Lecture 20: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 4, 2022

• The use of **Taylor's formula** leads to an extension of the formula that provides **polynomial approximations** of all orders for functions of two independent variables.

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$\begin{aligned} f(a+h, b+k) = {} & f(a, b) + (hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})\Big|_{(a,b)} \\ & + \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\Big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} \\ & + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}. \end{aligned} \tag{7}$$

• The **first n derivative terms** are evaluated at (a, b). The **last term** is evaluated at some point $(a + ch, b + ck)$ on the line segment joining (a, b) and $(a + h, b + k)$.

Taylor's Formula for Two Variables

• If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Equation (7) assumes the following form.

Taylor's Formula for $f(x, y)$ at the Origin

$$\begin{aligned} f(x,y) &= f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) \\ &+ \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\ &+ \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)} \end{aligned} \tag{8}$$

• The first n derivative terms are evaluated at (0, 0). The last term is evaluated at a point on the line segment joining the origin and (x, y).

• Taylor's formula provides **polynomial approximations** of two-variable functions. The first n derivative terms give the polynomial; the **last term** gives the **approximation error**.

• The first three terms of Taylor's formula give the **function's linearization**. To improve on the linearization, we add higher-power terms.

Taylor's Formula for Two Variables

Example 1

Find a quadratic approximation to $f(x,y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

Solution We take $n = 2$ in Equation (8):

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) + \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\Big|_{(cx, cy)}.$$

Calculating the values of the partial derivatives,

$$f(0,0) = \sin x \sin y\Big|_{(0,0)} = 0, \qquad f_{xx}(0,0) = -\sin x \sin y\Big|_{(0,0)} = 0,$$

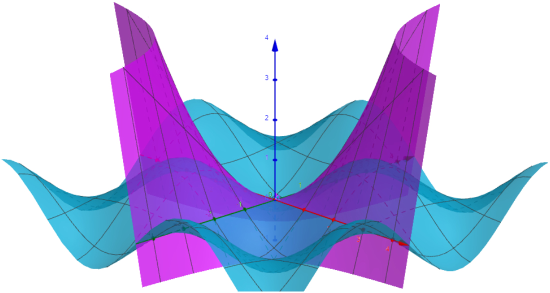
$$f_x(0,0) = \cos x \sin y\Big|_{(0,0)} = 0, \qquad f_{xy}(0,0) = \cos x \cos y\Big|_{(0,0)} = 1,$$

$$f_y(0,0) = \sin x \cos y\Big|_{(0,0)} = 0, \qquad f_{yy}(0,0) = -\sin x \sin y\Big|_{(0,0)} = 0,$$

we have the result

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)), \quad \text{or} \quad \sin x \sin y \approx xy.$$

Example 1



The error in the approximation is

$$E(x, y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,\, cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \le 0.1$ and $|y| \le 0.1$. Hence

$$|E(x, y)| \le \frac{1}{6}\left((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3\right) = \frac{8}{6}(0.1)^3 \le 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \le 0.1$ and $|y| \le 0.1$. ■

Taylor's Formula for Two Variables

Example 2

Use Taylor's formula for $f(x, y) = \ln(2x + y + 1)$ at the origin to find quadratic and cubic approximations of f near the origin.

$$f(x,y) = \ln(2x+y+1) \Rightarrow f_x = \frac{2}{2x+y+1},\ f_y = \frac{1}{2x+y+1},\ f_{xx} = \frac{-4}{(2x+y+1)^2},\ f_{xy} = \frac{-2}{(2x+y+1)^2},\ f_{yy} = \frac{-1}{(2x+y+1)^2}$$

$$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \tfrac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 0 + x\cdot 2 + y\cdot 1 + \tfrac{1}{2}\left[x^2\cdot(-4) + 2xy\cdot(-2) + y^2\cdot(-1)\right] = 2x + y + \tfrac{1}{2}\left(-4x^2 - 4xy - y^2\right)$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2,\ \text{quadratic approximation;}$$

$$f_{xxx} = \frac{16}{(2x+y+1)^3},\ f_{xxy} = \frac{8}{(2x+y+1)^3},\ f_{xyy} = \frac{4}{(2x+y+1)^3},\ f_{yyy} = \frac{2}{(2x+y+1)^3}$$

