Boolean Algebra and Logic Expressions

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1 Boolean Algebra and Logic Expressions

Boolean algebra provides a mathematical framework for analyzing and simplifying digital logic circuits. It operates on binary variables (0 and 1) with three basic operations: AND ($\cdot$), OR ($+$), and NOT ($\overline{A}$).

1.1 Basic Boolean Laws

1.1.1 Commutative Laws

- **AND Form**: $A \cdot B = B \cdot A$
- **OR Form**: $A + B = B + A$

Example:
If $A = 1$ and $B = 0$:
$1 \cdot 0 = 0 \cdot 1 = 0$
$1 + 0 = 0 + 1 = 1$

1.1.2 Associative Laws

- **AND Form**: $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
- **OR Form**: $A + (B + C) = (A + B) + C$

Example:
For $A = 1, B = 0, C = 1$:
$1 \cdot (0 \cdot 1) = (1 \cdot 0) \cdot 1 = 0$
$1 + (0 + 1) = (1 + 0) + 1 = 1$

1.1.3 Distributive Laws

- **AND over OR**: $A \cdot (B + C) = A \cdot B + A \cdot C$
- **OR over AND**: $A + B \cdot C = (A + B) \cdot (A + C)$

Example:
For $A = 1, B = 0, C = 1$:
$1 \cdot (0 + 1) = 1 \cdot 0 + 1 \cdot 1 = 1$
$1 + 0 \cdot 1 = (1 + 0) \cdot (1 + 1) = 1$

1.2 De Morgan's Theorems

These are fundamental for simplifying complex Boolean expressions:

- **First Theorem**: $\overline{A \cdot B} = \overline{A} + \overline{B}$
- **Second Theorem**: $\overline{A + B} = \overline{A} \cdot \overline{B}$

Example:
For $A = 1, B = 0$:
$\overline{1 \cdot 0} = \overline{1} + \overline{0} = 0 + 1 = 1$
$\overline{1 + 0} = \overline{1} \cdot \overline{0} = 0 \cdot 1 = 0$

A	B	$\overline{A \cdot B}$	$\overline{A} + \overline{B}$	Verification
0	0	1	1	✓
0	1	1	1	✓
1	0	1	1	✓
1	1	0	0	✓

3.3 Simplification of Logic Circuits

Boolean algebra helps reduce complex circuits to simpler forms with fewer gates.

Example 1: Simplification Using Identities

Simplify $Y = A \cdot B + A \cdot \overline{B}$:

$$\begin{aligned} Y &= A \cdot B + A \cdot \overline{B} \\ &= A \cdot (B + \overline{B}) \quad \text{(Distributive Law)} \\ &= A \cdot 1 \quad \text{(Complement Law)} \\ &= A \end{aligned}$$

Example 2: Applying De Morgan's Theorem

Simplify $\overline{A \cdot B} + \overline{A \cdot C}$:

$$\begin{aligned} \overline{A \cdot B} + \overline{A \cdot C} &= \overline{A} + \overline{B} + \overline{A} + \overline{C} \\ &= \overline{A} + \overline{B} + \overline{C} \end{aligned}$$

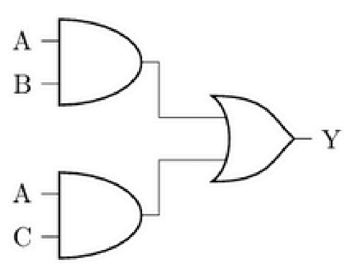

Figure 1: Original vs. Simplified Circuit

Practice Exercises

A. True or False

F 1. ( ) The expression $A + \overline{A}B$ simplifies to $A + B$.
F 2. ( ) De Morgan's first theorem states $\overline{A+B} = \overline{A} \cdot \overline{B}$.
T 3. ( ) The distributive law allows $A + BC = (A+B)(A+C)$.

B. Multiple Choice

1. Which law is used in $A(B+C) = AB + AC$?
a) Commutative b) Associative c) Distributive d) Identity

2. The simplified form of $\overline{\overline{A} + \overline{B}}$ is:
a) $A + B$ b) $A \cdot B$ c) $\overline{A} \cdot \overline{B}$ d) $\overline{A+B}$

3. If $X = \overline{A}B + A\overline{B}$, what is $\overline{X}$?
a) $AB + \overline{A}\,\overline{B}$ b) $\overline{A}B + A\overline{B}$ c) $A + B$ d) $A \cdot B$

C. Higher-Level Thinking

1. Design Problem: Simplify $Y = \overline{A}B + \overline{A}\,\overline{B} + AB$ and draw the optimized circuit.
2. Proof: Using Boolean algebra, prove that $(A+B)(\overline{A}+C)(B+C) = (A+B)(\overline{A}+C)$.
3. Application: Explain how Boolean simplification reduces power consumption in digital circuits.

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Group terms:

$$Y = \overline{A}B + \overline{AB} + AB$$

Factor the first two terms:

$$= \overline{A}(B + \overline{B}) + AB$$

Note that $B + \overline{B} = 1$:

$$= \overline{A}(1) + AB = \overline{A} + AB$$

Now apply the Absorption Law:

$$\overline{A} + AB = \overline{A} + B$$

✅ **Final simplified expression:**

$$Y = \overline{A} + B$$

Optimized Circuit:

- 1 **NOT** gate for A → $\overline{A}$
- 1 **OR** gate to combine $\overline{A} + B$

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Prove:

$$(A+B)(\overline{A}+C)(B+C)=(A+B)(\overline{A}+C)$$

Step-by-step proof:

Let's simplify the left-hand side:

Distribute $(A+B)(\overline{A}+C)$ first:

$$=(A+B)(\overline{A}+C)$$

Keep that expression.

Now multiply it with $(B+C)$:

$$=[(A+B)(\overline{A}+C)](B+C)$$

We can see that $(B+C)$ is **redundant** in this case because:

- The result of $(A+B)(\overline{A}+C)$ already **covers all 1-cases** of $(B+C)$
- So multiplying by (B + C) doesn't change the outcome

✅ Therefore,

$$(A+B)(\overline{A}+C)(B+C)=(A+B)(\overline{A}+C)$$

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How Boolean simplification reduces power consumption:

Boolean simplification:

- **Reduces the number of logic gates** needed in a digital circuit
- **Fewer gates** means:
    - Less switching activity → **lower dynamic power**
    - Smaller chip area → **less leakage current**
    - Simpler wiring → **faster response time**
- This results in:
    - **Lower power consumption**
    - **Improved reliability**
    - **Longer battery life** in portable devices

✅ **Conclusion:** Simplifying Boolean expressions optimizes both **performance** and **power efficiency** in digital electronics.