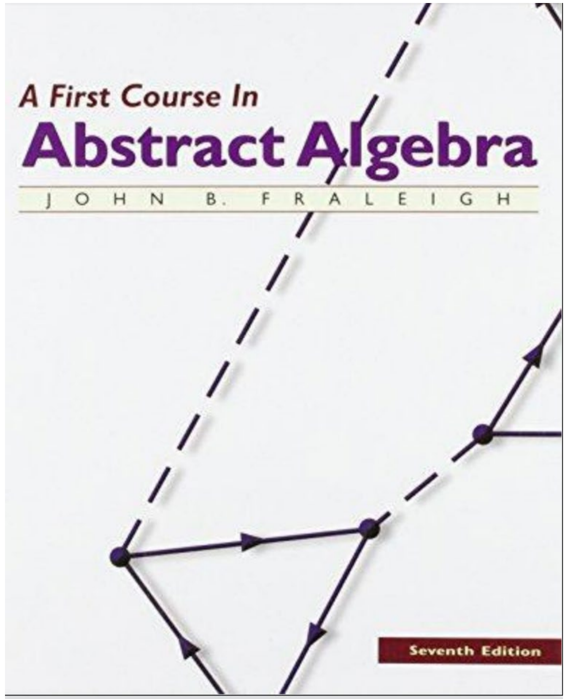
Abstract Algebra

Reference book



Final score

Final test 50%

Homework 10%

Test in each lesson 10%

Chapter test 10%×3

App score 10%

SECTION 0: SETS AND RELATIONS

Definition: A set is a well-defined collection of objects.

S is a set and a is some object, then either a is **definitely** in S (denoted by $a \in S$), or a is **definitely** not in S, denoted by $a \notin S$.

the set S of some positive numbers. ×

the T of all prime positive integers. ✓

SECTION 0: SETS AND RELATIONS

Definition: A set is a well-defined collection of objects.

A set S is made up of elements, and if a is one of these elements, we shall denote this fact by $a \in S$.

There is exactly one set with no elements. It is the empty set, denoted by Ø.

SECTION 0: SETS AND RELATIONS

2 ways:.

listing elements separated by commas, in braces.
{2,4,6,8}.

giving a characterizing property of the elements. $\{x \mid P(x)\}$

$\{x \mid x$ is an even whole positive number $\leq 8\}$

$\{2x \mid x = 1,2,3,4\}$

SECTION 0: SETS AND RELATIONS

0.1 Definition A set B is a subset of a set A, denoted by $B \subseteq A$ or $A \supseteq B$. if every element of B is in A.

The notations $B \subset A$ or $A \supset B$ will be used for $B \subset A$ but $B \neq A$.

Definition $A=B$ if $A \supseteq B$ and $B \supseteq A$.

Tips A itself and Ø are both subsets of A.

0.3 Example Let $S=\{1, 2, 3\}$.
This set S has a total of eight subsets,
namely $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{1,2,3\}$.

SECTION 0: SETS AND RELATIONS

0.4 Definition Let A and B be sets. The set $A\times B=\{(a,b)\mid a\in A$ and $b\in B\}$ is the Cartesian product(卡尔积) of A and B.

0.5 Example If $A=\{1,2,3\}$ and $B=\{3,4\}$, then we have
$A\times B=\{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$.

0.6 Example The set $R\times R$ is the familiar Euclidean plane.

SECTION 0: SETS AND RELATIONS

Notations

Z is the set of all integers(whole numbers: +, − and 0)

Q is the set of all rational numbers(that is, numbers that can be expressed as quotients m/n of integers, where n≠0).

R is the set of all real numbers.

C is the set of all complex numbers.

Z^+, Q^+, and R^+ are the sets of positive members of Z, Q, and R, resp.

Z^*, Q^*, R^* and C^* are the sets of nonzero members of Z, Q, R, and C, resp.

SECTION 0: SETS AND RELATIONS

0.7 Definition A relation between sets A and B is a subset R of $A \times B$. We read $(a, b) \in R$ as a is related to b and write $a \, R \, b$.

Definition A relation on a set A is a subset R of $A \times A$.

Example bignees relation R on a R

$R = \{ (a, b) \mid a > b \text{ and } a, b \in R \}$

"$>$" $= \{ (a, b) \mid a, b \in R \}$

$aR\, b$

$a > b$

SECTION 0: SETS AND RELATIONS

Example Equality Relation $=$ on a set X

"$=$" $=\{(a, b) \mid a=b\}=\{(x,x) \mid x\in X\}$

$a = b$

SECTION 0: SETS AND RELATIONS

0.9 Example The graph of the function f where $f(x)=x^3$ for all $x \in R$, is the subset $\{(x, x^3) \mid x^3 \in R\}$ of $R \times R$. It is a relation on R.

SECTION 0: SETS AND RELATIONS

0.10 Definition A **function** ϕ mapping X into Y is a relation between X and Y with the property that each $x \in X$ appears as the first member of exactly one ordered pair (x, y). Such a function is also called a map or mapping of X into Y. We write $\phi : X \to Y$ and express $(x, y) \in \phi$ by $\phi(x) = y$.

The **domain** of Φ is the set X and the set Y is the codomain of Φ.

The **range** of Φ is $\Phi[X] = \{\Phi(x) \mid x \in X\}$.

SECTION 0: SETS AND RELATIONS

0.11 Example We can view the addition of real numbers as a function $+:(R\times R)\to R$, that is, as a mapping of $R\times R$ into R.

$+$ on $(2,3)\in R\times R$

$+((2,3))=5.$

$((2,3),5)\in +.$

$2+3=5.$

SECTION 0: SETS AND RELATIONS

0.11 Definition The number of elements in a set X is the cardinality of X and is often denoted by $|X|$.

$$|\{2,5,7\}|=3. \quad |R|=?$$

0.11 Definition Two sets X and Y have the same cardinality, if there exist a pairing of each x in X with only one y in Y in such a way that each element of Y is also used only once in this pairing.

$X=\{2,5,7\}$ and $Y=\{?,!,\#\}$

$2\leftrightarrow ?,\quad 5\leftrightarrow !,\quad 7\leftrightarrow \#$ $\qquad\qquad \{(2,?),(5,!),(7,\#)\}$

SECTION 0: SETS AND RELATIONS

Example Shows that the sets Z and Z^+ have the same cardinality.

1	2	3	4	5	6	7	8	9	10
0	-1	1	-2	2	-3	3	-4	4	-5

Z and $Z+$ have the same cardinality.

SECTION 0: SETS AND RELATIONS

0.12 Definition A function $\phi: X \to Y$ is one to one (injection, 单射). if $\phi(x_1)=\phi(x_2)$ only when $x_1=x_2$

The function ϕ is onto (surjective, 满射) Y if the range of ϕ is Y.

A map that is both injective and surjective is a bijection(双射，一一映射).

SECTION 0: SETS AND RELATIONS

0.12 Definition If a subset of $X \times Y$ is a one-to-one function ϕ mapping X onto Y. If we interchange the first and second members of all ordered pairs (x, y) in ϕ to obtain a set of ordered pairs (y,x). we get a subset of $Y \times X$, which gives a one-to-one function mapping Y onto X. This function is called the **inverse function** of ϕ, and is denoted by ϕ^{-1}.

Summarizing If ϕ maps X one to one onto Y and $\phi(x)=y$, then ϕ^{-1} maps Y one to one onto X, and $\phi^{-1}(y)=x$.

SECTION 0: SETS AND RELATIONS

0.13 Definition Two sets X and Y have the **same cardinality** if there exists a one-to-one function mapping X onto Y, that is, if there exists a one-to-one correspondence between X and Y.

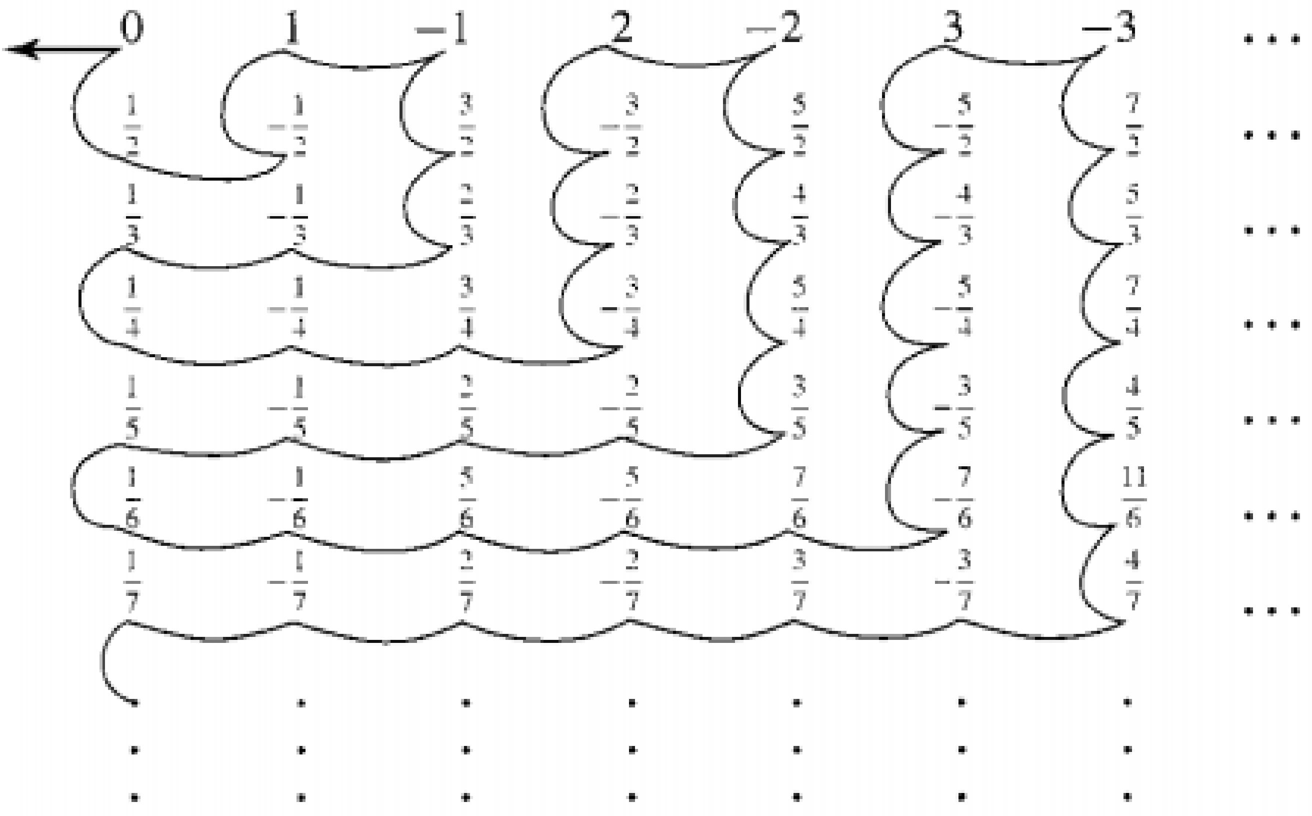
$$|Z|=|Z^{+}|=N_0$$

A set has cardinality N_0 if and only if all of its elements could be listed in an infinite row, so that we could "number them" using Z^{+}.

A set has cardinality N_0 if and only if all of its elements could be listed in an infinite row, so that we could "number them" using Z^{+}.

SECTION 0: SETS AND RELATIONS

$|\mathcal{Q}|=\aleph_0$



SECTION 0: SETS AND RELATIONS

R has too many elements to be paired with those in Z^+.

We just denote the cardinality of R by $|R|$.

There are infinitely many different cardinal numbers even greater than $|R|$.

SECTION 0: SETS AND RELATIONS

Definition Sets are **disjoint** if no two of them have any element in common.

0.16 Definition A **partition(分划，划分)** of a set S is a collection of nonempty subsets of S such that every element of S is in exactly one of the subsets.The subsets are the **cells** of the partition.

we denote by $\bar{x}$ the cell containing the element x of S.

SECTION 0: SETS AND RELATIONS

0.17 Example Splitting Z^+ into the subset of even positive integers and the subset of odd positive integers

$\overline{1}=\{1,3,5,7,....\}$

$\overline{1}4=\{2,3,4,6,....\}$

SECTION 0: SETS AND RELATIONS

Example We could also partition Z+ into three cells. one consisting of the positive integers divisible by 3. Another containing all positive integers leaving a remainder of 1 when divided by 3. The last containing positive integers leaving a remainder of 2 when divided by 3.

$\overline{3}=\{3,6,9,....\}$

$\overline{4}=\{1,4,7,10,....\}$

$\overline{2}=\{2,5,8,11,....\}$

SECTION 0: SETS AND RELATIONS

0.18 Definition An equivalence relation R on a set S is one that satisfies these three properties for all $x,y,z\in S$.

1.(Reflexive)(自反性) xRx.
2.(Symmetric)(对称性) If xRy,then yRx.
3.(Transitive)(传递性) If xRy and yRz then xRz.

SECTION 0: SETS AND RELATIONS

0.19 Example For any nonempty set S, the equality relation $=$ defined by the subset $\{(x,x) x \in S\}$ of $S \times S$ is an equivalence relation.

SECTION 0: SETS AND RELATIONS

0.20 Example (Congruence Modulo n) Let $n \in Z^+$. The equivalence relation on Z^+ corresponding to the partition of Z^+ into residue classes modulo n, discussed in Example 0.17, is congruence modulo n.

It is sometimes denoted by $\equiv_n$. Rather than write $a \equiv_n$ b, we usually write $a \equiv b$ (mod n), read,"a is congruent to b modulo n."

15 = 27(mod 4)

SECTION 0: SETS AND RELATIONS

0.21 Example Let a relation R on the set Z be defined by $nR\,m$ if and only if $nm \geq 0$. Is R an equivalence relation?

SECTION 0: SETS AND RELATIONS

SECTION 0: SETS AND RELATIONS

SECTION 0: SETS AND RELATIONS

Abstract Algebra

SECTION 1-2: BINARY OPERATIONS

2.1 Definition: A binary operation $*$ on a set S is a function mapping $S \times S$ into S. For each $(a,b) \in S \times S$, we will denote the element $*((a,b))$ of S by $a * b$.

2.2 Example Our usual addition + is a binary operation on the set R. Our usual multiplication • is adifferent binary operation on R.

Note We require a binary operation on a set S to be defined for every ordered pair (a, b) of elements from S.

SECTION 1-2: BINARY OPERATIONS

2.1 Definition: A binary operation * on a set S is a function mapping $S \times S$ into S. For each $(a,b) \in S \times S$, we will denote the element $*((a,b))$ of S by $a * b$.

2.3 Let $M(R)$ be the set of all matrices with real entries. The usual matrix addition + is not a binary operation on this set since A+B is not defined for an ordered pair (A, B) of matrices having different numbers of rows or of columns.

tip Let $M_n(R)$ be the set of all matrices of type $n \times n$ with real entries. The usual matrix addition + is a binary operation.

SECTION 1-2: BINARY OPERATIONS

2.1 Definition: Let $*$ be a binary operation on S and let H be a subset of S. The subset H is closed(封闭) under $*$ if for all $a, b \in H$ we also have $a*b \in H$. In this case, the binary operation on H given by restricting $*$ to H is the induced operation of $*$ on H.

Tip By our very definition of a binary operation$*$ on S, the set S is closed under $*$

2.5 Example Our usual addition + on the set R of real numbers does not induce a binary operation on the set R^* of nonzero real numbers

$$2+(-2)=0$$

SECTION 1-2: BINARY OPERATIONS

2.1 Definition: Let $*$ be a binary operation on S and let H be a subset of S. The subset H is closed(封闭) under $*$ if for all $a, b \in H$ we also have $a*b \in H$. In this case, the binary operation on H given by restricting $*$ to H is the induced operation of $*$ on H.

2.6 Example Let $+$ and $\cdot$ be the usual binary operations of addition and multiplication on the set Z, and let $H=\{n^2 \mid n \in Z^+\}$. Determine whether H is closed under (a) addition and (b) multiplication.

SECTION 1-2: BINARY OPERATIONS

2.7 Example Let F be the set of all real-valued functions f having as domain the set R of real numbers. We are familiar from calculus with the binary operations $+$, $-$, $\cdot$ and $\circ$ on F.

Namely, for each ordered pair (f,g) of functions in F, we define for each $x \in H$

$f+g$ by $(f+g)(x) = f(x) + g(x)$ addition,

$f-g$ by $(f-g)(x) = f(x) - g(x)$ subtraction,

$f \cdot g$ by $(f \cdot g)(x) = f(x)g(x)$ multiplication,

$f \circ g$ by $(f \circ g)(x) = f(x)g(x)$ composition.

SECTION 1-2: BINARY OPERATIONS

2.8 Example On Z^+, we define a binary operation $*$ by $a*b$ equals the smaller of a and b, or the common value if $a=b$.

$$2*11=2;\ 15*10=10;\ \text{and}\ 3*3=3.$$

SECTION 1-2: BINARY OPERATIONS

2.10 Example On Z^+, we define a binary operation $*''$ by $a *'' b = (a * '' b) + 2$, where $*$ is defined in Example 2.8.

$4 *'' 7 =$

$25 *'' 9 =$

$6 *'' 6 =$

SECTION 1-2: BINARY OPERATIONS

2.9 Example On Z^+, we define a binary operation $*'$ by $a*'b=a$.

$2*'3=2$
$25*'10=25$
$5*'5=5$

SECTION 1-2: BINARY OPERATIONS

2.11 Definition A binary operation $*$ on a set S is commutative if (and only if) $a*b=b*a$ for all $a,b\in S$.

Are $*$, $*'$ and $*''$ commutative?

SECTION 1-2: BINARY OPERATIONS

2.11 Definition A binary operation on a set S is **associative** if $(a*b)*c=a*(b*c)$ for all $a, b, c \in S$.

Are $*$, $*'$ and $*''$ associative?

SECTION 1-2: BINARY OPERATIONS

2.13 Theorem (Associativity of Composition) Let S be a set and let f, g, and h be functions mapping S into S. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

SECTION 1-2: BINARY OPERATIONS

2.14 Example Table 2.15 defines the binary operation $*$ on $S=\{a,b,c\}$ by the following rule:

2.15 Table

$*$	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

(ith entry on the left)$*$(jth entry on the top)
=(entry in the ith row and jth column of the table body)

SECTION 1-2: BINARY OPERATIONS

2.16 Example Table 2.17 Complete Table 2.17 so that* is a commutative binary operation on the set $S=\{a,b,c,d\}$.

2.17 Table

$*$	a	b	c	d
a	b			
b	d	a		
c	a	c	d	
d	a	b	b	c

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

2.19 Example On Q,let $a*b= a/b$.Here * is not everywhere defined on Q,for no rational number is assigned by this rule to the pair (2, 0).

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

2.20 Example On Q^+,let $a*b=a/b$.Here both Conditions 1 and 2 are satisfied,and$*$ is a binary
operation on Q^+.

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

2.21 Example On Z^+,let $a*b=a/b$.Here both Conditions 1 and 2 are satisfied,and$*$ is a binary
operation on Z^+.

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

2.22 Example Let F be the set of all real-valued functions with domain R as in Example 2.7.Suppose we define $*$ to give the usual quotient of f by g, that is,$f*g=h$, where $h(x)=f(x)/g(x)$.

SECTION 1-2: BINARY OPERATIONS

1. Exactly **one** element is assigned to each possible ordered pair of elements of S,
2. For each ordered pair of elements of S, the element assigned to it is again in

2.23 Example Let F be the set of all real-valued functions with domain R as in Example 2.7.Suppose we define $*$ to give the usual quotient of f by g, that is,$f*g=h$, where $h(x)=f(x)/g(x)$.

Abstract Algebra

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.7 Definition: Let $(S, *)$ and $(S', *')$ be binary algebraic structures. An isomorphism of S with S' is a one-to-one function ϕ mapping S onto S' such that $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in S$.

If such a map ϕ exists, then S and S' are isomorphic binary structures, which we denote by S □ S', omitting the $*$ and $*'$ from the notation.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

Question How to Show That Binary Structures Are Isomorphic

Step 1 Define the function ϕ that gives the isomorphism of S with S'.

Step 2 Show that ϕ is a one-to-one function(injective)(单设).

Step 3 Show that ϕ is onto(surjective)（满射） S'.

Step 4 Show that $\phi(x * y) = \phi(x) *' \phi(y)$

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.8 Example Let us show that the binary structure $(\mathbb{R}, +)$ with operation the usual addition is isomorphic to the structure $(\mathbb{R}^+, \cdot)$ where $\cdot$ is the usual multiplication.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.9 Example Let $2Z=\{2n|n \in Z\}$, so that 2Z is the set of all even integers, positive, negative, and zero. We claim that (Z, +) is isomorphic to (2Z, +), where + is the usual addition.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

Question How to Show That Binary Structures Are Not Isomorphic

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.10 Example Example The binary structures $(Q, +)$ and $(R, +)$ are not isomorphic because Q has cardinality $\aleph_0$ while $|R| \neq \aleph_0$.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.11 Example The sets Z and Z^+ both have cardinality $\aleph_0$, and there are lots of one-to-one functions mapping Z and Z^+. However, the binary structures $(Z, \cdot)$ and $(Z^+, \cdot)$, where $\cdot$ is the usual multiplication, are not isomorphic.

