2-3 Trees

CMSC 420

Problems with AVL Trees

Have to store height, compare subtree heights for balance

Rotations are expensive, especially RL and LR

Can we do better, while maintaining $\mathcal{O}(\log n)$ search?

Yes! 2-3 Trees, B-Trees, RB-Trees

We'll start with 2-3 Trees, which will lead into the others

Properties of 2-3 Trees

Perfectly Balanced

- All nodes have either 0 children or the maximum they support
- All leaf nodes are at the same *depth*
- That is, $\forall n \in T, B(n) = 0$ (for a suitable definition of B)

How Do We Achieve Perfect Balance?

This isn't possible for a BST, unless the number of nodes is $2^a - 1$

So what do we do?

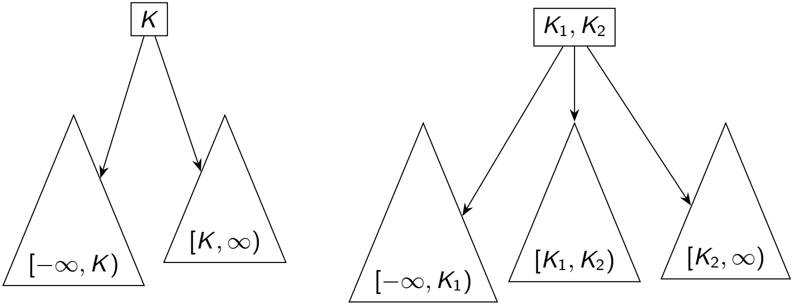
Define 2-nodes and 3-nodes

- 2-nodes have 2 children
- 3-nodes have 3 children

Node Types

A 2-node is a BST node:

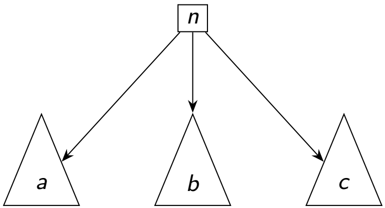
A 3-node has 2 keys:

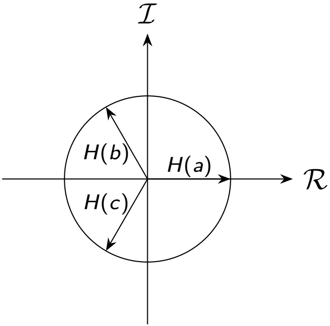


How do we define balance for a 3-node?

A Working Definition of Balance

You are only responsible for knowing the last line

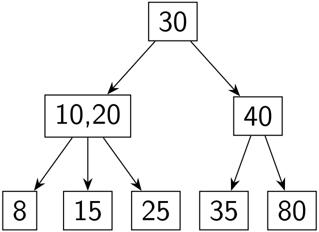




$$B(n) = H(a) + \frac{(-1+i\sqrt{3})}{2}H(b) + \frac{(-1-i\sqrt{3})}{2}H(c)$$

For our purposes, the invariant implies $H(a) = H(b) = H(c)$

An Example of a 2-3 Tree



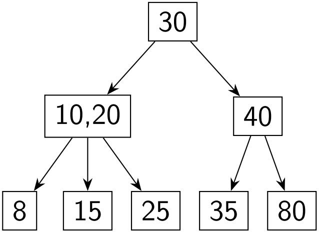
10,20 is a 3-node

The rest are 2-nodes

Search in a 2-3 Tree

This works (almost) identically to a BST

Let's say we're searching for 15



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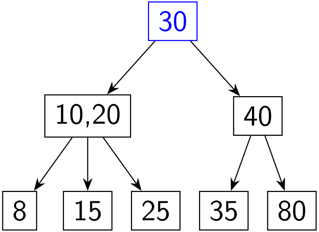

Start at the root

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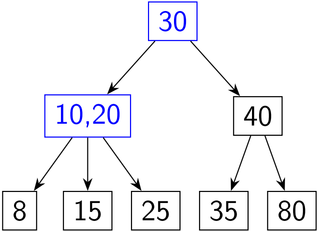

Start at the root

$15 < 30$

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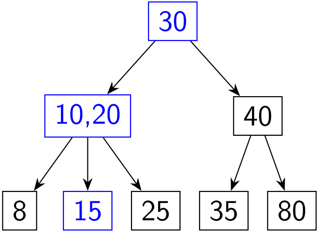

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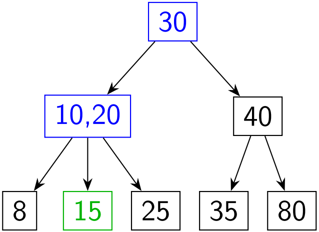

Start at the root

$15 < 30$

$10 \leq 15 < 20$

$15 = 15$

Insertion into a 2-3 Tree

Start with an empty tree

Insertion into a 2-3 Tree

Start with an empty tree

Insert an element
$\Rightarrow$ We have a 2-node

$\boxed{3}$

Insertion into a 2-3 Tree

Start with an empty tree

Insert an element
⇒ We have a 2-node

Insert another element
⇒ It expands to a 3-node

3,20

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Start with an empty tree

Insert an element
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Insert another element
$\Rightarrow$ It expands to a 3-node

Insert a third element
$\Rightarrow$ A 4-node isn't allowed!

