Chapter 4

Sequences

We have seen that limits are at the core of calculus and analysis: every single notion is a limit in one sense or another! Examples where limits are used include the slope of the tangent line, the length of a curve, the area under a graph. Moreover, Calculus is at the basis of most modern mathematics. In other words, limits are the building blocks for all your future endeavours in mathematics.

You have seen the notion of a limit of a function, but now we are looking at the simplest example of a function: a sequence. Sequences often appear in mathematics and scientific experiments: when trying to approximate a value, we usually construct a sequence of numbers approaching the value up to a given error. An example for this is Newton's method. In particular, we are interested in infinite sequences and in the questions whether the elements of a sequence approach a limit.

4.1 Definition of a sequence

Here we discuss the proper definition and the notion of a limit.

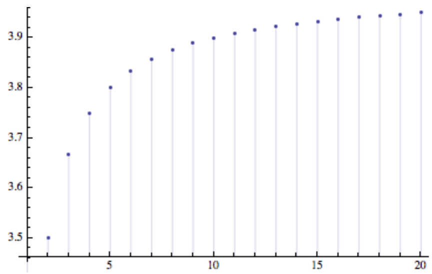
Definition 4.1. A *sequence* of real numbers is a real valued function defined on the natural numbers:

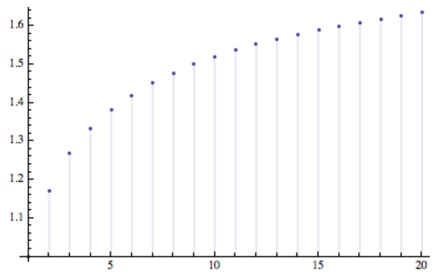
$$a : \mathbb{N} \to \mathbb{R}\,.$$

We write for short $a_n = a(n)$ and denote a sequence by $(a_n)_{n\in\mathbb{N}}$.

We often give sequences as follows:

$$(a_n)_{n\in\mathbb{N}},\ a_n = \frac{4n-1}{n}, \qquad \text{or} \qquad \left(\frac{2\sqrt{n}}{\sqrt{n}+1}\right)_{n\in\mathbb{N}}.$$





From our definition it is obvious that we can obtain new sequences by adding, subtracting, multiplying, dividing the n–th components of two sequences for all $n \in \mathbb{N}$ (after all, sequences are special cases of functions!):

Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we obtain a new sequence $(c_n)_{n\in\mathbb{N}}$ by setting

- $c_n = a_n + b_n$
- $c_n = a_n - b_n$.
- $c_n = a_n \cdot b_n$.
- $c_n = \frac{a_n}{b_n}$, provided $b_n \neq 0$ for all $n \in \mathbb{N}$.

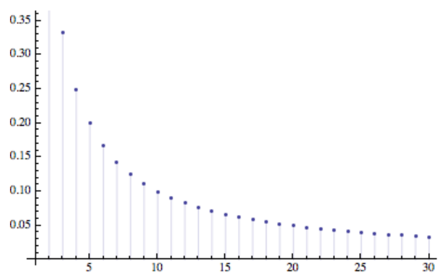
Our interest is to understand the behaviour of the sequence; what can we say about the shape of the graph of the function? Here are some useful notions:

Definition 4.2. A sequence $(a_n)_{n\in\mathbb{N}}$ is called

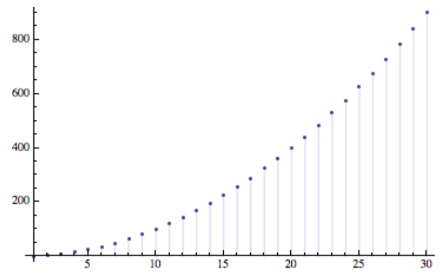
(i) *monotonic sequence* if
   - either $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ (increasing),
   - or $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ (decreasing).

(ii) *bounded sequence* if there exist $m, M \in \mathbb{R}$: $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.

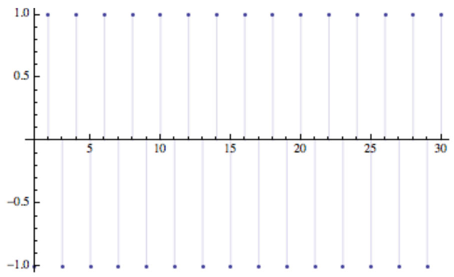
Example 4.3. The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is (strictly) decreasing and bounded.



