

Solutions to
Principles of Quantum Mechanics
(Second Edition)
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Chapter 1

Mathematical Introduction

Exercise 1. The null vector is $(0, 0, 0)$. The inverse under addition is $(-a, -b, -c)$. A vector of the form $(a, b, 1)$ does not form a vector space because it fails to satisfy the closure property under addition and multiplication:

$$(a, b, 1) + (d, e, 1) = (a + d, b + e, 2) \notin (a, b, 1)$$

$$\alpha(a, b, 1) = (\alpha a, \alpha b, \alpha) \notin (a, b, 1)$$

Exercise 2.

Exercise 3. The set of kets is not linearly independent, as $|3\rangle = |1\rangle - 2|2\rangle$.

Exercise 4. Arrange the row vectors into a matrix and find the determinant:

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0 \quad (1.2a)$$

Since the determinant is zero, one of the row vectors is expressible as a linear combination of the others. This is seen in $(3, 2, 1) = 2(1, 1, 0) + (1, 0, 1)$.

For the second set of vectors, we perform the same procedure:

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2 \quad (1.2b)$$

So there is no vector in this set that is expressible as a linear combination of the other vectors.

Exercise 5. Let $|I\rangle = \vec{A}$ and $|II\rangle = \vec{B}$. Following Gram-Schmidt orthonormalisation:

$$\hat{A} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} = |1\rangle \quad (1.3a)$$

$$|2'\rangle = |II\rangle - |1\rangle \langle 1|II\rangle = \frac{18}{5} \quad (1.3b)$$

Exercise 6.

Exercise 7.

Chapter 2

Review of Classical Mechanics

Exercise 1. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (2.1a)$$

Solving the Euler-Lagrange equation gives the equation of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + kx = 0 \quad (2.1b)$$

Exercise 2. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k(x_1 - x_2)^2 \quad (2.2a)$$

Solving the Euler-Lagrange equations gives the equations of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} - \frac{\partial \mathcal{L}}{\partial x_1} = m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad (2.2b)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_2} - \frac{\partial \mathcal{L}}{\partial x_2} = m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \quad (2.2c)$$

Rearranging, we get the same equations of motion:

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

Exercise 3. The Lagrangian in polar coordinates is:

$$\mathcal{L} = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - V(r) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right] - V(r) \quad (2.3a)$$

Solving the Euler-Lagrange equations gives the equations of motion:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - (mr\sin^2\theta)\dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \quad (2.3b)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = mr^2\ddot{\theta} + 2mrr\dot{\theta} - \frac{1}{2}(mr^2\sin 2\theta)\dot{\phi}^2 = 0 \quad (2.3c)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = (mr^2\sin^2\theta)\ddot{\phi} + 2m\dot{\phi}(r\dot{r}\sin^2\theta + r^2\dot{\theta}\sin 2\theta) = 0 \quad (2.3d)$$

Exercise 4. Substituting $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ with $\dot{\mathbf{r}}_{\text{CM}}$ and $\dot{\mathbf{r}}$:

$$\mathcal{L} = \frac{1}{2}m_1\left|\dot{\mathbf{r}}_{\text{CM}} + \frac{m_2\dot{\mathbf{r}}}{m_1 + m_2}\right|^2 + \frac{1}{2}m_2\left|\dot{\mathbf{r}}_{\text{CM}} - \frac{m_1\dot{\mathbf{r}}}{m_1 + m_2}\right|^2 - V(\mathbf{r}) \quad (2.4)$$

Expanding the squares:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_{\text{CM}}|^2 + \frac{1}{2}m_1\left|\frac{2m_2\dot{\mathbf{r}}_{\text{CM}}}{m_1 + m_2}\right|^2 + \frac{1}{2}m_1\left(\frac{m_2}{m_1 + m_2}\right)^2|\dot{\mathbf{r}}|^2 \\ & - \frac{1}{2}m_2\left|\frac{2m_1\dot{\mathbf{r}}_{\text{CM}}}{m_1 + m_2}\right|^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1 + m_2}\right)^2|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \end{aligned} \quad (2.5)$$

Which gives the final expression:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_{\text{CM}}|^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad (2.6)$$

Exercise 5.

