Eigenvectors and Eigenvalues

Daoping Zhang

The Chinese University of Hong Kong

dpzhang@math.cuhk.edu.hk

April 29, 2020

Outline

Eigenvectors and Eigenvalues

Let V be a vector space and let

$$A : V \to V$$

be a linear map of V into itself. An element $v \in V$ is called an eigenvector of A if there exists a number λ such that $Av = \lambda v$. If $v \neq O$ then λ is uniquely determined, because $\lambda_1 v = \lambda_2 v$ implies $\lambda_1 = \lambda_2$. In this case, we say that λ is an eigenvalue of A belonging to the eigenvector v. We also say that v is an eigenvector with the eigenvalue λ. Instead of eigenvector and eigenvalue, one also uses the terms characteristic vector and characteristic value.

If A is a square $n \times n$ matrix then an eigenvector of A is by definition an eigenvector of the linear map of $\mathbb{R}^n$ into itself represented by this matrix. Thus an eigenvector X of A is a (column) vector of $\mathbb{R}^n$ for which there exists $\lambda \in \mathbb{R}$ such that $AX = \lambda X$.

Let V be the vector space over $\mathbb{R}$ consisting of all infinitely differentiable functions. Let $\lambda \in \mathbb{R}$. Then the function f such that $f(t) = e^{\lambda t}$ is an eigenvector of the derivative d/dt because $df/dt = \lambda e^{\lambda t}$.
Let $A : V \to V$ is a linear map, and v is an eigenvector of A, then for any non-zero scalar c, cv is also an eigenvector of A, with the same eigenvalue.

Theorem

Let V be a vector space and let $A : V \to V$ be a linear map. Let $\lambda \in \mathbb{R}$. Let V_λ be the subspace of V generated by all eigenvectors of A having λ as eigenvalue. Then every non-zero element of V_λ is an eigenvector of A having λ as an eigenvalue.

Proof.

Let $v_1, v_2 \in V$ be such that $Av_1 = \lambda v_1$ and $Av_2 = \lambda v_2$. Then

$$A(v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2).$$

If $c \in K$ then $A(cv_1) = cAv_1 = c\lambda v_1 = \lambda cv_1$. This proves our theorem. $\square$

The subspace V_λ is called the eigenspace of A belonging to λ.

Theorem

Let V be a vector space and let $A : V \to V$ be a linear map. Let $v_1, ..., v_m$ be eigenvectors of A, with eigenvalues $\lambda_1, ..., \lambda_m$ respectively. Assume that these eigenvalues are distinct, i.e.

$$\lambda_i \neq \lambda_j \quad \text{if} \quad i \neq j.$$

Then $v_1, ..., v_m$ are linearly independent.

Suppose V is a vector space of dimension n and $A : V \to V$ is a linear map having n eigenvectors $v_1, ..., v_n$ whose eigenvalues $\lambda_1, ..., \lambda_n$ are distinct. Then $\{v_1, ..., v_n\}$ is a basis of V.

The Characteristic Polynomial

Theorem

Let V be a finite dimensional vector space, and let λ be a number. Let $A : V \to V$ be a linear map. Then λ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible.

Proof.

Assume that λ is an eigenvalue of A. Then there exists an element $v \in V, v \neq O$ such that $Av = \lambda v$. Hence $Av - \lambda v = O$, and $(A - \lambda I)v = O$. Hence $A - \lambda I$ has a non-zero kernel, and $A - \lambda I$ cannot be invertible. Conversely, assume that $A - \lambda I$ is not invertible. We see that $A - \lambda I$ must have a non-zero kernel, meaning that there exists an element $v \in V, v \neq O$ such that $(A - \lambda I)v = O$. Hence $Av - \lambda v = O$, and $Av = \lambda v$. Thus λ is an eigenvalue of A. This proves our theorem. $\square$

Let A be an $n \times n$ matrix, $A = (a_{ij})$. We define the characteristic polynomial P_A of A to be the determinant

$$P_A(t) = Det(tI - A).$$

We can also view A as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and we also say that $P_A(t)$ is the characteristic polynomial of this linear map.

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

is

$$\begin{vmatrix} t-1 & 1 & -3 \\ 2 & t-1 & -1 \\ 0 & -1 & t+1 \end{vmatrix}$$

which we expand according to the first column, to find

$$P_A(t) = t^3 - t^2 - 4t + 6.$$

For an arbitrary matrix $A = (a_{ij})$, the characteristic polynomial can be found by expanding according to the first column, and will always consist of a sum

$$(t - a_{11}) \cdots (t - a_{nn}) + \cdots .$$

Each term other than the one we have written down will have degree $< n$. Hence the characteristic polynomial is of type

$$P_A(t) = t^n + \text{ terms of lower degree.}$$

Theorem

Let A be an $n \times n$ matrix. A number λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A.

This Theorem gives us an explicit way of determining the eigenvalues of a matrix, provided that we can determine explicitly the roots of its characteristic polynomial.

Find the eigenvalues and a basis for the eigenspaces of the matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}.$$

Find the eigenvalues and a basis for the eigenspaces of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Find the eigenvalues and a basis for the eigenspaces of the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Theorem

Let A, B be two $n \times n$ matrices, and assume that B is invertible. Then the characteristic polynomial of A is equal to the characteristic polynomial of $B^{-1}AB$.

Eigenvalues and Eigenvectors of Symmetric Matrices

Theorem

Let A be a real symmetric matrix and let λ be an eigenvalue in $\mathbb{C}$. Then λ is real. If $Z \neq O$ is a complex eigenvector with eigenvalue λ, and $Z = X + iY$ where $X, Y \in \mathbb{R}^n$, then both X, Y are real eigenvectors of A with eigenvalue λ, and X or $Y \neq O$.

Proof

Let $Z = (z_1, ..., z_n)^t$ with complex coordinates z_i. Then

$$Z \cdot \bar{Z} = \bar{Z} \cdot Z = \bar{Z}^t Z = \bar{z}_1 z_1 + \cdots + \bar{z}_n z_n = |z_1|^2 + \cdots + |z_n|^2 > 0.$$

By hypothesis, we have $AZ = \lambda Z$. Then

$$\bar{Z}^t AZ = \bar{Z}^t \lambda Z = \lambda \bar{Z}^t Z.$$

The transpose of a 1×1 matrix is equal to itself, so we also get

Proof.

$$Z^t A^t \bar{Z} = Z^t A \bar{Z} = \lambda \bar{Z}^t Z.$$

But $\overline{AZ} = \bar{A}\bar{Z} = A\bar{Z}$ and $\overline{AZ} = \overline{\lambda Z} = \bar{\lambda}\bar{Z}$. Therefore

$$\lambda \bar{Z}^t Z = \bar{\lambda} Z^t \bar{Z}.\text{ Since } Z^t \bar{Z} \neq 0 \text{ it follows that } \lambda = \bar{\lambda}, \text{ so } \lambda \text{ is real.}$$

Now from $AZ = \lambda Z$ we get

$$AX + iAY = \lambda X + i\lambda Y,$$

and since A, X, Y are real it follows that $AX = \lambda X$ and $AY = \lambda Y$. This proves the theorem. □