Solutions to
Gravitation: Foundations and Frontiers
by T. Padmanabhan

Arjit Seth

Chapter 1

Special relativity

1. *Light clocks.* Perpendicular:

$$ct' = \sqrt{c^2 - v^2}t$$
$$t = \frac{t'}{\sqrt{1 - v^2/c^2}} = \gamma t'$$

Parallel: Using proper time -

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau$$
$$\frac{L'}{c+v} + \frac{L'}{c-v} = \gamma\frac{2L}{c}$$
$$L'\frac{2c}{c^2 - v^2} = \gamma\frac{2L}{c}$$
$$L' = L\frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma}$$

2. *Superluminal motion.*

$$\Delta t' = t_2' - t_1'$$
$$t_1' - t_1 \approx \frac{v}{c}\Delta t\cos\theta$$
$$t_2' - t_1 = L/c$$
$$\Delta t' = \Delta t(1 - (v/c)\cos\theta)$$
$$v_{app} = v\Delta t\sin\theta = \frac{v\sin\theta}{1 - (v/c)\cos\theta}$$

Rewriting this expression and plotting it for $v = 0.99c$:

$$v_{app} = \frac{v\sqrt{1-\cos^2\theta}}{1 - v\cos\theta}, \quad c = 1$$

3. *The strange world of four-vectors.*

(a) This is evident from taking the inner product, since the magnitudes add up.

$$\left(a^i + b^i\right)\left(a_i + b_i\right) = a^i a_i + b^i b_i + 2\left(a^i b_i\right)$$

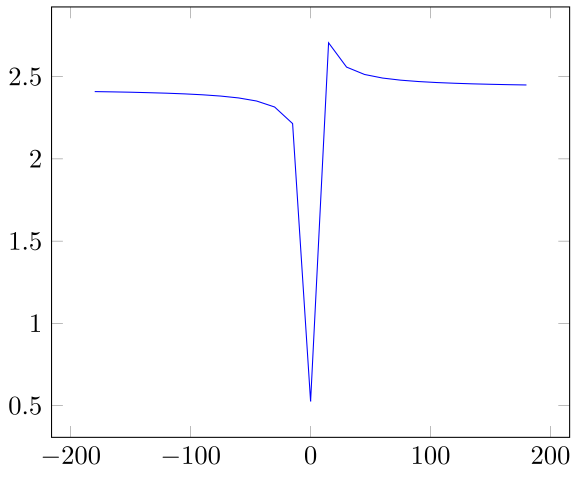
(b) Let k^i be a non-zero null vector. If non-zero a^i is a vector orthogonal to k^i, then $a_i k^i = 0$.

4. *Focused to the front.*

(a) Using a Lorentz transformation on the 'time' and 'space' components of the four-vector $k^a = (\omega, \omega\mathbf{n}/c)$:

$$\begin{aligned}
\omega_R &= \gamma\left[\omega_L - \frac{\omega_L v\cos\theta_L}{c}\right] = \gamma\omega_L(1 - (v/c)\cos\theta_L) \\
\omega_R\cos\theta_R &= \gamma\left[\omega_L\cos\theta_L - \frac{\omega_L v}{c}\right] = \gamma\omega_L[\cos\theta_L - (v/c)] \\
\frac{\omega_R\cos\theta_R}{\omega_R} &= \frac{\gamma\omega_L[\cos\theta_L - (v/c)]}{\gamma\omega_L[1 - (v/c)\cos\theta_L]} \\
\mu_R &= \frac{\mu_L - (v/c)}{1 - (v\mu_L/c)}
\end{aligned}$$

(b)