$\boxed{3,12,20}$

Insertion into a 2-3 Tree

Start with an empty tree

Insert an element
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Insert another element
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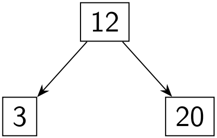
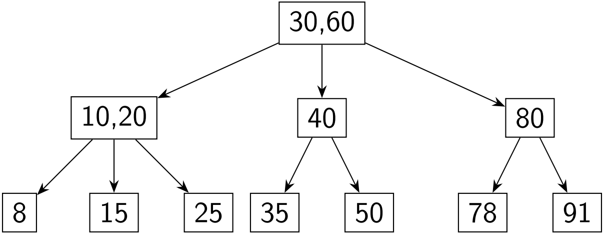
Insert a third element
⇒ A 4-node isn't allowed!

Split the node,
the middle element becomes a 2-node



How Does a 2-3 Tree Grow?

Let's suppose it were a fancy BST:



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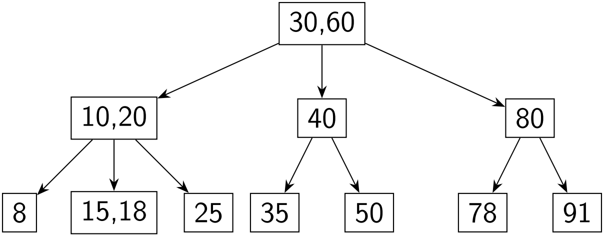

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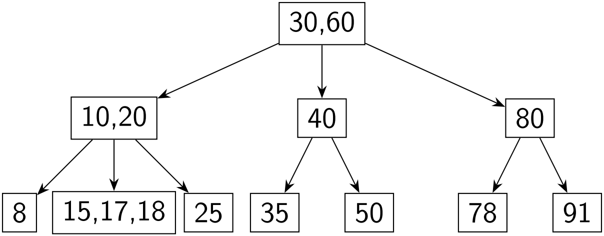

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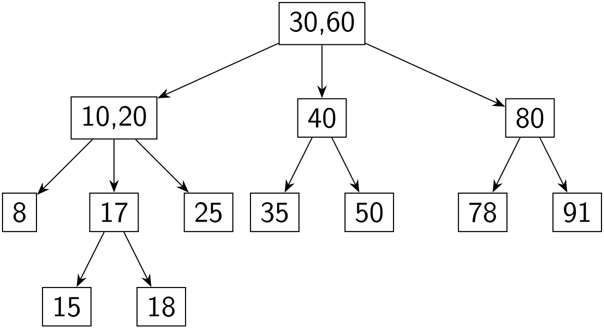

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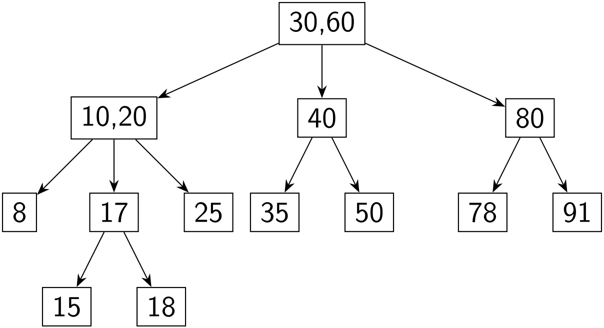
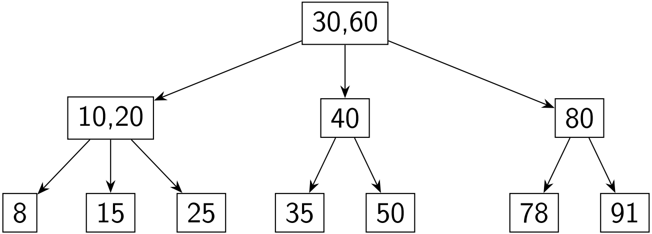


This violates our invariant!

A Key Distinction

BSTs grow *downwards*, but 2-3 trees grow *upwards*!



