

MATH3821 Assignment 1

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Question 1

For Simple Linear Regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$.

- Let $\beta_0 = \alpha - \beta_1 \bar{x}$. Then the SLR model can be expressed as $y_i = \alpha + \beta_1(x_i - \bar{x}) + \epsilon_i$.
- α equivalent to the mean response ($\alpha = \beta_0 + \beta_1 \bar{x}$) of the previous SLR model and is the intercept of the new model
- To find the closed form formula of the LSE,

$$RSS(\beta_1) = \sum_{i=1}^n [y_i - (\alpha + \beta_1(x_i - \bar{x}))]^2$$

$$\frac{\partial RSS(\beta_1)}{\partial \alpha} = -2 \sum_{i=1}^n (y_i - (\alpha + \beta_1(x_i - \bar{x}))) \quad (1)$$

$$\frac{\partial RSS(\beta_1)}{\partial \beta_1} = -2 \sum_{i=1}^n (y_i - (\alpha + \beta_1(x_i - \bar{x}))) (x_i - \bar{x}) \quad (2)$$

Let Equation (1) = 0

$$-2 \sum_{i=1}^n (y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))) = 0$$

$$\sum_{i=1}^n y_i - \sum_{i=1}^n \hat{\alpha} - \sum_{i=1}^n \hat{\beta}_1 x_i + \sum_{i=1}^n \hat{\beta}_1 \bar{x} = 0$$

$$n\hat{\alpha} = \sum_{i=1}^n y_i - \hat{\beta}_1 \sum_{i=1}^n x_i + n\hat{\beta}_1 \bar{x}$$

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^n y_i - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 \bar{x}$$

$$\hat{\alpha} = \bar{y}$$

Let Equation (2) = 0

$$-2 \sum_{i=1}^n (y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))) (x_i - \bar{x}) = 0$$

$$\sum_{i=1}^n y_i (x_i - \bar{x}) - \sum_{i=1}^n \hat{\alpha} (x_i - \bar{x}) - \sum_{i=1}^n \hat{\beta}_1 (x_i - \bar{x})^2 = 0$$

$$\hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (y_i - \hat{\alpha}) (x_i - \bar{x})$$

since $\hat{\alpha} = \bar{y}$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

d)

$$\begin{aligned} \text{Var}[\hat{\alpha}] &= \text{Var}\left[\frac{1}{n} \sum_{i=1}^n y_i\right] \\ &= \frac{1}{n^2} \text{Var}\left[\sum_{i=1}^n y_i\right] \end{aligned}$$

Since all y_i 's are uncorrelated

$$\begin{aligned} &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}[y_i] \\ &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{n\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Therefore $\text{Var}[\hat{\alpha}] = \frac{\sigma^2}{n}$.

We note that $\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^n (x_i - \bar{x})y_i$, since

$$\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^n y_i(x_i - \bar{x}) - \sum_{i=1}^n \bar{y}(x_i - \bar{x})$$

and we notice that

$$\sum_{i=1}^n (x_i - \bar{x}) = \sum_{i=1}^n x_i - \frac{1}{n}n \sum_{i=1}^n x_i = 0$$

To calculate $\text{Var}[\hat{\beta}_1]$,

$$\begin{aligned} \text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] \\ &= \frac{1}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \text{Var}\left[\sum_{i=1}^n (x_i - \bar{x})y_i\right] \end{aligned}$$

Since all y_i are independent

$$\begin{aligned} &= \frac{1}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \sum_{i=1}^n \text{Var}[(x_i - \bar{x})y_i] \\ &= \frac{1}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \sum_{i=1}^n (x_i - \bar{x})^2 \text{Var}[y_i] \\ &= \frac{1}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \sum_{i=1}^n (x_i - \bar{x})^2 \sigma^2 \\ &= \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{aligned}$$

