MA3071 – DLI
Financial Mathematics – Introduction

Dr. Ting Wei

University of Leicester

My details

Lecturer:	Dr Ting Wei
E-mail:	tw258@leicester.ac.uk
Appointments:	E-mail me to book appointments.

Blackboard Site

- Course material and course announcements will be available on Blackboard.
- Lecture slides posted on Bb before the lectures.
- Send me e-mail if I forget to upload something!
- Importance of attending classes.
- Some references are provided on Bb if you are interested in further reading, but it is not necessary to succeed in this course.

Prerequisites

- Essential prerequisites: basic probability (random variables and their distributions, mean, variance, covariance, etc.), basic calculus (derivatives and integrals, Lagrangian approach, etc.), and differential equations.
- We will not cover these in the course, and you are expected to work independently to revise them.
- Desirable skills: Programming (MATLAB, etc.), Excel
- If you have any doubts or questions about prerequisites, please get in touch with me via discussion board or email.

Assessment

- 30% Coursework
- 70% Written examination

Assessment

- *Coursework*: Consisting of 3 computer-based problem sheets, to be done individually. Hand in coursework electronically via Blackboard (one file per person). Deadlines are as follows,
  - Coursework 1: 08/Nov/2023, 16:00 (UK)/23:59 (China)
  - Coursework 2: 08/Dec/2023, 16:00 (UK)/23:59 (China)
  - Coursework 3: 20/Dec/2023, 16:00 (UK)/23:59 (China)
- *Exam*: 4 questions, 25 points each, exam date TBC.

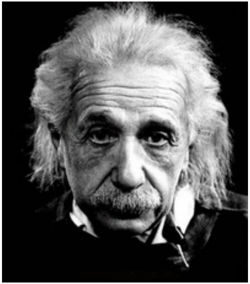
Question

- What do you think are the main topics of financial mathematics?

Answers

- There are many financial games. And in this module, we concentrate on two types of financial games.
  - Option games.
  - Portfolio optimization games.

Answers



"You have to learn the rules of the game. And then you have to play better than anyone else."

– Albert Einstein

Example: Option

- Suppose that a man wants to buy a house in a year from now. The current price is £$95,000$ and he believes the price will go up in a year. Therefore, he only wants to pay a guaranteed price of £$100,000$ even though the price may be higher than that. Are there any financial tools that would allow him to achieve this goal?

Example, cont.

- European call option contract tells you that it is possible for him to buy the house in a year from now, for a fixed price £$100,000$. However, to do that he needs to pay an extra fee, which is the price to buy the right.
- What is the price for him to buy the right? How much does he pay to have the current homeowner happily grant him this right?
- Obviously, the price should be fair enough for both parties to accept.

Option games

- One of the financial games is option games. Our task is to determine the option prices.

Example: Portfolio optimization

- Suppose that in a financial market, there are only two stocks (Tencent and Alibaba) available for investment. You have a lot of money and would like to invest your money in the two stocks to form a portfolio. Which of the following is a better portfolio?
  - A. 40% of money in Tencent, 60% of money in Alibaba;
  - B. 50% of money in Tencent, 50% of money in Alibaba;
  - C. 60% of money in Tencent, 40% of money in Alibaba.

Example, cont.

- We cannot answer this question intuitively.
- If both stocks have the same expected rates of return,
  - If the Tencent is more risky than the Alibaba, portfolio A would be better.
  - If the Alibaba is more risky than the Tencent, portfolio C would be better.
  - If the Tencent is as risky as the Alibaba, all the above portfolios are equivalent.
- What if both stocks are equivalently risky, but have different expected rates of return?

Example, cont.

- What about "20% of money in Tencent, 80% of money in Alibaba"?
- **Question 1**: Is there a better portfolio than all of the above?
- **Question 2**: What if both stocks have different expected rates of return and different risk levels?

Example, cont.

- Can we determine a pair of numbers (x, y) such that $x + y = 1$ and "x fraction of money in Tencent and y fraction of money in Alibaba" is the best portfolio among all feasible portfolios?

- Please note: x or y may be negative when short selling is allowed in the market.

Portfolio optimization

- Another financial game is portfolio optimization game. Our job is to identify the optimal portfolio.

Syllabus

1. Introduction to options;
2. Binomial tree models;
3. Brownian motion and stochastic differential equations;
4. Black-Scholes pricing models;
5. Monte-Carlo methods for option pricing;
6. Risk measures and Mean-variance portfolio theory;
7. Asset pricing models under equilibrium.

MA3071 – DLI
Financial Mathematics – Section 1
Introduction to options

Dr. Ting Wei

University of Leicester

Options - definitions

- An **option** gives the buyer of the option (or the person granted the option) the right, but not the obligation, to buy or sell a specified asset (the underlying asset) on a predetermined future date (the expiry/maturity date) for a predetermined price (the strike price, usually denoted by K).
  - A **call option** grants the right, but not the obligation, to **purchase** an underlying asset on a specified maturity date in the future for a predetermined strike price.
  - A **put option** grants the right, but not the obligation, to **sell** an underlying asset on a specified maturity date in the future for a predetermined strike price.
- A **European option** can only be exercised at maturity, while an **American option** can be exercised at any time before its maturity.

Example

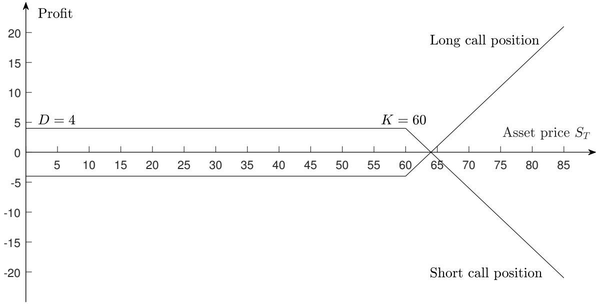
Let S_t denote the price of an underlying asset at time t, T denote the maturity, K be the strike price, and D be the option premium at time T.

