Solutions to
Gravitation: Foundations and Frontiers
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Chapter 1

Special relativity

1. *Light clocks.* Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$
$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau$$
$$\frac{L'}{c+v} + \frac{L'}{c-v} = \gamma\frac{2L}{c}$$
$$L'\frac{2c}{c^2 - v^2} = \gamma\frac{2L}{c}$$
$$L' = L\frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

2. *Superluminal motion.*

$$\Delta t' = t'_2 - t'_1$$
$$t'_1 - t_1 \approx \frac{v}{c}\Delta t\cos\theta$$
$$t'_2 - t_1 = L/c$$
$$\Delta t' = \Delta t(1 - (v/c)\cos\theta)$$
$$v_{app} = v\Delta t\sin\theta = \frac{v\sin\theta}{1 - (v/c)\cos\theta}$$

Rewriting this expression and plotting it for $v = 0.99c$:

$$v_{app} = \frac{v\sqrt{1 - \cos^2\theta}}{1 - v\cos\theta}, \quad c = 1$$

3. *The strange world of four-vectors.*

(a) This is evident from taking the inner product, since the magnitudes add up.

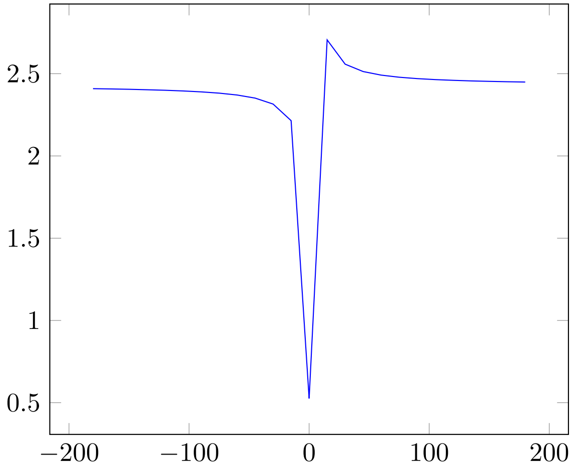
$$(a^i + b^i)(a_i + b_i) = a^i a_i + b^i b_i + 2(a^i b_i)$$

(b) Let k^i be a non-zero null vector. If non-zero a^i is a vector orthogonal to k^i, then $a_i k^i = 0$.

4. *Focused to the front.*

(a) Using a Lorentz transformation on the 'time' and 'space' components of the four-vector $k^a = (\omega, \omega \mathbf{n}/c)$:

$$\begin{aligned}
\omega_R &= \gamma\left[\omega_L - \frac{\omega_L v \cos\theta_L}{c}\right] = \gamma\omega_L(1 - (v/c)\cos\theta_L) \\
\omega_R \cos\theta_R &= \gamma\left[\omega_L \cos\theta_L - \frac{\omega_L v}{c}\right] = \gamma\omega_L[\cos\theta_L - (v/c)] \\
\frac{\omega_R \cos\theta_R}{\omega_R} &= \frac{\gamma\omega_L[\cos\theta_L - (v/c)]}{\gamma\omega_L[1 - (v/c)\cos\theta_L]} \\
\mu_R &= \frac{\mu_L - (v/c)}{1 - (v\mu_L/c)}
\end{aligned}$$

(b) 

(c) The solid angle is found by taking the differential of μ_R:

$$\begin{aligned}
\mathrm{d}\Omega &= \int \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = -\int \mathrm{d}(\cos\theta) \, \mathrm{d}\phi \\
\mathrm{d}(\cos\theta_R) &= \left[\frac{(v/c)[\cos\theta_L - (v/c)]}{[1 - (v/c)\cos\theta_L]^2} + \frac{1}{[1 - (v/c)\cos\theta_L]}\right] \mathrm{d}(\cos\theta_L) \\
\mathrm{d}\Omega' &= \frac{1}{\gamma^2} \frac{\mathrm{d}\Omega}{[1 - (v/c)\cos\theta]^2}
\end{aligned}$$

The energy is:

$$\mathrm{d}\mathcal{E}' = \hbar\omega = \gamma \, \mathrm{d}\mathcal{E} \, [1 - (v/c)\cos\theta]$$

