MH4514

Financial Mathematics

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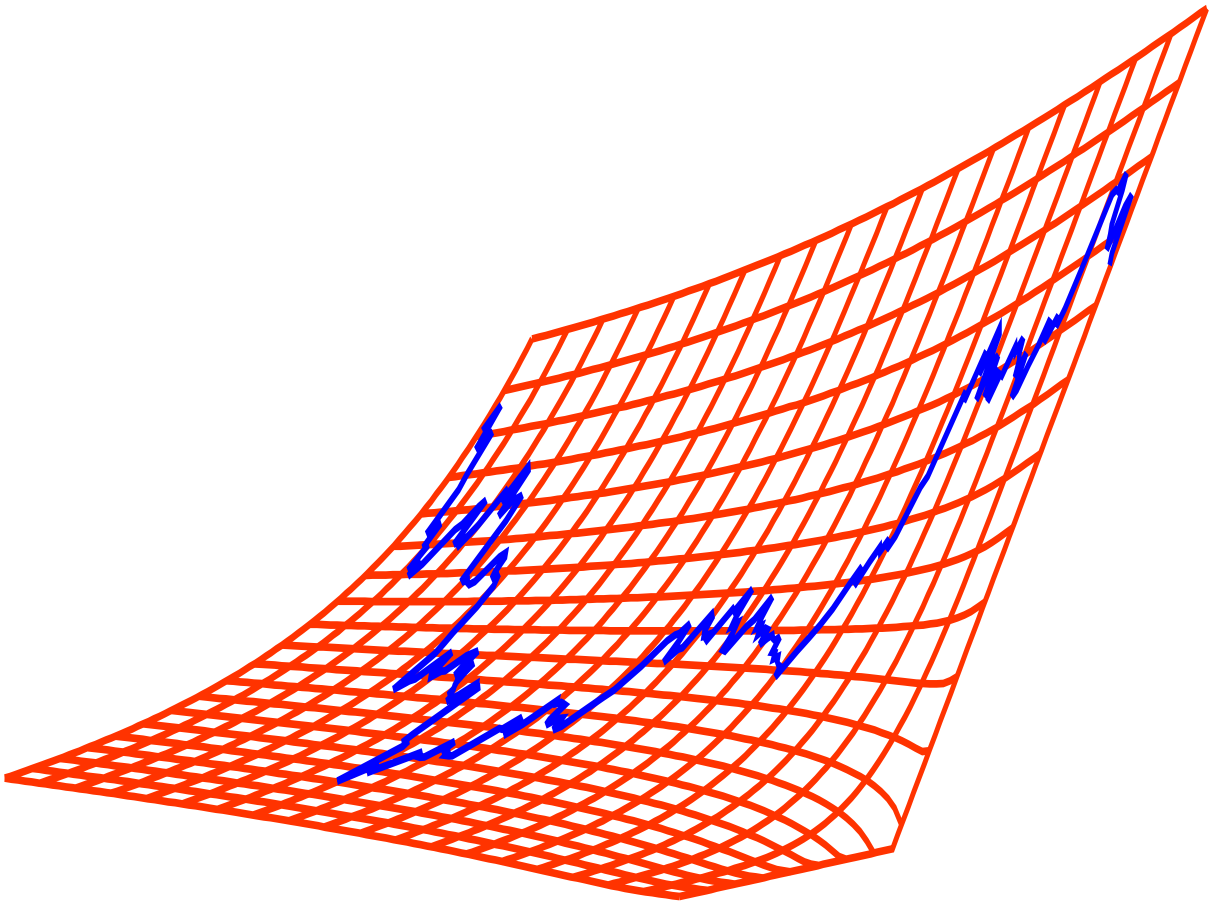

January 20, 2021 https://personal.ntu.edu.sg/nprivault/indext.html

This course is an introduction to the pricing and hedging of financial derivatives using stochastic calculus and partial differential equations. The presentation is done both in discrete and continuous-time financial models, with an emphasis on the complementarity between algebraic and probabilistic methods.

The descriptions of the asset model, self-financing portfolios, arbitrage and market completeness, are first given in Chapter 2 in a simple two time-step setting. These notions are then reformulated in discrete time in Chapter 3. The pricing and hedging of options in discrete time, particularly in the fundamental example of the Cox-Ross-Rubinstein model, are considered in Chapter 4, with a description of the passage from discrete to continuous time that prepares the transition to the subsequent chapters.

A presentation of Brownian motion, stochastic integrals and the associated Itô formula, is given in Chapter 5, with application to stochastic asset price modeling in Chapter 6. The Black-Scholes model is presented from the angle of partial differential equation (PDE) methods in Chapter 7, with the derivation of the Black-Scholes formula by transforming the Black-Scholes PDE into the standard heat equation, which is then solved by a heat kernel argument. The martingale approach to pricing and hedging is then presented in Chapter 8, and complements the PDE approach of Chapter 7 by recovering the Black-Scholes formula via a probabilistic argument. An introduction to stochastic volatility is given in Chapter 9, including by a presentation of volatility estimation tools including historical, local, and implied volatilities.

Chapter 10 contains an elementary introduction to finite difference methods for the numerical solution of PDEs and stochastic differential equations, dealing with the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for SDEs. The text is completed with an appendix containing the needed probabilistic background.

The document contains 105 solved exercises and 1 problem with solutions, and includes 10 Python codes on pages 74, 86, 89, 91, 126, 126, 179, 202 and 239, and 34 R codes on pages 125, 126, 128, 130, 167, 163, 179, 181, 193, 186, 200, 202, 213, 239, 240, 252, 252, 282, and 284. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on a problem number sends the reader to the corresponding solution, however this feature should not be misused.

This text also contains external links and 140 figures, including 24 animated figures, *e.g.* Figures 4.7, 4.9, 5.6, 5.7, 5.9, 5.10, 5.14, 6.6, 7.5 and S.3, 2 embedded videos Figures 0.1 and 9.3, and one interacting 3D graph in Figures 7.4 and 7.11, that may require using Acrobat Reader for viewing on the complete pdf file. The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European *put option* on that asset, expiring on October 05, 2009, cf. § 7.1.

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*Animated figures (work in Acrobat reader).

List of Tables

Introduction (soft opening)

Modern quantitative finance requires a strong background in fields such as stochastic calculus, optimization, partial differential equations (PDEs) and numerical methods, or even infinite dimensional analysis. In addition, the emergence of new complex financial instruments on the markets makes it necessary to rely on increasingly sophisticated mathematical tools. Not all readers of this book will eventually work in quantitative financial analysis, nevertheless they may have to interact with quantitative analysts, and becoming familiar with the tools they employ be an advantage. In addition, despite the availability of ready made financial calculators it still makes sense to be able oneself to understand, design and implement such financial algorithms. This can be particularly useful under different types of conditions, including an eventual lack of trust in financial indicators, possible unreliability of expert advice such as buy/sell recommendations, or other factors such as market manipulation. Instead of relying on predictions of stock price movements based on various tools (technical analysis, charting, "cup & handle" figures), we acknowledge that predicting the future is a difficult task and we rely on the Efficient Market Hypothesis. In this framework, the time evolution of the prices of risky assets will be modeled by random walks and stochastic processes.

Historical sketch

We start with a description of some of the main steps, ideas and individuals that played an important role in the development of the field over the last century.

Robert Brown, botanist, 1827

Brown observed the movement of pollen particles as described in his paper "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies." Phil. Mag. 4, 161-173, 1828.

Philosophical Magazine, first published in 1798, is a journal that "publishes articles in the field of condensed matter describing original results, theories and concepts relating to the structure and properties of crystalline materials, ceramics, polymers, glasses, amorphous films, composites and soft matter."

Louis Bachelier, Mathematician, PhD 1900

Bachelier, 1900 used Brownian motion for the modeling of stock prices in his PhD thesis "Théorie de la spéculation", Annales Scientifiques de l'Ecole Normale Supérieure 3 (17): 21-86, 1900.

Albert Einstein, physicist

Einstein received his 1921 Nobel Prize in part for investigations on the theory of Brownian motion: "... in 1905 Einstein founded a kinetic theory to account for this movement", presentation speech by S. Arrhenius, Chairman of the Nobel Committee, Dec. 10, 1922.

Einstein, 1905 "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen", Annalen der Physik 17.

Norbert Wiener, Mathematician, founder of cybernetics

Wiener is credited, among other fundamental contributions, for the mathematical foundation of Brownian motion, published in 1923. In particular he constructed the Wiener space and Wiener measure on $\mathscr{C}_0([0,1])$ (the space of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 0).

Wiener, 1923 "Differential space", Journal of Mathematics and Physics of the Massachusetts Institute of Technology, 2, 131-174, 1923.

Kiyoshi Itô (伊藤清), Mathematician, C.F. Gauss Prize 2006

Itô constructed the Itô integral with respect to Brownian motion, cf. Itô, Kiyoshi, Stochastic integral. Proc. Imp. Acad. Tokyo 20, (1944). 519-524. He also constructed the stochastic calculus with respect to Brownian motion, which laid the foundation for the development of calculus for random processes, see Itô, 1951 "On stochastic differential equations", *in* Memoirs of the American Mathematical Society.

"Renowned math wiz Itô, 93, dies." (The Japan Times, Saturday, Nov. 15, 2008)

> Kiyoshi Itô, an internationally renowned mathematician and professor emeritus at Kyoto University died Monday of respiratory failure at a Kyoto hospital, the university said Friday. He was 93. Itô was once dubbed "the most famous Japanese in Wall Street" thanks to his contribution to the founding of financial derivatives theory. He is known for his work on stochastic differential equations and the "Itô Formula", which laid the foundation for the Black and Scholes, 1973 model, a key tool for financial engineering. His theory is also widely used in fields like physics and biology.

Paul Samuelson, economist, Nobel Prize 1970

Samuelson, 1965 rediscovered Bachelier's ideas and proposed geometric Brownian motion as a model for stock prices. In an interview he stated "In the early 1950s I was able to locate by chance this unknown Bachelier, 1900 book, rotting in the library of the University of Paris, and when I opened it up it was as if a whole new world was laid out before me." We refer to "Rational theory of warrant pricing" by Paul Samuelson, *Industrial Management Review*, p. 13-32, 1965.

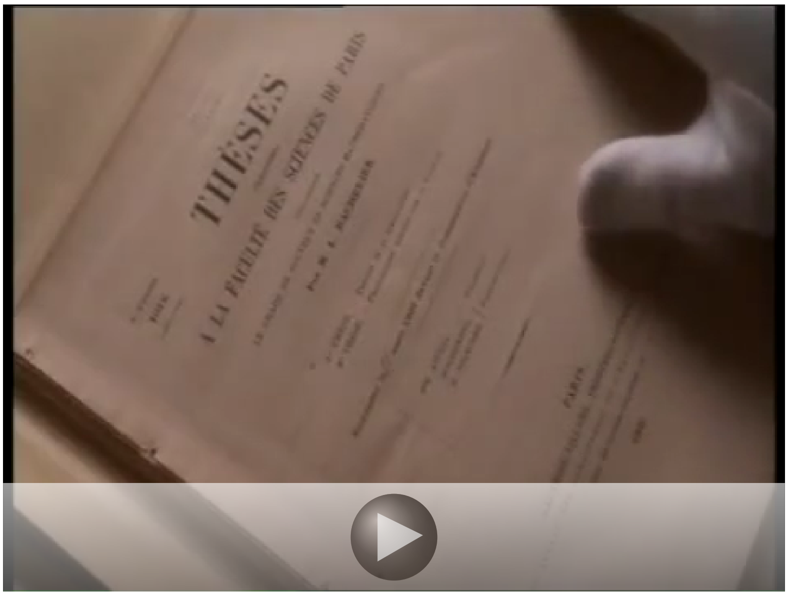


Figure 0.1: Clark, 2000 "As if a whole new world was laid out before me."[*]

In recognition of Bachelier's contribution, the Bachelier Finance Society was started in 1996 and now holds the World Bachelier Finance Congress every two years.

Robert Merton, Myron Scholes, economists

Robert Merton and Myron Scholes shared the 1997 Nobel Prize in economics: "In collaboration with Fisher Black, developed a pioneering formula for the valuation of stock options ... paved the way for economic valuations in many areas ... generated new types of financial instruments and facilitated more efficient risk management in society."[†]

Black and Scholes, 1973 "The Pricing of Options and Corporate Liabilities". Journal of Political Economy 81 (3): 637-654.

The development of options pricing tools contributed greatly to the expansion of option markets and led to development several ventures such as the "Long Term Capital Management" (LTCM), founded in 1994. The fund yielded annualized returns of over 40% in its first years, but registered lost US$ 4.6 billion in less than four months in 1998, which resulted into its closure in early 2000.

Oldřich Vašíček, economist, 1977

Interest rates behave differently from stock prices, notably due to the phenomenon of mean reversion, and for this reason they are difficult to model using geometric Brownian motion. Vašíček, 1977 was the first to suggest a mean-reverting model for stochastic interest rates, based on the Ornstein-Uhlenbeck process, in "An equilibrium characterization of the term structure", Journal of

[*] Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

[†] This has to be put in relation with the modern development of risk societies; "societies increasingly preoccupied with the future (and also with safety), which generates the notion of risk".

Financial Economics 5: 177-188.

David Heath, Robert Jarrow, Andrew Morton

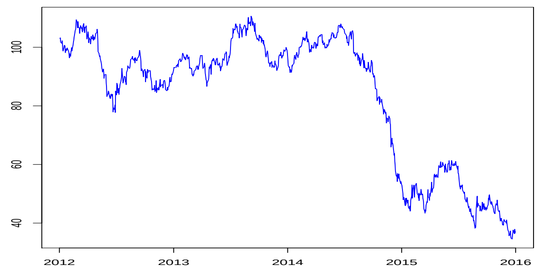
These authors proposed in 1987 a general framework to model the evolution of (forward) interest rates, known as the Heath-Jarrow-Morton (HJM) model, see Heath, Jarrow, and Morton, 1992 “Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation”, Econometrica, (January 1992), Vol. 60, No. 1, pp 77-105.

Alan Brace, Dariusz Gatarek, Marek Musiela (BGM)

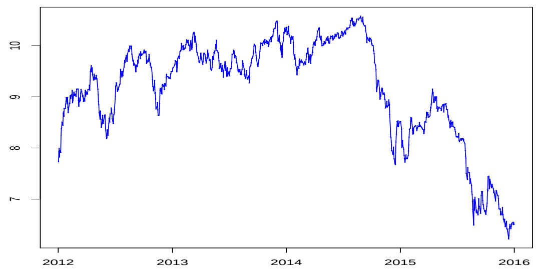
The Brace, Gatarek, and Musiela, 1997 model is actually based on geometric Brownian motion, and it is specially useful for the pricing of interest rate derivatives such as interest rate caps and swaptions on the LIBOR market, see “The Market Model of Interest Rate Dynamics”. Mathematical Finance Vol. 7, page 127. Blackwell 1997, by Alan Brace, Dariusz Gatarek, Marek Musiela.

Financial derivatives

The following graphs exhibit a correlation between commodity (oil) prices and an oil-related asset price.



(a) WTI price graph.



(b) Graph of Keppel Corp. stock price

Figure 0.2: Comparison of WTI *vs* Keppel price graphs.

The study of financial derivatives aims at finding functional relationships between the price of an underlying asset (a company stock price, a commodity price, etc) and the price of a related financial contract (an option, a financial derivative, etc.).

Option contracts

Option credit contracts appear to have been used as early as the 10*th* century by traders in the Mediterranean. Early accounts of option trades can also be found in *The Politics* Aristotle, 350 BCE by Aristotle (384-322 BC). Referring to the philosopher Thales of Miletus (c. 624 - c. 546 BC), Aristotle writes:

> “He (Thales) knew by his skill in the stars while it was yet winter that there would be a great harvest of olives in the coming year; so, having a little money, he gave deposits for the use of all the olive-presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest-time came, and many were wanted all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money”.

As of year 2015, the size of the financial derivatives market is estimated at over one quadrillion (or one million billions, or 10^{15}) USD, which is more than 10 times the size of the total Gross World Product (GWP).

We close this introduction with a description of (European) call and put options, which are at the basis of risk management.

European put option contracts

As previously mentioned, an important concern for the buyer of a stock at time t is whether its price S_T can decline at some future date T. The buyer of the stock may seek protection from a market crash by purchasing a contract that allows him to sell his asset at time T at a guaranteed price K fixed at time t. This contract is called a put option with strike price K and exercise date T.

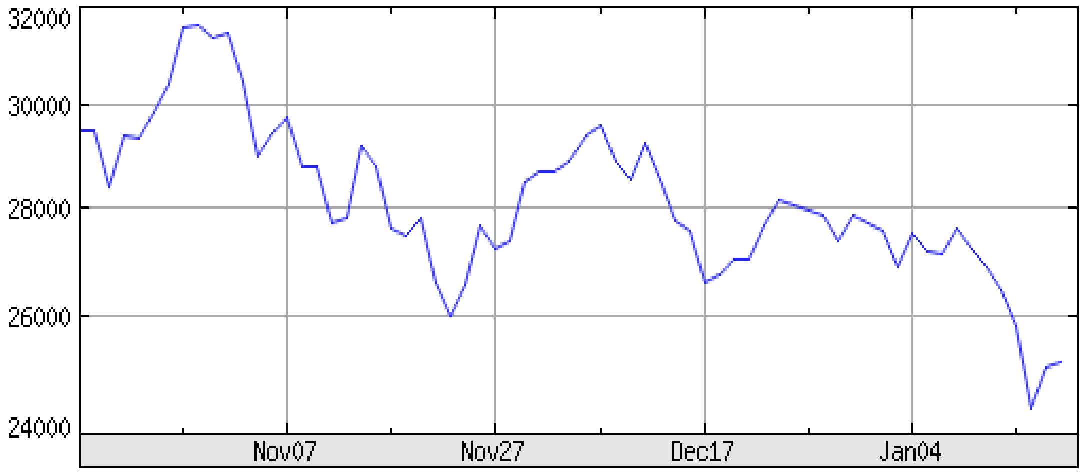


Figure 0.3: Graph of the Hang Seng index - holding a put option might be useful here.

Definition 0.1 A (European) *put* option is a contract that gives its holder the right (but not the obligation) to *sell* a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.

In case the price S_T falls down below the level K, exercising the contract will give the holder of the option a gain equal to $K - S_T$ in comparison to those who did not subscribe the option contract and have to sell the asset at the market price S_T. In turn, the issuer of the option contract will register a loss also equal to $K - S_T$ (in the absence of transaction costs and other fees).

If S_T is above K then the holder of the option contract will not exercise the option as he may choose to sell at the price S_T. In this case the profit derived from the option contract is 0.

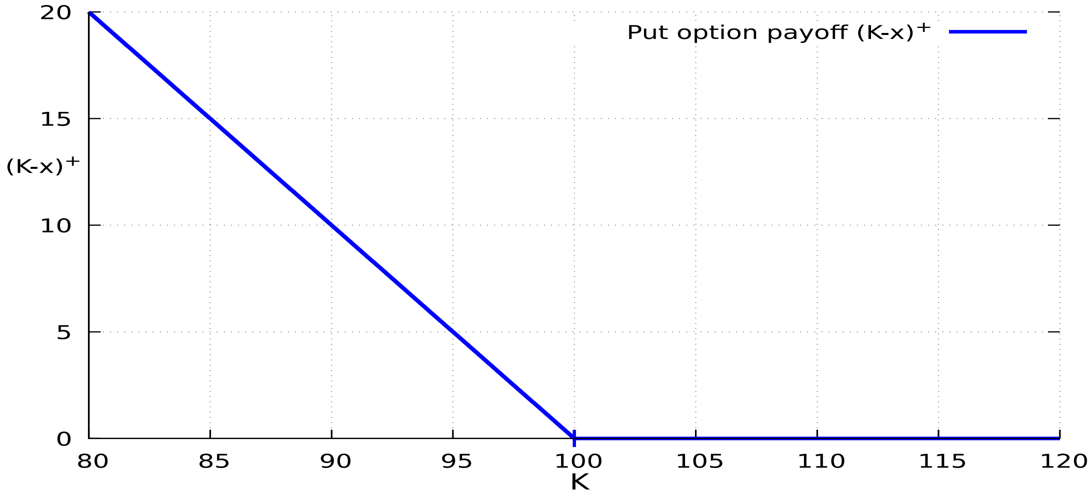


Figure 0.4: Payoff function of a put option with strike price $K = 100$.

See *e.g.* http://optioncreator.com/stwwxvz.

In general, the payoff of a (so called *European*) put option contract can be written as

$$\phi(S_T) = (K - S_T)^+ := \begin{cases} K - S_T, & S_T \leqslant K, \\ \\ 0, & S_T \geqslant K. \end{cases}$$

Two possible scenarios (S_T finishing above K or below K) are illustrated in Figure 0.5.

Figure 0.5: Sample price processes simulated by a geometric Brownian motion.

Example of put option: the **buy back guarantee**[*] in currency exchange is a common example of European put option.

European call option contracts

On the other hand, if the trader aims at buying some stock or commodity, his interest will be in prices not going up and he might want to purchase a call option, which is a contract allowing him to buy the considered asset at time T at a price not higher than a level K fixed at time t.

[*]Right-click to open or save the attachment.

Here, in the event that S_T goes above K, the buyer of the option contract will register a potential gain equal to $S_T - K$ in comparison to an agent who did not subscribe to the call option.

Definition 0.2 A (European) *call* option is a contract that gives its holder the right (but not the obligation) to *purchase* a quantity of assets at a predefined price K called the strike price, and at a predefined date T called the maturity.

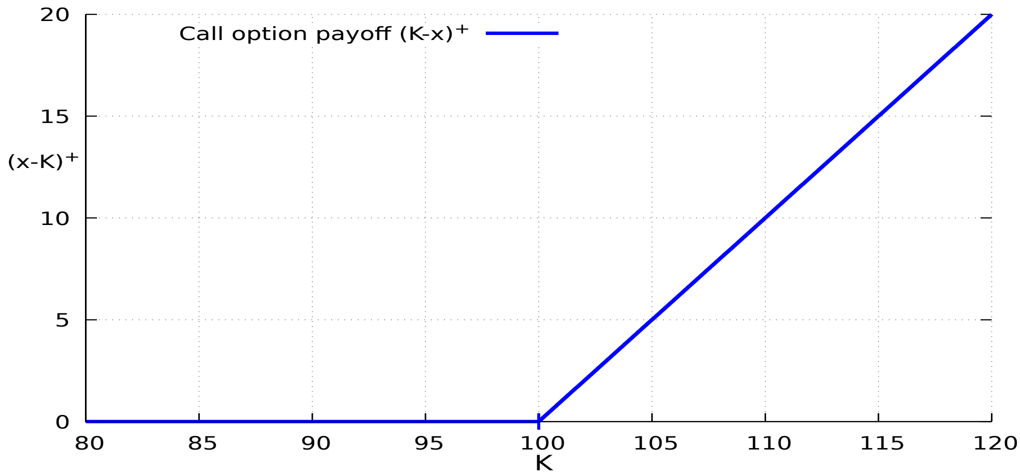


Figure 0.6: Payoff function of a call option with strike price $K = 100$.

See *e.g.* http://optioncreator.com/stqhbgn.

In general, the payoff of a (so called European) call option contract can be written as

$$\phi(S_T) = (S_T - K)^+ := \begin{cases} S_T - K, & S_T \geqslant K, \\ 0, & S_T \leqslant K. \end{cases}$$

In market practice, options are often divided into a certain number n of *warrants*, the (possibly fractional) quantity n being called the *entitlement ratio*.

Example of call option: the **price lock guarantee**[*] in online booking is a common example of European *call* option.

Cash settlement *vs* physical delivery

Cash settlement. In the case of a cash settlement, the option contract issuer will satisfy the option contract by selling $\alpha = 1$ stock at the price $S_1 \in \{\$2, \$5\}$, refund the initial \$2 loan, and hand in the remaining amount $C = (S_1 - K)^+$ to the option contract holder.

Physical delivery. In the case of physical delivery of the underlying asset, the option contract issuer will deliver $\alpha = 1$ stock to the option contract holder in exchange for $K = \$2$, which will be used to refund the initial \$2 loan subscribed by the option contract issuer.

The derivatives market

As of year 2015, the size of the derivatives market was estimated at more that \$1.2 quadrillion,[†] or more than 10 times the Gross World Product (GWP). See **here** or **here** for up-to-date data on

[*] Right-click to open or save the attachment.
[†] One thousand trillion, or one million billion, or 10^{15}.

notional amounts outstanding and gross market value from the Bank for International Settlements (BIS).

Option pricing

In order for an option contract to be fair, the buyer of the option contract should pay a fee (similar to an insurance fee) at the signature of the contract. The computation of this fee is an important issue, which is known as option *pricing*.

Option hedging

The second important issue is that of *hedging*, *i.e.* how to manage a given portfolio in such a way that it contains the required random payoff $(K - S_T)^+$ (for a put option) or $(S_T - K)^+$ (for a call option) at the maturity date T.

The next Figure 0.7 illustrates a sharp increase and sharp drop in asset price, making it valuable to hold a call option contract during the first half of the graph, whereas holding a put option contract would be recommended during the second half.

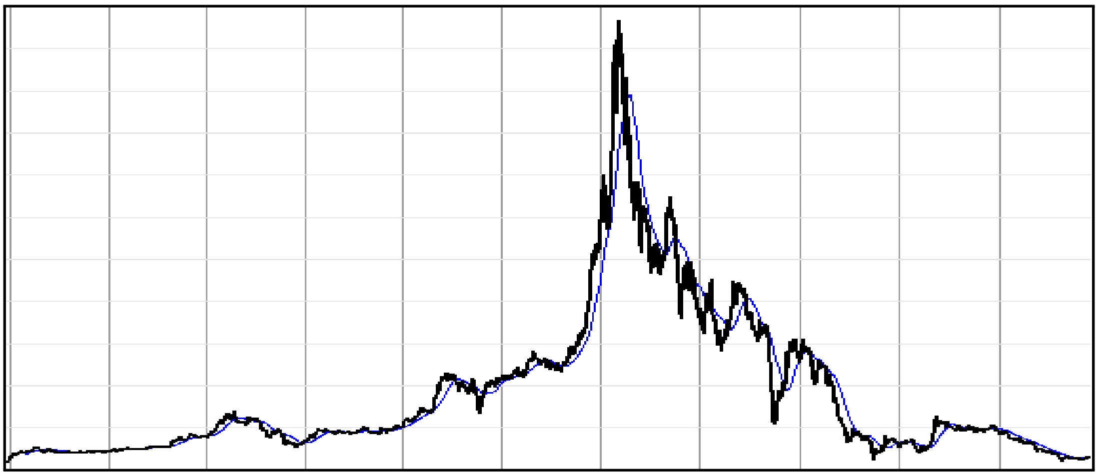
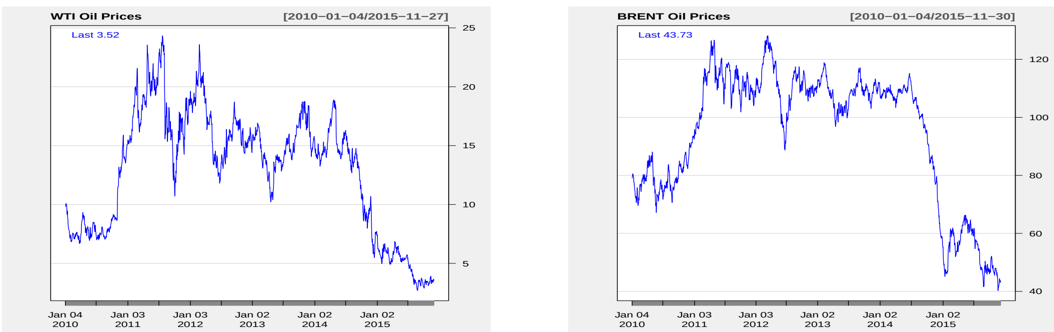


Figure 0.7: "Infogrames" stock price curve.

Example: Fuel hedging and the four-way zero-collar option



(a) WTI price graph.

(b) Brent price graph

Figure 0.8: Brent and WTI price graphs.

```r
install.packages("Quandl")
library(Quandl);library(quantmod)
getSymbols("DCOILBRENTEU", src="FRED")
chartSeries(DCOILBRENTEU,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
BRENT = Quandl("FRED/DCOILBRENTEU",start_date="2010-01-01",
      end_date="2015-11-30",type="xts")
chartSeries(BRENT,up.col="blue",theme="white",name = "BRENT Oil Prices",lwd=5)
getSymbols("WTI", from="2010-01-01", to="2015-11-30")
WTI <- Ad(`WTI`)
chartSeries(WTI,up.col="blue",theme="white",name = "WTI Oil Prices",lwd=5)
```

(April 2011)
Fuel hedge promises Kenya Airways smooth ride in volatile oil market.[*]

(November 2015)
A close look at the role of fuel hedging in Kenya Airways $259 million loss.[*]

The four-way call collar call option requires its holder to purchase the underlying asset (here, airline fuel) at a price specified by the blue curve in Figure 0.9, when the underlying asset price is represented by the red line.

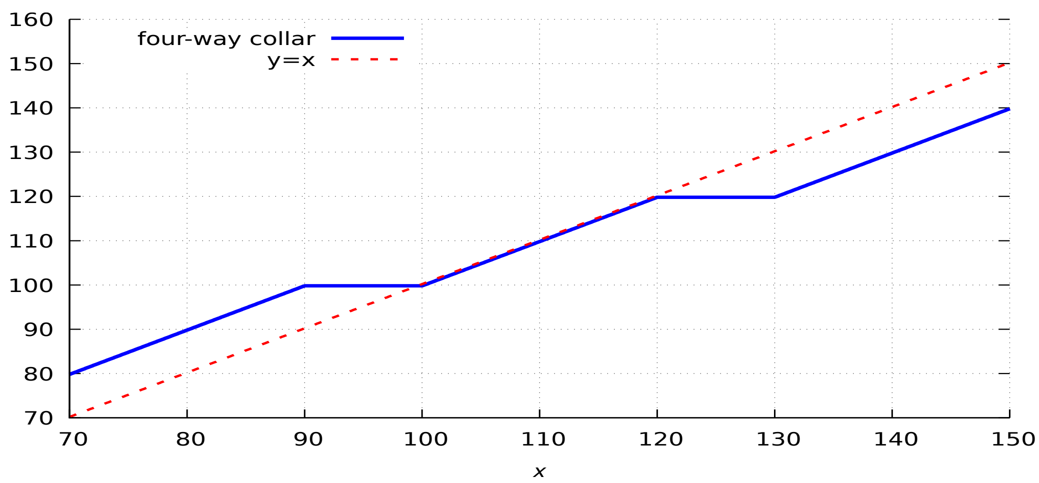


Figure 0.9: Price map of a four-way collar option.

The four-way call collar option contract will result into a positive or negative payoff depending on current fuel prices, as illustrated in Figure 0.10.

[*]Right-click to open or save the attachment.

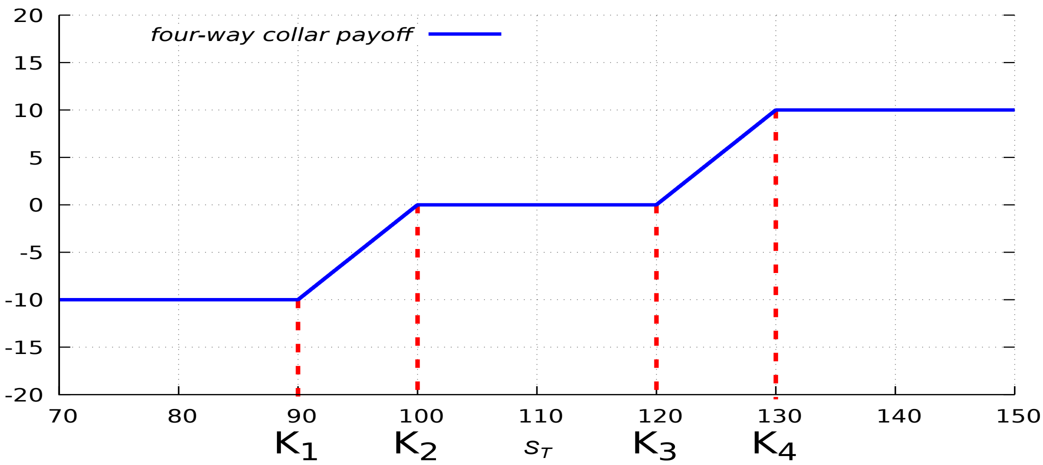


Figure 0.10: Payoff function of a four-way call collar option.

The four-way call collar payoff can be written as a linear combination

$$\phi(S_T) = (K_1 - S_T)^+ - (K_2 - S_T)^+ + (S_T - K_3)^+ - (S_T - K_4)^+$$

of call and put option payoffs with respective strike prices

$$K_1 = 90, \quad K_2 = 100, \quad K_3 = 120, \quad K_4 = 130,$$

see *e.g.* http://optioncreator.com/st5rf51.

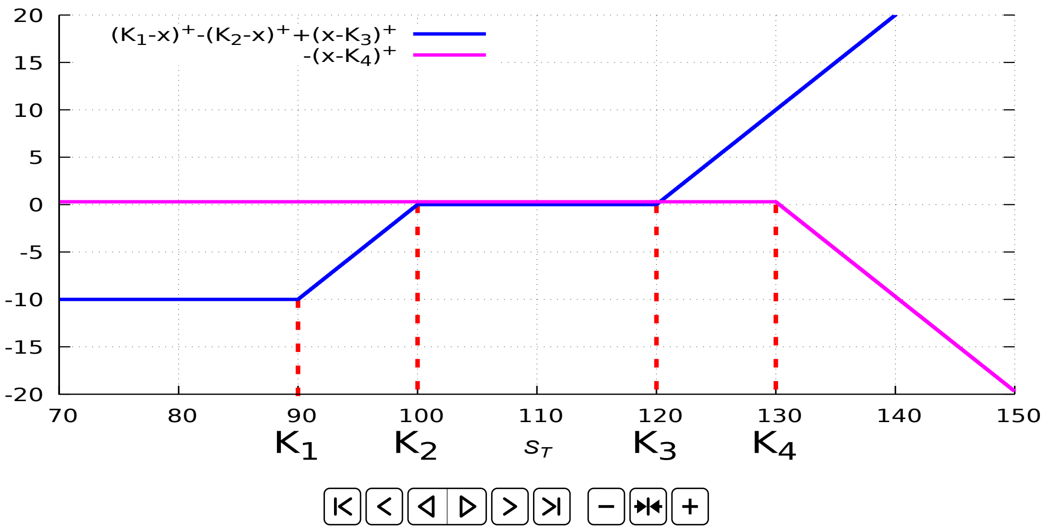


Figure 0.11: Four-way call collar payoff as a combination of call and put options.*

Therefore, the four-way call collar option contract can be *synthesized* by:

1. purchasing a *put option* with strike price $K_1 = \$90$, and
2. selling (or issuing) a *put option* with strike price $K_2 = \$100$, and
3. purchasing a *call option* with strike price $K_3 = \$120$, and
4. selling (or issuing) a *call option* with strike price $K_4 = \$130$.

*The animation works in Acrobat Reader on the entire pdf file.

Moreover, the call collar option contract can be made *costless* by adjusting the boundaries K_1, K_2, K_3, K_4, in which case it becomes a *zero-collar* option.

Example - the 4-5-2 model

We close this introduction with a simplified example of the pricing and hedging technique in a binary model. Consider a risky stock valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values

$$S_1 = \begin{cases} \$5 \\ \\ \$2 \end{cases}$$

at time $t = 1$. In addition, consider an option contract that promises a claim payoff C whose values are contingent to the market data of S_1:

$$C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$$

Question: Does C represent the payof of a put option contract? Of a call option contract? If yes, with which strike price K?

At time $t = 0$ the option contract issuer (or writer) chooses to invest α units in the risky asset S, while keeping $\$\beta$ on our bank account, meaning that we invest a total amount

$$\alpha S_0 + \$\beta \qquad\qquad \text{at } t = 0.$$

Here, the amount $\$\beta$ may be positive or negative, depending on whether it is corresponds to savings or to debt, and is interpreted as a *liability*.

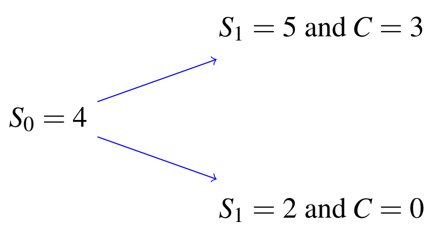
The following issues can be addressed:

a) Hedging: How to choose the portfolio allocation $(\alpha, \$\beta)$ so that the value

$$\alpha S_1 + \$\beta$$

of the portfolio matches the future payoff C at time $t = 1$?

b) Pricing: How to determine the amount $\alpha S_0 + \$\beta$ to be invested by the option contract issuer in such a portfolio at time $t = 0$?



$S_0 = 4$	$S_1 = 5$ and $C = 3$
	$S_1 = 2$ and $C = 0$

Hedging means that at time $t = 1$ the portfolio value matches the future payoff C, *i.e.*

$$\alpha S_1 + \$\beta = C.$$

This condition can be rewritten as

$$C = \begin{cases} \$3 = \alpha \times \$5 + \$\beta & \text{if } S_1 = \$5, \\ \\ \$0 = \alpha \times \$2 + \$\beta & \text{if } S_1 = \$2, \end{cases}$$

i.e.

$$\begin{cases} 5\alpha+\beta=3, \\ \\ 2\alpha+\beta=0, \end{cases} \quad \text{which yields} \quad \begin{cases} \alpha=1 \text{ stock}, \\ \\ \$\beta=-\$2. \end{cases}$$

In other words, the option contract issuer purchases 1 (one) unit of the stock S at the price $S_0 = \$4$, and borrows \$2 from the bank. The price of the option contract is then given by the portfolio value

$$\alpha S_0 + \$\beta = 1 \times \$4 - \$2 = \$2.$$

at time $t = 0$.

The above computation is implemented in the attached **IPython notebook** that can be run **here**. This algorithm is scalable and can be extended to recombining binary trees over multiple time steps.

Definition 0.3 The *arbitrage-free price* of the option contract is defined as the initial cost $\alpha S_0 + \$\beta$ of the portfolio hedging the claim payoff C.

Conclusion: in order to deliver the random payoff $C = \begin{cases} \$3 & \text{if } S_1 = \$5 \\ \\ \$0 & \text{if } S_1 = \$2. \end{cases}$ to the option contract holder at time $t = 1$, the option contract issuer (or writer) has to:

1. charge $\alpha S_0 + \$\beta = \$2$ (the option contract price) at time $t = 0$,
2. borrow $-\$\beta = \$2$ from the bank,
3. invest those $\$2 + \$2 = \$4$ into the purchase of $\alpha = 1$ unit of stock valued at $S_0 = \$4$ at time $t = 0$,
4. wait until time $t = 1$ to sell the stock at the price $S_1 = \$5$ or $S_1 = \$2$, and refund the \$2 loan.

   We find that the portfolio value evolved into

$$C = \begin{cases} \alpha \times \$5 + \$\beta = 1 \times \$5 - \$2 = \$3 & \text{if } S_1 = \$5, \\ \\ \alpha \times \$2 + \$\beta = 1 \times \$2 - \$2 = 0 & \text{if } S_1 = \$2, \end{cases}$$

   so that the option contract and the equality $C = \alpha S_1 + \$\beta$ can be fulfilled whatever the evolution of S, and allows us to break even.

Here, the option contract price $\alpha S_0 + \$\beta = \$2$ is interpreted as the cost of hedging the option. In Chapters 3 and 4 we will see that this model is scalable and extends to discrete time.

