MA1202
All candidates

Semester 2 Examinations 2019

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
Department	MATHEMATICS
Module Code	MA1202
Module Title	INTRODUCTORY STATISTICS
Exam Duration	Two hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	4
Number of Questions	3
Instructions to Candidates	Answer all questions. All marks gained will be counted. All questions carry equal weight.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	Statistical tables.
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	No

1. (a) Let $\hat{\theta}$ be an estimator of an unknown parameter θ.

   i. Define the Bias of $\hat{\theta}$, $\mathrm{Bias}(\hat{\theta})$; [2 marks]

   ii. Define the mean squared error of $\hat{\theta}$, $\mathrm{MSE}(\hat{\theta})$; [2 marks]

   (b) Let $X_1, \ldots, X_n$ be an independent random sample from a population X with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Let

$$\bar{X} = \frac{1}{n}(X_1 + \ldots + X_n)$$

   be an estimator of μ

   i. Calculate the Bias of $\bar{X}$, $\mathrm{Bias}(\bar{X})$; [4 marks]

   ii. Calculate the mean squared error of $\bar{X}$, $\mathrm{MSE}(\bar{X})$. [6 marks]

   (c) A continuous random variable Y has density function

$$f_Y(y) = \begin{cases} 2\alpha e^{-\alpha y}, & y > 0, \\ 0, & y \leq 0, \end{cases}$$

   where $\alpha > 0$ is an unknown parameter to be estimated. Suppose $y_1, \ldots, y_n$ are observations of independent sample random variables $Y_1, \ldots, Y_n$, respectively, all with the same distribution as Y.

   i. Find the maximum likelihood estimate for α. [8 marks]

   ii. Find the maximum likelihood estimator for α. [3 marks]

Total: 25 marks

2. (a) Suppose that $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$. Let $X_1, \ldots, X_5$ be an independent random sample from X, with sample variance S_x^2 and $Y_1, \ldots, Y_9$ be an independent random sample from Y, with sample variance S_y^2. Assume that X_i and Y_j are mutually independent for all $i = 1, ..., 5$ and $j = 1, ..., 9$.

   i. Using the results $\frac{4S_x^2}{\sigma_x^2} \sim \chi_4^2$ and $\frac{8S_y^2}{\sigma_y^2} \sim \chi_8^2$ show that

$$\frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2} \sim F_{4,8}.$$

[3 marks]

   ii. Find the critical value b such that $P(F_{4,8} > b) = 0.05$. [2 marks]

   iii. Find the critical value a such that $P(F_{4,8} > a) = 0.95$. [3 marks]

   iv. Hence, if $S_x^2 = 10$ and $S_y^2 = 5$, compute a 90% confidence interval for σ_x^2/σ_y^2. [5 marks]

(b) If there is no seasonal effect on human births, we would expect equal numbers of children to be born in each of the four seasons (winter, spring, summer and fall). A student took a survey from his 1st year class and found that, of the 120 students in the class, 25 were born in winter, 35 in spring, 32 in summer, and 28 in fall. He wondered if the excess in the spring was an indication that births were not uniform throughout the year.

   i. What is the expected number of births in each season if there is no seasonal effect on birth? [2 marks]

   ii. Compute the χ^2 statistic for the χ^2 goodness of fit test. [4 marks]

   iii. How many degrees of freedom does the χ^2 statistic have? [1 mark]

   iv. Perform a χ^2 goodness of fit test of the hypothesis that there is no seasonal effect on human births at the $\alpha = 0.05$ significance level. What do you conclude? [5 marks]

3. (a) Outline the steps in carrying out a statistical hypothesis test. [5 marks]

(b) Let X_1 and X_2 be two random variables with distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Assuming that X_1 and X_2 are independent then $X_1 + X_2 \sim N(\mu, \sigma^2)$.

i. What is μ in terms of μ_1 and μ_2? [1 mark]

ii. What is σ^2 in terms of σ_1^2 and σ_2^2? [1 mark]

(c) An animal park has two types of giraffe: the reticulated giraffe and the Masai giraffe. The height of adult female reticulated giraffes $X_r \sim N(\mu_r, \sigma_r^2)$ with mean $\mu_r = 438$cm, but unknown variance σ_r^2.

i. Suppose that there are n female reticulated giraffes in the park, with sample mean $\bar{X}_r$ and sample variance S_r^2, what distribution does the following statistic have?

$$\frac{\bar{X}_r - \mu_r}{S_r/\sqrt{n}}.$$

[2 marks]

ii. The heights of 9 adult female Masai giraffes in the park are measured and recorded below:

435cm, 440cm, 450cm, 425cm, 460cm, 465cm, 455cm, 425cm, 450cm

Calculate the sample mean $\bar{x}_m$ and the sample variance s_m^2 of this dataset.

[4 marks]

iii. Assuming the height of adult female Masai giraffes is also normally distributed, with mean μ_m and variance $\sigma_m^2 = \sigma_r^2$, perform a hypothesis test at the 5% significance level with the null hypothesis $\mu_m = \mu_r$ and the alternative hypothesis $\mu_m > \mu_r$. What is the conclusion?

[6 marks]

iv. A new collection of 6 female Cape giraffes is brought to the wildlife park. The heights of these giraffe are measured and the mean height and sample variance are found to be $\bar{x}_c = 455$cm and $s_c^2 = 200$cm, respectively. Using the same sample data as in part (ii), perform a suitable hypothesis test at the 5% significance level to test the null hypothesis $\sigma_c^2 = \sigma_m^2$ against the alternative hypothesis $\sigma_c^2 < \sigma_m^2$, where σ_c^2 is the variance of the height of Cape giraffes. What is the conclusion?

[6 marks]

Total: 25 marks