

Solutions to  
Quantum Field Theory for the Gifted Amateur  
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April 20, 2018

# Chapter 1

## Lagrangians

1. *Fermat's principle of least time.*

$$t = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{v_2} = \frac{\sqrt{x^2 + h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{c/n_2}$$
$$\frac{dt}{dx} = \frac{x}{c/n_1 \sqrt{x^2 + h_1^2}} - \frac{(l-x)}{c/n_2 \sqrt{(l-x)^2 + h_2^2}} = 0$$
$$n_1 \sin \theta = n_2 \sin \phi$$

2. *Practice with functional derivatives.*

(a)

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int G(x, y) [f(y) + \epsilon \delta(y - z)] dy - \int G(x, y) f(y) dy \right] = G(x, z)$$

(b)

$$\frac{\delta I[f^\alpha]}{\delta x_0} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int [f(x) + \epsilon \delta(x - x_0)]^\alpha - f(x)^\alpha dx \right] = \alpha [f(x_0)]^{\alpha-1}$$
$$\therefore \frac{\delta I[f^3]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int_{-1}^1 [f(x) + \epsilon \delta(x - x_0)]^3 dx - \int_{-1}^1 [f(x)]^3 dx \right] = 3[f(x_0)]^2$$
$$\frac{\delta^2 I[f^3]}{\delta f(x_0) \delta f(x_1)} = \lim_{\epsilon \rightarrow 0} \frac{3}{\epsilon} \left[ [f(x_0) + \epsilon \delta(x_0 - x_1)]^2 - f^2(x_0) \right] = 6f(x_1)$$

(c)

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int \left( \frac{\partial}{\partial y} [f(y) + \epsilon \delta(y - x)] \right)^2 dy - \int \left( \frac{\partial f}{\partial y} \right)^2 dy \right]$$
$$= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int \left[ \frac{\partial f}{\partial y} + \epsilon \delta'(y - x) \right]^2 dy - \int \left( \frac{\partial f}{\partial y} \right)^2 dy \right] = 2 \frac{\partial^2 f}{\partial x^2}$$

3. Euler-Lagrange equations using functional derivatives and more.

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int \left[ g(y, f) + \frac{\partial g(y, f)}{\partial f} \epsilon \delta(y - x) \right] dy - \int g(y, f) dy \right] = \frac{\partial g(x, f)}{\partial f(x)}$$

4. Results on Dirac Delta functions.

$$\begin{aligned} \frac{\delta \phi(x)}{\delta \phi(y)} &= \lim_{\epsilon \rightarrow 0} \frac{\phi(x) + \epsilon \delta(x - y) - \phi(x)}{\epsilon} = \delta(x - y) \\ \frac{\delta \dot{\phi}(t)}{\delta \phi(t_0)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \frac{d}{dt} [\phi(t) + \epsilon \delta(t - t_0)] - \dot{\phi}(t) \right] = \frac{d}{dt} \delta(t - t_0) \end{aligned}$$

5. Derivation of the wave equation.

$$\begin{aligned} S &= \int (T - V) dt = \frac{1}{2} \int \rho \left( \frac{\partial \psi}{\partial t} \right)^2 - \mathcal{T} (\nabla \psi)^2 dt \\ \frac{\delta S}{\delta \psi} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \left[ \int \rho \left( \frac{\partial}{\partial t} [\psi + \epsilon \delta(t - t_0)] \right)^2 - \mathcal{T} (\nabla [\psi + \epsilon \delta(\mathbf{x} - \mathbf{y})])^2 dt - \int \rho \left( \frac{\partial \psi}{\partial t} \right)^2 - \mathcal{T} (\nabla \psi)^2 dt \right] \\ &= \int \left[ \rho \frac{\partial}{\partial t} \delta(t - t_0) \frac{\partial \psi}{\partial t} - \mathcal{T} \nabla \delta(\mathbf{x} - \mathbf{y}) \nabla \psi \right] dt = 0 \\ &\implies \nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}, \quad v = \sqrt{\frac{\mathcal{T}}{\rho}} \end{aligned}$$

6. Functional derivative of a Wick expansion term in the generating functional.

$$\begin{aligned} Z_0[J] &= \exp \left( -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x - y) J(y) \right) \\ \frac{\delta Z_0[J]}{\delta J(z_1)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \exp \left( -\frac{1}{2} \int d^4x d^4y [J(x) + \epsilon \delta(x - z_1)] \Delta(x - y) [J(y) + \epsilon \delta(y - z_1)] \right) \right. \\ &\quad \left. - \exp \left( -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x - y) J(y) \right) \right] \end{aligned}$$

## Chapter 2

# Simple harmonic oscillators

### 1. Commutators of ladder operators.

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) - \frac{m\omega}{2\hbar} \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) \\ &= \frac{1}{2i\hbar} ([\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]) = 1 \end{aligned}$$

### 2. Perturbation theory and ladder operators.

The perturbative term  $\hat{H}_p = \lambda \hat{W} = \lambda \hat{x}^4$ . Its first-order correction is:

$$\begin{aligned} E_n &= E_n^{(0)} + \langle \phi_n | \hat{H}_p | \phi_n \rangle = \left( n + \frac{1}{2} \right) \hbar\omega + \langle n | \lambda \hat{x}^4 | n \rangle \\ &= \left( n + \frac{1}{2} \right) \hbar\omega + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (\hat{a} + \hat{a}^\dagger)^4 | n \rangle \\ &= \left( n + \frac{1}{2} \right) \hbar\omega + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | \text{painful} | n \rangle \end{aligned}$$

### 3. Fourier transform of $\hat{x}_k$ .

$$\begin{aligned} \hat{x}_j &= \frac{1}{\sqrt{N}} \sum_k \hat{x}_k e^{ikja}, \quad \hat{x}_k = \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{a}_k + \hat{a}_{-k}^\dagger) \\ \hat{x}_j &= \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{a}_k + \hat{a}_{-k}^\dagger) e^{ikja} = \sqrt{\frac{\hbar}{Nm}} \sum_k \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k e^{ikja} + \hat{a}_k^\dagger e^{-ikja}) \end{aligned}$$

## 4. Ground state of the harmonic oscillator.

$$\begin{aligned}
\sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) |0\rangle &= 0 \\
\langle x | \hat{x} | 0 \rangle + \frac{i}{m\omega} \langle x | \hat{p} | 0 \rangle &= 0 \\
\left( x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x | 0 \rangle &= 0 \\
\left( \frac{d}{dx} + \frac{m\omega}{\hbar} x \right) \langle x | 0 \rangle &= 0
\end{aligned}$$

