Solutions to Quantum Field Theory for the Gifted Amateur by Tom Lancaster, et al

Arjit Seth

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Chapter 1

Lagrangians

1. Fermat's principle of least time.

$$t = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{v_2} = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{c/n_1\sqrt{x^2 + h_1^2}} - \frac{(l-x)}{c/n_2\sqrt{(l-x)^2 + h_2^2}} = 0$$

$$n_1 \sin\theta = n_2 \sin\phi$$

2. Practice with functional derivatives.

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int G(x,y)[f(y) + \epsilon\delta(y-z)]\,\mathrm{d}y - \int G(x,y)f(y)\,\mathrm{d}y\right] = G(x,z)$$

$$\frac{\delta I[f^3]}{\delta f(x_0)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_{-1}^{1} [f(x) + \epsilon\delta(x-x_0)]^3\,\mathrm{d}x - \int_{-1}^{1} [f(x)]^3\,\mathrm{d}x\right] =$$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int\right]$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int \left(\frac{\partial}{\partial y}[f(y) + \epsilon\delta(y-x)]\right)^2 \mathrm{d}y - \int \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right]$$

$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int [f'(y) + \epsilon\delta'(y-x)]^2\,\mathrm{d}y - \int \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}y\right]$$

3. Euler-Lagrange equations using functional derivatives and more.

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int \left[g(y,f) + \frac{\partial g(y,f)}{\partial f}\epsilon\delta(y-x)\right]\mathrm{d}y - \int g(y,f)\,\mathrm{d}y\right] = \frac{\partial g(x,f)}{\partial f(x)}$$

4. Results on Dirac Delta functions.

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{\phi(x)+\epsilon\delta(x-y)-\phi(x)}{\epsilon} = \delta(x-y)$$

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{\mathrm{d}}{\mathrm{d}t}[\phi(t)+\epsilon\delta(t-t_0)]-\dot{\phi}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0)$$

5. Derivation of the wave equation.

$$S = \int T - V\ \mathrm{d}t = \frac{1}{2}\int \rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \mathcal{T}(\nabla\psi)^2\ \mathrm{d}t$$

$$\frac{\delta S}{\delta\psi} = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left[\int \rho\left(\frac{\partial}{\partial t}[\psi+\epsilon\delta(t-t_0)]\right)^2 - \mathcal{T}(\nabla[\psi+\epsilon\delta(\mathbf{x}-\mathbf{y})])^2\ \mathrm{d}t\right.$$

$$\left.-\int \rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \mathcal{T}(\nabla\psi)^2\ \mathrm{d}t\right] =$$

$$\int\left[\rho\frac{\partial}{\partial t}\delta(t-t_0)\frac{\partial\psi}{\partial t} - \mathcal{T}\nabla\delta(\mathbf{x}-\mathbf{y})\nabla\psi\right]\ \mathrm{d}t = 0$$

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2},\quad v = \sqrt{\frac{\mathcal{T}}{\rho}}$$

6. Functional derivative of a Wick expansion term in the generating functional.

$$Z_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ J(x)\Delta(x-y)J(y)\right)$$

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ [J(x)+\epsilon\delta(x-z_1)]\Delta(x-y)[J(y)+\epsilon\delta(y-z_1)]\right)\right.$$

$$\left.-\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y\ J(x)\Delta(x-y)J(y)\right)\right]$$

Chapter 2

Simple harmonic oscillators

1. Commutators of ladder operators.

$$[\hat{a}, \hat{a}^\dagger] = \frac{m\omega}{2\hbar}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) - \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$
$$= \frac{1}{2i\hbar}([\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]) = 1$$

2. Perturbation theory and ladder operators. The perturbative term $\hat{H}_p = \lambda\hat{W} = \lambda x^4$. Its first-order correction is:

$$E_n = E_n^{(0)} + \langle\phi_n|\hat{H}_p|\phi_n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega + \langle n|\lambda x^4|n\rangle$$

3. Fourier transform of $\hat{x}_k$.

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \hat{\tilde{x}}_k e^{ikja}, \quad \hat{\tilde{x}}_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k + \hat{a}^\dagger_{-k}\right)$$

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k + \hat{a}^\dagger_{-k}\right)e^{ikja} = \sqrt{\frac{h}{Nm}}\sum_k \frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_k e^{ikja} + \hat{a}^\dagger_k e^{-ikja}\right)$$

4. Ground state of the harmonic oscillator.

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|0\rangle = 0$$

$$\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle = 0$$

$$\left(x+\frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle = 0$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{m\omega}{\hbar}x\right)\langle x|0\rangle = 0$$