The time is:

$$\mathrm{d}t' = \gamma\,\mathrm{d}t$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is $(c = 1)$:

$$\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}} = \gamma^2(1 - v\cos\theta)^3\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\text{lab}}$$
$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\text{lab}} = \frac{\left(1 - v^2\right)^2}{(1 - v\cos\theta)^3}\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}}$$

If the emission is isotropic, $\mathrm{d}\Omega = \mathrm{d}\Omega' = 4\pi$:

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{\text{lab}} = \mathrm{d}\Omega\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\text{rest}} = \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'}\right)_{\text{rest}}$$

5. *Transformation of antisymmetric tensors.*

$$A^{i'k'} = L^{k'}_{k}L^{i'}_{i}A^{ik} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^{i'}}{\partial x^i}A^{ik}$$

If $A^{ik} = -A^{ki}$ then:

$$A^{k'i'} = -A^{i'k'}?$$

6. *Practice with completely antisymmetric tensors.*

(a)

(b) Multiplying each index by the metric tensor:

$$\epsilon^{abcd} = g^{ai}g^{bj}g^{ck}g^{dl}\epsilon_{ijkl} =$$

(c)

(d)

7. *A null curve in flat spacetime.*

$$\eta_{ij}x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

8. *Shadows are Lorentz invariant.*

9. *Hamiltonian form of action - Newtonian mechanics.*

$$\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\;[p\dot{q} - H(p,q)]$$
$$\delta\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t\;\left[\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right] = 0$$

Using $\delta\dot{q} = \dot{\mathrm{d}}(\delta q)$ and integrating by parts:

$$\int_{t_2}^{t_1} \mathrm{d}t\; p\,\delta\dot{q} = \cancel{|p\,\delta q|_{t_1}^{t_2}} - \int_{t_2}^{t_1} \mathrm{d}t\; \dot{p}\,\delta q$$
$$\therefore \int_{t_2}^{t_1} \mathrm{d}t\, \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right] = 0$$

Since $\delta q = 0$ at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

10. *Hamiltonian form of action - special relativity.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda\, \left[p_a\dot{x}^a - \frac{1}{2}C\left(\frac{H}{mc^2} + mc^2\right)\right], \quad H = \eta_{ab}p^a p^b$$
$$\delta\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \delta p_a\dot{x}^a + p_a\delta\dot{x}^a - \frac{1}{2}\left[\delta C\left(\frac{H}{mc^2} + mc^2\right) + C\left(\frac{2p^a\delta p_a}{mc^2}\right)\right]\mathrm{d}\lambda = 0$$
$$= \int_{\lambda_1}^{\lambda_2} \left[\dot{x}^a - \frac{C}{mc^2}p^a\right]\delta p_a - \dot{p}_a\delta x^a - \frac{1}{2}\left[\frac{H + m^2c^4}{mc^2}\right]\delta C\,\mathrm{d}\lambda = 0$$

Each term must individually be zero, giving the equations of motion:

$$\frac{p^a}{c^2} = \frac{m\dot{x}^a}{C} \implies \dot{p}_a = 0, \quad H = -m^2c^4 = \eta_{ab}p^a p^b, \quad C = 1$$

11. *Hitting a mirror.* Using $E = -p^i u_i$ in the rest frame of the mirror:

$$\vec{p}_1 = (h\nu_1/c,\, h\nu_1\cos\theta/c,\, h\nu_1\sin\theta/c,\, 0), \quad \vec{p}_2 = (h\nu_2/c,\, -h\nu_2\cos\phi/c,\, h\nu_2\sin\phi/c,\, 0)$$
$$E = -\frac{\gamma h\nu_1}{c}[1 + (v/c)\cos\theta](c) = -\frac{\gamma h\nu_2}{c}[1 - (v/c)\cos\phi](c)$$
$$\frac{\nu_2}{\nu_1} = \frac{c + v\cos\theta}{c - v\cos\phi}$$

12. *Photon-electron scattering.*

(a) *Compton scattering.* The four-momentum components of the photon and electron before the collision are:

$$\vec{p}_1 = (h\nu_i/c,\, h\nu_i/c,\, 0,\, 0), \quad \vec{p}_2 = (mc, 0, 0, 0)$$

