UNIVERSITY OF LEICESTER

MA7404
All Candidates

Semester 1 Examinations 2022

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	Computing and Mathematical Sciences
Module Code	MA7404
Module Title	Markov processes
Exam Duration	Two hours + 45 minutes upload time
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	8
Number of Questions	4
Instructions to Candidates	Answer all questions
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	Formulae and tables for actuarial examinations
Are students permitted to bring their own Books/Statutes/Notes?	Yes
Additional Stationery	Yes

1. An insurance company receives two types of claims, which we call type A and type B claims. The type A and type B claims arrive according to independent Poisson processes with parameters μ_A and μ_B claims per week, respectively. Parameters μ_A and μ_B are unknown and should be estimated from the past data. The claim sizes are independent. The sizes of type A claims follow gamma distribution with parameters $\alpha_A = 3$, $\lambda_A = 0.01$, while the sizes of type B claims follow Pareto distribution with parameters $\alpha_B = 4$, $\lambda_B = 600$.

(i) [5 marks] The numbers of type A claims during the last 7 weeks were

$$10, 12, 15, 6, 8, 13 \text{ and } 6.$$

Use the method of maximum likelihood to estimate parameter μ_A.

(ii) [5 marks] The numbers of type B claims during the last 7 weeks were

$$2, 3, 1, 2, 0, 1 \text{ and } 2.$$

Use the method of percentile at level $\alpha = e^{-2} \approx 0.135$ to estimate parameter μ_B.

For the questions below, use the values of μ_A and μ_B found in parts (i) and (ii), or use $\mu_A = 5$ and $\mu_B = 1$ if you cannot solve (i) or (ii).

(iii) [5 marks] Estimate the mean and standard deviation of the total size S_3 of all claims to be received in the next 3 weeks.

(iv) [5 marks] Estimate the probability that, during the next 2 weeks, the company will receive exactly 7 claims in total.

(v) [5 marks] Estimate the probability that the next three claims the company will receive will happen to be the type A claims.

Total: 25 marks

Answer:

(i) Application (Similar to seen)

Denote $\lambda = \mu_A$ The probability to receive n claims during a week is $\frac{\lambda^n e^{-\lambda}}{n!}$. Let $n_1, \dots, n_7$ be the numbers of claims in the past 7 weeks. The probability to receive exactly this numbers of claims is

$$L = \prod_{i=1}^{7} \frac{\lambda^{n_i} e^{-\lambda}}{n_i!} = \frac{\lambda^{\sum n_i} e^{-7\lambda}}{\prod(n_i!)}.$$

Hence

$$\log L = \sum_{i=1}^{7} n_i \log \lambda - 7\lambda - \log(\prod(n_i!)).$$

The derivative $\frac{1}{\lambda}\sum_{i=1}^{7} n_i - 7$ is equal to 0 if

$$\mu_A = \lambda = \frac{\sum_{i=1}^{7} n_i}{7} = 10.$$

(ii) Higher skills (Unseen)

From data, we get 0 claims with probability $1/7$. Because $\alpha = e^{-2} \approx 0.135 < 1/7$, the quantile at α is 0. Hence, the parameter μ_B should be chosen in such a way that the probability of 0 claims is equal to α. Hence, we have

$$\frac{\mu_B^0 e^{-\mu_B}}{0!} = \alpha = e^{-2}$$

from which we can find that $\mu_B = 2$.

(iii) Application (Similar to seen)

Let N_A be the number of type A claims in 3 weeks, X_A be the size of type A claim, and S_A be the total size of all type A claims. Then N_A follow a Poisson distribution with parameter $3\mu_A = 30$, hence

$$E[S_A] = 3\mu_A \cdot E[X_A] = 3\mu_A \cdot \frac{\alpha_A}{\lambda_A} = 30 \cdot \frac{3}{0.01} = 9,000.$$

and

$$Var[S_A] = 3\mu_A(Var[X_A] + E[X_A]^2) = 3\mu_A(\frac{\alpha_A}{\lambda_A^2} + E[X_A]^2) = 30(\frac{3}{(0.01)^2} + 300^2) = 3,600,000.$$

