Solutions to
Quantum Field Theory for the Gifted Amateur
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Chapter 1

Lagrangians

Problem 1.

$$(a) \quad t = \frac{\sqrt{x^2+h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2+h_2^2}}{v_2} = \frac{\sqrt{x^2+h_1^2}}{c/n_1} + \frac{\sqrt{(l-x)^2+h_2^2}}{c/n_2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{x}{c/n_1\sqrt{x^2+h_1^2}} - \frac{(l-x)}{c/n_2\sqrt{(l-x)^2+h_2^2}} = 0$$

$$n_1 \sin\theta = n_2 \sin\phi$$

Problem 2.

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int G(x,y)[f(y)+\epsilon\delta(y-z)]\,\mathrm{d}y - \int G(x,y)f(y)\,\mathrm{d}y\right] = G(x,z)$$

$$\frac{\delta I[f^3]}{\delta f(x_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{-1}^{1}[f(x)+\epsilon\delta(x-x_0)]^3\,\mathrm{d}x - \int_{-1}^{1}[f(x)]^3\,\mathrm{d}x\right] =$$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\right]$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int\left(\frac{\partial}{\partial y}[f(y)+\epsilon\delta(y-x)]\right)^2\mathrm{d}y - \int\left(\frac{\partial f}{\partial y}\right)^2\mathrm{d}y\right]$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int[f'(y)+\epsilon\delta'(y-x)]^2\,\mathrm{d}y - \int\left(\frac{\partial f}{\partial y}\right)^2\mathrm{d}y\right]$$

Problem 3.

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int \left[g(y,f) + \frac{\partial g(y,f)}{\partial f} \epsilon \delta(y-x) \right] \mathrm{d}y - \int g(y,f)\, \mathrm{d}y \right] = \frac{\partial g(x,f)}{\partial f(x)}$$

Problem 4.

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \lim_{\epsilon \to 0} \frac{\phi(x) + \epsilon \delta(x-y) - \phi(x)}{\epsilon} = \delta(x-y)$$

$$\frac{\delta \dot{\phi}(t)}{\delta \phi(t_0)} = \lim_{\epsilon \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}t}[\phi(t) + \epsilon \delta(t-t_0)] - \dot{\phi}(t)}{\epsilon} = \frac{\mathrm{d}}{\mathrm{d}t} \delta(t-t_0)$$

Problem 5.

$$S = \int T - V \ \mathrm{d}^3 x = \frac{1}{2} \int \rho \left(\frac{\partial \psi}{\partial t} \right)^2 - \mathcal{T} (\nabla \psi)^2 \ \mathrm{d}^3 x$$

$$\frac{\delta S}{\delta \psi} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} [\int \rho \left(\frac{\partial}{\partial t} [\psi + \epsilon \delta(t-t_0)] \right)^2 - \mathcal{T} (\nabla [\psi + \epsilon \delta(\mathbf{x} - \mathbf{y})])^2 \ \mathrm{d}^3 x$$

$$- \int \rho \left(\frac{\partial \psi}{\partial t} \right)^2 - \mathcal{T} (\nabla \psi)^2 \ \mathrm{d}^3 x] =$$

$$\int \left[\rho \frac{\partial}{\partial t} \delta(t-t_0) \frac{\partial \psi}{\partial t} - \mathcal{T} \nabla \delta(\mathbf{x} - \mathbf{y}) \nabla \psi \right] \mathrm{d}^3 x = 0$$

$$\nabla^2 \psi = \frac{1}{\nu^2} \frac{\partial^2 \psi}{\partial t^2}, \ \ \nu = \sqrt{\frac{\mathcal{T}}{\rho}}$$

Problem 6.

Chapter 2

Simple harmonic oscillators

Problem 1.

$$
\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) - \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \\
&= \frac{1}{2i\hbar}([\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]) = 1
\end{aligned}
$$

Problem 2.

$$
\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda x^4 \\
\hat{H} &= \lambda\left[\hat{x}^4 + \frac{m\omega^2}{2\lambda}\hat{x}^2 + \frac{\hat{p}^2}{2m\lambda}\right]
\end{aligned}
$$

Problem 3.

$$
\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \tilde{x}_k e^{ikja}, \quad \hat{x}_k = \sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k + \hat{a}^\dagger_{-k}\right)
$$

$$
\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \sqrt{\frac{\hbar}{2m\omega_k}}\left(\hat{a}_k + \hat{a}^\dagger_{-k}\right)e^{ikja} = \frac{1}{\sqrt{N}}\sqrt{\frac{h}{m}}\sum_k \frac{1}{\sqrt{2\omega_k}}\left(\hat{a}_k e^{ikja} + \hat{a}^\dagger_k e^{-ikja}\right)
$$

Problem 4.

