MA7404
All Candidates

Semester 1 Examinations 2021

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	Mathematics and Actuarial Science
Module Code	MA7404
Module Title	Markov processes
Exam Duration	Two hours + 45 minutes upload time
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	7
Number of Questions	4
Instructions to Candidates	Answer all questions
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	Formulae and tables for actuarial examinations
Are students permitted to bring their own Books/Statutes/Notes?	Yes
Additional Stationery	Yes

1. An insurance company receives claims of sizes $X_1, X_2, \dots$. If claim X_i exceeds £1000, the insurance company pays £1000, and the remaining part $X_i - 1000$ is covered by reinsurance. Hence, the total payments made by the insurance company during the next week is $S = Y_1 + Y_2 + \dots + Y_N$, where N is the (random) number of claims to be received next week, and $Y_i = \min\{X_i, 1000\}$ for $i = 1, 2, \dots, N$. The company assumes that N can take values from 0 to K with equal probabilities, X_i are independent from each other and from N, identically distributed, and follow the uniform distribution on $[0, B]$. Parameters K and B are unknown and should be estimated from the past data.

(i) [5 marks] The numbers of claims during the last 10 weeks were

$$4, 0, 5, 6, 1, 0, 3, 2, 5 \text{ and } 4.$$

Use the method of maximum likelihood to estimate parameter K.

(ii) [10 marks] The sizes of the last 10 claims were

$$700, 800, 500, 100, 1000+, 900, 800, 1000+, 200 \text{ and } 300,$$

where $1000+$ indicates that the claim size is over £1000 but the exact size is unknown.

(a) **[5 marks]**. Which of the following methods would be the best to use to estimate parameter B: method of moments, method of percentiles at level $\alpha = 0.75$, or method of percentiles at level $\alpha = 0.95$? Please justify your answer.

(b) **[5 marks]**. Use the method selected in Part (a) to estimate the parameter B.

(iii) [10 marks]

(a) **[4 marks]**. Estimate the mean and variance of N.

(b) **[4 marks]**. Estimate the mean and variance of Y_i.

(c) **[2 marks]**. Estimate the mean and variance of S.

Total: 25 marks

Answer:

(i) Application (Similar to seen)

The likelihood of the listed data is 0 for $K < 6$ (because in this case there is a 0 chance for $N = 6$ claims happened in week 4). For any $K \geq 6$, the likelihood of any data point is $\frac{1}{K+1}$, hence the likelihood of the data is $\left(\frac{1}{K+1}\right)^{10}$. This is maximized at $K = 6$.

(ii)

(a) Higher skills (Unseen)

Because we have two data points with unknown value $1000+$, we cannot estimate neither expectation (needed for the method of moments) nor percentile at level $\alpha = 0.95$. Hence, method of percentiles at level $\alpha = 0.75$ is the best one in this situation.

(b) Application (Similar to seen)

From the data, for any small ε, we see that $P[X_i < 900-\varepsilon] = 0.7 < 0.75$, while $P[X_i < 900+\varepsilon] = 0.8 > 0.75$, hence the best constant C such that $P[X_i \leq C] \approx 0.75$ is $C = 900$.

If X_i is uniformly distributed on $[0,B]$, then $P[X_i \leq 900]$ is $\frac{900}{B}$. Hence, $\frac{900}{B} = 0.75$, hence $B = \frac{900}{0.75} = 1200$.

