Solutions to
Gravitation: Foundations and Frontiers
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Chapter 1

Special relativity

1. *Light clocks.* Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$
$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau$$
$$\frac{L'}{c+v} + \frac{L'}{c-v} = \gamma\frac{2L}{c}$$
$$L'\frac{2c}{c^2 - v^2} = \gamma\frac{2L}{c}$$
$$L' = \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

2. *Superluminal motion.*

$$\Delta t' = t_2' - t_1'$$
$$t_1' - t_1 \approx \frac{v}{c}\Delta t\cos\theta$$
$$t_2' - t_1 = L/c$$
$$\Delta t' = \Delta t(1 - (v/c)\cos\theta)$$
$$v_{app} = v\Delta t\sin\theta = \frac{v\sin\theta}{1 - (v/c)\cos\theta}$$

3. *The strange world of four-vectors.*

(a)

$$\left(a^i + b^i\right)\left(a_i + b_i\right) = a^i a_i + b^i b_i + 2\left(a^i b_i\right)$$

4. *Focused to the front.*

(a) Using a Lorentz transformation on the 'time' component of the four-vector $k^a = (\omega, \omega \mathbf{n}/c)$:

$$\omega_R = \gamma\left(\omega_L - \frac{\omega_L v \cos\theta_L}{c}\right) = \gamma\omega_L(1 - (v/c)\cos\theta_L)$$

$$\frac{\omega_R \cos\theta_R}{\omega_R} = \frac{\gamma\omega_L(\cos\theta_L - (v/c))}{\gamma\omega_L(1 - (v/c)\cos\theta_L)}$$

$$\mu_R = \frac{\mu_L - (v/c)}{1 - (v\mu_L/c)}$$

(b)

(c)

$$\int \mathrm{d}(\cos\theta)\, \mathrm{d}\phi = -\int \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = -\, \mathrm{d}\Omega$$

$$\mathrm{d}(\cos\theta_R) = \left[\frac{(v/c)[\cos\theta_L - (v/c)]}{[1 - (v/c)\cos\theta_L]^2} + \frac{1}{[1 - (v/c)\cos\theta_L]}\right] \mathrm{d}(\cos\theta_L)$$

$$\mathrm{d}\Omega' = \frac{1}{\gamma^2} \frac{\mathrm{d}\Omega}{[1 - (v/c)\cos\theta]^2}$$

The energy is:

$$\mathrm{d}\mathcal{E}' = \hbar\omega = \gamma\, \mathrm{d}\mathcal{E}\, [1 - (v/c)\cos\theta]$$

The time is:

$$\mathrm{d}t' = \gamma\, \mathrm{d}t$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is ($c = 1$):

$$\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\, \mathrm{d}\Omega'}\right)_{\text{rest}} = \gamma^2 (1 - v\cos\theta)^3 \left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\, \mathrm{d}\Omega}\right)_{\text{lab}}$$

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\, \mathrm{d}\Omega}\right)_{\text{lab}} = \frac{\left(1 - v^2\right)^2}{(1 - v\cos\theta)^3} \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\, \mathrm{d}\Omega'}\right)_{\text{rest}}$$

If the emission is isotropic, $\mathrm{d}\Omega = \mathrm{d}\Omega' = 4\pi$:

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{\text{lab}} = \mathrm{d}\Omega \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}} = \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'}\right)_{\text{rest}}$$

5. *Transformation of antisymmetric tensors.*

$$A^{i'k'} = L^{k'}_{k} L^{i'}_{i} A^{ik} = \frac{\partial x^{k'}}{\partial x^{k}} \frac{\partial x^{i'}}{\partial x^{i}} A^{ik}$$

If $A^{ik} = -A^{ki}$ then:

$$A^{k'i'} = -A^{i'k'}?$$

6. *Practice with completely antisymmetric tensors.*

7. *A null curve in flat spacetime.*

$$\eta_{ij} x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

8. *Shadows are Lorentz invariant.*

9. *Hamiltonian form of action - Newtonian mechanics.*

$$\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\ [p\dot{q} - H(p,q)]$$

$$\delta\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\ \left[\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right] = 0$$

Using $\delta\dot{q} = \dot{\mathrm{d}}(\delta q)$ and integrating by parts:

$$\int_{t_2}^{t_1} \mathrm{d}t\ p\,\delta\dot{q} = \cancel{|p\,\delta q|_{t_1}^{t_2}} - \int_{t_2}^{t_1} \mathrm{d}t\ \dot{p}\,\delta q$$

$$\int_{t_2}^{t_1} \mathrm{d}t\ \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right] = 0$$

Since $\delta q = 0$ at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

10. *Hamiltonian form of action - special relativity.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \left[p_a \dot{x}^a - \frac{1}{2} C \left(\frac{H}{m} + m \right) \right]$$

11. *Hitting a mirror.*

12. *Photon-electron scattering.*

13. *More practice with collisions.*

14. *Relativistic rocket.*

15. *Practice with equilibrium distribution functions.*

16. *Projection effects.*

17. *Relativistic virial theorem.* The conservation law implies the following:

$$\begin{aligned}
\partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \partial_\mu \left[x^\alpha x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \left[\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha
\end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned}
\partial_0 \left(\partial_0 T^{00} \right) x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} \right] x^\alpha x^\beta \\
\partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad \left(\partial_i T^{i\alpha} = 0 \right) \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \right] + 2T^{\alpha\beta}, \quad \left(T^{\mu\alpha} \partial_\mu x^\beta = T^{\alpha\beta} \right)
\end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that $T^{ij} = 0$ outside a compact region in space:

$$\begin{aligned}
\int \mathrm{d}^3x \; \partial_0^2 T^{00} x^\alpha x^\beta &= \int \mathrm{d}^3x \left[\cancel{-\partial_\mu \left(\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \right)} + 2T^{\alpha\beta} \right] \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3x \; T^{00} x^\alpha x^\beta &= 2 \int \mathrm{d}^3x \; T^{\alpha\beta}
\end{aligned}$$

