MATH3821 Assignment 1

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Question 1

For Simple Linear Regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$.

a) Let $\beta_0 = \alpha - \beta_1 \bar{x}$. Then the SLR model can be expressed as $y_i = \alpha + \beta_1(x_i - \bar{x}) + \epsilon_i$.

b) α is the intercept of the new model?

c) To find the closed form formula of the LSE,

$$
\begin{aligned}
RSS(\beta_1) &= \sum_{i=1}^{n} [y_i - (\alpha + \beta_1(x_i - \bar{x}))]^2 \\
\frac{\mathrm{d}RSS(\beta_1)}{\mathrm{d}\alpha} &= -2\sum_{i=1}^{n} (y_i - (\alpha + \beta_1(x_i - \bar{x}))) \qquad (1) \\
\frac{\mathrm{d}RSS(\beta_1)}{\mathrm{d}\beta_1} &= -2\sum_{i=1}^{n} (y_i - (\alpha + \beta_1(x_i - \bar{x}))(x_i - \bar{x})) \qquad (2)
\end{aligned}
$$

Let Equation (1) = 0

$$
\begin{aligned}
&-2\sum_{i=1}^{n} (y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))) = 0 \\
&\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{\alpha}_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i + \sum_{i=1}^{n} \hat{\beta}_1 \bar{x} = 0 \\
&n\hat{\alpha}_i = \sum_{i=1}^{n} y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i + n\hat{\beta}_1 \bar{x} \\
&\hat{\alpha}_i = \frac{1}{n}\sum_{i=1}^{n} y_i - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 \bar{x} \\
&\hat{\alpha}_i = \bar{y}
\end{aligned}
$$

Let Equation (2) = 0

$$
\begin{aligned}
&-2\sum_{i=1}^{n} \Big(y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))(x_i - \bar{x})\Big) = 0 \\
&\sum_{i=1}^{n} y_i(x_i - \bar{x}) - \sum_{i=1}^{n} \hat{\alpha}(x_i - \bar{x}) - \sum_{i=1}^{n} \hat{\beta}_1(x_i - \bar{x})^2 = 0 \\
&\hat{\beta}_1 \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} (y_i - \hat{\alpha})(x_i - \bar{x})
\end{aligned}
$$

since $\hat{\alpha} = \bar{y}$

$$
\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}
$$

d)

$$\begin{aligned}
Var[\hat{\alpha}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] \\
&= \frac{1}{n^2} Var\left[\sum_{i=1}^{n} y_i\right]
\end{aligned}$$

Since all y_i's are uncorrelated

$$\begin{aligned}
&= \frac{1}{n^2}\sum_{i=1}^{n} Var[y_i] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{n\sigma^2}{n^2} \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Therefore $Var[\hat{\alpha}] = \frac{\sigma^2}{n}$.

We note that $\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i$, since

$$\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n} y_i(x_i - \bar{x}) - \sum_{i=1}^{n} \bar{y}(x_i - \bar{x})$$

and we notice that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \frac{1}{n} n \sum_{i=1}^{n} x_i = 0$$

To calculate $Var[\hat{\beta_1}]$,

$$\begin{aligned}
Var[\hat{\beta_1}] &= Var\left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})y_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] \\
&= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} Var\left[\sum_{i=1}^{n}(x_i - \bar{x})y_i\right] \\
&= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n} Var\left[(x_i - \bar{x})y_i\right] \\
&= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n}(x_i - \bar{x}) Var\left[y_i\right] \\
&= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n}(x_i - \bar{x})\sigma^2 \\
&= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}
\end{aligned}$$

Let $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})y_i$.
To calculate $Cov[\hat{\alpha}, \hat{\beta_1}]$

$$
\begin{aligned}
Cov[\hat{\alpha}, \hat{\beta_1}] &= Cov[\bar{y}, \hat{\beta_1}] \\
&= Cov\left[\bar{y}, \frac{S_{xy}}{S_{xx}}\right] \\
&= \frac{1}{S_{xx}} Cov\left[\bar{y}, S_{xy}\right] \\
&= \frac{1}{S_{xx}} Cov\left[\frac{1}{n}\sum_{i=1}^{n} y, S_{xy}\right] \\
&= \frac{1}{nS_{xx}} Cov\left[\sum_{i=1}^{n} y, \sum_{j=1}^{n}(x_j - \bar{x})y_j)\right] \\
&= \frac{1}{nS_{xx}} \sum_{i=1}^{n}\sum_{j=1}^{n}(x_j - \bar{x})Cov[y_i, y_j] \\
&= \frac{1}{nS_{xx}} \sum_{i=j}^{n}(x_i - \bar{x})\sigma^2 \\
&= 0
\end{aligned}
$$

e)

```r
set.seed(1234567)
x = runif(1000)
eps = rnorm(1000)
y = 5 + 10*x + eps
```

f)

g)

Question 2

Given n independent binary random variables $Y_1 \cdots Y_n$ with

$$P(Y_i = 1) = \pi_i \text{ and } P(Y_i = 0) = 1 - \pi_i$$

The probability function of Y_i is:

$$\pi_i^{Y_i}(1 - \pi_i)^{1-Y_i}$$

where $Y_i = 0$ or 1

a) For a probability function to belong to the exponential family of distributions, it must follow the formula:

$$f(y; \theta, \phi) = K(y, \frac{p}{\phi}) \exp\left(\frac{p}{\phi}\{y\theta - c(\theta)\}\right)$$

