Lecture notes for Calculus and Analysis

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Chapter 1

Precalculus

Let us start by reviewing some terminology, notation and formulas of elementary mathematics.

1.1 Sets

A *set* S is a collection of distinct objects, called *elements* of S.

Notations:

$x \in S$	:	x is element of the set S
$x \notin S$	:	x is not element of the set S
$\{x : P\}$	:	the set of all objects x satisfying the condition P.

Relations and operations:

$A \subset B$	:	A is a *subset* of B, that is, for all $x \in A$ we have also $x \in B$. We write for short: $\forall x \in A : x \in B$
$A \cup B$	$=$	$\{x : x \in A \text{ or } x \in B\}$ is the *union* of the sets A and B.
$A \cap B$	$=$	$\{x : x \in A \text{ and } x \in B\}$ is the *intersection* of the sets A and B.
$A \setminus B$	$=$	$\{x : x \in A \text{ but } x \notin B\}$ is the *difference* of the sets A and B.

For the phrase "there exists" we also write for short $\exists$.

1.2 Number systems

Natural numbers The set of all positive integers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

Some mathematicians and textbook writers denote this by $\mathbb{N}_0$, as by default they *do not include* 0 in the set of natural numbers $\mathbb{N}$.

Integers

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$$

Rational numbers

$$\mathbb{Q} = \{x : x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$$

Irrational numbers Numbers which cannot represented as a quotient of integers, e.g. $\sqrt{2}, \pi, \ldots$.

Real numbers

$$\mathbb{R} = \{x : x \text{ is rational or irrational}\}$$

Real numbers can be represented as decimals. e.g. $x = \frac{3}{5}$ is represented $x = 0.6$, and $x = \frac{1}{3}$ as $x = 0.333\ldots$. For irrational numbers this process never terminates or repeats,

$$\text{e.g. } \sqrt{2} = 1.41421356237\ldots.$$

1.3 Ordered fields

The real numbers, and the rational numbers, are example of special sets which are called *fields*. This just means that the real numbers, and also the rational numbers, have operations (addition $x + y$ and multiplication $x \cdot y$) satisfying the field axioms.

Axiom 1.1 (Field axioms)**.** *A field is a set F together with addition and multiplication operations, and elements $0 \in F$ and $1 \in F - \{0\}$, satisfying the field axioms:*

Name:	**additive**	**multiplicative**
Identity	$\forall x \in F,\ x + 0 = x$	$\forall x \in F,\ x \cdot 1 = x$
Inverse	$\forall x \in F,\ \exists(-x),\ x + (-x) = 0$	$\forall x \in F - \{0\},\ \exists x^{-1},\ x \cdot x^{-1} = 1$
Associativity	$\forall x, y, z \in F,\ x + (y + z) = (x + y) + z$	$\forall x, y, z \in F,\ x(yz) = (xy)z$
Commutativity	$\forall x, y \in F,\ x + y = y + x$	$\forall x, y \in F,\ xy = yx$
Distributivity	$\forall x, y, z \in F,\ x(y + z) = xy + xz$	

Associativity means we can sometimes leave out some brackets. Also, we usually abbreviate $x + (-y)$ to $x - y$ and abbreviate $(-x) + y$ to $-x + y$, but the latter is not the same as $-(x + y)$.

From the field axioms we can derive (that is, prove) a lot of basic properties, such as $0x = 0$, and $(-1)x = -x$, and also $xy = 0$ implies $x = 0$ or $y = 0$.

Geometrically, we represent real numbers as points on the real line. In particular, we can give an *order* on the real numbers.

Axiom 1.2 (Order axioms)**.** *The real numbers $\mathbb{R}$ are an ordered field, that is, the field $\mathbb{R}$ has a relation $<$ so that for all $a, b, c \in \mathbb{R}$:*

Total order *Either $a < b$, $b < a$ or $a = b$.*

Transitivity *If $a < b$ and $b < c$ then $a < c$.*

Compatibility *If $a < b$ then $a + c < b + c$. If $a < b$ and $c > 0$ then $ac < bc$.*

Obviously, we define "$a > b$" by saying that this means that "$b < a$", and "$a \leq b$" by saying that this means that either "$a < b$" or "$a = b$".

Remark 1.3. *The real numbers and the rational numbers are an ordered field, but the complex numbers are not.*

Let us formulate some conclusions from these axioms.

