Lecture 27: Integrals and Vector Fields.

MA2032 Vector Calculus

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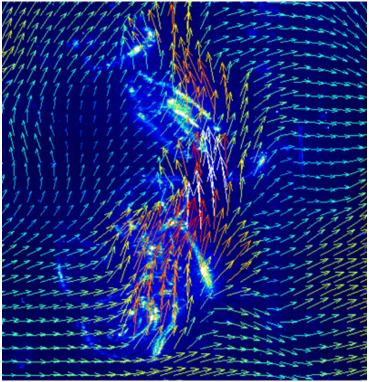
Vector Fields and Line Integrals: Work, Circulation, and Flux

• **Gravitational and electric forces** have both a **direction** and a **magnitude**.

They are represented by a **vector at each point in their domain**, producing a **vector field**.

• We show **how to compute the work done in moving an object through such a field** by using a **line integral** involving the vector field.

• We also discuss **velocity fields**, such as the vector field representing the velocity of a flowing fluid in its domain.



• A **line integral** can be used to find the **rate at which the fluid flows** along or across a curve within the domain.

Vector Fields

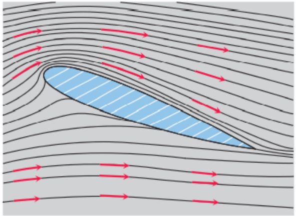


Figure - Velocity vectors of a flow around an airfoil in a wind tunnel.

- Suppose a **region** in the plane or in space **is occupied by a moving fluid**, such as air or water.
- The fluid is made up of a **large number of particles**, and at any instant of time, a **particle has a velocity** v.
- Such a fluid flow is an **example of a vector field**.
- Generally, a **vector field is a function** that **assigns a vector to each point in its domain**.
- A **vector field** on a three-dimensional domain in space **might have a formula** like

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

- The vector field is **continuous** if the component functions M, N, and P are continuous;
- It is **differentiable** if each of the component functions is differentiable.

Gradient Fields

• The **gradient vector** of a differentiable scalar-valued function at a point **gives the direction of greatest increase of the function**.

• An **important type of vector field** is formed by all the gradient vectors of the function.

• We define the gradient field of a differentiable function $f(x, y, z)$ to be the field of gradient vectors

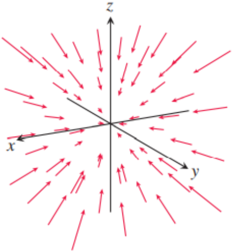
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

• At each point (x, y, z), the **gradient field gives a vector pointing in the direction of greatest increase** of f, with **magnitude being the value of the directional derivative** in that direction.

Gradient Fields

Example 1

Suppose that a material is heated, that the resulting temperature T at each point (x, y, z) in a region of space is given by $T = 100 - x^2 - y^2 - z^2$, and that $\mathbf{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\mathbf{F}$.



Solution The gradient field $\mathbf{F}$ is the field $\mathbf{F} = \nabla T = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$. At each point in the region, the vector field $\mathbf{F}$ gives the direction for which the increase in temperature is greatest. The vectors point toward the origin, where the temperature is greatest. See Figure 16.17.

• In previous lecture we defined the **line integral of a scalar function** $f(x, y, z)$ **over a path C** • **We turn our attention now to the idea of**

a line integral of a vector field F **along the curve** C.

• Assume that the vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

• has **continuous components**, and that the curve C has a **smooth parametrization (forward direction)**

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \leq t \leq b.$$

• At each point along the path C, the **tangent vector** $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$ is a unit vector tangent to the path and pointing in this forward direction.

• The line integral of the vector field is the line integral of the **scalar tangential component** of $\mathbf{F}$ along C

• This tangential component **is given by the dot product** $\mathbf{F} \cdot \mathbf{T} = \mathbf{F} \cdot d\mathbf{r}/ds$, so

DEFINITION Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{1}$$

Line Integrals of Vector Fields

We evaluate line integrals of vector fields in a following way

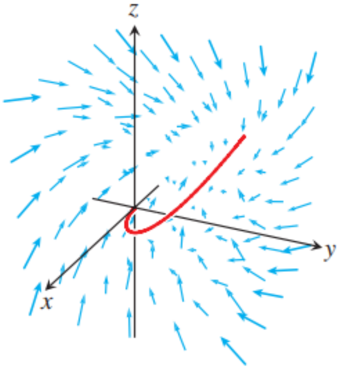
Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ along the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \leq t \leq b$, to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (2)$$

Example 2

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $F(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k}, 0 \leq t \leq 1$ and shown in Figure.