This is easily solved by separation of variables. Attempting a series solution for practice:

$$\begin{aligned}
\langle x | 0 \rangle &= \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} &= 0 \\
\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} \frac{m\omega}{\hbar} a_n x^{n+1} &= 0 \\
a_{n+2} &= -\frac{m\omega}{\hbar(n+2)} a_n, \quad a_0 = A, \quad a_1 = 0 \\
\langle x | 0 \rangle &= A \left[ 1 + \left( -\frac{m\omega}{2\hbar} \right) x^2 + \frac{1}{2} \left( -\frac{m\omega}{2\hbar} \right)^2 x^4 + \frac{1}{6} \left( -\frac{m\omega}{2\hbar} \right)^3 x^6 + \dots \right] \\
\langle x | 0 \rangle &= A \exp \left( -\frac{m\omega x^2}{2\hbar} \right) \\
A &= 1 / \left| \exp \left( -\frac{m\omega x^2}{2\hbar} \right) \right| \\
A &= 1 / \sqrt{\int_{-\infty}^{\infty} \exp \left( 2 \frac{m\omega}{2\hbar} x^2 \right) dx} = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \\
\langle x | 0 \rangle &= \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left( -\frac{m\omega x^2}{2\hbar} \right)
\end{aligned}$$

## Chapter 3

# Occupation number representation

1. Practice with exponentials and ladder operators.

$$\frac{1}{V} \sum_{\mathbf{p}\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] = \frac{1}{V} \sum_{\mathbf{p}\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \delta_{\mathbf{p}\mathbf{q}} = \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

2. Ladder operator identities.

(a)

$$\begin{aligned} [\hat{a}, (\hat{a}^\dagger)^n] &= [\hat{a}(\hat{a}^\dagger)^n - (\hat{a}^\dagger)^n \hat{a}] \\ &= [(1 + \hat{a}^\dagger \hat{a})(\hat{a}^\dagger)^{n-1} - (\hat{a}^\dagger)^{n-1} (\hat{a}^\dagger \hat{a})] = [(\hat{a}^\dagger)^{n-1} - [\hat{a}^\dagger \hat{a}, (\hat{a}^\dagger)^{n-1}]] \end{aligned}$$

(b)

$$\langle 0 | \hat{a}^n (\hat{a}^\dagger)^m | 0 \rangle = \sqrt{n!} \sqrt{m!} \langle n | m \rangle$$

(c)

(d)

3. Three-dimensional harmonic oscillator.

$$\begin{aligned} \hat{a}_i^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x}_i - \frac{i}{m\omega} \hat{p}_i \right) \\ [\hat{a}_i, \hat{a}_j^\dagger] &= \frac{m\omega}{2\hbar} \left[ \left( \hat{x}_i + \frac{i}{m\omega} \hat{p}_i \right) \left( \hat{x}_j - \frac{i}{m\omega} \hat{p}_j \right) - \left( \hat{x}_j - \frac{i}{m\omega} \hat{p}_j \right) \left( \hat{x}_i + \frac{i}{m\omega} \hat{p}_i \right) \right] \\ &= \frac{m\omega}{2\hbar} \left( [\hat{x}_i, \hat{x}_j] + \frac{1}{m^2\omega^2} [\hat{p}_i, \hat{p}_j] - \frac{i}{m\omega} ([\hat{x}_j, \hat{p}_i] + [\hat{x}_i, \hat{p}_j]) \right) = \delta_{ij} \\ \hat{a}_i^\dagger \hat{a}_i &= \frac{\hat{p}_i^2}{2m\hbar\omega} + \frac{1}{2\hbar\omega} m\omega^2 \hat{x}_i^2 + \frac{i}{2\hbar} [\hat{x}_i, \hat{p}_i] = \frac{1}{\hbar\omega} \left[ \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 - \frac{\hbar\omega}{2} \right] \\ \hat{H} &= \sum_{i=1}^3 \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 = \hbar\omega \sum_{i=1}^3 \left( \hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) \\ \hat{L}^i &\equiv -i\hbar\epsilon^{ijk} \hat{a}_j^\dagger \hat{a}_k \end{aligned}$$

4. *Slater determinant for fermions.* Consider an  $n$ -particle state:

$$\langle \mathbf{p}'_1 \mathbf{p}'_2 \mathbf{p}'_3 \dots \mathbf{p}'_n | \mathbf{p}_n \mathbf{p}_{n-1} \mathbf{p}_{n-2} \dots \mathbf{p}_1 \rangle = \langle 0 | \hat{a}_{\mathbf{p}'_1} \hat{a}_{\mathbf{p}'_2} \hat{a}_{\mathbf{p}'_3} \dots \hat{a}_{\mathbf{p}_n} \hat{a}_{\mathbf{p}_n}^\dagger \hat{a}_{\mathbf{p}_{n-1}}^\dagger \hat{a}_{\mathbf{p}_{n-2}}^\dagger \dots \hat{a}_{\mathbf{p}_1}^\dagger | 0 \rangle$$

## Chapter 4

# Making second quantization work

### 1. Commutation relations of density field operators.

$$\begin{aligned}
 [\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})]_\zeta &= \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})]_\zeta = 0 \\
 \hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}) &= \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y}) \\
 &= -\zeta\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \delta^{(3)}(\mathbf{x} - \mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \\
 &= -\zeta^2\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) + \delta^{(3)}(\mathbf{x} - \mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})
 \end{aligned}$$

So  $\zeta = \pm 1$  yields the same result regardless of bosons or fermions.

### 2. Single-particle density matrix in terms of ladder operators.

$$\begin{aligned}
 \hat{\rho}_1(\mathbf{x} - \mathbf{y}) &= \langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \rangle \\
 &= \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p} \cdot \mathbf{x}} \sum_{\mathbf{q}} \hat{a}_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{y}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{p}\mathbf{q}} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} \langle \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{q}} \rangle
 \end{aligned}$$

### 3. Hubble Hamiltonian. Solving its eigenvalue problem:

$$\begin{aligned}
 |\hat{H} - \lambda \hat{I}| &= \begin{vmatrix} U - \lambda & -t & -t & 0 \\ -t & -\lambda & 0 & -t \\ -t & 0 & -\lambda & -t \\ 0 & -t & -t & U - \lambda \end{vmatrix} = 0 \\
 \lambda_1 = 0, \lambda_2 = U, \lambda_{3,4} &= \frac{1}{2} \left[ U \pm \sqrt{16t^2 + U^2} \right] \\
 \nu_1 &= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \nu_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \nu_{3,4} = \begin{bmatrix} 1 \\ \frac{-U \pm \sqrt{16t^2 + U^2}}{4t} \\ \frac{-U \pm \sqrt{16t^2 + U^2}}{4t} \\ 0 \end{bmatrix}
 \end{aligned}$$



