A Strongly-coupled Non-parametric Integral Boundary Layer Method for Aerodynamic Analysis with Free Transition

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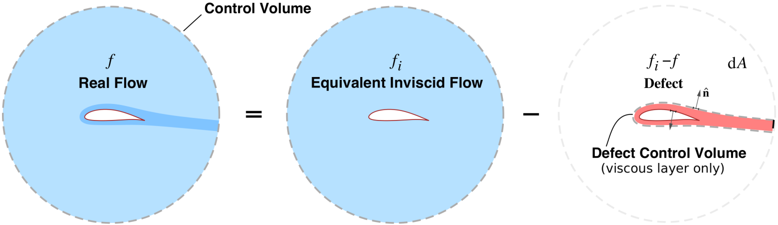
Motivation for Viscous-inviscid Interaction

"Divide & Conquer": viscous-inviscid zonal decomposition



Motivation for Viscous-inviscid Interaction

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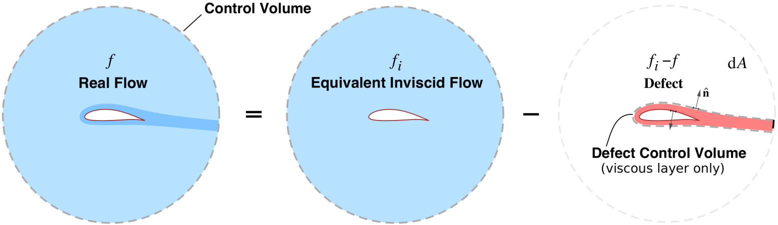
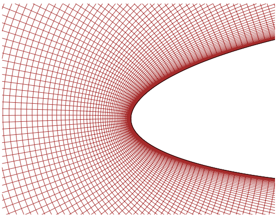
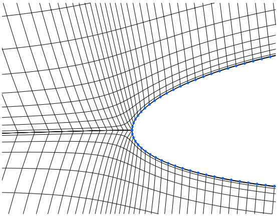




Navier-Stokes

DOF $\sim$ 500,000

runtime $\sim$ hours



Full potential+Integal BL

DOF $\sim$ 5,000

runtime $\sim$ seconds

Motivation for Coupled IBL

Integral Boundary Layer (IBL) + Inviscid = Coupled IBL

- \+ Faster than alternative viscous flow solvers (e.g. coupled BL, RANS)
- \+ Conducive to various inviscid flow solvers
- \+ Compatible with structural formulations (via virtual displacement)
- \+ Convenient for flow transition modeling (vs. RANS)

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Fidelity	Moderate	High(er)
Method	Coupled IBL	RANS, LES, ...
Example	2D: XFOIL, MSES Quasi-2D: TRANAIR	Industry/Research codes 2D/3D

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⇒ **Goal**: extend coupled IBL methods to general 3D application

Challenges for 3D IBL Development

Applicability for General 3D Configurations

Solution: Non-parametric IBL using Discontinuous Galerkin (DG) FEM

- Replace explicit curvilinear coordinates with local Cartesian basis

Robustness in Viscous-inviscid Coupling

Solution: Strong coupling via simultaneous solution

- Use global Newton solver & flexible coupling interface

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Reliability of Physical Modeling & Numerical Solution

Ongoing: Turbulence modeling, transition modeling, 3D flow effects etc.

Current focus: numerical treatment of free transition for DG-IBL

- Free transition introduces a solution-dependent ("moving") interface
- ... poses challenges in numerical discretization & nonlinear solution
- This talk uses the example of 2D steady-state incompressible flow

Outline

- Strongly-coupled IBL Formulation
- Numerical Discretization
- Nonlinear Solution
- Numerical Results

2D Coupled IBL Formulation

IBL equations (variable $Q_{\mathrm{IBL}} = \{\delta, \mathcal{A}\}$): (no transition yet)

$$\widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla} \mathbf{q}_{\mathrm{e}} - \boldsymbol{\tau}_{\mathrm{w}} = \mathbf{0}, \qquad \widetilde{\nabla} \cdot \mathbf{K} + \mathbf{D} \cdot \widetilde{\nabla} q_{\mathrm{e}}^2 - 2\mathcal{D} = 0$$

where $\mathbf{q}_{\mathrm{e}}$ depends on equivalent inviscid flow

Inviscid equation (variable Q_{inv}): e.g. incompressible potential flow

$$\nabla^2 \Phi = 0 \quad \text{(PDE)} \qquad \text{subject to} \qquad (\rho_{\mathrm{i}}\, \mathbf{q}_{\mathrm{i}})_{\mathrm{w}} \cdot \hat{\mathbf{n}}_{\mathrm{w}} = \Lambda \quad \text{(BC)}$$

where flow velocity $\mathbf{q}_{\mathrm{i}} \equiv \nabla \Phi$, and Λ depends on viscous layer

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Auxiliary viscous equation (variable: edge velocity $\mathbf{q}_{\mathrm{e}}$):

$$\mathbf{q}_{\mathrm{e}} - \mathbf{q}_{\mathrm{i}}(Q_{\mathrm{inv}}) = 0$$

Auxiliary inviscid equation (variable: transpiration source Λ):

$$\Lambda - \widetilde{\nabla} \cdot \mathbf{M}(Q_{\mathrm{IBL}}) = 0$$

Inviscid equation (variable Q_{inv}): e.g. incompressible potential flow

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Simultaneous Viscous-inviscid Solution

Global Newton-Raphson method for nonlinear equation $\boldsymbol{\mathcal{R}}(\boldsymbol{Q}) = \mathbf{0}$:

$$\boldsymbol{Q}^{n+1} = \boldsymbol{Q}^n + \Delta \boldsymbol{Q}^n, \qquad \boldsymbol{\mathcal{R}}(\boldsymbol{Q}^n) + \left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{Q}}\right]^n \Delta \boldsymbol{Q}^n = \mathbf{0}$$

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$$

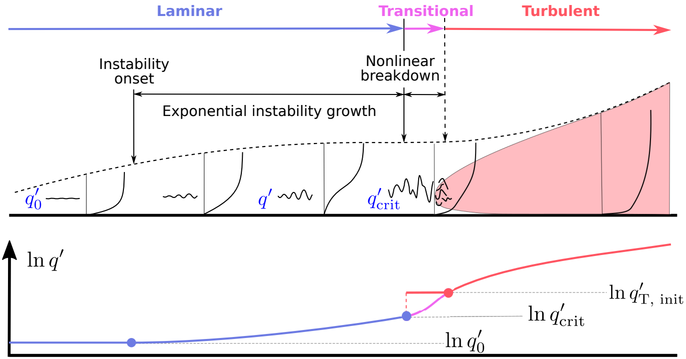
$$
\left[\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \boldsymbol{Q}}\right] =
\begin{array}{cl}
\begin{pmatrix}
[\mathbf{X}] & 0 & 0 & [\mathbf{X}] \\
[\mathbf{X}] & [\mathbf{X}] & [\mathbf{X}] & 0 \\
[\mathbf{X}] & 0 & [\mathbf{X}] & 0 \\
0 & [\mathbf{X}] & 0 & [\mathbf{X}]
\end{pmatrix}
&
\begin{array}{l}
\text{Auxiliary viscous} \\
\text{Auxiliary inviscid} \\
\text{IBL} \\
\text{Inviscid}
\end{array}
\\
\begin{array}{cccc}
\{\mathbf{q}_{\mathrm{e}}\} & \{\Lambda\} & \{Q_{\mathrm{IBL}}\} & \{Q_{\mathrm{inv}}\}
\end{array}
&
\end{array}
$$

✓ $[\partial \boldsymbol{\mathcal{R}} / \partial \boldsymbol{Q}]$ conveniently constructed via automatic differentiation

✓ Flexible swap of inviscid solver & extension to multi-disciplines

Flow Transition Model

Linear flow instability theory: Tollmien-Schlichting (TS) wave amplification



- Velocity fluctuation q': based on Reynolds decomposition
- Single variable $\mathcal{G} := \ln(q'/q_{\text{e}})$ unifies $\{\tilde{n}, c_\tau\}$