Let $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2$ and $S_{xy} = \sum_{i=1}^n (x_i - \bar{x})y_i$.
 To calculate $Cov[\hat{\alpha}, \hat{\beta}_1]$

$$\begin{aligned}
 Cov[\hat{\alpha}, \hat{\beta}_1] &= Cov[\bar{y}, \hat{\beta}_1] \\
 &= Cov\left[\bar{y}, \frac{S_{xy}}{S_{xx}}\right] \\
 &= \frac{1}{S_{xx}} Cov[\bar{y}, S_{xy}] \\
 &= \frac{1}{S_{xx}} Cov\left[\frac{1}{n} \sum_{i=1}^n y_i, S_{xy}\right] \\
 &= \frac{1}{nS_{xx}} Cov\left[\sum_{i=1}^n y_i, \sum_{j=1}^n (x_j - \bar{x})y_j\right] \\
 &= \frac{1}{nS_{xx}} \sum_{i=1}^n \sum_{j=1}^n (x_j - \bar{x}) Cov[y_i, y_j]
 \end{aligned}$$

When $i \neq j$, $Cov[y_i, y_j] = 0$ since all y_i are uncorrelated with each other, and $Cov[y_i, y_j] = Var[y_i]$ when $i = j$

$$\begin{aligned}
 &= \frac{1}{nS_{xx}} \sum_{i=1}^n (x_i - \bar{x}) \sigma^2 \\
 &= 0
 \end{aligned}$$

e)

```

set.seed(1234567)
x = runif(1000)
eps = rnorm(1000)
y = 5 + 10*x + eps
model <- y~x
RSS <- function(b) c(-2 * sum(y - b[1] - b[2] * (x-mean(x))), -2 * sum((y - b[1] - b[2] * (x-mean(x)))
                                * (x-mean(x))))

#This function gives us the gradient of the RSS
bn <- c(0,0)
gamma <- 0.00001
kmax <- 100000

for (k in 0:kmax) {
  bnp1 <- bn - gamma*RSS(bn)
  if(sum(RSS(bn)^2) <= 0.00001){
    cat("b(alpha, beta): ", bnp1, "-- RSS:", RSS(c(bnp1[1],bnp1[2])), "\n", "Iterations:", k, "\n")
    break
  }
  bn <- bnp1
}

```

```

## b(alpha, beta): 10.06591 10.07213 -- RSS: -8.663772e-11 -0.003155449
## Iterations: 8218

```

```

#This algorithm starts at alpha = 0 and b1 = 0
#I get the l2 norm of score since the RSS function gives the gradient which is the score
#thus the sum of squares of gradient (RSS) must be less than 0.00001

```

```

#alpha from minimisation
print('Alpha from minimisation')

## [1] "Alpha from minimisation"
bnp1[1]

## [1] 10.06591
#Using the closed form formula in c) we find alpha
print('Alpha from the closed form formula in 1c')

## [1] "Alpha from the closed form formula in 1c"
mean(y)

## [1] 10.06591
#Finding beta
print('Beta from minimisation')

## [1] "Beta from minimisation"
bnp1[2]

## [1] 10.07213
print('Beta from the closed form formula in 1c')

## [1] "Beta from the closed form formula in 1c"
sum(((y-mean(y))*(x-mean(x)))/(sum((x-mean(x))^2)))

## [1] 10.07215
#We can see that alpha has the same value as does the beta
#about 8218 iterations were required
#Below is the working to find the gradient function for the RSS

```

To get the gradient of the RSS we must get the first derivative of:

$$S(\beta_1, \beta_2) = \sum_{i=1}^n (y_i - \beta_1 - \beta_2 x_i)^2$$

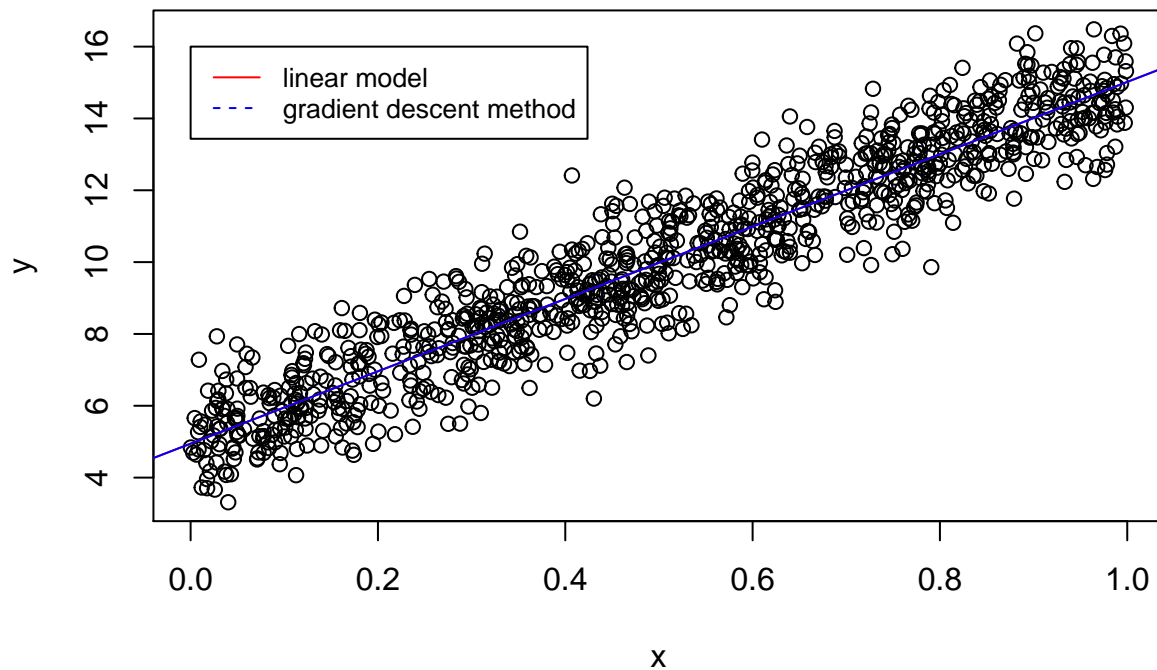
$$S'(\beta_1, \beta_2) = (-2 \sum_{i=1}^n (y_i - \beta_1 - \beta_2 x_i), -2 \sum_{i=1}^n x_i (y_i - \beta_1 - \beta_2 x_i))$$

f)

```

plot(x,y)
lm = lm(y~x)
abline(lm, col = "red")
abline(bnp1[1]-bnp1[1]*mean(x), bnp1[2], col = 'blue')
legend(0,16,legend=c("linear model", "gradient descent method"),
      col=c("red", "blue"), lty=1:2, cex=0.8)

```



Note that the lines overlap due to both methods achieving the same result

```
g)
beta = c(0,0)
imax = 100000
RSShess = function(b) cbind(c(2*length(x), 2*sum(x-mean(x))), c(2*sum(x-mean(x)), 2*sum((x-mean(x))^2)))
#This function gives us the hessian matrix for the RSS

for (i in 0:imax) {
  beta1 <- beta - solve(RSShess(beta))%*%RSS(beta)
  if(sum(RSS(beta)^2) <= 0.00001){
    cat("beta: ", beta1, "-- RSS:", RSS(c(beta1[1],beta1[2])), "\n", "Iterations:", i, "\n")
    break
  }
  beta <- beta1
}

## beta:  10.06591 10.07215 -- RSS: -1.37159e-12 -1.039592e-13
## Iterations: 1

#alpha from NR
print('Alpha from Newton-Raphson')

## [1] "Alpha from Newton-Raphson"
beta1[1]

## [1] 10.06591
```

```

#Using the closed form formula in c) we find alpha
print('Alpha from closed form formula in 1c')

## [1] "Alpha from closed form formula in 1c"
mean(y)

## [1] 10.06591
#Finding beta
print('Beta from Newton-Raphson')

## [1] "Beta from Newton-Raphson"
beta1[2]

## [1] 10.07215
print('Beta from closed form formula')

## [1] "Beta from closed form formula"
sum(((y-mean(y))*(x-mean(x)))/(sum((x-mean(x))^2)))

## [1] 10.07215
#We can see that values are the same
#1 iterations was required
#It took alot less iterations than the gradient descent method
#This could be due to the fact that the Newton-raphson method
#accounts for the curvature of the loglikelihood and as such should need less iterations
#Below is the working to find the hessian matrix for the RSS