# Chapter 5

## Continuous systems

### 1. Explicit time dependence of Lagrangian and Hamiltonian.

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ \frac{dL}{dt} &= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)\end{aligned}$$

The Hamiltonian is defined as the Legendre transformation with a canonical momentum  $p = \partial L / \partial \dot{q}$ . Therefore:

$$\frac{\partial L}{\partial t} = \frac{d(L - p\dot{q})}{dt} = -\frac{dH}{dt}$$

### 2. Commutation relations of Poisson brackets.

$$\begin{aligned}\{A, B\}_{PB} &= \sum_i \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \\ \{B, A\}_{PB} &= \sum_i \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial q_i} = -\{A, B\}_{PB}\end{aligned}$$

### 3. Commutation relations of Hermitian operators. Since A and B are Hermitian, $A = A^\dagger, B = B^\dagger$ .

$$[A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[B, A]$$

### 4. Investigating the non-relativistic limit of the relativistic free particle. This is easily found by Taylor expansions of $\gamma$ , then taking the low-velocity limit:

$$\begin{aligned}L &= -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \approx -mc^2 + \frac{1}{2}mv^2 \\ p &= \frac{\partial L}{\partial v} = \gamma mv \approx mv \\ H &= pv - L = \gamma mv^2 + \frac{mc^2}{\gamma} = \frac{1}{\gamma} \left[ \left(1 - \frac{v^2}{c^2}\right) mv^2 + mc^2 \right] = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2\end{aligned}$$

5. *Extremisation of the spacetime interval.*

$$\begin{aligned}\int_a^b ds &= \int_a^b \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} dt = \int_a^b \frac{dt}{\gamma} = \int_a^b L dt \\ \frac{\partial L^2}{\partial \mathbf{v}} &= \frac{2\mathbf{v}}{c^2} \\ \frac{d}{dt} \left( \frac{\partial L^2}{\partial \mathbf{v}} \right) - \frac{\partial L^2}{\partial \mathbf{x}} &= \frac{2\dot{\mathbf{v}}}{c^2} = 0\end{aligned}$$

Since the acceleration is zero, the velocity is constant. Hence a straight world-line path does minimise the interval.

6. *Electromagnetic Lagrangian.*

$$\begin{aligned}L &= \frac{-mc^2}{\gamma} + q\mathbf{A} \cdot \mathbf{v} - qV \\ \nabla L &= q[\nabla(\mathbf{A} \cdot \mathbf{v}) - \nabla V] \\ &= q[(\cancel{\mathbf{A} \cdot \nabla})\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{A} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{v}) - q\nabla V] \\ &= q[\mathbf{E} + \mathbf{v} \times \mathbf{B}], \quad \because \mathbf{E} = -q\nabla V, \mathbf{B} = \nabla \times \mathbf{A} \\ \frac{\partial L}{\partial \mathbf{v}} &= -\frac{mc^2}{2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \left( -\frac{2\mathbf{v}}{c^2} \right) = \gamma m\mathbf{v} \\ 0 &= \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} - \nabla L \\ \implies \frac{d}{dt}(\gamma m\mathbf{v}) &= q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]\end{aligned}$$

7. *Non-relativistic limit of the electromagnetic Lagrangian.*

$$\begin{aligned}L &= \frac{-mc^2}{\gamma} + q\mathbf{A} \cdot \mathbf{v} - qV \approx \frac{1}{2}m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - qV \\ \mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}\end{aligned}$$

Finding the Hamiltonian is equivalent to finding the energy in terms of momentum:

$$\begin{aligned}H &= \mathbf{p} \cdot \mathbf{v} - L = m\mathbf{v}^2 + q\mathbf{A} \cdot \mathbf{v} - L \\ &= mc^2 + \frac{1}{2}m\mathbf{v}^2 + qV = mc^2 + \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV, \quad \mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{m}\end{aligned}$$

Adjusting the zero of the Hamiltonian by subtracting  $mc^2$  gives the well-known result.

8. *Hunting for Lorentz invariants in electromagnetism.*

$$\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} =$$

9. *Deriving Maxwell's equations.* ( $\epsilon_0 = \mu_0 = c = 1$ ) The first equation is:

$$\begin{aligned}\partial_\mu F^{\mu 0} &= J^0 = \rho \\ \implies \nabla \cdot \mathbf{E} &= \rho\end{aligned}$$

The second equation is:

$$\begin{aligned}\partial_\mu F^{\mu i} &= J^i = \mathbf{J} \\ \Rightarrow -\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} &= \mathbf{J}\end{aligned}$$

The third equation is:

$$\begin{aligned}\partial_\lambda F_{\mu 0} + \partial_0 F_{\lambda \mu} + \partial_\mu F_{0\lambda} &= 0 \\ \Rightarrow \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0\end{aligned}$$

The fourth equation is:

$$\begin{aligned}\partial_\lambda F_{\mu i} + \partial_i F_{\lambda \mu} + \partial_\mu F_{i\lambda} &= 0 \\ \Rightarrow \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

**10. Deriving the continuity equation of electromagnetism.** Differentiating:

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \partial_\beta J^\beta$$

Since mixed partial derivatives are symmetric and  $F^{\alpha\beta}$  is antisymmetric, the operation obviously gives zero:

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \partial_\beta J^\beta = 0$$

The second equality can be interpreted as a continuity equation akin to fluid mechanics with the charge density  $\rho$  and the current density  $\mathbf{J}$ :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

## Chapter 6

# A first stab at relativistic quantum mechanics

1. *Massive scalar field Lagrangian.*

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\ \frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \\ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) &= 0 \\ (\partial^2 + m^2)\phi &= 0 \\ \pi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \partial^0\phi = \dot{\phi} \\ \mathcal{H} = \pi\dot{\phi} - \mathcal{L} &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\end{aligned}$$

## Chapter 7

# Examples of Lagrangians, or how to write down a theory

1. *Massive scalar field with a twist.*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1} = 0$$

$$(\partial^2 + m^2) \phi + \sum_{n=1}^{\infty} \lambda_n (2n+2) \phi^{2n+1} = 0$$

2. Massive scalar field with a source.

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}[\partial_\mu \phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2 + J(x)\phi(x) \\
 \frac{\partial \mathcal{L}}{\partial \phi(x)} &= -m^2\phi(x) + J(x) \\
 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} &= \partial^\mu \phi(x) \\
 \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) &= 0 \\
 \partial_\mu \partial^\mu \phi(x) + m^2\phi(x) - J(x) &= 0 \\
 (\partial^2 + m^2)\phi(x) &= J(x)
 \end{aligned}$$

