모두의연구소

수포자도 도전해 볼 만한

Linear Algebra in Mathematics

Decomposition

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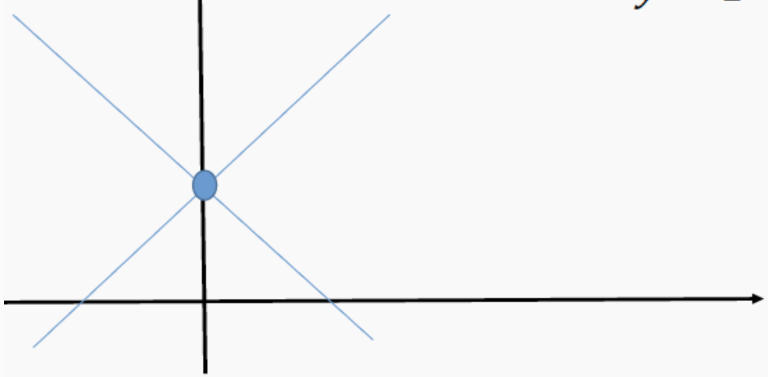
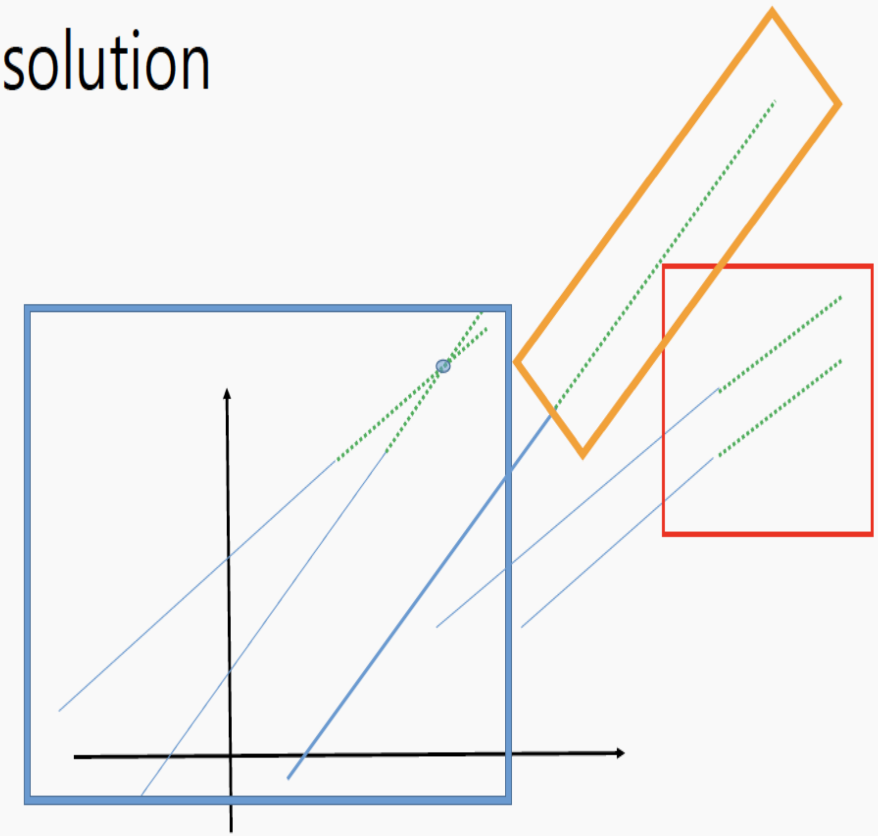
Equation

$$\begin{cases} x + y = 1 \\ -x + y = 1 \end{cases} \Rightarrow \begin{array}{r} x + y = 1 \\ 2y = 2 \end{array}$$

$$\Rightarrow \begin{array}{l} x = 0 \\ y = 1 \end{array}$$



solution



Linear system

ex. $\mathbf{A} = \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix}$

$$\Rightarrow \begin{array}{rrrl} x_1 & +3x_2 & -6x_3 & = 5 \\ -2x_1 & +4x_2 & +2x_3 & = 2 \\ x_1 & +x_2 & -x_3 & =10 \end{array}$$

$\Rightarrow$ **A**x=b

Note.

> Interchanging any two rows.
> Multiplying any row by a nonzero constant
> Adding a multiple of one row to another.

