Solutions to Gravitation: Foundations and Frontiers by T. Padmanabhan

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Chapter 1

Special relativity

Problem 1 (Light clocks)**.** Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$

$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$\mathrm{d}t = \gamma \, \mathrm{d}\tau$$

$$\frac{L'}{c+v} + \frac{L'}{c-v} = \gamma \frac{2L}{c}$$

$$L' \frac{2c}{c^2 - v^2} = \gamma \frac{2L}{c}$$

$$L' = \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

Problem 2 (Superluminal motion).

$$\Delta t' = t_2' - t_1'$$
$$t_1' - t_1 \approx \frac{v}{c} \Delta t \cos\theta$$
$$t_2' - t_1 = L/c$$
$$\Delta t' = \Delta t(1 - (v/c)\cos\theta)$$
$$v_{app} = v\Delta t \sin\theta = \frac{v\sin\theta}{1 - (v/c)\cos\theta}$$

Problem 3 (The strange world of four-vectors).

$$\left(a^i + b^i\right)\left(a_i + b_i\right) = a^i a_i + b^i b_i + 2\left(a^i b_i\right)$$

Problem 4 (Focused to the front). Using a Lorentz transformation on the 'time' component of the four-vector $k^a = (\omega, \omega\mathbf{n}/c)$:

$$(a) \quad \omega_R = \gamma\left(\omega_L - \frac{\omega_L v \cos\theta_L}{c}\right) = \gamma\omega_L(1 - (v/c)\cos\theta_L)$$
$$\frac{\omega_R \cos\theta_R}{\omega_R} = \frac{\gamma\omega_L(\cos\theta_L - (v/c))}{\gamma\omega_L(1 - (v/c)\cos\theta_L)}$$
$$\mu_R = \frac{\mu_L - (v/c)}{1 - (v\mu_L/c)}$$

$$(c) \int \mathrm{d}(\cos\theta)\,\mathrm{d}\phi = -\int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = -\,\mathrm{d}\Omega$$
$$\mathrm{d}(\cos\theta_R) = \left[\frac{(v/c)[\cos\theta_L - (v/c)]}{[1 - (v/c)\cos\theta_L]^2} + \frac{1}{[1 - (v/c)\cos\theta_L]}\right]\mathrm{d}(\cos\theta_L)$$
$$\mathrm{d}\Omega' = \frac{1}{\gamma^2}\frac{\mathrm{d}\Omega}{[1 - (v/c)\cos\theta]^2}$$

The energy is:

$$\mathrm{d}\mathcal{E}' = \hbar\omega = \gamma\,\mathrm{d}\mathcal{E}\,[1 - (v/c)\cos\theta]$$

The time is:

$$\mathrm{d}t' = \gamma\,\mathrm{d}t$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is ($c = 1$):

$$\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}} = \gamma^2(1 - v\cos\theta)^3 \left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\text{lab}}$$

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\text{lab}} = \frac{\left(1 - v^2\right)^2}{(1 - v\cos\theta)^3}\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}}$$

If the emission is isotropic, $\mathrm{d}\Omega = \mathrm{d}\Omega' = 4\pi$:

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{\text{lab}} = \mathrm{d}\Omega\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}} = \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'}\right)_{\text{rest}}$$

Problem 5 (Transformation of antisymmetric tensors).

$$A^{i'k'} = L^{k'}_{k} L^{i'}_{i} A^{ik} = \frac{\partial x^{k'}}{\partial x^{k}}\frac{\partial x^{i'}}{\partial x^{i}} A^{ik}$$

If $A^{ik} = -A^{ki}$ then:

$$A^{k'i'} = -A^{i'k'}?$$

Problem 6 (Practice with completely antisymmetric tensors).

Problem 7 (A null curve in flat spacetime).

$$\eta_{ij}x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

Problem 8 (Shadows are Lorentz invariant).

Problem 9 (Hamiltonian form of action - Newtonian mechanics).

$$\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\ [p\dot{q} - H(p,q)]$$

$$\delta\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\ \left[\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right] = 0$$

Using $\delta\dot{q} = \dot{\mathrm{d}}(\delta q)$ and integrating by parts:

$$\int_{t_2}^{t_1} \mathrm{d}t\ p\,\delta\dot{q} = \cancel{\left|p\,\delta q\right|_{t_1}^{t_2}} - \int_{t_2}^{t_1} \mathrm{d}t\ \dot{p}\,\delta q$$

$$\int_{t_2}^{t_1} \mathrm{d}t\ \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right] = 0$$

Since $\delta q = 0$ at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Problem 10 (Hamiltonian form of action - special relativity).

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left[p_a \dot{x}^a - \frac{1}{2} C \left(\frac{H}{m} + m \right) \right]$$

Problem 11 (Hitting a mirror).

Problem 12 (Photon-electron scattering).

Problem 13 (More practice with collisions).

Problem 14 (Relativistic rocket).

Problem 15 (Practice with equilibrium distribution functions).

Problem 16 (Projection effects).

