Loan Payment Deferments Due to Labor Market Shocks: A Case Study

Preliminary: Comments Welcome

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Abstract

This report analyzes loan payment deferment as a result of COVID-19 related shutdowns in the US. We focus on a portfolio of unsecured consumer loans originated by 2 different institutions. Our analysis focuses on a few key questions:

- what is the magnitude of COVID-related deferment?
- are there systematic relationships between loan attributes and payment deferment?
- how are labor market trends related to the probability of loan deferment?
- does the sensitivity to labor market shocks vary by region?

The model and results presented provide a general framework that can be applied not only to unsecured consumer loans but also more broadly to other lending sectors. While the data are still preliminary and the events they capture relatively recent, our conclusions are based on a rigorous and transparent statistical analysis and presented with confidence bounds that respect the intrinsic uncertainty of the data-generating process. The chief contribution of this paper, in terms of techniques, is the use of a "mixed-model" with random effects within a bayesian estimation framework which has enabled us to answer some of the questions we posed earlier and which would not have been possible using more traditional approaches.

This study should be useful to investors and policy-makers alike, allowing for data-driven estimates of potential deferment (and distress) rates on loan portfolios.

0.1 Introduction

Our reasons for undertaking this research project were driven by practical considerations — like many other investors in consumer and mortgage lending, we happen to be long these loans. As such, it is critical for us to evaluate future losses and prospective returns on these loans and make assessments about their "fundamental" value. We do this with the explicit recognition of the unprecedented nature of the COVID shock and the fact that in many ways, we are sailing through uncharted waters.

While our motivations were pragmatic, a natural question that arises in this context is the applicability of the analysis to a broader population if loans. While there is a natural tendency to always seek out more and greater amounts of data, in practice, investors in most cases hold narrow subsets of the overall population of loans. While larger datasets may give us more precise estimates (up to a point), the fact is that we want to make statements about OUR portfolio, not a fictional universe which is not owned by anyone in particular. The challenge then is to employ statistical methods that allow us to extract information from "small" not "big" data and turn these into useful

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insights for decision-making but which may nevertheless provide guidance about the broader population as well. This is where the bayesian methods we deploy in this report come in useful since they explicitly deal with inferential uncertainty in an intrinsic way and can be used to provide insights in other contexts as well.

There are two parts to our project. First, we describe the data set in some detail and present stratifications by different loan attributes. We also present the deferment rates within each strata in order to get intuition around the impact of loan attributes. We then provide statistics around the labor markets in various states. We look at the impact of the annual percentage change in initial claims, starting March 14th (which we peg as the start of the COVID crisis for our purposes) and through the week ending July 4, 2020. An open question that the modeling seeks to answer is the impact of the claims variables on deferment rates and whether these can be leveraged into a prediction framework going forward. A discussion of the statistical model that relates the observed outcome (did the loan defer: Yes/No?) to the various loan attributes is provided in the appendix. The framework employed is based on Survival Analysis, using a hierarchical bayes approach as in [Gelman and Hill(2007)] and [Gelman(2006)].

In the second part of our work, we develop a methodology for forecasting the path of initial claims at the national and state levels over the next few months. This analysis is unique in its own way and leverages a brief descriptive note put out by Federal Reserve Bank of NY researchers in a blog article. We use the claims forecast as inputs into the predictions for deferment rates at the end of second quarter of 2020, which is our forecast horizon. The model and the estimation results and forecasts are provided in an accompanying piece.

Before we dive into the details, there are three key technical aspects in this report that are worth highlighting. **First, the use of Survival or Hazard models** to estimate the marginal deferment probability, as a function of weeks elapsed since the crisis, is **key** to sensible projections of deferment [2]. As we show, these marginal hazards have a very strong "duration" component which impacts longer-term forecasts of the cumulative amount of deferment we expect in the future.

Second, we extend the survival model framework by incorporating **parameter hierarchies (within a bayesian framework)** that explicitly account for random variation in the impact of variables across loan clusters. This allows for the possibility of "unobserved heterogeneity" in the data by explicitly modeling a cluster-specific random variable that interacts with and modifies the hazards for loan clusters. This is an important enhancement since (i) there may be differences in the composition of the workforce across groups that impact the way in which a given volume of claims affect deferment rates, and (ii) the borrower base itself may differ across groups in both observable and unobservable ways. We control for the observed attributes explicitly but the hierarchical framework allows us to model unobserved factors as well.

Third, we develop a **statistical framework to model "decay" rates for weekly claims** and the role that labor markets play in determining deferment rates, building upon ideas first discussed by researchers at the NY Fed. The projections from this framework serve as inputs to our longer-term deferment forecasts and allows us to model the impact of different economic scenarios in the future, an important tool to have in the arsenal given the considerable uncertainties that still remain regarding the future path of the economy.

0.2 Data

In Table 1, we provided an overview of our data sample. In all, we have 3351 loans in our data, in roughly a 50/50 split (by count) across the 2 institutions.

The aggregate original amount issued is $58,044,797.00, with a weighted-average interest rate of 15.52%, a weighted-average FICO score of 705.48 and is 15.90 months seasoned. The weighted-average original-term is 44.81 months. **Overall, the deferment rate on this portfolio is 13.33%.**

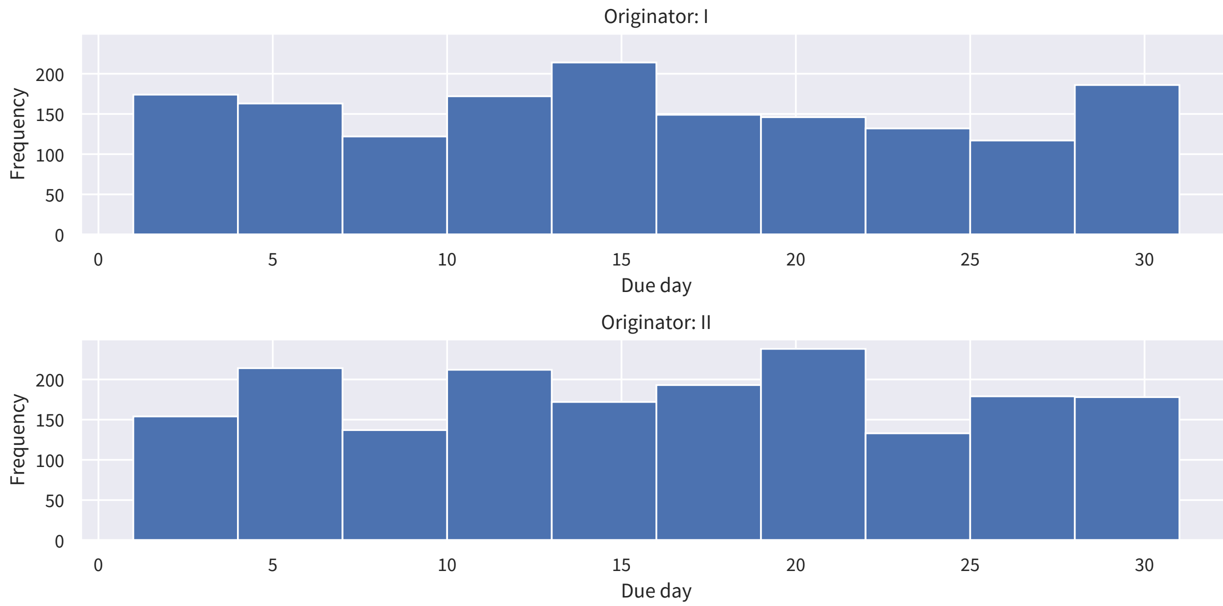
The portfolio statistics presented here are as of July 12, 2020 which is more than one month into the onset of the significant "shelter-at-home" orders across the country and resulting economic disruptions. Since most of these payment deferrals are for anywhere from 1 to 3 months, the deferment percentages can be viewed as the cumulative share of loans deferred or delinquent since the start of the COVID crisis. By way of comparison, we provide recent deferment figures for other related sectors such as mortgages. Approximately 8.46% of all mortgage loans were in forebearance as of May 24th, 2020 which is a roughly 49 days earlier than the cutoff date for our data set. In the Ginnie Mae sector, 11.82% of loans were in forebearance while the comparable figure for conventional mortgages was 6.39%.

[2] This is a benefit over and above the intrinsic gain from using this framework in the context of "censored" data where most of the observations have not yet experienced deferment

Table 1: Portfolio Summary

Originator	Grade	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	G0	214	$4,000,375.00	$1,529,839.54	7.80%	21.43	731.71	45.02	13.32	$ 11,327.98	7.91%	4.59%
	G1	499	$8,835,875.00	$3,718,544.17	10.98%	21.67	704.35	47.17	17.78	$ 8,901.95	11.01%	11.17%
	G2	534	$9,750,850.00	$4,091,295.15	14.52%	25.61	689.40	47.88	16.42	$ 8,289.64	17.15%	12.29%
	G3	261	$4,726,375.00	$2,084,725.82	20.34%	23.79	682.64	49.22	19.37	$ 7,831.40	18.84%	6.26%
	G4	33	$ 769,725.00	$ 472,367.87	25.11%	23.14	689.08	51.99	22.88	$ 6,359.15	29.16%	1.42%
II	G0	151	$3,106,200.00	$2,201,570.50	8.19%	9.17	765.96	42.82	22.11	$ 11,307.98	3.13%	6.61%
	G1	389	$6,266,985.00	$4,459,807.99	10.32%	10.29	726.25	41.67	22.54	$ 9,826.56	5.93%	13.39%
	G2	442	$7,833,416.00	$5,503,067.05	13.42%	11.79	707.97	41.36	24.81	$ 8,762.68	11.78%	16.53%
	G3	347	$6,667,180.00	$4,532,446.21	18.71%	14.69	697.60	44.42	27.57	$ 8,536.98	20.11%	13.61%
	G4	250	$4,173,601.00	$3,016,919.98	25.13%	13.81	683.89	46.07	27.58	$ 8,740.39	16.71%	9.06%
	G5	154	$1,378,065.00	$1,239,871.21	30.39%	6.38	677.11	48.94	25.74	$ 5,918.91	16.62%	3.72%
	G6	77	$ 536,150.00	$ 444,477.77	31.82%	8.09	664.36	36.00	24.06	$ 5,701.43	16.11%	1.33%
I	ALL	1541	$28,083,200.00	$11,896,772.55	13.99%	23.42	698.32	47.69	17.22	$ 8,714.79	14.82%	35.73%
II	ALL	1810	$29,961,597.00	$21,398,160.72	16.37%	11.71	709.46	43.22	25.07	$ 8,966.98	12.50%	64.27%
ALL	ALL	3351	$58,044,797.00	$33,294,933.27	15.52%	15.90	705.48	44.81	22.26	$ 8,876.87	13.33%	100.00%

Figure 1: Distribution of due dates



In figure 1, we present the frequency distribution of the payment dates on the loans in our sample. Since borrowers may have a tendency to hold off on requesting a deferral until they are close to or past their due day, and given the relatively short data window, this may lead to biases. A relatively uniform distribution of payment due dates would serve to assuage this concern. Thankfully, this is exactly what we find in the data presented here, eliminating this aspect of the data as a potential source of concern.

In Table 2, we provide a stratification of the portfolio by loan purpose. More than two-thirds of the loans are used for consolidating existing debt, mostly drawn on credit cards. The second largest category is for purchases, while less than 10% is used for expenses such as for education, wedding etc. ("LifeCyle").

In Table 3, a stratification across the borrower's employment status is provided. The "Self-employed" and "Other" categories generally comprise anywhere from 10% to 15% of the portfolio[3].

