Hopf Algebras and Combinatorics

Vinay Madhusudanan and Arjit Seth
Manipal Institute of Technology, Manipal University

Context

These are notes created based on the lectures by Prof. Federico Ardila at San Francisco State University. Hopf algebras seem to be a difficult topic to introduce as a simple definition. They apparently have applications in quantum field theory, pertinent to Feynman diagrams.

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1 Introduction

A Hopf algebra is a complicated mathematical structure with a definition involving lots of properties and scary commutative diagrams. It fundamentally invokes the tensor product, and it is difficult to construct as a complete definition in one sitting, so we must start with smaller definitions and build up the structure progressively.

Definition 1 (Hopf algebra)**.** A *Hopf algebra* H is a $\mathbb{K}$-vector space with five operations:

$$\begin{aligned}
\text{Multiplication:} \quad & m\colon H \otimes H \to H \\
\text{Unit:} \quad & u\colon \mathbb{K} \to H \\
\text{Comultiplication:} \quad & \Delta\colon H \to H \otimes H \\
\text{Counit:} \quad & \epsilon\colon H \to \mathbb{K} \\
\text{Antipode:} \quad & S\colon H \to H
\end{aligned}$$

These definitions invoke lots of commutative diagrams that are difficult to TeX, so they'll be shown later once a better understanding is developed. Now we must preliminarily define the tensor product.

Definition 2 (Tensor product)**.** Let $\mathbb{K}$ be a field, and let V and W be vector spaces over $\mathbb{K}$. The tensor product $V \otimes W$ is a vector space over $\mathbb{K}$ generated by vectors $v \otimes w$, $v \in V$, $w \in W$, and satisfying the following properties:

Distributivity over addition: $\quad (v + v') \otimes (w + w') = v \otimes w + v \otimes w' + v' \otimes w + v' \otimes w'$
Scalar multiplication independent of two arguments: $\quad \lambda(v \otimes w) = \lambda v \otimes w = v \otimes \lambda w$

The combination of these is a property called *bilinearity.* Note that this vector space is much larger than the product space $V \times W$:

$$\begin{aligned} \dim U \times V &= \dim U + \dim V \\ \dim U \otimes V &= \dim U \dim V \end{aligned}$$

This is intuitively obvious because the tensor product defines an *actual* product between elements of V and W, instead of a restricted component structure induced by a Cartesian product, which constrains manipulations to V and W independently.

Example (Tensor product of bases)**.** If $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ are bases of V and W, then $\{v_i \otimes w_j \mid i \in I,\ j \in J\}$ is a basis for $V \otimes W$. Let $\{v_i\}$ and $\{w_j\}$ be the standard bases in two and three dimensions, represented as column and row vectors respectively. One element of the basis for the tensor product is:

$$v_1 \otimes w_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is clear that this construction develops the standard basis for $V_2 \otimes W_3$.

Example (Polynomial ring and matrices)**.** Let $V = \mathbb{R}[x]$ and $W = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. A formal expression of an element in $V \otimes W$ is, for example, $(2 + 2x) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example (Dirac matrices)**.** The following anticommutation relations have a corresponding matrix representation called the Pauli matrices:

$$\sigma_i^2 = I_2, \ \ \sigma_i \sigma_j + \sigma_j \sigma_i = 0, \ i \neq j$$
$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To develop the relativistic theory of the electron, Dirac constructed 4×4 matrices γ^μ, $\mu = 0, 1, 2, 3$, out of the Pauli matrices using tensor products. The details are not relevant, so we will just display the results:

$$\gamma := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$\gamma^0 = \sigma_3 \otimes \mathbf{1}_2 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
$$\gamma^i = \gamma \otimes \sigma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \ \ i = 1, 2, 3$$

The Pauli and Dirac matrices form Clifford algebras respectively, which I don't want to talk about, Mark.

1.1 Motivating Examples for Hopf Algebras

Example 1 (Groups)**.** Let G be a *finite group* and $\mathbb{K}$ a *field*. To allow multiplication of scalars into the group, define the group ring:

$$H = \mathbb{K}G = \left\{ \sum_{i=1}^{n} \lambda_i g_i \;\middle|\; n \in \mathbb{N},\ \forall\, \lambda_i \in \mathbb{K},\ \forall\, g_i \in G \right\}$$

Multiplication: $m(g \otimes h) = gh$. Extending this linearly:

$$m\left(\sum_{i=1}^{n} \lambda_i g_i \otimes \sum_{j=1}^{n} \mu_j h_j\right) = \left(\sum_{i=1}^{n} \lambda_i g_i\right)\left(\sum_{j=1}^{n} \mu_j h_j\right) = \sum_{i,j}^{n,n} \lambda_i \mu_j (g_i h_j)$$

Comultiplication: $\Delta(g) = g \otimes g$. Extending this linearly:

$$\Delta\left(\sum_{i=1}^{n} \lambda_i g_i\right) = \sum_{i=1}^{n} \lambda_i (g_i \otimes g_i)$$

These definitions do give a Hopf algebra according to the lecturer.