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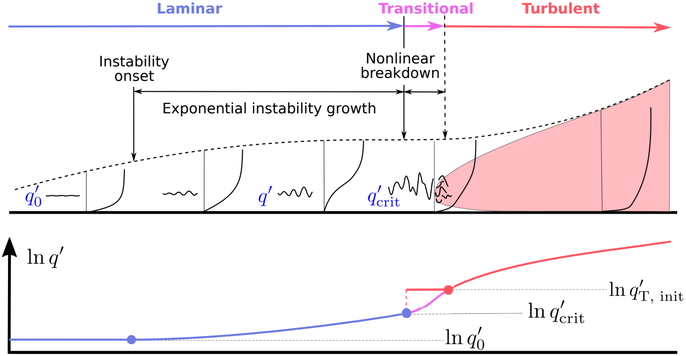
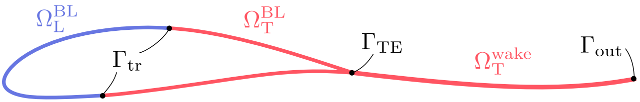


- Velocity fluctuation q': based on Reynolds decomposition
- Single variable $\mathcal{G} := \ln(q'/q_{\text{e}})$ unifies $\{\tilde{n}, c_\tau\}$
- Laminar: e^N envelope method
  - Amplification factor $\tilde{n}$ tracks max TS wave q' growth
- Turbulent: shear stress (coefficient c_τ) transport with "lag" equation

2D IBL Formulation with Transition



Generic Equation: $\widetilde{\nabla} \cdot \mathbf{F} + S = 0$ (transport, hyperbolic, manifold PDE)

Sub-domain	Unknown	Equation
Laminar $\Omega_{\mathrm{L}}^{\mathrm{BL}}$	$\{\delta, \mathcal{A}, \tilde{n}(\mathcal{G})\}$	{mom., k.e., TS amplification}
Turbulent $\Omega_{\mathrm{T}}^{\mathrm{BL}}, \Omega_{\mathrm{T}}^{\mathrm{wake}}$	$\{\delta, \mathcal{A}, c_\tau(\mathcal{G})\}$	{mom., k.e., shear stress lag}

Interface Conditions: conservation & compatibility

@ transition front Γ_{tr}, trailing edge Γ_{TE}

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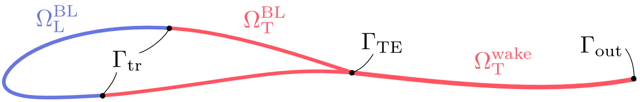

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Key Challenge: "Free" interface Γ_{tr}

- Transition criterion $\tilde{n}(\mathcal{G}) = \tilde{n}_{\mathrm{crit}} \rightarrow$ identifies Γ_{tr}
- Interface location Γ_{tr} interweaves with IBL solution $\{\delta, \mathcal{A}, \mathcal{G}\}$

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(Plain) Discontinuous Galerkin FEM

Approximation: solution $\boldsymbol{v}_h(\xi)$, geometry $\mathbf{r}_h(\xi)$, and local basis $\hat{\mathbf{e}}_h(\xi)$

$$\boldsymbol{v}_h = \sum_j \boldsymbol{v}_{h,j} \mathcal{W}_j(\xi), \qquad \mathbf{r}_h = \sum_j \mathbf{r}_{h,j} \mathcal{W}_j(\xi), \qquad \hat{\mathbf{e}}_h = \frac{\partial \mathbf{r}_h / \partial \xi}{\| \partial \mathbf{r}_h / \partial \xi \|}$$

Discretization: discrete weighted residuals (i.e. weak form)

$$\mathcal{R}_{\mathrm{IBL}} \left(\boldsymbol{v}_h, \mathcal{W}; \hat{\mathbf{e}}_h \right) := \sum_{K \in \mathcal{T}_h} \int_K \Big(\mathcal{W}\, S(\boldsymbol{v}_h) - \widetilde{\nabla} \mathcal{W} \cdot \mathbf{F}(\boldsymbol{v}_h) \Big) \, \mathrm{d}\ell + \sum_{\partial K \in \partial \mathcal{T}_h} \mathcal{W}\, \widehat{\mathbf{t} \cdot \mathbf{F}(\boldsymbol{v}_h)} = 0, \qquad \forall \mathcal{W} \in \mathcal{V}_h$$

with numerical flux $\widehat{\mathbf{t} \cdot \mathbf{F}(\boldsymbol{v}_h)}$ (e.g. upwinding, Lax-Friedrichs)

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However, plain DG FEM needs modification for *free* transition front Γ_{tr}

Discretization for Free Interface Problem

Existing Methods

- Examples for flow transition (in 2D finite-difference setting):
  - MTFLOW: implicit interface tracking
  - XFOIL, MSES: explicit interface tracking
- Adaptive conformal mesh
  - e.g. arbitrary Lagrangian-Eulerian (ALE)
- Fixed-mesh approach
  - Sharp interface representation: e.g. enriched FEM
  - Distributed interface representation: e.g. immersed interface method

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Proposed Methods for DG-IBL: fixed-mesh treatment

- Captured transition
  - Implicit Γ_{tr}; unified discretization for entire domain
- Fitted transition
  - Explicit Γ_{tr}; separate discretizations for sub-domains

Strategy I: Captured Transition

Generic Equation: $\widetilde{\nabla} \cdot \mathbf{F} + S = 0$ (transport, hyperbolic, manifold PDE)

- Unknowns: $\{\delta, \mathcal{A}, \mathcal{G}\}$. Equations: {mom., k.e., and TS amp./lag}

... defined on entire domain Ω, with different flux/source on sub-domains:

$$\mathbf{F} = \begin{cases} \mathbf{F}_{\mathrm{L}} & \text{on } \Omega_{\mathrm{L}} \\ \mathbf{F}_{\mathrm{T}} & \text{on } \Omega_{\mathrm{T}} \end{cases}, \qquad S = \begin{cases} S_{\mathrm{L}} & \text{on } \Omega_{\mathrm{L}} \\ S_{\mathrm{T}} & \text{on } \Omega_{\mathrm{T}} \end{cases}$$

Transition Interface Conditions: conservation & compatibility

- automatically & weakly imposed if flux is continuous

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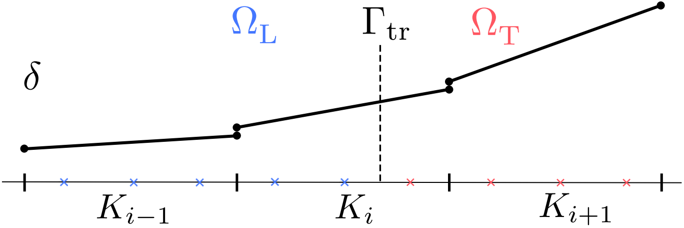
Discretization: reuses plain DG FEM (e.g. $p = 1$, 3-point quadrature)



Strategy II: Fitted Transition

Governing Equation:

Domain	Unknown	Generic Equation
Entire Ω	$\{\tilde{n}(\mathcal{G}_{\mathrm{L}})\}$	$\widetilde{\nabla} \cdot \mathbf{F}(\boldsymbol{v}) + S(\boldsymbol{v}) = 0$
Laminar $\Omega_{\mathrm{L}}^{\mathrm{BL}}$	$\{\delta_{\mathrm{L}}, \mathcal{A}_{\mathrm{L}}\}$	$\widetilde{\nabla} \cdot \mathbf{F}_{\mathrm{L}}(\boldsymbol{v}_{\mathrm{L}}) + S_{\mathrm{L}}(\boldsymbol{v}_{\mathrm{L}}) = 0$
Turbulent $\Omega_{\mathrm{T}}^{\mathrm{BL}}, \Omega_{\mathrm{T}}^{\mathrm{wake}}$	$\{\delta_{\mathrm{T}}, \mathcal{A}_{\mathrm{T}}, c_{\tau}(\mathcal{G}_{\mathrm{T}})\}$	$\widetilde{\nabla} \cdot \mathbf{F}_{\mathrm{T}}(\boldsymbol{v}_{\mathrm{T}}) + S_{\mathrm{T}}(\boldsymbol{v}_{\mathrm{T}}) = 0$

