MA3071 – DLI
Financial Mathematics – Section 2
Binomial tree models

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Background

- The binomial tree model is a numerical method for estimating option prices in a no-arbitrage framework.
- All such methods are by definition discrete in nature, but with small enough steps (in time and change in modelled variable) the result will converge towards the continuous equivalent (Black-Scholes).
- More steps usually mean a more accurate price, but are more computationally intensive.

Background, cont.

- The model is very flexible and can be used for pricing American options and exotic options where the payoff is path dependent (such as Asian options, barrier options).
- It also provides insight to key concepts in financial economic theory such as hedging portfolios and risk-neutral pricing. These are central to the development of the Black-Scholes formula and option pricing in general.

Financial Assumptions

- The option payoff/claim (C) is a function of the underlying asset price at time T (S_T), i.e. $C = f(S_T)$.
- The risk-free interest rate (ρ) is known and is a constant over a certain time period.
- The volatility/standard deviation (σ) of the return on the underlying asset is a constant.
- There are no transaction costs in buying or selling the underlying asset or the option.
- Short selling is allowed.
- There is no arbitrage opportunity.

Mathematical Assumptions

- **Markov Property**: Behaviour of asset prices satisfies Markov property. Given the present price, the future price does not depend on the past prices.
- **Martingale Property**: To achieve the no-arbitrage condition, we make the assumption that the discounted asset price is a martingale.

Conditional expectation

- If X and Y are two random variables, the conditional expectation of X given $Y = y$ is
  - Discrete: $\mathbb{E}[X|Y = y] = \sum_x xP(X = x|Y = y)$
  - Continuous: $\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$
- Y as a random variable has different possible values. Therefore, $\mathbb{E}[X|Y]$ **is still a random variable**, since its value depends on the values of the random variable Y.

Properties of conditional expectation

- If X is independent of Y, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X]$$

- Law of total expectation:

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X] \quad \text{and} \quad \mathbb{E}\left[\mathbb{E}[g(X)|Y]\right] = \mathbb{E}[g(X)]$$

- Linearity:

$$\mathbb{E}[aX_1 + bX_2|Y] = a\mathbb{E}[X_1|Y] + b\mathbb{E}[X_2|Y]$$

Markov Property

- **Definition**: Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_s, s \in I\}$, for some totally ordered index set I, and let $(S, \mathcal{S})$ be a measurable space. A stochastic process $\{X_t : \Omega \to S\}_{t \in I}$ defined on $(S, \mathcal{S})$ and adapted to the filtration is said to possess the Markov property if, for each $A \in \mathcal{S}$ and each $s, t \in I$ with $t > s \geq 0$,

$$P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s)$$

- Alternatively, the Markov property can be formulated as the following conditional expectation,

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] = \mathbb{E}[f(X_t) | X_s]$$

for all $t > s \geq 0$ and $f : S \to \mathbb{R}$ is a bounded and measurable function.

Markov Property

- A stochastic process satisfies the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present values) depends only upon the present state.

$$P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, X_{t_{n-2}} = x_{n-2}, \cdots, X_{t_0} = x_0) \\ = P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1})$$

  and

$$\mathbb{E}[f(X_{t_n}) | X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = \mathbb{E}[f(X_{t_n}) | X_{t_{n-1}}]$$

  for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- An important result is that any process with **independent increments** satisfies the Markov property.

Martingale Property

- **Definition**: A martingale is a stochastic process $\{X_t\}_{t\in I}$ that satisfies:

$$\mathbb{E}[|X_t|] < \infty \text{ for all } t \in I,$$
$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \text{ for all } s, t \in I \text{ and } t > s \geq 0$$

- For example,

$$\mathbb{E}[X_{t_n}|X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = X_{t_{n-1}}$$

for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- For our mathematical assumption, the **discounted asset price** is a stochastic process and a martingale.

Assumptions

- Please Note: All of the above financial and mathematical assumptions also apply to the **Black-Scholes model** and the **Monte-Carlo methods** of option pricing.

Single period binomial tree

- In the simplest version of the model, we consider only one time period. Over this time period, the price of the underlying asset (S_t) can only increase or decrease (i.e. is Bernoulli).
- The "increase" or "decrease" is proportional to the current value with factors u and d such that $u > d > 0$.

$$
\begin{array}{ccc}
 & & S_1 = S_0 u \\
 & \nearrow & \\
S_0 & & \\
 & \searrow & \\
 & & S_1 = S_0 d
\end{array}
$$

Two period binomial tree

- An example of a two period tree is shown here.

$$
\begin{array}{ccccccc}
 & & & & & & S_2 = S_0u^2 \\
 & & & & & \nearrow & \\
 & & S_1 = S_0u & & & & \\
 & \nearrow & & & \searrow & & \\
S_0 & & & & & & S_2 = S_0ud \\
 & \searrow & & & \nearrow & & \\
 & & S_1 = S_0d & & & & \\
 & & & & & \searrow & \\
 & & & & & & S_2 = S_0d^2
\end{array}
$$

- The above binomial tree satisfies the **Markov property**.

