Chapter 2

Limits and Continuity

The notion of a limit is a "can't afford to miss" topic — if you do, it will come back to haunt you.

Why is it so important? Limits are at the core of calculus and analysis: every single notion is a limit in one sense or another!

Examples, where limits are used, include the slope of the tangent line, the length of a curve, the area under a graph.

2.1 Definition of a limit

Let us start with the idea of a limit (and this needs to be refined later so that we can actually work with limits). If you are struggling with the topic, going to the online resource and investing time and work is particularly important; you cannot ignore the problem.

Given a function $f : D \to \mathbb{R}$ on an open interval, and $c \in \mathbb{R}$ then we mean by

$$\lim_{x \to c} f(x) = L$$

that

"as x approaches c, $f(x)$ approaches L"

or, formulated differently,

"$f(x)$ is close to L whenever $x \neq c$ is close to c."

By the one–sided limit

$$\lim_{x \to c^-} f(x) = L$$

we mean

"as x approaches c from the left, $f(x)$ approaches L"

and by

$$\lim_{x \to c^+} f(x) = L$$

we mean

"as x approaches c from the right, $f(x)$ approaches L".

Note that even if both one–sided limits exist, the limit might not: the left and right limits could have different values.

Let us now give a precise definition of a limit. Firstly, for saying $f(x)$ is close to L for all $x \neq c$ close to c, we need $f(x)$ to makes sense for $x \neq c$ near c (but not for c itself).

That is, we require that there is a (possibly very small) $p > 0$ such that the open intervals $(c - p, c), (c, c + p)$ are contained in the domain of f.

$f(x)$ is close to L means that the distance $|f(x) - L|$ is small, that is, as small as we choose it to be. So what we want is that given $\varepsilon > 0$ then $|f(x) - L| < \varepsilon$ if x is close enough to c.

But x is close enough to c, $x \neq c$, means that there exists a $\delta > 0$ such that $0 < |x-c| < \delta$.

Voilà, that's our definition:

Definition 2.1 (ε, δ–definition)**.** Let $f : D \to \mathbb{R}$ be a function with $(c-p, c) \cup (c, c+p) \subset D$ for some $p > 0$. Then we say that the *limit* of f at c is L, that is

$$\lim_{x \to c} f(x) = L\,,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that :

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon\,.$$

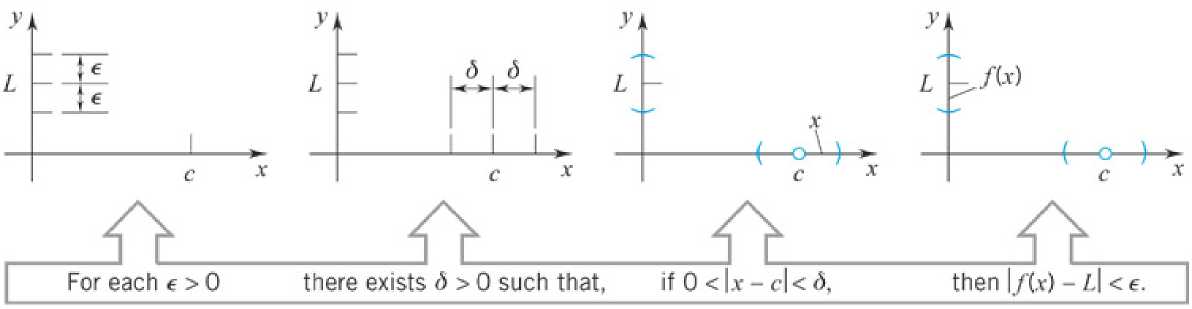

Figure 2.1: A picture of the definition of limit

Note that the order of ε and δ is crucial: we first choose how close we want to come to L and then have to find how close we have to be to c for this to be true.

Thus, δ depends on the choice of ε.

The definition of a limit is, as mentioned before, on the core of calculus and analysis. It may appear confusing in the beginning, so you should practise this notion with loads of examples.

The standard method to use the ε, δ–argument has two parts:

"Find δ" scratch work to find δ

"δ works" verify that your candidate for δ indeed satisfies the conditions.