We note that the initial option contract price of \$2 can be turned to $C = \$3$ (%50 profit) ... or into $C = \$0$ (total ruin).

Thinking further

1) The expected claim payoff at time $t = 1$ is

$$\begin{aligned} \mathbb{E}[C] &= \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) \\ &= \$3 \times \mathbb{P}(S_1 = \$5). \end{aligned}$$

In absence of arbitrage opportunities ("fair market"), this expected payoff $\mathbb{E}[C]$ should equal the initial amount \$2 invested in the option. In that case we should have

$$\begin{cases} \mathbb{E}[C] = \$3 \times \mathbb{P}(S_1 = \$5) = \$2 \\ \\ \mathbb{P}(S_1 = \$5) + \mathbb{P}(S_1 = \$2) = 1. \end{cases}$$

from which we can *infer* the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{2}{3} \\ \\ \mathbb{P}(S_1 = \$2) = \dfrac{1}{3}, \end{cases} \tag{0.1}$$

which are called *risk-neutral* probabilities. We see that under the risk-neutral probabilities, the stock S has twice more chances to go up than to go down in a "fair" market.

2) Based on the probabilities (0.1) we can also compute the expected value $\mathbb{E}[S_1]$ of the stock at time $t = 1$. We find

$$\begin{aligned} \mathbb{E}[S_1] &= \$5 \times \mathbb{P}(S_1 = \$5) + \$2 \times \mathbb{P}(S_1 = \$2) \\ &= \$5 \times \frac{2}{3} + \$2 \times \frac{1}{3} \\ &= \$4 \\ &= S_0. \end{aligned}$$

Here this means that, on average, no additional profit can be made from an investment on the risky stock. In a more realistic model we can assume that the riskles bank account yields an interest rate equal to r, in which case the above analysis is modified by letting $\$\beta$ become $\$(1+r)\beta$ at time $t = 1$, nevertheless the main conclusions remain unchanged.

Implied probabilities

By matching the theoretical price $\mathbb{E}[C]$ to an actual market price data $\$P$ as

$$\$P = \mathbb{E}[C] = \$3 \times \mathbb{P}(C = \$3) + \$0 \times \mathbb{P}(C = \$0) = \$3 \times \mathbb{P}(S_1 = \$5)$$

we can infer the probabilities

$$\begin{cases} \mathbb{P}(S_1 = \$5) = \dfrac{\$P}{3} \\ \\ \mathbb{P}(S_1 = \$2) = \dfrac{3 - \$P}{3}, \end{cases} \tag{0.2}$$

which are *implied probabilities* estimated from market data, as illustrated in Figure 0.12. We note that the conditions

$$0 < \mathbb{P}(S_1 = \$5) < 1, \qquad 0 < \mathbb{P}(S_1 = \$2) < 1$$

are equivalent to $0 < \$P < 3$, which is consistent with financial intuition in a non-deterministic market.

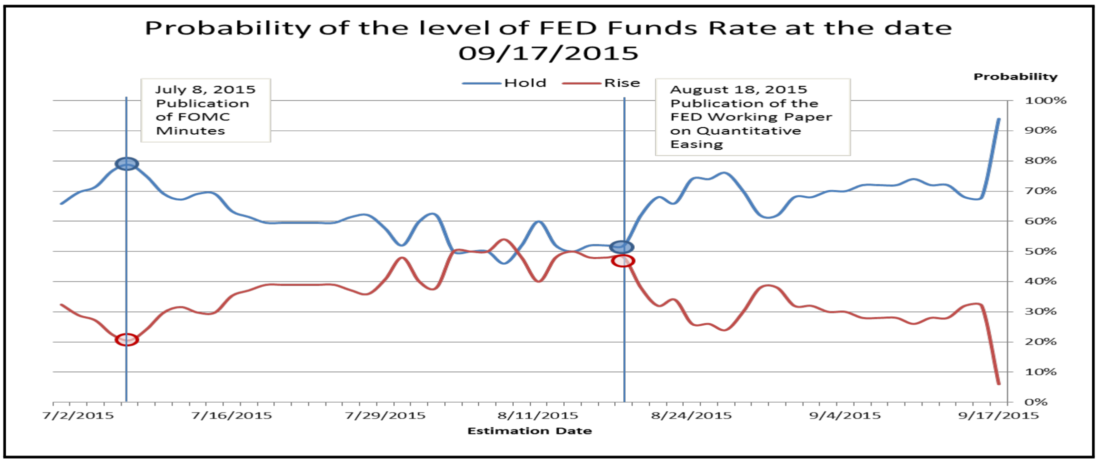


Figure 0.12: Implied probabilities.

Note that implied probabilities should also be used with caution, as shown in Figures 0.13-0.14.

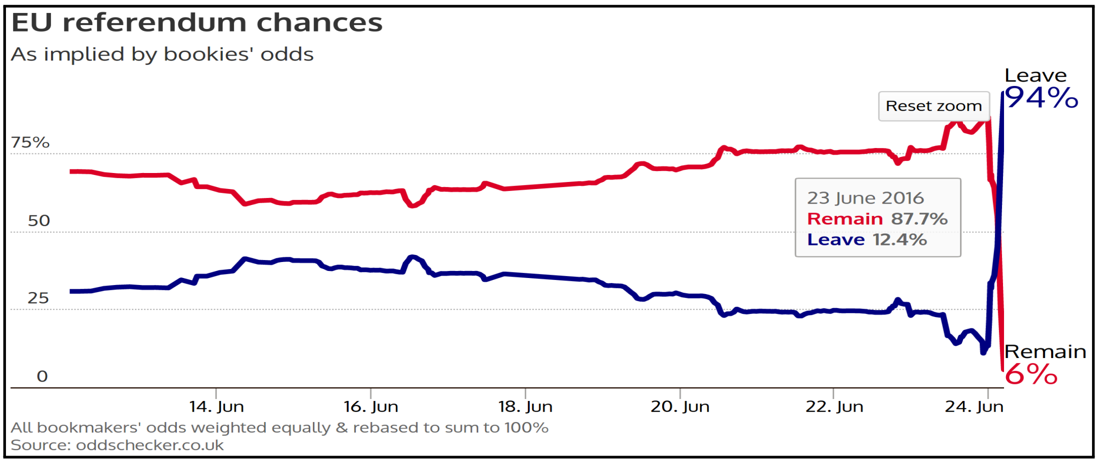


Figure 0.13: Implied probabilities according to bookmakers.

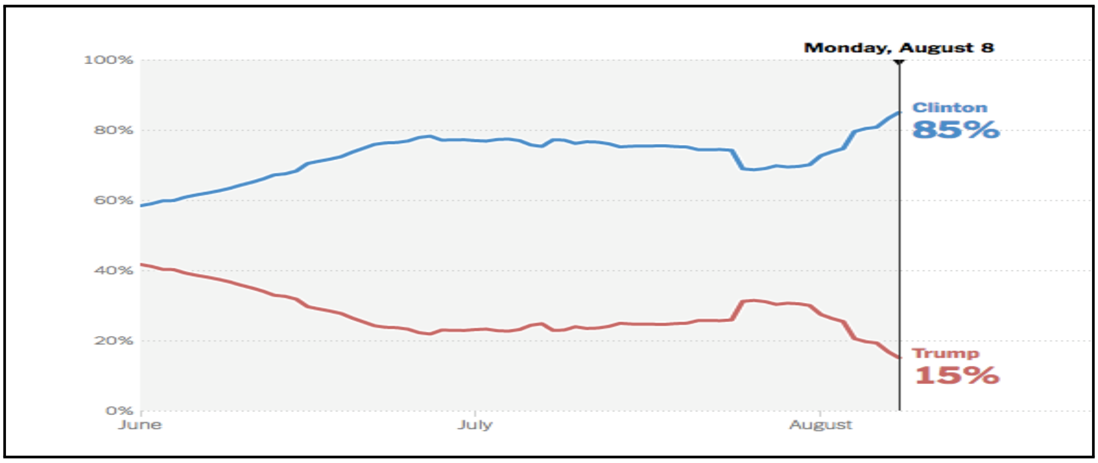


Figure 0.14: Implied probabilities according to polling.

Implied probabilities can be estimated using *e.g.* binary options, see for example Exercise 4.9.

The *Practitioner* expects a good model to be:

- *Robust* with respect to missing, spurious or noisy data,
- *Fast* - prices have to be delivered daily in the morning,
- *Easy* to calibrate - parameter estimation,
- *Stable* with respect to re-calibration and the use of new data sets.

Typically, a medium size bank manages 5,000 options and 10,000 deals daily over 1,000 possible scenarios and dozens of time steps. This can mean a hundred million computations of $\mathbb{E}[C]$ daily, or close to a billion such computations for a large bank.

The *Mathematician* tends to focus on more theoretical features, such as:

- *Elegance*,
- *Sophistication*,
- *Existence* of analytical (closed-form) solutions / error bounds,
- *Significance* to mathematical finance.

This includes:

- *Creating* new payoff functions and structured products,
- *Defining* new models for underlying asset prices,
- *Finding* new ways to compute expectations $\mathbb{E}[C]$ and hedging strategies.

The methods involved include:

- Monte Carlo (60%),
- PDEs and finite differences (30%),
- Other analytic methods and approximations (10%),

\+ AI and Machine Learning techniques.

1. Discrete-Time Martingales

As mentioned in the introduction, stochastic processes can be classified into two main families, namely *Markov processes* on the one hand, and *martingales* on the other hand. Markov processes have been our main focus of attention so far, and in this chapter we turn to the notion of martingale. In particular we will give a precise mathematical meaning to the description of martingales stated in the introduction, which says that when $(X_n)_{n\in\mathbb{N}}$ is a martingale, the best possible estimate at time $n \in \mathbb{N}$ of the future value X_m at time $m > n$ is X_n itself. The main application of martingales will be to recover in an elegant way previous results on gambling processes. The concept of martingale has many applications in stochastic modeling, for example in financial mathematics, where martingales are used to characterize the fairness and equilibrium of a market model.

1.1 Filtrations and Conditional Expectations

Before dealing with martingales we need to introduce the important notion of *filtration* generated by a discrete-time stochastic process $(S_n)_{n\in\mathbb{N}}$. The filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ generated by a stochastic process $(S_n)_{n\in\mathbb{N}}$ taking its values in a state space $\mathbb{S}$, is the family of σ-algebras

$$\mathscr{F}_n := \sigma(S_0, S_1, \ldots, S_n), \qquad n \geqslant 0,$$

which denote the collections of events generated by $S_0, S_1, \ldots, S_n$. Examples of such events include

$$\{S_0 \leqslant a_0,\ S_1 \leqslant a_1,\ \ldots,\ S_n \leqslant a_n\}$$

for $a_0, a_1, \ldots, a_n$ a given fixed sequence of real numbers. Note that we have the inclusion $\mathscr{F}_n \subset \mathscr{F}_{n+1}$, $n \in \mathbb{N}$, *i.e.* $(\mathscr{F}_n)_{n\in\mathbb{N}}$ is *non-decreasing*.

One refers to $\mathscr{F}_n$ as the *information* generated by $(S_k)_{k\in\mathbb{N}}$ up to time n, and to $(\mathscr{F}_n)_{n\in\mathbb{N}}$ as the *information flow* generated by $(S_n)_{n\in\mathbb{N}}$. We say that a random variable is $\mathscr{F}_n$-measurable whenever F can be written as a function $F = f(S_0, S_1, \ldots, S_n)$ of $(S_0, S_1, \ldots, S_n)$.

Example

1. Consider the simple random walk

$$S_n := X_1 + X_2 + \cdots + X_n, \qquad n \geqslant 0,$$

where $(X_k)_{k\geqslant 1}$ is a sequence of independent, identically distributed $\{-1,1\}$-valued random variables, and $S_0 := 0$. The filtration (or information flow) $(\mathscr{F}_n)_{n\in\mathbb{N}}$ generated by $(S_n)_{n\in\mathbb{N}}$ satisfies

$$\mathscr{F}_0 = \big\{\{S_0 \neq 0\}, \{S_0 = 0\}\big\} = \big\{\emptyset, \Omega\big\}, \quad \mathscr{F}_1 = \big\{\emptyset, \{X_1 = 1\}, \{X_1 = -1\}, \Omega\big\},$$

and

$$\mathscr{F}_2 = \sigma\big(\big\{\emptyset, \{X_1 = 1, X_2 = 1\}, \{X_1 = 1, X_2 = -1\}, \{X_1 = -1, X_2 = 1\},$$
$$\{X_1 = -1, X_2 = -1\}, \Omega\big\}\big).$$

The notation $\mathscr{F}_n$ is useful to represent a quantity of information available at time n, and various sub σ-algebras of $\mathscr{F}_n$ can be defined. For example, the σ-algebra $\mathscr{G}$ generated by S_2 satisfies

$$\begin{aligned}\mathscr{G} =& \sigma\big(\big\{\emptyset, \{S_2 = -2\}, \{S_2 = 2\}, \{S_2 = 0\}, \Omega\big\}\big)\\ =& \sigma\big(\big\{\emptyset, \{X_1 = -1, X_2 = -1\}, \{X_1 = 1, X_2 = 1\}\\ & \{X_1 = 1, X_2 = -1\} \cup \{X_1 = -1, X_2 = 1\}, \Omega\big\}\big),\end{aligned}$$

which contains less information than $\mathscr{F}_2$, as it only tells whether the increments X_1, X_2 are both equal to 1 or to -1.

We now review the definition of conditional expectation, see Section 11.6 for details. Given F a random variable with finite mean, the conditional expectation $\mathbb{E}[F \mid \mathscr{F}_n]$ refers to

$$\mathbb{E}[F \mid X_0, X_1, \ldots, X_n] = \mathbb{E}[F \mid X_0 = k_0, \ldots, X_n = k_n]_{k_0 = X_0, \ldots, k_n = X_n},$$

given that $X_0, X_1, \ldots, X_n$ are respectively equal to $k_0, k_1, \ldots, k_n \in \mathbb{S}$.

The conditional expectation $\mathbb{E}[F \mid \mathscr{F}_n]$ is itself a random variable that depends only on the values of $X_0, X_1, \ldots, X_n$, *i.e.* on the history of the process up to time $n \in \mathbb{N}$. It can also be interpreted as the best possible estimate of F in mean-square sense, given the values of $X_0, X_1, \ldots, X_n$, see Proposition 11.9.

Definition 1.1 A stochastic process $(Z_n)_{n\in\mathbb{N}}$ is said to be $(\mathscr{F}_n)_{n\in\mathbb{N}}$-adapted if the value of Z_n depends on no more than the information available up to time n in $\mathscr{F}_n$.

In other words, the value of an $\mathscr{F}_n$-adapted process Z_n is determined by a function of $X_0, X_1, \ldots, X_n$ for all $n \in \mathbb{N}$.

By point (ii)) page 292, any integrable $\mathscr{F}_n$*-adapted* process $(Z_n)_{n\in\mathbb{N}}$ satisfies

$$\mathbb{E}[Z_n \mid \mathscr{F}_n] = Z_n, \qquad n \geqslant 0.$$

1.2 Martingales - Definition and Properties

We now turn to the definition of *martingale*.

Definition 1.2 An integrable,[a] discrete-time stochastic process $(Z_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$ if $(Z_n)_{n\in\mathbb{N}}$ is $\mathscr{F}_n$-adapted and satisfies the property

$$\mathbb{E}[Z_{n+1} \mid \mathscr{F}_n] = Z_n, \qquad n \geqslant 0. \tag{1.1}$$

[a]Integrable means $\mathbb{E}[|Z_n|] < \infty$ for all $n \in \mathbb{N}$.

The process $(Z_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$ if, given the information $\mathscr{F}_n$ known up to time n, the best possible estimate of Z_{n+1} is simply Z_n.

Exercise. Using the *tower property* of conditional expectations, show that Definition 1.1 can be equivalently stated by saying that

$$\mathbb{E}[M_n \mid \mathscr{F}_k] = M_k, \qquad 0 \leqslant k < n.$$

A particular property of martingales is that their expectation is constant over time.

Proposition 1.1 Let $(Z_n)_{n\in\mathbb{N}}$ be a martingale. We have

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \qquad n \geqslant 0.$$

Proof. From the tower property (11.38) we have:

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1} \mid \mathscr{F}_n]] = \mathbb{E}[Z_n], \qquad n \geqslant 0,$$

hence by induction on $n \in \mathbb{N}$ we have

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_{n-1}] = \cdots = \mathbb{E}[Z_1] = \mathbb{E}[Z_0], \qquad n \geqslant 0.$$

□

Examples of martingales

1. Any centered[*] integrable process $(S_n)_{n\in\mathbb{N}}$ with independent increments is a martingale with respect to the filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ generated by $(S_n)_{n\in\mathbb{N}}$.

   Indeed, in this case we have
$$\begin{aligned}\mathbb{E}[S_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}[S_n \mid \mathscr{F}_n] + \mathbb{E}[S_{n+1} - S_n \mid \mathscr{F}_n] \\ &= \mathbb{E}[S_n \mid \mathscr{F}_n] + \mathbb{E}[S_{n+1} - S_n] \\ &= \mathbb{E}[S_n \mid \mathscr{F}_n] = S_n, \qquad n \geqslant 0.\end{aligned}$$
   In addition to being a martingale, a stochastic process $(X_n)_{n\in\mathbb{N}}$ with centered independent increments is also a Markov process.

   However, not all martingales have the Markov property, and not all Markov processes are martingales. In addition, there are martingales and Markov processes which do not have independent increments.

[*]A process $(S_n)_{n\in\mathbb{N}}$ is said to be centered if $\mathbb{E}[S_n] = 0$ for all $n \in \mathbb{N}$.

2. Given $F \in L^2(\Omega)$ a square-integrable random variable and $(\mathscr{F}_n)_{n\in\mathbb{N}}$ a filtration, the process $(X_n)_{n\in\mathbb{N}}$ defined by $X_n := \mathbb{E}[F \mid \mathscr{F}_n]$ is an $(\mathscr{F}_n)_{n\in\mathbb{N}}$-martingale under the probability measure $\mathbb{P}$, as follows from the tower property:

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = \mathbb{E}[\mathbb{E}[F \mid \mathscr{F}_{n+1}] \mid \mathscr{F}_n] = \mathbb{E}[F \mid \mathscr{F}_n] = X_n, \quad n \geqslant 0, \tag{1.2}$$

from the tower property (11.38).

The following Figure 1.1 illustrates various estimates $X_n = \mathbb{E}[F \mid \mathscr{F}_n]$ at time $n =$"Wed", "Thu", "Fri", "Sat", for a random outcome F known at time "Sat", *i.e.* $X_{\text{Wed}} = 26$, $X_{\text{Thu}} = 28$, $X_{\text{Fri}} = 26$, $X_{\text{Sat}} = 24$.

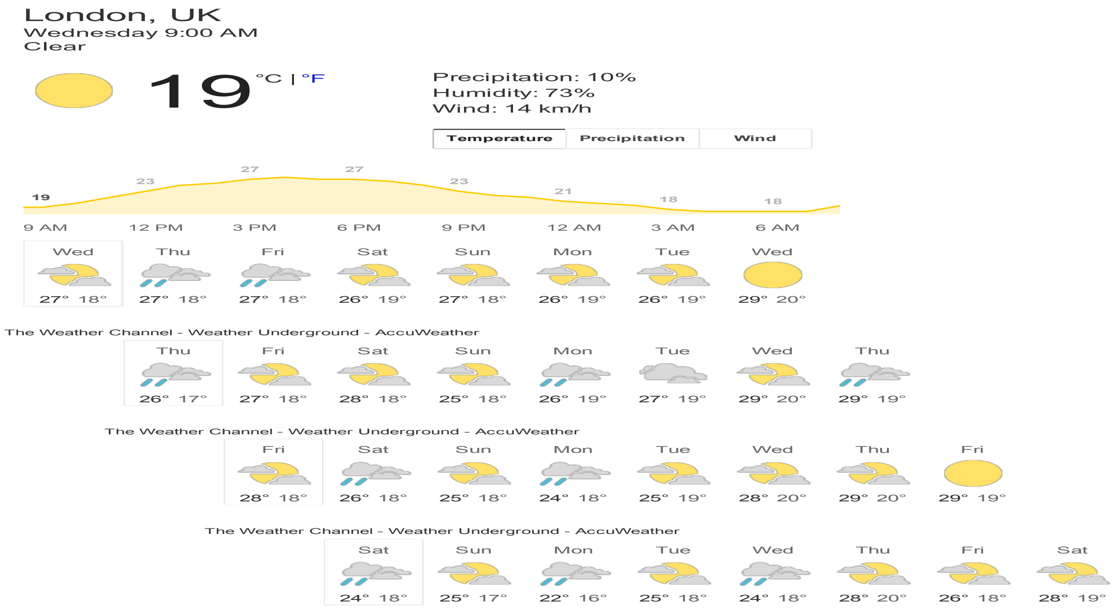


Figure 1.1: Updated weather forecast.

1.3 Stopping Times

Next, we turn to the definition of stopping time. If an event occurs at a (random) *stopping time*, it should be possible, at any time $n \in \mathbb{N}$, to determine whether the event has already occured, based on the information available up to time n. This idea is formalized in the next definition.

Definition 1.3 A random variable $\tau : \Omega \longrightarrow \mathbb{N} \cup \{+\infty\}$ is a stopping time with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$ if

$$\{\tau > n\} \in \mathscr{F}_n, \qquad n \geqslant 0. \tag{1.3}$$

The meaning of Relation (1.3) is that the knowledge of $\{\tau > n\}$ depends only on the information present in $\mathscr{F}_n$ up to time n, *i.e.* on the knowledge of $X_0, X_1, \ldots, X_n$.

Note that condition (1.3) is equivalent to the condition

$$\{\tau \leqslant n\} \in \mathscr{F}_n, \qquad n \geqslant 0,$$

since $\mathscr{F}_n$ is *stable by complement* and $\{\tau \leqslant n\} = \{\tau > n\}^c$.

Not every $\mathbb{N}$-valued random variable is a stopping time, however, hitting times provide natural examples of stopping times.

Proposition 1.2 The first hitting time

$$\tau_x := \inf\{k \geqslant 0 \ : \ X_k = x\}$$

of $x \in \mathbb{S}$ is a stopping time.

Proof. We have

$$\begin{aligned} \{\tau_x > n\} &= \{X_0 \neq x, \ X_1 \neq x, \ \ldots, X_n \neq x\} \\ &= \{X_0 \neq x\} \cap \{X_1 \neq x\} \cap \cdots \cap \{X_n \neq x\} \in \mathscr{F}_n, \quad n \geqslant 0, \end{aligned}$$

since

$$\{X_0 \neq x\} \in \mathscr{F}_0 \subset \mathscr{F}_n, \ \{X_1 \neq x\} \in \mathscr{F}_1 \subset \mathscr{F}_n, \ldots, \ \{X_n \neq x\} \in \mathscr{F}_n, \quad n \geqslant 0.$$

□

Hitting times can be used to trigger "buy limit" or "sell stop" orders in finance. On the other hand, the first time

$$\tau := \inf\left\{k \geqslant 0 \ : \ X_k = \operatorname*{Max}_{l=0,1,\ldots,N} X_l\right\}$$

the process $(X_k)_{k\in\mathbb{N}}$ reaches its maximum over $\{0,1,\ldots,N\}$ is not a stopping time. Indeed, it is not possible to decide whether $\{\tau \leqslant n\}$, *i.e.* the maximum has been reached before time n, based on the information available up to time n.

Exercise: Show from Definition 1.3 that the minimum $\tau \wedge \nu := \min(\tau, \nu)$ and the maximum $\tau \vee \nu := \operatorname{Max}(\tau, \nu)$ of two stopping times are themselves stopping times.

Definition 1.4 Given $(Z_n)_{n\in\mathbb{N}}$ a stochastic process and $\tau : \Omega \longrightarrow \mathbb{N}$ a stopping time, the stopped process

$$(Z_{\tau\wedge n})_{n\in\mathbb{N}} = (Z_{\min(\tau,n)})_{n\in\mathbb{N}}$$

is defined as

$$Z_{\tau\wedge n} = Z_{\min(\tau,n)} = \begin{cases} Z_n & \text{if } \ n < \tau, \\ Z_\tau & \text{if } \ n \geqslant \tau, \end{cases}$$

Using indicator functions we may also write

$$Z_{\tau\wedge n} = Z_n \mathbb{1}_{\{n<\tau\}} + Z_\tau \mathbb{1}_{\{n\geqslant\tau\}}, \qquad n \geqslant 0.$$

The following Figure 1.2 is an illustration of the path of a stopped process.

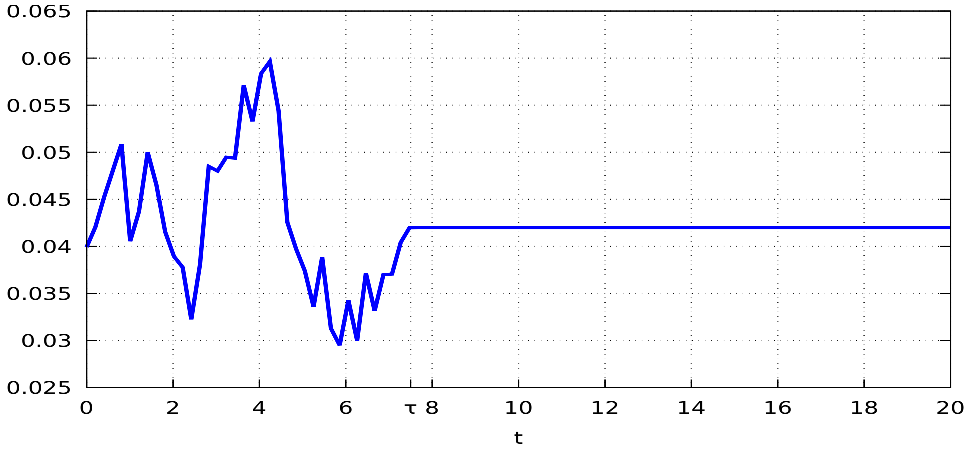


Figure 1.2: Stopped process.

The following Theorem 1.3 is called the Stopping Time Theorem, it is due to J.L. Doob (1910-2004).

Theorem 1.3 Assume that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$ and that $\tau:\Omega\longrightarrow\mathbb{N}\cup\{+\infty\}$ is a stopping time with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$. Then the *stopped process* $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

Proof. Writing

$$M_n = M_0 + \sum_{l=1}^{n}(M_l - M_{l-1}) = M_0 + \sum_{l\geqslant 1}\mathbb{1}_{\{l\leqslant n\}}(M_l - M_{l-1}),$$

we have

$$M_{\tau\wedge n} = M_0 + \sum_{l=1}^{\tau\wedge n}(M_l - M_{l-1}) = M_0 + \sum_{l=1}^{n}\mathbb{1}_{\{l\leqslant \tau\}}(M_l - M_{l-1}),$$

and for $k\leqslant n$ we find

$$\begin{aligned}
\mathbb{E}[M_{\tau\wedge n}\mid\mathscr{F}_k] &= \mathbb{E}\left[M_0 + \sum_{l=1}^{n}\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1})\,\Big|\,\mathscr{F}_k\right]\\
&= M_0 + \sum_{l=1}^{n}\mathbb{E}\left[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1})\,\big|\,\mathscr{F}_k\right]\\
&= M_0 + \sum_{l=1}^{k}\mathbb{E}\left[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1})\,\big|\,\mathscr{F}_k\right] + \sum_{l=k+1}^{n}\mathbb{E}\left[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1})\,\big|\,\mathscr{F}_k\right]\\
&= M_0 + \sum_{l=1}^{k}(M_l - M_{l-1})\,\mathbb{E}\left[\mathbb{1}_{\{l\leqslant\tau\}}\,\big|\,\mathscr{F}_k\right]\\
&\quad + \sum_{l=k+1}^{n}\mathbb{E}\left[\mathbb{E}\left[(M_l - M_{l-1})\mathbb{1}_{\{l-1<\tau\}}\,\big|\,\mathscr{F}_{l-1}\right]\,\big|\,\mathscr{F}_k\right]\\
&= M_0 + \sum_{l=1}^{k}(M_l - M_{l-1})\mathbb{1}_{\{l\leqslant\tau\}}\\
&\quad + \sum_{l=k+1}^{n}\mathbb{E}\left[\mathbb{1}_{\{l-1<\tau\}}\underbrace{\mathbb{E}\left[(M_l - M_{l-1})\,\big|\,\mathscr{F}_{l-1}\right]}_{=0}\,\Big|\,\mathscr{F}_k\right]
\end{aligned}$$

$$\begin{aligned} &= M_0 + \sum_{l=1}^{\tau\wedge k}(M_l - M_{l-1}) \\ &= M_{\tau\wedge k}, \end{aligned}$$

$k = 0, 1, \ldots, n$, where we used the tower property and the fact that

$$\{\tau \geqslant l\} = \{\tau > l-1\} \in \mathscr{F}_{l-1} \subset \mathscr{F}_l \subset \mathscr{F}_k, \qquad 1 \leqslant l \leqslant k.$$

□

By the Stopping Time Theorem 1.3 we know that the stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is a martingale, hence its expectation is constant by Proposition 1.1.

i) If τ is a stopping time bounded by a constant $N > 0$, *i.e.* $\tau \leqslant N$, we have

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau\wedge N}] = \mathbb{E}[M_{\tau\wedge 0}] = \mathbb{E}[M_0]. \tag{1.4}$$

ii) As a consequence of (1.4), if $(M_n)_{n\in\mathbb{N}}$ is a martingale and $\tau \leqslant N$ and $\nu \leqslant N$ are two *bounded* stopping times bounded by a constant $N > 0$, we have

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu] = \mathbb{E}[M_0]. \tag{1.5}$$

iii) In case τ is only *a.s.* finite, *i.e.* $\mathbb{P}(\tau < \infty) = 1$, we may also write

$$\mathbb{E}[M_\tau] = \mathbb{E}\left[\lim_{n\to\infty} M_{\tau\wedge n}\right] = \lim_{n\to\infty} \mathbb{E}[M_{\tau\wedge n}] = \mathbb{E}[M_0],$$

provided that the limit and expectation signs can be exchanged, however this may not be always the case.

In some situations the exchange of limit and expectation signs may not be valid.* Nevertheless, the exchange is possible when the stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is bounded in absolute value, *i.e.* $|M_{\tau\wedge n}| \leqslant K$ *a.s.*, $n \in \mathbb{N}$, for some constant $K > 0$, as a consequence of the *dominated convergence theorem*.

Analog statements can be proved for *sub*martingales, see *e.g.* Exercise 1.4 for this notion.

1.4 Ruin Probabilities

In the sequel we will show that, as an application of the Stopping Time Theorem 1.3, the ruin probabilities computed for random walks can be recovered in a simple and elegant way.

Consider the standard random walk (or gambling process) $(S_n)_{n\in\mathbb{N}}$ on $\{0, 1, \ldots, B\}$ with independent $\{-1, 1\}$-valued increments, and

$$\mathbb{P}(S_{n+1} - S_n = +1) = p \quad \text{and} \quad \mathbb{P}(S_{n+1} - S_n = -1) = q, \quad n \geqslant 0.$$

Let

$$\tau_{0,B} : \Omega \longrightarrow \mathbb{N}$$

be the first hitting time of the boundary $\{0, B\}$, defined by

$$\tau = \tau_{0,B} := \inf\{n \geqslant 0 \ : \ S_n = B \text{ or } S_n = 0\}.$$

*Consider for example the sequence $M_n := n\mathbb{1}_{\{U<1/n\}}$, $n \geqslant 1$, where $U \simeq U(0,1]$ is a uniformly distributed random variable on $(0,1]$, see Exercise 1.3.

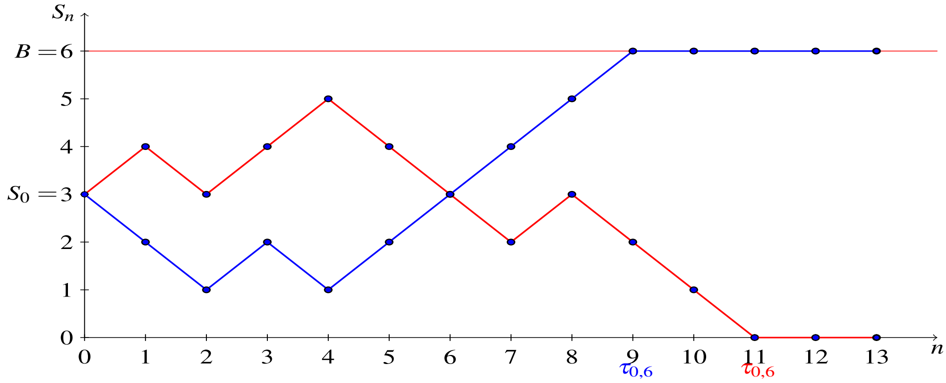


Figure 1.3: Sample paths of the random walk $(S_n)_{n\in\mathbb{N}}$.

One can easily check that the event $\{\tau > n\}$ depends only on the history of $(S_k)_{k\in\mathbb{N}}$ up to time n since for $k \in \{1,2,\ldots,B-1\}$ we have

$$\{\tau > n\} = \{0 < S_0 < B\} \cap \{0 < S_1 < B\} \cap \cdots \cap \{0 < S_n < B\} \subset \mathscr{F}_n,$$

hence τ is a stopping time.

We will recover the ruin probabilities

$$\mathbb{P}(S_\tau = 0 \mid S_0 = k), \qquad k = 0,1,\ldots,B,$$

in three steps, first in the unbiased case $p = q = 1/2$ (note that the hitting time τ can be shown to be *a.s.* finite, *i.e.* $\mathbb{P}(\tau < \infty) = 1$.

Unbiased case $p = q = 1/2$

Step 1. The process $(S_n)_{n\in\mathbb{N}}$ is a martingale.

We note that the process $(S_n)_{n\in\mathbb{N}}$ has independent increments, and in the unbiased case $p = q = 1/2$ those increments are centered:

$$\mathbb{E}[S_{n+1} - S_n] = 1 \times p + (-1) \times q = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0, \tag{1.6}$$

hence $(S_n)_{n\in\mathbb{N}}$ is a *martingale* by Point 1 page 19.

Step 2. The stopped process $(S_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale, as a consequence of the Stopping Time Theorem 1.3.

Step 3. Since the stopped process $(S_{\tau\wedge n})_{n\in\mathbb{N}}$ is a martingale by the Stopping Time Theorem 1.3, we find that its expectation $\mathbb{E}[S_{\tau\wedge n} \mid S_0 = k]$ is constant in $n \in \mathbb{N}$ by Proposition 1.1, which gives

$$k = \mathbb{E}[S_0 \mid S_0 = k] = \mathbb{E}[S_{\tau\wedge n} \mid S_0 = k], \qquad k = 0,1,\ldots,B.$$

Letting n go to infinity and noting that $\mathbb{P}(\tau < \infty) = 1$ and $|S_n| \leqslant B$, $n \in \mathbb{N}$, we get

$$\begin{aligned}
\mathbb{E}[S_\tau \mid S_0 = k] &= \mathbb{E}\left[\lim_{n\to\infty} S_{\tau\wedge n} \,\Big|\, S_0 = k\right] \\
&= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge n} \mid S_0 = k] \\
&= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge 0} \mid S_0 = k] \\
&= \lim_{n\to\infty} \mathbb{E}[S_0 \mid S_0 = k]
\end{aligned}$$

$$= \quad k,$$

where the exchange between limit and expectation is justified by the boundedness $|S_{\tau\wedge n}| \leqslant B$ *a.s.*, $n \in \mathbb{N}$. Hence we have

$$\begin{cases} 0 \times \mathbb{P}(S_\tau = 0 \mid S_0 = k) + B \times \mathbb{P}(S_\tau = B \mid S_0 = k) = \mathbb{E}[S_\tau \mid S_0 = k] = k \\ \\ \mathbb{P}(S_\tau = 0 \mid S_0 = k) + \mathbb{P}(S_\tau = B \mid S_0 = k) = \mathbb{P}(\tau < \infty) = 1, \end{cases}$$

which shows that

$$\mathbb{P}(S_\tau = B \mid S_0 = k) = \frac{k}{B} \quad \text{and} \quad \mathbb{P}(S_\tau = 0 \mid S_0 = k) = 1 - \frac{k}{B},$$

$k = 0, 1, \ldots, B$. Namely, the solution has been obtained in a simple way without solving any finite difference equation, demonstrating the power of the martingale approach.

Biased case $p \neq q$

Next, we turn to the biased case where $p \neq q$. In this case the process $(S_n)_{n\in\mathbb{N}}$ is no longer a martingale because its increments are not centered:

$$\mathbb{E}[S_{n+1} - S_n] = 1 \times p + (-1) \times q = p - q \neq 0. \tag{1.7}$$

In order to apply the Stopping Time Theorem 1.3, we need to construct a martingale of a different type. Here, we note that the process

$$M_n := \left(\frac{q}{p}\right)^{S_n}, \qquad n \geqslant 0,$$

is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

Step 1. The process $(M_n)_{n\in\mathbb{N}}$ is a martingale.

Indeed, we have

$$\begin{aligned}
\mathbb{E}[M_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \Big| \mathscr{F}_n\right] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n}\left(\frac{q}{p}\right)^{S_n} \Big| \mathscr{F}_n\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n} \Big| \mathscr{F}_n\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n}\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\mathbb{P}(S_{n+1} - S_n = 1) + \left(\frac{q}{p}\right)^{-1} \mathbb{P}(S_{n+1} - S_n = -1)\right) \\
&= \left(\frac{q}{p}\right)^{S_n} \left(p\frac{q}{p} + q\left(\frac{q}{p}\right)^{-1}\right) \\
&= \left(\frac{q}{p}\right)^{S_n} \left(\frac{pq^2 + p^2q}{pq}\right) \\
&= \left(\frac{q}{p}\right)^{S_n} (q+p) = \left(\frac{q}{p}\right)^{S_n} = M_n,
\end{aligned}$$

$n \in \mathbb{N}$. In particular, the expectation of $(M_n)_{n\in\mathbb{N}}$ is constant over time by Proposition 1.1 since it is a martingale, *i.e.* we have

$$\left(\frac{q}{p}\right)^k = \mathbb{E}[M_0 \mid S_0 = k] = \mathbb{E}[M_n \mid S_0 = k], \qquad k = 0,1,\ldots,B, \quad n \geqslant 0.$$

Step 2. The stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale, as a consequence of the Stopping Time Theorem 1.3.

Step 3. Since the stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ remains a martingale by the Stopping Time Theorem 1.3, its expected value $\mathbb{E}[M_{\tau\wedge n} \mid S_0 = k]$ is constant in $n \in \mathbb{N}$ by Proposition 1.1. This gives

$$\left(\frac{q}{p}\right)^k = \mathbb{E}[M_0 \mid S_0 = k] = \mathbb{E}[M_{\tau\wedge n} \mid S_0 = k].$$

Next, letting n go to infinity we find*

$$\begin{aligned}
\mathbb{E}[M_\tau \mid S_0 = k] &= \mathbb{E}\left[\lim_{n\to\infty} M_{\tau\wedge n} \,\middle|\, S_0 = k\right] \\
&= \lim_{n\to\infty} \mathbb{E}[M_{\tau\wedge n} \mid S_0 = k] \\
&= \mathbb{E}[M_0 \mid S_0 = k] \\
&= \left(\frac{q}{p}\right)^k,
\end{aligned}$$

hence

$$\begin{aligned}
\left(\frac{q}{p}\right)^k &= \mathbb{E}[M_\tau \mid S_0 = k] \\
&= \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \,\middle|\, S_0 = k\right) + \left(\frac{q}{p}\right)^0 \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^0 \,\middle|\, S_0 = k\right) \\
&= \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \,\middle|\, S_0 = k\right) + \mathbb{P}(M_\tau = 1 \mid S_0 = k).
\end{aligned}$$

Solving the system of equations

$$\begin{cases}
\left(\frac{q}{p}\right)^k = \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \,\middle|\, S_0 = k\right) + \mathbb{P}(M_\tau = 1 \mid S_0 = k) \\
\\
\mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \,\middle|\, S_0 = k\right) + \mathbb{P}(M_\tau = 1 \mid S_0 = k) = 1,
\end{cases}$$

gives

$$\begin{aligned}
\mathbb{P}(S_\tau = B \mid S_0 = k) &= \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \,\middle|\, S_0 = k\right) \qquad (1.8) \\
&= \frac{(q/p)^k - 1}{(q/p)^B - 1}, \qquad k = 0,1,\ldots,B,
\end{aligned}$$

*Using the fact that $\mathbb{P}(\tau < \infty) = 1$.

and

$$
\begin{aligned}
\mathbb{P}(S_\tau = 0 \mid S_0 = k) &= \mathbb{P}(M_\tau = 1 \mid S_0 = k) \\
&= 1 - \frac{(q/p)^k - 1}{(q/p)^B - 1}, \\
&= \frac{(q/p)^B - (q/p)^k}{(q/p)^B - 1},
\end{aligned}
$$

$k = 0, 1, \ldots, B$.