For the given probability density function:

$$
\begin{aligned}
f(y;\pi) &= \pi_i^y(1-\pi_i)^{1-y} \\
&= \exp\left(\log \pi_i^y(1-\pi_i)^{1-y}\right) \\
&= \exp\left(\log \pi_i^y + \log(1-\pi_i)^{1-y})\right) \\
&= \exp\left(y\log \pi_i + (1-y)\log(1-\pi_i)\right) \\
&= \exp\left(y\log(\frac{\pi}{1-\pi}) + \log(1-\pi)\right)
\end{aligned}
$$

With $p = 1$ and $\phi = 1$, the above equation follows the form of the exponential family of distribution where $K(y, \frac{p}{\phi}) = 1$, $\theta = \log(\frac{\pi}{1-\pi})$ and $c(\theta) = -\log(1-\pi) = -\log(1-\frac{e^\theta}{1+e^\theta})$ where $\pi = \frac{e^\theta}{1+e^\theta}$.

b) As seen in 2a, the naturalised parameter is $\theta = \log(\frac{\pi}{1-\pi})$

c) As seen in 2a, the cumulant generator is $c(\theta) = -\log(1-\frac{e^\theta}{1+e^\theta})$. Since $\mathrm{E}[Y] = c'(\theta)$, $c'(\theta) = -(\frac{e^\theta}{1+e^\theta}) = -(-\pi) = \pi$. Therefore, $\mathrm{E}[Y] = \pi$.

d) Given the link function:

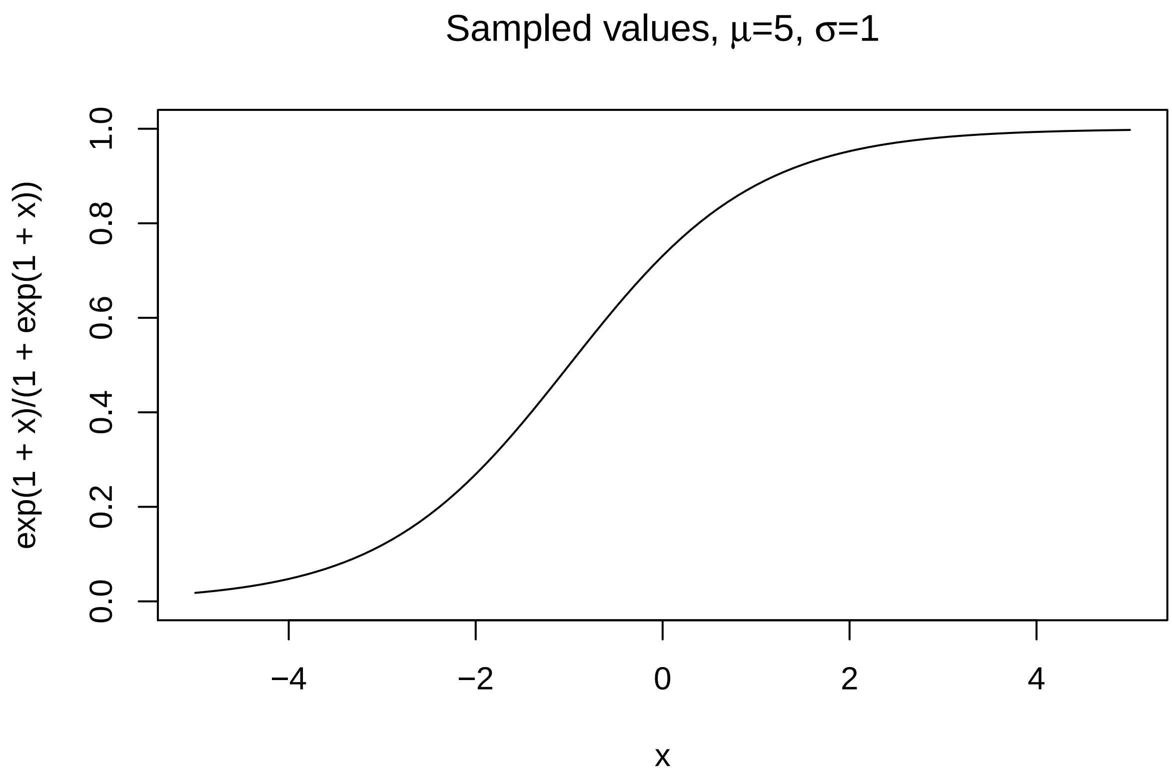
$$
g(\pi) = \log(\frac{\pi}{1-\pi}) = e^{x^T\beta}
$$

it can be rearranged in terms of the probability π,

$$
\begin{aligned}
e^{x^T\beta} &= \log(\frac{\pi}{1-\pi}) \\
e^{x^T\beta} - \pi e^{x^T\beta} &= \pi \\
\pi &= \frac{e^{x^T\beta}}{1+e^{x^T\beta}}
\end{aligned}
$$

e)

```
curve(exp(1+x)/(1+exp(1+x)), xlim = c(-5, 5), ylim = c(0, 1), main=expression(paste("Sampled values, ",
    "=1")))
```



It shows the log odds of the insecticide working with a given probability?

f) The following probability density function:

$$f(y;\theta) = \frac{1}{\phi}\exp\left(\frac{(y-\theta)}{\phi} - \exp\left[\frac{(y-\theta)}{\phi}\right]\right)$$

is NOT in the exponential family of distributions as it does not follow the form of a probability density function in the exponential family.