MA2404
All Candidates

Semester 2 Examinations 2021-2022

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	Leicester International Institute
Module Code	MA2404
Module Title	Markov processes
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	8
Number of Questions	4
Instructions to Candidates	Answer all questions
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Yes
Books/Statutes provided by the University	-
Are students permitted to bring their own Books/Statutes/Notes?	Yes
Additional Stationery	Yes

1. An insurance company receives claims of sizes $X_1, X_2, \dots$. Hence, the total size of all claims during a week is $S = X_1 + X_2 + \dots + X_N$, where N is the (random) number of claims during a week. The company assumes that N follows the geometric distribution with parameter p, X_i are independent from each other and from N, identically distributed, and follow the exponential distribution with parameter λ. Parameters p and λ are unknown and should be estimated from the past data.

   To protect itself from large claims, an insurance company arranged excess of loss reinsurance policy with retention level $M = 2000$, that is, if claim X_i exceeds £2000, the insurance company pays £2000, and the remaining part $X_i - 2000$ is covered by reinsurance.

   (i) [5 marks] The numbers of claims an insurance company received during the last 10 weeks were

   $$1, 3, 1, 1, 2, 4, 0, 3, 0 \text{ and } 5.$$

   Use the method of maximum likelihood to estimate parameter p.

   (ii) [5 marks] The last 10 non-zero payments a *reinsurance* company made are

   $$200, 700, 2000, 1500, 100, 3400, 400, 600, 5000 \text{ and } 1100.$$

   Use the method of moments to estimate parameter λ.

   (iii) [5 marks] Estimate the expectation and standard deviation of the total size S of all claims to be received by an insurance company during the next week.

   (iv) [5 marks] Let $S = I + R$, where I and R are the sums to be paid next week by insurance and reinsurance companies, respectively. Estimate the expectation of R.

   (v) [5 marks] Estimate the probability that $R = 0$.

   Total: 25 marks

Answer:

(i) Application (Similar to seen)

The probability to receive n claims during a week is $(1-p)^n p$. Let $n_1, \dots, n_{10}$ be the numbers of claims in the past 10 weeks. The probability to receive exactly this numbers of claims is

$$L = \prod_{i=1}^{10} (1-p)^n p = (1-p)^{\sum_i n_i} p^{10}.$$

Hence

$$\log L = \sum_{i=1}^{10} n_i \log(1-p) + 10 \log p$$

The derivative

$$-\frac{\sum_{i=1}^{10} n_i}{1-p} + \frac{10}{p} = -\frac{20}{1-p} + \frac{10}{p}$$

is equal to 0 if

$$p = \frac{10}{20+10} = \frac{1}{3}.$$

(ii) Application (Similar to seen)

The cdf and pdf of exponential distributions are

$$f(x) = \lambda e^{-\lambda x} \quad \text{and} \quad F(x) = 1 - e^{-\lambda x}.$$

Let W denotes the size of non-zero payments of the reinsurer. Then the pdf of W is

$$g(w) = \frac{f(w+M)}{1-F(M)} = \frac{\lambda e^{-\lambda(w+M)}}{1-(1-e^{-\lambda M})} = \lambda e^{-\lambda w}.$$

Hence W has an exponential distribution with the same parameter λ. Its expectation is $1/lambda$. The average of the data is

$$\frac{1}{10}(200+700+2000+1500+100+3400+400+600+5000+1100) = 1500.$$

Hence $\lambda = 1/1500$.

(iii) Application (Similar to seen)

Using the formulas for expectation and variance of geometric distribution, we get

$$\mu_N = E[N] = \frac{1-p}{p} = \frac{1-1/3}{1/3} = 2,$$

$$\sigma_N^2 = \frac{1-p}{p^2} = \frac{1-1/3}{(1/3)^2} = 6.$$

Using the formulas for expectation and variance of exponential distribution, we get

$$\mu_X = E[X] = \frac{1}{\lambda} = 1500,$$

$$\sigma_X^2 = \frac{1}{\lambda^2} = (1500)^2,$$

hence

$$\mu_S = E[S] = \mu_N \mu_X = 2 \cdot 1500 = 3000,$$

and

$$\sigma_S = \sqrt{\mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2} = \sqrt{2 \cdot (1500)^2 + 6 \cdot (1500)^2} = 1500\sqrt{8} = 3000\sqrt{2} \approx 4243.$$

(iv) Application (Similar to seen)