1.5 Mean Game Duration

In this section we show that the mean game durations $\mathbb{E}[\tau \mid S_0 = k]$ can also be recovered as a second application of the Stopping Time Theorem 1.3.

In the case of a fair game $p = q = 1/2$ the martingale method can be used by noting that $\left(S_n^2 - n\right)_{n \in \mathbb{N}}$ is also a martingale.

Step 1. The process $\left(S_n^2 - n\right)_{n \in \mathbb{N}}$ is a martingale.

We have

$$
\begin{aligned}
&\mathbb{E}[S_{n+1}^2 - (n+1) \mid \mathcal{F}_n] = \mathbb{E}[(S_{n+1} - S_n + S_n)^2 - (n+1) \mid \mathcal{F}_n] \\
&\quad = \mathbb{E}[S_n^2 + (S_{n+1} - S_n)^2 + 2S_n(S_{n+1} - S_n) - (n+1) \mid \mathcal{F}_n] \\
&\quad = \mathbb{E}[S_n^2 - n - 1 \mid \mathcal{F}_n] + \mathbb{E}[(S_{n+1} - S_n)^2 \mid \mathcal{F}_n] + 2\,\mathbb{E}[S_n(S_{n+1} - S_n) \mid \mathcal{F}_n] \\
&\quad = S_n^2 - n - 1 + \mathbb{E}[(S_{n+1} - S_n)^2 \mid \mathcal{F}_n] + 2S_n\,\mathbb{E}[S_{n+1} - S_n \mid \mathcal{F}_n] \\
&\quad = S_n^2 - n - 1 + \mathbb{E}[(S_{n+1} - S_n)^2] + 2S_n\,\mathbb{E}[S_{n+1} - S_n] \\
&\quad = S_n^2 - n - 1 + \mathbb{E}[(S_{n+1} - S_n)^2] \\
&\quad = S_n^2 - n, \quad n \geqslant 0.
\end{aligned}
$$

Step 2. The stopped process $\left(S_{\tau \wedge n}^2 - \tau \wedge n\right)_{n \in \mathbb{N}}$ is also a martingale, as a consequence of the Stopping Time Theorem 1.3.

Step 3. Since the stopped process $\left(S_{\tau \wedge n}^2 - \tau \wedge n\right)_{n \in \mathbb{N}}$ is also a martingale by the Stopping Time Theorem 1.3, its expectation $\mathbb{E}[S_{\tau \wedge n}^2 - \tau \wedge n \mid S_0 = k]$ is constant in $n \in \mathbb{N}$ by Proposition 1.1, hence we have

$$
k^2 = \mathbb{E}[S_0^2 - 0 \mid S_0 = k] = \mathbb{E}\left[S_{\tau \wedge n}^2 - \tau \wedge n \,\middle|\, S_0 = k\right],
$$

and since $\mathbb{P}(\tau < \infty) = 1$, after taking the limit as n tends to infinity we find

$$
\begin{aligned}
\mathbb{E}\left[S_\tau^2 - \tau \mid S_0 = k\right] &= \mathbb{E}\left[\lim_{n \to \infty} S_{\tau \wedge n}^2 - \lim_{n \to \infty} \tau \wedge n \,\middle|\, S_0 = k\right] \\
&= \mathbb{E}\left[\lim_{n \to \infty} S_{\tau \wedge n}^2 \mid S_0 = k\right] - \mathbb{E}\left[\lim_{n \to \infty} \tau \wedge n \mid S_0 = k\right] \\
&= \lim_{n \to \infty} \mathbb{E}\left[S_{\tau \wedge n}^2 \mid S_0 = k\right] - \lim_{n \to \infty} \mathbb{E}[\tau \wedge n \mid S_0 = k] \\
&= \lim_{n \to \infty} \mathbb{E}\left[S_{\tau \wedge n}^2 - \tau \wedge n \,\middle|\, S_0 = k\right] \\
&= k^2,
\end{aligned}
$$

since $S^2_{\tau\wedge n} \in [0, B^2]$ for all $n \in \mathbb{N}$ and the mapping $n \mapsto \tau \wedge n$ is non-decreasing in $n \in \mathbb{N}$, and this gives*†

$$\begin{aligned} k^2 &= \mathbb{E}\left[S^2_\tau - \tau \mid S_0 = k\right] \\ &= \mathbb{E}\left[S^2_\tau \mid S_0 = k\right] - \mathbb{E}[\tau \mid S_0 = k] \\ &= B^2 \mathbb{P}(S_\tau = B \mid S_0 = k) + 0^2 \times \mathbb{P}(S_\tau = 0 \mid S_0 = k) - \mathbb{E}[\tau \mid S_0 = k], \end{aligned}$$

i.e.

$$\begin{aligned} \mathbb{E}[\tau \mid S_0 = k] &= B^2 \mathbb{P}(S_\tau = B \mid S_0 = k) - k^2 \\ &= B^2 \frac{k}{B} - k^2 \\ &= k(B-k), \end{aligned}$$

$k = 0, 1, \ldots, B$.

Finally, we show how to recover the value of the mean game duration, *i.e.* the mean hitting time of the boundaries $\{0, B\}$ in the non-symmetric case $p \neq q$.

Step 1. The process $S_n - (p-q)n$ is a martingale.

In this case we note that although $(S_n)_{n\in\mathbb{N}}$ does not have centered increments and is not a martingale, the compensated process

$$S_n - (p-q)n, \qquad n \geqslant 0,$$

is a martingale because, in addition to being independent, its increments are centered random variables:

$$\mathbb{E}[S_{n+1} - S_n - (p-q)] = \mathbb{E}[S_{n+1} - S_n] - (p-q) = 0, \qquad n \geqslant 0,$$

by (1.7).

Step 2. The stopped process $(S_{\tau\wedge n} - (p-q)(\tau \wedge n))_{n\in\mathbb{N}}$ is also a martingale, as a consequence of the Stopping Time Theorem 1.3.

Step 3. The expectation $\mathbb{E}[S_{\tau\wedge n} - (p-q)(\tau\wedge n) \mid S_0 = k]$ is constant in $n \in \mathbb{N}$.

Step 4. Since the stopped process $(S_{\tau\wedge n} - (p-q)(\tau \wedge n))_{n\in\mathbb{N}}$ is a martingale by the Stopping Time Theorem 1.3, we have

$$k = \mathbb{E}[S_0 - 0 \mid S_0 = k] = \mathbb{E}[S_{\tau\wedge n} - (p-q)(\tau\wedge n) \mid S_0 = k],$$

and, since $\mathbb{P}(\tau < \infty) = 1$, after taking the limit as n goes to infinity we find

$$\begin{aligned} \mathbb{E}[S_\tau - (p-q)\tau \mid S_0 = k] &= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge n} - (p-q)(\tau\wedge n) \mid S_0 = k] \\ &= \mathbb{E}\left[\lim_{n\to\infty} S_{\tau\wedge n} - (p-q)\lim_{n\to\infty} \tau\wedge n \,\middle|\, S_0 = k\right] \\ &= k, \end{aligned}$$

which gives

$$k = \mathbb{E}[S_\tau - (p-q)\tau \mid S_0 = k]$$

*By application of the *dominated convergence* theorem.
†By application of the *monotone convergence* theorem.

$$\begin{aligned}
&= \mathbb{E}[S_\tau \mid S_0 = k] - (p-q)\,\mathbb{E}[\tau \mid S_0 = k] \\
&= B \times \mathbb{P}(S_\tau = B \mid S_0 = k) + 0 \times \mathbb{P}(S_\tau = 0 \mid S_0 = k) - (p-q)\,\mathbb{E}[\tau \mid S_0 = k],
\end{aligned}$$

i.e.

$$\begin{aligned}
(p-q)\,\mathbb{E}[\tau \mid S_0 = k] &= B \times \mathbb{P}(S_\tau = B \mid S_0 = k) - k \\
&= B\frac{(q/p)^k - 1}{(q/p)^B - 1} - k,
\end{aligned}$$

from (1.8), hence

$$\mathbb{E}[\tau \mid S_0 = k] = \frac{1}{p-q}\left(B\frac{(q/p)^k - 1}{(q/p)^B - 1} - k\right), \qquad k = 0, 1, \ldots, B.$$

In Table 1.1, we summarize the family of martingales used to treat the above problems.

Problem \ Probabilities	Unbiased	Biased
Ruin probability	S_n	$\left(\frac{q}{p}\right)^{S_n}$
Mean game duration	$S_n^2 - n$	$S_n - (p-q)n$

Table 1.1: List of martingales.

Exercises

Exercise 1.1 Consider a sequence $(X_n)_{n\geqslant 1}$ of independent Bernoulli random variables, with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}, \qquad n \geqslant 1,$$

and the process $(M_n)_{n\in\mathbb{N}}$ defined by $M_0 := 0$ and

$$M_n := \sum_{k=1}^{n} 2^{k-1} X_k, \qquad n \geqslant 1,$$

with in particular

$$\begin{cases}
M_1 = X_1, \\
M_2 = X_1 + 2X_2, \\
M_1 = X_1 + 2X_2 + 4X_3, \\
\vdots
\end{cases}$$

see Figure 1.4.

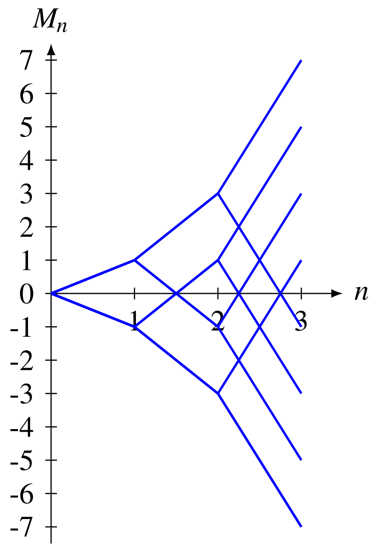


Figure 1.4: Possible paths of the process $(M_n)_{n\in\mathbb{N}}$.

Note that when $X_1 = X_2 = \cdots = X_{n-1} = -1$ and $X_n = 1$, we have

$$M_n = -\sum_{k=1}^{n-1} 2^{k-1} + 2^{n-1} = -\frac{1-2^{n-1}}{1-2} + 2^{n-1} = 1, \qquad n \geqslant 1,$$

while when $X_1 = X_2 = \cdots = X_{n-1} = X_n = -1$, we have

$$M_n = -\sum_{k=1}^{n} 2^{k-1} = -\frac{1-2^n}{1-2} = 1-2^n, \qquad n \geqslant 1.$$

a) Show that the process $(M_n)_{n\in\mathbb{N}}$ is a martingale.
b) Is the random time

$$\tau := \inf\{n \geqslant 1 \ : \ M_n = 1\}$$

a stopping time?
c) Consider the stopped process

$$M_{\tau\wedge n} := M_n \mathbb{1}_{\{n<\tau\}} + \mathbb{1}_{\{\tau\leqslant n\}} = \begin{cases} M_n = 1-2^n & \text{if } n < \tau, \\ \\ M_\tau = 1 & \text{if } n \geqslant \tau, \end{cases}$$

$n \in \mathbb{N}$, see Figure 1.5. Give an interpretation of $(M_{n\wedge\tau})_{n\in\mathbb{N}}$ in terms of betting strategy for a gambler starting a game at $M_0 = 0$.

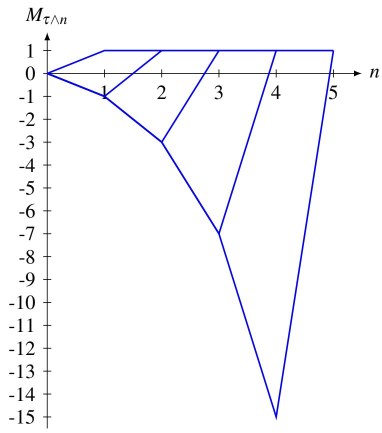


Figure 1.5: Possible paths of the stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$.

d) Determine the two possible values of $M_{\tau\wedge n}$ and the probability distribution of $M_{\tau\wedge n}$ at any time $n \geqslant 1$.

e) Show, using the result of Question (d)), that we have

$$\mathbb{E}[M_{\tau\wedge n}] = 0, \qquad n \geqslant 0.$$

f) Show that the result of Question (e)) can be recovered using the Stopping Time Theorem 1.3.

Exercise 1.2 Show that, as the filtration $(\mathscr{F}_n)_{n\geqslant 0}$ satisfies $\mathscr{F}_{n-1} \subset \mathscr{F}_n$, $n \geqslant 1$, the Condition (1.3) is also equivalent to

$$\{\tau = n\} \in \mathscr{F}_n, \qquad n \geqslant 0. \tag{1.9}$$

Exercise 1.3 Give an example of an *a.s.* converging and unbounded sequence $(X_n)_{n\in\mathbb{N}}$ of random variables for which expectation and limit cannot be exchanged, *i.e.*

$$\lim_{n\to\infty} \mathbb{E}[X_n] \neq \mathbb{E}[\lim_{n\to\infty} X_n].$$

Exercise 1.4 Let $(M_n)_{n\in\mathbb{N}}$ be a discrete-time *sub*martingale with respect to a filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$, with $\mathscr{F}_0 = \{\emptyset, \Omega\}$, *i.e.* we have

$$M_n \leqslant \mathbb{E}[M_{n+1} \mid \mathscr{F}_n], \qquad n \geqslant 0.$$

a) Show that for all $n \geqslant 0$ we have $\mathbb{E}[M_n] \leqslant \mathbb{E}[M_{n+1}]$, *i.e.* *sub*martingales have a *non-decreasing* expected value.

b) Show that independent increment processes whose increments have nonnegative expectation are examples of *sub*martingales.

c) (Doob-Meyer decomposition). Show that there exists two processes $(N_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ such that

i) $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$,

ii) $(A_n)_{n\in\mathbb{N}}$ is *non-decreasing*, *i.e.* $A_n \leqslant A_{n+1}$, *a.s.*, $n \in \mathbb{N}$,

iii) $(A_n)_{n\in\mathbb{N}}$ is predictable in the sense that A_n is $\mathscr{F}_{n-1}$-measurable, $n \in \mathbb{N}$, and

iv) $M_n = N_n + A_n$, $n \in \mathbb{N}$.
Hint: Let $A_0 := 0$, $A_1 := A_0 + \mathbb{E}[M_1 - M_0 \mid \mathscr{F}_0]$, and

$$A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n], \quad n \geqslant 0,$$

and define $(N_n)_{n\in\mathbb{N}}$ in such a way that it satisfies the four required properties.

d) Show that for all bounded stopping times σ and τ such that $\sigma \leqslant \tau$ *a.s.*, we have

$$\mathbb{E}[M_\sigma] \leqslant \mathbb{E}[M_\tau].$$

Hint: Use the Doob Stopping Time Theorem 1.3 for martingales and (1.5).

Exercise 1.5

a) Show that for any convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have the inequality

$$\phi\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leqslant \frac{\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n)}{n}, \tag{1.10}$$

$x_1, \ldots, x_n \in \mathbb{R}$, $n \geqslant 1$.

b) Consider a martingale $(S_n)_{n=0,1,\ldots,N}$ under the probability measure $\mathbb{P}$, with respect to the filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$, and let ϕ be a convex function. Show the inequality

$$\mathbb{E}\left[\phi\left(\frac{S_1 + S_2 + \cdots + S_N}{N}\right)\right] \leqslant \mathbb{E}[\phi(S_N)].$$

Hint: Use in the following order:
(i) the convexity inequality (1.10),
(ii) the martingale property of $(S_k)_{k\in\mathbb{N}}$,
(iii) the conditional Jensen inequality $\phi(\mathbb{E}[F \mid \mathscr{G}]) \leqslant \mathbb{E}[\phi(F) \mid \mathscr{G}]$,
(iv) the tower property of conditional expectations.

c) Given the (convex) function $\phi(x) := (x - K)^+$, show that the price of a discrete-time Asian option with payoff

$$\left(\frac{S_1 + S_2 + \cdots + S_N}{N} - K\right)^+$$

is upper bounded by the price of the European *call* option with payoff $(S_N - K)^+$ and maturity N.

Exercise 1.6 A stochastic process $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale if it satisfies

$$M_k \leqslant \mathbb{E}[M_n \mid \mathscr{F}_k], \qquad k = 0, 1, \ldots, n.$$

a) Show that the expected value $\mathbb{E}[M_n]$ of *sub*martingales is non-decreasing in time $n \in \mathbb{N}$.

b) Consider the random walk given by $S_0 := 0$ and

$$S_n := \sum_{k=1}^{n} X_k = X_1 + X_2 + \cdots + X_n, \qquad n \geqslant 1,$$

where $(X_n)_{n\geqslant 1}$ is an *i.i.d.* Bernoulli sequence of $\{0,1\}$-valued random variables with $\mathbb{P}(X_n = 1) = p$, $n \geqslant 1$. Under which condition on $\alpha \in \mathbb{R}$ is the process $(S_n - \alpha n)_{n\in\mathbb{N}}$ a *sub*martingale?

Exercise 1.7 Recall that a stochastic process $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale if it satisfies

$$M_k \leqslant \mathbb{E}[M_n \mid \mathscr{F}_k], \qquad k = 0, 1, \ldots, n.$$

a) Show that any convex function $(\phi(M_n))_{n\in\mathbb{N}}$ of a martingale $(M_n)_{n\in\mathbb{N}}$ is itself a *sub*martingale.
Hint: Use Jensen's inequality.
b) Show that any convex non-decreasing function $\phi(M_n)$ of a *sub*martingale $(M_n)_{n\in\mathbb{N}}$ remains a *sub*martingale.

Problem 1.8

a) Consider $(M_n)_{n\in\mathbb{N}}$ a *nonnegative martingale*. For any $x > 0$, let

$$\tau_x := \inf\{n \geqslant 0 \ : \ M_n \geqslant x\}.$$

Show that the random time τ_x is a stopping time.
b) Show that for all $n \geqslant 0$ we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[M_n]}{x}, \qquad x > 0. \tag{1.11}$$

Hint: Proceed as in the proof of the Markov inequality and use the Doob Stopping Time Theorem 1.3 1.3 for the stopping time τ_x.
c) Show that (1.11) remains valid for a nonnegative *sub*martingale.

Hint: Use the Doob Stopping Time Theorem 1.3 for *sub*martingales as in Exercise 1.4-(d)).
d) Show that for any $n \geqslant 0$ we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[(M_n)^2]}{x^2}, \qquad x > 0.$$

e) Show that more generally we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[(M_n)^p]}{x^p}, \qquad x > 0,$$

for all $n \geqslant 0$ and $p \geqslant 1$.
f) Given $(Y_n)_{n\geqslant 1}$ a sequence of centered independent random variables with same mean $\mathbb{E}[Y_n] = 0$ and variance $\sigma^2 = \mathrm{Var}[Y_n]$, $n \geqslant 1$, consider the random walk $S_n = Y_1 + Y_2 + \cdots + Y_n$, $n \geqslant 1$, with $S_0 = 0$.

Show that for all $n \geqslant 0$ we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} |S_k| \geqslant x\Big) \leqslant \frac{n\sigma^2}{x^2}, \qquad x > 0.$$

g) Show that for any (not necessarily nonnegative) *sub*martingale, we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[M_n^+]}{x}, \qquad x > 0,$$

where $z^+ = \mathrm{Max}(z, 0)$, $z \in \mathbb{R}$.
h) A stochastic process $(M_n)_{n\in\mathbb{N}}$ is a *super*martingale* if it satisfies

$$\mathbb{E}[M_n \mid \mathcal{F}_k] \leqslant M_k, \qquad k = 0, 1, \ldots, n.$$

Show that for any *nonnegative super*martingale we have

$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[M_0]}{x}, \qquad x > 0.$$

* "This obviously inappropriate nomenclature was chosen under the malign influence of the noise level of radio's SUPERman program, a favorite supper-time program of Doob's son during the writing of Doob, 1953", cf. Doob, 1984, historical notes, page 808.

i) Show that for any *nonnegative submartingale* $(M_n)_{n\in\mathbb{N}}$ and any convex non-decreasing nonnegative function ϕ we have

$$\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n}\phi(M_k)\geqslant x\right)\leqslant\frac{\mathbb{E}[\phi(M_n)]}{x},\qquad x>0.$$

Hint: Consider the stopping time

$$\tau_x^\phi:=\inf\{n\geqslant 0\ :\ \phi(M_n)\geqslant x\},$$

and use the result of Exercise 1.7-(b)).

j) Give an example of a nonnegative *super*martingale which is *not* a martingale.

Exercise 1.9 Consider the random walk $(S_n)_{n\in\mathbb{N}}$ on $\{0,1,\ldots,B\}$ with independent $\{-1,1\}$-valued increments, and

$$\mathbb{P}(S_{n+1}-S_n=+1)=p\quad\text{and}\quad\mathbb{P}(S_{n+1}-S_n=-1)=q,\quad n\geqslant 0,$$

and the martingale

$$M_n:=\left(\frac{q}{p}\right)^{S_n},\qquad n\geqslant 0,$$

where $p,q\in(0,1)$ are such that $p+q=1$. Show that for all $n\geqslant 0$ and $r\geqslant 1$ we have

$$\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n}M_k\geqslant x\right)\leqslant\frac{(p(q/p)^r+q(p/q)^r)^n}{x^r},\qquad x>0.$$

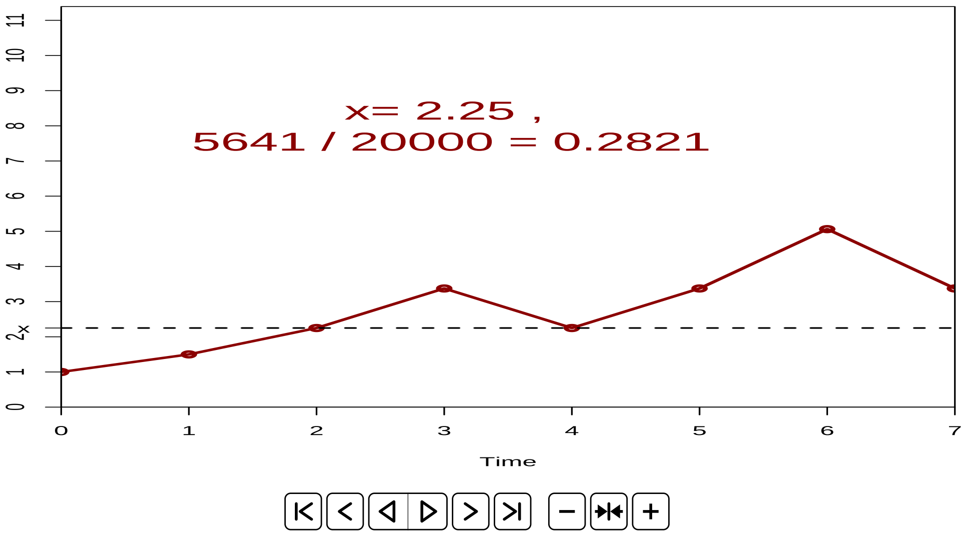


Figure 1.6: Random walk supremum.*

*The animation works in Acrobat Reader on the entire pdf file.

2. Assets, Portfolios, and Arbitrage

In this chapter, the concepts of portfolio, arbitrage, market completeness, pricing and hedging, are introduced in a simplified single-step financial model with only two time instants $t = 0$ and $t = 1$. A binary asset price model is considered as an example in Section 2.7.

2.1 Definitions and Notation

We will use the following notation. An element $\bar{x}$ of $\mathbb{R}^{d+1}$ is a vector

$$\bar{x} = (x^{(0)}, x^{(1)}, \ldots, x^{(d)})$$

made of $d+1$ components. The scalar product $\bar{x} \cdot \bar{y}$ of two vectors $\bar{x}, \bar{y} \in \mathbb{R}^{d+1}$ is defined by

$$\bar{x} \cdot \bar{y} := x^{(0)}y^{(0)} + x^{(1)}y^{(1)} + \cdots + x^{(d)}y^{(d)}.$$

The vector

$$\bar{S}_0 = (S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)})$$

denotes the prices at time $t=0$ of $d+1$ assets. Namely, $S_0^{(i)}>0$ is the price at time $t=0$ of asset n^o $i=0,1,\ldots,d$.

The asset values $S_1^{(i)}>0$ of assets No $i=0,1,\ldots,d$ at time $t=1$ are represented by the vector

$$\bar{S}_1=\big(S_1^{(0)},S_1^{(1)},\ldots,S_1^{(d)}\big),$$

whose components $\big(S_1^{(1)},\ldots,S_1^{(d)}\big)$ are random variables defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$.

In addition we will assume that asset n^o 0 is a riskless asset (of savings account type) that yields an interest rate $r>0$, *i.e.* we have

$$S_1^{(0)}=(1+r)S_0^{(0)}.$$

2.2 Portfolio Allocation and Short Selling

A *portfolio* based on the assets $0,1,\ldots,d$ is viewed as a vector

$$\bar{\xi}=\big(\xi^{(0)},\xi^{(1)},\ldots,\xi^{(d)}\big)\in\mathbb{R}^{d+1},$$

in which $\xi^{(i)}$ represents the (possibly fractional) quantity of asset n^o i owned by an investor, $i=0,1,\ldots,d$. The *price* of such a portfolio, or the cost of the corresponding investment, is given by

$$\bar{\xi}\bullet\bar{S}_0=\sum_{i=0}^d\xi^{(i)}S_0^{(i)}=\xi^{(0)}S_0^{(0)}+\xi^{(1)}S_0^{(1)}+\cdots+\xi^{(d)}S_0^{(d)}$$

at time $t=0$. At time $t=1$,the *value* of the portfolio has evolved into

$$\bar{\xi}\bullet\bar{S}_1=\sum_{i=0}^d\xi^{(i)}S_1^{(i)}=\xi^{(0)}S_1^{(0)}+\xi^{(1)}S_1^{(1)}+\cdots+\xi^{(d)}S_1^{(d)}.$$

There are various ways to construct a portfolio allocation $\big(\xi^{(i)}\big)_{i=0,1,\ldots,d}$.

i) If $\xi^{(0)}>0$, the investor puts the amount $\xi^{(0)}S_0^{(0)}>0$ on a savings account with interest rate r.

ii) If $\xi^{(0)}<0$, the investor borrows the amount $-\xi^{(0)}S_0^{(0)}>0$ with the same interest rate r.

iii) For $i=1,2,\ldots,d$, if $\xi^{(i)}>0$ then the investor purchases a (possibly fractional) quantity $\xi^{(i)}>0$ of the asset n^o i.

iv) If $\xi^{(i)}<0$, the investor borrows a quantity $-\xi^{(i)}>0$ of asset i and sells it to obtain the amount $-\xi^{(i)}S_0^{(i)}>0$.

In the latter case one says that the investor *short sells* a quantity $-\xi^{(i)}>0$ of the asset n^o i, which lowers the cost of the portfolio.

Definition 2.1 The *short selling ratio*, or percentage of daily turnover activity related to short selling, is defined as as the ratio of the number of daily short sold shares divided by daily volume.

Profits are usually made by first buying at a low price and then selling at a high price. Short sellers apply the same rule but in the reverse time order: first sell high, and then buy low if possible, by applying the following procedure.

1. Borrow the asset n^o i.

2. At time $t = 0$, sell the asset $n^o\ i$ on the market at the price $S_0^{(i)}$ and invest the amount $S_0^{(i)}$ at the interest rate $r > 0$.

3. Buy back the asset $n^o\ i$ at time $t = 1$ at the price $S_1^{(i)}$, with hopefully $S_1^{(i)} < (1+r)S_0^{(i)}$.

4. Return the asset to its owner, with possibly a (small) fee $p > 0$.*

At the end of the operation the profit made on share $n^o\ i$ equals

$$(1+r)S_0^{(i)} - S_1^{(i)} - p > 0,$$

which is positive provided that $S_1^{(i)} < (1+r)S_0^{(i)}$ and $p > 0$ is sufficiently small.

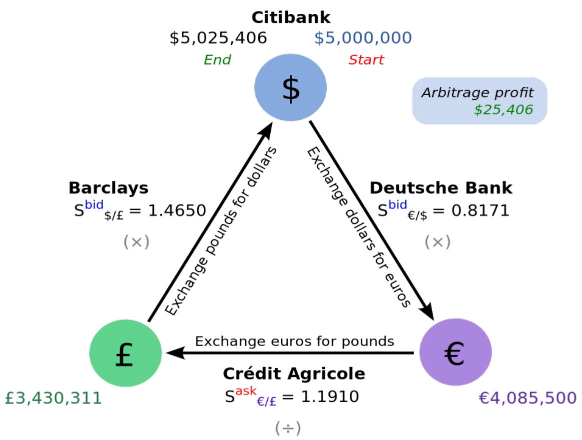
2.3 Arbitrage

Arbitrage can be described as:

"the purchase of currencies, securities, or commodities in one market for immediate resale in others in order to profit from unequal prices".†

In other words, an arbitrage opportunity is the possibility to make a strictly positive amount of money starting from zero, or even from a negative amount. In a sense, the existence of an arbitrage opportunity can be seen as a way to "beat" the market.

For example, triangular arbitrage is a way to realize arbitrage opportunities based on discrepancies in the cross exchange rates of foreign currencies, as seen in Figure 2.1.‡



(a) (Wikipedia).

XE Live Exchange Rates

USD	EUR	GBP
1.00000	0.89347	0.76988
1.11923	1.00000	0.86167
1.29891	1.16054	1.00000

(b) Cross exchange rates.

Figure 2.1: Examples of triangular arbitrage.

As an attempt to realize triangular arbitrage based on the data of Figure 2.1b, one could:

1. Change US$1.00 into €0.89347,
2. Change €0.89347 into $0.89347 \times 0.86167 = 0.769876295$,
3. Change back 0.769876295 into US$$0.769876295 \times 1.2981 =$ US$0.999376418,

which would actually result into a small loss. Alternatively, one could:

1. Change US$1.00 into 0.76988,

*The cost p of short selling will not be taken into account in later calculations.
†https://www.collinsdictionary.com/dictionary/english/arbitrage
‡https://en.wikipedia.org/wiki/Triangular_arbitrage

2. Change 0.76988 into €1.16054 $\times$ 0.76988 =€0.893476535,
3. Change back €0.893476535 into US\$0.893476535 $\times$ 1.11923 = US\$1.000005742,

which would result into a small gain, assuming the absence of transaction costs.

Next, we state a mathematical formulation of the concept of arbitrage.

Definition 2.2 A portfolio allocation $\bar{\xi} \in \mathbb{R}^{d+1}$ constitutes an arbitrage opportunity if the three following conditions are satisfied:

i) $\bar{\xi} \cdot \bar{S}_0 \leqslant 0$ at time $t = 0$, *[start from a zero-cost portfolio or in debt]*

ii) $\bar{\xi} \cdot \bar{S}_1 \geqslant 0$ at time $t = 1$, *[finish with a nonnegative amount]*

iii) $\mathbb{P}\big(\bar{\xi} \cdot \bar{S}_1 > 0\big) > 0$ at time $t = 1$. *[profit made with nonzero probability]*

Note that there exist multiple ways to break the assumptions of Definition 2.2 in order to achieve absence of arbitrage. For example, under absence of arbitrage, satisfying Condition (i) means that either $\bar{\xi} \cdot \bar{S}_1$ cannot be almost surely* nonnegative (*i.e.*, potential losses cannot be avoided), or $\mathbb{P}\big(\bar{\xi} \cdot \bar{S}_1 > 0\big) = 0$, (*i.e.*, no strictly positive profit can be made).

Realizing arbitrage

In the example below we realize arbitrage by buying and holding an asset.

1. Borrow the amount $-\xi^{(0)} S_0^{(0)} > 0$ on the riskless asset n^o 0.

2. Use the amount $-\xi^{(0)} S_0^{(0)} > 0$ to purchase a quantity $\xi^{(i)} = -\xi^{(0)} S_0^{(0)} / S_0^{(i)}$, of the risky asset n^o $i \geqslant 1$ at time $t = 0$ and price $S_0^{(i)}$ so that the initial portfolio cost is
$$\xi^{(0)} S_0^{(0)} + \xi^{(i)} S_0^{(i)} = 0.$$

3. At time $t = 1$, sell the risky asset n^o i at the price $S_1^{(i)}$, with hopefully $S_1^{(i)} > (1+r) S_0^{(i)}$.

4. Refund the amount $-(1+r)\xi^{(0)} S_0^{(0)} > 0$ with interest rate $r > 0$.

At the end of the operation the profit made is

$$\begin{aligned}
\xi^{(i)} S_1^{(i)} - \big(-(1+r)\xi^{(0)} S_0^{(0)}\big) &= \xi^{(i)} S_1^{(i)} + (1+r)\xi^{(0)} S_0^{(0)} \\
&= -\xi^{(0)} \frac{S_0^{(0)}}{S_0^{(i)}} S_1^{(i)} + (1+r)\xi^{(0)} S_0^{(0)} \\
&= -\xi^{(0)} \frac{S_0^{(0)}}{S_0^{(i)}} \big(S_1^{(i)} - (1+r) S_0^{(i)}\big) \\
&= \xi^{(i)} \big(S_1^{(i)} - (1+r) S_0^{(i)}\big) \\
&> 0,
\end{aligned}$$

or $S_1^{(i)} - (1+r) S_0^{(i)}$ per unit of stock invested, which is positive provided that $S_1^{(i)} > S_0^{(i)}$ and r is sufficiently small.

Arbitrage opportunities can be similarly realized using the short selling procedure described in Section 2.2.

*"Almost surely", or "*a.s.*", means "with probability one".

City	Currency	US$
Tokyo	38,800 yen	$346
Hong Kong	HK$2,956.67	$381
Seoul	378,533 won	$400
Taipei	NT$12,980	$404
New York		$433
Sydney	A$633.28	$483
Frankfurt	€399	$513
Paris	€399	$513
Rome	€399	$513
Brussels	€399.66	$514
London	£279.99	$527
Manila	29,500 pesos	$563
Jakarta	5,754,1676 rupiah	$627

Figure 2.2: Arbitrage: Retail prices around the world for the Xbox 360 in 2006.

There are many real-life examples of situations where arbitrage opportunities can occur, such as:
- assets with different returns (finance),

- servers with different speeds (queueing, networking, computing),

- highway lanes with different speeds (driving).

In the latter two examples, the absence of arbitrage is consequence of the fact that switching to a faster lane or server may result into congestion, thus annihilating the potential benefit of the shift.

六合彩投注換算表

MARK SIX INVESTMENT TABLE

複式 Multiple		一胆拖 One Banker with		两胆拖 Two Bankers with		三胆拖 Three Bankers with		四胆拖 Four Bankers with		五胆拖 Five Bankers with	
所選號碼總數 No. of Selections	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$	配腳數目 No. of Legs	HK$
7	35	6	30	5	25	4	20	3	15	2	10
8	140	7	105	6	75	5	50	4	30	3	15
9	420	8	280	7	175	6	100	5	50	4	20
10	1,050	9	630	8	350	7	175	6	75	5	25
11	2,310	10	1,260	9	630	8	280	7	105	6	30
12	4,620	11	2,310	10	1,050	9	420	8	140	7	35
13	8,580	12	3,960	11	1,650	10	600	9	180	8	40
14	15,015	13	6,435	12	2,475	11	825	10	225	9	45
15	25,025	14	10,010	13	3,575	12	1,100	11	275	10	50
49	69,919,080	48	8,561,520	47	891,825	46	75,900	45	4,950	44	220

Table 2.1: Absence of arbitrage - the Mark Six “Investment Table”.

In the table of Figure 2.1 the absence of arbitrage opportunities is materialized by the fact that the price of each combination is found to be proportional to its probability, thus making the game fair and disallowing any opportunity or arbitrage that would result of betting on a more profitable combination.

In the sequel, we will work under the assumption that arbitrage opportunities do not occur and we will rely on this hypothesis for the pricing of financial instruments.

Example: share rights

Let us give a market example of pricing by absence of arbitrage.

From March 24 to 31, 2009, HSBC issued *rights* to buy shares at the price of \$28. This *right* behaves similarly to an option in the sense that it gives the right (with no obligation) to buy the stock at the discount price $K = \$28$. On March 24, the HSBC stock price closed at \$41.70.

The question is: how to value the price $\$R$ of the right to buy one share? This question can be answered by looking for arbitrage opportunities. Indeed, the underlying stock can be purchased in two different ways:

1. Buy the stock directly on the market at the price of \$41.70. Cost: \$41.70,

   or:

2. First, purchase the right at price $\$R$, and then the stock at price \$28. Total cost: \$R+\$28.

a) In case

$$\$R + \$28 < \$41.70, \tag{2.1}$$

arbitrage would be possible for an investor who owns no stock and no rights, by
   i) Buying the right at a price $\$R$, and then
   ii) Buying the stock at price \$28, and
   iii) Reselling the stock at the market price of \$41.70.

The profit made by this investor would equal

$$\$41.70 - (\$R + \$28) > 0.$$

b) On the other hand, in case

$$\$R + \$28 > \$41.70, \tag{2.2}$$

arbitrage would be possible for an investor who owns the rights, by:
   i) Buying the stock on the market at \$41.70,
   ii) Selling the right by contract at the price $\$R$, and then
   iii) Selling the stock at \$28 to that other investor.

In this case, the profit made would equal

$$\$R + \$28 - \$41.70 > 0.$$

In the absence of arbitrage opportunities, the combination of (2.1) and (2.2) implies that $\$R$ should satisfy

$$\$R + \$28 - \$41.70 = 0,$$

i.e. the arbitrage-free price of the right is given by the equation

$$\$R = \$41.70 - \$28 = \$13.70. \tag{2.3}$$

Interestingly, the *market* price of the right was \$13.20 at the close of the session on March 24. The difference of \$0.50 can be explained by the presence of various market factors such as transaction costs, the time value of money, or simply by the fact that asset prices are constantly fluctuating over time. It may also represent a small arbitrage opportunity, which cannot be at all excluded. Nevertheless, the absence of arbitrage argument (2.3) prices the right at \$13.70, which is quite close to its market value. Thus the absence of arbitrage hypothesis appears as an accurate tool for pricing.

Exercise: A company issues share rights, so that ten rights allow one to purchase three shares at the price of €6.35. Knowing that the stock is currently valued at €8, estimate the price of the right by absence of arbitrage.

Answer: Letting R denote the price of one right, it will require $10R/3$ to purchase one stock at €6.35, hence absence of arbitrage tells us that

$$\frac{10}{3}R + 6.35 = 8,$$

from which it follows that

$$R = \frac{3}{10}(8 - 6.35) = €0.495.$$

Note that the actual share right was quoted at €0.465 according to market data.