In $(Z, \cdot)$ there are two elements x such that $x \cdot x = x$, namely, 0 and 1. However, in $(Z^+, \cdot)$, there is only the single element 1.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.15 Example We show that the binary structures $(\mathbb{Q}, +)$ and $(\mathbb{Z}, +)$ under the usual addition are not isomorphic.

The equation $x+x=c$ has a solution x for all $c \in Q$, but this is not the case in Z. For example, the equation $x+x=3$ has no solution in Z.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.16 Example The binary structures $(C, \cdot)$ and $(R, \cdot)$ under the usual multiplication are not isomorphic.

The equation $x \cdot x = c$ has a solution x for all $c \in C$, but $x \cdot x = -1$ has no solution in R.

A

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.17 Example The binary structure $\langle (M_2(R), \cdot \rangle$ of 2×2 real matrices with the usual matrix multiplication is not isomorphic to $(R, \cdot)$ with the usual number multiplication.

Multiplication of numbers is commutative, but multiplication of matrices is not.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.12 Definition Let $(S, *)$ be a binary structure. An element e of S is an **identity element(单位元)** for $*$ if

$$e*s = s*e = s \text{ for all } s \in S.$$

3.13 Theorem (Uniqueness of Identity Element) A binary structure $(S, *)$ has at most one identity element.

SECTION 1-3: ISOMORPHIC BINARY STRUCTURES

3.14 Theorem Suppose $(S, *)$ has an identity element e for $*$. If $\phi : S \to S'$ is an isomorphism of $(S, *)$ with $(S', *')$, then $\phi(e)$ is an identity element for the binary operation $*'$ on S'.

Abstract Algebra

SECTION 4: GROUPS

Example:

$(5+x)=2$

$-5+(5+x)=-5+2,$ adding -5

$(-5+5)+x=-5+2,$ associative law

$0+x=-5+2,$ computing $-5+5$

$x=-5+2,$ 0 is the identity element

$x=-3,$ computing $-5+2$

SECTION 4: GROUPS

4.1 Definition A **group$(G, *)$(群)** is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

R1: For all $a, b, c \in G$, we have

$(a*b)*c = a*(b*c)$ associativity(结合律) of $*$

R2: There is an element e in G such that for all $x \in G$,

$e*x=x*e=x.$ identity element(单位元) e for $*$

R3: Corresponding to each $a \in G$, there is an element a' in G such that $a*a'=a'*a=e$. inverse(逆元) a' of a.

SECTION 4: GROUPS

4.4 Example The set Z^+ under addition is not a group.

There is no identity element for $+$ in Z^+.

SECTION 4: GROUPS

4.5 Example The set of all nonnegative integers (including 0) under addition is still not a group.

Thereis an identity element 0.

But it has no inverse for 2.

SECTION 4: GROUPS

4.6 Example The familiar additive properties of integers and of rational, real, and complex numbersshow that Z, Q, R, and C under addition are abelian groups.

4.3 Definition A group G is abelian(阿贝尔，交换) if its binary operation is **commutative**.

SECTION 4: GROUPS

4.7 Example The set Z^+ under multiplication is not a group.

There is an identity 1, but no inverseof 3.

SECTION 4: GROUPS

SECTION 4: GROUPS

4.9 Example. The set of all real-valued functions with domain R under function addition is a group. This group is abelian.

SECTION 4: GROUPS

4.10 Example. (Linear Algebra) Those who have studied vector spaces should note that the axioms for a vector space V pertaining just to vector addition can be summarized by asserting that V under vector addition is an abelian group.

SECTION 4: GROUPS

4.11 Example. The set $M_{m\times n}(R)$ of all $m \times n$ matrices under matrix addition is a group.

The $m \times n$ matrix with all entries 0 is the identity matrix. This group is abelian.

SECTION 4: GROUPS

4.12 Example. The set $M_n(R)$ of all $n \times n$ matrices under matrix multiplication is not a group.

The $n \times n$ matrix with all entries 0 has no inverse.

4.13 Example. Show that the subset S of $M_n(R)$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group.

SECTION 4: GROUPS

4.14 Example. Let $*$ be defined on Q^+ by $a*b=ab/2$.

SECTION 4: GROUPS

4.15 Theorem If G is a group with binary operation $*$, then the left and right cancellation laws hold in G, that is, $a*b=a*c$ implies $b=c$, and $b*a=c*a$ implies $b=c$ for all $a,b,c \in G$.

SECTION 4: GROUPS

4.16 Theorem If G is a group with binary operation $*$, and if a and b are any elements of G, then the linear equations $a*x=b$ and $y*a=b$ have unique solutions x and y in G.

SECTION 4: GROUPS

4.17 Theorem In a group G with binary operation $*$, there is only one elemente in G such that $e*x=x*e=x$

for all $x \in G$. Likewise for each $a \in G$, there is only one element a' in G such that $a'*a=a*a'=e$.

The identity element and inverse of each element are **unique** in a group.

4.18 Corollary Let G be a group. For all $a,b \in G$, we have

$(a*b)'=b'*a'$.

SECTION 4: GROUPS

4.17 Finite groups:

A minimal set that might give rise to a group is a one-element set $\{e\}$. The only possible binary operation $*$ on $\{e\}$ is defined by $e*e=e$.

SECTION 4: GROUPS

Question: Is there a group structure on a set of two elements.

*	e	a
e	e	a
a	a	e

$Z_2 = \{0,1\}$ under addition modulo 2 is a group

SECTION 4: GROUPS

Question: Is there a group structure on a set of three elements.

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$Z_3 = \{0,1,2\}$ under addition modulo 3 is a group

There is only one group of three elements, up to isomorphism.

Abstract Algebra

SECTION 5: SUBGROUPS

Notation and Terminology:

For $a*b$,

denoted it by $a+b$ if * is commutative

ab if * is not commutative or not determinate

For identity element e,

denoted 0 if * is commutative

denoted 1 if * is not commutative or not determinate

SECTION 5: SUBGROUPS

Notation and Terminology:

*	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

For identity element e,

denoted 0 if * is commutative

denoted 1 if * is not commutative or not determinate

SECTION 5: SUBGROUPS

Notation and Terminology:

$*$	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

For identity element e,

denoted 0 if * is commutative

denoted 1 if * is not commutative or not determinate

SECTION 5: SUBGROUPS

Notation and Terminology:

For inverse of an element a'

$-a$ in additive notation.

a^{-1} in multiplicative notation

For identity element e,

denoted 0 if * is commutative

denoted 1 if * is not commutative or not determinate

SECTION 5: SUBGROUPS

Notation and Terminology:

In multiplicative notation

denote the product $aaa...a$ for n factors a by a^n

denote the product $a^{-1}a^{-1}a^{-1}...a^{-1}$ for n factors $-a$ by a^{-n}

In additive notation

denoted $a+a+a+...+a$ for n summands by na

denoted $(-a)+(-a)+(-)+...+(-a)$ for n summands by $-na$

SECTION 5: SUBGROUPS

5.3 Definition If G is a group, then the order $|G|$ of G is the number of elements in G.

SECTION 5: SUBGROUPS

5.4 Definition If a subset H of a group G is closed under the binary operation of G and if H with the induced operation from G is itself a group, then H is a subgroup(子群) of G. We shall let $H \leqslant G$ or $G \geqslant H$ denote that H is a subgroup of G, and $H<G$ or $G>H$ shall mean $H \leqslant G$ but $H \neq G$.

Thus $(Z, +) < (Q, +)$

$(Q, +)$ s not a subgroup of $(R, \cdot)$

SECTION 5: SUBGROUPS

5.5 Definition If G is a group, then the subgroup consisting of G itself is the improper subgroup of G. All other subgroups are proper subgroups. The subgroup $\{e\}$ is the trivial subgroup(平凡子群) of G. All other subgroups are nontrivial.

SECTION 5: SUBGROUPS

5.6 Example Let R^n be the additive group of all n-component row vectors with real number entries.

The subset consisting of all of these vectors having 0 as entry in the first component is a subgroup of R^n.

$$R^n = \{(x_1, x_2, \ldots, x_n) | x_1, x_2, \ldots x_n \in R\}$$

$$H = \{(0, x_2, \ldots, x_n) | x_2, \ldots x_n \in R\}$$

SECTION 5: SUBGROUPS

5.13 Example Q^+ under multiplication is a proper subgroup of R^+ under multiplication.

5.8 Example The nth roots of unity in C form a subgroup U_n of the group C^* of nonzero complex numbers under multiplication.

SECTION 5: SUBGROUPS

5.7 Example There are two different types of group structures of order 4

Z_4

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

V: Klein 4-group

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

SECTION 5: SUBGROUPS

$Z_4 = \{0,1,2,3\}$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The only nontrivial proper subgroup of Z_4 is $\{0,2\}$.

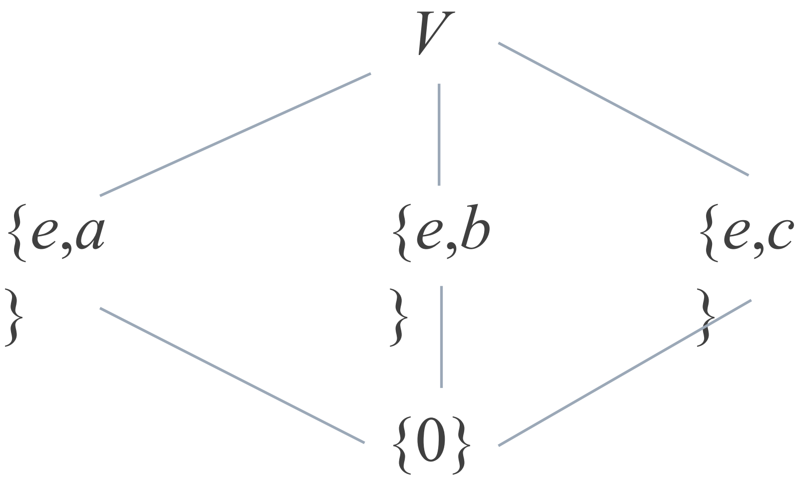
$$\begin{array}{c} Z_4 \\ | \\ \{0,2\} \\ | \\ \{0\} \end{array}$$

SECTION 5: SUBGROUPS

V:
Klein 4-group

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

V has three nontrivial proper subgroups



SECTION 5: SUBGROUPS

5.13 Example Let F be the group of all real-valued functions with domain R under addition. The subset of F consisting of those functions that are continuous is a subgroup of F.

SECTION 5: SUBGROUPS

5.14 Theorem A subset H of a group G is a subgroup of G if and only if

1. H is closed under the binary operation of G,
2. the identity element e of G is in H,
3. for all $a \in H$ it is true that $a^{-1} \in H$ also.

SECTION 5: SUBGROUPS

4.12 Example The set $M_n(R)$ of all $n \times n$ matrices under matrix multiplication is not a group.

4.13 Example Show that the subset S of $M_n(R)$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group.

5.16 Example Let T be the subset of G consisting of those matrices with determinant 1.

SECTION 5: SUBGROUPS

5.17 Theorem Let G be a group and let $a \in G$. Then

$H = \{a^n \mid n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a, that is, every subgroup containing a contains H.

SECTION 5: SUBGROUPS

5.17 Theorem Let G be a group and let $a \in G$. Then

$H=\{a^n | n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a, that is, every subgroup containing a contains H.

5.18 Definition Let G be a group and let $a \in G$. Then the subgroup $\{a^n | n \in Z\}$ of G, characterized in Theorem 5.17, is called the cyclic subgroup(循环群) of G generated(生成) by a, and denoted by $\langle a \rangle$.

SECTION 5: SUBGROUPS

5.19 Definition An element a of a group G **generates(生成)** G and is a **generator(生成元)** for G if $\langle a\rangle=G$. A group G is **cyclic** if there is some element a in G that **generates** G.

SECTION 5: SUBGROUPS

5.20 Example Let Z_4 and V be the groups of Example 5.9. Then Z_4 is cyclic and both 1 and 3 are generators, that is, $\langle 1\rangle=\langle 3\rangle=Z_4$. However, V is not cyclic, for $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are proper subgroups of two elements.

SECTION 5: SUBGROUPS

5.20 Example Let Z_4 and V be the groups of Example 5.9. Then Z_4 is cyclic and both 1 and 3 are generators, that is, $\langle 1\rangle=\langle 3\rangle=Z_4$. However, V is not cyclic, for $\langle a\rangle$, $\langle b\rangle$, and $\langle c\rangle$ are proper subgroups of two elements.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

SECTION 5: SUBGROUPS

5.21 Example The group Z under addition is a cyclic group. Both 1 and −1 are generators for this group, and they are the only generators. Also, for $n \in Z^+$, the group Z_n under addition modulo n is cyclic. If $n>1$, then both 1 and $n-1$ are generators, but there may be others.

SECTION 5: SUBGROUPS

5.22 Example Consider the group Z under addition. Let us find $\langle 3\rangle$. Here the notation is additive, and $\langle 3\rangle$ must contain

$$3, \quad 3+3=6, \quad 3+3+3=9, \quad 3+3+3+3=12, \quad \ldots$$

$$0, \quad -3, \quad -6, \quad -9, \quad -12, \quad \ldots$$

In other words, the cyclic subgroup generated by 3 consists of all multiples of 3, positive, negative, and zero. We denote this subgroup by $3Z$ as well as $\langle 3\rangle$. In a similar way, we shall let nZ be the cyclic subgroup $\langle n\rangle$ of Z. Note that $6Z<3Z$.

Abstract Algebra

SECTION 6: CYCLIC GROUPS

5.17 Theorem Let G be a group and let $a \in G$. Then

$H=\{a^n \mid n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a, that is, every subgroup containing a contains H.

5.18 Definition Let G be a group and let $a \in G$. Then the subgroup $\{a^n \mid n \in Z\}$ of G, characterized in Theorem 5.17, is called the cyclic subgroup(循环群) of G generated(生成) by a, and denoted by $\langle a \rangle$.

SECTION 6: CYCLIC GROUPS

5.19 Definition An element a of a group G **generates(生成)** G and is a **generator(生成元)** for G if $\langle a\rangle=G$. A group G is **cyclic** if there is some element a in G that **generates** G.

SECTION 6: CYCLIC GROUPS

6.1 Theorem Every cyclic group is abelian.

SECTION 6: CYCLIC GROUPS

Division Algorithm for Z If m is a positive integer and n is any integer, then there exist unique integers q and r such that.

$n = mg + r$

SECTION 6: CYCLIC GROUPS

6.4 Example Find the quotient g and remainder r when 38 is divided by 7 according to the division

$$38=35+3=7(5)+3$$

SECTION 6: CYCLIC GROUPS

6.5 Example Find the quotient g and remainder r when -38 is divided by 7 according to the division algorithm.

$$38=-42+4=7(-6)+4$$

SECTION 6: CYCLIC GROUPS

6.6 Theorem A subgroup of a cyclic group is cyclic.

SECTION 6: CYCLIC GROUPS

6.7 Corollary The subgroups of Z under addition are precisely the groups $n\mathrm{Z}$ under addition for $n \in Z$.

SECTION 6: CYCLIC GROUPS

6.8 Definition Let r and s be two positive integers. The positive generator d of the cyclic group $H = \{nr + ms \mid n, m \in Z\}$ under addition is the **greatest common divisor** (abbreviated **gcd**, 最大公约数) of r and s. We write $d = \gcd(r, s)$.

SECTION 6: CYCLIC GROUPS

6.9 Example Find the gcd of 42 and 72.

SECTION 6: CYCLIC GROUPS

Definition Two positive integers are **relatively prime(互素)** if their gcd is 1

SECTION 6: CYCLIC GROUPS

Result If r and s are relatively prime and if r divides sm, then r must divide m.

SECTION 6: CYCLIC GROUPS

6.10 Theorem Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $<Z,+>$. If G has finite order n, then G is isomorphic to $<Z_n,+_n>$.

SECTION 6: CYCLIC GROUPS

6.13 Example Motivated by our work with U_n, it is nice to visualize the elements $e = a^0, a^1, a^2 .., a^{n-1}$ of a cyclic group of order n as being distributed evenly on a circle(see Fig. 6.11).

SECTION 6: CYCLIC GROUPS

6.14 Theorem Let G be a cyclic group with n elements and generated by a. Let $b \in G$ and let $b=a^s$. Then b generates a cyclic subgroup H of G containing n/d elements, where d is the reatest common divisor of n and s. Also, $\langle a^s \rangle = \langle a^t \rangle$ if and only if gcd$(s,n)=$ gcd(t,n).

SECTION 6: CYCLIC GROUPS

6.15 Example For an example using additive notation, consider Z_2, with the generator a=1. Since the greatest common divisor of 3 and 12 is 3, 3=3·1 generates a subgroup of 12/3=4 elements, namely

$\langle 3\rangle=\{0,3,6,9\}$

SECTION 6: CYCLIC GROUPS

6.16 Corollary If a is a generator of a finite cyclic group G of order n, then the other generators of G are the elements of the form a^r, where r is relatively prime to n.

SECTION 6: CYCLIC GROUPS

6.17 Example Let us find all subgroups of Z_{18} and give their subgroup diagram.

Abstract Algebra

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

5.17 Theorem Let G be a group and let $a \in G$. Then

$H=\{a^n | n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a, that is, every subgroup containing a contains H.

5.18 Definition Let G be a group and let $a \in G$. Then the subgroup $\{a^n | n \in Z\}$ of G, characterized in Theorem 5.17, is called the **cyclic subgroup(循环群)** of G **generated(生成)** by a, and denoted by $\langle a \rangle$.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

The smallest subgroup of G that contains both a and b.

$$a^2b^4a^{-3}b^2a^5$$

$$a^{-5}b^{-2}a^3b^{-4}\,a^{-2}$$

We call a and b generators(生成元) of this subgroup.

If this subgroup should be all of G, then we say that $\{a,b\}$ generates(生成) G.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.1 Example The Klein 4-group $V=\{e,a,b,c\}$ of Example 5.9.

V is generated by $\{a,b\}$.

V is also generated by $\{a,c\},\{b,c\},\{a,b,c\}$

V is not generated by $\{e,a\}$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.2 Example The group Z_6.

It is generated by $\{1\},\{5\}$.

It is generated by $\{2,3\}$.

It is also generated by $\{3,4\},\{1,3,4\}\{1,3\}\{3,5\}$.

But it is not generated by $\{2,4\}$.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.3 Definition Let $\{S_i | i \in I\}$ be a collection of sets. Here I may be any set of indices. The intersection $\cap_{i\in I} S_i$ of the sets S_i is the set of all elements that are in all the sets S_i, that is

$$\cap_{i\in I} S_i = \{x | x \in S_i \text{ for all } i \in I\}.$$

If I is finite, $I = \{1, 2, ..., n\}$, we may denote $\cap_{i\in I} S_i$ by $S_1 \cap S_2 ... \cap S_1$

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.4 Theorem The intersection of some subgroups H_i of a group G for $i \in I$ is again a subgroup of G.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.5 Definition Let G be a group and let $a_i \in G$ for $i \in I$. The smallest subgroup of G containing $\{a_i | i \in I\}$ is the subgroup **generated(生成)** by $\{a_i | i \in I\}$. If this subgroup is all of G, then $\{a_i | i \in I\}$ generates G and the a_i are **generators(生成元)** of G. If there is a finite set $\{a_i | i \in I\}$ that generates G, then G is **finitely generated(有限生成)**.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

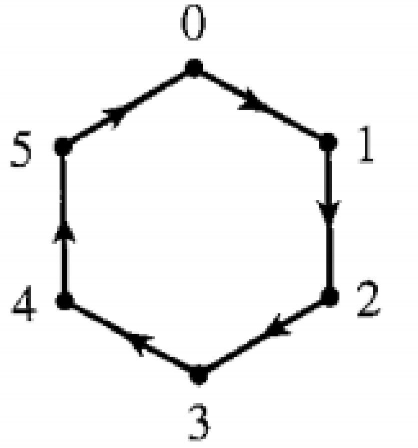
7.6 Theorem If G is a group and $a_i \in I$ for $i \in I$, then the subgroup H of G generated by $\{a_i | i \in I\}$ has as elements precisely those elements of G that are finite products of integral powers of the a_i, where powers of a fixed a_i may occur several times in the product.

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECTlON

Each element is represented by a dot,

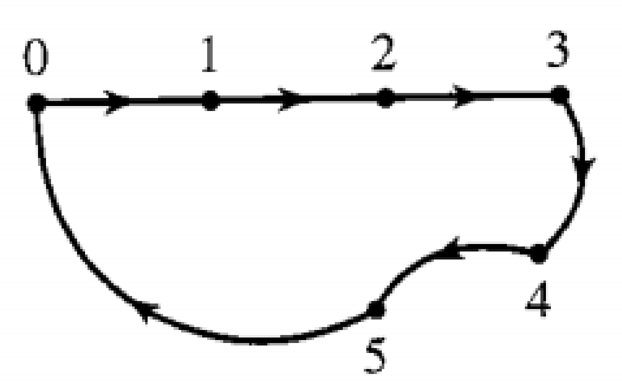
Each generatoris represented by one type of arc.

