

Solutions to  
Gravitation: Foundations and Frontiers  
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# Chapter 1

## Special relativity

**Problem 1** (Light clocks). Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$
$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$dt = \gamma d\tau$$
$$\frac{L'}{c + v} + \frac{L'}{c - v} = \gamma \frac{2L}{c}$$
$$L' \frac{2c}{c^2 - v^2} = \gamma \frac{2L}{c}$$
$$L' = \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

**Problem 2** (Superluminal motion).

$$\begin{aligned}
 \Delta t' &= t'_2 - t'_1 \\
 t'_1 - t_1 &\approx \frac{v}{c} \Delta t \cos \theta \\
 t'_2 - t_1 &= L/c \\
 \Delta t' &= \Delta t (1 - (v/c) \cos \theta) \\
 v_{app} &= v \Delta t \sin \theta = \frac{v \sin \theta}{1 - (v/c) \cos \theta}
 \end{aligned}$$

**Problem 3** (The strange world of four-vectors).

$$(a^i + b^i)(a_i + b_i) = a^i a_i + b^i b_i + 2(a^i b_i)$$

**Problem 4** (Focused to the front). Using a Lorentz transformation on the ‘time’ component of the four-vector  $k^a = (\omega, \omega \mathbf{n}/c)$ :

$$\begin{aligned}
 (a) \quad \omega_R &= \gamma \left( \omega_L - \frac{\omega_L v \cos \theta_L}{c} \right) = \gamma \omega_L (1 - (v/c) \cos \theta_L) \\
 \frac{\omega_R \cos \theta_R}{\omega_R} &= \frac{\gamma \omega_L (\cos \theta_L - (v/c))}{\gamma \omega_L (1 - (v/c) \cos \theta_L)} \\
 \mu_R &= \frac{\mu_L - (v/c)}{1 - (v \mu_L / c)} \\
 (c) \quad \int d(\cos \theta) d\phi &= - \int \sin \theta d\theta d\phi = - d\Omega \\
 d(\cos \theta_R) &= \left[ \frac{(v/c) [\cos \theta_L - (v/c)]}{[1 - (v/c) \cos \theta_L]^2} + \frac{1}{[1 - (v/c) \cos \theta_L]} \right] d(\cos \theta_L) \\
 d\Omega' &= \frac{1}{\gamma^2} \frac{d\Omega}{[1 - (v/c) \cos \theta]^2}
 \end{aligned}$$

The energy is:

$$d\mathcal{E}' = \hbar \omega = \gamma d\mathcal{E} [1 - (v/c) \cos \theta]$$

The time is:

$$dt' = \gamma dt$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is ( $c = 1$ ):

$$\begin{aligned}\left(\frac{d\mathcal{E}'}{dt' d\Omega'}\right)_{\text{rest}} &= \gamma^2 (1 - v \cos \theta)^3 \left(\frac{d\mathcal{E}}{dt d\Omega}\right)_{\text{lab}} \\ \left(\frac{d\mathcal{E}}{dt d\Omega}\right)_{\text{lab}} &= \frac{(1 - v^2)^2}{(1 - v \cos \theta)^3} \left(\frac{d\mathcal{E}'}{dt' d\Omega'}\right)_{\text{rest}}\end{aligned}$$

If the emission is isotropic,  $d\Omega = d\Omega' = 4\pi$ :

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{lab}} = d\Omega \left(\frac{d\mathcal{E}'}{dt' d\Omega'}\right)_{\text{rest}} = \left(\frac{d\mathcal{E}'}{dt'}\right)_{\text{rest}}$$

**Problem 5** (Transformation of antisymmetric tensors).

$$A^{i'k'} = L^{k'}_k L^{i'}_i A^{ik} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^{i'}}{\partial x^i} A^{ik}$$

If  $A^{ik} = -A^{ki}$  then:

$$A^{k'i'} = -A^{i'k'}?$$

**Problem 6** (Practice with completely antisymmetric tensors).

**Problem 7** (A null curve in flat spacetime).

$$\eta_{ij} x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

**Problem 8** (Shadows are Lorentz invariant).

**Problem 9** (Hamiltonian form of action - Newtonian mechanics).

$$\begin{aligned}\mathcal{A} &= \int_{t_2}^{t_1} dt [p\dot{q} - H(p, q)] \\ \delta\mathcal{A} &= \int_{t_2}^{t_1} dt \left[ \dot{q} \delta p + p \delta \dot{q} - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial q} \delta q \right] = 0\end{aligned}$$