The sequence $(n^2)_{n\in\mathbb{N}}$ is (strictly) increasing but not bounded.



The sequence $((-1)^n)_{n\in\mathbb{N}}$ is bounded but not monotonic.



4.2 Limits of sequences

Our main question is however to investigate how a sequence behaves when n gets very large: does it approach a particular value, does it disappear to infinity or is behaved very irregularly?

For example the sequence

$$(a_n)_{n\in\mathbb{N}}, a_n = \frac{1}{n}$$

seems to be nicely behaved; the larger n gets, the closer we get to 0. So should we say that $(a_n)_{n\in\mathbb{N}}$ converges to a limit L if a_n gets close to L?

This definition is too vague: consider the sequence $(a_n)_{n\in\mathbb{N}}$,

$$a_n = \begin{cases} n, & n \neq 20 \\ 0, & n = 20 \end{cases}.$$

It gets close to 0 since the twentieth element is actually equal to zero – but that's clearly not what we understand as the limit when n growth really large.

So we need to incorporate that the a_n's should get close not once but when we go out to infinity.

How about the sequence $(a_n)_{n\in\mathbb{N}}$

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ n & n \text{ even} \end{cases}.$$

When n growth large, the odd elements of the sequence get closer and closer to 0 – but the even ones disappear to infinity! Again, this should not be meant when saying that a number is the limit of a sequence.

What we want is that L is the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ if " a_n is arbitrary close to L for all n large enough".

What we need now is a mathematical way to formulate the above statement. When a_n is close to L then the distance between a_n and L is small. Recall that the *absolute value* measures the distance to 0, so we want that $|a_n - L|$ can be made arbitrarily small by taking n large enough. So given a small number ε we can find a number $K \in \mathbb{N}$ so that whenever n is bigger than K, the distance between a_n and L is smaller than ε, that is, $|a_n - L| < \varepsilon$.

Voilà, that's our definition:

Definition 4.4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. We say that a_n has limit L,

$$\lim_{n\to\infty} a_n = L\,,$$

if for all $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \qquad \text{for all } n \geq K\,.$$

If a sequence has a limit it is said to be a *convergent sequence*, otherwise it is called a *divergent sequence*.

Note that the order of ε and K is crucial: we first choose how close we want to come to L and then have to find how large we have to be with n for this to be true.

Convince yourself that the "beginning" of a sequence has no impact on the convergence of the sequence. What is important is what happens for large n: "All's Well That Ends Well". In particular, one cannot investigate the convergence of a sequence by computing finitely many elements with a computer. In other words, if you compute a few values of a sequence and then only guess the limit you will get zero marks; a proper argument is needed.

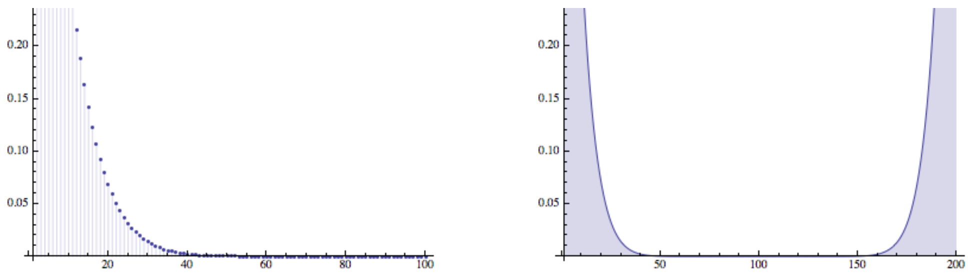


Figure 4.1: looks like it converges to 0 after 100 evaluations. After 200 the picture tells a different story!

Example 4.5. The sequence $(a_n)_{n\in\mathbb{N}}$, $a_n = \frac{4n-1}{n}$ converges to $L = 4$.

"Find K" Let us do some scratch work first to understand what we need to find.

