Lecture 5: Vectors and the Geometry of Space.

MA2032 Vector Calculus

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Lines and Planes in Space

- Today I will show how to use scalar and vector products to write **equations for lines, line segments, and planes in space**.
- We will use **these representations** throughout the rest of the course in studying the calculus of curves and surfaces in space.
- **In the plane**, a line is determined by a **point** and a number giving the **slope** of the line.
- **In space** a line is determined by a **point** and a **vector** giving the direction of the line.

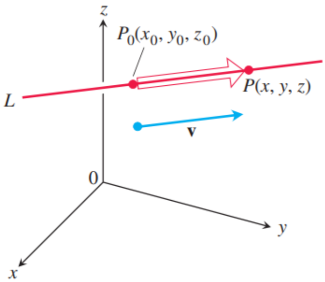
Lines and Line Segments in Space

• Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $v = \nu_1 i + \nu_2 j + \nu_3 k$.

• Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0 P}$ is parallel to v as shown in Figure.

• Thus, $\overrightarrow{P_0 P} = tv$ for some scalar parameter t.

• The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$.



• The expanded form of the equation $\overrightarrow{P_0 P} = tv$ is

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(\nu_1 i + \nu_2 j + \nu_3 k),$$

which can be rewritten as

$$xi + yj + zk = x_0 i + y_0 j + z_0 k + t(\nu_1 i + \nu_2 j + \nu_3 k).$$

Lines and Planes in Space

If $r(t)$ is the position vector of a point $P(x, y, z)$ on the line and r_0 is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation above gives the following vector form for the equation of a line in space

Vector Equation for a Line

A **vector equation for the line** L through $P_0(x_0, y_0, z_0)$ parallel to v is

$$r(t) = r_0 + tv, \quad -\infty < t < \infty$$

where r is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

The **standard parametrization of the line** through $P_0(x_0, y_0, z_0)$ parallel to $v = \nu_1 i + \nu_2 j + \nu_3 k$ is

$$x = x_0 + t\nu_1, y = y_0 + t\nu_2, z = z_0 + t\nu_3, \quad -\infty < t < \infty$$

Example

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution

• The vector $\overrightarrow{PQ} = (1-(-3))i + (-1-2)j + (4-(-3))k = 4i - 3j + 7k$ is parallel to the line, and the standard parametrization of the line equations with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

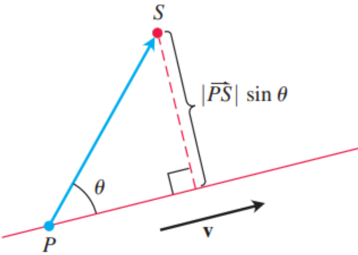
$$x = -3 + 4t,\ y = 2 - 3t,\ z = -3 + 7t.$$

• We could have chosen $Q(1, -1, 4)$ as the "base point" and written $x = 1 + 4t, y = -1 - 3t, z = 4 + 7t.$

• These equations serve as well as the first; they simply place you at a different point on the line for a given value of t.

The Distance from a Point to a Line in Space

To find the **distance from a point S to a line** that passes through a point P parallel to a vector v, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line, see Figure. In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}|\sin\theta$, which is



$$\frac{|\overrightarrow{PS}|\,|\mathbf{v}|\sin\theta}{|\mathbf{v}|} = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}.$$

Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Example

Find the distance from the point $S(1, 1, 5)$ to the line $L: \ x = 1 + t, y = 3 - t, z = 2t$.

Solution

We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $v = i - j + 2k$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a Plane in Space

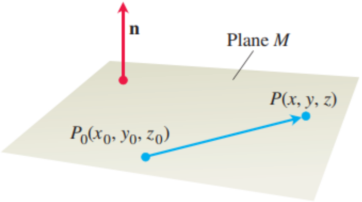
• A **plane in space is determined** by knowing a **point** on the plane and its **"tilt"** or orientation.

• This "tilt" is defined by specifying a vector that is perpendicular or **normal** to the plane.

• Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $n = Ai + Bj + Ck$.

• Vector from P_0 to any point P on the plane is orthogonal to n.

• Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to n.



• Thus, the dot product $n \cdot \overrightarrow{P_0P} = 0$.

• This equation is equivalent to

$$(Ai + Bj + Ck) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0,$$

so the plane M consists of the points (x, y, z) satisfying

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $n = Ai + Bj + Ck$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

Example

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution

We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane. The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6.$$

Lines of Intersection

- Just as lines are parallel if and only if they have the same direction, two **planes are parallel** if and only if their **normals are parallel**, or$n_1 = kn_2$ for some scalar k.
- Two planes that are not parallel **intersect in a line**.

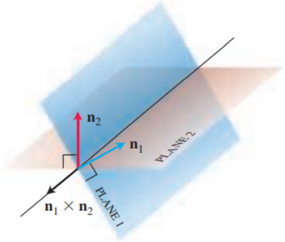
Example

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution

- The line of intersection of two planes is perpendicular to both planes' normal vectors n_1 and n_2 (see Figure) and therefore parallel to $n_1 \times n_2$.
- Turning this around, $n_1 \times n_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$



A line and a plane intersection

Example

Find the point where the line $x = \frac{8}{3} + 2t,\ y = -2t,\ z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

Solution

The point $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$ lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane

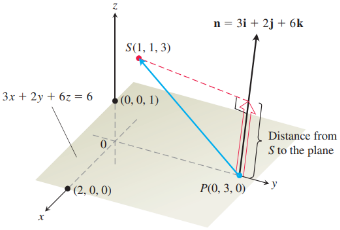
If P is a point on a plane with normal n, then the distance from any point S to the plane is the **length of the vector projection** of $\overrightarrow{PS}$ onto n, as given in the following formula.

Distance from a Point S to a Plane with Normal n at Point P

$$d = \left| \overrightarrow{PS} \cdot \frac{n}{|n|} \right|$$

Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.



The Distance from a Point to a Plane

Solution

- We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector n normal to the plane.
- The coefficients in the equation $3x + 2y + 6z = 6$ give $n = 3i + 2j + 6k$.
- The points on the plane easiest to find from the plane's equation are the intercepts.
- If we take P to be the y-intercept $(0, 3, 0)$, then

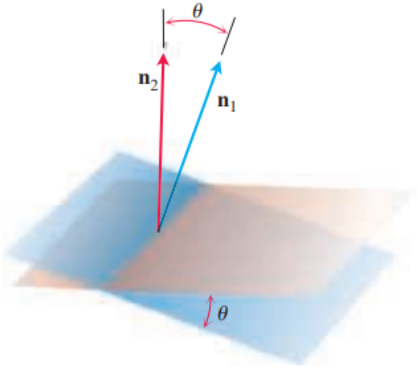
$$\overrightarrow{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$
$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

- Therefore, the distance from S to the plane is

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \qquad \text{Length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS}$$
$$= \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right|$$
$$= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.$$

Angles Between Planes

• The angle between two intersecting planes is defined to be the **acute angle between their normal vectors**



Angles Between Planes

Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution

The vectors $n_1 = 3i - 6j - 2k$, $n_2 = 2i + j - 2k$ are normals to the planes. The angle between them is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.} \qquad \text{About 79 degrees}$$