Solutions to
Quantum Field Theory for the Gifted Amateur
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Chapter 1

Lagrangians

1. *Fermat's principle of least time.*

$$
\begin{gathered}
t = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{v_2} = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2} \\
\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{c/n_1\sqrt{x^2 + h_1^2}} - \frac{(l-x)}{c/n_2\sqrt{(l-x)^2 + h_2^2}} = 0 \\
n_1 \sin\theta = n_2 \sin\phi
\end{gathered}
$$

2. *Practice with functional derivatives.*

(a)

$$
\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int G(x,y)[f(y) + \epsilon\delta(y-z)]\,\mathrm{d}y - \int G(x,y) f(y)\,\mathrm{d}y\right] = G(x,z)
$$

(b)

$$
\begin{aligned}
\frac{\delta I[f^\alpha]}{\delta x_0} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int [f(x) + \epsilon\delta(x - x_0)]^\alpha - f(x)^\alpha\,\mathrm{d}x\right] = \alpha[f(x_0)]^{\alpha-1} \\
\therefore \frac{\delta I[f^3]}{\delta f(x_0)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int_{-1}^{1} [f(x) + \epsilon\delta(x - x_0)]^3\,\mathrm{d}x - \int_{-1}^{1} [f(x)]^3\,\mathrm{d}x\right] = 3[f(x_0)]^2 \\
\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \lim_{\epsilon \to 0} \frac{3}{\epsilon}\left[[f(x_0) + \epsilon\delta(x_0 - x_1)]^2 - f^2(x_0)\right] = 6f(x_1)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int \left(\frac{\partial}{\partial y}[f(y) + \epsilon\delta(y-x)]\right)^2 \mathrm{d}y - \int \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int \left[\frac{\partial f}{\partial y} + \epsilon\delta'(y-x)\right]^2 \mathrm{d}y - \int \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right] = 2\frac{\partial^2 f}{\partial x^2}
\end{aligned}
$$

3. *Euler-Lagrange equations using functional derivatives and more.*

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left[g(y,f)+\frac{\partial g(y,f)}{\partial f}\epsilon\delta(y-x)\right]\mathrm{d}y - \int g(y,f)\,\mathrm{d}y\right] = \frac{\partial g(x,f)}{\partial f(x)}$$

4. *Results on Dirac Delta functions.*

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{\phi(x)+\epsilon\delta(x-y)-\phi(x)}{\epsilon} = \delta(x-y)$$

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{\mathrm{d}}{\mathrm{d}t}[\phi(t)+\epsilon\delta(t-t_0)]-\dot{\phi}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0)$$

5. *Derivation of the wave equation.*

$$S = \int (T-V)\,\mathrm{d}t = \frac{1}{2}\int \rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \mathcal{T}(\nabla\psi)^2\,\mathrm{d}t$$

$$\frac{\delta S}{\delta\psi} = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left[\int \rho\left(\frac{\partial}{\partial t}[\psi+\epsilon\delta(t-t_0)]\right)^2 - \mathcal{T}(\nabla[\psi+\epsilon\delta(\mathbf{x}-\mathbf{y})])^2\ \mathrm{d}t\ - \int\rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \mathcal{T}(\nabla\psi)^2\ \mathrm{d}t\right]$$

$$= \int\left[\rho\frac{\partial}{\partial t}\delta(t-t_0)\frac{\partial\psi}{\partial t} - \mathcal{T}\nabla\delta(\mathbf{x}-\mathbf{y})\nabla\psi\right]\ \mathrm{d}t = 0$$

$$\implies \nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2},\quad v=\sqrt{\frac{\mathcal{T}}{\rho}}$$

6. *Functional derivative of a Wick expansion term in the generating functional.*

$$Z_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ J(x)\Delta(x-y)J(y)\right)$$

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ [J(x)+\epsilon\delta(x-z_1)]\Delta(x-y)[J(y)+\epsilon\delta(y-z_1)]\right)\right.$$
$$\left. - \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ J(x)\Delta(x-y)J(y)\right)\right]$$

Chapter 2

Simple harmonic oscillators

1. *Commutators of ladder operators.*

$$[\hat{a},\hat{a}^\dagger] = \frac{m\omega}{2\hbar}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right) - \frac{m\omega}{2\hbar}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)$$
$$= \frac{1}{2i\hbar}([\hat{x},\hat{p}]+[\hat{x},\hat{p}]) = 1$$

2. *Perturbation theory and ladder operators.* The perturbative term $\hat{H}_p = \lambda\hat{W} = \lambda\hat{x}^4$. Its first-order correction is:

$$E_n = E_n^{(0)} + \langle\phi_n|\hat{H}_p|\phi_n\rangle = \left(n+\frac{1}{2}\right)\hbar\omega + \left\langle n\middle|\lambda\hat{x}^4\middle|n\right\rangle$$
$$= \left(n+\frac{1}{2}\right)\hbar\omega + \lambda\left(\frac{\hbar}{2m\omega}\right)^2\left\langle n\middle|(\hat{a}+\hat{a}^\dagger)^4\middle|n\right\rangle$$
$$= \left(n+\frac{1}{2}\right)\hbar\omega + \lambda\left(\frac{\hbar}{2m\omega}\right)^2\langle n|painful|n\rangle$$

3. *Fourier transform of $\hat{x}_k$.*

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \hat{\tilde{x}}_k e^{ikja}, \quad \hat{\tilde{x}}_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k+\hat{a}^\dagger_{-k}\right)$$

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k\sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k+\hat{a}^\dagger_{-k}\right)e^{ikja} = \sqrt{\frac{h}{Nm}}\sum_k\frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_k e^{ikja}+\hat{a}^\dagger_k e^{-ikja}\right)$$

4. *Ground state of the harmonic oscillator.*

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|0\rangle = 0$$
$$\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle = 0$$
$$\left(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle = 0$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{m\omega}{\hbar}x\right)\langle x|0\rangle = 0$$

