Computability, Complexity, and Languages

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CHAPTER 1

PRELIMINARIES

1 Sets and n-tuples

We shall often be dealing with *sets* of objects of some definite kind. Thinking of a collection iof entities as a *set* simply amounts to a decision to regard the whole collection as a single object. We shall use the word *class* as synonymous with *set*. In particular we write N for the set of *natural numbers* $0, 1, 2, 3 \cdots$.

It is useful to speak of the *empty set*, written $\varnothing$, which has no members. The equation $R = S$, where R and S are sets, means that R and S are *identical as sets*, that is, that they have exactly the same members. We write $R \subseteq S$ and speak of R as a *subset* of S to mean that every element of R is also an element of S. We write $R \subset S$ to indicate that $R \subseteq S$ but $R \neq S$. In this case R is called a *proper subset* of S. If R and S are set, we write $R \cup S$ for the *union* of R and S, which is the collection of all objects which are members of either R or S or both. $R \cap S$, the *intersection* of R and S, is the set of all objects that belong to both R and S. $R - S$, the set of all objects that belong to R and do not belong to S, is the *difference* between R and S. Often we will be working in contexts where all sets being considered are subsets of some fixed set D (sometimes called a *domain* or a *universe*). In such a case we write $\bar{S}$ for $D - S$, and call $\bar{S}$ the *complement* of S. We write

$$\{a_1, a_2, \cdots, a_n\}$$

for the set consisting of the n objects $a_1, a_2, \cdots, a_n$. Sets that can be written in this form as well as the empty set are called *finite.* Sets that are not finite are called *infinite.* Since two sets are equal if and only if they have the same members. That is, the order in which we may choose to write the members of a set is irrelevant. Where order is important, we speak instead of an n-tuple or a *list.* A 2-tuple is called an *ordered pair*, and a 3-tuple is called an *ordered triple.* Unlike the case for sets of one object, we *do not distinguish between the object a and the 1-tuple* (a). The crucial property of n-tuples is

$$(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$$

if and only if

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots, \quad \textit{and} \quad a_n = b_n.$$

If $S_1, S_2, \cdots, S_n$ are given sets, then we write $S_1 \times S_2 \times \cdots \times S_n$ for the set of all n-tuples such that $a_1 \in S_1, a_2 \in S_2, \cdots, a_n \in S_n$. $S_1 \times S_2 \times \cdots \times S_n$ is sometimes called the *Cartesian product* of $S_1, S_2, \cdots, S_n$.

2 Functions

For f a function, one writes $f(a) = b$ to mean that $(a, b) \in f$; the definition of function ensures that for each a there can be at most one such b. The set of all a such that $(a, b) \in f$ for some b is called the *domain* of f. The set of all $f(a)$ for a in the domain of f is called the *range* of f.

Functions f are often specified by *algorithms* that provide procedures for obtaining $f(a)$ from a. However, it is quite possible to possess an algorithm that specifies a function without being able to tell which elements belong to its domain. This makes the notion of a so-called *partial function* play a central role in computability theory. A *partial function on a set S* is simply a function whose domain is a subset of S. If f is a partial function on S and $a \in S$, then we write $f(a) \downarrow$ and say that $f(a)$ is *defined* to indicate that a is in the domain of f; if a is not in the domain of f, we write $f(a) \uparrow$ and say that $f(a)$ is *undefined*. If a partial function on S has the domain S, then it is called *total*. Finally, we should mention that the empty set $\varnothing$ is itself a function. Considered as a partial function on some set S, *it is nowhere defined*.

A partial function f on a set S^n is called an *n-ary partial function on* S, or a function of n variables on S. We use *unary* and *binary* for 1-ary and 2-ary, respectively.

A function f is *one-one* if, for all x, y in the doamin of f, $f(x) = f(y)$ implies $x = y$. If the range of f is the set S, then we say that f is an *onto* function with respect to S, or simply that f is *onto* S.

We will sometimes refer to the idea of *closure*. If S is a set and f is a partial function on S, then S is *closed under* f if the range of f is a subset of S.

3 Alphabets and Strings

An *alphabet* is simply some finite nonempty set A of objects called *symbols*. An n-tuple of symbols of A is called a *word* or a *string* on A. The set of all words on the alphabet A is written A^*. Any subset of A^* is called a *language on* A or a *language with alphabet* A. We do *not* distinguish between a symbol $a \in A$ and the word of length 1 consisting of that symbol.