This is easily solved by separation of variables. Attempting a series solution for practice:

$$\langle x|0\rangle = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty}\frac{m\omega}{\hbar}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}\frac{m\omega}{\hbar}a_n x^{n+1} = 0$$

$$a_{n+2} = -\frac{m\omega}{\hbar(n+2)}a_n,\;\; a_0 = A,\; a_1 = 0$$

$$\langle x|0\rangle = A\left[1+\left(-\frac{m\omega}{2\hbar}\right)x^2+\frac{1}{2}\left(-\frac{m\omega}{2\hbar}\right)^2x^4+\frac{1}{6}\left(-\frac{m\omega}{2\hbar}\right)^3x^6+\ldots\right]$$

$$\langle x|0\rangle = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$A = 1/\left|\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\right|$$

$$A = 1/\sqrt{\int_{-\infty}^{\infty}\exp\left(2\frac{m\omega}{2\hbar}x^2\right)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Chapter 3

Occupation number representation

1. Practice with exponentials and ladder operators.

$$\frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] = \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} = \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

2. Ladder operator identities.

(a)

$$\left[\hat{a}, \left(\hat{a}^{\dagger}\right)^{n}\right] = \left[\hat{a}\left(\hat{a}^{\dagger}\right)^{n} - \left(\hat{a}^{\dagger}\right)^{n}\hat{a}\right]$$
$$= \left[(1+\hat{a}^{\dagger}\hat{a})\left(\hat{a}^{\dagger}\right)^{n-1} - \left(\hat{a}^{\dagger}\right)^{n-1}\left(\hat{a}^{\dagger}\hat{a}\right)\right] = \left[\left(\hat{a}^{\dagger}\right)^{n-1} - \left[\hat{a}^{\dagger}\hat{a}, \left(\hat{a}^{\dagger}\right)^{n-1}\right]\right]$$

(b)

(c)

(d)

3. Three-dimensional harmonic oscillator.

$$\hat{a}_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right)$$

$$\begin{aligned}
\left[\hat{a}_i, \hat{a}_j^\dagger\right] &= \frac{m\omega}{2\hbar}\left[\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right) - \left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right)\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\right] \\
&= \frac{m\omega}{2\hbar}\left([\hat{x}_i, \hat{x}_j] + \frac{1}{m^2\omega^2}[\hat{p}_i, \hat{p}_j] - \frac{i}{m\omega}([\hat{x}_j.\hat{p}_i] + [\hat{x}_i, \hat{p}_j])\right) = \delta_{ij}
\end{aligned}$$

$$\hat{H} = \frac{1}{2m}$$

$$\hat{L}^i = -i\hbar\epsilon^{ijk}\hat{a}_j^\dagger\hat{a}_k$$

4. Slater determinant for fermions.

Chapter 4

Making Second Quantization Work

1. Commutation relations of density field operators.

$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^{\dagger}(\mathbf{y})\right]_{\zeta} = \delta^{(3)}(\mathbf{x} - y), \quad \left[\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right]_{\zeta} = 0$$

$$\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}) = \hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{y})\hat{\psi}(\mathbf{y})$$
$$= -\zeta\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \delta^{(3)}(\mathbf{x} - y)\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{y})$$
$$= -\zeta^2\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) + \delta^{(3)}(\mathbf{x} - y)\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{y})$$

So $\zeta = \pm 1$ yields the same result regardless of bosons or fermions.

2. Single-particle density matrix in terms of ladder operators.

$$\hat{\rho}_1(\mathbf{x} - \mathbf{y}) = \left\langle \hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{y}) \right\rangle$$
$$= \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{\mathbf{q}} \hat{a}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{pq}} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \left\langle \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{q}} \right\rangle$$

3. Hubble Hamiltonian.

$$|\hat{H} - \lambda\hat{I}| = \begin{vmatrix} U - \lambda & -t & -t & 0 \\ -t & -\lambda & 0 & -t \\ -t & 0 & -\lambda & -t \\ 0 & -t & -t & U - \lambda \end{vmatrix} =$$

Chapter 5

Continuous systems

1.

2.

3.

4.

5. Extremisation of proper length.

$$\int_a^b \mathrm{d}s = \int_a^b \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}t}{\gamma} = \int_a^b L\,\mathrm{d}t$$

$$\frac{\partial L^2}{\partial \mathbf{v}} = \frac{2\mathbf{v}}{c^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^2}{\partial \mathbf{v}}\right) - \frac{\partial L^2}{\partial \mathbf{x}} = \frac{2\dot{\mathbf{v}}}{c^2} = 0$$

Since the acceleration is zero, the velocity is constant. Hence a straight world-line path does minimise the interval.