3. Two coupled massive scalar fields.

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 - g(\phi_1^2 + \phi_2^2)^2 \\
 \frac{\partial \mathcal{L}}{\partial \phi_1} &= -m^2\phi_1 - 4g\phi_1(\phi_1^2 + \phi_2^2) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - 4g\phi_2(\phi_1^2 + \phi_2^2) = 0 \\
 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} &= \partial^\mu \phi_1, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2 \\
 \partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_1(\phi_1^2 + \phi_2^2) &= 0 \\
 \partial_\mu \partial^\mu \phi_2 + m^2\phi_2 + 4g\phi_2(\phi_1^2 + \phi_2^2) &= 0
 \end{aligned}$$

4. Introducing the conjugate momentum. Referring to Chapter 6's solution:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

## Chapter 8

# The passage of time

1. *Properties of a specific form of the time-evolution operator.* Let  $\hat{U}(t_1, t_2) = \exp \left[ i\hat{H}(t_2 - t_1) \right]$ :

$$\begin{aligned}\hat{U}(t_1, t_1) &= \exp \left[ i\hat{H}(t_1 - t_1) \right] = 1 \\ \hat{U}(t_3, t_2)\hat{U}(t_2, t_1) &= \exp \left[ i\hat{H}(t_3 - t_1) \right] = \hat{U}(t_3, t_1) \\ i \frac{d}{dt_2} \exp \left[ i\hat{H}(t_2 - t_1) \right] &= i^2 \exp \left( i\hat{H}t_2 \right) \hat{H} \exp \left( -i\hat{H}t_1 \right) = \hat{H}\hat{U}(t_2, t_1), \because \left[ \hat{U}, \hat{H} \right] = 0\end{aligned}$$

The time evolution operator is unitary, so  $\hat{U}^{-1} = \hat{U}^\dagger$ . Therefore:

$$\begin{aligned}\hat{U}^\dagger(t_2, t_1) &= \exp \left[ i\hat{H}(t_1 - t_2) \right] = \hat{U}(t_1, t_2) \\ \hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) &= \exp \left[ i\hat{H}(t_1 - t_2) \right] \exp \left[ i\hat{H}(t_2 - t_1) \right] = 1\end{aligned}$$

2. *Time-dependence of ladder operators.*

$$\begin{aligned}\hat{H} &= \sum_k E_k \hat{a}_k^\dagger \hat{a}_k \\ \hat{a}_k^\dagger(t) &= e^{i\hat{H}t/\hbar} \hat{a}_k^\dagger(0) e^{-i\hat{H}t/\hbar} \\ \frac{d\hat{a}_k^\dagger(t)}{dt} &= \frac{i}{\hbar} \left( e^{i\hat{H}t/\hbar} \left[ \hat{H}, \hat{a}_k^\dagger(0) \right] e^{-i\hat{H}t/\hbar} \right) \\ &= \frac{iE_k}{\hbar} \left( e^{i\hat{H}t/\hbar} \left[ \hat{n}_k, \hat{a}_k^\dagger(0) \right] e^{-i\hat{H}t/\hbar} \right) = \frac{iE_k}{\hbar} \hat{a}_k^\dagger(t) \\ \int \frac{d\hat{a}_k^\dagger(t)}{\hat{a}_k^\dagger(t)} &= \int \frac{iE_k}{\hbar} dt \implies \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0) e^{iE_k t/\hbar}\end{aligned}$$

3. *Time-dependence of an operator of the form  $\hat{X} = X_{lm} \hat{a}_l^\dagger \hat{a}_m$ .*

$$\begin{aligned}\hat{X}(t) &= e^{i\hat{H}t/\hbar} X_{lm} \hat{a}_l^\dagger \hat{a}_m e^{-i\hat{H}t/\hbar} \\ \frac{d\hat{X}}{dt} &= \end{aligned}$$

4. *Hamiltonian of a spin-1/2 particle in a magnetic field.*

$$\begin{aligned}\frac{d\hat{S}_H^z}{dt} &= \frac{1}{i\hbar} [\hat{S}_H^z, \omega \hat{S}_H^y] = \frac{\omega}{i\hbar} [\hat{S}_H^z, \hat{S}_H^y] = \frac{\omega}{i\hbar} (-i\hbar \hat{S}_H^x) = -\omega \hat{S}_H^x \\ \frac{d\hat{S}_H^x}{dt} &= \frac{1}{i\hbar} [\hat{S}_H^x, \omega \hat{S}_H^y] = \frac{\omega}{i\hbar} [\hat{S}_H^x, \hat{S}_H^y] = \frac{\omega}{i\hbar} (i\hbar \hat{S}_H^z) = \omega \hat{S}_H^z\end{aligned}$$

Spin behaves like angular momentum.



## Chapter 9

# Quantum mechanical transformations

### 1. Generators of the translation operator.

$$\begin{aligned}\hat{U}(\mathbf{a}) &= \exp[-i\hat{\mathbf{p}} \cdot \mathbf{a}] \\ \left. \frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{a}=0} &= -i\hat{\mathbf{p}} \exp[-i\hat{\mathbf{p}} \cdot \mathbf{0}] \\ \Rightarrow \hat{\mathbf{p}} &= -\frac{1}{i} \left. \frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}} \right|_{\mathbf{a}=0}\end{aligned}$$

### 2. Generators of the Lorentz group for four-vectors.

$$K = \left. \frac{1}{i} \frac{\partial \Lambda(\phi^1)}{\partial \phi^1} \right|_{\phi^1=0} = \frac{1}{i} \left. \begin{bmatrix} \sinh \phi^1 & \cosh \phi^1 & 0 & 0 \\ \cosh \phi^1 & \sinh \phi^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right|_{\phi^1=0} = -i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and similarly for  $\phi^i$ .

### 3. Infinitesimal Lorentz transformations. Going to the MCRF and composing boosts:

$$\Lambda^\mu_\nu = \lim_{\mathbf{v} \rightarrow 0} \begin{bmatrix} \gamma & \gamma v^1 & \gamma v^2 & \gamma v^3 \\ \gamma v^1 & \gamma & 0 & 0 \\ \gamma v^2 & 0 & \gamma & 0 \\ \gamma v^3 & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix}$$

For an infinitesimal counter-clockwise rotations, compose the matrices:

$$\Lambda^\mu_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & 0 \\ 0 & -\theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \theta^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta^1 \\ 0 & 0 & -\theta^1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Compose the boosts and rotation matrices:

$$\Lambda_\nu^\mu = \Lambda_{\bar{\nu}}^\mu \Lambda_\nu^{\bar{\nu}} = L_z R_z L_y R_y L_x R_x$$

$$\Lambda_\nu^\mu = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Extracting the identity matrix, the general infinitesimal Lorentz transformation can be written as:

$$\Lambda = \mathbf{1} + \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

The following tensors are indeed antisymmetric:

$$\omega^{\mu\nu} = \omega^\mu_\lambda g^{\lambda\nu} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega^\lambda_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & -\theta^3 & \theta^2 \\ -v^2 & \theta^3 & 0 & -\theta^1 \\ -v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