Corollary 1.4. *(i)* $x, y > 0 \implies xy > 0$

(ii) $-x < 0 \iff x > 0$

(here "$\iff$" means "if and only if", that is, $-x < 0 \implies x > 0$ and $x > 0 \implies -x < 0$).

(iii) $x \neq 0 \implies x^2 > 0$.

(iv) $1 > 0$.

(v) $0 < x < y \implies 0 < y^{-1} < x^{-1}$.

1.3.1 Intervals and absolute value

Using the order relation on the real numbers we can define *intervals* and the *absolute value*.

We denote by

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

the *open interval* from a to b, by

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

the *closed interval* from a to b, and by

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\}\,.$$

the half–open intervals. Moreover, we write

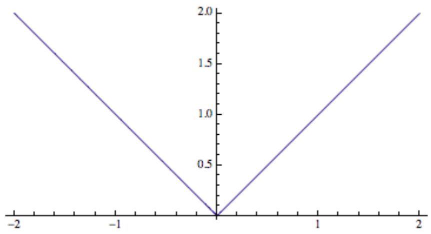
$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}, \qquad (a, \infty) = \{x \in \mathbb{R} : x > a\}\,.$$

An open interval does not contain the *endpoints* whereas the endpoints a, b are elements of the closed interval $[a, b]$. We call a point c an *interior point* of an interval, if it is not an endpoint.

The *absolute value* is an important function related to the order on $\mathbb{R}$,

Definition 1.5. The *absolute value* of $a \in \mathbb{R}$ is defined by

$$|a| = \max\{a, -a\} = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$



The absolute value measures the distance of a to 0, therefore $|a-b|$ is the distance between a and b.

Properties: For all $a, b \in \mathbb{R}$:

(i) $|a| = 0 \iff a = 0$

(ii) $|-a| = |a|$

(iii) $|ab| = |a| \cdot |b|$, $|a^2| = |a|^2$

Also we can prove that

$$|a + b| \leq |a| + |b| \qquad (\textit{triangle inequality})$$

and

$$|a - b| \geq ||a| - |b||.$$

The following, which we will meet a lot later, illustrates two definitions of two sets of real numbers, either with the use of interval notation or with the absolute value function:

$$\{x \in \mathbb{R} \, : \, |x - c| < \delta\} \quad = \quad (c - \delta, c + \delta)$$

$$\{x \in \mathbb{R} \, : \, 0 < |x - c| < \delta\} \quad = \quad (c - \delta, c + \delta) - \{c\} \quad = \quad (c - \delta, c) \cup (c, c + \delta)$$

1.4 Least Upper Bound Axiom

Intuitively, the *completeness axiom*, often called the *LUB axiom*, implies that there are no "gaps" in the real numbers. This contrasts with the rational numbers, whose corresponding number line has a "gap" at each irrational value.

Definition 1.6. A non–empty subset $S \subset \mathbb{R}$ of the real numbers is said to be

(i) *bounded above* if there exists $M \in \mathbb{R}$ so that $x \leq M$ for all $x \in S$; in this case M is called an *upper bound* of S;

(ii) *bounded below* if there exists $m \in \mathbb{R}$ so that $m \leq x$ for all $x \in S$; in this case m is called a *lower bound* of S;

(iii) *bounded* if S is both bounded above and below.

Note that if M is an upper bound for S then every $\tilde{M} \in [M, \infty]$ is also an upper bound for S.

So is there a best possible upper bound and how do we find it? This is easy, if we know that the upper bound M is an element of the set S; then we call it a *maximum*, denoted by $M = \max S$. Similarly, a lower bound m of S with $m \in S$ is called the *minimum* of S, denoted by $m = \min S$.

Consider the set $S = \{x \mid x = \frac{1}{n}, n \in \mathbb{N}\}$. We know that S is bounded below by 0 since $\frac{1}{n} > 0$ for all n. But is there a bigger lower bound? Assume that $m > 0$ is a lower bound of S then $\frac{1}{n} \geq m$ for all n. However, consider the number $\frac{1}{m} < \infty$. Then there exists $K \in \mathbb{N}$ which is bigger than $\frac{1}{m}$. But then $\frac{1}{K} < m$. Since $\frac{1}{K} \in S$ this gives a contradiction to m being a lower bound. Thus, 0 is the best lower bound, we call it the greatest lower bound.