6. Electromagnetic Lagrangian.

$$L = \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV$$
$$\nabla L = q[\nabla(\mathbf{A}\cdot\mathbf{v}) - \nabla V]$$
$$= q[\cancel{(A\cdot\nabla)\mathbf{v}} + \cancel{(\mathbf{v}\cdot\nabla)\mathbf{A}} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \cancel{\mathbf{A}\times(\nabla\times\mathbf{v})} - q\nabla V]$$
$$= q[\mathbf{E} + \mathbf{v}\times B], \quad \because \mathbf{E} = -q\nabla V,\ \mathbf{B} = \nabla\times\mathbf{A}$$
$$\frac{\partial L}{\partial \mathbf{v}} = -\frac{mc^2}{2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left(-\frac{2\mathbf{v}}{c^2}\right) = \gamma m\mathbf{v}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}} = \nabla L \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q[\mathbf{E} + \mathbf{v}\times\mathbf{B}]$$

7. Non-relativistic limit of the electromagnetic Lagrangian.

$$L = \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV \approx \frac{1}{2}m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - qV$$
$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}$$

Finding the Hamiltonian is equivalent to finding the energy in terms of momentum:

$$H = \mathbf{p}\cdot\mathbf{v} - L = m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - L$$
$$= mc^2 + \frac{1}{2}m\mathbf{v}^2 + qV = mc^2 + \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV, \quad \mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{m}$$

8. Hunting for Lorentz invariants in electromagnetism.

9. Deriving one of Maxwell's equations.

10. Deriving the continuity equation of electromagnetism.

Chapter 6

A first stab at relativistic quantum mechanics

1. Massive scalar field Lagrangian.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

$$\left(\partial^2 + m^2\right)\phi = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^0\phi = \dot{\phi}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Chapter 7

Examples of Lagrangians, or how to write down a theory

1. Massive scalar field with a twist.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\partial_\mu \partial^\mu \phi + m^2\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0$$

$$\left(\partial^2 + m^2\right)\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0$$

2. Massive scalar field with a source.

$$\mathcal{L} = \frac{1}{2}[\partial_\mu \phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2 + J(x)\phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} = -m^2\phi(x) + J(x)$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = \partial^\mu \phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) = 0$$
$$\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) - J(x) = 0$$
$$\left(\partial^2 + m^2\right)\phi(x) = J(x)$$

3. Two coupled massive scalar fields.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1\right)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}\left(\partial_\mu \phi_2\right)^2 - \frac{1}{2}m^2\phi_2^2 - g\left(\phi_1^2 + \phi_2^2\right)^2$$
$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2\phi_1 - 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial^\mu \phi_1, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$

4. Introducing the conjugate momentum. Referring to Chapter 5's solution:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Chapter 8

The passage of time

1.

2. Time-dependence of ladder operators.

$$\hat{H} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k$$

$$\hat{a}_k^\dagger(t) = e^{i\hat{H}t/\hbar}\hat{a}_k^\dagger(0)e^{-i\hat{H}t/\hbar}$$

$$\frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\mathrm{d}t} = \frac{i}{\hbar}\left(e^{i\hat{H}t/\hbar}\left[\hat{H}, \hat{a}_k^\dagger(0)\right]e^{-i\hat{H}t/\hbar}\right)$$

$$= \frac{iE_k}{\hbar}\left(e^{i\hat{H}t/\hbar}\left[\hat{n}_k, \hat{a}_k^\dagger(0)\right]e^{-i\hat{H}t/\hbar}\right) = \frac{iE_k}{\hbar}\hat{a}_k^\dagger(t)$$

$$\int \frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\hat{a}_k^\dagger(t)} = \int \frac{iE_k}{\hbar}\,\mathrm{d}t \longrightarrow \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0)e^{iE_kt/\hbar}$$

3. Time-dependence of an operator of the form $\hat{X} = X_{lm}\hat{a}_l^\dagger\hat{a}_m$.

$$\hat{X}(t) = e^{i\hat{H}t/\hbar}X_{lm}\hat{a}_l^\dagger\hat{a}_m e^{-i\hat{H}t/\hbar}$$

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} =$$

4. Hamiltonian of a spin-1/2 particle in a magnetic field.

$$\frac{\mathrm{d}\hat{S}_H^z}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^z, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(-i\hbar\hat{S}_H^x\right) = -\omega\hat{S}_H^x$$

$$\frac{\mathrm{d}\hat{S}_H^x}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^x, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(i\hbar\hat{S}_H^z\right) = \omega\hat{S}_H^z$$

Spin behaves like angular momentum.

Chapter 9

Quantum mechanical transformations

1. Generators of the translation operator.

$$\hat{U}(\mathbf{a}) = \exp[-i\hat{\mathbf{p}} \cdot \mathbf{a}]$$
$$\left.\frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}}\right|_{\mathbf{a}=0} = -i\hat{\mathbf{p}} \exp[-i\hat{\mathbf{p}} \cdot \mathbf{0}]$$
$$\hat{\mathbf{p}} = -\frac{1}{i} \left.\frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}}\right|_{\mathbf{a}=0}$$

2. Generators of the Lorentz group for four-vectors.

$$K = \frac{1}{i} \left.\frac{\partial \mathbf{\Lambda}(\phi^1)}{\partial \phi^1}\right|_{\phi^1=0} = \frac{1}{i} \left.\begin{vmatrix} \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ \cosh\phi^1 & \sinh\phi^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\right|_{\phi^1=0} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and similarly for ϕ^i.

