有限元基础及应用

讲义（英文）

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Lecture 1 FEM = Discretization method + Rayleigh-Ritz method

Since the advent of Newton's masterpiece *Mathematical Principles of Natural Philosophy* in 1687, **calculus** had been taking part in scientific findings and engineering applications in nature, with an increasingly more and more important role. Nowadays, it has been widely acknowledged that most (if not all) natural phenomena, whether mechanical, electrical, or biological, could be described in terms of ordinary or partial **differential equations** (DEs) that capture the laws of physics, e.g., Newton's law, Fourier's law, Fick's law, Ampere's law, Gauss's law, and Faraday's law. How to solve the DE-governed **field problems**, e.g., initial value problems (IVP), boundary value problems (BVPs) arising in scientific research and engineering practice is a fundamental issue that is common to scientists and engineers. To deal with this issue, there are mainly two different kinds of approaches: one is the exact (closed-form) solution approach (i.e., analytical techniques); the other is the approximate solution approach (i.e., numerical techniques). To date**,** as the advancement of computer technology, numerical techniques show more great advantages over analytical techniques, especially in tackling field problems with complex **boundary/initial conditions** and **loadings**. Among various numerical techniques, **finite element method** (FEM) is one of the most prestigious.

The history of FEM dates back to 1877 when the Rayleigh-Ritz method was first invented by Rayleigh and later improved by Ritz in 1908. The Rayleigh-Ritz method is also known as Galerkin method since Galerkin proposed the concept of function orthogonality in the Lebesgue-2 integral (an inner product defined for a function space) in 1915, which generalized the idea of Rayleigh-Ritz method. Herein, an example is presented to show the idea of Rayleigh-Ritz method. In the example, consider the following field problem to find the unknown **field variable** $u = \hat{u}(x)$ in the **field domain** $0 \le x \le 1$ subject to the **governing equation** and **boundary conditions** as following

$$\begin{aligned} &\text{G.E.} \quad \frac{d^2u}{dx^2} + u = x^4, \quad 0 \le x \le 1 \\ &\text{B.C.} \quad u(0) = 0, \quad u(1) = 0 \end{aligned} \tag{1}$$

Solution procedure 1

Step 1: Derive the **weak form** of the differential governing equation by integration over the whole field domain with a **test function** $v = \hat{v}(x)$.

$$\begin{aligned} &\int_0^1 \left(\frac{d^2u}{dx^2} + u \right) v dx = \int_0^1 x^4 v dx \\ &\Rightarrow \\ &\int_0^1 \frac{d^2u}{dx^2} v dx + \int_0^1 u v dx = \int_0^1 x^4 v dx \\ &\Rightarrow \\ &\left. \frac{du}{dx} v \right|_0^1 - \int_0^1 \frac{du}{dx}\frac{dv}{dx} dx + \int_0^1 u v dx = \int_0^1 x^4 v dx \end{aligned} \tag{2}$$

Step 2: Construct a **trial function** $\bar{u}$ to approximate the solution u by a linear combination of N basis functions that satisfy the boundary conditions

$$\boxed{\bar{u} = \sum_{j=1}^{N} c_j \phi_j} \tag{3}$$

where c_j are unknown constants called **Ritz coefficients**, and $\phi_j = \hat{\phi}_j(x)$ are basis functions that are constructed according to the boundary conditions

$$\begin{aligned}&\phi_1 = x(1-x)\\&\phi_2 = x^2(1-x)\\&\vdots\\&\phi_N = x^N(1-x)\end{aligned} \tag{4}$$

Step 3: Derive the system of N algebraic equations by letting $v = \phi_i$ respectively in Eq. (2).

$$\begin{aligned}&\sum_{j=1}^{N}\left[\int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i\phi_j dx\right]c_j = -\int_0^1 x^4\phi_i dx, \quad i = 1,2,\dots,N\\&\Rightarrow\\&\sum_{j=1}^{N} B(\phi_i,\phi_j)c_j = L(\phi_i)\end{aligned} \tag{5}$$

where $B(\phi_i,\phi_j)$ is called a **bilinear form**, and $L(\phi_i)$ is called a **linear form**.

Step 4: Solve the Ritz coefficients according to Eq. (5) by hands or computers, and further obtain the approximate solution by Eq. (3).

Solution procedure 2

Step 1: Derive the **variational form** of the differential governing equation and further the **functional** Π of the field variable by integration over the whole field domain with a variation δu of the field variable, where δ is known as the variation operator. δu is a new and independent field variable, but has a very small perturbation/deviation from the function u.

$$\int_0^1\left(\frac{d^2u}{dx^2}+u\right)\delta u dx=\int_0^1 x^4\delta u dx$$

$$\Rightarrow$$

$$\int_0^1\frac{d^2u}{dx^2}\delta u dx+\int_0^1 u\delta u dx=\int_0^1 x^4\delta u dx$$

$$\Rightarrow$$

$$\frac{du}{dx}\delta u\Big|_0^1-\int_0^1\frac{du}{dx}\frac{d\delta u}{dx}dx+\int_0^1 u\delta u dx=\int_0^1 x^4\delta u dx$$

$$\Rightarrow$$

$$\delta\left[\frac{1}{2}\int_0^1\left(\frac{du}{dx}\right)^2-u^2dx+\int_0^1 x^4 u dx\right]=0$$

$$\Rightarrow$$

$$\Pi(u)=\frac{1}{2}\int_0^1\left(\frac{du}{dx}\right)^2-u^2+2x^4 u dx \tag{6}$$

Note that $\delta u(x)=\delta u(1)=0$ is assumed and the differential operator and the variation operator are interchangeable, i.e.,

$$\frac{d\delta u}{dx}=\delta\left(\frac{du}{dx}\right)$$

$$\Rightarrow$$

$$\frac{du}{dx}\frac{d\delta u}{dx}=\frac{du}{dx}\delta\left(\frac{du}{dx}\right)=\delta\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2\right]$$

Step 2: This step is the same as the step 2 in the solution procedure 1.

Step 3: Derive the system of N algebraic equations by substituting Eq. (3) into Eq. (6) as

$$\Pi(c_j)=\frac{1}{2}\int_0^1\left(\sum_{j=1}^{N}\frac{d\phi_j}{dx}c_j\right)^2-\left(\sum_{j=1}^{N}\phi_j c_j\right)^2+2x^4\sum_{i=1}^{N}\phi_j c_j dx \tag{7}$$

and applying the necessary conditions for minimization of the functional $I(c_j)$

$$
\begin{aligned}
&\frac{\partial \Pi}{\partial c_1}=0,\quad \frac{\partial \Pi}{\partial c_2}=0,\quad \ldots,\quad \frac{\partial \Pi}{\partial c_N}=0 \\
&\Rightarrow \\
&\frac{\partial \Pi}{\partial c_i}=0=\int_0^1 \frac{d\phi_i}{dx}\sum_{j=1}^{N}\frac{d\phi_j}{dx}c_j-\phi_i\sum_{j=1}^{N}\phi_j c_j+x^4\phi_i dx,\quad i=1,2,\ldots,N \\
&\Rightarrow \\
&\sum_{j=1}^{N}\left[\int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}-\phi_i\phi_j dx\right]c_j=-\int_0^1 x^4\phi_i dx,\quad i=1,2,\ldots,N \\
&\Rightarrow \\
&\sum_{j=1}^{N}B\left(\phi_i,\phi_j\right)c_j=L\left(\phi_i\right)
\end{aligned}
\tag{8}
$$

Step 4: Solve the Ritz coefficients according to Eq. (8) by hands or computers, and further obtain the approximate solution by Eq. (3).

It is noted that the key of applying Rayleigh-Ritz method is the construction of the trial function (i.e., Eq. (3)). Numerically speaking, the construction process is an interpolation process by invoking a finite set of **degrees of freedom** (DoF). The most significant feature of Rayleigh-Ritz method is that the test function has to be chosen among the basis functions. If this requirement is removed, Rayleigh-Ritz method turns into a more general method called **weighted residual method** (e.g., co-location, co-volume, moment, least square). In addition, the Reyleigh-Ritz method considers the whole field domain when testing the residual of the differential equations. This limits its application to field problems with complex field domain. To overcome this, in 1943, R. Courant proposed a method where a complex field domain was first discretized by a finite number of triangle subdomains, and a trial function was then constructed with respect to each subdomain. Courant's method is deemed as the first FEM published. While his method was not popular then due to the low computation capability of that time, however, the idea of combining discretization method and

Reyleigh-Ritz method gradually showed up. In 1944, J.H. Argyris proposed a similar idea and apply it to aircraft design. And later, the power of this idea is enlarged by computer technology.

During the WWII, computer technology experienced a time of fast development due to various military demands, e.g., in 1946, the first modern electronic computer ENIAC was made for ballistic trajectory calculation in the USA. This extremely boosts the development of FEM and its applications. In 1950s, a large number of scholars (Clough, Wilson, Taylor, et.al.) from University of California at Berkeley in the USA endeavored themselves to research activities related to computer-aided design of aircraft, sponsored by Boeing company. Their work laid the foundation of the modern FEM. From then on, during 1960s-1980s, various FEM software emerged, e.g., SAP (Wilson, 1969, USA), NASTRAN (NASA, 1969, USA), ANSYS (Swanson, 1970, USA), MARC (Marcal, 1971, USA), ANIDA (Bathe, 1975, USA), DYNA3D (Hallquist, 1976, USA), ABAQUS (Hibbitt et.al., 1978, USA), COMSOL (Littmarck et.al., 1986, Sweden), RADIOSS (Mecalog, 1987, France). Most of these software products targeted the structural design/analysis market in the areas of civil engineering, mechanical engineering, aerospace engineering, etc. From the 1990s to the current, commercialization of these software products is the main trend since the **state-of-the-art FEM is very mature**. During the course, many small companies were acquired by large companies, e.g., ABAQUS was acquired by SIMULIA in 2005, ANSYS was acquired by SYNOSYS in 2024. Apart from commercial FEM software, these days, open-source FEM software become popular, such as MOOSE, GetFEM, FEniCS, FreeFEM, Code_Aster, and CalculiX, just to name a few.

Lecture 2 Mathematical preliminaries

2.1 Basic matrix notion

Since a FEM relies on **discretization methods** to transform a continuum field domain into a discrete field domain (i.e., elements connected by nodes) and **interpolation methods** to approximate a continuum field variable in terms of its nodal value, the resulting system of algebraic equations from continuum field problems could be finally described by matrices and vectors, which suit the computer operation well. For example, for the following system of algebraic equations

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3 \end{cases} \tag{1}$$

It can be rewritten in matrix form as

$$\mathbf{Ax} = \mathbf{b} \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3}$$

are called the system matrix, unknown vector, and right-hand-side vector respectively.

Since calculation based on matrix operation is intensively used in FEM. We will go through some important concepts about matrix algebra.

Definition 1-Matrix. For a general matrix, the number of rows "m" and number of columns "n" are usually not equal.

$$\mathbf{A}_{m\times n}=\begin{bmatrix} A_{11} & A_{12} & A_{13} & . & A_{1n} \\ A_{21} & A_{22} & A_{23} & . & A_{2n} \\ A_{31} & A_{32} & A_{33} & . & A_{3n} \\ . & . & . & . & . \\ A_{m1} & A_{m2} & A_{m3} & . & A_{mn} \end{bmatrix} \tag{4}$$

For an entry in the matrix, there is a unique row number and column number to name the entry. Example for the i-th row and j-th column entry, it is denoted as

$$A_{ij},\ 1\le i\le m,\ 1\le j\le n \tag{5}$$

Definition 2-Column matrix.

Example:

$$\mathbf{A}_{3\times 1}=\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \tag{6}$$

Definition 3-Row matrix.

Example:

$$\mathbf{A}_{1\times 3}=\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \tag{7}$$

Definition 4-Square matrices. When the number of rows and the number of columns are equal, the matrix is called a **square matrix**. For an $n\times n$ square matrix $\mathbf{A}_{n\times n}$, it is usually called an n-th order square matrix.

Example: a 3rd order square matrix

$$\mathbf{A}=\begin{bmatrix} 1 & 4 & 5 \\ 7 & 2 & 6 \\ 8 & 9 & 3 \end{bmatrix} \tag{8}$$

There are two special square matrices: one is called identity matrix $\mathbf{I}_{n\times n}$ whose diagonal entries are unit, and the other entries are null; the other is called null matrix $\mathbf{0}_{n\times n}$.

Example:

$$\mathbf{I}_{3\times3}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \mathbf{0}_{3\times3}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{9}$$

Definition 5-Matrix equality. For two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times n}$ of the same dimensions, they are equal if and only if their entries are equal $A_{ij}=B_{ij},\ 1\le i\le m,\ 1\le j\le n$ respectively.

Example:

$$\begin{aligned}&\mathbf{A}_{2\times3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&\mathbf{B}_{2\times3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&\Rightarrow\\&\mathbf{A}_{2\times3}=\mathbf{B}_{2\times3}\end{aligned} \tag{10}$$

Definition 6-Matrix addition and subtraction. For two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times n}$ of the same dimensions, their addition and subtraction are defined in terms of their entries

$$\begin{aligned}C_{ij}&=A_{ij}+B_{ij},\ 1\le i\le m,\ 1\le j\le n\\D_{ij}&=A_{ij}-B_{ij},\ 1\le i\le m,\ 1\le j\le n\end{aligned} \tag{11}$$

Example:

$$\begin{aligned}&\mathbf{A}_{2\times3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&\mathbf{B}_{2\times3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&\Rightarrow\\&\mathbf{C}_{2\times3}=\mathbf{A}_{2\times3}+\mathbf{B}_{2\times3}=\begin{bmatrix}2&4&6\\6&4&2\end{bmatrix}\\&\mathbf{D}_{2\times3}=\mathbf{A}_{2\times3}\text{-}\mathbf{B}_{2\times3}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\end{aligned} \tag{12}$$

Definition 7-Scalar multiplication. For a matrix $\mathbf{A}_{m\times n}$, a scalar times the matrix means the scalar times every entry of the matrix.

Example:

$$\begin{aligned}&\mathbf{A}_{2\times 3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&s=2\\&\Rightarrow\\&s\mathbf{A}_{2\times 3}=\begin{bmatrix}2&4&6\\6&4&2\end{bmatrix}\end{aligned}\tag{13}$$

Definition 8-Matrix multiplication. For two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{p\times q}$, their multiplication $\mathbf{C}_{m\times q}=\mathbf{A}_{m\times n}\mathbf{B}_{p\times q}$ takes place only when $n=p$, and the entry in $\mathbf{C}_{m\times q}$ is evaluated as

$$C_{ij}=\sum_{k=1}^{n}A_{ik}B_{kj},\ 1\le i\le m,\ 1\le j\le q\tag{14}$$

Example:

$$\begin{aligned}&\mathbf{A}_{2\times 3}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\\&\mathbf{B}_{3\times 2}=\begin{bmatrix}2&1\\3&2\\4&3\end{bmatrix}\\&\Rightarrow\\&\mathbf{C}_{2\times 2}=\mathbf{A}_{2\times 3}\mathbf{B}_{3\times 2}=\begin{bmatrix}20&14\\14&10\end{bmatrix}\end{aligned}\tag{15}$$

Note that matrix multiplication doesn't commute

$$\mathbf{AB}\neq\mathbf{BA}\tag{16}$$

Definition 9-Matrix transpose. For a matrix $\mathbf{A}_{m\times n}$, its transpose is denoted as $\mathbf{A}^{T}_{n\times m}$.

$$A^{T}_{ji}=A_{ij},\ 1\le i\le m,\ 1\le j\le n\tag{17}$$

Example:

$$\begin{aligned}
&\mathbf{A}_{2\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \\
&\Rightarrow \\
&\mathbf{A}_{3\times2}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}
\end{aligned} \tag{18}$$

For the transpose of a matrix multiplication, the following identity holds

$$\left(\mathbf{AB}\right)^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{19}$$

Example:

$$\begin{aligned}
&\mathbf{A}_{2\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \\
&\mathbf{B}_{3\times2} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 3 \end{bmatrix} \\
&\Rightarrow \\
&\mathbf{C}_{2\times2} = \left(\mathbf{A}_{2\times3}\mathbf{B}_{3\times2}\right)^{\mathrm{T}} = \mathbf{B}_{2\times3}^{\mathrm{T}}\mathbf{A}_{3\times2}^{\mathrm{T}} = \begin{bmatrix} 20 & 14 \\ 14 & 10 \end{bmatrix}
\end{aligned} \tag{20}$$

Definition 10-Symmetric matrix. If a square matrix satisfies the following condition, then it is called a symmetric matrix

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}} \tag{21}$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \tag{22}$$

Definition 11-Skew-symmetric matrix. If a square matrix satisfies the following condition, then it is called a skew-symmetric matrix

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}} \tag{23}$$

Example:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} \tag{24}$$

Definition 12-Determinant, minor, and cofactor of a square matrix.

For a 1st order square matrix, its determinant is defined as

$$\begin{aligned} &\mathbf{A} = \left[\mathrm{A}_{11} \right] \\ &\Rightarrow \\ &\det \mathbf{A} = \mathrm{A}_{11} \end{aligned} \tag{25}$$

For a 2nd order square matrix, its determinant is defined as

$$\begin{aligned} &\mathbf{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{bmatrix} \\ &\Rightarrow \\ &\det \mathbf{A} = \begin{vmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{vmatrix} = \mathrm{A}_{11}\mathrm{A}_{22} - \mathrm{A}_{12}\mathrm{A}_{21} \end{aligned} \tag{26}$$

For a 3rd order square matrix, its determinant is defined as

$$\begin{aligned} &\mathbf{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \\ &\Rightarrow \\ &\det \mathbf{A} = \begin{vmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{vmatrix} \\ &= \mathrm{A}_{11}\mathrm{A}_{22}\mathrm{A}_{33} + \mathrm{A}_{12}\mathrm{A}_{23}\mathrm{A}_{31} + \mathrm{A}_{13}\mathrm{A}_{21}\mathrm{A}_{32} \\ &-\mathrm{A}_{13}\mathrm{A}_{22}\mathrm{A}_{31} - \mathrm{A}_{11}\mathrm{A}_{23}\mathrm{A}_{32} - \mathrm{A}_{12}\mathrm{A}_{21}\mathrm{A}_{33} \end{aligned} \tag{27}$$

For higher order square matrix $\mathbf{A}_{n\times n}$, its determinant could be evaluated by the so-called Laplace expansion method recursively according to an arbitrary i-th row or an arbitrary j-th column as following

$$\det \mathbf{A} = \sum_{k=1}^{n} \mathrm{A}_{ik}\mathrm{C}_{ik} = \sum_{k=1}^{n} \mathrm{A}_{kj}\mathrm{C}_{kj} \tag{28}$$

where the component of the cofactor matrix $\mathbf{C}_{n\times n}$ is defined as

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{29}$$

where the minor M_{ij} corresponding to the A_{ij} is defined as the determinant of the sub-matrix after eliminating the i-th row and j-th column of the matrix $\mathbf{A}_{n\times n}$.

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix} \tag{30}$$

By Laplace expansion about the first row, one has

$$\begin{aligned}
&M_{11} = \det\begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix} = -10 \\
&M_{12} = \det\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} = -6 \\
&M_{13} = \det\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = 2 \\
&\Rightarrow \\
&C_{11} = (-1)^{1+1} \times M_{11} = -10 \\
&C_{12} = (-1)^{1+2} \times M_{12} = 12 \\
&C_{13} = (-1)^{1+3} \times M_{13} = 6 \\
&\Rightarrow \\
&\det \mathbf{A} = 1\times C_{11} + 2\times C_{12} + 3\times C_{13} = 8
\end{aligned} \tag{31}$$

Definition 13-Adjoint and inverse of a square matrix. The adjoint matrix of a square matrix $\mathbf{A}_{n\times n}$ could be evaluated by transposing the matrix of cofactors.

$$\mathrm{adj}\mathbf{A} = \mathbf{C}^{T} \tag{32}$$

The inverse of a square matrix $\mathbf{A}_{n\times n}$ could be evaluated as

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}\mathbf{A}}{\det \mathbf{A}} \tag{33}$$

Example:

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 2 & 2 & 4\\ 3 & 4 & 3\end{bmatrix} \tag{34}$$

Calculate all the minors

$$\mathrm{M}_{11}=\det\begin{bmatrix}2 & 4\\ 4 & 3\end{bmatrix}=-10, \mathrm{M}_{12}=\det\begin{bmatrix}2 & 4\\ 3 & 3\end{bmatrix}=-6, \mathrm{M}_{13}=\det\begin{bmatrix}2 & 2\\ 3 & 4\end{bmatrix}=2$$
$$\mathrm{M}_{21}=\det\begin{bmatrix}2 & 3\\ 4 & 3\end{bmatrix}=-6, \mathrm{M}_{22}=\det\begin{bmatrix}1 & 3\\ 3 & 3\end{bmatrix}=-6, \mathrm{M}_{23}=\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=-2$$
$$\mathrm{M}_{31}=\det\begin{bmatrix}2 & 3\\ 2 & 4\end{bmatrix}=2, \mathrm{M}_{32}=\det\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=-2, \mathrm{M}_{33}=\det\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=-2$$
$$\Rightarrow$$
$$\mathrm{C}_{11}=(-1)^{1+1}\times\mathrm{M}_{11}=-10, \mathrm{C}_{12}=(-1)^{1+2}\times\mathrm{M}_{12}=6, \mathrm{C}_{13}=(-1)^{1+3}\times\mathrm{M}_{13}=2$$
$$\mathrm{C}_{21}=(-1)^{2+1}\times\mathrm{M}_{21}=6, \mathrm{C}_{22}=(-1)^{2+2}\times\mathrm{M}_{22}=-6, \mathrm{C}_{23}=(-1)^{2+3}\times\mathrm{M}_{23}=2$$
$$\mathrm{C}_{31}=(-1)^{3+1}\times\mathrm{M}_{31}=2, \mathrm{C}_{32}=(-1)^{3+2}\times\mathrm{M}_{32}=6, \mathrm{C}_{33}=(-1)^{3+3}\times\mathrm{M}_{33}=-2$$
$$\Rightarrow$$
$$\mathbf{C}=\begin{bmatrix}-10 & 6 & 2\\ 6 & -6 & 2\\ 2 & 2 & -2\end{bmatrix}$$
$$\Rightarrow$$
$$\mathrm{adj}\mathbf{A}=\begin{bmatrix}-10 & 6 & 2\\ 6 & -6 & 2\\ 2 & 2 & -2\end{bmatrix}, \mathbf{A}^{-1}=\frac{1}{8}\begin{bmatrix}-10 & 6 & 2\\ 6 & -6 & 2\\ 2 & 2 & -2\end{bmatrix} \tag{35}$$

Definition 15-Orthogonal matrix

A square matrix $\mathbf{A}$ is called orthogonal if it satisfies the following condition

$$\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{-1}$$
or
$$\mathbf{A}\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}}\mathbf{A}=\mathrm{I} \tag{36}$$

2.2 Solving a system of linear equations.

Suppose we have a system of linear equations

$$\mathbf{Ax}=\mathbf{b} \tag{37}$$

where

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 2 & 2 & 4\\ 3 & 4 & 3\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \tag{38}$$

2.2.1 Cofactor method

We directly use the result in Eq. (35).

$$\begin{gathered}\mathbf{A}^{-1}=\frac{1}{8}\begin{bmatrix}-10 & 6 & 2\\ 6 & -6 & 2\\ 2 & 2 & -2\end{bmatrix}\\ \Rightarrow\\ \mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\begin{bmatrix}-1/4\\ 1/4\\ 1/4\end{bmatrix}\end{gathered} \tag{39}$$

2.2.2 Cramer's rule

Step 1: Evaluate the following determinants

$$\begin{gathered}\det\mathbf{A}=\begin{vmatrix}1 & 2 & 3\\ 2 & 2 & 4\\ 3 & 4 & 3\end{vmatrix}=8\\ \det\mathbf{A}_1=\begin{vmatrix}1 & 2 & 3\\ 1 & 2 & 4\\ 1 & 4 & 3\end{vmatrix}=-2\\ \det\mathbf{A}_2=\begin{vmatrix}1 & 1 & 3\\ 2 & 1 & 4\\ 3 & 1 & 3\end{vmatrix}=2\\ \det\mathbf{A}_3=\begin{vmatrix}1 & 2 & 1\\ 2 & 2 & 1\\ 3 & 4 & 1\end{vmatrix}=2\end{gathered} \tag{40}$$

Step 2: Evaluate the unknowns

$$\mathbf{x}=\frac{1}{\det\mathbf{A}}\begin{bmatrix}\det\mathbf{A}_1\\ \det\mathbf{A}_2\\ \det\mathbf{A}_3\end{bmatrix}=\begin{bmatrix}-1/4\\ 1/4\\ 1/4\end{bmatrix} \tag{41}$$

2.2.3 Gaussian elimination

Step 1: Construct an augmented matrix composed of $\mathbf{A}$ and $\mathbf{b}$.

$$\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 2 & 4 & 1\\ 3 & 4 & 3 & 1\end{bmatrix} \tag{42}$$

Step 2: Produce an upper triangle matrix

Row 2-2*Row 1.

Row 3-3*Row 1

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 0 & -2 & -2 & -1\\ 0 & -2 & -6 & -2\end{bmatrix} \tag{43}$$

Row 3-Row2

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 0 & -2 & -2 & -1\\ 0 & 0 & -4 & -1\end{bmatrix} \tag{44}$$

Step 3: Back substitution.

$$\begin{aligned}x_3 &= 1/4\\ x_2 &= \left[-1+(-2)\times x_3\right]/\left(-2\right)=1/4\\ x_1 &= 1-3\times x_3-2\times x_2=-1/4\end{aligned} \tag{45}$$

So

$$\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-1/4\\ 1/4\\ 1/4\end{bmatrix} \tag{46}$$

2.3 Integral relations (also known as **Green formulae**)

Integral of differential equations is essential in FEM during the derivation of the corresponding weak form or functional. Herein, just list some useful relations, these relations are known as Green formulae.

2.3.1 Integration by part.

For brevity, consider the formula for two functions u and v of one independent variable x

$$\int_a^b \frac{du}{dx} v dx = [uv]\Big|_a^b - \int_a^b u \frac{dv}{dx} dx \tag{47}$$

This could be understood by the well-known relationship

$$\frac{d}{dx}(uv) = \frac{du}{dx} v + u \frac{dv}{dx}$$

2.3.2 Gradient theorem.

The integral of the gradient of a scalar field over a domain is equal to the integral of outer normal times the scalar field over the boundary of the domain.

$$\int_\Omega \nabla u dV = \int_{\partial\Omega} \mathbf{n} u dA \tag{48}$$

where $\nabla = \sum_{i=1}^{n\dim} \frac{\partial}{\partial x_i} \mathbf{e}_i$ is the Nabla operator, ndim means the dimension of the space, and $\mathbf{n}$ denotes the unit outer normal direction.