- European call option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?
- European put option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?

Long and short positions

- A **long position** on an option is when the option has been purchased, while a **short position** is when the option is sold.
- Therefore, a long position on a European option gives the holder (i.e. the buyer, or owner of the option) the right but not the obligation to exercise the option. The holder of the short position (i.e. the seller, or writer of the option) will be obliged to sell, or buy, the underlying asset for the agreed price, if the option is exercised.

European call option profit at maturity

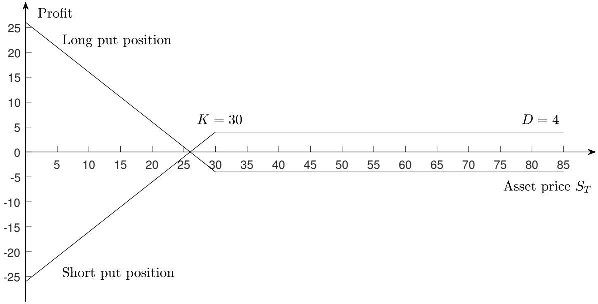


The profit (denoted by P) at maturity T for the holder of a call option is calculated using the formula:

$$P = (S_T - K)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(S_T - K)_+$ is called **call option claim/payoff**.

European put option profit at maturity



The profit P at maturity T for the holder of a put option is calculated using the formula:

$$P = (K - S_T)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(K - S_T)_+$ is called **put option claim/payoff**.

Some discussions

- The loss for a short position and the profit for a long position in a put option are both limited to the difference between the strike price and the premium (i.e. $K - D$).
- In contrast, with a call option, the loss for a short position and the profit for a long position can be positively infinite as the price of the underlying asset theoretically has no upper limit. Therefore, call options are more risky for short positions but can be more profitable for long positions.
- Asset price S_T is a random variable, thus the option profit P is also a random variable.
- The profit for the writer of a European option is always the opposite of the profit for the holder. Thus, only when $\mathbb{E}[P] = 0$ would it be considered fair for both the writer and the holder.

Question

- If you want to buy a European option that expires at T, how much do you need to pay at time 0?

Basic concept of option prices

- Under the assumption that $\mathbb{E}[P] = 0$, the price of a European option at time t (denoted as V_t) is the present value of the expected payoff at time T, that is
  - Call option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$

  where ρ represents the risk-free interest rate for the period $[0, 1]$, and $(1+\rho)^{-(T-t)}$ is the discount factor for the period $[t, T]$.
- In the case of continuous compounding, discount factor is modified as $e^{-\rho(T-t)}$, and
  - Call option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$
- The option premium at T is calculated by $D = V_t(1+\rho)^{T-t}$ or $D = V_t e^{\rho(T-t)}$.

Option pricing models

- Option pricing is one of main tasks of this module.
- We will talk about the following option pricing models in this course:
  - Binomial tree models.
  - Black–Scholes models.
  - Monte-Carlo methods.
- We focus more on the option pricing of European style options.

The concept of arbitrage

- We will encounter the concept of arbitrage when discussing those pricing models. An arbitrage opportunity arises when you can execute a series of transactions that result in a profit without incurring any risk. More formally, arbitrage occurs when:
  - The initial net investment is zero.
  - The probability of making a loss is zero.
  - The probability of making a profit is strictly greater than zero.
- Clearly, if such an opportunity was available then investors would trade as much as they could to take advantage of this "free lunch".

Example

In a financial market, bank A offers a one-year fixed-term bond with an interest rate of ρ_1, and bank B offers another one-year fixed-term bond with an interest rate of ρ_2. Assuming short sales are allowed in this market, are there any arbitrage opportunities when $\rho_1 \neq \rho_2$?

No-arbitrage principle in option pricing

- Since the value of options depends on the value of some other financial variables (price of underlying assets S_t, strike price K, maturity T, etc.), their price will be a function of those variables. If the option price is not consistent with the underlying, then arbitrage would be possible.
- This would result in a loss to one of the parties to the transaction as others traded to take advantage of the arbitrage opportunity. Everyone is therefore keen to avoid this outcome.
- Even if it were to happen, market activity would move prices such that they would fall back in line with the no arbitrage equivalent.
- It ensures that the calculated prices for the relevant options are consistent with the related market variables.

Factors affecting option prices

- The price of the underlying asset (S_t, usually referred to as the spot value) compared to the strike price (K).
- The volatility of the underlying (usually meaning the annualised standard deviation, σ), as a measure of how uncertain the value in future is.
- Interest rates (ρ), since we are estimating present values.
- The term to expiry (T).

Relationship between these key factors and option prices

- A longer term to maturity means a higher option value. The increased uncertainty regarding values further into the future implies a greater potential benefit for the option holder.
- A higher level of volatility also leads to a higher option value (this is the most significant variable because it has the greatest individual impact on the option value). This is due to the greater uncertainty.

Relationship between these key factors and option prices

- Higher interest rates lead to higher call option prices and lower put option prices. As interest rates rise in the market, the expected return demanded by investors in stocks tends to increase. Conversely, the present value of future cash flows generated by option contracts decreases. The overall impact of these two factors is an increase in call option value and a decrease in put option value.
- For fixed levels of the above variables, a higher spot price of the underlying asset increases the option value for call options due to their higher intrinsic value, while it decreases the option value for put options.

MA3071 – DLI
Financial Mathematics – Section 2
Binomial tree models

Dr. Ting Wei

University of Leicester

Background

- The binomial tree model is a numerical method for estimating option prices in a no-arbitrage framework.
- All such methods are by definition discrete in nature, but with small enough steps (in time and change in modelled variable) the result will converge towards the continuous equivalent (Black-Scholes).
- More steps usually mean a more accurate price, but are more computationally intensive.