4 Predicates

By a *predicate* or a *Boolean-valued function* on a set S we mean a *total* function P on S such that for each $a \in S$, either

$$P(a) = \text{TRUE} \quad \text{or} \quad P(a) = \text{FALSE},$$

where TRUE and FALSE are a pair of distinct objects called *truth values*. We often say $P(a)$ *is true* for $P(a)$ =TRUE, and $P(a)$ *is false* for $P(a)$ =FALSE. Given a predicate P on a set S, there is a corresponding subset R of S, namely, the set of all elements $a \in S$ for which $P(a) = 1$. The predicate P is called the *characteristic function* of the set R.

5 Quantifiers

In this section we will be concerned exclusively with predicates on N^m (or what is the same thing, m-ary predicates on N) for different values of m. Thus, let $P(t, x_1, \cdots, x_n)$ be an $(n+1)$-ary predicate. Consider the predicate $Q(y, x_1, \cdots, x_n)$ defined by

$$\begin{aligned} Q(y, x_1, \ldots, x_n) \Leftrightarrow & P(0, x_1, \ldots, x_n) \vee P(1, x_1, \ldots, x_n) \\ & \vee \cdots \vee P(y, x_1, \ldots, x_n). \end{aligned}$$

Thus the predicate $Q(y, x_1, \cdots, x_n)$ is true just in case there is value of $t \leq y$ such that $P(t, x_1, \cdots, x_n)$ is true. We write this predicate Q as

$$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n).$$

The expression "$(\exists t)_{\leq y}$" is called a *bounded existential quantifier*. Similarly, we write $(\forall t)_{\leq y} P(t, x_1, \ldots, x_n)$ for the predicate

$$P(0, x_1, \ldots, x_n) \& P(1, x_1, \ldots, x_n) \& \cdots \& P(y, x_1, \ldots, x_n).$$

The predicate is true just in case $P(t, x_1. \cdots, x_n)$ is true for *all* $t \leq y$. The expression "$(\forall t)_{\leq y}$" is called a *bounded universal quantifier*.

6 Proof by Contradiction

Recall that a number is called a *prime* if it has *exactly two distinct divisors*, itself and 1. Consider the following assertion:

$$n^2 - n + 41 \text{ is prime for all } n \in N.$$

This assertion is in fact *false.*

In a *proof by contradiction*, one begins by supposing that the assertion we wish to prove is false. In a proof by contradiction we look for a pair of statements developed in the course of the proof which *contradict* one another.

Theorem 6.1

Let $x \in \{a, b\}^*$ such that $xa = ax$. Then $x = a^{[n]}$ for some $n \in N$.

7 Mathematical Induction

Mathematical induction furnishes an important technique for proving statements of the form $(\forall n)P(n)$, where P is a predicate on N. One proceeds by proving a pair of auxiliary statements, namely, $P(0)$ and

$$(\forall n)(\textit{if } P(n) \textit{ then } P(n+1)). \tag{1.1}$$

Why is this helpful? Because sometimes it is much easier to prove (**??**) than to prove $(\forall n)P(n)$ in some other way. In proving this second auxiliary proposition one typically considers some fixed but arbitrary value k of n and shows that if we assume $P(k)$ we can prove $P(k+1)$. $P(k)$ is then called the *induction hypothesis.*

There are some paradoxical things about proofs by mathematical induction. One is assuming $P(k)$ for some *particular* k in order to show that $P(k+1)$ follows.

It is also paradoxical that in using induction (we shall often omit the word *mathematical*), it is sometimes easier to prove statements by first making them "stronger." We wish to prove $(\forall n)P(n)$. Instead we decide to prove the *stronger* assertion $(\forall n)(P(n)\&Q(n))$ (which of course implies the original statement). The technique of deliberately strengthening what is to be proved for the purpose of making proofs by induction easier is called *induction loading.*

Theorem 7.1

For all $n \in N$ we have $\sum_{i=0}^{n}(2i+1) = (n+1)^2$.

Another form of mathematical induction that is often very useful is called *course-of-values induction* or sometimes *complete induction.*

Theorem 7.2

There is no string $x \in \{a, b\}^*$ such that $ax = xb$.

CHAPTER 2

PROGRAMS AND COMPUTABLE FUNCTIONS

1 A Programming Language

In particular, the letters

$$X_1 \; X_2 \; X_3 \; \cdots$$

will be called the *input variables* of $\mathscr{L}$, the letter Y will be called the *output variable* of $\mathscr{L}$, and the letters

$$Z_1 \; Z_2 \; Z_3 \; \cdots$$

will be called the *local variables* of $\mathscr{L}$.