[3]In the case of Originator I, the employment category is really a dummy variable for the presence or absence of employment history — if there is information on this count, this field is coded as "Employed" otherwise it is coded as "Other"

Table 2: Portfolio summary, by purpose

Originator	Purpose	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	Debt Consolidation	1111	$21,441,150.00	$9,194,208.54	14.05%	23.04	696.58	47.52	18.25	$ 8,519.19	13.87%	77.28%
	Acquisition	236	$3,993,800.00	$1,621,943.45	13.24%	24.95	706.86	48.65	13.08	$ 10,339.80	15.30%	13.63%
	LifeCycle	36	$ 398,875.00	$ 152,596.87	14.20%	26.07	694.25	47.33	16.32	$ 7,071.20	4.97%	1.28%
	Other	158	$2,249,375.00	$ 928,023.70	14.59%	24.10	701.19	47.81	14.36	$ 8,082.77	24.98%	7.80%
II	Debt Consolidation	1225	$20,774,740.00	$14,812,866.22	16.06%	11.71	708.57	43.03	26.09	$ 8,875.70	11.40%	69.22%
	Acquisition	312	$5,304,752.00	$3,725,153.36	16.21%	12.29	714.17	44.26	23.01	$ 8,785.57	17.11%	17.41%
	LifeCycle	154	$2,026,354.00	$1,496,190.69	19.38%	10.95	704.55	44.36	24.09	$ 7,600.43	12.94%	6.99%
	Other	119	$1,855,751.00	$1,363,950.46	16.87%	11.05	711.62	41.14	20.68	$ 11,952.78	11.33%	6.37%

Table 3: Portfolio summary, by employment status

Originator	Employment	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	Employed	1388	$25,852,700.00	$10,883,109.73	14.03%	23.67	697.43	47.83	16.97	$ 8,939.22	15.04%	91.48%
	Self-employed	0	$ 0.00	$ 0.00	0.00%	0.00	0.00	0.00	0.00	$ 0.00	0.00%	0.00%
	Other	153	$2,230,500.00	$1,013,662.83	13.52%	20.75	707.85	46.23	19.88	$ 6,305.16	12.40%	8.52%
II	Employed	1510	$26,056,587.00	$18,566,888.57	16.25%	11.80	709.90	43.49	25.20	$ 9,130.02	12.22%	86.82%
	Self-employed	177	$2,104,815.00	$1,490,623.11	17.26%	11.32	702.21	41.62	19.33	$ 9,576.08	22.80%	6.97%
	Other	122	$1,780,195.00	$1,326,935.48	17.16%	10.86	711.84	41.00	29.67	$ 6,048.36	5.01%	6.21%

In Table 4, the portfolio is stratified across housing tenure. Across the 2 institutions, roughly a quarter to two-thirds of the borrowing is by renters.

Table 4: Portfolio summary, by homeownership

Originator	Housing	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	Own	1169	$21,906,100.00	$8,886,510.46	13.36%	24.56	699.89	47.64	17.10	$ 9,261.43	13.67%	74.70%
	Rent	372	$6,177,100.00	$3,010,262.10	15.85%	20.07	693.66	47.84	17.58	$ 7,101.06	18.20%	25.30%
II	Own	632	$11,708,252.00	$7,868,814.61	15.22%	13.76	716.79	45.77	24.83	$ 10,103.08	10.00%	36.77%
	Rent	1178	$18,253,345.00	$13,529,346.10	17.04%	10.52	705.19	41.73	25.21	$ 8,306.21	13.95%	63.23%

Finally, in Table 5, we stratify by loan term. Across the 2 institutions, roughly 50% - 70% of the loans are for 3-year amortization terms, with the remainder for a 5-year term.

An important question, with possible implications about the prospective cure rates for the group of defermented loans, is what they look like versus the subset of loans that were already delinquent before the crisis. This is presented in Table 6.

The deferment subset (labeled "Covid") has better credit quality (as measured) by their FICO scores than both "Current" and delinquent sub-population for Originator I. This may imply that the cure rate from deferments may be better on the deferred sub-population than has been the experience on the delinquent sub-population. In the case of Originator II, the deferment and delinquent sets have roughly the same FICO score which is lower than that on the set of loans that are "Current".

0.3 Employment

The economic disruption caused by COVID is in many ways unusual in that it strikes at the Consumption component of overall GDP. As such, the disruption is much broader than would be the case, say, for an investment led recession, caused by a contraction in an isolated segment of the economy.

In some ways, this resembles the 2008 recession which was caused by a massive asset writedowns in the banking sector (on a global basis) leading to an economy-wide credit crunch. To the extent that it strikes at almost two-thirds of overall economic output, the disruption is naturally even larger, as has become obvious in the labor market figures released over the last month. Labor markets are likely to be the key to explaining deferment, and both the full magnitude of job losses and how quickly they are reversed is going to be the driver of ultimate loan performance.

Table 5: Portfolio summary, by term

Originator	Term	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	36	1112	$18,200,850.00	$6,101,308.83	13.69%	19.53	691.56	36.00	16.47	$ 9,319.97	14.03%	51.29%
	60	429	$9,882,350.00	$5,795,463.73	14.30%	27.52	705.43	60.00	18.01	$ 8,077.67	15.65%	48.71%
II	36	1310	$21,109,942.00	$14,964,879.06	15.96%	9.58	704.45	36.00	26.21	$ 9,053.18	11.85%	69.94%
	60	500	$8,851,655.00	$6,433,281.65	17.32%	16.67	721.12	60.00	22.41	$ 8,766.45	14.01%	30.06%

Table 6: DQ *vs* Deferment profile

Originator	DQ Status	N	Orig. Bal.	Cur. Bal.	WAC	WALA	FICO	WAOT	DTI	Income	Defer	Share
I	Current	1438	$26,129,975.00	$11,055,345.93	13.80%	23.14	698.55	47.59	17.28	$ 8,730.69	14.42%	92.93%
	In Grace Period	22	$ 389,575.00	$ 139,108.58	17.71%	29.50	697.05	49.38	18.29	$ 8,415.17	51.71%	1.17%
	Late (16-30 Days)	16	$ 290,900.00	$ 92,887.06	16.34%	27.00	679.36	41.70	12.94	$ 10,046.62	13.76%	0.78%
	Late (31-120 Days)	56	$1,140,000.00	$ 540,088.98	16.22%	26.74	699.46	50.89	16.93	$ 8,706.56	15.57%	4.54%
	Default	9	$ 132,750.00	$ 69,342.00	15.22%	26.37	680.59	43.84	12.37	$ 5,060.57	0.00%	0.58%
II	Current	1729	$28,679,697.00	$20,435,019.43	16.18%	11.63	709.97	43.24	25.10	$ 9,004.18	12.03%	95.50%
	In Grace Period	12	$ 177,500.00	$ 145,983.08	20.79%	10.31	683.95	41.33	17.84	$ 11,757.59	40.27%	0.68%
	Late (16-30 Days)	12	$ 199,500.00	$ 154,931.50	22.18%	13.21	691.38	42.82	29.68	$ 7,415.03	56.87%	0.72%
	Late (31-120 Days)	57	$ 904,900.00	$ 662,226.71	19.99%	14.11	703.40	43.00	24.78	$ 7,566.93	10.56%	3.09%

The trend in annualized percentage change in weekly initial claims and their distribution is presented in Figure 2. The underlying data are the individual state/week observations on claims, merged with the appropriate loan histories starting March 14th. When we first did this figure, we thought we had made a mistake but the percentage changes depicted here are correct - on an year-over-year basis, initial claims did really increase by aproximately 8000% at their peak in early April! The annual percentage changes in claims are standardized by subtracting the mean and dividing by the standard deviation.

0.3.1 Claims

In Figures 3 and 4, we depict the relationship between payment deferment and the claims measure. The solid trend line is a robust fit for the scatter plot depicted here. The patterns appear to show a relatively small relationship (and possibly opposite in effect across the 2 originators) — in both cases, there is considerable variation across states.

The modeling exercise will seek to examine how much of the difference in slopes and the variability across states can be explained by individual loan attributes and unobservable effects modeled as "random effects".

0.4 Model

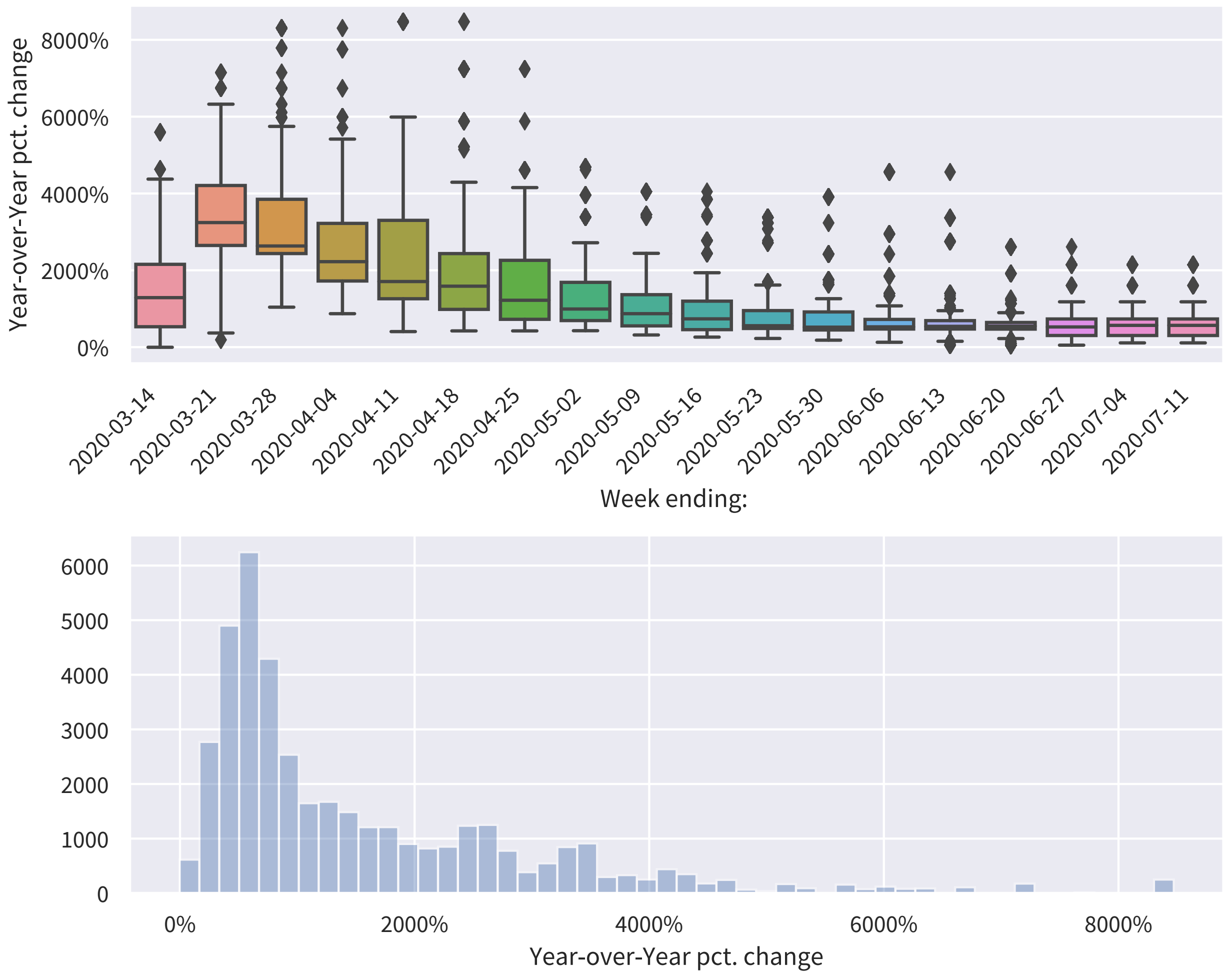
The modeling framework used in this report draws upon statistical tools used in the analysis of events with a "time-until" component to them. In our case, the time-until, or "lifetime" we are interested in predicting is the time until a borrower asks the servicer for a deferment or goes delinquent. Time in this context is measured from an assumed epoch start date of March 14th, 2020 which we assume as the start of the COVID-19 crisis for our purposes and is the same across all loans.

We incorporate state-based differences in the distribution of hazard rates that manifest themselves in both the pattern of duration dependence as well as the impact that changes in state-level initial claims have on hazards. As is detailed in the appendix, duration dependence is captured using a set of interval-specifc intercepts. All numerical covariates are standardized by subtracting the mean and dividing by the standard deviation. Categorical features are encoded using "dummy variables" where the first category is treated as the baseline or reference category.

