

Solutions to
Quantum Field Theory for the Gifted Amateur
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Chapter 1

Lagrangians

Problem 1.

$$(a) \quad t = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{v_2} = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2}$$
$$\frac{dt}{dx} = \frac{x}{c/n_1 \sqrt{x^2 + h_1^2}} - \frac{(l-x)}{c/n_2 \sqrt{(l-x)^2 + h_2^2}} = 0$$
$$n_1 \sin \theta = n_2 \sin \phi$$

Problem 2.

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int G(x, y) [f(y) + \epsilon \delta(y-z)] dy - \int G(x, y) f(y) dy \right] = G(x, z)$$
$$\frac{\delta I[f^3]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int_{-1}^1 [f(x) + \epsilon \delta(x-x_0)]^3 dx - \int_{-1}^1 [f(x)]^3 dx \right] =$$
$$\frac{\delta^2 I[f^3]}{\delta f(x_0) \delta f(x_1)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int \right]$$
$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int \left(\frac{\partial}{\partial y} [f(y) + \epsilon \delta(y-x)] \right)^2 dy - \int \left(\frac{\partial f}{\partial y} \right)^2 dy \right]$$
$$= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int [f'(y) + \epsilon \delta'(y-x)]^2 dy - \int \left(\frac{\partial f}{\partial y} \right)^2 dy \right]$$

Problem 3.

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int \left[g(y, f) + \frac{\partial g(y, f)}{\partial f} \epsilon \delta(y-x) \right] dy - \int g(y, f) dy \right] = \frac{\partial g(x, f)}{\partial f(x)}$$

Problem 4.

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\epsilon \rightarrow 0} \frac{\phi(x) + \epsilon\delta(x-y) - \phi(x)}{\epsilon} = \delta(x-y)$$

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \lim_{\epsilon \rightarrow 0} \frac{\frac{d}{dt}[\phi(t) + \epsilon\delta(t-t_0)] - \dot{\phi}(t)}{\epsilon} = \frac{d}{dt}\delta(t-t_0)$$

Problem 5.

$$S = \int T - V \, d^3x = \frac{1}{2} \int \rho \left(\frac{\partial\psi}{\partial t} \right)^2 - \mathcal{T}(\nabla\psi)^2 \, d^3x$$

$$\frac{\delta S}{\delta\psi} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \left[\int \rho \left(\frac{\partial}{\partial t} [\psi + \epsilon\delta(t-t_0)] \right)^2 - \mathcal{T}(\nabla[\psi + \epsilon\delta(\mathbf{x}-\mathbf{y})])^2 \, d^3x \right.$$

$$\left. - \int \rho \left(\frac{\partial\psi}{\partial t} \right)^2 - \mathcal{T}(\nabla\psi)^2 \, d^3x \right] =$$

$$\int \left[\rho \frac{\partial}{\partial t} \delta(t-t_0) \frac{\partial\psi}{\partial t} - \mathcal{T} \nabla \delta(\mathbf{x}-\mathbf{y}) \nabla\psi \right] \, d^3x = 0$$

$$\nabla^2\psi = \frac{1}{\nu^2} \frac{\partial^2\psi}{\partial t^2}, \quad \nu = \sqrt{\frac{\mathcal{T}}{\rho}}$$

Problem 6.

Chapter 2

Simple harmonic oscillators

Problem 1.

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) - \frac{m\omega}{2\hbar} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \\ &= \frac{1}{2i\hbar} ([\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]) = 1 \end{aligned}$$

Problem 2.

