Lecture 22: Multiple Integrals.

MA2032 Vector Calculus

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November 9, 2022

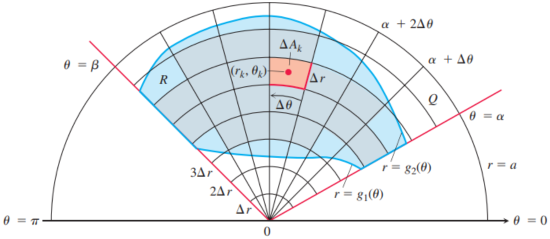
Double Integrals in Polar Form

• Double integrals are sometimes **easier to evaluate** if we change to polar coordinates.

• Today you will learn **how to accomplish the change** and how to evaluate double integrals over regions whose **boundaries are given by polar equations**.

• We consider a function $f(x, y)$ defined on region **R** that is bounded by the **rays** $\theta = \alpha$ and $\theta = \beta$ and by the continuous **curves** $r = g_1(\theta)$ and $r = g_2(\theta)$.

• Then R lies in a fan-shaped **region** Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$.



Double Integrals in Polar Form

• We cover **Q** by a grid of circular arcs and rays, where $\Delta r = a/m$ and $\Delta\theta = (\beta - \alpha)/m'$
• Partition **Q** into small patches called "polar rectangles".
• **A Riemann sum over R** is defined as

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k.$$

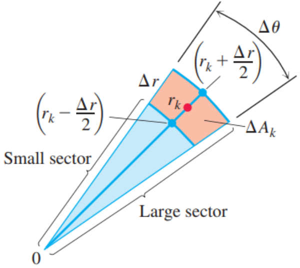
• When **a limit of the sums S_n exists**, is called the **double integral** of f over **R**, written as

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta) dA$$

• We first have to write the sum S_n in a way that **expresses** ΔA_k in terms of Δr and $\Delta\theta$.

Double Integrals in Polar Form

- The area of a wedge-shaped sector of a circle having radius r and angle θ is $A = 1/2\theta r^2$.



Inner radius: $\frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$

Outer radius: $\frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector}\\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta.\end{aligned}$$

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta. \qquad \lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta.$$

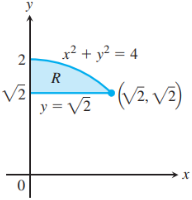
- Fubini's Theorem:

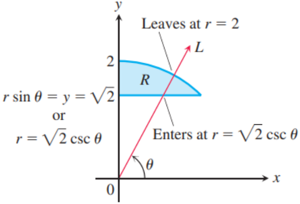
$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

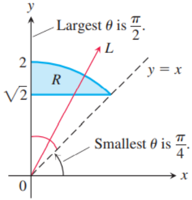
- **Step 1: Sketch** the region and label the bounding curves .

- **Step 2: Find the r-limits of integration**. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis

- **Step 3: Find the θ-limits of integration**. Find the smallest and largest θ-values that bound R.

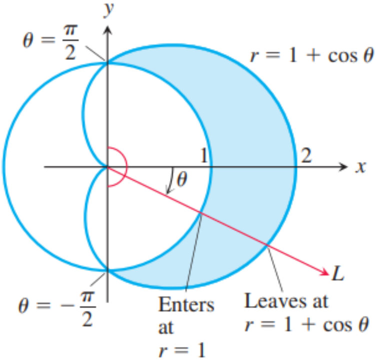






Example 1

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



Example 1

Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta. \qquad \blacksquare$$

Changing Cartesian Integrals into Polar Integrals

• The procedure for changing a Cartesian integral into a polar integral has **two steps**:

• **First substitute** $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\ dy$ by $r\ dr\ d\theta$ in the Cartesian integral.

• **Then supply** polar limits of integration for the boundary of R.

• The **Cartesian integral** then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

• where **G** denotes the same region of integration, but now **described in polar coordinates**.

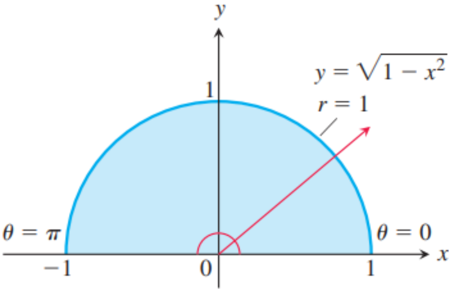
Changing Cartesian Integrals into Polar Integrals

Example 2

Evaluate

$$\iint_R e^{x^2+y^2}\,dy\;dx$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.



Example 2

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to evaluate it. Polar coordinates make this possible. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ give

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

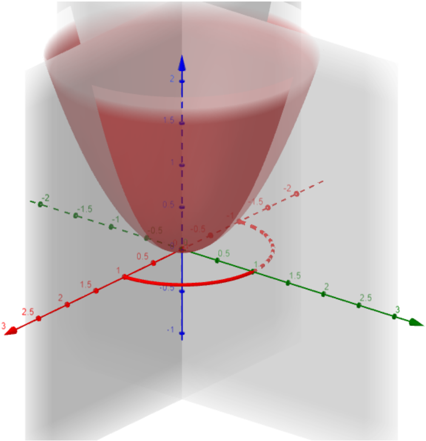
$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$

The r in the $r\,dr\,d\theta$ is what allowed us to integrate e^{r^2}. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

Changing Cartesian Integrals into Polar Integrals

Example 3

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy\, dx$.



Example 3

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

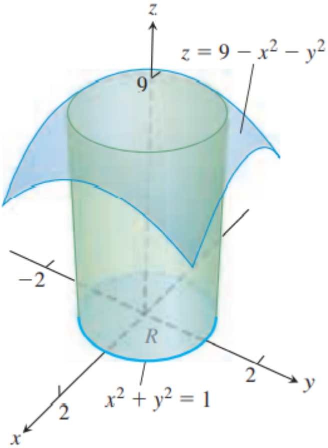
which is difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le \pi/2$, and $0 \le r \le 1$, and replacing $dy\,dx$ by $r\,dr\,d\theta$ in the double integral, we get

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx &= \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta \\
&= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.
\end{aligned}$$

The polar coordinate transformation is effective here because $x^2 + y^2$ simplifies to r^2 and the limits of integration become constants. ■

Example 4

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



Example 4

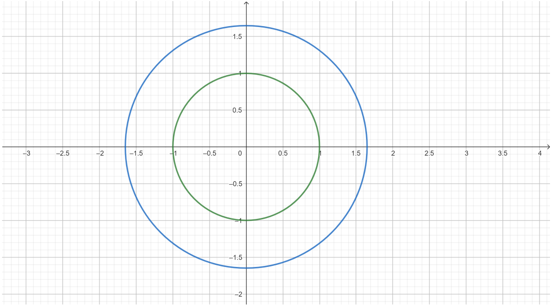
Solution The region of integration R is bounded by the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$
\begin{aligned}
\iint_R (9 - x^2 - y^2)\, dA &= \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta \qquad r^2 = x^2 + y^2, \quad dA = r\, dr\, d\theta. \\
&= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta \\
&= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}. \qquad \blacksquare
\end{aligned}
$$

Changing Cartesian Integrals into Polar Integrals

Example 5

Converting to a polar integral then integrate $f(x, y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ over the region $1 \leq x^2 + y^2 \leq e$.



$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\,dr\,d\theta = 2\int_0^{2\pi}\left[r\ln r - r\right]_1^{e^{1/2}} d\theta = 2\int_0^{2\pi}\sqrt{e}\left[\left(\frac{1}{2}-1\right)+1\right] d\theta = 2\pi\left(2-\sqrt{e}\right)$$