(iii)

(a) Application (Similar to seen)

$$E[N] = \frac{1}{7}\sum_{i=0}^{6} i = 3, \quad E[N^2] = \frac{1}{7}\sum_{i=0}^{6} i^2 = 13, \quad Var[N] = E[N^2] - (E[N])^2 = 13 - 3^2 = 4.$$

(b) Higher skills (Unseen)

$$E[Y_i] = E[Y_i|X_i < 1000]P[X_i < 1000] + E[Y_i|X_i \geq 1000]P[X_i \geq 1000] = 500 \cdot \frac{1000}{1200} + 1000\frac{200}{1200} = \frac{7000}{12} \approx 583$$

$$E[Y_i^2|X_i < 1000] = \frac{1}{1000}\int_0^1 000y^2 dy = \frac{1}{1000}\frac{1000^3}{3} = \frac{1000^2}{3},$$

$$E[Y_i^2] = E[Y_i^2|X_i < 1000]P[X_i < 1000] + E[Y_i^2|X_i \geq 1000]P[X_i \geq 1000] = \frac{1000^2}{3}\frac{1000}{1200} + 1000^2\frac{200}{1200} \approx 444,167$$

$$Var[Y_i] = E[Y_i^2] - (E[Y_i])^2 \approx 444,167 - 583^2 = 104,278.$$

(c) Application (Similar to seen)

$$E[S] = E[N] \cdot E[Y_i] = 1,750.$$

$$Var[S] = E[N]Var[Y_i] + Var[N](E[Y_i])^2 \approx 3 \cdot (104,278) + 4 \cdot (583)^2 = 1,672,390.$$

Syllabus cover

This question covers items 1.1 (loss distributions...) and 1.2 (compound distributions...) of the syllabus.

2. **(i) [10 marks]** A company receives claims from two policies. Let N and M be the numbers of claims from these policies, respectively. The company assume that there are three equally likely scenarios: (1) $N = 0$, $M = 2$, (2) $N = 1$, $M = 0$, and (3) $N = 2$, $M = 1$.

   (a) **[4 marks]**. Estimate Pearson correlation coefficient $Corr(N,M)$.

   (b) **[4 marks]**. Estimate Kendall coefficient of concordance.

   (c) **[2 marks]**. Based on the results in parts (a) and (b), discuss where N and M has direct dependence, inverse dependence, or independent.

   (ii) [10 marks] The sizes of claims from both policies follow the Gamma distribution, but with different parameters $\alpha > 0$ and $\gamma > 0$. Using the limiting density ratios test, determine whether the tail of the Gamma distribution becomes heavier or lighter if

(a) **[5 marks]**. λ is fixed and α increases;

(b) **[5 marks]**. α is fixed and λ increases.

(iii) [5 marks] The claim sizes X and Y are dependent with copula $C(u,v) = \min(u,v)$. Calculate the coefficient of upper tail dependence of X and Y.

Total: 25 marks

Answer:

(i) Application (Similar to seen)

(a) $E[M] = E[N] = \frac{0+1+2}{3} = 1$. $E[M^2] = E[N^2] = \frac{0^2+1^2+2^2}{3} = 5/3$. $Var[M] = Var[N] = 2/3$. $E[XY] = (0\cdot 2+1\cdot 0+2\cdot 1)/3 = 2/3$, $Cov(M,N) = 2/3-1^2 = -1/3$. $Corr(M,N) = \frac{-1/3}{\sqrt{2/3}\sqrt{2/3}} = -1/2$.

(b) When we go from state (1) to (2), N increases and M decreases, inverse dependence. When we go from (2) to (3), both increases. When we go from (3) to (1), N decreases but M increases. Hence we have $C = 1$ concordant pairs, and $D = 2$ discordant ones, and

$$\tau = \frac{C-D}{C+D} = \frac{1-2}{1+2} = -\frac{1}{3}.$$

(c) Because $Corr(M,N) < 0$ and $\tau < 0$, we have inverse dependence.

(ii) Application (Similar to seen)

If $\alpha_1 > \alpha_2$, then

$$\lim_{x\to\infty} \frac{x^{\alpha_1-1}e^{-\lambda x}}{x^{\alpha_2-1}e^{-\lambda x}} = \lim_{x\to\infty} x^{\alpha_1-\alpha_2} = +\infty,$$

hence increasing α makes tail heavier.