The model is calibrated to a "training" data set that consists of a random sample of 80% of the loans from the full dataset, stratified on state. The paramete estimates derived from the training set are then used to derive predictions for the remaining loans that constitute the "test" sample.

Further details are provided in the appendix.

Figure 2: Weekly Claims (Year-over-Year pct. change): trend and distribution



0.5 Estimates

We now turn to a discussion of the results. We use the PyMC3 Python package to estimate the model [Salvatier J.(2016)]. The parameter estimates are presented for each originator in turn.

0.5.1 Hazards

In this section, we present depictions of the marginal deferment probabilities (using Nelson-Aalen hazards), to set the stage for what we should expect our fully-specified hazards to look like. The hazard estimates depicted in Figure 5 were computed using the Lifelines Python package [Cam Davidson Pilon(2020)].

The hazards rose sharply in the first weeks after the start of the crisis, to between 2% and 2.5%, but have declined since then. They reveal a difference in operating protocols where it appears that in the case of Originator I, the initial flurry of claims were processed in a batch and then approved all at once in the second week. The overall pattern of events and censoring is presented in Table 7. The “observed” column indicates the count of loans where deferment was observed during the interval specified in the “event-at” column on the left. Since the study inception date is the same for all loans, a large fraction of the data show up as “censored” in the last interval, and reported under the censored column. The other entries in this column pertain to loans that either prepaid or were charged-off during the period starting March 14th, 2020 and the cutoff date. The removed column represents the portion of the “at-risk” population that is no longer at risk since the loan was either censored or experienced an event. The at-risk figure for the previous interval is decremented by the removed column for that interval to give a new at-risk number.

Figure 3: Originator I: deferment hazard

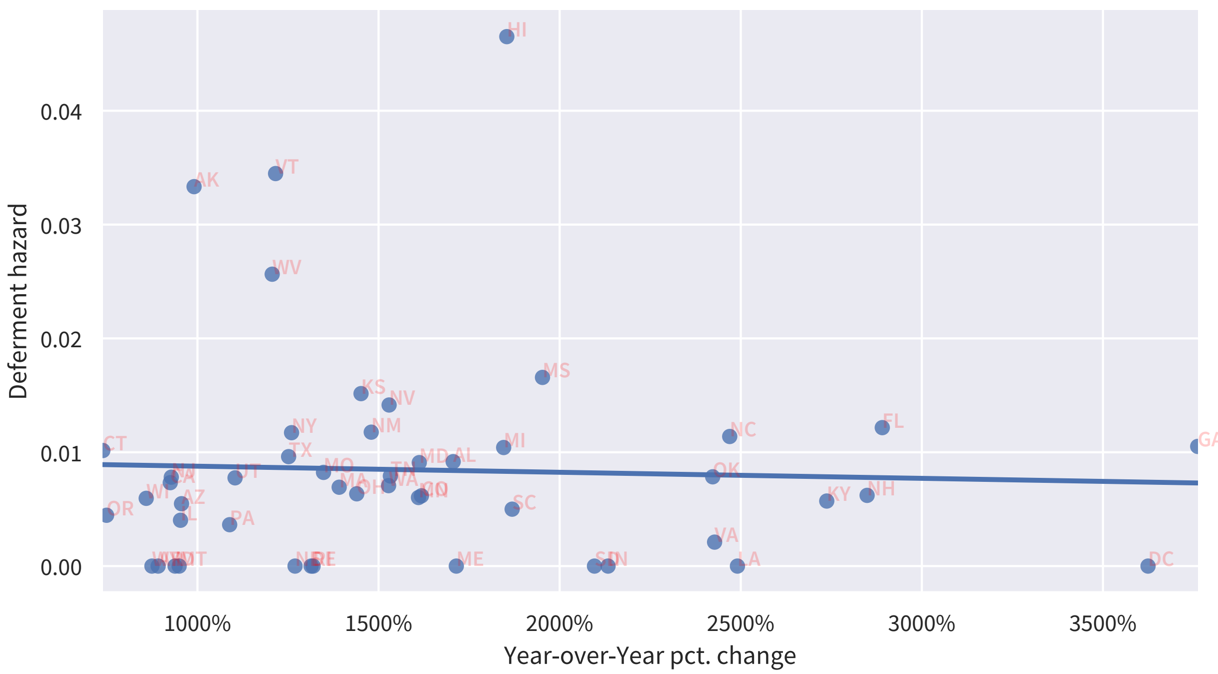
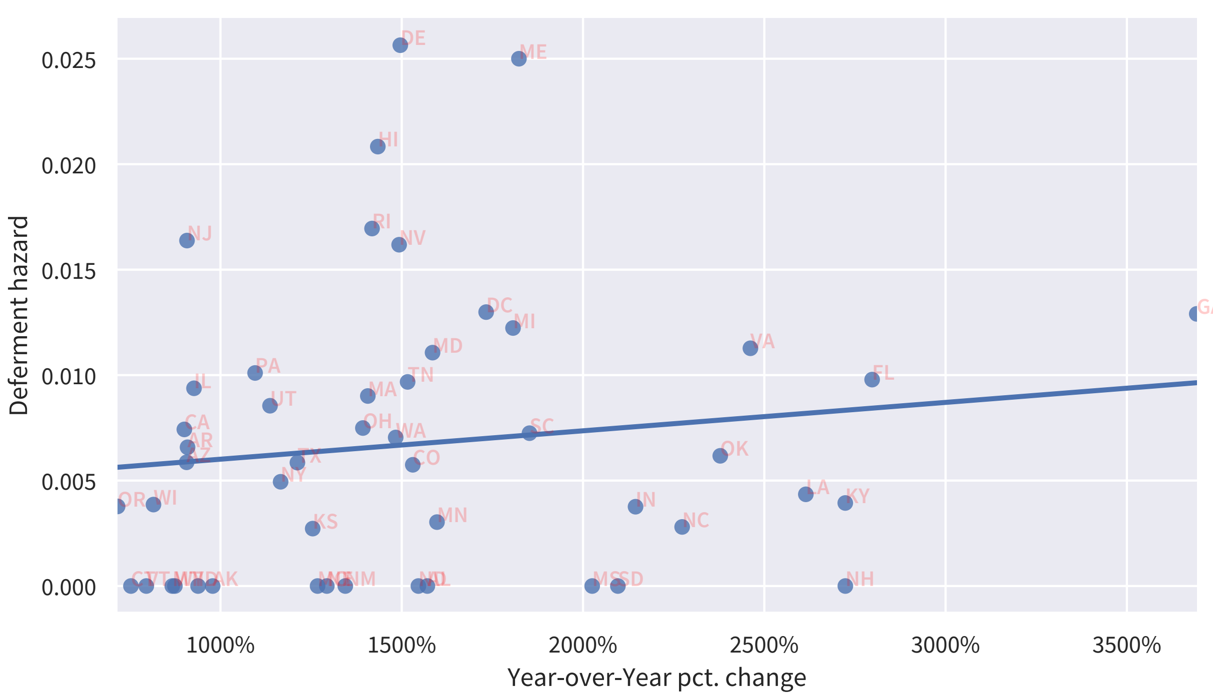


Figure 4: Originator II: deferment hazard



Nelson-Aalen hazards for the top 12 states by loan count are presented in Figure 6.

Figure 5: Hazards

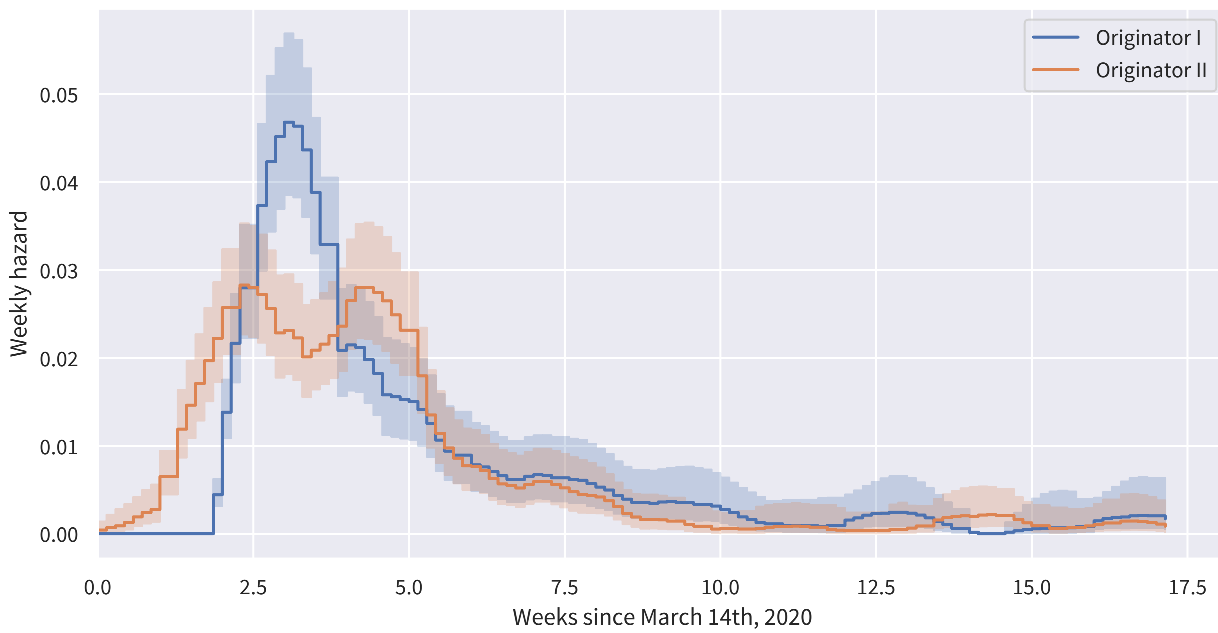


Table 7: Survival table: all loans

event_at	removed	observed	censored	at_risk
(0.0, 3.447]	272	174	98	3351
(3.447, 6.893]	265	161	104	3079
(6.893, 10.34]	134	31	103	2814
(10.34, 13.786]	106	9	97	2680
(13.786, 17.233]	2574	12	2562	2574