$x \longrightarrow y$ means that $xa = y$.



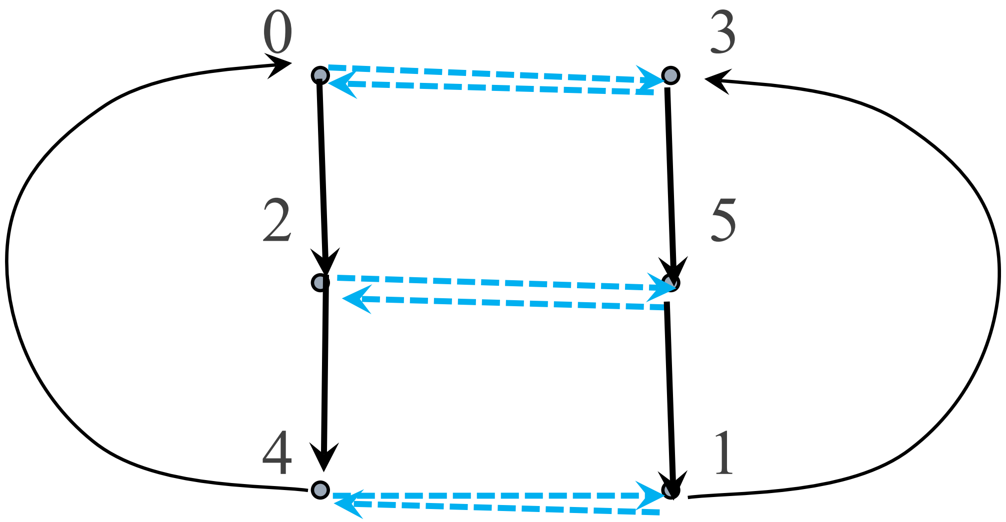
$\{1\}$ using $\longrightarrow$

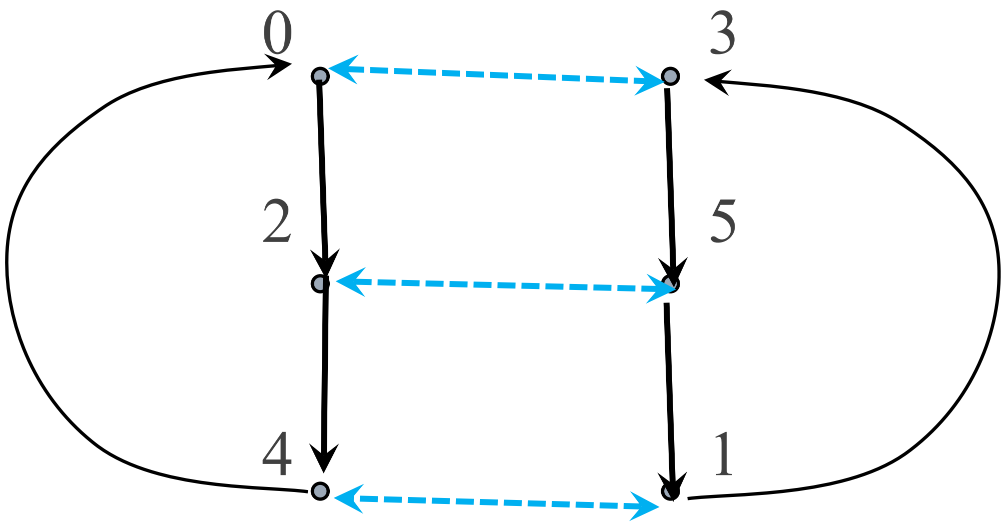
SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECTlON



$\{1\}$ using $\longrightarrow$

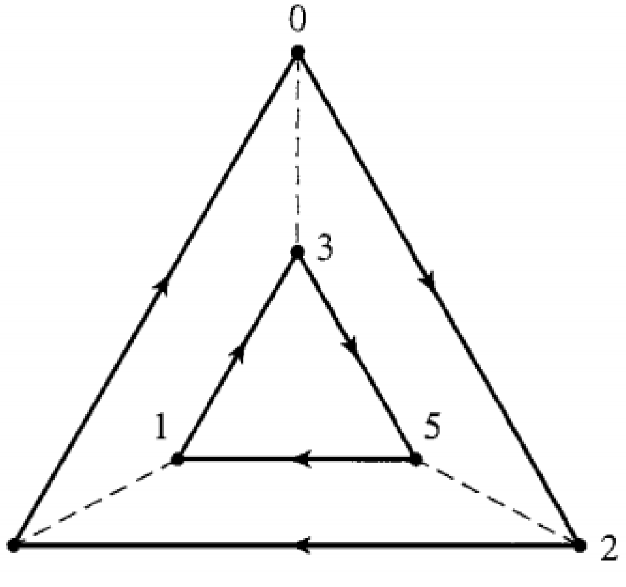
SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON





{2,3} using ⟶ and - - - →

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

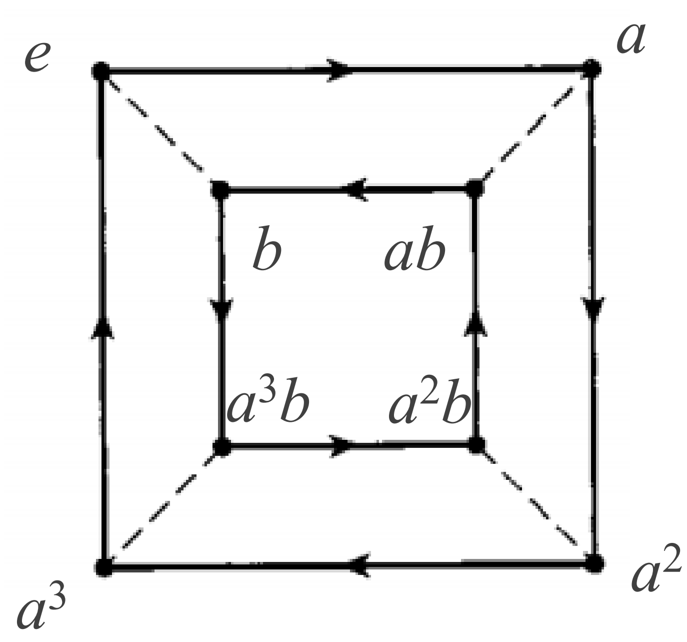


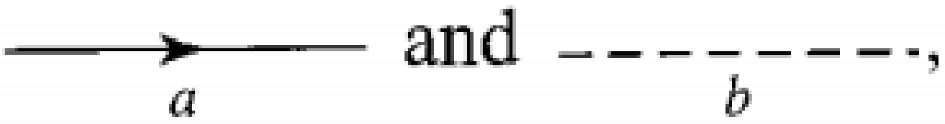
$\{2,3\}$ using $\longrightarrow$ and $\dashrightarrow$

1. The digraph is connected. $gx=h$ has a solution
2. At most one arc goes from a vertex g to a vertex h. $ga=h$ and $gb=h \Rightarrow a=b$
3. Each vertex g has exactly one arc of each type
4. sequences of arc from g to h $\Rightarrow$sequences of arc from u to v $gq=h$ and $gr=h \Rightarrow uq=v$ and $ur=v$

SECTION 7: GENERATING SETS AND CAYLEY DIGRAPHSSECT1ON

7.12 Example





Abstract Algebra

SECTION 8: GROUPS OF PERMUTATIONS

8.3 Theorem A **permutation(轮换,置换，变换)** of a set A is a function ϕ ： $A \rightarrow A$ that is both one to one and onto.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

SECTION 8: GROUPS OF PERMUTATIONS

Result Function composition $\circ$ is a binary operation on the collection of all permutations of a set A.

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[rr, bend right] & A \arrow[r, "\tau"] & A \end{tikzcd}$$

$$\sigma \circ \tau : A \to A \quad (\sigma\tau(a_1) = \sigma(\tau(a_1))$$

$\sigma\tau$ is one to one and onto.

SECTION 8: GROUPS OF PERMUTATIONS

8.4 Example Suppose that $A=\{1,2,3,4,5\}$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix}$$

SECTION 8: GROUPS OF PERMUTATIONS

8.4 Example Suppose that $A=\{1,2,3,4,5\}$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix}$$

SECTION 8: GROUPS OF PERMUTATIONS

8.5 Example Let A be a nonempty set, and let S_A be the collection of all permutations of A. Then S_A is a group under permutation multiplication.

SECTION 8: GROUPS OF PERMUTATIONS

8.5 Example Let A be a nonempty set, and let S_A be the collection of all permutations of A. Then S_A is a group under permutation multiplication.

8.6 Definition Let A be the finite set $\{1,2,..,n\}$. The group of all permutations of A is the symmetric group(对称群) on n letters, and is denoted by S_n. Note that S_n has $n!$ elements, where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$.

SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

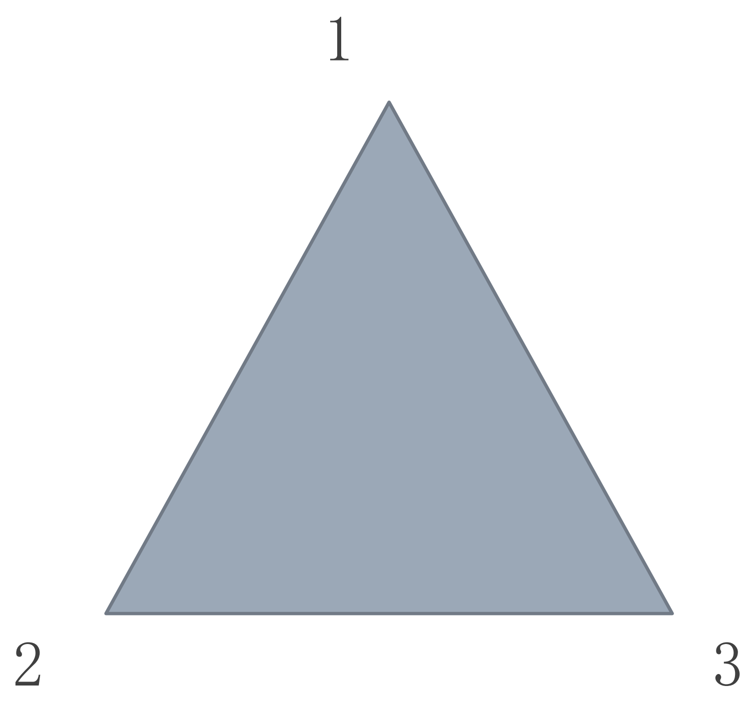
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

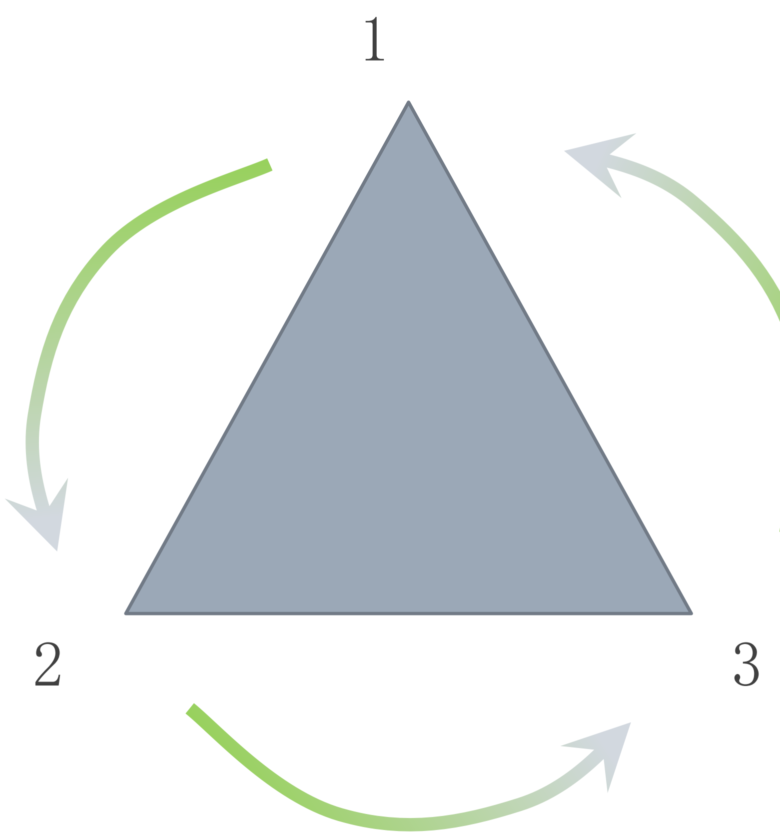


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

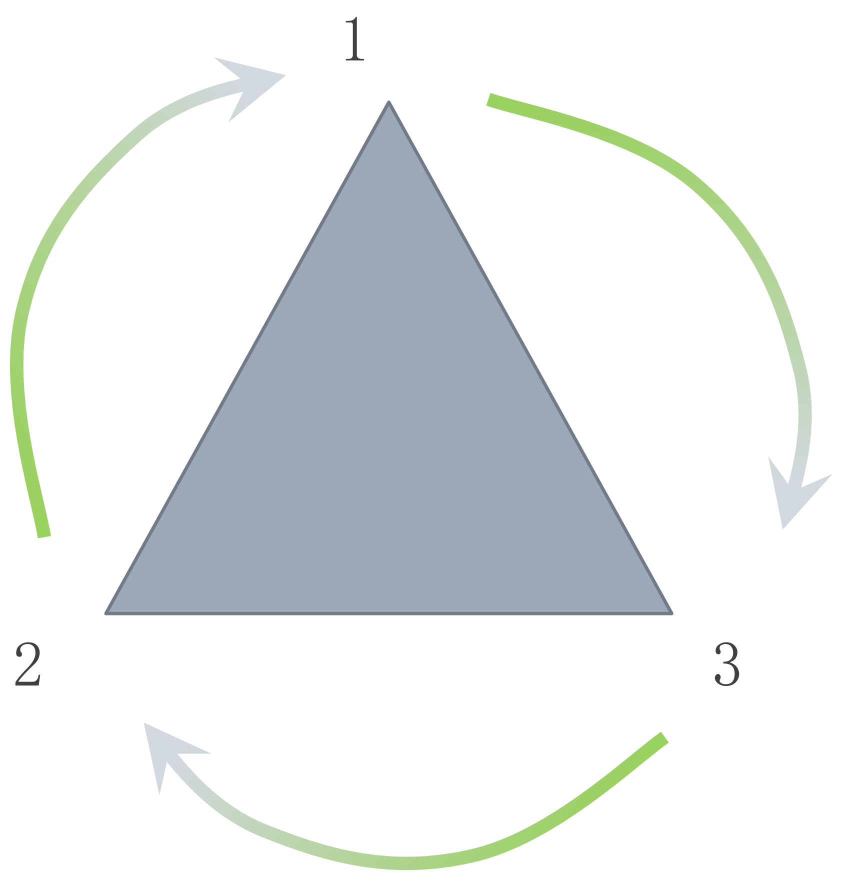


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

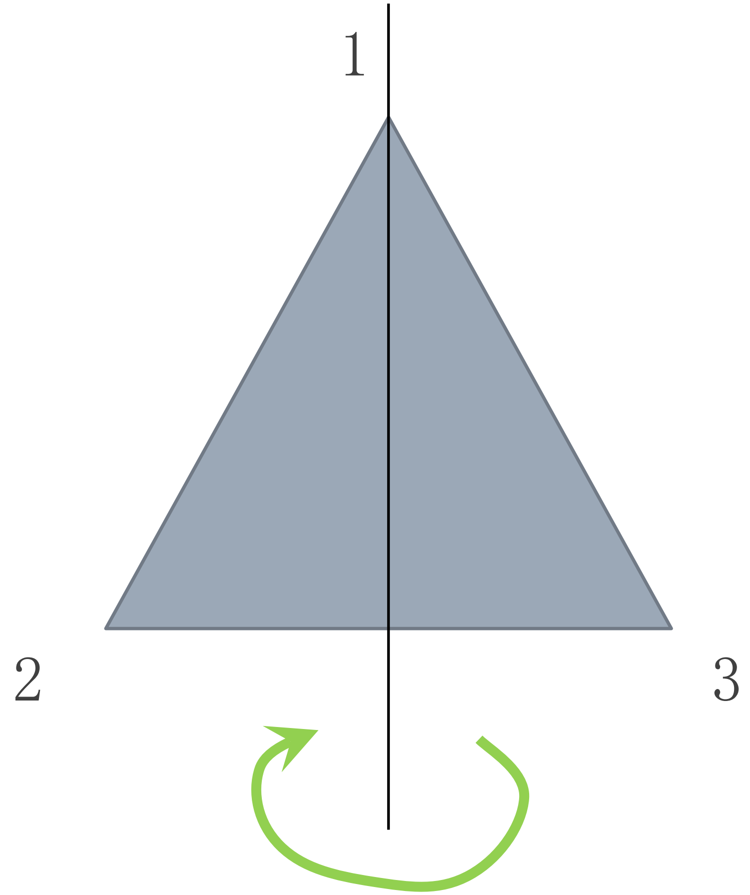


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

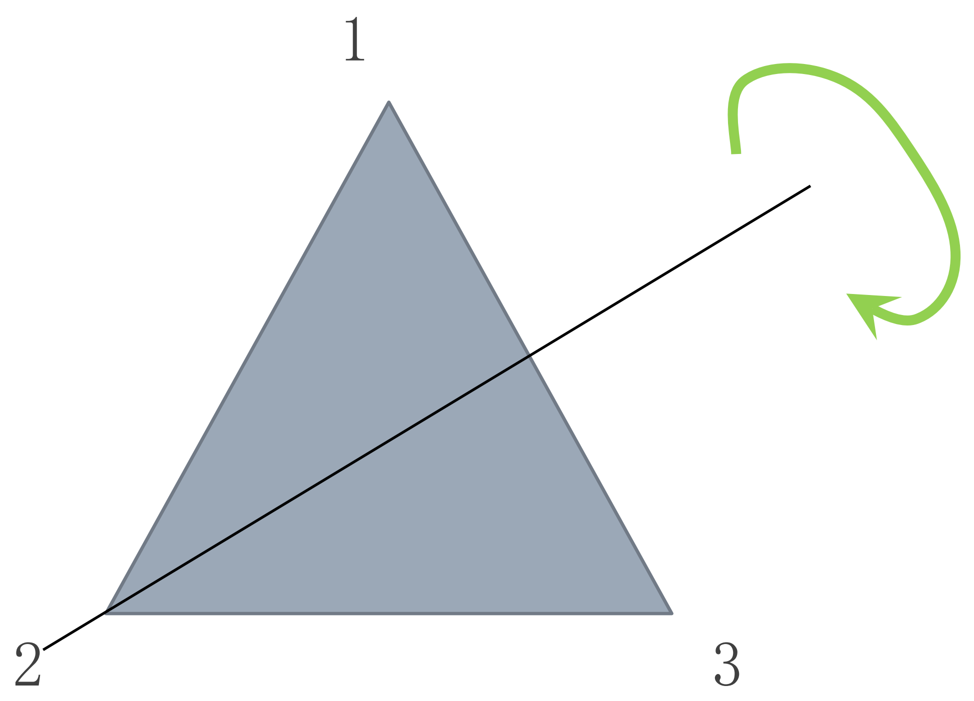


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

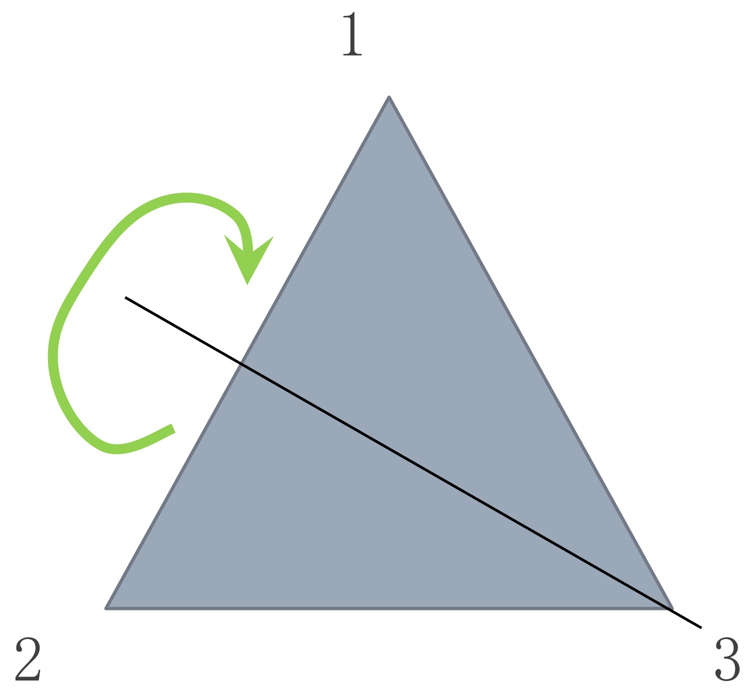


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

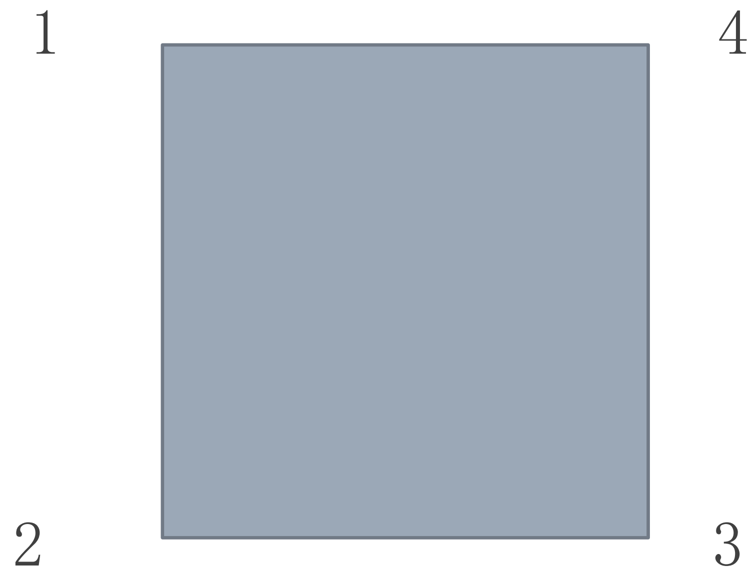
SECTION 8: GROUPS OF PERMUTATIONS

The notation D_3 stands for the third dihedral group. The nth dihedral group D_n is the group of symmetries of the regular n-gon.