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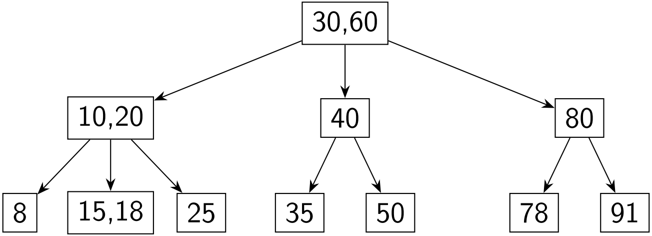

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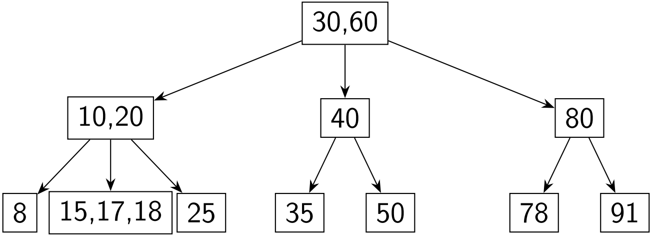

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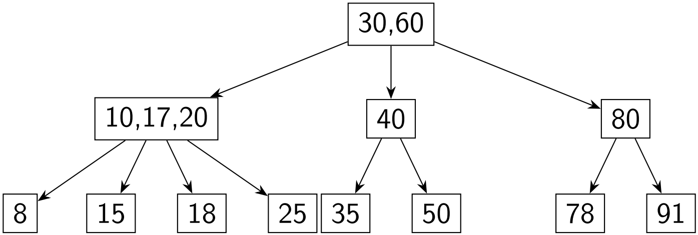

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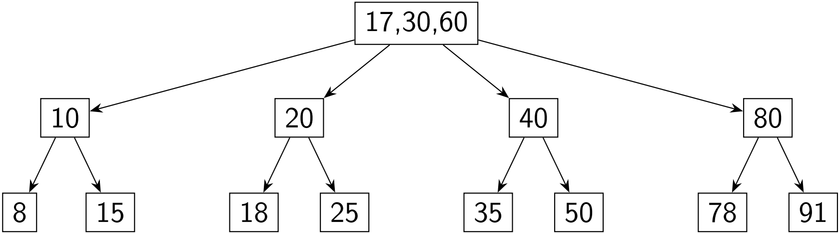

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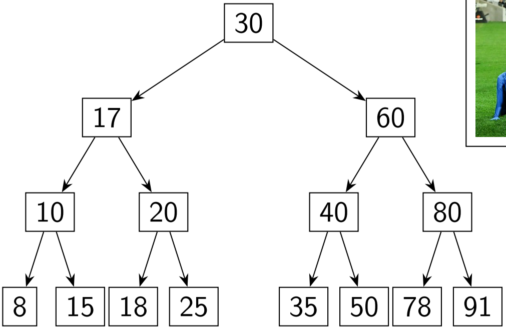


Insertion Abstracted

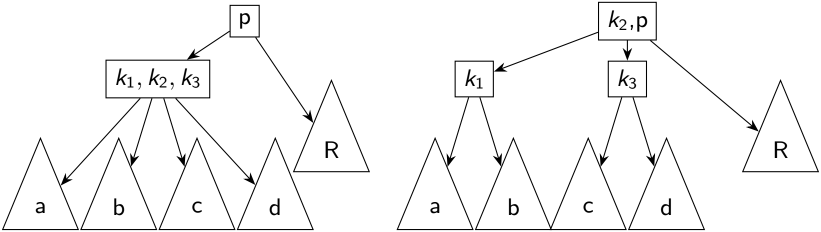
1. Search to find the appropriate leaf for this element
2. Is it a 2-node?

   yes Add the new element here, making it a 3-node, and terminate

   no Continue to the next step
3. Temporarily create a 4-node with three keys: $k_1 < k_2 < k_3$
4. Is this the root?

   yes Create a new root 2-node with k_2 and children k_1 (with children a and b) and k_3 (with children c and d); terminate

   no Create k_1 and k_3 as above; add k_2 to p; go to step 2



Keeping Trees Shorter

The previous technique works

Trees stay perfectly balanced

Tends to make more 2-nodes, which means taller trees
$\Rightarrow$ More steps to reach a leaf

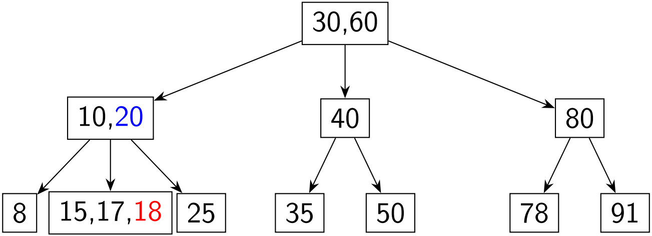
More 2-nodes $\Rightarrow \mathcal{O}(\log_2 n)$
More 3-nodes $\Rightarrow \mathcal{O}(\log_3 n)$

Note: Technically $\mathcal{O}(\log_2 n) = \mathcal{O}(\log_3 n)$

How do we avoid creating 2-nodes?

