MA2252 Introduction to Computing

Lecture 16
Root finding

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Learning outcomes

At the end of lecture, students will be able to

- understand root-finding methods
- implement these methods in MATLAB

Introduction

Finding roots of functions is something we've been doing since school.

A root of a function $f(x)$ is the value of x for which $f(x) = 0$.

zero

Example: The roots of $f(x) = x^2 - 3x + 2$ are 1 and 2.

$f(1) = 1^2 - 3 + 2 = 3 - 3 = 0$

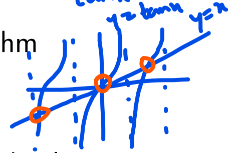
$f(2) = 0$

Introduction (contd.)

There are many ways to find the roots of a function:

- Try a guess value! → not a good idea because so many guess values
- Use a mathematical formula → e.g. quadratic formula
- Sketch the graph of the function
- Apply a root-finding method/algorithm

tan x = x



The last approach is what we'll study in this lecture.

Sometimes, finding roots is not that easy! For example, how can we find roots of transcendental equations?

$e^0 - 0 = 1$

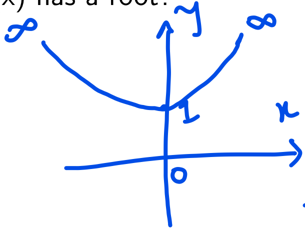
$e^x - x = 0$

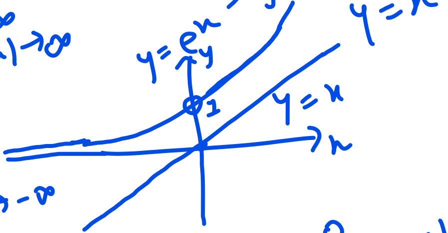
$\Rightarrow y = e^x$ & $y = x$

Consider the function $f(x) = e^x - x$.

$x \to \infty$
$f(x) \to \infty$

Does f(x) has a root?





$x \to -\infty$

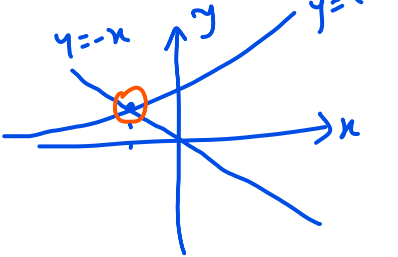
$f'(x) = e^x - 1 = 0 \Rightarrow x = 0$

$f''(x) = e^x \Rightarrow f''(0) = 1 > 0$

Now consider the function $f(x) = e^x + x$. Does f(x) has a root?

$$e^x + x = 0$$

$$e^x = -x$$



Introduction (contd.)

Some root-finding methods:

- Bisection Method
- Newton-Raphson Method → same as Newton's method

Activity

Let f(x) be a function defined in the interval [a,b]. Suppose f(a)f(b)<0. Choose the correct statement.

f(a) & f(b) are of different signs

1. f(x) always has a root in (a,b).
2. f(x) has exactly one root in (a,b) if f is continuous on [a,b].
3. f(x) has at least one root in (a,b) if f is continuous on [a,b].
4. f(x) doesn't have a root in (a,b).

To answer, please go to the mentimeter link provided in the chat.

Steps:

$[a, b]$ such that $r \in [a, b]$

1. Choose a guess interval which may contain the root.

$a \quad m \quad b$

2. Approximate the root by the mid-point of this interval.

$m = \frac{a+b}{2}$ $\quad (a, m)$ or (m, b)

3. Bisect the subsequent sub-intervals containing the root until the error is less than tolerance.

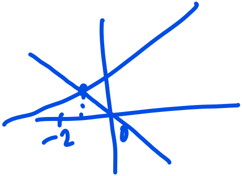
tolerance is upper bound on your error.

4. The root is given by the midpoint of the last bisected interval.

$|f(x_i)| < tol$

$tol = 0.1, 0.01, 0.001$ $\quad f(x_i) \approx 0$

Bisection Method (contd.)



Consider again the function $f(x) = e^x + x$. Find its root using bisection method.

$a = -2, \; b = 0$

Write a function in MATLAB which takes the values of end points a and b of the guess interval, the function f, the tolerance tol and outputs the root of function f.

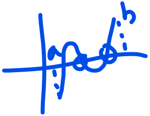
Homework: write bisection method function using while loop without recursion.

Demo

Bisection Method (contd.)

Limitations of bisection method:

- Requires knowledge of interval containing the root
- Cannot detect multiple roots → because (a) starting condition is $f(a) \cdot f(b) < 0$ (b) can converge to any root
- Fails if the function is discontinuous on the interval [a,b]



Isaac Newton Joseph Raphson

Simpson extended this method for non-linear eq^ns

Steps:

1. Choose a guess value x_0 for the root.
2. Find a linear approximation of f(x) around $x = x_0$.

   1st order Taylor polynomial

   $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ → straight line

3. Find the root (say x_1) of this linear approximation. The obtained x_1 is an improvement on the guess value x_0.

   $f(x) = f(x_0)$

4. Repeat the steps 2 and 3 to find improved guess values x_i $(i = 2, 3, ..., n)$ until the error is less than tolerance.

Newton-Raphson Method (contd.)

The iterative formula for Newton-Raphson method is derived as

$$x_i = x_{i-1} - \frac{f(x_i)}{f'(x_{i-1})} \tag{1}$$

Newton-Raphson Method (contd.)

Find the root of the function $f(x) = e^x + x$ using Newton-Raphson method.

Newton-Raphson Method (contd.)

Write a function in MATLAB which takes the guess value x0, the function f and its derivative df, the tolerance tol and outputs the root of function f.

Demo

Newton-Raphson Method (contd.)

Limitations of Newton-Raphson method:

- Fails if $f'(x_i) = 0$ for some x_i.
  Example: $f(x) = x^3 - x^2 - x - 1$ with $x_0 = 1$.

- Fails if $f'(x_i)$ gets closer to zero for successive x_i values.
  Example: Consider $f(x) = \frac{1}{x} - e^x$ for $x_0 < 0$.

- In case of multiple roots, a guess values may converge to a different root than the one which is required.
  Example: $f(x) = \tan^{-1} x - x^2$ has two roots but for $x_0 < 0$, the method only gives the root $x = 0$.

End of Lecture 16

Please provide your feedback here