Definition 1.7. Let $S \subset \mathbb{R}$ be non–empty. Then

(i) $M \in \mathbb{R}$ is called the *least upper bound* (or, also frequently used, *supremum*) of S, written as $\text{lub}(S)$ (or $\sup(S)$), if

  (a) M is an upper bound of S, and

  (b) if $\tilde{M}$ is an upper bound of S then $M \leq \tilde{M}$.

(ii) $m \in \mathbb{R}$ is called the *greatest lower bound* (or, also frequently used, *infimum*) of S, written as $\text{glb}(S)$ (or $\inf(S)$), if

  (a) m is a lower bound of S, and

  (b) if $\tilde{m}$ is a lower bound of S then $\tilde{m} \leq m$.

Example 1.8. The set $S = (0, \sqrt{2})$ does not have a least upper bound in $\mathbb{Q}$. However, it has a least upper bound in $\mathbb{R}$: $\sqrt{2} \in \mathbb{R}$.

And here is the property of the real numbers that guarantees that there are no gaps in the real line:

Axiom 1.9 (LUB–axiom)**.** *Every non–empty subset of the real numbers which has an upper bound has a least upper bound.*

Remark 1.10. *The least upper bound axioms together with the field axioms and the order axioms make the set of real numbers essentially unique; every other set satisfying these axioms is isomorphic to* $\mathbb{R}$*.*

Here is a property of the least upper bound we will use frequently:

Theorem 1.11. *Let* $S \subset \mathbb{R}$*. If* $M = \operatorname{lub} S$ *and* $\varepsilon > 0$ *then there exists* $s \in S$ *with*

$$M - \varepsilon < s \leq M\,.$$

Similar statements as for the least upper bound also hold for the greatest lower bound.

1.5 Elementary functions

Here is a quick introduction to functions, just a reminder to help set out the notation you should use in Calculus and Analysis.

Let $D \subset R$. Then f is a function on D if f assigns to each $x \in D$ a value $f(x) \in \mathbb{R}$. We often write

$$f : D \to \mathbb{R}, \quad x \mapsto f(x)\,.$$

We call $D = \operatorname{dom}(f)$ the *domain* of f and

$$\operatorname{range}(f) = \{f(x) : x \in D\}$$

the *range* of f, that is, all values f attains on D. We also write for a set $S \subset D$:

$$f(S) = \{y : y = f(x) \text{ for some } x \in S\}\,.$$

Thus, $\operatorname{range}(f) = f(\operatorname{dom}(f))$.

The most common domains we will look at are intervals: either a closed interval like $[a, b]$ or an open interval like (a, b) or (a, ∞) or $\mathbb{R} = (-\infty, \infty)$.

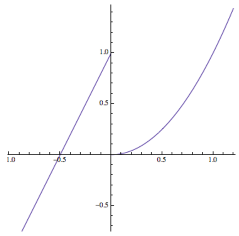
If not noted otherwise, we will usually use the biggest possible domain for the function, e.g, for

$$f(x) = \frac{x+1}{x-1}$$

we use $\operatorname{dom}(f) = \mathbb{R} \setminus \{1\}$. For a lot of applications it is important to keep track of the domain; the behaviour of a function might change on a bigger or smaller domain.

A function can also be defined piecewise:

$$f(x) = \begin{cases} 2x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases}.$$



A function is called *even* if $f(-x) = f(x)$ for all $x \in \mathrm{dom}(f)$. (Symmetric about the y–axis). A function is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathrm{dom}(f)$. (Symmetric about origin).

Here are some functions you probably are already familiar with:

Polynomials: A function

$$P(x) = a_n x^n + \ldots + a_1 x + a_0, \quad x \in \mathbb{R},$$

is called a *polynomial of degree* n with coefficients $a_0, \ldots, a_n \in \mathbb{R}$, $a_n \neq 0$. If $P(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $x = x_0$ is called a *zero* or *root* of $P(x)$, or a solution of $P(x) = 0$, and we can factorise

$$P(x) = (x - x_0)Q(x)\,,$$

where Q is a polynomial of degree $n-1$. A polynomial of degree n has at most n zeros. A polynomial of odd degree has at least one zero.

Rational Functions: A *rational function* is given as a quotient of two polynomials P, Q,

$$R(x) = \frac{P(x)}{Q(x)}\,.$$

The domain of R is given by $\mathrm{dom}(R) = \{x : Q(x) \neq 0\}$.

Trigonometric Functions: The sine function is an odd function, the cosine is even, and the tangent $\tan(x) = \frac{\sin x}{\cos x}$ is odd. The domain of tan is $\mathrm{dom}(\tan) = \{x : \cos x \neq 0\} = \mathbb{R} \setminus \{\pm\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. The range of tan is $\mathrm{range}(\tan) = \mathbb{R}$.