This is easily solved by separation of variables. Attempting a series solution for practice:

$$\langle x|0\rangle = \sum_{n=0}^{\infty} a_n x^n$$
$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} = 0$$
$$\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} = 0$$
$$a_{n+2} = -\frac{m\omega}{\hbar(n+2)}a_n, \quad a_0 = A,\ a_1 = 0$$
$$\langle x|0\rangle = A\left[1 + \left(-\frac{m\omega}{2\hbar}\right)x^2 + \frac{1}{2}\left(-\frac{m\omega}{2\hbar}\right)^2 x^4 + \frac{1}{6}\left(-\frac{m\omega}{2\hbar}\right)^3 x^6 + ...\right]$$
$$\langle x|0\rangle = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$
$$A = 1/\left|\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\right|$$
$$A = 1/\sqrt{\int_{-\infty}^{\infty}\exp\left(2\frac{m\omega}{2\hbar}x^2\right)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$
$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Chapter 3

Occupation number representation

1. *Practice with exponentials and ladder operators.*

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] = \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} = \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

2. *Ladder operator identities.*

(a)

$$\begin{aligned}\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] &= \left[\hat{a}\left(\hat{a}^\dagger\right)^n - \left(\hat{a}^\dagger\right)^n\hat{a}\right] \\ = \left[(1+\hat{a}^\dagger\hat{a})\left(\hat{a}^\dagger\right)^{n-1} - \left(\hat{a}^\dagger\right)^{n-1}\left(\hat{a}^\dagger\hat{a}\right)\right] &= \left[\left(\hat{a}^\dagger\right)^{n-1} - \left[\hat{a}^\dagger\hat{a}, \left(\hat{a}^\dagger\right)^{n-1}\right]\right]\end{aligned}$$

(b)

$$\langle 0|\hat{a}^n\left(\hat{a}^\dagger\right)^m|0\rangle = \sqrt{n!}\sqrt{m!}\,\langle n|m\rangle$$

(c)

(d)

3. *Three-dimensional harmonic oscillator.*

$$\begin{aligned}\hat{a}^\dagger_i &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right) \\ \left[\hat{a}_i, \hat{a}^\dagger_j\right] &= \frac{m\omega}{2\hbar}\left[\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right) - \left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right)\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\right] \\ &= \frac{m\omega}{2\hbar}\left([\hat{x}_i, \hat{x}_j] + \frac{1}{m^2\omega^2}[\hat{p}_i, \hat{p}_j] - \frac{i}{m\omega}([\hat{x}_j . \hat{p}_i] + [\hat{x}_i, \hat{p}_j])\right) = \delta_{ij} \\ \hat{a}^\dagger_i\hat{a}_i &= \frac{\hat{p}_i^2}{2m\hbar\omega} + \frac{1}{2\hbar\omega}m\omega^2\hat{x}_i^2 + \frac{i}{2\hbar}[\hat{x}_i, \hat{p}_i] = \frac{1}{\hbar\omega}\left[\frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i^2 - \frac{\hbar\omega}{2}\right] \\ \hat{H} &= \sum_{i=1}^{3}\frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_i^2 = \hbar\omega\sum_{i=1}^{3}\left(\hat{a}^\dagger_i\hat{a}_i + \frac{1}{2}\right) \\ \hat{L}^i &\equiv -i\hbar\epsilon^{ijk}\hat{a}^\dagger_j\hat{a}_k\end{aligned}$$

4. *Slater determinant for fermions.* Consider an n-particle state:

$$\langle \mathbf{p}'_1 \mathbf{p}'_2 \mathbf{p}'_3 \dots \mathbf{p}'_n | \mathbf{p}_n \mathbf{p}_{n-1} \mathbf{p}_{n-2} \dots \mathbf{p}_1 \rangle = \langle 0 | \hat{a}_{\mathbf{p}'_1} \hat{a}_{\mathbf{p}'_2} \hat{a}_{\mathbf{p}'_3} \dots \hat{a}_{\mathbf{p}_n} \hat{a}^\dagger_{\mathbf{p}_n} \hat{a}^\dagger_{\mathbf{p}_{n-1}} \hat{a}^\dagger_{\mathbf{p}_{n-2}} \dots \hat{a}^\dagger_{\mathbf{p}_1} | 0 \rangle$$

Chapter 4

Making second quantization work

1. *Commutation relations of density field operators.*

$$\left[\hat{\psi}(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{y})\right]_\zeta = \delta^{(3)}(\mathbf{x}-y), \quad \left[\hat{\psi}(\mathbf{x}),\hat{\psi}(\mathbf{y})\right]_\zeta = 0$$

$$\begin{aligned}
\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}) &= \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y}) \\
&= -\zeta\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \delta^{(3)}(\mathbf{x}-y)\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \\
&= -\zeta^2\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-y)\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})
\end{aligned}$$

So $\zeta = \pm 1$ yields the same result regardless of bosons or fermions.

2. *Single-particle density matrix in terms of ladder operators.*

$$\begin{aligned}
\hat{\rho}_1(\mathbf{x}-\mathbf{y}) &= \left\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \right\rangle \\
&= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{\mathbf{q}} \hat{a}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}} = \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left\langle \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}} \right\rangle
\end{aligned}$$

3. *Hubble Hamiltonian.* Solving its eigenvalue problem:

$$|\hat{H}-\lambda\hat{I}| = \begin{vmatrix} U-\lambda & -t & -t & 0 \\ -t & -\lambda & 0 & -t \\ -t & 0 & -\lambda & -t \\ 0 & -t & -t & U-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = 0,\ \lambda_2 = U,\ \lambda_{3,4} = \frac{1}{2}\left[U \pm \sqrt{16t^2+U^2}\right]$$

$$\nu_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},\ \nu_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ \nu_{3,4} = \begin{bmatrix} 1 \\ \frac{-U\pm\sqrt{16t^2+U^2}}{4t} \\ \frac{-U\pm\sqrt{16t^2+U^2}}{4t} \\ 0 \end{bmatrix}$$

Chapter 5

Continuous systems

1. *Explicit time dependence of Lagrangian and Hamiltonian.*

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

The Hamiltonian is defined as the Legendre transformation with a canonical momentum $p = \partial L/\partial \dot{q}$. Therefore:

$$\frac{\partial L}{\partial t} = \frac{\mathrm{d}(L - p\dot{q})}{\mathrm{d}t} = -\frac{\mathrm{d}H}{\mathrm{d}t}$$

2. *Commutation relations of Poisson brackets.*

$$\{A, B\}_{PB} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$
$$\{B, A\}_{PB} = \sum_i \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} = -\{A, B\}_{PB}$$