Key Rotation

Core idea:
We might have siblings who are 2-nodes, and can expand



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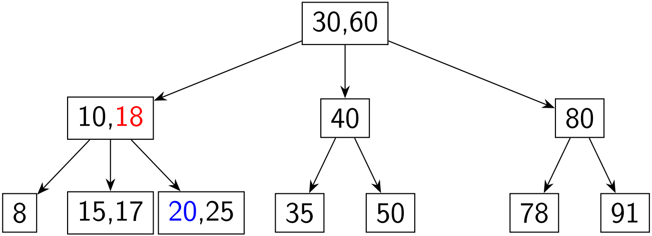
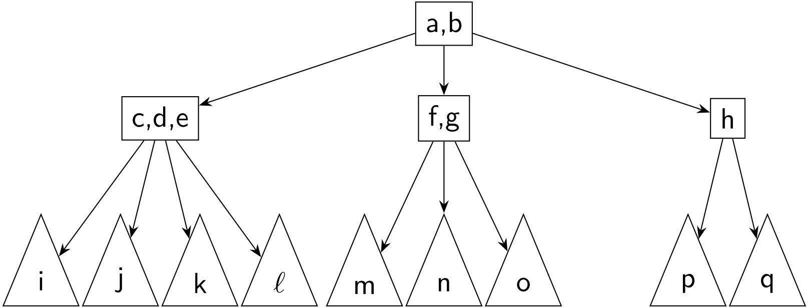


Key Rotation Abstracted

First, assign each node an *age*, increasing with keys
$\Rightarrow$ 15,17 is older than 8 and younger than 20,25

Prefer to rotate towards older siblings, starting with closest in age
$\Rightarrow$ Try younger siblings (closest first) if no available older ones

If we have a 2-node sibling:



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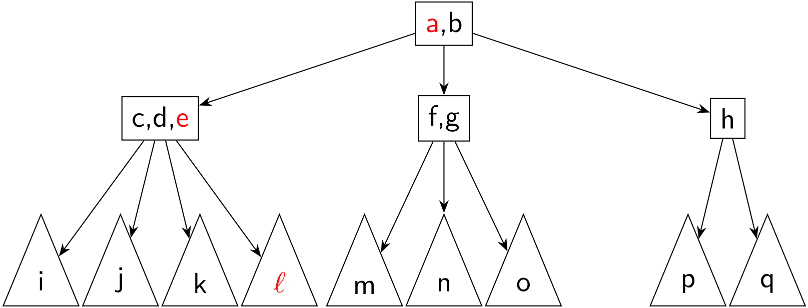

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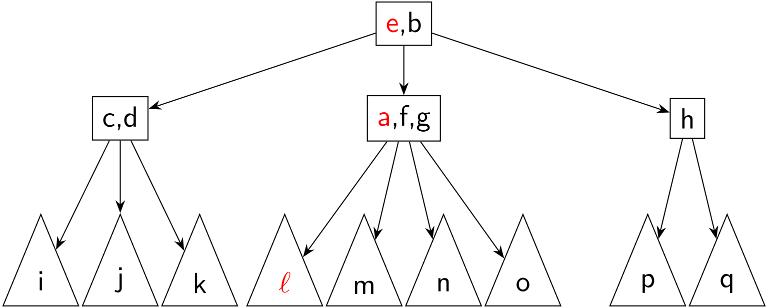

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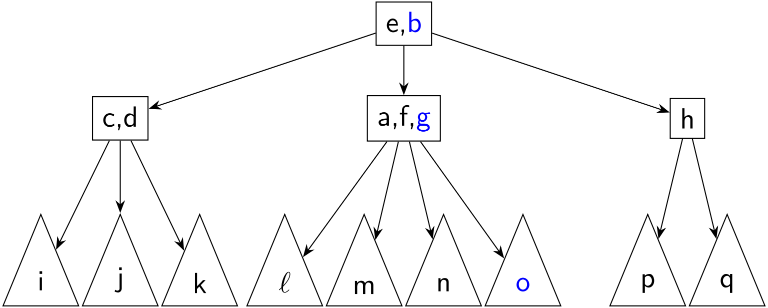