D_4 will then be the group of symmetries of the square. It is also called the octic group.

SECTION 8: GROUPS OF PERMUTATIONS

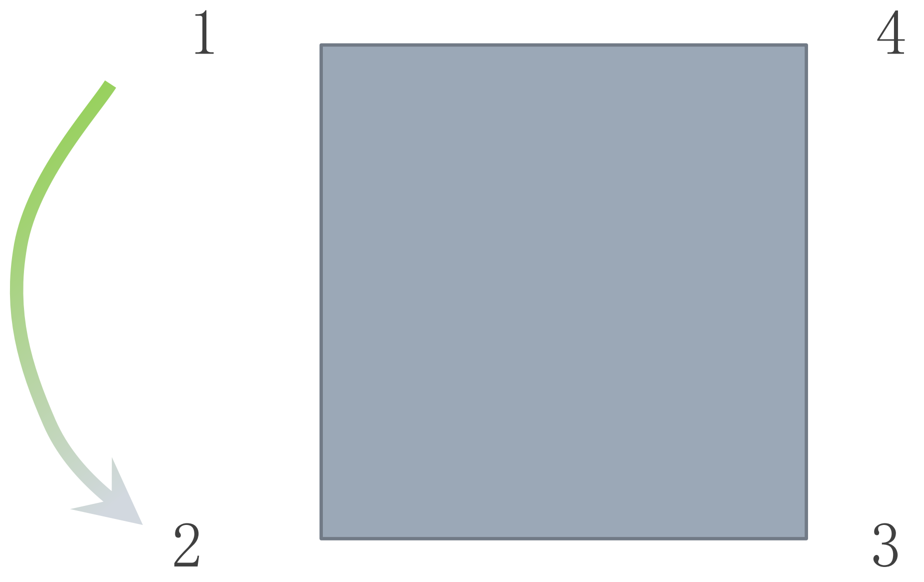
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

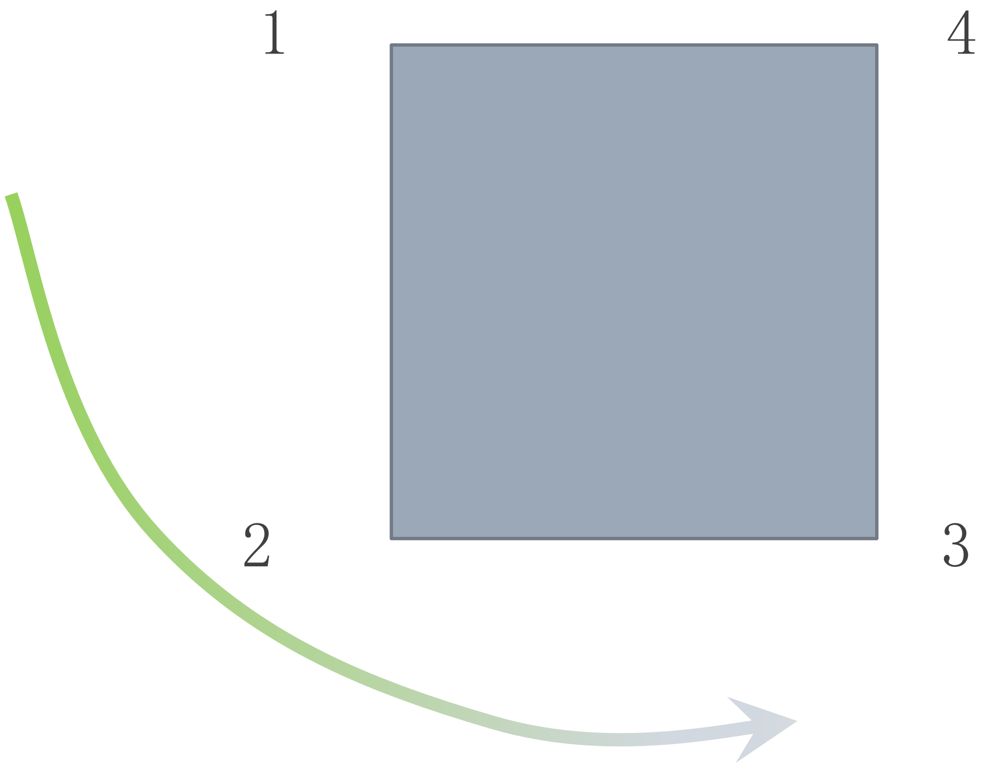


SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$



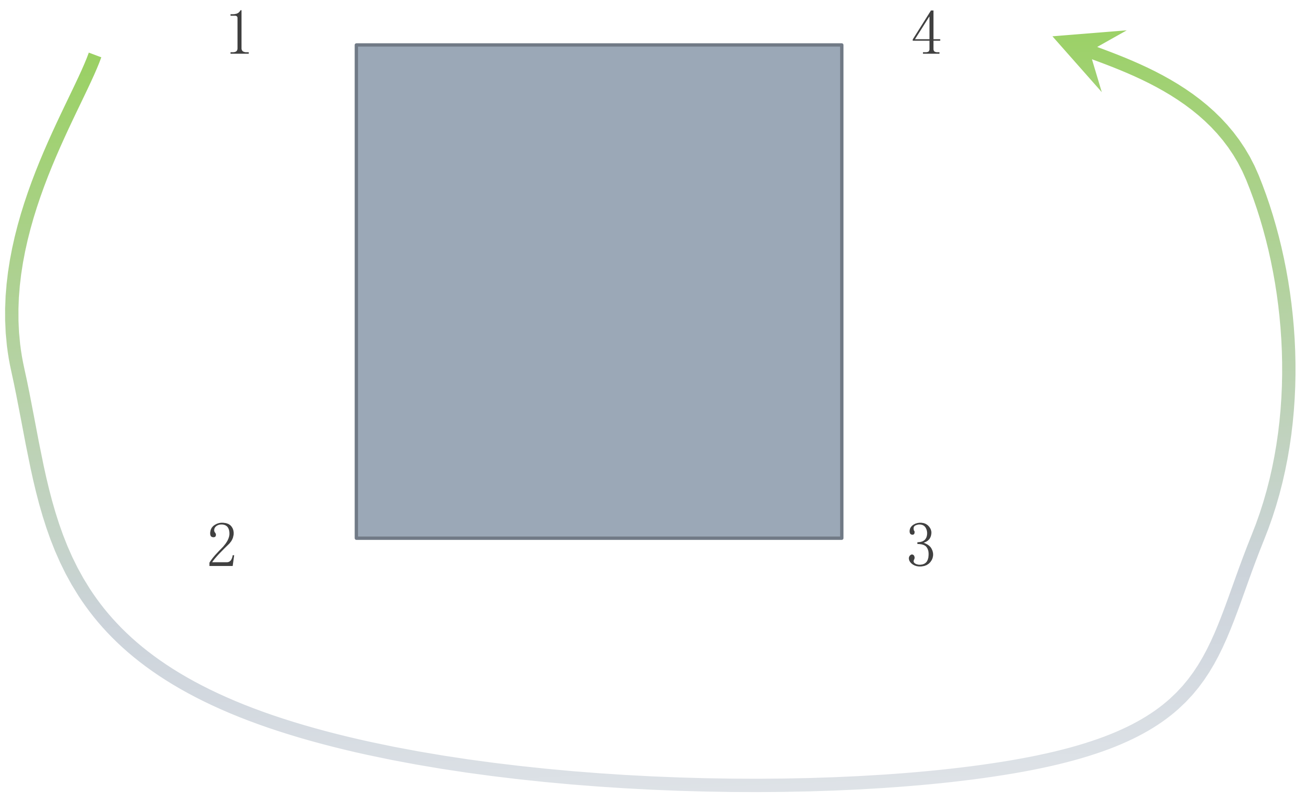
SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

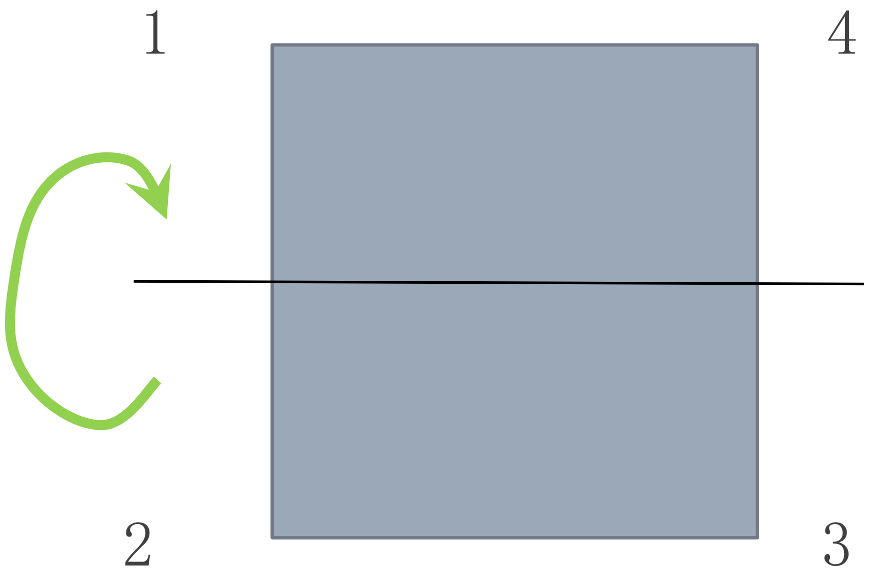
$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$



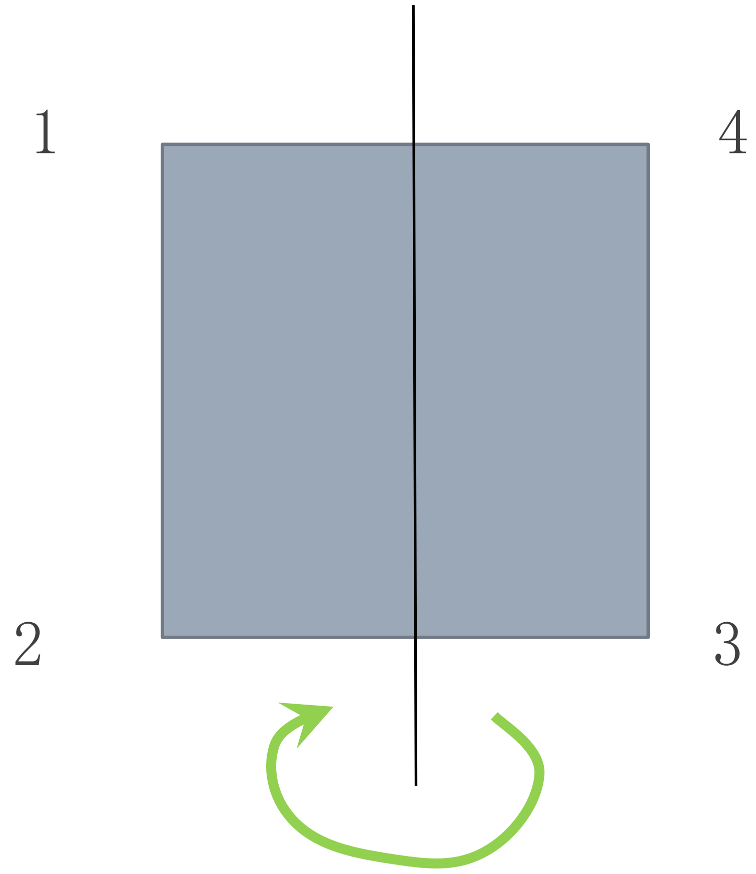
SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

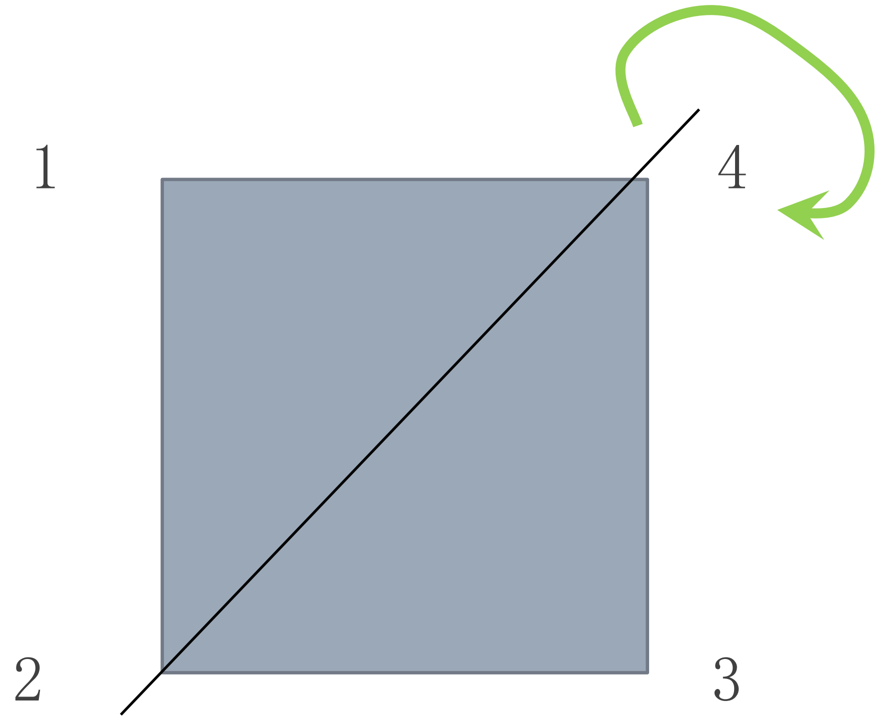
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$



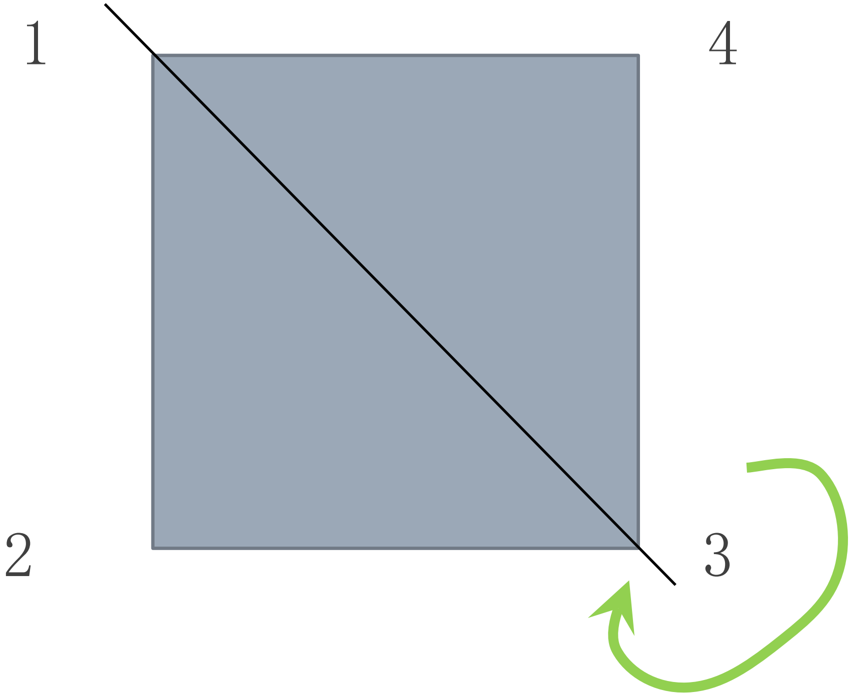
SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

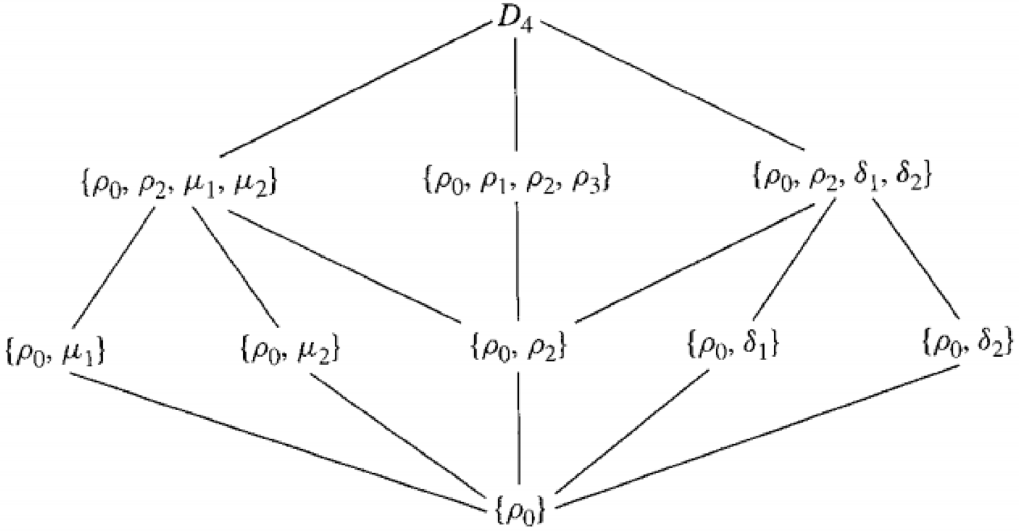
SECTION 8: GROUPS OF PERMUTATIONS

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix} \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \qquad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

8.14 Definition Let $f: A \to B$ be a function and let H be a subset of A. The image of H under f is $\{f(h) \mid h \in H\}$ and is denoted by $f[H]$.

8.15 Lemma Let G and G' be groups and let $\varphi: G \to G'$ be a one-to-one function such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G'$. Then $\varphi[G]$ is a subgroup of G' and φ provides an isomorphism of G with $\varphi[G]$.

SECTION 8: GROUPS OF PERMUTATIONS

8.15 Lemma Let G and G' be groups and let $\varphi: G \to G'$ be a one-to-one function such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G'$. Then $\varphi[G]$ is a subgroup of G' and φ provides an isomorphism of G with $\varphi[G]$.

SECTION 8: GROUPS OF PERMUTATIONS

8.16 Theorem (Cayley's Theorem) Every group is isomorphic to a group of permutations.

For $x \in G$ let $\lambda_x : G \rightarrow G$ be defined by $\lambda_x(g) = xg$ for all $g \in G$.

$$\phi : G \rightarrow S_G$$

$$\phi : x \rightarrow \lambda_x$$

SECTION 8: GROUPS OF PERMUTATIONS

8.16 Theorem (Cayley's Theorem) Every group is isomorphic to a group of permutations.

For $x \in G$ let $\lambda_x : G \to G$ be defined by $\lambda_x(g) = xg$ for all $g \in G$.

$\phi : G \to S_G$ $\quad\quad$ $\phi : x \to \lambda_x$ $\quad\quad$ ϕ is the left regular representation of G

For $x \in G$ let $\rho_x : G \to G$ be defined by $\rho_x(g) = gx$ for all $g \in G$.

$\mu : G \to S_G$ $\quad\quad$ $\mu : x \to \rho_x^{-1}$ $\quad\quad$ μ is the right regular representation of G

SECTION 8: GROUPS OF PERMUTATIONS

8.18 Example Let us compute the left regular representation of the group given by the group table.

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Abstract Algebra

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

Fact Each permutation σ of a set A determines a natural partition.

Each permutation σ of a set A determines an equivalence relation.

For $a,b\in A$, let $\boldsymbol{a\sim b}$ if and only if $b=\sigma^n(a)$ for some $n\in Z$. (1)

Reflexive Clearly $a\sim b$ since $a=\iota(a)=\sigma^0(a)$.