3. *Commutation relations of Hermitian operators.* Since A and B are Hermitian, $A = A^\dagger, B = B^\dagger$.

$$[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[B, A]$$

4. *Investigating the non-relativistic limit of the relativistic free particle.* This is easily found by Taylor expansions of γ, then taking the low-velocity limit:

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \approx -mc^2 + \frac{1}{2}mv^2$$
$$p = \frac{\partial L}{\partial v} = \gamma m v \approx mv$$
$$H = pv - L = \gamma m v^2 + \frac{mc^2}{\gamma} = \frac{1}{\gamma}\left[\left(1 - \frac{v^2}{c^2}\right)mv^2 + mc^2\right] = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2$$

5. *Extremisation of the spacetime interval.*

$$\int_a^b \mathrm{d}s = \int_a^b \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}t}{\gamma} = \int_a^b L\,\mathrm{d}t$$
$$\frac{\partial L^2}{\partial \mathbf{v}} = \frac{2\mathbf{v}}{c^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^2}{\partial \mathbf{v}}\right) - \frac{\partial L^2}{\partial \mathbf{x}} = \frac{2\dot{\mathbf{v}}}{c^2} = 0$$

Since the acceleration is zero, the velocity is constant. Hence a straight world-line path does minimise the interval.

6. *Electromagnetic Lagrangian.*

$$\begin{aligned}
L &= \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV \\
\nabla L &= q[\nabla(\mathbf{A}\cdot\mathbf{v}) - \nabla V] \\
&= q[\cancel{(A\cdot\nabla)\mathbf{v}} + \cancel{(\mathbf{v}\cdot\nabla)\mathbf{A}} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \cancel{\mathbf{A}\times(\nabla\times\mathbf{v})} - q\nabla V] \\
&= q[\mathbf{E} + \mathbf{v}\times B], \quad \because \mathbf{E} = -q\nabla V,\ \mathbf{B} = \nabla\times\mathbf{A} \\
\frac{\partial L}{\partial \mathbf{v}} &= -\frac{mc^2}{2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left(-\frac{2\mathbf{v}}{c^2}\right) = \gamma m\mathbf{v} \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}} - \nabla L \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q[\mathbf{E} + \mathbf{v}\times\mathbf{B}]
\end{aligned}$$

7. *Non-relativistic limit of the electromagnetic Lagrangian.*

$$\begin{aligned}
L &= \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV \approx \frac{1}{2}m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - qV \\
\mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}
\end{aligned}$$

Finding the Hamiltonian is equivalent to finding the energy in terms of momentum:

$$\begin{aligned}
H &= \mathbf{p}\cdot\mathbf{v} - L = m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - L \\
&= mc^2 + \frac{1}{2}m\mathbf{v}^2 + qV = mc^2 + \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV, \quad \mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{m}
\end{aligned}$$

Adjusting the zero of the Hamiltonian by subtracting mc^2 gives the well-known result.

8. *Hunting for Lorentz invariants in electromagnetism.*

$$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} =$$

9. *Deriving Maxwell's equations.* $(\epsilon_0 = \mu_0 = c = 1)$ The first equation is:

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= J^0 = \rho \\
&\implies \boldsymbol{\nabla}\cdot\mathbf{E} = \rho
\end{aligned}$$

The second equation is:

$$\partial_\mu F^{\mu i} = J^i = \mathbf{J}$$
$$\implies -\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J}$$

The third equation is:

$$\partial_\lambda F_{\mu 0} + \partial_0 F_{\lambda\mu} + \partial_\mu F_{0\lambda} = 0$$
$$\implies \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

The fourth equation is:

$$\partial_\lambda F_{\mu i} + \partial_i F_{\lambda\mu} + \partial_\mu F_{i\lambda} = 0$$
$$\implies \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

10. *Deriving the continuity equation of electromagnetism.* Differentiating:

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \partial_\beta J^\beta$$

Since mixed partial derivatives are symmetric and $F^{\alpha\beta}$ is antisymmetric, the operation obviously gives zero:

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \partial_\beta J^\beta = 0$$

The second equality can be interpreted as a continuity equation akin to fluid mechanics with the charge density ρ and the current density $\mathbf{J}$:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

Chapter 6

A first stab at relativistic quantum mechanics

1. *Massive scalar field Lagrangian.*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\left(\partial^2 + m^2\right)\phi = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^0 \phi = \dot{\phi}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Chapter 7

Examples of Lagrangians, or how to write down a theory

1. *Massive scalar field with a twist.*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty}\lambda_n\phi^{2n+2}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi - \sum_{n=1}^{\infty}\lambda_n(2n+2)\phi^{2n+1}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\partial_\mu\partial^\mu\phi + m^2\phi + \sum_{n=1}^{\infty}\lambda_n(2n+2)\phi^{2n+1} = 0$$

$$\left(\partial^2 + m^2\right)\phi + \sum_{n=1}^{\infty}\lambda_n(2n+2)\phi^{2n+1} = 0$$

2. *Massive scalar field with a source.*

$$\mathcal{L} = \frac{1}{2}[\partial_\mu \phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2 + J(x)\phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} = -m^2\phi(x) + J(x)$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = \partial^\mu \phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}\right) = 0$$
$$\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) - J(x) = 0$$
$$\left(\partial^2 + m^2\right)\phi(x) = J(x)$$

3. *Two coupled massive scalar fields.*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 - g\left(\phi_1^2 + \phi_2^2\right)^2$$
$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2\phi_1 - 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial^\mu \phi_1, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$

4. *Introducing the conjugate momentum.* Referring to Chapter 6's solution:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Chapter 8

The passage of time

1. *Properties of a specific form of the time-evolution operator.* Let $\hat{U}(t_1, t_2) = \exp\left[i\hat{H}(t_2 - t_1)\right]$:

$$\hat{U}(t_1, t_1) = \exp\left[i\hat{H}(t_1 - t_1)\right] = 1$$
$$\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \exp\left[i\hat{H}(t_3 - t_1)\right] = \hat{U}(t_3, t_1)$$
$$i\frac{\mathrm{d}}{\mathrm{d}t_2}\exp\left[i\hat{H}(t_2 - t_1)\right] = i^2\exp\left(i\hat{H}t_2\right)\hat{H}\exp\left(-i\hat{H}t_1\right) = \hat{H}\hat{U}(t_2, t_1), \ \because \left[\hat{U}, \hat{H}\right] = 0$$