Remark. The comultiplication $\Delta(g) = 1 \otimes g + g \otimes 1 + g \otimes g$ should also be valid, as the mapping *should* point to all possible information of where g can come from. As we will see later, this idea will be partially evident in future definitions of different structures.

Example 2 (Polynomial rings)**.** Let $H = \mathbb{K}[X]$, the *polynomial ring*.

Multiplication: $m\big(X^i \otimes X^j\big) = X^{i+j}$. Extending this linearly:

$$m\left(\sum_{i}^{k} \alpha_i X^i \otimes \sum_{j}^{l} \beta_j X^j\right) = \sum_{i,j}^{k,l} \alpha_i \beta_j X^{i+j}, \ \ \forall\, \alpha_i, \beta_j \in \mathbb{K},\ \forall\, k, l \in \mathbb{N}$$

Remark. This is a commutative multiplication because the ring forms a commutative algebra by definition, making the multiplication of the tensor product commutative in i and j. This linear extension is obvious because the ring is commutative in addition by definition.

Comultiplication: Extending the logic from groups:

$$\Delta(X) = (1 \otimes X) + (X \otimes 1)$$

Remark. The combination $(X \otimes X)$ is not included here probably because the addition of ring elements is analogous to multiplication of group elements, justifying $g \otimes g$ in the definition for groups.

This ring already has a multiplicative structure, which the coproduct should obey naturally:

$$\begin{aligned}\Delta\left(X^2\right) &= [1 \otimes X + X \otimes 1] \cdot [1 \otimes X + X \otimes 1] \\ &= 1 \otimes X^2 + 2(X \otimes X) + X^2 \otimes 1\end{aligned}$$

This suggests the following extension:

$$\Delta(X^n) = \sum_{i}^{n} \binom{n}{i} \left(X^i \otimes X^{n-i}\right)$$

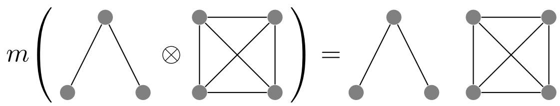
Notice the following, which shows the information about the coefficient of a comultiplication:

$$m(\Delta(X^n)) = \sum_{i=1}^{n} \binom{n}{i} X^n = 2^n X^n$$

Example 3 (Graphs)**.** Let $[G]$ be the *isomorphism class of the graph* $G = (V, E)$. The class of all isomorphic graphs is denoted by $L = \{\, [G] \mid\ G \in \mathcal{G}\,\}$, where $\mathcal{G}$ is the class of all graphs. Let the vector space $H = \mathbb{K}L$, in which the elements are linear combinations of graphs:

$$H = \left\{ \sum_i k_i [G_i] \,\middle|\, k \in \mathbb{K},\ G \in \mathcal{G} \right\}$$

Multiplication: Defined most naturally as a disjoint union of the arguments as components of the graph, which is commutative:

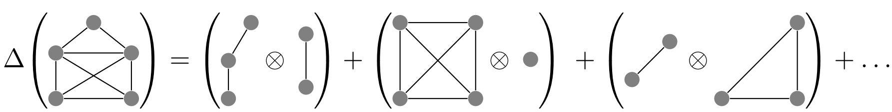


(1)

Comultiplication: Not entirely obvious, but a satisfactory definition can be deduced from the previous examples and observations. The comultiplication structure of an element is essentially a decomposition of the components by taking complementary combinations of the subelements of the algebraic structure (FIX THIS).
This idea can be implemented in this vector space as well by selecting a subset S of the vertex set of the graph $V(G)$, constructing the graph based on the existence of edges between the vertices of S, and taking a tensor product with its complement. This is shown by (dropping the isomorphism class notation for brevity):

$$\Delta(G) = \sum_{S \in V(G)} G\big|_S \otimes G\big|_{V(G)-S}$$

This is illustrated using the following example:



The number of tensor products is easily countable as the number of subsets of $V(G)$, which is 2^n, where n is the number of vertices of the graph G. For the above example, there are 5 vertices, so there are 32 tensor products to be summed over.

Remark. The product of the coproduct of a graph is:

$$m(\Delta(G)) \neq G$$

This indicates that the product of the coproduct of a graph does not bring back the original argument, so m and Δ are not inverse operations. (Note: This is also evident from the group and the polynomial ring examples.)