Example

Consider a single period binomial tree. Assume that $\rho = \frac{1}{3}$ over period $[0, 1]$, and the asset prices at times 0, 1 are defined by

$$S_0 = 6, S_1 = S_0 Y$$

where Y is a random variable with

$$P(Y = 3) + P(Y = 0.5) = 1$$

Write down the single-period binomial tree and calculate the option price V_0 of the option claim $C = (S_1 - 6)_+$.

Questions

- How to make sure the option price is arbitrage free?
- What are the probabilities of the asset price rising and falling?

Martingale

- As binomial tree model is a discrete model, the discounted asset price is $(1+\rho)^{-t}S_t$, where ρ is the risk-free interest rate over each period in the tree.

- According to the mathematical assumptions, $(1+\rho)^{-t}S_t$ is a martingale if

$$\mathbb{E}_Q\left[(1+\rho)^{-T}S_T|\mathcal{F}_t\right] = (1+\rho)^{-t}S_t, \quad T > t \geq 0$$

where $\mathbb{E}_Q[\bullet]$ denotes the expectation based on q-probabilities such that

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio

- Let C_u and C_d denote the option payoff at time 1 when the price of the underlying asset increases and decreases, respectively.

- Suppose we hold a portfolio of stocks and bonds at time 0, with ϕ units of stock and ψ units of bond. At time 1, this portfolio will be worth:
  - $\phi S_0 u + \psi(1+\rho)$ if the stock price increased,
  - $\phi S_0 d + \psi(1+\rho)$ if the stock price decreased.

Hedging Portfolio, cont.

- We now choose ϕ and ψ such that the value of the portfolio at time 1 is equal to the payoff of the option. Therefore,

$$\phi S_0 u + \psi(1+\rho) = C_u$$
$$\phi S_0 d + \psi(1+\rho) = C_d$$

- We then solve the simultaneous equations to get,

$$\phi = \frac{C_u - C_d}{S_0(u-d)} \quad \text{and} \quad \psi = \frac{uC_d - dC_u}{(1+\rho)(u-d)}$$

Hedging Portfolio, cont.

- Since the values of the hedging portfolio and the option are equal at time 1, they must be equal at time 0 to avoid arbitrage. The value of the hedging portfolio is then equal to the price of the option at time 0. Therefore,

$$V_0 = \phi S_0 + \psi = \frac{C_u q_u + C_d q_d}{1+\rho}$$
$$q_u + q_d = 1$$

- We then obtain,

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio, cont.

- The hedging portfolio (ϕ, ψ) is also called a replicating portfolio because it matches the option payoffs with no risk.
- This approach can also be employed for hedging purposes by the option seller/writer: that is an investment strategy which reduces the amount of risk carried by the seller of the option when used in conjunction with the short position in the option.

No-arbitrage condition

- The no-arbitrage condition must hold for the option game

$$d < 1 + \rho < u$$

- Moreover, it is easy to check that
  - $q_u + q_d = 1$
  - $0 < q_u < 1$, and $0 < q_d < 1$ iff the above no arbitrage condition holds.
  - $\mathbb{E}_Q[Y] = uq_u + dq_d = 1 + \rho$
- As long as the above arbitrage-free condition holds, equivalent martingale probabilities ($q-$probabilities) are in effect for arbitrage-free option pricing, and the value of the hedging portfolio equals the value of the option at all times.
- $(1+\rho)^{-t} S_t$ is a martingale $\Leftrightarrow d < 1 + \rho < u$.

Think about

- If $1 + \rho < d < u$, what's the arbitrage opportunity?
- What if $d < u < 1 + \rho$?

Two period binomial tree

- For the two period binomial model, the discrete time market consists of two assets: one non risky asset (bond) with fixed interest rate ρ and one risky asset such that the asset prices at times 0, 1 and 2 are defined by

$$S_0, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

  where Y, Y_1, Y_2 are iid random variables with

$$P(Y = u) + P(Y = d) = 1$$

- We want to determine the arbitrage free time 0 option price of the option claim $C = f(S_2)$ with expiry date $T = 2$.

Two period binomial tree, cont

- $q-$probabilities are the **same** as found from the single period binomial tree model.
- Then, the general binomial tree is defined by

$$
\begin{array}{ccccc}
 & & & & S_2 = S_0u^2\ , \quad \begin{array}{c} C_{uu} = f(S_0uu) \\ \text{with } \ q_u^2 \end{array} \\
 & & S_1 = S_0u & \begin{array}{c} \nearrow^{q_u} \\ \searrow^{q_d} \end{array} & \\
S_0 & \begin{array}{c} \nearrow^{q_u} \\ \searrow_{q_d} \end{array} & & & S_2 = S_0ud\ , \quad \begin{array}{c} C_{ud} = f(S_0ud) \\ \text{with } \ 2q_uq_d \end{array} \\
 & & S_1 = S_0d & \begin{array}{c} \nearrow^{q_u} \\ \searrow_{q_d} \end{array} & \\
 & & & & S_2 = S_0d^2\ , \quad \begin{array}{c} C_{dd} = f(S_0dd) \\ \text{with } \ q_d^2 \end{array}
\end{array}
$$

- The option price is

$$V_0 = (1+\rho)^{-2}\left[q_u^2 C_{uu} + 2q_uq_d C_{ud} + q_d^2 C_{dd}\right]$$

Example

Consider a discrete market with one risky asset and one risk free asset. The interest rate $\rho = 0.5$ is fixed over each period, and the asset prices at times 0, 1 and 2 are defined by

$$S_0 = 4, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

where Y, Y_1, Y_2 are iid random variables with

$$P(Y = 8) + P(Y = 0.5) = 1$$

(i) Determine the equivalent martingale probabilities.

