Hopf Algebras and Combinatorics

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Context

These are notes created based on the lectures from San Francisco State University under Prof. Federico Ardila. Hopf algebras seem to be a difficult topic to introduce as a simple definition. They apparently have applications in quantum field theory, pertinent to Feynman diagrams.

Contents

1 Introduction

A Hopf algebra is a complicated mathematical structure with a definition involving lots of properties and scary commutative diagrams. It fundamentally invokes the tensor product, and it is difficult to construct as a complete definition in one sitting, so we must do it in pieces.

Definition 1 (Hopf algebra)**.** A *Hopf algebra* H is a $\mathbb{K}$-vector space with five operations:

$$\begin{aligned} \text{Multiplication:} \quad & m\colon H \otimes H \to H \\ \text{Unit:} \quad & u\colon \mathbb{K} \to H \\ \text{Comultiplication:} \quad & \Delta\colon H \to H \otimes H \\ \text{Counit:} \quad & \epsilon\colon H \to \mathbb{K} \\ \text{Antipode:} \quad & S\colon H \to H \end{aligned}$$

These definitions invoke lots of commutative diagrams that are difficult to TeX, so they'll be shown later once a better understanding is developed. Now we must preliminarily define the tensor product.

Definition 2 (Tensor product)**.** Let $\mathbb{K}$ be a field, and let V and W be vector spaces over $\mathbb{K}$. The tensor product $V \otimes W$ is a vector space over $\mathbb{K}$ generated by vectors $v \otimes w$, $v \in V$, $w \in W$, and satisfying the following properties:

Distributivity over addition: $(v+v')\otimes(w+w') = v\otimes w + v\otimes w' + v'\otimes w + v'\otimes w'$
Scalar multiplication independent of two arguments: $\lambda(v\otimes w) = \lambda v\otimes w = v\otimes \lambda w$

The combination of these is a property called *bilinearity*. Note that this vector space is much larger than the product space $V\times W$:

$$\dim U\times V = \dim U + \dim V$$
$$\dim U\otimes V = \dim U \dim V$$

This is intuitively obvious because the tensor product defines an *actual* product between elements of V and W, instead of a restricted component structure induced by a Cartesian product, which constrains manipulations to V and W independently.

Example (Tensor product of bases)**.** If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are bases of V and W, then $\{v_i\otimes w_j \mid i\in I,\ j\in J\}$ is a basis for $V\otimes W$. Let $\{v_i\}$ and $\{w_j\}$ be the standard bases in two and three dimensions, represented as column and row vectors respectively. One element of the basis for the tensor product is:

$$v_1\otimes w_3 = \begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

It is clear that this construction develops the standard basis for $V_2\otimes W_3$.

Example (Polynomial ring and matrices)**.** Let $V = \mathbb{R}[x]$ and $W = \mathrm{Mat}_{2\times 2}(\mathbb{R})$. A formal expression of an element in $V\otimes W$ is, for example, $(2+2x)\otimes\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

Example (Dirac matrices)**.** The following anticommutation relations have a corresponding matrix representation called the Pauli matrices:

$$\sigma_i^2 = I_2,\ \ \sigma_i\sigma_j + \sigma_j\sigma_i = 0,\ i\neq j \tag{1}$$

$$\sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\ \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{2}$$

To develop the relativistic theory of the electron, Dirac constructed 4×4 matrices $\gamma^\mu,\ \mu = 0,1,2,3$, out of the Pauli matrices using tensor products. The details are not relevant, so we will just display the results:

$$\gamma \equiv \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{3}$$

$$\gamma^0 = \sigma_3\otimes \mathbf{1}_2 = \begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{4}$$

$$\gamma^i = \gamma\otimes\sigma_i = \begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix},\ \ i = 1,2,3 \tag{5}$$

The Pauli and Dirac matrices form Clifford algebras respectively, which I don't know anything about.