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|0\rangle = 0$$

$$\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle = 0$$

$$\left(x+\frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\langle x|0\rangle = 0$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{m\omega}{\hbar}x\right)\langle x|0\rangle = 0$$

This is easily solved by separation of variables. Attempting a series solution for practice:

$$\langle x|0\rangle = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty}\frac{m\omega}{\hbar}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}\frac{m\omega}{\hbar}a_n x^{n+1} = 0$$

$$a_{n+2} = -\frac{m\omega}{\hbar(n+2)}a_n, \quad a_0 = A,\ a_1 = 0$$

$$\langle x|0\rangle = A\left[1+\left(-\frac{m\omega}{2\hbar}\right)x^2+\frac{1}{2}\left(-\frac{m\omega}{2\hbar}\right)^2x^4+\frac{1}{6}\left(-\frac{m\omega}{2\hbar}\right)^3x^6+\ldots\right]$$

$$\langle x|0\rangle = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$A = 1/\left|\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\right|$$

$$A = 1/\sqrt{\int_{-\infty}^{\infty}\exp\left(2\frac{m\omega}{2\hbar}x^2\right)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Chapter 3

Occupation number representation

Problem 1.

$$(a) \quad \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] = \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\delta_{\mathbf{pq}} = \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

Problem 2.

$$\left[\hat{a}, \left(\hat{a}^{\dagger}\right)^{n}\right] = \left[\hat{a}\left(\hat{a}^{\dagger}\right)^{n} - \left(\hat{a}^{\dagger}\right)^{n}\hat{a}\right]$$

$$= \left[(1+\hat{a}^{\dagger}\hat{a})\left(\hat{a}^{\dagger}\right)^{n-1} - \left(\hat{a}^{\dagger}\right)^{n-1}\left(\hat{a}^{\dagger}\hat{a}\right)\right] = \left[\left(\hat{a}^{\dagger}\right)^{n-1} - \left[\hat{a}^{\dagger}\hat{a}, \left(\hat{a}^{\dagger}\right)^{n-1}\right]\right]$$

Problem 3.

$$\hat{a}^{\dagger}_{i} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_i - \frac{i}{m\omega}\hat{p}_i\right)$$

$$\left[\hat{a}_i, \hat{a}^{\dagger}_j\right] = \frac{m\omega}{2\hbar}\left[\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right) - \left(\hat{x}_j - \frac{i}{m\omega}\hat{p}_j\right)\left(\hat{x}_i + \frac{i}{m\omega}\hat{p}_i\right)\right]$$

$$= \frac{m\omega}{2\hbar}\left([\hat{x}_i, \hat{x}_j] + \frac{1}{m^2\omega^2}[\hat{p}_i, \hat{p}_j] - \frac{i}{m\omega}([\hat{x}_j.\hat{p}_i] + [\hat{x}_i, \hat{p}_j])\right) = \delta_{ij}$$

$$\hat{H} = \frac{1}{2m}$$

$$\hat{L}^i = -i\hbar\epsilon^{ijk}\hat{a}^{\dagger}_j\hat{a}_k$$

Chapter 4

Making Second Quantization Work

Problem 1.

$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right]_\zeta = \delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad \left[\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right]_\zeta = 0$$

$$\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}) = \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})$$

$$= -\zeta\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})$$

$$= -\zeta^2\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})$$

So $\zeta = \pm 1$ yields the same result regardless of bosons or fermions.

Problem 2.

$$\hat{\rho}_1(\mathbf{x}-\mathbf{y}) = \left\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \right\rangle$$

$$= \frac{1}{\mathcal{V}}\sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_{\mathbf{q}} \hat{a}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{y}} = \frac{1}{\mathcal{V}}\sum_{\mathbf{pq}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}$$

Problem 3.

$$|\hat{H} - \lambda\hat{I}| = \begin{vmatrix} U-\lambda & -t & -t & 0 \\ -t & -\lambda & 0 & -t \\ -t & 0 & -\lambda & -t \\ 0 & -t & -t & U-\lambda \end{vmatrix} =$$

Chapter 5

Continuous systems

Problem 1.

$$\int_a^b \mathrm{d}s = \int_a^b \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}t}{\gamma} = \int_a^b L\,\mathrm{d}t$$

$$\frac{\partial L^2}{\partial \mathbf{v}} = \frac{2\mathbf{v}}{c^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^2}{\partial \mathbf{v}}\right) - \frac{\partial L^2}{\partial \mathbf{x}} = \frac{2\dot{\mathbf{v}}}{c^2} = 0$$

Since the acceleration is zero, the velocity is constant. Hence a straight world-line path does minimise the interval.