0.5.2 Pooled

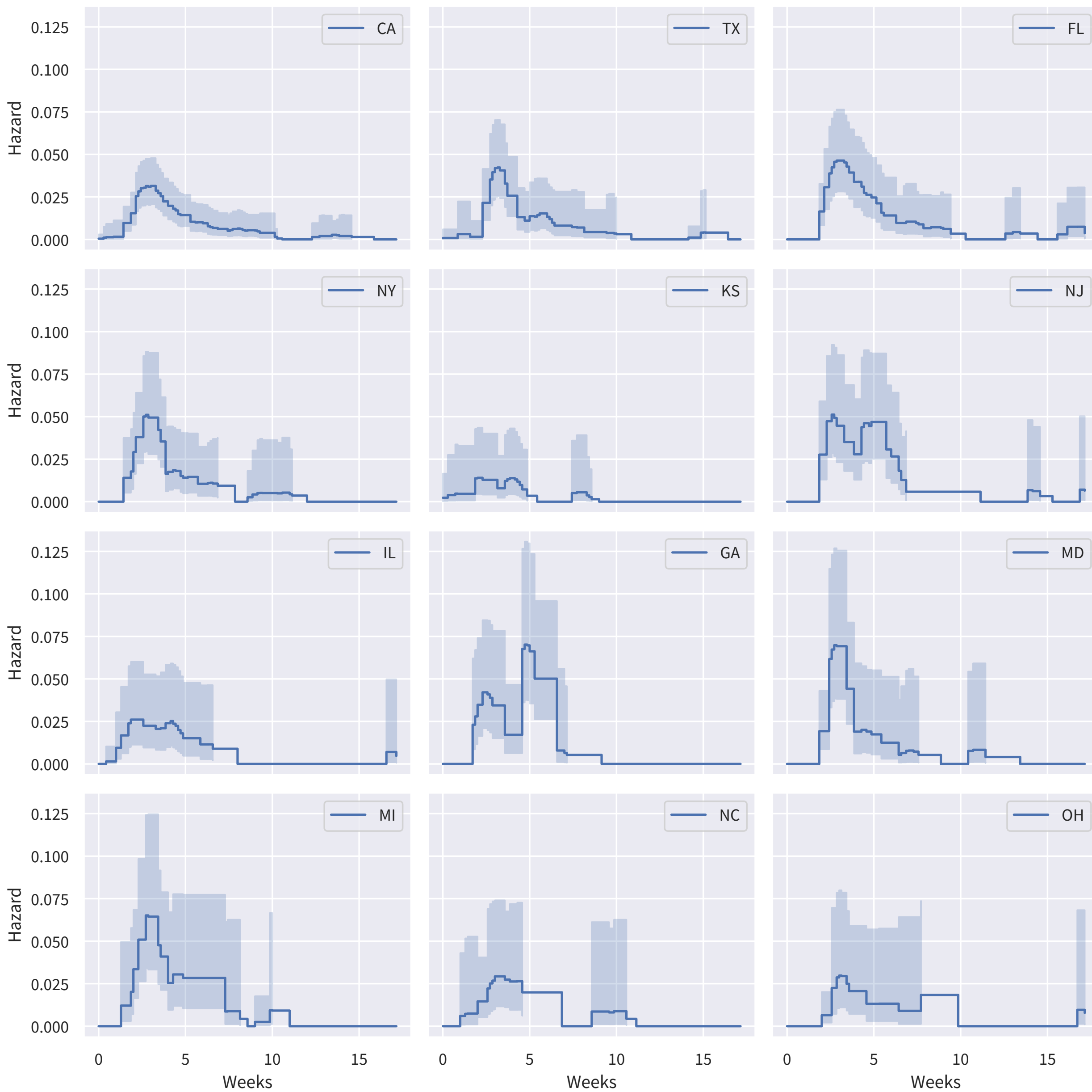
We first provide a summary of the estimates for the pooled model, in Table 8. The pooled model treats all observations as being derived from the same underlying distribution ignoring the impact of differences driven by loan clusters identified by either region or originator. The estimates serve to provide a baseline against which the results of the hierarchical model can be compared and contrasted.

0.5.2.1 Predictive distribution

We examine the posterior predictive distribution of the probability of the binary deferment outcome variable versus the mean of the observed outcome in the hold-out **test** data set. This is presented in Figure 7 where the vertical line represents the observed deferment percent while the barchart shows the distribution of posterior predicted probabilities in the sample, together with the 95% Highest Posterior Density (HPD) interval. Note that these are hazards and not unconditional probabilities.

The mean of the distribution of predicted hazards matches the average deferment rate in the sample quite well. We have also examined other standard metrics for measuring convergence for the MCMC sampler that support the the validity of the sampling results presented here but have witheld them in the interest of brevity.

Figure 6: Top 12 states: Nelson-Aalen hazards



0.5.2.2 Predicted hazard

We depict the fitted hazard and its 2 standard-deviation interval as a function of duration t in Figure 8 for loans in the test dataset. The observed hazards are also plotted. Again, when viewed in terms of emprical versus fitted hazards, the model seems to capture the hold-out data quite well, with most of the observations within the 2 standard-deviation intervals and the general pattern of duration dependence captured with the interval-specific intercepts.

0.5.2.3 Feature impact

The Average Marginal Effect (AME) of the features in the model is presented in Figure 9. The estimates are converted to basis points and represent the impact of a 1-unit change in the covariate on the weekly hazard rate. Since all our numerical covariates are standardized, the AME represents the impact of a 1 standard-deviation change in the

Table 8: Pooled model: population means

	mean	sd	hdi_3%	hdi_97%	r_hat
purpose[T.Acquisition]	0.258	0.110	0.065	0.478	1.000
purpose[T.LifeCycle]	0.150	0.131	-0.119	0.380	1.000
purpose[T.Other]	0.199	0.113	-0.015	0.425	1.001
employment_status[T.Self-employed]	0.407	0.181	0.105	0.748	1.000
employment_status[T.Other]	-0.312	0.238	-0.759	0.131	1.002
term[T.Y5]	0.194	0.091	0.028	0.372	1.000
home_ownership[T.Rent]	0.213	0.090	0.040	0.387	1.000
loanstatus[T.In Grace Period]	0.486	0.300	0.028	1.048	1.001
loanstatus[T.Late (16-30 Days)]	0.215	0.185	-0.123	0.601	1.001
loanstatus[T.Late (31-120 Days)]	0.079	0.188	-0.302	0.389	1.001
loanstatus[T.Default]	0.184	0.214	-0.229	0.595	0.999
std_fico_0	0.177	0.203	-0.221	0.592	1.001
std_fico_1	0.182	0.215	-0.256	0.588	1.001
std_fico_2	0.188	0.201	-0.232	0.553	1.001
std_original_balance_0	0.193	0.211	-0.227	0.581	1.001
std_original_balance_1	0.179	0.210	-0.212	0.622	1.001
std_original_balance_2	0.183	0.204	-0.200	0.618	1.000
std_dti_0	0.178	0.211	-0.233	0.596	1.000
std_dti_1	0.187	0.216	-0.239	0.622	1.001
std_dti_2	0.174	0.213	-0.247	0.601	1.000
std_stated_monthly_income_0	0.186	0.212	-0.241	0.606	0.999
std_stated_monthly_income_1	0.177	0.209	-0.271	0.556	1.001
std_stated_monthly_income_2	0.177	0.220	-0.225	0.633	1.001
std_age	0.002	0.062	-0.115	0.113	1.001
std_pct_ic	0.184	0.045	0.097	0.265	1.000

variable. In the case of categorical covariates, the AME measures the probability impact of a specific level versus the reference or “baseline” category for that variable.

The AME is calculated as follows:

$$\Delta P(y|X\beta) = \beta \exp(X\beta) \tag{1}$$

This is an N element vector where N is the number of rows in the dataset. The β coefficient is multiplied by the average of the term in square brackets to derive the AME for the covariate. For reference, the hazard is specified as $P(y|X\beta) = \exp(X\beta)$.

0.5.3 Hierarchical

We now present the results of the multi-level model, where loans are grouped into nested clusters of originators within states. Treating all loans without regard to the state they belong to papers over the regional nature of this crisis. In addition, modeling all loans within a state as the same without regard to the originator/servicer may also be misleading in that it may average any differences in the approach different originators may be taking with respect to the treatment of deferment requests. Further technical details are provided in the appendix.

0.5.3.1 Predictive distribution

In Figure 10, we present the Posterior Predictive Density for the hierarchical model. The fitted distribution of the outcome variable and the 95% interval is presented versus the mean of the observed outcome variable.

Figure 7: Pooled model: posterior predictive distribution

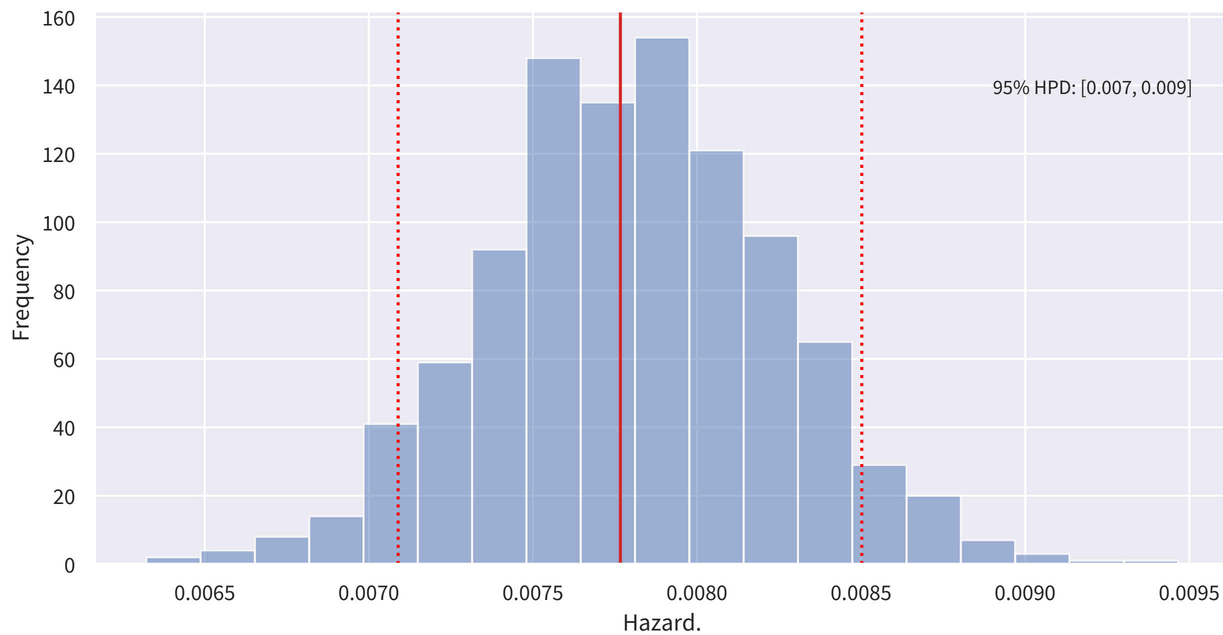
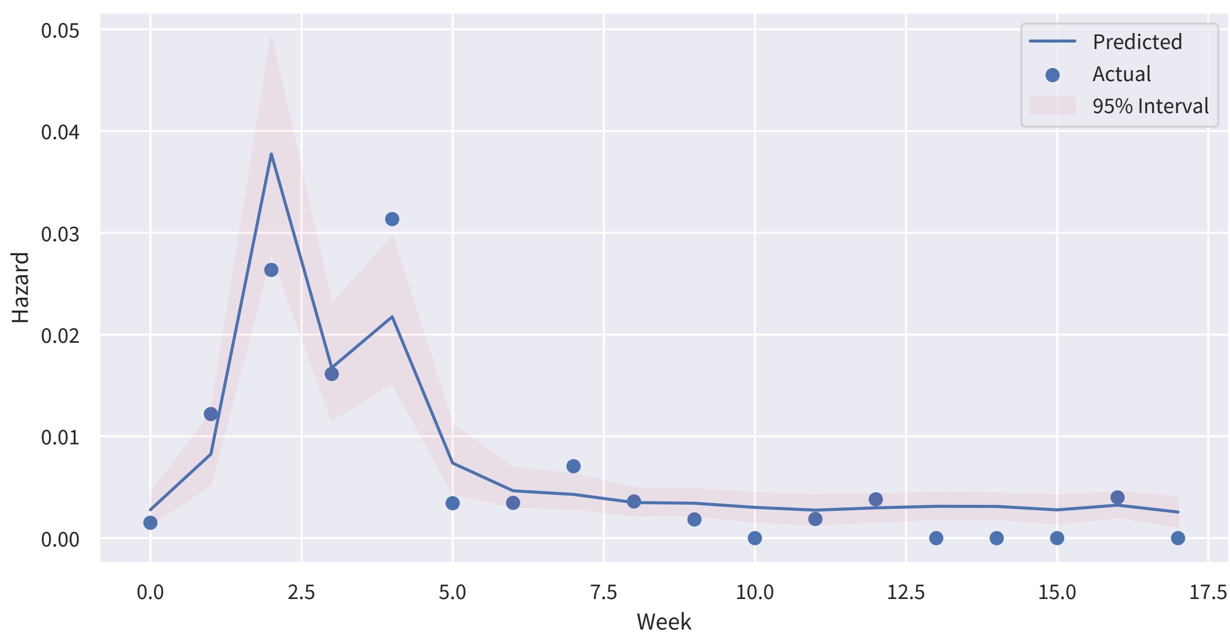


Figure 8: Pooled model: fitted hazard



0.5.3.2 Predicted hazard

We depict the fitted hazard and its 95% credibility interval as a function of duration t in Figure 11 for loans in the test dataset. The observed hazards are also plotted. The hierarchical estimates also model the variability in the test data quite well, with the average deferment rate well within the distribution of predicted hazards in the validation set.