- Extend $\tilde{n}(\mathcal{G}_{\mathrm{L}})$ from $\Omega_{\mathrm{L}}^{\mathrm{BL}}$ to Ω $\rightarrow$ Tracking Γ_{tr} is a local operation

Transition Interface Conditions:

- via DG weighted residuals (weak) or Lagrange multipliers (strong)

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Transition Interface Conditions:

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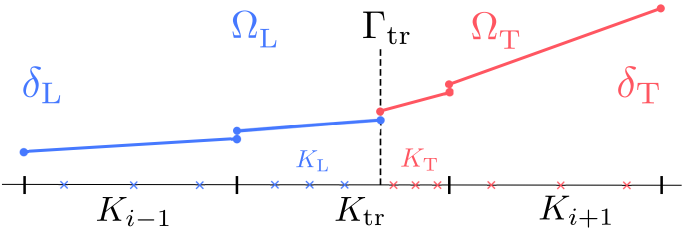
Discretization:

- Reuses plain DG FEM in:
  - entire domain for $\tilde{n}(\mathcal{G}_{\mathrm{L}})$
  - sub-domain interiors for $\{\delta_{\mathrm{L}}, \mathcal{A}_{\mathrm{L}}\}$ and $\{\delta_{\mathrm{T}}, \mathcal{A}_{\mathrm{T}}, c_\tau(\mathcal{G}_{\mathrm{T}})\}$
- Transitional element requires modification $\rightarrow$ cut-cell DG treatment

Cut-cell DG Treatment

Cut-cell DG: solution approximation & weighted residual

- Example: $p = 1$ Lagrange polynomial basis $\{\mathcal{W}_j\}$, 3-point quadrature
- Explicitly track Γ_{tr} location x_{tr} by $\tilde{n}(\mathcal{G}_{\mathrm{L}})$ and transition criterion



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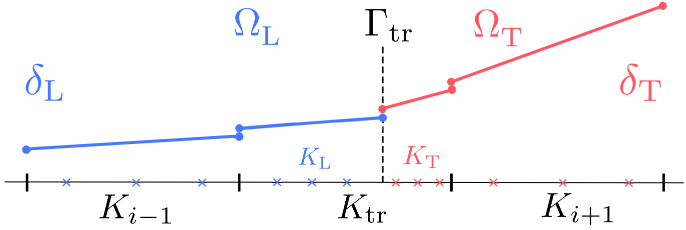

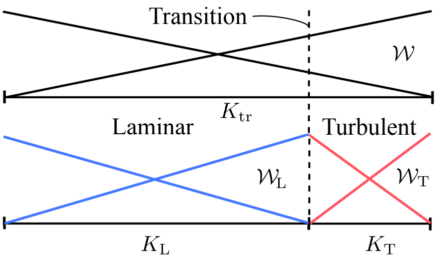

- K_{tr} uses standard basis $\mathcal{W}$
- Sub-cells $K_{\mathrm{L}}, K_{\mathrm{T}}$ use cut-cell basis $\mathcal{W}_{\mathrm{L}}, \mathcal{W}_{\mathrm{T}}$ respectively

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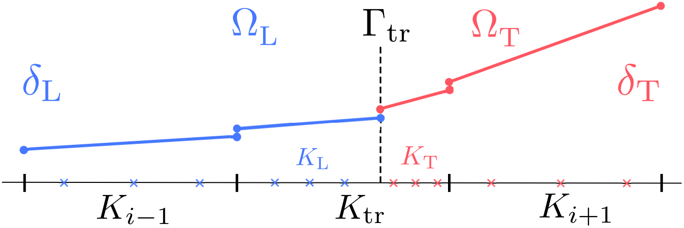

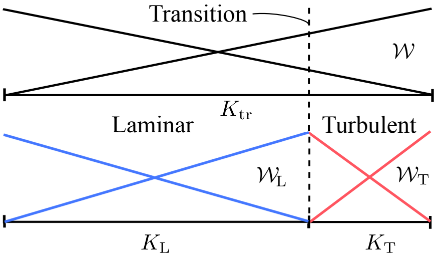

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Newton Solver Enhancement: Line Search

$$\boldsymbol{Q}^{n+1} = \boldsymbol{Q}^{n} + \beta\,\Delta\boldsymbol{Q}^{n} \qquad \text{(Line-search update)}$$

Adapt solution update step size $\beta \in (0, 1]$ so that

- residual $\boldsymbol{\mathcal{R}}$ decreases for each update
- state $\boldsymbol{Q}$ remains physically valid (e.g. positive BL thickness $\delta > 0$)

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✓ fine with captured transition, and fitted transition with fixed Γ_{tr}

× *but* free Γ_{tr} in fitted transition can produce discontinuous residual

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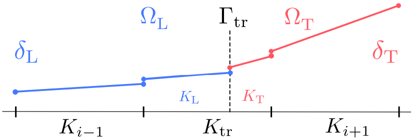

$$\mathcal{R} = [\quad \cdots \mathcal{R}_{\mathrm{L}} \cdots \quad | \quad \cdots \mathcal{R}_{\mathrm{T}} \cdots \quad]^T$$

$$Q = [\quad \cdots Q_{\mathrm{L}} \cdots \quad | \quad \cdots Q_{\mathrm{T}} \cdots \quad]^T$$

Initial state (iter: n)

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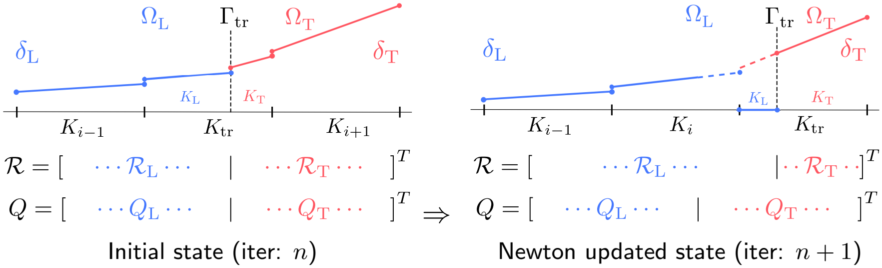
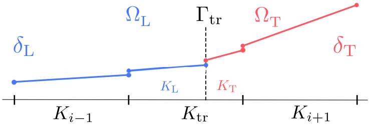


Newton Solver Enhancement: Under-relaxation



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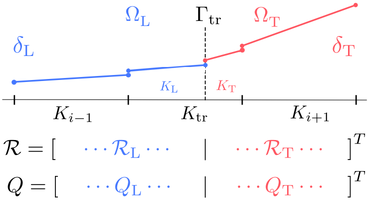
Initial state

Two-step Solution Update

Newton Solver Enhancement: Under-relaxation



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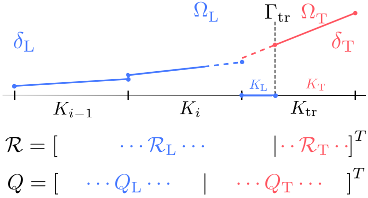


Under-relaxation

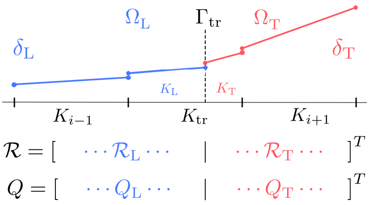
Two-step Solution Update

❶ Under-relaxed Newton update:

$$\widetilde{\boldsymbol{Q}}^{n+1} = \boldsymbol{Q}^n + \alpha\, \Delta \boldsymbol{Q}^n$$

- Relaxation factor α ensures physicality

Newton Solver Enhancement: Under-relaxation



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Two-step Solution Update

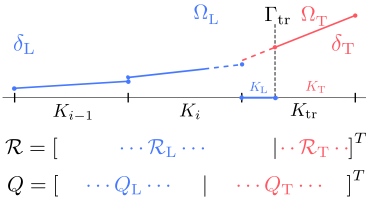
❶ Under-relaxed Newton update:

$$\widetilde{\boldsymbol{Q}}^{n+1} = \boldsymbol{Q}^n + \alpha\, \Delta \boldsymbol{Q}^n$$

- Relaxation factor α ensures physicality

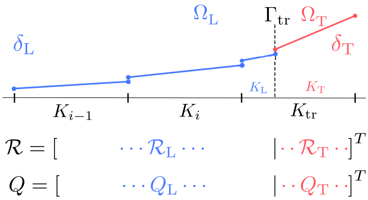
❷ Re-conditioning by forced-transition solution (fixed Γ_{tr} based on $\widetilde{\boldsymbol{Q}}^{n+1}$).