2.4 Risk-Neutral Probability Measures

In order to use absence of arbitrage in the general context of pricing financial derivatives, we will need the notion of *risk-neutral probability measure*.

The next definition says that under a risk-neutral probability measure, the risky assets n^o $1,2,\ldots,d$ have same *average* rate of return as the riskless asset n^o 0.

Definition 2.3 A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if

$$\mathbb{E}^*\left[S_1^{(i)}\right] = (1+r)S_0^{(i)}, \qquad i = 1,2,\ldots,d. \tag{2.4}$$

Here, $\mathbb{E}^*$ denotes the expectation under the probability measure $\mathbb{P}^*$. Note that for $i = 0$, the condition $\mathbb{E}^*\left[S_1^{(0)}\right] = (1+r)S_0^{(0)}$ is always satisfied by definition.

In other words, $\mathbb{P}^*$ is called "risk neutral" because taking risks under $\mathbb{P}^*$ by buying a stock $S_1^{(i)}$ has a neutral effect: on average the expected yield of the risky asset equals the risk-free interest rate obtained by investing on the savings account with interest rate r.

On the other hand, under a "risk premium" probability measure $\mathbb{P}^\#$, the expected return of the risky asset $S_1^{(i)}$ would be higher than r, *i.e.* we would have

$$\mathbb{E}^\#\left[S_1^{(i)}\right] > (1+r)S_0^{(i)}, \qquad i = 1,2,\ldots,d,$$

whereas under a "negative premium" measure $\mathbb{P}^\flat$, the expected return of the risky asset $S_1^{(i)}$ would be lower than r, *i.e.* we would have

$$\mathbb{E}^\flat\left[S_1^{(i)}\right] < (1+r)S_0^{(i)}, \qquad i = 1,2,\ldots,d.$$

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 2.4 A probability measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{P}^*(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}. \tag{2.5}$$

The following Theorem 2.1 can be used to check for the existence of arbitrage opportunities, and is known as the first fundamental theorem of asset pricing.

Theorem 2.1 A market is *without* arbitrage opportunity if and only if it admits at least one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. (i) Sufficiency. Assume that there exists a risk-neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$. Since $\mathbb{P}^*$ is risk neutral we have

$$\bar{\xi} \cdot \bar{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)} = \frac{1}{1+r} \sum_{i=0}^{d} \xi^{(i)} \mathbf{E}^*\left[S_1^{(i)}\right] = \frac{1}{1+r} \mathbf{E}^*\left[\bar{\xi} \cdot \bar{S}_1\right] \geqslant 0, \tag{2.6}$$

by Definition 2.2-(ii). We proceed by contradiction, and suppose that the market admits an arbitrage opportunity. In this case, Definition 2.2-(iii) implies $\mathbb{P}\big(\bar{\xi} \cdot \bar{S}_1 > 0\big) > 0$, hence $\mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 > 0\big) > 0$ because $\mathbb{P}$ is equivalent to $\mathbb{P}^*$. Since by Relation (11.13) we have

$$\begin{aligned}
0 &< \mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 > 0\big) \\
&= \mathbb{P}^*\Big(\bigcup_{n \geqslant 1} \{\bar{\xi} \cdot \bar{S}_1 > 1/n\}\Big) \\
&= \lim_{n \to \infty} \mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 > 1/n\big) \\
&= \lim_{\varepsilon \searrow 0} \mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 > \varepsilon\big),
\end{aligned}$$

there exists $\varepsilon > 0$ such that $\mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 \geqslant \varepsilon\big) > 0$, hence

$$\begin{aligned}
\mathbf{E}^*\left[\bar{\xi} \cdot \bar{S}_1\right] &\geqslant \mathbf{E}^*\left[\bar{\xi} \cdot \bar{S}_1 \mathbb{1}_{\{\bar{\xi} \cdot \bar{S}_1 \geqslant \varepsilon\}}\right] \\
&\geqslant \varepsilon \mathbf{E}^*\left[\mathbb{1}_{\{\bar{\xi} \cdot \bar{S}_1 \geqslant \varepsilon\}}\right] \\
&= \varepsilon \mathbb{P}^*\big(\bar{\xi} \cdot \bar{S}_1 \geqslant \varepsilon\big) \\
&> 0,
\end{aligned}$$

and by (2.6) we conclude that

$$\bar{\xi} \cdot \bar{S}_0 = \frac{1}{1+r} \mathbf{E}^*\left[\bar{\xi} \cdot \bar{S}_1\right] > 0,$$

which results into a contradiction by Definition 2.2-(i). We conclude that the market is without arbitrage opportunities.

(ii) The proof of necessity relies on the theorem of separation of convex sets by hyperplanes Proposition 2.2 below, see Theorem 1.6 in Föllmer and Schied, 2004. It can be briefly sketched as follows. Given two financial assets with net discounted gains X, Y and a portfolio made of one unit of X and c unit(s) of Y, the absence of arbitrage opportunity property of Definition 2.2 can be reformulated by saying that for any portfolio choice determined by $c \in \mathbb{R}$, we have

$$X + cY \geqslant 0 \quad \Longrightarrow \quad X + cY = 0, \qquad \mathbb{P} - a.s., \tag{2.7}$$

i.e. a risk-free portfolio with no loss cannot entail a stricly positive gain. In other words, if one wishes to make a strictly positive gain on the market, one has to accept the possibility of a loss.

To show that this implies the existence of a risk-neutral probability measure $\mathbb{P}^*$ under which the risky investments have zero discounted return, *i.e.*

$$\mathbb{E}_{\mathbb{P}^*}[X] = \mathbb{E}_{\mathbb{P}^*}[Y] = 0, \tag{2.8}$$

the convex separation Proposition 2.2 below is applied to the convex subset

$$\mathscr{C} = \{(\mathbb{E}_{\mathbb{Q}}[X], \mathbb{E}_{\mathbb{Q}}[Y]) \ : \ \mathbb{Q} \in \mathscr{P}\} \subset \mathbb{R}^2$$

of $\mathbb{R}^2$, where $\mathscr{P}$ is the family of probability measures $\mathbb{Q}$ on Ω equivalent to $\mathbb{P}$. If (2.8) does not hold under any $\mathbb{P}^* \in \mathscr{P}$ then $(0,0) \notin \mathscr{C}$ and the convex separation Proposition 2.2 below applied to the convex sets $\mathscr{C}$ and $\{(0,0)\}$ shows the existence of $c \in \mathbb{R}$ such that

$$\mathbb{E}_{\mathbb{Q}}[X] + c\,\mathbb{E}_{\mathbb{Q}}[Y] \geqslant 0 \text{ for all } \mathbb{Q} \in \mathscr{P}, \tag{2.9}$$

and

$$\mathbb{E}_{\mathbb{P}^*}[X] + c\,\mathbb{E}_{\mathbb{P}^*}[Y] > 0 \text{ for some } \mathbb{P}^* \in \mathscr{P}. \tag{2.10}$$

Condition (2.9) shows that $X + cY \geqslant 0$ almost surely* while Condition (2.10) implies $\mathbb{P}^*(X + cY > 0) > 0$, which contradicts absence of arbitrage by Definition 2.2-(iii). If the direction of the inequalities in (2.9) and (2.10) are reversed, we can reach the same conclusion by replacing the allocation $(1, c)$ with $(-1, -c)$. □

Next is a version of the separation theorem for convex sets, which relies on the more general Theorem 2.3 below.

Proposition 2.2 Let $\mathscr{C}$ be a convex set in $\mathbb{R}^2$ such that $(0,0) \notin \mathscr{C}$. Then there exists $c \in \mathbb{R}$ such that *e.g.*

$$x + cy \geqslant 0,$$

for all $(x,y) \in \mathscr{C}$, and

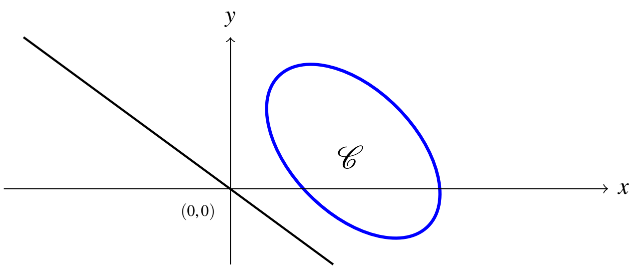
$$x^* + cy^* > 0,$$

for some $(x^*, y^*) \in \mathscr{C}$, up to a change of direction in both inequalities $\geqslant$ and $>$.

Proof. Theorem 2.3 below applied to $\mathscr{C}_1 := (0,0)$ and $\mathscr{C}_2 := \mathscr{C}$ shows that for some $a, c \in \mathbb{R}$ we have *e.g.*

$$0 + 0 \times c = 0 \leqslant a \leqslant x + cy$$

for all $(x,y) \in \mathscr{C}$.



On the other hand, if $x + cy = 0$ for all $(x,y) \in \mathscr{C}$ then the convex set $\mathscr{C}$ is contained in a line crossing $(0,0)$, for which there clearly exist $\tilde{c} \in \mathbb{R}$ such that $x + \tilde{c}y \geqslant 0$ for all $(x,y) \in \mathscr{C}$ and $x^* + \tilde{c}y^* > 0$ for some $(x^*, y^*) \in \mathscr{C}$, because $(0,0) \notin \mathscr{C}$. □

*"Almost surely", or "*a.s.*", means "with probability one".

The proof of Proposition 2.2 relies on the following result, see *e.g.* Theorem 4.14 in Hiriart-Urruty and Lemaréchal, 2001.

Theorem 2.3 Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two disjoint convex sets in $\mathbb{R}^2$. Then there exists $a, c \in \mathbb{R}$ such that

$$x^{(1)} + cy^{(1)} \leqslant a \qquad \text{and} \qquad a \leqslant x^{(2)} + cy^{(2)},$$

for all $(x^{(1)}, y^{(1)}) \in \mathscr{C}_1$ and $(x^{(2)}, y^{(2)}) \in \mathscr{C}_2$ (up to exchange of $\mathscr{C}_1$ and $\mathscr{C}_2$).

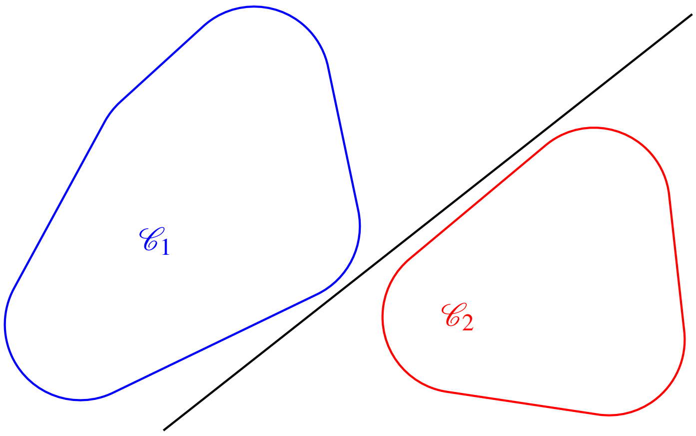


Figure 2.3: Separation of convex sets by the linear equation $x + cy = a$.

2.5 Hedging Contingent Claims

In this section we consider the notion of contingent claim. “Contingent” is an adjective that means:

1. Subject to chance.

2. Occurring or existing only if (certain circumstances) are the case; dependent on.

More generally, we will work according to the following broad definition.

Definition 2.5 A contingent claim is any nonnegative random variable $C : \Omega \longrightarrow [0, \infty)$.

In practice, the random variable C will represent the payoff of an (option) contract at time $t = 1$.

Referring to Definition 0.2, the European call option with maturity $t = 1$ on the asset n^o i is a contingent claim whose payoff C is given by

$$C = \left(S_1^{(i)} - K\right)^+ := \begin{cases} S_1^{(i)} - K & \text{if } S_1^{(i)} \geqslant K, \\ 0 & \text{if } S_1^{(i)} < K, \end{cases}$$

where K is called the *strike price*. The claim payoff C is called “contingent” because its value may depend on various market conditions, such as $S_1^{(i)} > K$. A contingent claim is also called a financial “derivative” for the same reason.

Similarly, referring to Definition 0.1, the European put option with maturity $t = 1$ on the asset

n^o i is a contingent claim with payoff

$$C = \left(K - S_1^{(i)}\right)^+ := \begin{cases} K - S_1^{(i)} & \text{if } S_1^{(i)} \leqslant K, \\ \\ 0 & \text{if } S_1^{(i)} > K, \end{cases}$$

Definition 2.6 A contingent claim with payoff C is said to be attainable if there exists a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)})$ such that

$$C = \bar{\xi} \cdot \bar{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} = \xi^{(0)} S_1^{(0)} + \xi^{(1)} S_1^{(1)} + \cdots + \xi^{(d)} S_1^{(d)},$$

with $\mathbb{P}$-probability one.

When a contingent claim with payoff C is attainable, a trader will be able to:

1. at time $t = 0$, build a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)}) \in \mathbb{R}^{d+1}$,

2. invest the amount
$$\bar{\xi} \cdot \bar{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}$$
in this portfolio at time $t = 0$,

3. at time $t = 1$, obtain the equality
$$C = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)}$$
and pay the claim amount C using the portfolio value
$$\bar{\xi} \cdot \bar{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} = \xi^{(0)} S_1^{(0)} + \xi^{(1)} S_1^{(1)} + \cdots + \xi^{(d)} S_1^{(d)}.$$

We note that in order to attain the claim payoff C, an initial investment $\bar{\xi} \cdot \bar{S}_0$ is needed at time $t = 0$. This amount, to be paid by the buyer to the issuer of the option (the option writer), is also called the *arbitrage-free price*, or option premium, of the contingent claim, and is denoted by

$$\pi_0(C) := \bar{\xi} \cdot \bar{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}. \tag{2.11}$$

The action of allocating a portfolio allocation $\bar{\xi}$ such that

$$C = \bar{\xi} \cdot \bar{S}_1 = \sum_{i=0}^{d} \xi^{(i)} S_1^{(i)} \tag{2.12}$$

is called *hedging*, or *replication*, of the contingent claim with payoff C.

Definition 2.7 In case the portfolio value $\bar{\xi} \cdot \bar{S}_1$ at time $t = 1$ exceeds the amount of the claim, *i.e.* when

$$\bar{\xi} \cdot \bar{S}_1 \geqslant C,$$

we say that the portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(d)})$ is *super-hedging*.

In this document we only focus on hedging, *i.e.* on *replication* of the contingent claim with payoff C, and we will not consider super-hedging.

As a simplified illustration of the principle of hedging, one may an buy oil-related asset in order to hedge oneself against a potential price rise of gasoline. In this case, any increase in the price of gasoline that would result in a higher value of the financial derivative C would be correlated to an increase in the underlying asset value, so that the equality (2.12) would be maintained.

2.6 Market Completeness

Market completeness is a strong property saying that any contingent claim can be perfectly hedged.

Definition 2.8 A market model is said to be *complete* if every contingent claim is attainable.

The next result is the second fundamental theorem of asset pricing.

Theorem 2.4 A market model without arbitrage opportunities is complete if and only if it admits only one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. See the proof of Theorem 1.40 in Föllmer and Schied, 2004. □

Theorem 2.4 will give us a concrete way to verify market completeness by searching for a unique solution $\mathbb{P}^*$ to Equation (2.4).

2.7 Example: Binary Market

In this section we work out a simple example that allows us to apply Theorem 2.1 and Theorem 2.4. We take $d = 1$, *i.e.* the portfolio is made of

- a riskless asset with interest rate r and priced $(1+r)S_0^{(0)}$ at time $t = 1$,

- and a risky asset priced $S_1^{(1)}$ at time $t = 1$.

We use the probability space

$$\Omega = \{\omega^-, \omega^+\},$$

which is the simplest possible nontrivial choice of probability space, made of only two possible outcomes with

$$\mathbb{P}(\{\omega^-\}) > 0 \quad \text{and} \quad \mathbb{P}(\{\omega^+\}) > 0,$$

in order for the setting to be nontrivial. In other words the behavior of the market is subject to only two possible outcomes, for example, one is expecting an important binary decision of "yes/no" type, which can lead to two distinct scenarios called ω^- and ω^+.

In this context, the asset price $S_1^{(1)}$ is given by a random variable

$$S_1^{(1)} : \Omega \longrightarrow \mathbb{R}$$

whose value depends on whether the scenario ω^-, resp. ω^+, occurs.

Precisely, we set

$$S_1^{(1)}(\omega^-) = a, \qquad \text{and} \qquad S_1^{(1)}(\omega^+) = b,$$

i.e., the value of $S_1^{(1)}$ becomes equal a under the scenario ω^-, and equal to b under the scenario ω^+, where $0 < a < b$. *

*The case $a = b$ leads to a trivial, constant market.

Arbitrage

The first natural question is:

- *Arbitrage*: Does the market allow for arbitrage opportunities?

We will answer this question using Theorem 2.1, by searching for a risk-neutral probability measure $\mathbb{P}^*$ satisfying the relation

$$\mathbb{E}^*\left[S_1^{(1)}\right] = (1+r)S_0^{(1)}, \tag{2.13}$$

where $r > 0$ denotes the risk-free interest rate, cf. Definition 2.3.

In this simple framework, any measure $\mathbb{P}^*$ on $\Omega = \{\omega^-, \omega^+\}$ is characterized by the data of two numbers $\mathbb{P}^*(\{\omega^-\}) \in [0,1]$ and $\mathbb{P}^*(\{\omega^+\}) \in [0,1]$, such that

$$\mathbb{P}^*(\Omega) = \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1. \tag{2.14}$$

Here, saying that $\mathbb{P}^*$ is *equivalent* to $\mathbb{P}$ simply means that

$$\mathbb{P}^*(\{\omega^-\}) > 0 \quad \text{and} \quad \mathbb{P}^*(\{\omega^+\}) > 0.$$

Although we should solve (2.13) for $\mathbb{P}^*$, at this stage it is not yet clear how $\mathbb{P}^*$ is involved in the equation. In order to make (2.13) more explicit we write the expected value as

$$\mathbb{E}^*\left[S_1^{(1)}\right] = a\mathbb{P}^*\big(S_1^{(1)} = a\big) + b\mathbb{P}^*\big(S_1^{(1)} = b\big),$$

hence Condition (2.13) for the existence of a risk-neutral probability measure $\mathbb{P}^*$ reads

$$a\mathbb{P}^*\big(S_1^{(1)} = a\big) + b\mathbb{P}^*\big(S_1^{(1)} = b\big) = (1+r)S_0^{(1)}.$$

Using the Condition (2.14) we obtain the system of two equations

$$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^+\}) = (1+r)S_0^{(1)} \\ \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases} \tag{2.15}$$

with *unique* risk-neutral solution

$$\boxed{\begin{cases} p^* := \mathbb{P}^*(\{\omega^+\}) = \mathbb{P}^*\big(S_1^{(1)} = b\big) = \dfrac{(1+r)S_0^{(1)} - a}{b-a} \\ \\ q^* := \mathbb{P}^*(\{\omega^-\}) = \mathbb{P}^*\big(S_1^{(1)} = a\big) = \dfrac{b-(1+r)S_0^{(1)}}{b-a}. \end{cases} \qquad (2.16)}$$

In order for a solution $\mathbb{P}^*$ to exist as a probability measure, the numbers $\mathbb{P}^*(\{\omega^-\})$ and $\mathbb{P}^*(\{\omega^+\})$ must be nonnegative. In addition, for $\mathbb{P}^*$ to be equivalent to $\mathbb{P}$ they should be strictly positive from (2.5).

We deduce that if a, b and r satisfy the condition

$$\boxed{a < (1+r)S_0^{(1)} < b, \qquad (2.17)}$$

then there exists a risk-neutral *equivalent* probability measure $\mathbb{P}^*$ which is unique, hence by Theorems 2.1 and 2.4 the market is without arbitrage and complete.

R

i) If $a = (1+r)S_0^{(1)}$, resp. $b = (1+r)S_0^{(1)}$, then $\mathbb{P}^*(\{\omega^+\}) = 0$, resp. $\mathbb{P}^*(\{\omega^-\}) = 0$, and $\mathbb{P}^*$ is not equivalent to $\mathbb{P}$.

Therefore, we check from (2.16) that the condition

$$a < b \leqslant (1+r)S_0^{(1)} \qquad \text{or} \qquad (1+r)S_0^{(1)} \leqslant a < b, \tag{2.18}$$

do not imply existence of an *equivalent* risk-neutral probability measure and absence of arbitrage opportunities in general.

ii) If $a = b = (1+r)S_0^{(1)}$ then (2.4) admits an infinity of solutions, hence the market is without arbitrage but it is not complete. More precisely, in this case both the riskless and risky assets yield a deterministic return rate r and the portfolio value becomes

$$\bar{\xi} \cdot \bar{S}_1 = (1+r)\bar{\xi} \cdot \bar{S}_0,$$

at time $t = 1$, hence the terminal value $\bar{\xi} \cdot \bar{S}_1$ is deterministic and this *single* value can not always match the value of a contingent claim with (random) payoff C, that could be allowed to take *two* distinct values $C(\omega^-)$ and $C(\omega^+)$. Therefore, market completeness does not hold when $a = b = (1+r)S_0^{(1)}$.

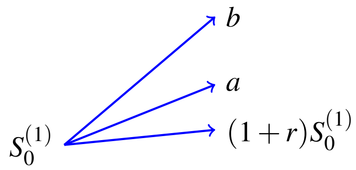
Let us give a financial interpretation of Condition (2.18).

1. If $(1+r)S_0^{(1)} \leqslant a < b$, let $\xi^{(1)} := 1$ and choose $\xi^{(0)}$ such that

$$\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} = 0$$

according to Definition 2.2-(i), *i.e.*

$$\xi^{(0)} = -\xi^{(1)} \frac{S_0^{(1)}}{S_0^{(0)}} < 0.$$



In particular, Condition (i)) of Definition 2.2 is satisfied, and the investor borrows the amount $-\xi^{(0)} S_0^{(0)} > 0$ on the riskless asset and uses it to buy one unit $\xi^{(1)} = 1$ of the risky asset. At time $t = 1$ he sells the risky asset $S_1^{(1)}$ at a price at least equal to a and refunds the amount $-(1+r)\xi^{(0)} S_0^{(0)} > 0$ that he borrowed, with interest. The profit of the operation is

$$\begin{aligned}
\bar{\xi} \cdot \bar{S}_1 &= (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} S_1^{(1)} \\
&\geqslant (1+r)\xi^{(0)} S_0^{(0)} + \xi^{(1)} a \\
&= -(1+r)\xi^{(1)} S_0^{(1)} + \xi^{(1)} a \\
&= \xi^{(1)}\big(-(1+r)S_0^{(1)} + a\big) \\
&\geqslant 0,
\end{aligned}$$

which satisfies Condition (ii)) of Definition 2.2. In addition, Condition (iii)) of Definition 2.2 is also satisfied because

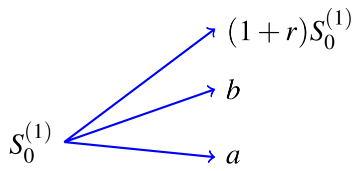
$$\mathbb{P}\big(\bar{\xi}\cdot\bar{S}_1>0\big)=\mathbb{P}\big(S_1^{(1)}=b\big)=\mathbb{P}(\{\omega^+\})>0.$$

2. If $a<b\leqslant(1+r)S_0^{(1)}$, let $\xi^{(0)}>0$ and choose $\xi^{(1)}$ such that

$$\xi^{(0)}S_0^{(0)}+\xi^{(1)}S_0^{(1)}=0,$$

according to Definition 2.2-(i), *i.e.*

$$\xi^{(1)}=-\xi^{(0)}\frac{S_0^{(0)}}{S_0^{(1)}}<0.$$



This means that the investor borrows a (possibly fractional) quantity $-\xi^{(1)}>0$ of the risky asset, sells it for the amount $-\xi^{(1)}S_0^{(1)}$, and invests this money on the riskless account for the amount $\xi^{(0)}S_0^{(0)}>0$. As mentioned in Section 2.2, in this case one says that the investor *shortsells* the risky asset. At time $t=1$ she obtains $(1+r)\xi^{(0)}S_0^{(0)}>0$ from the riskless asset, spends at most b to buy back the risky asset, and returns it to its original owner. The profit of the operation is

$$\begin{aligned}\bar{\xi}\cdot\bar{S}_1 &= (1+r)\xi^{(0)}S_0^{(0)}+\xi^{(1)}S_1^{(1)}\\ &\geqslant (1+r)\xi^{(0)}S_0^{(0)}+\xi^{(1)}b\\ &= -(1+r)\xi^{(1)}S_0^{(1)}+\xi^{(1)}b\\ &= \xi^{(1)}\big(-(1+r)S_0^{(1)}+b\big)\\ &\geqslant 0,\end{aligned}$$

since $\xi^{(1)}<0$. Note that here, $a\leqslant S_1^{(1)}\leqslant b$ became

$$\xi^{(1)}b\leqslant\xi^{(1)}S_1^{(1)}\leqslant\xi^{(1)}a$$

because $\xi^{(1)}<0$. We can check as in Part 1 above that Conditions (i))-(iii)) of Definition 2.2 are satisfied.

3. Finally if $a=b\neq(1+r)S_0^{(1)}$ then (2.4) admits no solution as a probability measure $\mathbb{P}^*$ hence arbitrage opportunities exist and can be constructed by the same method as above.

Under Condition (2.17) there is absence of arbitrage and Theorem 2.1 shows that no portfolio strategy can yield both $\bar{\xi}\cdot\bar{S}_1\geqslant 0$ and $\mathbb{P}\big(\bar{\xi}\cdot\bar{S}_1>0\big)>0$ starting from $\xi^{(0)}S_0^{(0)}+\xi^{(1)}S_0^{(1)}\leqslant 0$, however this is less simple to show directly.

Market completeness

The second natural question is:

- *Completeness*: Is the market complete, *i.e.*, are all contingent claims attainable?

In the sequel we work under the condition

$$a < (1+r)S_0^{(1)} < b,$$

under which Theorems 2.1 and 2.4 show that the market is without arbitrage and complete since the risk-neutral probability measure $\mathbb{P}^*$ exists and is unique.

Let us recover this fact by elementary calculations. For any contingent claim with payoff C we need to show that there exists a portfolio allocation $\bar{\xi} = (\xi^{(0)}, \xi^{(1)})$ such that $C = \bar{\xi} \cdot \bar{S}_1$, *i.e.*

$$\begin{cases} \xi^{(0)}(1+r)S_0^{(0)} + \xi^{(1)}b = C(\omega^+) \\ \\ \xi^{(0)}(1+r)S_0^{(0)} + \xi^{(1)}a = C(\omega^-). \end{cases} \tag{2.19}$$

These equations can be solved as

$$\xi^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{S_0^{(0)}(1+r)(b-a)} \quad \text{and} \quad \xi^{(1)} = \frac{C(\omega^+) - C(\omega^-)}{b-a}. \tag{2.20}$$

In this case we say that the portfolio allocation $(\xi^{(0)}, \xi^{(1)})$ *hedges* the contingent claim with payoff C. In other words, any contingent claim is attainable and the market is indeed complete. Here, the quantity

$$\xi^{(0)}S_0^{(0)} = \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)}$$

represents the amount invested on the riskless asset.

Note that if $C(\omega^+) \geqslant C(\omega^-)$ then $\xi^{(1)} \geqslant 0$ and there is not short selling. This occurs in particular if C has the form $C = h(S_1^{(1)})$ with $x \longmapsto h(x)$ a non-decreasing function, since

$$\begin{aligned} \xi^{(1)} &= \frac{C(\omega^+) - C(\omega^-)}{b-a} \\ &= \frac{h(S_1^{(1)}(\omega^+)) - h(S_1^{(1)}(\omega^-))}{b-a} \\ &= \frac{h(b) - h(a)}{b-a} \\ &\geqslant 0, \end{aligned}$$

thus there is no short selling. This applies in particular to European call options with strike price K, for which the function $h(x) = (x-K)^+$ is non-decreasing. Similarly we will find that $\xi^{(1)} \leqslant 0$, *i.e.* short selling always occurs when h is a non-increasing function, which is the case in particular for European put options with payoff function $h(x) = (K-x)^+$.

Arbitrage-free price

Definition 2.9 The *arbitrage-free price* $\pi_0(C)$ of the contingent claim with payoff C is defined in (2.11) as the initial value at time $t = 0$ of the portfolio hedging C, *i.e.*

$$\pi_0(C) = \bar{\xi} \cdot \bar{S}_0 = \sum_{i=0}^{d} \xi^{(i)} S_0^{(i)}, \tag{2.21}$$

where $(\xi^{(0)}, \xi^{(1)})$ are given by (2.20).

Arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value (mark to market).* Note that $\pi_0(C)$ cannot be 0 since this would entail the existence of an arbitrage opportunity according to Definition 2.2.

The next proposition shows that the arbitrage-free price $\pi_0(C)$ of the claim can be computed as the expected value of its payoff C under the risk-neutral probability measure, after discounting by the factor $1+r$ in order to account for the time value of money.

Proposition 2.5 The arbitrage-free price $\pi_0(C) = \bar{\xi} \cdot \bar{S}_0$ of the contingent claim with payoff C is given by

$$\pi_0(C) = \frac{1}{1+r} \mathbb{E}^*[C]. \tag{2.22}$$

Proof. Using the expressions (2.16) of the risk-neutral probabilities $\mathbb{P}^*(\{\omega^-\})$, $\mathbb{P}^*(\{\omega^+\})$, and (2.20) of the portfolio allocation $(\xi^{(0)}, \xi^{(1)})$, we have

$$\begin{aligned}
\pi_0(C) &= \bar{\xi} \cdot \bar{S}_0 \\
&= \xi^{(0)} S_0^{(0)} + \xi^{(1)} S_0^{(1)} \\
&= \frac{bC(\omega^-) - aC(\omega^+)}{(1+r)(b-a)} + S_0^{(1)} \frac{C(\omega^+) - C(\omega^-)}{b-a} \\
&= \frac{1}{1+r}\left(C(\omega^-) \frac{b - S_0^{(1)}(1+r)}{b-a} + C(\omega^+) \frac{(1+r)S_0^{(1)} - a}{b-a} \right) \\
&= \frac{1}{1+r}\left(C(\omega^-)\mathbb{P}^*\big(S_1^{(1)} = a\big) + C(\omega^+)\mathbb{P}^*\big(S_1^{(1)} = b\big) \right) \\
&= \frac{1}{1+r} \mathbb{E}^*[C].
\end{aligned}$$

$\square$

In the case of a European call option with strike price $K \in [a,b]$ we have $C = \big(S_1^{(1)} - K\big)^+$ and

$$\begin{aligned}
\pi_0\big((S_1^{(1)} - K)^+\big) &= \frac{1}{1+r} \mathbb{E}^*\left[(S_1^{(1)} - K)^+ \right] \\
&= \frac{1}{1+r} (b-K) \mathbb{P}^*\big(S_1^{(1)} = b\big) \\
&= \frac{1}{1+r} (b-K) \frac{(1+r)S_0^{(1)} - a}{b-a}. \\
&= \frac{b-K}{b-a} \left(S_0^{(1)} - \frac{a}{1+r} \right).
\end{aligned}$$

In the case of a European put option we have $C = \big(K - S_1^{(1)}\big)^+$ and

$$\begin{aligned}
\pi_0\big((K - S_1^{(1)})^+\big) &= \frac{1}{1+r} \mathbb{E}^*\left[(K - S_1^{(1)})^+ \right] \\
&= \frac{1}{1+r} (K-a) \mathbb{P}^*\big(S_1^{(1)} = a\big) \\
&= \frac{1}{1+r} (K-a) \frac{b - (1+r)S_0^{(1)}}{b-a}.
\end{aligned}$$

*Not to be confused with "market making".

$$= \frac{K-a}{b-a}\left(\frac{b}{1+r} - S_0^{(1)}\right).$$

Here, $\left(S_0^{(1)} - K\right)^+$, resp. $\left(K - S_0^{(1)}\right)^+$ is called the *intrinsic value* at time 0 of the call, resp. put option.

The simple setting described in this chapter raises several questions and remarks.

Remarks

1. The fact that $\pi_0(C)$ can be obtained by two different methods, *i.e.* an algebraic method via (2.20) and (2.21) and a probabilistic method from (2.22) is not a simple coincidence. It is actually a simple example of the deep connection that exists between probability and analysis.

   In a continuous-time setting, (2.20) will be replaced with a *partial differential equation* (PDE) and (2.22) will be computed via the *Monte Carlo* method. In practice, the quantitative analysis departments of major financial institutions can be split into a "*PDE team*" and a "*Monte Carlo* team", often trying to determine the same option prices by two different methods.

2. What if we have three possible scenarios, *i.e.* $\Omega = \{\omega^-, \omega^o, \omega^+\}$ and the random asset $S_1^{(1)}$ is allowed to take more than two values, *e.g.* $S_1^{(1)} \in \{a, b, c\}$ according to each scenario? In this case the system (2.15) would be rewritten as

   $$\begin{cases} a\mathbb{P}^*(\{\omega^-\}) + b\mathbb{P}^*(\{\omega^o\}) + c\mathbb{P}^*(\{\omega^+\}) = (1+r)S_0^{(1)} \\ \\ \mathbb{P}^*(\{\omega^-\}) + \mathbb{P}^*(\{\omega^o\}) + \mathbb{P}^*(\{\omega^+\}) = 1, \end{cases}$$

   and this system of two equations with three unknowns does not admit a unique solution, hence the market can be without arbitrage but it cannot be complete, cf. Exercise 2.4.

   Market completeness can be reached by adding a second risky asset, *i.e.* taking $d = 2$, in which case we will get three equations and three unknowns. More generally, when Ω contains $n \geqslant 2$ market scenarios, completeness of the market can be reached provided that we consider d risky assets with $d + 1 \geqslant n$. This is related to the Meta-Theorem 8.3.1 of Björk, 2004 in which the number d of traded risky underlying assets is linked to the number of random sources through arbitrage and market completeness.

Exercises

Exercise 2.1 Consider a risky asset valued $S_0 = \$3$ at time $t = 0$ and taking only two possible values $S_1 \in \{\$1, \$5\}$ at time $t = 1$, and a financial claim given at time $t = 1$ by

$$C := \begin{cases} \$0 \quad \text{if } S_1 = \$5 \\ \\ \$2 \quad \text{if } S_1 = \$1. \end{cases}$$

Is C the payoff of a call option or of a put option? Give the strike price of the option.

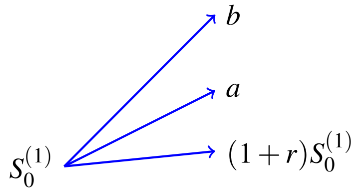
Exercise 2.2 Consider a risky asset valued $S_0 = \$4$ at time $t = 0$, and taking only two possible values $S_1 \in \{\$2, \$5\}$ at time $t = 1$. Compute the initial value $V_0 = \alpha S_0 + \$\beta$ of the portfolio hedging the claim payoff

$$C = \begin{cases} \$0 & \text{if } S_1 = \$5 \\ \\ \$6 & \text{if } S_1 = \$2 \end{cases}$$

at time $t = 1$, and find the corresponding risk-neutral probability measure $\mathbb{P}^*$.

Exercise 2.3

a) Consider the following market model:

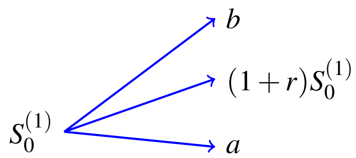


i) Does this model allow for arbitrage? Yes | No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing on savings | N.A. |

b) Consider the following market model:

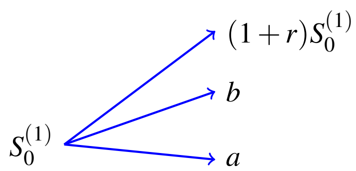


i) Does this model allow for arbitrage? Yes | No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing on savings | N.A. |

c) Consider the following market model:



i) Does this model allow for arbitrage? Yes | No |

ii) If this model allows for arbitrage opportunities, how can they be realized?

By shortselling | By borrowing on savings | N.A. |

Exercise 2.4 In a market model with two time instants $t = 0$ and $t = 1$, consider

- a riskless asset valued $S_0^{(0)}$ at time $t = 0$, and value $S_1^{(0)} = (1+r)S_0^{(0)}$ at time $t = 1$.

- a risky asset with price $S_0^{(1)}$ at time $t = 0$, and three possible values at time $t = 1$, with $a < b < c$, *i.e.*:

$$S_1^{(1)} = \begin{cases} S_0^{(1)}(1+a), \\ \\ S_0^{(1)}(1+b), \\ \\ S_0^{(1)}(1+c). \end{cases}$$

a) Show that this market is without arbitrage but not complete.

b) In general, is it possible to hedge (or replicate) a claim with three distinct claim payoff values C_a, C_b, C_c in this market?

Exercise 2.5 We consider a riskless asset valued $S_1^{(0)} = S_0^{(0)}$, at times $k = 0, 1$, where the risk-free interest rate is $\underline{r = 0}$, and a risky asset $S^{(1)}$ whose return $R_1 := (S_1^{(1)} - S_0^{(1)})/S_0^{(1)}$ can take three values $(-b, 0, b)$ at each time step, with $b > 0$ and

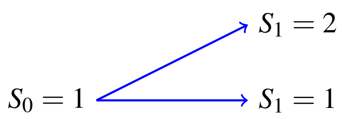
$$p^* := \mathbb{P}^*(R_1 = b) > 0, \quad \theta^* := \mathbb{P}^*(R_1 = 0) > 0, \quad q^* := \mathbb{P}^*(R_1 = -b) > 0,$$

a) Determine all possible risk-neutral probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in terms of the parameter $\theta^* \in (0, 1)$ from the condition $\mathbb{E}^*[R_1] = 0$.

b) We assume that the variance $\mathrm{Var}^*\left[\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}}\right] = \sigma^2 > 0$ of the asset return is known to be equal to σ^2. Show that this condition determines a unique risk-neutral probability measure $\mathbb{P}^*_\sigma$ under a certain condition on b and σ.

Exercise 2.6

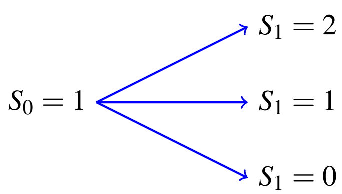
a) Consider the following binary one-step model $(S_t)_{t=0,1,2}$ with interest rate $\underline{r = 0}$ and $\mathbb{P}(S_1 = 2) = 1/3$.



i) Is the model without arbitrage? $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

ii) Does there exist a risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$? $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

b) Consider the following ternary one-step model with $\underline{r = 0}$, $\mathbb{P}(S_1 = 2) = 1/4$ and $\mathbb{P}(S_1 = 0) = 1/9$.



i) Does there exist a risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$? $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

ii) Is the model without arbitrage? $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

iii) Is the market complete? $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

iv) Does there exist a *unique* risk-neutral measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$?
$\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid}$

Exercise 2.7 Consider a one-step market model with two time instants $t = 0$ and $t = 1$ and two assets:
- a riskless asset π with price π_0 at time $t = 0$ and value $\pi_1 = \pi_0(1+r)$ at time $t = 1$,
- a risky asset S with price S_0 at time $t = 0$ and random value S_1 at time $t = 1$.