Let NR be the number of non-zero claims made by reinsurer. For any claim X_i received by insurance company, the probability that it goes to reinsurer is

$$\pi = P(X_i > M) = 1 - F(M) = e^{-\lambda M} = e^{-2000/1500} = e^{-4/3}.$$

Hence

$$E[NR] = \pi E[N] = e^{-4/3} \cdot 2.$$

As established in (ii), the size of non-zero claims are exponential with parameter $\lambda = \frac{1}{1500}$, hence the expected size is $1/\lambda = 1500$. Hence,

$$E[R] = 1500 \times (2e^{-4/3}) = 3000e^{-4/3} \approx 791.$$

(iv) Higher skills (Unseen)

Assume that $N = j$, that is, insurance company receives j claims. Then $R = 0$ is all these claims do not exceed M. For each claim X_i,

$$q = P[X_i \leq M] = 1 - P[X_i > m] = 1 - e^{-4/3}.$$

The probability that all j claims do not exceed M is then q^j. Hence

$$P[S = 0|N = j] = q^j.$$

Now by the law of total probability

$$P[S=0] = \sum_{j=0}^{\infty} P[S=0|N=j]P[N=j] = \sum_{j=0}^{\infty} q^j \cdot (1-p)^j p = p \sum_{j=0}^{\infty} (q(1-p))^j = \frac{p}{1-q(1-p)}$$

$$= \frac{1/3}{1-(1-e^{-4/3})(2/3)} \approx 0.655.$$

2. **(i)** An insurance company receives claims that follow Burr distribution Burr$(\alpha, \lambda, \gamma)$ with density

$$f(x) = \frac{\alpha\gamma\lambda^{\alpha}x^{\gamma-1}}{(\lambda + x^{\gamma})^{\alpha+1}}, \quad x > 0$$

where α, λ, γ are positive parameters. Using the limiting density ratios test, determine whether the tail of the Burr distribution becomes heavier or lighter if

   (a) **[5 marks]**. λ and γ are fixed and α increases;

   (b) **[5 marks]**. α and γ are fixed and λ increases;

   (c) **[5 marks]**. λ and α are fixed and γ increases.

   (ii) The random variables X and Y are dependent with the Clayton copula with parameter $\alpha = 1/2$.

   (a) **[5 marks]**. Calculate the coefficient of lower tail dependence of X and Y.

   (b) **[5 marks]**. Calculate the survival copula $\bar{C}(u, v)$.

   Total: 25 marks

Answer:

(i)

(a) Application (Similar to seen)

If $\alpha_1 > \alpha_2$, then

$$\lim_{x\to\infty} \frac{\alpha_1\gamma\lambda^{\alpha_1}x^{\gamma-1}}{(\lambda + x^{\gamma})^{\alpha_1+1}} : \frac{\alpha_2\gamma\lambda^{\alpha_2}x^{\gamma-1}}{(\lambda + x^{\gamma})^{\alpha_2+1}} = \lim_{x\to\infty} \frac{\alpha_1}{\alpha_2}(\lambda + x^{\gamma})^{\alpha_2-\alpha_1} = 0,$$

hence increasing α makes tail lighter.

(b) Application (Similar to seen)

If $\lambda_1 > \lambda_2$, then

$$\lim_{x\to\infty} \frac{\alpha\gamma\lambda_1^\alpha x^{\gamma-1}}{(\lambda_1+x^\gamma)^{\alpha+1}} : \frac{\alpha\gamma\lambda_2^\alpha x^{\gamma-1}}{(\lambda_2+x^\gamma)^{\alpha+1}} = \frac{\lambda_1^\alpha}{\lambda_2^\alpha} \lim_{x\to\infty} \left(\frac{\lambda_2+x^\gamma}{\lambda_1+x^\gamma}\right)^{\alpha+1} = \frac{\lambda_1^\alpha}{\lambda_2^\alpha} > 1,$$

hence increasing λ makes tail heavier.

