Solutions to
Quantum Field Theory for the Gifted Amateur
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Chapter 1

From Particles to Fields

1. WRONG

$$G(x_2;x_1) = \int d^D\mathbf{y} \int d^D\mathbf{p} \int d^D\mathbf{p} \frac{\theta(x_2^0 - y^0)\theta(y^0 - x_1^0)}{(2\pi)^{2D}} F(|\mathbf{p}|, x_2^0 - y^0) F(|\mathbf{p}|, y^0 - x_1^0) e^{i\mathbf{p}\cdot(\mathbf{x}_2 - \mathbf{y} + \mathbf{y} - \mathbf{x}_1)}$$
$$= \int d^D\mathbf{y} \int d^D\mathbf{p} \frac{\theta(x_2^0 - x_1^0)}{(2\pi)^D} F(|\mathbf{p}|, x_2^0 - x_1^0) e^{i\mathbf{p}\cdot(\mathbf{x}_2 - \mathbf{x}_1)}$$

2. Substituting into Eq. 1.7:

$$\int_{-\infty}^{\infty} d^D\mathbf{x}\, N(t) \exp\left(\frac{im|\mathbf{x}|^2}{2t} - i\mathbf{p}\cdot\mathbf{x}\right) = \theta(t) e^{-if(|\mathbf{p}|)t}$$

This is a product of D Gaussian integrals, with the evaluation of each nth component in Cartesian coordinates:

$$\int dx_n \exp\left(\frac{imx_n^2}{2t} - ip_n x_n\right) = \sqrt{\frac{2\pi it}{m}} \exp\left(-\frac{ip_n^2 t}{2m}\right)$$

Which is the propagator in momentum space, and gives the final result:

$$N(t) = \theta(t)\left(\frac{m}{2\pi it}\right)^{D/2} e^{i\psi t}, \quad \psi = \left(\frac{|\mathbf{p}|^2}{2m} - f(|\mathbf{p}|)\right)t$$

3. Unsatisfactorily, substitute the solution into $i\partial_t \hat{A} = [\hat{A}, \hat{H}]$.

$$i\partial_t\left[e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}\right] = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}\hat{H} - \hat{H}e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$$
$$-\hat{H}e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}\hat{H} = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}\hat{H} - \hat{H}e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$$

This works because the operator commutes with its exponential, easily verified by the Taylor series representation.

4. The ground state wavefunction of the quantum harmonic oscillator is:

$$\phi_0(\mathbf{x}) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left(-\frac{m\omega|\mathbf{x}|^2}{2}\right)$$

The action for the harmonic oscillator in Euclidean time with the boundary conditions $(\mathbf{0}, 0)$, $(\mathbf{x}, -it)$ is

$$S = \frac{m\omega|\mathbf{x}|^2 \cos -i\omega t}{2 \sin -i\omega t} = -\frac{m\omega|\mathbf{x}|^2 \cosh \omega t}{2 \sinh \omega t}$$
$$\phi_0(\mathbf{x}) \propto \int \mathcal{D}q \lim_{t_E \to \infty} \exp\left[-\frac{m\omega|\mathbf{x}|^2 \cosh \omega t}{2 \sinh \omega t}\right]$$

The limit is evaluated via L'Hôpital's rule, giving $\phi_0(\mathbf{x}) \propto \exp\left(m\omega|\mathbf{x}|^2/2\right)$.

5. The propagator for the harmonic oscillator of frequency ω_0 coupled to an external source is:

$$G^J(x_2; x_1) = A[\mathbf{x}_{cl}] \exp(iS), \quad S = \int dt \, \frac{m}{2} \left[\dot{x}^2 - \omega^2 x^2 + J(t)x\right]$$

The equation of motion is:

$$\ddot{x} + \omega^2 x = \frac{J(t)}{m}$$

Solving this with boundary conditions (x_i, t_i), (x_f, t_f):

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} + \int_{t_i}^{t_f} dt' \, G(t,t') \frac{J(t')}{m}, \quad G(t,t') = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - \omega_0^2}$$

where $G(t,t')$ is the Green function, evaluated as follows:

$$G(t,t') =$$