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 + \lambda x^4 \\ \hat{H} &= \lambda \left[\hat{x}^4 + \frac{m\omega^2}{2\lambda} \hat{x}^2 + \frac{\hat{p}^2}{2m\lambda} \right] \end{aligned}$$

Problem 3.

$$\begin{aligned} \hat{x}_j &= \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja}, \quad \hat{x}_k = \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{a}_k + \hat{a}_{-k}^\dagger) \\ \hat{x}_j &= \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{a}_k + \hat{a}_{-k}^\dagger) e^{ikja} = \frac{1}{\sqrt{N}} \sqrt{\frac{\hbar}{m}} \sum_k \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k e^{ikja} + \hat{a}_k^\dagger e^{-ikja}) \end{aligned}$$

Problem 4.

$$\begin{aligned} \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) |0\rangle &= 0 \\ \langle x | \hat{x} | 0 \rangle + \frac{i}{m\omega} \langle x | \hat{p} | 0 \rangle &= 0 \\ \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x | 0 \rangle &= 0 \\ \left(\frac{d}{dx} + \frac{m\omega}{\hbar} x \right) \langle x | 0 \rangle &= 0 \end{aligned}$$

This is easily solved by separation of variables. Attempting a series solution for practice:

$$\begin{aligned}
 \langle x|0\rangle &= \sum_{n=0}^{\infty} a_n x^n \\
 \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} &= 0 \\
 \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} &= 0 \\
 a_{n+2} &= -\frac{m\omega}{\hbar(n+2)} a_n, \quad a_0 = A, \quad a_1 = 0 \\
 \langle x|0\rangle &= A \left[1 + \left(-\frac{m\omega}{2\hbar} \right) x^2 + \frac{1}{2} \left(-\frac{m\omega}{2\hbar} \right)^2 x^4 + \frac{1}{6} \left(-\frac{m\omega}{2\hbar} \right)^3 x^6 + \dots \right] \\
 \langle x|0\rangle &= A \exp\left(-\frac{m\omega x^2}{2\hbar} \right) \\
 A &= 1 / \left| \exp\left(-\frac{m\omega x^2}{2\hbar} \right) \right| \\
 A &= 1 / \sqrt{\int_{-\infty}^{\infty} \exp\left(2\frac{m\omega}{2\hbar} x^2 \right) dx} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \\
 \langle x|0\rangle &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar} \right)
 \end{aligned}$$

Chapter 3

Occupation number representation

Problem 1.

$$() \quad \frac{1}{V} \sum_{\mathbf{p}\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}] = \frac{1}{V} \sum_{\mathbf{p}\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \delta_{\mathbf{p}\mathbf{q}} = \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

Problem 2.

$$\begin{aligned} [\hat{a}, (\hat{a}^{\dagger})^n] &= [\hat{a}(\hat{a}^{\dagger})^n - (\hat{a}^{\dagger})^n \hat{a}] \\ &= [(1 + \hat{a}^{\dagger} \hat{a})(\hat{a}^{\dagger})^{n-1} - (\hat{a}^{\dagger})^{n-1} (\hat{a}^{\dagger} \hat{a})] = [(\hat{a}^{\dagger})^{n-1} - [\hat{a}^{\dagger} \hat{a}, (\hat{a}^{\dagger})^{n-1}]] \end{aligned}$$

Problem 3.

$$\begin{aligned} \hat{a}_i^{\dagger} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x}_i - \frac{i}{m\omega} \hat{p}_i \right) \\ [\hat{a}_i, \hat{a}_j^{\dagger}] &= \frac{m\omega}{2\hbar} \left[\left(\hat{x}_i + \frac{i}{m\omega} \hat{p}_i \right) \left(\hat{x}_j - \frac{i}{m\omega} \hat{p}_j \right) - \left(\hat{x}_j - \frac{i}{m\omega} \hat{p}_j \right) \left(\hat{x}_i + \frac{i}{m\omega} \hat{p}_i \right) \right] \\ &= \frac{m\omega}{2\hbar} \left([\hat{x}_i, \hat{x}_j] + \frac{1}{m^2\omega^2} [\hat{p}_i, \hat{p}_j] - \frac{i}{m\omega} ([\hat{x}_j, \hat{p}_i] + [\hat{x}_i, \hat{p}_j]) \right) = \delta_{ij} \\ \hat{H} &= \frac{1}{2m} \\ \hat{L}^i &= -i\hbar \epsilon^{ijk} \hat{a}_j^{\dagger} \hat{a}_k \end{aligned}$$