3. Infinitesimal Lorentz transformations. Going to the MCRF and composing boosts:

$$\Lambda^{\mu}_{\ \nu} = \lim_{\mathbf{v}\to 0} \begin{bmatrix} \gamma & \gamma v^1 & \gamma v^2 & \gamma v^3 \\ \gamma v^1 & \gamma & 0 & 0 \\ \gamma v^2 & 0 & \gamma & 0 \\ \gamma v^3 & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix}$$

For an infinitesimal counter-clockwise rotations, compose the matrices:

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & 0 \\ 0 & -\theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \theta^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta^1 \\ 0 & 0 & -\theta^1 & 1 \end{bmatrix}$$

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Compose the boosts and rotation matrices:

$$\Lambda^{\mu}_{\ \nu} = \Lambda^{\mu}_{\ \bar{\nu}} \Lambda^{\bar{\nu}}_{\ \nu} = L_z R_z L_y R_y L_x R_x$$

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Extracting the identity matrix, the general infinitesimal Lorentz transformation can be written as:

$$\mathbf{\Lambda} = \mathbf{1} + \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

The following tensors are indeed antisymmetric:

$$\omega^{\mu\nu} = \omega^{\mu}{}_{\lambda} g^{\lambda\nu} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega^{\lambda}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & -\theta^3 & \theta^2 \\ -v^2 & \theta^3 & 0 & -\theta^1 \\ -v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

4.

Chapter 10

Symmetry

1. Commutation relations between scalar field and its conjugate momentum.

$$[\phi(x), P^\alpha] = \phi(x)P^\alpha - P^\alpha\phi(x) = \int \left[\phi(x)T^{0\alpha} - T^{0\alpha}\phi(x)\right] \mathrm{d}^3y$$

2. Noether current of N-field system.

3. Energy-momentum tensor and momentum of the massive scalar field.

$$T^{\mu\nu} = \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}$$

$$T^{00} = \Pi^0\partial^0\phi - g^{00}\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right] = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \partial_\mu[\partial^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}] \\ &= \partial^2\phi\partial^\nu\phi - \partial^\mu\phi\partial_\mu\partial^\nu\phi - \frac{1}{2}\left[\partial^\rho\phi\partial^\nu\partial_\rho\phi + \partial_\rho\phi\partial^\nu\partial^\rho\phi - 2m^2\phi\partial^\nu\phi\right] \\ &= \left(\partial^2 + m^2\right)\phi(\partial^\nu\phi) = 0\end{aligned}$$

$$P^i = \int T^{0i}\, dd = \int \left(\Pi^0\partial^i\phi - g^{0i}\mathcal{L}\right) dd = \int \partial^0\phi\partial^i\phi\, dd$$

The Klein-Gordon equation, which is the equation of motion for scalar field theory, satisfies the divergence of the energy-momentum tensor.

4. Energy-momentum tensor and momentum of the electromagnetic field.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu]$$

$$\frac{\partial(\partial_\mu A_\nu \partial^\mu A^\nu)}{\partial(\partial_\sigma A_\rho)} = \delta^\sigma_\mu \delta^\rho_\nu \partial^\mu A^\nu + \partial_\mu A_\nu g^{\alpha\sigma} g^{\rho\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\sigma A^\rho$$

$$\frac{\partial(\partial_\mu A_\nu \partial^\nu A^\mu)}{\partial(\partial_\sigma A_\rho)} = \delta^\sigma_\mu \delta^\rho_\nu \partial^\nu A^\mu + \partial_\mu A_\nu g^{\alpha\rho} g^{\sigma\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\rho A^\sigma$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\rho)} = -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho} = \Pi^{\sigma\rho}$$

$$T^\mu_\nu = \Pi^{\mu\sigma}\partial_\nu A_\sigma - \delta^\mu_\nu \mathcal{L}$$

$$T^{\mu\nu} = g^{\alpha\nu}T^\mu_\alpha = -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$X^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu = -F^{\lambda\mu}A^\nu = X^{\mu\lambda\nu}$$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\nu X^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\nu \left(F^{\mu\lambda}A^\nu\right)$$

$$= -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \cancel{\partial_\lambda F^{\mu\lambda} A^\nu} + F^{\mu\lambda}\partial_\lambda A^\nu$$

$$\overset{\lambda\to\sigma}{=} F^{\mu\sigma}(\partial_\sigma A^\nu - \partial^\nu A_\sigma) + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = F^{\mu\sigma}F^\nu_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$\tilde{T}^{00} = F^{0\sigma}F^0_\sigma + \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} = \mathbf{E}^2 + \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

$$\tilde{T}^{i0} = F^{i\sigma}F^0_\sigma + \cancel{\frac{1}{4}g^{i0}F_{\alpha\beta}F^{\alpha\beta}} = \epsilon^{ijk}E_j B_k = (\mathbf{E}\times\mathbf{B})^i$$

