_{x\to 1}\frac{x^2-1}{x^2-1+\ln x}$

$0=\ln 1$

$0=0$

$$\overset{\frac{0}{0}\text{ ind}}{=} \lim_{x\to 1}\frac{2x}{2x+\frac{1}{x}}=\frac{2}{3}$$

L'H Rule