1.1 Motivating Examples for Hopf Algebras

Example 1 (Groups)**.** Let G be a *finite group* and $\mathbb{K}$ a *field.* To allow multiplication of scalars into the group, define the group ring:

$$H = \mathbb{K}G = \left\{ \sum_{i=1}^{n} \lambda_i g_i \;\middle|\; n \in \mathbb{N},\ \forall\, \lambda_i \in \mathbb{K},\ \forall\, g_i \in G \right\} \tag{6}$$

<u>Multiplication</u>: $m(g \otimes h) = gh$. Extending this linearly:

$$m(g \otimes h) = \left(\sum_{i=1}^{n} \lambda_i g_i \right) \left(\sum_{j=1}^{n} \mu_j h_j \right) = \sum_{i,j}^{n,n} \lambda_i \mu_j (g_i h_j) \tag{7}$$

<u>Comultiplication</u>: $\Delta(g) = g \otimes g$. Extending this linearly:

$$\Delta \left(\sum_{i=1}^{n} \lambda_i g_i \right) = \sum_{i=1}^{n} \lambda_i (g_i \otimes g_i) \tag{8}$$

These definitions do give a Hopf algebra according to the lecturer.

Remark. The comultiplication $\Delta(g) = 1 \otimes g + g \otimes 1 + g \otimes g$ should also be valid, as the mapping *should* point to all possible information of where g can come from. As we will see later, this idea will be partially evident in future definitions of different structures.

Example 2 (Polynomial Rings)**.** Let $H = \mathbb{K}[X]$, the *polynomial ring.*

<u>Multiplication</u>: $m\big(X^i \otimes X^j\big) = X^{i+j}$. Extending this linearly:

$$m \left(\sum_{i}^{k} \alpha_i X^i \otimes \sum_{j}^{l} \beta_j X^j \right) = \sum_{i,j}^{k,l} \alpha_i \beta_j X^{i+j}, \quad \forall\, \alpha_i, \beta_j \in \mathbb{K},\ \forall\, k, l \in \mathbb{N} \tag{9}$$

Remark. This is a commutative multiplication because the ring forms a commutative algebra by definition, making the multiplication of the tensor product commutative in i and j. This linear extension is obvious because the ring is commutative in addition by definition.

<u>Comultiplication</u>: Extending the logic from groups:

$$\Delta(X) = (1 \otimes X) + (X \otimes 1) \tag{10}$$

Remark. The combination $(X \otimes X)$ is not included here probably because the addition of ring elements is analogous to multiplication of group elements, justifying $g \otimes g$ in the definition for groups.

This ring already has a multiplicative structure, which the coproduct should obey naturally:

$$\begin{aligned}\Delta\big(X^2\big) &= [1 \otimes X + X \otimes 1] \cdot [1 \otimes X + X \otimes 1] && (11)\\ &= 1 \otimes X^2 + 2(X \otimes X) + X^2 \otimes 1 && (12)\end{aligned}$$

This suggests the following extension:

$$\Delta(X^n) = \sum_{i}^{n} \binom{n}{i} \big(X^i \otimes X^{n-i}\big) \tag{13}$$

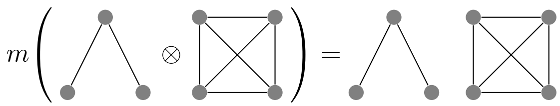
Notice the following, which shows the information about the coefficient of a comultiplication:

$$m(\Delta(X^n)) = \sum_{i}^{n} \binom{n}{i} X^n = 2^n X^n \tag{14}$$

Example 3 (Graphs)**.** Let $[G]$ be the *isomorphism class of the graph* $G = (V, E)$. The class of all isomorphic graphs is denoted by $L = \{\, [G] \mid\ G \in \mathcal{G}\,\}$, where $\mathcal{G}$ is the class of all graphs. Let the vector space $H = \mathbb{K}L$, in which the elements are linear combinations of graphs:

$$H = \left\{ \sum_{i} k_i [G_i] \,\middle|\, k \in \mathbb{K},\ G \in \mathcal{G} \right\} \tag{15}$$

Multiplication: Defined most naturally as a disjoint union of the arguments as components of the graph, which is commutative:



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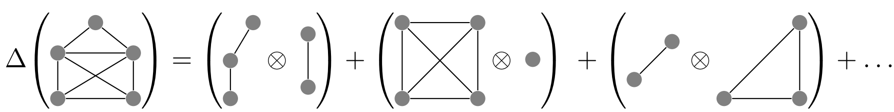
Comultiplication: Not entirely obvious, but a satisfactory definition can be deduced from the previous examples and observations. The comultiplication structure of an element is essentially a decomposition of the components by taking complementary combinations of the subelements of the algebraic structure (FIX THIS).
This idea can be implemented in this vector space as well by selecting a subset S of the vertex set of the graph $V(G)$, constructing the graph based on the existence of edges between the vertices of S, and taking a tensor product with its complement. This is shown by (dropping the isomorphism class notation for brevity):

$$\Delta(G) = \sum_{S \in V(G)} G\big|_S \otimes G\big|_{V(G)-S} \tag{17}$$

This is illustrated using the following example:



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The number of tensor products is easily countable as the number of subsets of $V(G)$, which is 2^n, where n is the number of vertices of the graph G. For the above example, there are 5 vertices, so there are 32 tensor products to be summed over.

Remark. The product of the coproduct of a graph is:

$$m(\Delta(G)) \neq G \tag{19}$$

This indicates that the product of the coproduct of a graph does not bring back the original argument, so m and Δ are not inverse operations. (Note: This is also evident from the group and the polynomial ring examples.)

Example 4 (Permutations)**.** Let P_n denote the set of *all permutations of the subsets* of $\mathbb{N}$ (mod n), where n is finite. To clarify with an example, take the set $\mathbb{N}$ (mod 5) $= \{1, 2, 3, 4, 5\}$. Here are two permutations of P_5:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{20}$$

Remark. Although they might have the 'same' action, they are fundamentally different elements (of S_5 and S_2, respectively in the above example) by definition. The set P_n consists of elements of the permutation groups S_1 to S_n. (CHECK)

The vector space is defined as:

$$H = \mathbb{K}P_n = \left\{ \sum_i^n k_i P_i \;\middle|\; \forall\, k_i \in \mathbb{K},\ \forall\, P_i \in P_n,\ \forall\, n \in \mathbb{N} \right\} \tag{21}$$

Note that the coefficients do not imply the number of times of application of their corresponding permutations; they just depict a formal linear extension for the vector space.

<u>Multiplication</u>: An example is $(1\,2) \otimes (1\,2\,3)$ using one-line notation:

$$m((1\,2) \otimes (3\,2\,1)) = 12543 + 15243 + 15423 + 15432 + 51243 + 51423 + \dots \tag{22}$$

Which shows that it respects the order of the arguments while shuffling and generates only *some* permutations of S_5 as a result. This might be represented by using U_i to denote the ith entry of the permutation U:

$$m(U \otimes V) = m((U_1 \dots U_j) \otimes (V_1 \dots V_k)) \tag{23}$$

$$= \sum p(U_1 \dots V_{j+1} \dots U_j \dots V_{j+k}),\ \ n(U_j) < n(U_{j+1}),\ n(V_{i+j}) < n(V_{i+j+1}) \tag{24}$$

Where $p(U_1 \dots V_{i+1} \dots U_i \dots V_{i+j})$ stands for the permutations of the resultant set with the values of V_i right-shifted by j, under the rule that the partial order of the elements is not lost, using $n(U_i)$ to denote placement. (NEEDS IMPROVEMENT)

<u>Comultiplication</u>: Similar to graphs, the coproduct of permutations is obtained by cutting a line between the numbers (in one-line notation, e.g. $(1\,2 \mid 3\,4\,5)$), and writing the

outer product with each piece as a component, then summing over all complementary combinations.

$$\Delta(U_1U_2\dots U_n) = \sum_{i=0}^{n} (U_1\dots U_i) \otimes (U_{i+1}\dots U_n) \tag{25}$$

An example using a permutation of S_5:

$$\Delta(4\,2\,5\,3\,1) = (\phi \otimes 42531) + (1 \otimes 2431) + (21 \otimes 321) + \dots \tag{26}$$

Note: Substrings like $(4\,5\,3)$ are just elements of S_3, so they can be rewritten as $(2\,3\,1)$, and so on for substrings of varying lengths with their corresponding symmetric groups.

Remark. Nothing so far.

2 Algebras Over a Field