Problem 2.

$$L = \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV$$

$$\nabla L = q[\nabla(\mathbf{A}\cdot\mathbf{v}) - \nabla V]$$

$$= q[\cancel{(\mathbf{A}\cdot\nabla)\mathbf{v}} + \cancel{(\mathbf{v}\cdot\nabla)\mathbf{A}} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \cancel{\mathbf{A}\times(\nabla\times\mathbf{v})} - q\nabla V]$$

$$= q[\mathbf{E} + \mathbf{v}\times\mathbf{B}], \quad \because \mathbf{E} = -q\nabla V,\ \mathbf{B} = \nabla\times\mathbf{A}$$

$$\frac{\partial L}{\partial \mathbf{v}} = -\frac{mc^2}{2\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left(-\frac{2\mathbf{v}}{c^2}\right) = \gamma m\mathbf{v}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}} = \nabla L \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\mathbf{v}) = q[\mathbf{E} + \mathbf{v}\times\mathbf{B}]$$

Problem 3.

$$L = \frac{-mc^2}{\gamma} + q\mathbf{A}\cdot\mathbf{v} - qV \approx \frac{1}{2}m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - qV$$
$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}$$

Finding the Hamiltonian is equivalent to finding the energy in terms of momentum:

$$H = \mathbf{p}\cdot\mathbf{v} - L = m\mathbf{v}^2 + q\mathbf{A}\cdot\mathbf{v} - L$$
$$= mc^2 + \frac{1}{2}m\mathbf{v}^2 + qV = mc^2 + \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV, \quad \mathbf{v} = \frac{\mathbf{p} - q\mathbf{A}}{m}$$

Chapter 6

A first stab at relativistic quantum mechanics

Problem 1.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\left(\partial^2 + m^2\right)\phi = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial^0 \phi = \dot{\phi}$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Chapter 7

Examples of Lagrangians, or how to write down a theory

Problem 1.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

$$\partial_\mu \partial^\mu \phi + m^2\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0$$

$$\left(\partial^2 + m^2\right)\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0$$

Problem 2.

$$\mathcal{L} = \frac{1}{2}[\partial_\mu \phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2 + J(x)\phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} = -m^2\phi(x) + J(x)$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = \partial^\mu \phi(x)$$
$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right) = 0$$
$$\partial_\mu \partial^\mu \phi(x) + m^2\phi(x) - J(x) = 0$$
$$\left(\partial_\mu \partial^\mu + m^2\right)\phi(x) = J(x)$$

Problem 3.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 - g\left(\phi_1^2 + \phi_2^2\right)^2$$
$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2\phi_1 - 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2\phi_2 - 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial^\mu \phi_1, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_2$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_1\left(\phi_1^2 + \phi_2^2\right) = 0$$
$$\partial_\mu \partial^\mu \phi_1 + m^2\phi_1 + 4g\phi_2\left(\phi_1^2 + \phi_2^2\right) = 0$$

Problem 4. Referring to Chapter 5's solution:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Chapter 8

The passage of time

Problem 1.

Problem 2.

$$\hat{H} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k$$

$$\hat{a}_k^\dagger(t) = e^{i\hat{H}t/\hbar} \hat{a}_k^\dagger(0) e^{-i\hat{H}t/\hbar}$$

$$\frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\mathrm{d}t} = \frac{i}{\hbar}\left(e^{i\hat{H}t/\hbar}\left[\hat{H}, \hat{a}_k^\dagger(0)\right] e^{-i\hat{H}t/\hbar}\right)$$

$$= \frac{iE_k}{\hbar}\left(e^{i\hat{H}t/\hbar}\left[\hat{n}_k, \hat{a}_k^\dagger(0)\right] e^{-i\hat{H}t/\hbar}\right) = \frac{iE_k}{\hbar}\hat{a}_k^\dagger(t)$$

$$\int \frac{\mathrm{d}\hat{a}_k^\dagger(t)}{\hat{a}_k^\dagger(t)} = \int \frac{iE_k}{\hbar}\,\mathrm{d}t \longrightarrow \hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0) e^{iE_k t/\hbar}$$

Problem 3.

$$\hat{X}(t) = e^{i\hat{H}t/\hbar} X_{lm} \hat{a}_l^\dagger \hat{a}_m e^{-i\hat{H}t/\hbar}$$

$$\frac{\mathrm{d}\hat{X}}{\mathrm{d}t} =$$

Problem 4.

$$\frac{\mathrm{d}\hat{S}_H^z}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^z, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(-i\hbar\hat{S}_H^x\right) = -\omega\hat{S}_H^x$$

$$\frac{\mathrm{d}\hat{S}_H^x}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{S}_H^x, \omega\hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left[\hat{S}_H^z, \hat{S}_H^y\right] = \frac{\omega}{i\hbar}\left(i\hbar\hat{S}_H^z\right) = \omega\hat{S}_H^z$$

Chapter 9

Quantum mechanical transformations

Problem 1.

$$\hat{U}(\mathbf{a}) = \exp[-i\hat{\mathbf{p}} \cdot \mathbf{a}]$$

$$\left.\frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}}\right|_{\mathbf{a}=0} = -i\hat{\mathbf{p}} \exp[-i\hat{\mathbf{p}} \cdot 0]$$

$$\hat{\mathbf{p}} = -\frac{1}{i}\left.\frac{\partial \hat{U}(\mathbf{a})}{\partial \mathbf{a}}\right|_{\mathbf{a}=0}$$

Problem 2.