Exercise 6. The conservation of energy in the harmonic oscillator states that:

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \quad (2.7a)$$

Dividing both sides by E :

$$\frac{p^2}{2mE} + \frac{kx^2}{2E} = 1 \longrightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{p}{b}\right)^2 = 1, \quad a^2 = \frac{2E}{k}, \quad b^2 = 2mE \quad (2.7b)$$

Exercise 7. The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k(x_1 - x_2)^2 \quad (2.8a)$$

Finding the generalised momenta:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m_i \dot{x}_i \longrightarrow \dot{x}_i = \frac{p_i}{m_i} \quad (2.8b)$$

The Hamiltonian is found by:

$$\mathcal{H} = \sum_i p_i \dot{x}_i - \mathcal{L} = \mathcal{T} + \mathcal{V} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k(x_1^2 + x_2^2) + \frac{1}{2}k(x_1 - x_2)^2 \quad (2.8c)$$

Solving Hamilton's equations gives the equations of motion:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{x}_i \longrightarrow \dot{x}_i = \frac{p_i}{m_i} \quad (2.8d)$$

$$-\frac{\partial \mathcal{H}}{\partial x_i} = \dot{p}_i \longrightarrow \dot{p}_1 = -2kx_1 + kx_2, \quad \dot{p}_2 = kx_1 - 2kx_2 \quad (2.8e)$$

But $\dot{p}_i = m\ddot{x}_i$, so we get the same equations of motion as before:

$$m\ddot{x}_1 = -2kx_1 + kx_2, \quad m\ddot{x}_2 = kx_1 - 2kx_2 \quad (2.8f)$$

Exercise 8. The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2) |\dot{\mathbf{r}}_{\text{CM}}|^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \quad (2.9a)$$

The generalised momenta are:

$$\mathbf{p}_{\text{CM}} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_{\text{CM}}} = (m_1 + m_2) |\dot{\mathbf{r}}_{\text{CM}}|, \quad \mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \left(\frac{m_1 m_2}{m_1 + m_2} \right) |\dot{\mathbf{r}}| \quad (2.9b)$$

Writing $m_1 + m_2 = M$ and $m_1 m_2 / M = \mu$, and finding the Hamiltonian:

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = \frac{1}{2}M \left| \frac{\mathbf{p}_{\text{CM}}}{M} \right|^2 + \frac{1}{2}\mu \left| \frac{\mathbf{p}}{\mu} \right|^2 + V(\mathbf{r}) = \frac{|\mathbf{p}_{\text{CM}}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + V(\mathbf{r}) \quad (2.9c)$$

Exercise 9.

$$\{\omega, \lambda\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) \quad (2.10a)$$

$$\{\lambda, \omega\} = \sum_i \left(\frac{\partial \lambda}{\partial q_i} \frac{\partial \omega}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \frac{\partial \omega}{\partial q_i} \right) = -\{\omega, \lambda\} \quad (2.10b)$$

$$\{\omega, \lambda + \sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial(\lambda + \sigma)}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial(\lambda + \sigma)}{\partial q_i} \right) = \quad (2.10c)$$

$$\sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) = \{\omega, \lambda\} + \{\omega, \sigma\} \quad (2.10d)$$

$$\{\omega, \lambda \sigma\} = \sum_i \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda \sigma}{\partial q_i} \right) = \quad (2.10e)$$

$$\sum_i \sigma \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right) + \sum_i \lambda \left(\frac{\partial \omega}{\partial q_i} \frac{\partial \sigma}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \sigma}{\partial q_i} \right) = \{\omega, \lambda\} \sigma + \lambda \{\omega, \sigma\} \quad (2.10f)$$

Exercise 10. (i)

$$\{q_i, q_j\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = 0, \quad \because \frac{\partial q_i}{\partial q_k} = \delta_{ik}, \quad \frac{\partial q_j}{\partial q_k} = \delta_{jk}, \quad \forall i, j, k \in \mathbb{N} \quad (2.11a)$$

$$\{p_i, p_j\} = \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = 0, \quad \because \frac{\partial p_i}{\partial p_k} = \delta_{ik}, \quad \frac{\partial p_j}{\partial p_k} = \delta_{jk}, \quad \forall i, j, k \in \mathbb{N} \quad (2.11b)$$