2.3.3 Divergence theorem.

The integral of the divergence of a vector field over a domain is equal to the integral of outer normal projection of the vector field over the boundary of the domain.

$$\int_\Omega \nabla \cdot \mathbf{v} dV = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{v} dA \tag{49}$$

2.3.4 Example (Integration by parts)

Suppose we are assigned to solve the following field problem

$$\begin{aligned} &\text{G.E.} \quad \frac{d^2u}{dx^2} = \frac{x(L-x)}{2}, \quad 0 \le x \le L \\ &\text{B.C.} \quad u(0) = 0, \quad u(L) = 0 \end{aligned} \tag{50}$$

Solution procedure 1

Step 1: Derive the **weak form** of the differential governing equation by integration over the whole field domain with a **test function** $v = \hat{v}(x)$.

$$\begin{aligned}
&\int_0^L \frac{d^2u}{dx^2} v dx = \int_0^L \frac{x(L-x)}{2} v dx \\
&\Rightarrow \\
&\left.\frac{du}{dx} v\right|_0^L - \int_0^L \frac{du}{dx}\frac{dv}{dx} dx = \int_0^L \frac{x(L-x)}{2} v dx
\end{aligned} \tag{51}$$

Step 2: Construct a **trial function** $\bar{u}$ to approximate the solution u by a linear combination of N basis functions that satisfy the boundary conditions

$$\boxed{\bar{u} = \sum_{j=1}^{N} c_j \phi_j} \tag{52}$$

where c_j are unknown constants called **Ritz coefficients**, and $\phi_j = \hat{\phi}_j(x)$ are basis functions that are constructed according to the boundary conditions. For brevity, herein, we consider two term approximation (i.e., N=2)

$$\begin{aligned}
&\phi_1 = \sin\left(\frac{\pi x}{L}\right) \\
&\phi_2 = x\sin\left(\frac{\pi x}{L}\right) \\
&\Rightarrow \\
&\bar{u} = c_1\phi_1 + c_2\phi_2 \\
&\quad = c_1 \sin\left(\frac{\pi x}{L}\right) + c_2 x \sin\left(\frac{\pi x}{L}\right)
\end{aligned} \tag{53}$$

Step 3: Introduce $u \approx \bar{u}$ into Eq. (51) and derive the system of 2 algebraic equations by letting $v = \phi_i, i = 1, 2$ respectively

For i=1, one has

$$\begin{aligned}
&\left.\frac{d\bar{u}}{dx}\phi_1\right|_0^L - \int_0^L \frac{d\bar{u}}{dx}\frac{d\phi_1}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_1 dx \\
&\Rightarrow \text{Apply B.C.} \\
&-\int_0^L \frac{d\bar{u}}{dx}\frac{d\phi_1}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_1 dx \\
&\Rightarrow \text{Plug } \bar{u} = c_1\phi_1 + c_2\phi_2 \\
&-\int_0^L \frac{d(c_1\phi_1 + c_2\phi_2)}{dx}\frac{d\phi_1}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_1 dx \\
&\Rightarrow \\
&\boxed{\left(-\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_1}{dx}dx\right)c_1 + \left(-\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_1}{dx}dx\right)c_2 = \int_0^L \frac{x(L-x)}{2}\phi_1 dx}
\end{aligned} \tag{54}$$

For i=2, similarly, one has

$$\begin{aligned}
&\left.\frac{d\bar{u}}{dx}\phi_2\right|_0^L - \int_0^L \frac{d\bar{u}}{dx}\frac{d\phi_2}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_2 dx \\
&\Rightarrow \text{Apply B.C.} \\
&-\int_0^L \frac{d\bar{u}}{dx}\frac{d\phi_2}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_2 dx \\
&\Rightarrow \text{Plug } \bar{u} = c_1\phi_1 + c_2\phi_2 \\
&-\int_0^L \frac{d(c_1\phi_1 + c_2\phi_2)}{dx}\frac{d\phi_2}{dx}dx = \int_0^L \frac{x(L-x)}{2}\phi_2 dx \\
&\Rightarrow \\
&\boxed{\left(-\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_2}{dx}dx\right)c_1 + \left(-\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_2}{dx}dx\right)c_2 = \int_0^L \frac{x(L-x)}{2}\phi_2 dx}
\end{aligned} \tag{55}$$

Step 4: Form a system of linear equations, evaluate integrals and solve for the Ritz coefficients

$$\begin{bmatrix} -\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_1}{dx}dx & -\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_1}{dx}dx \\ -\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_2}{dx}dx & -\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_2}{dx}dx \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_0^L \frac{x(L-x)}{2}\phi_1 dx \\ \int_0^L \frac{x(L-x)}{2}\phi_2 dx \end{bmatrix} \tag{56}$$

Solution procedure 2

Step 1: Derive the **variational form** of the differential governing equation and further the **functional** Π of the field variable by integration over the whole field domain with a variation δu of the field variable, where δ is known as the variation operator. δu is a new and independent field variable, but has a very small perturbation/deviation from the function u.

$$
\begin{aligned}
&\int_0^L \frac{d^2u}{dx^2}\delta u dx = \int_0^L \frac{x(L-x)}{2}\delta u dx \\
&\Rightarrow \\
&\frac{du}{dx}\delta u\Big|_0^L - \int_0^L \frac{du}{dx}\frac{d\delta u}{dx}dx = \int_0^L \frac{x(L-x)}{2}\delta u dx \\
&\Rightarrow \\
&\delta\left[\frac{1}{2}\int_0^L\left(\frac{du}{dx}\right)^2 dx + \int_0^L \frac{x(L-x)}{2}u dx\right] = 0 \\
&\Rightarrow \\
&\Pi(u) = \frac{1}{2}\int_0^L\left(\frac{du}{dx}\right)^2 + x(L-x)u dx
\end{aligned}
\tag{57}
$$

Note that $\delta u(x) = \delta u(L) = 0$ is assumed and the differential operator and the variation operator are interchangeable.

$$
\begin{aligned}
&\frac{d\delta u}{dx} = \delta\left(\frac{du}{dx}\right) \\
&\Rightarrow \\
&\frac{du}{dx}\frac{d\delta u}{dx} = \frac{du}{dx}\delta\left(\frac{du}{dx}\right) = \delta\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2\right]
\end{aligned}
$$

Step 2: This step is the same as the step 2 in the solution procedure 1.

Step 3: Derive the system of N algebraic equations by substituting Eq. (53) into Eq. (57) as

$$
\Pi(c_1, c_2) = \frac{1}{2}\int_0^L\left(\frac{d\phi_1}{dx}c_1 + \frac{d\phi_2}{dx}c_2\right)^2 + x(L-x)(\phi_1 c_1 + \phi_2 c_2)dx \tag{58}
$$

and applying the necessary conditions for minimization of the functional $I(c_j)$

$$\begin{aligned}
&\frac{\partial \Pi}{\partial c_1}=0, \quad \frac{\partial \Pi}{\partial c_2}=0 \\
&\Rightarrow \\
&\frac{\partial \Pi}{\partial c_1}=\int_0^L \frac{d\phi_1}{dx}\left(\frac{d\phi_1}{dx}c_1+\frac{d\phi_2}{dx}c_2\right)+\frac{x(L-x)}{2}\phi_1 dx \\
&\frac{\partial \Pi}{\partial c_2}=\int_0^L \frac{d\phi_2}{dx}\left(\frac{d\phi_1}{dx}c_1+\frac{d\phi_2}{dx}c_2\right)+\frac{x(L-x)}{2}\phi_2 dx \\
&\Rightarrow \\
&\begin{bmatrix} -\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_1}{dx}dx & -\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_1}{dx}dx \\ -\int_0^L \frac{d\phi_1}{dx}\frac{d\phi_2}{dx}dx & -\int_0^L \frac{d\phi_2}{dx}\frac{d\phi_2}{dx}dx \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} \int_0^L \frac{x(L-x)}{2}\phi_1 dx \\ \int_0^L \frac{x(L-x)}{2}\phi_2 dx \end{bmatrix}
\end{aligned} \tag{59}$$

Step 4: Solve the system for the Ritz coefficients.

Lecture 3 Energy method

3.1 Basic notions

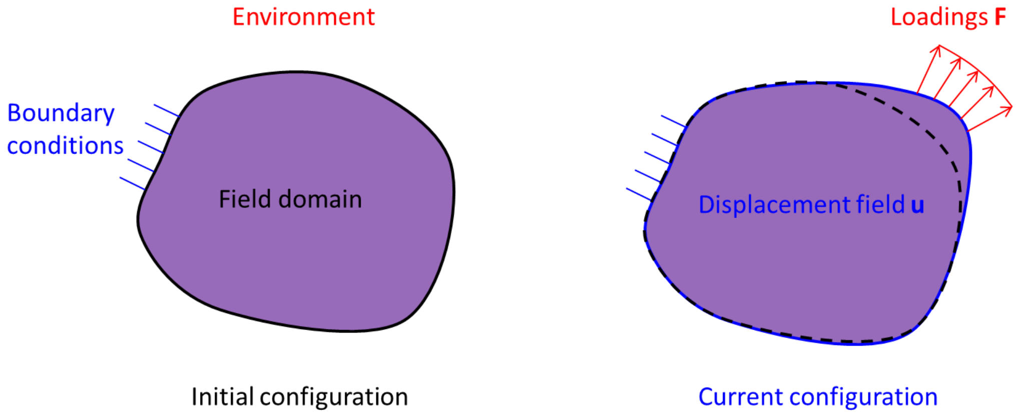


Fig. 1

Definition-1 Configuration

As shown in Fig. 1, the **configuration of a body** is the set of positions of all its constitutive particles. Usually, the **initial configuration** of a body is defined as the field domain of interest denoted as Ω. Its boundary is denoted as $\partial\Omega$ that includes the part $\partial\Omega_u$ for boundary conditions and the part $\partial\Omega_F$ for loadings. A displacement field **u** defined in the field domain leads to the **current configuration (deformed configuration)** of the body.

$$\mathbf{x}(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}), \mathrm{X} \in \Omega \tag{1}$$

Definition-2 Work of external forces and external energy

The **work** of an external force **F** from environment acting through a displacement field **u** in a physical body is also called the **external energy**, e.g, for a boundary loading, the external work is written as

$$\Pi_{\text{ext}}(\mathbf{u}) = \int_{\partial\Omega_F} \mathbf{F} \cdot \mathbf{u} \mathrm{d}A \tag{2}$$

Note that a force is a vector, a displacement is a vector, and the work is defined as their inner product, which is a scalar.

Definition-3 Work of internal forces and internal energy(Strain energy)

The displacement field $\mathbf{u}$ in the body results in internal forces due to material constitutive response. Similar to external forces, the work done by the internal forces is called i**nternal energy** or **strain energy**, e.g., the work could be stored in the form of strain energy as following

$$\Pi_{\text{int}}(\mathbf{u}) = \int_{\Omega} W dV \tag{3}$$

where W is a scalar function of the gradient of the displacement field $\mathbf{u}$ called strain energy density.

Definition-4 Conservative system

If work done by external forces due to environment and internal forces due to constitutive response of the body are independent of path from the initial configuration to the current configuration, we say the system including the body and the environment is a conservative system. That means there is no dissipation of energy, and the level of potential energy of the system is totally determined by the configuration of the body.

Definition-5 Quasi-static process

For a process of the configuration change of a body, if the velocity of all particles in the body is nearly zero (i.e., the inertial force is negligible), the process is called quasi-static.

3.2 Principle of stationary potential energy

For a conservative system in a quasi-static process, “Potential energy” point of view is an alternative of the "Free body diagram/Differential body" point of view for system equilibrium analysis (i.e., static analysis). Sometimes, the "Potential energy" point of view is easier to apply to derive the governing equations and boundary conditions for a field problem.

For a field problem arising from a conservative system in a quasi-static process, the **total potential energy** (or simply called **potential energy**) Π of the system could always be found by integrating differential governing equations by field variations as shown in the previous lectures, or by the so-called potential energy analysis).

In a potential energy analysis, the general form of the potential energy of a conservative system in a quasi-static process usually takes the following form

$$\Pi(\mathbf{u}) = \Pi_{\text{int}}(\mathbf{u}) - \Pi_{\text{ext}}(\mathbf{u}) \tag{4}$$

where Π_{int} is the **internal energy** of the body due to constitutive laws, Π_{ext} is the **external work** done by the environment.

Unlike physicists, mathematicians prefer to use a more general term "functional" instead of "potential energy". They define functional as **a scalar value function contains integral of a field variable over a field domain, e.g., the line, area, volume of interest**.

For an admissible and equilibrium configuration defined by a displacement field $\mathbf{u}$, if vary the configuration by such an incremental displacement field $\Delta\mathbf{u} = \varepsilon\delta\mathbf{u}$, where $\varepsilon \ll$ is a very small scalar and $\delta\mathbf{u}$ is defined as **a variation of the displacement field** (or called a **virtual displacement field**), which is required to be null(zero) at the boundary of the field domain, then the change of the total potential energy therefore is stated as

$$\Delta\Pi(\mathbf{u}+\Delta\mathbf{u})=\Pi(\mathbf{u}+\Delta\mathbf{u})-\Pi(\mathbf{u}) \tag{5}$$

By using Taylor's expansion in terms of the scalar ε about 0, one has

$$\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})=\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})\big|_{\varepsilon=0}+\left.\frac{\mathrm{d}\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\varepsilon+\mathrm{O}(\varepsilon^2) \tag{6}$$

Introducing Eq. (6) into Eq. (5), one has

$$\Delta\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})=\left.\frac{\mathrm{d}\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\varepsilon+\mathrm{O}(\varepsilon^2) \tag{7}$$

Since $\varepsilon \ll 1$, the higher order term $\mathrm{O}(\varepsilon^2)$ in the above equation is negligible. So, the change of potential energy is dominated by the first order term, whose coefficient is defined as the variation of the potential energy $\Pi(\mathbf{u})$ with respect to the variation of the displacement field $\delta\mathbf{u}$.

$$\delta\Pi(\mathbf{u})[\delta\mathbf{u}]=\left.\frac{\mathrm{d}\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \tag{8}$$

The principle of stationary potential energy states that the condition for an equilibrium configuration should satisfy the following condition

$$\boxed{\delta\Pi(\mathbf{u})[\delta\mathbf{u}]=0} \tag{9}$$

This statement means, for an arbitrary variation of the displacement field $\delta\mathbf{u}$, the potential energy $\Pi(\mathbf{u})$ would not change, i.e., the potential energy is stationary. Moreover, this means the body of interest is in an equilibrium configuration.

Therefore, once the potential energy is determined, the variational form of the governing differential equations could be readily obtained by Eq. (8). In addition, the equilibrium configuration could be achieved at the stationary point of the potential energy (minimum point, maximum point, or saddle point) by Eq. (9). When the stationary point is a minimum point, the principle of stationary potential energy is also called **principle of minimum potential energy**. In solid and structural mechanics, the principle is also known as **principle of virtual displacement/work**.

It is noted that, for a system with finite degrees of freedom (e.g., N), Eq. (8) can be written as

$$\begin{aligned}
&\left.\frac{d\Pi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{d\varepsilon}\right|_{\varepsilon=0} \\
&=\left.\frac{d\Pi(u_1+\varepsilon\delta u_1, u_2+\varepsilon\delta u_2, \dots, u_N+\varepsilon\delta u_N)}{d\varepsilon}\right|_{\varepsilon=0} \\
&=\sum_{i=1}^{N}\left[\left.\frac{\partial\Pi(u_1+\varepsilon\delta u_1, u_2+\varepsilon\delta u_2, \dots, u_N+\varepsilon\delta u_N)}{\partial(u_1+\varepsilon\delta u_1)}\delta u_i\right|_{\varepsilon=0}\right] \\
&=\sum_{i=1}^{N}\frac{\partial\Pi}{\partial u_i}\delta u_i
\end{aligned} \tag{10}$$

Thus, the principle of stationary potential energy could be stated as

$$\frac{\partial\Pi}{\partial u_i}=0, i=1,2,\dots,N \tag{11}$$

3.3 Total potential energy of a spring

As shown in **Fig. 2**, a spring is elastic and its stiffness is denoted as k. We use energy method to find it equilibrium displacement u subject to an external force F.

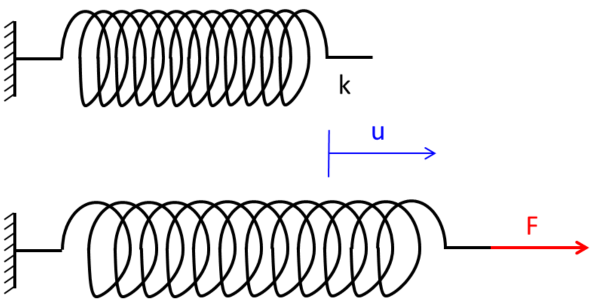


Fig. 2

After energy analysis, the total potential energy of the spring is directly written as

$$\Pi(\mathrm{u})=\frac{1}{2}\mathrm{ku}^2-\mathrm{Fu} \tag{12}$$

The variation of the total potential energy with respect to the variation of the displacement field is derived as

$$\begin{aligned}
&\delta\Pi(\mathrm{u})[\delta\mathrm{u}]=\left.\frac{\mathrm{d}\Pi(\mathrm{u}+\varepsilon\delta\mathrm{u})}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\\
&=\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{2}\mathrm{k}(\mathrm{u}+\varepsilon\delta\mathrm{u})^2-\mathrm{F}(\mathrm{u}+\varepsilon\delta\mathrm{u})\right]\right|_{\varepsilon=0}\\
&=\left[\mathrm{k}(\mathrm{u}+\varepsilon\delta\mathrm{u})\delta\mathrm{u}-\mathrm{F}\delta\mathrm{u}\right]\Big|_{\varepsilon=0}\\
&=(\mathrm{ku}-\mathrm{F})\delta\mathrm{u}
\end{aligned} \tag{13}$$

By the principle of stationary potential energy, one has the equilibrium equation

$$\begin{aligned}
&\delta\Pi(\mathrm{u})[\delta\mathrm{u}]=(\mathrm{ku}-\mathrm{F})\delta\mathrm{u}=0\\
&\Rightarrow\\
&\mathrm{ku}-\mathrm{F}=0
\end{aligned} \tag{14}$$

Alternatively, since the system is discrete, if directly uses the derivative of the potential energy, one has

$$\frac{\mathrm{d}\Pi(\mathrm{u})}{\mathrm{du}}=\mathrm{ku}-\mathrm{F}=0 \tag{15}$$

We see that both methods are equivalent.

3.4 Total potential energy of a bar

As shown in **Fig. 3**, a bar is of length L, cross section area A, and Young's modulus E subjected to a distributed force F(x)=kx. We use energy method and Rayleigh-Ritz method to approximate the equilibrium displacement field u(x).

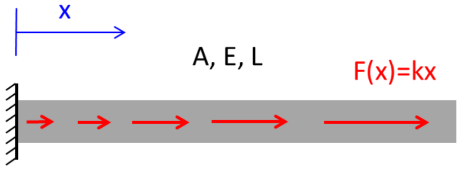


Fig. 3

The total potential energy of the bar is

$$\Pi(u)=\frac{1}{2}\int_0^L AE\left(\frac{du}{dx}\right)^2 dx-\int_0^L kxudx \tag{16}$$

Construct a trial function as

$$\bar{u}(x)=c_1x+c_2x^2 \tag{17}$$

Plugging Eq. (17) into Eq. (16), one has

$$\Pi(c_1,c_2)=\frac{AE}{2}\int_0^L (c_1+2c_2x)^2 dx-\int_0^L c_1kx^2+c_2kx^3dx \tag{18}$$

The equilibrium condition is therefore derived as

$$\begin{aligned}\frac{\partial\Pi(c_1,c_2)}{\partial c_1}&=AE\int_0^L (c_1+2c_2x)dx-\int_0^L kx^2dx=0\\ \frac{\partial\Pi(c_1,c_2)}{\partial c_2}&=AE\int_0^L (c_1+2c_2x)2xdx-\int_0^L kx^3dx=0\end{aligned} \tag{19}$$

Recast the system of linear equations in matrix form as

$$AE\begin{bmatrix}\int_0^L dx & \int_0^L 2xdx\\ \int_0^L 2xdx & \int_0^L 4x^2dx\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\int_0^L kx^2dx\\ \int_0^L kx^3dx\end{bmatrix} \tag{20}$$

After evaluating the integrals, one has

$$AE\begin{bmatrix}L & L^2\\ L^2 & \frac{4}{3}L^3\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}\frac{k}{3}L^3\\ \frac{k}{4}L^4\end{bmatrix} \tag{21}$$

The solution to the linear system is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{7kL^2}{12AE} \\ -\dfrac{kL}{4AE} \end{bmatrix} \tag{22}$$

Introducing Eq. (22) into Eq. (17), one obtains an approximate solution to the bar problem

$$\bar{u}(x) = \frac{7kL^2}{12AE}x - \frac{kL}{4AE}x^2 \tag{23}$$

Moreover, the tip displacement is given as

$$\bar{u}(L) = \frac{kL^3}{3AE} \tag{24}$$

3.5 Total potential energy of a beam

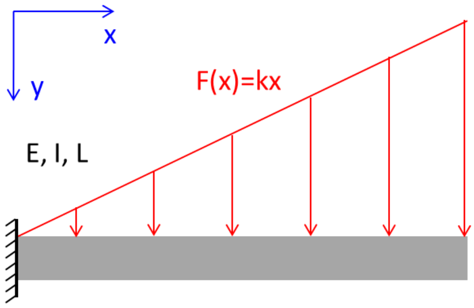


Fig. 4

As shown in **Fig. 4**, a beam is of length L, bending moment of inertial I, and Young's modulus E subjected to a distributed force F(x)=kx. We use energy method and Rayleigh-Ritz method to approximate the equilibrium deflection field v(x).

The total potential energy of the beam is

$$\Pi(v) = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L kxvdx \tag{25}$$

Construct a trial function as

$$\bar{v}(x) = c_1x^2 + c_2x^3 \tag{26}$$

Plugging Eq. (26) into Eq. (25), one has

$$\Pi(c_1, c_2) = \frac{EI}{2}\int_0^L (2c_1 + 6c_2 x)^2 dx - \int_0^L c_1 kx^3 + c_2 kx^4 dx \tag{27}$$

The equilibrium condition is therefore derived as

$$\begin{aligned}
\frac{\partial \Pi(c_1, c_2)}{\partial c_1} &= EI\int_0^L 2(2c_1 + 6c_2 x)dx - \int_0^L kx^3 dx = 0 \\
\frac{\partial \Pi(c_1, c_2)}{\partial c_2} &= EI\int_0^L (2c_1 + 6c_2 x)6x dx - \int_0^L kx^4 dx = 0
\end{aligned} \tag{28}$$

Recast the system of linear equations in matrix form as

$$EI\begin{bmatrix} \int_0^L 4dx & \int_0^L 12x dx \\ \int_0^L 12x dx & \int_0^L 36x^2 dx \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \int_0^L kx^3 dx \\ \int_0^L kx^4 dx \end{bmatrix} \tag{29}$$

After evaluating the integrals, one has

$$EI\begin{bmatrix} 4L & 6L^2 \\ 6L^2 & 12L^3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{4}L^4 \\ \frac{k}{5}L^5 \end{bmatrix} \tag{30}$$

The solution to the linear system is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3kL^3}{20EI} \\ -\frac{7kL^2}{120EI} \end{bmatrix} \tag{31}$$

Introducing Eq. (31) into Eq. (26), one obtains an approximate solution to the beam problem

$$\bar{v}(x) = \frac{3kL^3}{20EI}x^2 - \frac{7kL^2}{120EI}x^3 \tag{32}$$

Moreover, the tip deflection is given as

$$\bar{v}(L) = \frac{11kL^5}{120EI} \tag{33}$$

Appendix: analogy for 1-D linear elastic structural members

In this course, the unknown field variable is chosen as the displacement field. For 1-D linear elastic structural members, the axial strain of a bar and the curvature of a beam are related to the strain energy in a quadratic form that is similar to a spring. Their corresponding stiffness parameters are introduced.

	Constitutive variable	Strain energy	Parameter
Spring	u (displacement)	$\frac{1}{2}\mathrm{ku}^2$	k (spring stiffness)
Bar	$\varepsilon = \mathrm{du}/\mathrm{dx}$ (axial strain)	$\int_0^L \frac{1}{2}\mathrm{EA}\varepsilon^2\mathrm{dx}$	EA (axial stiffness)
Beam	$\kappa = \mathrm{d}^2\mathrm{v}/\mathrm{dx}^2$ (curvature)	$\int_0^L \frac{1}{2}\mathrm{EI}\kappa^2\mathrm{dx}$	EI (bending stiffness)

Lecture 4 Direct stiffness method (Spring system)

4.1 Example 1

Consider a system that consists of two springs in series subject to a force F as shown in **Fig. 1**

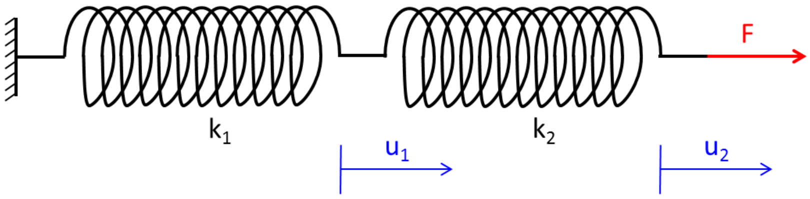


Fig. 1

The two springs has different stiffness denoted as k_1 and k_2. The question here is to find the equilibrium tip displacement.

Method 1: energy method

As before, we first try to use the energy method to solve this problem. The total potential energy is

$$\Pi(u_1, u_2) = \frac{1}{2}k_1 u_1^2 + \frac{1}{2}k_2(u_2 - u_1)^2 - Fu_2 \tag{1}$$

Applying the principle of stationary potential energy, one has

$$\begin{aligned} \frac{\partial \Pi(u_1, u_2)}{\partial u_1} &= k_1 u_1 - k_2(u_2 - u_1) = 0 \\ \frac{\partial \Pi(u_1, u_2)}{\partial u_2} &= k_2(u_2 - u_1) - F = 0 \end{aligned} \tag{2}$$

Recast the linear system of equations in matrix form as

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix} \tag{3}$$

Therefore, the displacement could be easily solved by inverting the **stiffness matrix**.

Method 2: direct stiffness method

By observation, it is a natural thought to partition the system into two springs since the two springs share similar elastic behavior except the stiffness value. Therefore, it will be convenient to first construct a representative spring element as shown in **Fig. 2** and then apply this element when needed, just like a template.

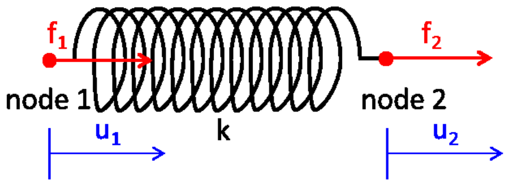


Fig. 2

For this representative spring element, one can also use the principal of stationary potential to obtain the equilibrium relationship between the **nodal displacement** (i.e., u_1 and u_2) and the **nodal force** (i.e., f_1 and f_2).

$$\Pi(u_1,u_2)=\frac{1}{2}k(u_2-u_1)^2-f_1u_1-f_2u_2 \tag{4}$$

Applying the principle of stationary potential energy, one has

$$\begin{aligned}\frac{\partial\Pi(u_1,u_2)}{\partial u_1}&=-k(u_2-u_1)-f_1=0\\ \frac{\partial\Pi(u_1,u_2)}{\partial u_2}&=k(u_2-u_1)-f_2=0\end{aligned} \tag{5}$$

Recast the linear system of equations in matrix form as

$$\boxed{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} \tag{6}$$

So far, we have obtained the stiffness matrix of a typical spring element. Next, let's apply it to solve the problem according to the following steps.

Step 1: Partition the spring system into pieces (elements), number the **Elements** and number the **Nodes** shown in **Fig. 3**. Herein, the first Capital letter means global numbering in contrast to the elemental local numbering shown in Fig. 2)

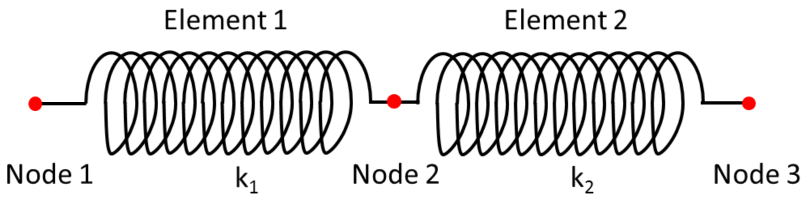


Fig. 3

Step 2: apply the template spring element stiffness relationship Eq. (6).

For Element 1, one has

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \tag{7}$$

Similarly, for Element 2, one has

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} \tag{8}$$

Herein, the superscript $^{(1)}$ and $^{(2)}$ represent element numbers. Note that the local node 2 of the Element 1 is just the local node 1 of the Element 2 (i.e., the Node 2 in the global numbering).

Step 3: By direct superposition of the local (elemental) stiffness matrices, one has the global system of equations in a matrix form as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{9}$$

Step 4: apply the boundary conditions to modify the global equation Eq. (9).

Introducing the boundary condition $u_1 = 0$ into Eq.(9), one has the reduced global system of equations

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} f_2 \\ f_3 \end{bmatrix} \tag{10}$$

Step 5: apply the loadings to solve the global system of equations Eq. (10).

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} 0 \\ F \end{bmatrix} \tag{11}$$

Obviously, Eq. (11) is just the same as Eq. (3). This means that the two methods lead to the same results. It seems that the first method that has less analysis steps is better. However, for computers, the second method is of more advantage since they are good at doing repeating labor works once given a template.

4.2 Example 2

Consider a system that consists of two springs in series subject to a known displacement Δ as shown in **Fig. 4**.

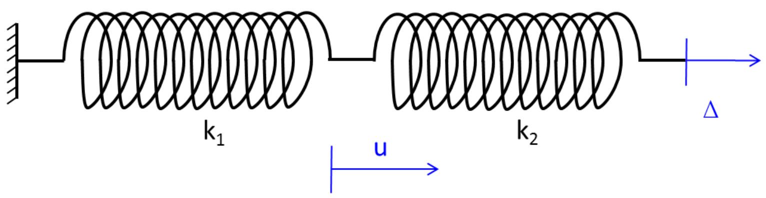


Fig. 4

The question here is to find the equilibrium displacement u and the reaction forces at both ends.