Example 2.2. Let $f(x) = 2x - 1$. Then $\lim_{x \to 2} f(x) = 3$:

"Find δ" Scratch work to find δ: For $\varepsilon > 0$ want $\delta > 0$ so that if x has distance at most δ from 2, i.e., $0 < |x - 2| < \delta$, then $f(x)$ has distance at most ε from 3, i. e., $|f(x) - 3| < \varepsilon$. In our case $|f(x) - 3| = |2x - 1 - 3| = 2|x - 2|$ so we choose $\delta = \frac{\varepsilon}{2}$.

"δ works" We now verify that your candidate for δ indeed satisfies the conditions: Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2}$. Then $\delta > 0$ and for x with $0 < |x - 2| < \delta$ we have

$$|f(x) - 3| = 2|x - 2| < 2\delta = \varepsilon\,.$$

Thus, indeed $\lim_{x \to 2} f(x) = 3$.

Example 2.3. Let $f(x) = \frac{x^2-1}{x-1}$. Then $\lim_{x \to 1} f(x) = 2$:

"Find δ" Scratch work to find δ: For $\varepsilon > 0$ want $\delta > 0$ so that if x has distance at most δ from c, i.e., $0 < |x - 1| < \delta$, then $f(x)$ has distance at most ε from 2, i. e., $|f(x) - 2| < \varepsilon$. In our case

$$|f(x) - 2| = \left|\frac{x^2 - 1}{x - 1} - 2\right| = \left|\frac{x^2 - 1 - 2(x - 1)}{x - 1}\right| = \left|\frac{x^2 - 2x + 1)}{x - 1}\right| = \left|\frac{(x - 1)^2)}{x - 1}\right| \overset{x \neq 1}{=} |x - 1|$$

so we choose $\delta = \varepsilon$.

"δ works" We now verify that our candidate for δ indeed satisfies the conditions: Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then $\delta > 0$ and for x with $0 < |x - 1| < \delta$ we have

$$|f(x) - 2| = \left|\frac{x^2 - 1}{x - 1} - 2\right| = \left|\frac{x^2 - 1 - 2(x - 1)}{x - 1}\right| = \left|\frac{x^2 - 2x + 1)}{x - 1}\right| = \left|\frac{(x - 1)^2)}{x - 1}\right| \overset{x \neq 1}{=} |x - 1| < \delta = \varepsilon.$$

Thus, indeed $\lim_{x \to 1} f(x) = 2$.

Note: we couldn't just plug in $x = 1$: in this case $x - 1 = 0$, and thus, f is not defined at $x = 1$. However, the limit is, as we have seen.

To get an idea of the value of the limit, observe that $f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$.

Common errors: You cannot plugin the value of the function to find the limit. Let us consider the function

$$f(x) = \begin{cases} 2 & x \neq 2 \\ 3 & x = 2 \end{cases}$$

We have $\lim_{x\to 2} f(x) \overset{x\neq 2}{=} \lim_{x\to 2} 2 \neq 3 = f(2)$.

Of course, we don't want to use the ε, δ–criterion every single time. So, we try to find some general rules which allow to derive a limit, once we know simpler ones. Here are some simple examples we can use later:

Example 2.4.

(i) $\forall c \in \mathbb{R} : \lim\limits_{x\to c} x = c$

(ii) $\forall c \in \mathbb{R} : \lim\limits_{x\to c} |x| = |c|$

(iii) If $f(x) = k$ is the constant function with constant $k \in \mathbb{R}$ then $\forall c \in \mathbb{R}$: $\lim\limits_{x\to c} f(x) = k$.

Could you explain these in terms ε, δ-arguments if you needed to in the exam?

Here are some other ways to verify that a number is the limit of f at c which will come handy in various situations.

Theorem 2.5 (Equivalent formulations)**.** *The following are equivalent:*

(i) $\lim\limits_{x\to c} f(x) = L$

(ii) $\lim\limits_{x\to c}(f(x) - L) = 0$

(iii) $\lim\limits_{h\to 0} f(c + h) = L$

(iv) $\lim\limits_{x\to c} |f(x) - L| = 0.$

In the same spirit the one–sided limits can be introduced, and corresponding statements proved. We will only touch quickly on this subject and leave it to you to fill in the details.