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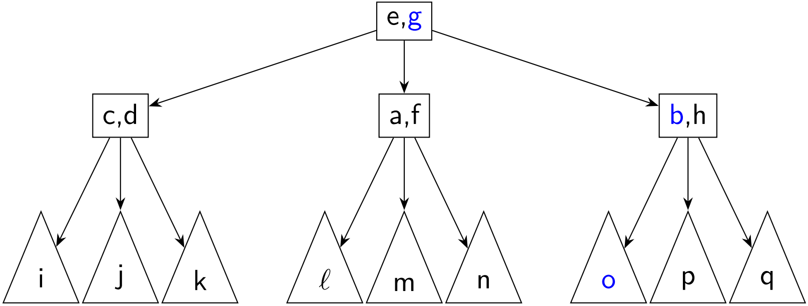


Deletion from a 2-3 Tree

We'll start with leaves

Inner node deletions will become leaf deletions

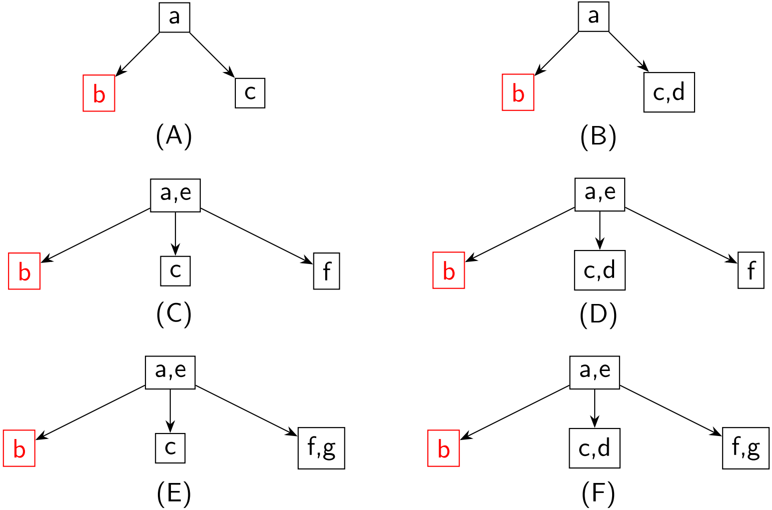
Deleting a key from a leaf 3-node is easy

$$\boxed{k_1, k_2} \Longrightarrow \boxed{k_1}$$

Deleting from a 2-node will be more complicated

Deletion from a 2-3 Tree

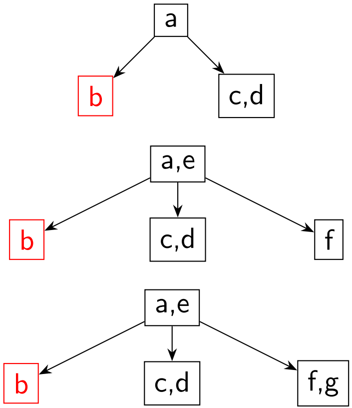
It's only interesting if this isn't the only element in the tree, so



Deletion from a 2-3 Tree

Cases B, D, and F

We can rotate keys to the left



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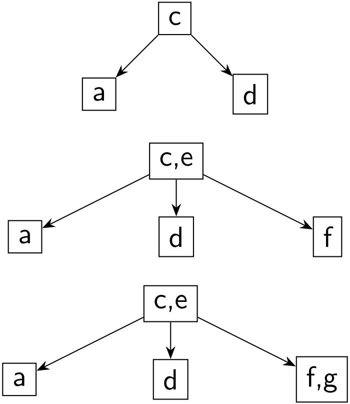
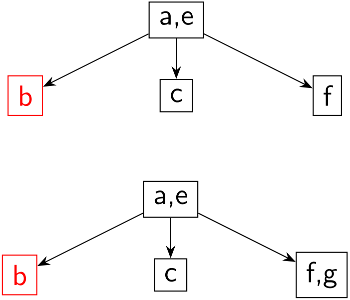


Deletion from a 2-3 Tree

Cases C and E

We can merge from the parent



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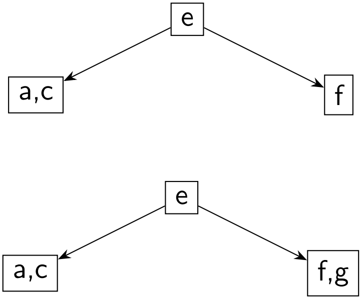
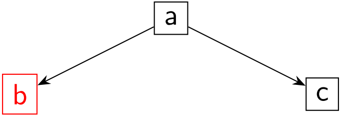


Deletion from a 2-3 Tree

Case A

There's nothing we can rotate in this case



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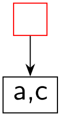
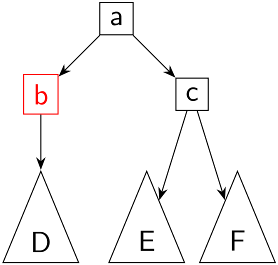


Empty node now has to be deleted, propagating upwards!