If $\lambda_1 > \lambda_2$, then

$$\lim_{x\to\infty} \frac{x^{\alpha-1}e^{-\lambda_1 x}}{x^{\alpha-1}e^{-\lambda_2 x}} = \lim_{x\to\infty} e^{(\lambda_2-\lambda_1)x} = 0,$$

hence increasing λ makes tail lighter.

(iii) Application (Similar to seen)

The coefficient of upper tail dependence is

$$\lambda_U = \lim_{u\to 0+} \frac{\bar{C}(u,u)}{u},$$

where $\bar{C}$ is the survival copula defined by $\bar{C}(1-u,1-v) = 1-u-v+C(u,v)$. This implies that $\bar{C}(u,u) = 1-(1-u)-(1-u)+C(1-u,1-u) = 2u-1-\min(1-u,1-u) = 2u-1-(1-u) = u$.

Hence,

$$\lambda_U = \lim_{u\to 0+} \frac{u}{u} = 1.$$

Syllabus cover

This question covers items 1.3 (introduction to copulas) and 1.4 (introduction to extreme value theory) of the syllabus.

3. An insurance company is considering investing their capital into a financial instrument whose current price is $P_0 = 1$ and future price after t weeks is modelled as

$$P_t = \exp(0.1t + 0.2X_t), \quad t = 0, 1, 2, \ldots,$$

where X_t is the symmetric simple unrestricted random walk on integers starting at $X_0 = 0$.

(i) [5 marks] State whether each of the following statements is true or false. The stochastic process X_t:

(a) **[1 mark]**. is stationary;

(b) **[1 mark]**. is weakly stationary;

(c) **[1 mark]**. has independent increments;

(d) **[1 mark]**. has stationary increments;

(e) **[1 mark]**. is a Markov chain.

(ii) [5 marks] Answer the same questions as in Part (i) for the stochastic process P_t.

(iii) [5 marks] Estimate the probability that $X_7 = 5$.

(iv) [5 marks] Find the expectation and variance of X_7.

(v) [5 marks] Estimate the probability that $P_7 < 2$.

Total: 25 marks

Answer:

(i)Application (Similar to seen)

Week stationarity implies that $E[X_t]$ and $Var[X_t]$ are constants independent from t. However, X_0 is a constant and $Var[X_0] = 0$, but $Var[X_1] > 0$, so $Var[X_1] \neq Var[X_0]$. Hence, the process is not weekly stationary. Hence, it is not stationary.

By definition of the symmetric simple unrestricted random walk, $X_t = \sum_{i=1}^{t} Y_i$, where Y_i are independent, and such that $Y_i = \pm 1$ with equal probabilities. The increments Y_i are i.i.d., hence X_t has stationary and independent increment. Independent increments imply Markov property, time and state space are discrete, so the process in the Markov chain. In summary, we have

(a) False, (b) False, (c) True, (d) True, (e) True.

(ii) Application (Similar to seen)

(a) False, (b) False, (c) True, (d) False, (e) True.

(iii) Application (Similar to seen)

A possible path from state $X_0 = 0$ to state $X_7 = 5$ is $0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 5$. We see that there are 6 steps "up" and 1 "down". Because the step "down" can be at any time, there are 7 such trajectories in total. The probability of each trajectory is $(1/2)^7$. Hence, the answer is $\frac{7}{128}$.

(iv) Higher skills (Unseen)

$P_7 < 2$ if $\exp(0.1 \cdot 7 + 0.2X_7) < 2$, or $0.7 + 0.2X_7 < \ln(2)$, or $X_7 < \frac{\ln(2)-0.7}{0.2} \approx -0.034$. Because X_7 is an integer, this is equivalent to $P(X_7 < 0)$. Because X_7 is never 0, and from symmetry, $P(X_7 < 0) = 0.5$.