Definition 2.6. (i) Let $f : D \to \mathbb{R}$ be a function with $(c - p, c) \subset D$ for some $p > 0$. Then we say that the *left–sided limit* of f at c is L, that is

$$\lim_{x\to c^-} f(x) = L\,,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that :

$$c - \delta < x < c \implies |f(x) - L| < \varepsilon\,.$$

(ii) Let $f : D \to \mathbb{R}$ be a function with $(c, c+p) \subset D$ for some $p > 0$. Then we say that the *right–sided limit* of f at c is L, that is

$$\lim_{x \to c^+} f(x) = L\,,$$

if for all $\varepsilon > 0$ there exists $\delta > 0$ such that :

$$c < x < c + \delta \implies |f(x) - L| < \varepsilon\,.$$

The one–sided limits are connected to the limit via

Theorem 2.7. *Let $f : D \to \mathbb{R}$, $c \in \mathbb{R}$. Then*

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L\,.$$

Of course it would not make sense for the same function at the same point to have two different limits, but this is not obvious from the definition so we should prove it:

Theorem 2.8. *If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$ then $L = M$.*

Proof. Suppose L and M were different, and try and prove this is imposible. For example, we could let $\varepsilon = |L - M|/4$, then if L and M we different we would have $\varepsilon > 0$ and by the definition of limit:

$$\exists \delta_L > 0 : 0 < |x - c| < \delta_L \Rightarrow |f(x) - L| < \varepsilon$$
$$\exists \delta_M > 0 : 0 < |x - c| < \delta_M \Rightarrow |f(x) - M| < \varepsilon$$

Now choose some x with $0 < |x - c| < \min(\delta_L, \delta_M)$, and by the triangle inequality

$$|L - M| = |(f(x) - M) - (f(x) - L)| \le |f(x) - M| + |f(x) - L| < \varepsilon + \varepsilon = |L - M|/2.$$

We have a contradiction to our hypothesis that $L \neq M$, so we must have $L = M$. $\square$

2.2 The pinching theorem for limits

Theorem 2.9 (Pinching Theorem)**.** *Let $f, g, h : D \to \mathbb{R}$. Suppose that for all $x \in D$*

$$h(x) \le f(x) \le g(x)\,.$$

If $\lim_{x\to c} h(x) = L$ *and* $\lim_{x\to c} g(x) = L$ *then*

$$\lim_{x\to c} f(x) = L\,.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to c} h(x) = L$ and $\lim_{x\to c} g(x) = L$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x-c| < \delta_1 \implies |h(x) - L| < \varepsilon, \text{ i.e., } L - \varepsilon < h(x) < L + \varepsilon,$$

and

$$0 < |x-c| < \delta_2 \implies |g(x) - L| < \varepsilon, \text{ i.e., } L - \varepsilon < g(x) < L + \varepsilon\,.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x-c| < \delta$ we have

$$L - \varepsilon < h(x) \le f(x) \le g(x) < L + \varepsilon\,,$$

that is $|f(x) - L| < \varepsilon$. Thus, $\lim_{x\to c} f(x) = L$.

□

2.3 Limit laws

Our goal is to find laws which simplify finding a limit.

We have already seen the pinching theorem allows us to prove results that might look hard. We also have basic results such as:

Theorem 2.10. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = 0$ *then* $\lim_{x\to c} f(x)g(x) = 0$.

The following theorem will allow us to compute lots of limits by using the examples we have seen before.

Theorem 2.11 (Limit Laws). *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *then*

(i) $\lim_{x\to c}[f(x) + g(x)] = L + M$

(ii) $\lim_{x\to c}[\alpha f(x)] = \alpha L$ *for* $\alpha \in \mathbb{R}$

(iii) $\lim_{x\to c}[f(x)g(x)] = LM$

Remark 2.12. *(i) For the limit laws it is important that the individual limits exist. For example, for* $f(x) = \frac{1}{x}, g(x) = x$

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} \frac{1}{x} x = \lim_{x\to 0} 1 = 1$$

whereas $\lim_{x\to 0} g(x) = 0$ *and* $\lim_{x\to 0} f(x)$ *does not exist; thus, in this case*

$$\lim_{x\to 0}[f(x)g(x)] \neq [\lim_{x\to 0} f(x)] \cdot [\lim_{x\to 0} g(x)]\,.$$

Thus, if you use the limit laws you have to state why you are allowed to so (mark reduction otherwise).