$$K = \frac{1}{i}\left.\frac{\partial \mathbf{\Lambda}(\phi^1)}{\partial \phi^1}\right|_{\phi^1=0} = \frac{1}{i}\left.\begin{vmatrix} \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ \cosh\phi^1 & \sinh\phi^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\right|_{\phi^1=0} = -i\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3. Going to the MCRF and composing boosts:

$$\Lambda^{\mu}_{\ \nu} = \lim_{\mathbf{v}\to 0}\begin{bmatrix} \gamma & \gamma v^1 & \gamma v^2 & \gamma v^3 \\ \gamma v^1 & \gamma & 0 & 0 \\ \gamma v^2 & 0 & \gamma & 0 \\ \gamma v^3 & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix}$$

For an infinitesimal counter-clockwise rotations, compose the matrices:

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & 0 \\ 0 & -\theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \theta^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta^1 \\ 0 & 0 & -\theta^1 & 1 \end{bmatrix}$$

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Compose the boosts and rotation matrices:

$$\Lambda^{\mu}_{\nu} = \Lambda^{\mu}_{\ \bar{\nu}}\Lambda^{\bar{\nu}}_{\ \ \nu} = L_z R_z L_y R_y L_x R_x$$

$$\Lambda^{\mu}_{\nu} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{bmatrix}$$

Extracting the identity matrix, the general infinitesimal Lorentz transformation can be written as:

$$\mathbf{\Lambda} = \mathbf{1} + \omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

The following tensors are indeed antisymmetric:

$$\omega^{\mu\nu} = \omega^{\mu}_{\ \lambda} g^{\lambda\nu} = \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & -\theta^3 & \theta^2 \\ v^2 & \theta^3 & 0 & -\theta^1 \\ v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega^{\lambda}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & -\theta^3 & \theta^2 \\ -v^2 & \theta^3 & 0 & -\theta^1 \\ -v^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix}$$

Chapter 10

Symmetry

Problem 1.

$$[\phi(x), P^{\alpha}] = \phi(x)P^{\alpha} - P^{\alpha}\phi(x) = \int \phi(x)T^{0\alpha}\ \mathrm{d}^3y - \int T^{0\alpha}\phi(x)\mathrm{d}^3y$$

Problem 2.

Problem 3.

$$T^{\mu\nu} = \Pi^{\mu}\partial^{\nu}\phi - g^{\mu\nu}\mathcal{L}$$

$$T^{00} = \Pi^0\partial^0\phi - g^{00}\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right] = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\partial_\mu T^{\mu\nu} = \partial_\mu[\partial^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}]$$

$$= \partial^2\phi\partial^\nu\phi - \partial^\mu\phi\partial_\mu\partial^\nu\phi - \frac{1}{2}\left[\partial^\rho\phi\partial^\nu\partial_\rho\phi + \partial_\rho\phi\partial^\nu\partial^\rho\phi - 2m^2\phi\partial^\nu\phi\right]$$

$$= \left(\partial^2 + m^2\right)\phi(\partial^\nu\phi) = 0$$

$$P^i = \int T^{0i}\ \mathrm{d}^3x = \int \left(\Pi^0\partial^i\phi - g^{0i}\mathcal{L}\right)\ \mathrm{d}^3x = \int \partial^0\phi\partial^i\phi\ \mathrm{d}^3x$$

The Klein-Gordon equation, which is the equation of motion for scalar field theory, satisfies the divergence of the energy-momentum tensor.

Problem 4.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu]$$

$$\frac{\partial(\partial_\mu A_\nu \partial^\mu A^\nu)}{\partial(\partial_\sigma A_\rho)} = \delta^\sigma_\mu \delta^\rho_\nu \partial^\mu A^\nu + \partial_\mu A_\nu g^{\alpha\sigma} g^{\rho\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\sigma A^\rho$$

$$\frac{\partial(\partial_\mu A_\nu \partial^\nu A^\mu)}{\partial(\partial_\sigma A_\rho)} = \delta^\sigma_\mu \delta^\rho_\nu \partial^\nu A^\mu + \partial_\mu A_\nu g^{\alpha\rho} g^{\sigma\beta} \delta^\mu_\alpha \delta^\nu_\beta = 2\partial^\rho A^\sigma$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\rho)} = -(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = -F^{\sigma\rho} = \Pi^{\sigma\rho}$$

$$T^\mu_\nu = \Pi^{\mu\sigma}\partial_\nu A_\sigma - \delta^\mu_\nu \mathcal{L}$$

$$T^{\mu\nu} = g^{\alpha\nu}T^\mu_\alpha = -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$X^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu = -F^{\lambda\mu}A^\nu = X^{\mu\lambda\nu}$$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\nu X^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\nu \left(F^{\mu\lambda}A^\nu\right)$$

$$= -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \cancel{\partial_\lambda F^{\mu\lambda} A^\nu} + F^{\mu\lambda}\partial_\lambda A^\nu$$

$$\overset{\lambda\to\sigma}{=} F^{\mu\sigma}(\partial_\sigma A^\nu - \partial^\nu A_\sigma) + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = F^{\mu\sigma}F^\nu_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$\tilde{T}^{00} = F^{0\sigma}F^0_\sigma + \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} = \mathbf{E}^2 + \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right) = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

$$\tilde{T}^{i0} = F^{i\sigma}F^0_\sigma + \cancel{\frac{1}{4}g^{i0}F_{\alpha\beta}F^{\alpha\beta}} = \epsilon^{ijk}E_j B_k = (\mathbf{E}\times\mathbf{B})^i$$