The time evolution operator is unitary, so $\hat{U}^{-1} = \hat{U}^\dagger$. Therefore:

$$\hat{U}^\dagger(t_2, t_1) = \exp\left[i\hat{H}(t_1 - t_2)\right] = \hat{U}(t_1, t_2)$$
$$\hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) = \exp\left[i\hat{H}(t_1 - t_2)\right]\exp\left[i\hat{H}(t_2 - t_1)\right] = 1$$

2. *Time-dependence of ladder operators.*

$$\hat{H} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k$$
$$\hat{a}_k^\dagger(t) = e^{i\hat{H}t/\hbar}\hat{a}_k^\dagger(0)e^{-i\hat{H}t/\hbar}$$
$$\frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\mathrm{d}t} = \frac{i}{\hbar}\Big(e^{i\hat{H}t/\hbar}\left[\hat{H}, \hat{a}_k^\dagger(0)\right]e^{-i\hat{H}t/\hbar}\Big)$$
$$= \frac{iE_k}{\hbar}\Big(e^{i\hat{H}t/\hbar}\left[\hat{n}_k, \hat{a}_k^\dagger(0)\right]e^{-i\hat{H}t/\hbar}\Big) = \frac{iE_k}{\hbar}\hat{a}_k^\dagger(t)$$
$$\int\frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\hat{a}_k^\dagger(t)} = \int\frac{iE_k}{\hbar}\,\mathrm{d}t \implies \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0)e^{iE_k t/\hbar}$$

3. *Time-dependence of an operator of the form $\hat{X} = X_{lm}\hat{a}_l^\dagger\hat{a}_m$.*

$$\hat{X}(t) = e^{i\hat{H}t/\hbar}X_{lm}\hat{a}_l^\dagger\hat{a}_m e^{-i\hat{H}t/\hbar}$$
$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} =$$

4. *Hamiltonian of a spin-1/2 particle in a magnetic field.*

$$\frac{\mathrm{d}\hat{S}_H^z}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^z, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(-i\hbar\hat{S}_H^x\right) = -\omega\hat{S}_H^x$$
$$\frac{\mathrm{d}\hat{S}_H^x}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^x, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(i\hbar\hat{S}_H^z\right) = \omega\hat{S}_H^z$$

Spin behaves like angular momentum.

Chapter 9

Quantum mechanical transformations

1. *Generators of the translation operator.*

$$\begin{aligned}\hat{U}(\mathbf{a}) &= \exp[-i\hat{\mathbf{p}}\cdot\mathbf{a}]\\ \left.\frac{\partial\hat{U}(\mathbf{a})}{\partial\mathbf{a}}\right|_{\mathbf{a}=0} &= -i\hat{\mathbf{p}}\exp[-i\hat{\mathbf{p}}\cdot\mathbf{0}]\\ \implies \hat{\mathbf{p}} &= -\frac{1}{i}\left.\frac{\partial\hat{U}(\mathbf{a})}{\partial\mathbf{a}}\right|_{\mathbf{a}=0}\end{aligned}$$

2. *Generators of the Lorentz group for four-vectors.*

$$K = \frac{1}{i}\left.\frac{\partial\boldsymbol{\Lambda}(\phi^1)}{\partial\phi^1}\right|_{\phi^1=0} = \frac{1}{i}\left.\begin{vmatrix}\sinh\phi^1 & \cosh\phi^1 & 0 & 0\\ \cosh\phi^1 & \sinh\phi^1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{vmatrix}\right|_{\phi^1=0} = -i\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

and similarly for ϕ^i.

3. *Infinitesimal Lorentz transformations.* Going to the MCRF and composing boosts:

$$\Lambda^\mu{}_\nu = \lim_{\mathbf{v}\to 0}\begin{bmatrix}\gamma & \gamma v^1 & \gamma v^2 & \gamma v^3\\ \gamma v^1 & \gamma & 0 & 0\\ \gamma v^2 & 0 & \gamma & 0\\ \gamma v^3 & 0 & 0 & \gamma\end{bmatrix} = \begin{bmatrix}1 & v^1 & v^2 & v^3\\ v^1 & 1 & 0 & 0\\ v^2 & 0 & 1 & 0\\ v^3 & 0 & 0 & 1\end{bmatrix}$$

For an infinitesimal counter-clockwise rotations, compose the matrices:

$$\Lambda^\mu{}_\nu = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & \theta^3 & 0\\ 0 & -\theta^3 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -\theta^2\\ 0 & 0 & 1 & 0\\ 0 & \theta^2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \theta^1\\ 0 & 0 & -\theta^1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & \theta^3 & -\theta^2\\ 0 & -\theta^3 & 1 & \theta^1\\ 0 & \theta^2 & -\theta^1 & 1\end{bmatrix}$$

Compose the boosts and rotation matrices:

$$\Lambda^\mu_\nu = \Lambda^\mu_{\ \bar{\nu}}\Lambda^{\bar{\nu}}_{\ \nu} = L_z R_z L_y R_y L_x R_x$$

$$\Lambda^\mu_\nu = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Extracting the identity matrix, the general infinitesimal Lorentz transformation can be written as:

$$\mathbf{\Lambda} = \mathbf{1} + \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

The following tensors are indeed antisymmetric:

$$\omega^{\mu\nu} = \omega^\mu_{\ \lambda} g^{\lambda\nu} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

$$\omega_{\mu\nu} = g_{\mu\lambda}\omega^\lambda_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & -\theta^3 & \theta^2 \\ -v^2 & \theta^3 & 0 & -\theta^1 \\ -v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

4. *Generators of the Poincaré group.*

Chapter 10

Symmetry

1. *Commutation relations between scalar field and its conjugate momentum.*

$$[\phi(x), P^\alpha] = \phi(x)P^\alpha - P^\alpha\phi(x) = \int \left[\phi(x)T^{0\alpha} - T^{0\alpha}\phi(x)\right] \mathrm{d}^3y$$

