MATH3821 Assignment 1

Stephen Sung

Question 1

For Simple Linear Regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma^2)$.

a) Let $\beta_0 = \alpha - \beta_1 \bar{x}$. Then the SLR model can be expressed as $y_i = \alpha + \beta_1(x_i - \bar{x}) + \epsilon_i$.

b) α equivalent to the mean response ($\alpha = \beta_0 + \beta_1 \bar{x}$) of the previous SLR model and is the intercept of the new model.

c) To find the closed form formula of the LSE,

$$
\begin{aligned}
RSS(\beta_1) &= \sum_{i=1}^{n}[y_i - (\alpha + \beta_1(x_i - \bar{x}))]^2 \\
\frac{\mathrm{d}RSS(\beta_1)}{\mathrm{d}\alpha} &= -2\sum_{i=1}^{n}(y_i - (\alpha + \beta_1(x_i - \bar{x}))) \qquad (1) \\
\frac{\mathrm{d}RSS(\beta_1)}{\mathrm{d}\beta_1} &= -2\sum_{i=1}^{n}(y_i - (\alpha + \beta_1(x_i - \bar{x}))(x_i - \bar{x})) \qquad (2)
\end{aligned}
$$

Let Equation (1) = 0

$$
\begin{aligned}
&-2\sum_{i=1}^{n}(y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))) = 0 \\
&\sum_{i=1}^{n} y_i - \sum_{i=1}^{n}\hat{\alpha}_i - \sum_{i=1}^{n}\hat{\beta}_1 x_i + \sum_{i=1}^{n}\hat{\beta}_1\bar{x} = 0 \\
&n\hat{\alpha}_i = \sum_{i=1}^{n} y_i - \hat{\beta}_1\sum_{i=1}^{n} x_i + n\hat{\beta}_1\bar{x} \\
&\hat{\alpha}_i = \frac{1}{n}\sum_{i=1}^{n} y_i - \hat{\beta}_1\bar{x} + \hat{\beta}_1\bar{x} \\
&\hat{\alpha}_i = \bar{y}
\end{aligned}
$$

Let Equation (2) = 0

$$
\begin{aligned}
&-2\sum_{i=1}^{n}\Big(y_i - (\hat{\alpha} + \hat{\beta}_1(x_i - \bar{x}))(x_i - \bar{x})\Big) = 0 \\
&\sum_{i=1}^{n} y_i(x_i - \bar{x}) - \sum_{i=1}^{n}\hat{\alpha}(x_i - \bar{x}) - \sum_{i=1}^{n}\hat{\beta}_1(x_i - \bar{x})^2 = 0 \\
&\hat{\beta}_1\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(y_i - \hat{\alpha})(x_i - \bar{x})
\end{aligned}
$$

since $\hat{\alpha} = \bar{y}$

$$
\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}
$$

d)

$$\begin{aligned} Var[\hat{\alpha}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] \\ &= \frac{1}{n^2} Var\left[\sum_{i=1}^{n} y_i\right] \end{aligned}$$

Since all y_i's are uncorrelated

$$\begin{aligned} &= \frac{1}{n^2}\sum_{i=1}^{n} Var[y_i] \\ &= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\ &= \frac{n\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Therefore $Var[\hat{\alpha}] = \frac{\sigma^2}{n}$.

We note that $\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i$, since

$$\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) = \sum_{i=1}^{n} y_i(x_i - \bar{x}) - \sum_{i=1}^{n} \bar{y}(x_i - \bar{x})$$

and we notice that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \frac{1}{n} n \sum_{i=1}^{n} x_i = 0$$

To calculate $Var[\hat{\beta_1}]$,

$$\begin{aligned} Var[\hat{\beta_1}] &= Var\left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})y_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] \\ &= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} Var\left[\sum_{i=1}^{n}(x_i - \bar{x})y_i\right] \\ &= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n} Var\left[(x_i - \bar{x})y_i\right] \\ &= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 Var\left[y_i\right] \\ &= \frac{1}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \sigma^2 \\ &= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \end{aligned}$$

