Chapter 2
Integral Sliding Mode Control

Variable Structure Control Systems (VSCS) are a class of systems where the control law, as a function of the system state, is deliberately changed (from one structure to another) according to some predefined rules: for example a relay system. During a sliding mode the closed-loop system response is constrained to evolve along a sliding surface in the state-space to an equilibrium point. In sliding mode schemes, a switching function typically dictates which structure of control law is to be used at a particular time instant, depending on the position of the state from the sliding surface. The set of points for which the switching function is zero is called the sliding surface. SMC has now become an established tool to design controllers for uncertain systems, and provides robustness properties against matched uncertainties i.e. uncertainties that affect the plant dynamics acting in the input channels. However this robustness against external disturbances and parameter variations matched to the control can only be achieved after the occurrence of the sliding mode. Before the occurrence of the sliding mode i.e. during the so-called reaching phase, the system is affected by external disturbances—even matched ones. In order to eliminate the reaching phase and to ensure robustness throughout the entire closed-loop system response (i.e. to enforce a sliding mode for all time) the idea of Integral Sliding Modes (ISM) was proposed. In this chapter a step-by-step design procedure is described for the synthesis of sliding mode controllers; then these ideas are extended to integral sliding modes in order to ensure robustness throughout the entire system response. Necessary conditions for the existence of sliding modes are also given. The properties of the system while in the sliding mode are also explained, and are examined through simulations.

2.1 Introduction

SMC is a useful robust technique to handle sudden and large changes in the system dynamics and has been applied to many areas—for example motor control, aircraft and spacecraft control, process control and power systems. The realisation of a sliding

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mode controller comprises two steps. The first step is to design a sliding (switching) surface on which the sliding motion will take place. The second step is to design a control law, which depends on the choice of switching function and forces the system state trajectories to reach and slide on the surface. An important condition in the sliding mode literature is the reachability condition, which guarantees the existence of the sliding mode. Once sliding is achieved and maintained, robustness against matched uncertainties is guaranteed. Details of the design procedures are given in the next sections.

2.2 Problem Statement and Equivalent Control

In order to explain the design procedure for a system where *full state information* is available, consider an uncertain linear time invariant (LTI) system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + M\xi(t, x) \tag{2.1}$$

where $A \in \mathrm{I\!R}^{n\times n}$, $B \in \mathrm{I\!R}^{n\times m}$.

Assumption 2.1 It is assumed that the matrix B has full rank i.e. $\text{rank}(B) = m$, where $1 \leq m < n$ and the pair (A, B) is controllable.

Assumption 2.2 The matrix $M \in \mathrm{I\!R}^{n\times l}$ is assumed to be known and lies in the range space of the input distribution matrix B i.e. $\mathscr{R}(M) \subset \mathscr{R}(B)$, therefore it is possible to write $M = BD$ for some $D \in \mathrm{I\!R}^{m\times l}$.

The function $\xi(t, x)$ represents an external disturbance or models uncertainty which is unknown but has a known upper bound for all x and t. Therefore the uncertain system in (2.1) can be rewritten as

$$\dot{x} = Ax(t) + Bu(t) + BD\xi(t, x) \tag{2.2}$$

Uncertainty of the form in (2.2), acting in the channel of the input distribution matrix, is referred to as matched uncertainty. As a first step, define a sliding surface as

$$\mathscr{S} = \{x \in \mathrm{I\!R}^n \quad : \quad \sigma(t) = 0\} \tag{2.3}$$

where $\sigma(t)$ is a linear switching function defined as

$$\sigma(t) = Gx(t) \tag{2.4}$$

where $G \in \mathrm{I\!R}^{m\times n}$ is a design matrix and is of full rank. Furthermore by design it is assumed that the square matrix GB is nonsingular i.e. $\det(GB) \neq 0$. It is important that the sliding motion on the sliding surface should be stable and robust against the uncertainty $\xi(t, x)$. Therefore in order to analyse the sliding motion associated with the sliding surface in (2.3), consider the time derivative of (2.4) given by

$$\dot{\sigma}(t) = G\dot{x}(t) \tag{2.5}$$

Substituting the open-loop system equations from (2.2) into (2.5) gives

$$\dot{\sigma}(t) = G\left(Ax(t) + Bu(t) + BD\xi(t, x)\right) \tag{2.6}$$

It is assumed that the system states are forced to reach the sliding surface at time t_s say, so that for all $t \geq t_s$ an ideal sliding motion can be obtained i.e.

$$\sigma(t) = \dot{\sigma}(t) = 0 \quad \text{for all } t \geq t_s$$

The control signal $u(t)$ such that the time derivative $\dot{\sigma}(t)$ along the state trajectories is equal to zero can be obtained by equating Eq. (2.6) to zero which yields

$$u_{eq}(t) = -(GB)^{-1}\left(GAx(t) + GBD\xi(t, x)\right) \quad \text{for } t \geq t_s \tag{2.7}$$

where the square matrix GB is nonsingular by design. The expression $u_{eq}(t)$ in (2.7) is termed the equivalent control and can be thought of as the average value which the control signal must take to maintain the sliding motion on the sliding surface. However, it is not the control law that is applied to the system to induce the sliding mode. In order to obtain an expression for the sliding motion (i.e. the motion while the system is in the sliding mode), substituting the value of $u_{eq}(t)$ from (2.7) into (2.2), yields

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\left(-(GB)^{-1}(GAx(t) + GBD\xi(t, x))\right) + BD\xi(t, x) \\ &= \underbrace{(I_n - B(GB)^{-1}G)}_{\Gamma}Ax(t) + (I_n - B(GB)^{-1}G)BD\xi(t, x) \end{aligned} \tag{2.8}$$

Note that the projection operator Γ has the property that

$$\Gamma B = 0 \tag{2.9}$$

As a result, Eq. (2.8) reduces to

$$\dot{x}(t) = \Gamma Ax(t) \quad \text{for } t \geq t_s \tag{2.10}$$

From (2.10), it is clear that the effect of the uncertainty $\xi(t, x)$ during the sliding mode is completely rejected i.e. the reduced order system motion is insensitive to matched uncertainties. Also the stability of the sliding motion depends on the choice of sliding surface, i.e. the choice of switching matrix G.