2. *Noether current of N-field system.*

3. *Energy-momentum tensor and momentum of the massive scalar field.*

$$\begin{aligned}
T^{\mu\nu} &= \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \\
T^{00} &= \Pi^0\partial^0\phi - g^{00}\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right] = \pi\dot{\phi} - \mathcal{L} \\
&= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \\
\partial_\mu T^{\mu\nu} &= \partial_\mu[\partial^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}] \\
&= \partial^2\phi\partial^\nu\phi - \partial^\mu\phi\partial_\mu\partial^\nu\phi - \frac{1}{2}\left[\partial^\rho\phi\partial^\nu\partial_\rho\phi + \partial_\rho\phi\partial^\nu\partial^\rho\phi - 2m^2\phi\partial^\nu\phi\right] \\
&= \left(\partial^2 + m^2\right)\phi[\partial^\nu\phi] = 0 \\
P^i &= \int T^{0i}\mathrm{d}^3x = \int \left(\Pi^0\partial^i\phi - g^{0i}\mathcal{L}\right)\mathrm{d}^3x \\
&= \int \partial^0\phi\partial^i\phi\,\mathrm{d}^3x
\end{aligned}$$

The Klein-Gordon equation, which is the equation of motion for scalar field theory, satisfies the divergence of the energy-momentum tensor.

4. *Energy-momentum tensor and momentum of the electromagnetic field.*

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu] \\
\frac{\partial(\partial_\mu A_\nu \partial^\mu A^\nu)}{\partial(\partial_\sigma A_\rho)} &= \delta^\sigma_\mu \delta^\rho_\nu \partial^\mu A^\nu + \partial_\mu A_\nu g^{\alpha\sigma} g^{\rho\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\sigma A^\rho \\
\frac{\partial(\partial_\mu A_\nu \partial^\nu A^\mu)}{\partial(\partial_\sigma A_\rho)} &= \delta^\sigma_\mu \delta^\rho_\nu \partial^\nu A^\mu + \partial_\mu A_\nu g^{\alpha\rho} g^{\sigma\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\rho A^\sigma \\
\frac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\rho)} &= -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho} = \Pi^{\sigma\rho} \\
T^\mu_\nu &= \Pi^{\mu\sigma}\partial_\nu A_\sigma - \delta^\mu_\nu \mathcal{L} \\
T^{\mu\nu} &= g^{\alpha\nu} T^\mu_\alpha = -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
X^{\lambda\mu\nu} &= F^{\mu\lambda} A^\nu = -F^{\lambda\mu} A^\nu = X^{\mu\lambda\nu} \\
\tilde{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\nu X^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\nu \big(F^{\mu\lambda} A^\nu\big) \\
&= -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \cancel{\partial_\lambda F^{\mu\lambda} A^\nu} + F^{\mu\lambda}\partial_\lambda A^\nu \\
[\lambda \to \sigma] &= F^{\mu\sigma}(\partial_\sigma A^\nu - \partial^\nu A_\sigma) + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = F^{\mu\sigma} F^\nu_\sigma + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
\tilde{T}^{00} &= F^{0\sigma} F^0_\sigma + \frac{1}{4} g^{00} F_{\alpha\beta} F^{\alpha\beta} = E^2 + \frac{1}{2}\big(B^2 - E^2\big) = \frac{1}{2}\big(E^2 + B^2\big) \\
\tilde{T}^{i0} &= F^{i\sigma} F^0_\sigma + \cancel{\frac{1}{4} g^{i0} F_{\alpha\beta} F^{\alpha\beta}} = \epsilon^{ijk} E_j B_k = (\mathbf{E} \times \mathbf{B})^i
\end{aligned}$$

Chapter 11

Canonical quantization of fields

1. *Commutation relations of quantum field position operators.* Let $\displaystyle\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} \equiv \int_{\mathbf{p}}$:

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\phi}(y)\right] &= \int_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right) \int_{\mathbf{q}} \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} + \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right) \\
&\quad - \int_{\mathbf{q}} \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} + \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right) \int_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right) \\
&= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-ip\cdot x} e^{iq\cdot y} + \left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] e^{ip\cdot x} e^{-iq\cdot y}\right) \\
&= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ip\cdot x} e^{iq\cdot y} - \delta^{(3)}(\mathbf{q}-\mathbf{p}) e^{ip\cdot x} e^{-iq\cdot y}\right) \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) = 0, \quad \mathbf{p} \mapsto -\mathbf{p}
\end{aligned}$$

2. *Commutation relations of quantum field position operator and its conjugate momentum.*

$$\begin{aligned}
\left[\hat{\phi}(x), \hat{\Pi}^0(y)\right] &= \int_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right) \int_{\mathbf{q}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} - \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right) \\
&\quad - \int_{\mathbf{q}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} - \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right) \int_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right) \\
&= i \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-ip\cdot x} e^{iq\cdot y} + \left[\hat{a}_{\mathbf{q}}, \hat{a}^\dagger_{\mathbf{p}}\right] e^{ip\cdot x} e^{-iq\cdot y}\right) \\
&= \frac{i}{2} \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ip\cdot x} e^{iq\cdot y} + \delta^{(3)}(\mathbf{q}-\mathbf{p}) e^{ip\cdot x} e^{-iq\cdot y}\right) \\
&= \frac{i}{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right) = i\,\delta^{(3)}(x-y), \quad \mathbf{p} \mapsto -\mathbf{p}
\end{aligned}$$