Example 2

Solution We have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k} \qquad z = \sqrt{t},\ xy = t^3,\ -y^2 = -t^2$$

and

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}.$$

Thus,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt \qquad \text{Eq. (2)}$$

$$= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right)dt$$

$$= \left[\left(\frac{3}{2}\right)\left(\frac{2}{5}t^{5/2}\right) + \frac{1}{4}t^4\right]_0^1 = \frac{17}{20}.$$

Line Integrals with Respect to dx, dy, or dz

- It is often useful to consider **each component direction separately**.
- So we want to **evaluate a line integral** of a scalar function **with respect to only one of the coordinates**, such as $\int_C M\,dx$.
- This type of integral **is not the same** as the arc length line integral $\int_C M\,ds$ we defined, since it picks out displacement in the direction of only one coordinate.
- To define the integral $\int_C M\,dx$ for the scalar function $M(x, y, z)$, we **specify a vector field** $F = M(x, y, z)i$ having a component **only in the** x-**direction**, and none in the y- or z-direction.
- Then, over the curve C parametrized by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \;\; a \leq t \leq b$, we have $x = g(t), \; dx = g'(t)\,dt$, and

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt &= M(x, y, z)\mathbf{i} \cdot \left(g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}\right) dt \\ &= M(x, y, z)\,g'(t)\,dt = M(x, y, z)\,dx.\end{aligned}$$

Line Integrals with Respect to dx, dy, or dz

Expressing everything in terms of the parameter t along the curve C, we have the **following formulas** for three integrals with respect to dx, dy, and dz:

$$\int_C M(x, y, z)\, dx = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt$$

$$\int_C N(x, y, z)\, dy = \int_a^b N(g(t), h(t), k(t))\, h'(t)\, dt$$

$$\int_C P(x, y, z)\, dz = \int_a^b P(g(t), h(t), k(t))\, k'(t)\, dt$$

Example 3

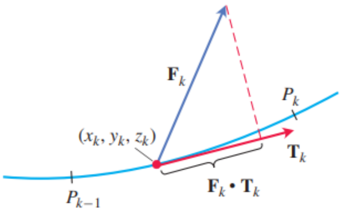
Evaluate the line integral $\int_C (-ydx + zdy + 2x)\, dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$.

Solution We express everything in terms of the parameter t, so $x = \cos t$, $y = \sin t$, $z = t$, and $dx = -\sin t\, dt$, $dy = \cos t\, dt$, $dz = dt$. Then,

$$
\begin{aligned}
\int_C -y\, dx + z\, dy + 2x\, dz &= \int_0^{2\pi} [(-\sin t)(-\sin t) + t\cos t + 2\cos t]\, dt \\
&= \int_0^{2\pi} [2\cos t + t\cos t + \sin^2 t]\, dt \\
&= \left[2\sin t + (t\sin t + \cos t) + \left(\frac{t}{2} - \frac{\sin 2t}{4} \right) \right]_0^{2\pi} \\
&= [0 + (0 + 1) + (\pi - 0)] - [0 + (0 + 1) + (0 - 0)] \\
&= \pi.
\end{aligned}
$$

Work Done by a Force over a Curve in Space

• Suppose that the vector field F represents a **force** throughout a region in space.
• For a curve C in space, **we define the work** done by a continuous force field F **to move an object** along C from a point A to another point B as follows.



$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k .$$

• As $n \to \infty$ and $\Delta s_k \to 0$, these sums approach the line integral

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and let $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (6)$$

Work Done by a Force over a Curve in Space

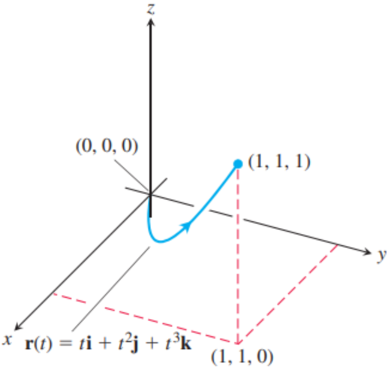
TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \big(Mg'(t) + Nh'(t) + Pk'(t)\big)\, dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

Work Done by a Force over a Curve in Space

Example 4

Find the work done by the force field
$\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ in moving an object along the curve
$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \leq t \leq 1$, from $(0, 0, 0)$ to $(1, 1, 1)$.



Example 4

Solution First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.\end{aligned}$$

Substitute $x = t, y = t^2, z = t^3$.

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= [(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8.\end{aligned}$$

Evaluate dot product.

So,

$$\begin{aligned}\text{Work} &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt \\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.\end{aligned}$$

Flow Integrals and Circulation for Velocity Fields

• Suppose that F represents the **velocity field of a fluid** flowing through a region in space

• Under these circumstances, the integral of $\mathbf{F} \cdot \mathbf{T}$ along a curve in the region gives the **fluid's flow** along, or circulation around, the curve.

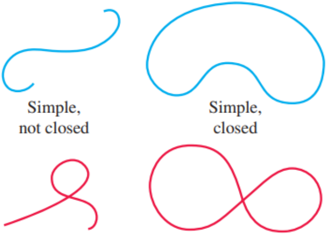
DEFINITION If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{7}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

Flux Across a Simple Closed Plane Curve

• A curve in the *xy*-plane is **simple** if it does not cross itself.
• When a curve starts and ends at the same point, it is a **closed curve** or loop.
• To find the **rate** at which a **fluid is entering or leaving a region** enclosed by a smooth simple closed curve C in the *xy*-plane,



• we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{8}$$

Flux Across a Simple Closed Plane Curve

Calculating Flux Across a Smooth Closed Plane Curve

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C = \oint_C M\,dy - N\,dx \tag{9}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.