Similarly, for type B claims,

$$E[S_B] = 3\mu_B \cdot E[X_B] = 3\mu_B \cdot \frac{\lambda_B}{\alpha_B - 1} = 6 \cdot \frac{600}{4-1} = 1,200.$$

and

$$Var[S_B] = 3\mu_B(Var[X_B] + E[X_B]^2) = 3\mu_B(\frac{\alpha_B \lambda_B^2}{(\alpha_B - 1)^2(\alpha_B - 2)} + E[X_B]^2)$$

$$= 6(\frac{4(600)^2}{(4-1)^2(4-2)} + 200^2) = 720,000.$$

Now, S_A and S_B are independent, and $S = S_A + S_B$, hence

$$E[S] = E[S_A] + E[S_B] = 9,000 + 1,200 = 10,200$$

and

$$\sigma[S] = \sqrt{Var[S]} = \sqrt{Var[S_A] + Var[S_B]} = \sqrt{4,320,000} \approx 2,078.$$

(iv) Application (Similar to seen)

The total number N_t of all claims over t weeks is again a Poisson process with rate $\mu = \mu_A + \mu_B = 12$ claims per week. Hence,

$$P(N_2 = 7) = e^{-2\mu}\frac{(2\mu)^7}{7!} == e^{-24}\frac{(24)^7}{7!} = \frac{31,850,496}{35}e^{-24} \approx 0.00003435.$$

(iv) Application (Similar to seen)

$$\left(\frac{\mu_A}{\mu_A + \mu_B}\right)^3 = \frac{10^3}{(10+2)^3} = \frac{125}{216} \approx 0.5787.$$

Syllabus cover

This question covers items 1.1 (loss distributions...) and 1.2 (compound distributions...) of the syllabus.

Feedback

Part (i) has been answered well. In part (ii), many students gave wrong answer $\mu_B = 0$. In part (iii) most students did some partial progress but very few finished until the end. Part (iv) has been answered reasonably well. In part (v), many students estimated the probability of at least one type A claim during some time period, but the question was that the next claim will be type A, which is different.

2. A company models their total expenses in 2022 as a random variable X that follows Weibull distribution with parameters $c = 0.002$ and $\gamma = 1/2$. They also model their total expenses in 2023 as Weibull distribution with parameters $c = 0.01$ and $\gamma = 1/3$. They consider three possible copulas to model the dependence of X and Y:

   (1) The independence copula $C(u, v) = uv$;

   (2) The co-monotonic copula $C(u, v) = \min(u, v)$, and

   (3) The Clayton copula with parameter $\alpha = 1$.

   (i) [5 marks] Using the limiting density ratios test, determine which random variable (X or Y) has distribution with a heavier tail.

   (ii) For each of the three models for copula listed above:

   (a) **[10 marks]**. Compute the survival copula $\bar{C}(u, v)$.

   (b) **[10 marks]**. Estimate the probability that both X and Y exceed a million pounds.

   Total: 25 marks

Answer:

(i) Application (Similar to seen)

If $\alpha_1 > \alpha_2$, then

$$\lim_{x\to\infty} \frac{f_X(x)}{f_Y(x)} = \lim_{x\to\infty} \frac{0.002 \cdot 1/2 \cdot x^{1/2-1} e^{-0.002x^{1/2}}}{0.01 \cdot 1/3 \cdot x^{1/3-1} e^{-0.01x^{1/3}}} = \lim_{x\to\infty} 0.3x^{1/6} \exp(0.01x^{1/3} - 0.002x^{1/2}) = 0,$$

hence Y has heavier tail.

(ii)

(a) Application (Similar to seen)

From lecture notes

$$\bar{C}(1-u, 1-v) = 1 - u - v + C(u, v),$$

or, equivalently,

$$\bar{C}(u, v) = 1 - (1-u) - (1-v) + C(1-u, 1-v) = -1 + u + v + C(1-u, 1-v).$$

Hence, for independence copula

$$\bar{C}(u,v) = -1+u+v+(1-u)(1-v) = uv.$$

For co-monotonic copula

$$\bar{C}(u,v) = -1+u+v+\min(1-u,1-v).$$

If $u \geq v$ then $\min(1-u,1-v) = 1-u$ and $\bar{C}(u,v) = -1+u+v+1-u = v$. Similarly, if $v \geq u$ then $\bar{C}(u,v) = u$. Hence, in general,

$$\bar{C}(u,v) = \min(u,v).$$

Finally, the Clayton copula with $\alpha = 1$ is

$$C(u,v) = (u^{-1}+v^{-1}+1)^{-1} = \frac{uv}{u+v-uv}.$$

Hence,

$$\bar{C}(u,v) = -1+u+v+C(1-u,1-v) = -1+u+v+\frac{(1-u)(1-v)}{(1-u)+(1-v)-(1-u)(1-v)} =$$

$$-1+u+v+\frac{(1-u)(1-v)}{1-uv} = \frac{(-1+u+v)(1-uv)+(1-u)(1-v)}{1-uv} = \frac{uv(2-u-v)}{1-uv}.$$