Background, cont.

- The model is very flexible and can be used for pricing American options and exotic options where the payoff is path dependent (such as Asian options, barrier options).
- It also provides insight to key concepts in financial economic theory such as hedging portfolios and risk-neutral pricing. These are central to the development of the Black-Scholes formula and option pricing in general.

Financial Assumptions

- The option payoff/claim (C) is a function of the underlying asset price at time T (S_T), i.e. $C = f(S_T)$.
- The risk-free interest rate (ρ) is known and is a constant over a certain time period.
- The volatility/standard deviation (σ) of the return on the underlying asset is a constant.
- There are no transaction costs in buying or selling the underlying asset or the option.
- Short selling is allowed.
- There is no arbitrage opportunity.

Mathematical Assumptions

- **Markov Property**: Behaviour of asset prices satisfies Markov property. Given the present price, the future price does not depend on the past prices.
- **Martingale Property**: To achieve the no-arbitrage condition, we make the assumption that the discounted asset price is a martingale.

Conditional expectation

- If X and Y are two random variables, the conditional expectation of X given $Y = y$ is
  - Discrete: $\mathbb{E}[X|Y = y] = \sum_x xP(X = x|Y = y)$
  - Continuous: $\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$
- Y as a random variable has different possible values. Therefore, $\mathbb{E}[X|Y]$ **is still a random variable**, since its value depends on the values of the random variable Y.

Properties of conditional expectation

- If X is independent of Y, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X]$$

- Law of total expectation:

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X] \quad \text{and} \quad \mathbb{E}\left[\mathbb{E}[g(X)|Y]\right] = \mathbb{E}[g(X)]$$

- Linearity:

$$\mathbb{E}[aX_1 + bX_2|Y] = a\mathbb{E}[X_1|Y] + b\mathbb{E}[X_2|Y]$$

Markov Property

- **Definition**: Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_s, s \in I\}$, for some totally ordered index set I, and let $(S, \mathcal{S})$ be a measurable space. A stochastic process $\{X_t : \Omega \to S\}_{t \in I}$ defined on $(S, \mathcal{S})$ and adapted to the filtration is said to possess the Markov property if, for each $A \in \mathcal{S}$ and each $s, t \in I$ with $t > s \geq 0$,

$$P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s)$$

- Alternatively, the Markov property can be formulated as the following conditional expectation,

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] = \mathbb{E}[f(X_t) | X_s]$$

for all $t > s \geq 0$ and $f : S \to \mathbb{R}$ is a bounded and measurable function.

Markov Property

- A stochastic process satisfies the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present values) depends only upon the present state.

$$
\begin{aligned}
P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, X_{t_{n-2}} = x_{n-2}, \cdots, X_{t_0} = x_0) \\
= P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1})
\end{aligned}
$$

  and

$$
\mathbb{E}[f(X_{t_n})|X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = \mathbb{E}[f(X_{t_n})|X_{t_{n-1}}]
$$

  for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- An important result is that any process with **independent increments** satisfies the Markov property.

Martingale Property

- **Definition**: A martingale is a stochastic process $\{X_t\}_{t \in I}$ that satisfies:

$$\mathbb{E}[|X_t|] < \infty \text{ for all } t \in I,$$
$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \text{ for all } s, t \in I \text{ and } t > s \geq 0$$

- For example,

$$\mathbb{E}[X_{t_n}|X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = X_{t_{n-1}}$$

for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- For our mathematical assumption, the **discounted asset price** is a stochastic process and a martingale.

Assumptions

- Please Note: All of the above financial and mathematical assumptions also apply to the **Black-Scholes model** and the **Monte-Carlo methods** of option pricing.

Single period binomial tree

- In the simplest version of the model, we consider only one time period. Over this time period, the price of the underlying asset (S_t) can only increase or decrease (i.e. is Bernoulli).
- The "increase" or "decrease" is proportional to the current value with factors u and d such that $u > d > 0$.

$$
\begin{array}{ccc}
 & & S_1 = S_0 u \\
 & \nearrow & \\
S_0 & & \\
 & \searrow & \\
 & & S_1 = S_0 d
\end{array}
$$

Two period binomial tree

- An example of a two period tree is shown here.

$$
\begin{array}{ccccc}
 & & & & S_2 = S_0u^2 \\
 & & & \nearrow & \\
 & & S_1 = S_0u & & \\
 & \nearrow & & \searrow & \\
S_0 & & & & S_2 = S_0ud \\
 & \searrow & & \nearrow & \\
 & & S_1 = S_0d & & \\
 & & & \searrow & \\
 & & & & S_2 = S_0d^2
\end{array}
$$

- The above binomial tree satisfies the **Markov property**.

Example

Consider a single period binomial tree. Assume that $\rho = \frac{1}{3}$ over period $[0, 1]$, and the asset prices at times 0, 1 are defined by

$$S_0 = 6, S_1 = S_0 Y$$

where Y is a random variable with

$$P(Y = 3) + P(Y = 0.5) = 1$$

Write down the single-period binomial tree and calculate the option price V_0 of the option claim $C = (S_1 - 6)_+$.

Questions

- How to make sure the option price is arbitrage free?
- What are the probabilities of the asset price rising and falling?