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -6 & 5 \\ -2 & 4 & 2 & 2 \\ 2 & 1 & -1 & 10 \end{array}\right)$$

$$\Rightarrow \mathbf{E}_2 + \mathbf{E}_3 \rightarrow \mathbf{E}_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -6 & 5 \\ -2 & 4 & 2 & 2 \\ 0 & 5 & 1 & 12 \end{array}\right)$$

Augmented matrix

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -6 & 5 \\ -2 & 4 & 2 & 2 \\ 0 & 5 & 1 & 12 \end{array}\right)$$

$$\Rightarrow 2E_1 + E_2 \rightarrow E_2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -6 & 5 \\ 0 & 10 & -10 & 12 \\ 0 & 5 & 1 & 12 \end{array}\right)$$

step 1. $R_2 + 2R_1 \longrightarrow R_2$

$$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

step 2. $R_3 - 2R_1 \longrightarrow R_2$.

$$\Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

step 3. $R_3 + 1/2R_2 \longrightarrow R_2$.

$$\Rightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}$$

LU Decomposition

$$(E_3E_2E_1)A = U \Leftrightarrow A = (E_3E_2E_1)^{-1}U$$
$$\Leftrightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Leftrightarrow A = LU$$

LU Decomposition

$$\begin{aligned} x_1 \quad +3x_2 \quad -6x_3 &= 5 \\ -2x_1 \quad +4x_2 \quad +2x_3 &= 2 \\ x_1 \quad +x_2 \quad -x_3 &= 10 \end{aligned}$$

$$\Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

LU Decomposition

$$(E_3E_2E_1)A = U \Leftrightarrow A = (E_3E_2E_1)^{-1}U$$

$$\Leftrightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Leftrightarrow A = LU$$

LU Decomposition

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

$$\Leftrightarrow \mathbf{LUx} = \mathbf{b}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

LU Decomposition

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

$\Leftrightarrow$ **Let Ux = y**

$\Leftrightarrow$ **Then Ly = b**

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

LU Decomposition

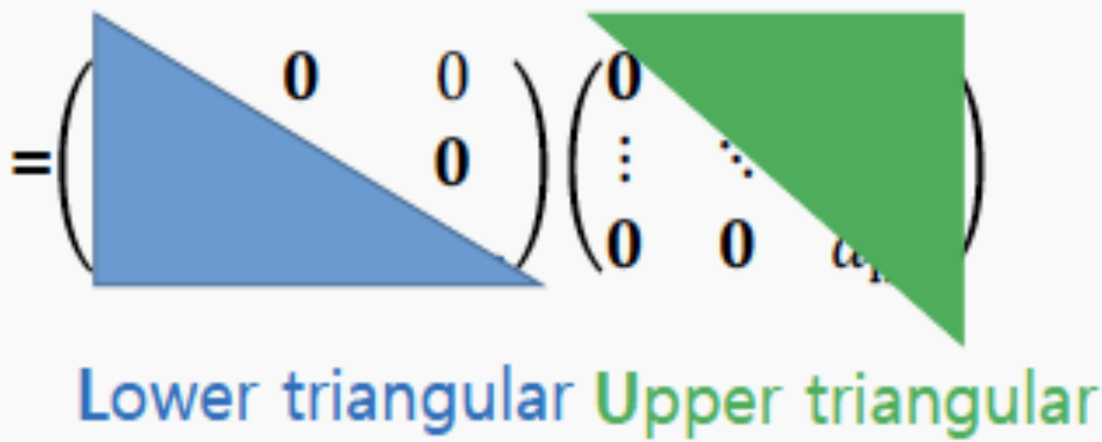
$$\Leftrightarrow \textbf{Solve } \mathbf{Ux} = \mathbf{y}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & -6 \\ 0 & 10 & -10 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 6 \end{pmatrix}$$

$$\Leftrightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 22/5 \\ 11/5 \\ 1 \end{pmatrix}$$

LU Decomposition

$$\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{\mathrm{m}1} & \cdots & a_{\mathrm{mn}} \end{pmatrix}=\begin{pmatrix} a_{11} & \mathbf{0} & 0 \\ \vdots & \ddots & \mathbf{0} \\ a_{\mathrm{m}1} & \cdots & a_{\mathrm{mn}} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & a_{\mathrm{mn}} \end{pmatrix}$$



QR Decomposition

$$\begin{aligned} x_1 \quad &+3x_2 \quad -6x_3 = 5 \\ -2x_1 \quad &+4x_2 \quad +2x_3 = 2 \\ x_1 \quad &+x_2 \quad -x_3 = 10 \end{aligned}$$

$$\Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

QR Decomposition

$$\mathbf{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}=(Q)\begin{pmatrix} 0 & & \\ \vdots & \ddots & \\ 0 & 0 & a_{mn} \end{pmatrix},$$