Chapter 4

Making Second Quantization Work

Problem 1.

$$\begin{aligned}\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right]_\zeta &= \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \left[\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right]_\zeta = 0 \\ \hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}) &= \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y}) \\ &= -\zeta\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \delta^{(3)}(\mathbf{x} - \mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \\ &= -\zeta^2\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) + \delta^{(3)}(\mathbf{x} - \mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})\end{aligned}$$

So $\zeta = \pm 1$ yields the same result regardless of bosons or fermions.

Problem 2.

$$\begin{aligned}\hat{\rho}_1(\mathbf{x} - \mathbf{y}) &= \left\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \right\rangle \\ &= \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{\mathbf{q}} \hat{a}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{p}\mathbf{q}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{q}} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})}\end{aligned}$$

Problem 3.

$$|\hat{H} - \lambda\hat{I}| = \begin{vmatrix} U - \lambda & -t & -t & 0 \\ -t & -\lambda & 0 & -t \\ -t & 0 & -\lambda & -t \\ 0 & -t & -t & U - \lambda \end{vmatrix} =$$

Chapter 5

Continuous systems

Problem 1.

$$\begin{aligned}\int_a^b ds &= \int_a^b \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} dt = \int_a^b \frac{dt}{\gamma} = \int_a^b L dt \\ \frac{\partial L^2}{\partial \mathbf{v}} &= \frac{2\mathbf{v}}{c^2} \\ \frac{d}{dt} \left(\frac{\partial L^2}{\partial \mathbf{v}} \right) - \frac{\partial L^2}{\partial \mathbf{x}} &= \frac{2\dot{\mathbf{v}}}{c^2} = 0\end{aligned}$$

Since the acceleration is zero, the velocity is constant. Hence a straight world-line path does minimise the interval.

Problem 2.

$$\begin{aligned}L &= \frac{-mc^2}{\gamma} + q\mathbf{A} \cdot \mathbf{v} - qV \\ \nabla L &= q[\nabla(\mathbf{A} \cdot \mathbf{v}) - \nabla V] \\ &= q[(\mathbf{A} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{A} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}) - q\nabla V] \\ &= q[\mathbf{E} + \mathbf{v} \times \mathbf{B}], \quad \because \mathbf{E} = -q\nabla V, \mathbf{B} = \nabla \times \mathbf{A} \\ \frac{\partial L}{\partial \mathbf{v}} &= -\frac{mc^2}{2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \left(-\frac{2\mathbf{v}}{c^2} \right) = \gamma m\mathbf{v} \\ \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} &= \nabla L \longrightarrow \frac{d}{dt}(\gamma m\mathbf{v}) = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]\end{aligned}$$

Problem 3.

$$\begin{aligned}L &= \frac{-mc^2}{\gamma} + q\mathbf{A} \cdot \mathbf{v} - qV \approx \frac{1}{2}m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - qV \\ \mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}\end{aligned}$$

Finding the Hamiltonian is equivalent to finding the energy in terms of momentum:

$$\begin{aligned}
 H &= \mathbf{p} \cdot \mathbf{v} - L = m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - L \\
 &= mc^2 + \frac{1}{2}m\mathbf{v}^2 + qV = mc^2 + \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV, \quad \mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{m}
 \end{aligned}$$

Chapter 6

A first stab at relativistic quantum mechanics

Problem 1.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 \\ \frac{\partial \mathcal{L}}{\partial \phi} &= -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \\ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) &= 0 \\ (\partial^2 + m^2)\phi &= 0 \\ \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^0 \phi = \dot{\phi} \\ \mathcal{H} = \pi \dot{\phi} - \mathcal{L} &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2\end{aligned}$$