Method 1: energy method

We first try to use the energy method to solve this problem. The total potential energy is

$$\Pi(u)=\frac{1}{2}k_1u^2+\frac{1}{2}k_2(\Delta-u)^2 \tag{12}$$

Applying the principle of stationary potential energy, one has

$$
\begin{aligned}
&\frac{\partial \Pi(\mathrm{u})}{\partial \mathrm{u}}=\mathrm{k}_1\mathrm{u}-\mathrm{k}_2(\Delta-\mathrm{u})=0 \\
&\Rightarrow \\
&\mathrm{u}=\frac{\mathrm{k}_2\Delta}{\mathrm{k}_1+\mathrm{k}_2}
\end{aligned}
\tag{13}
$$

The magnitude of reaction forces at both ends are

$$
\begin{aligned}
&\mathrm{F}_{\mathrm{left}}=\mathrm{k}_1\mathrm{u}=\frac{\mathrm{k}_1\mathrm{k}_2\Delta}{\mathrm{k}_1+\mathrm{k}_2} \\
&\mathrm{F}_{\mathrm{right}}=\mathrm{k}_2(\Delta-\mathrm{u})=\frac{\mathrm{k}_2(1-\mathrm{k}_2)\Delta}{\mathrm{k}_1+\mathrm{k}_2}
\end{aligned}
\tag{14}
$$

Method 2: direct stiffness method

Just as the Steps 1-3 in Example 1, the difference is at the step 4 where boundary conditions are applied. We have in this example that

$$
\begin{bmatrix} \mathrm{k}_1+\mathrm{k}_2 & -\mathrm{k}_2 \\ -\mathrm{k}_2 & \mathrm{k}_2 \end{bmatrix}\begin{bmatrix} \mathrm{u}_2 \\ \Delta \end{bmatrix}=\begin{bmatrix} \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix}
\tag{15}
$$

And by applying loading, one has

$$
\begin{bmatrix} \mathrm{k}_1+\mathrm{k}_2 & -\mathrm{k}_2 \\ -\mathrm{k}_2 & \mathrm{k}_2 \end{bmatrix}\begin{bmatrix} \mathrm{u}_2 \\ \Delta \end{bmatrix}=\begin{bmatrix} 0 \\ \mathrm{f}_3 \end{bmatrix}
\tag{16}
$$

From Eq.(16), one can solve for

$$
\begin{aligned}
&\mathrm{u}=\mathrm{u}_2=\frac{\mathrm{k}_2\Delta}{\mathrm{k}_1+\mathrm{k}_2} \\
&\mathrm{F}_{\mathrm{right}}=\mathrm{f}_3=\frac{\mathrm{k}_2(1-\mathrm{k}_2)\Delta}{\mathrm{k}_1+\mathrm{k}_2}
\end{aligned}
\tag{17}
$$

and from Eq. (9), one has

$$
\mathrm{F}_{\mathrm{left}}=\frac{\mathrm{k}_1\mathrm{k}_2\Delta}{\mathrm{k}_1+\mathrm{k}_2}
\tag{18}
$$

Remarks:

From Example 1 and Example 2, we introduce the direct stiffness method and compare it to the energy method. While the energy method is easy to apply when one uses hands to do computation, the direct stiffness where discretization is introduced is more general and routine for computers. The procedure for the direct stiffness method is summarized as following:

(1) Derive element stiffness matrix.

(2) Discretize the system into elements.

(3) Assemble element stiffness matrices to form system stiffness matrix.

(4) Apply boundary conditions and loadings to form equations.

(5) Solve equations.

4.3 Example 3

By this example, we further show the solution procedure of direct stiffness method. This can be deemed as a prototype of the finite element method.

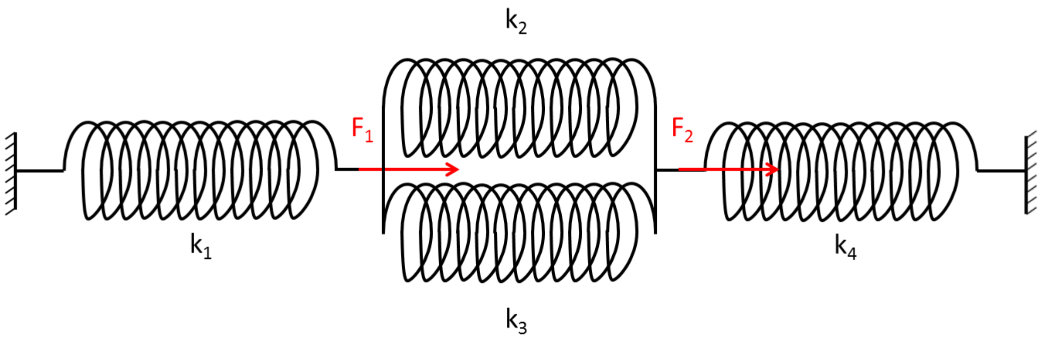


Fig. 5

Next, we consider the example shown in **Fig. 5**.

Step 1: Partition the spring system into elements and do numbering shown in **Fig. 6**.

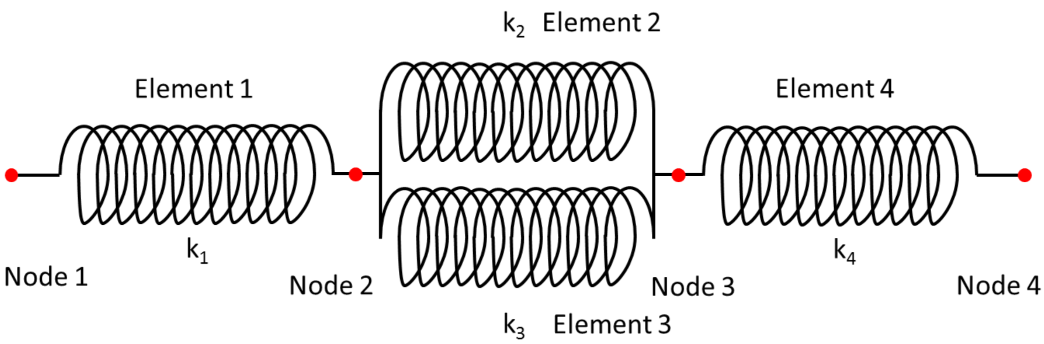


Fig. 6

Step 2: Derive element equations.

For Element 1, one has

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \tag{19}$$

For Element 2, one has

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} \tag{20}$$

For Element 3, one has

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{bmatrix} u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} = \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{bmatrix} \tag{21}$$

For Element 4, one has

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} = \begin{bmatrix} f_1^{(4)} \\ f_2^{(4)} \end{bmatrix} \tag{22}$$

Step 3: Assemble all element equations to form a global system of equations.

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2-k_3 & 0 \\ 0 & -k_2-k_3 & k_2+k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{23}$$

Step 4: Apply boundary conditions

$$\begin{bmatrix} k_1+k_2+k_3 & -k_2-k_3 \\ -k_2-k_3 & k_2+k_3+k_4 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} \tag{24}$$

Step 5: Apply loadings and solve the equations

$$\begin{bmatrix} k_1 + k_2 + k_3 & -k_2 - k_3 \\ -k_2 - k_3 & k_2 + k_3 + k_4 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{25}$$

Step 6: Once the displacement unknowns at Node 2 and Node 3 are solved for by Eq. (25), the magnitude of reaction forces at Node 1 and Node 4 could be evaluated by Eq. (23)

$$\begin{aligned} f_1 &= k_1 u_2 \\ f_4 &= k_4 u_3 \end{aligned} \tag{26}$$

Lecture 5 Bar system

5.1 Stiffness matrix of a bar element in 1-D space

Fig. 1

Consider the bar element as shown in **Fig. 1**. For this representative bar element, its total potential energy is

$$\Pi(u)=\int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx - f_1u_1 - f_2u_2 \tag{1}$$

Since we are given the following conditions

$$\begin{aligned} x_1 &= 0,\ u(x_1)=u_1 \\ x_2 &= L,\ u(x_2)=u_2 \end{aligned} \tag{2}$$

a linear polynomial is adopted to approximate the displacement field

$$u(x)\approx \bar{u}(x)=c_0+c_1x \tag{3}$$

Introducing Eq.(2) into Eq. (3) gives rise to

$$\begin{cases} \bar{u}(0)=c_0+c_1 0=u_1 \\ \bar{u}(L)=c_0+c_1L=u_2 \end{cases} \tag{4}$$

So,

$$\begin{aligned} c_0 &= u_1 \\ c_1 &= \frac{u_2-u_1}{L} \end{aligned} \tag{5}$$

Plugging Eq. (5) back to Eq. (3), one has

$$\bar{u}(x)=u_1+\frac{u_2-u_1}{L}x \tag{6}$$

Furthermore, Eq. (6) could be re-arranged in the following form

$$\bar{u}(x) = \frac{L-x}{L}u_1 + \frac{x}{L}u_2 = N_1(x)u_1 + N_2(x)u_2 \tag{7}$$

where $N_1(x)$ and $N_2(x)$ are called **shape functions** of the bar element with respect to the degree of freedom u_1 and u_2, respectively. It can be shown that

$$\begin{gathered} N_1(x) + N_2(x) = 1 \\ N_i(x_j) = \delta_{ij},\ i = 1,2;\ j = 1,2 \end{gathered} \tag{8}$$

where δ is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 1, \text{ if } i{=}j \\ 0, \text{ if } i \neq j \end{cases} \tag{9}$$

So far, we can introduce Eq. (7) into Eq. (1) to obtain the approximated total potential energy

$$\Pi(u_1, u_2) = \int_0^L \frac{1}{2} EA \left(\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \right)^2 dx - f_1u_1 - f_2u_2 \tag{10}$$

and further apply the principle of stationary potential energy

$$\begin{aligned} \frac{\partial \Pi(u_1,u_2)}{\partial u_1} &= \int_0^L EA \left(\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \right) \frac{dN_1}{dx} dx - f_1 = 0 \\ \frac{\partial \Pi(u_1,u_2)}{\partial u_2} &= \int_0^L EA \left(\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \right) \frac{dN_2}{dx} dx - f_2 = 0 \end{aligned} \tag{11}$$

Recast the linear system of equations in matrix form as

$$\begin{bmatrix} \int_0^L EA \frac{dN_1}{dx}\frac{dN_1}{dx} dx & \int_0^L EA \frac{dN_2}{dx}\frac{dN_1}{dx} dx \\ \int_0^L EA \frac{dN_1}{dx}\frac{dN_2}{dx} dx & \int_0^L EA \frac{dN_2}{dx}\frac{dN_2}{dx} dx \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{12}$$

Evaluate the integrals

$$\boxed{\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} \tag{13}$$

So far, we have obtained the stiffness matrix of a typical bar element.

5.2 Stiffness matrix of a bar element in 2D space

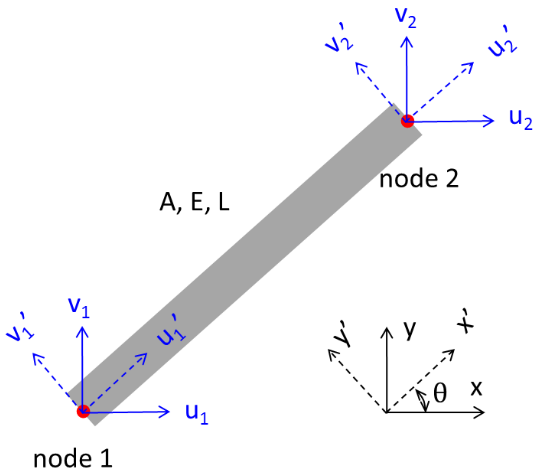


Fig. 2

By coordinate transformation for the nodal displacement as shown in **Fig. 2**, one has the following relationship

$$
\begin{cases}
u_1' = \cos\theta u_1 + \sin\theta v_1 \\
v_1' = -\sin\theta u_1 + \cos\theta v_1 \\
u_2' = \cos\theta u_2 + \sin\theta v_2 \\
v_2' = -\sin\theta u_2 + \cos\theta v_2
\end{cases}
$$
$$
\Rightarrow \tag{14}
$$
$$
\begin{bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}
$$

where C stands for $\cos\theta$ and S stands for $\sin\theta$. Similarly, for the nodal forces as shown in **Fig. 3**

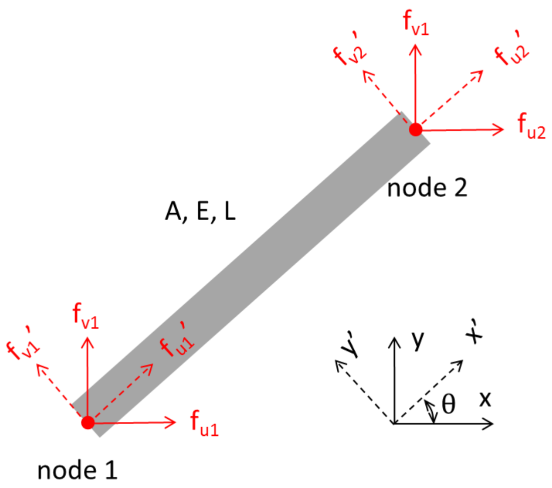


Fig. 3

one has the following relationship

$$\begin{cases} f'_{u1} = \cos\theta f_{u1} + \sin\theta f_{v1} \\ f'_{v1} = -\sin\theta f_{u1} + \cos\theta f_{v1} \\ f'_{u2} = \cos\theta f_{u2} + \sin\theta f_{v2} \\ f'_{v2} = -\sin\theta f_{u2} + \cos\theta f_{v2} \end{cases}$$

$$\Rightarrow \tag{15}$$

$$\begin{bmatrix} f'_{u1} \\ f'_{v1} \\ f'_{u2} \\ f'_{v2} \end{bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} f_{u1} \\ f_{v1} \\ f_{u2} \\ f_{v2} \end{bmatrix}$$

In addition, in the new coordinate system after rotation transformation, form Eq. (13), we know the stiffness matrix of a typical bar element could be recast as

$$\frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{bmatrix} = \begin{bmatrix} f'_{u1} \\ f'_{v1} \\ f'_{u2} \\ f'_{v2} \end{bmatrix} \tag{16}$$

By introducing Eq. (14) and Eq. (15) into Eq. (16), one has

$$\frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} f_{u1} \\ f_{v1} \\ f_{u2} \\ f_{v2} \end{bmatrix} \tag{17}$$

Note that

$$\begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}^T \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} = \mathbf{I}_{4\times 4} \tag{18}$$

So,

$$\frac{EA}{L}\begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_{u1} \\ f_{v1} \\ f_{u2} \\ f_{v2} \end{bmatrix} \tag{19}$$

Furthermore, one has the stiffness matrix for an inclined bar element in 2D space

$$\boxed{\frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} f_{u1} \\ f_{v1} \\ f_{u2} \\ f_{v2} \end{bmatrix}} \tag{20}$$

5.3 Stress in a bar element in 2D space

According to the definition of stress

$$\sigma = \frac{f'_{u2}}{A} \tag{21}$$

By Eq.(17), one has

$$\begin{bmatrix} f'_{u1} \\ f'_{v1} \\ f'_{u2} \\ f'_{v2} \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Cu_1 + Sv_1 \\ -Su_1 + Cv_1 \\ Cu_2 + Sv_2 \\ -Su_2 + Cv_2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} Cu_1 + Sv_1 - Cu_2 - Sv_2 \\ 0 \\ -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\ 0 \end{bmatrix} \tag{22}$$

Therefore, the stress could be evaluated as

$$\sigma = \frac{E}{L}\left(-Cu_1 - Sv_1 + Cu_2 + Sv_2\right) \tag{23}$$

5.4 Bar system in 2D space

A bar system in 2D space is also called a **plane truss**.

5.4.1 Example 1

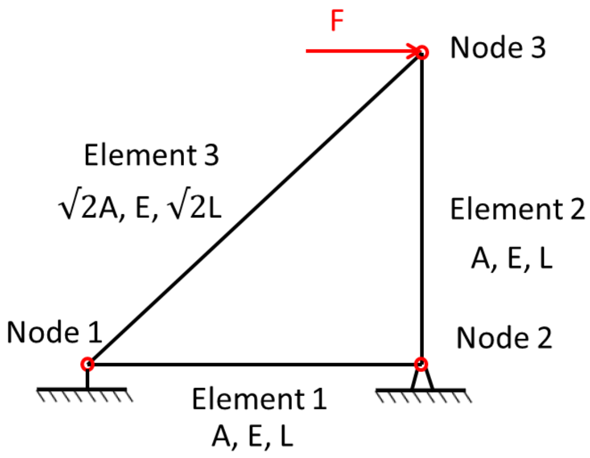


Fig. 4

Consider the plane truss shown in **Fig. 4**. Find the nodal displacement and stress distribution.

Step 1: Loop elements to evaluate element stiffness matrices

Element 1: Calculate direction cosines

$$\begin{aligned} &\theta = 0 \\ &\Rightarrow \\ &C = \cos\theta = 1 \\ &S = \sin\theta = 0 \end{aligned} \tag{24}$$

Evaluate element stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

Element 2: Calculate direction cosines

$$\begin{aligned} &\theta = 90^{o} \\ &\Rightarrow \\ &C = \cos\theta = 0 \\ &S = \sin\theta = 1 \end{aligned} \tag{26}$$

Evaluate element stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{27}$$

Element 3: Calculate direction cosines

$$\begin{aligned} &\theta = 45^{o} \\ &\Rightarrow \\ &C = \cos\theta = \sqrt{2}/2 \\ &S = \sin\theta = \sqrt{2}/2 \end{aligned} \tag{28}$$

Evaluate element stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \tag{29}$$

Step 2: Assemble element stiffness matrix directly and loading vectors to form system equations

$$\frac{EA}{L}\begin{bmatrix} 1+1/2 & 1/2 & -1 & 0 & -1/2 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1/2 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ -1/2 & -1/2 & 0 & -1 & 1/2 & 1+1/2 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F \\ 0 \end{bmatrix} \tag{30}$$

Step 3: Apply boundary conditions to reduce the system of equations

Since we have known that at the boundaries

$$v_1 = u_2 = v_2 = 0 \tag{31}$$

The reduced system of equations is

$$\frac{EA}{L}\begin{bmatrix} 3/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix} \tag{32}$$

Step 4: Solve the system of equations

$$\begin{bmatrix} u_1 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ FL/EA \\ 0 \end{bmatrix} = \begin{bmatrix} FL/EA \\ 4FL/EA \\ -FL/EA \end{bmatrix} \tag{33}$$

Step 5: Compute the stress in each element

Element 1: where C and S are computed in Step 1 correspondingly.

$$\sigma_1 = \frac{E}{L}\left(-Cu_1 - Sv_1 + Cu_2 + Sv_2\right) = -\frac{F}{A} \tag{34}$$

Element 2: where C and S are computed in Step 1 correspondingly.

$$\sigma_2 = \frac{E}{L}\left(-Cu_2 - Sv_2 + Cu_3 + Sv_3\right) = -\frac{F}{A} \tag{35}$$

Element 3: where C and S are computed in Step 1 correspondingly.

$$\sigma_3 = \frac{E}{\sqrt{2}L}\left(-Cu_1 - Sv_1 + Cu_3 + Sv_3\right) = \frac{2F}{A} \tag{36}$$

5.4.2 Example 2

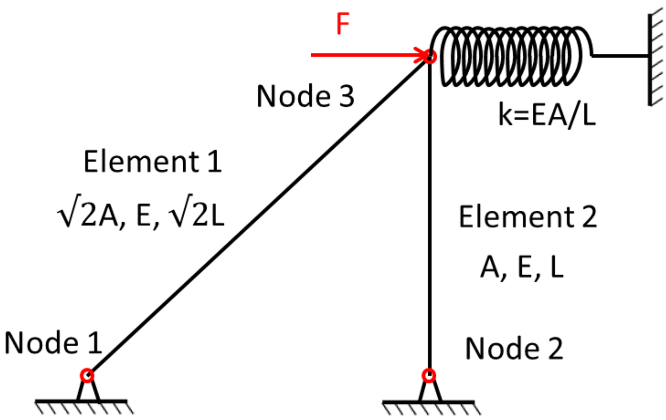


Fig. 5

Consider the plane truss shown in **Fig. 5**. Find the nodal displacement and stress distribution.

Step 1: Loop elements to evaluate element stiffness matrices

Element 1: Calculate direction cosines

$$\begin{aligned}
&\theta = 45^{o} \\
&\Rightarrow \\
&C = \cos\theta = \sqrt{2}/2 \\
&S = \sin\theta = \sqrt{2}/2
\end{aligned} \tag{37}$$

Evaluate element stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \tag{38}$$

Element 2: Calculate direction cosines

$$\begin{aligned}
&\theta = 90^{o} \\
&\Rightarrow \\
&C = \cos\theta = 0 \\
&S = \sin\theta = 1
\end{aligned} \tag{39}$$

Evaluate element stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{40}$$

Spring element:

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{41}$$

Step 2: Assemble element stiffness matrix directly and loading vectors to form system equations

$$\frac{EA}{L}\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 \\ 1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1/2 & -1/2 & 0 & 0 & 1/2+kL/EA & 1/2 \\ -1/2 & -1/2 & 0 & -1 & 1/2 & 1+1/2 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F \\ 0 \end{bmatrix} \tag{42}$$

Step 3: Apply boundary conditions to reduce the system of equations

Since we have known that at the boundaries

$$u_1 = v_1 = u_2 = v_2 = 0 \tag{43}$$

The reduced system of equations is

$$\frac{EA}{L}\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{44}$$

Step 4: Solve the system of equations

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.75 & -0.25 \\ -0.25 & 0.75 \end{bmatrix}\begin{bmatrix} FL/EA \\ 0 \end{bmatrix} = \begin{bmatrix} 3FL/4EA \\ -FL/4EA \end{bmatrix} \tag{45}$$

Step 5: Compute the stress in each element

Element 1: where C and S are computed in Step 1 correspondingly.

$$\sigma_1 = \frac{E}{\sqrt{2}L}\left(-Cu_1 - Sv_1 + Cu_3 + Sv_3\right) \tag{46}$$

Element 2: where C and S are computed in Step 1 correspondingly.

$$\sigma_2 = \frac{E}{L}\left(-Cu_2 - Sv_2 + Cu_3 + Sv_3\right) \tag{47}$$

Lecture 6 Beam system

6.1 Stiffness matrix of a (Euler) beam element in 2-D space

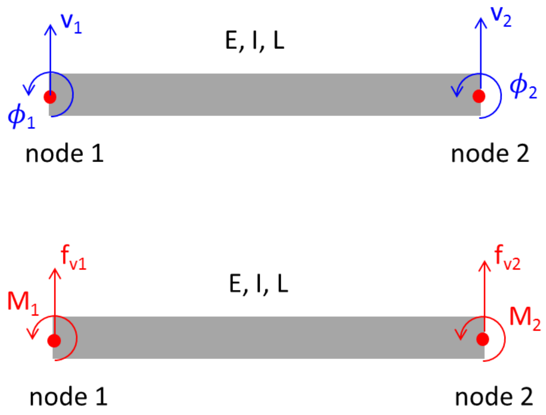


Fig. 1

Consider the beam element as shown in **Fig. 1**. For this representative beam element, its total potential energy is

$$\Pi(u)=\int_0^L \frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2 dx - f_{v1}v_1 - f_{v2}v_2 - M_1\phi_1 - M_2\phi_2 \tag{1}$$

Since we are given the following conditions

$$\begin{aligned} &x_1=0,\ v(x_1)=v_1 \\ &x_1=0,\ \frac{dv}{dx}(x_1)=\phi_1 \\ &x_2=L,\ v(x_2)=v_2 \\ &x_2=L,\ \frac{dv}{dx}(x_2)=\phi_2 \end{aligned} \tag{2}$$

a cubic polynomial is adopted to approximate the displacement field

$$v(x)\approx \bar{v}(x)=c_0+c_1x+c_2x^2+c_3x^3 \tag{3}$$

Introducing Eq.(2) into Eq. (3) gives rise to

$$\begin{cases} \bar{v}(0)=c_0+c_1 0+c_2 0+c_3 0=v_1 \\ d\bar{v}/dx(0)=c_1+c_2 0+c_3 0=\phi_1 \\ \bar{v}(L)=c_0+c_1L+c_2L^2+c_3L^3=v_2 \\ d\bar{v}/dx(L)=c_1+2c_2L+3c_3L^2=\phi_2 \end{cases} \tag{4}$$

So,

$$
\begin{aligned}
&c_0 = v_1 \\
&c_1 = \phi_1 \\
&\Rightarrow \\
&\begin{bmatrix} L^2 & L^3 \\ 2L & 3L^2 \end{bmatrix}\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v_2 - v_1 - \phi_1 L \\ \phi_2 - \phi_1 \end{bmatrix} \\
&\Rightarrow \\
&c_2 = \frac{1}{L^4}\left[\left(v_2 - v_1 - \phi_1 L\right)3L^2 - \left(\phi_2 - \phi_1\right)L^3\right] = \frac{3\left(v_2 - v_1\right)}{L^2} - \frac{2\phi_1 + \phi_2}{L} \\
&c_3 = \frac{1}{L^4}\left[\left(\phi_2 - \phi_1\right)L^2 - \left(v_2 - v_1 - \phi_1 L\right)2L\right] = \frac{\phi_2 + \phi_1}{L^2} - \frac{2\left(v_2 - v_1\right)}{L^3}
\end{aligned}
\tag{5}
$$

Plugging Eq. (5) back to Eq. (3), one has

$$
\bar{u}(x) = v_1 + \phi_1 x + \left[\frac{3\left(v_2 - v_1\right)}{L^2} - \frac{2\phi_1 + \phi_2}{L}\right]x^2 + \left[\frac{\phi_2 + \phi_1}{L^2} - \frac{2\left(v_2 - v_1\right)}{L^3}\right]x^3 \tag{6}
$$

Furthermore, Eq. (6) could be re-arranged in the following form

$$
\begin{aligned}
\bar{v}(x) &= \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)v_1 + \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\phi_1 \\
&+ \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)v_2 + \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)\phi_2 \\
&= N_1(x)v_1 + N_2(x)\phi_1 + N_3(x)v_2 + N_4(x)\phi_2
\end{aligned}
\tag{7}
$$

where $N_1(x)$, $N_2(x)$, $N_3(x)$ and $N_4(x)$ are called **shape functions** of the beam element with respect to the degree of freedom v_1, ϕ_1, v_2 and ϕ_2, respectively.

Evaluate the derivatives of Eq. (7)

$$
\begin{aligned}
\frac{d\bar{v}}{dx}(x) &= \left(-\frac{6x}{L^2} + \frac{6x^2}{L^3}\right)v_1 + \left(1 - \frac{4x}{L} + \frac{3x^2}{L^2}\right)\phi_1 \\
&+ \left(\frac{6x}{L^2} - \frac{6x^2}{L^3}\right)v_2 + \left(-\frac{2x}{L} + \frac{3x^2}{L^2}\right)\phi_2
\end{aligned}
\tag{8}
$$

and

$$\begin{aligned}\frac{d^2\overline{v}}{dx^2}(x)=&\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\&+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{aligned}\tag{9}$$

Introduce Eq. (9) into Eq. (1) to obtain the approximated total potential energy

$$\begin{aligned}\Pi(v_1,\phi_1,v_2,\phi_2)&=\int_0^L\frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2dx-f_{v1}v_1-f_{v2}v_2-M_1\phi_1-M_2\phi_2\\&=\int_0^L\frac{1}{2}EI\begin{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{bmatrix}^2dx\\&\quad-f_{v1}v_1-f_{v2}v_2-M_1\phi_1-M_2\phi_2\end{aligned}\tag{10}$$

Apply the principle of stationary potential energy

$$\begin{aligned}\frac{\partial\Pi}{\partial v_1}&=\int_0^L EI\begin{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)dx-f_{v1}=0\\
\frac{\partial\Pi}{\partial \phi_1}&=\int_0^L EI\begin{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{bmatrix}\left(-\frac{4}{L}+\frac{6x}{L^2}\right)dx-M_1=0\\
\frac{\partial\Pi}{\partial v_2}&=\int_0^L EI\begin{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{bmatrix}\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)dx-f_{v2}=0\\
\frac{\partial\Pi}{\partial \phi_2}&=\int_0^L EI\begin{bmatrix}\left(-\frac{6}{L^2}+\frac{12x}{L^3}\right)v_1+\left(-\frac{4}{L}+\frac{6x}{L^2}\right)\phi_1\\+\left(\frac{6}{L^2}-\frac{12x}{L^3}\right)v_2+\left(-\frac{2}{L}+\frac{6x}{L^2}\right)\phi_2\end{bmatrix}\left(-\frac{2}{L}+\frac{6x}{L^2}\right)dx-M_2=0\end{aligned}\tag{11}$$

Evaluate the integrals and recast the equations in matrix form

$$\boxed{\frac{EI}{L^3}\begin{bmatrix}12&6L&-12&6L\\6L&4L^2&-6L&2L^2\\-12&-6L&12&-6L\\6L&2L^2&-6L&4L^2\end{bmatrix}\begin{bmatrix}v_1\\\phi_1\\v_2\\\phi_2\end{bmatrix}=\begin{bmatrix}f_{v1}\\M_1\\f_{v2}\\M_2\end{bmatrix}}\tag{12}$$

So far, we have obtained the stiffness matrix of a typical beam element.

