Lecture 2: Vectors and the Geometry of Space.

MA2032 Vector Calculus

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Vectors

In this section we show how to represent things that have both **magnitude and direction** in the plane or in space.

Definition:

The **vector** represented by the directed line segment $\overrightarrow{AB}$ has **initial point A** and **terminal point B** and **its length** is denoted by $|\overrightarrow{AB}|$. Two **vectors are equal** if they have the same length and direction, regardless of the initial point.

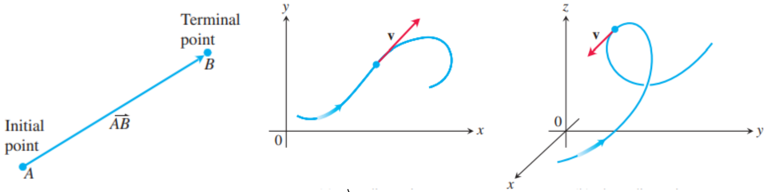


Figure 4: (left) The directed line segment $\overrightarrow{AB}$ is called a vector; (middle/right) Velocity vector v at a specific location for a particle moving along a path in the plane/space.

Component Form of Vectors

Definition:

If v is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin (vector in **standard position**) and terminal point (v_1, v_2), then the **component form** of v is $v = \langle v_1, v_2 \rangle$.

If v is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of v is $v = \langle v_1, v_2, v_3 \rangle$.

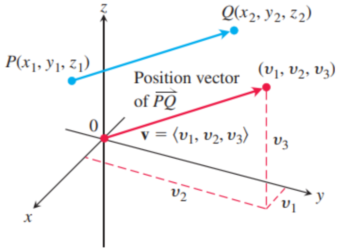


Figure 5: A vector $\overrightarrow{\mathrm{PQ}}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{\mathrm{PQ}}$ and v are parallel and have the same length.

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $v = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{\mathrm{PQ}}$ is $v = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Component Form of Vectors

Two vectors are **equal** if and only if their standard position vectors are identical. Thus $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

Definition:

The **magnitude** or **length** of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

$$||v|| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the **zero vector** $0 = \langle 0, 0 \rangle$ or $0 = \langle 0, 0, 0 \rangle$. This vector is also the **only vector with no specific direction**.

Example

Find the (a) component form and (b) length of the vector with initial point P(-3, 4, 1) and terminal point Q(-5, 2, 2).

Solution:

(a) The standard position vector v representing $\overrightarrow{\mathrm{PQ}}$ has components
$v_1 = x_2 - x_1 = -5 - (-3) = -2,\ \ v_2 = y_2 - y_1 = 2 - 4 = -2,$
and $v_3 = z_2 - z_1 = 2 - 1 = 1.$
The component form of $\overrightarrow{\mathrm{PQ}}$ is $v = \langle -2, -2, 1 \rangle$.

(b) The length or magnitude of $v = \overrightarrow{\mathrm{PQ}}$ is
$||v|| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$

Vector Algebra Operations

Definitions:

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ be vectors with κ a scalar.

Addition: $u + v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
Scalar multiplication: $\kappa u = \langle \kappa u_1, \kappa u_2, \kappa u_3 \rangle$
Difference: $u - v = \langle u_1 + (-1)v_1, u_2 + (-1)v_2, u_3 + (-1)v_3 \rangle$

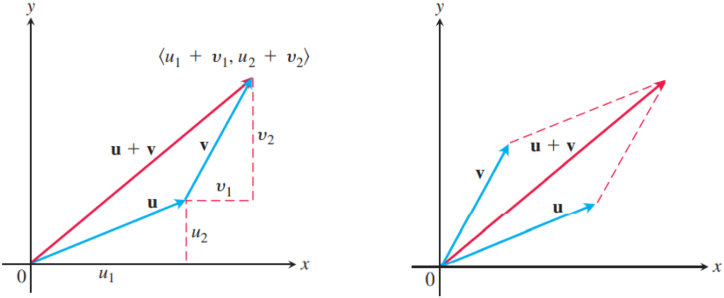


Figure 6: (Left) **Geometric interpretation** of the vector sum. (Right) The **parallelogram law** of vector addition in which both vectors are in standard position.

Vector Algebra Operations

Vector operations have many of the properties of ordinary arithmetic.

Properties of Vector Operations

Let u, v, w be vectors and a, b be scalars.