(b) Higher skills (Unseen)

By the definition of survival copula,

$$P(X>x, Y>y) = \bar{C}(P[X>x], P[Y>y]).$$

In our case, $x = y = 10^6$.

$$P[X>x] = 1-F_X(x) = 1-(1-e^{-cx^\gamma}) = e^{-cx^\gamma} = e^{-0.002(10^6)^{1/2}} = e^{-2},$$

$$P[Y>y] = 1-F_Y(y) = 1-(1-e^{-cy^\gamma}) = e^{-cy^\gamma} = e^{-0.01(10^6)^{1/3}} = e^{-1}.$$

Hence,

$$P(X>x, Y>y) = \bar{C}(e^{-2}, e^{-1}).$$

Now, for independence copula,

$$\bar{C}(e^{-2}, e^{-1}) = e^{-2}\cdot e^{-1} = e^{-3} \approx 0.050,$$

for co-monotonic copula

$$\bar{C}(e^{-2}, e^{-1}) = \min(e^{-2}, e^{-1}) = e^{-2} \approx 0.135,$$

and for the Clayton copula

$$\bar{C}(e^{-2}, e^{-1}) = \frac{e^{-3}(2-e^{-2}-e^{-1})}{1-e^{-3}} \approx 0.078.$$

Syllabus cover

This question covers items 1.3 (introduction to copulas) and 1.4 (introduction to extreme value theory) of the syllabus.

Feedback

In part (i), surprisingly many students claimed without proof that the limit is infinite. In part (ii) (a) many students even did not attempted the question with Clayton copula. In part (ii) (b) many students calculated probability that both X and Y are less than million and then subtract it from 1. But this is incorrect, see the solution why.

3. A credit rating agency assigns two possible ratings for a bond B: I - investment grade, and J - junk grade. The rating is reviewed once a year. From the past data, the agency noticed the following

   - Among the bonds with investment grades, 70% stay at the investment grade next year, 25% move to the junk grade, and only 5% default within a year.
   - Among the bonds that just moved from the investment grade to junk grade, 30% return to the investment grade next year, 60% stay at the junk grade, and 10% default.
   - Among the bonds that are in the junk grade for at least two consecutive years, 10% move to the investment grade next year, 70% stays at the junk grade, and 20% default.

   (i) [5 marks] Based on these data, can the credit rating dynamic be modelled as a Markov chain with states I - investment grade, J - junk grade, and D - default? If not, suggest an alternative state space.

   (ii) [5 marks] Model the process as a Markov chain with state space described in part (i). Write down the one-step transition matrix. Draw the transition graph.

   (iii) [5 marks] Estimate the probability that the bond currently at investment grade will not default within the next 2 years but then will default during the third year.

   (iv) [5 marks] Find the stationary distribution of this Markov chain.

   (v) (a) **[3 marks]** Is this Markov chain finite, irreducible, and aperiodic?

   (b) **[2 marks]** Does it converge to its stationary distribution?

   Total: 25 marks

Answer:

(i) Application (Similar to seen)

Not. An alternative state space is I - investment grade, J1 - junk grade for bonds that just moved from the investment grade, J2 - all other bounds in the junk grade, and D - default.

(ii) Application (Similar to seen)

The one-step transition matrix is

$$P = \begin{bmatrix} 0.7 & 0.25 & 0 & 0.05 \\ 0.3 & 0 & 0.6 & 0.1 \\ 0.1 & 0 & 0.7 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) Application (Similar to seen)

Calculating P^2 and looking at the correct entry, we get that the probability of default within the next 2 years is 0.11.

Similarly, calculating P^3 and looking at the correct entry, we get that the probability of default within the next 3 years is $\frac{743}{4000} = 0.18575$.

Hence, the probability of default during exactly the third year is

$$0.18575 - 0.11 = 0.0757.$$

(iv) Application (Similar to seen)

If $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ is stationary distribution, the forth equation is $0.05\pi_1 + 0.1\pi_2 + 0.2\pi_3 + \pi_4 = \pi_4$, from which we have $0.05\pi_1 + 0.1\pi_2 + 0.2\pi_3 = 0$, which implies that $\pi_1 = \pi_2 = \pi_3 = 0$ and then it must be $\pi_4 = 1$. Hence, the stationary distribution is $\pi = (\pi_1, \pi_2, \pi_3, \pi_4) = (0, 0, 0, 1)$.