(ii) The Hamiltonian with $a = b$ is:

$$\mathcal{H} = p_x^2 + p_y^2 + a(x^2 + y^2) \quad (2.11c)$$

The angular momentum about the z-axis $l_z = xp_y - yp_x$ is conserved because the potential $V(x, y)$ is expressible as $V(x^2 + y^2)$ and the z-coordinate is not present in the Hamiltonian. Explicit computation yields:

$$\{l_z, \mathcal{H}\} = \sum_i \left(\frac{\partial l_z}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial l_z}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) = \left(\frac{\partial l_z}{\partial x} \frac{\partial \mathcal{H}}{\partial p_x} - \frac{\partial l_z}{\partial p_x} \frac{\partial \mathcal{H}}{\partial x} \right) + \left(\frac{\partial l_z}{\partial y} \frac{\partial \mathcal{H}}{\partial p_y} - \frac{\partial l_z}{\partial p_y} \frac{\partial \mathcal{H}}{\partial y} \right) \quad (2.11d)$$

$$= (p_y \cdot 2p_x - y \cdot 2x) + (-p_x \cdot 2p_y - (-p_x) \cdot 2y) = 0 \quad (2.11e)$$

Exercise 11.

Exercise 12.

$$\{\bar{x}, \bar{y}\} = \sum_{k=x,y} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{y}}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{y}}{\partial q_k} \right) = 0 \quad (2.12a)$$

$$\{\bar{x}, \bar{p}_y\} = \sum_{k=x,y} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{p}_y}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{p}_y}{\partial q_k} \right) = 0 \quad (2.12b)$$

Exercise 13.

$$\{\rho, p_\rho\} = \sum_{k=x,y} \left(\frac{\partial \rho}{\partial q_k} \frac{\partial p_\rho}{\partial p_k} - \frac{\partial \rho}{\partial p_k} \frac{\partial p_\rho}{\partial q_k} \right) = \left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2 = 1 \quad (2.13a)$$

$$\{\rho, p_\phi\} = \sum_{k=x,y} \left(\frac{\partial \rho}{\partial q_k} \frac{\partial p_\phi}{\partial p_k} - \frac{\partial \rho}{\partial p_k} \frac{\partial p_\phi}{\partial q_k} \right) = -y \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + x \left(\frac{y}{\sqrt{x^2 + y^2}} \right) = 0 \quad (2.13b)$$

$$\begin{aligned} \{\phi, p_\rho\} &= \sum_{k=x,y} \left(\frac{\partial \phi}{\partial q_k} \frac{\partial p_\rho}{\partial p_k} - \frac{\partial \phi}{\partial p_k} \frac{\partial p_\rho}{\partial q_k} \right) = \left(\frac{-y}{x^2 + y^2} \right) \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \\ &\quad + \left(\frac{x}{x^2 + y^2} \right) \left(\frac{y}{\sqrt{x^2 + y^2}} \right) = 0 \end{aligned} \quad (2.13c)$$

$$\{\phi, p_\phi\} = \sum_{k=x,y} \left(\frac{\partial \phi}{\partial q_k} \frac{\partial p_\phi}{\partial p_k} - \frac{\partial \phi}{\partial p_k} \frac{\partial p_\phi}{\partial q_k} \right) = -y \left(\frac{-y}{x^2 + y^2} \right) + x \left(\frac{x}{x^2 + y^2} \right) = 1 \quad (2.13d)$$

$$\{\rho, \phi\} = \sum_{k=x,y} \left(\frac{\partial \rho}{\partial q_k} \frac{\partial \phi}{\partial p_k} - \frac{\partial \rho}{\partial p_k} \frac{\partial \phi}{\partial q_k} \right) = 0 \quad (2.13e)$$

$$\{p_\rho, p_\phi\} = \sum_{k=x,y} \left(\frac{\partial p_\rho}{\partial q_k} \frac{\partial p_\phi}{\partial p_k} - \frac{\partial p_\rho}{\partial p_k} \frac{\partial p_\phi}{\partial q_k} \right) = 0 \quad (2.13f)$$