Using  $\delta \dot{q} = \dot{d}(\delta q)$  and integrating by parts:

$$\begin{aligned}\int_{t_2}^{t_1} dt p \delta \dot{q} &= \cancel{p \delta q \Big|_{t_1}^{t_2}} - \int_{t_2}^{t_1} dt \dot{p} \delta q \\ \int_{t_2}^{t_1} dt \left[ \left( \dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left( \dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right] &= 0\end{aligned}$$

Since  $\delta q = 0$  at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

**Problem 10** (Hamiltonian form of action - special relativity).

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} d\lambda \left[ p_a \dot{x}^a - \frac{1}{2} C \left( \frac{H}{m} + m \right) \right]$$

**Problem 11** (Hitting a mirror).

**Problem 12** (Photon-electron scattering).

**Problem 13** (More practice with collisions).

**Problem 14** (Relativistic rocket).

**Problem 15** (Practice with equilibrium distribution functions).

**Problem 16** (Projection effects).

**Problem 17** (Relativistic virial theorem). The conservation law implies the following:

$$\begin{aligned} \partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\ &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\mu} \partial_\mu [x^\alpha x^\beta] \\ &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\mu} [\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta] \\ &= -\partial_\mu [T^{0\mu} x^\alpha x^\beta] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha \end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned} \partial_0 (\partial_0 T^{00}) x^\alpha x^\beta &= -\partial_\mu [\partial_0 T^{0\mu}] x^\alpha x^\beta \\ \partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\ &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad (\partial_i T^{i\alpha} = 0) \\ &= -\partial_\mu [\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha] + 2T^{\alpha\beta}, \quad (T^{\mu\alpha} \partial_\mu x^\beta = T^{\alpha\beta}) \end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that  $T^{ij} = 0$  outside a compact region in space:

$$\begin{aligned} \int d^3x \partial_0^2 T^{00} x^\alpha x^\beta &= \int d^3x \left[ -\partial_\mu (\partial_0 T^{0\mu} x^\alpha x^\beta + \cancel{T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha}) + 2T^{\alpha\beta} \right] \\ \frac{d^2}{dt^2} \int d^3x T^{00} x^\alpha x^\beta &= 2 \int d^3x T^{\alpha\beta} \end{aligned}$$

**Problem 18** (Explicit computation of spin precession). The four-velocity and four-acceleration are:

$$u^i = (\gamma, \gamma \vec{v}) = (\gamma, -\gamma r \omega \sin \omega t, \gamma r \omega \cos \omega t, 0)$$

$$a^i = \gamma \left( \dot{\gamma}, \dot{\gamma} \vec{v} + \ddot{\gamma} \right) = -\gamma^2 \omega^2 (0, x, y, 0)$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$\frac{dS^j}{d\tau} = u^j (S^k a_k)$$

$$S^k a_k = -\gamma^2 \omega^2 (x S^x + y S^y)$$

$$\frac{d}{dt} [(y S^x - x S^y)] = -\omega (x S^x + y S^y)$$

**Problem 19** (Little group of the Lorentz group).

## Chapter 2

# Scalar and electromagnetic fields in special relativity

**Problem 1** (Measuring the  $F^{ab}$ ).

**Problem 2** (Schrödinger equation and gauge transformation). The transformed Schrödinger equation is:

$$i\hbar\partial_t\psi = \left[ \frac{1}{2m} \left[ \mathbf{P} - \frac{q}{c}(\mathbf{A} + \nabla f) \right]^2 + q(\phi + \partial_t f) \right] \psi$$

