Solutions to Principles of Quantum Mechanics (Second Edition) by Ramamurti Shankar

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Chapter 1

Mathematical Introduction

Exercise 1. The null vector is $(0,0,0)$. The inverse under addition is $(-a,-b,-c)$. A vector of the form $(a,b,1)$ does not form a vector space because it fails to satisfy the closure property under addition and multiplication:

$$(a,b,1)+(d,e,1)=(a+d,b+e,2)\notin(a,b,1)$$

$$\alpha(a,b,1)=(\alpha a,\alpha b,\alpha)\notin(a,b,1)$$

Exercise 2.

Exercise 3. The set of kets is not linearly independent, as $|3\rangle=|1\rangle-2\,|2\rangle$.

Exercise 4. Arrange the row vectors into a matrix and find the determinant:

$$\begin{vmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 3 & 2 & 1\end{vmatrix}=0 \tag{1.2a}$$

Since the determinant is zero, one of the row vectors is expressible as a linear combination of the others. This is seen in $(3,2,1)=2(1,1,0)+(1,0,1)$.

For the second set of vectors, we perform the same procedure:

$$\begin{vmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{vmatrix}=2 \tag{1.2b}$$

So there is no vector in this set that is expressible as a linear combination of the other vectors.

Exercise 5. Let $|\mathrm{I}\rangle = \vec{\mathbf{A}}$ and $|\mathrm{II}\rangle = \vec{\mathbf{B}}$. Following Gram-Schmidt orthonormalisation:

$$\hat{\mathbf{A}} = \frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}} = |1\rangle \tag{1.3a}$$

$$|2'\rangle = |\mathrm{II}\rangle - |1\rangle \langle 1|\mathrm{II}\rangle = \frac{18}{5} \tag{1.3b}$$

Exercise 6.

Exercise 7.

Chapter 2

Review of Classical Mechanics

Exercise 1. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2.1a}$$

Solving the Euler-Lagrange equation gives the equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + kx = 0 \tag{2.1b}$$

Exercise 2. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k(x_1 - x_2)^2 \tag{2.2a}$$

Solving the Euler-Lagrange equations gives the equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} - \frac{\partial \mathcal{L}}{\partial x_1} = m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \tag{2.2b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} - \frac{\partial \mathcal{L}}{\partial x_2} = m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \tag{2.2c}$$

Rearranging, we get the same equations of motion:

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

Exercise 3. The Lagrangian in polar coordinates is:

$$\mathcal{L} = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(r) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right] - V(r) \tag{2.3a}$$

Solving the Euler-Lagrange equations gives the equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - (mr\sin^2\theta)\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \tag{2.3b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - \frac{1}{2}(mr^2\sin 2\theta)\dot{\phi}^2 = 0 \tag{2.3c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = (mr^2\sin^2\theta)\ddot{\phi} + 2m\dot{\phi}(r\dot{r}\sin^2\theta + r^2\dot{\theta}\sin 2\theta) = 0 \tag{2.3d}$$

Exercise 4. Substituting $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ with $\dot{\mathbf{r}}_{\mathrm{CM}}$ and $\dot{\mathbf{r}}$:

$$\mathcal{L} = \frac{1}{2}m_1\left|\dot{\mathbf{r}}_{\mathrm{CM}} + \frac{m_2\dot{\mathbf{r}}}{m_1+m_2}\right|^2 + \frac{1}{2}m_2\left|\dot{\mathbf{r}}_{\mathrm{CM}} - \frac{m_1\dot{\mathbf{r}}}{m_1+m_2}\right|^2 - V(\mathbf{r}) \tag{2.4}$$

Expanding the squares:

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(m_1+m_2)\left|\dot{\mathbf{r}}_{\mathrm{CM}}\right|^2 + \frac{1}{2}m_1\left|\frac{2m_2\dot{\mathbf{r}}\mathbf{r}_{\mathrm{CM}}}{m_1+m_2}\right| + \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2|\dot{\mathbf{r}}|^2 \\ -\frac{1}{2}m_2\left|\frac{2m_1\dot{\mathbf{r}}\mathbf{r}_{\mathrm{CM}}}{m_1+m_2}\right| + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2|\dot{\mathbf{r}}|^2 - V(\mathbf{r})\end{aligned} \tag{2.5}$$

Which gives the final expression:

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\left|\dot{\mathbf{r}}_{\mathrm{CM}}\right|^2 + \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{2.6}$$

Exercise 5.