(c) The solid angle is found by taking the differential of μ_R:

$$\begin{aligned}
\mathrm{d}\Omega &= \int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = -\int \mathrm{d}(\cos\theta)\,\mathrm{d}\phi \\
\mathrm{d}(\cos\theta_R) &= \left[\frac{(v/c)[\cos\theta_L - (v/c)]}{[1-(v/c)\cos\theta_L]^2} + \frac{1}{[1-(v/c)\cos\theta_L]}\right]\mathrm{d}(\cos\theta_L) \\
\mathrm{d}\Omega' &= \frac{1}{\gamma^2}\frac{\mathrm{d}\Omega}{[1-(v/c)\cos\theta]^2}
\end{aligned}$$

The energy is:

$$\mathrm{d}\mathcal{E}' = \hbar\omega = \gamma\,\mathrm{d}\mathcal{E}\,[1-(v/c)\cos\theta]$$

The time is:

$$\mathrm{d}t' = \gamma\,\mathrm{d}t$$

Therefore, the energy emitted per unit time into a given solid angle in the rest frame is $(c=1)$:

$$\begin{aligned}
\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\mathrm{rest}} &= \gamma^2(1-v\cos\theta)^3\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\mathrm{lab}} \\
\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t\,\mathrm{d}\Omega}\right)_{\mathrm{lab}} &= \frac{\left(1-v^2\right)^2}{(1-v\cos\theta)^3}\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\mathrm{rest}}
\end{aligned}$$

If the emission is isotropic, $\mathrm{d}\Omega = \mathrm{d}\Omega' = 4\pi$:

$$\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\right)_{\mathrm{lab}} = \mathrm{d}\Omega\left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'\,\mathrm{d}\Omega'}\right)_{\mathrm{rest}} = \left(\frac{\mathrm{d}\mathcal{E}'}{\mathrm{d}t'}\right)_{\mathrm{rest}}$$

5. *Transformation of antisymmetric tensors.*

$$A^{i'k'} = L_k^{k'} L_i^{i'} A^{ik} = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial x^{i'}}{\partial x^i} A^{ik}$$

If $A^{ik} = -A^{ki}$ then:

$$A^{k'i'} = -A^{i'k'}?$$

6. *Practice with completely antisymmetric tensors.*

(a)

(b) Multiplying each index by the metric tensor:

$$\epsilon^{abcd} = g^{ai} g^{bj} g^{ck} g^{dl} \epsilon_{ijkl} =$$

(c)

(d)

7. *A null curve in flat spacetime.*

$$\eta_{ij} x^i x^j = -t^2 + x^2 + y^2 + z^2 = 0$$

8. *Shadows are Lorentz invariant.*

9. *Hamiltonian form of action - Newtonian mechanics.*

$$\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t \; [p\dot{q} - H(p,q)]$$
$$\delta\mathcal{A} = \int_{t_2}^{t_1} \mathrm{d}t \; \left[\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right] = 0$$

Using $\delta\dot{q} = \dot{\mathrm{d}}(\delta q)$ and integrating by parts:

$$\int_{t_2}^{t_1} \mathrm{d}t \; p\,\delta\dot{q} = \cancel{|p\,\delta q|_{t_1}^{t_2}} - \int_{t_2}^{t_1} \mathrm{d}t \; \dot{p}\,\delta q$$
$$\therefore \int_{t_2}^{t_1} \mathrm{d}t \; \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right] = 0$$

Since $\delta q = 0$ at the endpoints, the second term is zero; therefore, the first term in parentheses should be zero, leading to the equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

10. *Hamiltonian form of action - special relativity.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \; \left[p_a\dot{x}^a - \frac{1}{2}C\left(\frac{H}{mc^2} + mc^2\right)\right], \quad H = \eta_{ab}p^a p^b$$
$$\delta\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \delta p_a\dot{x}^a + p_a\delta\dot{x}^a - \frac{1}{2}\left[\delta C\left(\frac{H}{m} + m\right) + C\left(\frac{2p^a\delta p_a}{mc^2}\right)\right] \mathrm{d}\lambda = 0$$
$$= \int_{\lambda_1}^{\lambda_2} \left[\dot{x}^a - \frac{C}{m}p^a\right]\delta p_a - \dot{p}_a\delta x^a - \frac{1}{2}\left[\frac{H + m^2}{m}\right]\delta C \,\mathrm{d}\lambda = 0, \quad c = 1$$