6.2 Equivalent nodal force/moment for distributed loads

For an arbitrary distributed force on the beam, its equivalent nodal force/moment could be evaluated by work equivalent

$$\int_0^L \bar{v}(x)w(x)dx = f_{v1}v_1 + f_{v2}v_2 + M_1\phi_1 + M_2\phi_2 \tag{13}$$

where we use the approximated displacement field from Eq. (7). For example,

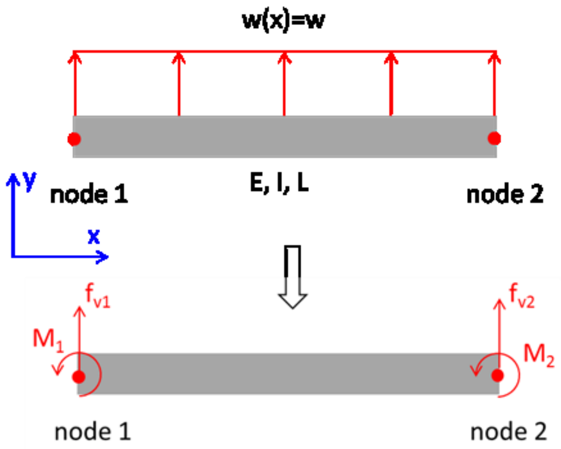


Fig. 2

As shown in **Fig. 2**, we consider a constant distributed load. Its nodal equivalent force is determined as

$$\begin{aligned}\int_0^L \bar{v}(x)wdx &= \left(w\int_0^L 1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}dx\right)v_1 + \left(w\int_0^L x-\frac{2x^2}{L}+\frac{x^3}{L^2}dx\right)\phi_1 \\ &+\left(w\int_0^L \frac{3x^2}{L^2}-\frac{2x^3}{L^3}dx\right)v_2 + \left(w\int_0^L -\frac{x^2}{L}+\frac{x^3}{L^2}dx\right)\phi_2\end{aligned} \tag{14}$$

Therefore,

$$
\begin{aligned}
f_{v1} &= w\int_0^L 1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}\,dx=\frac{wL}{2} \\
M_1 &= w\int_0^L x-\frac{2x^2}{L}+\frac{x^3}{L^2}\,dx=\frac{wL^2}{12} \\
f_{v2} &= w\int_0^L \frac{3x^2}{L^2}-\frac{2x^3}{L^3}\,dx=\frac{wL}{2} \\
M_2 &= w\int_0^L -\frac{x^2}{L}+\frac{x^3}{L^2}\,dx=-\frac{wL^2}{12}
\end{aligned} \tag{15}
$$

6.3 Stiffness matrix of an inclined beam element in 2-D space

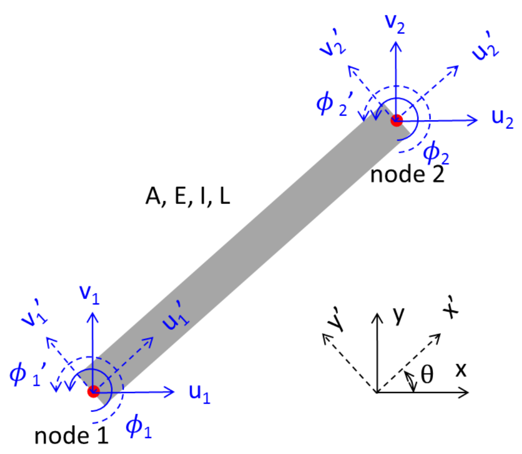


Fig. 3

By coordinate transformation for the nodal displacement as shown in **Fig. 3**, one has the following relationship

$$\begin{cases} u_1' = \cos\theta u_1 + \sin\theta v_1 \\ v_1' = -\sin\theta u_1 + \cos\theta v_1 \\ \phi_1' = \phi_1 \\ u_2' = \cos\theta u_2 + \sin\theta v_2 \\ v_2' = -\sin\theta u_2 + \cos\theta v_2 \\ \phi_2' = \phi_2 \end{cases}$$

$$\Rightarrow \tag{16}$$

$$\begin{bmatrix} u_1' \\ v_1' \\ \phi' \\ u_2' \\ v_2' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{bmatrix} = \mathbf{T}_{6\times 6} \cdot \begin{bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{bmatrix}$$

where C stands for $\cos\theta$ and S stands for $\sin\theta$. Similarly, for the nodal forces as shown in **Fig. 4**

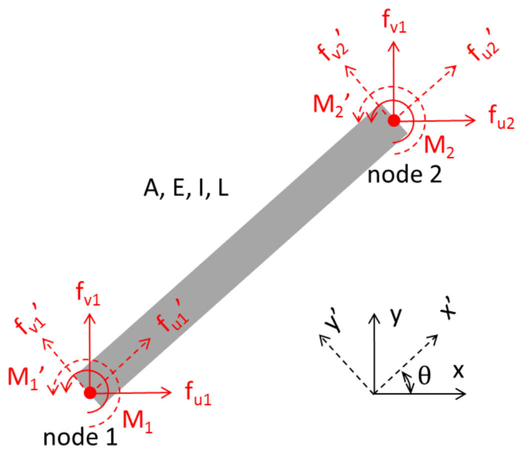


Fig. 4

one has the following relationship

$$
\begin{cases}
f'_{u1} = \cos\theta f_{u1} + \sin\theta f_{v1} \\
f'_{v1} = -\sin\theta f_{u1} + \cos\theta f_{v1} \\
M'_1 = M_1 \\
f'_{u2} = \cos\theta f_{u2} + \sin\theta f_{v2} \\
f'_{v2} = -\sin\theta f_{u2} + \cos\theta f_{v2} \\
M'_2 = M_2
\end{cases}
$$

$$
\Rightarrow \tag{17}
$$

$$
\begin{bmatrix} f'_{u1} \\ f'_{v1} \\ M'_1 \\ f'_{u2} \\ f'_{v2} \\ M'_2 \end{bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{u1} \\ f_{v1} \\ M_1 \\ f_{u2} \\ f_{v2} \\ M_2 \end{bmatrix} = \mathbf{T}_{6\times6} \cdot \begin{bmatrix} f_{u1} \\ f_{v1} \\ M_1 \\ f_{u2} \\ f_{v2} \\ M_2 \end{bmatrix}
$$

In addition, in the new coordinate system after rotation transformation, form Eq. (12), we know the stiffness matrix of a typical beam element could be recast as

$$
\frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & 6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} u'_1 \\ v'_1 \\ \phi'_1 \\ u'_2 \\ v'_2 \\ \phi'_2 \end{bmatrix} = \begin{bmatrix} f'_{u1} \\ f'_{v1} \\ M'_1 \\ f'_{u2} \\ f'_{v2} \\ M'_2 \end{bmatrix} \tag{18}
$$

By introducing Eq. (16) and Eq. (17) into Eq. (18), one has

$$
\frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \cdot \mathbf{T}_{6\times6} \cdot \begin{bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{bmatrix} = \mathbf{T}_{6\times6} \cdot \begin{bmatrix} f_{u1} \\ f_{v1} \\ M_1 \\ f_{u2} \\ f_{v2} \\ M_2 \end{bmatrix} \tag{19}
$$

Note that

$$
\mathbf{T}_{6\times6}^{T} \cdot \mathbf{T}_{6\times6} = \mathbf{I}_{6\times6} \tag{20}
$$

So,

$$\mathbf{T}_{6\times6}^{T}\cdot\frac{EI}{L^3}\begin{bmatrix}0&0&0&0&0&0\\0&12&6L&0&-12&6L\\0&6L&4L^2&0&-6L&2L^2\\0&0&0&0&0&0\\0&-12&-6L&0&12&-6L\\0&6L&2L^2&0&-6L&4L^2\end{bmatrix}\cdot\mathbf{T}_{6\times6}\begin{bmatrix}u_1\\v_1\\\phi_1\\u_2\\v_2\\\phi_2\end{bmatrix}=\begin{bmatrix}f_{u1}\\f_{v1}\\M_1\\f_{u2}\\f_{v2}\\M_2\end{bmatrix}\tag{21}$$

Furthermore, one has the stiffness matrix for an inclined beam element in 2D space

$$\boxed{\frac{EI}{L^3}\begin{bmatrix}12S^2&-12SC&-6LS&-12S^2&12SC&-6LS\\&12C^2&6LC&12SC&-12C^2&6LC\\&&4L^2&6LS&-6LC&2L^2\\&&&12S^2&-12SC&6LS\\&&&&12C^2&-6LS\\Sym&&&&&4L^2\end{bmatrix}\begin{bmatrix}u_1\\v_1\\\phi_1\\u_2\\v_2\\\phi_2\end{bmatrix}=\begin{bmatrix}f_{u1}\\f_{v1}\\M_1\\f_{u2}\\f_{v2}\\M_2\end{bmatrix}}\tag{22}$$

where ‘Sym’ means symmetry about the diagonal.

6.4 Stiffness matrix of an inclined column-beam element in 2-D space

If we take the bar element and beam element together into consideration, then in the rotated coordinate system, one has

$$\begin{bmatrix}C_1&0&0&-C_1&0&0\\0&12C_2&6LC_2&0&-12C_2&6LC_2\\0&6LC_2&4L^2C_2&0&-6LC_2&2L^2C_2\\-C_1&0&0&C_1&0&0\\0&-12C_2&-6LC_2&0&12C_2&-6LC_2\\0&6LC_2&2L^2C_2&0&-6LC_2&4L^2C_2\end{bmatrix}\begin{bmatrix}u_1'\\v_1'\\\phi_1'\\u_2'\\v_2'\\\phi_2'\end{bmatrix}=\begin{bmatrix}f_{u1}'\\f_{v1}'\\M_1'\\f_{u2}'\\f_{v2}'\\M_2'\end{bmatrix}\tag{23}$$

where

$$C_1=\frac{EA}{L},\quad C_2=\frac{EI}{L^3}\tag{24}$$

By using transformation relationships Eq. (16) and Eq. (17), one has

$$\mathbf{T}_{6\times6}^{T}\cdot\begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4L^2C_2 & 0 & -6LC_2 & 2L^2C_2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2L^2C_2 & 0 & -6LC_2 & 4L^2C_2 \end{bmatrix}\cdot\mathbf{T}_{6\times6}\begin{bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{bmatrix}=\begin{bmatrix} f_{u1} \\ f_{v1} \\ M_1 \\ f_{u2} \\ f_{v2} \\ M_2 \end{bmatrix} \tag{25}$$

After evaluating the matrix multiplications, one has the stiffness matrix for an inclined column-beam element in 2D space

$$\boxed{\frac{E}{L}\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & & & & AS^2+\frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ \text{Sym} & & & & & 4I \end{bmatrix}} \tag{26}$$

where ‘Sym’ means symmetry about the diagonal.

MATLAB code

```
syms C S L E A I
T=[C S 0 0 0 0;
   -S C 0 0 0 0;
   0 0 1 0 0 0;
   0 0 0 C S 0;
   0 0 0 -S C 0;
   0 0 0 0 0 1];
TT=transpose(T);
K=[0 0 0 0 0 0;
   0 12 6*L 0 -12 6*L;
   0 6*L 4*L^2 0 -6*L 2*L^2;
   0 0 0 0 0 0;
   0 -12 -6*L 0 12 -6*L;
   0 6*L 2*L^2 0 -6*L 4*L^2];
K=E*I/L^3*K;
K(1,1)=E*A/L;
K(4,4)=E*A/L;
```

```
K(1,4)=-E*A/L;
K(4,1)=-E*A/L;
K_new=TT*K*T
```

6.5 Beam system in 2D space

A beam system in 2D space is also called a **plane frame**.

6.5.1 Example 1

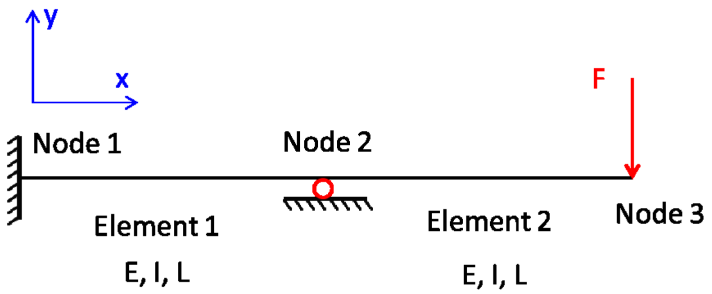


Fig. 5

Consider the plane frame shown in Fig. 5. Find the nodal displacements and slopes

Step 1: Loop elements

Element 1:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1^{(1)} \\ \phi_1^{(1)} \\ v_2^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} f_{v1}^{(1)} \\ M_1^{(1)} \\ f_{v2}^{(1)} \\ M_2^{(1)} \end{bmatrix} \tag{27}$$

Element 2:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1^{(2)} \\ \phi_1^{(2)} \\ v_2^{(2)} \\ \phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_{v1}^{(2)} \\ M_1^{(2)} \\ f_{v2}^{(2)} \\ M_2^{(2)} \end{bmatrix} \tag{28}$$

Step 2: Assemble element stiffness matrix directly and loading vectors to form system equations

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \end{bmatrix} \tag{29}$$

Step 3: Apply boundary conditions to reduce the system of equations

Since we have known that at the boundaries

$$v_1 = \phi_1 = v_2 = 0 \tag{30}$$

The reduced system of equations is

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -F \\ 0 \end{bmatrix} \tag{31}$$

Step 4: Solve the system of equations

$$\begin{bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{bmatrix} = \frac{L^3}{EI}\begin{bmatrix} \frac{1}{4L^2} & \frac{1}{4L} & \frac{1}{4L^2} \\ \frac{1}{4L} & \frac{7}{12} & \frac{3}{4L} \\ \frac{1}{4L^2} & \frac{3}{4L} & \frac{5}{4L^2} \end{bmatrix}\begin{bmatrix} 0 \\ -F \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{L^2F}{4EI} \\ -\frac{7L^3F}{12EI} \\ -\frac{3L^2F}{4EI} \end{bmatrix} \tag{32}$$

MATLAB code

```
syms L
K=[8*L^2, -6*L, 2*L^2;
    -6*L,   12,  -6*L;
   2*L^2, -6*L, 4*L^2];
inv(K)
```

6.5.2 Example 2

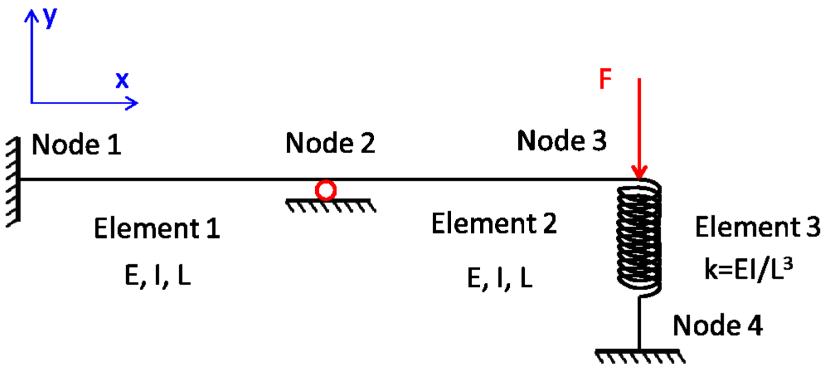


Fig. 6

Consider the plane truss shown in Fig. 6. Find the nodal displacement and stress distribution.

Step 1: Loop elements

Element 1:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1^{(1)} \\ \phi_1^{(1)} \\ v_2^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} f_{v1}^{(1)} \\ M_1^{(1)} \\ f_{v2}^{(1)} \\ M_2^{(1)} \end{bmatrix} \tag{33}$$

Element 2:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1^{(2)} \\ \phi_1^{(2)} \\ v_2^{(2)} \\ \phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_{v1}^{(2)} \\ M_1^{(2)} \\ f_{v2}^{(2)} \\ M_2^{(2)} \end{bmatrix} \tag{34}$$

Spring element:

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1^{(3)} \\ v_2^{(3)} \end{bmatrix} = \begin{bmatrix} f_{v1}^{(3)} \\ f_{v2}^{(3)} \end{bmatrix} \tag{35}$$

Step 2: Assemble element stiffness matrix directly and loading vectors to form system equations

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ 12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+1 & -6L & -1 \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -F \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

Step 3: Apply boundary conditions to reduce the system of equations

Since we have known that at the boundaries

$$v_1 = \phi_1 = v_2 = v_4 = 0 \tag{37}$$

The reduced system of equations is

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 13 & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -F \\ 0 \end{bmatrix} \tag{38}$$

Step 4: Solve the system of equations

$$\begin{bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{bmatrix} = \frac{L^3}{EI}\begin{bmatrix} \frac{4}{19L^2} & \frac{3}{19L} & \frac{5}{38L^2} \\ \frac{3}{19L} & \frac{7}{19} & \frac{9}{19L} \\ \frac{5}{38L^2} & \frac{9}{19L} & \frac{17}{19L^2} \end{bmatrix}\begin{bmatrix} 0 \\ -F \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3L^2F}{19EI} \\ -\frac{7L^3F}{19EI} \\ -\frac{9L^2F}{19EI} \end{bmatrix} \tag{39}$$

MATLAB code

```
syms L
K=[8*L^2, -6*L, 2*L^2;
    -6*L,   13,  -6*L;
   2*L^2, -6*L, 4*L^2];
inv(K)
```

Lecture 7 Linear elasticity

In the previous lectures, we have learnt the basic idea about the finite element method in terms of the spring system, bar system, and beam system. We see that there is a standard procedure to derive the element stiffness matrix based on energy method and a standard procedure to discretize and assemble the system of interest. The spring element, the bar element, and the beam element introduced there are all called **structural elements**, where simplified kinematic models are assumed for solids with special shapes. From this lecture on, we will learn **continuum elements** for solids with arbitrary shapes. Since Hooke's linear elastic model is the simplest material model for a continuum solid, in this lecture, we will talk about it first.

7.1 Global coordinate system

A global coordinate system is required to record the position of a particle, or the configuration of a body in a finite element analysis. For example, in a 3D space, a rectangular Cartesian coordinate is usually built by defining three orthogonal basis vectors and a fixed origin

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

7.2 Displacement field

When a body is subject to external forces, the particle inside the body would move in a translational or rotational manner to occupy a new position. If we denote the initial position of the particle as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2}$$

and the current position as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

Then, the displacement of the particle is denoted as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4}$$

For all the particles in the body, a displacement field is formed with respect to the initial configuration Ω_0 of the body.

$$\mathbf{u} = \hat{\mathbf{u}}(\mathbf{X}), \quad \mathbf{X} \in \Omega_0 \tag{5}$$

In addition, the current configuration that comprises the deformed positions of all the particles is denoted by Ω_t.

7.3 (Small/Engineering) strain field

For a differential line $\mathrm{d}\mathbf{X}$ in the initial configuration, it deforms to a differential line $\mathrm{d}\mathbf{x}$ in the current configuration. Then, the deformation of the differential line could be described by the displacement gradient

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \mathrm{d}\mathbf{X} = \left(\mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right) \mathrm{d}\mathbf{X} = \left(\mathbf{I} + \bar{\nabla} \mathbf{u} \right) \mathrm{d}\mathbf{X} \tag{6}$$

where the **gradient of the displacement field** is defined as

$$\bar{\nabla}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{7}$$

Note that $\bar{\nabla} * = \frac{\partial *}{\partial \mathbf{X}}$ is defined in the initial configuration.

The arc-length or the norm of the differential line in the current configuration is

$$\begin{aligned}
&dl = \|d\mathbf{x}\| = \sqrt{d\mathbf{x}^T d\mathbf{x}} = \sqrt{d\mathbf{X}^T \left(\mathbf{I} + \bar{\nabla}\mathbf{u}\right)^T \left(\mathbf{I} + \bar{\nabla}\mathbf{u}\right) d\mathbf{X}} \\
&\Rightarrow \\
&dl = \sqrt{d\mathbf{X}^T \left[\mathbf{I} + \bar{\nabla}\mathbf{u} + \left(\bar{\nabla}\mathbf{u}\right)^T + \left(\bar{\nabla}\mathbf{u}\right)^T \bar{\nabla}\mathbf{u}\right] d\mathbf{X}} \\
&\Rightarrow \\
&dl = \sqrt{d\mathbf{X}^T d\mathbf{X} + d\mathbf{X}^T \left[\bar{\nabla}\mathbf{u} + \left(\bar{\nabla}\mathbf{u}\right)^T\right] d\mathbf{X} + d\mathbf{X}^T \left(\bar{\nabla}\mathbf{u}\right)^T \bar{\nabla}\mathbf{u} d\mathbf{X}} \\
&\Rightarrow \\
&dl = \sqrt{\left(dL\right)^2 + d\mathbf{X}^T \left[\bar{\nabla}\mathbf{u} + \left(\bar{\nabla}\mathbf{u}\right)^T\right] d\mathbf{X} + d\mathbf{X}^T \left(\bar{\nabla}\mathbf{u}\right)^T \bar{\nabla}\mathbf{u} d\mathbf{X}} \\
&\Rightarrow \\
&dl = dL\sqrt{1 + d\mathbf{n}^T \left[\bar{\nabla}\mathbf{u} + \left(\bar{\nabla}\mathbf{u}\right)^T\right] d\mathbf{n} + d\mathbf{n}^T \left(\bar{\nabla}\mathbf{u}\right)^T \bar{\nabla}\mathbf{u} d\mathbf{n}}
\end{aligned} \tag{8}$$

where $d\mathbf{X} = \|d\mathbf{X}\|\mathbf{n} = dL\mathbf{n}$, $\mathbf{n}$ is the unit vector along the direction of $d\mathbf{X}$.

In this course, we assume the norm of the gradient of the displacement field is a small value, i.e.,

$$\|\bar{\nabla}\mathbf{u}\| = \max \frac{\|\bar{\nabla}\mathbf{u} \cdot d\mathbf{X}\|}{\|d\mathbf{X}\|} = \max \|\bar{\nabla}\mathbf{u} \cdot \mathbf{n}\| \ll 1, \quad \forall d\mathbf{X} \in \mathbb{R}^3 \tag{9}$$

Thus, the 2^{nd} order and higher order terms in Eq. (8) could be neglected

$$
\begin{aligned}
&\mathrm{dl} \approx \mathrm{dL}\sqrt{1+\mathbf{dn}^{\mathrm{T}}\left[\bar{\nabla}\mathbf{u}+\left(\bar{\nabla}\mathbf{u}\right)^{\mathrm{T}}\right]\mathbf{dn}} \\
&\Rightarrow \\
&\mathrm{dl} \approx \mathrm{dL}\left(1+\mathbf{dn}^{\mathrm{T}}\frac{\left[\bar{\nabla}\mathbf{u}+\left(\bar{\nabla}\mathbf{u}\right)^{\mathrm{T}}\right]}{2}\mathbf{dn}\right) \\
&\Rightarrow \\
&\varepsilon_{\mathrm{n}} = \frac{\mathrm{dl}-\mathrm{dL}}{\mathrm{dL}} = \mathbf{dn}^{\mathrm{T}}\frac{\left[\bar{\nabla}\mathbf{u}+\left(\bar{\nabla}\mathbf{u}\right)^{\mathrm{T}}\right]}{2}\mathbf{dn}
\end{aligned}
\tag{10}
$$

From Eq. (10), one could evaluate the strain ε_{n} along an arbitrary direction $\mathbf{n}$ when a **small strain field** is introduced as

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{\left[\bar{\nabla}\mathbf{u}+\left(\bar{\nabla}\mathbf{u}\right)^{\mathrm{T}}\right]}{2} \\
&= \begin{bmatrix}
\dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) \\
\dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) & \dfrac{\partial u_3}{\partial X_3}
\end{bmatrix} \\
&= \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
\end{aligned}
\tag{11}
$$

We notice that $\boldsymbol{\varepsilon}$ is symmetric. A standard eigenvalue analysis could be performed to find its principal strains λ^{ε} and corresponding principal directions $\mathbf{n}^{\varepsilon}$.

$$
\boldsymbol{\varepsilon}\cdot\mathbf{n}^{\varepsilon} = \lambda^{\varepsilon}\mathbf{n}^{\varepsilon}
\tag{12}
$$

In addition, there are only six independent components. So, for convenience of computation, a strain vector is usually introduced as

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \tilde{\varepsilon}_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} \\ \dfrac{\partial u_2}{\partial X_2} \\ \dfrac{\partial u_3}{\partial X_3} \\ \dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1} \\ \dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1} \\ \dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2} \end{bmatrix} \tag{13}$$

where $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$ and $\tilde{\varepsilon}_3$ are called **direct components** of the strain vector and γ_{12}, γ_{13} and γ_{23} are **shear component**s of the strain vector.

In linear elasticity, the small/engineering strain defined in the initial configuration is adopted as the **strain measure**.

7.4 (Cauchy) stress field

We already know the definition of stress for 1D case, i.e., force per unit area. Herein, we generalize the definition to 3D case.

In the current configuration, consider a differential surface whose area is denoted as da and whose **unit normal direction** is denoted as

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{14}$$

Note that $\|\mathbf{n}\| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1$. The differential surface in 3D space is therefore defined as

$$\boxed{\mathbf{S} = \mathbf{n}da} \tag{15}$$

Consider a **traction** (i.e., force per the differential area) that is imposed on the differential surface

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{16}$$

The force in 3D space is therefore defined as

$$\boxed{\mathbf{F} = \mathbf{t}da} \tag{17}$$

We define stress for the 3D case as

$$\boldsymbol{\sigma} = \frac{\mathbf{F}}{\mathbf{S}} \tag{18}$$

Then, a relationship between the traction and the unit normal direction could be derived as

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{19}$$

It is apparent that stress could be written in a matrix form as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{20}$$

Consider a body that is subject to surface traction in the current configuration.

(1) Force balance requires that

$$\begin{aligned} & \int_{\partial\Omega_t} \mathbf{t} da = 0 \\ & \Rightarrow \\ & \int_{\partial\Omega_t} \boldsymbol{\sigma} \cdot \mathbf{n} da = 0 \\ & \Rightarrow \\ & \int_{\Omega_t} \boldsymbol{\sigma} \cdot \nabla dv = 0 \\ & \Rightarrow \\ & \boldsymbol{\sigma} \cdot \nabla = 0 \end{aligned} \tag{21}$$

Note that $\nabla * = \frac{\partial *}{\partial \mathbf{x}}$ is defined in the current configuration.

(2) Moment balance requires that

$$
\begin{aligned}
&\int_{\partial\Omega_t} \mathbf{x}\times\mathbf{t}da = 0\\
&\Rightarrow\\
&\int_{\partial\Omega_t} \mathbf{x}\times\boldsymbol{\sigma}\cdot\mathbf{n}da = 0\\
&\Rightarrow\\
&\int_{\Omega_t} (\mathbf{x}\times\boldsymbol{\sigma})\cdot\nabla dv = 0\\
&\Rightarrow\\
&\int_{\Omega_t} \begin{bmatrix} \sigma_{32}-\sigma_{23}\\ \sigma_{13}-\sigma_{31}\\ \sigma_{21}-\sigma_{12} \end{bmatrix} dv + \int_{\Omega_t} \mathbf{x}\times\boldsymbol{\sigma}\cdot\nabla dv = 0\\
&\Rightarrow\\
&\sigma_{32}-\sigma_{23} = 0\\
&\sigma_{13}-\sigma_{31} = 0\\
&\sigma_{21}-\sigma_{12} = 0\\
&\Rightarrow\\
&\boldsymbol{\sigma}^{T} = \boldsymbol{\sigma}
\end{aligned}
\tag{22}
$$

From Eq. (22), we know that the stress matrix is symmetric. Similar to the eigenvalue analysis of a strain matrix, a standard eigenvalue analysis could be performed as following

$$
\boldsymbol{\sigma}\cdot\mathbf{n}^{\sigma} = \lambda^{\sigma}\mathbf{n}^{\sigma} \tag{23}
$$

where λ^{σ} and $\mathbf{n}^{\sigma}$ denote a principal stresses and its corresponding principal direction, respectively.