Exercise 6. The conservation of energy in the harmonic oscillator states that:

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \tag{2.7a}$$

Dividing both sides by E:

$$\frac{p^2}{2mE} + \frac{kx^2}{2E} = 1 \longrightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{p}{b}\right)^2 = 1,\ \ a^2 = \frac{2E}{k}\ \ ,\ \ b^2 = 2mE \tag{2.7b}$$

Exercise 7. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k(x_1 - x_2)^2 \tag{2.8a}$$

Finding the generalised momenta:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m_i \dot{x}_i \longrightarrow \dot{x}_i = \frac{p_i}{m_i} \tag{2.8b}$$

The Hamiltonian is found by:

$$\mathcal{H} = \sum_i p_i \dot{x}_i - \mathcal{L} = \mathcal{T} + \mathcal{V} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}k(x_1 - x_2)^2 \tag{2.8c}$$

Solving Hamilton's equations gives the equations of motion:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{x}_i \longrightarrow \dot{x}_i = \frac{p_i}{m_i} \tag{2.8d}$$

$$-\frac{\partial \mathcal{H}}{\partial x_i} = \dot{p}_i \longrightarrow \dot{p}_1 = -2kx_1 + kx_2, \;\; \dot{p}_2 = kx_1 - 2kx_2 \tag{2.8e}$$

But $\dot{p}_i = m\ddot{x}_i$, so we get the same equations of motion as befroe:

$$m\ddot{x}_1 = -2kx_1 + kx_2, \;\; m\ddot{x}_2 = kx_1 - 2kx_2 \tag{2.8f}$$

Exercise 8. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\left|\dot{\mathbf{r}}_{\mathrm{CM}}\right|^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \tag{2.9a}$$

The generalised momenta are:

$$\mathbf{p}_{\mathrm{CM}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_{\mathrm{CM}}} = (m_1 + m_2)|\dot{\mathbf{r}}_{\mathrm{CM}}|, \;\; \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \left(\frac{m_1 m_2}{m_1 + m_2}\right)|\dot{\mathbf{r}}| \tag{2.9b}$$

Writing $m_1 + m_2 = M$ and $m_1 m_2 / M = \mu$, and finding the Hamiltonian:

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = \frac{1}{2}M\left|\frac{\mathbf{p}_{\mathrm{CM}}}{M}\right|^2 + \frac{1}{2}\mu\left|\frac{\mathbf{r}}{\mu}\right|^2 + V(\mathbf{r}) = \frac{|\mathbf{p}_{\mathrm{CM}}|}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + V(\mathbf{r}) \tag{2.9c}$$

Exercise 9.

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}\right) \tag{2.10a}$$

$$\{\lambda, \omega\} = \sum_i \left(\frac{\partial \lambda}{\partial q_i}\frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i}\frac{\partial \omega}{\partial q_i}\right) = -\{\omega, \lambda\} \tag{2.10b}$$

$$\{\omega, \lambda + \sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial(\lambda + \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial(\lambda + \sigma)}{\partial q_i} \right) = \tag{2.10c}$$

$$\sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) = \{\omega, \lambda\} + \{\omega, \sigma\} \tag{2.10d}$$

$$\{\omega, \lambda\sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda\sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda\sigma}{\partial q_i} \right) = \tag{2.10e}$$

$$\sum_i \sigma \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \lambda \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) = \{\omega, \lambda\}\sigma + \lambda\{\omega, \sigma\} \tag{2.10f}$$

Exercise 10. (i)

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0, \quad \because \frac{\partial q_i}{\partial q_k} = \delta_{ik}, \quad \frac{\partial q_j}{\partial q_k} = \delta_{jk}, \quad \forall\, i, j, k \in \mathbb{N} \tag{2.11a}$$

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0, \quad \because \frac{\partial p_i}{\partial p_k} = \delta_{ik}, \quad \frac{\partial p_j}{\partial p_k} = \delta_{jk}, \quad \forall\, i, j, k \in \mathbb{N} \tag{2.11b}$$