Chapter 11

Canonical quantization of fields

Problem 1.

$$\left[\hat{\phi}(x), \hat{\phi}(y)\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip\cdot x}\right)$$

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq\cdot y}\right) - \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} + \hat{a}_{\mathbf{q}}^{\dagger} e^{iq\cdot y}\right)$$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}_{\mathbf{p}}^{\dagger} e^{ip\cdot x}\right)$$

$$= \int \mathrm{d}^3 p \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}\right] e^{-ip\cdot x} e^{iq\cdot y} + \left[\hat{a}_{\mathbf{p}}^{\dagger}, \hat{a}_{\mathbf{q}}\right] e^{ip\cdot x} e^{-iq\cdot y}\right)$$

$$= \int \mathrm{d}^3 p \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ip\cdot x} e^{iq\cdot y} - \delta^{(3)}(\mathbf{q}-\mathbf{p}) e^{ip\cdot x} e^{-iq\cdot y}\right)$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)}\right)$$

Problem 2.

$$\left[\hat{\phi}(x), \hat{\Pi}^0(y)\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^{\dagger}_{\mathbf{p}} e^{ip\cdot x}\right)$$
$$\int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} - \hat{a}^{\dagger}_{\mathbf{q}} e^{iq\cdot y}\right)$$
$$- \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} (-iE_{\mathbf{q}}) \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} - \hat{a}^{\dagger}_{\mathbf{q}} e^{iq\cdot y}\right) \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^{\dagger}_{\mathbf{p}} e^{ip\cdot x}\right)$$
$$= \frac{i}{2} \int \mathrm{d}^3 p \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{E_{\mathbf{q}}}{(E_{\mathbf{p}} E_{\mathbf{q}})^{\frac{1}{2}}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\right] e^{-ip\cdot x} e^{iq\cdot y} + \left[\hat{a}_{\mathbf{q}}, \hat{a}^{\dagger}_{\mathbf{p}}\right] e^{ip\cdot x} e^{-iq\cdot y}\right)$$
$$= \int \mathrm{d}^3 p \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{(4E_{\mathbf{p}} E_{\mathbf{q}})^{\frac{1}{2}}} \left(\delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip\cdot x} e^{iq\cdot y} + \delta^{(3)}(\mathbf{q} - \mathbf{p}) e^{ip\cdot x} e^{-iq\cdot y}\right)$$
$$= \frac{i}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right)$$

Chapter 12

Examples of canonical quantization

Problem 1.

$$\mathcal{H} = \partial^0\hat{\psi}^\dagger\hat{\psi} + \partial^0\hat{\psi}\hat{\psi}^\dagger + \nabla\hat{\psi}^\dagger\cdot\nabla\hat{\psi} + m^2\hat{\psi}^\dagger\hat{\psi}$$

$$= \int\frac{\mathrm{d}^3q}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}(-iE_{\mathbf{q}})\left(\hat{a}^\dagger_{\mathbf{q}}e^{-iq\cdot x} - \hat{b}_{\mathbf{q}}e^{iq\cdot x}\right)\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)$$

$$+\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}(-iE_{\mathbf{p}})\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} - \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\int\frac{\mathrm{d}^3q}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}e^{-iq\cdot x} + \hat{b}_{\mathbf{q}}e^{iq\cdot x}\right)$$

$$+\int\frac{\mathrm{d}^3q}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}(-i\mathbf{q})\left(\hat{a}^\dagger_{\mathbf{q}}e^{-iq\cdot x} - \hat{b}_{\mathbf{q}}e^{iq\cdot x}\right)\cdot\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}(i\mathbf{p})\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} - \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)$$

$$+m^2\int\frac{\mathrm{d}^3q}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}e^{-iq\cdot x} + \hat{b}_{\mathbf{q}}e^{iq\cdot x}\right)\int\frac{\mathrm{d}^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)$$

$$= \int\mathrm{d}^3p\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{-iE_{\mathbf{q}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(p+q)\cdot x} - \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(p+q)\cdot x}\right)$$

$$+\int\mathrm{d}^3p\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}e^{-i(p+q)\cdot x} - \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{q}}e^{i(p+q)\cdot x}\right)$$

$$+\int\mathrm{d}^3p\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathbf{p}\cdot\mathbf{q}}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(p+q)\cdot x} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(p+q)\cdot x}\right)$$

$$+m^2\int\mathrm{d}^3p\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{(4E_{\mathbf{p}}E_{\mathbf{q}})^{\frac{1}{2}}}\left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}e^{-i(p+q)\cdot x} + \hat{b}_{\mathbf{q}}\hat{b}^\dagger_{\mathbf{p}}e^{i(p+q)\cdot x}\right)$$