Chapter 12

Examples of canonical quantization

1. *Complex scalar field theory.*

$$
\begin{aligned}
\hat{\mathcal{H}} &= \partial^0 \hat{\psi}^\dagger \hat{\psi} + \partial^0 \hat{\psi} \hat{\psi}^\dagger + \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} + m^2 \hat{\psi}^\dagger \hat{\psi} \\
&= \int_{\mathbf{q}} (iE_{\mathbf{q}}) \Big(\hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot x} - \hat{b}_{\mathbf{q}} e^{-iq\cdot x} \Big) \int_{\mathbf{p}} \Big(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x} \Big) \\
&+ \int_{\mathbf{q}} (-iE_{\mathbf{q}}) \Big(\hat{a}_{\mathbf{q}} e^{-iq\cdot x} - \hat{b}^\dagger_{\mathbf{q}} e^{iq\cdot x} \Big) \int_{\mathbf{p}} \Big(\hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x} + \hat{b}_{\mathbf{p}} e^{-ip\cdot x} \Big) \\
&+ \int_{\mathbf{q}} (i\mathbf{q}) \Big(\hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot x} - \hat{b}_{\mathbf{q}} e^{-iq\cdot x} \Big) \cdot \int_{\mathbf{p}} (i\mathbf{p}) \Big(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} - \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x} \Big) \\
&+ m^2 \int_{\mathbf{q}} \Big(\hat{a}^\dagger_{\mathbf{q}} e^{-iq\cdot x} + \hat{b}_{\mathbf{q}} e^{iq\cdot x} \Big) \int_{\mathbf{p}} \Big(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x} \Big) \\
&= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \Big(\hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} e^{i(q-p)\cdot x} - \hat{b}_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{p}} e^{-i(q-p)\cdot x} \Big) \\
&+ \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{-iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \Big(\hat{a}_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} e^{i(q-p)\cdot x} - \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{q}} e^{-i(q-p)\cdot x} \Big) \\
&\int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{\mathbf{p}\cdot\mathbf{q}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \Big(\hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} e^{i(q-p)\cdot x} + \hat{b}_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{p}} e^{-i(q-p)\cdot x} \Big) \\
&+ m^2 \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \Big(\hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} e^{i(q-p)\cdot x} + \hat{b}_{\mathbf{q}} \hat{b}^\dagger_{\mathbf{p}} e^{-i(q-p)\cdot x} \Big)
\end{aligned}
$$

2. *Commutation relations of complex scalar fields.*

(a)

$$
\begin{aligned}
\Big[\hat{\psi}(x), \hat{\psi}^\dagger(y) \Big] &= \int_{\mathbf{p}} \Big(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x} \Big) \int_{\mathbf{q}} \Big(\hat{a}^\dagger_{\mathbf{q}} e^{-iq\cdot x} + \hat{b}_{\mathbf{q}} e^{iq\cdot x} \Big) \\
&- \int_{\mathbf{q}} \Big(\hat{a}^\dagger_{\mathbf{q}} e^{-iq\cdot x} + \hat{b}_{\mathbf{q}} e^{iq\cdot x} \Big) \int_{\mathbf{p}} \Big(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x} \Big)
\end{aligned}
$$

(b)

3. *Commutation relations of Noether charges for two scalar fields.*

(a)

$$\begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$
$$\left[\hat{Q}_N,\hat{\phi}_1\right]=-iD\hat{\phi}_1=i\hat{\phi}_2$$

(b)

$$\left[\hat{Q}_N,\hat{\phi}_2\right]=-iD\hat{\phi}_2=-i\hat{\phi}_1$$

(c)

$$\left[\hat{Q}_N,\hat{\psi}\right]=\frac{1}{\sqrt{2}}\left[\hat{Q}_N,\hat{\phi}_1\right]+\frac{i}{\sqrt{2}}\left[\hat{Q}_N,\hat{\phi}_2\right]=\frac{i}{\sqrt{2}}\hat{\phi}_2+\frac{1}{\sqrt{2}}\hat{\phi}_1=\hat{\psi}$$
$$\begin{bmatrix}\phi_1'\\ \phi_2'\end{bmatrix}=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$
$$\left[\hat{Q}_N,\hat{\phi}_1\right]=-iD\hat{\phi}_1=i\hat{\phi}_2,\quad \left[\hat{Q}_N,\hat{\phi}_2\right]=-iD\hat{\phi}_2=-i\hat{\phi}_1$$
$$\left[\hat{Q}_N,\hat{\psi}\right]=\frac{1}{\sqrt{2}}\left[\hat{Q}_N,\hat{\phi}_1\right]+\frac{i}{\sqrt{2}}\left[\hat{Q}_N,\hat{\phi}_2\right]=\frac{i}{\sqrt{2}}\hat{\phi}_2+\frac{1}{\sqrt{2}}\hat{\phi}_1=\hat{\psi}$$

4. *Using Noether's theorem to derive the number-phase uncertainty relation.* Note: $D\hat{\theta}=\pm 1$. Substituting:

$$\left[\hat{Q}_N,\hat{\theta}\right]=-iD\hat{\theta}=i$$
$$\left[\int\rho(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x},\theta(\mathbf{x},t)\right]=\int\mathrm{d}^3\mathbf{x}\;[\rho,\theta]=i$$

5. *Equations of motion of non-relativistic complex scalar field theory.*

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \Pi^\mu_\psi = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -V(x)\psi^\dagger(x), \quad \Pi^0_\psi = i\psi^\dagger$$

$$\partial_0 \Pi^0_\psi = i\partial_0 \psi^\dagger, \quad \partial_i \Pi^i_\psi = -\frac{1}{2m}\nabla^2 \psi^\dagger$$

$$\therefore i\partial_0 \psi^\dagger - \frac{1}{2m}\partial_i \partial^i \psi^\dagger - V(x)\psi^\dagger(x) = 0$$

$$\implies i\partial_0 \psi^\dagger = \hat{H}\psi^\dagger, \quad \hat{H} = -\frac{1}{2m}\nabla^2 + \hat{V}$$

$$V = 0 \implies i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2 \psi$$

$$iT'(t)X(x) = -\frac{1}{2m}X''(x)T(t)$$

$$\frac{T'}{T} = -iE \implies T(t) = Ae^{-iEt}$$

$$X'' + 2mEX = 0 \implies X(x) = Be^{ipx} + Ce^{-ipx}, \; p = \sqrt{2mE}$$

$$T(t)X(x) = Ae^{i(px-Et)} + Be^{-i(px-Et)}$$

6. *Noether current for non-relativistic complex scalar field theory.*

$$J^0_N = i\Psi^\dagger(i\Psi) + i\Psi(-i\Psi^\dagger)$$

$$Q_{N_c} = \int \left[\hat{\Psi}\hat{\Psi}^\dagger - \hat{\Psi}^\dagger\hat{\Psi}\right] \mathrm{d}^3\mathbf{x}$$

$$= \int \mathrm{d}^3x \left[\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}^\dagger_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}} - \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}^\dagger_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} \right]$$

$$\frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{x} \left[\int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{q} \; \hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q}\hat{a}_\mathbf{p} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \right]$$

$$= \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{q} \; \left(\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} \delta^3(\mathbf{p}-\mathbf{q}) - \hat{a}^\dagger_\mathbf{q}\hat{a}_\mathbf{p} \delta^3(\mathbf{q}-\mathbf{p}) \right)$$

$$= \int \mathrm{d}^3\mathbf{p} \left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{p}\right] = \mathbf{p}$$

So momentum is conserved, naturally.