2.2.1 Sliding Mode Control Laws

The second step is to design a control law such that the sliding motion on the surface $\mathscr{S}$ is guaranteed in finite time. A sliding mode controller for a system of the form in (2.1) typically consists of two parts; a linear part and a nonlinear part so that

$$u(t) = u_l(t) + u_n(t) \tag{2.11}$$

where the nonlinear part contains a discontinuous component and is responsible for inducing a sliding motion on $\mathscr{S}$, whereas the linear part, which is normally the nominal equivalent control, is responsible for helping to maintain sliding. Specifically

$$u(t) = -(GB)^{-1}GAx(t) - \rho(t,x)(GB)^{-1}\frac{\sigma(t)}{\|\sigma(t)\|} \quad \text{for } \sigma(t) \neq 0 \tag{2.12}$$

where $\frac{\sigma(t)}{\|\sigma(t)\|}$ is the unit vector component and $\rho(t,x)$ is a scalar gain chosen large enough (i.e. greater than the size of the uncertainty present in the system) to enforce the sliding motion.

Remark 2.1. For single input systems, the sliding mode controller in (2.12) becomes

$$u(t) = -(GB)^{-1}GAx(t) - \rho(t,x)(GB)^{-1}\mathrm{sign}(\sigma(t)) \quad \text{for } \sigma(t) \neq 0 \tag{2.13}$$

where sign(.) is the signum function and has the property that $\sigma \,\mathrm{sign}(\sigma) = |\sigma|$.

2.3 Reachability Problem

In the sliding mode literature the controller $u(t)$ is designed so that the so-called reachability condition is satisfied, which is a sufficient condition to ensure that at each time instant, the system state trajectories will converge towards the sliding surface. Mathematically this can be expressed for the case of single input systems as

$$\lim_{\sigma(t)\to 0+} \dot{\sigma}(t) < 0 \qquad \lim_{\sigma(t)\to 0-} \dot{\sigma}(t) > 0 \tag{2.14}$$

or in a compact form as

$$\sigma(t)\dot{\sigma}(t) < 0 \tag{2.15}$$

near the sliding surface $\sigma(t) = 0$. A stronger condition which ensures an ideal sliding motion in finite time, even in the presence of external disturbances or uncertainty, is given by

$$\sigma(t)\dot{\sigma}(t) \leq -\eta|\sigma(t)| \tag{2.16}$$

where η represents a positive design scalar. The expression in (2.16) is often called the η-reachability condition.

For multi-input systems, a natural multivariable version of the reachability condition in (2.16) is

$$\sigma^T(t)\dot{\sigma}(t) \leq -\eta\|\sigma(t)\| \tag{2.17}$$

This is a sufficient condition to show that the sliding surface $\mathscr{S}$ is attractive.

In order to demonstrate that the controller designed in (2.12) satisfies the η-reachability condition (2.17), substituting the value of (2.12) into (2.6) gives

$$\begin{aligned}\dot{\sigma}(t) &= GAx(t) + GB\left(-(GB)^{-1}GAx(t) - \rho(t,x)(GB)^{-1}\frac{\sigma(t)}{\|\sigma(t)\|}\right) + GBD\xi(t,x)\\ &= -\rho(t,x)\frac{\sigma(t)}{\|\sigma(t)\|} + GBD\xi(t,x)\end{aligned} \tag{2.18}$$

Pre-multiplying both sides of (2.18) by $\sigma^T(t)$ yields

$$\sigma^T(t)\dot{\sigma}(t) = -\rho(t,x)\frac{\sigma^T(t)\sigma(t)}{\|\sigma(t)\|} + \sigma^T(t)GBD\xi(t,x) \tag{2.19}$$

and using the property that $\sigma^T\sigma = \|\sigma\|^2$, Eq. (2.19) becomes

$$\begin{aligned}\sigma^T(t)\dot{\sigma}(t) &= -\rho(t,x)\|\sigma(t)\| + \sigma^T(t)GBD\xi(t,x)\\ &\leq \|\sigma(t)\|(-\rho(t,x) + \|GBD\xi(t,x)\|)\end{aligned} \tag{2.20}$$

For any particular choice of scalar gain $\rho(t,x)$ such that

$$\rho(t,x) \geq \|GBD\xi(t,x)\| + \eta \tag{2.21}$$

where η is a positive scalar, the inequality in (2.20) becomes

$$\sigma^T(t)\dot{\sigma}(t) \leq -\eta\|\sigma(t)\| \tag{2.22}$$

From (2.22), it is clear that the η-reachability condition is satisfied, which ensures the existence of an ideal sliding motion on the sliding surface $\mathscr{S}$.

2.4 A Simple Simulation Example

In this section, the design procedure for the typical sliding mode controller discussed in the previous sections is applied to a simulation example, to offer insight into the design procedure.