The wider range of predicted outcomes is also evident in Figure 11 in that the 95% credibility interval is wider than what we observed in the model where the data were pooled across states and originators.

Figure 9: Pooled Model: Average Marginal Effects

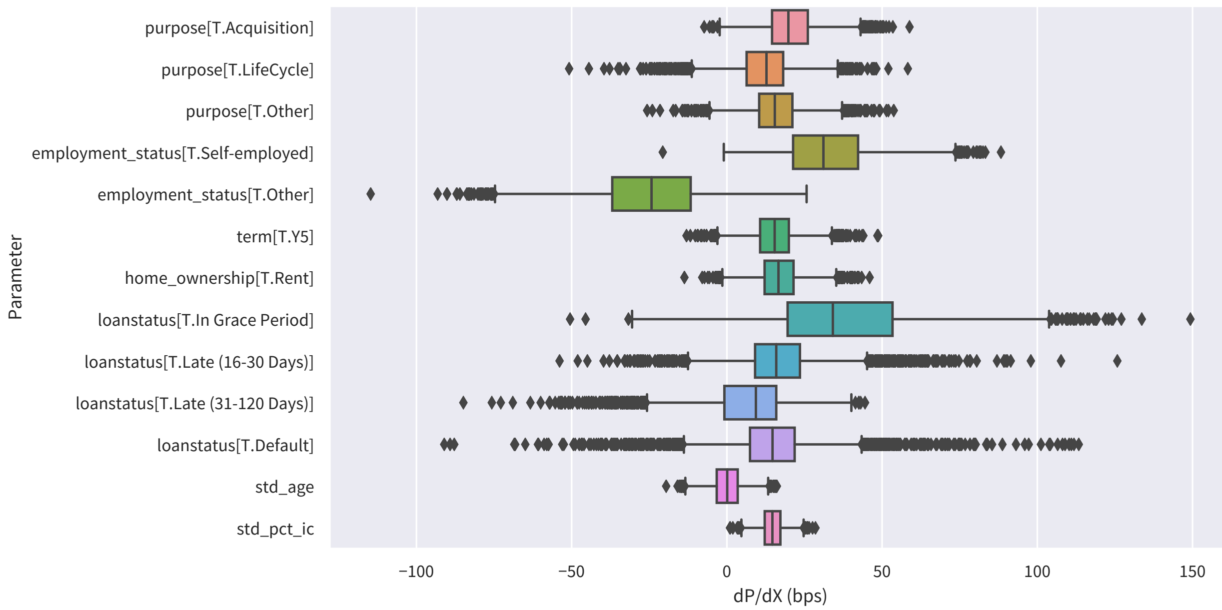
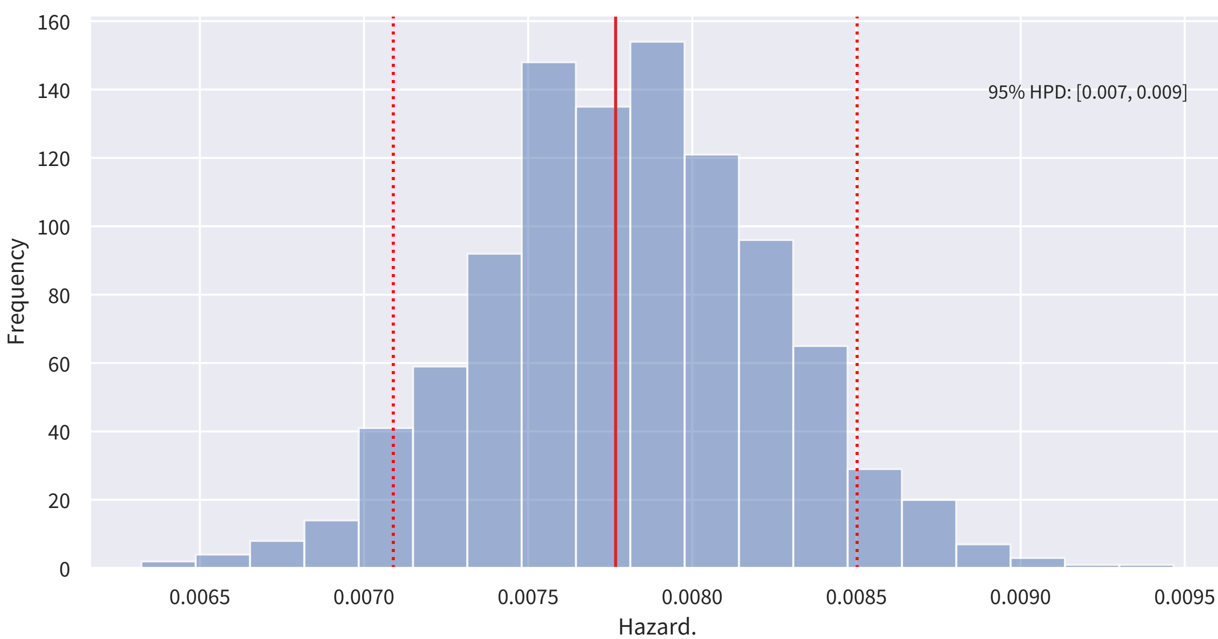


Figure 10: Hierarchical model: posterior predictive distribution



0.5.3.3 Feature impact

The fixed-effect estimates are presented in Table 9.

Average Marginal Effects for the hierarchical model are presented in Figure 12. The variation in the the impact of claims across states is presented in Figure 14.

The uncertainty around the impact of claims on deferment rates is visually presented in Figure 13.

Figure 11: Hierarchical model: fitted hazard

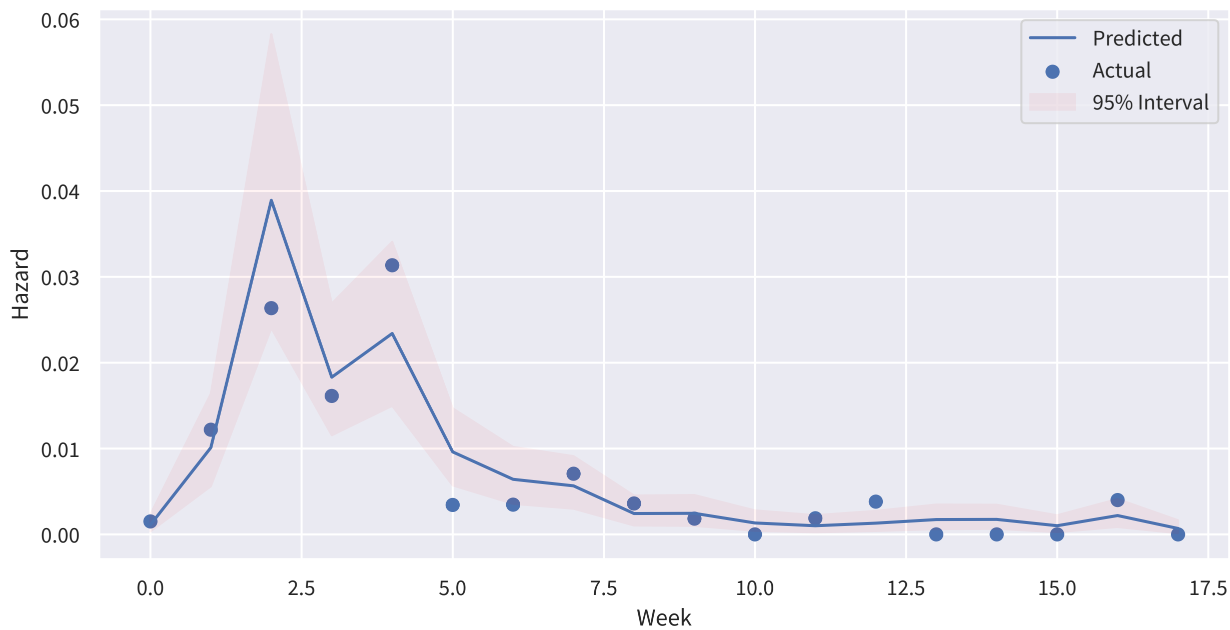
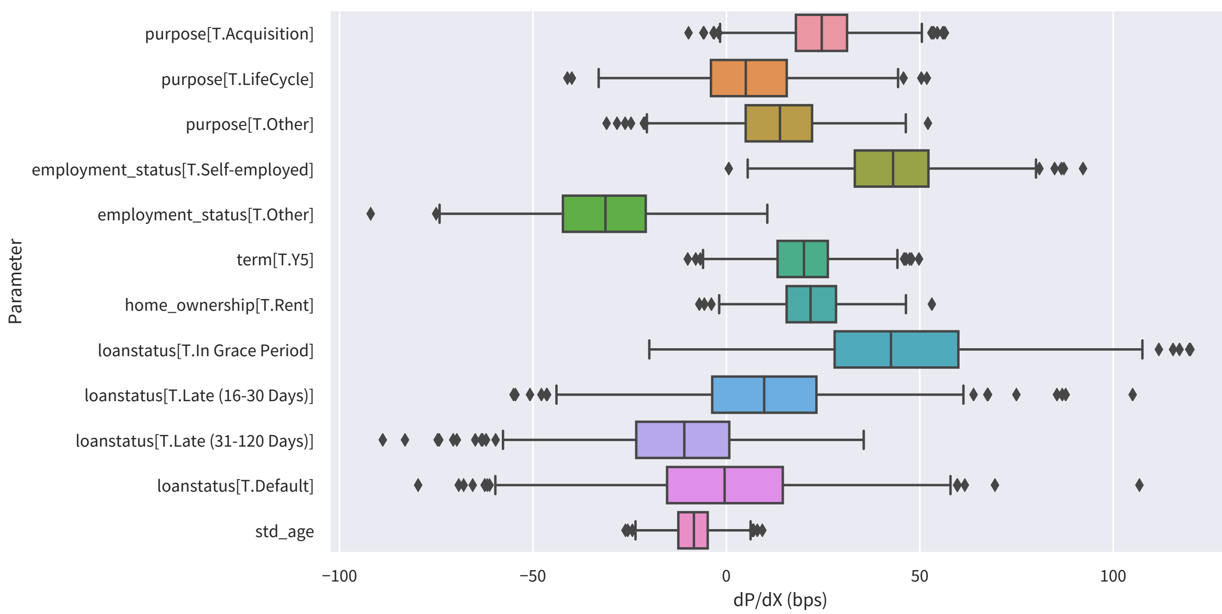


Figure 12: Hierarchical model: Average Marginal Effects (population)



0.5.4 Originator effect

In this section, we provide estimates of the effect that the identity of the originator on the hazards. These are provided in Table 10.

Confirming the results of the non-parametric Aalen fits presented earlier, the first originator has a higher estimated baseline hazard than the second, as evidenced by its larger γ parameter.