18. *Explicit computation of spin precession.* The four-velocity and four-acceleration are:

$$
\begin{aligned}
u^i &= (\gamma, \gamma\vec{v}) = (\gamma, -\gamma r\omega \sin\omega t, \gamma r\omega \cos\omega t, 0) \\
a^i &= \gamma\left(\dot{\gamma}, \dot{\gamma}\vec{v} + \dot{\vec{v}}\gamma\right) = -\gamma^2\omega^2(0, x, y, 0)
\end{aligned}
$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$
\begin{aligned}
\frac{\mathrm{d}S^j}{\mathrm{d}\tau} &= u^j\left(S^k a_k\right) \\
S^k a_k &= -\gamma^2\omega^2(xS^x + yS^y) \\
\frac{\mathrm{d}}{\mathrm{d}t}[(yS^x - xS^y)] &= -\omega(xS^x + yS^y)
\end{aligned}
$$

19. *Little group of the Lorentz group.*

Chapter 2

Scalar and electromagnetic fields in special relativity

1. *Measuring the F^{ab}.*

2. *Schrödinger equation and gauge transformation.* The transformed Schrödinger equation is:

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}\left[\mathbf{P} - \frac{q}{c}(\mathbf{A} + \nabla f)\right]^2 + q(\phi + \partial_t f)\right]\psi$$

3. *Four-vectors leading to electric and magnetic fields.*

4. *Hamiltonian form of action - charged particle.*

5. *Three-dimensional form of the Lorentz force.* Using the electromagnetic tensor equation of motion:

$$m\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = qF^{ik}u_k \implies m\frac{\mathrm{d}u^i}{\mathrm{d}t} = qF^{i\alpha}u_\alpha$$

$$m\frac{\mathrm{d}u^0}{\mathrm{d}t} = q(\mathbf{E}\cdot\mathbf{v}) \implies \frac{m\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(\mathbf{E}\cdot\mathbf{v})$$

$$m\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} = qF^{\alpha\beta}u_\beta = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$m\left[\frac{\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right] = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\implies \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m\gamma}\left[\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{v})\mathbf{v}\right]$$

6. *Pure gauge impostors.* This can be transformed into polar coordinates ($x = \cos\theta, y = \sin\theta$) and evaluated ($A_r$ is obviously 0):

$$A_\theta = \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = 1,$$
$$\oint_{C=x^2+y^2} \mathbf{A}\cdot \mathrm{d}\mathbf{s} = \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

The reason why $\mathbf{A}$ is not a pure gauge mode is because f is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

7. *Pure electric or magnetic fields.*

8. *Elegant solution to non-relativistic Coulomb motion.*

   (a) Since the angular momentum is conserved, $\mathrm{d}\mathbf{J}\,/\,\mathrm{d}t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J}) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\times\mathbf{J} + \cancel{\mathbf{p}\times\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}}$$
$$f(r)\hat{\mathbf{r}}\times\mathbf{r}\times m\mathbf{v} = mf(r)[\mathbf{r}(\cancel{\hat{\mathbf{r}}\cdot\mathbf{v}}) - \mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{r})]$$
$$= -mf(r)r\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -mf(r)r^2\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}$$

   Therefore, if $f(r)r^2 = -|\alpha|$, then:

$$\int \frac{1}{m|\alpha|}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J})\,\mathrm{d}t = \int \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}\,\mathrm{d}t$$
$$\implies \frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} = \mathbf{e}$$

   where $\mathbf{e}$ is a conserved vector, arising as a constant of the integration.

   (b) Note that $\mathbf{p}$ and $\mathbf{J}$ are perpendicular:

$$\mathbf{e}\cdot\mathbf{e} = \left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right]\cdot\left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right]$$
$$\implies |\mathbf{e}|^2 = \frac{|\mathbf{p}\times\mathbf{J}|^2}{m^2|\alpha|^2} - 2\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|}\right) + 1$$
$$= \frac{p^2J^2}{m^2|\alpha|^2} - 2\left(\frac{J^2}{m|\alpha|r}\right) + 1, \quad J^2 = (\mathbf{r}\times\mathbf{p})\cdot\mathbf{J}$$
$$= 1 + \frac{2EJ^2}{m|\alpha|^2}, \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r}$$

(c)

$$
\begin{aligned}
\mathbf{e}\cdot\mathbf{r} &= \frac{\mathbf{r}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \mathbf{r}\cdot\hat{\mathbf{r}} \\
er\cos\theta &= \frac{J^2}{m|\alpha|} - r \\
\implies r(\theta) &= \frac{J^2/m|\alpha|}{1+e\cos\theta}
\end{aligned}
$$

(d)

$$
E = \frac{p^2}{2m} - \frac{|\alpha|}{r} - \frac{|\beta|}{r^2}
$$

9. *More on uniformly accelerated motion.*