**Problem 3** (Four-vectors leading to electric and magnetic fields).

**Problem 4** (Hamiltonian form of action - charged particle).

**Problem 5** (Three-dimensional form of the Lorentz force). Using the electromagnetic

tensor equation of motion:

$$\begin{aligned}
m \frac{du^i}{d\tau} &= q F^{ik} u_k \implies m \frac{du^i}{dt} = q F^{i\alpha} u_\alpha \\
m \frac{du^0}{dt} &= q(\mathbf{E} \cdot \mathbf{v}) \implies \frac{m\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} = q(\mathbf{E} \cdot \mathbf{v}) \\
m \frac{du^\alpha}{dt} &= q F^{\alpha\beta} u_\beta = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
m \left[ \frac{\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} + \gamma \frac{d\mathbf{v}}{dt} \right] &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
\implies \frac{d\mathbf{v}}{dt} &= \frac{q}{m\gamma} \left[ \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2}(\mathbf{E} \cdot \mathbf{v})\mathbf{v} \right]
\end{aligned}$$

**Problem 6** (Pure gauge impostors). This can be transformed into polar coordinates ( $x = \cos \theta, y = \sin \theta$ ) and evaluated ( $A_r$  is obviously 0):

$$\begin{aligned}
A_\theta &= \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = 1, \\
\oint_{C=x^2+y^2} \mathbf{A} \cdot d\mathbf{s} &= \int_0^{2\pi} d\theta = 2\pi
\end{aligned}$$

The reason why  $\mathbf{A}$  is not a pure gauge mode is because  $f$  is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

**Problem 7** (Pure electric or magnetic fields).

**Problem 8** (Elegant solution to non-relativistic Coulomb motion). Since the angular momentum is conserved,  $d\mathbf{J}/dt = 0$ :

$$\begin{aligned}
(a) \quad \frac{d}{dt}(\mathbf{p} \times \mathbf{J}) &= \frac{d\mathbf{p}}{dt} \times \mathbf{J} + \mathbf{p} \times \frac{d\mathbf{J}}{dt} \\
f(r)\hat{\mathbf{r}} \times \mathbf{r} \times m\mathbf{v} &= mf(r)[\mathbf{r}(\hat{\mathbf{r}} \times \mathbf{v}) - \mathbf{v}(\hat{\mathbf{r}} \cdot \mathbf{r})] \\
&= -mf(r)r \frac{d\mathbf{r}}{dt} = -mf(r)r^2 \frac{d\hat{\mathbf{r}}}{dt}
\end{aligned}$$

Therefore, if  $f(r)r^2 = -|\alpha|$ , then:

$$\begin{aligned}
\int \frac{1}{m|\alpha|} \frac{d}{dt}(\mathbf{p} \times \mathbf{J}) dt &= \int \frac{d\hat{\mathbf{r}}}{dt} dt \\
\implies \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} &= \mathbf{e}
\end{aligned}$$



where  $\mathbf{e}$  is a conserved vector, arising as a constant of the integration.

(b) Note that  $\mathbf{p}$  and  $\mathbf{J}$  are perpendicular:

$$\begin{aligned}
 \mathbf{e} \cdot \mathbf{e} &= \left[ \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} \right] \cdot \left[ \frac{\mathbf{p} \times \mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} \right] \\
 \Rightarrow |\mathbf{e}|^2 &= \frac{|\mathbf{p} \times \mathbf{J}|^2}{m^2|\alpha|^2} - 2 \left( \frac{\hat{\mathbf{r}} \cdot \mathbf{p} \times \mathbf{J}}{m|\alpha|} \right) + 1 \\
 &= \frac{p^2 J^2}{m^2|\alpha|^2} - 2 \left( \frac{J^2}{m|\alpha|r} \right) + 1, \quad J^2 = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{J} \\
 &= 1 + \frac{2EJ^2}{m|\alpha|^2}, \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r}
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \mathbf{e} \cdot \mathbf{r} &= \frac{\mathbf{r} \cdot \mathbf{p} \times \mathbf{J}}{m|\alpha|} - \mathbf{r} \cdot \hat{\mathbf{r}} \\
 er \cos \theta &= \frac{J^2}{m|\alpha|} - r \\
 \Rightarrow r(\theta) &= \frac{J^2/m|\alpha|}{1 + e \cos \theta}
 \end{aligned}$$

$$(d) \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r} - \frac{|\beta|}{r^2}$$