#### 4. Generators of the Poincaré group.

# Chapter 10

## Symmetry

1. *Commutation relations between scalar field and its conjugate momentum.*

$$[\phi(x), P^\alpha] = \phi(x)P^\alpha - P^\alpha\phi(x) = \int [\phi(x)T^{0\alpha} - T^{0\alpha}\phi(x)] d^3y$$

2. *Noether current of N-field system.*

3. *Energy-momentum tensor and momentum of the massive scalar field.*

$$\begin{aligned} T^{\mu\nu} &= \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \\ T^{00} &= \Pi^0 \partial^0 \phi - g^{00} \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] = \pi \dot{\phi} - \mathcal{L} \\ &= \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \\ \partial_\mu T^{\mu\nu} &= \partial_\mu [\partial^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L}] \\ &= \partial^2 \phi \partial^\nu \phi - \partial^\mu \phi \partial_\mu \partial^\nu \phi - \frac{1}{2} [\partial^\rho \phi \partial^\nu \partial_\rho \phi + \partial_\rho \phi \partial^\nu \partial^\rho \phi - 2m^2 \phi \partial^\nu \phi] \\ &= (\partial^2 + m^2) \phi [\partial^\nu \phi] = 0 \\ P^i &= \int T^{0i} d^3x = \int (\Pi^0 \partial^i \phi - g^{0i} \mathcal{L}) d^3x \\ &= \int \partial^0 \phi \partial^i \phi d^3x \end{aligned}$$

The Klein-Gordon equation, which is the equation of motion for scalar field theory, satisfies the divergence of the energy-momentum tensor.

## 4. Energy-momentum tensor and momentum of the electromagnetic field.

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu] \\
\frac{\partial(\partial_\mu A_\nu \partial^\mu A^\nu)}{\partial(\partial_\sigma A_\rho)} &= \delta_\mu^\sigma \delta_\nu^\rho \partial^\mu A^\nu + \partial_\mu A_\nu g^{\alpha\sigma} g^{\rho\beta} \delta_\alpha^\mu \delta_\beta^\nu = 2\partial^\sigma A^\rho \\
\frac{\partial(\partial_\mu A_\nu \partial^\nu A^\mu)}{\partial(\partial_\sigma A_\rho)} &= \delta_\mu^\sigma \delta_\nu^\rho \partial^\nu A^\mu + \partial_\mu A_\nu g^{\alpha\rho} g^{\sigma\beta} \delta_\alpha^\mu \delta_\beta^\nu = 2\partial^\rho A^\sigma \\
\frac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\rho)} &= -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho} = \Pi^{\sigma\rho} \\
T_\nu^\mu &= \Pi^{\mu\sigma} \partial_\nu A_\sigma - \delta_\nu^\mu \mathcal{L} \\
T^{\mu\nu} &= g^{\alpha\nu} T_\alpha^\mu = -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
X^{\lambda\mu\nu} &= F^{\mu\lambda} A^\nu = -F^{\lambda\mu} A^\nu = X^{\mu\lambda\nu} \\
\tilde{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda X^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\nu (F^{\mu\lambda} A^\nu) \\
&= -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \cancel{\partial_\lambda F^{\mu\lambda} A^\nu} + F^{\mu\lambda} \partial_\lambda A^\nu \\
[\lambda \rightarrow \sigma] &= F^{\mu\sigma} (\partial_\sigma A^\nu - \partial^\nu A_\sigma) + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = F^{\mu\sigma} F_\sigma^\nu + \frac{1}{4}g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
\tilde{T}^{00} &= F^{0\sigma} F_\sigma^0 + \frac{1}{4}g^{00} F_{\alpha\beta} F^{\alpha\beta} = E^2 + \frac{1}{2}(B^2 - E^2) = \frac{1}{2}(E^2 + B^2) \\
\tilde{T}^{i0} &= F^{i\sigma} F_\sigma^0 + \cancel{\frac{1}{4}g^{i0} F_{\alpha\beta} F^{\alpha\beta}} = \epsilon^{ijk} E_j B_k = (\mathbf{E} \times \mathbf{B})^i
\end{aligned}$$

# Chapter 11

## Canonical quantization of fields

1. *Commutation relations of quantum field position operators.* Let  $\int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} \equiv \int_{\mathbf{p}}:$

$$\begin{aligned}
 [\hat{\phi}(x), \hat{\phi}(y)] &= \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \int_{\mathbf{q}} (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) \\
 &\quad - \int_{\mathbf{q}} (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \\
 &= \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} ([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}] e^{-ip \cdot x} e^{iq \cdot y} + [\hat{a}_{\mathbf{p}}^{\dagger}, \hat{a}_{\mathbf{q}}] e^{ip \cdot x} e^{-iq \cdot y}) \\
 &= \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip \cdot x} e^{iq \cdot y} - \delta^{(3)}(\mathbf{q} - \mathbf{p}) e^{ip \cdot x} e^{-iq \cdot y}) \\
 &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}) = 0, \quad \mathbf{p} \mapsto -\mathbf{p}
 \end{aligned}$$

2. *Commutation relations of quantum field position operator and its conjugate momentum.*

$$\begin{aligned}
 [\hat{\phi}(x), \hat{\Pi}^0(y)] &= \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \int_{\mathbf{q}} (-iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} - \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) \\
 &\quad - \int_{\mathbf{q}} (-iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}} e^{-iq \cdot y} - \hat{a}_{\mathbf{q}}^{\dagger} e^{iq \cdot y}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) \\
 &= i \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} ([\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}] e^{-ip \cdot x} e^{iq \cdot y} + [\hat{a}_{\mathbf{q}}, \hat{a}_{\mathbf{p}}^{\dagger}] e^{ip \cdot x} e^{-iq \cdot y}) \\
 &= \frac{i}{2} \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip \cdot x} e^{iq \cdot y} + \delta^{(3)}(\mathbf{q} - \mathbf{p}) e^{ip \cdot x} e^{-iq \cdot y}) \\
 &= \frac{i}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} (e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}) = i \delta^{(3)}(x - y), \quad \mathbf{p} \mapsto -\mathbf{p}
 \end{aligned}$$