The four-momentum components of the scattered photon and electron after the collision are:

$$\vec{p}_3 = (h\nu_f/c,\, h\nu_f\cos\theta/c,\, h\nu_f\sin\theta/c,\, 0), \quad \vec{p}_4 = (p^0, p^1, p^2, p^3)$$

Let $c = 1$. Using the conservation of four-momentum, the four-momentum of the scattered electron is:

$$\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3 = \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos\theta \\ -h\nu_f \sin\theta \\ 0 \end{bmatrix}$$

The four-momentum of a particle is $\vec{p} \cdot \vec{p} = -m^2$. Finding the magnitude of the scattered electron's four-momentum:

$$\vec{p}_4 \cdot \vec{p}_4 = \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos\theta \\ -h\nu_f \sin\theta \\ 0 \end{bmatrix} \cdot \begin{bmatrix} h(\nu_i - \nu_f) + m \\ h\nu_i - h\nu_f \cos\theta \\ -h\nu_f \sin\theta \\ 0 \end{bmatrix}$$
$$= -(h(\nu_i - \nu_f) + m)^2 + (h\nu_i - h\nu_f \cos\theta)^2 + (-h\nu_f \sin\theta)^2 = -m^2$$

Expanding the expression and cancelling terms:

$$-2h^2 \nu_i \nu_f (1 - \cos\theta) + 2hm(\nu_i - \nu_f) = 0$$

Rearranging the terms, the final expression for the wavelength shift in Compton scattering is:

$$\lambda' - \lambda = \frac{h}{m}(1 - \cos\theta), \quad \lambda' = \frac{1}{\nu_f},\ \lambda = \frac{1}{\nu_i}$$

(b) *Inverse Compton scattering.*

13. *More practice with collisions.*

14. *Relativistic rocket.*

15. *Practice with equilibrium distribution functions.*

16. *Projection effects.*

17. *Relativistic virial theorem.* The conservation law implies the following:

$$\begin{aligned}
\partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \partial_\mu \left[x^\alpha x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\mu} \left[\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta \right] \\
&= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta \right] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha
\end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned}
\partial_0 \left(\partial_0 T^{00} \right) x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} \right] x^\alpha x^\beta \\
\partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta \right] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad \left(\partial_i T^{i\alpha} = 0 \right) \\
&= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \right] + 2T^{\alpha\beta}, \quad \left(T^{\mu\alpha} \partial_\mu x^\beta = T^{\alpha\beta} \right)
\end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that $T^{ij} = 0$ outside a compact region in space:

$$\int \mathrm{d}^3x \; \partial_0^2 T^{00} x^\alpha x^\beta = \int \mathrm{d}^3x \big[-\partial_\mu \big(\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha \big) + 2T^{\alpha\beta} \big]$$

$$\therefore \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3x \; T^{00} x^\alpha x^\beta = 2 \int \mathrm{d}^3x \; T^{\alpha\beta}$$

18. *Explicit computation of spin precession.* The four-velocity and four-acceleration are:

$$u^i = (\gamma, \gamma \mathbf{v}) = (\gamma, -\gamma r\omega \sin\omega t, \gamma r \omega \cos\omega t, 0)$$
$$a^i = \gamma(\dot{\gamma}, \dot{\gamma}\mathbf{v} + \dot{\mathbf{v}}\gamma) = -\gamma^2\omega^2(0, x, y, 0)$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$\frac{\mathrm{d}S^j}{\mathrm{d}\tau} = u^j \left(S^k a_k \right)$$
$$S^k a_k = -\gamma^2 \omega^2 (x S^x + y S^y)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[(yS^x - xS^y)] = -\omega(xS^x + yS^y)$$

19. *Little group of the Lorentz group.*

Chapter 2

Scalar and electromagnetic fields in special relativity

1. *Measuring the F^{ab}.*

2. *Schrödinger equation and gauge transformation.* The transformed Schrödinger equation is:

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}\Big[\mathbf{P} - \frac{q}{c}(\mathbf{A} + \nabla f)\Big]^2 + q(\phi + \partial_t f)\right]\psi$$

3. *Four-vectors leading to electric and magnetic fields.*

(a) Taking dot products:

$$E^i u_i = F^{ij} u_j u_i = 0$$

$$B^a u_a = \frac{1}{2}\epsilon^{abcd} F_{cd} u_a u_b = \frac{1}{2}(*F)^{ab} u_a u_b = 0$$

since $u_i u_j$ is symmetric and F^{ij} and its dual are antisymmetric.

