Lecture 4: Vectors and the Geometry of Space.

MA2032 Vector Calculus

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The Cross Product

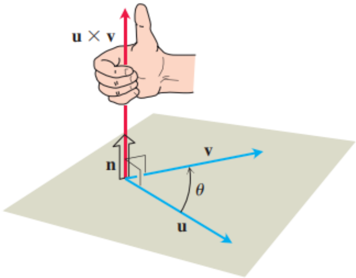
- In studying lines in the plane, when we needed to describe **how a line was tilting**, we used the notions of slope and angle of inclination.
- In space, we want a way to describe **how a plane is tilting**.
- We accomplish this by **multiplying two vectors** in the plane together to get a third **vector perpendicular to the plane**.
- The **direction of this third vector tells us the "inclination" of the plane**.
- The product we use to multiply the vectors together is the vector or **cross product**, the second of the two vector multiplication methods.
- The cross product gives us a simple **way to find a variety of geometric quantities**, including volumes, areas, and perpendicular vectors.

The Cross Product of Two Vectors in Space

• Two vectors are **parallel** if one is a nonzero multiple of the other.

• If u and v are not parallel, they **determine a plane**.

• The vectors in this plane are **linear combinations of** u **and** v, so they can be written as a sum $au + bv$.



• We select the **unit vector** n **perpendicular to the plane** by the right-hand rule, see Figure.

• Then we define a new vector as follows.

Definition:

The cross product $u \times v$ ("u cross v") is the vector

$$u \times v = (|u|\ |v| \sin\theta) n.$$

The Cross Product of Two Vectors in Space

- Unlike the dot product, **the cross product is a vector**.
- For this reason it is also called the **vector product** of u and v, and can be applied only to vectors in space.
- The **vector** $u \times v$ **is orthogonal to both** u **and** v because it is a scalar multiple of n.

Parallel Vectors

Nonzero vectors u and v are **parallel** ($\theta = 0$ or $\theta = \pi \Rightarrow \sin = 0$) if and only if $u \times v = 0$.

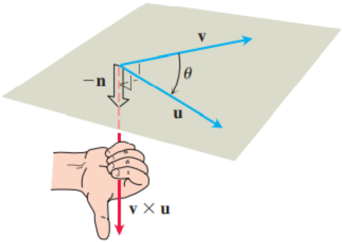
Properties of the Cross Product

If u, v, and w are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$	**2.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$	**4.** $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$	**6.** $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

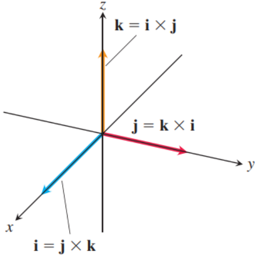
Property 3

- Visualization of Property 3: when the fingers of your right hand curl through the angle θ from v to u, your thumb points the opposite way.
- The unit vector we choose in forming $v \times u$ is the negative of the one we choose in forming $u \times v$, see Figure.



When we apply the definition and Property 3 to calculate the **pairwise cross products of** i**,** j**, and** k, we find

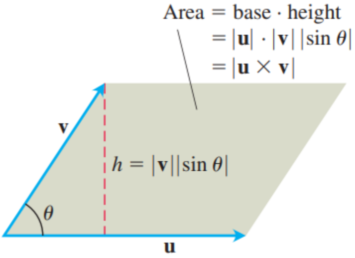
$$
\begin{aligned}
i \times j &= -(j \times i) = k, \\
j \times k &= -(k \times j) = i, \\
k \times i &= -(i \times k) = j, \\
i \times i = j \times j &= k \times k = 0.
\end{aligned}
$$



Area of a Parallelogram

• Because n is a unit vector, the magnitude of $u \times v$ is $|u \times v| = |u|\,|v|\,|\sin\theta|\,|n| = |u|\,|v|\sin\theta.$

• This is the **area of the parallelogram** determined by u and v, see Figure $|u|$ being the base of the parallelogram and $|v|\,|\sin\theta|$ the height.



Determinant Formula for $u \times v$

• Our next objective is to calculate $u \times v$ **from the components** of u and v relative to a Cartesian coordinate system.

Suppose that $u = u_1 i + u_2 j + u_3 k$ and $v = \nu_1 i + \nu_2 j + \nu_3 k$.
Then the distributive laws and the rules for multiplying i, j, and k tell us that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\
&= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\
&= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.
\end{aligned}$$

The component terms in the last line are the same as the terms in the expansion of the symbolic **determinant**

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

Determinant Formula for $u \times v$

Calculating the Cross Product as a Determinant

If $u = u_1 i + u_2 j + u_3 k$ and $v = \nu_1 i + \nu_2 j + \nu_3 k$, then

$$u \times v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

Example: Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$, see Figure.

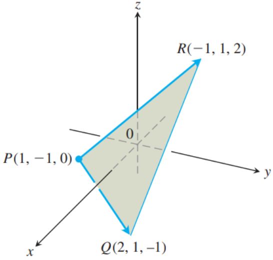
Solution:

The vector $\overrightarrow{\mathrm{PQ}} \times \overrightarrow{\mathrm{PR}}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

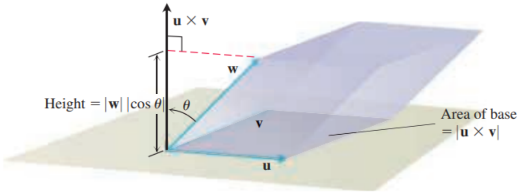
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$



Triple Scalar or Box Product

- The product $(u \times v) \cdot w$ is called the **triple scalar product** of u, v, and w (in that order).
- As you can see from the formula
$|(u \times v) \cdot w| = |u \times v||w||\cos\theta|$,
the absolute value of this product is the **volume of the parallelepiped** (parallelogram-sided box) determined by u, v, and w, see Figure.
- The number $|u \times v|$ is the **area of the base** parallelogram.
- The number $|w||\cos\theta|$ is the **parallelepiped's height**.
- Because of **this geometry**, $(u \times v) \cdot w$ is also called the box product of u, v, and w.



Volume = area of base · height
= $|\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,|\cos\theta|$

Triple Scalar Product

The triple scalar product can be evaluated **as a determinant**:

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right) \cdot \mathbf{w} \\
&= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\end{aligned}
$$

Calculating the Triple Scalar Product as a Determinant

$$
(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ \nu_1 & \nu_2 & \nu_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
$$