(ii) The Hamiltonian with $a = b$ is:

$$\mathcal{H} = p_x^2 + p_y^2 + a(x^2 + y^2) \tag{2.11c}$$

The angular momentum about the z-axis $l_z = xp_y - yp_x$ is conserved because the potential $V(x, y)$ is expressible as $V(x^2 + y^2)$ and the z-coordinate is not present in the Hamiltonian. Explicit computation yields:

$$\{l_z, \mathcal{H}\} = \sum_i \left(\frac{\partial l_z}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial l_z}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) = \left(\frac{\partial l_z}{\partial x} \frac{\partial \mathcal{H}}{\partial p_x} - \frac{\partial l_z}{\partial p_x} \frac{\partial \mathcal{H}}{\partial x} \right) + \left(\frac{\partial l_z}{\partial y} \frac{\partial \mathcal{H}}{\partial p_y} - \frac{\partial l_z}{\partial p_y} \frac{\partial \mathcal{H}}{\partial y} \right) \tag{2.11d}$$

$$= (p_y \cdot 2p_x - y \cdot 2x) + (-p_x \cdot 2p_y - (-p_x) \cdot 2_y) = 0 \tag{2.11e}$$

Exercise 11.

Exercise 12.

$$\{\bar{x},\bar{y}\} = \sum_{k=x,y}\left(\frac{\partial \bar{x}}{\partial q_k}\frac{\partial \bar{y}}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k}\frac{\partial \bar{y}}{\partial q_k}\right) = 0 \tag{2.12a}$$

$$\{\bar{x},\bar{p}_y\} = \sum_{k=x,y}\left(\frac{\partial \bar{x}}{\partial q_k}\frac{\partial \bar{p}_y}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k}\frac{\partial \bar{p}_y}{\partial q_k}\right) = 0 \tag{2.12b}$$

Exercise 13.

$$\{\rho,p_\rho\} = \sum_{k=x,y}\left(\frac{\partial \rho}{\partial q_k}\frac{\partial p_\rho}{\partial p_k} - \frac{\partial \rho}{\partial p_k}\frac{\partial p_\rho}{\partial q_k}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 = 1 \tag{2.13a}$$

$$\{\rho,p_\phi,=\} \sum_{k=x,y}\left(\frac{\partial \rho}{\partial q_k}\frac{\partial p_\phi}{\partial p_k} - \frac{\partial \rho}{\partial p_k}\frac{\partial p_\phi}{\partial q_k}\right) = -y\left(\frac{x}{\sqrt{x^2+y^2}}\right) + x\left(\frac{y}{\sqrt{x^2+y^2}}\right) = 0 \tag{2.13b}$$

$$\begin{aligned}\{\phi,p_\rho\} = \sum_{k=x,y}\left(\frac{\partial \phi}{\partial q_k}\frac{\partial p_\rho}{\partial p_k} - \frac{\partial \phi}{\partial p_k}\frac{\partial p_\rho}{\partial q_k}\right) &= \left(\frac{-y}{x^2+y^2}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) \\ &+\left(\frac{x}{x^2+y^2}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) = 0\end{aligned} \tag{2.13c}$$

$$\{\phi,p_\phi\} = \sum_{k=x,y}\left(\frac{\partial \phi}{\partial q_k}\frac{\partial p_\phi}{\partial p_k} - \frac{\partial \phi}{\partial p_k}\frac{\partial p_\phi}{\partial q_k}\right) = -y\left(\frac{-y}{x^2+y^2}\right) + x\left(\frac{x}{x^2+y^2}\right) = 1 \tag{2.13d}$$

$$\{\rho,\phi\} = \sum_{k=x,y}\left(\frac{\partial \rho}{\partial q_k}\frac{\partial \phi}{\partial p_k} - \frac{\partial \rho}{\partial p_k}\frac{\partial \phi}{\partial q_k}\right) = 0 \tag{2.13e}$$

$$\{p_\rho,p_\phi\} = \sum_{k=x,y}\left(\frac{\partial p_\rho}{\partial q_k}\frac{\partial p_\phi}{\partial p_k} - \frac{\partial p_\rho}{\partial p_k}\frac{\partial p_\phi}{\partial q_k}\right) = 0 \tag{2.13f}$$

Exercise 14.

Exercise 15.