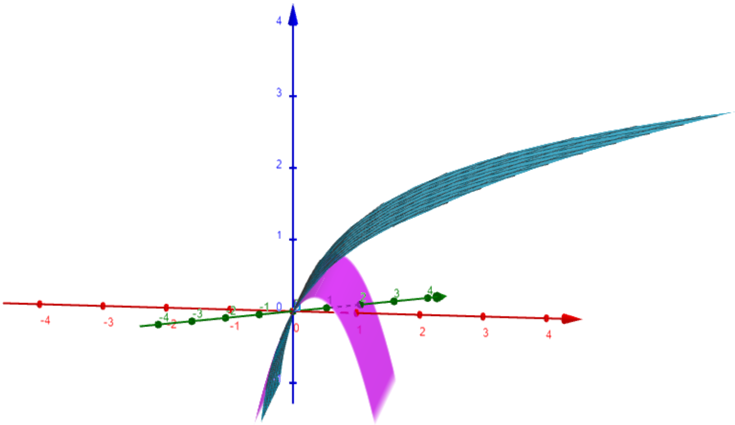
$$\Rightarrow f(x,y) \approx \text{quadratic} + \tfrac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 + \tfrac{1}{6}\left(x^3\cdot 16 + 3x^2 y\cdot 8 + 3xy^2\cdot 4 + y^3\cdot 2\right)$$

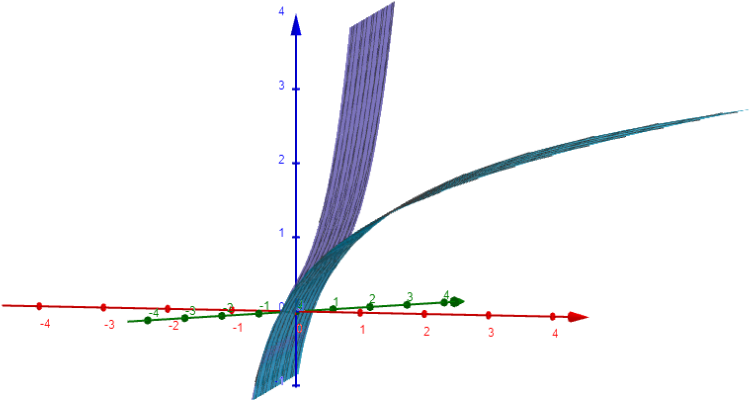
$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 - \tfrac{1}{3}\left(8x^3 + 12x^2 y + 6xy^2 + y^2\right)$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 + \tfrac{1}{3}(2x+y)^3,\ \text{cubic approximation}$$

Example 2



Example 2



Example 3

Use Taylor's formula for $f(x,y) = \frac{1}{1-x-y+xy}$ at the origin to find quadratic and cubic approximations of f near the origin.

$$f(x,y)=\frac{1}{1-x-y+xy} \Rightarrow f_x = \frac{1-y}{(1-x-y+xy)^2},\ f_y = \frac{1-x}{(1-x-y+xy)^2},\ f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3},\ f_{xy} = \frac{1}{(1-x-y+xy)^2},$$

$$f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3} \Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 1\cdot x\cdot 1 + y\cdot 1 + \frac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 1 + y^2\cdot 2\right) = 1 + x + y + x^2 + xy + y^2,\ \text{quadratic approximation;}$$