```

To get the Hessian matrix of the RSS we need to further derive the gradient found in Question 1e with respect to α and β_1 .

$$\begin{aligned}
 S'(\alpha, \beta_1) &= (-2 \sum_{i=1}^n (y_i - \alpha - \beta_1(x_i - \bar{x})), -2 \sum_{i=1}^n (x_i - \bar{x})(y_i - \alpha - \beta_1(x_i - \bar{x}))) \\
 \frac{\partial^2 S(\alpha, \beta_1)}{\partial \alpha^2} &= 2 \sum_{i=1}^n 1 \\
 &= 2n \\
 \frac{\partial^2 S(\alpha, \beta_1)}{\partial \beta_1^2} &= 2 \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x}) \\
 &= 2 \sum_{i=1}^n (x_i - \bar{x})^2 \\
 \frac{\partial^2 S(\alpha, \beta_1)}{\partial \alpha \partial \beta_1} &= 2 \sum_{i=1}^n (x_i - \bar{x}) \\
 &= \frac{\partial^2 S(\alpha, \beta_1)}{\partial \beta_1 \partial \alpha}
 \end{aligned}$$

Question 2

Given n independent binary random variables $Y_1 \cdots Y_n$ with

$$P(Y_i = 1) = \pi_i \text{ and } P(Y_i = 0) = 1 - \pi_i$$

The probability function of Y_i is:

$$\pi_i^{Y_i} (1 - \pi_i)^{1-Y_i}$$

where $Y_i = 0$ or 1

- a) For a probability function to belong to the exponential family of distributions, it must follow the formula:

$$f(y; \theta, \phi) = K(y, \frac{p}{\phi}) \exp \left(\frac{p}{\phi} \{y\theta - c(\theta)\} \right)$$

For the given probability density function:

$$\begin{aligned} f(y; \pi) &= \pi^y (1 - \pi)^{1-y} \\ &= \exp(\log \pi^y (1 - \pi)^{1-y}) \\ &= \exp(\log \pi^y + \log(1 - \pi)^{1-y}) \\ &= \exp(y \log \pi + (1 - y) \log(1 - \pi)) \\ &= \exp \left(y \log \left(\frac{\pi}{1 - \pi} \right) + \log(1 - \pi) \right) \end{aligned}$$

With $p = 1$ and $\phi = 1$, the above equation follows the form of the exponential family of distribution where $K(y, \frac{p}{\phi}) = 1$, $\theta = \log(\frac{\pi}{1 - \pi})$ and $c(\theta) = -\log(1 - \pi) = -\log(1 - \frac{e^\theta}{1 + e^\theta})$ where $\pi = \frac{e^\theta}{1 + e^\theta}$.

- b) As seen in 2a, the naturalised parameter is $\theta = \log(\frac{\pi}{1 - \pi})$
- c) As seen in 2a, the cumulant generator is $c(\theta) = -\log(1 - \frac{e^\theta}{1 + e^\theta})$. Since $E[Y] = c'(\theta)$, $c'(\theta) = -(\frac{e^\theta}{1 + e^\theta}) = -(-\pi) = \pi$. Therefore, $E[Y] = \pi$.
- d) Given the link function:

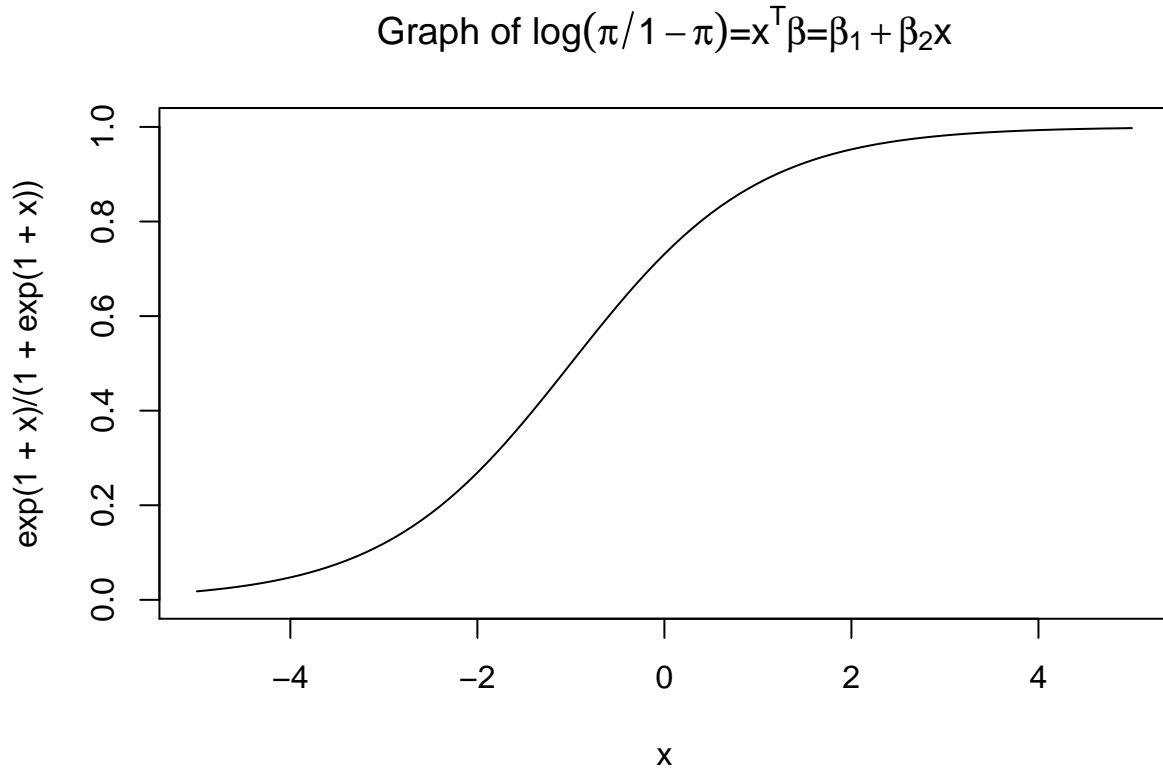
$$g(\pi) = \log\left(\frac{\pi}{1 - \pi}\right) = e^{x^T \beta}$$

it can be rearranged in terms of the probability π ,

$$\begin{aligned} e^{x^T \beta} &= \log\left(\frac{\pi}{1 - \pi}\right) \\ e^{x^T \beta} - \pi e^{x^T \beta} &= \pi \\ \pi &= \frac{e^{x^T \beta}}{1 + e^{x^T \beta}} \end{aligned}$$

- e)

```
curve(exp(1+x)/(1+exp(1+x)), xlim = c(-5, 5), ylim = c(0, 1),
      main=expression(paste("Graph of ", log(pi/1-pi), '=', x^{T}, beta, '=', beta[1]+beta[2], 'x'))
)
```



It shows the log odds of the insecticide working with a given probability, with increasing dosage increasing the probability of insecticide working.

f) The following probability density function:

$$f(y; \theta) = \frac{1}{\phi} \exp \left(\frac{(y - \theta)}{\phi} - \exp \left[\frac{(y - \theta)}{\phi} \right] \right)$$

is NOT in the exponential family of distributions as it does not follow the form of a probability density function in the exponential family. There is not a way to rearrange the probability density function such that it follows the form of a function in the exponential family, given in 2a, as isolating y from the inner exponential function is not possible and there is no θ component/naturalised parameter.