Example 4 (Permutations)**.** Let P_n denote the set of *all permutations of the subsets* of $\mathbb{N} \pmod{n}$, where n is finite. To clarify with an example, take the set $\mathbb{N} \pmod 5 = \{1, 2, 3, 4, 5\}$. Here are two permutations of P_5:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Remark. Although they might have the 'same' action, they are fundamentally different elements (of S_5 and S_2, respectively in the above example) by definition. The set P_n consists of elements of the permutation groups S_1 to S_n. (CHECK)

The vector space is defined as:

$$H = \mathbb{K}P_n = \left\{ \sum_i^n k_i P_i \;\middle|\; \forall\, k_i \in \mathbb{K},\ \forall\, P_i \in P_n,\ \forall n \in \mathbb{N} \right\}$$

Note that the coefficients do not imply the number of times of application of their corresponding permutations; they just depict a formal linear extension for the vector space.

<u>Multiplication</u>: An example is $(1\,2) \otimes (1\,2\,3)$ using one-line notation:

$$m((1\,2) \otimes (3\,2\,1)) = 12543 + 15243 + 15423 + 15432 + 51243 + 51423 + \ldots$$

Which shows that it respects the order of the arguments while shuffling and generates only *some* permutations of S_5 as a result. This might be represented by using U_i to denote the ith entry of the permutation U:

$$m(U \otimes V) = m((U_1 \ldots U_j) \otimes (V_1 \ldots V_k))$$
$$= \sum p(U_1 \ldots V_{j+1} \ldots U_j \ldots V_{j+k}),\ \ n(U_j) < n(U_{j+1}),\ n(V_{i+j}) < n(V_{i+j+1})$$

Where $p(U_1 \ldots V_{i+1} \ldots U_i \ldots V_{i+j})$ stands for the permutations of the resultant set with the values of V_i right-shifted by j, under the rule that the partial order of the elements is not lost, using $n(U_i)$ to denote placement. (NEEDS IMPROVEMENT)

<u>Comultiplication</u>: Similar to graphs, the coproduct of permutations is obtained by cutting a line between the numbers (in one-line notation, e.g. $(1\,2 \mid 3\,4\,5)$), and writing the

outer product with each piece as a component, then summing over all complementary combinations.

$$\Delta(U_1U_2\dots U_n) = \sum_{i=0}^{n} (U_1\dots U_i)\otimes(U_{i+1}\dots U_n)$$

An example using a permutation of S_5:

$$\Delta(4\,2\,5\,3\,1) = (\phi\otimes 42531) + (1\otimes 2431) + (21\otimes 321) + \dots$$

Note: Substrings like $(4\,5\,3)$ are just elements of S_3, so they can be rewritten as $(2\,3\,1)$, and so on for substrings of varying lengths with their corresponding symmetric groups.

Remark. Nothing so far.

2 Algebras Over a Field/Ring

An *algebra over a field* is a structure that is simultaneously a *vector space* and a *ring*. The setup will be a field $\mathbb{K}$ and a ring A with a multiplicative identity 1, called a $\mathbb{K}$-algebra. Three definitions are introduced that will be shown to be equivalent progressively.

Definition 3 ($\mathbb{K}$-algebra)**.** A forms a $\mathbb{K}$-*algebra* if $\mathbb{K}\subset Z(A)$ and $1_{\mathbb{K}} = 1_A$, where $Z(A)$ is the centre of the ring.

Example (Polynomial rings)**.** Let $A = \mathbb{K}[x]$. This forms a $\mathbb{K}$-algebra because the ring is commutative and the elements of the field in the ring are the constant polynomials, and $1_{\mathbb{K}}$ is the same as 1_A. This directly generalises to the multivariate polynomial ring $B = \mathbb{K}[x_1,\dots,x_n]$, $n\in\mathbb{N}$, where n is finite.

An example of a ring that is a vector space, but does not form a $\mathbb{K}$-algebra is $A = \mathrm{Mat}_{n\times n}(\mathbb{K})$. This is because $\mathbb{K}\not\subset A$. However, copies of $\mathbb{K}$ exist as matrices in the form $\mathbb{K}I_{n\times n}$, where I is the identity matrix of corresponding dimensions. This structure should form a $\mathbb{K}$-algebra, which motivates the second definition:

Definition 4 ($\mathbb{K}$-algebra)**.** A is a $\mathbb{K}$-*algebra* if there is an embedding $u:\mathbb{K}\to A$ such that $u(\mathbb{K})\subseteq Z(A)$ and $u(1_{\mathbb{K}}) = 1_A$.

Remark. This allows $A = \mathrm{Mat}_{n\times n}(\mathbb{K})$ to form a $\mathbb{K}$-algebra because $u(\lambda) = \lambda I$, $\lambda\in\mathbb{K}$. An observation is that any A now forms a $\mathbb{K}$-vector space because scalar multiplication is implemented as $\lambda\cdot a := u(\lambda)a$, $\lambda\in\mathbb{K}$, $a\in A$. The polynomial ring $\mathbb{K}[x]$'s construction as a $\mathbb{K}$-algebra also follows directly from this definition.