Symmetric If $a\sim b$, then $b=\sigma^n(a)$ for some $n\in Z$. But then $a=\sigma^{-n}(b)$

Transitive Suppose $a\sim b$ and $b\sim c$, then $b=\sigma^n(a)$ and $c=\sigma^m(b)$.

we find that $c=\sigma^m(\sigma^n(a))=\sigma^{m+n}(a)$.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.1 Definition Let σ be a permutation of a set A. The equivalence classes in A determined by the equivalence relation (1) are the **orbits(轨道)** of σ.

9.2 Example Since the identity permutation of A leaves each element of A fixed, the orbits of ι are the one-element subsets of A.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

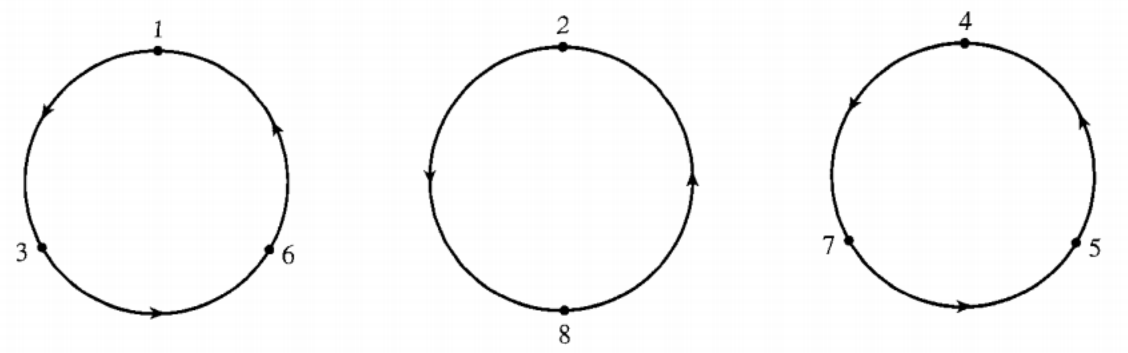
9.3 Example Find the orbits of the permutation σ in S_8.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.3 Example Find the orbits of the permutation σ in S_8.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

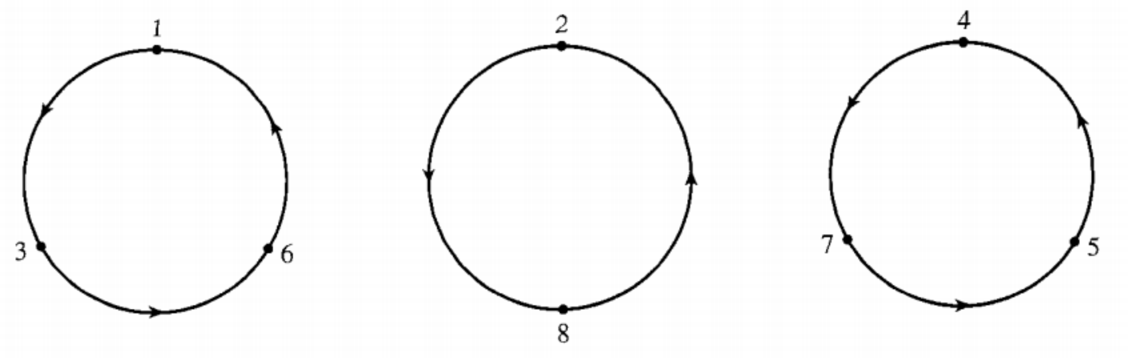


SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

Fact Each individual circle also defines, by itself, a permutation in S_n.

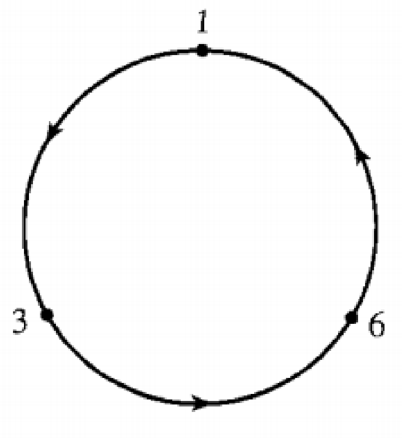
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 6 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$$



SECTION 8: GROUPS OF PERMUTATIONS

9.6 Definition A permutation $\sigma \in S_n$, is a **cycle(轮换)** if it has at most one orbit containing more than one element. The length of a cycle is the number of elements in its largest orbit.



$\cdots\cdots$

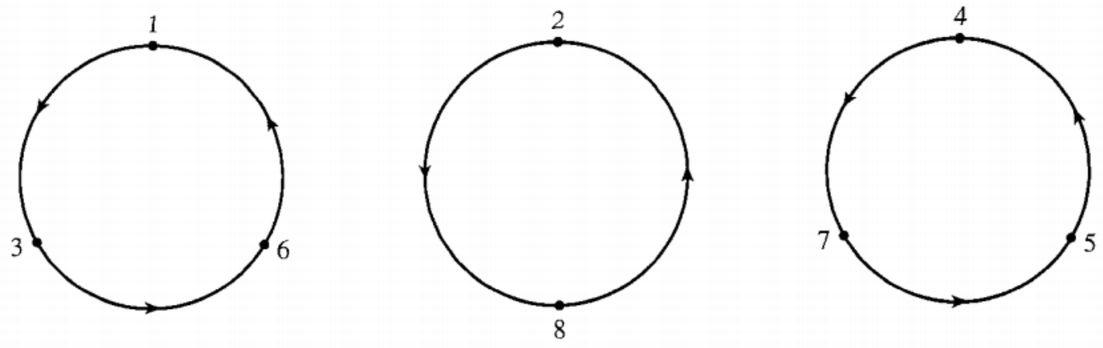
$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 6 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$$

$$=(1,3,6)$$

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.3 Example Find the orbits of the permutation σ in S_8.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

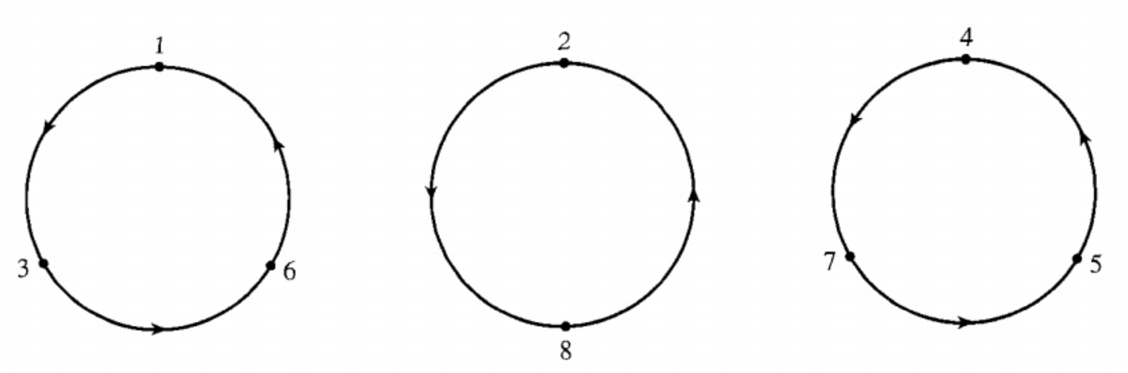


=(1,3,6) =(2,8) =(4,7,5)

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

=(1,3,6) (2,8) (4,7,5)



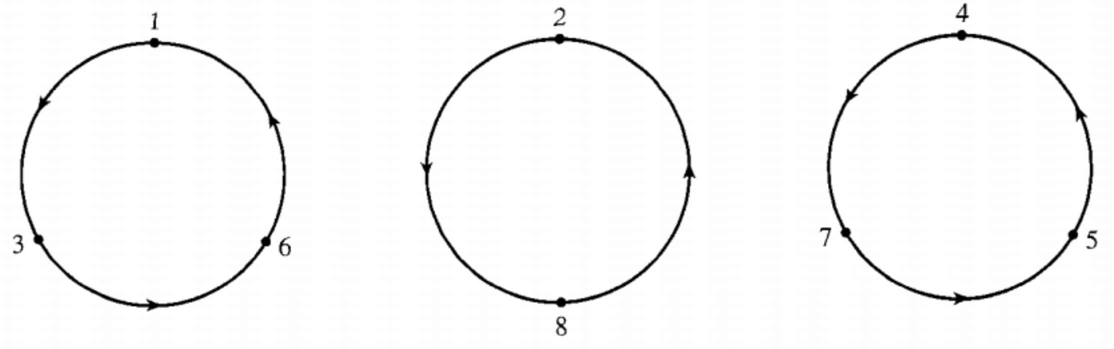
=(1,3,6) **=(2,8)** **=(4,7,5)**

9.8 Theorem Every permutation σ of a finite set is a product of disjoint cycles.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$$

=(1,3,6) (2,8) (4,7,5)



=(1,3,6) =(2,8) =(4,7,5)

9.8 Theorem Every permutation σ of a finite set is a product of disjoint cycles.

Note These cycles are disjoint.
Any integer is in at most one of these cycles.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.9 Example Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$

and write it as a product of disjoint cycles.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.10 Example Consider the cycles $(1, 4, 5, 6)$ and $(2, 1, 5)$ in S_6. Give the product of them.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.11 Definition A cycle of length 2 is a **transposition(对换)**.

$$(a_1,a_2,\dots, a_n)= (a_1,a_n)(a_1,a_{n-1})\dots(a_1,a_3)(a_1,a_2).$$

$$(1,2)=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 2 & 1 & 3 & 4 & * & * & * & *\end{pmatrix}$$

$$(1,3)(1,2)=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 2 & 3 & 1 & 4 & * & * & * & *\end{pmatrix}$$

$$(1,2,3,4)=(1,4)(1,3)(1,2)=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ 2 & 3 & 4 & 1 & * & * & * & *\end{pmatrix}$$

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.12 Corollary Any permutation of a finite set of at least two elements is a product of transpositions.

9.13 Example (1,6)(2, 5, 3) is the product (1,6)(2,3)(2,5) of transpositions.

9.14 Example In S_n, for $n \geqslant 2$, the identity permutation (1) is the product (1,2)(1,2) of transpositions.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.15 Theorem No permutation in S_n, can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Proof From the cycles.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.15 Theorem No permutation in S_n, can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Proof From linear algebra.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.15 Theorem No permutation in S_n, can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Proof Let $\sigma \in S_n$, and let $\tau = (i,j)$ be a transposition in S_n. We claim that the number of orbits of σ and of $\tau\sigma$ differ by 1.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.19 Example The identity permutation in S_n, is an even permutation.

the permutation $(1,4,5,6)(2,1,5)$ in S_6 is an odd permutation.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

Let τ be any fixed transposition in S_n (it exists since $n \geq 2$).

λ_τ is one to one and onto.

SECTION 9: ORBITS, CYCLES, AND THE ALTERNATING GROUPS

9.20 Theorem If $n \geq 2$, then the collection of all even permutations of $\{1,2,3,..,n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n:

9.21 Definition The subgroup of S_n, consisting of the even permutations of n letters is the **alternating group(交错群)** A_n, on n letters.

Abstract Algebra

Observation The order of a subgroup H of a finite group G seems always to be a divisor of the order of G.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.1 Theorem Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by $a \sim_L b$ if and only if $a^{-1}b \in H$. Let the relation $\sim_R$ be defined on G by $a \sim_R b$ if and only if $ab^{-1} \in H$. Then $\sim_L$ and $\sim_R$ are a equivalence relations on G.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.1 Theorem Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by $a \sim_L b$ if and only if $a^{-1}b \in H$. Then $\sim_L$ is a equivalence relation on G.

Proof

Symmetric

Transitive

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.2 Definition Let H be a subgroup of a group G. The subset $aH=\{ah|\in H\}$ of G is the left coset(左陪集) of G containing a, while the subset $Ha=\{ha|\in H\}$ is the right coset(右陪集) of H containing a.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.2 Definition Let H be a subgroup of a group G. The subset $aH=\{ah|\in H\}$ of G is the left coset(左陪集) of G containing a, while the subset $Ha=\{ha|\in H\}$ is the right coset(右陪集) of H containing a.

Fact aH is the cell of $\sim_L$ containing a.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.2 Definition Let H be a subgroup of a group G. The subset $aH=\{ah|\in H\}$ of G is the left coset(左陪集) of G containing a, while the subset $Ha=\{ha|\in H\}$ is the right coset(右陪集) of H containing a.

10.3 Example Exhibit the left cosets and the right cosets of the subgroup $3Z$ of Z.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.2 Definition Let H be a subgroup of a group G. The subset $aH=\{ah|\in H\}$ of G is the **left coset(左陪集)** of G containing a, while the subset $Ha=\{ha|\in H\}$ is the **right coset(右陪集)** of H containing a.

10.3 Example Exhibit the left cosets and the right cosets of the subgroup $3Z$ of Z.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.4 Example The group Z_6 is abelian. Find the partition of Z_6 into cosets of the subgroup $H=\{0,3\}$.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.4 Example The group Z_6 is abelian. Find the partition of Z_6 into cosets of the subgroup $H=\{0,3\}$.

$+_6$	0	3	1	4	2	5
0						
3						
1						
4						
2						
5						

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.4 Example The group Z_6 is abelian. Find the partition of Z_6 into cosets of the subgroup $H=\{0,3\}$.

$+_6$	0	3	1	4	2	5
0	0	3	1	4	2	5
3	3	0	4	1	5	2
1	1	4	2	5	3	0
4	4	1	5	2	0	3
2	2	5	3	0	4	1
5	5	2	0	3	1	4

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.4 Example The group Z_6 is abelian. Find the partition of Z_6 into cosets of the subgroup $H=\{0,3\}$.

$+_6$	0	3	1	4	2	5
0	0	3	1	4	2	5
3	3	0	4	1	5	2
1	1	4	2	5	3	0
4	4	1	5	2	0	3
2	2	5	3	0	4	1
5	5	2	0	3	1	4

$+_6$	0	3	1	4	2	5
0						
3						
1						
4						
2						
5						

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.7 Example Let H be the subgroup $\langle\mu_1\rangle=\{\rho_0, \mu_1\}$ of S_3. Find the partitions of S_3 into left cosets of H, and the partition into right cosets of H.

left cosets $H=\{\rho_0, \mu_1\}$

$\rho_1 H=\{\rho_1\rho_0, \rho_1\mu_1\}=\{\qquad\}$

$\rho_2 H=\{\rho_2\rho_0, \rho_2\mu_1\}=\{\qquad\}$

right cosets $H=\{\rho_0, \mu_1\}$

$H\rho_1=\{\rho_0\rho_1, \mu_1\rho_1\}=\{\qquad\}$

$H\rho_2=\{\rho_0\rho_2, \mu_1\rho_2\}=\{\qquad\}$

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.7 Example Let H be the subgroup $<\mu_1>=\{\rho_0, \mu_1\}$ of S_3. Find the partitions of S_3 into left cosets of H, and the partition into right cosets of H.

left cosets $H=\{\rho_0, \mu_1\}$

$$\rho_1 H=\{\rho_1\rho_0, \rho_1\mu_1\}=\{\rho_1, \mu_3\}$$

$$\rho_2 H=\{\rho_2\rho_0, \rho_2\mu_1\}=\{\rho_2, \mu_2\}$$

right cosets $H=\{\rho_0, \mu_1\}$

$$H\rho_1=\{\rho_0\rho_1, \mu_1\rho_1\}=\{\rho_1, \mu_2\}$$

$$H\rho_2=\{\rho_0\rho_2, \mu_1\rho_2\}=\{\rho_2, \mu_3\}$$

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

Claim Every left coset and every right coset of H have the same number of elements as H.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.10 Theorem(Theorem of Lagrange) Let H be a subgroup of a finite group G. Then the order of H is a divisor of the order of G.

10.11 Corollary Every group of prime order is cyclic.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

Definition The **order(阶)** of an element is the same as the order of the cyclic subgroupgenerated by the element.

SECTION 10: COSETS AND THE THEOREM OF LAGRANGE

10.13 Definition Let H be a subgroup of a group G. The number of left cosets of H in G is the index $(G:H)$ of H in G.

Fact $(G:H)=|G| / |H|$ if $|G|$ is finite.

10.14 Theorem Suppose H and K are subgroups of a group G such that $K \leqslant H \leqslant G$, and suppose $(H:K)$ and $(G:H)$ are both finite. Then $(G:K)$ is finite, and $(G:K)=(G:H)(H:K)$.

Abstract Algebra

11.1 Definition The Cartesian product of sets S_1, S_2, . . . , S_n, is the set of all ordered n-tuples $(a_1, a_2, . . . , a_n)$, where $a_i \in S_i$ for $i = 1,2, . . . , n$. The Cartesian product is denoted by either

$$S_1 \times S_2 \times, \ldots, \times S_n$$

or by

$$\prod_{i=1}^{n} S_i$$

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.2 Theorem Let $G_1, G_2, \ldots, G_n$ be groups. For $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\Pi_1{}^n G_i$, define $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$ to be the element $(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Then $\Pi_1{}^n G_i$ is a group, **the direct product(直积)** of the groups G_i, under this binary operation.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

11.3 Example Consider the group $Z_2 \times Z_3$. It is a cyclic group.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.4 Example Consider the group $Z_3 \times Z_3$.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.5 Theorem The group $Z_m \times Z_n$ is cyclic and is isomorphic to Z_{mn} if and only if m and n are relatively prime, that is, the gcd of m and n is 1.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.6 Corollary The group $\Pi_1{}^n Z_{m_i}$ is cyclic and isomorphic to $Z_{m_1 m_2 \ldots m_n}$, if and only if the numbers m_i for $i = 1, \ldots, n$ are such that the gcd of any two of them is 1.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.7 Example The preceding corollary shows that if n is written as a product of powers of distinct prime numbers, as in

$$n = (p_1)^{n_1}(p_2)^{n_2}\ldots(p_r)^{n_r},$$

then Z_n is isomorphic to $Z_{p_1}{}^{n_1} \times Z_{p_2}{}^{n_2} \times \ldots \times Z_{p_r}{}^{n_r}$.

Example Z_{72} is isomorphic to $Z_8 \times Z_9$.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.8 Definition Let $r_1, r_2, \ldots, r_n$ be positive integers. Their **least common multiple(abbreviated lcm 最小公倍数)** is the positive generator of the cyclic group of all common multiples of the r_i, that is, the cyclic group of all integers divisible by each r_i; for $i = 1, 2, \ldots, n$.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.9 Theorem Let $(a_1, a_2, . . . , a_n) \in \Pi_1{}^n G_i$ If a_i is of finite order r_i in G_i, then the order of $(a_1, a_2, . . . , a_n)$ in $\Pi_1{}^n G_i$ is equal to the least common multiple of all the r_i.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.10 Example Find the order of $(8,4,10)$ in the group $Z_{12} \times Z_{60} \times Z_{24}$.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.11 Example The group $Z \times Z_2$ is generated by the elements (1,0) and (0,1). More generally, the direct product of n cyclic groups, each of which is either Z or Z_m for some positive integer m, is generated by the n n-tuples

$$(1,0,0...,0), (0,1,0,...,0), (0,0,1,...,0),...,(0,0,0,...,n).$$

Note Such a direct product might also be generated by fewer elements. For example, $Z_3 \times Z_4 \times Z_{35}$ is generated by the single element (1,1,1).

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

Note If $\Pi_1{}^n G_i$ is the direct product of groups G_i, the subset

$\bar{G}_i = \{(e_1, e_2, \ldots, a_i, \ldots, e_n) \mid a_i \in G_i\}$.

is a subgroup of $\Pi_1{}^n G_i$.

It is also clear that this subgroup $\bar{G}_i$ is naturally isomorphic to G_i.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

11.12 Theorem (Fundamental Theorem of Finitely Generated Abelian Groups) Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form

$$Z_{p_1}{}^{n1} \times Z_{p_2}{}^{n2} \times \ \ldots \times Z_{p_r}{}^{n\mathrm{r}} \times Z \times Z \times \ldots \times Z$$

where the p_i are primes, not necessarily distinct, and the r_i are positive integers.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

11.13 Example Find all abelian groups, up to isomorphism, of order 360.

Solution $360=2^3\times 3^2\times 5.$

1. $Z_2\times Z_2\times Z_2\times Z_3\times Z_3\times Z_5$
2. $Z_2\times Z_4\times Z_3\times Z_3\times Z_5$
3. $Z_8\times Z_3\times Z_3\times Z_5$
4. $Z_2\times Z_2\times Z_2\times Z_9\times Z_5$
5. $Z_2\times Z_4\times Z_9\times Z_5$
6. $Z_8\times Z_9\times Z_5$

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

11.14 Definition A group G is **decomposable** if it is isomorphic to a directproduct of two proper nontrivial subgroups. Otherwise G is indecomposable.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

11.15 Theorem The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELIAN GROUPS

11.16 Theorem If m divides the order of a finite abelian group G, then G has a subgroup of order m.

SECTION 11: DIRECT PRODUCTS AND FINITELY GENERATED ABELAN GROUPS

Abstract Algebra

13.1 Definition A map φ of a group G into a group G' is a **homomorphism(同态)** if the homomorphismproperty

$$\varphi(ab)=\varphi(a)\varphi(b)$$

holds for all $a,b \in G$.

$$\varphi(a *_1 b)=\varphi(a) *_2 \varphi(b)$$

13.2 Example Let $\varphi: G \to G'$ be a group homomorphism of G onto G'. We claim that if G is abelian, then G' must be abelian.