Table 9: Hierarchical model: population means

	mean	sd	hdi_3%	hdi_97%	r_hat
purpose[T.Acquisition]	0.226	0.172	-0.097	0.561	1.000
purpose[T.LifeCycle]	0.048	0.191	-0.343	0.392	1.000
purpose[T.Other]	0.122	0.180	-0.219	0.454	1.001
employment_status[T.Self-employed]	0.322	0.213	-0.087	0.710	1.003
employment_status[T.Other]	-0.253	0.207	-0.615	0.170	1.001
term[T.Y5]	0.176	0.171	-0.150	0.484	1.002
home_ownership[T.Rent]	0.202	0.163	-0.108	0.495	1.000
loanstatus[T.In Grace Period]	0.294	0.230	-0.145	0.728	1.001
loanstatus[T.Late (16-30 Days)]	0.085	0.227	-0.335	0.508	1.000
loanstatus[T.Late (31-120 Days)]	-0.105	0.212	-0.508	0.287	1.002
loanstatus[T.Default]	-0.005	0.251	-0.463	0.475	1.002
std_fico_0	-0.016	0.251	-0.461	0.476	1.000
std_fico_1	-0.001	0.252	-0.480	0.454	1.003
std_fico_2	0.003	0.252	-0.456	0.466	1.001
std_original_balance_0	0.022	0.255	-0.438	0.504	1.000
std_original_balance_1	-0.012	0.249	-0.497	0.447	1.003
std_original_balance_2	-0.005	0.247	-0.473	0.446	1.000
std_dti_0	-0.001	0.254	-0.465	0.472	1.001
std_dti_1	0.005	0.260	-0.477	0.505	1.001
std_dti_2	-0.008	0.245	-0.461	0.452	1.001
std_stated_monthly_income_0	0.006	0.260	-0.456	0.511	1.001
std_stated_monthly_income_1	-0.013	0.253	-0.473	0.467	1.002
std_stated_monthly_income_2	-0.004	0.248	-0.477	0.456	1.000
std_age	-0.083	0.139	-0.336	0.200	1.001

Table 10: Originator effect

	mean	sd	hdi_3%	hdi_97%	r_hat
γ_μ[0]	-0.643	0.637	-1.782	0.645	1.000
γ_μ[1]	-0.853	0.698	-2.068	0.525	1.001
γ_σ[0]	0.726	0.547	0.000	1.702	1.000
γ_σ[1]	0.784	0.580	0.000	1.809	1.000
γ[0]	-0.963	0.294	-1.502	-0.413	1.001
γ[1]	-1.294	0.303	-1.852	-0.727	1.000

0.6 Forecasts

In this section, we take the test portfolio and project the deferment hazards forward until the end of the third quarter. We do this by using the estimated hazard model and feed into it the claims forecast we generated in the previous section. The claims forecast for each state is generated by taking the US decay factor projection and applying it to the peak claims figure for the state. The projections are depicted in Figure 15. The cumulative amount of loans in deferment between the start of the projections (the last observed date for the data collection) and the end of the projection period is shown in the first panel of the figure. Note that a basic assumption underlying our analysis everywhere is the absence of loans “curing” from deferment — we prefer to handle that separately rather than in a unified but substantially more complex model with different states.

Figure 13: Uncertainty in claims impact

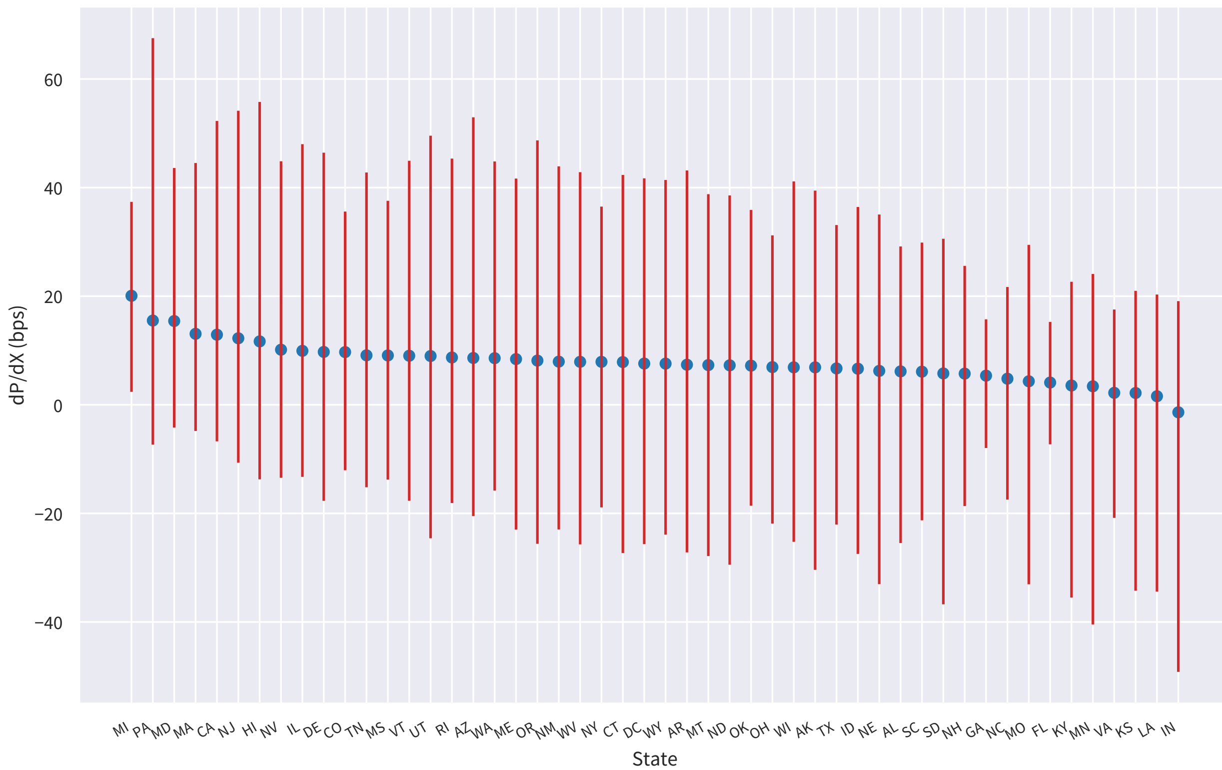


Figure 14: Claims sensitivity

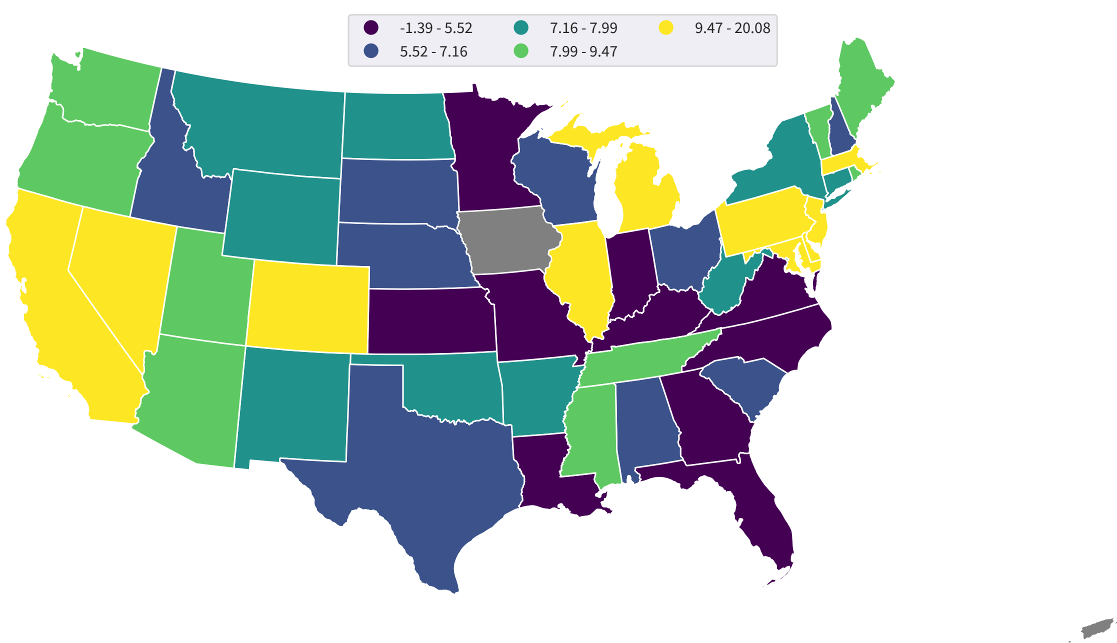


Figure 15: Deferment hazards

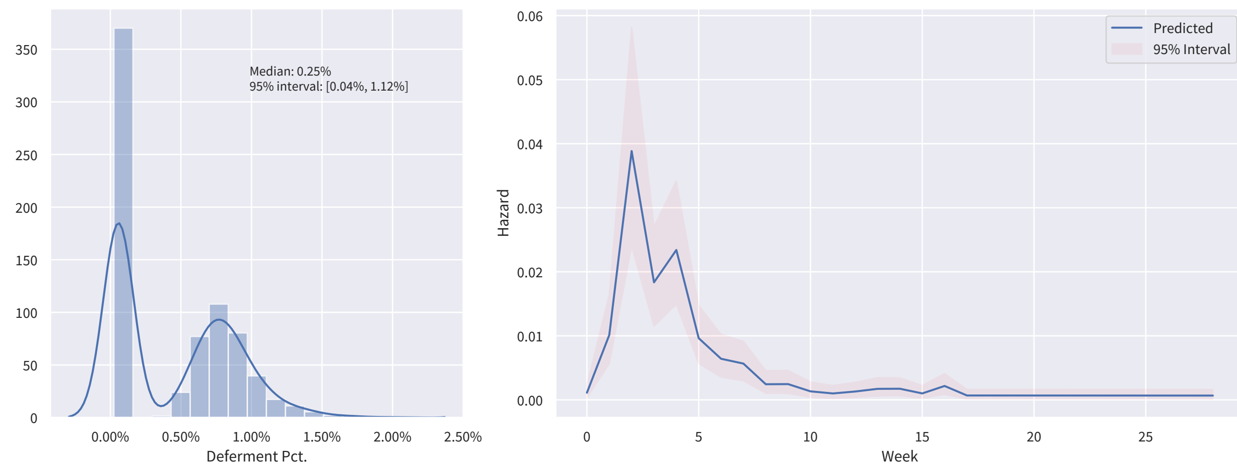


Table 11: Leave-One-Out (LOO) comparison

	rank	loo	p_loo	d_loo	weight	se	dse	warning	loo_scale
hierarchical	0	-1612.76	31.27	0.00	1.00	78.87	0.00	False	log
pooled	1	-1636.78	15.42	24.02	0.00	79.00	5.99	False	log

0.7 Comparison

In this section, we examine a basic question — given the additional complexity of mixed (or multi-level) models and the bayesian estimation framework, is any of it worth it? We demonstrate the utility of the approach adopted here by asking an important question, one of many that can be asked.

We approach the model comparison exercise in two ways. First, we use Leave-One-Out (LOO) cross validation to derive an estimate of out-of-sample fit for both models. Second, we use a more traditional probability calibration framework to compare predicted and observed survival outcomes at a defined event-time.

0.7.1 Leave-One-Out (LOO)

In the LOO comparison framework, the training dataset is repeatedly partitioned into training and testing sets by leaving a single observation out, using an efficient algorithm developed by [Vehtari et al.(2017)Vehtari, Gelman, and Gabry] that uses posterior samples from the MCMC algorithm. The computation uses Pareto-smoothed Importance Sampling (PSIS) to calculate point-wise out-of-sample prediction accuracy. These results are provided in Table 11.

0.7.2 Probability calibration

Another important real-world justification for any probabilistic model, be it binary or otherwise is its ability to generate probability predictions that match the observed frequency (or rate) of the outcomes under study. In our case, given a fixed time horizon say t_0, we would like to compare the predicted probability of deferment at that horizon versus the actual rates observed. This is what is depicted in Figure 17.

The framework used is laid out in [Peter C. Austin(2020)]. Each (x, y) point in the figure represents a mapping between the predicted probability of the outcome at t_0 versus the smoothed observed frequency of the outcome. An unresolved aspect of this framework being applied in a Bayesian context (at least in my mind) is the interaction between the "shrinkage" inherent in a mixed-model framewrk where cluster means are shrunk towards the grand mean depending on the strength of the data, and the requirement of fealty to the empirical estimates — the 2 approaches seem to be working at cross-purposes. We cannot resolve this here and leave this as a topic for future research. **Regardless, both PSIS-LOO and the ICI comparisons seem to provide a definitive thumbs-up for the**

hierarchical model, especially given that the comparison was performed using an out-of-sample dataset in the latter exercise.