Chapter 7

Examples of Lagrangians, or how to write down a theory

Problem 1.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1} = 0$$

$$(\partial^2 + m^2) \phi + \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1} = 0$$

Problem 2.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}[\partial_\mu \phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2 + J(x)\phi(x) \\
\frac{\partial \mathcal{L}}{\partial \phi(x)} &= -m^2\phi(x) + J(x) \\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} &= \partial^\mu \phi(x) \\
\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) &= 0 \\
\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) - J(x) &= 0 \\
(\partial_\mu \partial^\mu + m^2)\phi(x) &= J(x)
\end{aligned}$$

Problem 3.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 - g(\phi_1^2 + \phi_2^2)^2 \\
\frac{\partial \mathcal{L}}{\partial \phi_1} &= -m^2\phi_1 - 4g\phi_1(\phi_1^2 + \phi_2^2) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - 4g\phi_2(\phi_1^2 + \phi_2^2) = 0 \\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} &= \partial^\mu \phi_1, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2 \\
\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_1(\phi_1^2 + \phi_2^2) &= 0 \\
\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_2(\phi_1^2 + \phi_2^2) &= 0
\end{aligned}$$

Problem 4. Referring to Chapter 5's solution:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Chapter 8

The passage of time

Problem 1.

Problem 2.

$$\begin{aligned}\hat{H} &= \sum_k E_k \hat{a}_k^\dagger \hat{a}_k \\ \hat{a}_k^\dagger(t) &= e^{i\hat{H}t/\hbar} \hat{a}_k^\dagger(0) e^{-i\hat{H}t/\hbar} \\ \frac{d\hat{a}_k^\dagger(t)}{dt} &= \frac{i}{\hbar} \left(e^{i\hat{H}t/\hbar} [\hat{H}, \hat{a}_k^\dagger(0)] e^{-i\hat{H}t/\hbar} \right) \\ &= \frac{iE_k}{\hbar} \left(e^{i\hat{H}t/\hbar} [\hat{n}_k, \hat{a}_k^\dagger(0)] e^{-i\hat{H}t/\hbar} \right) = \frac{iE_k}{\hbar} \hat{a}_k^\dagger(t) \\ \int \frac{d\hat{a}_k^\dagger(t)}{\hat{a}_k^\dagger(t)} &= \int \frac{iE_k}{\hbar} dt \longrightarrow \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0) e^{iE_k t/\hbar}\end{aligned}$$

Problem 3.

$$\begin{aligned}\hat{X}(t) &= e^{i\hat{H}t/\hbar} X_{lm} \hat{a}_l^\dagger \hat{a}_m e^{-i\hat{H}t/\hbar} \\ \frac{d\hat{X}}{dt} &= \end{aligned}$$

Problem 4.

$$\begin{aligned}\frac{d\hat{S}_H^z}{dt} &= \frac{1}{i\hbar} [\hat{S}_H^z, \omega \hat{S}_H^y] = \frac{\omega}{i\hbar} [\hat{S}_H^z, \hat{S}_H^y] = \frac{\omega}{i\hbar} (-i\hbar \hat{S}_H^x) = -\omega \hat{S}_H^x \\ \frac{d\hat{S}_H^x}{dt} &= \frac{1}{i\hbar} [\hat{S}_H^x, \omega \hat{S}_H^y] = \frac{\omega}{i\hbar} [\hat{S}_H^x, \hat{S}_H^y] = \frac{\omega}{i\hbar} (i\hbar \hat{S}_H^z) = \omega \hat{S}_H^z\end{aligned}$$

Chapter 9

Quantum mechanical transformations

Problem 1.

$$\begin{aligned}\hat{U}(\mathbf{a}) &= \exp[-i\hat{\mathbf{p}} \cdot \mathbf{a}] \\ \left. \frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{a}=0} &= -i\hat{\mathbf{p}} \exp[-i\hat{\mathbf{p}} \cdot 0] \\ \hat{\mathbf{p}} &= -\frac{1}{i} \left. \frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{a}=0}\end{aligned}$$