Martingale

- As binomial tree model is a discrete model, the discounted asset price is $(1+\rho)^{-t}S_t$, where ρ is the risk-free interest rate over each period in the tree.
- According to the mathematical assumptions, $(1+\rho)^{-t}S_t$ is a martingale if

$$\mathbb{E}_Q\left[(1+\rho)^{-T}S_T|\mathcal{F}_t\right] = (1+\rho)^{-t}S_t, \quad T > t \geq 0$$

where $\mathbb{E}_Q[\bullet]$ denotes the expectation based on q-probabilities such that

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio

- Let C_u and C_d denote the option payoff at time 1 when the price of the underlying asset increases and decreases, respectively.
- Suppose we hold a portfolio of stocks and bonds at time 0, with ϕ units of stock and ψ units of bond. At time 1, this portfolio will be worth:
  - $\phi S_0 u + \psi(1+\rho)$ if the stock price increased,
  - $\phi S_0 d + \psi(1+\rho)$ if the stock price decreased.

Hedging Portfolio, cont.

- We now choose ϕ and ψ such that the value of the portfolio at time 1 is equal to the payoff of the option. Therefore,

$$\phi S_0 u + \psi(1+\rho) = C_u$$
$$\phi S_0 d + \psi(1+\rho) = C_d$$

- We then solve the simultaneous equations to get,

$$\phi = \frac{C_u - C_d}{S_0(u-d)} \quad \text{and} \quad \psi = \frac{uC_d - dC_u}{(1+\rho)(u-d)}$$

Hedging Portfolio, cont.

- Since the values of the hedging portfolio and the option are equal at time 1, they must be equal at time 0 to avoid arbitrage. The value of the hedging portfolio is then equal to the price of the option at time 0. Therefore,

$$V_0 = \phi S_0 + \psi = \frac{C_u q_u + C_d q_d}{1+\rho}$$

$$q_u + q_d = 1$$

- We then obtain,

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio, cont.

- The hedging portfolio (ϕ, ψ) is also called a replicating portfolio because it matches the option payoffs with no risk.
- This approach can also be employed for hedging purposes by the option seller/writer: that is an investment strategy which reduces the amount of risk carried by the seller of the option when used in conjunction with the short position in the option.

No-arbitrage condition

- The no-arbitrage condition must hold for the option game

$$d < 1 + \rho < u$$

- Moreover, it is easy to check that
  - $q_u + q_d = 1$
  - $0 < q_u < 1$, and $0 < q_d < 1$ iff the above no arbitrage condition holds.
  - $\mathbb{E}_Q[Y] = uq_u + dq_d = 1 + \rho$
- As long as the above arbitrage-free condition holds, equivalent martingale probabilities ($q-$probabilities) are in effect for arbitrage-free option pricing, and the value of the hedging portfolio equals the value of the option at all times.
- $(1 + \rho)^{-t} S_t$ is a martingale $\Leftrightarrow d < 1 + \rho < u$.

Think about

- If $1 + \rho < d < u$, what's the arbitrage opportunity?
- What if $d < u < 1 + \rho$?

Two period binomial tree

- For the two period binomial model, the discrete time market consists of two assets: one non risky asset (bond) with fixed interest rate ρ and one risky asset such that the asset prices at times 0, 1 and 2 are defined by

$$S_0, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

where Y, Y_1, Y_2 are iid random variables with

$$P(Y = u) + P(Y = d) = 1$$

- We want to determine the arbitrage free time 0 option price of the option claim $C = f(S_2)$ with expiry date $T = 2$.

Two period binomial tree, cont

- $q-$probabilities are the **same** as found from the single period binomial tree model.

- Then, the general binomial tree is defined by

$$
\begin{array}{ccccc}
 & & & & S_2 = S_0u^2\ , \quad C_{uu} = f(S_0uu) \\
 & & & \nearrow^{q_u} & \text{with } \ q_u^2 \\
 & & S_1 = S_0u & & \\
 & \nearrow^{q_u} & & \searrow^{q_d} & \\
S_0 & & & & S_2 = S_0ud\ , \quad C_{ud} = f(S_0ud) \\
 & \searrow_{q_d} & & \nearrow^{q_u} & \text{with } \ 2q_uq_d \\
 & & S_1 = S_0d & & \\
 & & & \searrow_{q_d} & \\
 & & & & S_2 = S_0d^2\ , \quad C_{dd} = f(S_0dd) \\
 & & & & \text{with } \ q_d^2
\end{array}
$$

- The option price is

$$V_0 = (1+\rho)^{-2}\left[q_u^2 C_{uu} + 2q_uq_dC_{ud} + q_d^2 C_{dd}\right]$$

Example

Consider a discrete market with one risky asset and one risk free asset. The interest rate $\rho = 0.5$ is fixed over each period, and the asset prices at times 0, 1 and 2 are defined by

$$S_0 = 4, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

where Y, Y_1, Y_2 are iid random variables with

$$P(Y = 8) + P(Y = 0.5) = 1$$

(i) Determine the equivalent martingale probabilities.

(ii) Write down the two-period binomial tree and find the arbitrage free time 0 option price of the European put option with strike price $K = 5$ and expiry day $T = 2$.

(iii) Determine the hedging portfolio for the two period tree.

Extending to n periods

An n period binomial tree is introduced as follows,

(I) $S_{t_i} = S_{t_{i-1}} Y_{t_i}$ for $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i - t_{i-1} = \frac{T}{n}$, $i = 1, 2, \cdots, n$, where S_{t_i} is the time t_i asset price and S_0 is a positive constant;

(II) Y and Y_{t_i}, $i = 1, 2, \cdots, n$ are independent and identically distributed random variables (iid) with

$$P(Y = u) + P(Y = d) = 1$$

where up factor u and down factor d are constants such that $u > d > 0$ and possible outcomes of Y;

(III) Y_{t_i} is independent of $S_{t_1}, \cdots, S_{t_{i-1}}$.

Extending to n periods, cont.