Propagating Deleted Nodes

We only need to consider three cases, corresponding to cases A, B, and C

Case A:



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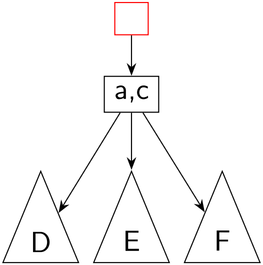

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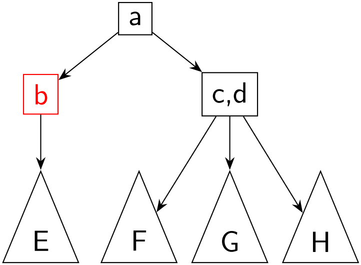

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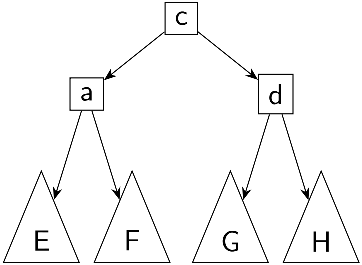

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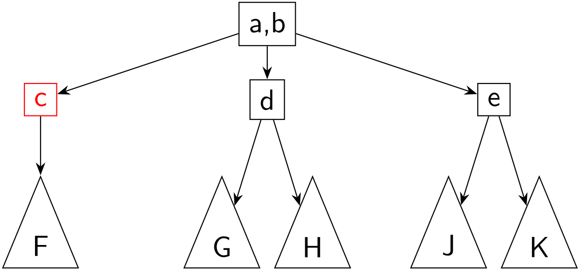

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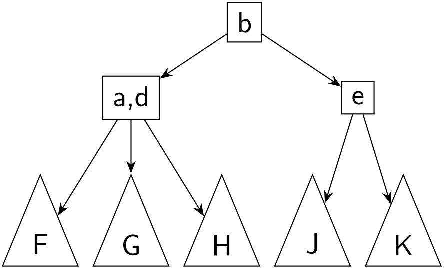
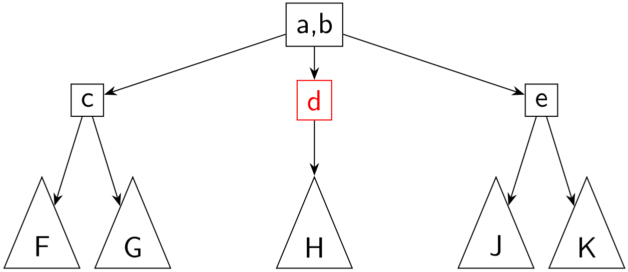


Propagating Deleted Nodes

There's one additional wrinkle:



Do we merge *a* and *c* or *b* and *e*?

Propagating Deleted Nodes

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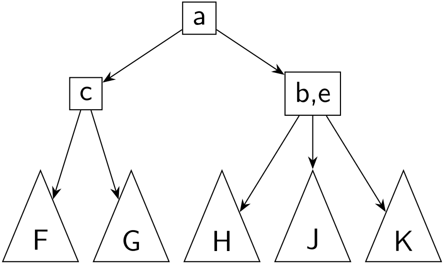


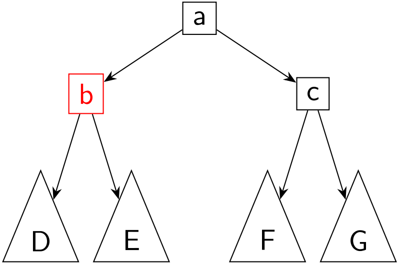
Do we merge *a* and *c* or *b* and *e*?

Always merge **right** when given the option!

$\Rightarrow$ This will be important when we cover B-Trees

Deleting Interior Keys

We haven't looked at any deletions like:



Why not?

In-Order Successors!

Like with BSTs, 2-3 Trees replace a removed interior item with its in-order successor

If the successor is still in an interior node, we continue with its successor

This will ultimately result in reaching a leaf node

Deletion is Expensive

Deleting a single item can cause a cascade of deletions

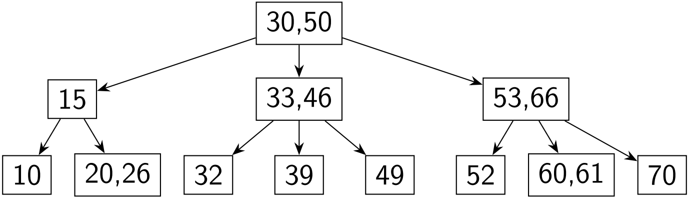
This might go all the way back to the root!