Exercise 14.**Exercise 15.****Exercise 16.**

Exercise 17. All we have to do is check the infinitesimal changes for $p = p_1 + p_2$:

$$\delta x_1 = \epsilon \frac{\partial p}{\partial p_1} = \epsilon, \quad \delta x_2 = \epsilon \frac{\partial p}{\partial p_2} = \epsilon \quad (2.14a)$$

$$\delta p_1 = -\epsilon \frac{\partial p}{\partial x_1} = 0, \quad \delta p_2 = -\epsilon \frac{\partial p}{\partial x_2} = 0 \quad (2.14b)$$

Therefore, the generator $g(q, p) = p$.

Exercise 18. The infinitesimal transformations are:

$$\bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i}, \quad \bar{p}_j = p_j - \epsilon \frac{\partial g}{\partial q_j}$$

Checking the Poisson brackets:

$$\{\bar{q}_i, \bar{p}_j\} = \sum_k \left(\frac{\partial \bar{q}_i}{\partial q_k} \frac{\partial \bar{p}_j}{\partial p_k} - \frac{\partial \bar{q}_i}{\partial p_k} \frac{\partial \bar{p}_j}{\partial q_k} \right) = \quad (2.15a)$$

$$\sum_k \left[\left(\delta_{ik} + \epsilon \frac{\partial^2 g}{\partial p_i \partial q_k} \right) \left(\delta_{jk} - \epsilon \frac{\partial^2 g}{\partial q_j \partial p_k} \right) - \left(\epsilon \frac{\partial^2 g}{\partial p_i \partial p_k} \right) \left(-\epsilon \frac{\partial^2 g}{\partial q_i \partial q_k} \right) \right] \quad (2.15b)$$

$$= \sum_k \left[\delta_{ik} \delta_{jk} + \epsilon \delta_{jk} \frac{\partial^2 g}{\partial p_i \partial q_k} - \epsilon \delta_{ik} \frac{\partial^2 g}{\partial q_j \partial p_k} + \epsilon^2 \frac{\partial^2 g}{\partial p_i \partial p_k} \frac{\partial^2 g}{\partial q_i \partial q_k} \right] \quad (2.15c)$$

$$= \delta_{ij} + \epsilon \left(\frac{\partial^2 g}{\partial p_i \partial q_j} - \frac{\partial^2 g}{\partial q_j \partial p_i} \right) + \mathcal{O}(\epsilon^2) = \delta_{ij}, \quad \because \frac{\partial^2 g}{\partial p_i \partial q_j} = \frac{\partial^2 g}{\partial q_j \partial p_i}, \quad \mathcal{O}(\epsilon^2) \approx 0 \quad (2.15d)$$

Exercise 19. The Hamiltonian under rotated coordinates is:

$$\mathcal{H}_R = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m \omega^2 [(x \cos \theta - y \sin \theta)^2 + (x \sin \theta + y \cos \theta)^2] = \mathcal{H} \quad (2.16a)$$

For the transformation to be noncanonical, the Poisson bracket $\{\bar{x}, \bar{p}_x, \neq 1$:

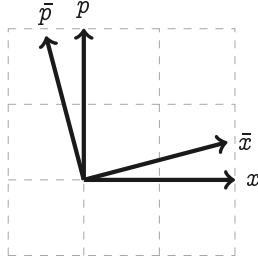
$$\{\bar{x}, \bar{p}_x\} = \sum_{k=x,y} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{p}_x}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{p}_x}{\partial q_k} \right) = \cos \theta \quad (2.16b)$$

To show that no conservation law follows:

$$\bar{q}_i = q_i + \epsilon \frac{\partial g}{\partial p_i} = q_i + \delta q_i, \quad \bar{p}_i = p_i \rightarrow \delta p_i = 0 \quad (2.16c)$$

$$\delta \mathcal{H} = \sum_i \frac{\partial \mathcal{H}}{\partial q_i} \left(\epsilon \frac{\partial g}{\partial p_i} \right) \neq \epsilon \{\mathcal{H}, g\} \quad (2.16d)$$

Exercise 20. A rotation in phase space can be shown via the following diagram:



The infinitesimal transformation is as follows:

$$\bar{x} = x \cos \epsilon - p \sin \epsilon \approx x - \epsilon p \quad (2.17a)$$

$$\bar{p} = x \sin \epsilon + p \cos \epsilon \approx \epsilon x + p \quad (2.17b)$$

We must verify if this transformation is canonical:

$$\{\bar{x}, \bar{p}\} = \sum_{k=x,p} \left(\frac{\partial \bar{x}}{\partial q_k} \frac{\partial \bar{p}}{\partial p_k} - \frac{\partial \bar{x}}{\partial p_k} \frac{\partial \bar{p}}{\partial q_k} \right) = 0 \quad (2.17c)$$

To find the generator, we must solve the following partial differential equations:

$$\frac{\partial g}{\partial p} = -p \longrightarrow g = -\frac{p^2}{2} + f(x) \quad (2.17d)$$

$$\frac{\partial g}{\partial x} = -x \longrightarrow g = -\frac{x^2}{2} + h(p) \quad (2.17e)$$

$$\therefore g = -\left(\frac{p^2}{2} + \frac{x^2}{2}\right) = -\mathcal{H}, \quad \text{so the generator is the negative of the Hamiltonian!} \quad (2.17f)$$

Exercise 21.

Chapter 3

The Postulates - A General Discussion

Exercise 1. (1) The possible values are the eigenvalues of L_z . Since L_z is already diagonal, its eigenvalues are the diagonal elements 1, 0 and -1.

(2)

$$\langle L_x \rangle = \langle 1 | L_x | 1 \rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \quad (3.1a)$$

$$\langle L_x^2 \rangle = \langle 1 | L_x^2 | 1 \rangle = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \quad (3.1b)$$

$$\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \frac{1}{\sqrt{2}} \quad (3.1c)$$

(3) We must solve the eigenvalue problem for L_x :

$$\frac{1}{\sqrt{2}} \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = \frac{1}{\sqrt{2}} [-\lambda(\lambda^2 - 1) - (-\lambda)] = 0 \longrightarrow \lambda = 1, 0, -1 \quad (3.1d)$$

The eigenstates are found by substituting the eigenvalues and solving:

$$|L_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad |L_x = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix}, \quad |L_x = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \end{bmatrix} \quad (3.1e)$$

(4) The eigenstate $|\psi\rangle$ for the eigenvalue $L_z = -1$ is:

$$|\psi\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.1f)$$

The probabilities are found by dotting the ket with the eigenbras corresponding to the eigenstates of L_z :

$$P(L_x = 0) = |\langle L_x = 0 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{2} \quad (3.1g)$$

$$P(L_x = 1) = |\langle L_x = 1 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{4} \quad (3.1h)$$

$$P(L_x = -1) = |\langle L_x = -1 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right|^2 = \frac{1}{4} \quad (3.1i)$$

(5) L_z^2 is a degenerate matrix with eigenvalues 0, 1, 1, so when the state is measured to be $L_z^2 = 1$, the state after the measurement is an eigenspace in \mathcal{V}^2 . The linearly independent eigenkets describing this eigenspace corresponding to the eigenvalue $L_z^2 = 1$ are:

$$|\omega, 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |\omega, 2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.1j)$$

Constructing the projection operator for these eigenkets to find the normalised state and its probability after measurement:

$$\mathbb{P}_\omega = \sum_i |\omega, i\rangle\langle\omega, i| \quad (3.1k)$$

$$|\psi'\rangle = \frac{\mathbb{P}_\omega |\psi\rangle}{|\langle\mathbb{P}_\omega \psi|\mathbb{P}_\omega \psi\rangle|} = \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad (3.1l)$$

$$P(L_z^2 = 1) = \langle\psi|\mathbb{P}_\omega|\psi\rangle = \left\langle \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \left| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right| \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} \right\rangle \quad (3.1m)$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix} = \frac{3}{4} \quad (3.1n)$$