(b)

$$u^a E^b - u^b E^a = \big(F^{bi} u^a - F^{ai} u^b\big) u_i$$

$$\epsilon^{ab}{}_{cd} u^c B^d = \frac{1}{2}\epsilon^{ab}{}_{cd}\epsilon^{di}{}_{jk} u^c u_i F^{jk}$$

4. *Hamiltonian form of action - charged particle.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2}\left[P_a\dot{x}^a - \frac{1}{2}C\left(\frac{H}{mc^2} + mc^2\right)\right], \quad H = \eta_{ij}\big(P^i - qA^i\big)\big(P^j - qA^j\big)$$

$$\delta\mathcal{A} = \int_{\lambda_1}^{\lambda_2}\delta P_a\dot{x}^a + P_a\delta\dot{x}^a - \frac{1}{2}\left[\delta C\left(\frac{H}{mc^2} + mc^2\right) + C\left(\frac{2\big(P^i - qA^i\big)(\delta P_i - q\delta A_i)}{mc^2}\right)\right]\mathrm{d}\lambda = 0$$

5. *Three-dimensional form of the Lorentz force.* Using the electromagnetic tensor equation of motion $u^i = (\gamma c, \gamma \mathbf{v})$:

$$
\begin{aligned}
m\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = qF^{ik}u_k &\implies mc\gamma\frac{\mathrm{d}u^0}{\mathrm{d}t} = qF^{0\alpha}u_\alpha, \quad \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}u^0}{\mathrm{d}t} \\
mc\gamma\frac{\mathrm{d}\gamma}{\mathrm{d}t} = q\gamma[(\mathbf{E}/c)\cdot\mathbf{v}] &\implies \frac{1}{\gamma^3}\left(m\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = q(\mathbf{E}\cdot\mathbf{v}), \quad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{\gamma^3}\left(\frac{\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) \\
m\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} &= qF^{\alpha\beta}u_\beta = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
\left[\frac{1}{\gamma^3}\left(\frac{m\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\mathbf{v} + m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right] &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \\
\implies \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{q}{m\gamma}\left[\mathbf{E}+\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{v})\mathbf{v}\right]
\end{aligned}
$$

6. *Pure gauge impostors.* This can be transformed into polar coordinates ($x = \cos\theta, y = \sin\theta$) and evaluated ($A_r = 0$, obviously):

$$
\begin{aligned}
A_\theta = \frac{\partial f}{\partial\theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta} = 1, \\
\oint_{C=x^2+y^2} \mathbf{A}\cdot\mathrm{d}\mathbf{s} &= \int_0^{2\pi} \mathrm{d}\theta = 2\pi
\end{aligned}
$$

The reason why $\mathbf{A}$ is not a pure gauge mode is because f is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

7. *Pure electric or magnetic fields.*

8. *Elegant solution to non-relativistic Coulomb motion.*

(a) Since the angular momentum is conserved, $\mathrm{d}\mathbf{J}\,/\,\mathrm{d}t = 0$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J}) &= \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\times\mathbf{J} + \cancel{\mathbf{p}\times\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}} \\
f(r)\hat{\mathbf{r}}\times\mathbf{r}\times m\mathbf{v} &= mf(r)[\mathbf{r}(\cancel{\hat{\mathbf{r}}\cdot\mathbf{v}}) - \mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{r})] \\
&= -mf(r)r\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -mf(r)r^2\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}
\end{aligned}
$$

Therefore, if $f(r)r^2 = -|\alpha|$, then:

$$
\begin{aligned}
\int \frac{1}{m|\alpha|}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J})\,\mathrm{d}t &= \int \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}\,\mathrm{d}t \\
\implies \frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} &= \mathbf{e}
\end{aligned}
$$

where $\mathbf{e}$ is a conserved vector independent of t, arising as a constant of the integration.