Problem 2.

$$\left[\hat{\psi}(x), \hat{\psi}^\dagger(y)\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right)$$

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}^\dagger_{\mathbf{q}} e^{-iq\cdot x} + \hat{b}_{\mathbf{q}} e^{iq\cdot x}\right) - \int \frac{\mathrm{d}^3 q}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{q}})^{\frac{1}{2}}} \left(\hat{a}^\dagger_{\mathbf{q}} e^{-iq\cdot x} + \hat{b}_{\mathbf{q}} e^{iq\cdot x}\right)$$

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_{\mathbf{p}})^{\frac{1}{2}}} \left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right)$$

Problem 3.

$$(a) \quad \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\left[\hat{Q}_N, \hat{\phi}_1\right] = -iD\hat{\phi}_1 = i\hat{\phi}_2$$

$$(b) \quad \left[\hat{Q}_N, \hat{\phi}_2\right] = -iD\hat{\phi}_1 = -i\hat{\phi}_1$$

$$(c) \quad \left[\hat{Q}_N, \hat{\psi}\right] = \frac{1}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi}_1\right] + \frac{i}{\sqrt{2}}\left[\hat{Q}_N, \hat{\phi}_2\right] = \frac{i}{\sqrt{2}}\hat{\phi}_2 + \frac{1}{\sqrt{2}}\hat{\phi}_1 = \hat{\psi}$$

Problem 4. Note: $D\hat{\theta} = \pm 1$. Substituting:

$$\left[\hat{Q}_N, \hat{\theta}\right] = -iD\hat{\theta} = i$$

$$\left[\int \rho(\mathbf{x}, t)\, \mathrm{d}^3 x, \theta(\mathbf{x}, t)\right] = \int \mathrm{d}^3\mathbf{x}\, [\rho, \theta]$$

Problem 5.

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\Pi^\mu_\psi = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -V(x)\psi^\dagger(x), \;\; \Pi^0_\psi = i\psi^\dagger$$

$$\partial_0\Pi^0_\psi = i\partial_0\psi^\dagger, \;\; \partial_i\Pi^i_\psi = -\frac{1}{2m}\nabla^2\psi^\dagger$$

$$\therefore i\partial_0\psi^\dagger - \frac{1}{2m}\partial_i\partial^i\psi^\dagger - V(x)\psi^\dagger(x) = 0$$

$$\longrightarrow i\partial_0\psi^\dagger = \hat{H}\psi^\dagger, \;\; \hat{H} = -\frac{1}{2m}\nabla^2 + \hat{V}$$

$$V = 0 \longrightarrow i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

$$iT'(t)X(x) = -\frac{1}{2m}X''(x)T(t)$$

$$\frac{T'}{T} = -iE \longrightarrow T(t) = Ae^{-iEt}$$

$$X'' + 2mEX = 0 \longrightarrow X(x) = Be^{ipx} + Ce^{-ipx}, \; p = \sqrt{2mE}$$

$$T(t)X(x) = Ae^{i(px-Et)} + Be^{-i(px-Et)}$$

Problem 6.

$$J^0_N = i\Psi^\dagger(i\Psi) + i\Psi(-i\Psi^\dagger)$$

$$Q_{N_c} = \int \hat{\Psi}\hat{\Psi}^\dagger - \hat{\Psi}^\dagger\hat{\Psi}\;\mathrm{d}^3x$$

$$= \int \mathrm{d}^3x\left[\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\hat{a}_\mathbf{p}e^{-ip\cdot x}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\hat{a}^\dagger_\mathbf{q}e^{iq\cdot x} - \int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}}\hat{a}^\dagger_\mathbf{q}e^{iq\cdot x}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\hat{a}_\mathbf{p}e^{-ip\cdot x}\right]$$

$$\frac{1}{(2\pi)^3}\int\mathrm{d}^3x\left[\int\mathrm{d}^3\mathbf{p}\int\mathrm{d}^3\mathbf{q}\;\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q}e^{i(p-q)\cdot x} - \hat{a}^\dagger_\mathbf{q}\hat{a}_\mathbf{p}e^{-i(p-q)\cdot x}\right]$$

$$= \int\mathrm{d}^3\mathbf{p}\int\mathrm{d}^3\mathbf{q}\;\left[\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q}\delta^3(\mathbf{p}-\mathbf{q}) - \hat{\mathbf{a}}^\dagger_\mathbf{q}\hat{\mathbf{a}}_\mathbf{p}\delta^\mathbf{3}(\mathbf{q}-\mathbf{p})\right]$$

$$= \int\mathrm{d}^3\mathbf{p}\left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{p}\right] = \mathbf{p}$$

Chapter 13

Fields with many components and massive electromagnetism

Problem 1. $\vec{J}$ represents the Levi-Civita tensor as a vector of matrices.

$$(a) \quad \hat{\vec{\mathbf{Q}}}_{N_c} = \int \mathrm{d}^3p \; \hat{\mathbf{A}}^\dagger \vec{J} \hat{\mathbf{A}}$$