Problem 2.

$$K = \frac{1}{i} \left. \frac{\partial \Lambda(\phi^1)}{\partial \phi^1} \right|_{\phi^1=0} = \frac{1}{i} \left. \begin{vmatrix} \sinh \phi^1 & \cosh \phi^1 & 0 & 0 \\ \cosh \phi^1 & \sinh \phi^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \right|_{\phi^1=0} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3. Going to the MCRF and composing boosts:

$$\Lambda_{\nu}^{\mu} = \lim_{\mathbf{v} \rightarrow 0} \begin{bmatrix} \gamma & \gamma v^1 & \gamma v^2 & \gamma v^3 \\ \gamma v^1 & \gamma & 0 & 0 \\ \gamma v^2 & 0 & \gamma & 0 \\ \gamma v^3 & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix}$$

For an infinitesimal counter-clockwise rotations, compose the matrices:

$$\Lambda_\nu^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & 0 \\ 0 & -\theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \theta^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta^1 \\ 0 & 0 & -\theta^1 & 1 \end{bmatrix}$$

$$\Lambda_\nu^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Compose the boosts and rotation matrices:

$$\Lambda_\nu^\mu = \Lambda_\nu^\mu \Lambda_\nu^\nu = L_z R_z L_y R_y L_x R_x$$

$$\Lambda_\nu^\mu = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Extracting the identity matrix, the general infinitesimal Lorentz transformation can be written as:

$$\Lambda = \mathbf{1} + \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

The following tensors are indeed antisymmetric:

$$\omega^{\mu\nu} = \omega_\lambda^\mu g^{\lambda\nu} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega_\nu^\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & -\theta^3 & \theta^2 \\ -v^2 & \theta^3 & 0 & -\theta^1 \\ -v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

Chapter 10

Symmetry

Problem 1.

$$[\phi(x), P^\alpha] = \phi(x)P^\alpha - P^\alpha\phi(x) = \int \phi(x)T^{0\alpha} d^3y - \int T^{0\alpha}\phi(x)d^3y$$

Problem 2.

Problem 3.

$$\begin{aligned} T^{\mu\nu} &= \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \\ T^{00} &= \Pi^0\partial^0\phi - g^{00}\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right] = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \\ \partial_\mu T^{\mu\nu} &= \partial_\mu[\partial^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}] \\ &= \partial^2\phi\partial^\nu\phi - \partial^\mu\phi\partial_\mu\partial^\nu\phi - \frac{1}{2}[\partial^\rho\phi\partial^\nu\partial_\rho\phi + \partial_\rho\phi\partial^\nu\partial^\rho\phi - 2m^2\phi\partial^\nu\phi] \\ &= (\partial^2 + m^2)\phi(\partial^\nu\phi) = 0 \\ P^i &= \int T^{0i} d^3x = \int (\Pi^0\partial^i\phi - g^{0i}\mathcal{L}) d^3x = \int \partial^0\phi\partial^i\phi d^3x \end{aligned}$$

The Klein-Gordon equation, which is the equation of motion for scalar field theory, satisfies the divergence of the energy-momentum tensor.

Problem 4.