The outcomes of measuring L_z are its eigenvalues, which are $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. Their respective probabilities are found by dotting the current state with their eigenvectors:

$$P(L_z = 0) = |\langle\omega_1|\psi'\rangle|^2 = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = 0 \quad (3.1o)$$

$$P(L_z = 1) = |\langle\omega_2|\psi'\rangle|^2 = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \frac{1}{3} \quad (3.1p)$$

$$P(L_z = -1) = |\langle\omega_3|\psi'\rangle|^2 = \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \frac{2}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right)^2 = \frac{2}{3} \quad (3.1q)$$

(6) The probabilities for each of the eigenvalues of L_z is:

$$P(L_z = 0) = |\langle\omega_1|\psi\rangle|^2 = \frac{1}{2} = |\alpha|^2 \quad (3.1r)$$

$$P(L_z = 1) = |\langle\omega_2|\psi\rangle|^2 = \frac{1}{4} = |\beta|^2 \quad (3.1s)$$

$$P(L_z = 1) = |\langle\omega_3|\psi\rangle|^2 = \frac{1}{4} = |\gamma|^2 \quad (3.1t)$$

The normalised state is thus:

$$|\psi\rangle = \frac{\alpha |L_z = 0\rangle + \beta |L_z = 1\rangle + \gamma |L_z = -1\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}} = \alpha |L_z = 0\rangle + \beta |L_z = 1\rangle + \gamma |L_z = -1\rangle \quad (3.1u)$$

However, the most general state is:

$$|\psi\rangle = \frac{e^{i\delta_1}}{\sqrt{2}} |L_z = 0\rangle + \frac{e^{i\delta_2}}{2} |L_z = 1\rangle + \frac{e^{i\delta_3}}{2} |L_z = -1\rangle \quad (3.1v)$$

This is because when performing measurements of other variables, interference terms come into play. For example, if we measure $L_x = 0$ in the given state:

$$P(L_x = 0) = |\langle L_x = 0 | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} e^{i\delta_1} \\ \sqrt{2}e^{i\delta_2} \\ e^{i\delta_3} \end{bmatrix} \right|^2 = \frac{1}{2} \sin^2 \left(\frac{\delta_3 - \delta_1}{2} \right) \quad (3.1w)$$

Evidently the state will depend on the phase difference $(\delta_3 - \delta_1)$. Clearly the exponential phase factors are relevant in measuring probabilities.

Exercise 2. The expectation value is given by:

$$\langle P \rangle = \langle \psi | P | \psi \rangle = \int_{-\infty}^{\infty} \langle \psi | k \rangle \langle k | P | \psi \rangle dk = \int_{-\infty}^{\infty} p \psi^*(k) \psi(k) dk \quad (3.2a)$$

$$\psi(k) = \langle k | \psi \rangle = \int_{-\infty}^{\infty} \langle k | x \rangle \langle x | \psi \rangle dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \quad (3.2b)$$

$$\psi^*(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \psi^*(x) dx = \psi(-k), \quad \therefore \psi^*(x) = \psi(x) \quad (3.2c)$$

$$\langle P \rangle = \int_{-\infty}^{\infty} \hbar k \psi(-k) \psi(k) dk = 0, \quad \therefore \text{the integral is odd} \quad (3.2d)$$

Exercise 3.

Exercise 4.

Chapter 4

Simple Problems in One Dimension

Exercise 1. This can be solved by substitution:

$$p = \pm \sqrt{2mE} \longrightarrow dp = \pm \frac{m}{\sqrt{2mE}} dE \quad (4.1a)$$

Since there are two values that p can take, we must expand the integral to include the values with respect to E :

$$U(t) = \int_{-\infty}^0 -\frac{m}{\sqrt{2mE}} |E, -\rangle \langle E, -| e^{-iEt/\hbar} dE + \int_0^{\infty} \frac{m}{\sqrt{2mE}} |E, +\rangle \langle E, +| e^{-iEt/\hbar} dE \quad (4.1b)$$

$$U(t) = \sum_{\alpha=\pm} \int_0^{\infty} \frac{m}{\sqrt{2mE}} |E, \alpha\rangle \langle E, \alpha| e^{-iEt/\hbar} dE \quad (4.1c)$$

Exercise 2. Using $|x\rangle$ as a trial solution:

$$\frac{P^2}{2m} |x\rangle = E |x\rangle \quad (4.2a)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} |x\rangle = E |x\rangle \quad (4.2b)$$

$$\left(\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + E \right) |x\rangle = 0 \quad (4.2c)$$