An *algebra over a ring*, by analogy, is a module and a ring with a multiplicative identity defined similarly:

Definition 5 (R-algebra)**.** M is an R-*algebra* if there is a ring homomorphism $u: R\to M$ such that $u(R)\subseteq Z(A)$ and $u(1_R) = 1_A$ with a left or right multiplication specified.

Example ($\mathbb{Z}$-algebra)**.** Let Q be the set of quaternions, which forms a $\mathbb{Z}$-algebra because its center consists of all real quaternions, which contains a copy of $\mathbb{Z}$ as a subring.

2.1 Algebra Homomorphism

Definition 6. Let A_1, A_2 be $\mathbb{K}$-algebras. An *algebra homomorphism* is $\phi\colon A_1 \to A_2$ which is a ring homomorphism and linear map simultaneously.

Remark. Observations:

- $B \subseteq A$ is a subalgebra if B is both a subspace and subring.
- A structure that quotients the algebra must quotient the vector space and the ring simultaneously. A subspace quotients the vector space, and an ideal quotients the ring. All ideals form subspaces, therefore ideals quotient the algebra.
- If $\phi\colon A \to B$ is a $\mathbb{K}$-algebra homomorphism, then $\operatorname{Im}\phi \cong A/\ker\phi$, where $\ker\phi$ forms an ideal I. This is analogous to the first isomorphism theorem from groups, rings, vector spaces, etc.
- The direct product of two $\mathbb{K}$-algebras is also a $\mathbb{K}$-algebra, as the vector space and ring structures are independently inherited. The tensor product also forms a $\mathbb{K}$-algebra.

Definition 7 ($\mathbb{K}$-algebra)**.** A $\mathbb{K}$-algebra is a vector space A over a field $\mathbb{K}$ equipped with linear maps $m\colon A\otimes A \to A$, $u\colon \mathbb{K} \to A$ such that the following diagrams commute:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}\otimes A \arrow[r, "u\otimes\mathrm{id}"] \arrow[dr, "f"'] & A\otimes A \arrow[d, "m"] & A\otimes\mathbb{K} \arrow[l, "\mathrm{id}\otimes u"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

Remark. For the first diagram: $(m\otimes\mathrm{id})(f\otimes g\otimes h) = fg\otimes h$, $m(fg\otimes h) = fgh$ and $(\mathrm{id}\otimes m)(f\otimes g\otimes h) = f\otimes gh$, $m(f\otimes gh) = fgh$, which shows that the tensor product is *associative*:

$$m(m(f\otimes g)\otimes h) = m(f\otimes m(g\otimes h))$$

For the second diagram: $f(k\otimes v) = f(1\otimes kv) = kv$ and $u(1) = 1_A$, $(u\otimes\mathrm{id})(1\otimes kv) = 1_A\otimes kv$, $m(1_A\otimes kv) = k_A v$. This implies $u(k) = k_A$, implying copies of the field exist in the vector space as an embedding. Following a similar procedure for the other half, we find that the tensor product is *unitary*, so there is a multiplicative identity on the left and right under the multiplication m:

$$m(1_A\otimes v) = m(v\otimes 1_A) = v$$

This definition is equivalent to the previous one, which can be verified by checking if the structure shares the remaining property of a ring, *distributivity*, which is satisfied because of the linearity of m:

$$m(u\otimes(v+w)) = m(u\otimes v) + m(u\otimes w) = uv + uw$$

Remark. To show that the previous definition implies the current one, the only non-trivial construction required is m, which is defined as a multiplication under the previous definition: $m\colon A \otimes A \to A$ such that $m(a \otimes b) := ab$, this remark is a little troubling because we do not have a clear definition of the tensor product and its relations with other algebraic structures, which will be investigated now.

3 Tensor Products

Definition 8 (Free vector space)**.** Let $F(V \times W) = \mathbb{K} - \operatorname{span}\{(v, w) \mid v \in V,\, w \in W\}$, called the *free vector space* on $V \times W$.

Definition 9 (Tensor product)**.** The tensor product space is the quotient:

$$V \otimes W = F(V \times W)/I$$

where I is generated by linear relations:

$$\begin{gathered}(v_1 + v_2, w) \sim (v_1, w) + (v_2, w)\\ (v, w_1 + w_2) \sim (v, w_1) + (v, w_2)\\ (cv, w) \sim (v, cw) \sim c(v, w)\end{gathered}$$

Remark. The tensor $a \otimes b = \overline{(a, b)}$, an equivalence class, and $a' \otimes b'$ can be equal to $a \otimes b$, $a' \neq a$, $b' \neq b$.

To define a linear $m\colon V \otimes W \to W$, we would have to define an $n\colon F(V \times W) \to W$ such that $n(I) = 0$, which implies that n is a bilinear map.