# LINEAR ALGEBRA II

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# 线性代数II（B.YU）

Ch. IX Polynomials and Matrices

## §1. POLYNOMIALS

- By a polynomial over $K$, we shall mean a formal expression

$$f(t) = \boxed{a_n t^n} + \cdots + \boxed{a_0}$$

$a_i t^i$

- Coefficients
- Degree

  If $a_n \neq 0$. $n = \deg(f)$

- Degree of zero polynomial $\deg(0) = -\infty$
- The leading coefficient : $a_n$
- The constant term $a_0$
- $K[t] = \{$ all polynomials over $K \}$

# §1. POLYNOMIALS

$(f+g)(t) = \sum_{k=0}^{\max\{m,n\}} (a_k+b_k)t^k$ $\left(\begin{array}{l} a_k=0,\ k>n \\ b_k=0\ \ k>m \end{array}\right)$

$n>m$

- **Theorem 1.1.** *Let $f$, $g$ be polynomials with coefficients in $K$. Then*

$$\deg(fg) = \deg f + \deg g.$$

$\deg(f\cdot g) = m+n$

$f(t) = \boxed{a_n t^n} + \cdots + a_0$, $g(t) = \boxed{b_m t^m} + \cdots + b_0$

$(fg)(t) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i t^i b_j t^j = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j t^{i+j} = \sum_{k=0}^{m+n} c_k t^k$, $\boxed{c_k = \sum_{i=0}^{k} a_i b_{k-i}}$ $\quad c_{m+n} = a_n \cdot b_m$

- **Theorem 1.2.** *Let $f$ be a polynomial with complex coefficients, of degree $\geqq 1$. Then $f$ has a root in* $\mathbf{C}$.

$c \neq 0$ $(cf)(t) = \sum_{i=1}^{n} c a_i t^i$, $\deg(cf) = \deg(f)$

- **Theorem 1.3.** *Let $f$ be a polynomial with complex coefficients, leading coefficient $1$, and $\deg f = n \geqq 1$. Then there exist complex numbers $\alpha_1, \dots, \alpha_n$ such that*

monic polynomial

$$f(t) = (t-\alpha_1)\cdots(t-\alpha_n).$$

$f(\alpha_i) = 0$

*The numbers $\alpha_1, \dots, \alpha_n$ are uniquely determined up to a permutation. Every root $\alpha$ of $f$ is equal to some $\alpha_i$, and conversely.*

- $f(t) = (t-\alpha_1)^{m_1} \cdots (t-\alpha_r)^{m_r}$, multiplicity

$\alpha_i \neq \alpha_j$ $(i \neq j)$

# §2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

operators

an operator of V

- Let $A$ be a square matrix with coefficients in $K$. Let $f \in K[t]$, and write

$$f(t) = a_n t^n + \cdots + a_0$$

with $a_i \in K$. We define

$$f(A) = a_n A^n + \cdots + a_0 I.$$

- **Example 1.** *Let* $f(t) = 3t^2 - 2t + 5$. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. Then

$$f(A) = 3\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^2 - \begin{pmatrix} 2 & -2 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}.$$

# §2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

an operator of V (over K)

**Theorem 2.1.** *Let $f, g \in K[t]$. Let $A$ be a square matrix with coefficients in $K$. Then*

$$(f+g)(A) = f(A) + g(A),$$

$$(fg)(A) = f(A)g(A).$$

$f(t) = a_n t^n + \cdots + a_0$

$g(t) = b_m t^m + \cdots + b_0$

*If $c \in K$, then $(cf)(A) = cf(A)$.*

Proof. $(f+g)(A) = \sum_{k=0}^{\max\{m,n\}} (a_k + b_k) A^k$ $\quad \left( \begin{array}{ll} a_k = 0 & \text{if } k > n \\ b_k = 0 & \text{if } k > m \end{array} \right)$

$f(A) + g(A) = (a_n A^n + \cdots + a_0 I) + (b_m A^m + \cdots + b_0 I) = \sum_{k=0}^{\max\{m,n\}} (a_k + b_k) A^k$

$(cf)(A) = \sum_{k=0}^{n} c a_k A^k$, $\quad c f(A) = c \cdot \sum_{k=0}^{n} a_k A^k = \sum_{k=0}^{n} c a_k A^k$

$(fg)(A) = \sum_{k=0}^{m+n} c_k A^k$ $\quad \left( c_k = \sum_{i=0}^{k} a_i b_{k-i} \right)$

$f(A) \cdot g(A) = \left( \sum_{i=0}^{n} a_i A^i \right) \left( \sum_{j=0}^{m} b_j A^j \right) = \sum_{k=0}^{m+n} c_k A^k$

# §2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

- **Example 2.** Let $f(t) = (t - 1)(t + 3) = t^2 + 2t - 3$. Then

$$f(A) = A^2 + 2A - 3I = (A - I)(A + 3I).$$

- **Example 3.** Let $\alpha_1, \ldots, \alpha_n$ be numbers. Let

$$f(t) = (t - \alpha_1) \cdots (t - \alpha_n).$$

Then

$$f(A) = (A - \alpha_1 I) \cdots (A - \alpha_n I).$$

## §2. POLYNOMIALS OF MATRICES AND LINEAR MAPS

operator of $n$-dimensional Vector space $V$

- **Theorem 2.2.** *Let $A$ be an $n \times n$ matrix in a field $K$. Then there exists a non-zero polynomial $f \in K[t]$ such that $f(A) = O$.*

Proof. Consider $I, A, A^2, \cdots, A^{n^2} \in \mathrm{Mat}_{n\times n}(K) \to n^2$-dimensional V. S.

They must be L. I., so there exist $c_0, c_1, \cdots, c_{n^2} \in K$, s.t.

$$c_0 I + c_1 A + \cdots + c_{n^2} A^{n^2} = O$$

Let $f(t) = c_{n^2} t^{n^2} + \cdots + c_1 t + c_0$.

We have $f(A) = O$

$\mathcal{L}(V, V)$ is $n^2$-dimensional

$\cong \mathrm{Mat}_{n\times n}(K)$

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- Homework:
  - P236: 3, 5,