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu] \\
\frac{\partial(\partial_\mu A_\nu \partial^\mu A^\nu)}{\partial(\partial_\sigma A_\rho)} &= \delta_\mu^\sigma \delta_\nu^\rho \partial^\mu A^\nu + \partial_\mu A_\nu g^{\alpha\sigma} g^{\rho\beta} \delta_\alpha^\mu \delta_\beta^\nu = 2\partial^\sigma A^\rho \\
\frac{\partial(\partial_\mu A_\nu \partial^\nu A^\mu)}{\partial(\partial_\sigma A_\rho)} &= \delta_\mu^\sigma \delta_\nu^\rho \partial^\nu A^\mu + \partial_\mu A_\nu g^{\alpha\rho} g^{\sigma\beta} \delta_\alpha^\mu \delta_\beta^\nu = 2\partial^\rho A^\sigma \\
\frac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\rho)} &= -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho} = \Pi^{\sigma\rho} \\
T_\nu^\mu &= \Pi^{\mu\sigma} \partial_\nu A_\sigma - \delta_\nu^\mu \mathcal{L} \\
T^{\mu\nu} &= g^{\alpha\nu} T_\alpha^\mu = -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
X^{\lambda\mu\nu} &= F^{\mu\lambda} A^\nu = -F^{\lambda\mu} A^\nu = X^{\mu\lambda\nu} \\
\tilde{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\nu X^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\nu (F^{\mu\lambda} A^\nu) \\
&= -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \cancel{\partial_\lambda F^{\mu\lambda} A^\nu} + F^{\mu\lambda} \partial_\lambda A^\nu \\
&\stackrel{\lambda \rightarrow \sigma}{=} F^{\mu\sigma} (\partial_\sigma A^\nu - \partial^\nu A_\sigma) + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = F^{\mu\sigma} F_\sigma^\nu + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
\tilde{T}^{00} &= F^{0\sigma} F_\sigma^0 + \frac{1}{4}g^{00} F_{\alpha\beta} F^{\alpha\beta} = \mathbf{E}^2 + \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \\
\tilde{T}^{i0} &= F^{i\sigma} F_\sigma^0 + \cancel{\frac{1}{4}g^{i0} F_{\alpha\beta} F^{\alpha\beta}} = \epsilon^{ijk} E_j B_k = (\mathbf{E} \times \mathbf{B})^i
\end{aligned}$$

Chapter 11

Canonical quantization of fields

Problem 1.

$$\begin{aligned} [\hat{\phi}(x), \hat{\phi}(y)] &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \\ &\quad \int \frac{d^3q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) - \int \frac{d^3q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) \\ &\quad \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \\ &= \int d^3p \int \frac{d^3q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} ([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}] e^{-ip \cdot x} e^{iq \cdot y} + [\hat{a}_{\mathbf{p}}^{\dagger}, \hat{a}_{\mathbf{q}}] e^{ip \cdot x} e^{-iq \cdot y}) \\ &= \int d^3p \int \frac{d^3q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip \cdot x} e^{iq \cdot y} - \delta^{(3)}(\mathbf{q} - \mathbf{p}) e^{ip \cdot x} e^{-iq \cdot y} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip \cdot (x-y)} - e^{-ip \cdot (y-x)} \right) \end{aligned}$$

Problem 2.

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\Pi}^0(y) \right] &= \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \\
&\quad \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} - \hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot y}) \\
&- \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} - \hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot y}) \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \\
&= \frac{i}{2} \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(E_{\mathbf{p}} E_{\mathbf{q}})^{\frac{1}{2}}} ([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] e^{-ip \cdot x} e^{iq \cdot y} + [\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}^\dagger] e^{ip \cdot x} e^{-iq \cdot y}) \\
&= \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}} E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip \cdot x} e^{iq \cdot y} + \delta^{(3)}(\mathbf{q} - \mathbf{p}) e^{ip \cdot x} e^{-iq \cdot y} \right) \\
&= \frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} \left(e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)} \right)
\end{aligned}$$

Chapter 12

Examples of canonical quantization

Problem 1.

