Lecture 21: Multiple Integrals.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

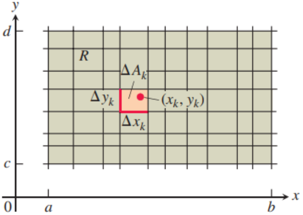
November 7, 2022

Double and Iterated Integrals over Rectangles

• In integral calculus, we defined the **Definite Integral** of a continuous function $f(x)$ over an interval $[a, b]$ as a **limit of Riemann sums**.

• In this section, we **extend this idea to define the double integral** of a continuous function of two variables $f(x, y)$ over a bounded rectangle R in the plane.

• We consider a function $f(x, y)$ defined on a **rectangular region R**,
$R: \ a \leq x \leq b, \ c \leq y \leq d$.
• We **subdivide R** into n small rectangles which form a **partition of R**.



• **A Riemann sum over R** is defined as

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Double and Iterated Integrals over Rectangles

• Sometimes the **Riemann sums converge** as the rectangle widths and heights both go to zero and whose number **n goes to infinity**.

• The resulting limit is then

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

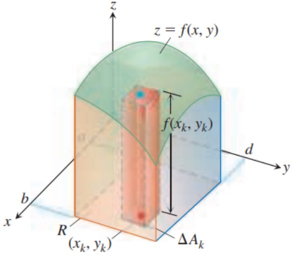
• When **a limit of the sums S_n exists**, giving the same limiting value **no matter what choices of an arbitrary point** (x_k, y_k) are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over **R**, written as

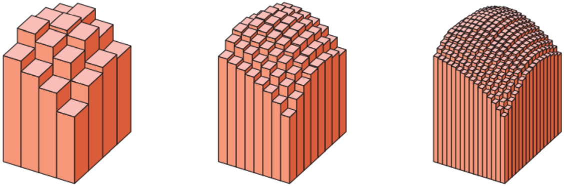
$$\iint_R f(x,y)dA \quad \text{or} \quad \iint_R f(x,y)dxdy$$

Double Integrals as Volumes

• When $f(x, y)$ is a **positive function** over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the **volume of the 3-dimensional solid** region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)dA$$



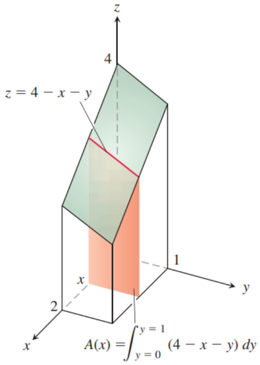


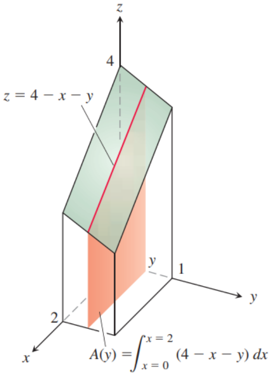
(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Fubini's Theorem for Calculating Double Integrals

• If we apply the **method of slicing**, with **slices perpendicular to the x-axis**, then the **volume** is $\int_{x=0}^{x=2} A(x)dx$ where $A(x)$ is the **cross-sectional area at x**.

• If we apply the method of slicing, with **slices perpendicular to the y-axis**, then the volume is $\int_{y=0}^{y=1} A(y)dy$ where $A(y)$ is the **cross-sectional area at y**.





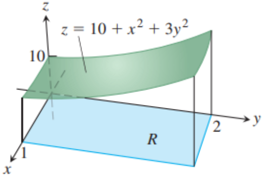
• A **theorem** published in 1907 by **Guido Fubini** says that the double integral of any continuous function **over a rectangle** can be calculated as an iterated integral **in either order of integration**.

THEOREM 1—Fubini's Theorem (First Form)
If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example 1

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: \; 0 \le x \le 1, \; 0 \le y \le 2$.



Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\,dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.$$

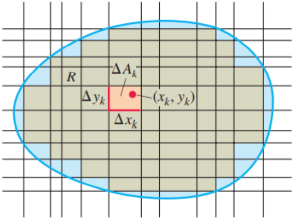
Double Integrals over General Regions

• Now we define and evaluate double integrals over bounded regions in the plane which are **more general** than rectangles.

• These double integrals are also evaluated as **iterated integrals**, with the main practical problem being that of **determining the limits of integration**.

• Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the **limits of integration often involve variables**, not just constants.