Let $S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and $S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})y_i$.
To calculate $Cov[\hat{\alpha}, \hat{\beta_1}]$

$$
\begin{aligned}
Cov[\hat{\alpha}, \hat{\beta_1}] &= Cov[\bar{y}, \hat{\beta}_1] \\
&= Cov\left[\bar{y}, \frac{S_{xy}}{S_{xx}}\right] \\
&= \frac{1}{S_{xx}} Cov\left[\bar{y}, S_{xy}\right] \\
&= \frac{1}{S_{xx}} Cov\left[\frac{1}{n}\sum_{i=1}^{n} y_i, S_{xy}\right] \\
&= \frac{1}{nS_{xx}} Cov\left[\sum_{i=1}^{n} y_i, \sum_{j=1}^{n}(x_j - \bar{x})y_j)\right] \\
&= \frac{1}{nS_{xx}} \sum_{i=1}^{n}\sum_{j=1}^{n}(x_j - \bar{x}) Cov[y_i, y_j]
\end{aligned}
$$

When $i \neq j$, $Cov[y_i, y_j] = 0$ since all y_i are uncorrelated with each other, and $Cov[y_i, y_j] = Var[y_i]$ when $i = j$

$$
\begin{aligned}
&= \frac{1}{nS_{xx}} \sum_{i=j}^{n}(x_i - \bar{x})\sigma^2 \\
&= 0
\end{aligned}
$$

e)

```r
set.seed(1234567)
x = runif(1000)
eps = rnorm(1000)
y = 5 + 10*x + eps
model <- y~x
RSS <- function(b) c(-2 * sum(y - b[1] - b[2] * x), -2 * sum((y - b[1] - b[2] * x) * x))
#This function gives us the gradient of the RSS
bn <- c(0,0)
gamma <- 0.00001
kmax <- 100000

for (k in 0:kmax) {
  bnp1 <- bn - gamma*RSS(bn)
  if(sum(RSS(bn)^2) <= 0.00001){
    cat("b: ", bnp1, "-- RSS:", RSS(c(bnp1[1],bnp1[2])),"\n","Iterations:",k,"\n")
    break
    }
  bn <- bnp1
}
```

```
## b:  4.943193 10.07213 -- RSS: 0.001505396 -0.002774549
##  Iterations: 9943
```

```r
#This algorithm starts at b0 = 0 and b1 = 0 which is the same as alpha = 0 and b1 = 0
#I get the l2 norm of score since the RSS function gives the gradient which is the score
#thus the sum of squares of gradient (RSS) must b less than 0.00001
#alpha from minimisation
bnp1[1]+bnp1[2]*mean(x)
```

```
## [1] 10.06591
```

```r
#Using the closed form formula in c) we find alpha
mean(y)
```

```
## [1] 10.06591
```

```r
#Finding beta
bnp1[2]
```

```
## [1] 10.07213
```

```r
sum(((y-mean(y))*(x-mean(x)))/(sum((x-mean(x))^2)))
```

```
## [1] 10.07215
```

```r
#We can see that alpha has the same value as does the beta
#about 9943 iterations were required
#Below is my working to find the gradient function for the RSS
```

To get the gradient of the RSS we must get the first derivative of:

$$S(\beta_1, \beta_2) = \sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i)^2$$
$$S'(\beta_1, \beta_2) = (-2\sum_{i=1}^{n}(y_i - \beta_1 - \beta_2 x_i), -2\sum_{i=1}^{n} x_i(y_i - \beta_1 - \beta_2 x_i))$$

f)

g)

Question 2

Given n independent binary random variables $Y_1 \cdots Y_n$ with

$$P(Y_i = 1) = \pi_i \text{ and } P(Y_i = 0) = 1 - \pi_i$$

The probability function of Y_i is:

$$\pi_i^{Y_i}(1 - \pi_i)^{1-Y_i}$$

where $Y_i = 0$ or 1

a) For a probability function to belong to the exponential family of distributions, it must follow the formula:

$$f(y; \theta, \phi) = K(y, \frac{p}{\phi}) \exp\left(\frac{p}{\phi}\{y\theta - c(\theta)\}\right)$$

For the given probability density function:

$$
\begin{aligned}
f(y;\pi) &= \pi_i^y(1-\pi_i)^{1-y} \\
&= \exp\left(\log \pi_i^y(1-\pi_i)^{1-y}\right) \\
&= \exp\left(\log \pi_i^y + \log(1-\pi_i)^{1-y})\right) \\
&= \exp\left(y\log \pi_i + (1-y)\log(1-\pi_i)\right) \\
&= \exp\left(y\log(\frac{\pi}{1-\pi}) + \log(1-\pi)\right)
\end{aligned}
$$