Chapter 11

Canonical quantization of fields

1. Commutation relations of quantum field position operators.

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{p})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{q})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{y}} + \hat{a}^\dagger_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{y}}\right) - \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{q})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{y}} + \hat{a}^\dagger_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{y}}\right)$$

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{p})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_\mathbf{p}E_\mathbf{q})^{\frac{1}{2}}} \left(\left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}\right] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} + \left[\hat{a}^\dagger_\mathbf{p}, \hat{a}_\mathbf{q}\right] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{y}}\right)$$

$$= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_\mathbf{p}E_\mathbf{q})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} - \delta^{(3)}(\mathbf{q}-\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{y}}\right)$$

$$= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\right)$$

2. Commutation relation of quantum field operator and its conjugate momentum.

$$
\begin{aligned}
&\left[\hat{\phi}(x), \hat{\Pi}^0(y)\right] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{y}} - \hat{a}^{\dagger}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}}\right) \\
&- \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{y}} - \hat{a}^{\dagger}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}}\right) \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \frac{i}{2} \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} + \left[\hat{a}_{\mathbf{q}}, \hat{a}^{\dagger}_{\mathbf{p}}\right] e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{y}}\right) \\
&= \int \mathrm{d}^3\mathbf{p} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} + \delta^{(3)}(\mathbf{q}-\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{y}}\right) \\
&= \frac{i}{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)
\end{aligned}
$$

Chapter 12

Examples of canonical quantization

1. Complex scalar field theory.

$$
\begin{aligned}
\mathcal{H} &= \partial^0\hat{\psi}^\dagger\hat{\psi} + \partial^0\hat{\psi}\hat{\psi}^\dagger + \nabla\hat{\psi}^\dagger\cdot\nabla\hat{\psi} + m^2\hat{\psi}^\dagger\hat{\psi} \\
&= \int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}(-iE_{\mathbf{q}})\left(\hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} - \hat{b}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right)\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right) \\
&+ \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}(-iE_{\mathbf{p}})\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - \hat{b}^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right)\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right) \\
&+ \int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}(-i\mathbf{q})\left(\hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} - \hat{b}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right)\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}(i\mathbf{p})\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} - \hat{b}^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right) \\
&+ m^2\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right)\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \int\mathrm{d}^3\mathbf{p}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{-iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} - \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right) \\
&+ \int\mathrm{d}^3\mathbf{p}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} - \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right) \\
&+ \int\mathrm{d}^3\mathbf{p}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{q}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right) \\
&+ m^2\int\mathrm{d}^3\mathbf{p}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right)
\end{aligned}
$$

2. Commutation relations of complex scalar fields.

(a)

$$\left[\hat{\psi}(x), \hat{\psi}^\dagger(y)\right] = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{p})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{q})^{\frac{1}{2}}} \left(\hat{a}^\dagger_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}}\right) - \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{q})^{\frac{1}{2}}} \left(\hat{a}^\dagger_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{x}} + \hat{b}_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}}\right)$$

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_\mathbf{p})^{\frac{1}{2}}} \left(\hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{b}^\dagger_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}\right)$$

(b)

3. (a)

$$\begin{bmatrix}\phi_1' \\ \phi_2'\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix} \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}$$

$$\left[\hat{Q}_N, \hat{\phi_1}\right] = -iD\hat{\phi_1} = i\hat{\phi_2}$$

(b)

$$\left[\hat{Q}_N, \hat{\phi_2}\right] = -iD\hat{\phi_1} = -i\hat{\phi_1}$$

(c) Commutation relations of Noether charges for two scalar fields.

$$\left[\hat{Q}_N, \hat{\psi}\right] = \frac{1}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi_1}\right] + \frac{i}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi_2}\right] = \frac{i}{\sqrt{2}}\hat{\phi_2} + \frac{1}{\sqrt{2}}\hat{\phi_1} = \hat{\psi}$$

$$\begin{bmatrix}\phi_1' \\ \phi_2'\end{bmatrix} = \begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix} \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}$$

$$\left[\hat{Q}_N, \hat{\phi_1}\right] = -iD\hat{\phi_1} = i\hat{\phi_2}, \quad \left[\hat{Q}_N, \hat{\phi_2}\right] = -iD\hat{\phi_1} = -i\hat{\phi_1}$$

$$\left[\hat{Q}_N, \hat{\psi}\right] = \frac{1}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi_1}\right] + \frac{i}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi_2}\right] = \frac{i}{\sqrt{2}}\hat{\phi_2} + \frac{1}{\sqrt{2}}\hat{\phi_1} = \hat{\psi}$$