$$\begin{aligned}
\mathcal{H} &= \partial^0 \hat{\psi}^\dagger \hat{\psi} + \partial^0 \hat{\psi} \hat{\psi}^\dagger + \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} + m^2 \hat{\psi}^\dagger \hat{\psi} \\
&= \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} - \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \\
&+ \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (-iE_{\mathbf{p}}) \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} - \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) \\
&+ \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-i\mathbf{q}) \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} - \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) \cdot \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} (i\mathbf{p}) \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} - \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \\
&+ m^2 \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \\
&= \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{-iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{-i(p+q) \cdot x} - \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x} \right) \\
&+ \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{-iE_{\mathbf{p}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger e^{-i(p+q) \cdot x} - \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{q}} e^{i(p+q) \cdot x} \right) \\
&+ \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{\mathbf{p} \cdot \mathbf{q}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{-i(p+q) \cdot x} + \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x} \right) \\
&+ m^2 \int d^3 p \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{-i(p+q) \cdot x} + \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x} \right)
\end{aligned}$$

Problem 2.

$$\begin{aligned}
\left[\hat{\psi}(x), \hat{\psi}^\dagger(y) \right] &= \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \\
&\int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) - \int \frac{d^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x} \right) \\
&\int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x} \right)
\end{aligned}$$

Problem 3.

$$\begin{aligned}
(a) \quad \begin{bmatrix} \phi'_1 \\ \phi'_2 \end{bmatrix} &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \\
\left[\hat{Q}_N, \hat{\phi}_1 \right] &= -iD\hat{\phi}_1 = i\hat{\phi}_2
\end{aligned}$$

$$(b) \quad \left[\hat{Q}_N, \hat{\phi}_2 \right] = -iD\hat{\phi}_1 = -i\hat{\phi}_1$$

$$(c) \quad \left[\hat{Q}_N, \hat{\psi} \right] = \frac{1}{\sqrt{2}} \left[\hat{Q}_N, \hat{\phi}_1 \right] + \frac{i}{\sqrt{2}} \left[\hat{Q}_N, \hat{\phi}_2 \right] = \frac{i}{\sqrt{2}} \hat{\phi}_2 + \frac{1}{\sqrt{2}} \hat{\phi}_1 = \hat{\psi}$$

Problem 4. Note: $D\hat{\theta} = \pm 1$. Substituting:

$$\begin{aligned}
\left[\hat{Q}_N, \hat{\theta} \right] &= -iD\hat{\theta} = i \\
\left[\int \rho(\mathbf{x}, t) d^3 x, \theta(\mathbf{x}, t) \right] &= \int d^3 \mathbf{x} [\rho, \theta]
\end{aligned}$$

Problem 5.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) &= \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \Pi_\psi^\mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \psi} &= -V(x) \psi^\dagger(x), \quad \Pi_\psi^0 = i \psi^\dagger \\
\partial_0 \Pi_\psi^0 &= i \partial_0 \psi^\dagger, \quad \partial_i \Pi_\psi^i = -\frac{1}{2m} \nabla^2 \psi^\dagger \\
\therefore i \partial_0 \psi^\dagger - \frac{1}{2m} \partial_i \partial^i \psi^\dagger - V(x) \psi^\dagger(x) &= 0 \\
\longrightarrow i \partial_0 \psi^\dagger &= \hat{H} \psi^\dagger, \quad \hat{H} = -\frac{1}{2m} \nabla^2 + \hat{V} \\
V = 0 \longrightarrow i \frac{\partial \psi}{\partial t} &= -\frac{1}{2m} \nabla^2 \psi \\
iT''(t)X(x) &= -\frac{1}{2m} X''(x)T(t) \\
\frac{T'}{T} &= -iE \longrightarrow T(t) = Ae^{-iEt} \\
X'' + 2mEX &= 0 \longrightarrow X(x) = Be^{ipx} + Ce^{-ipx}, \quad p = \sqrt{2mE} \\
T(t)X(x) &= Ae^{i(p x - Et)} + Be^{-i(p x - Et)}
\end{aligned}$$

Problem 6.