With $p = 1$ and $\phi = 1$, the above equation follows the form of the exponential family of distribution where $K(y, \frac{p}{\phi}) = 1$, $\theta = \log(\frac{\pi}{1-\pi})$ and $c(\theta) = -\log(1-\pi) = -\log(1-\frac{e^\theta}{1+e^\theta})$ where $\pi = \frac{e^\theta}{1+e^\theta}$.

b) As seen in 2a, the naturalised parameter is $\theta = \log(\frac{\pi}{1-\pi})$

c) As seen in 2a, the cumulant generator is $c(\theta) = -\log(1-\frac{e^\theta}{1+e^\theta})$. Since $\mathrm{E}[Y] = c'(\theta)$, $c'(\theta) = -(\frac{e^\theta}{1+e^\theta}) = -(-\pi) = \pi$. Therefore, $\mathrm{E}[Y] = \pi$.

d) Given the link function:

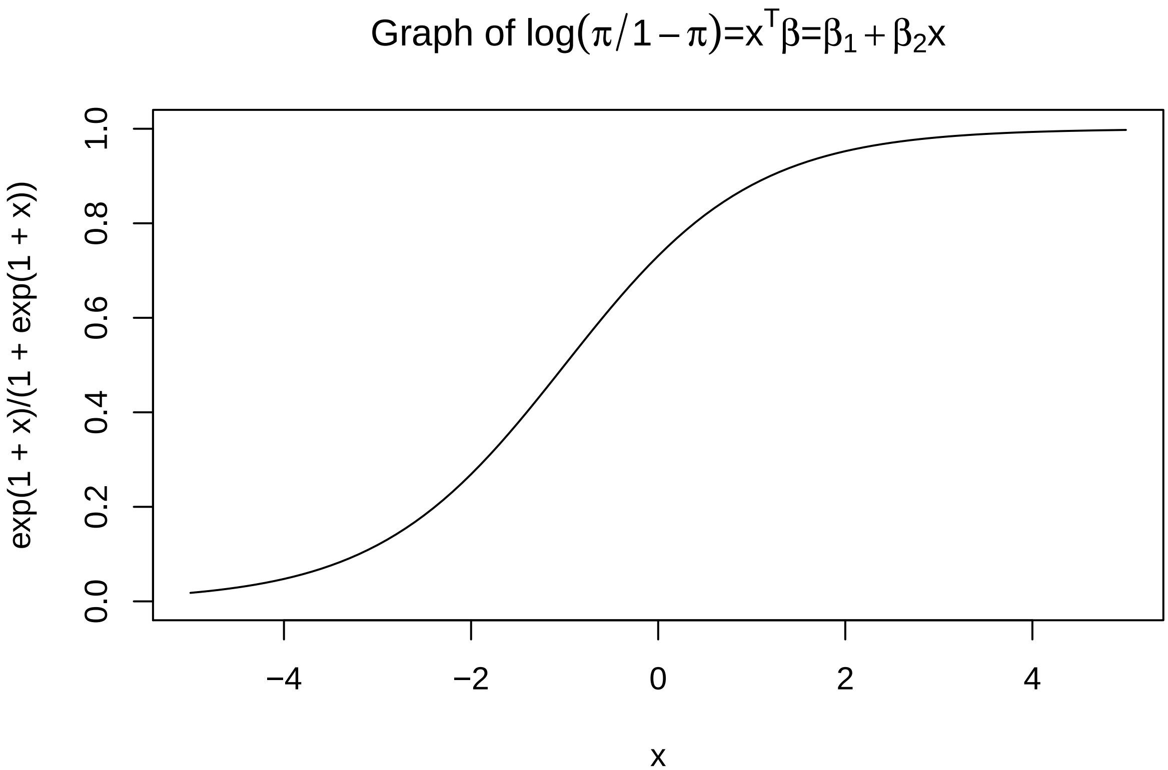
$$
g(\pi) = \log(\frac{\pi}{1-\pi}) = e^{x^T\beta}
$$

it can be rearranged in terms of the probability π,

$$
\begin{aligned}
e^{x^T\beta} &= \log(\frac{\pi}{1-\pi}) \\
e^{x^T\beta} - \pi e^{x^T\beta} &= \pi \\
\pi &= \frac{e^{x^T\beta}}{1+e^{x^T\beta}}
\end{aligned}
$$

e)

```r
curve(exp(1+x)/(1+exp(1+x)), xlim = c(-5, 5), ylim = c(0, 1),
      main=expression(paste("Graph of ", log(pi/1-pi),'=',x^{T},beta,'=',beta[1]+beta[2],'x'))
      )
```