- Distribution of S_{t_n} is defined by

$$S_{t_n} = S_0 u^j d^{n-j}$$

$$P(S_{t_n} = S_0 u^j d^{n-j}) = \binom{n}{j} q_u^j q_d^{n-j}$$

for $j = 0, 1, \cdots, n$.

- The arbitrage free time 0 option price of the option claim $C = f(S_{t_n})$ is calculated by

$$V_0 = (1+\rho)^{-n} \sum_{j=0}^{n} f(S_0 u^j d^{n-j}) \binom{n}{j} q_u^j q_d^{n-j}$$

where ρ is the interest rate over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.

Example

Let Y and Y_{t_i}, $i = 1, 2, \cdots, n$ be iid random variables with distribution

$$P(Y = 1) = 0.5, \quad P(Y = 3) = 0.5$$

and the underlying asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$.

Let $\rho = 0.25$ over each time period. Find the arbitrage free time 0 option price of the option claim $C = S_T^{10}$, when $S_0 = 4$, $n = 5$ and $T = t_n$.

Time varying binomial tree models

- Time varying interest rates ρ_i over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.

- Asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$, $i = 1, 2, \cdots, n$, where Y_{t_i} are independent, but in general not identically distributed

$$P(Y_{t_i} = u_i) + P(Y_{t_i} = d_i) = 1$$

with time varying factors u_i and d_i.

- The time varying $q-$probabilities are

$$q_u^{(i)} = \frac{1 + \rho_i - d_i}{u_i - d_i}, \quad q_d^{(i)} = 1 - q_u^{(i)}, \quad i = 1, 2, \cdots, n$$

Example

- Two period time varying binomial tree model:

$$
\begin{array}{ccccc}
 & & & \overset{q''_u}{\nearrow} & S_2 = S_0 u_1 u_2\,, \quad C_{u_1u_2} \text{ with } q'_u q''_u \\
 & & S_1 = S_0 u_1 & & \\
 & \overset{q'_u}{\nearrow} & & \overset{q''_d}{\searrow} & S_2 = S_0 u_1 d_2\,, \quad C_{u_1d_2} \text{ with } q'_u q''_d \\
S_0 & & & & \\
 & \overset{q'_d}{\searrow} & & \overset{q''_u}{\nearrow} & S_2 = S_0 d_1 u_2\,, \quad C_{d_1u_2} \text{ with } q'_d q''_u \\
 & & S_1 = S_0 d_1 & & \\
 & & & \overset{q''_d}{\searrow} & S_2 = S_0 d_1 d_2\,, \quad C_{d_1d_2} \text{ with } q'_d q''_d
\end{array}
$$

- The option price is

$$V_0 = \frac{C_{u_1u_2} q'_u q''_u + C_{u_1d_2} q'_u q''_d + C_{d_1u_2} q'_d q''_u + C_{d_1d_2} q'_d q''_d}{(1+\rho_1)(1+\rho_2)}$$

Calibrating binomial trees

- It is convenient to calibrate the tree such that the asset price follows a log-normal distribution,

$$\log\left(\frac{S_T}{S_t}\right) \sim N\left[\left(\rho - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$$

- In this case, as the number of time steps increases, the estimated option price will converge towards the Black-Scholes price.

Calibrating binomial trees

- Let Δ_t be the size of the time step, we have
  - $q_u = \frac{e^{\rho\Delta_t} - d}{u - d}$,
  - $u = e^{\sigma\sqrt{\Delta_t}}$,
  - $d = e^{-\sigma\sqrt{\Delta_t}}$.

MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part I

Dr. Ting Wei

University of Leicester

Background

- The option pricing models that we will encounter in future sections are based on a form of modelling known as stochastic calculus. This is the approach used to model financial variables in **continuous time**.
- This can result, under certain conditions, in closed-form solutions that are therefore very **computationally efficient**.

Brownian motion

- A continuous time process that has proved useful for financial modelling purposes is Brownian motion. This was first developed as an idea in the physical sciences, with botanist Robert Brown noting the erratic motion of pollen particles in water in 1827.
- Norbert Wiener developed a mathematically rigorous construction of the stochastic process in a more abstract, general sense. You will sometimes therefore see it called the Wiener process.
- One way to think about this process is the continuous time equivalent of a binomial tree (i.e. a discrete time process, where at each time point the value can increase or decrease with equal probability).

Standard Brownian motion

$\{B_t,\ t \geqslant 0\}$ is called a standard Brownian motion (SBM), if

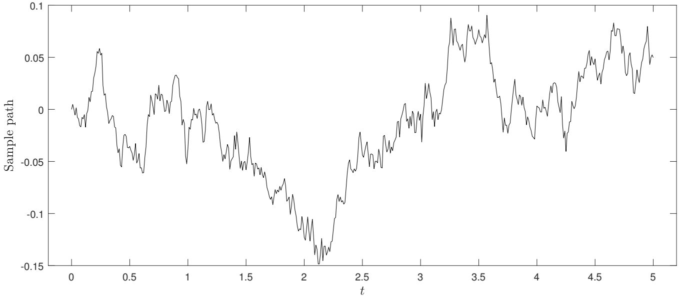
(1) $B_0 = 0$;

(2) $B_t - B_s \backsim N(0, t-s)$ for all $t > s$;

(3) B_t has independent increments such that $B_{t_1},\ B_{t_2} - B_{t_1},\ \cdots,\ B_{t_k} - B_{t_{k-1}}$ are independent for all $t_1 < t_2 < \cdots < t_{k-1} < t_k$;

Sample path of SBM

- An example of a sample path of an SBM is shown in the graph below,



- Notice how jagged the process is. In fact, it is possible to show that the sample path is nowhere differentiable: the limit for a mathematical derivative does not exist because the variable is not bounded, even on a short interval, so no unique tangent can be found.