(ii) *Note that the same limit laws apply for one–sided limits.*

By induction we conclude from the Limit Laws:

Corollary 2.13. *Let* f_k *be functions and* $a_k \in \mathbb{R}$*, for* $k = 1, \ldots, n$*. If each* $\lim\limits_{x\to c} f_k(x)$ *exists then*

$$\lim_{x\to c}\left(\sum_{k=1}^{n} a_k f_k(x)\right) = \sum_{k=1}^{n} a_k (\lim_{x\to c} f_k(x))$$

and

$$\lim_{x\to c}\left(\prod_{k=1}^{n} f_k(x)\right) = \prod_{k=1}^{n} \lim_{x\to c} f_k(x)$$

Applying the second result to the functions $f_k(x) = x$ we see that $\lim\limits_{x\to c} x^n = c^n$, and applying the first result to $f_k(x) = x^k$ we see:

Corollary 2.14. *If* $P(x) = a_n x^n + \ldots + a_0$ *is a real polynomial and* $c \in \mathbb{R}$ *then*

$$\lim_{x\to c} P(x) = P(c)\,.$$

Functions with $\lim\limits_{x\to c} f(x) = f(c)$ are called *continuous* at c. We will discuss this very soon!

Theorem 2.15. *If* $\lim\limits_{x\to c} g(x) = M$*,* $M \neq 0$*,* $\lim\limits_{x\to c} f(x) = L$*, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}\,.$$

Applying this to the quotient of polynomials we have:

Corollary 2.16. *For rational functions* $R = \frac{P}{Q}$ *with polynomials* P *and* Q *we have*

$$\lim_{x\to c} R(x) = R(c)$$

provided that $Q(c) \neq 0$*.*

Another consequence is that we obtain a criterion for a limit not to exist:

Theorem 2.17. *If* $\lim_{x\to c} f(x) = L$, $L \neq 0$, *and* $\lim_{x\to c} g(x) = 0$ *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *does not exist.*

The assumption $L \neq 0$ is crucial; for example, for $f(x) = x^2$ and $g(x) = x$ the limit exists and is $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

Proof. Proof by contradiction: Assume that $\lim_{x\to c} \frac{f(x)}{g(x)} = K$ for some $K \in \mathbb{R}$. Then by the limit laws

$$L = \lim_{x\to c} f(x) = \lim_{x\to c} g(x)\frac{f(x)}{g(x)} = \lim_{x\to c} g(x) \lim_{x\to c} \frac{f(x)}{g(x)} = 0 \cdot K = 0\,.$$

Since $L \neq 0$ this is a contradiction. □

We can only apply the limit laws if the limits exist. So there is no contradiction in the above argument if $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist!

2.4 Trigonometric examples

Example 2.18.

$$\lim_{x\to 0} \sin x = 0, \qquad \lim_{x\to 0} \cos x = 1\,.$$

Use $0 < |\sin x| < x$ for x near 0 and the Pinching Theorem. For the second statement use $\cos x = \sqrt{1 - \sin^2 x}$ for x near 0.

Another important consequence is:

Corollary 2.19.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\,.$$

2.5 Continuity

In ordinary language, “continuous” means that a process goes on without interruptions and abrupt changes. The mathematical notion is more precise.

Definition 2.20. Let $f : D \to \mathbb{R}$ a function and $c \in \mathbb{R}$ such that there exists $p > 0$ such that $(c - p, c + p) \in D$. Then f is called *continuous* at c if

$$\lim_{x\to c} f(x) = f(c)$$

In contrast to the definition of a limit, for continuity the function has to be defined at c. Thus, a function can only fail to be continuous at c if

(i) $\lim_{x\to c} f(x)$ exists but is not $f(c)$, or

(ii) $\lim_{x\to c} f(x)$ does not exist (but $f(c)$ does).