From the Pythagorean theorem we have

$$\sin^2(x) + \cos^2(x) = 1 \quad \forall x \in \mathbb{R}\,.$$

We also have

$$\sin(x+y) = \sin x \cos y + \sin y \cos x, \quad \cos(x+y) = \cos x \cos y - \sin x \sin y\,.$$

To construct new functions out of given functions f, g on a common domain D we have the algebraic operations:

$$(f \pm g)(x) = f(x) \pm g(x), \quad (fg)(x) = f(x)g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

where in the last case $\mathrm{dom}\left(\frac{f}{g}\right) = \{x \in D : g(x) \neq 0\}$.

Composition: If $f : D_f \to \mathbb{R}, g : D_g \to \mathbb{R}$ such that range$(g) \subset D_f = \mathrm{dom}(f)$ then we can define the composition $f \circ g$ on $D_g = \mathrm{dom}(g)$ by setting

$$(f \circ g)(x) = f(g(x)), \quad x \in D_g\,.$$

Note: the composition is not commutative, that is, in general $f \circ g$ is not the same as $g \circ f$.

Example 1.12. $f(x) = x + 3, g(x) = x^2$, $x \in \mathbb{R}$. The

$$(f \circ g)(x) = f(x^2) = x^2 + 3$$

whereas

$$(g \circ f)(x) = g(x + 3) = (x + 3)^2\,.$$

Since $(f \circ g)(1) = 4 \neq 16 = (g \circ f)(1)$ we see that $f \circ g \neq g \circ f$.

Remark 1.13. *To prove that two functions f, g on a common domain are equal, you have to verify $f(x) = g(x)$ for* every *$x \in \mathrm{dom}(f) = \mathrm{dom}(g)$. However, to show that two functions are not the same, it is enough to show that $f(x_0) \neq g(x_0)$ for one $x_0 \in \mathrm{dom}(f)$.*

Since this is a common beginner error, let us emphasise: to prove a statement it is not enough to verify it for a special example.

For example, consider the statement "$x^2 = 1$ for all $x \in \mathbb{R}$".

Here is a "proof": for $x = 1$ we have $x^2 = 1^2 = 1$ thus the statement is correct.

Of course, in this simple example we see immediately that this is rubbish!

Never prove a statement by giving an example!

Definition 1.14. A function f is said to be *one–to–one* (or *injective*) if for all $x_1, x_2 \in \mathrm{dom}(f)$:

$$f(x_1) = f(x_2) \implies x_1 = x_2\,.$$

Thus, if f is one–to–one then $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Common errors when showing that f is one–to–one:

(i) To show that $f(x_1) = f(x_2)$ if $x_1 = x_2$ (no marks: after all this is true for any function!)

(ii) To show that $f(x_1) \neq f(x_2)$ for some x_1, x_2. (No marks: recall that it is never a good idea to show a statement by an example!)

Example 1.15. The function $f(x) = \frac{2x+1}{x-1}$, $x \neq 1$, is one-to-one: Let $x_1, x_2 \in \mathbb{R} \setminus \{1\}$. Then

$$\begin{aligned} f(x_1) = f(x_2) &\implies \frac{2x_1+1}{x_1-1} = \frac{2x_2+1}{x_2-1} \\ &\implies (2x_1+1)(x_2-1) = (2x_2+1)(x_1-1) \\ &\implies x_2 - 2x_1 = x_1 - 2x_2 \implies x_1 = x_2 \,. \end{aligned}$$

Thus, f is one-to-one.

Theorem 1.16. *If f is one–to–one, then there exists a unique function $g : \mathrm{range}(f) \to \mathrm{dom}(f)$ such that*

$$f(g(x)) = x$$

for all $x \in \mathrm{range}(f)$.

Definition 1.17. The function g in the previous theorem is called the *inverse function* of f, and is denoted by $g = f^{-1}$.

Remark 1.18. *Note that*

$$f^{-1} \neq \frac{1}{f} \,.$$

For example, $f(x) = x$ has inverse function $f^{-1}(x) = x$ but $\frac{1}{f(x)} = \frac{1}{x}$.