13.3 Example Let S_n be the symmetric group on n ltters, and let $\varphi: S_n \to Z_2$ be defined by

$$\varphi(\sigma) = \begin{cases} 0 & if\ a\ is\ an\ even\ permutation \\ 1 & if\ \sigma\ is\ an\ odd\ permutation. \end{cases}$$

Show that φ is a homomorphism.

13.4 Example (Evaluation Homomorphism求值同态) Let F be the additive group of all functions mapping R into R, let R be the additive group of real numbers, and let c be any real number. Let $\varphi_c: F \to R$ be the evaluation homomorphism defined by

$$\varphi_c(f) = f(\mathrm{c}) \text{ for } f \in F.$$

13.5 Example Let R^n be the additive group of column vectors with n real-number components. Let A be an $m \times n$ matrix of real numbers. Let $\varphi: R^n \to R^n$ be defined by $\varphi(v) = Av$ for each column vector $v \in R^n$. Then φ is a homomorphism.

13.6 Example Let $GL(n,R)$ be the multiplicative group of all invertible $n \times n$ matrices. Recall that a matrix A is invertible if and only if its determinant, $\det(A)$, is nonzero. Recall also that for matrices $A,B \in GL(n.R)$ we have $\det(AB) = \det(A)\det(B)$.

This means that det is a homomorphism mapping $GL(n,R)$ into the multiplicative group R^* of nonzero real numbers.

13.7 Example Let $r \in Z$ and let $\varphi_r : Z \to Z$ be defined by $\varphi_r(n) = rn$ for all $n \in Z$. Then φ_r is a homomorphism.

Note that φ_0 is the trivial homomorphism.

φ_1 is the identity map, and φ_{-1} maps Z onto Z.

For all other r in Z, the map φ_r is not onto Z.

13.8 Example Let $G=G_1\times G_2\times\ldots\times G_n$, be a direct product of groups. The **projection map(投射)** $\pi_i: G\rightarrow G_i$ where $\pi_{\mathrm{i}}:(g_1,g_2\ldots g_n)=g$ is a homomorphism for each $i=1,2\ldots.,n$.

13.9 Example Let F be the additive group of continuous functions with domain [0, 1] and let R be the additive group of real numbers. The map σ: $F \to R$ defined by $\sigma(f)=\int_0^1 f(x)\mathrm{d}x$ for $f \in F$ is a homomorphism.

13.10 Example(Reduction Modulo n) Let γ be the natural map of Z into Z_n given by $\gamma(m)=r$, where r is the remainder given by the division algorithm when m is divided by n. Then γ is a homomorphism.

13.11 Definition Let φ be a mapping of a set X into a set Y, and let $A \subseteq X$ and $B \subseteq Y$. The **image** $\varphi[A]$ of A in Y under φ is $\{\varphi(a) \mid a \in A\}$. The set $\varphi[X]$ is the **range** of φ. The **inverse image** $\varphi^{-1}[B]$ of B in X is $\{x \in X \mid \varphi(x) \in B\}$.

13.12 Theorem Let φ be a homomorphism of a group G into a group G'.

1. If e is the identity element in G, then $\varphi(e)$ is the identity element e' in G'.

2. If $a \in G$, then $\varphi(a^{-1}) = \varphi(a)^{-1}$.

3. If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.

4. If K' is a subgroup of G', then $\varphi^{-1}[K']$ is a subgroup of G.

Tip φ **preserves** the **identity element**, **inverses**, and **subgroups**.

13.13 Definition Let $\varphi: G \to G'$ be a homomorphism of groups. The subgroup $\varphi^{-1}[\{e'\}] = \{x \in G \mid \varphi(x) = e'\}$ is the **kernel(核)** of φ, denoted by $\mathrm{Ker}(\varphi)$.

13.15 Theorem Let $\varphi: G \rightarrow G'$ be a group homomorphism, and let $H = \mathrm{Ker}(\varphi)$. Let $a \in G$. Then the set

$$\varphi^{-1}[\varphi(a)] = \{x \in G \mid \varphi(x) = \varphi(a)\}$$

is the left coset aH of H, and is also the right coset Ha of H. Consequently, the two partitions of G into left cosets and into right cosets of H are the same.

13.16 Example The absolute value function $|\ |$ is a homomorphism of the group C^* of nonzero complex numbers under multiplication onto the group R^+ of positive real numbers under multiplication.

13.17 Example Let D be the additive group of all differentiable functions mapping R into R, and let F be the additive group of all functions mapping R into R. Then differentiation gives us a map φ: $D \to F$, where $\varphi(f)=f'$ for $f \in F$. We easily see that φ is a homomorphism.

13.18 Corollary A group homomorphism $\varphi: G \to G'$ is a one-to-one map if and only if $\operatorname{Ker}(\varphi) = \{e\}$.

13.18 Corollary A group homomorphism $\varphi: G \longrightarrow G'$ is a one-to-one map if and only if Ker$(\varphi)=\{e\}$.

Corollary To Show $\varphi: G \longrightarrow G'$ is an isomorphism.

Step 1 Show φ is a homomorphism.

Step 2 Show Ker$(\varphi)=\{\mathrm{e}\}$.

Step 3 Show φ maps G onto G'.

Abstract Algebra

14.1 Theorem Let $\varphi : G \to G'$ be a group homomorphism with kernel H. Then the cosets of H form a **factor group**, $\boldsymbol{G/H}$, where $(aH)(bH)=(ab)H$. Also, the map $\mu : G/H \to \varphi[G]$ defined by $\mu(aH)=\varphi(a)$ is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices a and b from the cosets.

14.2 Example The map $\gamma: Z \to Z_n$, where $\gamma(m)$ is the remainder when m is divided by n in accordance with the division algorithm is a homomorphism. Of course, $\mathrm{Ker}(\gamma) = nZ$, the factor group Z/nZ is isomorphic to Z_n.

Definition A subgroup H of a group G is **normal(正规)** if $aH=Ha$ for any $a \in G$.

14.7 Example Since Z is an abelian group, nZ is a normal subgroup. Corollary 14.5 allows us toconstruct the factor group Z/nZ with no reference to a homomorphism. As we observed in Example 14.2, Z/nZ is isomorphic to Zn.

14.8 Example Consider the abelian group R under addition, and let $c \in R^+$. $R/\langle c \rangle$ is then also isomorphic to the circle group U of complex numbers of magnitude 1 under multiplication.

14.9 Theorem Let H be a normal subgroup of G. Then $\gamma: G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

14.11 Theorem (The Fundamental Homomorphism Theorem) Let $\varphi: G \to G'$ be a group homomorphism with kernel H. Then $\varphi[G]$ is a group, and $\mu: G/H \to \varphi[G]$ given by $\mu(gH)=\varphi[g]$ is an isomorphism. If $\gamma: G \to G/H$ is the homomorphism given by $\gamma(g)=gH$, then $\varphi(g)=\mu\gamma(g)$ for each $g \in G$.

14.12 Example Classify the group $(Z_4 \times Z_2)/(\{0\} \times Z_2)$ according to the fundamental theorem of finitely generated abelian groups (Theorem 11.12).

14.13 Theorem The following are three equivalent conditions for a subgroup H of a group G to be a normal subgroup of G.

1. $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
2. $gHg^{-1} = H$ for all $g \in G$.
3. $gH = Hg$ for all $g \in G$.

14.14 Example Every subgroup H of an abelian group G is normal.

Abstract Algebra

15.2 Example The trivial subgroup $N=\{0\}$ of Z is, of course, a normal subgroup. Compute $Z/\{0\}$.

15.3 Example Let n be a positive integer. The set $nR=\{nr|r\in R\}$ is a subgroup of R under addition, and it is normal since R is abelian. Compute R/nR.

15.4 Example S_n/A_n.

	A_n	σA_n
A_n	A_n	σA_n
σA_n	σA_n	A_n

15.4 Example S_n/A_n.

	even	odd
even	even	odd
odd	odd	even

15.5 Example (Falsity of the Converse of the Theorem of Lagrange)

A_4, which has order 12, contains no subgroup of order 6.

15.7 Example $(Z_4 \times Z_6)/\langle(0,1)\rangle$.

15.8 Theorem Let $G=H\times K$ be the direct product of groups H and K. Then $\overline{H}=\{(h,e)\mid h\in H\}$ is a normal subgroup of G. Also $G/\overline{H}$ is isomorphic to K in a natural way. Similarly, $G/\overline{K}\simeq H$ in a natural way.

15.9 Theorem A factor group of a cyclic group is cyclic.

15.10 Example $(Z_4 \times Z_6)/\langle(0,2)\rangle$.

15.11 Example $(Z_4 \times Z_6)/\langle(2,3)\rangle$.

15.12 Example Let us compute (that is, classify as in Theorem 11.12 the group $(Z\times Z)/\langle(1,1)\rangle$.

15.14 Definition A group is simple if it is nontrivial and has no proper nontrivial normal subgroups.

15.15 Definition The alternating group A_n, is simple for $n \geqslant 5$.

15.16 Theorem Let $\varphi: G \to G'$ be a group homomorphism. If N is a normal subgroup of G, then $\varphi[N]$ is a normal subgroup of $\varphi[G]$. Also, if N' is a normal subgroup of $\varphi[G]$, then $\varphi^{-1}[G]$ is a normal subgroup of G.

15.17 Definition A maximal normal subgroup of a group G is a normal subgroup M not equal to G such that there is no proper normal subgroup N of G properly containing M.

15.18 Theorem M is a maximal normal subgroup of G if and only if G/M is simple.

Definition The center(中心) $Z(G)$ is defined by $Z(G)=\{z \in G \mid zg = gz$ for all $g \in G\}$.

15.19 Example The center of agroup G always contains the identity elemente. It may be that $Z(G)=\{e\}$, in which case we say that the **center of G is trivial**.

$Z(S_3)=\{\rho_0\}$

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

15.19 Example The center of agroup G always contains the identity elemente. It may be that $Z(G)=\{e\}$, in which case we say that the **center of G is trivial**.

$Z(S_3)=\{\rho_0\}$

The center of every nonabelian group of order pq for primes p and q is trivial.

The center of $S_3 \times Z_5$ must be $\{\rho_0\} \times Z_5$, which is isomorphic to Z_5.

Definition An element $aba^{-1}b^{-1}$ in a group is a commutator of the group.

15.20 Theorem The set of all commutators $aba^{-1}b^{-1}$ for $a,b\in G$ generates a subgroup C(**the commutator subgroup**) of G. This subgroup C is a normal subgroup of G. Furthermore, if N is a normal subgroup of G, then G/N is abelian if and only if $C\leqslant N$.

15.20 Theorem The set of all commutators $aba^{-1}b^{-1}$ for $a,b \in G$ generates a subgroup C(**the commutator subgroup**) of G. This subgroup C is a normal subgroup of G. Furthermore, if N is a normal subgroup of G, then G/N is abelian if and only if $C \leqslant N$.

15.21 Example The commutator subgroup of C_3 is A_3.

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_3	μ_1	μ_2
ρ_2	ρ_2	ρ_0	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	ρ_0	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	ρ_0

Abstract Algebra

18.1 Definition A **ring(环)** $<R,+,\cdot>$ is a set R together with two binary operations + and $\cdot$, which we calladdition and multiplication, defined on R such that the following axioms are satisfied:

R1. $(R,+)$is an abelian group.

R2. Multiplication is associative.

R3. For all $a,b,c\in R$, the left distributive law, $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$ and the right distributive law $(a+b)\cdot c=(a\cdot c)+(b\cdot c)$ hold.

18.2 Example $<Z,+,\cdot>$, $<Q,+,\cdot>$, $<R,+,\cdot>$, and $<C,+,\cdot>$ are rings.

18.3 Example Let R be any ring and let $M_n(R)$ be the collection of all $n \times n$ matrices having elements of R as entries.

18.4 Example Let F be the set of all functions $f{:}R\rightarrow R$. We know that $(F, +)$ is an abelian group under the usual function addition, $(f+g)(x)=f(x)+g(x)$.

We define multiplication on F by $(f\cdot g)(x)=f(x)g(x)$.

Then $\langle F, +, \cdot\rangle$ is a ring.

18.5 Example $<nZ, +, \cdot>$ is a ring.

18.6 Example $<Z_n, +, \cdot>$ is a ring.

18.7 Example If $R_1, R_2 \dots, R_n$, are rings, we can form the set $R_1 \times R_2 \times \dots \times R_n$, of all ordered n-tuples $(r_1, r_2 \dots, r_n)$, where $r_i \in R_i$. Defining addition and multiplication of n-tuples by components (just as for groups), we see at once from the ring axioms in each component that the set of all these n-tuples forms a ring under addition and multiplication by components. The ring $R_1 \times R_2 \times \dots \times R_n$ is the direct product of the rings R_i.

18.8 Theorem If R is a ring with additive identity 0, then for any $a,b \in R$ we have

1. $0a=a0=0$,
2. $a(-b)=(-a)b=-(ab)$,
3. $(-a)(-b)= ab$.

18.9 Definition For rings R and R', a map $\varphi: R \to R'$ is a homomorphism if the following two conditions are satisfied for all $a, b \in R$:

1. $\varphi(a+b) = \varphi(a) + \varphi(b)$
2. $\varphi(ab) = \varphi(a)\varphi(b)$.

18.10 Example For each $a \in R$, we have the evaluation homomorphism $\varphi_a : F \to R$, where $\varphi_a(f) = f(a)$ $f \in F$.

18.11 Example The map $\varphi: Z \to Z_n$, where $\varphi(a)$ is the remainder of a modulo n is a ring homomorphism for each positive integer n.

18.12 Definition An isomorphism $\varphi: R \to R'$ from a ring R to a ring R' is a that is one to one and onto R'. The rings R and R' are then **isomorphic(同构)**.

18.13 Example As abelian groups $\langle Z,+\rangle$ and $\langle 2Z,+\rangle$ are isomorphic under the map $\varphi: Z \to 2Z$, with $\varphi(x)=2x$ for $x \in Z$.

But φ is not a ring isomorphism, for $\varphi(xy)=2xy$ while $\varphi(x)\varphi(y)=2x2y=4xy$.

18.14 Definition A ring in which the multiplication is **commutative(交换)** is a commutative ring. A ring with a multiplicative identity element is a ring with **unity(有一的环，幺环)**; the multiplicative identity element 1 is called **unity(幺元，单位元)**.

(1+1+⋯+1)(1+1+.⋯.+1)=1+1+.⋯.+1

$$(n\cdot 1)(m\cdot 1)=(nm)\cdot 1$$

18.15 Example We claim that for integers r and s where $\gcd(r,s)=1$, the rings Z_{rs} and $Z_r \times Z_s$ are isomorphic. Additively, they are both cyclic abelian groups of order rs with generators 1 and (1, 2) respectively. Thus $\varphi: Z_{rs} \to Z_r \times Z_s$, defined by $\varphi: (n \cdot 1) = n(1,1)$ is an ring isomorphism.

$$\varphi: (nm) = (nm)\ (1,1) = [n(1,1)][m(1,1)] = \varphi(n)\varphi(m)$$

Note In a ring R with unity $1 \neq 0$, the set R^* of nonzero elements, if closed under the ring multiplication, will be a multiplicative group if multiplicative inverses exist.

A **multiplicative inverse** of an element a in a ring R with unity $1 \neq 0$ is an element $a^{-1} \in R$ such that $aa^{-1} = a^{-1}a = 1$.

18.16 Definition Let R be a ring with unity $1 \neq 0$. An element u in R is a unit of R if it has a multiplicative inverse in R. If every non zero element of R is a unit, then R is a division **ring (除环 or skew field)**. A **field(域)** is a commutative division ring. A noncommutative division ring is called a **strictly skew field**.

18.17 Example Let us find the units in Z_{14}. Of course, 1 and $-1=13$ are units. Since $(3)(5)=1$ we see that 3 and 5 are units; therefore $-3=11$ and $-5=9$ are also units.

None of the remaining elements of Z_{14} can be units, since no multiple of 2, 4, 6, 7, 8, or 10 can be one more than a multiple of 14.

18.18 Example Z is not a field, because 2, for example, has no multiplicative inverse, so 2 is not a unit.

However, Q and R are fields.

Definition A **subring(子环)** of a ring is a subset of the ring that is a ring under induced operations from the whole ring; a **subfield (子域)** is defined similarly for a subset of a field.

Abstract Algebra

19.7 Example Z and Z_p, for any prime p are integral domains.

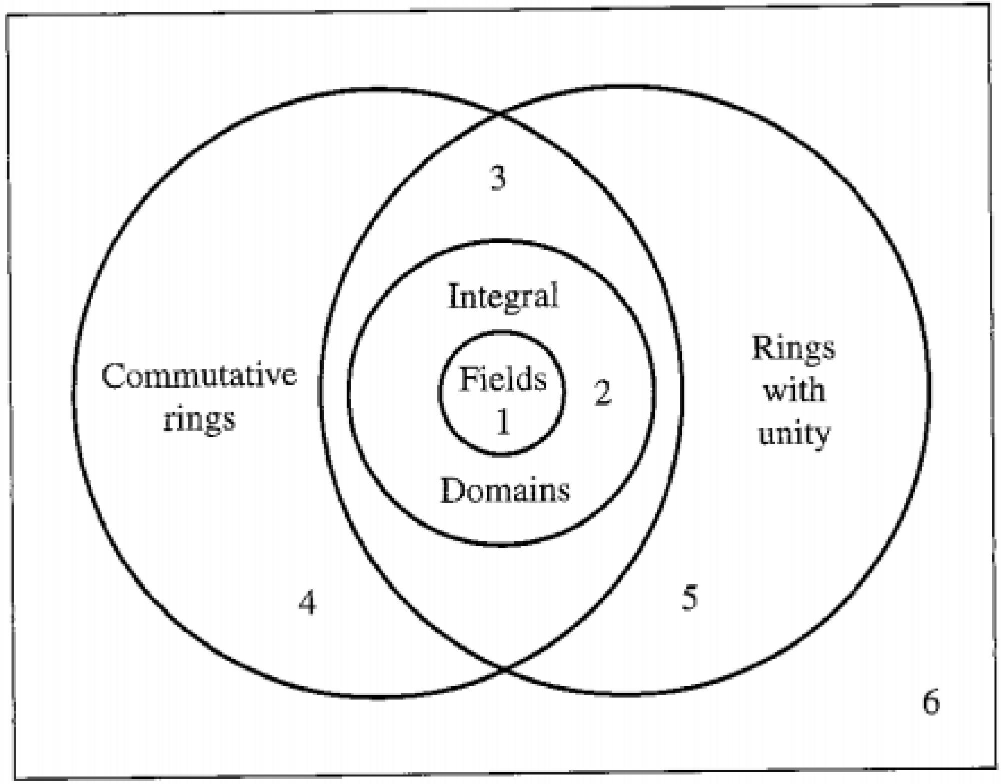
Z_p, is not an integral domain if p is not prime.

The direct product $R\times S$ of two nonzero rings R and S is not an integral domain. (for $r\in R$ and $s\in S$ both nonzero, we have $(r,0)(0,s)=(0,0)$.)

19.8 Example Show that although Z_2 is an integral domain, the matrix ring $M_2(Z_2)$ has divisors of zero.

19.9 Theorem Every field F is an integral domain.

Venn diagram



19.10 Figure A collection of rings.

19.11 Theorem Every finite integral domain is a field.

19.12 Corollary If p is a prime, then Z_p is a field..

19.13 Definition If for a ring R a positive integer n exists such that $n \cdot a=0$ for all $a \in R$, then the least such positive integer is the characteristic of the ring R. If no such positive integer exists, then R is of characteristic 0. ($n \cdot a=a+a+\ldots+a$)

19.14 Example The ring Z_n is of characteristic n, while Z,Q,R and C all have characteristic 0.

19.15 Theorem Let R be a ring with unity. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, then R has characteristic 0. If $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, then the smallest such integer n is the characteristic of R.

Abstract Algebra

SECTION 20: FERMAT'S AND EULER'S THEOREMS

Lemma For any field, the nonzero elements form a group under the field multiplication.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.1 Theorem (Little Theorem of Fermat) If $a \in Z$ and p is a prime not dividing a, then p divides $a^{p-1}-1$, that is, $a^{p-1} \equiv 1 \pmod{p}$ for $a \not\equiv 0 \pmod{p}$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.2 Corollary If $a \in Z$, then $a^p \equiv a \pmod{p}$ for any prime p.

