Lecture 13: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

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Limits and Continuity in Higher Dimensions

• In this section we develop **limits and continuity for multivariable functions**.

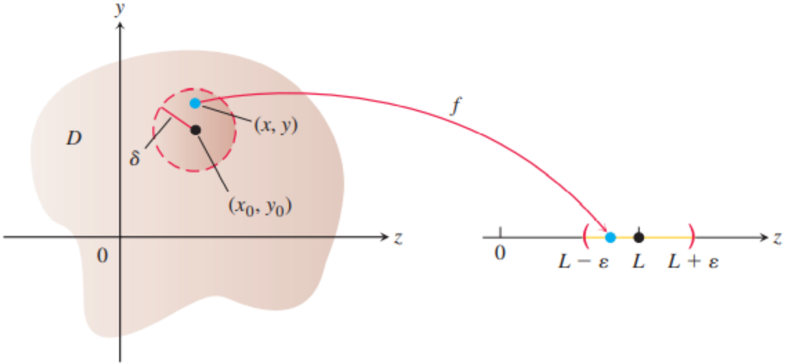
• The theory is **similar** to that developed for single-variable functions, but since we now have more than one independent variable, there is **additional complexity** that requires some **new ideas**.

• **Limits for Functions of Two Variables**: If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0).

• This is similar to the **informal definition** for the limit of a function of a single variable.

• Notice, however, that when (x_0, y_0) lies in the interior of f's domain, (x, y) **can approach** (x_0, y_0) **from any direction**, not just from the left or the right.

• **For the limit to exist**, the **same limiting value** must be obtained **whatever direction** of approach is taken.



Limits for Functions of Two Variables

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

- The definition of limit says that the distance between $f(x, y)$ and L **becomes** arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is **made** sufficiently small (but not 0).
- The definition applies to interior points (x_0, y_0) as well as boundary points of the domain of f, although a boundary point need not lie within the domain.
- The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f.

Limits for Functions of Two Variables

• As with single-variable functions, the limit of the sum of two functions is the sum of their limits **(when they both exist)**, with similar results for the limits of the differences, constant multiples, products, quotients, powers, and roots.

• These facts are summarized in **Theorem 1**.

THEOREM 1—Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M},\quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

   n a positive integer, and if n is even, we assume that $L > 0$.

Example 1

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

.

Solution:

Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we can find:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \qquad \text{Multiply by a form equal to 1.}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \qquad \text{Algebra}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) \qquad \text{Cancel the nonzero factor } (x - y).$$

$$= 0(\sqrt{0} + \sqrt{0}) = 0 \qquad \text{Known limit values}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (where we would have $x - y = 0$) is not in the domain of the given function.

• As with functions of a single variable, continuity is defined **in terms of limits**.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

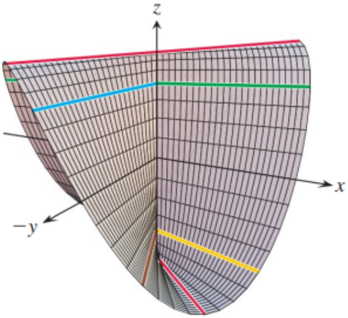
1. f is defined at (x_0, y_0),
2. $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Show that $f(x, y)$ is continuous at every point except the origin.

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Solution:



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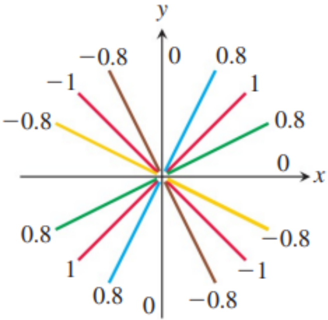