We assume that S_1 can take only the values $S_0(1+a)$ and $S_0(1+b)$, where $-1 < a < r < b$. The *return* of the risky asset is defined as

$$R = \frac{S_1 - S_0}{S_0}.$$

a) What are the possible values of R?

b) Show that under the probability measure $\mathbb{P}^*$ defined by

$$\mathbb{P}^*(R = a) = \frac{b-r}{b-a}, \qquad \mathbb{P}^*(R = b) = \frac{r-a}{b-a},$$

the expected return $\mathbb{E}^*[R]$ of S is equal to the return r of the riskless asset.

c) Does there exist arbitrage opportunities in this model? Explain why.

d) Is this market model complete? Explain why.

e) Consider a contingent claim with payoff C given by

$$C = \begin{cases} \alpha & \text{if } R = a, \\ \beta & \text{if } R = b. \end{cases}$$

Show that the portfolio allocation (η, ξ) defined* by

$$\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_0(1+r)(b-a)} \quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0(b-a)},$$

hedges the contingent claim with payoff C, *i.e.* show that at time $t = 1$ we have

$$\eta \pi_1 + \xi S_1 = C.$$

Hint: distinguish two cases $R = a$ and $R = b$.

f) Compute the arbitrage-free price $\pi_0(C)$ of the contingent claim payoff C using η, π_0, ξ, and S_0.

g) Compute $\mathbb{E}^*[C]$ in terms of a, b, r, α, β.

h) Show that the arbitrage-free price $\pi_0(C)$ of the contingent claim with payoff C satisfies

$$\pi_0(C) = \frac{1}{1+r}\mathbb{E}^*[C]. \tag{2.23}$$

i) What is the interpretation of Relation (2.23) above?

j) Let C denote the payoff at time $t = 1$ of a put option with strike price K=\$11 on the risky asset. Give the expression of C as a function of S_1 and K.

k) Letting $\pi_0 = S_0 = 1$, $r = 5\%$ and $a = 8$, $b = 11$, $\alpha = 2$, $\beta = 0$, compute the portfolio allocation (ξ, η) hedging the contingent claim with payoff C.

l) Compute the arbitrage-free price $\pi_0(C)$ of the claim payoff C.

Exercise 2.8 Consider a stock valued $S_0 = \$180$ at the beginning of the year. At the end of the year, its value S_1 can be either \$152 or \$203 and the risk-free interest rate is $r = 3\%$ per year. Given a put option with strike price K on this underlying asset, find the value of K for which the price of

*Here, η is the (possibly fractional) quantity of asset π and ξ is the quantity held of asset S.

the option at the beginning of the year is equal to the intrinsic option payoff. This value of K is called the break-even strike price. In other words, the break-even price is the value of K for which immediate exercise of the option is equivalent to holding the option until maturity.

How would a decrease in the interest rate r affect this break-even strike price?

3. Discrete-Time Model

The single-step model considered in Chapter 2 is extended to a discrete-time model with $N+1$ time instants $t = 0, 1, \ldots, N$. A basic limitation of the one-step model is that it does not allow for trading until the end of the first time period is reached, while the multistep model allows for multiple portfolio re-allocations over time. The Cox-Ross-Rubinstein (CRR) model, or binomial model, is considered as an example whose importance also lies with its computer implementability.

3.1 Discrete-Time Compounding

Investment plan

We invest an amount m each year in an investment plan that carries a constant interest rate r. At the end of the N-th year, the value of the amount m invested at the beginning of year $k = 1, 2, \ldots, N$ has turned into $(1+r)^{N-k+1}m$ and the value of the plan at the end of the N-th year becomes

$$\begin{aligned} A_N : &= m \sum_{k=1}^{N} (1+r)^{N-k+1} \qquad (3.1) \\ &= m \sum_{k=1}^{N} (1+r)^{k} \end{aligned}$$

$$= \quad m(1+r)\frac{(1+r)^N-1}{r},$$

i.e.

$$\boxed{\frac{A_N}{m} = \frac{(1+r)^{N+1}-(1+r)}{r},}$$

and

$$\boxed{N+1 = \frac{1}{\log(1+r)}\log\left(1+r+\frac{rA_N}{m}\right).}$$

Loan repayment

At time $t=0$ one borrows an amount $A_1 := A$ over a period of N years at the constant interest rate r per year.

Proposition 3.1 Constant repayment. Assuming that the loan is completely repaid at the beginning of year $N+1$, the amount m refunded every year is given by

$$\boxed{m = \frac{r(1+r)^N A}{(1+r)^N-1} = \frac{r}{1-(1+r)^{-N}}A.} \tag{3.2}$$

Proof. Denoting by A_k the amount owed by the borrower at the beginning of year n^o $k=1,2,\ldots,N$ with $A_1=A$, the amount m refunded at the end of the first year can be decomposed as

$$m = rA_1 + (m-rA_1),$$

into rA_1 paid in interest and $m-rA_1$ in principal repayment, *i.e.* there remains

$$\begin{aligned} A_2 &= A_1-(m-rA_1) \\ &= (1+r)A_1-m, \end{aligned}$$

to be refunded. Similarly, the amount m refunded at the end of the second year can be decomposed as

$$m = rA_2 + (m-rA_2),$$

into rA_2 paid in interest and $m-rA_2$ in principal repayment, *i.e.* there remains

$$\begin{aligned} A_3 &= A_2-(m-rA_2) \\ &= (1+r)A_2-m \\ &= (1+r)((1+r)A_1-m)-m \\ &= (1+r)^2A_1-m-(1+r)m \end{aligned}$$

to be refunded. After repeating the argument we find that at the beginning of year k there remains

$$\begin{aligned} A_k &= (1+r)^{k-1}A_1 - m\big(1+(1+r)+\cdots+(1+r)^{k-2}\big) \\ &= (1+r)^{k-1}A_1 - m\sum_{i=0}^{k-2}(1+r)^i \\ &= (1+r)^{k-1}A_1 + m\frac{1-(1+r)^{k-1}}{r} \end{aligned}$$

to be refunded, *i.e.*

$$\boxed{A_k = \frac{m-(1+r)^{k-1}(m-rA)}{r}, \qquad k=1,2,\ldots,N.} \tag{3.3}$$

We also note that the repayment at the end of year k can be decomposed as

$$m = rA_k + (m - rA_k),$$

with

$$rA_k = m + (1+r)^{k-1}(rA_1 - m)$$

in interest repayment, and

$$m - rA_k = (1+r)^{k-1}(m - rA_1)$$

in principal repayment. At the beginning of year $N+1$, the loan should be completely repaid, hence $A_{N+1} = 0$, which reads

$$(1+r)^N A + m\frac{1-(1+r)^N}{r} = 0,$$

and yields (3.2). □

We also have

$$\boxed{\frac{A}{m} = \frac{1-(1+r)^{-N}}{r}.} \tag{3.4}$$

and

$$\boxed{N = \frac{1}{\log(1+r)} \log\frac{m}{m-rA} = -\frac{\log(1-rA/m)}{\log(1+r)}.}$$

Remark: One needs $m > rA$ in order for N to be finite.

The next proposition is a direct consequence of (3.2) and (3.3).

Proposition 3.2 The *k-th* interest repayment can be written as

$$\boxed{rA_k = m\left(1 - \frac{1}{(1+r)^{N-k+1}}\right) = mr\sum_{l=1}^{N-k+1}(1+r)^{-l},}$$

and

the *k-th* principal repayment is

$$\boxed{m - rA_k = \frac{m}{(1+r)^{N-k+1}}, \qquad k=1,2,\ldots,N.}$$

Note that the sum of *discounted* payments at the rate r is

$$\sum_{l=1}^{N} \frac{m}{(1+r)^l} = m\frac{1-(1+r)^{-N}}{r} = A.$$

In particular, the first interest repayment satisfies

$$rA = rA_1 = mr\sum_{l=1}^{N}\frac{1}{(1+r)^l} = m\left(1-(1+r)^{-N}\right),$$

and the first principal repayment is

$$m - rA = \frac{m}{(1+r)^N}.$$

3.2 Arbitrage and Self-Financing Portfolios

Stochastic processes

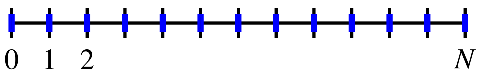
A *stochastic process* on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a family $(X_t)_{t\in T}$ of random variables $X_t : \Omega \longrightarrow \mathbb{R}$ indexed by a set T. Examples include:

- the two-instant model: $T = \{0,1\}$,
- the discrete-time model with finite horizon: $T = \{0,1,\ldots,N\}$,
- the discrete-time model with infinite horizon: $T = \mathbb{N}$,
- the continuous-time model: $T = \mathbb{R}_+$.

For real-world examples of stochastic processes one can mention:

- the time evolution of a risky asset, *e.g.* X_t represents the price of the asset at time $t \in T$.
- the time evolution of a physical parameter - for example, X_t represents a temperature observed at time $t \in T$.

In this chapter, we focus on the finite horizon discrete-time model with $T = \{0,1,\ldots,N\}$.



Asset price modeling

The prices at time $t = 0$ of $d+1$ assets numbered $0,1,\ldots,d$ are denoted by the *random* vector

$$\bar{S}_0 = \left(S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(d)}\right)$$

in $\mathbb{R}^{d+1}$. Similarly, the values at time $t = 1,2,\ldots,N$ of assets n^o $0,1,\ldots,d$ are denoted by the *random* vector

$$\bar{S}_t = \left(S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}\right)$$

on Ω, which forms a stochastic process $\left(\bar{S}_t\right)_{t=0,1,\ldots,N}$.

In the sequel we assume that asset n^o 0 is a riskless asset (of savings account type) yielding an interest rate r, *i.e.* we have

$$S_t^{(0)} = (1+r)^t S_0^{(0)}, \qquad t = 0,1,\ldots,N.$$

Portfolio strategies

Definition 3.1 A portfolio strategy is a stochastic process $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N} \subset \mathbb{R}^{d+1}$ where $\xi_t^{(k)}$ denotes the (possibly fractional) quantity of asset n^o k held in the portfolio over the time interval $(t-1,t]$, $t = 1,2,\ldots,N$.

Note that the portfolio allocation

$$\bar{\xi}_t = \left(\xi_t^{(0)}, \xi_t^{(1)}, \ldots, \xi_t^{(d)}\right)$$

is decided at time $t-1$ and remains constant over the interval $(t-1,t]$ while the stock price changes from $S_{t-1}^{(k)}$ to $S_t^{(k)}$ over this time interval.

In other words, the quantity

$$\xi_t^{(k)} S_{t-1}^{(k)}$$

represents the amount invested in asset n^o k at the beginning of the time interval $(t-1,t]$, and

$$\xi_t^{(k)} S_t^{(k)}$$

represents the value of this investment at the end of the time interval $(t-1,t]$, $t=1,2,\ldots,N$.

Self-financing portfolio strategies

The price of the porfolio at the beginning of the time interval $(t-1,t]$ is

$$\bar{\xi}_t \bullet \bar{S}_{t-1} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t-1}^{(k)},$$

when the market "opens" at time $t-1$, and when the market "closes"at the end of the time interval $(t-1,t]$ it becomes

$$\bar{\xi}_t \bullet \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \tag{3.5}$$

$t=1,2,\ldots,N$. After the new portfolio allocation $\bar{\xi}_{t+1}$ is designed, the portfolio value becomes

$$\bar{\xi}_{t+1} \bullet \bar{S}_t = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \tag{3.6}$$

at the beginning of the next trading session $(t,t+1]$, $t=0,1,\ldots,N-1$. Note that here the stock price $\bar{S}_t$ is assumed to remain constant "overnight", *i.e.* from the end of $(t-1,t]$ to the beginning of $(t,t+1]$, $t=1,2,\ldots,N-1$.

In case (3.5) coincides with (3.6) for $t=0,1,\ldots,N-1$ we say that the portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ is *self-financing*.

Note that a non self-financing portfolio could be either bleeding money, or burning cash, for no good reason.

Definition 3.2 A portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ is said to be *self-financing* if

$$\bar{\xi}_t \bullet \bar{S}_t = \bar{\xi}_{t+1} \bullet \bar{S}_t, \qquad t=1,2,\ldots,N-1, \tag{3.7}$$

i.e.

$$\sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \qquad t=1,2,\ldots,N-1.$$

The meaning of the self-financing condition (3.7) is simply that one cannot take any money in or out of the portfolio during the "overnight" transition period at time t. In other words, at the beginning of the new trading session $(t,t+1]$ one should re-invest the totality of the portfolio value obtained at the end of the interval $(t-1,t]$.

The next figure is an illustration of the self-financing condition.

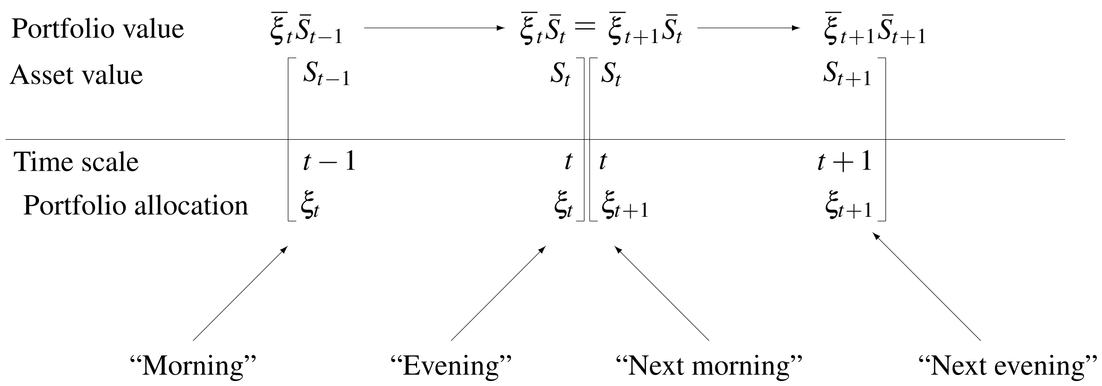


Figure 3.1: Illustration of the self-financing condition (3.7).

By (3.5) and (3.6) the self-financing condition (3.7) can be rewritten as

$$\sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \qquad t = 0,1,\ldots,N-1,$$

or

$$\sum_{k=0}^{d} \big(\xi_{t+1}^{(k)} - \xi_t^{(k)}\big) S_t^{(k)} = 0, \qquad t = 0,1,\ldots,N-1.$$

Note that any portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ which is constant over time, *i.e.* $\bar{\xi}_t = \bar{\xi}_{t+1}$, $t = 1,2,\ldots,N-1$, is self-financing by construction.

Here, portfolio re-allocation happens "overnight", during which time the global portfolio value remains the same due to the self-financing condition. The portfolio allocation ξ_t remains the same throughout the day, however the portfolio value changes from morning to evening due to a change in the stock price. Also, $\bar{\xi}_0$ is not defined and its value is actually not needed in this framework.

In case $\underline{d=1}$ we are only trading $d+1=2$ assets $\bar{S}_t = \big(S_t^{(0)}, S_t^{(1)}\big)$ and the portfolio allocation reads $\bar{\xi}_t = \big(\xi_t^{(0)}, \xi_t^{(1)}\big)$. In this case, the self-financing condition means that:

- In the event of an increase in the stock position $\xi_t^{(1)}$, the corresponding cost of purchase $\big(\xi_{t+1}^{(1)} - \xi_t^{(1)}\big) S_t^{(0)} > 0$ has to be *deducted from* the savings account value $\xi_t^{(0)} S_t^{(0)}$, which becomes updated as
$$\xi_{t+1}^{(0)} S_t^{(0)} = \xi_t^{(0)} S_t^{(0)} - \big(\xi_{t+1}^{(1)} - \xi_t^{(1)}\big) S_t^{(0)},$$
recovering (3.7).
- In the event of a decrease in the stock position $\xi_t^{(1)}$, the corresponding sale profit $\big(\xi_t^{(1)} - \xi_{t+1}^{(1)}\big) S_t^{(0)} > 0$ has to be *added to* from the savings account value $\xi_t^{(0)} S_t^{(0)}$, which becomes updated as
$$\xi_{t+1}^{(0)} S_t^{(0)} = \xi_t^{(0)} S_t^{(0)} + \big(\xi_t^{(1)} - \xi_{t+1}^{(1)}\big) S_t^{(0)},$$
recovering (3.7).

Clearly, the chosen unit of time may not be the day and it can be replaced by weeks, hours, minutes, or fractions of seconds in high-frequency trading.

Portfolio value

Definition 3.3 The portfolio value at times $t = 0, 1, \ldots, N$ is defined

$$V_N := \bar{\xi}_N \cdot \bar{S}_N \quad \text{and} \quad V_t := \bar{\xi}_{t+1} \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_{t+1}^{(k)} S_t^{(k)}, \quad t = 0, 1, \ldots, N-1.$$

Under the self-financing condition (3.7), the portfolio value V_t rewrites as

$$V_0 = \bar{\xi}_1 \cdot \bar{S}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t = 1, 2, \ldots, N, \tag{3.8}$$

as summarized in the following table.

V_0	V_1	V_2	$\cdots\cdots$	V_{N-1}	V_N
$\bar{\xi}_1 \cdot \bar{S}_0$	$\bar{\xi}_2 \cdot \bar{S}_1$	$\bar{\xi}_3 \cdot \bar{S}_2$	$\cdots\cdots$	$\bar{\xi}_N \cdot \bar{S}_{N-1}$	$\bar{\xi}_N \cdot \bar{S}_N$
$\bar{\xi}_1 \cdot \bar{S}_0$	$\bar{\xi}_1 \cdot \bar{S}_1$	$\bar{\xi}_2 \cdot \bar{S}_2$	$\cdots\cdots$	$\bar{\xi}_{N-1} \cdot \bar{S}_{N-1}$	$\bar{\xi}_N \cdot \bar{S}_N$

Table 3.1: Self-financing portfolio value process.

Discounting

My portfolio S_t grew by $b = 5\%$ this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My portfolio S_t grew by $b = 5\%$ this year.

The risk-free or inflation rate is $r = 10\%$.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

Figure 3.2: Why apply discounting?

Definition 3.4 Let

$$\bar{X}_t := \left(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\right)$$

denote the vector of discounted asset prices, defined as:

$$\widetilde{S}_t^{(i)} = \frac{1}{(1+r)^t} S_t^{(i)}, \qquad i = 0, 1, \ldots, d, \quad t = 0, 1, \ldots, N.$$

We can also write

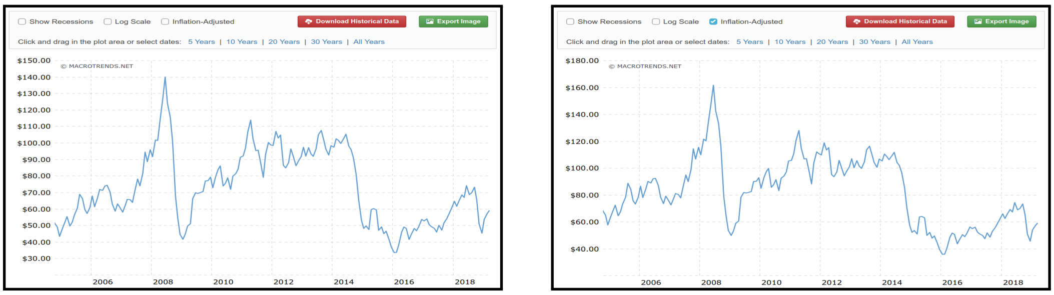
$$\bar{X}_t := \frac{1}{(1+r)^t} \bar{S}_t, \qquad t = 0, 1, \ldots, N.$$

The *discounted* value at time 0 of the portfolio is defined by

$$\widetilde{V}_t = \frac{1}{(1+r)^t} V_t, \qquad t = 0, 1, \ldots, N.$$

For $t = 1, 2, \ldots, N$ we have

$$\widetilde{V}_t = \frac{1}{(1+r)^t} \bar{\xi}_t \cdot \bar{S}_t$$



(a) *Without* inflation adjustment. (b) *With* inflation adjustment.

Figure 3.3: Are oil prices higher in 2019 compared to 2005?

$$
\begin{aligned}
&= \frac{1}{(1+r)^t} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\
&= \sum_{k=0}^{d} \xi_t^{(k)} \widetilde{S}_t^{(k)} \\
&= \overline{\xi}_t \cdot \overline{X}_t,
\end{aligned}
$$

while for $t = 0$ we get

$$
\widetilde{V}_0 = \overline{\xi}_1 \cdot \overline{X}_0 = \overline{\xi}_1 \cdot \bar{S}_0.
$$

The effect of discounting from time t to time 0 is to divide prices by $(1+r)^t$, making all prices comparable at time 0.

Arbitrage

The definition of arbitrage in discrete time follows the lines of its analog in the one-step model.

Definition 3.5 A portfolio strategy $(\overline{\xi}_t)_{t=1,2,\ldots,N}$ constitutes an arbitrage opportunity if all *three* following conditions are satisfied:

i) $V_0 \leqslant 0$ at time $t = 0$, *[start from a zero-cost portfolio or in debt]*

ii) $V_N \geqslant 0$ at time $t = N$, *[finish with a nonnegative amount]*

iii) $\mathbb{P}(V_N > 0) > 0$ at time $t = N$. *[profit made with nonzero probability]*

3.3 Contingent Claims

Recall that from Definition 2.5, a contingent claim is given by the nonnegative random payoff C of an option contract at maturity time $t = N$. For example, in the case of the European call option of Definition 0.2, the payoff C is given by $C = \big(S_N^{(i)} - K\big)^+$ where K is called the strike (or exercise) price of the option, while in the case of the European put option of Definition 0.1 we have $C = \big(K - S_N^{(i)}\big)^+$.

The list given below is somewhat restrictive and there exists many more option types, with new ones appearing constantly on the markets.

Physical delivery *vs* cash settlement

The cash settlement realized through the payoff $C = \big(S_N^{(i)} - K\big)^+$ can be replaced by the *physical delivery* of the underlying asset in exchange for the strike price K. Physical delivery occurs only when $S_N^{(i)} > K$, in which case the underlying asset can be sold at the price $S_N^{(i)}$ by the option holder, for a payoff $S_N^{(i)} - K$. When $S_N^{(i)} > K$, no delivery occurs and the payoff is 0, which is consistent with the expression $C = \big(S_N^{(i)} - K\big)^+$. A similar procedure can be applied to other option contracts.

Vanilla options

i) *European options.*

The payoff of the European call option on the underlying asset n^o i with maturity N and strike price K is

$$C = \big(S_N^{(i)} - K\big)^+ = \begin{cases} S_N^{(i)} - K & \text{if } S_N^{(i)} \geqslant K, \\ \\ 0 & \text{if } S_N^{(i)} < K. \end{cases}$$

The *moneyness* at time $t = 0, 1, \ldots, N$ of the European call option with strike price K on the asset n^o i is the ratio

$$\mathrm{M}_t^{(i)} := \frac{S_t^{(i)} - K}{S_t^{(i)}}, \qquad t = 0, 1, \ldots, N.$$

The option is said to be "*out of the money*" (OTM) when $\mathrm{M}_t^{(i)} < 0$, "*in the money*" (ITM) when $\mathrm{M}_t^{(i)} > 0$, and "*at the money*" (ATM) when $\mathrm{M}_t^{(i)} = 0$.

The payoff of the European put option on the underlying asset n^o i with exercise date N and strike price K is

$$C = \big(K - S_N^{(i)}\big)^+ = \begin{cases} K - S_N^{(i)} & \text{if } S_N^{(i)} \leqslant K, \\ \\ 0 & \text{if } S_N^{(i)} > K. \end{cases}$$

The *moneyness* at time $t = 0, 1, \ldots, N$ of the European put option with strike price K on the asset n^o i is the ratio

$$\mathrm{M}_t^{(i)} := \frac{K - S_t^{(i)}}{S_t^{(i)}}, \qquad t = 0, 1, \ldots, N.$$

ii) *Binary options.*

Binary (or digital) options, also called cash-or-nothing options, are options whose payoffs are of the form

$$C = \mathbb{1}_{[K,\infty)}\big(S_N^{(i)}\big) = \begin{cases} \$1 & \text{if } S_N^{(i)} \geqslant K, \\ \\ 0 & \text{if } S_N^{(i)} < K, \end{cases}$$

for binary call options, and

$$C = \mathbb{1}_{(-\infty,K]}\big(S_N^{(i)}\big) = \begin{cases} \$1 & \text{if } S_N^{(i)} \leqslant K, \\ \\ 0 & \text{if } S_N^{(i)} > K, \end{cases}$$

for binary put options.

Exotic options

i) *Asian options.*

The payoff of an Asian call option (also called option on average) on the underlying asset n^o i with exercise date N and strike price K is

$$C = \left(\frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} - K \right)^+ .$$

The payoff of an Asian put option on the underlying asset n^o i with exercise date N and strike price K is

$$C = \left(K - \frac{1}{N+1} \sum_{t=0}^{N} S_t^{(i)} \right)^+ .$$

It can be shown, cf. Exercise 4.12 that Asian call option prices can be upper bounded by European call option prices.

Other examples of such options include weather derivatives (based on averaged temperatures) and volatility derivatives (based on averaged volatilities).

ii) *Barrier options.*

The payoff of a down-an-out (or knock-out) barrier call option on the underlying asset n^o i with exercise date N, strike price K and barrier level B is

$$\begin{aligned} C &= \left(S_N^{(i)} - K\right)^+ \mathbb{1}_{\left\{ \min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B \right\}} \\ &= \begin{cases} \left(S_N^{(i)} - K\right)^+ & \text{if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} > B, \\ \\ 0 & \text{if } \min\limits_{t=0,1,\ldots,N} S_t^{(i)} \leqslant B. \end{cases} \end{aligned}$$

This option is also called a Callable Bull Contract with no residual value, or turbo warrant with no rebate, in which B denotes the call price $B \geqslant K$.

The payoff of an up-and-out barrier put option on the underlying asset n^o i with exercise date N, strike price K and barrier level B is

$$\begin{aligned} C &= \left(K - S_N^{(i)}\right)^+ \mathbb{1}_{\left\{ \operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} < B \right\}} \\ &= \begin{cases} \left(K - S_N^{(i)}\right)^+ & \text{if } \operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} < B, \\ \\ 0 & \text{if } \operatorname*{Max}\limits_{t=0,1,\ldots,N} S_t^{(i)} \geqslant B. \end{cases} \end{aligned}$$

This option is also called a Callable Bear Contract with no residual value, in which the call price B usually satisfies $B \leqslant K$. See J. Eriksson and Persson, 2006 and Wong and Chan, 2008 for the pricing of type R Callable Bull/Bear Contracts, or CBBCs, also called turbo warrants, which involve a rebate or residual value computed as the payoff of a down-and-in lookback option.

iii) *Lookback options.*

The payoff of a floating strike lookback call option on the underlying asset n^o i with exercise date N is

$$C = S_N^{(i)} - \min_{t=0,1,\ldots,N} S_t^{(i)}.$$

The payoff of a floating strike lookback put option on the underlying asset n^o i with exercise date N is

$$C = \left(\operatorname*{Max}_{t=0,1,\ldots,N} S_t^{(i)} \right) - S_N^{(i)}.$$

Options in insurance and investment

Such options are involved in the statements of Exercises 3.1 and 3.2.

Vanilla *vs* exotic options

Vanilla options such as European or binary options, have a payoff $\phi(S_N^{(i)})$ that depends only on the terminal value $S_N^{(i)}$ of the underlying asset at maturity, as opposed to exotic or path-dependent options such as Asian, barrier, or lookback options, whose payoff may depend on the whole path of the underlying asset price until expiration time.

Exotic *vs* Vanilla Options

Vanilla options are called that way because:

(A) They were first used for the trading of vanilla by the Maya beginning around the 14th century.

(B) "Plain vanilla" is the most standard and common of all ice cream flavors.

(C) To meet FDA standards, pure vanilla extract must contain 13.35 ounces of vanilla beans per gallon.

(D) Sir Charles C. Vanilla, FLS, was the early discoverer of the properties of Brownian motion in asset pricing.

Figure 3.4: Take the Quiz.

3.4 Martingales and Conditional Expectation

Before proceeding to the definition of risk-neutral probability measures in discrete time we need to introduce more mathematical tools such as conditional expectations, filtrations, and martingales.

Conditional expectations

Clearly, the expected value of any risky asset or random variable is dependent on the amount of available information. For example, the expected return on a real estate investment typically depends on the location of this investment.

In the probabilistic framework the available information is formalized as a collection $\mathscr{G}$ of events, which may be smaller than the collection $\mathscr{F}$ of all available events, *i.e.* $\mathscr{G} \subset \mathscr{F}$.[*]

The notation $\mathbb{E}[F \mid \mathscr{G}]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathscr{G}$, and it is read "the conditional expectation of F given $\mathscr{G}$". In a certain sense, $\mathbb{E}[F \mid \mathscr{G}]$ represents the best possible estimate of F in the mean-square sense, given

[*] The collection $\mathscr{G}$ is also called a σ-algebra, cf. Section 11.1.

the information contained in $\mathscr{G}$.

The conditional expectation satisfies the following five properties, cf. Section 11.7 for details and proofs.

i) $\mathbb{E}[FG \mid \mathscr{G}] = G\mathbb{E}[F \mid \mathscr{G}]$ if G depends only on the information contained in $\mathscr{G}$.

ii) $\mathbb{E}[G \mid \mathscr{G}] = G$ when G depends only on the information contained in $\mathscr{G}$.

iii) $\mathbb{E}[\mathbb{E}[F \mid \mathscr{H}] \mid \mathscr{G}] = \mathbb{E}[F \mid \mathscr{G}]$ if $\mathscr{G} \subset \mathscr{H}$, called the *tower property*, cf. also Relation (11.38).

iv) $\mathbb{E}[F \mid \mathscr{G}] = \mathbb{E}[F]$ when F "does not depend" on the information contained in $\mathscr{G}$ or, more precisely stated, when the random variable F is *independent* of the σ-algebra $\mathscr{G}$.

v) If G depends only on $\mathscr{G}$ and F is independent of $\mathscr{G}$, then

$$\mathbb{E}[h(F,G) \mid \mathscr{G}] = \mathbb{E}[h(F,x)]_{x=G}.$$

When $\mathscr{H} = \{\emptyset, \Omega\}$ is the trivial σ-algebra we have

$$\mathbb{E}[F \mid \mathscr{H}] = \mathbb{E}[F], \qquad F \in L^1(\Omega).$$

See (11.38) and (11.44) for illustrations of the tower property by conditioning with respect to discrete and continuous random variables.

Filtrations

The total amount of "information" available on the market at times $t = 0, 1, \ldots, N$ is denoted by $\mathscr{F}_t$. We assume that

$$\mathscr{F}_t \subset \mathscr{F}_{t+1}, \qquad t = 0, 1, \ldots, N-1,$$

which means that the amount of information available on the market increases over time.

Usually, $\mathscr{F}_t$ corresponds to the knowledge of the values $S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)}$, $i = 1, 2, \ldots, d$, of the risky assets up to time t. In mathematical notation we say that $\mathscr{F}_t$ is generated by $S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)}$, $i = 1, 2, \ldots, d$, and we usually write

$$\mathscr{F}_t = \sigma\left(S_0^{(i)}, S_1^{(i)}, \ldots, S_t^{(i)},\ i = 1, 2, \ldots, d\right), \qquad t = 0, 1, \ldots, N,$$

with $\mathscr{F}_0 = \{\emptyset, \Omega\}$.

<u>Example:</u> Consider the simple random walk

$$Z_t := X_1 + X_2 + \cdots + X_t, \qquad t \geqslant 0,$$

where $(X_t)_{t\geqslant 1}$ is a sequence of independent, identically distributed $\{-1, 1\}$ valued random variables. The filtration (or information flow) $(\mathscr{F}_t)_{t\geqslant 0}$ generated by $(Z_t)_{t\geqslant 0}$ is given by $\mathscr{F}_0 = \big\{\emptyset, \Omega\big\}$, $\mathscr{F}_1 = \big\{\emptyset, \{X_1 = 1\}, \{X_1 = -1\}, \Omega\big\}$, and

$$\mathscr{F}_2 = \sigma\Big(\big\{\emptyset, \{X_1 = 1, X_2 = 1\}, \{X_1 = 1, X_2 = -1\}, \{X_1 = -1, X_2 = 1\},$$
$$\{X_1 = -1, X_2 = -1\}, \Omega\big\}\Big).$$

The notation $\mathscr{F}_t$ is useful to represent a quantity of information available at time t. Note that different agents or traders may work with different filtrations. For example, an insider may have access to a filtration $(\mathscr{G}_t)_{t=0,1,\ldots,N}$ which is larger than the ordinary filtration $(\mathscr{F}_t)_{t=0,1,\ldots,N}$ available to an ordinary agent, in the sense that

$$\mathscr{F}_t \subset \mathscr{G}_t, \qquad t = 0,1,\ldots,N.$$

The notation $\mathbb{E}[F \mid \mathscr{F}_t]$ represents the expected value of a random variable F given (or conditionally to) the information contained in $\mathscr{F}_t$. Again, $\mathbb{E}[F \mid \mathscr{F}_t]$ denotes the best possible estimate of F in mean-square sense, given the information known up to time t.

We will assume that no information is available at time $t = 0$, which translates as

$$\mathbb{E}[F \mid \mathscr{F}_0] = \mathbb{E}[F]$$

for any integrable random variable F. As above, the conditional expectation with respect to $\mathscr{F}_t$ satisfies the following five properties:

i) $\mathbb{E}[FG \mid \mathscr{F}_t] = F\,\mathbb{E}[G \mid \mathscr{F}_t]$ if F depends only on the information contained in $\mathscr{F}_t$.

ii) $\mathbb{E}[F \mid \mathscr{F}_t] = F$ when F depends only on the information known at time t and contained in $\mathscr{F}_t$.

iii) $\mathbb{E}[\mathbb{E}[F \mid \mathscr{F}_{t+1}] \mid \mathscr{F}_t] = \mathbb{E}[F \mid \mathscr{F}_t]$ if $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ (by the *tower property*, cf. also Relation (8.1) below).

iv) $\mathbb{E}[F \mid \mathscr{F}_t] = \mathbb{E}[F]$ when F does not depend on the information contained in $\mathscr{F}_t$.

v) If F depends only on $\mathscr{F}_t$ and G is independent of $\mathscr{F}_t$, then

$$\mathbb{E}[h(F,G) \mid \mathscr{F}_t] = \mathbb{E}[h(x,G)]_{x=F}.$$

Note that by the tower property (iii) the process $t \longmapsto \mathbb{E}[F \mid \mathscr{F}_t]$ is a martingale, cf. *e.g.* Relation (8.1) for details.

Martingales

A martingale is a stochastic process whose value at time $t+1$ can be estimated using conditional expectation given its value at time t. Recall that a stochastic process $(M_t)_{t=0,1,\ldots,N}$ is said to be $(\mathscr{F}_t)_{t=0,1,\ldots,N}$-adapted if the value of M_t depends only on the information available at time t in $\mathscr{F}_t$, $t = 0,1,\ldots,N$.

Definition 3.6 A stochastic process $(M_t)_{t=0,1,\ldots,N}$ is called a discrete-time *martingale* with respect to the filtration $(\mathscr{F}_t)_{t=0,1,\ldots,N}$ if $(M_t)_{t=0,1,\ldots,N}$ is $(\mathscr{F}_t)_{t=0,1,\ldots,N}$-adapted and satisfies the property

$$\mathbb{E}[M_{t+1} \mid \mathscr{F}_t] = M_t, \qquad t = 0,1,\ldots,N-1.$$

Note that the above definition implies that $M_t \in \mathscr{F}_t$, $t = 0,1,\ldots,N$. In other words, a random process $(M_t)_{t=0,1,\ldots,N}$ is a martingale if the best possible prediction of M_{t+1} in the mean-square sense given $\mathscr{F}_t$ is simply M_t.

In discrete-time finance, the martingale property can be used to characterize risk-neutral probability measures, and for the computation of conditional expectations.

Exercise. Using the *tower property* (11.38) of conditional expectation, show that Definition 3.6 can be equivalently stated by saying that

$$\mathbb{E}[M_n \mid \mathcal{F}_k] = M_k, \qquad 0 \leqslant k < n.$$

A particular property of martingales is that their expectation is constant over time.

Proposition 3.3 Let $(Z_n)_{n\in\mathbb{N}}$ be a martingale. We have

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \qquad n \in \mathbb{N}.$$

Proof. From the tower property (11.38) of expectation we have:

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n]] = \mathbb{E}[Z_n], \qquad n \in \mathbb{N},$$

hence by induction on $n \in \mathbb{N}$ we have

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[Z_n] = \mathbb{E}[Z_{n-1}] = \cdots = \mathbb{E}[Z_1] = \mathbb{E}[Z_0], \qquad n \in \mathbb{N}.$$

□

Weather forecasting can be seen as an example of application of martingales. If M_t denotes the random temperature observed at time t, this process is a martingale when the best possible forecast of tomorrow's temperature M_{t+1} given the information known up to time t is simply today's temperature M_t, $t = 0, 1, \ldots, N-1$.

Definition 3.7 A stochastic process $(\xi_k)_{k\geqslant 1}$ is said to be *predictable* if ξ_k depends only on the information in $\mathcal{F}_{k-1}$, $k \geqslant 1$.

When $\mathcal{F}_0$ simply takes the form $\mathcal{F}_0 = \{\emptyset, \Omega\}$ we find that ξ_1 is a constant when $(\xi_t)_{t=1,2,\ldots,N}$ is a predictable process. Recall that on the other hand, the process $\big(S_t^{(i)}\big)_{t=0,1,\ldots,N}$ is *adapted* as $S_t^{(i)}$ depends only on the information in $\mathcal{F}_t$, $t = 0, 1, \ldots, N$, $i = 1, 2, \ldots, d$.

The discrete-time stochastic integral (3.9) will be interpreted as the sum of discounted profits and losses $\xi_k\big(\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}\big)$, $k = 1, 2, \ldots, t$, in a portfolio holding a quantity ξ_k of a risky asset whose price variation is $\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}$ at time $k = 1, 2, \ldots, t$.

An important property of martingales is that the discrete-time stochastic integral (3.9) of a predictable process is itself a martingale, see also Proposition 8.1 for the continuous-time analog of the following proposition, which will be used in the proof of Theorem 4.2 below.*

In the sequel, the martingale (3.9) will be interpreted as a discounted portfolio value, in which $\widetilde{S}_k^{(1)} - \widetilde{S}_{k-1}^{(1)}$ represents the increment in the discounted asset price and ξ_k is the amount invested in that asset, $k = 1, 2 \ldots, N$.

Theorem 3.4 *Martingale transform.* Given $(X_k)_{k=0,1,\ldots,N}$ a martingale and $(\xi_k)_{k=1,2,\ldots,N}$ a (bounded) predictable process, the discrete-time process $(M_t)_{t=0,1,\ldots,N}$ defined by

$$M_t = \sum_{k=1}^{t} \underbrace{\xi_k (X_k - X_{k-1})}_{\text{profit/loss}}, \qquad t = 0, 1, \ldots, N, \tag{3.9}$$

*See *here* for a related discussion of martingale strategies in a particular case.

is a martingale.