(v) Higher skills (Unseen)

If $Y_i = \pm 1$ with equal probabilities, then $E[Y_i] = 0$, $E[Y_i^2] = 1$, hence $Var[Y_i] = 1$. Now,

$$E[X_7] = \sum_{i=1}^{7} E[Y_i] = 0,$$

and

$$Var[X_7] = \sum_{i=1}^{7} Var[Y_i] = 7.$$

Syllabus cover

This question covers item item 3.1 of the syllabus: Describe and classify stochastic processes, and item 3.2: Define and apply a Markov chain.

4. To model COVID-19 sickness and mortality, a life insurance company uses a time-inhomogeneous sickness-death model, with states H (healthy), S (sick), C (critically sick) and D (dead), and age-dependent transition rates $\sigma(t)$, $\rho(t)$, $\mu(t)$, $v(t)$ and $w(t)$ for transitions $H \to S$, $S \to C$, $C \to D$, $C \to S$, and $S \to H$, respectively. Here t is an age measured in years. All other transition rates are assumed to be 0.

   (i) [10 marks] Write down

   (a) **[5 marks]** the generator matrix and the transition graph.

   (b) **[5 marks]** the Kolmogorov forward equation for transition probability $p_{HH}(s,t)$ $\left(\frac{\partial p_{HH}(s,t)}{\partial t} = \dots\right)$.

   (ii) [5 marks] Assuming that $\rho(t) = 0.1\sqrt{t}$, and $w(t) = 200/t$, estimate the probability that a person aged 50, currently in state S, will remain there continuously for the next year.

   (iii) [5 marks] Is the transition probability $p_{CC}(50,51)$ greater than, less than, or equal to the answer computed in Part (ii)? Justify your answer.

   (iv) [5 marks] To apply this model, you need to be able to classify patients as those considered to be "sick" and those considered to be "critically sick". You have a sample of $10,000$ patients you want to classify. The doctors can classify for you any 100 patients on your choice, but no more because they are too busy. They are, however, ready to give you all the information about all patients, such as their temperature, blood pressure, etc. Explain how you can use the nearest-neighbour method in supervised machine learning to classify the remaining patients.

   Total: 25 marks

Answer:

(i)

(a) Application (Similar to seen)

$$Q(t) = \begin{pmatrix} -\sigma(t) & \sigma(t) & 0 & 0 \\ w(t) & -w(t)-\rho(t) & \rho(t) & 0 \\ 0 & v(t) & -v(t)-\mu(t) & \mu(t) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(b) Application (Similar to seen)

$$\frac{\partial p_{HH}(s,t)}{\partial t} = p_{HH}(s,t)(-\sigma(t)) + p_{HS}(s,t)w(t).$$

(ii) Application (Similar to seen)

The residual holding time at state S is

$$P(R_s > h-s | X_s = S) = e^{-\int_s^h (\rho(u)+w(u))du}.$$

In our case,

$$P(R_{50} > 1 | X_{50} = S) = e^{-\int_{50}^{51}(0.1\sqrt{u}+200/u)du} = e^{-0.1(51^{3/2}-50^{3/2})/(3/2)-200(\ln(51)-\ln(50))} \approx e^{-4.67} \approx 0.0094.$$

(iii) Application (Similar to seen)

$p_{CC}(50,51)$ is greater than the probability to stay sick continuously, because it also include the probability to move to some other state and then back to sick state.

(iv) Higher skills (Unseen)

Assume that for each patient you have n results of various measurements, for example, x_1 is her/his temperature, x_2 is the blood pressure, and so on. All the results $(x_1,\dots,x_n)$ can be represented as a point in n-dimensional space. You colour points corresponding to patients which doctors classify as “critically sick” as red, and points corresponding to patients which doctors classify as “sick” as blue. Then you select any uncoloured point X at random, find the nearest blue point to it and the nearest red one. If the blue point is closer, you colour X in blue, otherwise in red. Then repeat this procedure.

Syllabus cover

This question covers item 3.3 Define and apply a Markov process, and item 5 Machine Learning.