The inverse transformations and resultant computations are as follows:

$$(b) \quad \hat{a}_1 = \frac{1}{\sqrt{2}}\left(\hat{b}_{-1} - \hat{b}_1\right),\; \hat{a}_2 = -\frac{i}{\sqrt{2}}\left(\hat{b}_{-1} + \hat{b}_1\right),\; \hat{a}_3 = \hat{b}_0$$

$$\hat{Q}^2_{N_c} =$$

$$\hat{Q}^3_{N_c} = -i \int \mathrm{d}^3p \left(\hat{a}^\dagger_{1\mathbf{p}}\hat{a}_{2\mathbf{p}} - \hat{a}^\dagger_{2\mathbf{p}}\hat{a}_{1\mathbf{p}}\right) = \int \mathrm{d}^3p \left(\hat{b}^\dagger_{1\mathbf{p}}\hat{b}_{1\mathbf{p}} - \hat{b}^\dagger_{-1\mathbf{p}}\hat{b}_{-1\mathbf{p}}\right)$$

$$J^1_{\hat{b}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\; J^2_{\hat{b}} = -\frac{i}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\; J^3_{\hat{b}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 2.

Chapter 14

Propagators and Green's functions

Problem 1.

$$(a) \quad \langle x|\hat{H}|\psi\rangle = E\,\langle x|\psi\rangle$$
$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + E\psi = 0, \ \ V = 0$$
$$\psi_n(x) = Ae^{ikx} + Be^{-ikx}, \ \ k = \sqrt{2mE}/\hbar$$
$$\psi(0) = \psi(a) = 0 \implies B = -A$$
$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

$$(b) \quad E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$
$$G^+(n, t_2, t_1) = \theta(t_2 - t_1)e^{-iE_n(t_2-t_1)}$$

$$(c) \quad G^+(n, \hbar\omega) = \frac{i}{\hbar\omega - E_n + i\epsilon}$$

Problem 2.

$$G_0^+(x,t,y,0) = \theta(t)\langle x(t)|y(t)\rangle$$
$$= \theta(t)\langle x|e^{-i\hat{H}t}|y\rangle$$
$$= \theta(t)\sum_n e^{iE_n t}\langle x|n\rangle\langle n|y\rangle = \theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_n t}$$
$$G_0^+(x,y,E) = \int G_0^+(x,t,y,0)\,\mathrm{d}t = \int_{-\infty}^{\infty}\theta(t)\sum_n \phi_n(x)\phi_n^*(y)e^{-iE_n t}e^{iEt}\,\mathrm{d}t$$

Using a damping factor $e^{-\epsilon t}$ to ensure convergence, switching the order of summation and integration, then integrating by parts ($\theta'(t) = \delta(t)$):

$$G_0^+(x,y,E) = \sum_n \int_{-\infty}^{\infty}\theta(t)\phi_n(x)\phi_n^*(y)e^{i(E-E_n+i\epsilon)t}\,\mathrm{d}t$$
$$= \sum_n \frac{i\phi_n(x)\phi_n^*(y)}{E-E_n+i\epsilon}$$

(b) The integral definition of the Heaviside step function is:

$$\theta(t) = i\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}\frac{e^{-izt}}{z+i\epsilon}$$

Substituting this into the original expression and changing the order of integration:

$$G_0^+(p,t,0) = \theta(t)e^{-iE_p t}$$
$$G_0^+(p,E) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i}{2\pi(z+i\epsilon)}e^{i(E-E_p-z)t}\,\mathrm{d}t\,\mathrm{d}z$$
$$= \int_{-\infty}^{\infty}\frac{i}{(z+i\epsilon)}\delta(E-E_p-z)\,\mathrm{d}z = \frac{i}{E-E_p+i\epsilon}$$

Problem 3. (a) The one-dimensional harmonic oscillator with the corresponding forcing function $f(t)$ has the following solution for the particular integral:

$$m\frac{\partial^2}{\partial t^2}A(t-u)+m\omega_0^2A(t-u)=\tilde{F}(\omega)e^{-i\omega(t-u)}$$

$$\begin{aligned}
A_P(t-u)&=\frac{1}{\left(D^2+\omega_0^2\right)}\frac{\tilde{F}(\omega)}{m}e^{-i\omega(t-u)}=\left(1+\frac{D^2}{\omega_0^2}\right)^{-1}\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)},\quad D=\frac{\mathrm{d}}{\mathrm{d}t}\\
&=\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{i\omega u}\left[\sum_{k=0}^{\infty}\left(\frac{iD}{\omega_0}\right)^{2k}e^{-i\omega t}\right]=\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\sum_{k=0}^{\infty}\left(\frac{\omega}{\omega_0}\right)^{2k}\\
&=\frac{\tilde{F}(\omega)}{m\omega_0^2}e^{-i\omega(t-u)}\left[\frac{1}{1-\omega^2/\omega_0^2}\right]=-\frac{\tilde{F}(\omega)}{m\left(\omega^2-\omega_0^2\right)}e^{-i\omega(t-u)}
\end{aligned}$$