We want to find $K \in \mathbb{N}$ such that $|\frac{4n-1}{n} - 4| < \varepsilon$ whenever $n \geq K$. Simplifying we get $|\frac{4n-1}{n} - 4| = \frac{|4n-1-4n|}{n} = \frac{1}{n}$. Thus, we take any $K \in \mathbb{N}$ with $\frac{1}{K} < \varepsilon$. (Note: we cannot say $\frac{1}{K} = \varepsilon$ because ε might not be rational).

"K works" Verify that our candidate for K indeed satisfies the conditions. This is the actual argument.

Let $\varepsilon > 0$ and let $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$. Then for all $n \geq K$ we have

$$|\frac{4n-1}{n} - 4| = \frac{|4n-1-4n|}{n} = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$$

since $n \geq K$ implies $\frac{1}{n} \leq \frac{1}{K}$. This shows that $\lim_{n\to\infty} \frac{4n-1}{n} = 4$.

If you are unfamiliar with the manipulations of inequalities, see the appendix in this Chapter.

Similarly, we show that the sequence $\left(\frac{2\sqrt{n}}{\sqrt{n}+1}\right)_{n\in\mathbb{N}}$ converges to 2.

The standard method to use has two parts:

" Find K" scratch work to find K.

"K works" verify that your candidate for K indeed satisfies the conditions.

Common errors:

(i) Argument only contains the "find" part. (This will result in major reduction in marks).

(ii) Argument does not contain the "find" part. (Usually, this will result in major reduction in marks unless it is kind of obvious how to find the K. If in doubt, show the work).

(iii) ε is not introduced or appears only after the K. (This will result in a zero mark: you can not find K if the ε is not given in advance).

Just to clarify this rather abstract question: can there be a sequence converging to two different limits? The answer is no:

Theorem 4.6. *If $(a_n)_{n\in\mathbb{N}}$ is a sequence with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ then $L = M$.*

Proof. Theorem 11.3.2 in the supplement to section 11.3. □

Theorem 4.7. *Every convergent sequence is bounded.*

The converse is false, there are bounded sequences which are divergent. For example $a_n = (-1)^n$.

However, the theorem implies that if a sequence is unbounded, then it is divergent.

Geometrically, we would expect a monotonic sequence to be convergent if it is bounded: it cannot disappear to ∞ but also cannot jump around between different values. More precisely:

Theorem 4.8. *If a sequence is bounded and monotonic then it is convergent.*

It is not so easy to *prove* a limit exists: it we will need the Least Upper Bound axiom for the real numbers that we saw in chapter 1.

4.3 Theorems for limits of sequences

Theorem 4.9. *If a sequence is bounded and monotonic then it is convergent.*

Of course, we don't want to use the ε–K criterion every single time. So, we try to find some general rules which allow to derive a limit, once we know simpler ones.

Limit laws If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences then $(a_n \pm b_n)_{n\in\mathbb{N}}$, $(a_n b_n)_{n\in\mathbb{N}}$ and, provided $b_n \neq 0 \forall n$, $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ are convergent, and

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n, \quad \lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n), \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Note that we also assume $\lim_{n\to\infty} b_n \neq 0$.

For the limit laws it is important that the individual limits exist. For example, for $a_n = \frac{1}{n}, b_n = n$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{1}{n} n = \lim_{n\to\infty} 1 = 1$$

whereas $\lim_{n\to\infty} a_n = 0$ and you might have thought that $\lim_{n\to\infty} a_n b_n = 0$. But $\lim_{n\to\infty} b_n$ does not exist so therefore, you cannot use the limit laws to compute the limit. In fact, in this case

$$\lim_{n\to\infty} a_n b_n \neq (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)\,.$$

A similar example is given by $a_n = \frac{1}{n}, b_n = (-1)^n$. You might be tempted to write

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = 0$$

since $\lim_{n\to\infty} a_n = 0$. However, you are not allowed to use the limit laws, since $(b_n)_{n\in\mathbb{N}}$ is not convergent. You need to use a more refined argument to find whether the limit of $(a_n b_n)_{n\in\mathbb{N}}$ exists.

Thus, if you use the limit laws you have to state why you are allowed to so (mark reduction otherwise).