# Chapter 12

## Examples of canonical quantization

### 1. Complex scalar field theory.

$$\begin{aligned}
 \hat{\mathcal{H}} &= \partial^0 \hat{\psi}^\dagger \hat{\psi} + \partial^0 \hat{\psi} \hat{\psi}^\dagger + \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} + m^2 \hat{\psi}^\dagger \hat{\psi} \\
 &= \int_{\mathbf{q}} (iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot x} - \hat{b}_{\mathbf{q}} e^{-iq \cdot x}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \\
 &\quad + \int_{\mathbf{q}} (-iE_{\mathbf{q}}) (\hat{a}_{\mathbf{q}} e^{-iq \cdot x} - \hat{b}_{\mathbf{q}}^\dagger e^{iq \cdot x}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger e^{ip \cdot x} + \hat{b}_{\mathbf{p}} e^{-ip \cdot x}) \\
 &\quad + \int_{\mathbf{q}} (i\mathbf{q}) (\hat{a}_{\mathbf{q}}^\dagger e^{iq \cdot x} - \hat{b}_{\mathbf{q}} e^{-iq \cdot x}) \cdot \int_{\mathbf{p}} (i\mathbf{p}) (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} - \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \\
 &\quad + m^2 \int_{\mathbf{q}} (\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \\
 &= \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{i(q-p) \cdot x} - \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{-i(q-p) \cdot x}) \\
 &\quad + \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{-iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger e^{i(q-p) \cdot x} - \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{q}} e^{-i(q-p) \cdot x}) \\
 &\quad + \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{\mathbf{p} \cdot \mathbf{q}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{i(q-p) \cdot x} + \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{-i(q-p) \cdot x}) \\
 &\quad + m^2 \int d^3\mathbf{p} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} (\hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{i(q-p) \cdot x} + \hat{b}_{\mathbf{q}} \hat{b}_{\mathbf{p}}^\dagger e^{-i(q-p) \cdot x})
 \end{aligned}$$

### 2. Commutation relations of complex scalar fields.

(a)

$$\begin{aligned}
 [\hat{\psi}(x), \hat{\psi}^\dagger(y)] &= \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot x}) \int_{\mathbf{q}} (\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot y} + \hat{b}_{\mathbf{q}} e^{iq \cdot y}) \\
 &\quad - \int_{\mathbf{q}} (\hat{a}_{\mathbf{q}}^\dagger e^{-iq \cdot x} + \hat{b}_{\mathbf{q}} e^{iq \cdot x}) \int_{\mathbf{p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot y} + \hat{b}_{\mathbf{p}}^\dagger e^{ip \cdot y})
 \end{aligned}$$

(b)

**3. Commutation relations of Noether charges for two scalar fields.**

(a)

$$\begin{bmatrix} \phi'_1 \\ \phi'_2 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$[\hat{Q}_N, \hat{\phi}_1] = -iD\hat{\phi}_1 = i\hat{\phi}_2$$

(b)

$$[\hat{Q}_N, \hat{\phi}_2] = -iD\hat{\phi}_2 = -i\hat{\phi}_1$$

(c)

$$[\hat{Q}_N, \hat{\psi}] = \frac{1}{\sqrt{2}} [\hat{Q}_N, \hat{\phi}_1] + \frac{i}{\sqrt{2}} [\hat{Q}_N, \hat{\phi}_2] = \frac{i}{\sqrt{2}} \hat{\phi}_2 + \frac{1}{\sqrt{2}} \hat{\phi}_1 = \hat{\psi}$$

$$\begin{bmatrix} \phi'_1 \\ \phi'_2 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$[\hat{Q}_N, \hat{\phi}_1] = -iD\hat{\phi}_1 = i\hat{\phi}_2, \quad [\hat{Q}_N, \hat{\phi}_2] = -iD\hat{\phi}_2 = -i\hat{\phi}_1$$

$$[\hat{Q}_N, \hat{\psi}] = \frac{1}{\sqrt{2}} [\hat{Q}_N, \hat{\phi}_1] + \frac{i}{\sqrt{2}} [\hat{Q}_N, \hat{\phi}_2] = \frac{i}{\sqrt{2}} \hat{\phi}_2 + \frac{1}{\sqrt{2}} \hat{\phi}_1 = \hat{\psi}$$

**4. Using Noether's theorem to derive the number-phase uncertainty relation. Note:  $D\hat{\theta} = \pm 1$ . Substituting:**

$$[\hat{Q}_N, \hat{\theta}] = -iD\hat{\theta} = i$$

$$\left[ \int \rho(\mathbf{x}, t) d^3\mathbf{x}, \theta(\mathbf{x}, t) \right] = \int d^3\mathbf{x} [\rho, \theta] = i$$

## 5. Equations of motion of non-relativistic complex scalar field theory.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) &= \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \Pi_\psi^\mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \psi} &= -V(x) \psi^\dagger(x), \quad \Pi_\psi^0 = i\psi^\dagger \\
\partial_0 \Pi_\psi^0 &= i\partial_0 \psi^\dagger, \quad \partial_i \Pi_\psi^i = -\frac{1}{2m} \nabla^2 \psi^\dagger \\
\therefore i\partial_0 \psi^\dagger - \frac{1}{2m} \partial_i \partial^i \psi^\dagger - V(x) \psi^\dagger(x) &= 0 \\
\Rightarrow i\partial_0 \psi^\dagger &= \hat{H} \psi^\dagger, \quad \hat{H} = -\frac{1}{2m} \nabla^2 + \hat{V} \\
V = 0 &\Rightarrow i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi \\
iT'(t)X(x) &= -\frac{1}{2m} X''(x)T(t) \\
\frac{T'}{T} &= -iE \Rightarrow T(t) = Ae^{-iEt} \\
X'' + 2mEX = 0 &\Rightarrow X(x) = Be^{ipx} + Ce^{-ipx}, \quad p = \sqrt{2mE} \\
T(t)X(x) &= Ae^{i(px-Et)} + Be^{-i(px-Et)}
\end{aligned}$$

## 6. Noether current for non-relativistic complex scalar field theory.

$$\begin{aligned}
J_N^0 &= i\Psi^\dagger(i\Psi) + i\Psi(-i\Psi^\dagger) \\
Q_{N_c} &= \int [\hat{\Psi} \hat{\Psi}^\dagger - \hat{\Psi}^\dagger \hat{\Psi}] d^3\mathbf{x} \\
&= \int d^3x \left[ \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \int \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}} - \int \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}} \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}} \hat{a}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \right] \\
&\quad \frac{1}{(2\pi)^3} \int d^3\mathbf{x} \left[ \int d^3\mathbf{p} \int d^3\mathbf{q} \hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} - \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \right] \\
&= \int d^3\mathbf{p} \int d^3\mathbf{q} (\hat{a}_{\mathbf{p}} \hat{a}_{\mathbf{q}}^\dagger \delta^3(\mathbf{p}-\mathbf{q}) - \hat{a}_{\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}} \delta^3(\mathbf{q}-\mathbf{p})) \\
&= \int d^3\mathbf{p} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}}^\dagger] = \mathbf{p}
\end{aligned}$$

So momentum is conserved, naturally.