(c) Higher skills (Unseen)

If $\gamma_1 > \gamma_2$, then

$$\lim_{x\to\infty} \frac{\alpha\gamma_1\lambda^\alpha x^{\gamma_1-1}}{(\lambda+x^{\gamma_1})^{\alpha+1}} : \frac{\alpha\gamma_2\lambda^\alpha x^{\gamma_2-1}}{(\lambda+x^{\gamma_2})^{\alpha+1}} = \lim_{x\to\infty} \frac{\alpha\gamma_1\lambda^\alpha x^{\gamma_1-1}}{(x^{\gamma_1})^{\alpha+1}} : \frac{\alpha\gamma_2\lambda^\alpha x^{\gamma_2-1}}{(x^{\gamma_2})^{\alpha+1}} =$$

$$= \frac{\gamma_1}{\gamma_2} \lim_{x\to\infty} x^{\alpha(\gamma_2-\gamma_1)} = 0$$

hence increasing γ makes tail lighter.

(The equation is marked as higher skills because the limit is slightly trickier)

(ii)

(a) Application (Similar to seen)

The Clayton copula with $\alpha = 1/2$ is

$$C(u,v) = \max\{u^{-1/2} + v^{-1/2} - 1, 0\}^{-2} = \left(\frac{\sqrt{uv}}{\sqrt{u}+\sqrt{v}-\sqrt{uv}}\right)^2,$$

provided that $\sqrt{u}+\sqrt{v}-\sqrt{uv} > 0$. Hence

$$C(u,u) = \left(\frac{\sqrt{uu}}{\sqrt{u}+\sqrt{u}-\sqrt{uu}}\right)^2 = \left(\frac{u}{\sqrt{u}(2-\sqrt{u})}\right)^2 = \frac{u}{(2-\sqrt{u})^2},$$

provided that $\sqrt{u}+\sqrt{u}-\sqrt{uu} > 0$, which is true for $u < 4$. Hence the coefficient of lower tail dependence is

$$\lambda_L = \lim_{u\to 0+} \frac{C(u,u)}{u} = \lim_{u\to 0+} \frac{1}{(2-\sqrt{u})^2} = 0.25.$$

(b) Application (Similar to seen)

From lecture notes

$$\bar{C}(1-u,1-v) = 1-u-v+C(u,v),$$

or, equivalently,

$$\bar{C}(u,v) = 1-(1-u)-(1-v)+C(1-u,1-v) = -1+u+v+C(1-u,1-v).$$

Hence,

$$\bar{C}(u,v) = -1+u+v+C(1-u,1-v) =$$

$$= -1+u+v+\left(\frac{\sqrt{(1-u)(1-v)}}{\sqrt{1-u}+\sqrt{1-v}-\sqrt{(1-u)(1-v)}}\right)^2,$$

provided that $\sqrt{1-u}+\sqrt{1-v}-\sqrt{(1-u)(1-v)} > 0.$

3. Consider a no claims discount (NCD) model for car-insurance premiums. The insurance company offers discounts of 0%, 25% and 50% of the full premium $C = 1000$, determined by the following rules:

   (a) All new policyholders start at the 0% level.

   (b) If no claim is made during the current year the policyholder moves up one discount level, or remains at the 50% level.

   (c) If one or more claims are made the policyholder moves to the 0% level.

   The insurance company believes that the probability of making a claim each year depends on the current discount level and is equal to 0.3, 0.2 and 0.1 for drivers at discount levels 0%, 25% and 50%, respectively.

   (i) [5 marks] Explain why can this process be modelled as a Markov chain. Determine the state space and transition matrix.

   (ii) [5 marks] Calculate the 3-step transition matrix for this NCD system.

   (iii) [5 marks] A policyholder currently has no discount and pays the full premium. Calculate the expectation of the price of her insurance contract after 3 years.

   (iv) [5 marks] Compute the stationary distribution for this NCD system.

   (v) [5 marks] Prove that the n-step transition probabilities of this Markov chain converge to the stationary distribution.

   Total: 25 marks

Answer:

(i) Application (Similar to seen)

The model can be considered as a Markov chain since the future discount depends only on the current level, not the entire history. The state space is $\mathcal{S} = \{0\%, 25\%, 50\%\}$, which is convenient to denote as $\mathcal{S} = \{0, 1, 2\}$ (where state 0 is the 0% state, 1 is the 20% state and 2 is the 40% state). The transition probability matrix between two states in a unit time is given by

$$\mathbf{P} = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.2 & 0 & 0.8 \\ 0.1 & 0 & 0.9 \end{pmatrix}. \qquad (1)$$

(ii) Application (Similar to seen)

$$\mathbf{P}^{(3)} = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.2 & 0 & 0.8 \\ 0.1 & 0 & 0.9 \end{pmatrix}^3 = \begin{pmatrix} 0.167 & 0.161 & 0.672 \\ 0.142 & 0.098 & 0.76 \\ 0.131 & 0.084 & 0.785 \end{pmatrix}.$$