1. u + v = v + u
2. (u + v) + w = u + (v + w)
3. u + 0 = u
4. u + (-u) = 0
5. $0 \times u = 0$
6. $1 \times u = u$
7. $a \times (bu) = (ab) \times u$
8. $a \times (u + v) = au + av$
9. $(a + b) \times u = au + bu$

When three or more space vectors lie in the same plane, we say they are **coplanar vectors**.
For **example**, the vectors u, v, and u + v are always coplanar.

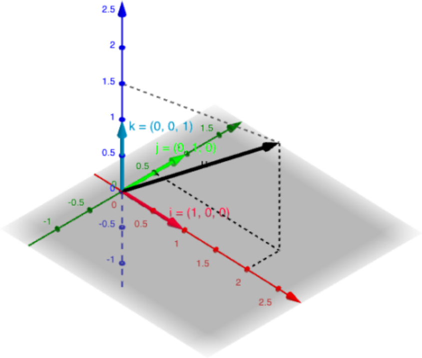
Unit Vectors

A vector v of length 1 is called a **unit vector**.

The **standard unit vectors** are $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$.

Any vector $v = \langle v_1, v_2, v_3 \rangle$ can be written as a **linear combination** of the standard unit vectors as follows:

$$v = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle = v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle = v_1 i + v_2 j + v_3 k.$$



Unit Vectors

If $v \neq 0$, then its length $||v||$ is not zero and $\left|\left|\frac{1}{||v||}v\right|\right| = \frac{1}{||v||}||v|| = 1$.

That is, $\frac{v}{||v||}$ is a unit vector in the direction of v, called the **direction** of the nonzero vector v.

We can express any nonzero vector v in terms of its two important features, **length** and **direction**, by writing $v = ||v||\frac{v}{||v||}$.

Example: If $v = 3i - 4j$ is a velocity vector, express v as a product of its speed times its direction of motion.

Solution: Speed is the magnitude (length) of v:
$||v|| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5$.
The unit vector $v/||v||$ is the direction of v:
$\frac{v}{||v||} = \frac{3}{5}i - \frac{4}{5}j$
So $v = 3i - 4j = 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$.

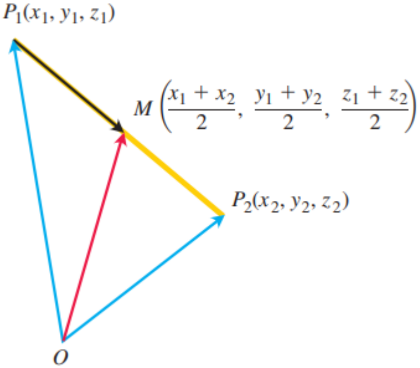
Midpoint of a Line Segment

The midpoint M of the line segment

joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

To see why, observe Figure at the right that $\overrightarrow{\mathrm{OM}} = \overrightarrow{\mathrm{OP}_1} + \frac{1}{2}(\overrightarrow{\mathrm{P}_1\mathrm{P}_2}) = \overrightarrow{\mathrm{OP}_1} + \frac{1}{2}(\overrightarrow{\mathrm{OP}_2} - \overrightarrow{\mathrm{OP}_1}) = \frac{1}{2}(\overrightarrow{\mathrm{OP}_1} + \overrightarrow{\mathrm{OP}_2}) = \frac{x_1+x_2}{2}i + \frac{y_1+y_2}{2}j + \frac{z_1+z_2}{2}k$.



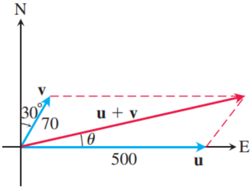
An important application of vectors occurs in navigation.

Example

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new **ground speed** and **direction**. What are they?

Solution:

If u is the velocity of the airplane alone and v is the velocity of the tailwind, then $|u| = 500$ and $|v| = 70$. The velocity of the airplane with respect to the ground is given by the magnitude and direction of the resultant vector u + v.



Solution

If we let the positive x-axis represent east and the positive y-axis represent north, then the component forms of u and v are $u = \langle 500, 0 \rangle$ and $v = \langle 70\cos 60^\circ, 70\sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle$.

Therefore,

$$u + v = \langle 535, 35\sqrt{3} \rangle = 535i + 35\sqrt{3}j$$
$$|u + v| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ$.
The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east.