The solution to this differential equation is readily found by substituting $D = \frac{d}{dx}$ and solving the algebraic equation, giving:

$$D = \pm \frac{ip}{\hbar} \longrightarrow \psi_E(x) = \frac{\beta}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip}{\hbar}x\right) + \frac{\gamma}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ip}{\hbar}x\right), \quad p = \sqrt{2mE} \quad (4.2d)$$

If $E \leq 0$, then the function consists of real exponentials that blow up at large values of x , and are thus not in the Hilbert space.

Exercise 3. We have the propagator and initial state:

$$U(t) = \exp\left[\frac{i}{\hbar}\left(\frac{\hbar^2 t}{2m} \frac{d^2}{dx^2}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar t}{2m}\right)^n \frac{d^{2n}}{dx^{2n}}, \quad \psi(x, 0) = \frac{e^{-x^2/2}}{\sqrt[4]{\pi}} \quad (4.3a)$$

$$\text{Expanding the initial state as a power series:} \quad \psi(x, 0) = \frac{1}{\sqrt[4]{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(2)^n} \quad (4.3b)$$

$$\psi(x, t) = \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar t}{2m}\right)^k \frac{d^{2k}}{dx^{2k}}\right) \left(\sum_{n=0}^{\infty} \frac{1}{\sqrt[4]{\pi}} \frac{(-1)^n x^{2n}}{n!(2)^n}\right) \quad (4.3c)$$

$$\psi(x, t) = \frac{1}{\sqrt[4]{\pi}} \left[\sum_{n=0}^{\infty} \sum_{k=0}^n \left(\frac{i\hbar t}{m}\right)^k \frac{(-1)^n}{n!k!(2)^{n+k}} \frac{(2n)!}{(2n-2k)!} x^{2(n-k)}\right] \quad (4.3d)$$

The coefficients of the x^{2n} terms are:

$$x^0 : \frac{(-1)^0}{0!} \left[1 - \frac{1}{2} \frac{i\hbar t}{m} + \frac{3}{8} \left(\frac{i\hbar t}{m}\right)^2 - \frac{5}{16} \left(\frac{i\hbar t}{m}\right)^3 + \frac{35}{128} \left(\frac{i\hbar t}{m}\right)^4 - \dots\right] \frac{x^{2 \times 0}}{2^0} \quad (4.3e)$$

$$x^2 : \frac{(-1)^1}{1!} \left[1 - \frac{3}{2} \left(\frac{i\hbar t}{m}\right) + \frac{15}{8} \left(\frac{i\hbar t}{m}\right)^2 - \frac{35}{16} \left(\frac{i\hbar t}{m}\right)^3 + \dots\right] \frac{x^{2 \times 1}}{2^1} \quad (4.3f)$$

$$x^4 : \frac{(-1)^2}{2!} \left[1 - \frac{5}{2} \left(\frac{i\hbar t}{m}\right) + \frac{35}{8} \left(\frac{i\hbar t}{m}\right)^2 - \dots\right] \frac{x^{2 \times 2}}{2^2} \quad (4.3g)$$

$$\psi(x, t) = \frac{1}{\sqrt[4]{\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{1 - \left(n + \frac{1}{2}\right) \left(\frac{i\hbar t}{m}\right) + \frac{1}{2!} \left(n + \frac{1}{2}\right) \left(n + \frac{3}{2}\right) \left(\frac{i\hbar t}{m}\right)^2 - \dots\right\} \frac{x^{2n}}{2^n}\right] \quad (4.3h)$$

$$\psi(x, t) = \frac{1}{\sqrt[4]{\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(1 + \frac{i\hbar t}{m}\right)^{-n-\frac{1}{2}} \frac{x^{2n}}{2^n}\right] \quad (4.3i)$$

$$= \frac{1}{\sqrt{\sqrt{\pi} \left(1 + \frac{i\hbar t}{m}\right)}} \exp\left[-\frac{x^2}{2 \left(1 + \frac{i\hbar t}{m}\right)}\right] \quad (4.3j)$$

Exercise 4.