7. *Transformation of the complex scalar field.*

Chapter 13

Fields with many components and massive electromagnetism

1. Angular momentum form of internal symmetries.

(a) $\vec{J}$ represents the Levi-Civita tensor as a vector of matrices.

$$\hat{\vec{\mathbf{Q}}}_{N_c} = \int \mathrm{d}^3\mathbf{p}\ \hat{\mathbf{A}}^\dagger \vec{J} \hat{\mathbf{A}}$$

(b) The inverse transformations and resultant computations are as follows:

$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left(\hat{b}_{-1} - \hat{b}_1\right),\ \hat{a}_2 = -\frac{i}{\sqrt{2}}\left(\hat{b}_{-1} + \hat{b}_1\right),\ \hat{a}_3 = \hat{b}_0$$

$$\hat{Q}^2_{N_c} =$$

$$\hat{Q}^3_{N_c} = -i\int \mathrm{d}^3\mathbf{p}\ \left(\hat{a}^\dagger_{1\mathbf{p}}\hat{a}_{2\mathbf{p}} - \hat{a}^\dagger_{2\mathbf{p}}\hat{a}_{1\mathbf{p}}\right) = \int \mathrm{d}^3\mathbf{p}\ \left(\hat{b}^\dagger_{1\mathbf{p}}\hat{b}_{1\mathbf{p}} - \hat{b}^\dagger_{-1\mathbf{p}}\hat{b}_{-1\mathbf{p}}\right)$$

$$J^1_{\hat{b}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ J^2_{\hat{b}} = -\frac{i}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ J^3_{\hat{b}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2. Lorentz boosting and circular polarization.

(a)

(b)

(c)

3. Projection tensors.

4. Playing with projection tensors.

Chapter 14

Gauge fields and gauge theory

1. Quantizing the electromagnetic field tensor.

2. The spin of the photon.

(a)

(b)

Chapter 15

Discrete transformations

1. *Gamma decay of a pion.*

2. *Classification of physical quantities.*

(a) Magnetic flux: Vector.

(b) Angular momentum: Pseudovector.

(c) Charge: Scalar.

(d) Scalar product of vector and pseudovector: Pseudoscalar.

(e) Scalar product of two vectors: Scalar.

(f) Scalar product of two pseudovectors: Scalar.

3. *Representations of spinors.*

(a) $\mathbf{R}(\hat{\mathbf{x}}, \theta)$

(b) $\mathbf{R}(\hat{\mathbf{y}}, \theta)$

(c) $\mathbf{R}(\hat{\mathbf{z}}, \theta)$

Chapter 16

Propagators and Green's functions

1. *Green's function for a particle in an infinite potential well.*

(a) The Schrödinger equation is:

$$\langle x|\hat{H}|\psi\rangle = \langle x|\frac{\hat{P}^2}{2m} + \hat{V}(x)|\psi\rangle = \langle x|-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \hat{V}(x)|\psi\rangle = E\,\langle x|\psi\rangle$$

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + [E - V(x)]\psi(x) = 0$$

$$V = 0 \implies \psi(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \sqrt{2mE}/\hbar$$

$$\psi(0) = \psi(a) = 0 \rightarrow B = -A$$

$$\implies A\sin(ka) = 0 \implies k = \frac{n\pi}{a}$$

$$\int_{-a}^{a} |\psi(x)|^2\,\mathrm{d}x = 1$$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

(b)

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$$G^+(n, t_2, t_1) = \theta(t_2 - t_1)e^{-iE_n(t_2 - t_1)}$$

(c) Let $t_2 = t$ and $t_1 = 0$, then taking the Fourier transform with a damping factor:

$$G^+(n, \hbar\omega) = \int_{-\infty}^{\infty} \theta(t)e^{-iE_n t}e^{i(\hbar\omega + i\epsilon)t}\,\mathrm{d}t$$

$$G^+(n, \hbar\omega) = \frac{i}{\hbar\omega - E_n + i\epsilon}$$

2. *Green's function in the energy expression.*

(a)

$$
\begin{aligned}
G_0^+(x,t,y,0) &= \theta(t)\,\langle x(t)|y(t)\rangle \\
&= \theta(t)\,\langle x|e^{-i\hat{H}t}|y\rangle \\
&= \theta(t)\sum_n e^{iE_n t}\,\langle x|n\rangle\,\langle n|y\rangle = \theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_n t} \\
G_0^+(x,y,E) &= \int G_0^+(x,t,y,0)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_n t}e^{iEt}\,\mathrm{d}t \\
&= \int_0^{\infty}\sum_n \phi_n(x)\phi_n^*(y)e^{-i(E-E_n)t}\,\mathrm{d}t
\end{aligned}
$$

Using a damping factor $e^{-\epsilon t}$ to ensure convergence, then switching the order of summation and integration:

$$
\begin{aligned}
G_0^+(x,y,E) &= \sum_n \int_0^{\infty} \phi_n(x)\phi_n^*(y)e^{i(E-E_n+i\epsilon)t}\,\mathrm{d}t \\
&= \sum_n \frac{i\phi_n(x)\phi_n^*(y)}{E-E_n+i\epsilon}
\end{aligned}
$$

(b) The integral definition of the Heaviside step function is:

$$
\theta(t) := i\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}\frac{e^{-izt}}{z+i\epsilon}
$$

Substituting this into the original expression and changing the order of integration:

$$
\begin{aligned}
G_0^+(p,t,0) &= \theta(t)e^{-iE_p t} \\
G_0^+(p,E) &= \int_{-\infty}^{\infty}\int_0^{\infty}\frac{i}{2\pi(z+i\epsilon)}e^{i(E-E_p-z)t}\,\mathrm{d}t\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty}\frac{i}{(z+i\epsilon)}\delta(E-E_p-z)\,\mathrm{d}z = \frac{i}{E-E_p+i\epsilon}
\end{aligned}
$$

3. *Green's function for the harmonic oscillator.*

(a) The one-dimensional harmonic oscillator with the corresponding forcing function $f(t)$ has the following differential equation:

$$m\frac{\partial^2}{\partial t^2}A(t-u) + m\omega_0^2 A(t-u) = \tilde{F}(\omega)e^{-i\omega(t-u)}$$