Therefore the solution is:

$$A(t-u)=c_1\cos\omega_0t+c_2\sin\omega_0t-\frac{\tilde{F}(\omega)}{m\left(\omega^2-\omega_0^2\right)}e^{-i\omega(t-u)}$$

(b) The differential equation that satisfies the Green's function is:

$$\left[m\frac{\partial^2}{\partial t^2}+m\omega_0^2\right]G(t,k)=\delta(t-k)$$

Taking the Fourier transform, rearranging and then taking its inverse:

$$\begin{aligned}
-m(\omega^2-\omega_0^2)G(\omega,k)&=\int_{-\infty}^{\infty}\delta(t-k)e^{i\omega t}\,\mathrm{d}t=e^{i\omega k}\\
G(t,k)&=-\frac{1}{m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-k)}}{\omega^2-\omega_0^2}
\end{aligned}$$

Using the previous result to verify the solution:

$$\begin{aligned}
A(t-u)&=-\frac{1}{2\pi m}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)}{\omega^2-\omega_0^2}e^{-i\omega(t-u+k)}\,\mathrm{d}\omega\,\mathrm{d}k\\
&=\frac{1}{2\pi i m}\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)}{\omega(\omega_0^2-\omega^2)}e^{-i\omega(t-u)}\,\mathrm{d}\omega\\
&=\frac{1}{m\omega_0^2}\int_{-\infty}^{\infty}\frac{1}{2\pi i}\left[\frac{1}{\omega}+\frac{1}{\omega_0^2-\omega^2}\right]\tilde{F}(\omega)e^{-i\omega(t-u)}\,\mathrm{d}\omega
\end{aligned}$$

(c) Taking the Laplace transform of the differential equation form of the Green's function:

$$G(s,u) = \frac{e^{us}}{m(s^2+\omega_0^2)}$$

Using convolution to find the inverse:

$$G^+(t,u) = \frac{1}{m\omega_0}\int_0^t \delta(k-u)\sin\omega_0(t-k)\,\mathrm{d}k = \frac{1}{m\omega_0}\sin\omega_0(t-u)$$

(d)

Problem 4. (a) Taking the three-dimensional Fourier transform:

$$\int_{-\infty}^{\infty} (\nabla^2 + \mathbf{k}^2) G_{\mathbf{k}}(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} = 1$$

$$\tilde{G}_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2 - \mathbf{q}^2}$$

(b) The Fourier transform of $G_{\mathbf{k}}^+(\mathbf{x})$ with a damping factor is:

$$\begin{aligned}\tilde{G}_{\mathbf{k}}^+(\mathbf{q}) &= \int_{-\infty}^{\infty} -\frac{e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}}{4\pi|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} = -\frac{1}{2}\int_{-1}^{1}\int_0^{\infty} |\mathbf{x}| e^{-i(|\mathbf{q}|\cos\theta - |\mathbf{k}| - i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}|\,\mathrm{d}(\cos\theta) \\ &= \frac{i}{2|\mathbf{q}|}\int_{-\infty}^{\infty}\left[e^{i|\mathbf{q}||\mathbf{x}|} - e^{-i|\mathbf{q}||\mathbf{x}|}\right] e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}\,\mathrm{d}|\mathbf{x}|\end{aligned}$$

Chapter 15

Propagators and fields

Problem 1.

$$\langle x|0\rangle$$

Chapter 16

Path integrals: I said to him, 'You're crazy'

Problem 1.

$$(a)\quad I(a) = -2\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = -2\sqrt{\frac{2\pi}{a}}$$

$$I'(a) = \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = \sqrt{\frac{2\pi}{a^3}}$$

$$(b) \quad J_n(a) = (-2)^{\frac{n}{2}} \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x = (-2)^{\frac{n}{2}} \sqrt{\frac{2\pi}{a}}$$

$$\frac{\mathrm{d}^k J_n(a)}{\mathrm{d}a^k} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{(-1/2)!}{(-1/2-k)!} a^{-\frac{1}{2}-k}$$

$$\frac{\mathrm{d}^{n/2} J_n(a)}{\mathrm{d}a^{n/2}} = (-2)^{\frac{n+1}{2}} \sqrt{\pi} \frac{\Gamma(1/2)}{\Gamma\left(\frac{1-n}{2}\right)} a^{-\left(\frac{n+1}{2}\right)} = \frac{i^n \pi}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{\frac{n+1}{2}}$$

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x}{\int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) \mathrm{d}x} = \frac{i^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2}{a}\right)^{n/2}$$

$$= \begin{cases} 0 & \forall\ n \in 2\mathbb{Z}^+ + 1 \\ a^{-n/2} \prod_{k=1}^{n/2} (2k-1) & \forall\ n \in 2\mathbb{Z}^+ \end{cases}$$

$$\therefore \frac{\mathrm{d}^n J_{2n}(a)}{\mathrm{d}a^n} = \frac{1}{a^n} \prod_{k=1}^{n} (2k-1)$$