Figure 16: Probability calibration: Deciles

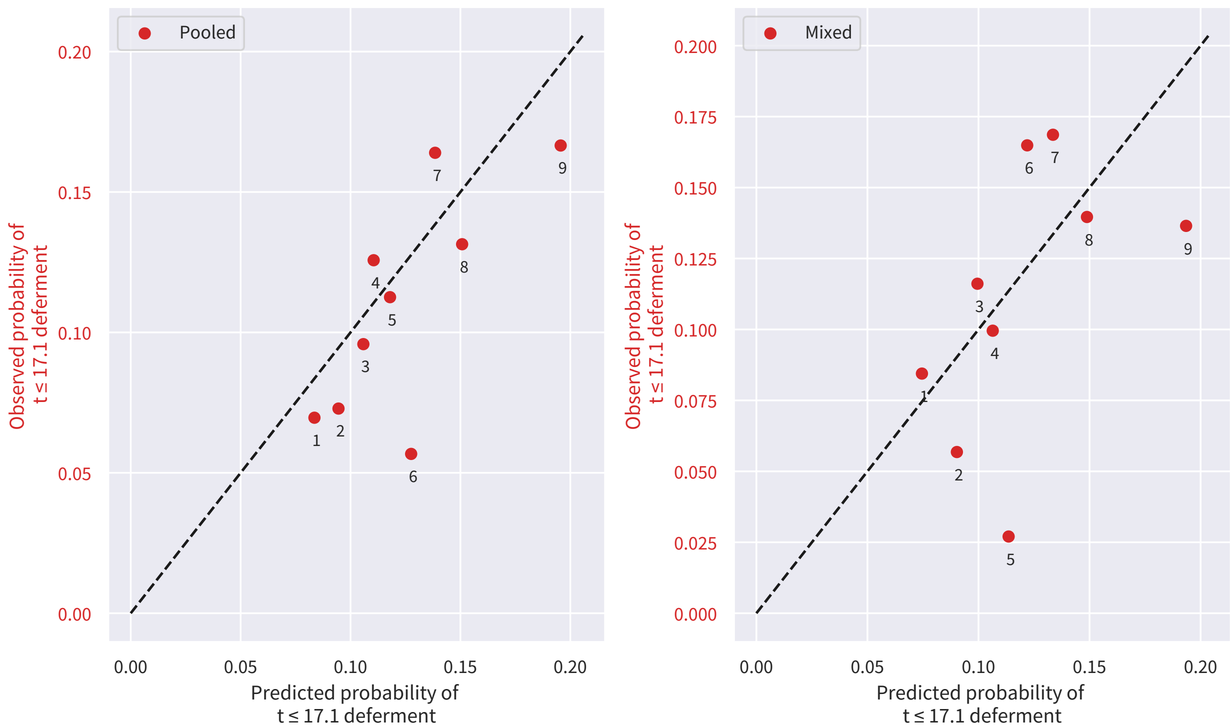
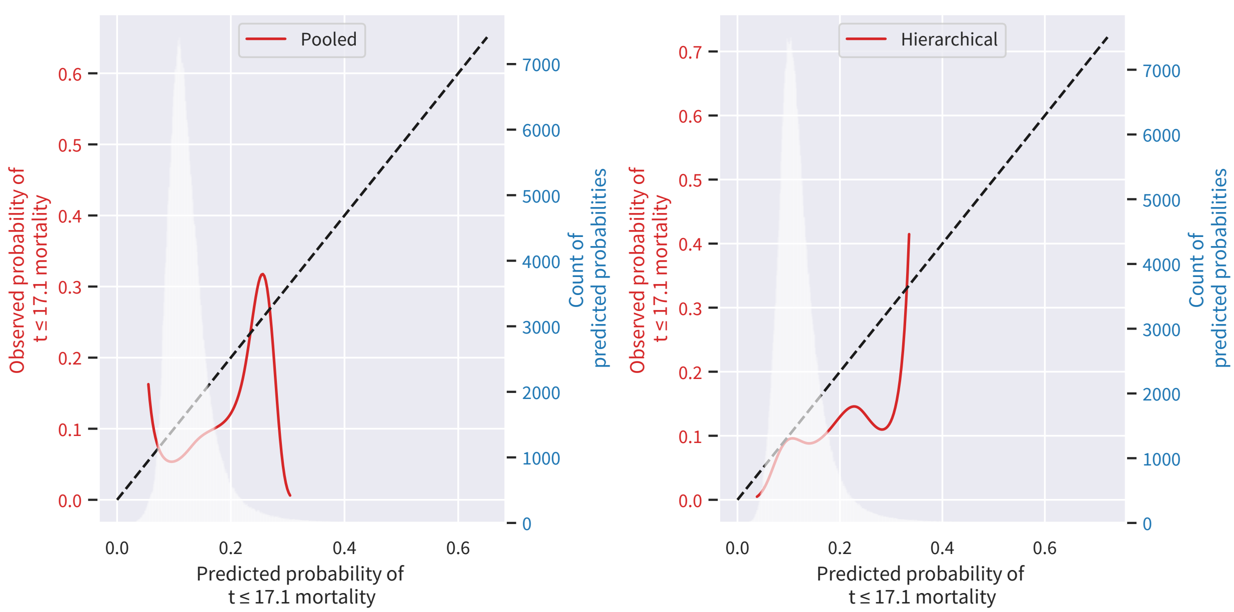


Figure 17: Probability calibration: Integrated Calibration Index (ICI)



A perfectly calibrated model would have all pairs of points line up with the 45-degree line. The Integrated Calibration Index (ICI) for the two models, is the mean of the absolute value of the differences between predicted probabilities and observed rates.

Table 12: Integrated Calibration Index

	Pooled	Hierarchical
E50	0.018398	0.012843
E95	0.094922	0.078902
Mean	0.026385	0.021013

$$ICI = \frac{1}{N}\sum |P^o_{t_0} - P^p_{t_0}| \tag{2}$$

We also report the the median and 95th percentiles of the absolute differences for each observation in the "test" dataset. The hierarchical model is unambibuously better than the pooled model in terms of the ICI, the median of the absolute differences and the predictions in the tail of the probability distribution of outcomes, offering substantial improvement over the pooled model.

Taken together, the two model comparison frameworks support our view that there is potential for meaningful improvement in forecasting ability using a mixed models approach relative to a more traditional approach which pools the data across states and originators.

0.8 Conclusion

Our goals in this report were two-fold. First, we wanted to set out a rigorous and transparent statistical framework for analyzing and forecasting deferment rates due to the COVID shock to the economy. While the data in the study are from the consumer loan sector, the framework established here has general applicability to a much broader range of assets. Second, we wanted to establish quantitative bounds around what we should expect for deferment rates over the next quarter. An important derivative work to this report is the development of a cashflow engine and valuation model to incorporate these assessments into loan pricing. A key element of that analysis will be a determination of the "cure" rate from deferment — a complicated excercse in its own right. For example, Groshen [Groshen(2020)] writes that

> *…83 percent of the increase in unemployment in March come*[sic] *from workers on temporary furloughs, not permanent layoffs.*

This may be significant in determining whether we are more likely to have a speedier recovery rather than a more traditional, drawn out slog from a classical recession. We see hints of this in our data as well — for example, in the vast majority of cases, borrowers have cited "Curtailment of Income" as the reason for seeking hardship deferments. On the other hand, if the nature of the employment shock turns into a more permanent layoffs instead of temporary furloughs, the outlook could be more dire.

These aspects need to be explored in greater detail and we leave these considerations for another paper, preferring to deal with our problem in manageable bite-sized chunks. Regardless, the forecasts presented in this report should still serve to provide a lower bound on valuation if one is willing to assume that all borrowers granted deferrals are likely to simply go delinquent when their deferment term expires.

0.9 Appendix

0.9.1 Model

The modeling framework used in this report draws upon statistical tools used in the analysis of events with a "time-until" component to them. In our case, the time-until, or "lifetime" we are interested in predicting is the time until a borrower asks the servicer for a deferment or goes delinquent. Time in this context is measured from an assumed epoch start date of March 14th, 2020 which we assume as the start of the COVID-19 crisis for our purposes and is the same across all loans.

0.9.1.1 Pooled

We first describe the general setup of the model, assuming a "pooled" setup where the concept of clusters and correlation of outcomes for loans within clusters is initially ignored. This allows us to provide a basic framework which is then extended to the hierarchical setting. The exposition here is based on and closely follows the excellent series of lecture notes by [Rodríguez(2007)].

Let the time-to-event T be a continuous variable with Cumulative Distribution Function (CDF), $F(t)$ and density $f(t)$. $F(t)$ defines $P(T < t)$ or the cumulative probability that the lifetime lasts until duration t. An alternative representation more commonly employed in the survival literature is the "Survival Function" $S(t)$ which is the complement of the CDF:

$$S(t) = P(T >= t) = 1 - F(t) = \int_t^\infty f(x)\,\mathrm{d}x. \tag{3}$$

The density function of lifetimes can be represented as $F'(t)$, the derivative of the CDF.

An alternative characterization of the distribution of T is given by the hazard function, or instantaneous rate of occurrence of the event, defined as:

$$\lambda(t) = \lim_{\mathrm{d}t \to 0} \frac{P(t \leq T < t + \mathrm{d}t | T \geq t)}{\mathrm{d}t} \tag{4}$$

The numerator in equation 4 is the conditional probability of an event ocurring in the interval $t, t + \mathrm{d}t)$, conditional on it not having occured before t. The denominator is the length of the time interval, which means the hazard represents a rate of ocurrence. In the limit, as $\mathrm{d}t$ goes to 0 — the hazard represents the instantaneous rate of occurence of the event.

The conditional probability in the numerator may be written as the ratio of the joint probability that T is in the interval $[t, t + \mathrm{d}t)$ and $T \geq t$ (which is, of course, the same as the probability that t is in the interval), to the probability that $T \geq t$. The former may be written as $f(t)\mathrm{d}t$ for small $\mathrm{d}t$, while the latter is $S(t)$ by definition. Dividing by $\mathrm{d}t$ and passing to the limit gives the useful result:

$$\lambda(t) = \frac{f(t)}{S(t)} \tag{5}$$

Since $-f(t)$ is the derivative of $S(t)$, equation 5 can also be written as:

$$\lambda(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln(S(t)) \tag{6}$$

Integrating from 0 to t and imposing the known boundary condition $S(0) = 1$ (since by construction, the event cannot have occurred at time 0), we obtain the following expression:

$$S(t) = \exp\left\{ -\int_0^t \lambda(x)\mathrm{d}x \right\} \tag{7}$$

The integration term in equation 7 is referred to as the *cumulative hazard* and is denoted

$$\Lambda(t) = \int_0^t \lambda(x)\mathrm{d}x \tag{8}$$

One can think of the cumulative hazard as the total risk of the event happening from the time of entry into the state to time t.

In the simplest case, the hazard is constant over time spent in the state, i.e., $\lambda(t) = \lambda$ for all t. The corresponding survival function is $S(t) = \exp(-\lambda t)$ which is an exponential distribution with parameter λ. The density is obtained as the product of the hazard and the survivor function, or:

$$f(t) = \lambda \exp\{-\lambda t\} \tag{9}$$

the mean of which is $1/\lambda$.

As an aside, a few interest points are worth noting here:

- There is a close connection between industry practice around the modeling of cash flows on loans and hazard rates. There are two operative concepts here - SMM and MDR which stand for Single Monthly Mortality and Monthly Default Rate respectively.
- Both represent unscheduled principal that is paid or written down, as a fraction of the balance outstanding. In this respect, the equivalence to the hazard function is clear - the unscheduled principal paid at time t is the density $f(t)$ while the balance still outstanding is $S(t)$.
- Typically these concepts are applied to groups of loans and aggregations of principal paid and balance outstanding, but the analytical apparatus of hazard rates is directly application at each individual loan level.

In analyzing cash flows, we are interested in events that can lead to a change in state and change the profile of scheduled cashflows. Once a loan is created, its lifetime (and/or its state) can change or be terminated through a multiplicity of ways. These events can be modeled using **cause-specific hazard functions**. The cause-specific hazard function for the jth failure type is:

$$\lambda_j(t) = \lim_{\mathrm{d}t \to 0} \frac{P(t \le T < t + \mathrm{d}t, J = j | T \ge t)}{\mathrm{d}t} \tag{10}$$

The total hazard of leaving the state can be written as the sum of cause-specific hazards, $\lambda(t, x) = \sum_{j=1}^{J} \lambda_j(t, x)$. Here, x represents covariates that impact the cause-specific hazards in a systematic way and allows us to write the hazard as:

$$\lambda_j(t) = \lambda_{0,j} \exp\{x_t \beta\} \tag{11}$$

where $\lambda_{0,j}$ represents an unspecified baseline hazard.