Exercise 16.

Exercise 17. All we have to do is check the infinitesimal changes for $p = p_1 + p_2$:

$$\delta x_1 = \epsilon \frac{\partial p}{\partial p_1} = \epsilon, \quad \delta x_2 = \epsilon \frac{\partial p}{\partial p_2} = \epsilon \tag{2.14a}$$

$$\delta p_1 = -\epsilon \frac{\partial p}{\partial x_1} = 0, \quad \delta p_2 = -\epsilon \frac{\partial p}{\partial x_2} = 0 \tag{2.14b}$$

Therefore, the generator $g(q,p) = p$.

Exercise 18. The infinitesimal transformations are:

$$\bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i}, \quad \bar{p}_j = p_j - \epsilon \frac{\partial g}{\partial q_j}$$

Checking the Poisson brackets:

$$\{\bar{q}_i, \bar{p}_j\} = \sum_k \left(\frac{\partial \bar{q}_i}{\partial q_k} \frac{\partial \bar{q}_j}{\partial p_k} - \frac{\partial \bar{q}_i}{\partial p_k} \frac{\partial \bar{p}_j}{\partial q_k} \right) = \tag{2.15a}$$

$$\sum_k \left[\left(\delta_{ik} + \epsilon \frac{\partial^2 g}{\partial p_i \partial q_k} \right) \left(\delta_{jk} - \epsilon \frac{\partial^2 g}{\partial q_j \partial p_k} \right) - \left(\epsilon \frac{\partial^2 g}{\partial p_i \partial p_k} \right) \left(-\epsilon \frac{\partial^2 g}{\partial q_i \partial q_k} \right) \right] \tag{2.15b}$$

$$= \sum_k \left[\delta_{ik}\delta_{jk} + \epsilon \delta_{jk} \frac{\partial^2 g}{\partial p_i \partial q_k} - \epsilon \delta_{ik} \frac{\partial^2 g}{\partial q_j \partial p_k} + \epsilon^2 \frac{\partial^2 g}{\partial p_i \partial p_k} \frac{\partial^2 g}{\partial q_i \partial q_k} \right] \tag{2.15c}$$

$$= \delta_{ij} + \epsilon \left(\frac{\partial^2 g}{\partial p_i \partial q_j} - \frac{\partial^2 g}{\partial q_j \partial p_i} \right) + \mathcal{O}(\epsilon^2) = \delta_{ij}, \quad \because \frac{\partial^2 g}{\partial p_i \partial q_j} = \frac{\partial^2 g}{\partial q_j \partial p_i}, \quad \mathcal{O}(\epsilon^2) \approx 0 \tag{2.15d}$$

Exercise 19. The Hamiltonian under rotated coordinates is:

$$\mathcal{H}_R = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 [(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2] = \mathcal{H} \tag{2.16a}$$

For the transformation to be noncanonical, the Poisson bracket $\{\bar{x}, \bar{p}_x\} \neq 1$:

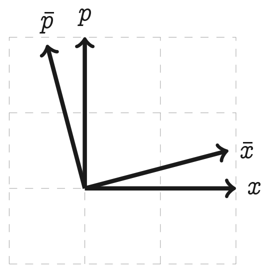
$$\{\bar{x}, \bar{p}_x\} = \sum_{k=x,y} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{p}_x}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{p}_x}{\partial q_k} \right) = \cos\theta \tag{2.16b}$$

To show that no conservation law follows:

$$\bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i} = q_i + \delta q_i, \quad \bar{p}_i = p_i \to \delta p_i = 0 \tag{2.16c}$$

$$\delta \mathcal{H} = \sum_i \frac{\partial \mathcal{H}}{\partial q_i} \left(\epsilon \frac{\partial g}{\partial p_i} \right) \neq \epsilon \{\mathcal{H}, g\} \tag{2.16d}$$

Exercise 20. A rotation in phase space can be shown via the following diagram:



The infinitesimal transformation is as follows:

$$\bar{x} = x\cos\epsilon - p\sin\epsilon \approx x - \epsilon p \tag{2.17a}$$
$$\bar{p} = x\sin\epsilon + p\cos\epsilon \approx \epsilon x + p \tag{2.17b}$$