In $\mathscr{L}$ we will be able to write "instructions" of various sorts; a "program" of $\mathscr{L}$ will then consist of a *list* (i.e., a finite sequence) of instructions.

Table 2.1

Insturction	Interpretation
$V \leftarrow V + 1$	Increase by 1 the value of the variable V.
$V \leftarrow V - 1$	If the value of V is 0, leave it unchanged; otherwise decrease by 1 the value of V.
IF $V \neq 0$ GOTO L	If the value of V is nonzero, perform the instruction with label L next; otherwise proceed to the next instruction in the list

We give in Table **??** a complete list of our instructions. In this list V stands for any variable and L stands for any label.

These instructions will be called the *increment*, *decrement*, and *conditional branch* instructions, respectively.

We will use the special convention that *the output variable Y and the local variables Z_i initially have the value* 0.

2 Some Examples of Programs

Our first example is the program

```
[A]    X ← X − 1
       Y ← Y + 1
       IF X ≠ 0 GOTO A
```

If the initial value x of X is not 0, the effect of this program is to copy x into Y and to decrement the value of X down to 0. We will say that this program *computes* the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ x & \text{otherwise.} \end{cases}$$

Although the preceding program is a perfectly well-defined program of our language $\mathscr{L}$, we may think of it as having arisen in an attempt to write a program that copies the value of X into Y, and therefore containing a "bug" because it does not handle 0 correctly. The following slightly more complicated example remedies this situation.

```
[A]  IF $X \neq 0$ GOTO $B$
     $Z \leftarrow Z + 1$
     IF $Z \neq 0$ GOTO $E$
[B]  $X \leftarrow X - 1$
     $Y \leftarrow Y + 1$
     $Z \leftarrow Z + 1$
     IF $Z \neq 0$ GOTO $A$
```

At first glance Z's role in the computation may not be obvious. It is used simply to allow us to code an *unconditional branch.* That is, the program segment

$$\begin{aligned}&Z \leftarrow Z + 1\\&\text{IF } Z \neq 0 \text{ GOTO } L\end{aligned} \tag{2.1}$$

has the effect (ignoring the effect on the value of Z) of an instruction

$$\text{GOTO } L$$

such as is available in most programming languages. Now GOTO L is not an instruction in our language $\mathscr{L}$, but since we will frequently have use for such an instruction, we can use it as an abbreviation for the program segment (**??**). Such an abbreviating pseudoinstruction will be called a *macro* and the program or program segment which it abbreviates will be called it *macro expansion.*

For our final example, we take the program

```
     $Y \leftarrow X_1$
     $Z \leftarrow X_2$
[C]  IF $Z \neq 0$ GOTO $A$
     GOTO $E$
[A]  IF $Y \neq 0$ GOTO $B$
     GOTO $A$
[B]  $Y \leftarrow Y - 1$
     $Z \leftarrow Z - 1$
     GOTO $C$
```

What happens if we begin with a value of X_1 less than the value of X_2? At this point the computation enters the "loop":

```
[A]  IF $Y \neq 0$ GOTO $B$
     GOTO $A$
```

Since $y = 0$, there is no way out of this loop and the computation will continue "forever." Thus, if we begin with $X_1 = m$, $X_2 = n$, where $m < n$, the computation will never terminate. In this case (and in similar cases) we will say that the program computes the *partial function*

$$g(x_1, x_2) = \begin{cases} x_1 - x_2 & \text{if } \ x_1 \geq x_2 \\ \uparrow & \text{if } \ x_1 < x_2. \end{cases}$$

3 Syntax

The symbols

$$X_1 \; X_2 \; X_3 \; \cdots$$

are called *input variables*,

$$Z_1 \; Z_2 \; Z_3 \; \cdots$$

are called *local variables*, and Y is called the *output variable* of $\mathscr{L}$. The symbols

$$A_1,\ B_1\ C_1\ D_1\ E_1\ A_2\ B_2\ \cdots$$

are called *labels* of $\mathscr{L}$. A *statement* is one of the following:

$$\begin{aligned}&V \leftarrow V+1\\&V \leftarrow V-1\\&V \leftarrow V\\&\text{IF } V \neq 0 \text{ GOTO } L\end{aligned}$$

where V may be any variable and L may be any label.

Next, an *instruction* is either a statement (in which case it is also called an *unlabeled* instruction) or $[L]$ followed by a statement (in which case the instruction is said to have L as its label or to be labeled L). A *program* is a list (i.e., a finite sequence) of instructions. The length of this list is called the *Length* of the progra. It is useful to include the *empty program* of length 0, which of course contains no instructions.