- Line search works well for fixed Γ_{tr}



Under-relaxation

$\Rightarrow$



Re-conditioning

Outline

- Strongly-coupled IBL Formulation
- Numerical Discretization
- Nonlinear Solution
- Numerical Results

Viscous-inviscid Analysis Demo

Test Case: NACA 0004, $\alpha = 0^\circ, Re = 10^5$, incompressible flow

- $\tilde{n}_{\text{crit}} = 0.6 \rightarrow$ triggers natural transition on airfoil surface

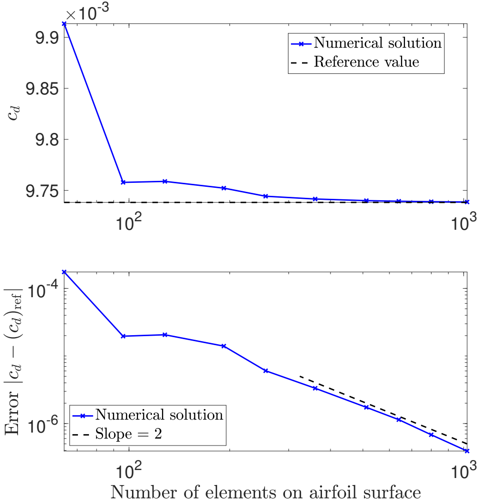
Implementation: Coupled IBL-panel Method

- Compare (1) XFOIL, (2) captured transition, and (3) fitted transition
- Focus on numerics (despite differences in closure models)

Demonstration

- Grid convergence study
- Output sensitivity w.r.t. parameter (e.g. c_d-$\tilde{n}_{\text{crit}}$)

Grid Convergence Study



XFOIL

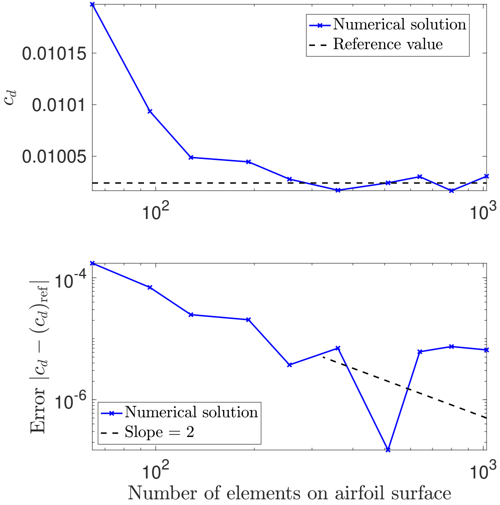
Setup

- $p = 1$ DG-IBL
- Output: c_d, with $(c_d)_{\mathrm{ref}}$ from 2048-element grid

Numerical Results

- XFOIL (baseline)
  - Grid-converged
  - 2nd-order accurate

Grid Convergence Study



Captured transition

Setup

- $p = 1$ DG-IBL
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Numerical Results

- XFOIL (baseline)
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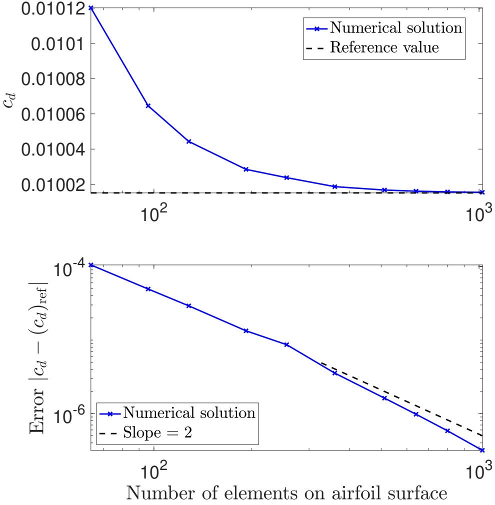
  - 2nd-order accurate
- Captured Transition
  - Not grid-converged
  - Oscillatory solution

Grid Convergence Study



Fitted transition

Setup

- $p = 1$ DG-IBL
- Output: c_d, with $(c_d)_{\mathrm{ref}}$ from 2048-element grid

Numerical Results

- XFOIL (baseline)
  - Grid-converged
  - 2nd-order accurate
- Captured Transition
  - Not grid-converged
  - Oscillatory solution
- Fitted Transition
  - Grid-converged
  - 2nd-order accurate

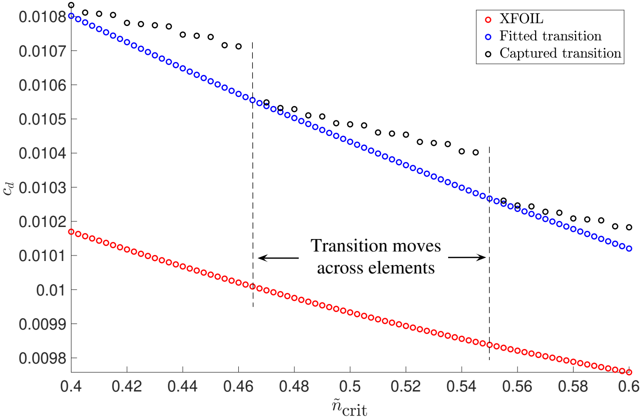
Output Sensitivity Analysis

Setup:

- Vary parameter $\tilde{n}_{\mathrm{crit}}$ with fixed grid $\rightarrow$ different transition location Γ_{tr}
- Desired outcome: smooth output-parameter relation (e.g. c_d-$\tilde{n}_{\mathrm{crit}}$)

Numerical Results:

- XFOIL (✓), captured transition ($\times$), and fitted transition (✓)



Summary

Numerical Treatment of Free Transition for DG-IBL

Approach I: Captured Transition

- $\times$ Numerical regularization required for suppressing discontinuities
- $+$ Convenient implementation using plain DG FEM

Approach II: Fitted Transition

- $+$ Accurate & robust solution for free-transition problem
- $\times$ Complex implementation necessitated for 3D extension

Summary

Numerical Treatment of Free Transition for DG-IBL

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Ongoing/future work

- Implement both transition treatments for 3D IBL
- Improve closure models: transition, turbulence etc.
- Apply various inviscid solvers: full potential and Euler
- Extend to aero-structural coupling: hybrid shell model (HSM)
  - $\rightarrow$ HSM by Drela et al., AIAA 2019-2227, 10:30am, Friday, Jan 11

Acknowledgments

- Funding source: NASA Grant Award NNX15AU41A
- Technical monitors: Michael Aftosmis and David Rodriguez

Q & A

Thank you!