We must verify if this transformation is canonical:

$$\{\bar{x}, \bar{p}\} = \sum_{k=x,p} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{p}}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{p}}{\partial q_k} \right) = 0 \tag{2.17c}$$

To find the generator, we must solve the following partial differential equations:

$$\frac{\partial g}{\partial p} = -p \longrightarrow g = -\frac{p^2}{2} + f(x) \tag{2.17d}$$
$$\frac{\partial g}{\partial x} = -x \longrightarrow g = -\frac{x^2}{2} + h(p) \tag{2.17e}$$

$$\therefore g = -\left(\frac{p^2}{2} + \frac{x^2}{2} \right) = -\mathcal{H}, \quad \text{so the generator is the negative of the Hamiltonian!} \tag{2.17f}$$

Exercise 21.

Chapter 3

The Postulates - A General Discussion

Exercise 1. (1) The possible values are the eigenvalues of L_z. Since L_z is already diagonal, its eigenvalues are the diagonal elements 1, 0 and -1.

(2)

$$\langle L_x \rangle = \langle 1|L_x|1\rangle = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \tag{3.1a}$$

$$\left\langle L_x^2 \right\rangle = \langle 1|L_x^2|1\rangle = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \frac{1}{2} \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{2} \tag{3.1b}$$

$$\Delta L_x = \sqrt{\langle L_x^2\rangle - \langle L_x\rangle^2} = \frac{1}{\sqrt{2}} \tag{3.1c}$$

(3) We must solve the eigenvalue problem for L_x:

$$\frac{1}{\sqrt{2}} \begin{vmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{vmatrix} = \frac{1}{\sqrt{2}}[-\lambda(\lambda^2 - 1) - (-\lambda)] = 0 \longrightarrow \lambda = 1, 0, -1 \tag{3.1d}$$

The eigenstates are found by substituting the eigenvalues and solving:

$$|L_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad |L_x = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix}, \quad |L_x = -1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \end{bmatrix} \tag{3.1e}$$

(4) The eigenstate $|\psi\rangle$ for the eigenvalue $L_z = -1$ is:

$$|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.1f}$$

The probabilities are found by dotting the ket with the eigenbras corresponding to the eigenstates of L_z:

$$P(L_x = 0) = |\langle L_x = 0|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{2} \tag{3.1g}$$

$$P(L_x = 1) = |\langle L_x = 1|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{4} \tag{3.1h}$$

$$P(L_x = -1) = |\langle L_x = -1|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{4} \tag{3.1i}$$

(5) L_z^2 is a degenerate matrix with eigenvalues $0, 1, 1$, so when the state is measured to be $L_z^2 = 1$, the state after the measurement is an eigenspace in $\mathcal{V}^2$. The linearly independent eigenkets describing this eigenspace corresponding to the eigenvalue $L_z^2 = 1$ are:

$$|\omega, 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |\omega, 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.1j}$$

Constructing the projection operator for these eigenkets to find the normalised state and its probability after measurement:

$$\mathbb{P}_\omega = \sum_i |\omega, i\rangle\langle\omega, i| \tag{3.1k}$$

$$|\psi'\rangle = \frac{\mathbb{P}_\omega |\psi\rangle}{|\langle\mathbb{P}_\omega\psi|\mathbb{P}_\omega\psi\rangle|} = \frac{2}{\sqrt{3}}\begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix} \tag{3.1l}$$

$$P\big(L_z^2 = 1\big) = \langle\psi|\mathbb{P}_\omega|\psi\rangle = \left\langle \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix} \middle| \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix} \middle| \begin{bmatrix}1/2\\1/2\\1/\sqrt{2}\end{bmatrix}\right\rangle \tag{3.1m}$$

$$= \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1/2\\1/2\\1/\sqrt{2}\end{bmatrix} = \frac{3}{4} \tag{3.1n}$$