Problem 17 (Relativistic virial theorem). The conservation law implies the following:

$$\begin{aligned}
\partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \partial_\mu \left[x^\alpha x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \left[\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha
\end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned}
\partial_0 \left(\partial_0 T^{00} \right) x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} \right] x^\alpha x^\beta \\
\partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad \left(\partial_i T^{i\alpha} = 0 \right) \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \right] + 2T^{\alpha\beta}, \quad \left(T^{\mu\alpha} \partial_\mu x^\beta = T^{\alpha\beta} \right)
\end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that $T^{ij} = 0$ outside a compact region in space:

$$\int \mathrm{d}^3 x \; \partial_0^2 T^{00} x^\alpha x^\beta = \int \mathrm{d}^3 x \left[\cancel{-\partial_\mu \left(\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \right)} + 2T^{\alpha\beta} \right]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3 x \; T^{00} x^\alpha x^\beta = 2 \int \mathrm{d}^3 x \; T^{\alpha\beta}$$

Problem 18 (Explicit computation of spin precession)**.** The four-velocity and four-acceleratio
are:

$$u^i = (\gamma, \gamma\vec{v}) = (\gamma, -\gamma r\omega \sin\omega t, \gamma r\omega \cos\omega t, 0)$$
$$a^i = \gamma\left(\dot{\gamma}, \dot{\gamma}\vec{v} + \dot{\vec{v}}\gamma\right) = -\gamma^2\omega^2(0, x, y, 0)$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$\frac{\mathrm{d}S^j}{\mathrm{d}\tau} = u^j\left(S^k a_k\right)$$
$$S^k a_k = -\gamma^2\omega^2(xS^x + yS^y)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[(yS^x - xS^y)] = -\omega(xS^x + yS^y)$$

Problem 19 (Little group of the Lorentz group)**.**

Chapter 2

Scalar and electromagnetic fields in special relativity

Problem 1 (Measuring the F^{a_b}).

Problem 2 (Schrödinger equation and gauge transformation). The transformed Schrödinger equation is:

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}\left[\mathbf{P} - \frac{q}{c}(\mathbf{A} + \nabla f)\right]^2 + q(\phi + \partial_t f)\right]\psi$$

Problem 3 (Four-vectors leading to electric and magnetic fields).

Problem 4 (Hamiltonian form of action - charged particle).

Problem 5 (Three-dimensional form of the Lorentz force). Using the electromagnetic tensor equation of motion:

$$\begin{aligned}
m\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = qF^{ik}u_k &\implies m\frac{\mathrm{d}u^i}{\mathrm{d}t} = qF^{i\alpha}u_\alpha \\
m\frac{\mathrm{d}u^0}{\mathrm{d}t} = q(\mathbf{E}\cdot\mathbf{v}) &\implies \frac{m\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}\cdot\mathbf{v}) \\
m\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} &= qF^{\alpha\beta}u_\beta = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
m\left[\frac{\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right] &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
\implies \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{q}{m\gamma}\left[\mathbf{E}+\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{v})\mathbf{v}\right]
\end{aligned}$$

Problem 6 (Pure gauge impostors)**.** This can be transformed into polar coordinates ($x = \cos\theta, y = \sin\theta$) and evaluated ($A_r$ is obviously 0):

$$A_\theta = \frac{\partial f}{\partial\theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta} = 1,$$

$$\oint_{C=x^2+y^2} \mathbf{A}\cdot\mathrm{d}\mathbf{s} = \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

The reason why $\mathbf{A}$ is not a pure gauge mode is because f is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

Problem 7 (Pure electric or magnetic fields)**.**

Problem 8 (Elegant solution to non-relativistic Coulomb motion)**.** Since the angular momentum is conserved, $\mathrm{d}\mathbf{J}\,/\,\mathrm{d}t = 0$:

$$\begin{aligned}
(a)\quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J}) &= \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\times\mathbf{J} + \cancel{\mathbf{p}\times\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}} \\
f(r)\hat{\mathbf{r}}\times\mathbf{r}\times m\mathbf{v} &= mf(r)[\mathbf{r}(\cancel{\hat{\mathbf{r}}\cdot\mathbf{v}}) - \mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{r})] \\
&= -mf(r)r\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -mf(r)r^2\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}
\end{aligned}$$

Therefore, if $f(r)r^2 = -|\alpha|$, then:

$$\int \frac{1}{m|\alpha|}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J})\,\mathrm{d}t = \int \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}\,\mathrm{d}t$$

$$\implies \frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} = \mathbf{e}$$

where $\mathbf{e}$ is a conserved vector, arising as a constant of the integration.
(b) Note that $\mathbf{p}$ and $\mathbf{J}$ are perpendicular:

$$\mathbf{e}\cdot\mathbf{e} = \left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right]\cdot\left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right]$$

$$\implies |\mathbf{e}|^2 = \frac{|\mathbf{p}\times\mathbf{J}|^2}{m^2|\alpha|^2} - 2\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|}\right) + 1$$

$$= \frac{p^2J^2}{m^2|\alpha|^2} - 2\left(\frac{J^2}{m|\alpha|r}\right) + 1, \quad J^2 = (\mathbf{r}\times\mathbf{p})\cdot\mathbf{J}$$

$$= 1 + \frac{2EJ^2}{m|\alpha|^2}, \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r}$$

$$(c) \quad \mathbf{e}\cdot\mathbf{r} = \frac{\mathbf{r}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \mathbf{r}\cdot\hat{\mathbf{r}}$$

$$er\cos\theta = \frac{J^2}{m|\alpha|} - r$$

$$\implies r(\theta) = \frac{J^2/m|\alpha|}{1+e\cos\theta}$$

$$(d) \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r} - \frac{|\beta|}{r^2}$$