20.3 Example Let us compute the remainder of 8^{103} when divided by 13.

Solution Using Fermat's theorem, we have

$$8^{103} \equiv (8^{12})^{8}(8^{7}) \equiv (1^{8})(8^{7}) \equiv 8^{7} \equiv (-5)^{7} \equiv ((-5)^{2})^{3}(-5)$$

$$\equiv (25)^{3}(-5) \equiv (-1)^{3}(-5) \equiv 5 \equiv (\text{mod } 13).$$

SECTION 20: FERMAT'S AND EULER'S THEOREMS

Solution By Fermat's theorem,

$2^{10} \equiv 1 \pmod{11}$,

so $2^{11213}-1 \equiv [(2^{10})^{1121} \cdot 2^3]-1 \equiv [1^{1121} \cdot 2^3]-1 \equiv 2^3-1 \equiv 8-1 \equiv 7 \pmod{11}$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.5 Example Show that for every integer n, the number $n^{33}-n$ is divisible by 15.

Solution Note that $n^{33}-n = n(n^{32}-1)$.

If 3 divides n, then surely 3 divides $n(n^{32}-1)$. If 3 does not divide n, then by Fermat's theorem, $n^2 \equiv 1 \pmod 3$ so

$n^{32}-1 \equiv (n^2)^{16}-1 \equiv 1^6-1 \equiv 0 \pmod 3$,

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.5 Example Show that for every integer n, the number $n^{33}-n$ is divisible by 15.

Solution Note that $n^{33}-n = n(n^{32}-1)$.

If 5 divides n, then surely 5 divides $n(n^{32}-1)$. If 5 does not divide n, then by Fermat's theorem, $n^4 \equiv 1 \pmod 5$ so

$n^{32}-1 \equiv (n^4)^8-1 \equiv 1^8-1 \equiv 0 \pmod 5$,

SECTION 20: FERMAT'S AND EULER'S THEOREMS

Lemma For any field, the nonzero elements form a group under the field multiplication.

20.6 Theorem The set G, of nonzero elements of Z_p that are not 0 divisors forms a group under multiplication modulo n.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

Defintion Let $\varphi(n)$ be defined as the number of positive integers less than or equal to n and relatively prime to n. This function $\varphi: Z^+ \rightarrow Z^+$ is the **Euler phi-function**.

Note that $\varphi(1) = 1$.

20.7 Example Let $n = 12$. The positive integers less than or equal to 12 and relatively prime to 12 are 1, 5, 7 and 11, $\varphi(12) = 4$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.8 Theorem (Euler's Theorem) If a is an integer relatively prime to n, then $a^{\varphi(n)}-1$ is divisible by n, that is, $a^{\varphi(n)} \equiv a \pmod{n}$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.8 Theorem (Euler's Theorem) If a is an integer relatively prime to n, then $a^{\varphi(n)}-1$ is divisible by n, that is, $a^{\varphi(n)} \equiv a \pmod{n}$.

20.1 Theorem (Little Theorem of Fermat) If $a \in Z$ and p is a prime not dividing a, then p divides $a^{p-1}-1$, that is, $a^{p-1} \equiv 1 \pmod{p}$ for $a \not\equiv 0 \pmod{p}$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.9 Example If a is relatively prime to 12, then $a^4 = 1 (\mathrm{mod}\ 12)$.
For example, with a= 7, we have $7^4 \equiv 1 \pmod{12}$

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.10 Theorem Let m be a positive integer and let $a \in Z_m$ be relatively prime to m. For each $b \in Z_m$, the equation $ax=b$ has a unique solution in Z_m.

20.11 Corollary If a and m are relatively prime intergers, then for any integer b, the congruence $ax \equiv b \pmod{m}$ has as solutions all integers in precisely one residue class modulo m.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.12 Theorem Let m be a positive integer and let $a,b \in Z_m$. Let d be the gcd of a and m. The equation $ax=b$ has a solution in Z_m if and only if d divides b. When d divides b, the equation has exactly d solutions in Z_m.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.13 Corollary Let d be the gcd of positive integers a and m. The congruence $ax \equiv b \pmod{m}$ has a solution if and only if d divides b. When this is the case, the solutions are the integers in exactly d distinct residue classes modulo m.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.13 Corollary Let d be the gcd of positive integers a and m. The congruence $ax \equiv b \pmod{m}$ has a solution if and only if d divides b. When this is the case, the solutions are the integers in exactly d distinct residue classes modulo m.

20.11 Corollary If a and m are relatively prime intergers, then for any integer b, the congruence $ax \equiv b \pmod{m}$ has as solutions all integers in precisely one residue class modulo m.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.14 Example Find all solutions of the congruence $12x \equiv 27 \pmod{18}$.

SECTION 20: FERMAT'S AND EULER'S THEOREMS

20.15 Example Find all solutions of the congruence $15x \equiv 27 \pmod{18}$.

Abstract Algebra

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Let D be an integral domain that we desire to enlarge to a field of quotients F.

1. Define what the elements of F are to be.

2. Define the binary operations of addition and multiplication on F.

3. Check all the field axioms to show that F is a field under these operations.

4. Show that F can be viewed as containing D as an integral subdomain.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Step 1 Let D be a given integral domain, and form the Cartesian product $D \times D = \{(a,b) | a,b \in D\}$.

Let S be the subset of $D \times D$ given by $S = \{(a,b) | a,b \in D, b \neq 0\}$.

21.1 Definition Two elements (a,b) and (c,d) in S are equivalent, denoted by $(a,b) \sim (c, d)$, if and only if $ad = bc$.

21.2 Lemma The relation $\sim$ between elements of the set S as just described is an equivalence relation.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.2 Lemma The relation $\sim$ between elements of the set S as just described is an equivalence relation.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.2 Lemma The relation $\sim$ between elements of the set S as just described is an equivalence relation.

Definie F to be the set of all equivalence classes $[(a,b)]$ for $(a,b) \in S$.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Step 2 Define operations of addition and multiplication on F.

$$[(a,b)]+[(c,d)] = [(ad+bc,\ bd)]$$

$$\text{and}\quad [(a,b)][(c,d)] = [(ac,\ bd)]$$

21.3 Lemma The operations is well-defined operations of addition and multiplication on F.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.3 Lemma The operations is well-defined operations of addition and multiplication on F.

$$[(a,b)]+[(c,d)] = [(ad + bc,\ bd)]$$

and $[(a,b)][(c,d)] = [(ac,\ bd)]$

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Step 3 F is a field.

1. Addition in F is commutative.
2. Addition is associative.
3. $[(0,1)]$ is an identity element for addition in F.
4. $[(-a,b)]$ is an additive inverse for $[(a,b)]$ in F.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Step 3 F is a field.

5. Multiplication in F is associative.

6. Multiplication in F is commutative.

7. The distributive laws hold in F.

8. $[(1,1)]$ is a multiplicative identity element in F.

9. If $[(a,b)] \in F$ is not the additive identity element, then $a \neq 0$ in D and $[(b,a)]$ is a multiplicative inverse for $[(a,b)]$.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.4 Lemma The map $i: D \to F$ given by $i(a) = [(a,1)]$ is an isomorphism of D with a subring of F.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.5 Theorem Any integral domain D can be enlarged to (or embedded in) a field F such that every element of F can be expressed as a quotient of two elements of D. (Such a field F is a **field of quotients of (分式域)** D.)

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.6 Theorem Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a subfield of L such that $\psi(a) = a$ for $a \in D$.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

21.8 Corollary Every field L containing an integral domain D contains a field of quotients of D.

SECTION 21: THE FIELD OF QUOTIENTS OF AN INTEGRAL DOMAIN

Abstract Algebra

SECTION 22: RINGS OF POLYNOMIALS

22.1 Definition Let R be a ring. A polynomial $f(x)$ with coefficients in R is an infinite formal sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \ldots + a_n x^n + \ldots$$

where $a \in R$ and $a_i = 0$ for all but a finite number of values of i.

The a_i are **coefficients** of $f(x)$. If for some $i \geq 0$ it is true that $a_i \neq 0$, the largest such value of i is the **degree** of $f(x)$. If all $a_i = 0$, then the degree of $f(x)$ is undefined.

SECTION 22: RINGS OF POLYNOMIALS

Let $f(x)=a_0+a_1x+\cdots+a_nx^n+\cdots$

and $g(x)=b_0+b_1x+\cdots+b_nx^n+\cdots$.

Addition

$f(x)+g(x)=c_0+c_1x+\cdots+c_nx^n+\cdots$., where $c_n= a_n+b_n$.

Multiplication

$f(x)g(x)=d_0+d_1x+\cdots+d_nx^n+\cdots$., where $d_n= \Sigma_{i=0}^{n} a_ib_{n-i}$.

SECTION 22: RINGS OF POLYNOMIALS

22.2 Theorem The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is $R[x]$, and if R has unity $1 \neq 0$, then 1 is also unity for $R[x]$.

SECTION 22: RINGS OF POLYNOMIALS

22.3 Example In $Z_2[x]$, we have

$$(x+1)^2=(x+1)(x+1)=x^2+(1+1)x+1=x^2+1.$$

Still working in $Z_2[x]$, we obtain

$$(x+1)+(x+1)=(x+1)x+(1+1)=0x+0=0.$$

SECTION 22: RINGS OF POLYNOMIALS

Definition Let $R[x,y]=R[x][y]$ the ring of polynomials in two indeterminates x and y with coefficients in R. The ring $R[x_1, x_2, \cdots, x_n]$ of polynomials in the n indeterminates x_i with coefficients in R is similarly defined.

SECTION 22: RINGS OF POLYNOMIALS

Fact If D is an integral domain then so is $D[x]$. In particular, if F is a field, then $F[x]$ is an integral domain but not a field.

We similarly define $F(x_1, x_2, \cdots, x_n)$ to be the field of quotients of $F[x_1, x_2, \cdots, x_n]$. This field $F(x_1, x_2, \cdots, x_n)$ is the field of rational functions in n indeterminates over F.

SECTION 22: RINGS OF POLYNOMIALS

22.4 Theorem (The Evaluation Homomorphisms for Field Theory) Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. The map ϕ_α: $F[x] \to E$ defined by

$\phi_\alpha(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1 \alpha + \cdots + a_n \alpha^n$, for $(a_0 + a_1 x + \cdots + a_n x^n) \in F[x]$ is a homomorphism of $F[x]$ into E.

Also, $\phi_\alpha(x) = \alpha$, and $\phi_\alpha(a) = a$.

SECTION 22: RINGS OF POLYNOMIALS

22.6 Example Let F be Q and E be R in Theorem 22.4, and consider the evaluation homomorphism

ϕ_0: $Q[x] \to R$.

Here

$\phi_0(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1 0 + \cdots + a_n 0^n = a_0.$

SECTION 22: RINGS OF POLYNOMIALS

22.7 Example Let F be Q and E be R in Theorem 22.4, and consider the evaluation homomorphism

ϕ_2: $Q[x] \to R$.

Here

$\phi_2(x^2+x-6)=2^2+2-6=0.$

Thus $x^2+x-6)$ is in the kernel of ϕ_2.

SECTION 22: RINGS OF POLYNOMIALS

22.8 Example Let F be Q and E be C in Theorem 22.4, and consider the evaluation homomorphism

ϕ_i: $Q[x] \to C$.

Here

$\phi_i(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1i + \cdots + a_ni^n$.

$\phi_i(x) = i$.

$\phi_i(x^2+1) = i^2 + 1 = 0$,

so x^2+1 is in the kernel ϕ_i.

SECTION 22: RINGS OF POLYNOMIALS

22.9 Example Let F be Q and E be R in Theorem 22.4, and consider the evaluation homomorphism

ϕ_π: $Q[x] \rightarrow R$.

Here

$\phi_\pi(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1 \pi + \cdots + a_n \pi^n$.

It can be proved that $a_0 + a_1 \pi + \cdots + a_n \pi^n = 0$ if and only if $a_i = 0$ for $i = 0, 1, \ldots n$. Thus the kernel of ϕ_π is $\{0\}$, and ϕ_π is a one-to-one map. so x^2+1 is in the kernel ϕ_i.

SECTION 22: RINGS OF POLYNOMIALS

22.10 Definition Let $f(x)=a_0+a_1x+\cdots+a_nx^n$ be in $F[x]$, and let $\phi_\alpha : F[x]\rightarrow E$ be the evaluation homomorphism of Theorem 22.4. Let $f(\alpha)$ denote

$\phi_\alpha(f(x))=a_0+a_1\alpha+\cdots+a_n\alpha^n$.

If $f(a)=0$, then α is a zero of $f(x)$.

SECTION 22: RINGS OF POLYNOMIALS

22.11 Theorem The polynomial x^2-2 has no zeros in the rational numbers. Thus $\sqrt{2}$ is not a rational bumber.

Abstract Algebra

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.1 Definition (Division Algorithm for $F[x]$) Let R be a ring. A polynomial $f(x)$ with coefficients in R is an infinite formal sum Let $f(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x)=b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be two elements of $F[x]$, with a_n and b_m both nonzero elements of F and $m>0$. Then there are unique polynomials $g(x)$ and $r(x)$ in $F[x]$ such that $f(x)=g(x)q(x)+r(x)$, where either $r(x)=0$ or the degree of $r(x)$ is less than the degree m of $g(x)$.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.2 Example Let us work with polynomials in $Z_5[x]$ and divide

$f(x)=x^4-3x^3+2x^2+4x-1$ by $g(x)=x^2-2x+3$ to find $q(x)$ and $r(x)$ of Theorem23.1.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.3 Corollary(Factor Theorem) An element $a \in F$ is a zero of $f(x) \in F[x]$ if and only if $x-a$ is a factor of $f(x)$ in $F[x]$.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.4 Example Working again in $Z_5[x]$, note that 1 is a zero of $f(x)=x^4+3x^3+2x+4$.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.5 Corollary A nonzero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.6 Corollary If G is a finite subgroup of the multiplicative group $<F^*, \cdot>$ of a field F, then G is cyclic. In particular, the multiplicative group of all nonzero elements of a finite field is cyclic.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.7 Definition A nonconstant polynomial $f(x) \in F[x]$ is irreducible over F or is an **irreducible polynomial** in $F[x]$ if $f(x)$ cannot be expressed as a product $g(x)h(x)$ of two polynomials $g(x)$ and $h(x)$ in $F[x]$ both of lower degree than the degree of $f(x)$. If $f(x) \in F[x]$ is a nonconstant polynomial that is not irreducible over F, then $f(x)$ is **reducible** over F.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.8 Example Theorem 22.11 shows that x^2-2 viewed in $Q[x]$ has no zeros in Q. This shows that x^2-2 is irreducible over Q. However, x^2-2 viewed in $R[x]$ is not irreducible over R.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.9 Example $f(x)=x^3+3x+2$ viewed in $Z_5[x]$ is irreducible over Z_5.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.10 Theorem Let $f(x) \in F[x]$, and let $f(x)$ be of degree 2 or 3. Then $f(x)$ is reducible over F if and only if it has a zero in F.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.11 Theorem If $f(x) \in Z[x]$, then $f(x)$ factors into a product of two polynomials of lower degrees r and s in $Q[x]$ if and only if it has such a factorization with polvnomials of the same degrees r and s in $Z[x]$.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.11 Theorem If $f(x) \in Z[x]$, then $f(x)$ factors into a product of two polynomials of lower degrees r and s in $Q[x]$ if and only if it has such a factorization with polvnomials of the same degrees r and s in $Z[x]$.

23.12 Corollary If $f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_0$ is in $Z[x]$ with $a_0 \neq 0$, and if $f(x)$ has a zero in Q, then it has a zero m in Z, and m must divide a_0.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.13 Example Corollary 23.12 gives us another proof of the irreducibility of x^2-2 over Q.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.14 Example Let us use Theorem 23.11 to show that

$f(x) = x^4 - 2x2 + 8x + 1$ viewed in $Q[x]$ is irreducible over Q.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.15 Theorem (Eisenstein Criterion) Let $p \in Z$ be a prime. Suppose that $f(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is in $Z[x]$, and $a_n \not\equiv 0 \pmod{p}$, but $a_i \equiv 0 \pmod{p}$ for all $i<n$, with $a_0 \not\equiv 0 \pmod{p^2}$. Then $f(x)$ is irreducible over Q.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.16 Example Taking $p=3$, we see by Theorem 23.15 that

$25x^5-9x^4-3x^2-12$ is irreducible over Q.

23.15 Theorem (Eisenstein Criterion) Let $p\in Z$ be a prime. Suppose that $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ is in $Z[x]$, and $a_n\neq 0 \pmod p$, but $a_i=0 \pmod p$ for all $i<n$, with $a_0\neq 0 \pmod{p^2}$. Then $f(x)$ is irreducible over Q.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.17 Corollary The polynomial

$$\frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over Q for any prime p.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.18 Theorem Let $p(x)$ be an irreducible polynomial in $F[x]$. If $p(x)$ divides $r(x)s(x)$ for $r(x), s(x) \in F[x]$, then either $p(x)$ divides $r(x)$ or $p(x)$ divides $s(x)$.

23.19 Corollary If $p(x)$ is irreducible in $F[x]$ and $p(x)$ divides the product $r_1(x)r_2(x) \ldots r_n(x)$ for $r_i(x) \in F[x]$, then $p(x)$ divides $r_i(x)$ for at least one i.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.20 Theorem If F is a field, then every nonconstant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials, the irreducible polynomials being unique except for order and for unit (that is, nonzero constant) factors in F.

SECTION 23: FACTORIZATION OF POLYNOMALS OVER A FIELD

23.21 Example A factorization of x^4+3x^3+2x+4 in $Z_5[x]$ is $(x-1)^3(x+1)$.

$(x-1)^3(x+1)=(x-1)^2(2x-2)(3x+3)$.

Abstract Algebra

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.1 Definition A map ϕ of a ring R into a ring R' is a homomorphism(同态)
if

$\phi(a+b)=\phi(a)+\phi(b)$

and

$\phi(ab)=\phi(a)\phi(b)$

for all elements a and b in R.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.2 Example (Projection Homomorphisms) Let R_1, R_2, . . . , R_n be rings. For each i, the map π_i: $R_1 \times R_2 \times \ldots \times R_n \to R_i$ defined by $\pi_i(r_1, r_2, \ldots, r_n) = r_i$ is a homomorphism, projection onto the ith component.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.3 Theorem (Analogue of Theorem 13.12) Let ϕ be a homomorphism of a ring R into a ring R'. If 0 is the additive identity in R, then $\phi(0)=0'$ is the additive identity in R', and if $a \in R$, then $\phi(-a)=-\phi(a)$. If S is a subring of R, then $\phi[S]$ is a subring of R'. Going the other way, if S' is a subring of R', then $\phi^{-1}[S]$ is a subring of R. Finally, if R has unity 1, then $\phi(1)$ is unity for $\phi[R]$. Loosely speaking, subrings correspond to subrings, and rings with unity correspond to rings with unity under a ring homomorphism.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.4 Definition Let a map $\phi: R \to R'$ be a homomorphism of rings. The subring $\phi^{-1}[0]=\{r \in R \mid \phi[R]=0'\}$ is the **kernel(核)** of ϕ, denoted by $\mathrm{Ker}(\phi)$

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.5 Theorem (Analogue of Theorem 13.15) Let a map ϕ: $R \to R'$ be a homomorphism of rings, and let H=Ker(ϕ). Then $\phi^{-1}[\phi[a]]=a+H=H+a$, where $a+H=H+a$ is the coset containing a of the commutative additive group $\langle H,+\rangle$.

13.15 Theorem Let φ: $G \to G'$ be a group homomorphism, and let H=Ker(φ). Let $a \in G$. Then the set

$$\varphi^{-1}[\varphi(a)] = \{x \in G \mid \varphi(x)=\varphi(a)\}$$

is the left coset aH of H, and is also the right coset Ha of H.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.6 Corollary (Analogue of Corollary 13.18) A ring homomorphism ϕ: $R \to R'$ is a one-to-one map if and only if $\mathrm{Ker}(\phi)=\{0\}$.

13.18 Corollary A group homomorphism $\phi: G \to G'$ is a one-to-one map if and only if $\mathrm{Ker}(\phi)=\{e\}$.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.7 Theorem （Analogue of Theorem14.1）Let ϕ: $R \longrightarrow R'$ be a ring homomorphism with kernel H. Then the additive cosets of H form a ring R/H whose binary operations are defined by choosing representatives. That is, the sum of two cosets is defined by

$$(a+H)+(b+H)=(a+b)$$

and the product of the cosets is defined by

$$(a+H)(b+H)=(ab)+H.$$

Also, the map $\mu: R/H \rightarrow \phi[R]$ defined by $\mu(a+H)=\phi(a)$ is an isomorphism.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

14.1 Theorem Let $\varphi: G \to G'$ be a group homomorphism with kernel H. Then the cosets of H form a **factor group**, $\boldsymbol{G/H}$, where $(aH)(bH)=(ab)H$. Also the map $\mu: G/H \to \varphi[G]$ defined by $\mu(aH)=\varphi(a)$ is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices a and b from the cosets.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.8 Example The map ϕ: $Z \to Z_n$ defined by $\phi(m)=r$, wherer is the remainder of m when divided by n, is a homomorphism. Since Ker(ϕ)= nZ. Theorem 26.7 shows that Z/nZ is a ring where operations on residue classes can be computed by choosing representatives and performing the corresponding operation in Z. The theorem also shows that this ring Z/nZ is isomorphic to Z_n.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.9 Theorem (Analogue of Theorem 14.4) Let H be a subring of the ring R. Multiplication of additive cosets of H is well defined by the equation $(a+H)(b+H)=(ab)+H$.

if and only if $ah\in H$ and $hb\in H$ for all $a,b\in R$ and $h\in H$.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.10 Definition An addive subgroup N of a ring R satisfying the properties

$aN\in N$ and $Nb\in N$ for all $a,b\in R$

is an ideal(理想).