In the first case, we can "remove" the discontinuity by changing the value of f at c to be L, and we call it a *removable discontinuity*. The second case is called an *essential discontinuity*: however one changes finitely many values of f, the discontinuity will not go away.

Example 2.21. The *Dirichlet function*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

has an essential discontinuity at every point.

We already have a small list of continuous functions: polynomials, rational functions $\frac{P}{Q}$ provided $Q(c) \neq 0$, absolute value function, square root function.

As before, we can construct new continuous functions by algebraic operations

Theorem 2.22. *If $f, g : D \to \mathbb{R}$ are continuous at $c \in D$, $\alpha \in \mathbb{R}$, then*

$$f + g, \quad f - g, \quad f \cdot g, \quad \alpha f, \quad \frac{f}{g} \quad \text{(provided } g(c) \neq 0\text{)}$$

are continuous at c.

Proof. Apply the Limit Laws! □

We now apply the ε, δ–criterion to get:

Theorem 2.23. *f is continuous at c if and only if $\forall \varepsilon > 0 \exists \delta > 0$:*

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon\,.$$

Proof. In difference to the definition of a limit, we can now allow $x = c$ because then $f(x) - f(c) = 0$, thus, $|f(x) - f(c)| < \varepsilon$ for all $\varepsilon > 0$! With this observation, the theorem follows immediately from the ε, δ definition of a limit. □

A form of this ε, δ definition of continuity was first given by Bernard Bolzano in 1817. Preliminary forms of a related definition of the limit were given by Cauchy. The formal definition and the distinction between pointwise continuity and uniform continuity were first given by Bolzano in the 1830s but the work wasn't published until the 1930s. Heine provided the first published definition of uniform continuity in 1872, but based these ideas on lectures given by Dirichlet in 1854.

Theorem 2.24. *If g is continuous at c, and f is continuous at $g(c)$ then $f \circ g$ is continuous at c.*

Proof. Let $\varepsilon > 0$. Since f is continous at $g(c)$ there exists $\delta_1 > 0$ such that

$$|t - g(x)| < \delta_1 \implies |f(t) - f(g(c))| < \varepsilon\,.$$

Since g is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |g(x) - g(c)| < \delta_1\,.$$

But then the previous statement shows (with $t = g(x)$) that

$$|x - c| < \delta \implies |g(x) - g(c)| < \delta_1 \implies |f(g(x)) - f(g(c))| < \varepsilon\,.$$

This shows that $f \circ g$ is continuous at c. $\square$

Example 2.25. $F(x) = \sqrt{\frac{x^2+1}{x-3}}$ is continuous for all $x \in (3, \infty)$: use $F = f \circ g$ with $f(x) = \sqrt{x}, g(x) = \frac{x^2+1}{x-3}$.

Definition 2.26. A function f is said to be *continuous on an interval* if it is continuous for all interior points of the interval, and one–sidedely continuous at the endpoints.

2.6 The Intermediate–Value Theorem

From our intuition, we think a continuous function on an interval shouldn't have holes or jumps. This is the content of the very important Intermediate Value Theorem. We will not prove this result here but it can be shown using the completeness of $\mathbb{R}$, that is, the Least Upper Bound Axiom.

Theorem 2.27 (Intermediate–Value Theorem)**.** *Let f be continuous on $[a, b]$ and K any number between $f(a)$ and $f(b)$. Then there exists at least one $c \in (a, b)$ with*

$$f(c) = K\,.$$

For $K = 0$, the statement is also known as Bolzano's theorem.

Note: c is in the open interval (a, b): it cannot take the values $c = a$ or $c = b$ since $f(a) \neq K$ and $f(b) \neq K$. Stating the Intermediate–Value Theorem as "... at least one $c \in [a, b]$..." gives mark reduction: although the statement is not false, the implication does not contain all information.

2.7 The Extreme Value Theorem

We first note that, if a function is continuous at a point, then in some small interval around that point it must be bounded.