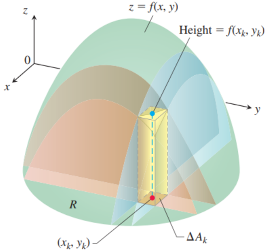
• When **a limit of the Riemann sums S_n exists**, then the function f is said to be **integrable** and the limit is called the **double integral** of f over **R**, written as

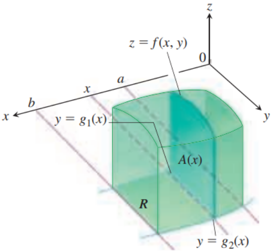


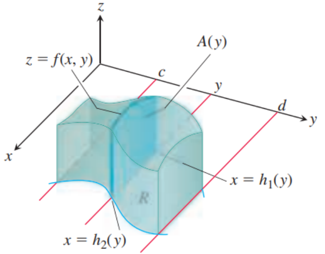
$$\lim_{n\to\infty} S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy$$

Volumes

• If $f(x, y)$ is **positive and continuous over R**, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)dA$.







• We may again calculate the volume by the method of slicing:

1) If R is a region bounded by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a, x = b$, then $V = \int_a^b A(x)dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$

2) If R is a region bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and on the sides by the lines $y = c, y = d$, then $V = \int_c^d A(y)dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$

Fubini's Theorem (Stronger Form)

THEOREM 2—Fubini's Theorem (Stronger Form)
Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

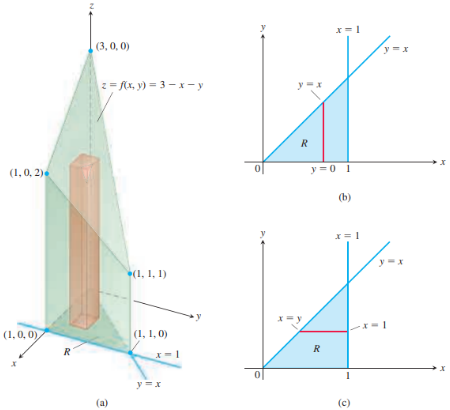
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 2

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.



Example 2

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

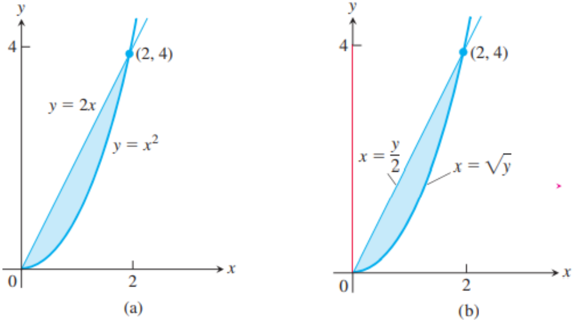
The two integrals are equal, as they should be. ■

Example 3

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$$

and write an equivalent integral with the order of integration reversed.



Example 3

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.16a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure 15.16b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R \big(f(x, y) \pm g(x, y)\big)\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. Domination:

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

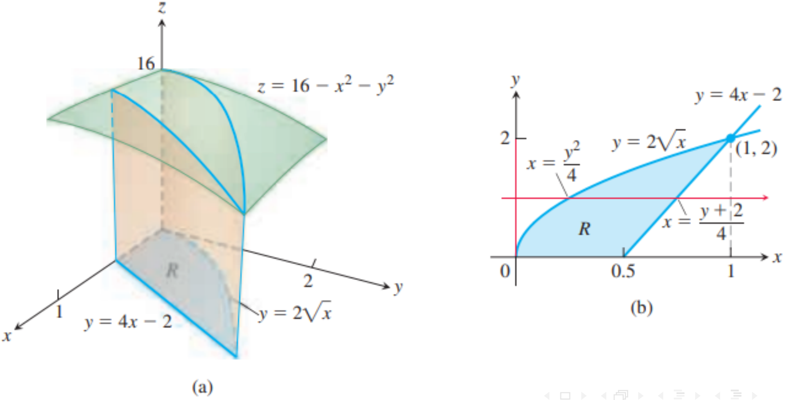
(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$

4. *Additivity:* If R is the union of two nonoverlapping regions R_1 and R_2, then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

Example 4

Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



Example 4

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$
\begin{aligned}
&\iint_R (16 - x^2 - y^2)\, dA \\
&\quad = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy \\
&\quad = \int_0^2 \left[16x - \frac{x^3}{3} - xy^2 \right]_{x=y^2/4}^{x=(y+2)/4} dx \\
&\quad = \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3 \cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] dy \\
&\quad = \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344} \right]_0^2 = \frac{20803}{1680} \approx 12.4.
\end{aligned}
$$

■