The likelihood function for this model can be derived as follows. Suppose we have observations t_i on N lifetimes of interest (e.g., time to a deferral request) and we also have, for each loan, an indicator variable d_i which is 1 if t_i denotes an observed time to a deferral request, and 0 otherwise. In the latter situation, the observation is considered “censored”.

The likelihood for an observation that is known to have deferred at time t_i can be written as the product of the survivor and hazard functions:

$$L_i = f(t_i) = S(t_i)\lambda(t_i). \tag{12}$$

If we consider a censored observation, we have:

$$L_i = S(t_i), \tag{13}$$

Since we have both types of observations, we know all observations lived at least until time t_i, but an observed deferral time with $d_i = 1$ implies we need to multiply the survivor funtion by the hazard at time t_i. The combined likelihood across all observations is:

$$L = \prod_{i=1}^{n} L_i = \prod_i \lambda(t_i)^{d_i} S(t_i). \tag{14}$$

Taking logs and simplifying, we have:

$$\log L = \sum_{i=1}^{n} \{d_i \log \lambda(t_i) - \Lambda(t_i)\}. \tag{15}$$

0.9.1.2 Piecewise-exponential

In our specific case, we know the exact day on which the borrower's deferment request was approved, or they were declared delinquent. Thus, we can take advantage of this extra bit of information by treating time t_i as continous, but in a setup where the observed times are bucketed into discrete intervals. This also allows us to accomodate covariates that may change over the course of time t_i but are external to the process and known at the start of each interval. Ths is referred to as Piece-Wise Exponential (PWE) for the underlying baseline hazard λ_t in the literature.

Under this specification, the time t_i can be grouped into J intervals, $[\tau_0, \tau_1, \tau_{j-1}, \tau_j, \ldots, \tau_J]$. If $t_i > \tau_j$, then the time spent in the interval is simply $\tau_j - \tau_{j-1}$. If the individual experienced an event or was censored in the interval j, then the time spent in that interval is $t_{ij} = t_i - \tau_{j-1}$. For each of these intervals, we can create a binary variable $d_{ij} = 0/1$ depending on whether the loan survived that interval and went into the next interval.

The first term in the log-likelihood can now be written as:

$$d_i \log \lambda_i(t_i) = d_{ij(i)} \log \lambda_{ij(i)}, \tag{16}$$

The second term in the log-likelihood can be expressed as:

$$\Lambda_i(t_i) = \int_0^{t_i} \lambda_i(t)dt = \sum_{j=1}^{j(i)} t_{ij}\lambda_{ij}, \tag{17}$$

Here, we rely on the fact that the cumulative hazard integral is composed of $j(i)$ terms and can be represented as the sum of these "mini" integrals. Each interval contributes the hazard λ_i multiplied by the time spent in the interval, which is 1 for all except the last where it is t_{ij}. Since the d_{ij} is zero for all intervals except the last, we can consolidate both terms in the log-likelihood under one summation:

$$\log L_i = \sum_{j=1}^{j(i)} \{d_{ij} \log \lambda_{ij} - t_{ij}\lambda_{ij}\}. \tag{18}$$

The contribution of each interval, which is a sub-observation for each loan, to the log-likelihood, is equivalent to the d_{ij} being drawn from a Poisson distribution with mean $\mu_{ij} = t_{ij}\lambda_{ij}$. The likelihood for this would be written as:

$$\log L_{ij} = d_{ij} \log \mu_{ij} - \mu_{ij} = d_{ij} \log(t_{ij}\lambda_{ij}) - t_{ij}\lambda_{ij}. \tag{19}$$

If we substitute $t_{ij}\lambda_{ij}$ for μ_{ij}, we get

$$\log L_{ij} = d_{ij} \log \mu_{ij} - \mu_{ij} = d_{ij} \log(t_{ij}\lambda_{ij}) - t_{ij}\lambda_{ij}. \tag{20}$$

This is the same as the previous formulation, with the exception of the $d_{ij} \log(t_{ij})$ term — this is only valid for the last interval and so contributes $\log(t_{ij})$ to the log-likelihood. This is equivalent to a Poisson model with an offset term equal to $\log(t_{ij})$. This is the final model specification we use in this report.

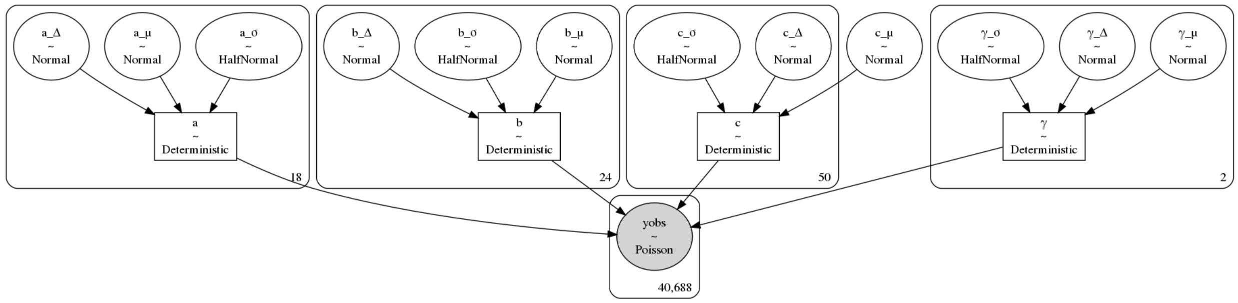
0.9.1.3 Hierarchical

In order to extend this to a hierarchical bayesian framework, we make the additional assumption that the terms in β are cluster-specific. In other words, the impact of these variables on the deferment hazard is different for loans in different clusters. We treat each geographic region, more specifically, a state, as a cluster, and within each state, each originator as a sub-cluster. Thus, there are three levels in the model, with the bottom-most level being the repeated observations for the individual loan, which are grouped within originator which is itself grouped within states.

All loans from the same state have a close relationship to each other, which may be the result of state-specific laws, regulations and customs. In the case of COVID-19, given the state-specific nature of restrictions, the grouping is a natural one to consider. In addition, different policies related to how deferment requests are handled may also lead to differences in how observationally similar loans respond to the same stimuli.

The schematic for the hierarchical model (incorporating additional tweaks to aid in the sampling) is presented in Figure 18. The "hierarchical" nature of the parameters is clearly drawn as one moves from the top to the bottom of the graph, with the final step being the simulation from the posteriors using the observed value of the dichotomous deferment variable as the observed outcome.

Figure 18: Hierarchical bayes model



In our model, the impact of the year-over-year percentage change in weekly initial claims is treated this way.

$$\begin{aligned} c_{ij} &\sim \mathcal{N}(\mu_c^j, \sigma_c^j) \\ \mu_c^j &\sim \mathcal{N}(\mu_c, \sigma_c) \end{aligned} \tag{21}$$

By now, we have the basic outlines of the task at hand. We have a portfolio of loans, and on some of these, borrowers have asked the servicer for a deferral of payment. This is the result of an economic shock generated by the widespread shutdowns in response to the COVID-19 virus. We want to understand how the probability of a borrower requesting a deferral depends on the attributes of the borrower and the loan, as well as state level metrics of labor market deferment.

The components of the x matrices encapsulate follows:

- loan-level: FICO, original balance, Debt-To-Income (DTI) ratio, stated monthly income, loan age. In addition, we have categorical variables for loan credit grade, purpose, employment status, and original term.
- state-level: Year-over-year percentage change in weekly claims (reported as of the Saturday of the week). Since the weekly intervals in which t is observed coincides with the claims reporting period, we are assuming that these are observed contemporaneously. In other word, the trigger for the decision to file a claim and ask for a deferment occurred sometime in the past, but both are recorded within the same week. The parameter measuring the impact of weekly claims for each loan is state-specific.

All right-hand-side variables are standardized by subtracting the mean and dividing by the standard deviation. In addition, all the categorical variables are encoded as dummmy variables with the first level of the category being treated as the reference category. This simplifies the interpretation of the model coefficients. The intercept term for each state can then be used to compute the probability of deferment for a "reference" loan.

In our case, this loan can be defined as a **Grade 1** loan, with the following additional characterstics: **FICO:** 700.80, **Original Balance:** $17,227.14, **DTI:** 20.51%, **Monthly Income:** $7,641.74, **Age:** 19.15 months.

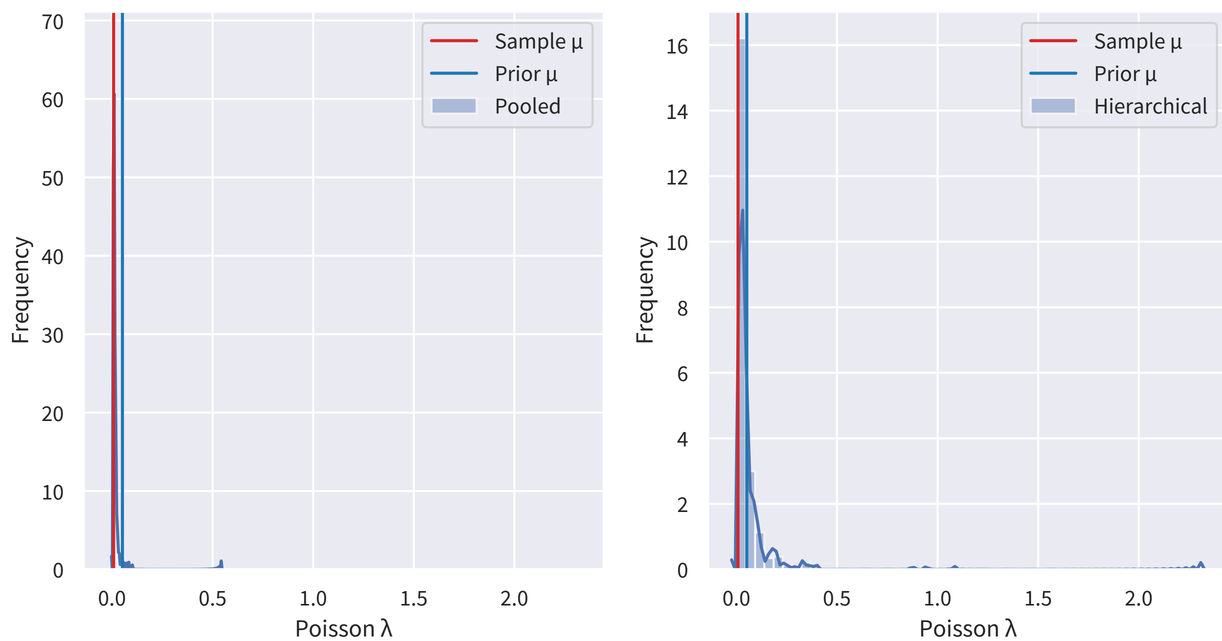
It is further assumed that the borrower used the loan proceeds for **consolidating debt**, that the borrower was either **employed** at the time of loan application or employment history information was available at the application date, the **amortization term** of the loan was 3 years, and that the borrower **owned** their home. In effect, for a loan that matches these attributes, the x term is a zero matrix, leaving the intercept as the only term on the right-hand side.

As a final iteration on the basic model, we have an originator-specific "frailty" term that multiplies the hazard for each observation, scaling it up or down. This is the γ term in the hierarchical model illustration.

0.9.2 Priors

A combination of "generic weakly informative priors" (to use the language employed in the STAN Wiki) and domain knowledge guided our choice of priors for both models. In particular, different values of the variance priors generated very different shapes for the prior-predictive distribution of the outcome variable. Our knowledge about processing capacity for deferment requests at the servicers allowed us to think of thresholds above which the predictive distribution tail values did not make sense.

Figure 19: Prior Predictive Distributions



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