LINEAR ALGEBRA II

Ch. II MATRICES

Ch. II Matrices

- A $m \times n$ matrix in K

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- A_i: the i-th row vector of A.
- A^j: the j-th column vector of A.
- Square matrix.
- Zero matrix $O_{m\times n}$, Unit (identity) matrix I_n.
- ${}^{\mathrm{t}}A$ or A^{T}: Transpose of A.
- For a complex matrix $A = A_1 + \mathrm{i}A_2$. $\bar{A} = (\bar{a}_{ij}) = A_1 - \mathrm{i}A_2$.
  ${}^{\mathrm{t}}(\bar{A}) = \overline{{}^{\mathrm{t}}A} = {}^{\mathrm{t}}\bar{A} = A^* = \bar{A}^{T}$

Ch. II Matrices

- $\mathrm{Mat}_{m\times n}(K)$ ($K^{m\times n}$) is a VS over K of dimension $m \times n$. K^{mn}

  $\{ E_{ij} \mid (i=1,\cdots,m,\ j=1,\cdots,n) \}$ is a basis of it.

- All symmetric matrices (${}^{t}A = A$). $n\times n$ S_n, $\dim S_n = \frac{n(n+1)}{2}$

  $\{ E_{ij}+E_{ji},\ E_{ii} \mid i=1,\cdots,n,\ j=1,\cdots,n \}$

- All diagonal matrices.

- All upper-triangular (lower-triangular). $\frac{n(n+1)}{2}$

  $\{ E_{ij} \mid i \le j \ (\ge) \}$

- All strictly upper-triangular (lower-triangular).

  $\{ \ \cdots \mid i<j \}$ $\frac{n(n-1)}{2}$

Ch. II Matrices

- A system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- $AX = B$. $= (A^1, \cdots, A^n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B$
- $x_1A^1 + \cdots + x_nA^n = B$.
- If $n > m$, then $AX = O$ has a non-trivial solution in K.

$x_1 A^1 + \cdots + x_n A^n = 0$ $\quad A^i \in K^m$

Ch. II Matrices

- If $n = m$ and the vectors $A^1, \ldots, A^n$ are linearly independent, then
  - $AX = B$ has a unique solution in K.

    $x_1 A^1 + \cdots + x_n A^n = B$
  - The only solution of $AX = O$ is the trivial solution.
- The set of solutions of $AX = O$ is a vector space over K.

Ch. II Matrices

- (Do this here and now!) Let $A^1, \ldots, A^n$ be column vectors of size m. Assume that they have entries in R, and that they are linearly independent over R. Show that they are linearly independent over C.

Ch. II Matrices

- Let $AX = O$ be a system of homogeneous linear equations with coefficients in R. If it has a non-trivial solution in C, show that it has a non-trivial solution in R.

Proof. Let $Z = U + iW$ be a nontrivial solution of $AZ=0$.

$AZ=0 \Rightarrow A(U+iW)=0 \Rightarrow AU + iAW=0$

$AU=0$, $AW=0$, $U \neq 0$ or $W \neq 0 \Rightarrow AZ=0$ has a nontrivial solution in R.

Ch. II Matrices

- Multiplication of matrices (associativity, distributivity, commutativity, cancellation, transpose and inverse of the product of matrices).

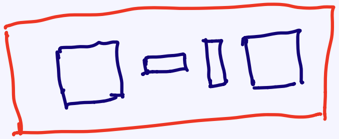
$$AB = 0,\quad A \neq 0 \not\Rightarrow B = 0$$

$$^{t}(AB) = {}^{t}B \cdot {}^{t}A, \qquad (AB)^{-1} = B^{-1}A^{-1}$$

- $(A + \mathrm{i}B)(C + \mathrm{i}D)$ $= AC - BD + i(AD + BC)$

- A 1×1 matrix vs. a scalar.

$$x = (a_1, \cdots, a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (x)_{1\times 1}$$



- Homework: P42, 36, 39.