Properties of SBM

- B_t has stationary increments: the distribution of $(B_t - B_s)$ depends only on $t - s$, where $t > s$.
- $\{B_t,\ t \geqslant 0\}$ is a Markov process.
- $\{B_t,\ t \geqslant 0\}$ returns infinitely often to any level, including 0.
- B_t has continuous sample paths: $t \to B_t$.

Properties of SBM, cont.

- $\mathbb{E}[B_t] = 0$.
- $\mathrm{Var}(B_t) = t$.
- $Cov(aB_t, bB_s) = ab\mathbb{E}(B_t B_s) = ab\min(t, s)$, where a and b are constants.
- $B_t - B_s$ has the same distribution as B_{t-s}, $t > s$. But they are different,
  - $B_t - B_s$ is the difference between two different random variables.
  - B_{t-s} is a single random variable.

Conditional expectation of SBM

The conditional expectation encountered in this course is no longer conditioning on a single random variable, but many (including a filtration, in the abstract sense).

Let $X_t = f(t, B_t)$ be a stochastic process, then we have

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[f(t, B_t)|\mathcal{F}_s] = \mathbb{E}[f(t, B_s + (B_t - B_s))|\mathcal{F}_s]$$

where $\{\mathcal{F}_s, 0 \leq s < t\}$ is a filtration of X_t.

Specifically, when $s = 0$,

$$\mathbb{E}[X_t|\mathcal{F}_0] = \mathbb{E}[X_t]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(s, B_s)|\mathcal{F}_s] = f(s, B_s)$,
- $\mathbb{E}[(B_t - B_s)^{2m+1}|\mathcal{F}_s] = 0,\ m = 0, 1, \ldots$
- $\mathbb{E}[(B_t - B_s)^{2m}|\mathcal{F}_s] = (t - s)^m(2m - 1)!!,\ m = 0, 1, \ldots,$
  where $(2m - 1)!! = 1 \times 3 \times 5 \times (2m - 1)$.
- If $f(t, B_t) = g(t)h(B_t)$, $\mathbb{E}[g(t)h(B_t)|\mathcal{F}_s] = g(t)\mathbb{E}[h(B_t)|\mathcal{F}_s]$
- $\mathbb{E}[g(t)|\mathcal{F}_s] = g(t)$ and $\mathrm{Var}[g(t)] = 0$.

for all $t > s \geqslant 0$.

Examples

- Show that B_t is a martingale.
- Show that $B_t^2 - t$ is a martingale.
- Is B_t^3 a martingale?
- Find $\mathbb{E}[t^2 B_t | \mathcal{F}_s]$, $t > s$.

Geometric Brownian Motion (preview)

- One of the shortcomings of standard Brownian motion, from the point of view of financial modelling, is that it allows negative values. It is not helpful when modelling equity prices, since these cannot be worth less than nothing.
- This can be addressed by assuming that the log of a stochastic process follows standard Brownian motion. The process itself will then be log-normally distributed.

Geometric Brownian Motion (preview)

- The asset price at time t can be modelled by

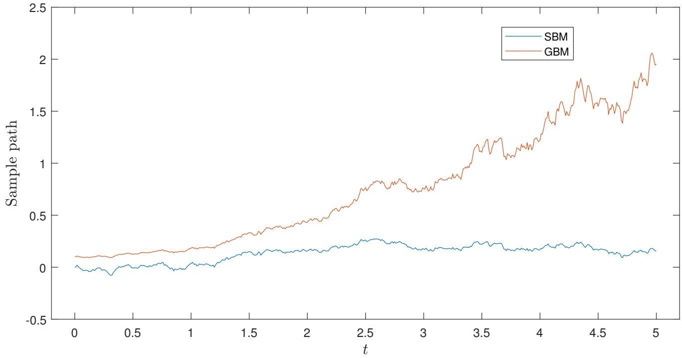
$$S_t = S_0 e^{at+\sigma B_t}$$

  where S_0 is the price at time 0, a is a constant and σ is the standard deviation/volatility of the return.

- This model is referred to as geometric Brownian motion (GBM) and is central to the development of many models in mathematical finance.

Sample path of GBM

- An example sample path, with a comparison to standard Brownian motion is shown below,



Stochastic calculus

- Since the sample paths of Brownian motion are not differentiable, that might suggest that the powerful applications that calculus can bring are not available.
- However, it is in fact possible to develop a definition of integrals of the form $\int_0^t Y_u dB_u$ for suitable random integrands Y_u (i.e. those that are adapted to the filtration of the Brownian motion).
- These stochastic integrals are also referred to as "Ito integrals", after the Japanese mathematician Kiyoshi Ito who first developed the idea in the 1940s.

Stochastic integrals

- The basic principle is that, while Brownian motion is "too erratic" to allow for a sensible definition of a mathematical derivative, there is enough regularity that it is possible to define a sequence of sums that will converge to a finite value under certain conditions. These can be considered an integral, in the same way as a standard integral is the limit of a sequence of Riemann sums.
- These stochastic integrals are themselves random variables, since B_t is random. We will therefore be interested in their distribution, expectation and variance.

Properties of stochastic integrals

(1) $\left(\int_0^t Y_u dB_u, t \geqslant 0\right)$ is a martingale.

(2) $\mathbb{E}[\int_0^t Y_u dB_u] = 0$.

(3) Ito isometry: $\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$.

(4) $\int_0^t Y_u dB_u$ follows a normal distribution with mean 0 and variance $\mathrm{Var}(\int_0^t Y_u dB_u) = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$, that is

$$\int_0^t Y_u dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t Y_u^2 du\right]\right)$$

Example

- State the distribution of the Ito integral $\int_0^t \sqrt{u} dB_u$.