2.4.1 Spring Mass Damper System

A simple example of a spring-mass-damper system (SMDS), driven by a force $u(t)$, is considered here as shown in Fig. 2.1. It is assumed that at $t = 0$ the mass m is pulled down from the equilibrium position, such that $y(0) = 0.1\,\mathrm{m}$ and $\dot{y}(0) = 0.05\,\mathrm{m/s}$. The dynamical equation of the mechanical system (Fig. 2.1) can be written as

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \tag{2.23}$$

where k is the spring constant, c is the viscous-friction coefficient and m is the mass. A disturbance signal $a \sin(y)$ is added to the control input channel to demonstrate the invariance against a disturbance while in the sliding mode. The values of these constants are chosen as $m = 1\,\mathrm{kg}$, $c = 3\,\mathrm{N \cdot s/m}$, $k = 2\,\mathrm{N/m}$ and $a = 0.1$. In order to write the differential equation in (2.23) in state-space form, define the state variables as $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, which represent the position and velocity of the mass m. Equation (2.23) can be written in terms of the state variables as

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t) \tag{2.24}$$

$$\dot{x}_2(t) = \ddot{y}(t) = -\frac{k}{m}x_1(t) - \frac{c}{m}x_2(t) + \frac{1}{m}(u(t) + a\sin(x_1(t))) \tag{2.25}$$

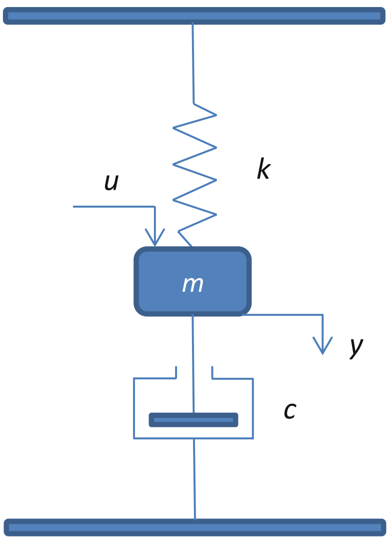

Fig. 2.1 Spring mass damper system

By substituting values for the spring constant k, viscous friction coefficient c, and mass m;

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} (u(t) + 0.1\sin(x_1(t))) \tag{2.26}$$

2.4.2 Simulation Objective and SMC Design

In the simulation it is assumed that at $t = 0$ the mass m is pulled down from the equilibrium position such that $y(0) = 0.1\,\text{m}$ and $\dot{y}(0) = 0.05\,\text{m/s}$. The objective here is to design a sliding mode controller to bring the system back to the equilibrium position from the initial conditions without overshooting in terms of displacement, and with a settling time of not more than 6 s.

The first step is to design a sliding surface. The switching function in (2.4) can be written in terms of the states $x_1(t)$ and $x_2(t)$ as

$$\sigma(t) = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = G_1 x_1(t) + G_2 x_2(t) \tag{2.27}$$

where $G_1 \in \mathbb{R}$ and $G_2 \in \mathbb{R}$. Here it is assumed $G_2 \neq 0$. While sliding, the switching function $\sigma(t) = 0$, and Eq. (2.27) can be written as

$$x_2(t) = -G_2^{-1} G_1 x_1(t) \tag{2.28}$$

It is clear from (2.28) that once x_1(t) is known, the state $x_2(t)$ can be easily determined, therefore substituting the value of (2.28) into (2.24), the sliding motion is given by

$$\dot{x}_1(t) = -G_2^{-1} G_1 x_1(t) \tag{2.29}$$

From (2.29) it is clear that during sliding the system behaves as a reduced order system. Choosing the value of $G_2 = 1$, the switching matrix G takes the form

$$G = \begin{bmatrix} G_1 & 1 \end{bmatrix}$$

In this example the value of $G_1 = 0.9$ is chosen. Using the fact that $GB = 1$, the sliding mode control law defined in (2.13) becomes

$$u(t) = \begin{bmatrix} 2 & 2.1 \end{bmatrix} x(t) - \rho(t, x)\,\text{sign}(\sigma(t)) \tag{2.30}$$

Finally in order to verify that the control law $u(t)$ in (2.30) satisfies the reachability condition (2.16), by substituting (2.30) and (2.26) into the time derivative of (2.27):

$$\begin{aligned}\dot{\sigma}(t) &= G_1\dot{x}_1(t) + G_2\dot{x}_2(t)\\ &= -\rho(t,x)\text{sign}(\sigma(t)) + 0.1\sin(x_1(t))\end{aligned} \tag{2.31}$$

Multiplying (2.31) with $\sigma(t)$ and choosing $\rho(t,x) \geq |0.1\sin(x_1(t))| + \eta = 0.1 + \eta$, it is clear the reachability condition in (2.16) has been established and

$$\sigma(t)\dot{\sigma}(t) \leq -\eta|\sigma(t)| \tag{2.32}$$

which ensures the existence of an ideal sliding mode.

2.4.3 Simulation Results

The sliding mode controller in (2.30) based on the nominal system (2.26) is now tested in simulation using the MATLAB/SIMULINK environment. In the simulations, the value of $\rho(t,x)$ is selected as $\rho(t,x) = 0.15$. From Fig. 2.2 it is clear that the disturbance has no effect on the system performance, which means that the design requirements of the displacement reaching the equilibrium position with no overshoot