(iii) Higher skills (Unseen)

With discount of 0%, 25% or 50% from the full premium 1000, a policyholder would pay 1000, 750 or 500, respectively. From (ii), the probabilities of these scenarios are 0.167, 0.161 and 0.672, respectively. Hence, the expectation is

$$1000 \cdot 0.167 + 750 \cdot 0.161 + 500 \cdot 0.672 = 623.75.$$

(iv) Application (Similar to seen)

The conditions for a stationary distribution lead to the following expressions

$$\begin{aligned}
\pi_0 &= 0.3\pi_0 + 0.2\pi_1 + 0.1\pi_2, \\
\pi_1 &= 0.7\pi_0, \\
\pi_2 &= 0.8\pi_1 + 0.9\pi_2, \\
1 &= \pi_0 + \pi_1 + \pi_2,
\end{aligned}$$

From the third equation, $0.1\pi_2 = 0.8\pi_1$. Hence, $\pi_2 = 8\pi_1 = 8(0.7\pi_0) = 5.6\pi_0$. Substituting into the last equation, we get $1 = \pi_0 + 0.7\pi_0 + 5.6\pi_0 = 7.3\pi_0$. Hence, $\pi_0 = \frac{10}{73}$, $\pi_1 = \frac{7}{73}$, $\pi_2 = \frac{56}{73}$.

(iv) Application (Similar to seen)

There are 3 states - a finite number, so the Markov chain is finite. We can move from any state to any other state in at most 2 steps, so it is irreducible. From any state, we can return back in either one step, or in 2 and in 3 steps, so no period works, hence the Markov chain is aperiodic. Because the Markov chain is finite, irreducible and aperiodic, it converges to its stationary distribution.

4. A company provides sick pay to its employees who are unable to work. They decided to ignore the mortality rates and use the two-state, time-inhomogeneous Markov jump process with states Healthy (H) that means fit to work and Sick (S) that means unable to work. The transition rate form H to S is $\sigma(t)$, while the transition rate from S to H is $\rho(t)$.

   (i) [5 marks] Write down the generator matrix and Kolmogorov's forward equations in matrix form for this process.

   (ii) [10 marks] Given an employee is sick at the time t_1, write down an expression for the probability that he or she will stay sick continuously until time $t_2 > t_1$. Estimate this probability for $t_1 = 40$, $t_2 = 40.5$ and $\rho(t) = 100/t$.

   (iii) [10 marks] The company assumes that $\sigma(t) = at$ and would like to use linear regression with least square error to find parameter a. Data shows that $\sigma(20) \approx 0.04$, $\sigma(40) \approx 0.08$ and $\sigma(60) \approx 0.1$. Find parameter a which best approximates these data.

   Total: 25 marks

Answer:

(i) Application (Similar to seen)

The Kolmogorov's forward equations in matrix form can be written as

$$\frac{\partial \mathbf{P}(s,t)}{\partial t} = \mathbf{P}(s,t)\mathbf{Q}(t),$$

where $\mathbf{P}(s,t)$ is the matrix whose entries are transition probabilities $p_{ij}(s,t)$, and $\mathbf{Q}(t)$ is the generator matrix:

$$\mathbf{Q}(t) = \begin{bmatrix} -\sigma(t) & \sigma(t) \\ \rho(t) & -\rho(t) \end{bmatrix}.$$

(ii) Application (Similar to seen)

The expression is

$$\exp\left(-\int_{t_1}^{t_2}\rho(t)dt\right)$$

For $t_1 = 40$, $t_2 = 40.5$ and $\rho(t) = 100/t$, this becomes

$$\exp\left(-\int_{40}^{40.5}\frac{100}{t}dt\right) = \exp(-100(\ln(40.5)-\ln(40))) = \left(\frac{40.5}{40}\right)^{-100} \approx 0.29.$$

(iii) Application (Similar to seen)

The least square error is

$$e(t) = (0.04-20a)^2 + (0.08-40a)^2 + (0.1-60a)^2.$$

Its derivative is

$$-2(20(0.04-20a)+40(0.08-40a)+60(0.1-60a)) = 0$$

if

$$a = \frac{20\cdot 0.04+40\cdot 0.08+60\cdot 0.1}{20^2+40^2+60^2} = \frac{1}{560} \approx 0.0018.$$