We search a *lot* more than we delete items, so efficient search is more valuable than efficient deletion

Many implementations use *Mark-and-Sweep*, both for 2-3 trees and AVL trees

- Delete less often
- May be able to combine deletions for efficiency gains

Deletion Example



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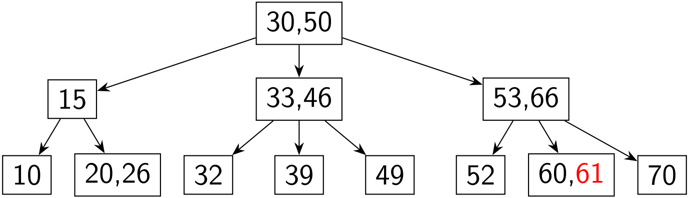

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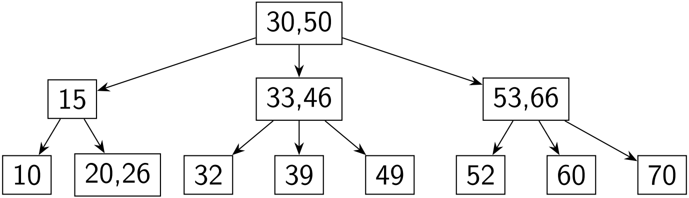

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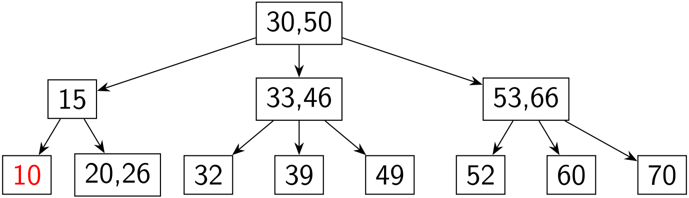

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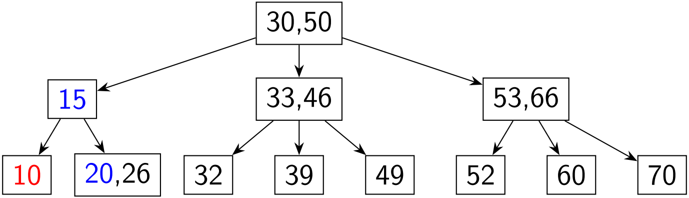

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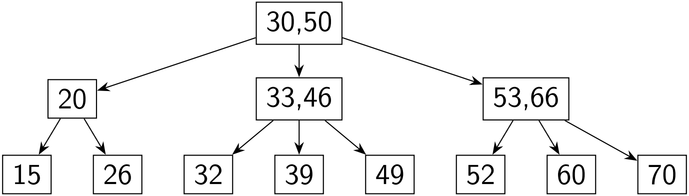
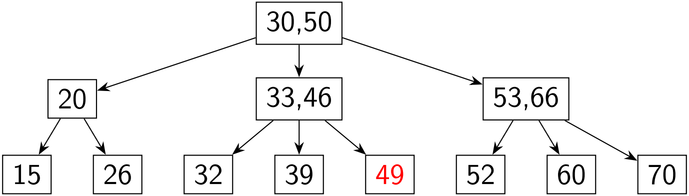
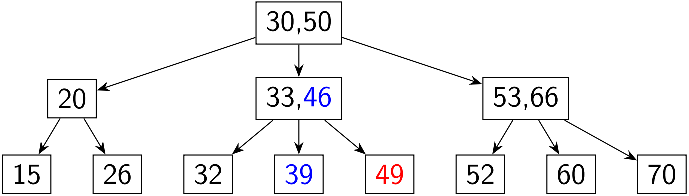


Deletion Example



Deletion Example



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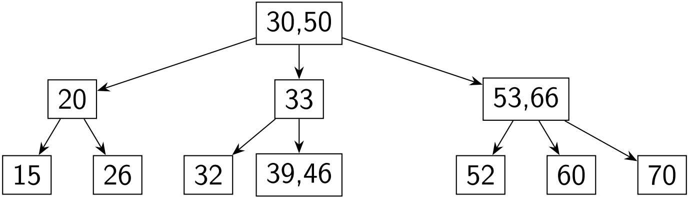

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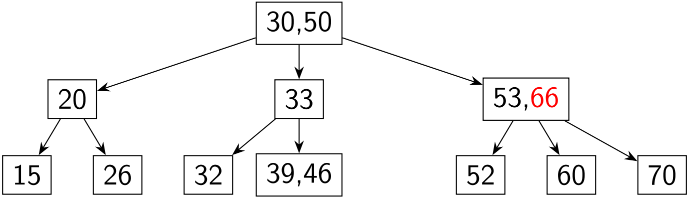

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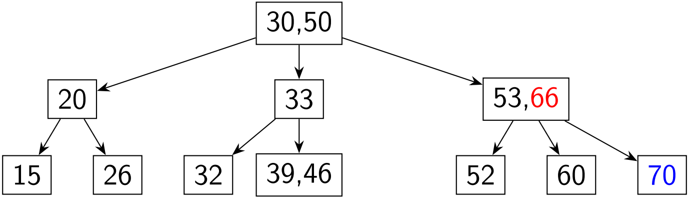