## 7. Transformation of the complex scalar field.



## Chapter 13

# Fields with many components and massive electromagnetism

1. Angular momentum form of internal symmetries.

(a)  $\vec{J}$  represents the Levi-Civita tensor as a vector of matrices.

$$\hat{\vec{Q}}_{N_c} = \int d^3\mathbf{p} \hat{\mathbf{A}}^\dagger \vec{J} \hat{\mathbf{A}}$$

(b) The inverse transformations and resultant computations are as follows:

$$\begin{aligned} \hat{a}_1 &= \frac{1}{\sqrt{2}}(\hat{b}_{-1} - \hat{b}_1), \quad \hat{a}_2 = -\frac{i}{\sqrt{2}}(\hat{b}_{-1} + \hat{b}_1), \quad \hat{a}_3 = \hat{b}_0 \\ \hat{Q}_{N_c}^2 &= \\ \hat{Q}_{N_c}^3 &= -i \int d^3\mathbf{p} (\hat{a}_{1\mathbf{p}}^\dagger \hat{a}_{2\mathbf{p}} - \hat{a}_{2\mathbf{p}}^\dagger \hat{a}_{1\mathbf{p}}) = \int d^3\mathbf{p} (\hat{b}_{1\mathbf{p}}^\dagger \hat{b}_{1\mathbf{p}} - \hat{b}_{-1\mathbf{p}}^\dagger \hat{b}_{-1\mathbf{p}}) \\ J_{\hat{b}}^1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_{\hat{b}}^2 = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_{\hat{b}}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

2. Lorentz boosting and circular polarization.

(a)

(b)

(c)

3. Projection tensors.

4. Playing with projection tensors.

## Chapter 14

# Gauge fields and gauge theory

1. Quantizing the electromagnetic field tensor.
2. The spin of the photon.
  - (a)
  - (b)

# Chapter 15

## Discrete transformations

1. *Gamma decay of a pion.*
2. *Classification of physical quantities.*
  - (a) Magnetic flux: Vector.
  - (b) Angular momentum: Pseudovector.
  - (c) Charge: Scalar.
  - (d) Scalar product of vector and pseudovector: Pseudoscalar.
  - (e) Scalar product of two vectors: Scalar.
  - (f) Scalar product of two pseudovectors: Scalar.
3. *Representations of spinors.*
  - (a)  $\mathbf{R}(\hat{\mathbf{x}}, \theta)$
  - (b)  $\mathbf{R}(\hat{\mathbf{y}}, \theta)$
  - (c)  $\mathbf{R}(\hat{\mathbf{z}}, \theta)$

## Chapter 16

# Propagators and Green's functions

1. *Green's function for a particle in an infinite potential well.*

(a) The Schrödinger equation is:

$$\langle x|\hat{H}|\psi\rangle = \langle x|\frac{\hat{P}^2}{2m} + \hat{V}(x)|\psi\rangle = \langle x|-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \hat{V}(x)|\psi\rangle = E\langle x|\psi\rangle$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + [E - V(x)]\psi(x) = 0$$

$$V = 0 \implies \psi(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \sqrt{2mE}/\hbar$$

$$\psi(0) = \psi(a) = 0 \rightarrow B = -A$$

$$\implies A \sin(ka) = 0 \implies k = \frac{n\pi}{a}$$

$$\int_{-a}^a |\psi(x)|^2 dx = 1$$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

(b)

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$$G^+(n, t_2, t_1) = \theta(t_2 - t_1) e^{-iE_n(t_2 - t_1)}$$

(c) Let  $t_2 = t$  and  $t_1 = 0$ , then taking the Fourier transform with a damping factor:

$$G^+(n, \hbar\omega) = \int_{-\infty}^{\infty} \theta(t) e^{-iE_n t} e^{i(\hbar\omega + i\epsilon)t} dt$$

$$G^+(n, \hbar\omega) = \frac{i}{\hbar\omega - E_n + i\epsilon}$$

## 2. Green's function in the energy expression.

(a)

$$\begin{aligned}
G_0^+(x, t, y, 0) &= \theta(t) \langle x(t) | y(t) \rangle \\
&= \theta(t) \langle x | e^{-i\hat{H}t} | y \rangle \\
&= \theta(t) \sum_n e^{iE_n t} \langle x | n \rangle \langle n | y \rangle = \theta(t) \sum_n \phi_n(x) \phi_n^*(y) e^{-iE_n t} \\
G_0^+(x, y, E) &= \int G_0^+(x, t, y, 0) dt \\
&= \int_{-\infty}^{\infty} \theta(t) \sum_n \phi_n(x) \phi_n^*(y) e^{-iE_n t} e^{iEt} dt \\
&= \int_0^{\infty} \sum_n \phi_n(x) \phi_n^*(y) e^{-i(E-E_n)t} dt
\end{aligned}$$

Using a damping factor  $e^{-\epsilon t}$  to ensure convergence, then switching the order of summation and integration:

$$\begin{aligned}
G_0^+(x, y, E) &= \sum_n \int_0^{\infty} \phi_n(x) \phi_n^*(y) e^{i(E-E_n+i\epsilon)t} dt \\
&= \sum_n \frac{i\phi_n(x) \phi_n^*(y)}{E - E_n + i\epsilon}
\end{aligned}$$

(b) The integral definition of the Heaviside step function is:

$$\theta(t) := i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z + i\epsilon}$$

Substituting this into the original expression and changing the order of integration:

$$\begin{aligned}
G_0^+(p, t, 0) &= \theta(t) e^{-iE_p t} \\
G_0^+(p, E) &= \int_{-\infty}^{\infty} \int_0^{\infty} \frac{i}{2\pi(z + i\epsilon)} e^{i(E-E_p-z)t} dt dz \\
&= \int_{-\infty}^{\infty} \frac{i}{(z + i\epsilon)} \delta(E - E_p - z) dz = \frac{i}{E - E_p + i\epsilon}
\end{aligned}$$

### 3. Green's function for the harmonic oscillator.

- (a) The one-dimensional harmonic oscillator with the corresponding forcing function  $f(t)$  has the following differential equation:

$$m \frac{\partial^2}{\partial t^2} A(t-u) + m\omega_0^2 A(t-u) = \tilde{F}(\omega) e^{-i\omega(t-u)}$$