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.11 Example We see that nZ is an ideal in the ring Z since we know it is a subring, and $s(nm)=(nm)s = n(ms) \in nZ$ for all $s \in Z$.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.12 Example Let F be the ring of all functions mapping R into R, and let C be the subring of F consisting of all the constant functions in F. Is C an ideal in F or Why?

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.13 Example Let F be the ring of all functions mapping R into R, and let N be the subring of all functions f such that $f(2)=0$. Is N an ideal in F? Why or why not?

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.14 Corollary (Analogue of Corollary 14.5) Let N be an ideal of a ring R. Then the additive cosets of N form a ring R/N with the binary operations defined by

$(a+N)+(b+N)=(a+b)+N$

and

$(a+N)(b+N)=ab+N.$

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.15 Definition The ring R/N in the preceding corollary is the **factor ring (or quotient ring, 商环)** of R by N.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.16 Theorem (Analogue of Theorem 14.9) Let N be an ideal of a ring R. Then $\gamma: R \to R/N$ given by $\gamma(x)=x+N$ is a ring homomorphism with kernel N.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.17 Theorem （Fundamental Homomorphism Theorem; Analogue of Theorem 14.11） Let $\phi: R \to R'$ be a ring homomorphism with kernel N. Then $\phi[R]$ is a ring， and the map μ: $R/N \to \phi[R]$ given by $\mu(x+N)=\phi(x)$ is an isomorphism. If $\gamma: R \to R/N$ is the homomorphism given by $\gamma(x)=x+N$， then for each $x \in R$， we have $\phi(x)= \mu\gamma(x)$.

SECTION 26: HOMOMORPHSMS AND FACTOR RINGS

26.19 Example If $\mu : Z/nZ \to Zn$ is an isomorphisim.

Abstract Algebra

SECTION 27: PRIME AND MAXIMAL IDEALS

27.1 Example A factor ring of an integral domain may be a field.

$Z/pZ=Zp$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.2 Example A factor ring of a ring may be an integral domain, even though the original ring is not. $(Z \times Z)/N$, where $N = \{(0,n) | n \in Z\}$

SECTION 27: PRIME AND MAXIMAL IDEALS

27.3 Example If R is not even an integral domain, that is, if R has zero divisors, it is still possible for R/N to be a field. Z_6/N, where $N = \{0,3\}$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.4 Example The structure of a factor ring may seem worse than that of the original ring. $Z/6Z=Z_6$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.5 Theorem If R is a ring with unity, and N is an ideal of R containing a unit, then $N=R$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.6 Corollary A field contains no proper nontrivial ideals.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.7 Definition A **maximal ideal(极大理想)** of a ring R is an ideal M different from R such that there is no proper ideal N of R properly containing M.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.8 Example If p is prime, pZ is a maximal ideal of Z.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.9 Theorem(Analogue of Theorem 15.18) Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if R/M is a field.

15.18 Theorem M is a maximal normal subgroup of G if and only if G/M is simple.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.10 Example Since Z/nZ is isomorphic to Z_n, and Z_n is a field if and only if n is a prime, we see that the maximal ideals of Z are precisely the ideals pZ for prime positive integers p.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.11 Corollary A commutative ring with unity is a field if and only if it has no proper nontrivial ideals.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.13 Definition An ideal $N \neq R$ in a commutative ring R is a prime ideal if $ab \in N$ implies that either $a \in N$ or $b \in N$ for $a, b \in R$.

Note $\{0\}$ is a prime ideal in Z, and indeed, in any integral domain.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.14 Example $Z \times \{0\}$ is a prime ideal of $Z \times Z$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.15 Theorem Let R be a commutative ring with unity, and let $N \neq R$ be an ideal in R. Then R/N is an integral domain if and only if N is a prime ideal in R.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.16 Corollary Every maximal ideal in a commutative ring R with unity is a prime ideal.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.17 Theorem If R is a ring with unity 1, then the map $\phi: Z \rightarrow R$ given by $\phi(n) =$ n • 1 for $n \in Z$ is a homomorphism of Z into R.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.18 Corollary If R is a ring with unity and characteristic $n>1$, then R contains a subring isomorphic to Zn. If R has characteristic 0, then R contains a subring isomorphic to Z.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.19 Theorem A field F is either of prime characteristic p and contains a subfield isomorphic to Z_p or of characteristic 0 and contains a subfield isomorphic to Q.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.20 Definition The fields Zp and Q are prime fields.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.21 Definition If R is a commutative ring with unity and $a \in R$, the ideal $\{ra | r \in \mathrm{R}\}$ of all multiples of a is the principal ideal generated by a and is denoted by $\langle a \rangle$. An ideal N of R is a principal ideal(主理想) if $N = \langle a \rangle$ for some $a \in R$.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.22 Example Every ideal of the ring Z is of the form nZ, which is generated by n, so every ideal of Z is a principal ideal.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.23 Example The ideal $<x>$ in $F[x]$ consists of all polynomials in $F[x]$ having zero constant term.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.24 Theorem If F is a field, every ideal in $F[x]$ is principal.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.25 Theorem An ideal $\langle p(x)\rangle \neq \{0\}$ of $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

SECTION 27: PRIME AND MAXIMAL IDEALS

27.26 Example Example 23.9 shows that x^3+3x+2 is irreducible in $Z_5[x]$, so $Z_5[x]/\langle x^3+3x+2\rangle$ is a field. Similarly, Theorem 22.11 shows that x^2-2 is irreducible in $Q[x]$, so $Q[x]/\langle x^2-2\rangle$ is a field. We shall examine such fields in more detail later.

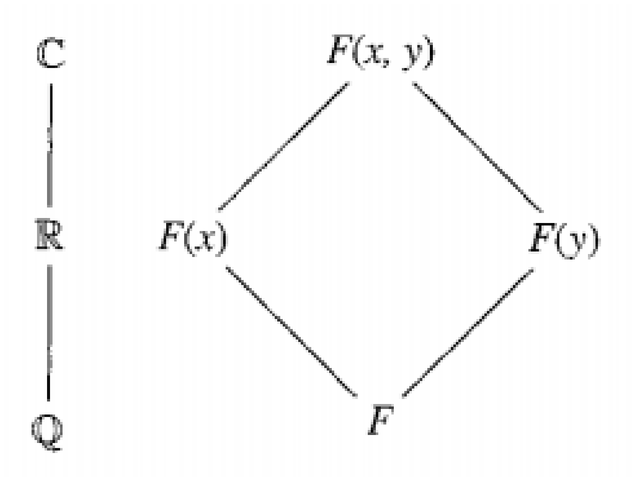
SECTION 27: PRIME AND MAXIMAL IDEALS

27.27 Theorem Let $p(x)$ be an irreducible polynomial in $F[x]$. If $p(x)$ divides $r(x)s(x)$ for $r(x), s(x) \in F[x]$, then either $p(x)$ divides $r(x)$ or $p(x)$ divides $s(x)$.

Abstract Algebra

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.1 Definition A field E is an extension field of a field F if $F \leqslant E$.



SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.3 Theorem (Kronecker's Theorem)(Basic Goal) Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then there exists an extension field E of F and an $\alpha \in E$ such that $f(a)=0$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.4 Example Let $F=R$, and let $f(x)=x^2+1$, which is well known to have no zeros in R and thus is irreducible over R by Theorem 23.10. Then $<x^2+1>$ is a maximal ideal in $R[x]$, so $R[x]/<x^2+1>$ is a field. Identifying $r \in R$ with $r+<x^2+1>$ in $R[x]/<x^2+1>$, we can view R as a subfield of $E=R[x]/<x^2+1>$. Let

$a=x+<x^2+1>$.

Thus α is a zero of x^2+1.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.5 Example Let $F=Q$, and consider $f(x)=x^4-5x^2+6$. This time $f(x)$ factors in Q[x] into $(x^2-2)(x^2-3)$, both factors being irreducible over Q, as we have seen. We can start with x^2-2 and construct an extension field E of Q containing α such that $\alpha^2-2=0$, or we can construct anextension field K of Q containing an element β such that $\beta^2-3=0$. The construction in either case is just as in Example 29.4.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.6 Definition An element α of an extension field E of a field F is algebraic over F if $f(a)=0$ for some nonzero $f(x)\in F[x]$. If α is not algebraic over F, then α is transcendental over F.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.7 Example C is an extension field of Q. Since $\sqrt{2}$ is a zero of x^2-2, we see that $\sqrt{2}$ is an algebraic element over Q.

Also, i is an algebraic element over Q, being a zero of x^2+1.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.8 Example It is well known (but not easy to prove) that the real numbers π and e are transcendental over Q.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.9 Example The real number π is transcendental over Q, as we stated in Example 29.8. However, π is algebraic over R, for it is a zero of $(x-\pi \in R[x]$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.10 Example It is easy to see that the real number $\sqrt{1+\sqrt{3}}$ is algebraic over Q.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.11 Definition An element of C that is algebraic over Q is an **algebraic number.** A **transcendental number** is an element of C that is transcendental over Q.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.12 Theorem Let E be an extension field of a field F and let $\alpha \in E$. Let $\phi_\alpha : F[x] \to E$ be the evaluation homomorphism of $F[x]$ into E such tha $\phi_\alpha(a) = a$ for $a \in F$ and $\phi_\alpha(x) = \alpha$. Then α is transcendental over F if and only if ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E, that is, if and only if ϕ_α is a one-to-one map.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.13 Theorem Let E be an extension field of a field F, and let $\alpha \in E$, where α is algebraic over F. Then there is an irreducible polynomial $p(x) \in F[x]$ such that $p(a)=0$. This irreducible polynomial $p(x)$ is uniquely determined up to a constant factor in F and is a polynomial of minimal degree⩾1 in $F[x]$ having α as a zero. If $f(\alpha)=0$ for $f(x) \in F[x]$, with $f(x)=0$. then $p(x)$ divides $f(x)$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.14 Definition Let E be an extension field of a field F, and let $\alpha \in E$, where α is algebraic over F. The unique monic polynomial $p(x)$ having the property described in Theorem 29.13 is the irreducible polynomial for α over F and will be denoted by $\operatorname{irr}(\alpha, F)$. The degree of $\operatorname{irr}(\alpha, F)$ is the degree of α over F, denoted by $\deg(\alpha, F)$.

Definition A polynomial having 1 as the coefficient of the highest power of x appearing is a **monic polynomial**.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.15 Example We know that $\mathrm{irr}(\sqrt{2}, Q) = x^2 - 2$. Referring to Example 29.10, we see that for $\alpha = \sqrt{1+\sqrt{3}}$ in R, α is a zero of $x^4 - 2x^2 - 2$, which is in $Q[x]$. Since $x^4 - 2x^2 - 2$ is irreducible over Q, we see that $\mathrm{irr}(\sqrt{1+\sqrt{3}}, Q) = x^4 - 2x^2 - 2$.

Thus $\sqrt{1+\sqrt{3}}$ is algebraic of degree 4 over Q.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

Let E be an extension field of a field F, and let $\alpha \in E$. Let ϕ_α be the evaluation homomorphism of $F[x]$ into E with $\phi_\alpha(a)=a$ for $a \in F$ and $\phi_\alpha(x)=\alpha$, as in Theorem 22.4. We consider two cases.

Case I Suppose α is algebraic over F. Then as in Theorem 29.13, the kernel of ϕ_α is $\langle \mathrm{irr}(\alpha, F) \rangle$ and by Theorem 27.25, $\langle \mathrm{irr}(\alpha, F) \rangle$ is a maximal ideal of $F[x]$. Therefore, $F[x]/\langle \mathrm{irr}(\alpha, F) \rangle$ is a field and is isomorphic to the image $\phi_\alpha[F[x]]$ in E. This subfield $\phi_\alpha[F[x]]$ of E is then the smallest subfield of E containing F and α. We shall denote this field by $F(\alpha)$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

Let E be an extension field of a field F, and let $\alpha \in E$. Let ϕ_α be the evaluation homomorphism of $F[x]$ into E with $\phi_\alpha(a)=a$ for $a \in F$ and $\phi_\alpha(x)=\alpha$, as in Theorem 22.4. We consider two cases.

Case II Suppose α is transcendental over F. Then by Theorem 29.12, ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E. Thus in this case $\phi_\alpha[F[x]]$ is not a field but an integral domain that we shall denote by $F[\alpha]$. By Corollary 21.8, E contains a field of quotients of $F[\alpha]$, which is thus the smallest subfield of E containing F and α. As in Case I, we denote this field by $F(\alpha)$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.16 Example Since π is transcendental over Q, the field $Q(\pi)$ is isomorphic to the field $Q(x)$ of rational functions over Q in the indeterminate x.

Thus from a structural viewpoint, an element that is transcendental over a field F behaves as though it were an indeterminate over F.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.17 Definition An extension field E of a field F is a simple extension of F if $E = F(\alpha)$ for some $\alpha \in E$.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.18 Theorem Let E be a simple extension $F(\alpha)$ of a field F, and let α be algebraic over F. Let the degree of $\mathrm{irr}(\alpha, F)$ be $n \geqslant 1$. Then every element β of $E = F(\alpha)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1\alpha + \ldots b_{n-1}\alpha^{n-1},$$

where the b_i are in F.

SECTION 29: INTRODUCTION TO EXTENSION FELDS

29.19 Example The polynomial $p(x)=x^2+x+1$ in $Z_2[x]$ is irreducible over Z_2. An extension field E of Z_2 containing a zero a of x^2+x+1. By Theorem 29.18, $Z_2(a)$ has as elements $0+0\alpha$, $1+0\alpha$, $0+1\alpha$, and $1+1\alpha$, that is, 0, 1, α and $1+\alpha$. This gives us a new finite field, of four elements! The addition and multiplication tables for this field are shown in Tables.

$+$	0	1	α	$1+\alpha$
0	0	1	α	$1+\alpha$
1	1	0	$1+\alpha$	α
α	α	$1+\alpha$	0	1
$1+\alpha$	$1+\alpha$	α	1	0

	0	1	α	$1+\alpha$
0	0	0	0	0
1	0	1	α	$1+\alpha$
α	0	α	$1+\alpha$	1
$1+\alpha$	0	$1+\alpha$	1	α

SECTION 29: INTRODUCTION TO EXTENSION FELDS

Example Finally, we can use Theorem 29.18 to fulfill our promise of Example 29.4 and show that $R[x]/<x^2+1>$ is isomorphic to the field C of complex numbers. We saw in Example 29.4 that we can view $R[x]/<x^2+1>$ as an extension field of R. Let $\alpha = x+<x^2+1>$.

Then $R[x]/<x^2+1>$ consists of all elements of the form $a+b\alpha$ for $a,b \in R$, by Theorem 29.18. Thus $R(\alpha) \approx C$.

Abstract Algebra

SECTION 34: ISOMORPHISM THEOREMS

34.2 Theorem (First Isomorphism Theorem) Let $\phi{:}\,G \to G'$ be a homomorphism with kernel K, and let $\gamma_K{:}G \to G/K$ be the canonical homomorphism. There is a unique isomorphism $\mu{:}\ G/K \to \phi{:}[G]$ such that $\phi{:}(x) = \mu(\gamma_K(x))$ for each $x \in G$. an extension field of a field F if $F \leqslant E$.

$K = der\phi$

$$
\begin{tikzcd}
x \quad G \arrow[rr, "\phi"] \arrow[d, "\gamma_K"'] & & G' \quad \phi(x) \\
x+K \quad G/K \arrow[urr, "\mu"'] & &
\end{tikzcd}
$$

SECTION 34: ISOMORPHISM THEOREMS

34.3 Lemma Let N be a normal subgroup of a group G and let $\gamma: G \to G/N$ be the canonical homomorphism. Then the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroups of G/N given by $\phi(L) = \gamma[L]$ is one to one and onto.

SECTION 34: ISOMORPHISM THEOREMS

Definition If H and N are subgroups of a group G, then we let

$\boldsymbol{HN} = \{hn \mid h \in H, n \in N\}$.

Definition We define the join $\boldsymbol{H \vee N}$ of H and N as the intersection of all subgroups of G that contain HN; thus $H \vee N$ is the smallest subgroup of G containing HN. Of course $H \vee N$ is also the smallest subgroup of G containing both H and N, since any such subgroup must contain HN.

Note In general, HN need not be a subgroup of G. However, we have the following lemma.

SECTION 34: ISOMORPHISM THEOREMS

34.4 Lemma If N is a normal subgroup of G, and if H is any subgroup of G, then $H \vee N = HN = NH$. Furthermore, if H is also normal in G, then HN is normal in G.

SECTION 34: ISOMORPHISM THEOREMS

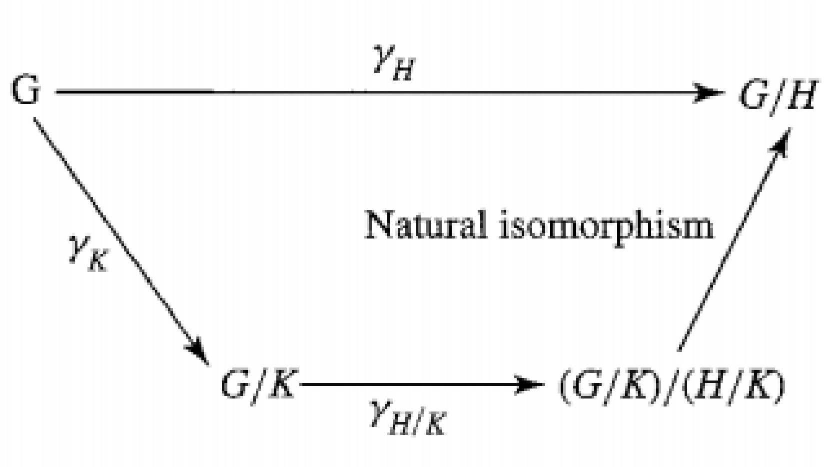
34.5 Theorem (Second Isomorphism Theorem) Let H be a subgroup of G and let N be a normal subgroup of G. Then $(HN)/N \approx H/(H \cap N)$.

SECTION 34: ISOMORPHISM THEOREMS

34.6 Example Let $G = \mathrm{Z}\times\mathrm{Z}\times\mathrm{Z}$, $H = \mathrm{Z}\times\mathrm{Z}\times\{0\}$, and $N = \{0\}\times Z\times Z$. Then clearly $HN = \mathrm{Z}\times\mathrm{Z}\times\mathrm{Z}$ and $H\cap N = \{0\}\times\mathrm{Z}\times\{0\}$. We have $(HN)/N \approx Z$ and we also have $H/(H\cap N) \approx Z$.

SECTION 34: ISOMORPHISM THEOREMS

34.7 Theorem (Third Isomorphism Theorem) Let H and K be normal subgroups of a group G with $K<H$. Then $G/H \approx (G/K)/(H/K)$.



Abstract Algebra

SECTION 36: SYLOW THEOREMS

36.2 Definition Let p be a prime. A group G is a p-group if every element in G has order a power of the prime p. A subgroup of a group G is a p-subgroup of G if the subgroup is itself a p-group.

SECTION 36: SYLOW THEOREMS

36.3 Theorem (Cauchy's Theorem) Let p be a prime. Let G be a finite group and let p divide $|G|$. Then G has an element of order p and, consequently, a subgroup of order p.

SECTION 36: SYLOW THEOREMS

36.8(First Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where $n \geqslant 1$ and where p does not divide m. Then

1. G contains a subgroup of order p^i for each i where $1 \leq i \leq n$,

2. Every subgroup H of G of order p^i is a normal subgroup of a subgroup of order p^i+1 for $1 \leqslant i \leqslant n$.

SECTION 36: SYLOW THEOREMS

36.9 Definition A Sylow p-subgroup P of a group G is amaximal p-subgroup of G, thatis, a p-subgroup contained in no larger p-subgroup.

SECTION 36: SYLOW THEOREMS

36.10 Theorem (Second Sylow Theorem) Let P_1 and P_2 be Sylow p-subgroups of a finite group G. Then P_1 and P_2 are conjugate subgroups of G.

SECTION 36: SYLOW THEOREMS

36.11 Theorem (Third Sylow Theorem) If G is a finite group and p divides $|G|$, then the number of Sylow p-subgroups is congruent to 1 modulo p and divides $|G|$.

SECTION 36: SYLOW THEOREMS

36.12 Example The Sylow 2-subgroups of S_3 have crder 2.

The subgroups of order 2 in S_3 in Example 8.7

are $\{\rho_0, \mu_1\}$, $\{\rho_0, \mu_2\}$, $\{\rho_0, \mu_3\}$

SECTION 36: SYLOW THEOREMS

36.12 Example No group of order 15 is simple.