A *state of a program* $\mathscr{P}$ is a list of equations of the form $V = m$, where V is a variable and m is a number, including an equation for each variable that occurs in $\mathscr{P}$ and including no two equations with the same variable. As an example, let $\mathscr{P}$ be the program which contains the variables $X\ Y\ Z$. (The definition of *state* does not require that the state can actually be "attained" from some initial state.) The list

$$X = 3, \quad Z = 3$$

is *not* a state of $\mathscr{P}$ since no equation in Y occurs. Likewise, the list

$$X = 3, \quad X = 4, \quad Y = 2, \quad Z = 2$$

is *not* a state of $\mathscr{P}$: there are two equations in X.

Let σ be a state of $\mathscr{P}$ and let V be a variable that occurs in σ. The *value of* V *at* σ is then the (unique) number q such that the equation $V = q$ is one of the equations making up σ.

Suppose we have a program $\mathscr{P}$ and a state σ of $\mathscr{P}$. In order to say what happens "next," we also need to know which instruction of $\mathscr{P}$ is about to be executed. We therefore define a *snapshot* or *instantaneous description* of a program $\mathscr{P}$ of length n to be a pair (i, σ) where $1 \leq i \leq n+1$, and σ is a state of $\mathscr{P}$.

If $s = (i, \sigma)$ is a snapshot of $\mathscr{P}$ and V is a variable of $\mathscr{P}$, then the *value* of V at s just means the value of V at σ.

A snapshot (i, σ) of a program $\mathscr{P}$ of length n is called *terminal* if $i = n+1$. If (i, σ) is a nonterminal snapshot of $\mathscr{P}$, we define the *successor* of (i, σ) to be the snapshot (j, τ) defined as follows:

Case 1. The ith instruction of $\mathscr{P}$ is $V \leftarrow V+1$ and σ contains the equation $V = m$. Then $j = i+1$ and τ is obtained from σ by replacing the equation $V = m$ by $V = m+1$ (i.e., the value of V at τ is $m+1$).

Case 2. The ith instruction of $\mathscr{P}$ is $V \leftarrow V-1$ and σ contains the equation $V = m$. Then $j = i+1$ and τ is obtained from σ by replacing the equation $V = m$ by $V = m-1$ if $m \neq 0$; if $m = 0$, $\tau = \sigma$.

Case 3. The ith instruction of $\mathscr{P}$ is $V \leftarrow V$. Then $\tau = \sigma$ and $j = i+1$.

Case 4. The ith instruction of $\mathscr{P}$ is IF $V \neq 0$ GOTO L. Then $\tau = \sigma$, and there are two subcases:

Case 4a. σ contains the equation $V = 0$. Then $j = i+1$.

Case 4b. σ contains the equation $V = m$ where $m \neq 0$. Then, if there is an instruction of $\mathscr{P}$ labeled L, j is the *least number* such that the jth instruction of $\mathscr{P}$ is labeled L. Otherwise, $j = n+1$.

A *computation* of a program $\mathscr{P}$ is defined to be a sequence (i.e., a list) $s_1, s_2, \ldots, s_k$ of snapshots of $\mathscr{P}$ such that s_{i+1} is the successor of s_i for $i = 1, 2, \cdots, k-1$ and s_k is terminal.

Note that we have not forbidden a program to contain more than one instruction having the same label. However, our definition of successor of a snapshot, in effect, interprets a branch instruction as always referring to the *first* statement in the program having the label in question.

4 Computable Functions

One would expect a program that computes a function of m variables to contain the input variables $X_1, X_2, \ldots, X_m$, and the output variable Y, and to have all other variables (if any) in the program to be local.

Thus, let $\mathscr{P}$ be any program in the language $\mathscr{L}$ and let $r_1, \ldots, r_m$ be m given numbers. We form the state σ of $\mathscr{P}$ which consists of the equations

$$X_1 = r_1, \quad X_2 = r_2, \quad \ldots, \quad X_m = r_m, \quad Y = 0$$

together with the equations $V = 0$ for each variable V in $\mathscr{P}$ other than $X_1, \ldots, X_m, Y$. We will call this the *initial state*, and the snapshot $(1, \sigma)$, the *initial snapshot.*

Case 1. *There is a computation* $s_1, s_2, \ldots, s_k$ of $\mathscr{P}$ *beginning with the initial snapshot.* Then we write $\psi_{\mathscr{P}}^{(m)}(r_1, \ldots, r_m)$ for the value of the variable Y at the (terminal) snapshot s_k.