Ito process

- The Ito process X_t is a stochastic process, which can be defined by the stochastic differential equation (SDE)

$$dX_t = A_t dt + Y_t dB_t$$

where A_t and Y_t are two expressions in terms of B_t and t.

- It is common to write the Ito process X_t in integral form

$$X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where X_0 is a constant and the initial value of X_t.

Ito process, cont.

- Note that $\int_0^t A_u du$ is deterministic and is called a deterministic integral. Therefore, the Ito process X_t has a deterministic mean and a random component, such that

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t A_u du\right] + \mathbb{E}\left[\int_0^t Y_u dB_u\right] \\
&= X_0 + \mathbb{E}\left[\int_0^t A_u du\right] \\
\mathrm{Var}[X_t] &= \mathrm{Var}[X_0] + \mathrm{Var}\left[\int_0^t A_u du\right] + \mathrm{Var}\left[\int_0^t Y_u dB_u\right] \\
&= \mathbb{E}\left[\int_0^t Y_u^2 du\right]
\end{aligned}$$

since the deterministic integral $\int_0^t A_u du$ is independent of the stochastic integral $\int_0^t Y_u dB_u$.

Properties of deterministic integral

(1) Fubini theorem: $\mathbb{E}\left[\int_0^t A_u du\right] = \int_0^t \mathbb{E}[A_u] du$

(2) $\mathrm{Var}\left[\int_0^t A_u du\right] = 0$

(3) If $X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$, then

$$X_t \sim N\left(X_0 + \int_0^t \mathbb{E}\left[A_u\right] du, \int_0^t \mathbb{E}\left[Y_u^2\right] du\right)$$

Example

- If $dX_t = -2tdt + 5\sqrt{t}dB_t$, state the distribution of the stochastic integral $\int_0^t \sqrt{u}dX_u$.

MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part II

Dr. Ting Wei

University of Leicester

Ito's lemma: core concept of stochastic calculus

Classical Ito's Lemma:

Let $f(t,x)$ be a twice partially differentiable function and $X_t = f(t, B_t)$ be a stochastic process. Then the stochastic differential $dX_t = df(t, B_t)$ is defined by the Ito formula

$$df(t, B_t) = f'_t dt + f'_{B_t} dB_t + \frac{1}{2} f''_{B_t B_t} dt$$

Ito rules: $(dB_t)^2 = dt$, $(dt)^2 = 0$, $dB_t dt = 0$

Remark

- Using the Ito's Lemma, we can write the stochastic process $X_t = f(t, B_t)$ in integral form

$$X_t = f(t, B_t) = f(0, B_0) + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where $A_u = f'_u + \frac{1}{2} f''_{B_u B_u}$ and $Y_u = f'_{B_u}$.

- And hence,

$$f(t, B_t) \sim N\left(f(0, B_0) + \int_0^t \mathbb{E}\left[f'_u + \frac{1}{2} f''_{B_u B_u} \right] du, \int_0^t \mathbb{E}\left[\left(f'_{B_u} \right)^2 \right] du \right)$$

Examples

- Find the stochastic differential $df(t, B_t)$ for the following stochastic processes
  - $f(t, B_t) = B_t^2$,
  - $f(t, B_t) = 7e^{tB_t}$.
- Find the integral forms of the stochastic processes B_t^2 and $7e^{tB_t}$.
- Compute $\mathbb{E}[B_t^2]$ using the Ito calculus.

Ito martingale

- The stochastic process $X_t = f(t, B_t)$ is a Ito martingale if

$$df(t, B_t) = Y_t dB_t$$

which has no term with dt (zero drift).

Examples

- Show that B_t is an Ito martingale.
- Show that $B_t^2 - t$ is an Ito martingale.
- Is B_t^3 an Ito martingale?

- For any stochastic process X_t, if it is a martingale, then zero drift (Ito martingale) is equivalent to the martingale definition $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leqslant s < t$.

Conditional expectation via Ito calculus

- Conditional Ito integral:

$$\mathbb{E}\left[\int_s^t Y_u dB_u \middle| \mathcal{F}_s\right] = 0$$

- Conditional Fubini theorem:

$$\mathbb{E}\left[\int_s^t A_u du \middle| \mathcal{F}_s\right] = \int_s^t \mathbb{E}[A_u|\mathcal{F}_s] du$$

where $0 \leqslant s < t$.

Examples

- Given $t > s \geqslant 0$, compute the following conditional expectations via the Ito calculus,
  - $\mathbb{E}[t^2 B_t | \mathcal{F}_s]$.
  - $\mathbb{E}[B_t^3 | \mathcal{F}_s]$.

Ito's Lemma, cont.

General Ito's Lemma:

Let $f(t,x)$ be a twice partially differentiable function and $f(t,X_t)$ be a stochastic process with $dX_t = A_t dt + Y_t dB_t$. Then, the general Ito's lemma states that

$$df(t,X_t) = f'_t dt + f'_{X_t} dX_t + \frac{1}{2} f''_{X_t X_t} (dX_t)^2$$

where Ito rules also applied,

$$(dX_t)^2 = Y_t^2 dt \quad \text{since} \quad (dB_t)^2 = dt,\ (dt)^2 = 0,\ dB_t dt = 0$$

Ito's Lemma, cont.

- For the convenience of calculation, one can also express the general Ito's lemma as follows,

$$df(t, X_t) = \left(f'_t + A_t f'_{X_t} + \frac{1}{2} Y_t^2 f''_{X_t X_t} \right) dt + Y_t f'_{X_t} dB_t$$

where A_t and Y_t are given in the SDE of dX_t.

Example

- Let X_t be defined by $dX_t = 2B_t dB_t$, find $d\left[t^3 X_t^5\right]$.