Lemma 2.28. *If $f : D \to \mathbb{R}$ is continuous at $x = c$ there exist $\delta, B > 0$ such that, for all $x \in (c - \delta, c + \delta)$, $|f(x)| < B$.*

The extreme value theorem extends the previous result, to say a function is bounded on all closed domains. In fact it says more: every continuous function attains its maximum and minimum on closed intervals. Recall

Definition 2.29. f takes on a *maximum* on I if there exists $x \in I$ with $f(x) = \max f(I)$, that is, $f(t) \leq f(x)$ for all $t \in I$.

f takes on a *minimum* on I if there exists $x \in I$ with $f(x) = \min f(I)$, that is, $f(t) \geq f(x)$ for all $t \in I$.

(Note: later we will call these global maximum and global minimum to distinguish from the local extrema).

There are examples of functions on $[a, b]$ which are bounded but do not attain their maximum or minimum. However, if the function is continuous, this cannot happen. We will not prove this theorem since the proof, as in the case of the Intermediate-Value Theorem needs the Least Upper Bound Axiom.

Theorem 2.30 (Extreme–Value Theorem)**.** *Let f be continuous on a bounded closed interval $[a, b]$. Then f takes on both its maximum and its minimum on $[a, b]$.*

The extreme value theorem was originally proved by Bernard Bolzano in the 1830s in a work Function Theory but the work remained unpublished until 1930. Bolzano's proof consisted of showing that a continuous function on a closed interval was bounded, and then showing that the function attained its maximum and minimum value. Both proofs involved what is known today as the Bolzano–Weierstrass theorem, see Chapter 6.

Thus, continuous functions map bounded closed intervals $[a, b]$ onto bounded closed intervals $[m, M]$.

All assumptions of the Extreme–Value Theorem are needed!

- If the interval is not bounded, then a continuous function may not attain its extreme values: $f(x) = x^3$ on $[0, \infty)$.
- If the interval is not closed, then a continuous function may not attain its extreme values: $f(x) = x^3$ on $[0, 1)$.
- If the function is not continuous on $[a, b]$ then f may not attain its extreme values: $$f(x) = \begin{cases} 3, & x = 1 \\ x, & 1 < x < 5 \\ 3, & x = 5 \end{cases}.$$

2.8 Continuity and inverse functions

Definition 2.31. A function f is said to be

(strictly) ***increasing*** on an interval I if for all $x_1, x_2 \in I$, $x_1 < x_2$: $f(x_1) < f(x_2)$.

(strictly) ***decreasing*** on an interval I if for all $x_1, x_2 \in I$, $x_1 < x_2$: $f(x_1) > f(x_2)$.

Example 2.32. If f is increasing then f is one–to–one.

Lemma 2.33. *If f is continuous on (a, b) and one–to–one, then f is increasing or decreasing.*

Proof. Workbook. □

Theorem 2.34. *If f is one–to–one and continuous on (a, b) so is f^{-1}.*

2.9 Study guide

Go through the material of chapter 2, and identify the main topics and statements. Have a look some more examples: could you do reproduce them without looking at your notes? Do you understand what they are examples of? Are they counterexamples? Check the red text: be aware of the most commonly made errors!

To deepen your understanding, think about the following:

- What is the difference between the following statements:
  - for all $\varepsilon > 0$ there exists $\delta > 0$: $0 < |x-c| < \delta \implies |f(x)-f(c)| < \varepsilon$.
  - there exists $\alpha > 0$ so that for all $\beta > 0$: $0 < |x-c| < \alpha \implies |f(x)-f(c)| < \beta$.
  - for all $\alpha > 0$ there exists $\beta > 0$: $0 < |x-c| < \alpha \implies |f(x)-f(c)| < \beta$.

  Give in each case an example of a function satisfying the condition.

- Why are the following statements not characterising a limit?

  (i) If x gets closer to c then $f(x)$ gets closer to $f(c)$.

  (ii) By choosing x close to c we can have $f(x)$ arbitrarily close to $f(c)$.

  Give in both cases an example which satisfies the condition but doesn't have a limit at c.

- What is the difference between a function which has a limit at c and a function which is continuous at c?

- Find a continuous function without zeros. Is your example contradicting the Intermediate–Value Theorem? Does $f(x) = \sin(x^3) - 1, x \in \mathbb{R}$, attain the value -1.5? If so, find c with $f(c) = -1.5$ (you may want to use your Excel-skills).