In practice, for convenience of computation, a stress vector is usually introduced as

$$
\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \tilde{\sigma}_1\\ \tilde{\sigma}_2\\ \tilde{\sigma}_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23} \end{bmatrix} = \begin{bmatrix} \sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23} \end{bmatrix} \tag{24}
$$

where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ are called the **direct components** of the stress vector and

τ_{12}, τ_{13} and τ_{23} are the **shear components** of the stress vector.

In linear elasticity, the Cauchy stress that defined in the current configuration is adopted as the **stress measure**. Nevertheless, when the small strain assumption holds, it is applicable.

7.5 Hooke's law (Linear elasticity)

The Hooke's law defines a linear relationship between the stress components and strain components for an isotropic linear elastic material.

$$
\begin{aligned}
\tilde{\varepsilon}_1 &= \frac{\tilde{\sigma}_1}{E} - \nu\frac{\tilde{\sigma}_2}{E} - \nu\frac{\tilde{\sigma}_3}{E} \\
\tilde{\varepsilon}_2 &= \frac{\tilde{\sigma}_2}{E} - \nu\frac{\tilde{\sigma}_1}{E} - \nu\frac{\tilde{\sigma}_3}{E} \\
\tilde{\varepsilon}_3 &= \frac{\tilde{\sigma}_3}{E} - \nu\frac{\tilde{\sigma}_1}{E} - \nu\frac{\tilde{\sigma}_2}{E} \\
\gamma_{12} &= \frac{1}{G}\tau_{12} \\
\gamma_{13} &= \frac{1}{G}\tau_{13} \\
\gamma_{23} &= \frac{1}{G}\tau_{23}
\end{aligned}
\tag{25}
$$

where E is called **Young's modulus**, ν is called **Poisson's ratio**, and G is called **shear modulus** defined as

$$G = \frac{E}{2(1+\nu)} \tag{26}$$

They are all material parameters. It could be written in the matrix form as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix}\tilde{\boldsymbol{\sigma}}=\mathbf{C}\tilde{\boldsymbol{\sigma}} \tag{27}$$

where $\mathbf{C}$ is called **compliance matrix**, which is symmetric and positive definite.

Alternatively, from Eq. (27), one has

$$\tilde{\boldsymbol{\sigma}}=\mathbf{D}\tilde{\boldsymbol{\varepsilon}} \tag{28}$$

where $\mathbf{D}=\mathbf{C}^{-1}$ is called **stiffness matrix**, which is also symmetric.

$$\mathbf{D}=\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{29}$$

where μ and λ are called **Lame constant**s.

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)}=G \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{30}$$

7.6 Total potential energy of a linear elastic body

The **strain energy density** for a Hooke material is introduced as

$$W=\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\tilde{\boldsymbol{\sigma}}=\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\mathbf{D}\tilde{\boldsymbol{\varepsilon}} \tag{31}$$

The internal energy of the body is written as

$$\Pi_{\text{int}} = \int_{\Omega_0} \mathrm{W}\,\mathrm{d}V = \int_{\Omega_0} \frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}V \tag{32}$$

The external energy of the body is written as

$$\Pi_{\text{ext}} = \int_{\partial\Omega_0} \mathbf{u}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}A + \int_{\Omega_0} \mathbf{u}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}V \tag{33}$$

where $\mathbf{t}$ is the **surface traction vector** and $\mathbf{f}$ is the **body force vector**.

So, the total potential energy is written as

$$\begin{aligned}\Pi(\mathbf{u}) &= \Pi_{\text{int}}(\mathbf{u}) - \Pi_{\text{ext}}(\mathbf{u}) \\ &= \int_{\Omega_0} \frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}V - \int_{\partial\Omega_0} \mathbf{u}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}A - \int_{\Omega_0} \mathbf{u}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}V\end{aligned} \tag{34}$$

Applying the principle of stationary potential energy, one has

$$\boxed{\delta\Pi(\mathbf{u})[\delta\mathbf{u}] = \int_{\Omega_0} \delta\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\varepsilon}}\,\mathrm{d}V - \int_{\partial\Omega_0} \delta\mathbf{u}^{\mathrm{T}}\mathbf{t}\,\mathrm{d}A - \int_{\Omega_0} \delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}V = 0} \tag{35}$$

Appendix

Method 1

$$\begin{aligned}(\mathbf{x}\times\boldsymbol{\sigma})\cdot\nabla &= \left(x_j\mathbf{e}_j \times \sigma_{kl}\mathbf{e}_k \otimes \mathbf{e}_l\right)\cdot\frac{\partial}{\partial x_m}\mathbf{e}_m \\ &= \left(\varepsilon_{ijk}x_j\sigma_{kl}\right)_{,l}\mathbf{e}_i \\ &= \varepsilon_{ijk}x_j\sigma_{kl,l}\mathbf{e}_i + \varepsilon_{ijk}x_{j,l}\sigma_{kl}\mathbf{e}_i \\ &= \varepsilon_{ijk}\delta_{jl}\sigma_{kl}\mathbf{e}_i \\ &= \varepsilon_{ijk}\sigma_{kj}\mathbf{e}_i \\ &= (\sigma_{32}-\sigma_{23})\mathbf{e}_1 + (\sigma_{13}-\sigma_{31})\mathbf{e}_2 + (\sigma_{21}-\sigma_{12})\mathbf{e}_3\end{aligned}$$

Method 2

$$
(\mathbf{x}\times\boldsymbol{\sigma})\cdot\nabla=\left(\begin{bmatrix}0 & -x_3 & x_2\\ x_3 & 0 & -x_1\\ -x_2 & x_1 & 0\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\right)\begin{bmatrix}\dfrac{\partial}{\partial x_1}\\ \dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_3}\end{bmatrix}
$$

$$
=\begin{bmatrix}-\sigma_{21}x_3+\sigma_{31}x_2 & -\sigma_{22}x_3+\sigma_{32}x_2 & -\sigma_{23}x_3+\sigma_{33}x_2\\ \sigma_{11}x_3-\sigma_{31}x_1 & \sigma_{12}x_3-\sigma_{32}x_1 & \sigma_{13}x_3-\sigma_{33}x_1\\ -\sigma_{11}x_2+\sigma_{21}x_1 & -\sigma_{12}x_2+\sigma_{22}x_1 & -\sigma_{13}x_2+\sigma_{23}x_1\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial x_1}\\ \dfrac{\partial}{\partial x_2}\\ \dfrac{\partial}{\partial x_3}\end{bmatrix}
$$

$$
=\begin{bmatrix}-\sigma_{21,1}x_3+\sigma_{31,1}x_2-\sigma_{22,2}x_3+\sigma_{32,2}x_2+\sigma_{32}-\sigma_{23,3}x_3-\sigma_{23}+\sigma_{33,3}x_2\\ \sigma_{11,1}x_3-\sigma_{31,1}x_1-\sigma_{31}+\sigma_{12,2}x_3-\sigma_{32,2}x_1+\sigma_{13,3}x_3+\sigma_{13}-\sigma_{33,3}x_1\\ -\sigma_{11,1}x_2+\sigma_{21,1}x_1+\sigma_{21}-\sigma_{12,2}x_2-\sigma_{12}+\sigma_{22,2}x_1-\sigma_{13,3}x_2+\sigma_{23,3}x_1\end{bmatrix}
$$

$$
=\begin{bmatrix}-\sigma_{21,1}x_3-\sigma_{22,2}x_3-\sigma_{23,3}x_3\\ -\sigma_{31,1}x_1-\sigma_{32,2}x_1-\sigma_{33,3}x_1\\ -\sigma_{11,1}x_2-\sigma_{12,2}x_2-\sigma_{13,3}x_2\end{bmatrix}+\begin{bmatrix}\sigma_{31,1}x_2+\sigma_{32,2}x_2+\sigma_{33,3}x_2\\ \sigma_{11,1}x_3+\sigma_{12,2}x_3+\sigma_{13,3}x_3\\ \sigma_{21,1}x_1+\sigma_{22,2}x_1+\sigma_{23,3}x_1\end{bmatrix}+\begin{bmatrix}\sigma_{32}-\sigma_{23}\\ \sigma_{13}-\sigma_{31}\\ \sigma_{21}-\sigma_{12}\end{bmatrix}
$$

Lecture 8 2D plane stress/strain

8.1 Basic notions

Definition-1 Plane stress

Plane stress is a stress state where the direct component and shear component normal to a plane are all zero. For example, for a plane expanded by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and it normal direction $\mathbf{e}_3$, the plane stress assumption means

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ 0 \\ \tau_{12} \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Definition-2 Plane strain

Plane strain is a strain state where the direct component and shear component normal to a plane are all zero. For example, for a plane expanded by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and it normal direction $\mathbf{e}_3$, the plane strain assumption means

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \tilde{\varepsilon}_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ 0 \\ \gamma_{12} \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

8.2 Hooke's law for plane stress

We introduce the plane stress assumption into the Hooke's law by the compliance matrix form.

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ 0 \\ \tau_{12} \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Therefore, we have

$$\begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tau_{12} \end{bmatrix}$$

$$\tilde{\varepsilon}_3 = -\frac{\nu}{E}\left(\tilde{\sigma}_1 + \tilde{\sigma}_2\right) \tag{4}$$

Moreover, the inverse of the compliance matrix in Eq. (4) is

$$\boxed{\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tau_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix}} \tag{5}$$

8.3 Hooke's law for plane strain

We introduce the plane strain assumption into the Hooke's law by the stiffness matrix form.

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ 0 \\ \gamma_{12} \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Therefore, we have

$$\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix}$$

$$\tilde{\sigma}_3 = \lambda\left(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2\right) \tag{7}$$

Note that

$$\begin{aligned} 2\mu+\lambda &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{8}$$

Introducing Eq. (8) into Eq. (7), one has

$$\boxed{\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tau_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix}} \tag{9}$$

8.4 Total potential energy and variational form

We denote the displacement vector, strain vector and stress vector at a material point in a 2D plane stress/strain linear elastic problem as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ \tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} \\ \dfrac{\partial u_2}{\partial X_2} \\ \dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1} \end{bmatrix},\ \tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tau_{12} \end{bmatrix} = \mathbf{D}\tilde{\boldsymbol{\varepsilon}} \tag{10}$$

For the **plane stress** case, we have

$$\mathbf{D}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix} \tag{11}$$

For the **plane strain** case, we have

$$\mathbf{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix} \tag{12}$$

The **strain energy density** is therefore introduced as

$$W=\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\tilde{\boldsymbol{\sigma}}=\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\mathbf{D}\tilde{\boldsymbol{\varepsilon}} \tag{13}$$

The internal energy of the body is written as

$$\Pi_{int}=\int_{\Omega_0} W dV = h\int_{\Omega_0}\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\mathbf{D}\tilde{\boldsymbol{\varepsilon}} dA \tag{14}$$

where h is the thickness, which is a constant.

The external energy of the body is written as

$$\Pi_{ext}=h\int_{\partial\Omega}\mathbf{u}^{T}\mathbf{t}dL + h\int_{\Omega}\mathbf{u}^{T}\mathbf{f}dA \tag{15}$$

where $\mathbf{t}$ is the **surface traction vector** and $\mathbf{f}$ is the **body force vector**.

$$\mathbf{t}=\begin{bmatrix}t_1\\ t_2\end{bmatrix},\ \mathbf{f}=\begin{bmatrix}f_1\\ f_2\end{bmatrix} \tag{16}$$

The total potential energy is written as

$$\begin{aligned}\Pi(\mathbf{u}) &= \Pi_{int}(\mathbf{u})-\Pi_{ext}(\mathbf{u})\\ &= h\int_{\Omega_0}\frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^{T}\mathbf{D}\tilde{\boldsymbol{\varepsilon}} dA - h\int_{\partial\Omega_0}\mathbf{u}^{T}\mathbf{t}dL - h\int_{\Omega_0}\mathbf{u}^{T}\mathbf{f}dA\end{aligned} \tag{17}$$

Applying the principle of stationary potential energy, one has

$$\boxed{\begin{aligned}&\delta\Pi(\mathbf{u})[\delta\mathbf{u}]=0\\&\Rightarrow\\&\int_{\Omega_0}\delta\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\varepsilon}}\mathrm{dA}-\int_{\partial\Omega_0}\delta\mathbf{u}^{\mathrm{T}}\mathbf{t}\mathrm{dL}-\int_{\Omega_0}\delta\mathbf{u}^{\mathrm{T}}\mathbf{f}\mathrm{dA}=0\end{aligned}} \tag{18}$$

This is just the variational form of equilibrium equations.

8.5 Constant Strain Triangular (CST) Element

8.5.1 Stiffness matrix and elemental equations for a CST element

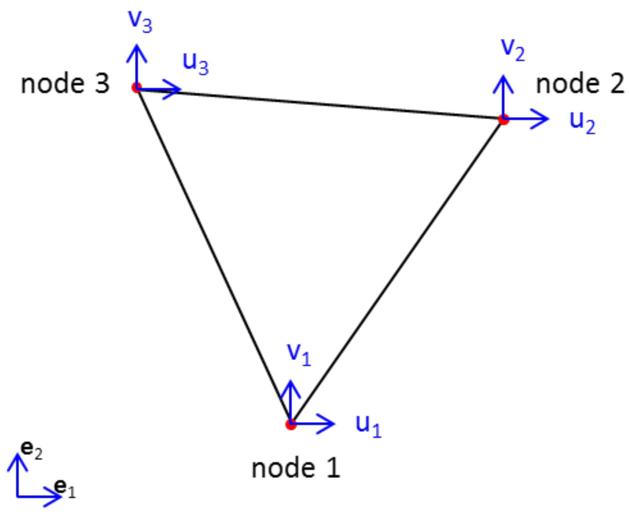


Fig. 1

Consider a triangular element with thickness denoted as h shown in **Fig. 1**. There are 3 nodes defined at the 3 vertices of the triangle. The coordinate of the 3 nodes are listed as

$$\mathbf{X}_1=\begin{bmatrix}x_1\\y_1\end{bmatrix},\ \mathbf{X}_2=\begin{bmatrix}x_2\\y_2\end{bmatrix},\ \mathbf{X}_3=\begin{bmatrix}x_3\\y_3\end{bmatrix} \tag{19}$$

At each node, a displacement vector with 2 degrees of freedom is defined

$$\mathbf{u}_1=\begin{bmatrix}u_1\\v_1\end{bmatrix},\ \mathbf{u}_2=\begin{bmatrix}u_2\\v_2\end{bmatrix},\ \mathbf{u}_3=\begin{bmatrix}u_3\\v_3\end{bmatrix} \tag{20}$$

In addition, as shown in **Fig. 2**, corresponding nodal forces are defined

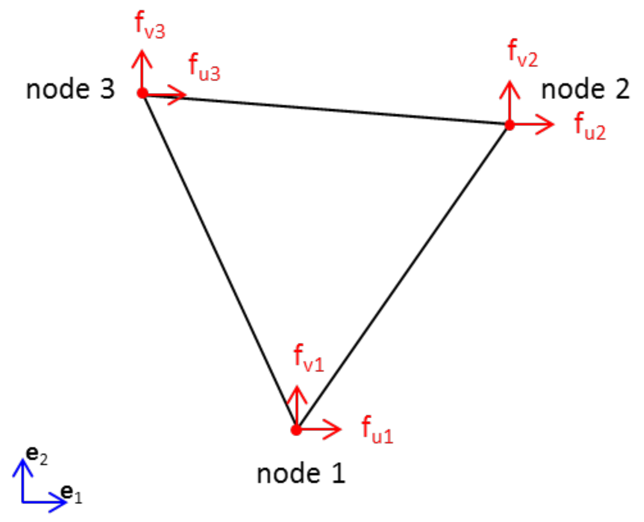


Fig. 2

Step 1: Write down the total potential energy

$$
\begin{aligned}
\Pi(\mathbf{u}) &= \Pi_{\text{int}}(\mathbf{u}) - \Pi_{\text{ext}}(\mathbf{u}) \\
&= h\int_{\Omega_0} \frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^T \mathbf{D}\tilde{\boldsymbol{\varepsilon}} dA - hf_{u1}u_1 - hf_{v1}v_1 - hf_{u2}u_2 - hf_{v2}v_2 - hf_{u3}u_3 - hf_{v3}v_3 \\
&= h\int_{\Omega_0} \frac{1}{2}\tilde{\boldsymbol{\varepsilon}}^T \mathbf{D}\tilde{\boldsymbol{\varepsilon}} dA - h\mathbf{u}_e^T \mathbf{f}_e
\end{aligned}
\tag{21}
$$

where the **nodal displacement vector** is

$$
\mathbf{u}_e = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \tag{22}
$$

and the nodal force vector is

$$
\mathbf{f}_e = \begin{bmatrix} f_{u1} \\ f_{v1} \\ f_{u2} \\ f_{v2} \\ f_{u3} \\ f_{v3} \end{bmatrix} \tag{23}
$$

Step 2: Assume the trial function for each component of the displacement

field

$$\begin{aligned} u(x,y) &= c_1 + c_2 x + c_3 y \\ v(x,y) &= c_4 + c_5 x + c_6 y \end{aligned} \tag{24}$$

where $c_0 \sim c_6$ are unknown coefficients to be determined.

Step 3: Express the unknown coefficients in terms of the nodal degrees of freedom (or Find the shape functions)

$$\begin{aligned} u(x_1,y_1) &= c_1 + c_2 x_1 + c_3 y_1 = u_1 \\ u(x_2,y_2) &= c_1 + c_2 x_2 + c_3 y_2 = u_2 \\ u(x_3,y_3) &= c_1 + c_2 x_3 + c_3 y_3 = u_3 \\ v(x_1,y_1) &= c_4 + c_5 x_1 + c_6 y_1 = v_1 \\ v(x_2,y_2) &= c_4 + c_5 x_2 + c_6 y_2 = v_2 \\ v(x_3,y_3) &= c_4 + c_5 x_3 + c_6 y_3 = v_3 \end{aligned} \tag{25}$$

By rewriting the above equations in matrix form, one has

$$\begin{gathered} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{gathered} \tag{26}$$

Furthermore, by applying Cramer’s rule to solve the two systems of linear equations respectively, one has

$$
\begin{aligned}
c_1 &= \frac{\begin{vmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+1}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} u_1 + (-1)^{1+2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} u_2 + (-1)^{1+3}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} u_3\right] \\
&= \frac{1}{\Delta}(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \\
c_2 &= \frac{\begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+2}\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} u_1 + (-1)^{2+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} u_2 + (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} u_3\right] \\
&= \frac{1}{\Delta}(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3) \\
c_3 &= \frac{\begin{vmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+3}\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} u_1 + (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} u_2 + (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} u_3\right] \\
&= \frac{1}{\Delta}(\chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3) \\
c_4 &= \frac{\begin{vmatrix} v_1 & x_1 & y_1 \\ v_2 & x_2 & y_2 \\ v_3 & x_3 & y_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+1}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} v_1 + (-1)^{1+2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} v_2 + (-1)^{1+3}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} v_3\right] \\
&= \frac{1}{\Delta}(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) \\
c_5 &= \frac{\begin{vmatrix} 1 & v_1 & y_1 \\ 1 & v_2 & y_2 \\ 1 & v_3 & y_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+2}\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} v_1 + (-1)^{2+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} v_2 + (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} v_3\right] \\
&= \frac{1}{\Delta}(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3) \\
c_6 &= \frac{\begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix}}{\Delta} = \frac{1}{\Delta}\left[(-1)^{1+3}\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} v_1 + (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} v_2 + (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} v_3\right] \\
&= \frac{1}{\Delta}(\chi_1 v_1 + \chi_2 v_2 + \chi_3 v_3)
\end{aligned}
\tag{27}
$$

where

$$
\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{28}
$$

and

$$\boxed{\begin{aligned}&\alpha_1 = x_2 y_3 - x_3 y_2,\ \alpha_2 = x_3 y_1 - x_1 y_3,\ \alpha_3 = x_1 y_3 - x_3 y_1\\&\beta_1 = y_2 - y_3,\ \beta_2 = y_3 - y_1,\ \beta_3 = y_1 - y_2\\&\chi_1 = x_3 - x_2,\ \chi_2 = x_1 - x_3,\ \chi_3 = x_2 - x_1\end{aligned}} \tag{29}$$

Introducing Eq. (27) into Eq. (24), one has

$$\begin{aligned}
u =& \frac{1}{\Delta}\left[(-1)^{1+1}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} u_1 + (-1)^{1+2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} u_2 + (-1)^{1+3}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} u_3\right]\\
&+\frac{x}{\Delta}\left[(-1)^{1+2}\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} u_1 + (-1)^{2+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} u_2 + (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} u_3\right]\\
&+\frac{y}{\Delta}\left[(-1)^{1+3}\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} u_1 + (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} u_2 + (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} u_3\right]\\
=& \frac{u_1}{\Delta}\left[(-1)^{1+1}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + (-1)^{1+2}\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} x + (-1)^{1+3}\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} y\right]\\
&+\frac{u_2}{\Delta}\left[(-1)^{1+2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + (-1)^{2+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} x + (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} y\right]\\
&+\frac{u_3}{\Delta}\left[(-1)^{1+3}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} x + (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} y\right]
\end{aligned} \tag{30}$$

Moreover, we notice that

$$\begin{aligned}
&(-1)^{1+1}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + (-1)^{1+2}\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} x + (-1)^{1+3}\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} y = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \Delta_1\\
&(-1)^{1+2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + (-1)^{2+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} x + (-1)^{2+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} y = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix} = \Delta_2\\
&(-1)^{1+3}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} x + (-1)^{3+3}\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} y = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} = \Delta_3
\end{aligned} \tag{31}$$

So, we have

$$u = \frac{\Delta_1}{\Delta} u_1 + \frac{\Delta_2}{\Delta} u_2 + \frac{\Delta_3}{\Delta} u_3 \tag{32}$$

Similarly, we have

$$v = \frac{\Delta_1}{\Delta} v_1 + \frac{\Delta_2}{\Delta} v_2 + \frac{\Delta_3}{\Delta} v_3 \tag{33}$$

In addition, we notice that these determinants can be interpreted by the area of the triangle.

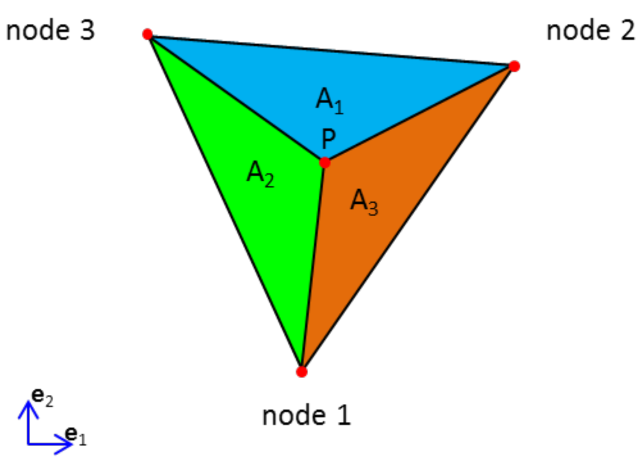


Fig. 3

As shown in **Fig. 3**, a point P(x,y) inside the triangle partitions the triangle into 3 three smaller triangles by connecting the point to the 3 vertices (nodes).

Triangle Area Theorem (TAT)

If the three vertices of a triangle is numbered in a counterclockwise manner and the coordinates of the three vertices are given as (x_i, y_i), $i = 1, 2, 3$, then the area of the triangle is determined as

$$A_{T_{123}} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_3 & y_3 \end{vmatrix} \tag{34}$$

Proof:

One can define two vectors as (See **Fig. 4**)

$$\begin{aligned} \mathbf{v}_1 &= (x_2 - x_1)\mathbf{e}_1 + (y_2 - y_1)\mathbf{e}_2 \\ \mathbf{v}_2 &= (x_3 - x_1)\mathbf{e}_1 + (y_3 - y_1)\mathbf{e}_2 \end{aligned} \tag{35}$$

Then the area of the parallelogram is

$$
\begin{aligned}
A_P &= \mathbf{v}_1 \times \mathbf{v}_2 \\
&= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x_3 - x_1 & y_3 - y_1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_3 & y_3 \end{vmatrix}
\end{aligned}
\tag{36}
$$

So, the area of the triangle is half of the area of the parallelogram

$$
A_{T_{123}} = \frac{1}{2} A_P = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_3 & y_3 \end{vmatrix}
\tag{37}
$$

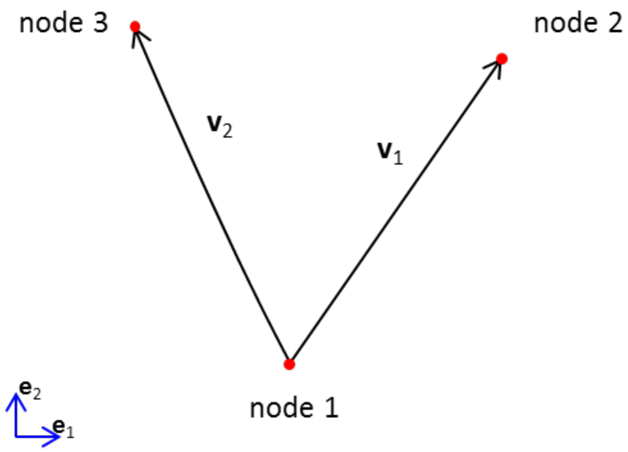


Fig. 4

By using the TAT, we can evaluate the areas of triangles shown in **Fig. 3**.

$$
\begin{aligned}
A_1 = A_{T_{P23}} &= \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\Delta_1 \\
A_2 = A_{T_{1P3}} &= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\Delta_2 \\
A_3 = A_{T_{12P}} &= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} = \frac{1}{2}\Delta_3 \\
A = A_{T_{123}} &= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & y_3 & y_3 \end{vmatrix} = \frac{1}{2}\Delta
\end{aligned}
\tag{38}
$$

Introducing Eq. (38) into Eq. (32) and Eq. (33), one has

$$
\begin{aligned}
u &= \frac{A_1}{A}u_1 + \frac{A_2}{A}u_2 + \frac{A_3}{A}u_3 = N_1u_1 + N_2u_2 + N_3u_3 \\
v &= \frac{A_1}{A}v_1 + \frac{A_2}{A}v_2 + \frac{A_3}{A}v_3 = N_1v_1 + N_2v_2 + N_3v_3
\end{aligned}
\tag{39}
$$

where $\frac{A_1}{A}, \frac{A_2}{A}, \frac{A_3}{A}$ are known as **area coordinates**.

Eq. (39) could be rewritten in matrix form as

$$
\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \mathbf{N}\mathbf{u}_e
\tag{40}
$$

Remarks:

(1) From Eq. (39), we know that the shape functions could be directly derived by the so-called **area coordinates** of a point inside the triangle.

Note that

$$
N_1 + N_2 + N_3 = \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1
\tag{41}
$$

So, there are only two independent area coordinates. In addition, we notice that

$$\begin{aligned} 0 \le N_1 \le 1 \\ 0 \le N_2 \le 1 \\ 0 \le N_3 \le 1 \end{aligned} \tag{42}$$

and

$$N_i\left(\mathbf{X}_j\right) = \delta_{ij} \tag{43}$$

Step 4: Evaluate the strain components in terms of the trial function

Method-1:

We start with the Eq. (40)

$$\begin{aligned} \tilde{\varepsilon}_1 &= \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\ \tilde{\varepsilon}_2 &= \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\ \gamma_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ &= \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \end{aligned} \tag{44}$$

Eq. (44) could be rewritten in matrix form as

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \mathbf{B}\mathbf{u}_e \tag{45}$$

Method-2:

We start with Eq. (24) and Eq. (27)

$$
\begin{aligned}
\tilde{\varepsilon}_1 &= \frac{\partial u}{\partial x} = c_2 = \frac{1}{\Delta}\left(\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3\right) \\
\tilde{\varepsilon}_2 &= \frac{\partial v}{\partial y} = c_6 = \frac{1}{\Delta}\left(\chi_1 v_1 + \chi_2 v_2 + \chi_3 v_3\right) \\
\gamma_{12} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = c_3 + c_5 \\
&= \frac{1}{\Delta}\left(\chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3\right) + \frac{1}{\Delta}\left(\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3\right)
\end{aligned}
\tag{46}
$$

Eq. (46) could be rewritten in matrix form as

$$
\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \chi_1 & 0 & \chi_2 & 0 & \chi_3 \\ \chi_1 & \beta_1 & \chi_2 & \beta_2 & \chi_3 & \beta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \mathbf{B}\mathbf{u}_e \tag{47}
$$

Remarks:

(1) From Eq. (46), it is apparent that all the stain components are constant. That is why the element is called *constant strain triangular element*.