General Ito Isometry

- Given $\int_0^t Y_u dB_u$ and $\int_0^t Q_u dB_u$ are two stochastic integrals,

$$\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)\left(\int_0^t Q_u dB_u\right)\right] = \mathbb{E}\left[\int_0^t (Y_u Q_u) du\right]$$

Examples

- Let X_t be defined by $dX_t = 2B_t dB_t$ and $X_0 = 0$, find $\mathbb{E}\left[B_t X_t\right]$ and $\mathbb{E}\left[X_t^2\right]$.
- Find $\mathbb{E}[(B_t^3 - 3tB_t + 1)^2]$ by applying the Ito isometry.

Ito's Lemma versus GBM

Let $\{B_t,\ t \geqslant 0\}$ be a standard Brownian motion. Consider a continuous-time market with one risk-free bond offering a fixed interest rate of ρ and one risky asset. The asset price is modelled by a stochastic process $\{S_t,\ t \geqslant 0\}$, which is defined as a solution to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Ito's Lemma versus GBM, cont.

I) $\{S_t,\ t \geqslant 0\}$ is a geometric Brownian motion with mean parameter $a = \mu - \frac{\sigma^2}{2}$ and volatility parameter σ.

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

II) Moreover, $e^{-\rho t} S_t$ is a martingale if and only if $\mu = \rho$.

III) The discounted financial derivative $e^{-\rho t} g(t, S_t)$ is a Martingale iff the Black-Scholes equation holds:

$$g'_t + \mu S_t g'_{S_t} + \frac{1}{2}\sigma^2 {S_t}^2 g''_{S_t S_t} = \rho g$$

GBM, cont.

- S_t follows a log-normal distribution, such that

$$\log\left(\frac{S_t}{S_0}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

- GBM has the following expected value and variance:
  - $\mathbb{E}[S_t] = S_0 e^{\mu t}$
  - $Var[S_t] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Conditional expectation of GBM

Let $S_t = S_0 e^{at+\sigma B_t}$ be a GBM, then we have

$$\begin{aligned}\mathbb{E}[f(S_T)|\mathcal{F}_t] &= \mathbb{E}\left[f\left(S_t \cdot \frac{S_T}{S_t}\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\Big|\mathcal{F}_t\right]\end{aligned}$$

where $\{\mathcal{F}_t, 0 \leq t < T\}$ is a filtration of S_t.

Specifically, when $t = 0$,

$$\mathbb{E}[f(S_T)|\mathcal{F}_0] = \mathbb{E}[f(S_T)]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(S_t)|\mathcal{F}_t] = f(S_t)$,
- $\mathbb{E}[S_T|\mathcal{F}_t] = S_t e^{\mu(T-t)}$
- $\mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle|\mathcal{F}_t\right] = e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}$, where k is a constant.
- $S_t \perp e^{a(T-t)+\sigma(B_T-B_t)}$ since $B_t \perp B_T - B_t$.

for all $T > t \geqslant 0$.

Examples

- Find $\mathbb{E}[6S_T^3|\mathcal{F}_t]$ with $S_t = S_0 e^{2t+B_t}$.
- Find $\mathbb{E}[\log(S_T)|\mathcal{F}_t]$ with $S_t = S_0 e^{at+\sigma B_t}$.

Multivariate Ito's Lemma

Multivariate Ito's Lemma:

Let $f(x_1, \cdots, x_k)$ be a twice partially differentiable function and $f\left(X_t^{(1)}, \cdots, X_t^{(k)}\right)$ be a stochastic process with $dX_t^{(i)} = A_t^{(i)} dt + Y_t^{(i)} dB_t,\ i = 1, \cdots, k.$ Then, the multivariate Ito's lemma states that

$$df\left(X_t^{(1)}, \cdots, X_t^{(k)}\right) = \sum_{i=1}^{k} f'_{X_t^{(i)}} dX_t^{(i)} + \frac{1}{2}\sum_{i=1}^{k} f''_{X_t^{(i)} X_t^{(i)}} \left(dX_t^{(i)}\right)^2 + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f''_{X_t^{(i)} X_t^{(j)}} dX_t^{(i)} dX_t^{(j)}$$

where $\left(dX_t^{(i)}\right)^2 = \left(Y_t^{(i)}\right)^2 dt$ and $dX_t^{(i)} dX_t^{(j)} = Y_t^{(i)} Y_t^{(j)} dt.$

Bivariate Ito's Lemma

In particular, when $k = 2$, $dX_t = A_t dt + Y_t dB_t$ and $dZ_t = G_t dt + Q_t dB_t$. Then, the bivariate Ito formula is

$$\begin{aligned} df(X_t, Z_t) = f'_{X_t} dX_t + f'_{Z_t} dZ_t + \frac{1}{2} f''_{X_t X_t} (dX_t)^2 \\ + \frac{1}{2} f''_{Z_t Z_t} (dZ_t)^2 + f''_{X_t Z_t} dX_t dZ_t \end{aligned}$$

And in the integral form,

$$\begin{aligned} f(X_t, Z_t) = f(X_0, Z_0) + \int_0^t f'_{X_s} dX_s + \int_0^t f'_{Z_s} dZ_s \\ + \frac{1}{2} \int_0^t f''_{X_s X_s} (dX_s)^2 + \frac{1}{2} \int_0^t f''_{Z_s Z_s} (dZ_s)^2 + \int_0^t f''_{X_s Z_s} dX_s dZ_s \end{aligned}$$

Examples

- Find $d[M_t S_t]$ through the bivariate Ito formula.
- If $dM_t = B_t^4 dB_t$ and $dS_t = B_t^2 dB_t$, $M_0 = S_0 = 0$, find $\mathbb{E}[M_t S_t]$.
- If $dM_t = (1 + t^2 B_t) dB_t$ and $M_0 = 0$, let $Y_t = t B_t M_t$. Is Y_t a martingale?