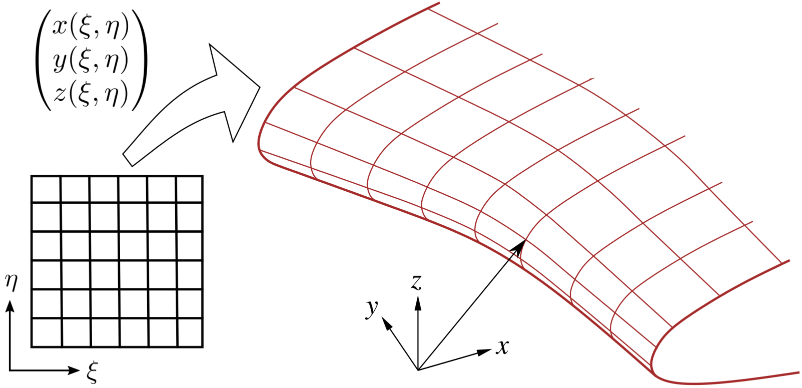
Contact: Shun Zhang, shunz@mit.edu

Outline

- Problem Statement
- 2D IBL Formulation
- Numerical Discretization for Coupled IBL
- Numerical Results

Issue for Existing 3D IBL Formulations

- Requires explicit curvilinear coordinates to parametrize surfaces

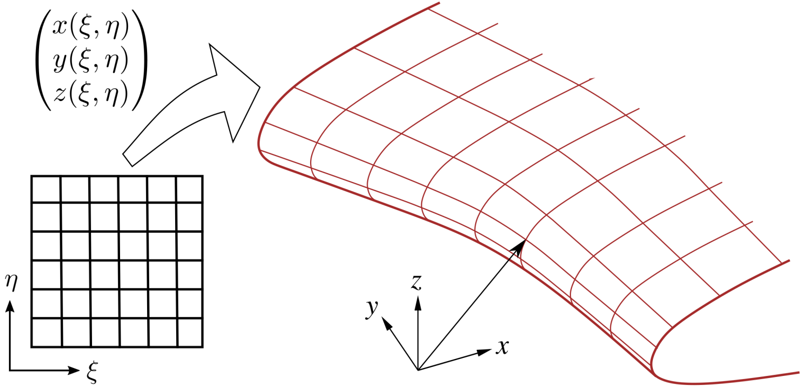


Issue for Existing 3D IBL Formulations

- Requires explicit curvilinear coordinates to parametrize surfaces

$\rightarrow$ Inapplicable to non-smooth features

$\rightarrow$ Cumbersome for complex geometries



Robustness Issue for BL Solvers

Bottleneck: viscous/inviscid coupling

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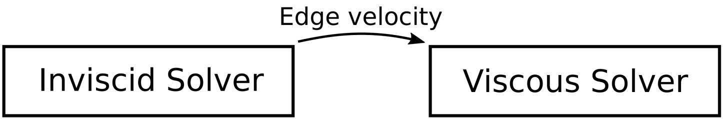
Classical one-way coupling

- Goldstein Singularity $\rightarrow$ Fail upon flow separation



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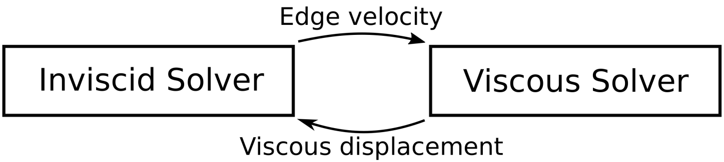
Two-way iteration

- Examples: Le Balleur (1981), Veldman (2009), Lokatt et al. (2017)
- Varied robustness, compromised efficiency



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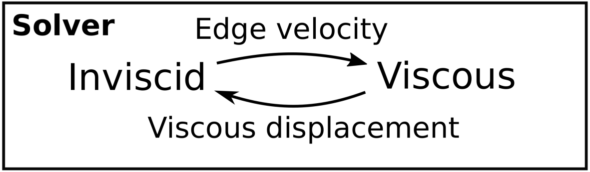

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Strong viscous/inviscid coupling

- Examples: XFOIL (Drela, 1989), MSES (Drela, 1987)
- Most reliable

Outline

- Problem Statement
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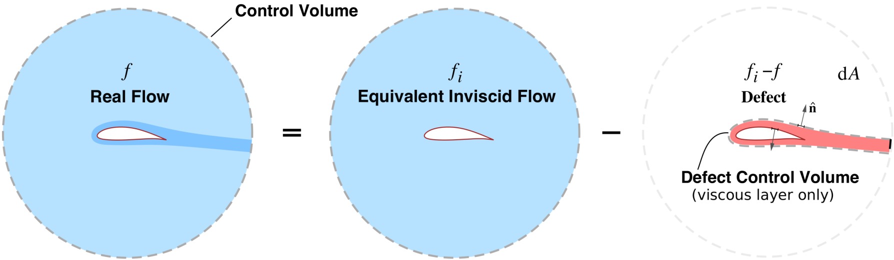
Viscous-inviscid Zonal Decomposition

- Equivalent inviscid flow (EIF)

  $\Rightarrow$ Inviscid flow equations on $\mathbf{f}_\mathrm{i}$: e.g. full potential, Euler
- Defect control volume (DCV)
  - Defect $\mathbf{f}_\mathrm{i} - \mathbf{f}$ vanishes outside DCV
  - BL approximations $\rightarrow$ Thin DCV
  - Dimension reduction: e.g. $\iint (\text{defect})\, \mathrm{d}A \Rightarrow \int (\text{defect integral})\, \mathrm{d}\ell$

  $\Rightarrow$ IBL equations on defect integral $\int (\mathbf{f}_\mathrm{i} - \mathbf{f})\, \mathrm{d}n$



Defect Integral Equations

Conservation laws: 2D $\rightarrow$ 1D

$$\iint (\mathsf{mass}_{\mathrm{i}} - \mathsf{mass})\, \mathrm{d}A \qquad \rightarrow \quad \widetilde{\nabla} \cdot \mathbf{M} - (\rho_{\mathrm{i}}\, \mathbf{q}_{\mathrm{i}})_{\mathrm{w}} \cdot \hat{\mathbf{n}}_{\mathrm{w}} = 0$$

$$\iint (\boldsymbol{\mathsf{mom}}_{\mathrm{i}} - \boldsymbol{\mathsf{mom}})\, \mathrm{d}A \qquad \rightarrow \quad \widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla} \mathbf{q}_{\mathrm{i}} - \boldsymbol{\tau}_{\mathrm{w}} = \mathbf{0}$$

$$\iint (\mathsf{k.e.}_{\mathrm{i}} - \mathsf{k.e.})\, \mathrm{d}A \qquad \rightarrow \quad \widetilde{\nabla} \cdot \mathbf{K} + \mathbf{D} \cdot \widetilde{\nabla} q_{\mathrm{i}}^2 - 2\mathcal{D} = 0$$

with in-plane operator $\widetilde{\nabla} \equiv \partial\,(\cdot)/\partial\xi\ \hat{\mathbf{e}}$

Defect Integral Equations

Conservation laws: 2D $\rightarrow$ 1D

$$\iint (\mathsf{mass_i} - \mathsf{mass})\,\mathrm{d}A \qquad \rightarrow \qquad \widetilde{\nabla} \cdot \mathbf{M} - (\rho_\mathrm{i}\,\mathbf{q}_\mathrm{i})_\mathrm{w} \cdot \hat{\mathbf{n}}_\mathrm{w} = 0$$

$$\iint (\mathbf{mom_i} - \mathbf{mom})\,\mathrm{d}A \qquad \rightarrow \qquad \widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla}\mathbf{q}_\mathrm{i} - \boldsymbol{\tau}_\mathrm{w} = \mathbf{0}$$

$$\iint (\mathsf{k.e._i} - \mathsf{k.e.})\,\mathrm{d}A \qquad \rightarrow \qquad \widetilde{\nabla} \cdot \mathbf{K} + \mathbf{D} \cdot \widetilde{\nabla} q_\mathrm{i}^2 - 2\mathcal{D} = 0$$

with in-plane operator $\widetilde{\nabla} \equiv \partial\,(\cdot)/\partial\xi\;\hat{\mathbf{e}}$, and defect integrals etc.