(ii) Write down the two-period binomial tree and find the arbitrage free time 0 option price of the European put option with strike price $K = 5$ and expiry day $T = 2$.

(iii) Determine the hedging portfolio for the two period tree.

Extending to n periods

An n period binomial tree is introduced as follows,

(I) $S_{t_i} = S_{t_{i-1}} Y_{t_i}$ for $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i - t_{i-1} = \frac{T}{n}$, $i = 1, 2, \cdots, n$, where S_{t_i} is the time t_i asset price and S_0 is a positive constant;

(II) Y and Y_{t_i}, $i = 1, 2, \cdots, n$ are independent and identically distributed random variables (iid) with

$$P(Y = u) + P(Y = d) = 1$$

where up factor u and down factor d are constants such that $u > d > 0$ and possible outcomes of Y;

(III) Y_{t_i} is independent of $S_{t_1}, \cdots, S_{t_{i-1}}$.

Extending to n periods, cont.

- Distribution of S_{t_n} is defined by

$$S_{t_n} = S_0 u^j d^{n-j}$$
$$P(S_{t_n} = S_0 u^j d^{n-j}) = \binom{n}{j} q_u^j q_d^{n-j}$$

  for $j = 0, 1, \cdots, n$.

- The arbitrage free time 0 option price of the option claim $C = f(S_{t_n})$ is calculated by

$$V_0 = (1+\rho)^{-n} \sum_{j=0}^{n} f(S_0 u^j d^{n-j}) \binom{n}{j} q_u^j q_d^{n-j}$$

  where ρ is the interest rate over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.

Example

Let Y and Y_{t_i}, $i = 1, 2, \cdots, n$ be iid random variables with distribution

$$P(Y = 1) = 0.5, \quad P(Y = 3) = 0.5$$

and the underlying asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$.

Let $\rho = 0.25$ over each time period. Find the arbitrage free time 0 option price of the option claim $C = S_T^{10}$, when $S_0 = 4$, $n = 5$ and $T = t_n$.

Time varying binomial tree models

- Time varying interest rates ρ_i over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.
- Asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$, $i = 1, 2, \cdots, n$, where Y_{t_i} are independent, but in general not identically distributed
$$P(Y_{t_i} = u_i) + P(Y_{t_i} = d_i) = 1$$
with time varying factors u_i and d_i.
- The time varying $q-$probabilities are
$$q_u^{(i)} = \frac{1 + \rho_i - d_i}{u_i - d_i}, \quad q_d^{(i)} = 1 - q_u^{(i)}, \quad i = 1, 2, \cdots, n$$

Example

- Two period time varying binomial tree model:

$$
\begin{array}{ccccl}
 & & & \nearrow^{q''_u} & S_2 = S_0 u_1 u_2\,, \quad C_{u_1 u_2} \ \text{ with } q'_u q''_u \\
 & & S_1 = S_0 u_1 & & \\
 & \nearrow^{q'_u} & & \searrow^{q''_d} & S_2 = S_0 u_1 d_2\,, \quad C_{u_1 d_2} \ \text{ with } q'_u q''_d \\
S_0 & & & & \\
 & \searrow^{q'_d} & & \nearrow^{q''_u} & S_2 = S_0 d_1 u_2\,, \quad C_{d_1 u_2} \ \text{ with } q'_d q''_u \\
 & & S_1 = S_0 d_1 & & \\
 & & & \searrow^{q''_d} & S_2 = S_0 d_1 d_2\,, \quad C_{d_1 d_2} \ \text{ with } q'_d q''_d
\end{array}
$$

- The option price is

$$V_0 = \frac{C_{u_1 u_2} q'_u q''_u + C_{u_1 d_2} q'_u q''_d + C_{d_1 u_2} q'_d q''_u + C_{d_1 d_2} q'_d q''_d}{(1+\rho_1)(1+\rho_2)}$$

Calibrating binomial trees

- It is convenient to calibrate the tree such that the asset price follows a log-normal distribution,

$$\log\left(\frac{S_T}{S_t}\right) \sim N\left[\left(\rho - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$$

- In this case, as the number of time steps increases, the estimated option price will converge towards the Black-Scholes price.

Calibrating binomial trees

- Let Δ_t be the size of the time step, we have
  - $q_u = \frac{e^{\rho \Delta_t} - d}{u - d}$,
  - $u = e^{\sigma \sqrt{\Delta_t}}$,
  - $d = e^{-\sigma \sqrt{\Delta_t}}$.