11. *Hitting a mirror.*

12. *Photon-electron scattering.*

13. *More practice with collisions.*

14. *Relativistic rocket.*

15. *Practice with equilibrium distribution functions.*

16. *Projection effects.*

17. *Relativistic virial theorem.* The conservation law implies the following:

$$\begin{aligned}\partial_0 T^{00} x^\alpha x^\beta &= -\partial_\mu T^{0\mu} x^\alpha x^\beta \\ &= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta\right] + T^{0\mu}\partial_\mu \left[x^\alpha x^\beta\right] \\ &= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta\right] + T^{0\mu}\left[\partial_\mu x^\alpha x^\beta + x^\alpha \partial_\mu x^\beta\right] \\ &= -\partial_\mu \left[T^{0\mu} x^\alpha x^\beta\right] + T^{0\alpha} x^\beta + T^{0\beta} x^\alpha\end{aligned}$$

Taking the time derivative and using the fact that partial derivatives commute:

$$\begin{aligned}\partial_0 \left(\partial_0 T^{00}\right) x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu}\right] x^\alpha x^\beta \\ \partial_0^2 T^{00} x^\alpha x^\beta &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta\right] + \partial_0 T^{0\alpha} x^\beta + \partial_0 T^{0\beta} x^\alpha \\ &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta\right] - \partial_\nu T^{\nu\alpha} x^\beta - \partial_\nu T^{\nu\beta} x^\alpha, \quad \left(\partial_i T^{i\alpha} = 0\right) \\ &= -\partial_\mu \left[\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha\right] + 2T^{\alpha\beta}, \quad \left(T^{\mu\alpha}\partial_\mu x^\beta = T^{\alpha\beta}\right)\end{aligned}$$

Integrating both sides of the equation and using the divergence theorem for the condition that $T^{ij} = 0$ outside a compact region in space:

$$\begin{aligned}\int \mathrm{d}^3x\ \partial_0^2 T^{00} x^\alpha x^\beta &= \int \mathrm{d}^3x \left[-\partial_\mu \left(\partial_0 T^{0\mu} x^\alpha x^\beta + T^{\mu\alpha} x^\beta + T^{\mu\beta} x^\alpha\right) + 2T^{\alpha\beta}\right] \\ \therefore \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int \mathrm{d}^3x\ T^{00} x^\alpha x^\beta &= 2\int \mathrm{d}^3x\ T^{\alpha\beta}\end{aligned}$$

18. *Explicit computation of spin precession.* The four-velocity and four-acceleration are:

$$\begin{aligned}u^i &= (\gamma, \gamma\mathbf{v}) = (\gamma, -\gamma r\omega \sin\omega t, \gamma r\omega \cos\omega t, 0) \\ a^i &= \gamma(\dot\gamma, \dot\gamma\mathbf{v} + \dot{\mathbf{v}}\gamma) = -\gamma^2\omega^2(0, x, y, 0)\end{aligned}$$

Using the equation of motion for a moving particle with spin and separating the space and time components:

$$\begin{aligned}
\frac{\mathrm{d}S^j}{\mathrm{d}\tau} &= u^j \left(S^k a_k\right) \\
S^k a_k &= -\gamma^2 \omega^2 (x S^x + y S^y) \\
\frac{\mathrm{d}}{\mathrm{d}t}[(y S^x - x S^y)] &= -\omega (x S^x + y S^y)
\end{aligned}$$

19. *Little group of the Lorentz group.*

Chapter 2

Scalar and electromagnetic fields in special relativity

1. *Measuring the F^{ab}.*

2. *Schrödinger equation and gauge transformation.* The transformed Schrödinger equation is:

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}\left[\mathbf{P} - \frac{q}{c}(\mathbf{A} + \nabla f)\right]^2 + q(\phi + \partial_t f)\right]\psi$$

3. *Four-vectors leading to electric and magnetic fields.*

(a) Taking dot products:

$$E^i u_i = F^{ij} u_j u_i = 0$$
$$B^a u_a = \frac{1}{2}\epsilon^{abcd} F_{cd} u_a u_b = \frac{1}{2}(*F)^{ab} u_a u_b = 0$$

since $u_i u_j$ is symmetric and F^{ij} and its dual are antisymmetric.

(b)

4. *Hamiltonian form of action - charged particle.*

$$\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \left[P_a \dot{x}^a - \frac{1}{2}C\left(\frac{H}{mc^2} + mc^2\right)\right], \quad H = \eta_{ij}\left(P^i - qA^i\right)\left(P^j - qA^j\right)$$
$$\delta\mathcal{A} = \int_{\lambda_1}^{\lambda_2} \delta P_a \dot{x}^a + P_a \delta\dot{x}^a - \frac{1}{2}\left[\delta C\left(\frac{H}{m} + m\right) + C\left(\frac{\left(P^i - qA^i\right)\left(\delta P_i - q\delta A_j\right)}{mc^2}\right)\right] \mathrm{d}\lambda = 0$$

5. *Three-dimensional form of the Lorentz force.* Using the electromagnetic tensor equation of motion $u^i = (\gamma c, \gamma \mathbf{v})$:

$$m\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = qF^{ik}u_k \implies mc\gamma\frac{\mathrm{d}u^0}{\mathrm{d}t} = qF^{0\alpha}u_\alpha, \quad \frac{\mathrm{d}u^0}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}u^0}{\mathrm{d}t}$$
$$mc\gamma\frac{\mathrm{d}\gamma}{\mathrm{d}t} = q\gamma[(\mathbf{E}/c)\cdot\mathbf{v}] \implies \frac{1}{\gamma^3}\left(m\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = q(\mathbf{E}\cdot\mathbf{v}), \quad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{\gamma^3}\left(\frac{\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)$$
$$m\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} = qF^{\alpha\beta}u_\beta = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$
$$\left[\frac{1}{\gamma^3}\left(\frac{m\mathbf{v}}{c^2}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\mathbf{v} + m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right] = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$
$$\implies \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m\gamma}\left[\mathbf{E}+\mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{v})\mathbf{v}\right]$$

6. *Pure gauge impostors.* This can be transformed into polar coordinates ($x = \cos\theta, y = \sin\theta$) and evaluated ($A_r = 0$, obviously):

$$A_\theta = \frac{\partial f}{\partial\theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta} = 1,$$
$$\oint_{C=x^2+y^2} \mathbf{A}\cdot\mathrm{d}\mathbf{s} = \int_0^{2\pi}\mathrm{d}\theta = 2\pi$$

The reason why $\mathbf{A}$ is not a pure gauge mode is because f is singular at the origin, making it non-differentiable at that point. It is also a non-removable singularity, so no analytic continuation can be performed.

7. *Pure electric or magnetic fields.*

8. *Elegant solution to non-relativistic Coulomb motion.*

(a) Since the angular momentum is conserved, $\mathrm{d}\mathbf{J}\,/\,\mathrm{d}t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J}) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\times\mathbf{J} + \cancel{\mathbf{p}\times\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}}$$
$$f(r)\hat{\mathbf{r}}\times\mathbf{r}\times m\mathbf{v} = mf(r)[\mathbf{r}\cancel{(\hat{\mathbf{r}}\cdot\mathbf{v})} - \mathbf{v}(\hat{\mathbf{r}}\cdot\mathbf{r})]$$
$$= -mf(r)r\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -mf(r)r^2\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}$$

Therefore, if $f(r)r^2 = -|\alpha|$, then:

$$\int \frac{1}{m|\alpha|}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\times\mathbf{J})\,\mathrm{d}t = \int \frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t}\,\mathrm{d}t$$
$$\implies \frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}} = \mathbf{e}$$

where $\mathbf{e}$ is a conserved vector, arising as a constant of the integration.