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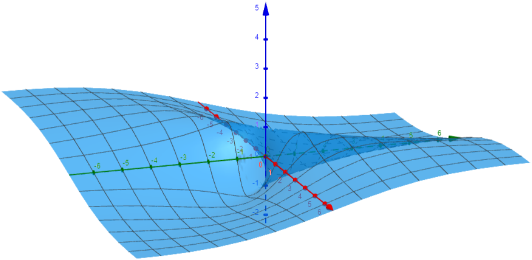
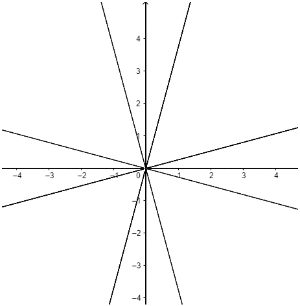
Continuity. Example 2





Continuity. Example 2

• The function f is **continuous at every point** (x, y) **except** $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of x and y, and therefore at those points the limiting value is simply obtained by **substituting** the values of x and y into that rational expression.

• At $(0, 0)$, the value of f is **defined**, but **f has no limit as** $(x, y) \to (0, 0)$.

• The reason is that **different paths** of approach to the origin can lead to **different results**, as we now see.

Continuity. Example 2

• For every value of m, the function f has a **constant value on the line** $y = mx, x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

• Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

• This **limit changes** with each value of the slope m.

• There is therefore **no single number** we may call the limit of f as (x, y) approaches the origin.

• The **limit fails to exist**, and the function is **not continuous** at the origin.

Two-Path Test for Nonexistence of a Limit

- Examples 2 illustrate an **important point** about limits of functions of two or more variables.
- For a limit to exist at a point, the **limit must be the same along every approach path**.
- This result is **analogous to the single-variable case** where both the left- and right-sided limits had to have the same value.
- For functions of two or more variables, if we ever find **paths with different limits**, we know the **function has no limit** at the point they approach.

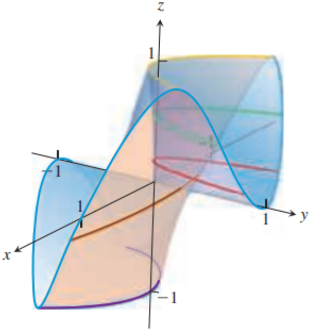
Two-Path Test for Nonexistence of a Limit
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0).

Continuity. Example 3

Show that the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as (x,y) approaches $(0,0)$.

Solution:



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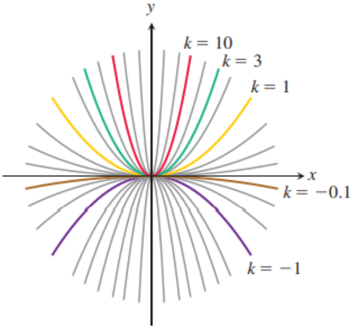

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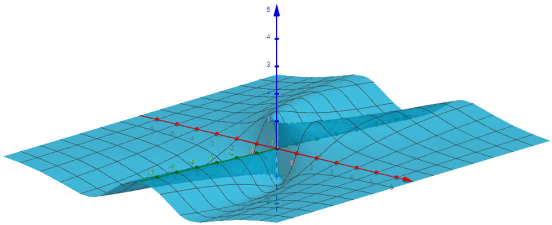
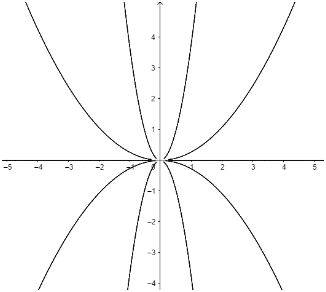
Continuity. Example 3





Continuity. Example 3

Solution:

• The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$.

• We examine the values of f along parabolic curves that end at $(0, 0)$.

• Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

• This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1.

• If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0.

• By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

Continuity of Compositions
If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

• **Functions of More Than Two Variables**:
The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and compositions all extend to functions of three or more variables.

• **Extreme Values of Continuous Functions on Closed, Bounded Sets**: A function of a single variable that is continuous at every point of a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$.

Extreme Values

• The same holds true of a function $z = (x, y)$ that is continuous on a closed, bounded set R in the plane.

• The function takes on **an absolute maximum** value at some point in R and an **absolute minimum value** at some point in R.

• The function may take on a maximum or minimum value **more than once over R**.

• **Similar results hold** for functions of three or more variables.