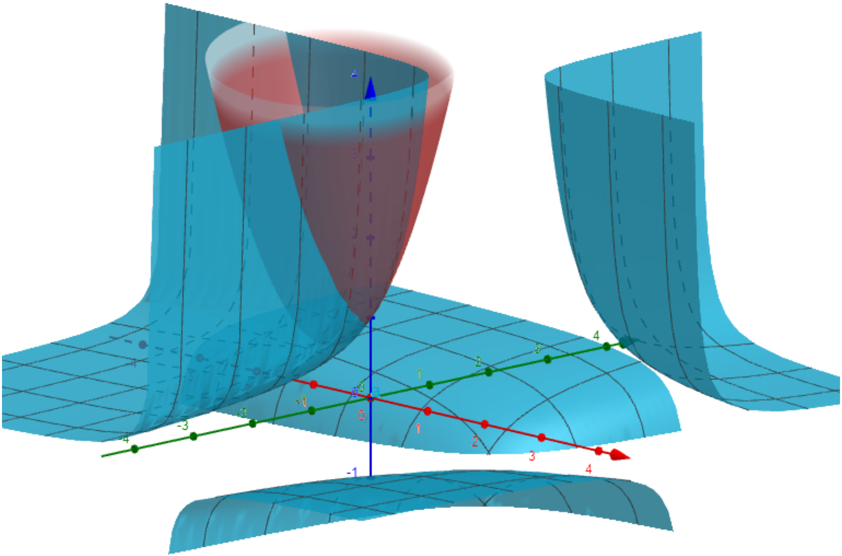
$$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4},\ f_{xxy} = \frac{[-4(1-x-y+xy)+6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4},\ f_{xyy} = \frac{[-4(1-x-y+xy)+6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4},$$

$$f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4} \Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

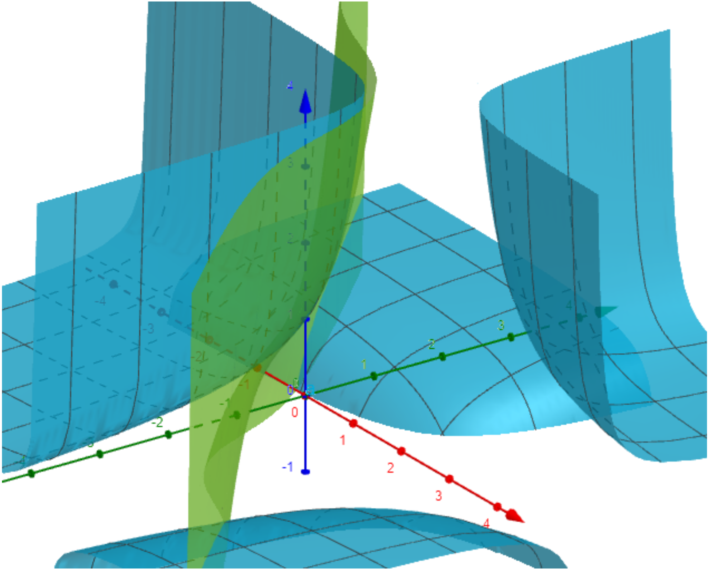
$$= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6}\left[x^3\cdot 6 + 3x^2 y\cdot 2 + 3xy^2\cdot 2 + y^3\cdot 6\right]$$

$$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + xy^2 + y^3,\ \text{cubic approximation}$$

Example 3



Example 3



Example 4

Use Taylor's formula to find a quadratic approximation of $f(x,y) = e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

$$f(x,y) = e^x \sin y \Rightarrow f_x = e^x \sin y,\ f_y = e^x \cos y,\ f_{xx} = e^x \sin y,\ f_{xy} = e^x \cos y,\ f_{yy} = -e^x \sin y$$

$$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \tfrac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 0 + x\cdot 0 + y\cdot 1 + \tfrac{1}{2}\left(x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right) = y + xy, \text{ quadratic approximation.}$$

Now, $f_{xxx} = e^x \sin y$, $f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, and $f_{yyy} = -e^x \cos y$.

Since $|x| \le 0.1$, $|e^x \sin y| \le e^{0.1} \sin 0.1| \approx 0.11$ and $|e^x \cos y| \le |e^{0.1} \cos 0.1| \approx 1.11$.

Therefore, $E(x,y) \le \frac{1}{6}\left[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3\right] \le 0.000814.$

Example 4