MathFinance 345/Stat390
Homework 98
Due December 5

Problem 1: Show that if M_t is a positive martingale and $f(t)$ is a continuous, nonrandom function of t, then $f(t)M_t$ is a martingale if and only if $f(t)$ is constant .

Problem 2: Prove Proposition 2 in the Lecture 9 notes.

Problem 3: Time-varying short rates and volatility. Let S_t be the share price at time t of a risky asset STOCK. Suppose that, under a risk-neutral probability measure P, the share price S_t of STOCK obeys a stochastic differential equation of the form

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_t \tag{1}$$

where μ_t and σ_t are continuous but nonrandom functions of time t. Suppose also that the market has a riskless asset MONEYMARKET whose share price B_t obeys the ordinary differential equation

$$dB_t = r_t B_t \, dt, \tag{2}$$

where the "short rate" r_t is again a continuous but nonrandom function of t.

(A) Solve the differential equations (1) and (2). HINT for (1): Guess and use Itô; iterate this process until a solution emerges.

(B) Prove that, under any risk-neutral measure, $\mu_t = r_t$.

(C) Prove that $\int_0^t \sigma \, dW_s$ is normally distributed, and find its mean and variance.

(D) Find a formula for the arbitrage price of a European CALL option on STOCK with strike price K and expiration T. HINT: Your answer should be of the same form as the Black-Scholes formula. The quantities

$$\int_0^T r_t \, dt \qquad \text{and} \qquad \int_0^T \sigma_t^2 \, dt$$

should figure prominently in the answer.