(b) Note that $\mathbf{p}$ and $\mathbf{J}$ are perpendicular:

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{e} &= \left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right]\cdot\left[\frac{\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \hat{\mathbf{r}}\right] \\
\implies |\mathbf{e}|^2 &= \frac{|\mathbf{p}\times\mathbf{J}|^2}{m^2|\alpha|^2} - 2\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|}\right) + 1 \\
&= \frac{p^2J^2}{m^2|\alpha|^2} - 2\left(\frac{J^2}{m|\alpha|r}\right) + 1, \quad J^2 = (\mathbf{r}\times\mathbf{p})\cdot\mathbf{J} \\
&= 1 + \frac{2EJ^2}{m|\alpha|^2}, \quad E = \frac{p^2}{2m} - \frac{|\alpha|}{r}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{e}\cdot\mathbf{r} &= \frac{\mathbf{r}\cdot\mathbf{p}\times\mathbf{J}}{m|\alpha|} - \mathbf{r}\cdot\hat{\mathbf{r}} \\
er\cos\theta &= \frac{J^2}{m|\alpha|} - r \\
\implies r(\theta) &= \frac{J^2/m|\alpha|}{1+e\cos\theta}
\end{aligned}$$

(d)

$$E = \frac{p^2}{2m} - \frac{|\alpha|}{r} - \frac{|\beta|}{r^2}$$

9. *More on uniformly accelerated motion.*

Chapter 3

Gravity and spacetime geometry: the inescapable connection

Chapter 4

Metric tensor, geodesics and covariant derivative

1. *Practice with metrics.*

(a) $x = 2\tan(\theta/2)\cos\phi, y = 2\tan(\theta/2)\sin\phi$. Since the coordinates are spherical:

$$\begin{aligned}
g_{ab} &= \eta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}, \quad X^i = \theta,\ \phi,\ x^a = x,\ y,\ \eta_{ij} = \begin{bmatrix}1 & 0\\ 0 & \sin^2\theta\end{bmatrix}\\
\frac{\partial\theta}{\partial x^a} &= \frac{\partial}{\partial x^a}\left[2\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{2}\right)\right], \quad \frac{\partial\phi}{\partial x^a} = \frac{\partial}{\partial x^a}\left[\tan^{-1}(y/x)\right]\\
\frac{\partial\theta}{\partial x^a} &= \frac{4x^a}{\sqrt{x^2+y^2}(x^2+y^2+4)} = \cos^2(\theta/2)\cos\phi,\ \cos^2(\theta/2)\sin\phi\\
\frac{\partial\phi}{\partial x} &= -\frac{y}{x^2+y^2} = -\frac{\sin\phi}{2\tan(\theta/2)}, \quad \frac{\partial\phi}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{2\tan(\theta/2)}\\
g_{xx} &= \left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial x}\right)^2\\
&= \left[\cos^2(\theta/2)\cos^2\phi + \frac{\sin^2\theta\sin^2\phi}{4\sin^2(\theta/2)}\right]\cos^2(\theta/2) = \cos^4(\theta/2)\\
g_{yy} &= \left(\frac{\partial\theta}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial y}\right)^2\\
&= \left[\cos^2(\theta/2)\sin^2\phi + \frac{\sin^2\theta\cos^2\phi}{4\sin^2\theta/2}\right]\cos^2(\theta/2) = \cos^4(\theta/2)\\
\therefore \mathrm{d}s^2 &= \cos^4\left(\frac{\theta}{2}\right)\left[\mathrm{d}x^2 + \mathrm{d}y^2\right]
\end{aligned}$$