MA1013
All candidates

Midsummer Examinations 2017

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
Department	Mathematics
Module Code	MA1013
Module Title	Calculus and Analysis II
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	3
Number of Questions	4
Instructions to Candidates	Answer all questions. All marks gained will be counted. All questions carry equal weight.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	No
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

1. (a) Define *convergence* of a sequence of real numbers $(a_n)_{n\geq 0}$ to a limit L. [3 marks]

   (b) Prove, directly from the ε–N definition, that $\dfrac{n+2}{n+1} \to 1$ as $n \to \infty$. [5 marks]

   (c) Determine whether the following sequences converge, and find their limits if they do. (You must justify your answers, but ε–N arguments are not required).

   $$\text{(i) } (\cos(n\pi))_{n\geq 0}, \qquad \text{(ii) } (\sin(n\pi))_{n\geq 0}, \qquad \text{(iii) } \left(\frac{\cos n}{1+n}\right)_{n\geq 0}.$$

   [9 marks]

   (d) Prove that every Cauchy sequence is bounded. [6 marks]

   (e) Give an example of a bounded sequence that is not a Cauchy sequence.

   [2 marks]

2. (a) i. Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ that is not integrable.

   ii. Prove that every monotonic decreasing function $f : [a,b] \to \mathbb{R}$ is bounded and is integrable.

   [5 marks]

   (b) Suppose $f : [0,1] \to \mathbb{R}$ is an integrable function for which $\int_0^1 f(x)dx$ is zero.

   i. Prove if $f(x)$ is continuous then there exists a point $c \in (0,1)$ such that $f(c) = 0$.

   ii. Give an example that shows the word 'continuous' here is necessary.

   [5 marks]

   (c) Solve the Initial Value Problem $\quad y'' + y = x, \quad y(0) = y'(0) = 2.$ [5 marks]

   (d) Use an appropriate substitution to evaluate $\displaystyle\int_2^x \frac{dt}{t\,(\ln t)^p}$ for any $p > 1$. [5 marks]

   (e) Prove that the infinite series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^p}$ converges if $p > 1$. [5 marks]

3. Consider the function of two variables $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

   (a) Prove f is continuous at the origin. [5 marks]

   (b) i. Compute the first partial derivatives $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.
   ii. Prove, directly from the definition, that $f_x(0,0) = 1$ and $f_y(0,0) = 0$.
   [6 marks]

   (c) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(1,1,\frac{1}{2})$. [4 marks]

   (d) Prove that neither f_x nor f_y are continuous at the origin. [6 marks]

   (e) The *directional derivative* of f at $\underline{x}$ in the direction of the unit vector $\underline{\hat{u}} = (p,q)$ is
   $$f_{\underline{\hat{u}}}(\underline{x}) = \lim_{h\to 0} \frac{f(\underline{x}+h\underline{\hat{u}}) - f(\underline{x})}{h}.$$
   Prove that for the function f above the directional derivative at the origin in the direction $\underline{\hat{u}} = (p,q)$ is given by $f_{(p,q)}(0,0) = p^3$. [4 marks]

4. (a) Define the notion of a *critical point* of a function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$. [3 marks]

   (b) State the Extreme Value Theorem for functions $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$. [4 marks]

   (c) i. Find the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
   $$f(x,y) = 3x^2 + 6xy + 6y^2 - 2x + 4y.$$
   ii. Use the Hessian to classify these critical points.
   [6 marks]

   (d) i. Find the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
   $$f(x,y) = \frac{xy}{1+x^2+y^2}.$$
   ii. Classify them as maximum, minimum or saddle *without* using the Hessian.
   [6 marks]

   (e) Find the (x,y,z)-coordinates of the global maximum and the global minimum of
   $$z = 4y - \frac{2}{3}y^3 - 4x^2y$$
   when the point (x,y) is restricted to the curve C given by the ellipse
   $$\{(x,y) \in \mathbb{R}^2 : x^2 + (\tfrac{y}{2})^2 = 1\}.$$
   [6 marks]