(2) By comparing Eq. (45) and Eq. (47), we have the following relationships

$$
\begin{aligned}
\frac{\partial N_1}{\partial x} &= \frac{\beta_1}{\Delta}, \quad \frac{\partial N_1}{\partial y} = \frac{\chi_1}{\Delta} \\
\frac{\partial N_2}{\partial x} &= \frac{\beta_2}{\Delta}, \quad \frac{\partial N_2}{\partial y} = \frac{\chi_2}{\Delta} \\
\frac{\partial N_3}{\partial x} &= \frac{\beta_3}{\Delta}, \quad \frac{\partial N_3}{\partial y} = \frac{\chi_3}{\Delta}
\end{aligned}
\tag{48}
$$

Step 5: Express the total potential energy in terms of nodal displacement vector

By introducing Eq. (45) into Eq. (21), one arrives at

$$
\Pi\left(\mathbf{u}_e\right) = h\mathbf{u}_e^T \int_{\Omega_0} \frac{1}{2}\mathbf{B}^T\mathbf{D}\mathbf{B}\,dA\,\mathbf{u}_e - h\mathbf{u}_e^T\mathbf{f}_e \tag{49}
$$

Step 6: Derive the equilibrium equations and stiffness matrix

By the principle of stationary potential energy, one has

$$\begin{aligned}&\delta\Pi(\mathbf{u}_e)[\delta\mathbf{u}_e]=0\\&\Rightarrow\\&\boxed{\int_{\Omega_0}\mathbf{B}^T\mathbf{D}\mathbf{B}dA\mathbf{u}_e=\mathbf{f}_e}\end{aligned} \tag{50}$$

Therefore, the stiffness matrix could be evaluated by

$$\boxed{\begin{aligned}\mathbf{K}_{CST}&=\int_{\Omega_0}\mathbf{B}^T\mathbf{D}\mathbf{B}dA\\&=\mathbf{B}^T\mathbf{D}\mathbf{B}\int_{\Omega_0}dA\\&=\mathbf{B}^T\mathbf{D}\mathbf{B}A\\&=\frac{1}{2}\mathbf{B}^T\mathbf{D}\mathbf{B}\Delta\end{aligned}} \tag{51}$$

which is a 6×6 matrix.

Remarks:

(1) For plane stress problems, we have the stiffness matrix as

$$\mathbf{K}_{CST}=\frac{E}{2\Delta(1-\nu^2)}\begin{bmatrix}\beta_1&0&\chi_1\\0&\chi_1&\beta_1\\\beta_2&0&\chi_2\\0&\chi_2&\beta_2\\\beta_3&0&\chi_3\\0&\chi_3&\beta_3\end{bmatrix}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&\frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\beta_1&0&\beta_2&0&\beta_3&0\\0&\chi_1&0&\chi_2&0&\chi_3\\\chi_1&\beta_1&\chi_2&\beta_2&\chi_3&\beta_3\end{bmatrix} \tag{52}$$

For plane strain problems, we have the stiffness matrix as

$$\mathbf{K}_{CST}=\frac{E}{2\Delta(1+\nu)(1-2\nu)}\begin{bmatrix}\beta_1&0&\chi_1\\0&\chi_1&\beta_1\\\beta_2&0&\chi_2\\0&\chi_2&\beta_2\\\beta_3&0&\chi_3\\0&\chi_3&\beta_3\end{bmatrix}\begin{bmatrix}1-\nu&\nu&0\\\nu&1-\nu&0\\0&0&\frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\beta_1&0&\beta_2&0&\beta_3&0\\0&\chi_1&0&\chi_2&0&\chi_3\\\chi_1&\beta_1&\chi_2&\beta_2&\chi_3&\beta_3\end{bmatrix} \tag{53}$$

8.5.2 Equivalent nodal forces for distributed surface traction

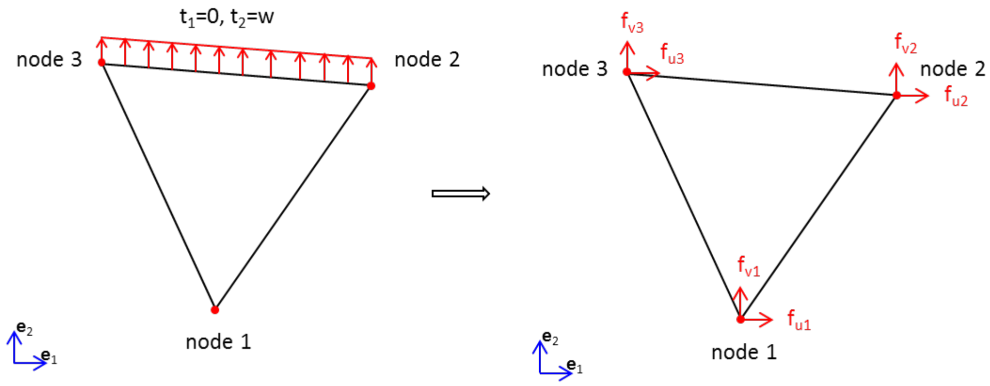


Fig. 5

See the example shown in Fig. 5. The area coordinate of the point on the edge that is subjected to surface traction is

$$\begin{aligned} N_1 &= 0 \\ N_2 &= \xi \\ N_3 &= 1 - N_1 - N_2 = 1 - \xi \end{aligned} \tag{54}$$

where $\xi \in [0,1]$ is a parameter. Introducing Eq. (54) into Eq. (40), one has

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi & 0 & 1-\xi & 0 \\ 0 & 0 & 0 & \xi & 0 & 1-\xi \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \mathbf{N}\mathbf{u}_e \tag{55}$$

The surface traction is

$$\mathbf{t} = \begin{bmatrix} 0 \\ w \end{bmatrix} \tag{56}$$

Therefore, by using the following work equivalent identity, one has

$$\begin{aligned} &\int_0^1 \mathbf{u}^T \mathbf{t} d\xi = \mathbf{u}_e^T \mathbf{f}_e \\ &\Rightarrow \\ &\int_0^1 \mathbf{N}^T \mathbf{t} d\xi = \mathbf{f}_e \end{aligned} \tag{57}$$

8.5.3 Example

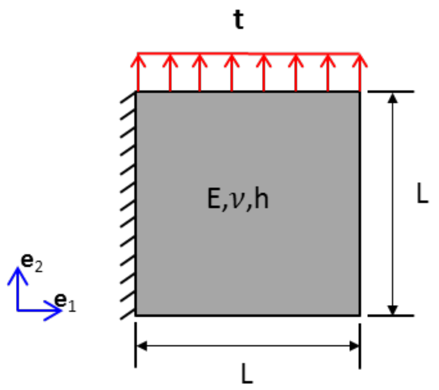


Fig. 6

Consider a 2D plane stress problem as shown in Fig. 6. A 2D square elastic domain with edge length L is subject to a constant traction t on its top surface and a fixed B.C. on its left edge. The thickness is 1. We would like to use finite element method to find the displacement field. For simplicity and illustration, we apply the CST element first.

Step 1 Discretize domain; Numbering nodes and elements

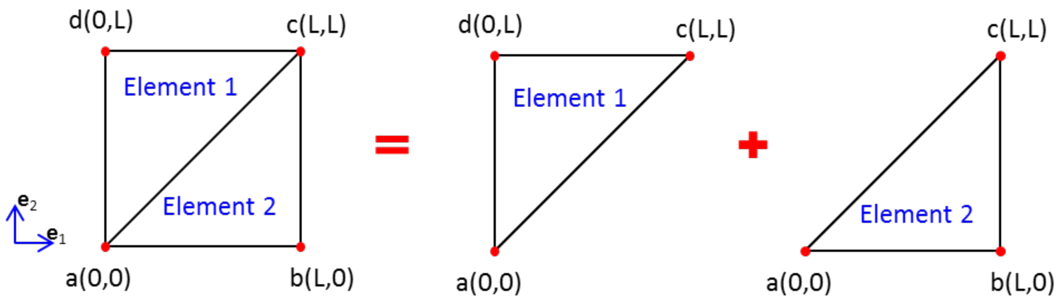
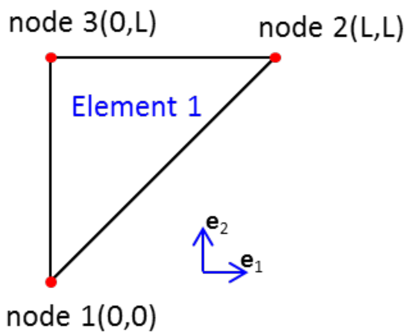


Fig. 7

Step 2 Evaluate element stiffness matrix and equations

For Element 1:



The nodal coordinates are

$$\begin{aligned} & x_1 = 0,\ y_1 = 0 \\ & x_2 = L,\ y_2 = L \\ & x_3 = 0,\ y_3 = L \end{aligned} \tag{58}$$

The area of the triangle is

$$\begin{aligned} & A = \frac{1}{2}L^2 \\ & \Rightarrow \\ & \Delta = L^2 \end{aligned} \tag{59}$$

The B-matrix parameters are evaluated as

$$\begin{aligned} & \beta_1 = 0,\ \beta_2 = L,\ \beta_3 = -L \\ & \chi_1 = -L,\ \chi_2 = 0,\ \chi_3 = L \end{aligned} \tag{60}$$

The stiffness matrix is evaluated as

$$\mathbf{K}^{(1)} =$$

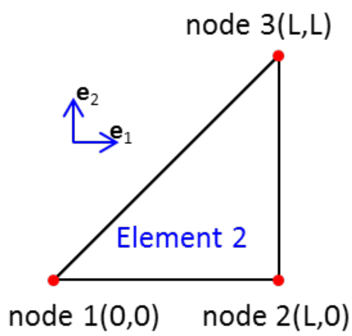
$$\frac{E}{2L^2\left(1-\nu^2\right)}\begin{bmatrix} 0 & 0 & -L \\ 0 & -L & 0 \\ L & 0 & 0 \\ 0 & 0 & L \\ -L & 0 & L \\ 0 & L & -L \end{bmatrix}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & L & 0 & -L & 0 \\ 0 & -L & 0 & 0 & 0 & L \\ -L & 0 & 0 & L & L & -L \end{bmatrix}$$

$$= \frac{E}{2L^2\left(1-\nu^2\right)}\begin{bmatrix} \frac{L^2}{2}(1-\nu) & 0 & 0 & -\frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) & \frac{L^2}{2}(1-\nu) \\ 0 & L^2 & -L^2\nu & 0 & L^2\nu & -L^2 \\ & & L^2 & 0 & -L^2 & L^2\nu \\ & & & \frac{L^2}{2}(1-\nu) & \frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) \\ & & & & \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) \\ \text{Sym} & & & & & \frac{L^2}{2}(3-\nu) \end{bmatrix} \quad (61)$$

In addition, the equivalent nodal force is evaluated as

$$\mathbf{f}_e = \int_0^1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \xi & 0 \\ 0 & \xi \\ 1-\xi & 0 \\ 0 & 1-\xi \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} d\xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}w \\ 0 \\ \frac{1}{2}w \end{bmatrix} \quad (62)$$

For Element 2:



The nodal coordinates are

$$\begin{aligned} x_1 &= 0,\ y_1 = 0 \\ x_2 &= L,\ y_2 = 0 \\ x_3 &= L,\ y_3 = L \end{aligned} \tag{63}$$

The area of the triangle is

$$\begin{aligned} &A = \frac{1}{2}L^2 \\ &\Rightarrow \\ &\Delta = L^2 \end{aligned} \tag{64}$$

The B-matrix parameters are evaluated as

$$\begin{aligned} &\beta_1 = -L,\ \beta_2 = L,\ \beta_3 = 0 \\ &\chi_1 = 0,\ \chi_2 = -L,\ \chi_3 = L \end{aligned} \tag{65}$$

The stiffness matrix is evaluated as

$$\mathbf{K}^{(2)} =$$

$$\frac{E}{2L^2\left(1-\nu^2\right)} \begin{bmatrix} -L & 0 & 0 \\ 0 & 0 & -L \\ L & 0 & -L \\ 0 & -L & L \\ 0 & 0 & L \\ 0 & L & 0 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -L & 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & -L & 0 & L \\ 0 & -L & -L & L & L & 0 \end{bmatrix}$$

$$= \frac{E}{2L^2\left(1-\nu^2\right)} \begin{bmatrix} L^2 & 0 & -L^2 & L^2\nu & 0 & -L^2\nu \\ & \frac{L^2}{2}(1-\nu) & \frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) & 0 \\ & & \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) & -\frac{L^2}{2}(1-\nu) & L^2\nu \\ & & & \frac{L^2}{2}(3-\nu) & \frac{L^2}{2}(1-\nu) & -L^2 \\ & & & & \frac{L^2}{2}(1-\nu) & 0 \\ \text{Sym} & & & & & L^2 \end{bmatrix} \tag{66}$$

Step 3 Assemble all the elemental stiffness matrices and force vectors

By direct stiffness method, the global stiffness matrix is assembled as

$$\mathbf{K}=\frac{E}{2L^2\left(1-\nu^2\right)}\begin{bmatrix} L^2+\frac{L^2}{2}(1-\nu) & 0 & -L^2 & L^2\nu & 0 & -L^2\nu-\frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) & \frac{L^2}{2}(1-\nu) \\ & \frac{L^2}{2}(1-\nu)+L^2 & \frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu)-L^2\nu & 0 & L^2\nu & -L^2 \\ & & \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) & -\frac{L^2}{2}(1-\nu) & L^2\nu & 0 & 0 \\ & & & \frac{L^2}{2}(3-\nu) & \frac{L^2}{2}(1-\nu) & -L^2 & 0 & 0 \\ & & & & \frac{L^2}{2}(1-\nu)+L^2 & 0 & -L^2 & L^2\nu \\ & & & & & L^2+\frac{L^2}{2}(1-\nu) & \frac{L^2}{2}(1-\nu) & -\frac{L^2}{2}(1-\nu) \\ & & & & & & \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) \\ \text{Sym} & & & & & & & \frac{L^2}{2}(3-\nu) \end{bmatrix} \tag{67}$$

The nodal force is assembled as

$$\mathbf{f}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{w}{2} \\ 0 \\ \frac{w}{2} \end{bmatrix} \tag{68}$$

Step 4 Apply boundary conditions

Since the left edge is fixed, the boundary conditions are

$$u_1=v_1=u_4=v_4=0 \tag{69}$$

After deleting corresponding rows and columns in the global stiffness, one obtains

$$\bar{\mathbf{K}}=\frac{E}{2L^2\left(1-\nu^2\right)}\begin{bmatrix} \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) & -\frac{L^2}{2}(1-\nu) & L^2\nu \\ & \frac{L^2}{2}(3-\nu) & \frac{L^2}{2}(1-\nu) & -L^2 \\ & & \frac{L^2}{2}(1-\nu)+L^2 & 0 \\ \text{Sym} & & & L^2+\frac{L^2}{2}(1-\nu) \end{bmatrix} \tag{70}$$

So

$$\frac{E}{2L^2\left(1-\nu^2\right)}\begin{bmatrix} \frac{L^2}{2}(3-\nu) & -\frac{L^2}{2}(1+\nu) & -\frac{L^2}{2}(1-\nu) & L^2\nu \\ & \frac{L^2}{2}(3-\nu) & \frac{L^2}{2}(1-\nu) & -L^2 \\ & & \frac{L^2}{2}(1-\nu)+L^2 & 0 \\ \text{Sym} & & & L^2+\frac{L^2}{2}(1-\nu) \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{w}{2} \end{bmatrix} \quad (71)$$

The solution is

$$\begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}=\frac{w\left(1-\nu^2\right)}{E\left(3\nu^4+8\nu^3-146\nu^2+368\nu-265\right)}\begin{bmatrix} 3\nu^3-18\nu^2+29\nu-5 \\ -3\nu^3+3\nu^2+27\nu-35 \\ -3\nu^3+5\nu^2+3\nu-5 \\ 24\nu^2-100\nu-100 \end{bmatrix} \quad (72)$$

Lecture 9 2D plane linear elements

In the previous lecture, we have learnt how to derive the stiffness matrix of a constant strain triangular element and use direct stiffness method to solve 2D plane stress/strain problems. In this lecture, we will talk about the standardization of the method in terms of 2D plane elements, i.e., the standard 3 node triangular element and the standard 4 node quadrilateral element.

9.1 Natural coordinates and Standard computation domain

9.1.1 Example

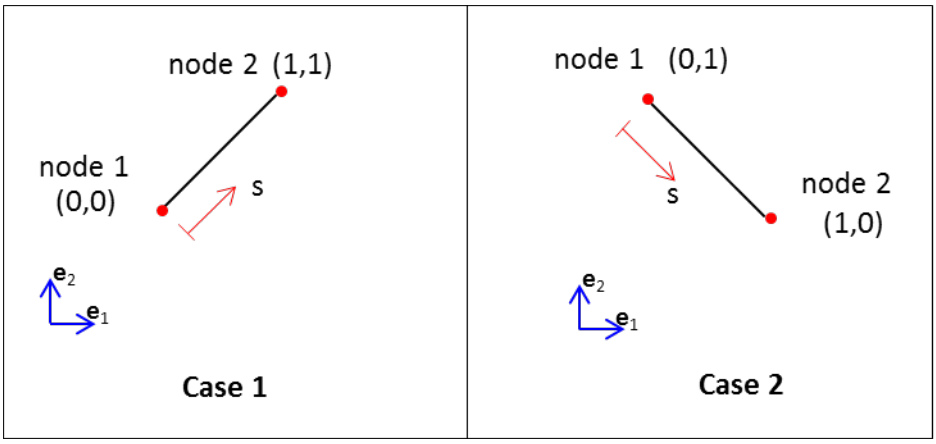


Fig. 1

Let's consider such a field problem defined in the 2D space. The field domain is a line segment. The problem is to find the following integral over the field domain

$$I = \int_0^L x + y^2 ds \tag{1}$$

where s is the arc-length of the line segment, or called the **natural coordinate**, (x, y) is the coordinate of a point in the line segment.

For Case 1 as shown in **Fig. 1**, we first do parameterization for the coordinates

$$\begin{cases} x = \hat{x}(s) = \dfrac{\sqrt{2}}{2}s \\ y = \hat{y}(s) = \dfrac{\sqrt{2}}{2}s \end{cases} \tag{2}$$

The length of the line segment is $L = \sqrt{2}$. Therefore, we have

$$I = \int_0^{\sqrt{2}} \frac{\sqrt{2}}{2}s + \frac{1}{2}s^2 ds = \frac{\sqrt{2}}{4}s^2\Big|_0^{\sqrt{2}} + \frac{1}{6}s^3\Big|_0^{\sqrt{2}} = \frac{5\sqrt{2}}{6} \tag{3}$$

For Case 2 as shown in **Fig. 1**, still, we first do parameterization for the coordinates

$$\begin{cases} x = \hat{x}(s) = \dfrac{\sqrt{2}}{2}s \\ y = \hat{y}(s) = 1 - \dfrac{\sqrt{2}}{2}s \end{cases} \tag{4}$$

The length of the line segment is $L = \sqrt{2}$. Therefore, we have

$$\begin{aligned} I &= \int_0^{\sqrt{2}} \frac{\sqrt{2}}{2}s + \left(1 - \frac{\sqrt{2}}{2}s\right)^2 ds = \int_0^{\sqrt{2}} -\frac{\sqrt{2}}{2}s + 1 + \frac{1}{2}s^2 ds \\ &= -\frac{\sqrt{2}}{4}s^2\Big|_0^{\sqrt{2}} + \sqrt{2} + \frac{1}{6}s^3\Big|_0^{\sqrt{2}} = \frac{5\sqrt{2}}{6} \end{aligned} \tag{5}$$

9.1.2 Standard computation domain

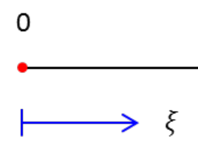


Fig. 2

Now, we consider mapping the line segment shown in **Fig. 1** (Case 1) to the **standard computation domain** shown in **Fig. 2**, where ξ is a natural coordinate.

Step 1 Assume a mapping function

$$\begin{cases} x = c_1 + c_2\xi \\ y = c_3 + c_4\xi \end{cases} \tag{6}$$

Step 2 Apply known nodal coordinate and solve for the unknown coefficients

$$\begin{cases} x(0) = c_1 = 0 \\ x(1) = c_1 + c_2 = 1 \\ y(0) = c_3 = 0 \\ y(1) = c_4 = 1 \end{cases} \tag{7}$$

Step 3 Obtain the parameterized coordinates

$$\begin{cases} x = \xi \\ y = \xi \end{cases} \tag{8}$$

Step 4 Evaluate the differentials

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{2}d\xi \tag{9}$$

Step 5 Evaluate the integral

$$I = \int_0^1 \left(\xi + \xi^2\right)\sqrt{2}d\xi = \frac{5\sqrt{2}}{6} \tag{10}$$

Similarly, the line segment shown in **Fig. 1** (Case 2) can also be mapped to the standard computation domain to evaluate the integral.

9.2 Standard triangular linear element (C2D3 element)

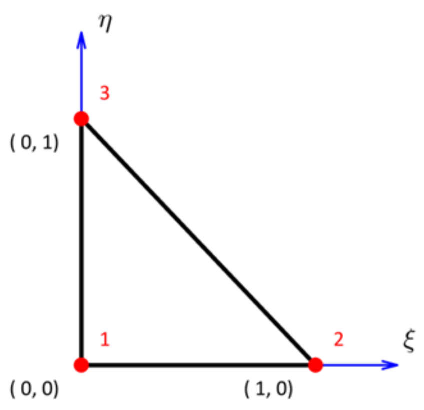


Fig. 3 C2D3 element

The shape of the element is shown in Fig. 3. About the name C2D3, "C" is short for "Continuum", 2D is short for "2D space", 3 stands for "3 nodes".

Convention for the edge numbering:

edge-1: node1-node2

edge-2: node2-node3

edge-3: node3-node1

Step 1: Derive the shape functions of C2D3:

$$\begin{aligned}\hat{\varphi}_1(\xi,\eta)&=1-\xi-\eta\\ \hat{\varphi}_2(\xi,\eta)&=\xi\\ \hat{\varphi}_3(\xi,\eta)&=\eta\end{aligned} \tag{11}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N}=\begin{bmatrix}\hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0\\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3\end{bmatrix} \tag{12}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u}=\begin{bmatrix}u\\ v\end{bmatrix}=\mathbf{N}\mathbf{u}_e=\mathbf{N}\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\end{bmatrix} \tag{13}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X}=\begin{bmatrix}x\\ y\end{bmatrix}=\mathbf{N}\mathbf{X}_e=\mathbf{N}\begin{bmatrix}x_1\\ y_1\\ x_2\\ y_2\\ x_3\\ y_3\end{bmatrix} \tag{14}$$

Remarks:

Note that both the field variables (u, v) and the field coordinates (x, y) are parameterized by the same natural coordinates (ξ, η). This kind of interpolation is also known as **iso-parametric** interpolation. The advantage is that the computation tasks (e.g., integration) are all changed to the computation domain. This facilitates standardization.

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{15}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H} = \begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} \\ \frac{\partial\hat{\varphi}_3}{\partial\xi} & \frac{\partial\hat{\varphi}_3}{\partial\eta} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

Step 5: Create the **J** matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} \end{bmatrix} = \mathbf{CH} \tag{17}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$dxdy = |\mathbf{J}| d\xi d\eta \tag{18}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\boldsymbol{\Gamma}=\begin{bmatrix}\dfrac{\partial\hat{\varphi}_1}{\partial x} & \dfrac{\partial\hat{\varphi}_1}{\partial y}\\ \dfrac{\partial\hat{\varphi}_2}{\partial x} & \dfrac{\partial\hat{\varphi}_2}{\partial y}\\ \dfrac{\partial\hat{\varphi}_3}{\partial x} & \dfrac{\partial\hat{\varphi}_3}{\partial y}\end{bmatrix}=\mathbf{H}\mathbf{J}^{-1} \tag{19}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & \Gamma_{21} & 0 & \Gamma_{31} & 0\\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & 0 & \Gamma_{32}\\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \Gamma_{32} & \Gamma_{31}\end{bmatrix} \tag{20}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \gamma_{12}\end{bmatrix}=\mathbf{B}\mathbf{u}_e \tag{21}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

For the **plane stress** case, we have

$$\mathbf{D}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix} \tag{22}$$

For the **plane strain** case, we have

$$\mathbf{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix} \tag{23}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K}=h\int_0^1\int_0^{1-\xi}\mathbf{B}^{T}\mathbf{D}\mathbf{B}\left|\mathbf{J}\right|d\xi d\eta} \tag{24}$$

where h denotes the thickness.

9.3 Standard quadrilateral linear element (C2D4 element)

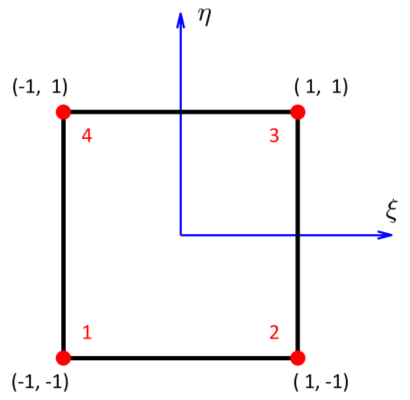


Fig. 4 C2D4 element

The shape of the element is shown in **Fig. 4**. About the name C2D4, “C” is short for “Continuum”, 2D is short for “2D space”, 4 stands for “4 nodes”.

Convention for the edge numbering:

Edge-1: node1-node2

Edge-2: node2-node3

Edge-3: node3-node4

Edge-4: node4-node1

Step 1: Derive the shape functions of C2D3:

$$\begin{aligned}\hat{\varphi}_1(\xi,\eta)&=(1-\xi)(1-\eta)/4\\ \hat{\varphi}_2(\xi,\eta)&=(1+\xi)(1-\eta)/4\\ \hat{\varphi}_3(\xi,\eta)&=(1+\xi)(1+\eta)/4\\ \hat{\varphi}_4(\xi,\eta)&=(1-\xi)(1+\eta)/4\end{aligned} \tag{25}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N}=\begin{bmatrix}\hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0 & \hat{\varphi}_4 & 0\\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0 & \hat{\varphi}_4\end{bmatrix} \tag{26}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \tag{27}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{N}\mathbf{X}_e = \mathbf{N}\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \tag{28}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \tag{29}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H} = \begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} \\ \frac{\partial\hat{\varphi}_3}{\partial\xi} & \frac{\partial\hat{\varphi}_3}{\partial\eta} \\ \frac{\partial\hat{\varphi}_4}{\partial\xi} & \frac{\partial\hat{\varphi}_4}{\partial\eta} \end{bmatrix} = \begin{bmatrix} -(1-\eta)/4 & -(1-\xi)/4 \\ (1-\eta)/4 & -(1+\xi)/4 \\ (1+\eta)/4 & (1+\xi)/4 \\ -(1+\eta)/4 & (1-\xi)/4 \end{bmatrix} \tag{30}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta}\\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta}\end{bmatrix}=\mathbf{CH} \tag{31}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$dxdy=|\mathbf{J}|d\xi d\eta \tag{32}$$

Step 6: Create the $\boldsymbol{\Gamma}$ matrix

$$\boldsymbol{\Gamma}=\begin{bmatrix}\frac{\partial\hat{\varphi}_1}{\partial x} & \frac{\partial\hat{\varphi}_1}{\partial y}\\ \frac{\partial\hat{\varphi}_2}{\partial x} & \frac{\partial\hat{\varphi}_2}{\partial y}\\ \frac{\partial\hat{\varphi}_3}{\partial x} & \frac{\partial\hat{\varphi}_3}{\partial y}\\ \frac{\partial\hat{\varphi}_4}{\partial x} & \frac{\partial\hat{\varphi}_4}{\partial y}\end{bmatrix}=\mathbf{HJ}^{-1} \tag{33}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & \Gamma_{21} & 0 & \Gamma_{31} & 0 & \Gamma_{41} & 0\\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & 0 & \Gamma_{32} & 0 & \Gamma_{42}\\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \Gamma_{32} & \Gamma_{31} & \Gamma_{42} & \Gamma_{41}\end{bmatrix} \tag{34}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \gamma_{12}\end{bmatrix}=\mathbf{Bu}_e \tag{35}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

For the **plane stress** case, we have

$$\mathbf{D}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} \tag{36}$$

For the **plane strain** case, we have

$$\mathbf{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix} \tag{37}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K}=h\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{T}\mathbf{D}\mathbf{B}\left|\mathbf{J}\right|d\xi d\eta} \tag{38}$$

where h denotes the thickness.

Remarks:

In the above, we have introduced two types of continuum element (C2D3 and C2D4 element) in 2D space: one is of triangular shape; the other is of quadrilateral shape.