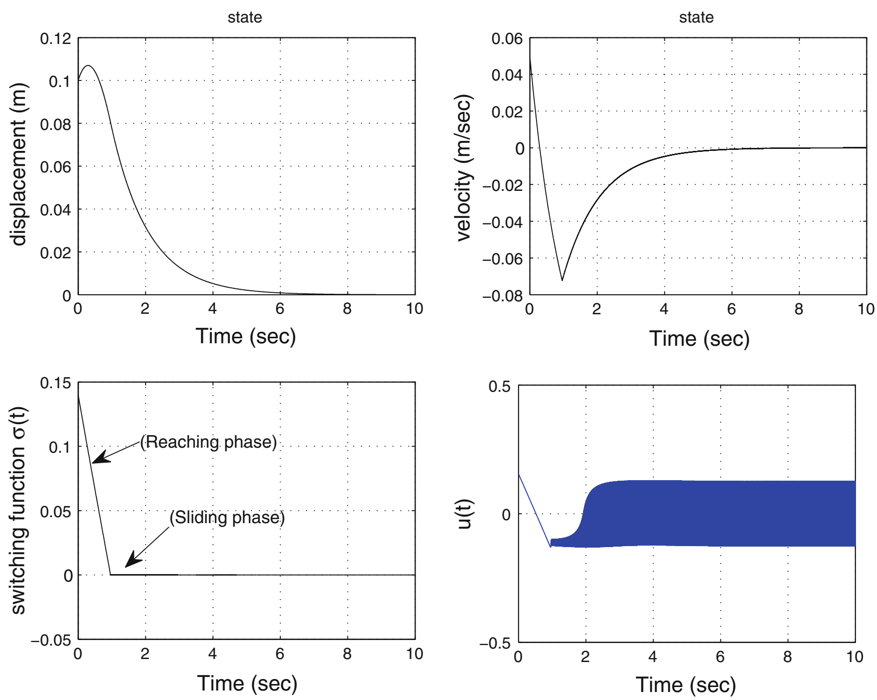

Fig. 2.2 Simulation results for the SMDS with disturbance

and within 6 s is met. The switching function plot in Fig. 2.2 shows that the sliding surface is attained in 1 s, i.e. the sliding motion starts after $t \geq 1$ s. However the discontinuous control signal exhibits high frequency switching which is undesired in most systems due to high wear on moving mechanical components.

2.5 Practical Sliding Mode Control Law

The discontinuity associated with the nonlinear discontinuous part of the control law in (2.12) is the main hurdle in a practical implementation—especially in mechanical systems. Traditionally this has been circumvented by "smoothing" the discontinuity. After doing this the state trajectories no longer slide on the sliding surface, and instead they evolve in the vicinity of the sliding surface: this is termed as pseudo-sliding. However this means total invariance against matched uncertainties is not guaranteed. Nevertheless a good (point-wise) approximation of the discontinuous control term ensures a certain level of robustness against matched uncertainties still remains.[1] One possibility is to use a sigmoidal approximation, where the unit vector term in (2.12) is replaced by

$$u_n(t) = -\rho(t, x)(GB)^{-1} \frac{\sigma(t)}{||\sigma(t)|| + \delta} \tag{2.33}$$

where δ is a small positive design scalar. In this book the sigmoidal approximation given in (2.33), as shown in Fig. 2.3, is used. Here, the value of δ is chosen as $\delta = 0.0001$, and the control law in (2.30) becomes

$$u(t) = \begin{bmatrix} 2 & 2.1 \end{bmatrix} x(t) - 0.15 \frac{\sigma(t)}{|\sigma(t)| + 0.0001}$$

From Fig. 2.4, it is clear that the chattering or high frequency switching of the control signal has been removed. Due to this approximation, the sliding motion will be in the vicinity of the sliding surface and will be termed pseudo-sliding instead of ideal sliding. The design requirements however are still met in the presence of the external disturbance as can be seen in Fig. 2.4.

[1] An alternative approach to smoothing the discontinuity which leads to chattering is to use a higher order sliding mode control approach [2]. Now the sliding motion takes place on the constraint set $\sigma = \dot{\sigma} = \cdots = \sigma^{r-1} = 0$ and is called an rth order sliding mode. Furthermore if it is possible to steer σ to zero using the discontinuous control based on $\dot{u}(t)$, then the actual control signal $u(t)$ will be continuous and the unwanted chattering effects can be alleviated [3].

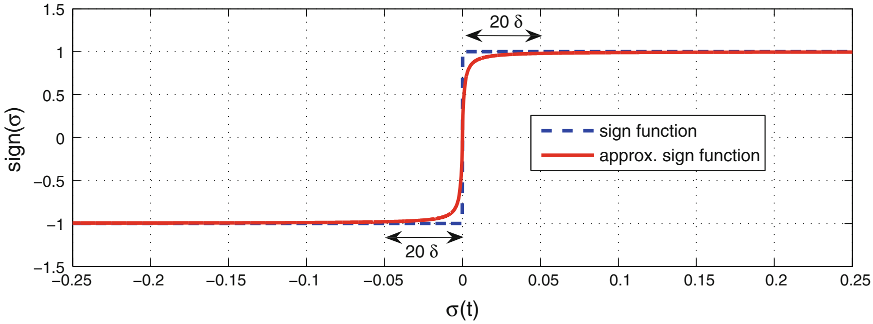

Fig. 2.3 An approximation of the sign function [1]

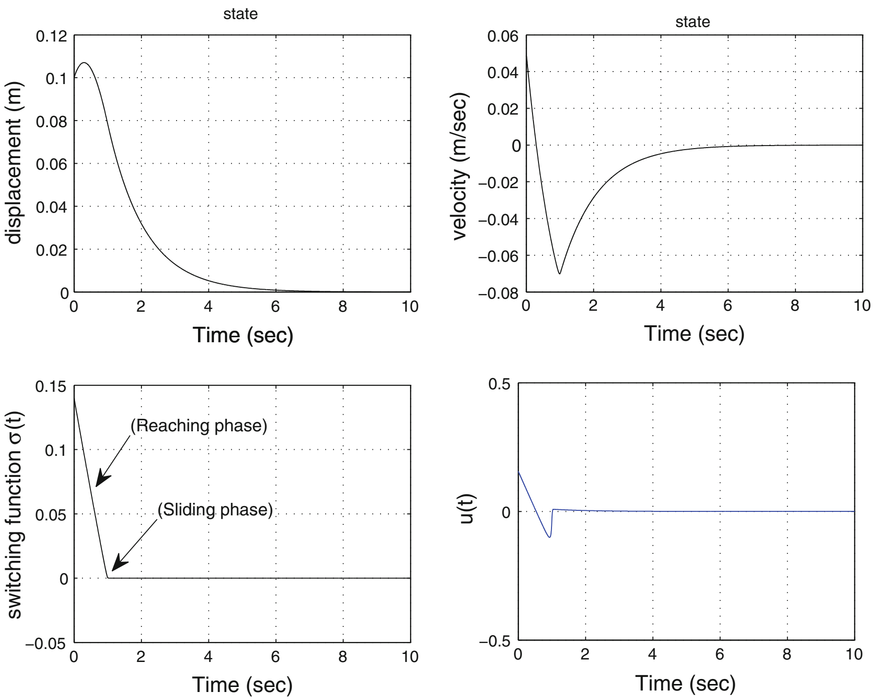

Fig. 2.4 Simulation results for the SMDS with modified control law

2.6 Properties of the Sliding Mode

The properties of conventional (1st order) siding modes are summarised below:

- during a sliding mode, the order of the sliding motion is $n - m$, where n and m represent the number of states and the number of inputs respectively;

- the stability of the closed-loop sliding motion depends only on these $n - m$ non-negative eigenvalues;
- the performance of the closed-loop sliding motion depends on the choice of sliding surface;
- during sliding, the sliding motion is invariant to matched uncertainties;

It should be noted that the robustness to uncertainty is only achieved once sliding takes place. In the sequel, Integral Sliding Mode Control (ISMC) schemes are discussed which eliminate the reaching phase associated with the classical SMC approach discussed in the previous sections, and induce a sliding mode for the entire closed-loop system response.

2.7 Integral Sliding Mode Control (ISMC)

The basic idea of ISMC was initially proposed to enforce a sliding mode from the beginning of the system response, which means a controller based on ISMC ideas can provide compensation to matched uncertainties throughout the entire system response. In this section, a step-by-step design procedure for Integral Sliding Mode (ISM) controllers is explained, and the special features associated with ISMC design are discussed. Again in this section, it is assumed that state information is available for the controller design.

2.7.1 Introduction

In ISMC, it is assumed that there exists a nominal plant, for which a properly designed state feedback controller has already been designed to ensure asymptotic stability of the closed-loop system, and to satisfy predefined performance specifications. A discontinuous controller is 'added' to the nominal state feedback controller to ensure the nominal performance is maintained, and the system is insensitive to external disturbances (faults/failures from a FTC perspective). This design philosophy provides the opportunity to retro-fit an ISM to the existing baseline controller to compensate for the matched uncertainties and external disturbances throughout the system response. As demonstrated in this chapter, when using sliding mode based schemes, the system state trajectories are insensitive to matched uncertainties while in the sliding mode. However no discussion has been made regarding unmatched uncertainties i.e. uncertainties which are not in the range space of the input distribution matrix. This will be addressed in the remainder of the chapter.

2.7.2 Problem Statement and ISM Controller Design

To explain the design procedure, consider an uncertain LTI system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + M\xi(t, x) + f_u(t, x) \tag{2.34}$$

subject to Assumptions 2.1 and 2.2, where $\xi(t, x)$ is a bounded unknown disturbance and the matrix M satisfies the matching condition and can be written as $M = BD$, for some $D \in \mathrm{I\!R}^{m\times l}$.

Assumption 2.3 The function $f_u(t, x)$ represents unmatched uncertainty i.e. it does not lie within the range space of matrix B, but is assumed to be bounded with known upper bound.

The nominal linear system associated with Eq. (2.34) can be written as

$$\dot{x}(t) = Ax(t) + Bu_o(t) \tag{2.35}$$

where $u_o(t)$ is a nominal control law which can be designed by any suitable state feedback paradigm to achieve desired nominal performance. Since it is assumed that the pair (A, B) is controllable, then there exists a state feedback controller of the form

$$u_o(t) = -Fx(t) \tag{2.36}$$

where $F \in \mathrm{I\!R}^{m\times n}$ is a state feedback gain to be designed, so that the state trajectories of the nominal system (2.35), say $x_o(t)$, are stable and meet the performance specifications. The matrix F can be designed using any state feedback design approach. The objective is to design a control law $u(t)$, such that the state trajectories $x(t)$ of (2.34), while in the sliding mode satisfy the condition $x(t) \equiv x_o(t)$ for all time if $f_u(.) = 0$, starting from the initial time instant i.e. when $x(0) \equiv x_o(0)$. To achieve $x(0) \equiv x_o(0)$ the order of the sliding dynamics should be the same as the nominal system.

2.7.3 Design Principles

Define a control law $u(t)$ of the form

$$u(t) = u_o(t) + u_n(t) \tag{2.37}$$

where $u_o(t)$ is the nominal controller and $u_n(t)$ is a nonlinear injection to induce a sliding mode. Then using (2.37), (2.34) can be written as

$$\dot{x}(t) = Ax(t) + Bu_o(t) + Bu_n(t) + BD\xi(t, x) + f_u(t, x) \tag{2.38}$$

where $u_o(t)$ is the state feedback controller defined in (2.36), and where $u_n(t)$ is chosen to reject the disturbance term $\xi(t,x)$ while in the sliding mode. Here the switching function is defined as

$$\sigma(t) = Gx(t) + z(t) \tag{2.39}$$

where $G \in \mathbb{R}^{m\times n}$ is design freedom and $z(t)$ is to be specified. Since the matrix B is of full rank, the switching matrix G can be chosen so that the matrix GB is nonsingular i.e. $\det(GB) \neq 0$. During sliding $\sigma(t) = \dot{\sigma}(t) = 0$ and therefore

$$\dot{\sigma}(t) = G\dot{x}(t) + \dot{z}(t) = 0 \tag{2.40}$$

In order to ensure that the equivalent control term associated with $u_n(t)$ rejects the effect of the matched disturbance term $\xi(t,x)$ (in the case when $f_u(t,x) = 0$), so that the condition $x(t) \equiv x_o(t)$ is satisfied for all $t > 0$, substituting the value of (2.38) into (2.40) gives:

$$\dot{\sigma}(t) = G\left(Ax(t) + Bu_o(t) + Bu_n(t) + BD\xi(t,x)\right) + \dot{z}(t) = 0 \tag{2.41}$$