4. Using Noether's theorem to derive the number-phase uncertainty relation. Note: $D\hat{\theta} = \pm 1$. Substituting:

$$\left[\hat{Q}_N, \hat{\theta}\right] = -iD\hat{\theta} = i$$

$$\left[\int \rho(\mathbf{x}, t)\; dd, \theta(\mathbf{x}, t)\right] = \int \mathrm{d}^3\mathbf{x}\; [\rho, \theta] = i$$

5. Equations of motion of non-relativistic complex scalar field theory.

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \Pi^\mu_\psi = 0$$
$$\frac{\partial \mathcal{L}}{\partial \psi} = -V(x)\psi^\dagger(x), \quad \Pi^0_\psi = i\psi^\dagger$$
$$\partial_0 \Pi^0_\psi = i\partial_0 \psi^\dagger, \quad \partial_i \Pi^i_\psi = -\frac{1}{2m}\nabla^2 \psi^\dagger$$
$$\therefore i\partial_0 \psi^\dagger - \frac{1}{2m}\partial_i \partial^i \psi^\dagger - V(x)\psi^\dagger(x) = 0$$
$$\longrightarrow i\partial_0 \psi^\dagger = \hat{H}\psi^\dagger, \quad \hat{H} = -\frac{1}{2m}\nabla^2 + \hat{V}$$
$$V = 0 \longrightarrow i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2 \psi$$
$$iT'(t)X(x) = -\frac{1}{2m}X''(x)T(t)$$
$$\frac{T'}{T} = -iE \longrightarrow T(t) = Ae^{-iEt}$$
$$X'' + 2mEX = 0 \longrightarrow X(x) = Be^{ipx} + Ce^{-ipx}, \ p = \sqrt{2mE}$$
$$T(t)X(x) = Ae^{i(px-Et)} + Be^{-i(px-Et)}$$

6. Noether current for non-relativistic complex scalar field theory.

$$J^0_N = i\Psi^\dagger(i\Psi) + i\Psi(-i\Psi^\dagger)$$
$$Q_{N_c} = \int \hat{\Psi}\hat{\Psi}^\dagger - \hat{\Psi}^\dagger\hat{\Psi} \ dd$$
$$= \int dd \left[\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}^\dagger_\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}} - \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}^\dagger_\mathbf{q} e^{iq\cdot x} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} \right]$$
$$\frac{1}{(2\pi)^3} \int dd \left[\int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{q} \ \hat{a}_\mathbf{p} \hat{a}^\dagger_\mathbf{q} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q} \hat{a}_\mathbf{p} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \right]$$
$$= \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}^3\mathbf{q} \ \left[\hat{a}_\mathbf{p} \hat{a}^\dagger_\mathbf{q} \delta^3(\mathbf{p}-\mathbf{q}) - \hat{a}^\dagger_\mathbf{q} \hat{a}_\mathbf{p} \delta^3(\mathbf{q}-\mathbf{p}) \right]$$
$$= \int \mathrm{d}^3\mathbf{p} \left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{p} \right] = \mathbf{p}$$

So momentum is conserved, naturally.

7. Transformation of the complex scalar field.

Chapter 13

Fields with many components and massive electromagnetism

1. Angular momentum form of internal symmetries.

(a) $\vec{J}$ represents the Levi-Civita tensor as a vector of matrices.

$$\hat{\vec{\mathbf{Q}}}_{N_c} = \int \mathrm{d}^3\mathbf{p}\ \hat{\mathbf{A}}^\dagger \vec{J} \hat{\mathbf{A}}$$

(b) The inverse transformations and resultant computations are as follows:

$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left(\hat{b}_{-1} - \hat{b}_1\right),\ \hat{a}_2 = -\frac{i}{\sqrt{2}}\left(\hat{b}_{-1} + \hat{b}_1\right),\ \hat{a}_3 = \hat{b}_0$$

$$\hat{Q}^2_{N_c} =$$

$$\hat{Q}^3_{N_c} = -i\int \mathrm{d}^3\mathbf{p}\ \left(\hat{a}^\dagger_{1\mathbf{p}}\hat{a}_{2\mathbf{p}} - \hat{a}^\dagger_{2\mathbf{p}}\hat{a}_{1\mathbf{p}}\right) = \int \mathrm{d}^3\mathbf{p}\ \left(\hat{b}^\dagger_{1\mathbf{p}}\hat{b}_{1\mathbf{p}} - \hat{b}^\dagger_{-1\mathbf{p}}\hat{b}_{-1\mathbf{p}}\right)$$

$$J^1_{\hat{b}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ J^2_{\hat{b}} = -\frac{i}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ J^3_{\hat{b}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2. Lorentz boosting and circular polarization.

(a)

(b)

(c)

3. Projection tensors.

4. Playing with projection tensors.

Chapter 14

Gauge fields and gauge theory

1. Quantizing the electromagnetic field tensor.

2. The spin of the photon.

(a)

(b)

Chapter 15

Discrete transformations

1. Gamma decay of a pion.

2. Classification of physical quantities.

(a) Magnetic flux: Vector.

