Solutions for Tutorial Problem Sheet 11, December 8.

(Infinite Sequences and Series)

Problem 1. Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

a) $\{a_n\} = \frac{1-n^3}{70-4n^2}$,

b) $\{a_n\} = \frac{\sin^2 n}{2^n}$,

c) $\{a_n\} = \frac{n}{2^n}$.

Solution:

$$\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)-n}{\left(\frac{70}{n^2}\right)-4} = \infty \Rightarrow \text{ diverges}$$

$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0 \text{ because } 0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n} \Rightarrow \text{ converges by the Sandwich Theorem for sequences}$$

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow \text{ converges (using l'Hôpital's rule)}$$

Problem 2. Use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$,

b) $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$,

c) $\sum_{n=0}^{\infty} \cos(n\pi)$.

Solution:

$$\lim_{n\to\infty} \frac{n}{n^2+3} = \lim_{n\to\infty} \frac{1}{2n} = 0 \Rightarrow \text{ test inconclusive}$$

$$\lim_{n\to\infty} \frac{e^n}{e^n+n} = \lim_{n\to\infty} \frac{e^n}{e^n+1} = \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \Rightarrow \text{ diverges}$$

$$\lim_{n\to\infty} \cos n\pi = \text{ does not exist } \Rightarrow \text{ diverges}$$

Problem 3. Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$,

b) $\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$,

c) $\sum_{n=1}^{\infty}\frac{2^n+4^n}{3^n+4^n}$.

Solution:

convergent geometric series with sum $\dfrac{\left(\frac{3}{2}\right)}{1-\left(-\frac{1}{2}\right)}=1$

$$\lim_{n\to\infty}\ln\frac{1}{3^n}=-\infty\neq 0\Rightarrow \text{ diverges}$$

$$\lim_{n\to\infty}\frac{2^n+4^n}{3^n+4^n}=\lim_{n\to\infty}\frac{\frac{2^n}{4^n}+1}{\frac{3^n}{4^n}+1}=\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{3}{4}\right)^n+1}=\frac{1}{1}=1\neq 0\Rightarrow \text{ diverges by } n\text{th-Term Test for divergence}$$

Problem 4. Use the Integral Test to determine if the series converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

a) $\sum_{n=1}^{\infty} \frac{1}{n+4}$,

b) $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$,

c) $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$.

Solution:

$f(x) = \frac{1}{x+4}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^{\infty} \frac{1}{x+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+4}\,dx = \lim_{b\to\infty} \left[\ln|x+4|\right]_1^b$

$= \lim_{b\to\infty} \left(\ln|b+4| - \ln 5\right) = \infty \Rightarrow \int_1^{\infty} \frac{1}{x+4}\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+4}$ diverges

$f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous, and decreasing for $x \ge 2$; $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$

$= \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b = \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow \int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

$f(x) = \frac{1}{\sqrt{x+4}}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^{\infty} \frac{1}{\sqrt{x+4}}\,dx = \lim_{b\to\infty} \int_1^b (x+4)^{-1/2}\,dx$

$= \lim_{b\to\infty} \left[2(x+4)^{1/2}\right]_1^b = \lim_{b\to\infty} \left(2(b+4)^{1/2} - 2\sqrt{5}\right) = \infty \Rightarrow \int_1^{\infty} \frac{1}{\sqrt{x+4}}\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$ diverges

Problem 5. Use the Limit Comparison Test to determine if each series converges or diverges.

a) $\sum_{n=2}^{\infty} \frac{n\,(n+1)}{(n^2+1)(n-1)}$,

b) $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$,

c) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$.

Solution:

Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series since $p = 1 \le 1$. Both series have positive terms for $n \ge 2$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{\left(n^2+1\right)(n-1)}}{1/n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1} = \lim_{n\to\infty} \frac{3n^2+2n}{3n^2-2n+1} = \lim_{n\to\infty} \frac{6n+2}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ diverges.

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series since $p = \frac{1}{2} \le 1$. Both series have positive terms for $n \ge 1$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\cdot 4^n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\cdot 4^n}$ diverges.

Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric series since $|r| = \left|\frac{2}{5}\right| < 1$. Both series have positive terms for $n \ge 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{(2/5)^n} = \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} \ln\left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} n \ln\left(\frac{10n+15}{10n+8}\right)$

$= \exp \lim_{x\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n} = \exp \lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{-1/n^2} = \exp \lim_{n\to\infty} \frac{70n^2}{(10n+15)(10n+8)} = \exp \lim_{n\to\infty} \frac{70n^2}{100n^2+230n+120}$

$= \exp \lim_{n\to\infty} \frac{140n}{200n+230} = \exp \lim_{n\to\infty} \frac{140}{200} = e^{7/10} > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ converges.