Example 1.19. The function $f(x) = \frac{2x+1}{x-1}$, $x \neq 1$, is one-to-one. Put $y = f^{-1}(x)$. Then $f(y) = x$ can be solved for y:

$$x = \frac{2y+1}{y-1} \implies x(y-1) = 2y+1 \implies y(x-2) = 1+x \implies y = \frac{1+x}{x-2} \,.$$

Thus, $f^{-1}(x) = \frac{1+x}{x-2}$, $x \neq 2$. The range of f is thus $\mathbb{R} \setminus \{2\}$.
To verify our result we compute

$$f(\frac{x+1}{x-2}) = \frac{2\frac{x+1}{x-2}+1}{\frac{x+1}{x-2}-1} = \frac{3x}{3} = x \,,$$

as it should be.

The graph of the inverse function f^{-1} is the graph of f reflected in the line $x = y$. From this geometric picture, it is intuitively clear that the inverse of a function is also one–to–one, and its inverse is the original function.

Lemma 1.20. *Let f be one–to–one and $g = f^{-1}$ its inverse function. Then*

$$g(f(x)) = x$$

for all $x \in \mathrm{dom}(f)$. In particular, f^{-1} is one–to–one and the inverse function of f^{-1} is $(f^{-1})^{-1} = f$.

1.6 Mathematical induction

There are various axioms needed to describe the set of real numbers. For the set of natural numbers, we follow a rather informal, intuitive view; the main axiom we impose is the axiom of induction.

Axiom 1.21 (Axiom of induction (see also Peano axioms))**.** *Let* $S \subset \mathbb{N}$*. If*

(i) $1 \in S$

(ii) $k \in S \implies k+1 \in S$

then $S = \mathbb{N}$

Think about domino theory: to get a chain of dominoes to fall, you have to kick the first one (base case) but it only works if every domino's fall guarantees the next one is falling too (induction step).

Mathematical induction: Let $P(n)$ be a statement depending on a natural number n. We want to show that $P(n)$ holds for all $n \in \mathbb{N}$. Thus, put $S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. If we can prove the *base case*, i.e. $1 \in S$, and show the *induction step*, that is, $(k \in S \implies k+1 \in S)$, then $S = \mathbb{N}$ by the axiom of induction. Therefore, in this case, $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem 1.22. *For all* $n \in \mathbb{N}$*:*

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

We introduce the $\sum$ notation for sums: if $a_m, \ldots, a_n \in \mathbb{R}, m \leq n, m, n \in \mathbb{N}$, then we denote by

$$\sum_{l=m}^{n} a_l = a_m + a_{m+1} + \ldots + a_n \,.$$

For example,

$$\sum_{l=1}^{n} l = 1 + 2 + \ldots + n$$

Similarly, we define the $\prod$ notation for products: if $a_m, \ldots, a_n \in \mathbb{R}, m \leq n, m, n \in \mathbb{N}$, then we denote by

$$\prod_{l=m}^{n} a_l = a_m \cdot a_{m+1} \cdot \ldots \cdot a_n \,.$$

For example,

$$\prod_{l=1}^{n} l = 1 \cdot 2 \cdot \ldots \cdot n = n!$$

Thus, the previous theorem reads as

Theorem 1.23. *For all $n \in \mathbb{N}$:*

$$\sum_{l=1}^{n} l = \frac{n(n+1)}{2}$$

Theorem 1.24. $\forall x \geq -1$:

$$(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}\,.$$

Common errors:

- The base case is not verified (heavy mark reduction).
- In the induction step the order of argument is wrong (mild to heavy mark reduction, depending on the error); in particular, showing that the induction claim implies a true statement.

  Consider the argument:

  $1 = 0$ implies $0 = 0 \cdot 1 = 0 \cdot 0$

  Even though the last statement "$0 = 0$" is correct, obviously this does not prove that $1 = 0$.
- A final sentence referring to the principle of induction (e.g., "from (1) and (2) we see...") is missing (mild mark reduction).

Example 1.25 (Quantum Numbers)**.** One can prove by induction that the sum of the first n odd numbers is n^2, that is,

$$\forall n \in \mathbb{N}: \quad \sum_{l=0}^{n-1} (2l+1) = n^2\,.$$

This equation appears in connection with quantum numbers: an electron has

- main quantum number n
- orbital angular momentum quantum number l which can attain the values $0, 1, \ldots, n-1$
- magnetic quantum number with possible values from $-l$ to l, that is, with $2l+1$ possible values.

Thus, by given n, the number of possible combinations of orbital angular momentum quantum number and magnetic quantum number is given by n^2.