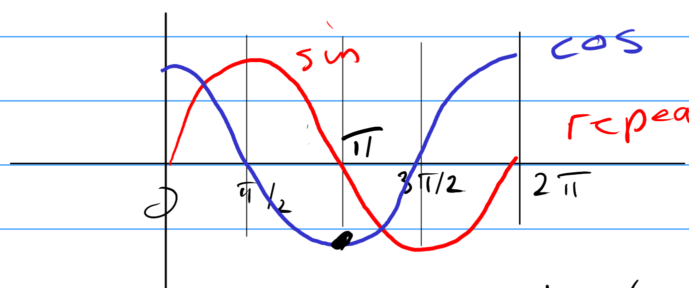
Calculus and Analysis 6/12/21

Taylor polynomial for a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous and differentiable n times in some open interval around $x = c$

$$P_n(x) = f(c) + \frac{f'(c)}{1}(x-c) + \frac{f''(c)}{2}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

if you know the values of $f, f', f'', f''', \dots, f^{(n)}$ at one point $x = c$, then $P_n(x)$ is a polynomial of degree n which is an approximation to $f(x)$ near $x = c$.

Example $f(x) = \sin(x)$ around $x = \pi$



$n=$	1	2	3	4
$f^{(n)}(x)$	$\cos(x)$	$-\sin(x)$	$-\cos(x)$	$f(x)$
$f^{(n)}(\pi)$	-1	0	1	0

$$P_{10}(x) = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \frac{1}{9!}(x-\pi)^9$$

Theorem If f continuous and differentiable $n+1$ times in an open interval around $x=c$ then for all x in this interval

$$f(x) = P_n(x) + f^{(n+1)}(\xi)\frac{(x-c)^{n+1}}{(n+1)!}$$

for some ξ between x and c

Examples for small values of n

$\underline{n=0}$ $f(x) = f(c) + f'(\xi)(x-c)$

for some ξ between x and c

This is just the Mean Value Theorem

$\exists\ \xi$ between x & c with $\dfrac{f(x)-f(c)}{x-c} = f'(\xi)$

$\underline{n=1}$

for some ξ between x & c

$$f(x) = \underbrace{f(c) + f'(c)(x-c)}_{P_1(x)} + \underbrace{f''(\xi)\frac{(x-c)^2}{2}}_{\text{Error term}}$$

$$= \underline{f'(c).x + f(c) - c.f'(c)} + E$$

Numerical approximation to solving $f(x)=0$ by Newton's Method

If the error is small

suppose $x = \alpha$ is a solution

& $c = x_n$ an approximation

$$0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + E$$

$$0 \simeq f(x_n) + f'(x_n)(\alpha - x_n)$$

$$\alpha \simeq x_n - \frac{f(x_n)}{f'(x_n)}$$

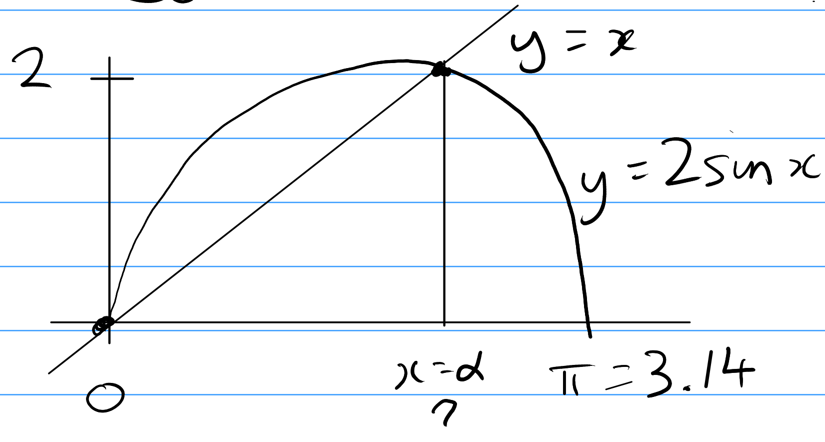
So we define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

& this, we assume, is a better approximation to the solution

<u>Example</u> $f = x - 2\sin(x)$ $\underline{f(x) = 0?}$

Solve $2\sin(x) = x$ with $x > 0$



Intermediate value theorem

$$f(1) = 1 - 2\sin(1) \simeq -0.7$$
$$f(2) = 2 - 2\sin(2) \simeq 0.2$$

$$1 < x < 2$$

Newton's method starting with $x_0 = 1.5$

$f'(x) = 1 - 2\cos(x)$ $\quad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_0 = 1.5$
$x_1 = 2.0766$
$x_2 = 1.9105$
$x_3 = 1.8956$
$x_4 = 1.8955$

$x_n \to \underline{\alpha = 1.895}\ldots$

Choose a different starting point:

$x_0 = 1.1$
$x_1 = 8.453$
$x_2 = 5.256$
$x_3 = 203.4$
$x_4 = 118$
$x_5 = -87$
$\vdots$
$x_{35} = -64000000$
-64 million

$f'(x_0)$ very small

$\dfrac{f(x_0)}{f'(x_0)}$ large

If f is a "smooth" (i.e we know lots of derivatives exist) then f can be well approximated around $x = c$ just by knowing $\dfrac{d^k f}{dx^k}$ at $x = c$ for $k = 0, 1, 2, 3, \ldots$

Even if f is not differentiable at all, continuity can still help

Bisection method : if f continuous on $[a_0, b_0]$ and $f(a_0), f(b_0)$ have <u>different signs</u>

we can find a very good approximation $x \in [a_0, b_0]$ to $f(x) = 0$

I.V.T. says $\exists$ solution between a_0 & b_0

Replace $[a_0, b_0]$ by an interval of half the length $[a_0, m_0]$ or $[m_0, b_0]$ where $m_0 = \frac{1}{2}(a_0 + b_0)$ midpoint

$[a_1, b_1]$ $= [a_0, m_0]$ if $f(a_0), f(m_0)$ have different signs

$= [m_0, b_0]$ if $f(m_0), f(b_0)$ have different signs.

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then repeat : choose

$$[a_n, b_n] = \text{either } [a_{n-1}, m_{n-1}] \text{ or } = [m_{n-1}, b_{n-1}] \quad \forall n$$

$$\& \quad b_n - a_n = \frac{1}{2^n}(b_0 - a_0) \to 0$$

α solution $\in [a_n, b_n]$

If $c=0$, Taylor around $x=0$ is called Maclaurin polynomial, Maclaurin's Theorem

Example Maclaurin Series for sin.

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

because $y=\sin(x)$

$$\frac{d^n y}{dx^n} \text{ at } x=0 = \begin{cases} 0 & n=0,4,8 \\ 1 & n=1,5,9, \\ 0 & n=2,6,10, \\ -1 & n=3,7,11,\dots \end{cases}$$

Next semester: series $\boxed{\sum_{n=0}^{d} c_n x^n} = P(x)$

polynomials of degree $\longrightarrow \infty$