$\mathbf{M}$	$\equiv$	$\int (\rho_\mathrm{i}\,\mathbf{q}_\mathrm{i} - \rho\,\mathbf{q})\,\mathrm{d}n$	mass flux defect
$\mathbf{p}$	$\equiv$	$\int (\mathbf{q}_\mathrm{i} - \mathbf{q})\,\rho\,\mathrm{d}n$	momentum defect
$\overline{\overline{\mathbf{P}}}$	$\equiv$	$\int (\mathbf{q}_\mathrm{i} - \mathbf{q})\,\rho\,\mathbf{q}\,\mathrm{d}n$	momentum defect flux
$\mathbf{K}$	$\equiv$	$\int (q_\mathrm{i}^2 - q^2)\,\rho\,\mathbf{q}\,\mathrm{d}n$	kinetic energy defect flux
$\mathbf{D}$	$\equiv$	$\int (\rho_\mathrm{i} - \rho)\,\mathbf{q}\,\mathrm{d}n$	density defect flux
	$\vdots$		

TS Wave Amplification Equation

$$\boxed{\mathbf{q}_\mathrm{e} \cdot \widetilde{\nabla} \mathcal{G} - \frac{q_\mathrm{e}}{\theta_{11}} \, \mathrm{f}_N + (1-R) \frac{1}{t_\mathrm{ref}} \exp\left(- \frac{q_\mathrm{e}^2}{q_\mathrm{ref}^2} \right) (\tilde{n} - \tilde{n}_0) = 0}$$

where the growth rate function f_N is given as follows,

$$f_N(H; Re_{\theta_{11}}) = R \, \frac{\mathrm{d}\tilde{n}}{\mathrm{d}Re_{\theta_{11}}} \, \theta_{11} \frac{\mathrm{d}Re_{\theta_{11}}}{\mathrm{d}x}$$

where $$R = \frac{1}{2} + \frac{1}{2} \tanh\left[10 \left(\ln Re_{\theta_{11}} - \ln Re_{\theta_{11,0}}\right)\right]$$

$$\ln Re_{\theta_{11,0}} = \frac{5.738}{(H-1)^{0.43}} + 1.612 \left[\tanh\left(\frac{14}{H-1} - 9.24\right) + 1\right]$$

$$\frac{\mathrm{d}\tilde{n}}{\mathrm{d}Re_{\theta_{11}}} = 0.028(H-1) - 0.0345 \exp\left[-\left(\frac{3.87}{H-1} - 2.52\right)^2\right]$$

$$\theta_{11} \frac{\mathrm{d}Re_{\theta_{11}}}{\mathrm{d}x} = -0.05 + \frac{2.7}{H-1} - \frac{5.5}{(H-1)^2} + \frac{3.0}{(H-1)^3} + 0.1 \exp\left(\frac{-20}{H-1}\right)$$

Shear Stress Transport ("Lag") Equation

Lag Equation:

$$\mathbf{q}_\mathrm{e} \cdot \widetilde{\nabla}\mathcal{G} - \frac{q_\mathrm{e}}{2\,\widetilde{\delta}}\left[5.6\left((c_\tau)_\mathrm{eq}^{1/2} - {c_\tau}^{1/2}\right)\right] - \frac{q_\mathrm{e}}{B_\mathrm{eq}\delta_1^*}\left[\frac{C_{f_1}}{2} - \left(\frac{H-1}{A_\mathrm{eq}K_\mathrm{dl}H}\right)^2\right] + \widetilde{\nabla}\cdot\mathbf{q}_\mathrm{e} = 0$$

with miscellaneous closure relations.

The unifying variable $\mathcal{G}$ relates to $\{\tilde{n}, c_\tau\}$:

$$\tilde{n} = \mathcal{G} - \ln Q'_\mathrm{crit} + \tilde{n}_\mathrm{crit}$$

$${c_\tau}^{1/2} = \exp\mathcal{G} = Q'_\mathrm{crit}\exp\left(\tilde{n} - \tilde{n}_\mathrm{crit}\right)$$

Transition Interface Conditions

At transition front $\Gamma_{\rm tr}$ (i.e. turbulent BL inlet),

$$\delta^*_{1\rm BC} = (\delta^*_1)_{\rm L,\Gamma_{tr}} \qquad \text{(mass conservation)}$$

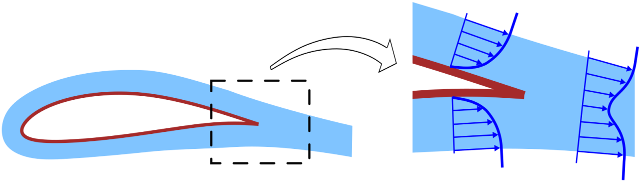
$$\theta_{11\rm BC} = (\theta_{11})_{\rm L,\Gamma_{tr}} \qquad \text{(momentum conservation)}$$

$$c_{\tau\rm BC} = c_{\tau\rm T,\ init} \qquad \text{(initial turbulent shear stress condition)}$$

where $c_{\tau\rm T,\ init} = c_{\tau\rm crit}$ is used in the current implementation.

2D IBL Trailing-edge Matching Conditions

Issue: Two IBL equations, but (seemingly) three conservation laws. Which pair of {mass, mom., k.e.} should be conserved?



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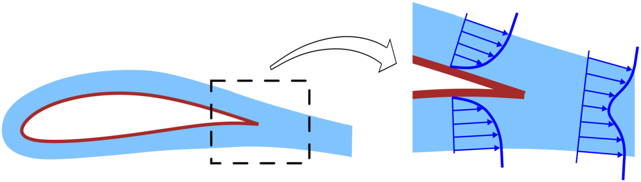

Introduce two equations (i.e. matching conditions)

$$\delta^*_{\text{TE, upper}} + \delta^*_{\text{TE, lower}} + h_{\text{TE}} = \delta^*_{\text{wake, inlet}} \qquad \text{(mass conservation)}$$

$$\theta_{\text{TE, upper}} + \theta_{\text{TE, lower}} = \theta_{\text{wake, inlet}} \qquad \text{(momentum conservation)}$$

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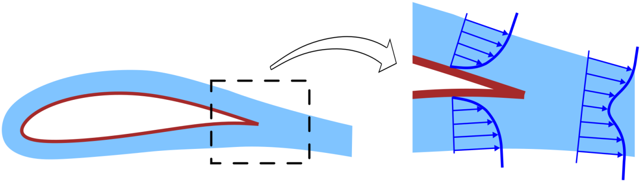

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$$\theta_{\text{TE, upper}} + \theta_{\text{TE, lower}} = \theta_{\text{wake, inlet}} \qquad \text{(momentum conservation)}$$

and two unknowns $\{F_\theta, F_{\theta^*}\}$ (Lagrange multipliers) with

$$\widehat{\mathbf{f}\cdot\mathbf{t}}_{\text{wake, inlet}} = [F_\theta, F_{\theta^*}]^T$$

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DG FEM for manifold PDE

How to discretize vectorial PDEs defined on manifolds?

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- **Example**: IBL momentum equation: vector

$$0 = \qquad \widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla} \mathbf{q}_{\mathrm{e}} - \boldsymbol{\tau}_{\mathrm{w}} \qquad \text{(PDE strong form)}$$

DG FEM for manifold PDE

How to discretize vectorial PDEs defined on manifolds?

- **Idea**: resolution in local basis (confinement to manifold)
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$$0 = \qquad \hat{\mathbf{e}} \cdot \left\{ \widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla} \mathbf{q}_{\mathrm{e}} - \boldsymbol{\tau}_{\mathrm{w}} \right\} \qquad \text{(resolved in local basis)}$$

DG FEM for manifold PDE

How to discretize vectorial PDEs defined on manifolds?