The outcomes of measuring L_z are its eigenvalues, which are $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. Their respective probabilities are found by dotting the current state with their eigenvectors:

$$P(L_z = 0) = |\langle\omega_1|\psi'\rangle|^2 = \left(\begin{bmatrix}0 & 1 & 0\end{bmatrix}\frac{2}{\sqrt{3}}\begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix}\right)^2 = 0 \tag{3.1o}$$

$$P(L_z = 1) = |\langle\omega_2|\psi'\rangle|^2 = \left(\begin{bmatrix}1 & 0 & 0\end{bmatrix}\frac{2}{\sqrt{3}}\begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix}\right)^2 = \frac{1}{3} \tag{3.1p}$$

$$P(L_z = -1) = |\langle\omega_3|\psi'\rangle|^2 = \left(\begin{bmatrix}0 & 0 & 1\end{bmatrix}\frac{2}{\sqrt{3}}\begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix}\right)^2 = \frac{2}{3} \tag{3.1q}$$

(6) The probabilities for each of the eigenvalues of L_z is:

$$P(L_z = 0) = |\langle\omega_1|\psi\rangle|^2 = \frac{1}{2} = |\alpha|^2 \tag{3.1r}$$

$$P(L_z = 1) = |\langle\omega_2|\psi\rangle|^2 = \frac{1}{4} = |\beta|^2 \tag{3.1s}$$

$$P(L_z = 1) = |\langle\omega_3|\psi\rangle|^2 = \frac{1}{4} = |\gamma|^2 \tag{3.1t}$$

The normalised state is thus:

$$|\psi\rangle = \frac{\alpha\,|L_z = 0\rangle + \beta\,|L_z = 1\rangle + \gamma\,|L_z = -1\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}} = \alpha\,|L_z = 0\rangle + \beta\,|L_z = 1\rangle + \gamma\,|L_z = -1\rangle \tag{3.1u}$$

However, the most general state is:

$$|\psi\rangle = \frac{e^{i\delta_1}}{\sqrt{2}}\,|L_z = 0\rangle + \frac{e^{i\delta_2}}{2}\,|L_z = 1\rangle + \frac{e^{i\delta_3}}{2}\,|L_z = -1\rangle \tag{3.1v}$$

This is because when performing measurements of other variables, interference terms come into play. For example, if we measure $L_x = 0$ in the given state:

$$P(L_x = 0) = |\langle L_x = 0|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} e^{i\delta_1} \\ \sqrt{2}e^{i\delta_2} \\ e^{i\delta_3} \end{bmatrix} \right|^2 = \frac{1}{2}\sin^2\left(\frac{\delta_3 - \delta_1}{2}\right) \tag{3.1w}$$

Evidently the state will depend on the phase difference $(\delta_3 - \delta_1)$. Clearly the exponential phase factors are relevant in measuring probabilities.

Exercise 2. The expectation value is given by:

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} \langle\psi|k\rangle\ \langle k|P|\psi\rangle\,\mathrm{d}k = \int_{-\infty}^{\infty} p\psi^*(k)\psi(k)\,\mathrm{d}k \tag{3.2a}$$

$$\psi(k) = \langle k|\psi\rangle = \int_{-\infty}^{\infty} \langle k|x\rangle\ \langle x|\psi\rangle\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\psi(x)\,\mathrm{d}x \tag{3.2b}$$

$$\psi^*(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\psi^*(x)\,\mathrm{d}x = \psi(-k), \quad \because \psi^*(x) = \psi(x) \tag{3.2c}$$

$$\langle P\rangle = \int_{-\infty}^{\infty} \hbar k\ \psi(-k)\psi(k)\,\mathrm{d}k = 0, \quad \because \text{the integral is odd} \tag{3.2d}$$

Exercise 3.

Exercise 4.

Chapter 4

Simple Problems in One Dimension

Exercise 1. This can be solved by substitution:

$$p = \pm\sqrt{2mE} \longrightarrow \mathrm{d}p = \pm\frac{m}{\sqrt{2mE}}\,\mathrm{d}E \tag{4.1a}$$

Since there are two values that p can take, we must expand the integral to include the values with respect to E:

$$U(t) = \int_{-\infty}^{0} -\frac{m}{\sqrt{2mE}}\,|E,-\rangle\langle E,-|\,e^{-iEt/\hbar}\,\mathrm{d}E + \int_{0}^{\infty} \frac{m}{\sqrt{2mE}}\,|E,+\rangle\langle E,+|\,e^{-iEt/\hbar}\,\mathrm{d}E \tag{4.1b}$$

$$U(t) = \sum_{\alpha=\pm}\int_{0}^{\infty} \frac{m}{\sqrt{2mE}}\,|E,\alpha\rangle\langle E,\alpha|\,e^{-iEt/\hbar}\,\mathrm{d}E \tag{4.1c}$$

