MATH3821 Assignment 1

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Question 1

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Question 2

Given n independent binary random variables $Y_1 \cdots Y_n$ with

$$P(Y_i = 1) = \pi_i \text{ and } P(Y_i = 0) = 1 - \pi_i$$

The probability function of Y_i is:

$$\pi_i^{Y_i}(1-\pi_i)^{1-Y_i}$$

where $Y_i = 0$ or 1

(a) For a probability function to belong to the exponential family of distributions, it must follow the formula:

$$f(y;\theta,\phi) = K(y, \frac{p}{\phi}) \exp\left(\frac{p}{\phi}\{y\theta - c(\theta)\}\right)$$

For the given probability density function:

$$\begin{aligned}
f(y;\pi) &= \pi_i^y (1-\pi_i)^{1-y} \\
&= \exp\left(\log \pi_i^y (1-\pi_i)^{1-y}\right) \\
&= \exp\left(\log \pi_i^y + \log(1-\pi_i)^{1-y})\right) \\
&= \exp\left(y \log \pi_i + (1-y)\log(1-\pi_i)\right) \\
&= \exp\left(y \log(\frac{\pi}{1-\pi}) + \log(1-\pi)\right)
\end{aligned}$$

With $p = 1$ and $\phi = 1$, the above equation follows the form of the exponential family of distribution where $K(y, \frac{p}{\phi}) = 1$, $\theta = \log(\frac{\pi}{1-\pi})$ and $c(\theta) = -\log(1-\pi) = -\log(1 - \frac{e^\theta}{1+e^\theta})$ where $\pi = \frac{e^\theta}{1+e^\theta}$

(b) As seen in 2a, the naturalised parameter is $\theta = \log(\frac{\pi}{1-\pi})$

(c) As seen in 2a, the cumulant generator is $c(\theta) = -\log(1 - \frac{e^\theta}{1+e^\theta})$. Since $\mathrm{E}[Y] = c'(\theta)$, $c'(\theta) = -(\frac{e^\theta}{1+e^\theta}) = -(-\pi) = \pi$. Therefore, $\mathrm{E}[Y] = \pi$.

(d) Given the link function

$$g(\pi) = \log(\frac{\pi}{1-\pi}) = e^{x^T\beta}$$

it can be rearranged in terms of the probability π,

$$\begin{aligned} e^{x^T\beta} &= \log(\frac{\pi}{1-\pi}) \\ e^{x^T\beta} - \pi e^{x^T\beta} &= \pi \\ \pi &= \frac{e^{x^T\beta}}{1+e^{x^T\beta}} \end{aligned}$$

(e) INSERT GRAPH

It shows the log odds of the insecticide working at a given probability?

(f) The following probability density function

$$f(y;\theta) = \frac{1}{\phi}\exp\left(\frac{(y-\theta)}{\phi} - \exp\left[\frac{(y-\theta)}{\phi}\right]\right)$$

is NOT in the exponential family of distributions as it does not follow the form of a probability density function in the exponential family.

Question 3

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)

Question 4

(a)

(b)

(c)

(d)

(e)

Question 5

(a)

(b)

(c)

(d)