Upper triangular. R

$\mathbf{Q}^{-1}=\mathbf{Q}^{\mathrm{T}}$, orthogonal matrix

QR Decomposition

$$A = \begin{pmatrix} 1 & 3 & -6 \\ -2 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} \approx \begin{pmatrix} 0.33 & 0.67 & -0.67 \\ -0.67 & 0.67 & 0.33 \\ 0.67 & 0.33 & 0.67 \end{pmatrix} \begin{pmatrix} 3 & -1 & -4 \\ 0 & 5 & -3 \\ 0 & 0 & 4 \end{pmatrix}$$

step 1. $\mathbf{A} \rightarrow \mathbf{U}$

Using Gram Schmidt Orthogonalization

step 2. $\mathbf{U} \rightarrow \mathbf{Q}$

Normalize

step 3. $\mathbf{Q}^{\mathbf{T}}\mathbf{A} = \mathbf{Q}^{\mathbf{T}}(\mathbf{QR}) = \mathbf{R}.$

Decomposition based on λ

ex. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{\lambda}_1 = 1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \boldsymbol{\lambda}_2 = -1,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \overset{\text{matrix}}{\boxed{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}} \overset{\text{vector}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \underset{\text{scalar}}{\boxed{-1}} \overset{\text{vector}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \qquad \Rightarrow \overset{\text{matrix}}{\boxed{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}} \overset{\text{vector}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \boxed{-1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \overset{\text{vector}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Decomposition based on λ

For A

$\Rightarrow Ax=\lambda x$

$\Rightarrow Ax=\lambda I x$

$\Rightarrow (A-\lambda I)x=0$

$\Rightarrow \boxed{\det(A-\lambda I)=0}$ Characteristic Polynomial

ex. $\mathbf{A}=\begin{pmatrix}\mathbf{1} & \mathbf{2}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix}$

$\Rightarrow \det\left(\begin{pmatrix}\mathbf{1} & \mathbf{2}\\ \mathbf{0} & -\mathbf{1}\end{pmatrix}-\lambda\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}\right)=\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$

$\Rightarrow \det\left(\begin{pmatrix}\mathbf{1}-\lambda & \mathbf{2}\\ \mathbf{0} & -\mathbf{1}-\lambda\end{pmatrix}\right)=\mathbf{0}$

$\Rightarrow (\mathbf{1}-\lambda)(-\mathbf{1}-\lambda)=0$

$\Rightarrow \lambda^{2}-\mathbf{1}=\mathbf{0}$

$\Rightarrow \lambda=\mathbf{1},\ \lambda=-\mathbf{1}$

Decomposition based on λ

$$\Rightarrow \lambda = 1, \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x + 2y \\ -y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow y = 0$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for any } x$$

$$\Rightarrow \lambda = -1, \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x + 2y \\ -y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$$\Rightarrow 2x + 2y = 0 \Rightarrow y = -x$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for any } x$$

Dia gonalization

ex. $\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$

$\Longrightarrow \mathbf{A} = \mathbf{PDP^{-1}}$

$\Leftrightarrow \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$

Spectral Decomposition

ex. $\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$

$\Longrightarrow$ $\mathbf{A} = \mathbf{PDP^{-1}}$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} v_{\lambda_1} & v_{\lambda_2} \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} v_{\lambda_1} & v_{\lambda_2} \end{pmatrix}^{-1}$$

$s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$

ex. $\mathbf{A}$ = symmetric

$\Longrightarrow$ $\mathbf{A} = \mathbf{QDQ^T}$

$$\Leftrightarrow A = \begin{pmatrix} v_{\lambda_1} & v_{\lambda_2} \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} v_{\lambda_1} & v_{\lambda_2} \end{pmatrix}^T$$

$s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$

Singular value

SVD using Eigenvalue

$$\mathbf{A}^{\mathbf{T}}\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

$\Longrightarrow$ step 1. $\mathbf{A}^{\mathbf{T}}\mathbf{A} \Longrightarrow \mathbf{s}_1 \geq \boldsymbol{s}_2 \Longrightarrow \mathbf{v}_1, \mathbf{v}_2$

$$\mathbf{A}\mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 10 \end{pmatrix}$$

$\Longrightarrow$ step 2. $\mathbf{A}^{\mathbf{T}}\mathbf{A} \Longrightarrow \mathbf{s}_1 \geq \boldsymbol{s}_2 \geq \boldsymbol{s}_3 \Longrightarrow \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$

SVD

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathbf{T}}, \quad \mathbf{A}^{-1} = \mathbf{V}(\mathbf{D}^{-1})^{\mathbf{T}}\ \mathbf{U}^{\mathbf{T}}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \boxed{(\mathbf{u_1} \quad \mathbf{u_2} \quad \mathbf{u_3})} \begin{pmatrix} \mathbf{s_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{s_2} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \boxed{(\mathbf{v_1} \quad \mathbf{v_2})^T}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = (\mathbf{v_1} \quad \mathbf{v_2}) \begin{pmatrix} \mathbf{1/s_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/s_2} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}^T (\mathbf{u_1} \quad \mathbf{u_2} \quad \mathbf{u_3})^T$$