(b) Angular momentum: Pseudovector.

(c) Charge: Pseudoscalar.

(d) Scalar product of vector and pseudovector: Scalar?

(e) Scalar product of two vectors: Pseudoscalar?

(f) Scalar product of two pseudovectors:

3. Representations of spinors.

(a) $\mathbf{R}(\hat{\mathbf{x}}, \theta)$

(b) $\mathbf{R}(\hat{\mathbf{y}}, \theta)$

(c) $\mathbf{R}(\hat{\mathbf{z}}, \theta)$

Chapter 16

Propagators and Green's functions

1. Green's function for a particle in an infinite potential well.

(a)

$$\langle x|\hat{H}|\psi\rangle = E\langle x|\psi\rangle$$
$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + E\psi = 0, \quad V = 0$$
$$\psi_n(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \sqrt{2mE}/\hbar$$
$$\psi(0) = \psi(a) = 0 \implies B = -A$$
$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

(b)

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$
$$G^+(n, t_2, t_1) = \theta(t_2 - t_1)e^{-iE_n(t_2 - t_1)}$$

(c)

$$G^+(n, \hbar\omega) = \frac{i}{\hbar\omega - E_n + i\epsilon}$$

2. Green's function in the energy expression.

(a)

$$
\begin{aligned}
G_0^+(x,t,y,0) &= \theta(t)\langle x(t)|y(t)\rangle \\
&= \theta(t)\langle x|e^{-i\hat{H}t}|y\rangle
\end{aligned}
$$

$$= \theta(t)\sum_n e^{iE_nt}\langle x|n\rangle\langle n|y\rangle = \theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_nt}$$

$$G_0^+(x,y,E) = \int G_0^+(x,t,y,0)\,\mathrm{d}t = \int_{-\infty}^{\infty}\theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_nt}e^{iEt}\,\mathrm{d}t$$

Using a damping factor $e^{-\epsilon t}$ to ensure convergence, switching the order of summation and integration, then integrating by parts ($\theta'(t) = \delta(t)$):

$$
\begin{aligned}
G_0^+(x,y,E) &= \sum_n \int_{-\infty}^{\infty}\theta(t)\phi_n(x)\phi_n^*(y)e^{i(E-E_n+i\epsilon)t}\,\mathrm{d}t \\
&= \sum_n \frac{i\phi_n(x)\phi_n^*(y)}{E-E_n+i\epsilon}
\end{aligned}
$$

(b) The integral definition of the Heaviside step function is:

$$\theta(t) = i\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}\frac{e^{-izt}}{z+i\epsilon}$$

Substituting this into the original expression and changing the order of integration:

$$G_0^+(p,t,0) = \theta(t)e^{-iE_pt}$$

$$G_0^+(p,E) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i}{2\pi(z+i\epsilon)}e^{i(E-E_p-z)t}\,\mathrm{d}t\,\mathrm{d}z$$

$$= \int_{-\infty}^{\infty}\frac{i}{(z+i\epsilon)}\delta(E-E_p-z)\,\mathrm{d}z = \frac{i}{E-E_p+i\epsilon}$$

3. Green's function for the harmonic oscillator.

(a) The one-dimensional harmonic oscillator with the corresponding forcing function $f(t)$ has the following solution for the particular integral:

$$m\frac{\partial^2}{\partial t^2}A(t-u) + m\omega_0^2 A(t-u) = \tilde{F}(\omega)e^{-i\omega(t-u)}$$

$$\begin{aligned}
A_P(t-u) &= \frac{1}{\left(D^2+\omega_0^2\right)}\frac{\tilde{F}(\omega)}{m}e^{-i\omega(t-u)} = \left(1+\frac{D^2}{\omega_0^2}\right)^{-1}\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}, \quad D = \frac{\mathrm{d}}{\mathrm{d}t} \\
&= \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{i\omega u}\left[\sum_{k=0}^{\infty}\left(\frac{iD}{\omega_0}\right)^{2k}e^{-i\omega t}\right] = \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\sum_{k=0}^{\infty}\left(\frac{\omega}{\omega_0}\right)^{2k} \\
&= \frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\left[\frac{1}{1-\omega^2/\omega_0^2}\right] = -\frac{\tilde{F}(\omega)}{m\left(\omega^2-\omega_0^2\right)}e^{-i\omega(t-u)}
\end{aligned}$$

Therefore the solution is:

$$A(t-u) = c_1\cos\omega_0 t + c_2\sin\omega_0 t - \frac{\tilde{F}(\omega)}{m\left(\omega^2-\omega_0^2\right)}e^{-i\omega(t-u)}$$

(b) The differential equation that satisfies the Green's function is:

$$\left[m\frac{\partial^2}{\partial t^2} + m\omega_0^2\right]G(t,k) = \delta(t-k)$$