During sliding it is expected that $u_{n_{eq}}(t) = -D\xi(t,x)$, i.e. it should compensate for the uncertainty, then selecting

$$\dot{z}(t) = -G(Ax(t) + Bu_o(t)), \quad z(0) = -Gx(0) \tag{2.42}$$

ensures

$$\dot{\sigma}(t) = GBu_n(t) + GBD\xi(t,x) \tag{2.43}$$

and so during sliding $u_{n_{eq}}(t) = -D\xi(t,x)$. Substituting the value of $u_{n_{eq}}(t)$ into (2.38) means the integral sliding mode is governed by

$$\dot{x}(t) = Ax(t) + Bu_o(t) \tag{2.44}$$

which confirms that the condition $x(t) \equiv x_o(t)$ is satisfied if $f_u(t,x) = 0$ and $x(0) = x_o(0)$. In the case when $f_u(t,x) \neq 0$ in (2.38), the equivalent control obtained from (2.41) can be written as

$$\begin{aligned} u_{n_{eq}}(t) &= -(GB)^{-1}GBD\xi(t,x) - (GB)^{-1}Gf_u(t,x) \\ &= -D\xi(t,x) - (GB)^{-1}Gf_u(t,x) \end{aligned} \tag{2.45}$$

Substituting the value of equivalent control $u_{n_{eq}}(t)$ from (2.45) into (2.38) and simplifying, the expression for the integral sliding mode dynamics can be written as

$$\dot{x}(t) = Ax(t) + Bu_o(t) + \underbrace{(I - B(GB)^{-1}G)}_{\Gamma} f_u(t,x) \tag{2.46}$$

From Eq. (2.46), it is clear that the effect of the matched uncertainty has been completely rejected while in the sliding mode. However the matrix Γ in (2.46) can amplify the effect of unmatched uncertainty $f_u(t,x)$. Therefore the objective in the next section is to design the integral sliding surface design parameter G to avoid any amplification of the unmatched uncertainty.

2.7.4 Integral Switching Surface

Using Eqs. (2.39) and (2.42), an integral switching function which eliminates the reaching phase is

$$\sigma(t) = Gx(t) - Gx(0) - G\int_0^t (Ax(\tau) + Bu_o(\tau))\, d(\tau) \tag{2.47}$$

The term $-Gx(0)$ ensures that $\sigma(0) = 0$, so the reaching phase is eliminated. The sliding mode will exist from time $t = 0$ and the system will be robust throughout the entire closed-loop system response against matched uncertainties.

From the previous analysis, it is clear that in the case of only matched uncertainty, then any choice of G which ensures GB is invertible is sufficient for the ISM design, but for unmatched uncertainty, a specific choice of G is needed. Here it will be argued that

$$G = B^+ = (B^T B)^{-1} B^T \tag{2.48}$$

is an appropriate choice. Note G in (2.48) is the Moore–Penrose left pseudo-inverse of the input distribution matrix B. The particular choice of G in (2.48) brings two advantages:

1. the modulation gain associated with $u_n(t)$ in (2.37) is minimised which means the amplitude of the chattering can be reduced;
2. it avoids amplifying the effect of the unmatched disturbance.

This choice of G also has the simplifying property that

$$GB = \underbrace{(B^T B)^{-1} B^T}_{G} B = I_m$$

and ensures that the square matrix GB is nonsingular. With the choice of G in (2.48), the projection operator Γ in (2.46) becomes

$$\Gamma = I_n - B(B^T B)^{-1} B^T \tag{2.49}$$

Notice that the projection operator Γ in (2.49) is symmetric and idempotent i.e. $\Gamma^2 = \Gamma$. The properties of symmetry and idempotency imply that $\|\Gamma\| = 1$, which means that the effect of f_u is not amplified since $\|\Gamma f_u\| \leq \|f_u\|$. In fact, it can be proved that $\|I - B(GB)^{-1}G\| \geq 1$ for any G, and so the choice of G in (2.48) is an optimal one in the sense of non-amplification of the unmatched uncertainty.

2.7.5 Integral Sliding Mode Control Laws

An integral sliding mode controller will now be designed based on the nominal system in (2.35). The control law has a structure given by

$$u(t) = u_o(t) + u_n(t) \tag{2.50}$$

where $u_o(t)$ is the linear part of the controller, and $u_n(t)$ is the discontinuous part to enforce a sliding mode along the sliding surface in (2.47). One choice of $u(t)$ is

$$u(t) = -Fx(t) - \rho(t,x)(GB)^{-1}\frac{\sigma(t)}{\|\sigma(t)\|} \quad \text{for } \sigma(t) \neq 0 \tag{2.51}$$

where F is the state feedback controller which is responsible for the performance of the nominal system and $\rho(t,x)$ is the modulation gain to enforce the sliding mode—whose precise value is given in the next subsection.

2.7.6 The Reachability Condition

To justify that the controller designed in (2.51) satisfies the η-reachability condition (2.22), which is a sufficient condition to ensure the existence of an ideal sliding motion, it can be shown from (2.34) and (2.36) that

$$\dot{\sigma}(t) = G\left(Ax(t) + Bu(t) + BD\xi(t,x) + f_u(t,x)\right) - GAx(t) + GBFx(t)$$

then substituting from (2.51), and after some simplification

$$\begin{aligned}\dot{\sigma}(t) &= GAx(t) + GB(-Fx(t) + u_n(t)) + GBD\xi(\cdot) + Gf_u(\cdot) - GAx(t) + GBFx(t)\\ &= -\rho(t,x)\frac{\sigma(t)}{\|\sigma(t)\|} + GBD\xi(t,x) + Gf_u(t,x)\end{aligned} \tag{2.52}$$

Then

$$\begin{aligned}\sigma^T(t)\dot{\sigma}(t) &= -\rho(t,x)\|\sigma(t)\| + \sigma^T(t)D\xi(t,x) + \sigma^T(t)Gf_u(t,x)\\ &\leq \|\sigma(t)\|(-\rho(t,x) + \|D\xi(t,x)\| + \|Gf_u(t,x)\|)\end{aligned} \tag{2.53}$$

where the fact that $GB = I_m$, has been used. In order to enforce a sliding mode the value of the modulation gain $\rho(t, x)$ should be greater than any disturbance or uncertainty in the system, and therefore for any choice of $\rho(t, x)$ which satisfies

$$\rho(t, x) \geq \|D\| \|\xi(t, x)\| + \|G\| \|f_u(t, x)\| + \eta \tag{2.54}$$

where η is some positive scalar, the η-reachability condition

$$\sigma^T(t)\dot{\sigma}(t) \leq -\eta||\sigma(t)|| \tag{2.55}$$

is satisfied.