Proof. Given $n > t \geqslant 0$ we have

$$\begin{aligned}\mathbb{E}[M_n \mid \mathscr{F}_t] &= \mathbb{E}\left[\sum_{k=1}^{n} \xi_k (X_k - X_{k-1}) \,\Big|\, \mathscr{F}_t\right] \\ &= \sum_{k=1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] \\ &= \sum_{k=1}^{t} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] \\ &= \sum_{k=1}^{t} \xi_k (X_k - X_{k-1}) + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] \\ &= M_t + \sum_{k=t+1}^{n} \mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right].\end{aligned}$$

In order to conclude to $\mathbb{E}[M_n \mid \mathscr{F}_t] = M_t$ it suffices to show that

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] = 0, \qquad t+1 \leqslant k \leqslant n.$$

First we note that when $0 \leqslant t \leqslant k-1$ we have $\mathscr{F}_t \subset \mathscr{F}_{k-1}$, hence by the "tower property" of conditional expectations we get

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] = \mathbb{E}\left[\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_{k-1}\right] \mid \mathscr{F}_t\right].$$

Next, since the process $(\xi_k)_{k\geqslant 1}$ is predictable, ξ_k depends only on the information in $\mathscr{F}_{k-1}$, and using Property (ii) of conditional expectations we may pull out ξ_k out of the expectation since it behaves as a constant parameter given $\mathscr{F}_{k-1}$, $k = 1,2,\ldots,n$. This yields

$$\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_{k-1}\right] = \xi_k \mathbb{E}\left[X_k - X_{k-1} \mid \mathscr{F}_{k-1}\right] = 0 \tag{3.10}$$

since

$$\begin{aligned}\mathbb{E}\left[X_k - X_{k-1} \mid \mathscr{F}_{k-1}\right] &= \mathbb{E}\left[X_k \mid \mathscr{F}_{k-1}\right] - \mathbb{E}\left[X_{k-1} \mid \mathscr{F}_{k-1}\right] \\ &= \mathbb{E}\left[X_k \mid \mathscr{F}_{k-1}\right] - X_{k-1} \\ &= 0, \qquad k = 1,2,\ldots,N,\end{aligned}$$

because $(X_k)_{k=0,1,\ldots,N}$ is a martingale. By (3.10), it follows that

$$\begin{aligned}\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\xi_k (X_k - X_{k-1}) \mid \mathscr{F}_{k-1}\right] \mid \mathscr{F}_t\right] \\ &= \mathbb{E}\left[\xi_k \mathbb{E}\left[X_k - X_{k-1} \mid \mathscr{F}_{k-1}\right] \mid \mathscr{F}_t\right] \\ &= 0,\end{aligned}$$

for $k = t+1,\ldots,n$. $\square$

3.5 Market Completeness and Risk-Neutral Measures

As in the two time step model, the concept of risk-neutral probability measure (or martingale measure) will be used to price financial claims under the absence of arbitrage hypothesis.*

*See also the Efficient Market Hypothesis.

Definition 3.8 A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral probability measure if under $\mathbb{P}^*$, the expected return of each risky asset equals the return r of the riskless asset, that is

$$\mathbb{E}^*\left[S_{t+1}^{(i)} \,\middle|\, \mathcal{F}_t\right] = (1+r)S_t^{(i)}, \qquad t = 0,1,\ldots,N-1, \tag{3.11}$$

$i = 0,1,\ldots,d$. Here, $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

Since $S_t^{(i)} \in \mathcal{F}_t$, denoting by

$$R_{t+1}^{(i)} := \frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}}$$

the return of asset n^o i over the time interval $(t, t+1]$, $t = 0,1,\ldots,N-1$, Relation (3.11) can be rewritten as

$$\begin{aligned}
\mathbb{E}^*\left[R_{t+1}^{(i)} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}^*\left[\frac{S_{t+1}^{(i)} - S_t^{(i)}}{S_t^{(i)}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\frac{S_{t+1}^{(i)}}{S_t^{(i)}} \,\middle|\, \mathcal{F}_t\right] - 1 \\
&= 1 + r, \qquad t = 0,1,\ldots,N-1,
\end{aligned}$$

which means that the average of the return $(S_{t+1}^{(i)} - S_t^{(i)})/S_t^{(i)}$ of asset n^o i under the risk-neutral probability measure $\mathbb{P}^*$ is equal to the risk-free interest rate r.

In other words, taking risks under $\mathbb{P}^*$ by buying the risky asset n^o i has a neutral effect, as the expected return is that of the riskless asset. The measure $\mathbb{P}^\sharp$ would yield a *positive* risk premium if we had

$$\mathbb{E}^\sharp\left[S_{t+1}^{(i)} \,\middle|\, \mathcal{F}_t\right] = (1+\tilde{r})S_t^{(i)}, \qquad t = 0,1,\ldots,N-1,$$

with $\tilde{r} > r$, and a *negative* risk premium if $\tilde{r} < r$.

In the next proposition we reformulate the definition of risk-neutral probability measure using the notion of martingale.

Proposition 3.5 A probability measure $\mathbb{P}^*$ on Ω is a risk-neutral measure if and only if the discounted price process

$$\widetilde{S}_t^{(i)} := \frac{S_t^{(i)}}{(1+r)^t}, \qquad t = 0,1,\ldots,N,$$

is a martingale *under* $\mathbb{P}^*$, *i.e.*

$$\mathbb{E}^*\left[\widetilde{S}_{t+1}^{(i)} \,\middle|\, \mathcal{F}_t\right] = \widetilde{S}_t^{(i)}, \qquad t = 0,1,\ldots,N-1, \tag{3.12}$$

$i = 0,1,\ldots,d$.

Proof. It suffices to check that by the relation $S_t^{(i)} = (1+r)^t \widetilde{S}_t^{(i)}$, Condition (3.11) can be rewritten as

$$(1+r)^{t+1}\mathbb{E}^*\left[\widetilde{S}_{t+1}^{(i)} \,\middle|\, \mathcal{F}_t\right] = (1+r)(1+r)^t \widetilde{S}_t^{(i)},$$

$i = 1,2,\ldots,d$, which is clearly equivalent to (3.12) after division by $(1+r)^t$, $t = 0,1,\ldots,N-1$. $\square$

Note that, as a consequence of Propositions 3.3 and 3.5, the discounted price process $\widetilde{S}_t^{(i)} := S_t^{(i)}/(1+r)^t$, $t = 0,1,\ldots,n$, has constant expectation under the risk-neutral probability measure $\mathbb{P}^*$, *i.e.*

$$\mathbb{E}^*\left[\widetilde{S}_t^{(i)}\right] = \widetilde{S}_0^{(i)}, \qquad t = 1,2,\ldots,N,$$

for $i = 0,1,\ldots,d$.

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 3.9 A probability measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{P}^*(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}. \tag{3.13}$$

Next, we restate in discrete time the first fundamental theorem of asset pricing, which can be used to check for the existence of arbitrage opportunities.

Theorem 3.6 A market is *without* arbitrage opportunity if and only if it admits at least one *equivalent* risk-neutral probability measure.

Proof. See Harrison and Kreps, 1979 and Theorem 5.17 of Föllmer and Schied, 2004. □

Next, we turn to the notion of *market completeness*, starting with the definition of attainability for a contingent claim.

Definition 3.10 A contingent claim with payoff C is said to be attainable (at time N) if there exists a self-financing portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ such that

$$C = \bar{\xi}_N \bullet \bar{S}_N = \sum_{k=0}^{d} \xi_N^{(k)} S_N^{(k)}, \qquad \mathbb{P}-a.s. \tag{3.14}$$

In case $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ is a portfolio that attains the claim payoff C at time N, *i.e.* if (3.14) is satisfied, we also say that $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ *hedges* the claim payoff C. In case (3.14) is replaced by the condition

$$\bar{\xi}_N \bullet \bar{S}_N \geqslant C,$$

we talk of super-hedging.

When a self-financing portfolio $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ hedges a claim payoff C, the arbitrage-free price $\pi_t(C)$ of the claim at time t is given by the value

$$\pi_t(C) = \bar{\xi}_t \bullet \bar{S}_t$$

of the portfolio at time $t = 0,1,\ldots,N$. Recall that arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value (mark to market). Note that at time $t = N$ we have

$$\pi_N(C) = \bar{\xi}_N \bullet \bar{S}_N = C,$$

i.e. since exercise of the claim occurs at time N, the price $\pi_N(C)$ of the claim equals the value C of the payoff.

Definition 3.11 A market model is said to be *complete* if every contingent claim is attainable.

The next result can be viewed as the second fundamental theorem of asset pricing in discrete time.

Theorem 3.7 A market model without arbitrage opportunities is complete if and only if it admits only one *equivalent* risk-neutral probability measure.

Proof. See Harrison and Kreps, 1979 and Theorem 5.38 of Föllmer and Schied, 2004. □

3.6 The Cox-Ross-Rubinstein (CRR) Market Model

We consider the discrete-time Cox-Ross-Rubinstein model Cox, Ross, and Rubinstein, 1979, also called the *binomial model*, with $N+1$ time instants $t = 0,1,\ldots,N$ and $d = 1$ risky asset, see Sharpe, 1978. In this setting, the price $S_t^{(0)}$ of the riskless asset evolves as

$$S_t^{(0)} = S_0^{(0)}(1+r)^t, \qquad t = 0,1,\ldots,N.$$

Let the *return* of the risky asset $S^{(1)}$ be defined as

$$R_t := \frac{S_t^{(1)} - S_{t-1}^{(1)}}{S_{t-1}^{(1)}}, \qquad t = 1,2,\ldots,N.$$

In the CRR model the return R_t is random and allowed to take only two values a and b at each time step, *i.e.*

$$R_t \in \{a,b\}, \qquad t = 1,2,\ldots,N,$$

with $-1 < a < b$. That means, the evolution of $S_{t-1}^{(1)}$ to $S_t^{(1)}$ is random and given by

$$S_t^{(1)} = \left\{ \begin{array}{ll} (1+b)S_{t-1}^{(1)} & \text{if } R_t = b \\ \\ (1+a)S_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = (1+R_t)S_{t-1}^{(1)}, \qquad t = 1,\ldots,N,$$

and

$$S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t} (1+R_k), \qquad t = 0,1,\ldots,N.$$

Note that the price process $\left(S_t^{(1)}\right)_{t=0,1,\ldots,N}$ evolves on a binary recombining (or binomial) tree of the following type:*

$$\begin{tikzcd}
 & & S_2 = S_0(1+b)^2 \\
 & S_1 = S_0(1+b) \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S_2 = S_0(1+a)(1+b) \\
 & S_1 = S_0(1+a) \arrow[ur] \arrow[dr] & \\
 & & S_2 = S_0(1+a)^2.
\end{tikzcd}$$

*Download the corresponding **IPython notebook1** and **IPython notebook2** that can be run **here**.

The discounted asset price is

$$\widetilde{S}_t^{(1)} = \frac{S_t^{(1)}}{(1+r)^t}, \qquad t = 0,1,\ldots,N,$$

with

$$\widetilde{S}_t^{(1)} = \left\{ \begin{array}{ll} \dfrac{1+b}{1+r}\widetilde{S}_{t-1}^{(1)} & \text{if } R_t = b \\ \\ \dfrac{1+a}{1+r}\widetilde{S}_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = \frac{1+R_t}{1+r}\widetilde{S}_{t-1}^{(1)}, \qquad t = 1,2,\ldots,N,$$

and

$$\widetilde{S}_t^{(1)} = \frac{S_0^{(1)}}{(1+r)^t}\prod_{k=1}^{t}(1+R_k) = \widetilde{S}_0^{(1)}\prod_{k=1}^{t}\frac{1+R_k}{1+r}.$$

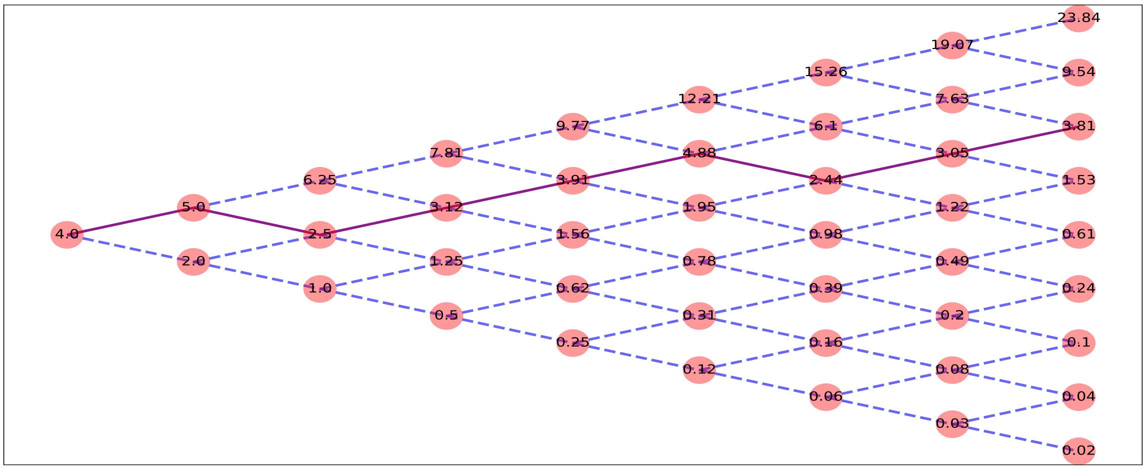


Figure 3.5: Discrete-time asset price tree in the CRR model.

In this model, the discounted value at time t of the portfolio is given by

$$\bar{\xi}_t \cdot \overline{X}_t = \xi_t^{(0)}\widetilde{S}_0^{(0)} + \xi_t^{(1)}\widetilde{S}_t^{(1)}, \qquad t = 1,2,\ldots,N.$$

The information $\mathscr{F}_t$ known in the market up to time t is given by the knowledge of $S_1^{(1)}, S_2^{(1)}, \ldots, S_t^{(1)}$, which is equivalent to the knowledge of $\widetilde{S}_1^{(1)}, \widetilde{S}_2^{(1)}, \ldots, \widetilde{S}_t^{(1)}$ or $R_1, R_2, \ldots, R_t$, *i.e.* we write

$$\mathscr{F}_t = \sigma\big(S_1^{(1)}, S_2^{(1)}, \ldots, S_t^{(1)}\big) = \sigma\big(\widetilde{S}_1^{(1)}, \widetilde{S}_2^{(1)}, \ldots, \widetilde{S}_t^{(1)}\big) = \sigma(R_1, R_2, \ldots, R_t),$$

$t = 0,1,\ldots,N$, where, as a convention, S_0 is a constant and $\mathscr{F}_0 = \{\emptyset, \Omega\}$ contains no information.

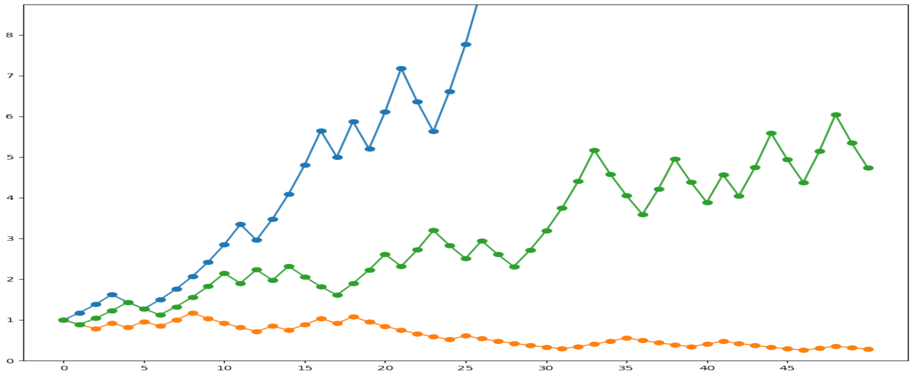


Figure 3.6: Discrete-time asset price graphs in the CRR model.

Theorem 3.8 The CRR model is without arbitrage opportunities if and only if $a < r < b$. In this case the market is complete and the *equivalent* risk-neutral probability measure $\mathbb{P}^*$ is given by

$$\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = \frac{b-r}{b-a}, \tag{3.15}$$

$t = 0,1,\ldots,N-1$. In particular, $(R_1, R_2, \ldots, R_N)$ forms a sequence of independent and identically distributed (*i.i.d.*) random variables under $\mathbb{P}^*$, with

$$p^* := \mathbb{P}^*(R_t = b) = \frac{r-a}{b-a} \quad \text{and} \quad q^* := \mathbb{P}^*(R_t = a) = \frac{b-r}{b-a}, \tag{3.16}$$

$t = 1,2,\ldots,N$.

Proof. In order to check for arbitrage opportunities we may use Theorem 3.6 and look for a risk-neutral probability measure $\mathbb{P}^*$. According to the definition of a risk-neutral measure this probability $\mathbb{P}^*$ should satisfy Condition (3.11), *i.e.*

$$\mathbb{E}^*\big[S_{t+1}^{(1)} \mid \mathcal{F}_t\big] = (1+r)S_t^{(1)}, \qquad t = 0,1,\ldots,N-1.$$

Rewriting $\mathbb{E}^*\big[S_{t+1}^{(1)} \mid \mathcal{F}_t\big]$ as

$$\begin{aligned}\mathbb{E}^*\big[S_{t+1}^{(1)} \mid \mathcal{F}_t\big] &= \mathbb{E}^*\big[S_{t+1}^{(1)} \mid S_t^{(1)}\big] \\ &= (1+a)S_t^{(1)}\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + (1+b)S_t^{(1)}\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t),\end{aligned}$$

it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases} (1+b)S_t^{(1)}\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + (1+a)S_t^{(1)}\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = (1+r)S_t^{(1)} \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

i.e.

$$\begin{cases} b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = r \\ \mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) + \mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) = 1, \end{cases}$$

with solution

$$\mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$. Since the values of $\mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t)$ and $\mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t)$ computed in (3.15) are non random, they are independent* of the information contained in $\mathscr{F}_t$ up to time t. As a consequence, under $\mathbb{P}^*$, the random variable R_{t+1} is independent of $R_1, R_2, \ldots, R_t$, hence the sequence of random variables $(R_t)_{t=0,1,\ldots,N}$ is made of *mutually independent* random variables under $\mathbb{P}^*$, and by (3.15) we have

$$\mathbb{P}^*(R_{t+1} = b) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a) = \frac{b-r}{b-a}.$$

Clearly, $\mathbb{P}^*$ can be equivalent to $\mathbb{P}$ only if $r - a > 0$ and $b - r > 0$. In this case the solution $\mathbb{P}^*$ of the problem is unique by construction, hence the market is complete by Theorem 3.7. $\square$

As a consequence of Proposition 3.5, letting $p^* := (r-a)/(b-a)$, when $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{a, b\}^N$ we have

$$\mathbb{P}^*(R_1 = \varepsilon_1, R_2 = \varepsilon_2, \ldots, R_N = \varepsilon_n) = (p^*)^l (1-p^*)^{N-l},$$

where l, resp. $N-l$, denotes the number of times the term "b", resp. "a", appears in the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N) \in \{a, b\}^N$.

Exercises

Exercise 3.1 Today I went to the Furong Peak mall. After exiting the Poon Way MTR station, I was met by a friendly investment consultant from NTRC Input, who recommended that I subscribe to the following investment plan. The plan requires to invest \$ 2,550 per year over the first 10 years. No contribution is required from year 11 until year 20, and the total projected surrender value is \$30,835 at maturity $N = 20$. The plan also includes a death benefit which is not considered here.

Year	Total Premiums Paid To-date (\$S)	Guaranteed (\$S)	Surrender Value Projected at 3.25% Non-Guaranteed (\$S)	Total (\$S)
1	2,550	0	0	0
2	5,100	2,460	140	2,600
3	7,650	4,240	240	4,480
4	10,200	6,040	366	6,406
5	12,750	8,500	518	9,018
10	25,499	19,440	1,735	21,175
15	25,499	22,240	3,787	26,027
20	25,499	24,000	6,835	30,835

Table 3.2: NTRC Input investment plan.

a) Compute the constant interest rate over 20 years corresponding to this investment plan.
b) Compute the projected value of the plan at the end of year 20, if the annual interest rate is $r = 3.25\%$ over 20 years.
c) Compute the projected value of the plan at the end of year 20, if the annual interest rate $r = 3.25\%$ is paid only over the first 10 years.

*The relation $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ is equivalent to the independence relation $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ of the events A and B.

Exercise 3.2 Today I went to the East mall. After exiting the Bukit Kecil MTR station, I was approached by a friendly investment consultant from Avenda Insurance, who recommended me to subscribe to the following investment plan. The plan requires me to invest \$ 3,581 per year over the first 10 years. No contribution is required from year 11 until year 20, and the total projected surrender value is \$50,862 at maturity $N = 20$. The plan also includes a death benefit which is not considered here.

Year	Total Premiums Paid To-date ($S)	Guaranteed ($S)	Surrender Value Projected at 3.25% Non-Guaranteed ($S)	Surrender Value Projected at 3.25% Total ($S)
1	3,581	0	0	0
2	7,161	1,562	132	1,694
3	10,741	3,427	271	3,698
4	14,321	5,406	417	5,823
5	17,901	6,992	535	7,527
10	35,801	19,111	1,482	20,593
15	35,801	29,046	3,444	32,490
20	35,801	43,500	7,362	50,862

Table 3.3: Avenda Insurance investment plan.

a) Using the following graph, compute the constant interest rate over 20 years corresponding to this investment.

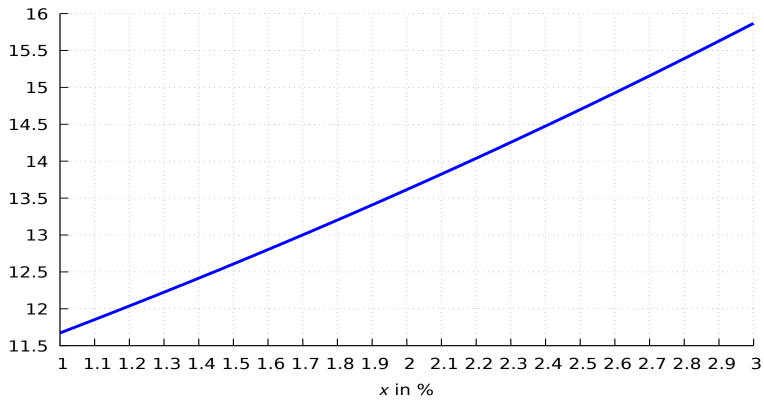


Figure 3.7: Graph of the function $x \mapsto ((1+x)^{21} - (1+x)^{11})/x$.

b) Compute the projected value of the plan at the end of year 20, if the annual interest rate is $r = 3.25\%$ over 20 years.

c) Compute the projected value of the plan at the end of year 20, if the annual interest rate $r = 3.25\%$ is paid only over the first 10 years.

Exercise 3.3 Consider a two-step trinomial (or ternary) market model $(S_t)_{t=0,1,2}$ with $r = 0$ and three possible return rates $R_t \in \{-1, 0, 1\}$. Show that the probability measure $\mathbb{P}^*$ given by

$$\mathbb{P}^*(R_t = -1) := \frac{1}{4}, \quad \mathbb{P}^*(R_t = 0) := \frac{1}{2}, \quad \mathbb{P}^*(R_t = 1) := \frac{1}{4}$$

is risk-neutral.

Exercise 3.4 We consider a riskless asset valued $S_k^{(0)} = S_0^{(0)}$, $k = 0, 1, \ldots, N$, where the risk-free interest rate is $\underline{r = 0}$, and a risky asset $S^{(1)}$ whose returns $R_k := \dfrac{S_k^{(1)} - S_{k-1}^{(1)}}{S_{k-1}^{(1)}}$, $k = 1, 2, \ldots, N$, form a sequence of independent identically distributed random variables taking three values $\{-b < 0 < b\}$ at each time step, with

$$p^* := \mathbb{P}^*(R_k = b) > 0, \quad \theta^* := \mathbb{P}^*(R_k = 0) > 0, \quad q^* := \mathbb{P}^*(R_k = -b) > 0,$$

$k = 1, 2, \ldots, N$. The information known to the market up to time k is denoted by $\mathscr{F}_k$.

a) Determine all possible risk-neutral probability measures $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in terms of the parameter $\theta^* \in (0,1)$.
b) Assume that the conditional variance

$$\operatorname{Var}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \,\middle|\, \mathscr{F}_k\right] = \sigma^2 > 0, \qquad k = 0, 1, \ldots, N-1, \tag{3.17}$$

of the asset return is constant and equal to σ^2. Show that this condition defines a unique risk-neutral probability measure $\mathbb{P}^*_\sigma$ under a certain condition on b and σ, and determine $\mathbb{P}^*_\sigma$ explicitly.

Exercise 3.5 We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$, with a riskless asset whose price π_t evolves as $\pi_t = \pi_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of S_{t-1} to S_t is given by

$$S_t = \begin{cases} (1+b)S_{t-1} \\ \\ (1+a)S_{t-1} \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset S is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N,$$

and $\mathscr{F}_t$ is generated by $R_1, R_2, \ldots, R_t$, $t = 1, 2, \ldots, N$.

a) What are the possible values of R_t?
b) Show that, under the probability measure $\mathbb{P}^*$ defined by

$$p^* = \mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t) = \frac{r-a}{b-a}, \qquad q^* = \mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$, the expected return $\mathbb{E}^*[R_{t+1} \mid \mathscr{F}_t]$ of S is equal to the return r of the riskless asset.
c) Show that under $\mathbb{P}^*$ the process $(S_t)_{t=0,1,\ldots,N}$ satisfies

$$\mathbb{E}^*[S_{t+k} \mid \mathscr{F}_t] = (1+r)^k S_t, \qquad t = 0, 1, \ldots, N-k, \quad k = 0, 1, \ldots, N.$$

Exercise 3.6 We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$, with a riskless asset whose price π_t evolves as $\pi_t = \pi_0(1+r)^t$, and a risky asset whose price S_t is given by

$$S_t = S_0 \prod_{k=1}^{t} (1 + R_k), \qquad t = 0, 1, \ldots, N,$$

where the *market return* R_k are independent random variables taking two possible values a and b with $-1 < a < r < b$, and $\mathbb{P}^*$ is the probability measure defined by

$$p^* = \mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t) = \frac{r-a}{b-a}, \qquad q^* = \mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t) = \frac{b-r}{b-a},$$

$t = 0, 1, \ldots, N-1$, where $(\mathscr{F}_t)_{t=0,1,\ldots,N}$ is the filtration generated by $(R_t)_{t=1,2,\ldots,N}$.

a) Compute the conditional expected return $\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t]$ under $\mathbb{P}^*$, $t = 0, 1, \ldots, N-1$.

b) Show that the discounted asset price process

$$\big(\widetilde{S}_t\big)_{t=0,1,\ldots,N} := (S_t / \pi_t)_{t=0,1,\ldots,N}$$

is a (nonnegative) $(\mathcal{F}_t)$-martingale under $\mathbb{P}^*$.
Hint: Use the independence of market returns $(R_t)_{t=1,2,\ldots,N}$ under $\mathbb{P}^*$.

c) Compute the moment $\mathbb{E}^*[(S_N)^\beta]$ for all $\beta > 0$.
Hint: Use the independence of market returns $(R_t)_{t=1,2,\ldots,N}$ under $\mathbb{P}^*$.

d) For any $\alpha > 0$, find an upper bound for the probability

$$\mathbb{P}^*\Big(S_t \geqslant \alpha \pi_t \ \text{ for some } t \in \{0, 1, \ldots, N\}\Big).$$

Hint: Use the fact that when $(M_t)_{t=0,1,\ldots,N}$ is a nonnegative martingale we have

$$\mathbb{P}\Big(\operatorname*{Max}_{t=0,1,\ldots,N} M_t \geqslant x\Big) \leqslant \frac{\mathbb{E}[(M_N)^\beta]}{x^\beta}, \qquad x > 0, \quad \beta \geqslant 1. \tag{3.18}$$

e) For any $x > 0$, find an upper bound for the probability

$$\mathbb{P}^*\Big(\operatorname*{Max}_{t=0,1,\ldots,N} S_t \geqslant x\Big).$$

Hint: Note that (3.18) remains valid for any nonnegative *sub*martingale.

4. Pricing and Hedging in Discrete Time

We consider the pricing and hedging of financial derivatives in the N-step Cox-Ross-Rubinstein (CRR) model with $N+1$ time instants $t = 0, 1, \ldots, N$. Vanilla options are priced and hedged using backward induction, and exotic options with arbitrary claim payoffs are dealt with using the Clark-Ocone formula in discrete time.

4.1 Pricing Contingent Claims

Let us consider an attainable contingent claim with (random) claim payoff $C \geqslant 0$ and maturity N. Recall that by the Definition 3.10 of attainability there exists a (self-financing) portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ that *hedges* the claim with payoff C, in the sense that

$$\bar{\xi}_N \cdot \bar{S}_N = \sum_{k=0}^{d} \xi_N^{(k)} S_N^{(k)} = C \tag{4.1}$$

at time N. If (4.1) holds at time N, then investing the amount

$$V_0 = \bar{\xi}_1 \cdot \bar{S}_0 = \sum_{k=0}^{d} \xi_1^{(k)} S_0^{(k)} \tag{4.2}$$

at time $t = 0$, resp.

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \tag{4.3}$$

at times $t = 1, 2, \ldots, N$ into a self-financing hedging portfolio $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ will allow one to hedge the option and to reach the perfect replication equality (4.1) at time $t = N$.

Definition 4.1 The value (4.2)-(4.3) at time t of a self-financing portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ hedging an attainable claim payoff C will be called an *arbitrage-free price* of the claim payoff C at time t and denoted by $\pi_t(C)$, $t = 0, 1, \ldots, N$.

Recall that arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value (mark to market).

Next we develop a second approach to the pricing of contingent claims, based on conditional expectations and martingale arguments. We will need the following lemma, in which $\widetilde{V}_t := V_t/(1+r)^t$ denotes the discounted portfolio value, $t = 0, 1, \ldots, N$.

Relation (4.4) in the following lemma has a natural interpretation by saying that when a portfolio is self-financing the value $\widetilde{V}_t$ of the (discounted) portfolio at time t is given by summing up the (discounted) profits and losses registered over all trading time periods from time 0 to time t. Note that in (4.4), the use of the vector of discounted asset prices

$$\bar{X}_t := \big(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\big), \qquad t = 0, 1, \ldots, N,$$

allows us to add up the discounted profits and losses $\bar{\xi}_t \cdot \big(\bar{X}_t - \bar{X}_{t-1}\big)$ since they are expressed in units of currency "at time 0". Indeed, in general, \$1 at time $t = 0$ cannot be added to \$1 at time $t = 1$ without proper discounting.

Lemma 4.1 The following statements are equivalent:

(i) The portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ is self-financing, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t, \qquad t = 1, 2, \ldots, N-1.$$

(ii) We have $\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t$ for all $t = 1, 2, \ldots, N-1$.

(iii) The discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic summation

$$\widetilde{V}_t = \widetilde{V}_0 + \underbrace{\sum_{k=1}^{t} \bar{\xi}_k \cdot \big(\bar{X}_k - \bar{X}_{k-1}\big)}_{\text{sum of profits and losses}}, \qquad t = 0, 1, \ldots, N, \tag{4.4}$$

of discounted profits and losses.

Proof. First, the self-financing condition (i)

$$\bar{\xi}_{t-1} \cdot \bar{S}_{t-1} = \bar{\xi}_t \cdot \bar{S}_{t-1}, \qquad t = 2, 3, \ldots, N,$$

is clearly equivalent to (ii) by division of both sides by $(1+r)^{t-1}$.

Assuming now that (ii) holds, by (3.8) we have

$$V_0 = \bar{\xi}_1 \cdot \bar{S}_0 \quad \text{and} \quad V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \quad t = 1, 2, \ldots, N.$$

which shows that (4.4) is satisfied for $t = 1$, in addition to being satisfied for $t = 0$. Next, for $t = 2,3,\ldots,N$ we have the telescoping identity

$$\begin{aligned}
\widetilde{V}_t &= \widetilde{V}_1 + \sum_{k=2}^{t} (\widetilde{V}_k - \widetilde{V}_{k-1}) \\
&= \widetilde{V}_1 + \sum_{k=2}^{t} (\bar{\xi}_k \cdot \bar{X}_k - \bar{\xi}_{k-1} \cdot \bar{X}_{k-1}) \\
&= \widetilde{V}_1 + \sum_{k=2}^{t} (\bar{\xi}_k \cdot \bar{X}_k - \bar{\xi}_k \cdot \bar{X}_{k-1}) \\
&= \widetilde{V}_1 + \sum_{k=2}^{t} \bar{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1}), \qquad t = 2,3,\ldots,N.
\end{aligned}$$

Finally, assuming that (iii) holds, we get

$$\widetilde{V}_t - \widetilde{V}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}), \qquad t = 1,2,\ldots,N,,$$

which rewrites as

$$\bar{\xi}_t \cdot \bar{X}_t - \bar{\xi}_{t-1} \cdot \bar{X}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}), \qquad t = 2,3,\ldots,N,,$$

or

$$\bar{\xi}_{t-1} \cdot \bar{X}_{t-1} = \bar{\xi}_t \cdot \bar{X}_{t-1}, \qquad t = 2,3,\ldots,N,$$

which implies (ii). □

In Relation (4.4), the term $\bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1})$ represents the (discounted) profit and loss

$$\widetilde{V}_t - \widetilde{V}_{t-1} = \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}),$$

of the self-financing portfolio strategy $(\bar{\xi}_j)_{j=1,2,\ldots,N}$ over the time interval $(t-1,t]$, computed by multiplication of the portfolio allocation $\bar{\xi}_t$ with the change of price $\bar{X}_t - \bar{X}_{t-1}$, $t = 1,2,\ldots,N$.

R As a consequence of Lemma 4.1, if a contingent claim with payoff C is attainable by a self-financing portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$, then the discounted claim payoff

$$\widetilde{C} := \frac{C}{(1+r)^N}$$

rewrites as the sum of discounted profits and losses

$$\widetilde{C} = \widetilde{V}_N = \bar{\xi}_N \cdot \bar{X}_N = \widetilde{V}_0 + \sum_{t=1}^{N} \bar{\xi}_t \cdot (\bar{X}_t - \bar{X}_{t-1}). \tag{4.5}$$

The sum (4.4) is also referred to as a discrete-time *stochastic integral* of the portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ with respect to the random process $(\bar{X}_t)_{t=0,1,\ldots,N}$.

R By Proposition 3.5, the process $(\overline{X}_t)_{t=0,1,\ldots,N}$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, hence by the martingale transform Theorem 3.4 and Lemma 4.1, $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ in (4.4) is also martingale under $\mathbb{P}^*$, provided that $(\overline{\xi}_t)_{t=1,2,\ldots,N}$ is a self-financing and predictable process.

The above remarks will be used in the proof of the next Theorem 4.2.

Theorem 4.2 The arbitrage-free price $\pi_t(C)$ of any (integrable) attainable contingent with claim payoff C is given by

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathscr{F}_t], \qquad t = 0,1,\ldots,N, \tag{4.6}$$

where $\mathbb{P}^*$ denotes any risk-neutral probability measure.

Proof. a) Short proof. Since the claim payoff C is attainable, there exists a self-financing portfolio strategy $(\xi_t)_{t=1,2,\ldots,N}$ such that $C = V_N$, *i.e.* $\widetilde{C} = \widetilde{V}_N$. In addition, by Theorem 3.4 Lemma 4.1 the process $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ is a martingale under $\mathbb{P}^*$, hence we have

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{V}_N \,\big|\, \mathscr{F}_t\big] = \mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathscr{F}_t\big], \qquad t = 0,1,\ldots,N, \tag{4.7}$$

which shows (4.8). To conclude, we note that by Definition 4.1 the arbitrage-free price $\pi_t(C)$ of the claim at time t is equal to the value V_t of the self-financing hedging C.

b) Long proof. For completeness, we include a self-contained, step-by-step derivation of (4.7) by following the argument of Theorem 3.4, as follows. By Remark 4.1 we have

$$\begin{aligned}
\mathbb{E}^*\big[\widetilde{C} \,\big|\, \mathscr{F}_t\big] &= \mathbb{E}^*\big[\widetilde{V}_N \,\big|\, \mathscr{F}_t\big] \\
&= \mathbb{E}^*\left[\widetilde{V}_0 + \sum_{k=1}^{N} \overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{E}^*\big[\widetilde{V}_0 \,\big|\, \mathscr{F}_t\big] + \sum_{k=1}^{N} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big] \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big] + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big] \\
&= \widetilde{V}_0 + \sum_{k=1}^{t} \overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big] \\
&= \widetilde{V}_t + \sum_{k=t+1}^{N} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big],
\end{aligned}$$

where we used Relation (4.4) of Lemma 4.1. In order to obtain (4.8) we need to show that

$$\sum_{k=t+1}^{N} \mathbb{E}^*\big[\overline{\xi}_k \cdot (\overline{X}_k - \overline{X}_{k-1}) \,\big|\, \mathscr{F}_t\big] = 0,$$

or

$$\mathbb{E}^*\big[\overline{\xi}_j \cdot (\overline{X}_j - \overline{X}_{j-1}) \,\big|\, \mathscr{F}_t\big] = 0,$$

for all $j = t+1,\ldots,N$. Since $0 \leqslant t \leqslant j-1$ we have $\mathscr{F}_t \subset \mathscr{F}_{j-1}$, hence by the "tower property" of conditional expectations we get

$$\mathbb{E}^*\big[\overline{\xi}_j \cdot (\overline{X}_j - \overline{X}_{j-1}) \,\big|\, \mathscr{F}_t\big] = \mathbb{E}^*\big[\mathbb{E}^*\big[\overline{\xi}_j \cdot (\overline{X}_j - \overline{X}_{j-1}) \,\big|\, \mathscr{F}_{j-1}\big] \,\big|\, \mathscr{F}_t\big],$$

therefore it suffices to show that

$$\mathbb{E}^*\left[\overline{\xi}_j \cdot \left(\overline{X}_j - \overline{X}_{j-1}\right) \,\middle|\, \mathscr{F}_{j-1}\right] = 0, \quad j = 1,2,\ldots,N.$$

We note that the porfolio allocation $\overline{\xi}_j$ over the time period $[j-1, j]$ is predictable, *i.e.* it is decided at time $j-1$, and it thus depends only on the information $\mathscr{F}_{j-1}$ known up to time $j-1$, hence

$$\mathbb{E}^*\left[\overline{\xi}_j \cdot \left(\overline{X}_j - \overline{X}_{j-1}\right) \,\middle|\, \mathscr{F}_{j-1}\right] = \overline{\xi}_j \cdot \mathbb{E}^*\left[\overline{X}_j - \overline{X}_{j-1} \,\middle|\, \mathscr{F}_{j-1}\right].$$

Finally we note that

$$\begin{aligned}
\mathbb{E}^*\left[\overline{X}_j - \overline{X}_{j-1} \,\middle|\, \mathscr{F}_{j-1}\right] &= \mathbb{E}^*\left[\overline{X}_j \,\middle|\, \mathscr{F}_{j-1}\right] - \mathbb{E}^*\left[\overline{X}_{j-1} \,\middle|\, \mathscr{F}_{j-1}\right] \\
&= \mathbb{E}^*\left[\overline{X}_j \,\middle|\, \mathscr{F}_{j-1}\right] - \overline{X}_{j-1} \\
&= 0, \qquad j = 1,2,\ldots,N,
\end{aligned}$$

because $\left(\overline{X}_t\right)_{t=0,1,\ldots,N}$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, and this concludes the proof of (4.7). Let

$$\widetilde{C} = \frac{C}{(1+r)^N}$$

denote the discounted payoff of the claim C. We will show that under any risk-neutral probability measure $\mathbb{P}^*$ the discounted value of any self-financing portfolio hedging C is given by

$$\widetilde{V}_t = \mathbb{E}^*\left[\widetilde{C} \,\middle|\, \mathscr{F}_t\right], \qquad t = 0,1,\ldots,N, \tag{4.8}$$

which shows that

$$V_t = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathscr{F}_t]$$

after multiplication of both sides by $(1+r)^t$. Next, we note that (4.8) follows from the martingale transform result of Theorem 3.4.

□

Note that (4.6) admits an interpretation in an insurance framework, in which $\pi_t(C)$ represents an insurance premium and C represents the random value of an insurance claim made by a subscriber. In this context, the premium of the insurance contract reads as the average of the values (4.6) of the random claims after discounting for the time value of money.