Taking the Fourier transform, rearranging and then taking its inverse:

$$\begin{aligned}
-m(\omega^2-\omega_0^2)G(\omega,k) &= \int_{-\infty}^{\infty}\delta(t-k)e^{i\omega t}\,\mathrm{d}t = e^{i\omega k} \\
G(t,k) &= -\frac{1}{m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-k)}}{\omega^2-\omega_0^2}
\end{aligned}$$

Using the previous result to verify the solution:

$$\begin{aligned}
A(t-u) &= -\frac{1}{2\pi m}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)}{\omega^2-\omega_0^2}e^{-i\omega(t-u+k)}\,\mathrm{d}\omega\,\mathrm{d}k \\
&= \frac{1}{2\pi i m}\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)}{\omega(\omega_0^2-\omega^2)}e^{-i\omega(t-u)}\,\mathrm{d}\omega \\
&= \frac{1}{m\omega_0^2}\int_{-\infty}^{\infty}\frac{1}{2\pi i}\left[\frac{1}{\omega}+\frac{1}{\omega_0^2-\omega^2}\right]\tilde{F}(\omega)e^{-i\omega(t-u)}\,\mathrm{d}\omega
\end{aligned}$$

(c) Taking the Laplace transform of the differential equation form of the Green's function:

$$G(s,u) = \frac{e^{us}}{m(s^2+\omega_0^2)}$$

Using convolution to find the inverse:

$$G^+(t,u) = \frac{1}{m\omega_0}\int_0^t \delta(k-u)\sin\omega_0(t-k)\,\mathrm{d}k = \frac{1}{m\omega_0}\sin\omega_0(t-u)$$

(d) The trajectory is:

4. Green's function of the Klein-Gordon equation.

(a) Taking the three-dimensional Fourier transform:

$$\int_{-\infty}^{\infty}(\nabla^2+\mathbf{k}^2)G_{\mathbf{k}}(\mathbf{x})e^{-i\mathbf{q}\cdot x}\,\mathrm{d}^3\mathbf{x} = 1$$

$$\tilde{G}_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2-\mathbf{q}^2}$$

(b) The Fourier transform of $G^+_{\mathbf{k}}(\mathbf{x})$ with a damping factor is:

$$\begin{aligned}
\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) &= \int_{-\infty}^{\infty} -\frac{e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}}{4\pi|\mathbf{x}|}e^{-i\mathbf{q}\cdot x}\,\mathrm{d}^3\mathbf{x}\\
&= -\frac{1}{2}\int_{-1}^{1}\int_0^{\infty}|\mathbf{x}|e^{-i(|\mathbf{q}|\cos\theta-|\mathbf{k}|-i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}|\,\mathrm{d}(\cos\theta)\\
&= \frac{i}{2|\mathbf{q}|}\int_0^{\infty}\left[e^{i|\mathbf{q}||\mathbf{x}|}-e^{-i|\mathbf{q}||\mathbf{x}|}\right]e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}|\\
&= \frac{1}{2|\mathbf{q}|}\left[\frac{1}{(|\mathbf{k}|+|\mathbf{q}|+i\epsilon)}-\frac{1}{(|\mathbf{k}|-|\mathbf{q}|+i\epsilon)}\right]
\end{aligned}$$

(c)

Chapter 17

Propagators and fields

1. Retarded field propagator for a free particle.

Chapter 18

Path integrals: I said to him, 'You're crazy'

1. Physicist's treatment of operators.
2. Path integral derivation of Wick's theorem.

   (a) Let

$$I(a) = -2\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = -2\sqrt{\frac{2\pi}{a}}$$

   Differentiating under the integral sign:

$$I'(a) = \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = \sqrt{\frac{2\pi}{a^3}}$$

(b)

$$J_n(a) = (-2)^{\frac{n}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = (-2)^{\frac{n}{2}} \sqrt{\frac{2\pi}{a}}$$

$$\frac{\mathrm{d}^k J_n(a)}{\mathrm{d}a^k} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{(-1/2)!}{(-1/2-k)!} a^{-\frac{1}{2}-k}$$

$$\frac{\mathrm{d}^{n/2} J_n(a)}{\mathrm{d}a^{n/2}} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{\Gamma(1/2)}{\Gamma\left(\frac{1-n}{2}\right)} a^{-\left(\frac{n+1}{2}\right)} = \frac{i^n \pi}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{\frac{n+1}{2}}$$

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x}{\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x} = \frac{i^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{n/2}$$

$$= \begin{cases} 0 & \forall\, n \in 2\mathbb{Z}^+ + 1 \\ a^{-n/2} \prod_{k=1}^{n/2} (2k-1) & \forall\, n \in 2\mathbb{Z}^+ \end{cases}$$

$$\therefore \frac{\mathrm{d}^n J_{2n}(a)}{\mathrm{d}a^n} = \frac{1}{a^n} \prod_{k=1}^{n} (2k-1)$$