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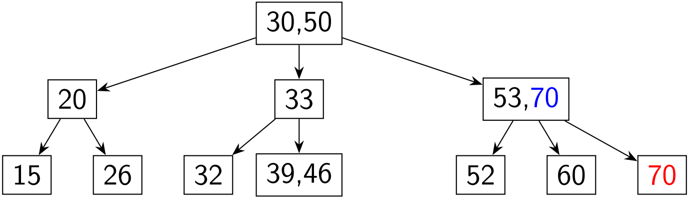

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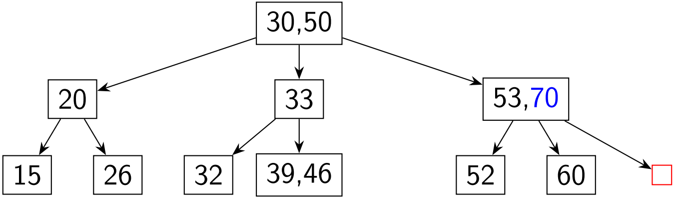

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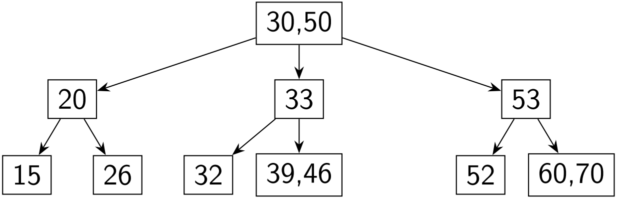

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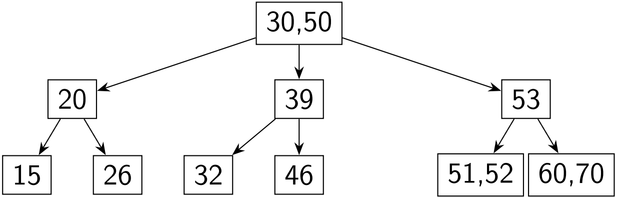

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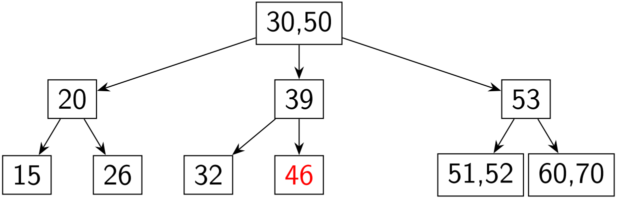

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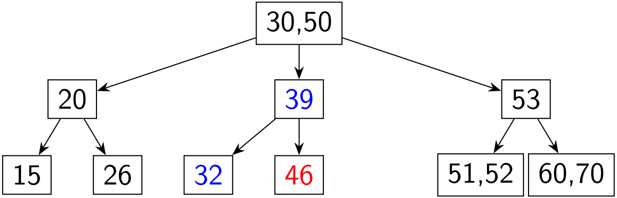

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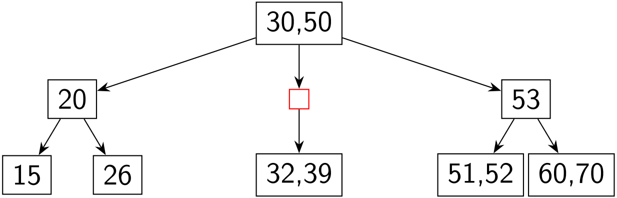

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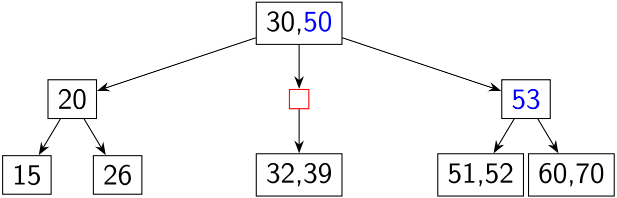

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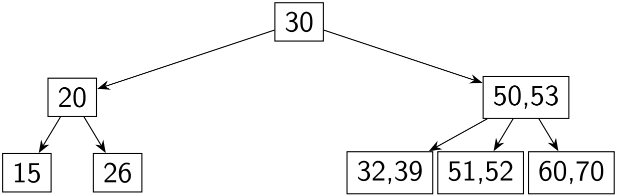

Implementing 2-3 Trees as Binary Trees

We can implement a 2-3 tree as a binary tree!

This is called a Red-Black Binary Search Tree

- Also known as an RBBST
- Also known as an RB-Tree

This will be our next topic