By induction we conclude from the Limit Laws:

Corollary 4.10. *Let* $(a_n^{(k)})_{n\in\mathbb{N}}$ *be convergent sequences of real numbers for* $k = 1, \ldots, r$. *Then*

$$\lim_{n\to\infty} \left(\sum_{l=1}^{r} \alpha_l a_n^{(l)} \right) = \sum_{l=1}^{r} \alpha_l (\lim_{n\to\infty} a_n^{(l)})$$

and

$$\lim_{n\to\infty} \left(\prod_{l=1}^{r} a_n^{(l)} \right) = \prod_{l=1}^{r} (\lim_{n\to\infty} a_n^{(l)})$$

Assume we have a situation where we know the behaviour of two sequences and have a third sequence lying in between like in the following picture:

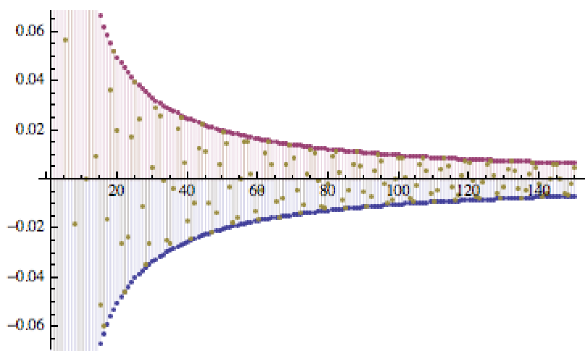


Figure 4.2: If the limits of the blue and red sequence are known, can we conclude the convergence of the green sequence?

Theorem 4.11 (Pinching theorem for sequences). *Let* $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ *be sequences such that there exists* $N \in \mathbb{N}$ *with*

$$a_n \leq b_n \leq c_n \quad \textit{for all } n \geq N\,.$$

If $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are convergent with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $(b_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} b_n = L$.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ there exists $K_1 \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq K_1$ and $K_2 \in \mathbb{N}$ such that $|c_n - L| < \varepsilon$ for all $n \geq K_2$. Let $K = \max\{N, K_1, K_2\}$ then we have

$$L - \varepsilon < a_n < L + \varepsilon, \quad L - \varepsilon < c_n < L + \varepsilon, \quad \text{and } a_n \leq b_n \leq c_n$$

for all $n \geq K$. Thus, combining all these inequalities we have

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

which shows that $L - \varepsilon < b_n < L + \varepsilon$ and thus $|b_n - L| < \varepsilon$ for all $n \geq K$. This shows that $\lim_{n\to\infty} b_n = L$.

□

In particular, if

$|b_n| \leq c_n$ and $c_n \to 0$ then $b_n \to 0$.

4.4 Cauchy sequences and the Bolzano–Weierstrass theorem

We have seen that every convergent sequence is bounded, but the converse is not true. However, if we only consider selected (but infinitely many) elements one can always obtain a convergent subsequence. For example, $(-1)^n$ not convergent, but it is bounded, and contains two obvious convergent (constant!) subsequences.

Definition 4.12. Let $(a_n)_{n\in\mathbb{N}}$ a sequence. A subsequence of $(a_n)_{n\in\mathbb{N}}$ is a sequence $(b_n)_{n\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$ there exists a $n_k \in \mathbb{N}$, $n_1 < n_2 < \ldots$, with $b_k = a_{n_k}$.

To understand better the relation between convergent subsequences and convergent sequences, we consider the notion of Cauchy sequences. Notice that if a sequence $(a_n)_{n\in\mathbb{N}}$ is convergent, then all elements a_n are close to the limit for large enough n. But then the distance between two elements also gets small. Sequences with this property are important in mathematics.

Definition 4.13. A sequence $(a_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \forall m, n \geq N\,.$$

This definition does not mention any limits, so we are not just repeating the definition of convergent sequence.