Remark 2.2. Inequality (2.55) can also be interpreted from a Lyapunov perspective. Define $V(t) = \frac{1}{2}\sigma^T(t)\sigma(t)$, then $\dot{V}(t) = \sigma^T(t)\dot{\sigma}(t)$ and from the inequalities in (2.53)–(2.55) it follows

$$\dot{V}(t) \leq -\eta||\sigma(t)|| = -\eta\sqrt{2V(t)} \tag{2.56}$$

Integrating both sides of (2.56) yields

$$\sqrt{2V(t)} - \sqrt{2V(0)} \leq -\eta t$$

which implies $V(t) \equiv 0$ in less than $\frac{\eta}{\sqrt{2V(0)}}$ units of time.

2.7.7 *Properties of Integral Sliding Mode*

The properties of integral sliding modes can be summarised as follows:

- there is no reaching phase and a sliding mode is enforced throughout the entire system response;
- during sliding, the order of the motion is the same as the original system;
- by a suitable choice of sliding surface, the effect of unmatched uncertainty can be ameliorated;
- during the sliding mode, the system motion is invariant to matched uncertainties;
- the ISM approach has the ability to be retro-fitted to an existing feedback controller;

2.7.8 *Simulation Example*

Here in this section, to make a direct comparison, the simulation scenario of the spring-mass-damper system from Sect. 2.4 will be simulated. Recall the system was represented as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} (u(t) + 0.1\sin(x_1(t)))$$

The objective is to design an ISM controller to bring the system back to the equilibrium position from the initial conditions without overshooting in terms of displacement, and with a settling time not more than 6 s. The integral switching function from Eq. (2.47) is

$$\sigma(t) = Gx(t) - Gx(0) - G\int_0^t (A - BF)\,x(\tau)d(\tau)$$

where the value of G is chosen as in (2.48), and here is equal to

$$G = (B^T B)^{-1} B^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{2.57}$$

The gain F in this example has been designed using the linear quadratic regulator (LQR) method,[2] and aims to regulate the system states to the origin by minimising the cost function

$$J = \int_0^\infty (x(t)^T Q x(t) + u_o(t)^T R u_o(t))dt \tag{2.58}$$

where Q is a symmetric positive definite ($s.p.d.$) matrix and R is a positive scalar. These matrices penalise the magnitude of the control signal $u_o(t)$ and the deviation of the system states from the origin. Here the values of Q and R are chosen as $Q = \text{diag}(1, 0.5)$ and $R = 1$, which results in the matrix

$$F = \begin{bmatrix} 0.2361 & 0.1579 \end{bmatrix}$$

The ISM control law is

$$u(t) = -Fx(t) - \rho\,\text{sign}(\sigma(t)) \quad \text{for } \sigma(t) \neq 0 \tag{2.59}$$

since the choice of G in (2.57) makes $GB = 1$. Here ρ is a fixed scalar satisfying $\rho = 0.1 + \eta$ where $\eta > 0$. It is easy to check that the control law $u(t)$ satisfies the reachability condition $\sigma(t)\dot{\sigma}(t) \leq -\eta|\sigma(t)|$. Here the sigmoidal approximation given in (2.33) and shown in Fig. 2.3 is used, and therefore the ISM control law in (2.59) is modified to become

$$u(t) = -Fx(t) - \rho\frac{\sigma(t)}{|\sigma(t)| + \delta} \tag{2.60}$$

where the value of the small positive scalar δ is chosen as $\delta = 0.0001$, to eliminate chattering. The control law $u(t)$, after substituting for the value of F, can be written as

[2]This is a well-known 'classical' state-space technique: for details see for example [4].

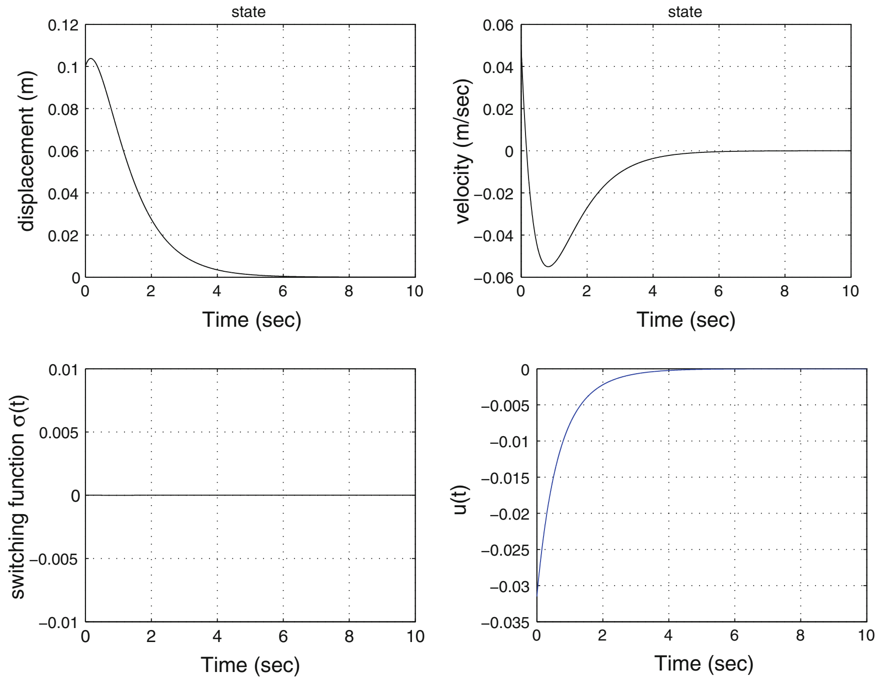

Fig. 2.5 Simulation results for the SMDS nominally (with ISMC)

$$u(t) = -0.2361x_1(t) - 0.2361x_2(t) - \rho \frac{\sigma(t)}{|\sigma(t)| + \delta}$$

The displacement plots in Figs. 2.5 and 2.6 show that the design requirements are met both nominally (without any disturbance) and in the presence of a disturbance term. From Fig. 2.6, it is clear that the effect of the disturbance $0.1 \sin(x_1(t))$ has been completely rejected. From the switching function plots in Figs. 2.5 and 2.6, it is clear there is no reaching phase i.e. the sliding mode starts from time $t = 0$.