We see that once the shape and node placement of an element is determined in the standard parameterized space, shape functions defined in terms of the natural coordinates should be determined correspondingly to each degree of freedom. We have shown how to obtain these shape functions in the previous lectures by defining a trial function with unknown coefficients and desired properties, e.g., a linear polynomial.

As the start point of a standard procedure for computing the stiffness matrix, shape functions play an important role. If the set of shape functions of an element satisfy the following two conditions

$$\sum_{i=1}^{N}\hat{\varphi}_i=1$$

$$\hat{\varphi}_i\left(\mathbf{X}_j\right)=\delta_{ij}$$

the element belongs to the genre of so-called **compatible elements**. Otherwise, it is called incompatible element.

9.4 Gaussian quadrature

As we can see from Eq. (24) and Eq. (38), the evaluation of the stiffness matrix of an element is finally attributed to computing integrals. This could be easily done by a numerical integration, e.g., Gaussian quadrature.

For example, if we would like to compute the integration of a function $f(\xi)$ over the domain $[-1,1]$

$$\int_{-1}^{1} f(\xi)d\xi \tag{39}$$

Step 1: Evaluate the following basic integration

$$\begin{aligned}
&f_0(\xi)=1 \Rightarrow \int_{-1}^{1} 1d\xi = 2 \\
&f_1(\xi)=\xi \Rightarrow \int_{-1}^{1} \xi d\xi = 0 \\
&f_2(\xi)=\xi^2 \Rightarrow \int_{-1}^{1} \xi^2 d\xi = \frac{2}{3} \\
&f_3(\xi)=\xi^3 \Rightarrow \int_{-1}^{1} \xi^3 d\xi = 0 \\
&f_4(\xi)=\xi^4 \Rightarrow \int_{-1}^{1} \xi^4 d\xi = \frac{2}{5} \\
&f_5(\xi)=\xi^5 \Rightarrow \int_{-1}^{1} \xi^5 d\xi = 0 \\
&\vdots
\end{aligned} \tag{40}$$

Step 2: Assume a Gauss integration formula

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{N} w_i f(\xi_i) \tag{41}$$

where N is the number of Gauss point in the domain, w_i is the weight and ξ_i is the natural coordinate of the i-th Gauss point.

Step 3: Determine the weight and position of Gauss points for various N

When $N=1$, we call the formula one point integration. The weight and position for each Gauss point could be determined as

$$\begin{cases} \int_{-1}^{1} f_0(\xi)d\xi = 2 = w_1 f_0(\xi_1) = w_1 \\ \int_{-1}^{1} f_1(\xi)d\xi = 0 = w_1 f_1(\xi_1) = w_1\xi_1 \end{cases}$$
$$\Rightarrow \tag{42}$$
$$\xi_1 = 0,\ w_1 = 2$$

This formula is suitable for integrating a polynomial of highest degree 1.

When $N = 2$, we call the formula two point integration. The weight and position for each Gauss point could be determined as

$$\begin{cases} \int_{-1}^{1} f_0(\xi)d\xi = 2 = w_1 f_0(\xi_1) + w_2 f_0(\xi_2) = w_1 + w_2 \\ \int_{-1}^{1} f_1(\xi)d\xi = 0 = w_1 f_1(\xi_1) + w_2 f_1(\xi_2) = w_1\xi_1 + w_2\xi_2 \\ \int_{-1}^{1} f_2(\xi)d\xi = \frac{2}{3} = w_1 f_2(\xi_1) + w_2 f_2(\xi_2) = w_1\xi_1^2 + w_2\xi_2^2 \\ \int_{-1}^{1} f_3(\xi)d\xi = 0 = w_1 f_3(\xi_1) + w_2 f_3(\xi_2) = w_1\xi_1^3 + w_2\xi_2^3 \end{cases} \tag{43}$$
$$\Rightarrow$$
$$\xi_1 = -\frac{\sqrt{3}}{3},\ w_1 = 1$$
$$\xi_2 = \frac{\sqrt{3}}{3},\ w_2 = 1$$

This formula is suitable for integrating a polynomial of highest degree 3.

When $N = 3$, we call the formula three point integration. The weight and position for each Gauss point could be determined as

$$\begin{cases} \int_{-1}^{1} f_0(\xi)d\xi = 2 = \sum_{i=}^{3} w_i f_0(\xi_i) = w_1 + w_2 + w_3 \\ \int_{-1}^{1} f_1(\xi)d\xi = 0 = \sum_{i=}^{3} w_i f_1(\xi_i) = w_1\xi_1 + w_2\xi_2 + w_3\xi_3 \\ \int_{-1}^{1} f_2(\xi)d\xi = \frac{2}{3} = \sum_{i=}^{3} w_i f_2(\xi_i) = w_1\xi_1^2 + w_2\xi_1^2 + w_3\xi_1^2 \\ \int_{-1}^{1} f_3(\xi)d\xi = 0 = \sum_{i=}^{3} w_i f_3(\xi_i) = w_1\xi_1^3 + w_2\xi_1^3 + w_3\xi_1^3 \\ \int_{-1}^{1} f_4(\xi)d\xi = \frac{2}{5} = \sum_{i=}^{3} w_i f_4(\xi_i) = w_1\xi_1^4 + w_2\xi_1^4 + w_3\xi_1^4 \\ \int_{-1}^{1} f_5(\xi)d\xi = 0 = \sum_{i=}^{3} w_i f_5(\xi_i) = w_1\xi_1^5 + w_2\xi_1^5 + w_3\xi_1^5 \end{cases}$$

$$\Rightarrow$$

$$\xi_1 = -\frac{\sqrt{15}}{5},\ w_1 = \frac{5}{9} \tag{44}$$

$$\xi_2 = 0,\ w_2 = \frac{8}{9}$$

$$\xi_3 = \frac{\sqrt{15}}{5},\ w_3 = \frac{5}{9}$$

This formula is suitable for integrating a polynomial of highest degree 5.

Lecture 10 2D plane quadratic elements

In this lecture, we go on to introduce 2D continuum elements. The differences mainly lie in the number of nodes in these elements and their corresponding shape functions.

10.1 C2D6 element

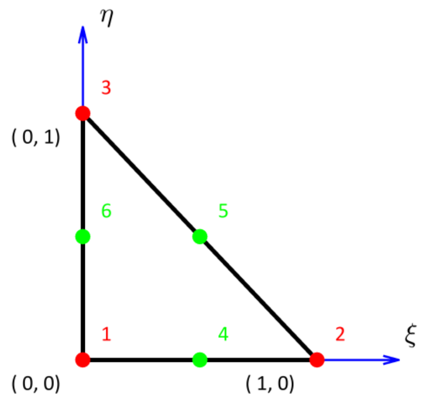


Fig. 1 C2D6

The shape of the element is shown in **Fig. 1**. About the name C2D6, “C” is short for “Continuum”, 2D is short for “2 dimension”, 6 stands for “6 nodes”.

Convention for the edge numbering:

edge-1: node1-node4-node2

edge-2: node2-node5-node3

edge-3: node3-node6-node1

Step 1: Derive the shape functions of C2D6:

$$
\begin{aligned}
\hat{\varphi}_1(\xi,\eta) &= (1-\xi-\eta)(1-2\xi-2\eta) \\
\hat{\varphi}_2(\xi,\eta) &= \xi(2\xi-1) \\
\hat{\varphi}_3(\xi,\eta) &= \eta(2\eta-1) \\
\hat{\varphi}_4(\xi,\eta) &= 4\xi(1-\xi-\eta) \\
\hat{\varphi}_5(\xi,\eta) &= 4\xi\eta \\
\hat{\varphi}_6(\xi,\eta) &= 4\eta(1-\xi-\eta)
\end{aligned}
\tag{1}
$$

Step 2: Create the **N** matrix for interpolation

$$
\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \cdots & \hat{\varphi}_6 & 0 \\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & \cdots & 0 & \hat{\varphi}_6 \end{bmatrix} \tag{2}
$$

Thereby, the field variable could be interpolated as

$$
\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_6 \\ v_6 \end{bmatrix} \tag{3}
$$

In addition, the field domain coordinate could be interpolated as

$$
\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{N}\mathbf{X}_e = \mathbf{N} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_6 \\ y_6 \end{bmatrix} \tag{4}
$$

Step 3: Create the **C** matrix by the nodal coordinates

$$
\mathbf{C} = \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \\ y_1 & y_2 & \cdots & y_6 \end{bmatrix} \tag{5}
$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix}\frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta}\\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta}\\ \frac{\partial\hat{\varphi}_3}{\partial\xi} & \frac{\partial\hat{\varphi}_3}{\partial\eta}\\ \frac{\partial\hat{\varphi}_4}{\partial\xi} & \frac{\partial\hat{\varphi}_4}{\partial\eta}\\ \frac{\partial\hat{\varphi}_5}{\partial\xi} & \frac{\partial\hat{\varphi}_5}{\partial\eta}\\ \frac{\partial\hat{\varphi}_6}{\partial\xi} & \frac{\partial\hat{\varphi}_6}{\partial\eta}\end{bmatrix}=\begin{bmatrix}-(3-4\xi-4\eta) & -(3-4\xi-4\eta)\\ 4\xi-1 & 0\\ 0 & 4\eta-1\\ 4(1-2\xi-\eta) & -4\xi\\ 4\eta & 4\xi\\ -4\eta & 4(1-\xi-2\eta)\end{bmatrix} \tag{6}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta}\\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta}\end{bmatrix}=\mathbf{CH} \tag{7}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{d}x\mathrm{d}y=|\mathbf{J}|\mathrm{d}\xi\mathrm{d}\eta \tag{8}$$

Step 6: Create the $\boldsymbol{\Gamma}$ matrix

$$\boldsymbol{\Gamma}=\mathbf{HJ}^{-1} \tag{9}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & \Gamma_{21} & 0 & \cdots & \Gamma_{61} & 0\\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & \cdots & 0 & \Gamma_{62}\\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \cdots & \Gamma_{62} & \Gamma_{61}\end{bmatrix} \tag{10}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \gamma_{12}\end{bmatrix}=\mathbf{Bu}_\mathrm{e} \tag{11}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

For the **plane stress** case, we have

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{12}$$

For the **plane strain** case, we have

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{13}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K} = h\int_0^1\int_0^{1-\xi} \mathbf{B}^T\mathbf{D}\mathbf{B}\left|\mathbf{J}\right| d\xi d\eta} \tag{14}$$

where h denotes the thickness.

10.2 C2D8 element

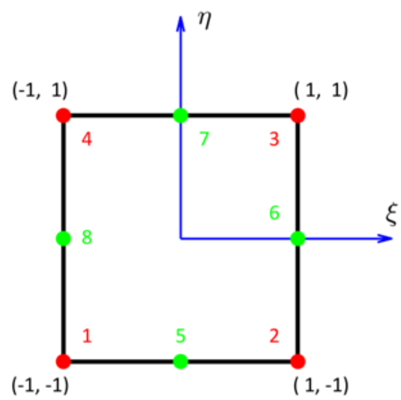


Fig. 1 C2D8

The shape of the element is shown in **Fig. 1**. About the name C2D8, “C” is short for “Continuum”, 2D is short for “2 dimension”, 8 stands for “8 nodes”. It is also called an 8-node **serendipity element**

Convention for the edge numbering:

Edge-1: node1-node5-node2

Edge-2: node2-node6-node3

Edge-3: node3-node7-node4

Edge-4: node4-node8-node1

Step 1: Derive the shape functions of C2D8:

$$\begin{aligned}
\hat{\varphi}_1(\xi,\eta) &= (1-\xi)(1-\eta)(-1-\xi-\eta)/4 \\
\hat{\varphi}_2(\xi,\eta) &= (1+\xi)(1-\eta)(-1+\xi-\eta)/4 \\
\hat{\varphi}_3(\xi,\eta) &= (1+\xi)(1+\eta)(-1+\xi+\eta)/4 \\
\hat{\varphi}_4(\xi,\eta) &= (1-\xi)(1+\eta)(-1-\xi+\eta)/4 \\
\hat{\varphi}_5(\xi,\eta) &= (1-\xi^2)(1-\eta)/2 \\
\hat{\varphi}_6(\xi,\eta) &= (1+\xi)(1-\eta^2)/2 \\
\hat{\varphi}_7(\xi,\eta) &= (1-\xi^2)(1+\eta)/2 \\
\hat{\varphi}_8(\xi,\eta) &= (1-\xi)(1-\eta^2)/2
\end{aligned} \tag{15}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \cdots & \hat{\varphi}_8 & 0 \\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & \cdots & 0 & \hat{\varphi}_8 \end{bmatrix} \tag{16}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_8 \\ v_8 \end{bmatrix} \tag{17}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X}=\begin{bmatrix} x \\ y \end{bmatrix}=\mathbf{N}\mathbf{X}_e=\mathbf{N}\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_8 \\ y_8 \end{bmatrix} \tag{18}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C}=\begin{bmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_2 & \cdots & y_8 \end{bmatrix} \tag{19}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} \\ \frac{\partial\hat{\varphi}_3}{\partial\xi} & \frac{\partial\hat{\varphi}_3}{\partial\eta} \\ \frac{\partial\hat{\varphi}_4}{\partial\xi} & \frac{\partial\hat{\varphi}_4}{\partial\eta} \\ \frac{\partial\hat{\varphi}_5}{\partial\xi} & \frac{\partial\hat{\varphi}_5}{\partial\eta} \\ \frac{\partial\hat{\varphi}_6}{\partial\xi} & \frac{\partial\hat{\varphi}_6}{\partial\eta} \\ \frac{\partial\hat{\varphi}_7}{\partial\xi} & \frac{\partial\hat{\varphi}_7}{\partial\eta} \\ \frac{\partial\hat{\varphi}_8}{\partial\xi} & \frac{\partial\hat{\varphi}_8}{\partial\eta} \end{bmatrix}=\begin{bmatrix} \frac{1}{4}(1-\eta)(2\xi+\eta) & \frac{1}{4}(1-\xi)(\xi+2\eta) \\ \frac{1}{4}(1-\eta)(2\xi-\eta) & \frac{1}{4}(1+\xi)(-\xi+2\eta) \\ \frac{1}{4}(1+\eta)(2\xi+\eta) & \frac{1}{4}(1+\xi)(\xi+2\eta) \\ \frac{1}{4}(1+\eta)(2\xi-\eta) & \frac{1}{4}(1-\xi)(-\xi+2\eta) \\ -\xi(1-\eta) & -\frac{1}{2}(1-\xi^2) \\ \frac{1}{2}(1-\eta^2) & -(1+\xi)\eta \\ -\xi(1+\eta) & \frac{1}{2}(1-\xi^2) \\ -\frac{1}{2}(1-\eta^2) & -(1-\xi)\eta \end{bmatrix} \tag{20}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} \end{bmatrix}=\mathbf{CH} \tag{21}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{dxdy} = |\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta \tag{22}$$

Step 6: Create the $\boldsymbol{\Gamma}$ matrix

$$\boldsymbol{\Gamma} = \mathbf{H} \cdot \mathbf{J}^{-1} \tag{23}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B} = \begin{bmatrix} \Gamma_{11} & 0 & \Gamma_{21} & 0 & \cdots & \Gamma_{61} & 0 \\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & \cdots & 0 & \Gamma_{62} \\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \cdots & \Gamma_{62} & \Gamma_{61} \end{bmatrix} \tag{24}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ \gamma_{12} \end{bmatrix} = \mathbf{Bu}_\mathrm{e} \tag{25}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

For the **plane stress** case, we have

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{26}$$

For the **plane strain** case, we have

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{27}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K} = h\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^\mathrm{T}\mathbf{D}\mathbf{B}\,|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta} \tag{28}$$

where h denotes the thickness.

Remarks:

The reason to choose quadratic elements

While the linear elements have fewer DoFs and thus is faster in computation, its accuracy is not good. C2D3 element should be avoided as much as possible in stress analysis since it is too stiff and has a low convergence rate even when mesh is fine. In contrast, a quadratic element has a higher accuracy.

In some application, using fine mesh for linear elements is not able to improve the accuracy due to that the field variable essentially requires higher order terms in the trial function to approximate its behavior.

In addition, quadratic elements can model curved surface with fewer elements and is good for bending dominant problems

Lecture 11 3D solid elements

In this lecture, we will go through the linear 3D solid elements (i.e., C3D4 element and C3D8 element) and the quadratic 3D solid elements (i.e., C3D10 element and C3D20 element). In contrast to the 2D elements introduced in the previous lectures, the 3D elements have more DoFs and more complex shape functions.

11.1 C3D4 element

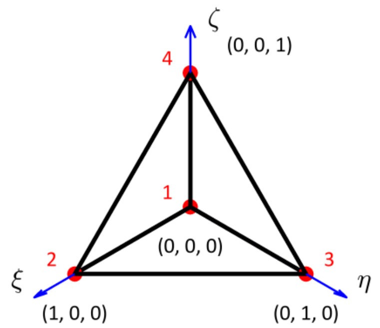


Fig.1 C3D4

The shape of the element is shown in **Fig. 1**. About the name C3D4, “C” is short for “Continuum”, 3D is short for “3 dimension”, 4 stands for “4 nodes”.

Convention for the edge numbering:

surface-1： node1-node2-node3

surface-2： node1-node2-node4

surface-3： node2-node3-node4

surface-4： node1-node4-node3

Step 1: Derive the shape functions of C3D4:

$$\begin{aligned}
\hat{\varphi}_1(\xi,\eta,\zeta) &= 1-\xi-\eta-\zeta \\
\hat{\varphi}_2(\xi,\eta,\zeta) &= \xi \\
\hat{\varphi}_3(\xi,\eta,\zeta) &= \eta \\
\hat{\varphi}_4(\xi,\eta,\zeta) &= \zeta
\end{aligned} \tag{1}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & 0 & \cdots & \hat{\varphi}_4 & 0 & 0 \\ 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & \cdots & 0 & \hat{\varphi}_4 & 0 \\ 0 & 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & \ldots & 0 & 0 & \hat{\varphi}_4 \end{bmatrix} \tag{2}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_4 \\ v_4 \\ w_4 \end{bmatrix} \tag{3}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{N}\mathbf{X}_e = \mathbf{N}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ \vdots \\ x_4 \\ y_4 \\ z_4 \end{bmatrix} \tag{4}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C}=\begin{bmatrix} x_1 & x_2 & \cdots & x_4 \\ y_1 & y_2 & \cdots & y_4 \\ z_1 & z_2 & \cdots & z_4 \end{bmatrix} \tag{5}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} & \frac{\partial\hat{\varphi}_1}{\partial\zeta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} & \frac{\partial\hat{\varphi}_2}{\partial\zeta} \\ \vdots & \vdots & \vdots \\ \frac{\partial\hat{\varphi}_4}{\partial\xi} & \frac{\partial\hat{\varphi}_4}{\partial\eta} & \frac{\partial\hat{\varphi}_4}{\partial\zeta} \end{bmatrix}=\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta} \end{bmatrix}=\mathbf{CH} \tag{7}$$

whose determinant $\left|\mathbf{J}\right|$ relates the differentials between the field domain and the computation domain.

$$dxdydz=\left|\mathbf{J}\right|d\xi d\eta d\zeta \tag{8}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\mathbf{\Gamma}=\begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial x} & \frac{\partial\hat{\varphi}_1}{\partial y} & \frac{\partial\hat{\varphi}_1}{\partial z} \\ \frac{\partial\hat{\varphi}_2}{\partial x} & \frac{\partial\hat{\varphi}_2}{\partial y} & \frac{\partial\hat{\varphi}_2}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial\hat{\varphi}_4}{\partial x} & \frac{\partial\hat{\varphi}_4}{\partial y} & \frac{\partial\hat{\varphi}_4}{\partial z} \end{bmatrix}=\mathbf{HJ}^{-1} \tag{9}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & 0 & \Gamma_{21} & 0 & 0 & \cdots & \Gamma_{41} & 0 & 0\\ 0 & \Gamma_{12} & 0 & 0 & \Gamma_{22} & 0 & \cdots & 0 & \Gamma_{42} & 0\\ 0 & 0 & \Gamma_{13} & 0 & 0 & \Gamma_{23} & \cdots & 0 & 0 & \Gamma_{43}\\ \Gamma_{12} & \Gamma_{11} & 0 & \Gamma_{22} & \Gamma_{21} & 0 & \cdots & \Gamma_{42} & \Gamma_{41} & 0\\ \Gamma_{13} & 0 & \Gamma_{11} & \Gamma_{23} & 0 & \Gamma_{21} & \cdots & \Gamma_{43} & 0 & \Gamma_{41}\\ 0 & \Gamma_{13} & \Gamma_{12} & 0 & \Gamma_{23} & \Gamma_{22} & \cdots & 0 & \Gamma_{43} & \Gamma_{42}\end{bmatrix} \tag{10}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{\varepsilon}_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}=\mathbf{B}\mathbf{u}_e \tag{11}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

$$\mathbf{D}=\begin{bmatrix}2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix} \tag{12}$$

where μ and λ are called **Lame constant**s.

$$\begin{aligned}\mu &= \frac{E}{2(1+\nu)} = G\\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}\end{aligned} \tag{13}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K}=\int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta}\mathbf{B}^T\mathbf{D}\mathbf{B}\left|\mathbf{J}\right|d\xi d\eta d\zeta} \tag{14}$$

Remarks:

To derive the shape functions for C3D4 element, one might directly use the so called volume coordinates, which are similar to the area-coordinates for C2D3 element that is introduced in the previous lecture.

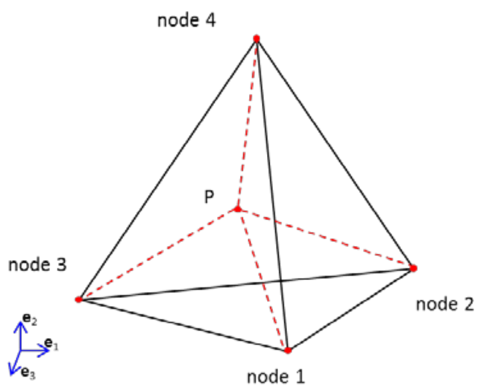


Fig. 2 Volume coordinates of a point in a tetrahedral element

Tetrahedron Volume Theorem (TVT)

If the four vertices of a tetrahedron is numbered per the right hand rule as shown in **Fig. 2** and the coordinates of the four vertices are given as (x_i, y_i, z_i), $i = 1,2,3,4$, then the volume of the tetrahedron is determined as

$$V_{T_{1234}} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \tag{15}$$

So, for an arbitrary point $P(x, y, z)$ inside the tetrahedron, four smaller constitutive tetrahedrons could be formed, i.e, $T_{P234}, T_{1P34}, T_{12P4}, T_{123P}$

$$\begin{aligned} V_{T_{P234}} &= \frac{1}{6}\begin{vmatrix} 1 & x & y & z \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \quad V_{T_{1P34}} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x & y & z \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \\ V_{T_{12P4}} &= \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x & y & z \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \quad V_{T_{123P}} = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x & y & z \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \end{aligned} \tag{16}$$

The volume-coordinates are defined as

$$L_1 = \frac{V_{T_{P234}}}{V_{T_{1234}}}, L_2 = \frac{V_{T_{1P34}}}{V_{T_{1234}}}, L_3 = \frac{V_{T_{12P4}}}{V_{T_{1234}}}, L_4 = \frac{V_{T_{123P}}}{V_{T_{1234}}} \tag{17}$$

It is noted that the volume-coordinates obey the following relationships that shape functions should satisfy

$$\begin{aligned} &\sum_{i=1}^{4} L_i = 1 \\ &L_i\left(\mathbf{X}_j\right) = \delta_{ij} \end{aligned} \tag{18}$$

It is easy to derive the shape functions given in Eq. (1)by using the volume-coordinates, i.e.,

$$\begin{aligned} \hat{\varphi}_1 &= L_1 \\ \hat{\varphi}_2 &= L_2 \\ \hat{\varphi}_3 &= L_3 \\ \hat{\varphi}_4 &= L_4 \end{aligned} \tag{19}$$

11.2 C3D8 element

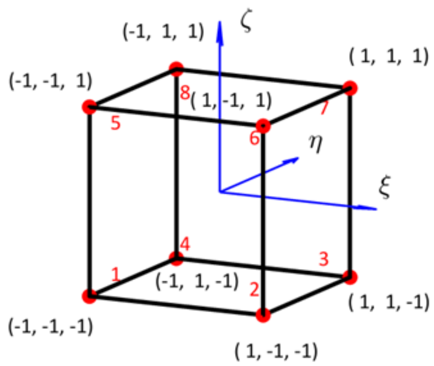


Fig. 3 C3D8

The shape of the element is shown in **Fig. 3**. About the name C3D8, "C" is short for "Continuum", 3D is short for "3 dimension", 8 stands for "8 nodes".

Convention for the edge numbering:

Surface-1： node1-node4-node3-node2

Surface-2： node5-node6-node7-node8

Surface-3： node1-node2-node6-node5

Surface-4： node2-node3-node7-node6

Surface-5： node3-node4-node8-node7

Surface-6： node1-node5-node8-node4

Step 1: Derive the shape functions of C3D8:

$$\begin{aligned}
\hat{\varphi}_1(\xi,\eta,\zeta) &= (1-\xi)(1-\eta)(1-\zeta)/8 \\
\hat{\varphi}_2(\xi,\eta,\zeta) &= (1+\xi)(1-\eta)(1-\zeta)/8 \\
\hat{\varphi}_3(\xi,\eta,\zeta) &= (1+\xi)(1+\eta)(1-\zeta)/8 \\
\hat{\varphi}_4(\xi,\eta,\zeta) &= (1-\xi)(1+\eta)(1-\zeta)/8 \\
\hat{\varphi}_5(\xi,\eta,\zeta) &= (1-\xi)(1-\eta)(1+\zeta)/8 \\
\hat{\varphi}_6(\xi,\eta,\zeta) &= (1+\xi)(1-\eta)(1+\zeta)/8 \\
\hat{\varphi}_7(\xi,\eta,\zeta) &= (1+\xi)(1+\eta)(1+\zeta)/8 \\
\hat{\varphi}_8(\xi,\eta,\zeta) &= (1-\xi)(1+\eta)(1+\zeta)/8
\end{aligned} \tag{20}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & 0 & \cdots & \hat{\varphi}_8 & 0 & 0 \\ 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & \cdots & 0 & \hat{\varphi}_8 & 0 \\ 0 & 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & \ldots & 0 & 0 & \hat{\varphi}_8 \end{bmatrix} \tag{21}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u}=\begin{bmatrix}u\\v\\w\end{bmatrix}=\mathbf{N}\mathbf{u}_e=\mathbf{N}\begin{bmatrix}u_1\\v_1\\w_1\\u_2\\v_2\\w_2\\\vdots\\u_8\\v_8\\w_8\end{bmatrix} \tag{22}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{N}\mathbf{X}_e=\mathbf{N}\begin{bmatrix}x_1\\y_1\\z_1\\x_2\\y_2\\z_2\\\vdots\\x_8\\y_8\\z_8\end{bmatrix} \tag{23}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C}=\begin{bmatrix}x_1 & x_2 & \cdots & x_8\\ y_1 & y_2 & \cdots & y_8\\ z_1 & z_2 & \cdots & z_8\end{bmatrix} \tag{24}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix}\frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} & \frac{\partial\hat{\varphi}_1}{\partial\zeta}\\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} & \frac{\partial\hat{\varphi}_2}{\partial\zeta}\\ \vdots & \vdots & \vdots\\ \frac{\partial\hat{\varphi}_8}{\partial\xi} & \frac{\partial\hat{\varphi}_8}{\partial\eta} & \frac{\partial\hat{\varphi}_8}{\partial\zeta}\end{bmatrix} \tag{25}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta}\\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}=\mathbf{CH} \tag{26}$$

whose determinant $\left|\mathbf{J}\right|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{dxdydz}=\left|\mathbf{J}\right|\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \tag{27}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\mathbf{\Gamma}=\begin{bmatrix}\frac{\partial\hat{\varphi}_1}{\partial x} & \frac{\partial\hat{\varphi}_1}{\partial y} & \frac{\partial\hat{\varphi}_1}{\partial z}\\ \frac{\partial\hat{\varphi}_2}{\partial x} & \frac{\partial\hat{\varphi}_2}{\partial y} & \frac{\partial\hat{\varphi}_2}{\partial z}\\ \vdots & \vdots & \vdots\\ \frac{\partial\hat{\varphi}_8}{\partial x} & \frac{\partial\hat{\varphi}_8}{\partial y} & \frac{\partial\hat{\varphi}_8}{\partial z}\end{bmatrix}=\mathbf{HJ}^{-1} \tag{28}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & 0 & \Gamma_{21} & 0 & 0 & \cdots & \Gamma_{81} & 0 & 0\\ 0 & \Gamma_{12} & 0 & 0 & \Gamma_{22} & 0 & \cdots & 0 & \Gamma_{82} & 0\\ 0 & 0 & \Gamma_{13} & 0 & 0 & \Gamma_{23} & \cdots & 0 & 0 & \Gamma_{83}\\ \Gamma_{12} & \Gamma_{11} & 0 & \Gamma_{22} & \Gamma_{21} & 0 & \cdots & \Gamma_{82} & \Gamma_{81} & 0\\ \Gamma_{13} & 0 & \Gamma_{11} & \Gamma_{23} & 0 & \Gamma_{21} & \cdots & \Gamma_{83} & 0 & \Gamma_{81}\\ 0 & \Gamma_{13} & \Gamma_{12} & 0 & \Gamma_{23} & \Gamma_{22} & \cdots & 0 & \Gamma_{83} & \Gamma_{82}\end{bmatrix} \tag{29}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{\varepsilon}_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}=\mathbf{Bu}_e \tag{30}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

$$\mathbf{D}=\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{31}$$

where μ and λ are called **Lame constants**.

$$\mu=\frac{E}{2(1+\nu)}=G$$
$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)} \tag{32}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K}=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{T}\mathbf{D}\mathbf{B}\left|\mathbf{J}\right|d\xi d\eta d\zeta} \tag{33}$$

11.3 C3D10 element

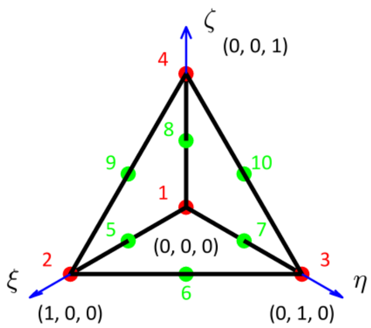


Fig. 4 C3D10

The shape of the element is shown in **Fig. 4**. About the name C3D10, “C” is short for “Continuum”, 3D is short for “3D dimension”, 10 stands for “10 nodes”.