$$\begin{aligned}
J_N^0 &= i\Psi^\dagger(i\Psi) + i\Psi(-i\Psi^\dagger) \\
Q_{N_c} &= \int \hat{\Psi} \hat{\Psi}^\dagger - \hat{\Psi}^\dagger \hat{\Psi} \, d^3x \\
&= \int d^3x \left[\int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{p}} e^{-ip \cdot x} \int \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot x} - \int \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot x} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{p}} e^{-ip \cdot x} \right] \\
&\quad \frac{1}{(2\pi)^3} \int d^3x \left[\int d^3\mathbf{p} \int d^3\mathbf{q} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger e^{i(p-q) \cdot x} - \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{-i(p-q) \cdot x} \right] \\
&= \int d^3\mathbf{p} \int d^3\mathbf{q} [\hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger \delta^3(\mathbf{p} - \mathbf{q}) - \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} \delta^3(\mathbf{q} - \mathbf{p})] \\
&= \int d^3\mathbf{p} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}}^\dagger] = \mathbf{p}
\end{aligned}$$

Chapter 13

Fields with many components and massive electromagnetism

Problem 1. \vec{J} represents the Levi-Civita tensor as a vector of matrices.

$$(a) \quad \hat{\vec{Q}}_{N_c} = \int d^3p \, \hat{\mathbf{A}}^\dagger \vec{J} \hat{\mathbf{A}}$$

The inverse transformations and resultant computations are as follows:

$$(b) \quad \hat{a}_1 = \frac{1}{\sqrt{2}}(\hat{b}_{-1} - \hat{b}_1), \quad \hat{a}_2 = -\frac{i}{\sqrt{2}}(\hat{b}_{-1} + \hat{b}_1), \quad \hat{a}_3 = \hat{b}_0$$

$$\hat{Q}_{N_c}^2 =$$

$$\hat{Q}_{N_c}^3 = -i \int d^3p \left(\hat{a}_{1\mathbf{p}}^\dagger \hat{a}_{2\mathbf{p}} - \hat{a}_{2\mathbf{p}}^\dagger \hat{a}_{1\mathbf{p}} \right) = \int d^3p \left(\hat{b}_{1\mathbf{p}}^\dagger \hat{b}_{1\mathbf{p}} - \hat{b}_{-1\mathbf{p}}^\dagger \hat{b}_{-1\mathbf{p}} \right)$$

$$J_b^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_b^2 = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_b^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 2.

Chapter 14

Path integrals: I said to him, 'You're crazy'

Problem 1.

$$(a) \quad I(a) = -2 \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = -2\sqrt{\frac{2\pi}{a}}$$

$$I'(a) = \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{ax^2}{2}\right) dx = \sqrt{\frac{2\pi}{a^3}}$$

$$(b) \quad J_n(a) = (-2)^{\frac{n}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = (-2)^{\frac{n}{2}} \sqrt{\frac{2\pi}{a}}$$

$$\frac{d^k J_n(a)}{da^k} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{(-1/2)!}{(-1/2 - k)!} a^{-\frac{1}{2} - k}$$

$$\frac{d^{n/2} J_n(a)}{da^{n/2}} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{\Gamma(1/2)}{\Gamma(\frac{1-n}{2})} a^{-(\frac{n+1}{2})} = \frac{i^n \pi}{\Gamma(\frac{1-n}{2})} \left(\frac{2}{a}\right)^{\frac{n+1}{2}}$$

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n \exp\left(-\frac{ax^2}{2}\right) dx}{\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx} = \frac{i^n \sqrt{\pi}}{\Gamma(\frac{1-n}{2})} \left(\frac{2}{a}\right)^{n/2}$$

$$= \begin{cases} 0 & \forall n \in 2\mathbb{Z}^+ + 1 \\ a^{-n/2} \prod_{k=1}^{n/2} (2k-1) & \forall n \in 2\mathbb{Z}^+ \end{cases}$$

$$\therefore \frac{d^n J_{2n}(a)}{da^n} = \frac{1}{a^n} \prod_{k=1}^n (2k-1)$$