However, We can prove all usual properties of convergent sequences for Cauchy sequences too. For example:

Theorem 4.14. *Every Cauchy sequence is bounded.*

Proof. If $(a_n)_{n\in\mathbb{N}}$ is Cauchy then (for $\varepsilon = 1$) there exists some $N \in \mathbb{N}$ such that

$$|a_n - a_m| < 1 \quad \forall n, m \geq N\,.$$

In particular

$$|a_n - a_N| < 1, \text{ and so } |a_n| < 1 + |a_N|, \quad \forall n \geq N\,.$$

So $M = \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\}$ is a bound for $(a_n)_{n\in\mathbb{N}}$. □

Indeed:

Theorem 4.15. *Every convergent sequence is a Cauchy–sequence.*

Proof. Let $\varepsilon > 0$ and $L = \lim_{n\to\infty} a_n$. Then there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\varepsilon}{2}, \qquad \text{for all } n \geq N\,.$$

Therefore,

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all } n, m \geq N\,.$$

This shows that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

In fact converse is true as well! We will see this in a little while.

Theorem 4.16 (Bolzano–Weierstrass theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then there exists a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$.*

Proof. The proof is a constructive one, that is, it not only gives the existence of this subsequence but also shows how a bisection method can be used to find it.

First note that, since $(a_n)_{n\in\mathbb{N}}$ is bounded there exists $M > 0$ with $|a_n| \leq M$ for all $n \in \mathbb{N}$. Next we show that there exists, for all $n \in \mathbb{N}$, a closed interval $I_n = [\alpha_n, \beta_n]$ such that

- the length $|I_n| = \beta_n - \alpha_n = \frac{1}{2^{n-1}}M$
- $I_n = [\alpha_n, \beta_n]$ contains infinitely many terms of the sequence $(a_n)_{n\in\mathbb{N}}$
- α_n is an increasing sequence
- β_n is a decreasing sequence

We prove this by induction:

Base case: Let $I_0 = [\alpha_0, \beta_0] = [-M, M]$, of length $2M = \frac{1}{2^{-1}}M$.

Inductive step: If we know the result is true for some $n = k$, we can divide I_k into two equal parts

$$[\alpha_k, \tfrac{1}{2}(\alpha_k + \beta_k] \text{ and } [\tfrac{1}{2}(\alpha_k + \beta_k, \beta_k]$$

One of these will contain infinitely terms of the sequence, call it $I_{k+1} = [\alpha_{k+1}, \beta_{k+1}]$. In either case, as $[\alpha_{k+1}, \beta_{k+1}] \subset [\alpha_k, \beta_k]$, we have $\alpha_k \leq \alpha_{k+1} < \beta_{k+1} \leq \beta_k$. The length is $|I_{k+1}| = \frac{1}{2}|I_k| = \frac{1}{2}\frac{1}{2^{k-1}}M = \frac{1}{2^k}M$.

Thus, the result for $n = k$ implies the result for $n = k + 1$.

From these bisections of intervals we can construct a subsequence. Since each I_k has infinitely terms of the sequence, just pick one from each, $a_{n_k} \in I_k$, making sure that $n_0 < n_1 < n_2 < \ldots$.

To see this subseqence converges, we note that

- the sequences (α_k) and (β_k) converge, as they are bounded and monotonic.
- they converge to the same limit as $\beta_k - \alpha_k \to 0$.
- a_{n_k} also converges to the same limit, by the Pinching theorem, as $\alpha_k \leq a_{n_k} \leq \beta_k$.

□

Using the Bolzano–Weierstrass theorem we can prove that indeed all Cauchy–sequences are convergent.

Theorem 4.17. *Every Cauchy sequence is convergent.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. We have seen that that $(a_n)_{n\in\mathbb{N}}$ is bounded. By Bolzano–Weierstrass we know also that it has a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converging to some limit $a \in \mathbb{R}$. Using the triangle inequality, we will prove that the Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ also converges, to the same limit a.

Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence there is $N \in \mathbb{N}$ with

$$|a_n - a_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N \,.$$

Since $(a_{n_k})_{k\in\mathbb{N}}$ converges to a there exists $k \in \mathbb{N}, k \geq N$ with

$$|a_{n_k} - a| < \frac{\varepsilon}{2} \,.$$

Since $n_k \geq k \geq N$ we have for $n \geq N$

$$|a_n - a_{n_k}| < \frac{\varepsilon}{2} \,.$$

But then

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \varepsilon \quad \forall n \geq N \,.$$

□