Using operator methods to solve the differential equation:

$$\begin{aligned}
A_P(t-u) &= \frac{1}{(D^2+\omega_0^2)}\frac{\tilde{F}(\omega)}{m}e^{-i\omega(t-u)} = \left(1+\frac{D^2}{\omega_0^2}\right)^{-1}\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}, \quad D = \frac{\mathrm{d}}{\mathrm{d}t} \\
&= \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{i\omega u}\left[\sum_{k=0}^{\infty}\left(\frac{iD}{\omega_0}\right)^{2k}e^{-i\omega t}\right] = \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\sum_{k=0}^{\infty}\left(\frac{\omega}{\omega_0}\right)^{2k} \\
&= \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\left[\frac{1}{1-\omega^2/\omega_0^2}\right] = -\frac{\tilde{F}(\omega)}{m(\omega^2-\omega_0^2)}e^{-i\omega(t-u)}
\end{aligned}$$

Therefore the solution is:

$$A(t-u) = c_1\cos\omega_0(t-u) + c_2\sin\omega_0(t-u) - \frac{\tilde{F}(\omega)}{m(\omega^2-\omega_0^2)}e^{-i\omega(t-u)}$$

(b) The differential equation that satisfies the Green's function is:

$$\left[m\frac{\partial^2}{\partial t^2} + m\omega_0^2\right]G(t,t') = \delta(t-t')$$

Taking the Fourier transform, rearranging and then taking its inverse:

$$\begin{aligned}
-m(\omega^2-\omega_0^2)G(\omega,t') &= \int_{-\infty}^{\infty}\delta(t-t')e^{i\omega t}\,\mathrm{d}t = e^{i\omega t'} \\
G(t,t') &= -\frac{1}{m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{\omega^2-\omega_0^2}
\end{aligned}$$

Using the previous result to verify the solution:

$$\begin{aligned}
A(t) &= \int G(t,t')f(t')\,\mathrm{d}t' \\
&= -\frac{1}{2\pi m}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)}{\omega^2-\omega_0^2}e^{i\omega t'}\,\mathrm{d}\omega\,\mathrm{d}t'
\end{aligned}$$

(c) Taking the Laplace transform of the differential equation form of the Green's function:

$$G(s,u) = \frac{e^{us}}{m(s^2+\omega_0^2)}$$

Using convolution to find the inverse:

$$G^+(t,u) = \frac{1}{m\omega_0}\int_0^t \delta(k-u)\sin\omega_0(t-k)\,\mathrm{d}k = \frac{1}{m\omega_0}\sin\omega_0(t-u)$$

(d) The trajectory is:

4. *Green's function of the Klein-Gordon equation.*

(a) Taking the three-dimensional Fourier transform:

$$\int_V (\nabla^2 + \mathbf{k}^2) G_{\mathbf{k}}(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} = \int_V \delta^3(\mathbf{x})\mathrm{d}^3\mathbf{x}$$
$$\tilde{G}_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2 - \mathbf{q}^2}$$

(b) The Fourier transform of $G^+_{\mathbf{k}}(\mathbf{x})$ with a damping factor is:

$$\begin{aligned}
\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) &= \int_{-\infty}^{\infty} -\frac{e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}}{4\pi|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} \\
&= -\frac{1}{2}\int_{-1}^{1}\int_0^{\infty} |\mathbf{x}| e^{-i(|\mathbf{q}|\cos\theta - |\mathbf{k}| - i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}|\,\mathrm{d}(\cos\theta) \\
&= -\frac{i}{2|\mathbf{q}|}\int_0^{\infty}\left[e^{i|\mathbf{q}||\mathbf{x}|} - e^{-i|\mathbf{q}||\mathbf{x}|}\right] e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}| \\
&= \frac{1}{2|\mathbf{q}|}\left[\frac{1}{(|\mathbf{k}| + |\mathbf{q}| + i\epsilon)} - \frac{1}{(|\mathbf{k}| - |\mathbf{q}| + i\epsilon)}\right] \\
&= \frac{1}{|\mathbf{k}|^2 - |\mathbf{q}|^2 + 2|\mathbf{k}|i\epsilon?}
\end{aligned}$$

(c)

Chapter 17

Propagators and fields

1. *Retarded field propagator for a free particle.*

Chapter 18

The S-matrix

Chapter 19

Expanding the S-matrix: Feynman diagrams

Chapter 20

Scattering theory

Chapter 21

Statistical physics: a crash course

Chapter 22

The generating functional for fields

Chapter 23

Path integrals: I said to him, 'You're crazy'

1. *Physicist's treatment of operators.*

2. *Path integral derivation of Wick's theorem.*

(a) Let

$$I(a) = -2\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = -2\sqrt{\frac{2\pi}{a}}$$

Differentiating under the integral sign:

$$I'(a) = \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = \sqrt{\frac{2\pi}{a^3}}$$

(b)

$$J_n(a) = (-2)^{\frac{n}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = (-2)^{\frac{n}{2}} \sqrt{\frac{2\pi}{a}}$$

$$\frac{\mathrm{d}^k J_n(a)}{\mathrm{d}a^k} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{(-1/2)!}{(-1/2-k)!} a^{-\frac{1}{2}-k}$$

$$\frac{\mathrm{d}^{n/2} J_n(a)}{\mathrm{d}a^{n/2}} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{\Gamma(1/2)}{\Gamma\left(\frac{1-n}{2}\right)} a^{-\left(\frac{n+1}{2}\right)} = \frac{i^n \pi}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{\frac{n+1}{2}}$$

$$\langle x^n \rangle = \frac{\displaystyle\int_{-\infty}^{\infty} x^n \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x} = \frac{i^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{n/2}$$

$$= \begin{cases} 0 & \forall\, n \in 2\mathbb{Z}^+ + 1 \\ a^{-n/2} \displaystyle\prod_{k=1}^{n/2} (2k-1) & \forall\, n \in 2\mathbb{Z}^+ \end{cases}$$

$$\therefore \frac{\mathrm{d}^n J_{2n}(a)}{\mathrm{d}a^n} = \frac{1}{a^n} \prod_{k=1}^{n} (2k-1)$$