Case 2. *There is no such computation*; i.e., there is an *infinite* sequence $s_1, s_2, s_3, \ldots$ beginning with the initial snapshot where each s_{i+1} is the successor of s_i. In this case $\psi_{\mathscr{P}}^{(m)}(r_1, \ldots, r_m)$ is undefined.

For any program $\mathscr{P}$ and any positive integer m, the function $\psi_{\mathscr{P}}^{(m)}(r_1, \ldots, r_m)$ is said to be *computed* by $\mathscr{P}$. A given partial function g (of one or more variables) is said to be *partially computable* if it is computed by some program.

A given function g of m variables is called *total* if $g(r_1, \ldots, r_m)$ is defined for *all* $r_1, \ldots, r_m$. A function is said to be *computable* of it is both partially computable and total.

Partially computable functions are also called *partial recursive*, and computable functions, i.e., functions that are both total and partial recursive, are called *recursive.*

5 More about Macros

We now see how to augment our language to include macros of the form

$$\text{IF } P(V_1, \ldots, V_n) \text{ GOTO } L$$

where $P(x_1, \ldots, x_n)$ is a computable predicate. Here we are making use of the convention that

$$\text{TRUE} = 1, \quad \text{FALSE} = 0.$$

Hence predicates are just total functions whose values are always either 0 or 1. And therefore, it makes perfect sense to say that some given *predicate* is or is not computable.

CHAPTER 3

PRIMITIVE RECURSIVE FUNCTIONS

1 Composition

We want to combine computable functions in such a way that the output of one becomes an input to another. In the simplest case we combine functions f and g to obtain the function

$$h(x) = f(g(x)).$$

More generally, for functions of several variables:

Definition 1.1

Let f be a function of k variables and let $g_1, \ldots, g_k$ be functions of n variables. Let

$$h(x_1, \ldots, x_n) = f(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)).$$

Then h is said to be obtained from f and $g_1, \ldots, g_k$ by *composition*.

Theorem 1.2

If h is obtained from the (partially) computable functions $f, g_1, \ldots, g_k$ by composition, then h is (partially) computable.
The word *partially* is placed in parentheses in order to assert the correctness of the statement with the word included or omitted in both places.

2 Recursion

Suppose k is some fixed number and

$$\begin{aligned} h(0) &= k, \\ h(t+1) &= g(t, h(t)), \end{aligned} \tag{3.1}$$

where g is some given *total* function of two variables. Then h is said to be obtained from g by *primitive recursion*, or simply *recursion*.

Theorem 2.1

Let h be obtained from g as in (**??**), and let g be computable. Then h is also computable.

A slightly more complicated kind of recursion is involved when we have

$$\begin{aligned} h(x_1, \ldots, x_n, 0) &= f(x_1, \ldots, x_n), \\ h(x_1, \ldots, x_n, t+1) &= g(t, h(x_1, \ldots, x_n, t), x_1, \ldots, x_n). \end{aligned} \tag{3.2}$$

Here the function h of $n+1$ variables is said to be obtained by *primitive recursion*, or simply *recursion*, from the total functions f (of n variables) and g (of $n+2$ variables). Again we have

Theorem 2.2

Let h be obtained from f and g as in (**??**) and let f, g be computable. Then h is also computable.

3 PRC Classes

Now we need some functions on which to get started. These will be

$$s(x) = x + 1,$$
$$n(x) = 0,$$

and the *projection functions*

$$u_i^n(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n.$$

The functions s, n, and u_i^n are called the *initial functions*.

Definition 3.1

A class of total functions $\mathscr{C}$ is called a *PRC class* if

1. the initial functions belong to $\mathscr{C}$.
2. a function obtained from functions belonging to $\mathscr{C}$ by either composition or recursion also belongs to $\mathscr{C}$.

Then we have

Theorem 3.2

The class of computable functions is a PRC class.

Definition 3.3

A function is called *primitive recursive* if it can be obtained from the initial functions by a finite number of applications of composition and recursion.

It is obvious from this definition that

Corollary 3.4

The class of primitive recursive functions is a PRC class.

Actually we can say more:

Theorem 3.5

A function is primitive recursive if and only if it belongs to every PRC class.

Corollary 3.6

Every primitive recursive function is computable.

4 Some Primitive Recursive Functions

The *predecessor function* $p(x)$ is defined as follows:

$$p(x) = \begin{cases} x - 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$