Convention for the edge numbering:

surface-1： node1-node2-node3

surface-2： node1-node2-node4

surface-3： node2-node3-node4

surface-4： node1-node4-node3

Step 1: Derive the shape functions of C3D10:

$$\begin{aligned}
\hat{\varphi}_1(\xi,\eta,\zeta) &= (2t_1-1)t_1 \\
\hat{\varphi}_2(\xi,\eta,\zeta) &= (2t_2-1)t_2 \\
\hat{\varphi}_3(\xi,\eta,\zeta) &= (2t_3-1)t_3 \\
\hat{\varphi}_4(\xi,\eta,\zeta) &= (2t_4-1)t_4 \\
\hat{\varphi}_5(\xi,\eta,\zeta) &= 4t_1t_2 \\
\hat{\varphi}_6(\xi,\eta,\zeta) &= 4t_2t_3 \\
\hat{\varphi}_7(\xi,\eta,\zeta) &= 4t_1t_3 \\
\hat{\varphi}_8(\xi,\eta,\zeta) &= 4t_1t_4 \\
\hat{\varphi}_9(\xi,\eta,\zeta) &= 4t_2t_4 \\
\hat{\varphi}_{10}(\xi,\eta,\zeta) &= 4t_3t_4
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
t_1(\xi,\eta,\zeta) &= 1-\xi-\eta-\zeta \\
t_2(\xi,\eta,\zeta) &= \xi \\
t_3(\xi,\eta,\zeta) &= \eta \\
t_4(\xi,\eta,\zeta) &= \zeta
\end{aligned} \tag{35}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & 0 & \cdots & \hat{\varphi}_{10} & 0 & 0 \\ 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & \cdots & 0 & \hat{\varphi}_{10} & 0 \\ 0 & 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & \ldots & 0 & 0 & \hat{\varphi}_{10} \end{bmatrix} \tag{36}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u}=\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\mathbf{N}\mathbf{u}_e=\mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_{10} \\ v_{10} \\ w_{10} \end{bmatrix} \tag{37}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\mathbf{N}\mathbf{X}_e=\mathbf{N}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ \vdots \\ x_{10} \\ y_{10} \\ z_{10} \end{bmatrix} \tag{38}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C}=\begin{bmatrix} x_1 & x_2 & \cdots & x_{10} \\ y_1 & y_2 & \cdots & y_{10} \\ z_1 & z_2 & \cdots & z_{10} \end{bmatrix} \tag{39}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} & \frac{\partial\hat{\varphi}_1}{\partial\zeta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} & \frac{\partial\hat{\varphi}_2}{\partial\zeta} \\ \vdots & \vdots & \vdots \\ \frac{\partial\hat{\varphi}_4}{\partial\xi} & \frac{\partial\hat{\varphi}_4}{\partial\eta} & \frac{\partial\hat{\varphi}_4}{\partial\zeta} \end{bmatrix}=\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{40}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta}\\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}=\mathbf{CH} \tag{41}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{dxdydz}=|\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \tag{42}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\mathbf{\Gamma}=\begin{bmatrix}\frac{\partial \hat{\varphi}_1}{\partial x} & \frac{\partial \hat{\varphi}_1}{\partial y} & \frac{\partial \hat{\varphi}_1}{\partial z}\\ \frac{\partial \hat{\varphi}_2}{\partial x} & \frac{\partial \hat{\varphi}_2}{\partial y} & \frac{\partial \hat{\varphi}_2}{\partial z}\\ \vdots & \vdots & \vdots\\ \frac{\partial \hat{\varphi}_{10}}{\partial x} & \frac{\partial \hat{\varphi}_{10}}{\partial y} & \frac{\partial \hat{\varphi}_{10}}{\partial z}\end{bmatrix}=\mathbf{HJ}^{-1} \tag{43}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & 0 & \Gamma_{21} & 0 & 0 & \cdots & \Gamma_{101} & 0 & 0\\ 0 & \Gamma_{12} & 0 & 0 & \Gamma_{22} & 0 & \cdots & 0 & \Gamma_{102} & 0\\ 0 & 0 & \Gamma_{13} & 0 & 0 & \Gamma_{23} & \cdots & 0 & 0 & \Gamma_{103}\\ \Gamma_{12} & \Gamma_{11} & 0 & \Gamma_{22} & \Gamma_{21} & 0 & \cdots & \Gamma_{102} & \Gamma_{101} & 0\\ \Gamma_{13} & 0 & \Gamma_{11} & \Gamma_{23} & 0 & \Gamma_{21} & \cdots & \Gamma_{103} & 0 & \Gamma_{101}\\ 0 & \Gamma_{13} & \Gamma_{12} & 0 & \Gamma_{23} & \Gamma_{22} & \cdots & 0 & \Gamma_{103} & \Gamma_{102}\end{bmatrix} \tag{44}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{\varepsilon}_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}=\mathbf{Bu}_e \tag{45}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

$$\mathbf{D}=\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{46}$$

where μ and λ are called **Lame constants**.

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)} = G \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{47}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K}=\int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta}\mathbf{B}^T\mathbf{D}\mathbf{B}|\mathbf{J}|d\xi d\eta d\zeta} \tag{48}$$

11.4 C3D20 element

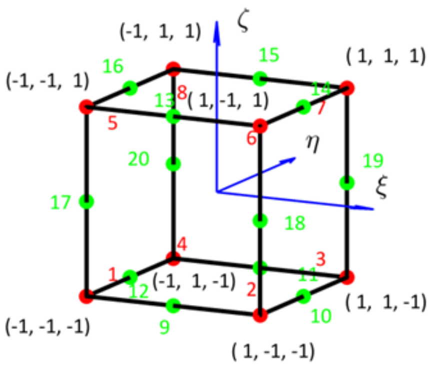


Fig. 5 C3D20

The shape of the element is shown in **Fig. 5**. About the name C3D20, “C” is short for “Continuum”, 3D is short for “3 dimension”, 20 stands for “20 nodes”.

Convention for the edge numbering:

Surface-1： node1-node4-node3-node2

Surface-2： node5-node6-node7-node8

Surface-3： node1-node2-node6-node5

Surface-4： node2-node3-node7-node6

Surface-5： node3-node4-node8-node7

Surface-6： node1-node5-node8-node4

Step 1: Derive the shape functions of C3D20:

$$
\begin{aligned}
&\hat{\varphi}_1(\xi,\eta,\zeta)=(1-\xi)(1-\eta)(1-\zeta)(-2-\xi-\eta-\zeta)/8\\
&\hat{\varphi}_2(\xi,\eta,\zeta)=(1+\xi)(1-\eta)(1-\zeta)(-2+\xi-\eta-\zeta)/8\\
&\hat{\varphi}_3(\xi,\eta,\zeta)=(1+\xi)(1+\eta)(1-\zeta)(-2+\xi+\eta-\zeta)/8\\
&\hat{\varphi}_4(\xi,\eta,\zeta)=(1-\xi)(1+\eta)(1-\zeta)(-2-\xi+\eta-\zeta)/8\\
&\hat{\varphi}_5(\xi,\eta,\zeta)=(1-\xi)(1-\eta)(1+\zeta)(-2-\xi-\eta+\zeta)/8\\
&\hat{\varphi}_6(\xi,\eta,\zeta)=(1+\xi)(1-\eta)(1+\zeta)(-2+\xi-\eta+\zeta)/8\\
&\hat{\varphi}_7(\xi,\eta,\zeta)=(1+\xi)(1+\eta)(1+\zeta)(-2+\xi+\eta+\zeta)/8\\
&\hat{\varphi}_8(\xi,\eta,\zeta)=(1-\xi)(1+\eta)(1+\zeta)(-2-\xi+\eta+\zeta)/8\\
&\hat{\varphi}_9(\xi,\eta,\zeta)=(1-\xi^2)(1-\eta)(1-\zeta)/4 \quad \hat{\varphi}_{10}(\xi,\eta,\zeta)=(1+\xi)(1-\eta^2)(1-\zeta)/4\\
&\hat{\varphi}_{11}(\xi,\eta,\zeta)=(1-\xi^2)(1+\eta)(1-\zeta)/4 \quad \hat{\varphi}_{12}(\xi,\eta,\zeta)=(1-\xi)(1-\eta^2)(1-\zeta)/4\\
&\hat{\varphi}_{13}(\xi,\eta,\zeta)=(1-\xi^2)(1-\eta)(1+\zeta)/4 \quad \hat{\varphi}_{14}(\xi,\eta,\zeta)=(1+\xi)(1-\eta^2)(1+\zeta)/4\\
&\hat{\varphi}_{15}(\xi,\eta,\zeta)=(1-\xi^2)(1+\eta)(1+\zeta)/4 \quad \hat{\varphi}_{16}(\xi,\eta,\zeta)=(1-\xi)(1-\eta^2)(1+\zeta)/4\\
&\hat{\varphi}_{17}(\xi,\eta,\zeta)=(1-\xi)(1-\eta)(1-\zeta^2)/4 \quad \hat{\varphi}_{18}(\xi,\eta,\zeta)=(1+\xi)(1-\eta)(1-\zeta^2)/4\\
&\hat{\varphi}_{19}(\xi,\eta,\zeta)=(1+\xi)(1+\eta)(1-\zeta^2)/4 \quad \hat{\varphi}_{20}(\xi,\eta,\zeta)=(1-\xi)(1+\eta)(1-\zeta^2)/4
\end{aligned}
\tag{49}
$$

Step 2: Create the **N** matrix for interpolation

$$
\mathbf{N}=\begin{bmatrix}
\hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & 0 & \cdots & \hat{\varphi}_{20} & 0 & 0\\
0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & 0 & \cdots & 0 & \hat{\varphi}_{20} & 0\\
0 & 0 & \hat{\varphi}_1 & 0 & 0 & \hat{\varphi}_2 & \ldots & 0 & 0 & \hat{\varphi}_{20}
\end{bmatrix}
\tag{50}
$$

Thereby, the field variable could be interpolated as

$$\mathbf{u}=\begin{bmatrix} u \\ v \\ w \end{bmatrix}=\mathbf{Nu}_e=\mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_{20} \\ v_{20} \\ w_{20} \end{bmatrix} \tag{51}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\mathbf{NX}_e=\mathbf{N}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \\ \vdots \\ x_{20} \\ y_{20} \\ z_{20} \end{bmatrix} \tag{52}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C}=\begin{bmatrix} x_1 & x_2 & \cdots & x_{20} \\ y_1 & y_2 & \cdots & y_{20} \\ z_1 & z_2 & \cdots & z_{20} \end{bmatrix} \tag{53}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H}=\begin{bmatrix} \frac{\partial\hat{\varphi}_1}{\partial\xi} & \frac{\partial\hat{\varphi}_1}{\partial\eta} & \frac{\partial\hat{\varphi}_1}{\partial\zeta} \\ \frac{\partial\hat{\varphi}_2}{\partial\xi} & \frac{\partial\hat{\varphi}_2}{\partial\eta} & \frac{\partial\hat{\varphi}_2}{\partial\zeta} \\ \vdots & \vdots & \vdots \\ \frac{\partial\hat{\varphi}_{20}}{\partial\xi} & \frac{\partial\hat{\varphi}_{20}}{\partial\eta} & \frac{\partial\hat{\varphi}_{20}}{\partial\zeta} \end{bmatrix} \tag{54}$$

Step 5: Create the **J** matrix

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta}\\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}\end{bmatrix}=\mathbf{CH} \tag{55}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z=|\mathbf{J}|\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \tag{56}$$

Step 6: Create the $\boldsymbol{\Gamma}$ matrix

$$\boldsymbol{\Gamma}=\begin{bmatrix}\frac{\partial \hat{\varphi}_1}{\partial x} & \frac{\partial \hat{\varphi}_1}{\partial y} & \frac{\partial \hat{\varphi}_1}{\partial z}\\ \frac{\partial \hat{\varphi}_2}{\partial x} & \frac{\partial \hat{\varphi}_2}{\partial y} & \frac{\partial \hat{\varphi}_2}{\partial z}\\ \vdots & \vdots & \vdots\\ \frac{\partial \hat{\varphi}_{20}}{\partial x} & \frac{\partial \hat{\varphi}_{20}}{\partial y} & \frac{\partial \hat{\varphi}_{20}}{\partial z}\end{bmatrix}=\mathbf{HJ}^{-1} \tag{57}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & 0 & \Gamma_{21} & 0 & 0 & \cdots & \Gamma_{201} & 0 & 0\\ 0 & \Gamma_{12} & 0 & 0 & \Gamma_{22} & 0 & \cdots & 0 & \Gamma_{202} & 0\\ 0 & 0 & \Gamma_{13} & 0 & 0 & \Gamma_{23} & \cdots & 0 & 0 & \Gamma_{203}\\ \Gamma_{12} & \Gamma_{11} & 0 & \Gamma_{22} & \Gamma_{21} & 0 & \cdots & \Gamma_{202} & \Gamma_{201} & 0\\ \Gamma_{13} & 0 & \Gamma_{11} & \Gamma_{23} & 0 & \Gamma_{21} & \cdots & \Gamma_{203} & 0 & \Gamma_{201}\\ 0 & \Gamma_{13} & \Gamma_{12} & 0 & \Gamma_{23} & \Gamma_{22} & \cdots & 0 & \Gamma_{203} & \Gamma_{202}\end{bmatrix} \tag{58}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_1\\ \tilde{\varepsilon}_2\\ \tilde{\varepsilon}_3\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix}=\mathbf{Bu}_e \tag{59}$$

Step 8: Create the **D** matrix (plane stress/plane strain)

$$\mathbf{D} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{60}$$

where μ and λ are called **Lame constant**s.

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)} = G \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{61}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\mathbf{K} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{T}\mathbf{D}\mathbf{B}\left|\mathbf{J}\right| d\xi d\eta d\zeta} \tag{62}$$

Remarks:

In engineering practice, 3D analysis is usually expensive. It is common to simplify a 3D problem to a 2D one. In addition, coarse mesh and low order elements are preferred at the initial prediction. As the computer technology progresses, 3D solid elements get more and more popular in engineering practice. In some circumstances, 3D solid element show more advantage, especially for those 2D elements are not suitable.

Lecture 12 Axially symmetric elements

In this lecture, we will briefly introduce the axially symmetric solid elements. These elements are used for axially symmetric structures, e.g. a hollowed cylinder as shown in **Fig. 1**. These elements are essentially 3D solid elements but have similar shape functions as that of 2D case.

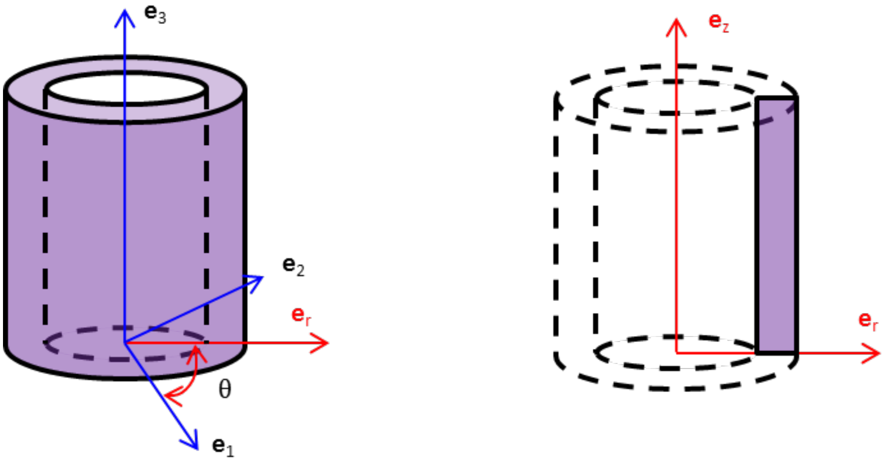


Fig.1 A hollowed cylinder and its coordinate system

12.1 CAX3 element

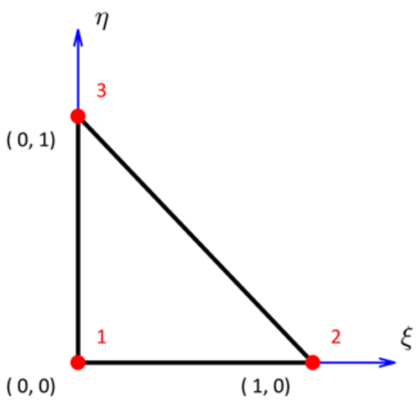


Fig.2 CAX3

The shape of the element is shown in **Fig. 2**. About the name CAX3, “C” is short for "Continuum", AX is short for "axial symmetric", 3 stands for "3 nodes".

Convention for the edge numbering:

edge-1: node1-node2

edge-2: node2-node3

edge-3: node3-node1

Step 1: Derive the shape functions of CAX3:

$$\begin{aligned}\hat{\varphi}_1(\xi,\eta) &= 1-\xi-\eta \\ \hat{\varphi}_2(\xi,\eta) &= \xi \\ \hat{\varphi}_3(\xi,\eta) &= \eta\end{aligned} \tag{1}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0 \\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 \end{bmatrix} \tag{2}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} \tag{3}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X} = \begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{N}\mathbf{X}_e = \mathbf{N}\begin{bmatrix} r_1 \\ z_1 \\ r_2 \\ z_2 \\ r_3 \\ z_3 \end{bmatrix} \tag{4}$$

Remarks:

This step is quite similar to that of the plane stress/plane strain case. Notice the coordinate is denoted as (r, z).

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C} = \begin{bmatrix} r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \tag{5}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H} = \begin{bmatrix} \frac{\partial \hat{\varphi}_1}{\partial \xi} & \frac{\partial \hat{\varphi}_1}{\partial \eta} \\ \frac{\partial \hat{\varphi}_2}{\partial \xi} & \frac{\partial \hat{\varphi}_2}{\partial \eta} \\ \frac{\partial \hat{\varphi}_3}{\partial \xi} & \frac{\partial \hat{\varphi}_3}{\partial \eta} \end{bmatrix} \tag{6}$$

Step 5: Create the **J** matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \mathbf{CH} \tag{7}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{d}r\mathrm{d}z = |\mathbf{J}|\mathrm{d}\xi\mathrm{d}\eta \tag{8}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{\partial \hat{\varphi}_1}{\partial r} & \frac{\partial \hat{\varphi}_1}{\partial z} \\ \frac{\partial \hat{\varphi}_2}{\partial r} & \frac{\partial \hat{\varphi}_2}{\partial z} \\ \frac{\partial \hat{\varphi}_3}{\partial r} & \frac{\partial \hat{\varphi}_3}{\partial z} \end{bmatrix} = \mathbf{HJ}^{-1} \tag{9}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B}=\begin{bmatrix}\Gamma_{11} & 0 & \Gamma_{21} & 0 & \Gamma_{31} & 0\\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & 0 & \Gamma_{32}\\ \hat{\varphi}_1/r & 0 & \hat{\varphi}_2/r & 0 & \hat{\varphi}_3/r & 0\\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \Gamma_{32} & \Gamma_{31}\end{bmatrix} \tag{10}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}}=\begin{bmatrix}\tilde{\varepsilon}_r\\ \tilde{\varepsilon}_z\\ \tilde{\varepsilon}_\theta\\ \gamma_{rz}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial u}{\partial r}\\ \dfrac{\partial v}{\partial z}\\ \dfrac{u}{r}\\ \dfrac{\partial u}{\partial z}+\dfrac{\partial v}{\partial r}\end{bmatrix}=\mathbf{B}\mathbf{u}_e \tag{11}$$

Step 8: Create the **D** matrix (axially symmetric case)

$$\mathbf{D}=\begin{bmatrix}2\mu+\lambda & \lambda & \lambda & 0\\ \lambda & 2\mu+\lambda & \lambda & 0\\ \lambda & \lambda & 2\mu+\lambda & 0\\ 0 & 0 & 0 & \mu\end{bmatrix} \tag{12}$$

where μ and λ are called **Lame constants**.

$$\begin{aligned}\mu&=\frac{E}{2(1+\nu)}=G\\ \lambda&=\frac{E\nu}{(1+\nu)(1-2\nu)}\end{aligned} \tag{13}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\begin{aligned}\mathbf{K}&=\int_0^{2\pi}\int_0^1\int_0^{1-\xi}\mathbf{B}^T\mathbf{D}\mathbf{B}|\mathbf{J}|r\,d\theta\,d\xi\,d\eta\\ &=2\pi\int_0^1\int_0^{1-\xi}\mathbf{B}^T\mathbf{D}\mathbf{B}|\mathbf{J}|r\,d\xi\,d\eta\end{aligned}} \tag{14}$$

12.2 CAX4 element

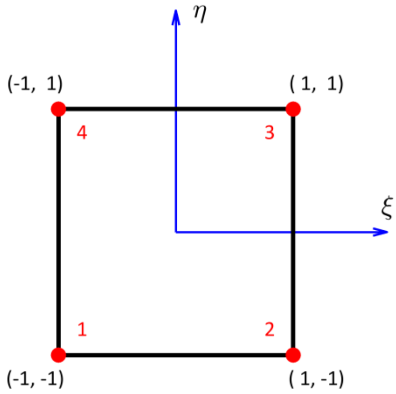


Fig. 3 CAX4 element

The shape of the element is shown in **Fig. 3**. About the name CAX4, "C" is short for "Continuum", 2D is short for "axially symmetric", 4 stands for "4 nodes".

Convention for the edge numbering:

Edge-1： node1-node2

Edge-2： node2-node3

Edge-3： node3-node4

Edge-4： node4-node1

Step 1: Derive the shape functions of CAX4:

$$\begin{aligned}
\hat{\varphi}_1(\xi,\eta) &= (1-\xi)(1-\eta)/4 \\
\hat{\varphi}_2(\xi,\eta) &= (1+\xi)(1-\eta)/4 \\
\hat{\varphi}_3(\xi,\eta) &= (1+\xi)(1+\eta)/4 \\
\hat{\varphi}_4(\xi,\eta) &= (1-\xi)(1+\eta)/4
\end{aligned} \tag{15}$$

Step 2: Create the **N** matrix for interpolation

$$\mathbf{N} = \begin{bmatrix} \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0 & \hat{\varphi}_4 & 0 \\ 0 & \hat{\varphi}_1 & 0 & \hat{\varphi}_2 & 0 & \hat{\varphi}_3 & 0 & \hat{\varphi}_4 \end{bmatrix} \tag{16}$$

Thereby, the field variable could be interpolated as

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{N}\mathbf{u}_e = \mathbf{N}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \tag{17}$$

In addition, the field domain coordinate could be interpolated as

$$\mathbf{X} = \begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{N}\mathbf{X}_e = \mathbf{N}\begin{bmatrix} r_1 \\ z_1 \\ r_2 \\ z_2 \\ r_3 \\ z_3 \\ r_4 \\ z_4 \end{bmatrix} \tag{18}$$

Step 3: Create the **C** matrix by the nodal coordinates

$$\mathbf{C} = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \tag{19}$$

Step 4: Create the **H** matrix by the derivatives of the shape functions

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial\hat{\varphi}_1}{\partial\xi} & \dfrac{\partial\hat{\varphi}_1}{\partial\eta} \\ \dfrac{\partial\hat{\varphi}_2}{\partial\xi} & \dfrac{\partial\hat{\varphi}_2}{\partial\eta} \\ \dfrac{\partial\hat{\varphi}_3}{\partial\xi} & \dfrac{\partial\hat{\varphi}_3}{\partial\eta} \\ \dfrac{\partial\hat{\varphi}_4}{\partial\xi} & \dfrac{\partial\hat{\varphi}_4}{\partial\eta} \end{bmatrix} \tag{20}$$

Step 5: Create the **J** matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial r}{\partial \eta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \mathbf{CH} \tag{21}$$

whose determinant $|\mathbf{J}|$ relates the differentials between the field domain and the computation domain.

$$\mathrm{d}r\mathrm{d}z = |\mathbf{J}|\,\mathrm{d}\xi\mathrm{d}\eta \tag{22}$$

Step 6: Create the $\mathbf{\Gamma}$ matrix

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{\partial \hat{\varphi}_1}{\partial r} & \frac{\partial \hat{\varphi}_1}{\partial z} \\ \frac{\partial \hat{\varphi}_2}{\partial r} & \frac{\partial \hat{\varphi}_2}{\partial z} \\ \frac{\partial \hat{\varphi}_3}{\partial r} & \frac{\partial \hat{\varphi}_3}{\partial z} \\ \frac{\partial \hat{\varphi}_4}{\partial r} & \frac{\partial \hat{\varphi}_4}{\partial z} \end{bmatrix} = \mathbf{HJ}^{-1} \tag{23}$$

Step 7: Create the **B** matrix for interpolation

$$\mathbf{B} = \begin{bmatrix} \Gamma_{11} & 0 & \Gamma_{21} & 0 & \Gamma_{31} & 0 & \Gamma_{41} & 0 \\ 0 & \Gamma_{12} & 0 & \Gamma_{22} & 0 & \Gamma_{32} & 0 & \Gamma_{42} \\ \hat{\varphi}_1/r & 0 & \hat{\varphi}_2/r & 0 & \hat{\varphi}_3/r & 0 & \hat{\varphi}_4/r & 0 \\ \Gamma_{12} & \Gamma_{11} & \Gamma_{22} & \Gamma_{21} & \Gamma_{32} & \Gamma_{31} & \Gamma_{42} & \Gamma_{41} \end{bmatrix} \tag{24}$$

Thereby, the derivative of field variable such as strain could be interpolated as

$$\tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \tilde{\varepsilon}_r \\ \tilde{\varepsilon}_z \\ \tilde{\varepsilon}_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{bmatrix} = \mathbf{Bu}_e \tag{25}$$

Step 8: Create the **D** matrix (axially symmetric case)

$$\mathbf{D}=\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \tag{26}$$

where μ and λ are called **Lame constants**.

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)} = G \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \end{aligned} \tag{27}$$

Step 9: Create the **K** matrix (i.e., the stiffness matrix)

$$\boxed{\begin{aligned} \mathbf{K} &= \int_0^{2\pi}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B}\left|\mathbf{J}\right| r\,d\theta\, d\xi\, d\eta \\ &= 2\pi\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B}\left|\mathbf{J}\right| r\,d\xi\, d\eta \end{aligned}} \tag{28}$$

Remarks:

In the above, it is shown that the procedure to evaluate the stiffness matrix is quite similar to that of 2D cases. In contrast to 2D plane stress/strain element, we see that the **B** matrix and **D** matrix are apparently different. In addition, the stiffness matrix is also different. Herein, we just show the axially symmetric linear elements CAX3 and CAX4. It is easy to generalize the procedure and apply it to the quadratic elements CAX6 and CAX8.