2.8 Sliding Modes as a Candidate for FTC

Sliding mode based control schemes are a strong candidate for fault tolerant control because of their inherent robustness to matched uncertainties. As argued in Sect. 1.3.1, actuator faults can be effectively modelled as matched uncertainties and therefore sliding mode based control schemes have an inherent capability to directly deal with actuator faults. However actuator failures cannot be handled directly by sliding modes schemes because the complete loss of effectiveness in a channel destroys

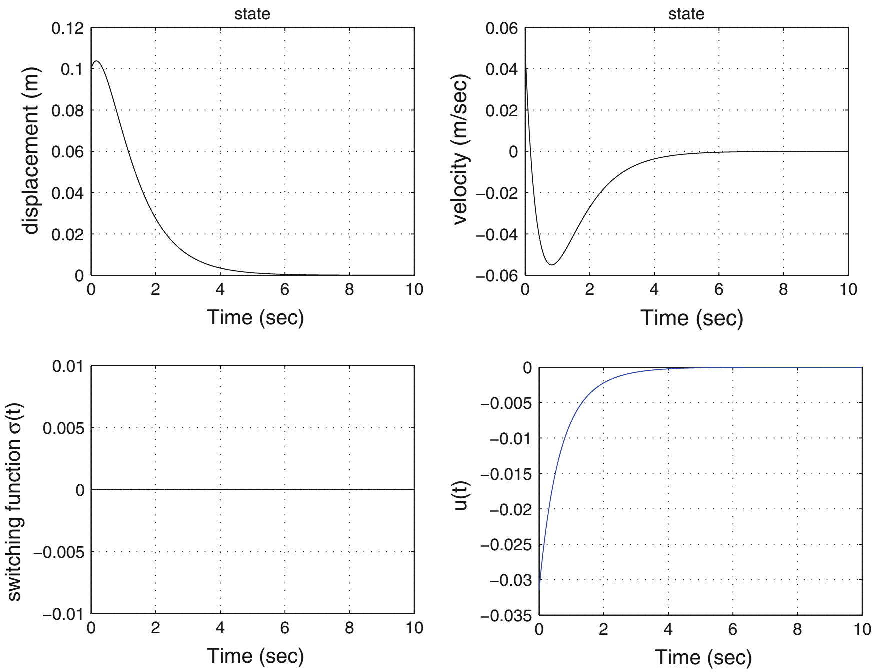

Fig. 2.6 Simulation results for the SMDS with disturbance (with ISMC)

the regularity of the sliding mode, and a unique equivalent control signal can no longer be determined.

In the subsequent chapters, Control Allocation (as discussed in Sect. 1.3.3) is considered as a potential candidate to be combined with ISM control to deal with actuator faults or failures due to its ability to effectively manage the actuator redundancy and to redistribute the control signals to the healthy actuators in the case of an actuator failure. The use of integral sliding modes ensures robustness for all time by eliminating the reaching phase associated with 'classical' SMC based methods. Furthermore integral sliding modes have the capacity to be retro-fitted to the existing controller design to introduce fault tolerance without changing or altering the existing control loops, which is advantageous from an industrial perspective.

2.9 Notes and References

The term sliding mode was first used in the literature in the context of relay systems [5]. Sliding mode control (SMC) is a particular class of variable structure control systems (VSCS) [6]. VSCS evolved from work in Russia in the early 1960s and spread around the world in the late 1970s after the publication of the survey paper by

Utkin [7]. SMC design paradigms [1, 2, 5] have become mature techniques for the control of uncertain systems and provide effective solutions against matched uncertainties; however to provide compensation against matched uncertainties throughout the entire system response, the idea of integral sliding mode control was initially proposed in [5, 8–10]. In [11–13], integral sliding mode control ideas were used for uncertain systems considering both matched and unmatched uncertainties and demonstrated that the system dynamics while on the sliding surface meet the performance specifications in the presence of matched uncertainties. In [13], it was first demonstrated that the effect of mismatched uncertainties can be minimised by the suitable choice of an integral sliding surface (the specific choice of G is given in Sect. 2.7.4). Details of the integral sliding mode approach in the context of robust LQ output control (but not in the context of FTC) can be found in [14]. Different methods have been used in the literature to smooth the transition near the sliding surface to remove chattering—see for example Chap. 3 in [1, 15]. An alternative approach to smoothing the discontinuous switching control law (which leads to chattering) is to use a higher order sliding mode control approach [3]. Many researchers have identified SMC as a potential candidate for FTC, see for example [16–20]. Researchers in [21] have focused on fault reconstruction and fault tolerant control schemes for aerospace applications using traditional SMC approaches. In [16, 17], it was argued that SMC could deal with significant and sudden changes in the system dynamics due to actuator faults and has the capability to become an alternative to reconfigurable control systems. In [18], a 'hedging' based SMC design is used to reduce the effect of neglected parasitic dynamics in a longitudinal control system for an aircraft. In [16, 19] the authors have demonstrated the combination of SMC with control allocation for FTC purposes. Recently in [22], a continuous integral sliding mode FTC scheme was proposed using a higher order sliding mode observer by incorporating fixed control allocation.

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