Using operator methods to solve the differential equation:

$$\begin{aligned} A_P(t-u) &= \frac{1}{(D^2 + \omega_0^2)} \frac{\tilde{F}(\omega)}{m} e^{-i\omega(t-u)} = \left(1 + \frac{D^2}{\omega_0^2}\right)^{-1} \frac{\tilde{F}(\omega)}{m\omega_0^2} e^{-i\omega(t-u)}, \quad D = \frac{d}{dt} \\ &= \frac{\tilde{F}(\omega)}{m\omega_0^2} e^{i\omega u} \left[ \sum_{k=0}^{\infty} \left(\frac{iD}{\omega_0}\right)^{2k} e^{-i\omega t} \right] = \frac{\tilde{F}(\omega)}{m\omega_0^2} e^{-i\omega(t-u)} \sum_{k=0}^{\infty} \left(\frac{\omega}{\omega_0}\right)^{2k} \\ &= \frac{\tilde{F}(\omega)}{m\omega_0^2} e^{-i\omega(t-u)} \left[ \frac{1}{1 - \omega^2/\omega_0^2} \right] = -\frac{\tilde{F}(\omega)}{m(\omega^2 - \omega_0^2)} e^{-i\omega(t-u)} \end{aligned}$$

Therefore the solution is:

$$A(t-u) = c_1 \cos \omega_0(t-u) + c_2 \sin \omega_0(t-u) - \frac{\tilde{F}(\omega)}{m(\omega^2 - \omega_0^2)} e^{-i\omega(t-u)}$$

- (b) The differential equation that satisfies the Green's function is:

$$\left[ m \frac{\partial^2}{\partial t^2} + m\omega_0^2 \right] G(t, t') = \delta(t - t')$$

Taking the Fourier transform, rearranging and then taking its inverse:

$$\begin{aligned} -m(\omega^2 - \omega_0^2) G(\omega, t') &= \int_{-\infty}^{\infty} \delta(t - t') e^{i\omega t} dt = e^{i\omega t'} \\ G(t, t') &= -\frac{1}{m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - \omega_0^2} \end{aligned}$$

Using the previous result to verify the solution:

$$\begin{aligned} A(t) &= \int G(t, t') f(t') dt' \\ &= -\frac{1}{2\pi m} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{F}(\omega)}{\omega^2 - \omega_0^2} e^{i\omega t'} d\omega dt' \end{aligned}$$

- (c) Taking the Laplace transform of the differential equation form of the Green's function:

$$G(s, u) = \frac{e^{us}}{m(s^2 + \omega_0^2)}$$

Using convolution to find the inverse:

$$G^+(t, u) = \frac{1}{m\omega_0} \int_0^t \delta(k-u) \sin \omega_0(t-k) dk = \frac{1}{m\omega_0} \sin \omega_0(t-u)$$

(d) The trajectory is:

4. Green's function of the Klein-Gordon equation.

(a) Taking the three-dimensional Fourier transform:

$$\int_V (\nabla^2 + \mathbf{k}^2) G_{\mathbf{k}}(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}} d^3\mathbf{x} = \int_V \delta^3(\mathbf{x}) d^3\mathbf{x}$$

$$\tilde{G}_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2 - \mathbf{q}^2}$$

(b) The Fourier transform of  $G_{\mathbf{k}}^+(\mathbf{x})$  with a damping factor is:

$$\begin{aligned} \tilde{G}_{\mathbf{k}}^+(\mathbf{q}) &= \int_{-\infty}^{\infty} -\frac{e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}}{4\pi|\mathbf{x}|} e^{-i\mathbf{q} \cdot \mathbf{x}} d^3\mathbf{x} \\ &= -\frac{1}{2} \int_{-1}^1 \int_0^{\infty} |\mathbf{x}| e^{-i(|\mathbf{q}| \cos \theta - |\mathbf{k}| - i\epsilon)|\mathbf{x}|} d|\mathbf{x}| d(\cos \theta) \\ &= -\frac{i}{2|\mathbf{q}|} \int_0^{\infty} \left[ e^{i|\mathbf{q}||\mathbf{x}|} - e^{-i|\mathbf{q}||\mathbf{x}|} \right] e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|} d|\mathbf{x}| \\ &= \frac{1}{2|\mathbf{q}|} \left[ \frac{1}{(|\mathbf{k}| + |\mathbf{q}| + i\epsilon)} - \frac{1}{(|\mathbf{k}| - |\mathbf{q}| + i\epsilon)} \right] \\ &= \frac{1}{|\mathbf{k}|^2 - |\mathbf{q}|^2 + 2|\mathbf{k}|i\epsilon} \end{aligned}$$

(c)

## Chapter 17

# Propagators and fields

1. *Retarded field propagator for a free particle.*



## **Chapter 18**

# **The S-matrix**

## **Chapter 19**

# **Expanding the S-matrix: Feynman diagrams**

## **Chapter 20**

# **Scattering theory**

## **Chapter 21**

# **Statistical physics: a crash course**

## **Chapter 22**

# **The generating functional for fields**

## Chapter 23

# Path integrals: I said to him, 'You're crazy'

1. *Physicist's treatment of operators.*
2. *Path integral derivation of Wick's theorem.*

(a) Let

$$I(a) = -2 \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = -2\sqrt{\frac{2\pi}{a}}$$

Differentiating under the integral sign:

$$I'(a) = \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{ax^2}{2}\right) dx = \sqrt{\frac{2\pi}{a^3}}$$

(b)

$$\begin{aligned}
J_n(a) &= (-2)^{\frac{n}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx = (-2)^{\frac{n}{2}} \sqrt{\frac{2\pi}{a}} \\
\frac{d^k J_n(a)}{da^k} &= (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{(-1/2)!}{(-1/2-k)!} a^{-\frac{1}{2}-k} \\
\frac{d^{n/2} J_n(a)}{da^{n/2}} &= (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{\Gamma(1/2)}{\Gamma(\frac{1-n}{2})} a^{-(\frac{n+1}{2})} = \frac{i^n \pi}{\Gamma(\frac{1-n}{2})} \left(\frac{2}{a}\right)^{\frac{n+1}{2}} \\
\langle x^n \rangle &= \frac{\int_{-\infty}^{\infty} x^n \exp\left(-\frac{ax^2}{2}\right) dx}{\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx} = \frac{i^n \sqrt{\pi}}{\Gamma(\frac{1-n}{2})} \left(\frac{2}{a}\right)^{n/2} \\
&= \begin{cases} 0 & \forall n \in 2\mathbb{Z}^+ + 1 \\ a^{-n/2} \prod_{k=1}^{n/2} (2k-1) & \forall n \in 2\mathbb{Z}^+ \end{cases} \\
\therefore \frac{d^n J_{2n}(a)}{da^n} &= \frac{1}{a^n} \prod_{k=1}^n (2k-1)
\end{aligned}$$