R By Remark 4.1 the self-financing discounted portfolio value process

$$\left(\widetilde{V}_t\right)_{t=0,1,\ldots,N} = \left((1+r)^{-t}\pi_t(C)\right)_{t=0,1,\ldots,N}$$

hedgin the claim C is a martingale under the risk-neutral probability measure $\mathbb{P}^*$. This fact can be recovered from Theorem 4.2 as in Remark 4.1, since from the "tower property" (11.38) of conditional expectations we have

$$\begin{aligned}
\widetilde{V}_t &= \mathbb{E}^*\left[\widetilde{C} \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[\widetilde{C} \,\middle|\, \mathscr{F}_{t+1}\right] \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\widetilde{V}_{t+1} \,\middle|\, \mathscr{F}_t\right], \qquad t = 0,1,\ldots,N-1.
\end{aligned} \tag{4.9}$$

This will also allow us to compute V_t by backward induction on $t = 0,1,\ldots,N-1$, starting from $V_N = C$, see (4.14) below.

In particular, for $t = 0$ we obtain the price at time 0 of the contingent claim with payoff C, *i.e.*

$$\pi_0(C) = \mathbb{E}^*\left[\widetilde{C} \,\middle|\, \mathscr{F}_0\right] = \mathbb{E}^*\left[\widetilde{C}\right] = \frac{1}{(1+r)^N} \mathbb{E}^*[C].$$

4.2 Pricing Vanilla Options in the CRR Model

In this section we consider the pricing of contingent claims in the discrete-time Cox, Ross, and Rubinstein, 1979 model of Section 3.6, with $d = 1$ and

$$S_t^{(0)} = S_0^{(0)}(1+r)^t, \qquad t = 0,1,\ldots,N,$$

and

$$S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t}(1+R_k) = \left\{ \begin{array}{ll} (1+b)S_{t-1}^{(1)} & \text{if } R_t = b \\ \\ (1+a)S_{t-1}^{(1)} & \text{if } R_t = a \end{array} \right\} = (1+R_t)S_{t-1}^{(1)},$$

$t = 1,\ldots,N$. More precisely we are concerned with vanilla options whose payoffs depend on the terminal value of the underlying asset, as opposed to exotic options whose payoffs may depend on the whole path of the underlying asset price until expiration time.

Recall that the portfolio value process $(V_t)_{t=0,1,\ldots N}$ and the discounted portfolio value process respectively satisfy

$$V_t = \bar{\xi}_t \cdot \bar{S}_t \text{ and } \widetilde{V}_t = \frac{1}{(1+r)^t}V_t = \frac{1}{(1+r)^t}\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_t \cdot \bar{X}_t, \quad t = 0,1,\ldots,N.$$

Here we will be concerned with the pricing of vanilla options with payoffs of the form

$$C = f(S_N^{(1)}),$$

e.g. $f(x) = (x-K)^+$ in the case of a European call option. Equivalently, the discounted claim payoff

$$\widetilde{C} = \frac{C}{(1+r)^N}$$

satisfies $\widetilde{C} = \widetilde{f}(S_N^{(1)})$ with $\widetilde{f}(x) = f(x)/(1+r)^N$. For example in the case of the European call option with strike price K we have

$$\widetilde{f}(x) = \frac{1}{(1+r)^N}(x-K)^+.$$

From Theorem 4.2, the discounted value of a portfolio hedging the attainable (discounted) claim payoff $\widetilde{C}$ is given by

$$\widetilde{V}_t = \mathbb{E}^*\left[\widetilde{f}(S_N^{(1)}) \mid \mathscr{F}_t\right], \qquad t = 0,1,\ldots,N,$$

under the risk-neutral probability measure $\mathbb{P}^*$. As a consequence, we have the following proposition.

Proposition 4.3 The arbitrage-free price $\pi_t(C)$ at time $t = 0,1,\ldots,N$ of the contingent claim with payoff $C = f(S_N^{(1)})$ is given by

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*\left[f(S_N^{(1)}) \mid \mathscr{F}_t\right], \tag{4.10}$$

$t = 0,1,\ldots,N$.

In the next proposition we implement the calculation of (4.10).*

*Download the corresponding (non-recursive) **IPython notebook** that can be run **here**.

Proposition 4.4 The price $\pi_t(C)$ of the contingent claim with payoff $C = f\big(S_N^{(1)}\big)$ satisfies

$$\pi_t(C) = v\big(t, S_t^{(1)}\big), \qquad t = 0, 1, \ldots, N,$$

where the function $v(t,x)$ is given by

$$\begin{aligned} v(t,x) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[f\left(x \prod_{j=t+1}^{N} (1+R_j) \right) \right] \\ &= \frac{1}{(1+r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (q^*)^{N-t-k} f\big(x(1+b)^k(1+a)^{N-t-k}\big), \end{aligned} \tag{4.11}$$

where the risk-neutral probabilities p^*, q^* are defined as

$$p^* := \frac{r-a}{b-a} \qquad \text{and} \qquad q^* := 1 - p^* = \frac{b-r}{b-a}. \tag{4.12}$$

Proof. From the relations

$$S_N^{(1)} = S_t^{(1)} \prod_{j=t+1}^{N} (1+R_j),$$

and (4.10) we have, using Property (v)) of the conditional expectation, see page 68, and the independence of the market returns $\{R_1, \ldots, R_t\}$ and $\{R_{t+1}, \ldots, R_N\}$,

$$\begin{aligned} \pi_t(C) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[f\big(S_N^{(1)}\big) \mid \mathscr{F}_t \big] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[f\left(S_t^{(1)} \prod_{j=t+1}^{N} (1+R_j) \right) \,\middle|\, S_t^{(1)} \right] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\left[f\left(x \prod_{j=t+1}^{N} (1+R_j) \right) \right]_{x=S_t^{(1)}}, \end{aligned}$$

where we used Property (v)) of the conditional expectation, see page 68, and the independence of the market returns. Next, we note that the number of times R_j is equal to b for $j \in \{t+1, \ldots, N\}$, has a binomial distribution with parameter $(N-t, p^*)$ since the set of paths from time $t+1$ to time N containing j times "$(1+b)$" has cardinality $\binom{N-t}{j}$ and each such path has probability

$$(p^*)^j (q^*)^{N-t-j}, \qquad j = 0, \ldots, N-t.$$

Hence we have

$$\begin{aligned} \pi_t(C) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[f\big(S_N^{(1)}\big) \mid \mathscr{F}_t \big] \\ &= \frac{1}{(1+r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (q^*)^{N-t-k} f\big(S_t^{(1)} (1+b)^k (1+a)^{N-t-k}\big). \end{aligned}$$

□

In the above proof we have also shown that $\pi_t(C)$ is given by the conditional expected value

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[f\big(S_N^{(1)}\big) \mid \mathscr{F}_t \big] = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*\big[f\big(S_N^{(1)}\big) \mid S_t^{(1)} \big]$$

given the value of $S_t^{(1)}$ at time $t = 0,1,\ldots,N$, due to the Markov property of $\big(S_t^{(1)}\big)_{t=0,1,\ldots,N}$. In particular, the price of the claim with payoff C is written as the average (path integral) of the values of the contingent claim over all possible paths starting from $S_t^{(1)}$.

Market terms and data

Intrinsic value. The *intrinsic value* at time $t = 0,1,\ldots,N$ of the option with payoff $C = h\big(S_N^{(1)}\big)$ is given by the immediate exercise payoff $h\big(S_t^{(1)}\big)$. The *extrinsic value* at time $t = 0,1,\ldots,N$ of the option is the remaining difference $\pi_t(C) - h\big(S_t^{(1)}\big)$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h\big(S_t^{(1)}\big)$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = \underbrace{h\big(S_t^{(1)}\big)}_{\text{intrinsic value}} + \underbrace{\pi_t(C) - h\big(S_t^{(1)}\big)}_{\text{extrinsic value}}, \qquad t = 0,1,\ldots,N.$$

Gearing. The *gearing* at time $t = 0,1,\ldots,N$ of the option with payoff $C = h\big(S_N^{(1)}\big)$ is defined as the ratio

$$\mathrm{G}_t := \frac{S_t^{(1)}}{\pi_t(C)} = \frac{S_t^{(1)}}{v\big(t,S_t^{(1)}\big)}, \qquad t = 0,1,\ldots,N.$$

Break-even price. The *break-even* price BEP_t of the underlying asset at time $t = 0,1,\ldots,N$ is the value of S for which the intrinsic option value $h\big(S_t^{(1)}\big)$ equals the option price $\pi_t(C)$. In other words, BEP_t represents the price of the underlying asset for which we would break even if the option was exercised immediately. For European call options it is given by

$$\mathrm{BEP}_t := K + \pi_t(C) = K + v\big(t,S_t^{(1)}\big), \qquad t = 0,1,\ldots,N.$$

whereas for European put options it is given by

$$\mathrm{BEP}_t := K - \pi_t(C) = K - v\big(t,S_t^{(1)}\big), \qquad t = 0,1,\ldots,N.$$

Premium. The option *premium* OP_t can be defined as the variation required from the underlying asset price in order to reach the break-even price for which the intrinsic option payoff equals the current option price, *i.e.* we have

$$\mathrm{OP}_t := \frac{\mathrm{BEP}_t - S_t^{(1)}}{S_t^{(1)}} = \frac{K + v\big(t,S_t^{(1)}\big) - S_t^{(1)}}{S_t^{(1)}}, \qquad t = 0,1,\ldots,N,$$

for European call options, and

$$\mathrm{OP}_t := \frac{S_t^{(1)} - \mathrm{BEP}_t}{S_t^{(1)}} = \frac{S_t^{(1)} + v\big(t,S_t^{(1)}\big) - K}{S_t^{(1)}}, \qquad t = 0,1,\ldots,N,$$

for European put options. The term "premium" is sometimes also used to denote the arbitrage-free price $v\big(t,S_t^{(1)}\big)$ of the option.

Pricing by backward induction

In the CRR model, the discounted portfolio value $\widetilde{V}_t$ can be computed by *backward induction* as in (4.9), using the martingale property of the discounted portfolio value process $(\widetilde{V}_t)_{t=0,1,\ldots,N}$ under the risk-neutral probability measure $\mathbb{P}^*$. Namely, by the "tower property" of conditional expectations, letting

$$\widetilde{v}\big(t,S_t^{(1)}\big) := \frac{1}{(1+r)^t} v\big(t,S_t^{(1)}\big), \qquad t = 0,1,\ldots,N,$$

we have

$$\begin{aligned}
\widetilde{v}\big(t,S_t^{(1)}\big) &= \widetilde{V}_t \\
&= \mathbb{E}^*\big[\widetilde{f}\big(S_N^{(1)}\big) \,\big|\, \mathscr{F}_t\big] \\
&= \mathbb{E}^*\big[\mathbb{E}^*\big[\widetilde{f}\big(S_N^{(1)}\big) \,\big|\, \mathscr{F}_{t+1}\big] \,\big|\, \mathscr{F}_t\big] \\
&= \mathbb{E}^*\big[\widetilde{V}_{t+1} \,\big|\, \mathscr{F}_t\big] \\
&= \mathbb{E}^*\big[\widetilde{v}\big(t+1,S_{t+1}^{(1)}\big) \,\big|\, S_t\big] \\
&= \widetilde{v}\big(t+1,(1+a)S_t^{(1)}\big)\mathbb{P}^*(R_{t+1}=a) + \widetilde{v}\big(t+1,(1+b)S_t^{(1)}\big)\mathbb{P}^*(R_{t+1}=b) \\
&= q^*\widetilde{v}\big(t+1,(1+a)S_t^{(1)}\big) + p^*\widetilde{v}\big(t+1,(1+b)S_t^{(1)}\big),
\end{aligned}$$

which shows that $\widetilde{v}(t,x)$ satisfies the backward recursion*

$$\widetilde{v}(t,x) = q^*\widetilde{v}\big(t+1,x(1+a)\big) + p^*\widetilde{v}\big(t+1,x(1+b)\big), \tag{4.13}$$

while the terminal condition $\widetilde{V}_N = \widetilde{f}\big(S_N^{(1)}\big)$ implies

$$\widetilde{v}(N,x) = \widetilde{f}(x), \qquad x > 0.$$

For non-discounted option prices $v(t,S_t)$, the function $v(t,x)$ satisfies the relation

$$v(t,x) = \frac{q^*}{1+r} v\big(t+1,x(1+a)\big) + \frac{p^*}{1+r} v\big(t+1,x(1+b)\big), \tag{4.14}$$

with the terminal condition

$$v(N,x) = f(x), \qquad x > 0.$$

The next Figure 4.1 presents a tree-based implementation of the pricing recursion (4.14).

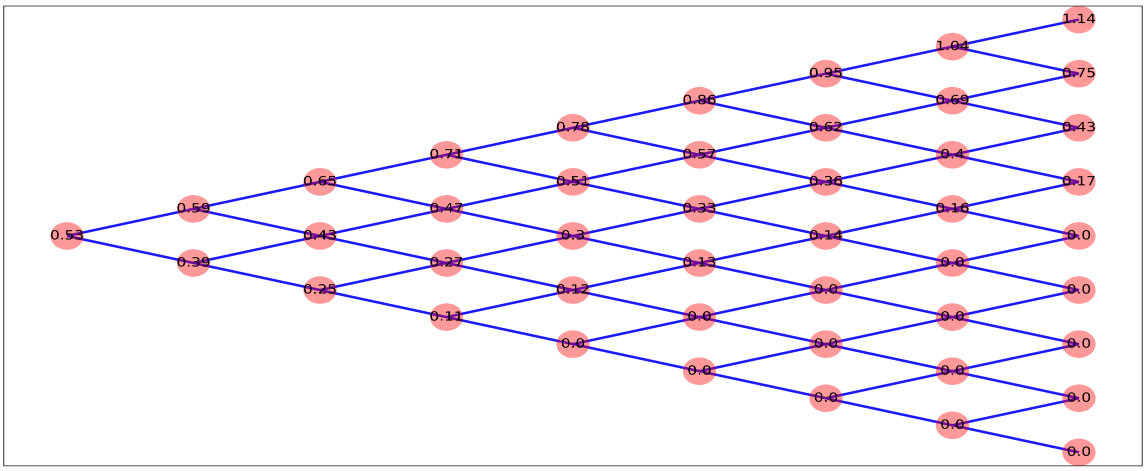


Figure 4.1: Discrete-time call option pricing tree.

*Download the corresponding (backward recursive) **IPython notebook** that can be run **here**.

Note that the discrete-time recursion (4.14) can be connected to the continuous-time Black-Scholes PDE (7.2), cf. Exercises 7.14.

4.3 Hedging Contingent Claims

The basic idea of hedging is to allocate assets in a portfolio in order to protect oneself from a given risk. For example, a risk of increasing oil prices can be hedged by buying oil-related stocks, whose value should be positively correlated with the oil price. In this way, a loss connected to increasing oil prices could be compensated by an increase in the value of the corresponding portfolio.

In the setting of this chapter, hedging an attainable contingent claim with payoff C means computing a self-financing portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$ such that

$$\bar{\xi}_N \cdot \bar{S}_N = C, \quad i.e. \quad \bar{\xi}_N \cdot \bar{X}_N = \widetilde{C}. \tag{4.15}$$

Price, then hedge.
The portfolio allocation $\bar{\xi}_N$ can be computed by first solving (4.15) for $\bar{\xi}_N$ from the payoff values C, based on the fact that the allocation $\bar{\xi}_N$ depends only on information up to time $N-1$, by the predictability of $\left(\bar{\xi}_k\right)_{1\leqslant k\leqslant N}$.

If the self-financing portfolio value V_t is known, for example from (4.6), *i.e.*

$$V_t = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0,1,\ldots,N, \tag{4.16}$$

we may similarly compute $\bar{\xi}_t$ by solving $\bar{\xi}_t \cdot \bar{S}_t = V_t$ for all $t = 1,2,\ldots,N-1$.

Hedge, then price.
If $V_t = \pi_t(C)$ has not been computed, we can use *backward induction* to compute a self-financing portfolio strategy. Starting from the values of $\bar{\xi}_N$ obtained by solving

$$\bar{\xi}_N \cdot \bar{S}_N = C,$$

we use the self-financing condition to solve for $\bar{\xi}_{N-1}$, $\bar{\xi}_{N-2}$, $\ldots$, $\bar{\xi}_4$, down to $\bar{\xi}_3$, $\bar{\xi}_2$, and finally $\bar{\xi}_1$.

In order to implement this algorithm we can use the $N-1$ self-financing equations

$$\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t, \qquad t = 1,2,\ldots,N-1, \tag{4.17}$$

allowing us in principle to compute the portfolio strategy $\left(\bar{\xi}_t\right)_{t=1,2,\ldots,N}$.

Based on the values of $\bar{\xi}_N$ we can solve

$$\bar{\xi}_{N-1} \cdot \bar{S}_{N-1} = \bar{\xi}_N \cdot \bar{S}_{N-1}$$

for $\bar{\xi}_{N-1}$, then

$$\bar{\xi}_{N-2} \cdot \bar{S}_{N-2} = \bar{\xi}_{N-1} \cdot \bar{S}_{N-2}$$

for $\bar{\xi}_{N-2}$, and successively $\bar{\xi}_2$ down to $\bar{\xi}_1$. In Section 4.4 the backward induction (4.17) will be implemented in the CRR model, see the proof of Proposition 4.5, and Exercises 4.15 and 4.4 for an application in a two-step model.

The discounted value $\widetilde{V}_t$ at time t of the portfolio claim can then be obtained from

$$\widetilde{V}_0 = \overline{\xi}_1 \cdot \overline{X}_0 \qquad \text{and} \qquad \widetilde{V}_t = \overline{\xi}_t \cdot \overline{X}_t, \qquad t = 1,2,\ldots,N. \tag{4.18}$$

In addition we have shown in the proof of Theorem 4.2 that the price $\pi_t(C)$ of the claim payoff C at time t coincides with the value V_t of any self-financing portfolio hedging the claim payoff C, *i.e.*

$$\pi_t(C) = V_t, \qquad t = 0,1,\ldots,N,$$

as given by (4.18). Hence the price of the claim can be computed either algebraically by solving (4.15) and (4.17) using backward induction and then using (4.18), or by a probabilistic method by a direct evaluation of the discounted expected value (4.16).

The development of hedging algorithms has increased *credit exposure* and counterparty risk when one party is unable to deliver the option payoff stated in the contract.

4.4 Hedging Vanilla Options in the CRR model

In this section we implement the backward induction (4.17) of Section 4.3 for the hedging of contingent claims in the discrete-time Cox-Ross-Rubinstein model. Our aim is to compute a self-financing portfolio strategy hedging a vanilla option with payoff of the form

$$C = h\big(S_N^{(1)}\big).$$

Since the discounted price $\widetilde{S}_t^{(0)}$ of the riskless asset satisfies

$$\widetilde{S}_t^{(0)} = (1+r)^{-t} S_t^{(0)} = S_0^{(0)},$$

we may sometimes write $S_0^{(0)}$ in place of $\widetilde{S}_t^{(0)}$. In Propositions 4.5 and 4.6 we present two different approaches to hedging and to the computation of the predictable process $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$, which is also called the *Delta*.

Proposition 4.5 *Price, then hedge.*[a] The self-financing replicating portfolio strategy

$$\big(\xi_t^{(0)}, \xi_t^{(1)}\big)_{t=1,2,\ldots,N} = \big(\xi_t^{(0)}\big(S_{t-1}^{(1)}\big), \xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\big)_{t=1,2,\ldots,N}$$

hedging the contingent claim with payoff $C = h\big(S_N^{(1)}\big)$ is given by

$$\begin{aligned}
\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) &= \frac{v\big(t,(1+b)S_{t-1}^{(1)}\big) - v\big(t,(1+a)S_{t-1}^{(1)}\big)}{(b-a)S_{t-1}^{(1)}} \\
&= \frac{\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big) - \widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)},
\end{aligned} \tag{4.19}$$

where the function $v(t,x)$ is given by (4.11), and

$$\begin{aligned}
\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) &= \frac{(1+b)v\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)v\big(t,(1+b)S_{t-1}^{(1)}\big)}{(b-a)S_t^{(0)}} \\
&= \frac{(1+b)\widetilde{v}\big(t,(1+a)S_{t-1}^{(1)}\big) - (1+a)\widetilde{v}\big(t,(1+b)S_{t-1}^{(1)}\big)}{(b-a)S_0^{(0)}},
\end{aligned} \tag{4.20}$$

$t = 1, 2, \ldots, N$, where the function $\widetilde{v}(t,x) = (1+r)^{-t} v(t,x)$ is given by (4.11).

[a] Download the corresponding "price, then hedge" **IPython notebook** that can be run **here**.

Proof. We first compute the self-financing hedging strategy $\left(\overline{\xi}_t\right)_{t=1,2,\ldots,N}$ by solving

$$\overline{\xi}_t \cdot \overline{X}_t = \widetilde{V}_t, \qquad t = 1, 2, \ldots, N,$$

from which we deduce the two equations

$$\begin{cases} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) S_0^{(0)} + \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) \dfrac{1+a}{1+r} \widetilde{S}_{t-1}^{(1)} = \widetilde{v}\big(t, (1+a) S_{t-1}^{(1)}\big) \\[2ex] \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) S_0^{(0)} + \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) \dfrac{1+b}{1+r} \widetilde{S}_{t-1}^{(1)} = \widetilde{v}\big(t, (1+b) S_{t-1}^{(1)}\big), \end{cases}$$

which can be solved as

$$\begin{cases} \xi_t^{(0)}\big(S_{t-1}^{(1)}\big) = \dfrac{(1+b)\widetilde{v}\big(t, (1+a) S_{t-1}^{(1)}\big) - (1+a)\widetilde{v}\big(t, (1+b) S_{t-1}^{(1)}\big)}{(b-a) S_0^{(0)}} \\[3ex] \xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = \dfrac{\widetilde{v}\big(t, (1+b) S_{t-1}^{(1)}\big) - \widetilde{v}\big(t, (1+a) S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)} / (1+r)}, \end{cases}$$

$t = 1, 2, \ldots, N$, which only depends on $S_{t-1}^{(1)}$, as expected. This is consistent with the fact that $\xi_t^{(1)}$ represents the (possibly fractional) quantity of the risky asset to be present in the portfolio over the time period $[t-1, t]$ in order to hedge the claim payoff C at time N, and is decided at time $t-1$. $\square$

By applying (4.19) to the function $v(t,x)$ in (4.11) we find

$$\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) = \frac{1}{(1+r)^{N-t}} \sum_{k=0}^{N-t} \binom{N-t}{k} (p^*)^k (q^*)^{N-t-k}$$
$$\times \frac{f\big(S_{t-1}^{(1)}(1+b)^{k+1}(1+a)^{N-t-k}\big) - f\big(S_{t-1}^{(1)}(1+b)^k(1+a)^{N-t-k+1}\big)}{(b-a) S_t^{(1)}},$$

$t = 0, 1, \ldots, N$.

The next Figure 4.2 presents a tree-based implementation of the risky hedging component (4.19).

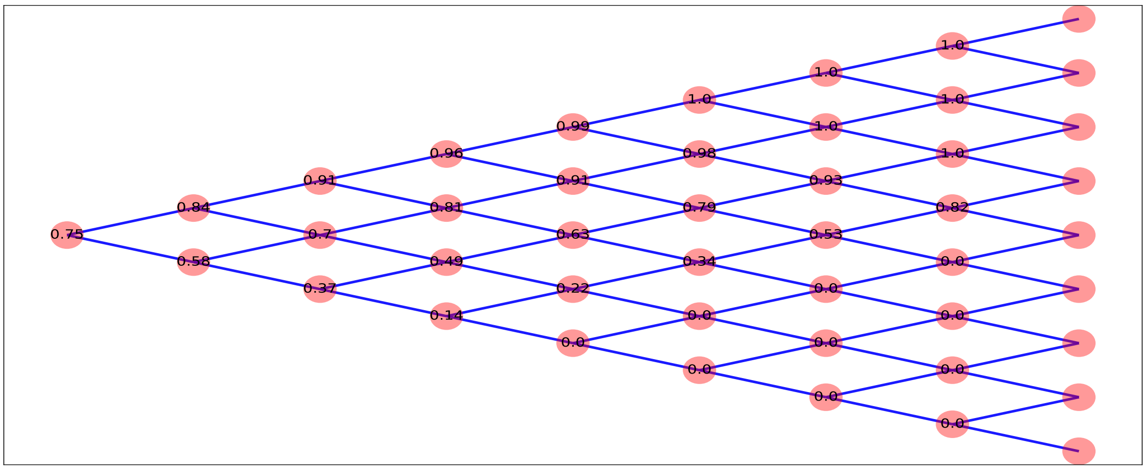


Figure 4.2: Discrete-time call option hedging strategy (risky component).

The next Figure 4.3 presents a tree-based implementation of the riskless hedging component (4.20).

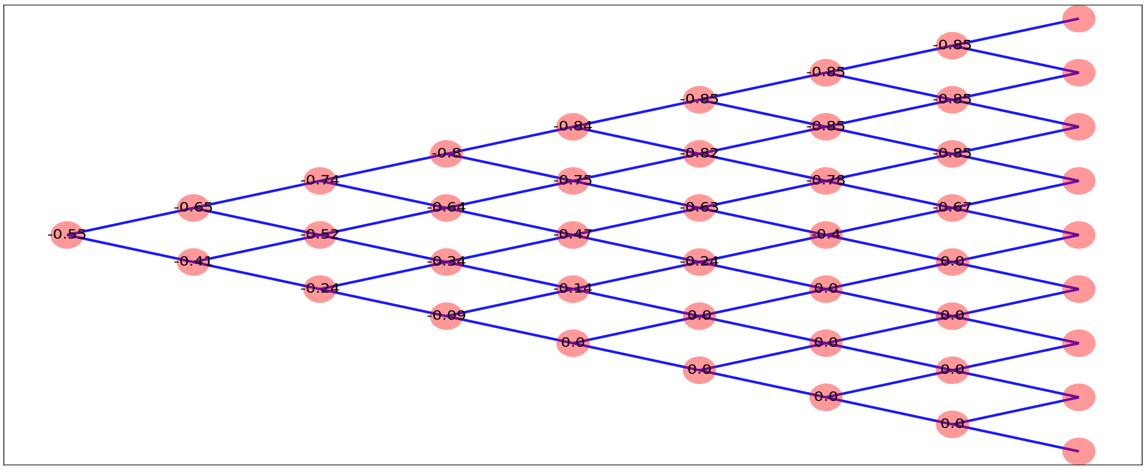


Figure 4.3: Discrete-time call option hedging strategy (riskless component).

Market terms and data

Delta. The *Delta* represents the quantity of underlying risky asset $S_t^{(1)}$ held in the portfolio over the time interval $[t-1,t]$. Here, it is denoted by $\xi_t^{(1)}\big(S_{t-1}^{(1)}\big)$ for $t=1,2,\ldots,N$.

Effective gearing. The *effective gearing* at time $t=1,2,\ldots,N$ of the option with payoff $C=h\big(S_N^{(1)}\big)$ is defined as the ratio

$$\begin{aligned} \mathrm{EG}_t &:= \mathrm{G}_t\xi_t^{(1)} \\ &= \frac{S_t^{(1)}}{\pi_t(C)}\xi_t^{(1)} \\ &= \frac{S_t^{(1)}\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)}{S_{t-1}^{(1)}v\big(t,S_t^{(1)}\big)(b-a)} \\ &= \frac{\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)/v\big(t,S_t^{(1)}\big)}{S_{t-1}^{(1)}(b-a)/S_t^{(1)}}, \qquad t=1,2,\ldots,N. \end{aligned}$$

The effective gearing $\mathrm{EG}_t=\xi_t S_t^{(1)}/\pi_t(C)$ can be interpreted as the *hedge ratio*, *i.e.* the percentage of the portfolio which is invested on the risky asset. It also allows one to represent the percentage change in the option price in terms of the potential percentage change $S_{t-1}^{(1)}(b-a)/S_t^{(1)}$ in the underlying asset price when the market return switches from a to b, as

$$\frac{\big(v\big(t,(1+b)S_{t-1}^{(1)}\big)-v\big(t,(1+a)S_{t-1}^{(1)}\big)\big)}{v\big(t,S_t^{(1)}\big)}=\mathrm{EG}_t\times\frac{S_{t-1}^{(1)}(b-a)}{S_t^{(1)}}.$$

R

i) If the function $x\mapsto h(x)$ is non-decreasing, *e.g.* in the case of European call options, then the function $x\mapsto\widetilde{v}(t,x)$ is also non-decreasing for all fixed $t=0,1,\ldots,N$, hence the portfolio strategy $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ defined by (4.11) or (4.19) satisfies $\xi_t^{(1)}\geqslant 0$, $t=1,2,\ldots,N$ and there is not short selling.

ii) Similarly, we can show that when $x\mapsto h(x)$ is a non-increasing function, *e.g* in the case of European put options, the portfolio allocation $\xi_t^{(1)}\leqslant 0$ is negative, $t=1,2,\ldots,N$, *i.e.* short selling always occurs.

iii) We can check that the portfolio strategy

$$(\bar{\xi}_t)_{t=1,2,\ldots,N} = (\xi_t^{(0)}, \xi_t^{(1)})_{t=1,2,\ldots,N} = (\xi_t^{(0)}(S_{t-1}^{(1)}), \xi_t^{(1)}(S_{t-1}^{(1)}))_{t=1,2,\ldots,N}$$

is self-financing, as by (4.19)-(4.20) we have

$$\begin{aligned}
\bar{\xi}_{t+1} \cdot \bar{X}_t &= \xi_{t+1}^{(0)}(S_t^{(1)}) S_0^{(0)} + \xi_{t+1}^{(1)}(S_t^{(1)}) \widetilde{S}_t^{(1)} \\
&= S_0^{(0)} \frac{(1+b)\widetilde{v}(t+1,(1+a)S_t^{(1)}) - (1+a)\widetilde{v}(t+1,(1+b)S_t^{(1)})}{(b-a)S_0^{(0)}} \\
&\quad + \widetilde{S}_t^{(1)} \frac{\widetilde{v}(t+1,(1+b)S_t^{(1)}) - \widetilde{v}(t+1,(1+a)S_t^{(1)})}{(b-a)\widetilde{S}_t^{(1)}/(1+r)} \\
&= \frac{(1+b)\widetilde{v}(t+1,(1+a)S_t^{(1)}) - (1+a)\widetilde{v}(t+1,(1+b)S_t^{(1)})}{b-a} \\
&\quad + \frac{\widetilde{v}(t+1,(1+b)S_t^{(1)}) - \widetilde{v}(t+1,(1+a)S_t^{(1)})}{(b-a)/(1+r)} \\
&= \frac{r-a}{b-a}\widetilde{v}(t+1,(1+b)S_t^{(1)}) + \frac{b-r}{b-a}\widetilde{v}(t+1,(1+a)S_t^{(1)}) \\
&= p^*\widetilde{v}(t+1,(1+b)S_t^{(1)}) + q^*\widetilde{v}(t+1,(1+a)S_t^{(1)}) \\
&= \widetilde{v}(t,S_t^{(1)}) \\
&= \xi_t^{(0)}(S_t^{(1)}) S_0^{(0)} + \xi_t^{(1)}(S_t^{(1)}) \widetilde{S}_t^{(1)} \\
&= \bar{\xi}_t \cdot \bar{X}_t, \qquad t = 0,1,\ldots,N-1,
\end{aligned}$$

where we used (4.13) or the martingale property of the discounted portfolio value process $(\tilde{v}(t,S_t^{(1)}))_{t=0,1,\ldots,N}$, cf. Lemma 4.1.

As a consequence of (4.20), the discounted amounts $\xi_t^{(0)} S_0^{(0)}$ and $\xi_t^{(1)} \widetilde{S}_t^{(1)}$ respectively invested on the riskless and risky assets are given by

$$S_0^{(0)} \xi_t^{(0)}(S_{t-1}^{(1)}) = \frac{(1+b)\widetilde{v}(t,(1+a)S_{t-1}^{(1)}) - (1+a)\widetilde{v}(t,(1+b)S_{t-1}^{(1)})}{b-a} \tag{4.21}$$

and

$$\widetilde{S}_t^{(1)} \xi_t^{(1)}(S_{t-1}^{(1)}) = (1+R_t)\frac{\widetilde{v}(t,(1+b)S_{t-1}^{(1)}) - \widetilde{v}(t,(1+a)S_{t-1}^{(1)})}{b-a},$$

$t = 1,2,\ldots,N$.

Regarding the quantity $\xi_t^{(0)}$ of the riskless asset in the portfolio at time t, from the relation

$$\widetilde{V}_t = \bar{\xi}_t \cdot \bar{X}_t = \xi_t^{(0)} \widetilde{S}_t^{(0)} + \xi_t^{(1)} \widetilde{S}_t^{(1)}, \qquad t = 1,2,\ldots,N,$$

we also obtain

$$\begin{aligned}
\xi_t^{(0)} &= \frac{\widetilde{V}_t - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{\widetilde{S}_t^{(0)}} \\
&= \frac{\widetilde{V}_t - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{S_0^{(0)}} \\
&= \frac{\widetilde{v}(t,S_t^{(1)}) - \xi_t^{(1)} \widetilde{S}_t^{(1)}}{S_0^{(0)}},
\end{aligned}$$

$t=1,2,\ldots,N$. In the next proposition we compute the hedging strategy by backward induction, starting from the relation

$$\xi_N^{(1)}\big(S_{N-1}^{(1)}\big)=\frac{h\big((1+b)S_{N-1}^{(1)}\big)-h\big((1+a)S_{N-1}^{(1)}\big)}{(b-a)S_{N-1}^{(1)}},$$

and

$$\xi_N^{(0)}\big(S_{N-1}^{(1)}\big)=\frac{(1+b)h\big((1+a)S_{N-1}^{(1)}\big)-(1+a)h\big((1+b)S_{N-1}^{(1)}\big)}{(b-a)S_0^{(0)}(1+r)^N},$$

that follow from (4.19) and (4.20) applied to the claim payoff function $h(\cdot)$.

Proposition 4.6 *Hedge, then price.*[a] The self-financing replicating portfolio strategy

$$\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}=\big(\xi_t^{(0)}\big(S_{t-1}^{(1)}\big),\xi_t^{(1)}\big(S_{t-1}^{(1)}\big)\big)_{t=1,2,\ldots,N}$$

hedging the contingent claim with payoff $C=h\big(S_N^{(1)}\big)$ is given from (4.19) at time $t=N$ by

$$\xi_N^{(1)}\big(S_{N-1}^{(1)}\big)=\frac{h\big((1+b)S_{N-1}^{(1)}\big)-h\big((1+a)S_{N-1}^{(1)}\big)}{(b-a)S_{N-1}^{(1)}}, \tag{4.22}$$

where the function $v(t,x)$ is given by (4.11), and

$$\xi_N^{(0)}\big(S_{N-1}^{(1)}\big)=\frac{(1+b)h\big((1+a)S_{N-1}^{(1)}\big)-(1+a)h\big((1+b)S_{N-1}^{(1)}\big)}{(b-a)S_N^{(0)}}, \tag{4.23}$$

and then inductively by

$$\begin{aligned}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big) &= \frac{(1+b)\xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)-(1+a)\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big)}{b-a}\\ &\quad +S_0^{(0)}\frac{\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)-\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)}{(b-a)\widetilde{S}_{t-1}^{(1)}/(1+r)},\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big) &= \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}\big(\xi_{t+1}^{(1)}\big((1+a)S_{t-1}^{(1)}\big)-\xi_{t+1}^{(1)}\big((1+b)S_{t-1}^{(1)}\big)\big)}{(b-a)(1+r)S_0^{(0)}}\\ &\quad +\frac{(1+b)\xi_{t+1}^{(0)}\big((1+a)S_{t-1}^{(1)}\big)-(1+a)\xi_{t+1}^{(0)}\big((1+b)S_{t-1}^{(1)}\big)}{b-a},\end{aligned} \tag{4.25}$$

$t=1,2,\ldots,N-1$.

The pricing function $\widetilde{v}(t,x)=(1+r)^{-t}v(t,x)$ is then given by

$$\widetilde{v}\big(t,S_t^{(1)}\big)=S_0^{(0)}\xi_t^{(0)}\big(S_{t-1}^{(1)}\big)+\widetilde{S}_t^{(1)}\xi_t^{(1)}\big(S_{t-1}^{(1)}\big),\quad t=1,2,\ldots,N.$$

[a] Download the corresponding "hedge, then price" **IPython notebook** that can be run **here**.

Proof. Relations (4.22)-(4.23) follow from (4.19)-(4.20) at time $t=N$. Next, by the self-financing condition (4.17) we have

$$\bar{\xi}_t\cdot\bar{X}_t=\bar{\xi}_{t+1}\cdot\bar{X}_t$$

i.e.

$$\begin{cases} S_0^{(0)}\xi_t^{(0)}(S_{t-1}^{(1)})+\widetilde{S}_{t-1}^{(1)}\xi_t^{(1)}(S_{t-1}^{(1)})\dfrac{1+b}{1+r} \\ \quad =\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)})S_0^{(0)}+\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)})\widetilde{S}_{t-1}^{(1)}\dfrac{1+b}{1+r} \\ \\ S_0^{(0)}\xi_t^{(0)}(S_{t-1}^{(1)})+\widetilde{S}_{t-1}^{(1)}\xi_t^{(1)}(S_{t-1}^{(1)})\dfrac{1+a}{1+r} \\ \quad =\xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})S_0^{(0)}+\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})\widetilde{S}_{t-1}^{(1)}\dfrac{1+a}{1+r}, \end{cases}$$

which can be solved as

$$\begin{aligned} \xi_t^{(1)}(S_{t-1}^{(1)}) &= \frac{(1+b)\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)})-(1+a)\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})}{b-a} \\ &\quad +(1+r)S_0^{(0)}\frac{\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)})-\xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})}{(b-a)\widetilde{S}_{t-1}^{(1)}}, \end{aligned}$$

and

$$\begin{aligned} \xi_t^{(0)}(S_{t-1}^{(1)}) &= \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}(\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})-\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)}))}{(b-a)(1+r)S_0^{(0)}} \\ &\quad +\frac{(1+b)\xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})-(1+a)\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)})}{b-a}, \end{aligned}$$

$t=1,2,\ldots,N-1$. □

R By (4.24)-(4.25) we can check that the corresponding discounted portfolio value process

$$(\widetilde{V}_t)_{t=1,2,\ldots,N}=\big(\overline{\xi}_t\cdot\overline{X}_t\big)_{t=1,2,\ldots,N}$$

is a martingale under $\mathbb{P}^*$:

$$\begin{aligned} \widetilde{V}_t &= \overline{\xi}_t\cdot\overline{X}_t \\ &= S_0^{(0)}\xi_t^{(0)}(S_{t-1}^{(1)})+\widetilde{S}_t^{(1)}\xi_t^{(1)}(S_{t-1}^{(1)}) \\ &= \frac{(1+a)(1+b)\widetilde{S}_{t-1}^{(1)}(\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})-\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)}))}{(b-a)(1+r)} \\ &\quad +S_0^{(0)}\frac{(1+b)\xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})-(1+a)\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)})}{(b-a)} \\ &\quad +\widetilde{S}_t^{(1)}\frac{(1+b)\xi_{t+1}^{(1)}((1+b)S_{t-1}^{(1)})-(1+a)\xi_{t+1}^{(1)}((1+a)S_{t-1}^{(1)})}{b-a} \\ &\quad +(1+r)\widetilde{S}_t^{(1)}S_0^{(0)}\frac{\xi_{t+1}^{(0)}((1+b)S_{t-1}^{(1)})-\xi_{t+1}^{(0)}((1+a)S_{t-1}^{(1)})}{(b-a)\widetilde{S}_{t-1}^{(1)}} \\ &= \frac{r-a}{b-a}S_0^{(0)}\xi_{t+1}^{(0)}(S_t^{(1)})+\frac{b-r}{b-a}S_0^{(0)}\xi_{t+1}^{(0)}(S_t^{(1)}) \\ &\quad +\frac{(r-a)(1+b)}{(b-a)(1+r)}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}(S_t^{(1)})+\frac{(b-r)(1+a)}{(b-a)(1+r)}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}(S_t^{(1)}) \\ &= p^*S_0^{(0)}\xi_{t+1}^{(0)}(S_t^{(1)})+q^*S_0^{(0)}\xi_{t+1}^{(0)}(S_t^{(1)}) \\ &\quad +p^*\frac{1+b}{1+r}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}(S_t^{(1)})+q^*\frac{1+a}{1+r}\widetilde{S}_t^{(1)}\xi_{t+1}^{(1)}(S_t^{(1)}) \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}^*\left[S_0^{(0)}\xi_{t+1}^{(0)}\big(S_t^{(1)}\big)+\widetilde{S}_{t+1}^{(1)}\xi_{t+1}^{(1)}\big(S_t^{(1)}\big)\,\big|\,\mathscr{F}_t\right]\\
&= \mathbf{E}^*\left[\widetilde{V}_{t+1}\,\big|\,\mathscr{F}_t\right],
\end{aligned}
$$

$t=1,2,\ldots,N-1$, as in Remark 4.1.