Exercise 2. Using $|x\rangle$ as a trial solution:

$$\frac{P^2}{2m}\,|x\rangle = E\,|x\rangle \tag{4.2a}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\,|x\rangle = E\,|x\rangle \tag{4.2b}$$

$$\left(\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + E\right)|x\rangle = 0 \tag{4.2c}$$

The solution to this differential equation is readily found by substituting $D = \frac{\mathrm{d}}{\mathrm{d}x}$ and solving the algebraic equation, giving:

$$D = \pm\frac{ip}{\hbar} \longrightarrow \psi_E(x) = \frac{\beta}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip}{\hbar}x\right) + \frac{\gamma}{\sqrt{2\pi\hbar}}\exp\left(-\frac{ip}{\hbar}x\right), \quad p = \sqrt{2mE} \tag{4.2d}$$

If $E \leq 0$, then the function consists of real exponentials that blow up at large values of x, and are thus not in the Hilbert space.

Exercise 3. We have the propagator and initial state:

$$U(t) = \exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2 t}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar t}{2m}\right)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}, \quad \psi(x,0) = \frac{e^{-x^2/2}}{\sqrt[4]{\pi}} \tag{4.3a}$$

Expanding the initial state as a power series: $\psi(x,0) = \frac{1}{\sqrt[4]{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!(2)^n}$ (4.3b)

$$\psi(x,t) = \left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar t}{2m}\right)^k\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\right)\left(\sum_{n=0}^{\infty}\frac{1}{\sqrt[4]{\pi}}\frac{(-1)^n x^{2n}}{n!(2)^n}\right) \tag{4.3c}$$

$$\psi(x,t) = \frac{1}{\sqrt[4]{\pi}}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{i\hbar t}{m}\right)^k\frac{(-1)^n}{n!k!(2)^{n+k}}\frac{(2n)!}{(2n-2k)!}x^{2(n-k)}\right] \tag{4.3d}$$

The coefficients of the x^{2n} terms are:

$$x^0 : \frac{(-1)^0}{0!}\left[1 - \frac{1}{2}\frac{i\hbar t}{m} + \frac{3}{8}\left(\frac{i\hbar t}{m}\right)^2 - \frac{5}{16}\left(\frac{i\hbar t}{m}\right)^3 + \frac{35}{128}\left(\frac{i\hbar t}{m}\right)^4 - \ldots\right]\frac{x^{2\times 0}}{2^0} \tag{4.3e}$$

$$x^2 : \frac{(-1)^1}{1!}\left[1 - \frac{3}{2}\left(\frac{i\hbar t}{m}\right) + \frac{15}{8}\left(\frac{i\hbar t}{m}\right)^2 - \frac{35}{16}\left(\frac{i\hbar t}{m}\right)^3 + \ldots\right]\frac{x^{2\times 1}}{2^1} \tag{4.3f}$$

$$x^4 : \frac{(-1)^2}{2!}\left[1 - \frac{5}{2}\left(\frac{i\hbar t}{m}\right) + \frac{35}{8}\left(\frac{i\hbar t}{m}\right)^2 - \ldots\right]\frac{x^{2\times 2}}{2^2} \tag{4.3g}$$

$$\psi(x,t) = \frac{1}{\sqrt[4]{\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{1 - \left(n+\frac{1}{2}\right)\left(\frac{i\hbar t}{m}\right) + \frac{1}{2!}\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(\frac{i\hbar t}{m}\right)^2 - \ldots\right\}\frac{x^{2n}}{2^n}\right] \tag{4.3h}$$

$$\psi(x,t) = \frac{1}{\sqrt[4]{\pi}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(1 + \frac{i\hbar t}{m}\right)^{-n-\frac{1}{2}}\frac{x^{2n}}{2^n}\right] \tag{4.3i}$$

$$= \frac{1}{\sqrt{\sqrt{\pi}\left(1+\frac{i\hbar t}{m}\right)}}\exp\left[-\frac{x^2}{2\left(1+\frac{i\hbar t}{m}\right)}\right] \tag{4.3j}$$

Exercise 4.