(v) Higher skills (Unseen)

The Markov chain is finite and aperiodic but not irreducible. Because it is not irreducible, the Theorem (stating that every finite irreducible aperiodic Markov chain converges to the stationary distribution) is not applicable. However, it is clear that in the limit every bond ends up in the Default state, hence the limiting distribution is $(0, 0, 0, 1)$. Because the stationary distribution is also $(0, 0, 0, 1)$, the Markov chain converges to the stationary distribution (despite being not irreducible).

Syllabus cover

This question covers item 3.2 of the syllabus: Define and apply a Markov chain.

Feedback

In part (i) some students claimed that three states suffices, which obviously affected the whole question. Those student who did part (i) correctly usually also correctly did (ii) and (iii). In (iv), some students were not able to correctly solve the resulting system of linear equations. In (v), some students noticed that the Markov chain is not irreducible and incorrectly concluded that it cannot converge to stationary distribution.

4. Assume that the marriage rate for an unmarried person aged t is $\alpha(t)$, the divorce rate for a married person aged t is $d(t)$, and the mortality rate for a person aged t is $\mu(t)$, independently of the martial status.

   (i) [5 marks] Build the marriage model that consists on three states: N - not married, M - married, and D - dead. Write down the generator matrix and the transition graph. You may assume that every married couple has the same age.

   (ii) [10 marks] Assume that $\mu(t) = at + b$ for some constants a and b. Assume that we have data points $\mu(20) = 0.002$, $\mu(40) = 0.006$, $\mu(60) = 0.02$, and $\mu(80) = 0.08$. Use simple linear regression to find constants a and b such that $\mu(t)$ best approximates the data.

   (iii) [10 marks] Using $\mu(t)$ from part (ii) (or use $\mu(t) = 0.001t - 0.03$ for $t \geq 30$ if you cannot solve (ii)) and $d(t) = 0.02$, find the probability that a person aged 40, currently married, will stay in the same marriage more than 10 years but less than 20 years.

Total: 25 marks

Answer:

(i) Application (Similar to seen)

$$Q(t) = \begin{pmatrix} -\alpha(t) - \mu(t) & \alpha(t) & \mu(t) \\ d(t) + \mu(t) & -d(t) - 2\mu(t) & \mu(t) \\ 0 & 0 & 0 \end{pmatrix},$$

Note that the transition rate from married to unmarried is $d(t) + \mu(t)$, because there are two ways to become unmarried: divorce (with rate $d(t)$), and the death of a partner (with rate $\mu(t)$).

(ii) Application (Similar to seen)

If (t_i, μ_i) are data points, and we are looking for a linear function in the form $\mu(t) = at + b$, then the mean-square error is $e(a,b) = \sum_{i=1}^n (\mu_i - at_i - b_i)^2$. We can minimize this error by calculating partial derivatives and equating them to 0. The solution is

$$a^* = \frac{\sum_{i=1}^n (t_i - \bar{t})(\mu_i - \bar{\mu})}{\sum_{i=1}^n (t_i - \bar{t})^2}, \quad b^* = \bar{\mu} - a^* \bar{t},$$

where $\bar{t} = \frac{1}{n} \sum_{i=1}^n t_i$, $\bar{\mu} = \frac{1}{n} \sum_{i=1}^n \mu_i$. In our case, computation gives

$$\mu(t) = 0.00124t - 0.035.$$

(iii) Application (Similar to seen)

The residual holding time at state M is

$$P(R_s > h - s | X_s = M) = e^{-\int_s^h (d(t) + 2\mu(t))dt}.$$

In our case,

$$P(R_{30} > 10 | X_{30} = M) = e^{-\int_{30}^{40} (0.02 + 2(0.00124t - 0.035))dt}$$

$$= \exp(-(0.00248(40^2 - 30^2)/2 + (0.02 - 2 \cdot 0.035)(40 - 30))) = e^{-0.368} \approx 0.69.$$

$$P(R_{30} > 20 | X_{30} = M) = e^{-\int_{30}^{50} (0.02 + 2(0.00124t - 0.035))dt}$$

$$= \exp(-(0.00248(50^2 - 30^2)/2 + (0.02 - 2 \cdot 0.035)(50 - 30))) = e^{-0.984} \approx 0.37.$$

Hence,

$$P(20 > R_{30} > 10) = P(R_{30} > 10) - P(R_{30} > 20) \approx 0.69 - 0.37 = 0.32.$$

Syllabus cover

This question covers item 3.3 Define and apply a Markov process, and item 5 Machine Learning.

Feedback

In part (i), essentially all students missed that one may move from married to unmarried not only by divorce but also but the death of a partner. Just one point has been subtracted for this. Parts (ii) and (iii) have been answered reasonably well.