- **Idea**: resolution in local basis (confinement to manifold)
- **Example**: IBL momentum equation: vector

$$0 = \int_K \mathcal{W}\,\hat{\mathbf{e}} \cdot \left\{ \widetilde{\nabla} \cdot \overline{\overline{\mathbf{P}}} + \mathbf{M} \cdot \widetilde{\nabla} \mathbf{q}_\mathrm{e} - \boldsymbol{\tau}_\mathrm{w} \right\} \mathrm{d}\ell \qquad \text{(elemental weighted residual)}$$

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$$= \ldots \qquad \text{(integration by parts} \rightarrow \text{weak form)}$$

- IBL kinetic energy equation: scalar

$$0 = \int_K \mathcal{W} \left\{ \widetilde{\nabla} \cdot \mathbf{K} + \mathbf{D} \cdot \widetilde{\nabla} q_\mathrm{e}^2 - 2\mathcal{D} \right\} \mathrm{d}\ell \qquad \text{(elemental weighted residual)}$$

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IBL DG Residuals

Notation:

$$(\boldsymbol{a}, \boldsymbol{b})_{\mathcal{T}_h} \equiv \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}\ell, \quad \langle \boldsymbol{c}, \boldsymbol{d} \rangle_{\partial \mathcal{T}_h} \equiv \sum_{\partial K \in \partial \mathcal{T}_h} \int_{\partial K} \boldsymbol{c} \cdot \boldsymbol{d}$$

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Assemble DG global weak form: $\forall \, \mathcal{W}$

$$\mathcal{R}^m_{\mathrm{IBL}}\left(\boldsymbol{v}, \mathcal{W}\right) \equiv \left(\mathcal{W}, \, -\left(\overline{\overline{\mathbf{P}}} \cdot \widetilde{\nabla}\right) \cdot \hat{\mathbf{e}} \, + \, \mathbf{M} \cdot \left(\widetilde{\nabla} \mathbf{q}_{\mathrm{e}} \cdot \hat{\mathbf{e}}\right) \, - \, \boldsymbol{\tau}_{\mathrm{w}} \cdot \hat{\mathbf{e}}\right)_{\mathcal{T}_h}$$

$$- \left(\widetilde{\nabla} \mathcal{W}, \, \overline{\overline{\mathbf{P}}}^T \cdot \hat{\mathbf{e}}\right)_{\mathcal{T}_h} + \left\langle \mathcal{W}, \, \widehat{\mathbf{t} \cdot \overline{\overline{\mathbf{P}}}^T \cdot \mathbf{e}} \right\rangle_{\partial \mathcal{T}_h}$$

$$\mathcal{R}^e_{\mathrm{IBL}}\left(\boldsymbol{v}, \mathcal{W}\right) \equiv \left(\mathcal{W}, \, \mathbf{D} \cdot \widetilde{\nabla} q_{\mathrm{e}}^2 \, - \, 2\mathcal{D}\right)_{\mathcal{T}_h} - \left(\widetilde{\nabla} \mathcal{W}, \, \mathbf{K}\right)_{\mathcal{T}_h} + \left\langle \mathcal{W}, \, \widehat{\mathbf{K} \cdot \mathbf{t}} \right\rangle_{\partial \mathcal{T}_h}$$

with numerical flux $\widehat{\mathbf{f} \cdot \mathbf{t}}$ where $\mathbf{f} \equiv \left\{\overline{\overline{\mathbf{P}}}^T \cdot \hat{\mathrm{e}}, \mathbf{K}\right\}^T$.

2D IBL DG Numerical Flux

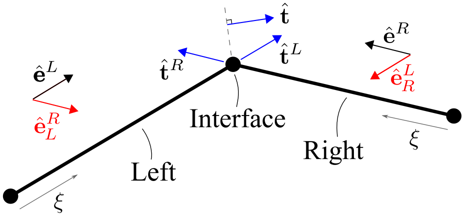
Manifold discretization issues

- Discontinuous $\hat{\mathrm{e}}$ at interface

$\rightarrow$ modify $\hat{\mathrm{e}}$ in $\widehat{\mathbf{f}\cdot\mathbf{t}}$

- $\hat{\mathbf{t}}^L, \hat{\mathbf{t}}^R$ not co-linear

$\rightarrow$ define unique $\hat{\mathbf{t}}$



Interface flux in Lax-Friedrichs formulation:

$$\widehat{\mathbf{f}\cdot\mathbf{t}}^L \equiv \frac{1}{2}\left\{\mathbf{f}(\boldsymbol{v}^L;\,\hat{\mathrm{e}}^L) + \mathbf{f}(\boldsymbol{v}^R;\,\hat{\mathrm{e}}^R_L)\right\}\cdot\hat{\mathbf{t}} + \frac{\alpha}{2}\left\{\boldsymbol{u}(\boldsymbol{v}^L;\,\hat{\mathrm{e}}^L) - \boldsymbol{u}(\boldsymbol{v}^R;\,\hat{\mathrm{e}}^R_L)\right\}$$

$$\widehat{\mathbf{f}\cdot\mathbf{t}}^R \equiv \frac{1}{2}\left\{\mathbf{f}(\boldsymbol{v}^L;\,\hat{\mathrm{e}}^L_R) + \mathbf{f}(\boldsymbol{v}^R;\,\hat{\mathrm{e}}^R)\right\}\cdot\hat{\mathbf{t}} + \frac{\alpha}{2}\left\{\boldsymbol{u}(\boldsymbol{v}^L;\,\hat{\mathrm{e}}^L_R) - \boldsymbol{u}(\boldsymbol{v}^R;\,\hat{\mathrm{e}}^R)\right\}$$

- Conservative variable $\boldsymbol{u} \equiv [\mathbf{p}\cdot\hat{\mathrm{e}}, k]^T$.
- Dissipation coefficient $\alpha \equiv \max\left\{\left|\mathbf{q}_{\mathrm{e}}^L\cdot\hat{\mathbf{t}}^L\right|, \left|\mathbf{q}_{\mathrm{e}}^R\cdot\hat{\mathbf{t}}^R\right|\right\}$

Discrete Residuals for Non-IBL Equations

Auxiliary inviscid residual: wall transpiration

$$\mathcal{R}_{\mathrm{auxi}} \equiv \left(\mathcal{W},\, \Lambda - \widetilde{\nabla}\cdot\mathbf{M}\right)_{\mathcal{T}_h} + \left\langle \phi,\, \left(\widehat{\mathbf{M}} - \mathbf{M}_{\mathrm{BC}}\right)\cdot\hat{\mathbf{n}} \right\rangle_{\partial\Omega}$$

$$= (\mathcal{W},\, \Lambda)_{\mathcal{T}_h} + \left(\widetilde{\nabla}\mathcal{W},\, \mathbf{M}\right)_{\mathcal{T}_h} - \left\langle \mathcal{W},\, \mathbf{M}\cdot\hat{\mathbf{t}} \right\rangle_{\partial\mathcal{T}_h\setminus\Gamma_{\mathrm{in}}} - \left\langle \mathcal{W},\, \mathbf{M}_{\mathrm{in}}\cdot\hat{\mathbf{t}} \right\rangle_{\Gamma_{\mathrm{in}}}$$

Inviscid equations: e.g. panel method

$$\mathcal{R}_{\mathrm{inv}}^{\Psi} \equiv \Psi_{(\gamma,\Lambda)} - \Psi_0 \quad \text{(flow tangency)}$$

$$\mathcal{R}_{\mathrm{inv}}^{\mathrm{K}} \equiv \sum_{j\in\mathrm{TE}} \gamma_j \quad \text{(Kutta condition)}$$

Auxiliary viscous residual (incompressible): edge velocity projection

$$\mathcal{R}_{\mathrm{auxv}} \equiv (\mathcal{W},\, \mathbf{q}_{\mathrm{e}} - \mathbf{q}_{\mathrm{i}})_{\mathcal{T}_h}$$

Outline

- Problem Statement
- 2D IBL Formulation
- Numerical Discretization for Coupled IBL
- Numerical Results

Manifold DG Verification

Verify: Applicability to general manifold PDE and high-order solution

Case: 2D shallow water equations on elliptical curve

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$$\widetilde{\nabla} \cdot (H\boldsymbol{v}) = 0 \qquad \text{(continuity)}$$

$$\widetilde{\nabla} \cdot (H\boldsymbol{v}\boldsymbol{v}) + gH\,\widetilde{\nabla}(H - b) = \mathbf{0} \qquad \text{(momentum conservation)}$$

water depth H, flow velocity $\boldsymbol{v} \equiv v_s\,\hat{\mathbf{s}}$, streamwise unit vector $\hat{\mathbf{s}}$