(b) Note that $\mathbf{p}$ and $\mathbf{J}$ are perpendicular:

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{e} &= \left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|}-\hat{\mathbf{r}}\right]\cdot\left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|}-\hat{\mathbf{r}}\right] \\
\implies |\mathbf{e}|^2 &= \frac{|\mathbf{p}\times\mathbf{J}|^2}{m^2|\alpha|^2}-2\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|}\right)+1 \\
&= \frac{p^2J^2}{m^2|\alpha|^2}-2\left(\frac{J^2}{m|\alpha|r}\right)+1, \quad J^2=(\mathbf{r}\times\mathbf{p})\cdot\mathbf{J} \\
&= 1+\frac{2EJ^2}{m|\alpha|^2}, \quad E=\frac{p^2}{2m}-\frac{|\alpha|}{r}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{r} &= \frac{\mathbf{r}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|}-\mathbf{r}\cdot\hat{\mathbf{r}} \\
er\cos\theta &= \frac{J^2}{m|\alpha|}-r \\
\implies r(\theta) &= \frac{J^2/m|\alpha|}{1+e\cos\theta}
\end{aligned}$$

(d)

$$E=\frac{p^2}{2m}-\frac{|\alpha|}{r}-\frac{|\beta|}{r^2}$$

9. *More on uniformly accelerated motion.*

Chapter 3

Gravity and spacetime geometry: the inescapable connection

Chapter 4

Metric tensor, geodesics and covariant derivative

1. *Practice with metrics.*

(a) $x = 2\tan(\theta/2)\cos\phi,\ y = 2\tan(\theta/2)\sin\phi$. Since the coordinates are spherical:

$$
\begin{aligned}
g_{ab} &= \eta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}, \quad X^i = \theta,\ \phi,\ x^a = x,\ y,\ \eta_{ij} = \begin{bmatrix}1 & 0\\ 0 & \sin^2\theta\end{bmatrix}\\
\frac{\partial\theta}{\partial x^a} &= \frac{\partial}{\partial x^a}\left[2\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{2}\right)\right], \quad \frac{\partial\phi}{\partial x^a} = \frac{\partial}{\partial x^a}\left[\tan^{-1}(y/x)\right]\\
\frac{\partial\theta}{\partial x^a} &= \frac{4x^a}{\sqrt{x^2+y^2}(x^2+y^2+4)} = \cos^2(\theta/2)\cos\phi,\ \cos^2(\theta/2)\sin\phi\\
\frac{\partial\phi}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\phi}{2\tan(\theta/2)}, \quad \frac{\partial\phi}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{2\tan(\theta/2)}\\
g_{xx} &= \left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial x}\right)^2\\
&= \left[\cos^2(\theta/2)\cos^2\phi + \frac{\sin^2\theta\sin^2\phi}{4\sin^2(\theta/2)}\right]\cos^2(\theta/2) = \cos^4(\theta/2)\\
g_{yy} &= \left(\frac{\partial\theta}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial y}\right)^2\\
&= \left[\cos^2(\theta/2)\sin^2\phi + \frac{\sin^2\theta\cos^2\phi}{4\sin^2\theta/2}\right]\cos^2(\theta/2) = \cos^4(\theta/2)\\
\therefore \mathrm{d}s^2 &= \cos^4\left(\frac{\theta}{2}\right)\left[\mathrm{d}x^2 + \mathrm{d}y^2\right]
\end{aligned}
$$

(b) $\phi = x,\ \theta = 2\tan^{-1} e^{-y}$.

$$g_{ab} = \eta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}, \quad X^i = \theta,\, \phi,\ x^a = x,\, y$$
$$\frac{\partial\theta}{\partial x} = 0,\ \frac{\partial\phi}{\partial x} = 1,\ \frac{\partial\theta}{\partial y} = -\frac{2}{e^y + e^{-y}} = -\operatorname{sech} y,\ \frac{\partial\phi}{\partial y} = 0$$