It shows the log odds of the insecticide working with a given probability?

f) The following probability density function:

$$f(y;\theta) = \frac{1}{\phi}\exp\left(\frac{(y-\theta)}{\phi} - \exp\left[\frac{(y-\theta)}{\phi}\right]\right)$$

is NOT in the exponential family of distributions as it does not follow the form of a probability density function in the exponential family.

Question 3

a)

```r
titanic <- read.table('titanic.txt', header=TRUE)
head(titanic)
```

```
##                                          Name PClass   Age    Sex
## 1                 Allen, Miss Elisabeth Walton    1st 29.00 female
## 2                  Allison, Miss Helen Loraine    1st  2.00 female
## 3          Allison, Mr Hudson Joshua Creighton    1st 30.00   male
## 4 Allison, Mrs Hudson JC (Bessie Waldo Daniels)   1st 25.00 female
## 5                Allison, Master Hudson Trevor    1st  0.92   male
## 6                           Anderson, Mr Harry    1st 47.00   male
##   Survived
## 1        1
## 2        0
## 3        0
## 4        0
```

```
## 5        1
## 6        1
```

```r
summary(titanic)
```

```
##                              Name      PClass         Age            Sex
##  Carlsson, Mr Frans Olof      :  2   1st:226   Min.   : 0.17   female:288
##  Connolly, Miss Kate          :  2   2nd:212   1st Qu.:21.00   male  :468
##  Kelly, Mr James              :  2   3rd:318   Median :28.00
##  Abbing, Mr Anthony           :  1             Mean   :30.40
##  Abbott, Master Eugene Joseph:  1             3rd Qu.:39.00
##  Abbott, Mr Rossmore Edward   :  1             Max.   :71.00
##  (Other)                      :747
##     Survived
##  Min.   :0.000
##  1st Qu.:0.000
##  Median :0.000
##  Mean   :0.414
##  3rd Qu.:1.000
##  Max.   :1.000
##
```

b)

```r
attach(titanic)
table(titanic$Sex)
```

```
##
## female   male
##    288    468
```

```r
tapply(titanic$Survived,titanic$Sex,mean)
```

```
##    female      male
## 0.7534722 0.2051282
```

```r
summary(lm(titanic$Survived~titanic$Sex))
```

```
##
## Call:
## lm(formula = titanic$Survived ~ titanic$Sex)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -0.7535 -0.2051 -0.2051  0.2465  0.7949
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)      0.75347    0.02445   30.82   <2e-16 ***
## titanic$Sexmale -0.54834    0.03107  -17.65   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4149 on 754 degrees of freedom
## Multiple R-squared:  0.2923, Adjusted R-squared:  0.2913
## F-statistic: 311.4 on 1 and 754 DF,  p-value: < 2.2e-16
```

c)

```r
titanic.glm <- glm(titanic$Survived~titanic$Age,family=binomial('logit'))
summary(titanic.glm)
```

```
##
## Call:
## glm(formula = titanic$Survived ~ titanic$Age, family = binomial("logit"))
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -1.1418  -1.0489  -0.9792   1.3039   1.4801
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -0.081428   0.173862  -0.468   0.6395
## titanic$Age -0.008795   0.005232  -1.681   0.0928 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 1025.6  on 755  degrees of freedom
## Residual deviance: 1022.7  on 754  degrees of freedom
## AIC: 1026.7
##
## Number of Fisher Scoring iterations: 4
```

```r
exp(titanic.glm$coefficients[2])
```

```
## titanic$Age
##   0.9912439
```

```r
exp(titanic.glm$coefficients[2])
```

```
## titanic$Age
##   0.9912439
```

d)

e)

f)

g)

h)

i)