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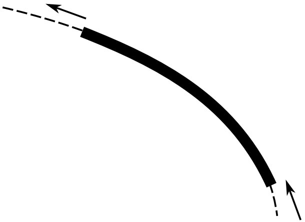
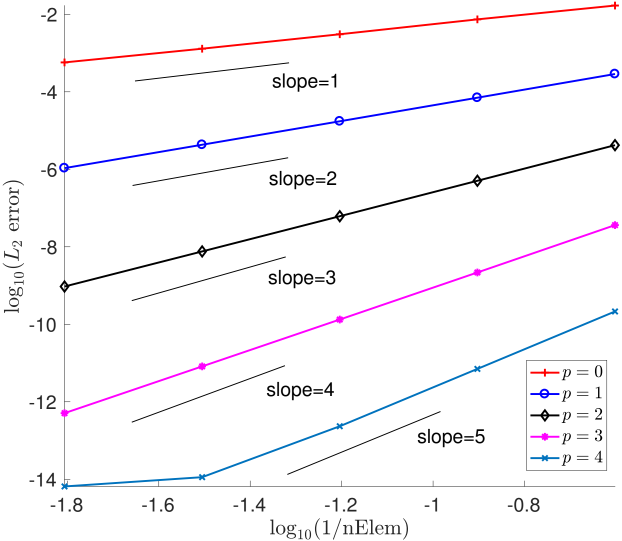


- Elliptical curve segment
- "Manufacture" analytic solution $\{H, v_s\}$ by prescribing b

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = \left(1 - \frac{v_s^2}{gH}\right)\frac{\mathrm{d}H}{\mathrm{d}\theta}$$

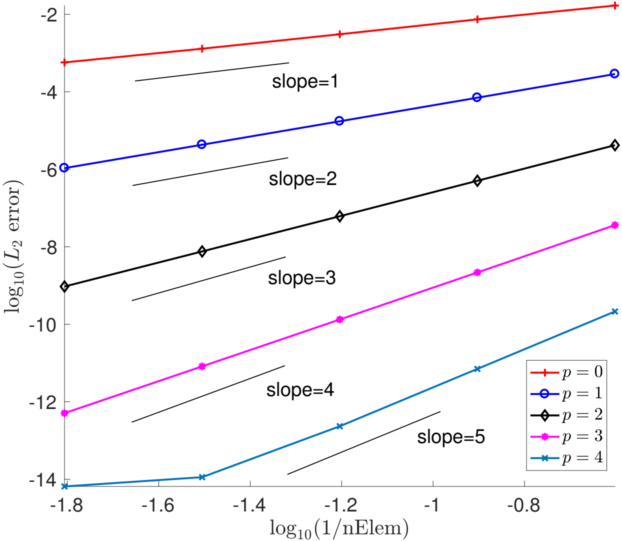
Manifold DG Verification

- Verified optimal convergence: solution L_2 error $\sim \mathcal{O}(h^{p+1})$



Manifold DG Verification

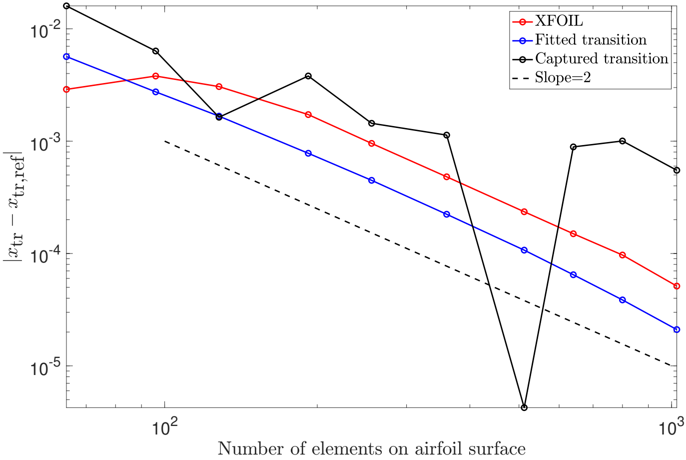
- Verified optimal convergence: solution L_2 error $\sim \mathcal{O}(h^{p+1})$
- Note that high-order solution requires high-order mesh



Grid Convergence Study

Numerical Results

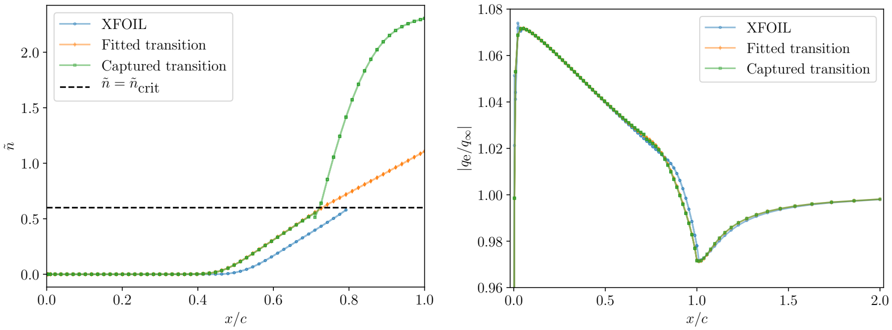
- Output: transition location x_{tr}, with $x_{\mathrm{tr,ref}}$ from 2048-element grid
- 2nd-order: XFOIL (✓), captured ($\times$) and fitted transition (✓)



Sample IBL Solution

Setup

- All methods use the same grid: 128 elements on airfoil
- Different closure relations → *Not* expected to match XFOIL exactly
- Qualitative agreement between all the methods under consideration



Left: amplification factor $\tilde{n}(\mathcal{G})$ **Right**: normalized edge speed $|q_e/q_\infty|$

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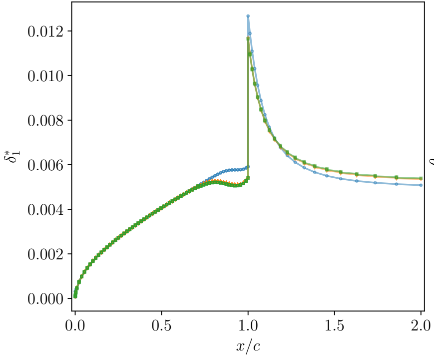

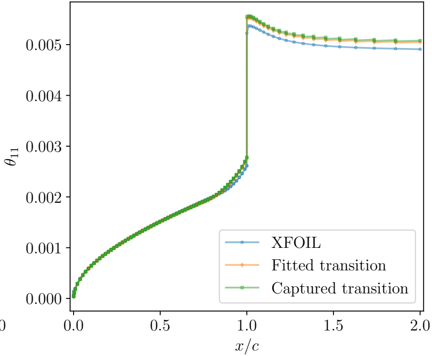

Left: displacement thickness δ_1^* **Right**: momentum defect thickness θ_{11}

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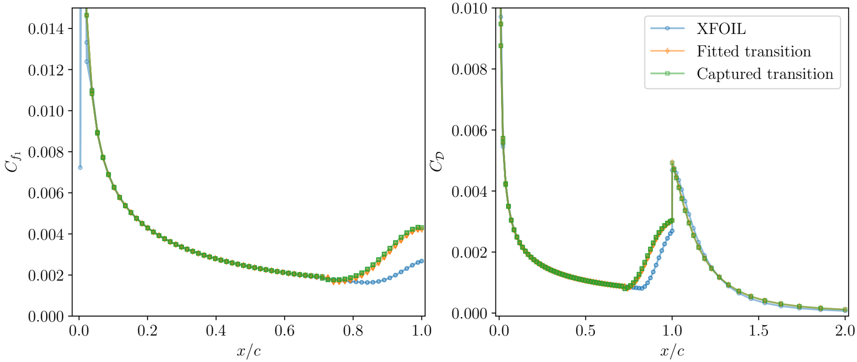

Left: skin friction coefficient C_{f_1} **Right**: dissipation coefficient $C_{\mathcal{D}}$