The next Figure 4.4 presents a tree-based implementation of the riskless hedging component (4.20).*

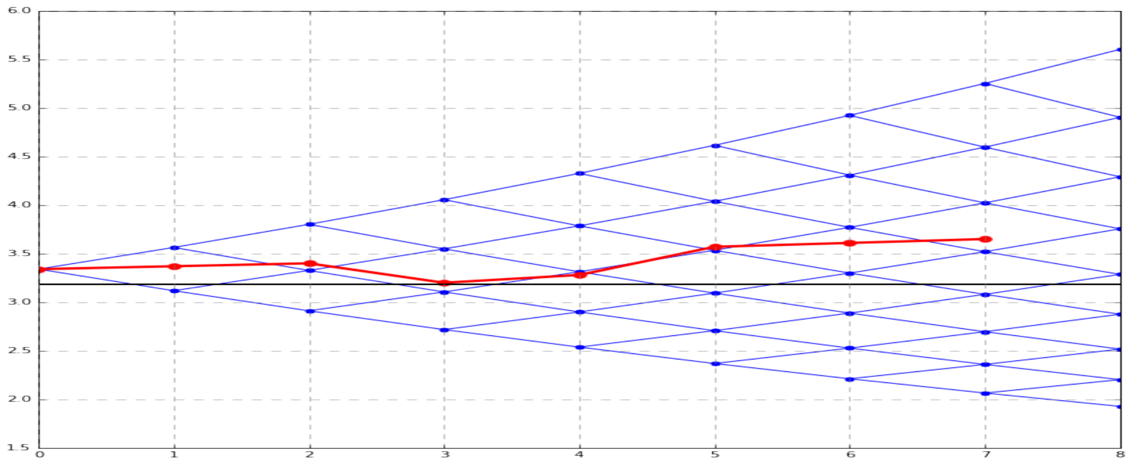


Figure 4.4: Tree of asset prices in the CRR model.

The next Figure 4.5 presents a tree-based implementation of option prices in the CRR model.

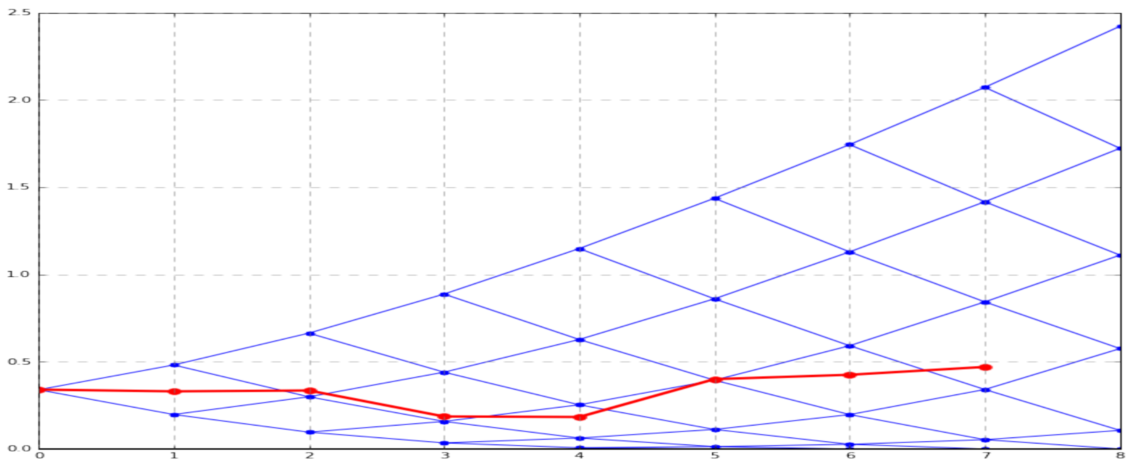


Figure 4.5: Tree of option prices in the CRR model.

The next Figure 4.6 presents a tree-based implementation of risky hedging portfolio allocation in the CRR model.

*Download the corresponding pricing and hedging **IPython notebook** that can be run **here**.

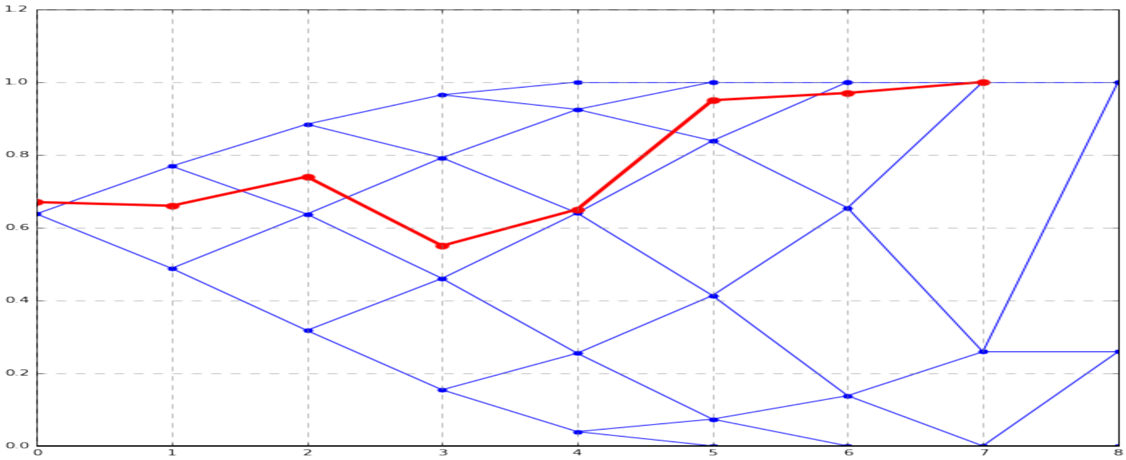


Figure 4.6: Tree of hedging portfolio allocations in the CRR model.

4.5 Hedging Exotic Options in the CRR Model

In this section we take $p = p^*$ given by (4.12) and we consider the hedging of path-dependent options. Here we choose to use the finite difference gradient and the discrete Clark-Ocone formula of stochastic analysis, see also Föllmer and Schied, 2004, Lamberton and Lapeyre, 1996, Privault, 2008, Chapter 1 of Privault, 2009, Ruiz de Chávez, 2001, or §15-1 of Williams, 1991. See Nunno, Øksendal, and Proske, 2009 and Section 8.2 of Privault, 2009 for a similar approach in continuous time. Given

$$\omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \Omega = \{-1, 1\}^N,$$

and $r = 1, 2, \ldots, N$, let

$$\omega_+^t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, +1, \omega_{t+1}, \ldots, \omega_N)$$

and

$$\omega_-^t := (\omega_1, \omega_2, \ldots, \omega_{t-1}, -1, \omega_{t+1}, \ldots, \omega_N).$$

We also assume that the return $R_t(\omega)$ is constructed as

$$R_t(\omega_+^t) = b \qquad \text{and} \qquad R_t(\omega_-^t) = a, \qquad t = 1, 2, \ldots, N, \quad \omega \in \Omega.$$

Definition 4.2 The operator D_t is defined on any random variable F by

$$D_t F(\omega) = F(\omega_+^t) - F(\omega_-^t), \qquad t = 1, 2, \ldots, N. \tag{4.26}$$

We define the centered and normalized return Y_t by

$$Y_t := \frac{R_t - r}{b - a} = \begin{cases} \dfrac{b-r}{b-a} = q, & \omega_t = +1, \\[2ex] \dfrac{a-r}{b-a} = -p, & \omega_t = -1, \end{cases} \qquad t = 1, 2, \ldots, N.$$

Note that under the risk-neutral probability measure $\mathbb{P}^*$ we have

$$\begin{aligned} \mathbb{E}^*[Y_t] &= \mathbb{E}^*\left[\frac{R_t - r}{b - a}\right] \\ &= \frac{a-r}{b-a}\mathbb{P}^*(R_t = a) + \frac{b-r}{b-a}\mathbb{P}^*(R_t = b) \end{aligned}$$

$$
\begin{aligned}
&= \frac{a-r}{b-a} \times \frac{b-r}{b-a} + \frac{b-r}{b-a} \times \frac{r-a}{b-a} \\
&= 0,
\end{aligned}
$$

and

$$
\operatorname{Var}[Y_t] = pq^2 + qp^2 = pq, \qquad t = 1,2,\ldots,N.
$$

In addition, the discounted asset price increment reads

$$
\begin{aligned}
\widetilde{S}_t^{(1)} - \widetilde{S}_{t-1}^{(1)} &= \widetilde{S}_{t-1}^{(1)} \frac{1+R_t}{1+r} - \widetilde{S}_{t-1}^{(1)} \\
&= \frac{R_t - r}{1+r} \widetilde{S}_{t-1}^{(1)} \\
&= \frac{b-a}{1+r} Y_t \widetilde{S}_{t-1}^{(1)}, \qquad t = 1,2,\ldots,N.
\end{aligned}
$$

We also have

$$
D_t Y_t = \frac{b-r}{b-a} + \frac{r-a}{b-a} = 1, \qquad t = 1,2,\ldots,N,
$$

and

$$
\begin{aligned}
D_t S_N^{(1)} &= S_0^{(1)} (1+b) \prod_{\substack{k=1 \\ k \neq t}}^{N} (1+R_k) - S_0^{(1)} (1+a) \prod_{\substack{k=1 \\ k \neq t}}^{N} (1+R_k) \\
&= (b-a) S_0^{(1)} \prod_{\substack{k=1 \\ k \neq t}}^{N} (1+R_k) \\
&= S_0^{(1)} \frac{b-a}{1+R_t} \prod_{k=1}^{N} (1+R_k) \\
&= \frac{b-a}{1+R_t} S_N^{(1)}, \qquad t = 1,2,\ldots,N.
\end{aligned}
$$

The following stochastic integral decomposition formula for the functionals of the binomial process is known as the Clark-Ocone formula in discrete time, cf. *e.g.* Privault, 2009, Proposition 1.7.1.

Proposition 4.7 For any square-integrable random variables F on Ω we have

$$
F = \mathbb{E}^*[F] + \sum_{k=1}^{\infty} Y_k \, \mathbb{E}^*[D_k F \mid \mathscr{F}_{k-1}]. \tag{4.27}
$$

The Clark-Ocone formula has the following consequence.

Corollary 4.8 Assume that $(M_k)_{k\in\mathbb{N}}$ is a square-integrable $(\mathscr{F}_k)_{k\in\mathbb{N}}$-martingale. Then we have

$$
M_N = \mathbb{E}^*[M_N] + \sum_{k=1}^{N} Y_k D_k M_k, \qquad N \geqslant 0.
$$

Proof. We have

$$
\begin{aligned}
M_N &= \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k \, \mathbb{E}^*[D_k M_N \mid \mathscr{F}_{k-1}] \\
&= \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k D_k \, \mathbb{E}^*[M_N \mid \mathscr{F}_k]
\end{aligned}
$$

$$= \mathbb{E}^*[M_N] + \sum_{k=1}^{\infty} Y_k D_k M_k$$
$$= \mathbb{E}^*[M_N] + \sum_{k=1}^{N} Y_k D_k M_k.$$

□

In addition to the Clark-Ocone formula we also state a discrete-time analog of Itô's change of variable formula, which can be useful for option hedging. The next result extends Proposition 1.13.1 of Privault, 2009 by removing the unnecessary martingale requirement on $(M_t)_{n\in\mathbb{N}}$.

Proposition 4.9 Let $(Z_n)_{n\in\mathbb{N}}$ be an $(\mathscr{F}_n)_{n\in\mathbb{N}}$-adapted process and let $f : \mathbb{R}\times\mathbb{N} \longrightarrow \mathbb{R}$ be a given function. We have

$$\begin{aligned} f(Z_t,t) &= f(Z_0,0) + \sum_{k=1}^{t} D_k f(Z_k,k) Y_k \\ &\quad + \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] - f(Z_{k-1},k-1) \big). \end{aligned} \tag{4.28}$$

Proof. First, we note that the process

$$t \longmapsto f(Z_t,t) - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] - f(Z_{k-1},k-1) \big)$$

is a martingale under $\mathbb{P}^*$. Indeed, we have

$$\begin{aligned} &\mathbb{E}^*\left[f(Z_t,t) - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] - f(Z_{k-1},k-1) \big) \,\Big|\, \mathscr{F}_{t-1} \right] \\ &= \mathbb{E}^*[f(Z_t,t) \mid \mathscr{F}_{t-1}] \\ &\quad - \sum_{k=1}^{t} \big(\mathbb{E}^*[\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] \mid \mathscr{F}_{t-1}] - \mathbb{E}^*[\mathbb{E}^*[f(Z_{k-1},k-1) \mid \mathscr{F}_{k-1}] \mid \mathscr{F}_{t-1}] \big) \\ &= \mathbb{E}^*[f(Z_t,t) \mid \mathscr{F}_{t-1}] - \sum_{k=1}^{t} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] - f(Z_{k-1},k-1) \big) \\ &= f(Z_{t-1},t-1) - \sum_{k=1}^{t-1} \big(\mathbb{E}^*[f(Z_k,k) \mid \mathscr{F}_{k-1}] - f(Z_{k-1},k-1) \big), \qquad t \geqslant 1. \end{aligned}$$

□

Note that if $(Z_t)_{t\in\mathbb{N}}$ is a discrete-time $(\mathscr{F}_t)_{t\in\mathbb{N}}$-martingale in $L^2(\Omega)$ written as

$$Z_t = Z_0 + \sum_{k=1}^{t} u_k Y_k, \qquad t \in \mathbb{N},$$

where $(u_t)_{t\in\mathbb{N}}$ is an $(\mathscr{F}_t)_{t\in\mathbb{N}}$-predictable process locally in $L^2(\Omega\times\mathbb{N})$, (*i.e.* $u(\cdot)\mathbb{1}_{[0,N]}(\cdot) \in L^2(\Omega\times\mathbb{N})$ for all $N > 0$), then we have

$$D_t f(Z_t,t) = f\big(Z_{t-1} + q u_t, t\big) - f\big(Z_{t-1} - p u_t, t\big), \tag{4.29}$$

$t = 1,2,\ldots,N$. On the other hand, the term

$$\mathbb{E}[f(Z_t,t) - f(Z_{t-1},t-1) \mid \mathscr{F}_{t-1}]$$

is analog to the finite variation part in the continuous-time Itô formula, and can be written as

$$pf\big(Z_{t-1}+qu_t,t\big)+qf\big(Z_{t-1}-pu_t,t\big)-f\big(Z_{t-1},t-1\big).$$

Naturally, if $(f(Z_t,t))_{t\in\mathbb{N}}$ is a martingale we recover the decomposition

$$\begin{aligned} f(Z_t,t) &= f(Z_0,0) \\ &\quad + \sum_{k=1}^{t}\big(f\big(Z_{k-1}+qu_k,k\big)-f\big(Z_{k-1}-pu_k,k\big)\big)Y_k \\ &= f(Z_0,0)+\sum_{k=1}^{t} Y_k D_k f(Z_k,k). \end{aligned} \tag{4.30}$$

This identity follows from Corollary 4.8 as well as from Proposition 4.7. In this case the Clark-Ocone formula (4.27) and the change of variable formula (4.30) both coincide and we have in particular

$$D_k f(Z_k,k)=\mathbb{E}[D_k f(Z_N,N)\mid\mathscr{F}_{k-1}],$$

$k=1,2,\ldots,N$. For example this recovers the martingale representation

$$\begin{aligned} \widetilde{S}_t^{(1)} &= S_0^{(1)}+\sum_{k=1}^{t} Y_k D_k \widetilde{S}_k^{(1)} \\ &= S_0^{(1)}+\frac{b-a}{1+r}\sum_{k=1}^{t}\widetilde{S}_{k-1}^{(1)}Y_k \\ &= S_0^{(1)}+\sum_{k=1}^{t}\widetilde{S}_{k-1}^{(1)}\frac{R_k-r}{1+r} \\ &= S_0^{(1)}+\sum_{k=1}^{t}\big(\widetilde{S}_k^{(1)}-\widetilde{S}_{k-1}^{(1)}\big), \end{aligned}$$

of the discounted asset price.

Our goal is to hedge an arbitrary claim payoff C on Ω, *i.e.* given an $\mathscr{F}_N$-measurable random variable C we search for a portfolio strategy $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ such that the equality

$$C=V_N=\xi_N^{(0)}S_N^{(0)}+\xi_N^{(1)}S_N^{(1)} \tag{4.31}$$

holds, where $S_N^{(0)}=S_0^{(0)}(1+r)^N$ denotes the value of the riskless asset at time $N\in\mathbb{N}$.

The next proposition is the main result of this section, and provides a solution to the hedging problem under the constraint (4.31).

Proposition 4.10 Given a contingent claim with payoff C, let

$$\xi_t^{(1)}=\frac{(1+r)^{-(N-t)}}{(b-a)S_{t-1}^{(1)}}\mathbb{E}^*[D_tC\mid\mathscr{F}_{t-1}], \qquad t=1,2,\ldots,N, \tag{4.32}$$

and

$$\xi_t^{(0)}=\frac{1}{S_t^{(0)}}\big((1+r)^{-(N-t)}\,\mathbb{E}^*[C\mid\mathscr{F}_t]-\xi_t^{(1)}S_t^{(1)}\big), \tag{4.33}$$

$t = 1,2,\ldots,N$. Then the portfolio strategy $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is self financing and satisfies

$$V_t = \xi_t^{(0)}S_t^{(0)} + \xi_t^{(1)}S_t^{(1)} = (1+r)^{-(N-t)}\,\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 1,2,\ldots,N,$$

in particular we have $V_N = C$, hence $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is a hedging strategy leading to C.

Proof. Let $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ be defined by (4.32), and consider the process $\big(\xi_t^{(0)}\big)_{t=0,1,\ldots,N}$ defined by

$$\xi_0^{(0)} = (1+r)^{-N}\frac{\mathbb{E}^*[C]}{S_0^{(1)}} \quad \text{and} \quad \xi_{t+1}^{(0)} = \xi_t^{(0)} - \frac{\big(\xi_{t+1}^{(1)} - \xi_t^{(1)}\big)S_t^{(1)}}{S_t^{(0)}},$$

$t = 0,1,\ldots,N-1$. Then $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ satisfies the self-financing condition

$$S_t^{(0)}\big(\xi_{t+1}^{(0)} - \xi_t^{(0)}\big) + S_t^{(1)}\big(\xi_{t+1}^{(1)} - \xi_t^{(1)}\big) = 0, \qquad t = 1,2,\ldots,N-1.$$

Let now

$$V_0 := \frac{1}{(1+r)^N}\mathbb{E}^*[C], \qquad V_t := \xi_t^{(0)}S_t^{(0)} + \xi_t^{(1)}S_t^{(1)}, \quad t = 1,2,\ldots,N,$$

and

$$\widetilde{V}_t = \frac{V_t}{(1+r)^t} \qquad t = 0,1,\ldots,N.$$

Since $\big(\xi_t^{(0)},\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ is self-financing, by Lemma 4.1 we have

$$\widetilde{V}_t = \widetilde{V}_0 + (b-a)\sum_{k=1}^{t}\frac{1}{(1+r)^k}Y_k\xi_k^{(1)}S_{k-1}^{(1)}, \tag{4.34}$$

$t = 1,2,\ldots,N$. On the other hand, from the Clark-Ocone formula (4.27) and the definition of $\big(\xi_t^{(1)}\big)_{t=1,2,\ldots,N}$ we have

$$\begin{aligned}
&\frac{1}{(1+r)^N}\mathbb{E}^*[C \mid \mathcal{F}_t] \\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*\left[\mathbb{E}^*[C] + \sum_{k=0}^{N}Y_k\,\mathbb{E}^*[D_kC \mid \mathcal{F}_{k-1}] \,\Big|\, \mathcal{F}_t\right] \\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*[C] + \frac{1}{(1+r)^N}\sum_{k=0}^{t}\mathbb{E}^*[D_kC \mid \mathcal{F}_{k-1}]Y_k \\
&\quad = \frac{1}{(1+r)^N}\mathbb{E}^*[C] + (b-a)\sum_{k=0}^{t}\frac{1}{(1+r)^k}\xi_k^{(1)}S_{k-1}^{(1)}Y_k \\
&\quad = \widetilde{V}_t
\end{aligned}$$

from (4.34). Hence

$$\widetilde{V}_t = \frac{1}{(1+r)^N}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0,1,\ldots,N,$$

and

$$V_t = (1+r)^{-(N-t)}\,\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad t = 0,1,\ldots,N. \tag{4.35}$$

In particular, (4.35) shows that we have $V_N = C$. To conclude the proof we note that from the relation $V_t = \xi_t^{(0)}S_t^{(0)} + \xi_t^{(1)}S_t^{(1)}$, $t = 1,2,\ldots,N$, the process $\big(\xi_t^{(0)}\big)_{t=1,2,\ldots,N}$ coincides with $\big(\xi_t^{(0)}\big)_{t=1,2,\ldots,N}$ defined by (4.33). $\square$

From Proposition 4.4, the price $\pi_t(C)$ of the contingent claim with payoff $C = f\big(S_N^{(1)}\big)$ is given by

$$\pi_t(C) = v\big(t, S_t^{(1)}\big),$$

where the function $v(t,x)$ is given by

$$\begin{aligned} v\big(t, S_t^{(1)}\big) &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid \mathcal{F}_t] \\ &= \frac{1}{(1+r)^{N-t}} \mathbb{E}^* \left[f\left(x \prod_{j=t+1}^{N} (1+R_j) \right) \right]_{x=S_t^{(1)}}. \end{aligned}$$

Note that in this case we have $C = v\big(N, S_N^{(1)}\big)$, $\mathbb{E}[C] = v(0, M_0)$, and the discounted claim payoff $\widetilde{C} = C/(1+r)^N = \widetilde{v}\big(N, S_N^{(1)}\big)$ satisfies

$$\begin{aligned} \widetilde{C} &= \mathbb{E}\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t \, \mathbb{E}\big[D_t \widetilde{v}\big(N, S_N^{(1)}\big) \mid \mathcal{F}_{t-1}\big] \\ &= \mathbb{E}\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t D_t \widetilde{v}\big(t, S_t^{(1)}\big) \\ &= \mathbb{E}\big[\widetilde{C}\big] + \sum_{t=1}^{N} \frac{1}{(1+r)^t} Y_t D_t v\big(t, S_t^{(1)}\big) \\ &= \mathbb{E}\big[\widetilde{C}\big] + \sum_{t=1}^{N} Y_t D_t \, \mathbb{E}\big[\widetilde{v}\big(N, S_N^{(1)}\big) \mid \mathcal{F}_t\big] \\ &= \mathbb{E}\big[\widetilde{C}\big] + \frac{1}{(1+r)^N} \sum_{t=1}^{N} Y_t D_t \, \mathbb{E}[C \mid \mathcal{F}_t], \end{aligned}$$

hence we have

$$\mathbb{E}\big[D_t v\big(N, S_N^{(1)}\big) \mid \mathcal{F}_{t-1}\big] = (1+r)^{N-t} D_t v\big(t, S_t^{(1)}\big), \quad t = 1, 2, \ldots, N,$$

and by Proposition 4.10 the hedging strategy for $C = f\big(S_N^{(1)}\big)$ is given by

$$\begin{aligned} \xi_t^{(1)} &= \frac{(1+r)^{-(N-t)}}{(b-a) S_{t-1}^{(1)}} \mathbb{E}\big[D_t v\big(N, S_N^{(1)}\big) \mid \mathcal{F}_{t-1}\big] \\ &= \frac{1}{(b-a) S_{t-1}^{(1)}} D_t v\big(t, S_t^{(1)}\big) \\ &= \frac{1}{(b-a) S_{t-1}^{(1)}} \Big(v\big(t, S_{t-1}^{(1)}(1+b)\big) - v\big(t, S_{t-1}^{(1)}(1+a)\big) \Big) \\ &= \frac{1}{(b-a) \widetilde{S}_{t-1}^{(1)}/(1+r)} \Big(\widetilde{v}\big(t, S_{t-1}^{(1)}(1+b)\big) - \widetilde{v}\big(t, S_{t-1}^{(1)}(1+a)\big) \Big), \end{aligned}$$

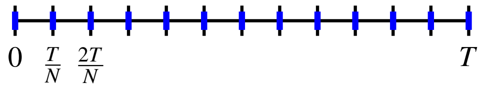
$t = 1, 2, \ldots, N$, which recovers Proposition 4.5 as a particular case. Note that $\xi_t^{(1)}$ is nonnegative (*i.e.* there is no short selling) when f is a non-decreasing function, because $a < b$. This is in particular true in the case of the European call option, for which we have $f(x) = (x-K)^+$.

4.6 Convergence of the CRR Model

As the pricing formulas (4.11) in the CRR model can be difficult to implement for large values on N, in this section we consider the convergence of the discrete-time model to the continuous-time Black Scholes model.

Continuous compounding - riskless asset

Consider the discretization
$$\left[0, \frac{T}{N}, \frac{2T}{N}, \ldots, \frac{(N-1)T}{N}, T\right]$$
of the time interval $[0, T]$ into N time steps.



Note that
$$\lim_{N\to\infty}(1+r)^N = \infty,$$
when $r > 0$, thus we need to renormalize r so that the interest rate on each time interval becomes r_N, with $\lim_{N\to\infty} r_N = 0$. It turns out that the correct renormalization is

$$r_N := r\frac{T}{N}, \tag{4.36}$$

so that for $T \geqslant 0$,

$$\begin{aligned}\lim_{N\to\infty}(1+r_N)^N &= \lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^N \\ &= \lim_{N\to\infty}\exp\left(N\log\left(1+r\frac{T}{N}\right)\right) \\ &= \mathrm{e}^{rT}. \end{aligned} \tag{4.37}$$

Hence the price $S_t^{(0)}$ of the riskless asset is given by

$$S_t^{(0)} = S_0^{(0)}\,\mathrm{e}^{rt}, \qquad t \in \mathbb{R}_+, \tag{4.38}$$

which solves the differential equation

$$\frac{dS_t^{(0)}}{dt} = rS_t^{(0)}, \qquad S_0^{(0)} = 1, \quad t \in \mathbb{R}_+. \tag{4.39}$$

We can also write

$$dS_t^{(0)} = rS_t^{(0)}dt, \qquad \text{or} \qquad \frac{dS_t^{(0)}}{S_t^{(0)}} = rdt, \tag{4.40}$$

and using $dS_t^{(0)} \simeq S_{t+dt}^{(0)} - S_t^{(0)}$ we can discretize this equation by saying that the *infinitesimal return* $(S_{t+dt}^{(0)} - S_t^{(0)})/S_t^{(0)}$ of the riskless asset equals rdt on the small time interval $[t, t+dt]$, *i.e.*

$$\frac{S_{t+dt}^{(0)} - S_t^{(0)}}{S_t^{(0)}} = rdt.$$

In this sense, the rate r is the instantaneous interest rate per unit of time.

The same equation rewrites in *integral form* as

$$S_T^{(0)} - S_0^{(0)} = \int_0^T dS_t^{(0)} = r\int_0^T S_t^{(0)}dt.$$

Continuous compounding - risky asset

The Galton board simulation of Figure 4.7 shows the convergence of the binomial random walk to a Gaussian distribution in large time.

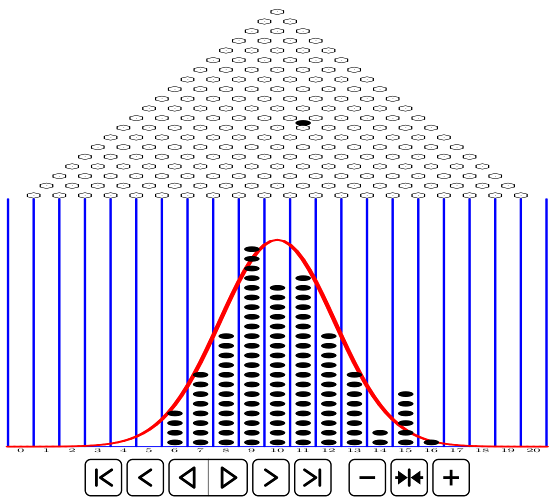


Figure 4.7: Galton board simulation.*

Figure 4.8 pictures a real-life Galton board.

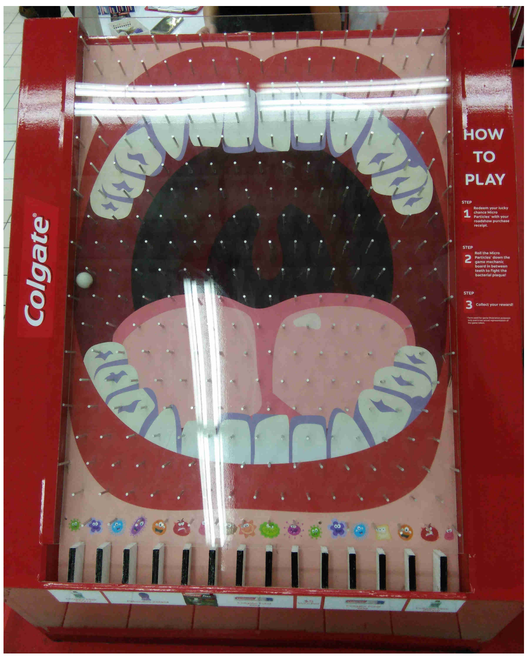


Figure 4.8: A real-life Galton board at Jurong Point # 03-01.

In the CRR model we need to replace the standard Galton board by its multiplicative version, which shows that as N tends to infinity the distribution of $S_N^{(1)}$ converges to the *lognormal distribution*

*The animation works in Acrobat Reader on the entire pdf file.

with probability density function of the form

$$x \longmapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left(-\frac{\left(-\left(r-\sigma^2/2\right)T + \log\left(x/S_0^{(1)}\right)\right)^2}{2\sigma^2 T} \right),$$

$x>0$, with location parameter $(r-\sigma^2/2)T+\log S_0^{(1)}$ and scale parameter $\sigma\sqrt{T}$, or log-variance $\sigma^2 T$, as illustrated in the modified Galton board of Figure 4.9 below, see also Figure 6.7.

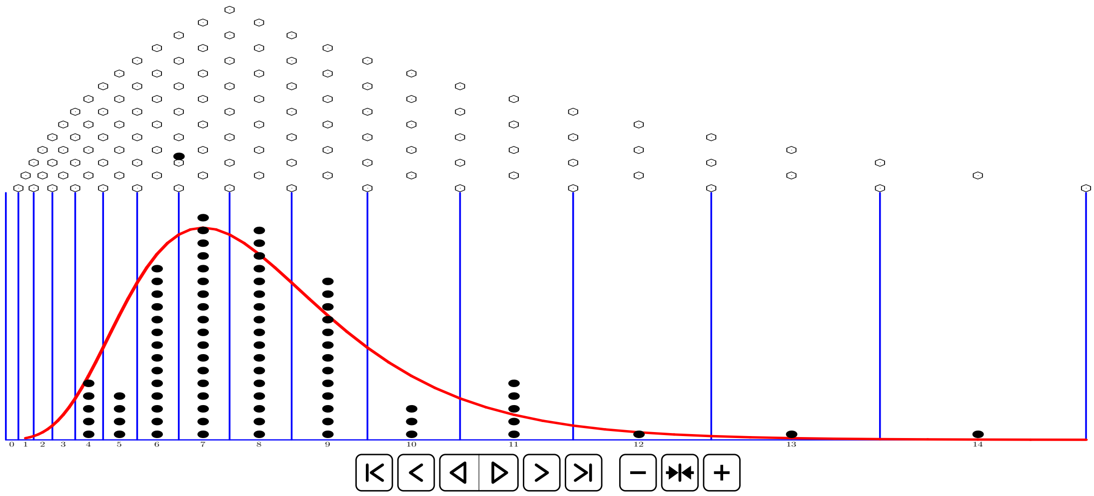


Figure 4.9: Multiplicative Galton board simulation.*

In addition to the renormalization (4.36) for the interest rate $r_N := rT/N$, we need to apply a similar renormalization to the coefficients a and b of the CRR model. Let $\sigma > 0$ denote a positive parameter called the volatility, which quantifies the range of random fluctations, and let a_N, b_N be defined from

$$\frac{1+a_N}{1+r_N} = 1-\sigma\sqrt{\frac{T}{N}} \quad \text{and} \quad \frac{1+b_N}{1+r_N} = 1+\sigma\sqrt{\frac{T}{N}}$$

i.e.

$$a_N = (1+r_N)\left(1-\sigma\sqrt{\frac{T}{N}}\right)-1 \quad \text{and} \quad b_N = (1+r_N)\left(1+\sigma\sqrt{\frac{T}{N}}\right)-1.$$

Consider the random return $R_k^{(N)} \in \{a_N, b_N\}$ and the price process defined as

$$S_{t,N}^{(1)} = S_0^{(1)} \prod_{k=1}^{t} \left(1+R_k^{(N)}\right), \qquad t = 1,2,\ldots,N.$$

Note that the risk-neutral probabilities are given by

$$\begin{aligned} \mathbb{P}^*\left(R_t^{(N)} = a_N\right) &= \frac{b_N - r_N}{b_N - a_N} \\ &= \frac{(1+r_N)\left(1+\sigma\sqrt{T/N}\right)-1-r_N}{(1+r_N)\left(1+\sigma\sqrt{T/N}\right)-(1+r_N)\left(1-\sigma\sqrt{T/N}\right)} \\ &= \frac{1}{2}, \quad t = 1,2,\ldots,N, \end{aligned}$$

*The animation works in Acrobat Reader on the entire pdf file.

and

$$\begin{aligned}
\mathbb{P}^*\big(R_t^{(N)} = b_N\big) &= \frac{r_N - a_N}{b_N - a_N} \\
&= \frac{r_N - (1+r_N)\big(1-\sigma\sqrt{T/N}\big) + 1}{(1+r_N)\big(1+\sigma\sqrt{T/N}\big) - (1+r_N)\big(1-\sigma\sqrt{T/N}\big)} \\
&= \frac{1}{2}, \qquad t = 1,2,\ldots,N.
\end{aligned}$$

Continuous-time limit in distribution

We have the following convergence result.

Proposition 4.11 Let f be a continuous and bounded function on $\mathbb{R}$. The price at time $t=0$ of a contingent claim with payoff $C = h\big(S_{N,N}^{(1)}\big)$ converges as follows:

$$\lim_{N\to\infty} \frac{1}{(1+rT/N)^N} \mathbb{E}^*\left[f\big(S_{N,N}^{(1)}\big)\right] = \mathrm{e}^{-rT}\,\mathbb{E}\left[h\big(S_0^{(1)}\,\mathrm{e}^{\sigma X + rT - \sigma^2 T/2}\big)\right] \tag{4.41}$$

where $X \simeq \mathcal{N}(0,T)$ is a centered Gaussian random variable with variance $T>0$.

Proof. This result is consequence of the weak convergence in distribution of the sequence $\big(S_{N,N}^{(1)}\big)_{N\geqslant 1}$ to a lognormal distribution, see *e.g.* Theorem 5.53 page 261 of Föllmer and Schied, 2004. Informally, using the Taylor expansion of the log function, we have

$$\begin{aligned}
\log S_N^{(1)} &= \log S_0^{(1)} + \sum_{k=1}^N \log\big(1+R_k^{(N)}\big) \\
&= \log S_0^{(1)} + \sum_{k=1}^N \log(1+r_N) + \sum_{k=1}^N \log\frac{1+R_k^{(N)}}{1+r_N} \\
&= \log S_0^{(1)} + \sum_{k=1}^N \log\left(1+\frac{rT}{N}\right) + \sum_{k=1}^N \log\left(1\pm\sigma\sqrt{\frac{T}{N}}\right) \\
&= \log S_0^{(1)} + \sum_{k=1}^N \frac{rT}{N} + \sum_{k=1}^N \left(\pm\sigma\sqrt{\frac{T}{N}} - \frac{\sigma^2 T}{2N} + o\left(\frac{T}{N}\right)\right) \\
&= \log S_0^{(1)} + rT - \frac{\sigma^2 T}{2} + \frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T} + o(1).
\end{aligned}$$

Next, we note that by the Central Limit Theorem (CLT), the normalized sum

$$\frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T}$$

of independent Bernoulli random variables, with variance

$$\begin{aligned}
\mathrm{Var}\left[\frac{1}{\sqrt{N}}\sum_{k=1}^N \pm\sqrt{\sigma^2 T}\right] &= \frac{\sigma^2 T}{N}\sum_{k=1}^N \left(1 - \big(\mathbb{P}^*\big(R_t^{(N)} = b_N\big) - \mathbb{P}^*\big(R_t^{(N)} = a_N\big)^2\big)\right) \\
&\simeq \sigma^2 T
\end{aligned}$$

converges in distribution to a centered $\mathcal{N}(0,\sigma^2 T)$ Gaussian random variable with variance $\sigma^2 T$. Finally, the convergence of the discount factor $(1+rT/N)^N$ to e^{-rT} follows from (4.37). □

Note that the expectation (4.41) can be written as the Gaussian integral

$$\mathrm{e}^{-rT}\,\mathbb{E}\left[f\left(S_0^{(1)}\,\mathrm{e}^{\sigma X+rT-\sigma^2T/2}\right)\right]=\mathrm{e}^{-rT}\int_{-\infty}^{\infty}f\left(S_0^{(1)}\,\mathrm{e}^{\sigma\sqrt{T}x+rT-\sigma^2T/2}\right)\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}dx,$$

see also Lemma 8.6 in Chapter 8, hence we have

$$\lim_{N\to\infty}\frac{1}{(1+rT/N)^N}\mathbb{E}^*\left[f\left(S_{N,N}^{(1)}\right)\right]=\mathrm{e}^{-rT}\int_{-\infty}^{\infty}f\left(S_0^{(1)}\,\mathrm{e}^{\sigma x\sqrt{T}+rT-\sigma^2T/2}\right)\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}dx.$$

It is a remarkable fact that in case $f(x)=(x-K)^+$, *i.e.* when

$$C=\left(S_T^{(1)}-K\right)^+$$

is the payoff of the European call option with strike price K, the above integral can be computed according to the *Black-Scholes formula*, as

$$\mathrm{e}^{-rT}\,\mathbb{E}\left[\left(S_0^{(1)}\,\mathrm{e}^{\sigma X+rT-\sigma^2T/2}-K\right)^+\right]=S_0^{(1)}\Phi(d_+)-K\,\mathrm{e}^{-rT}\Phi(d_-),$$

where

$$d_-=\frac{(r-\sigma^2/2)T+\log\left(S_0^{(1)}/K\right)}{\sigma\sqrt{T}},\qquad d_+=d_-+\sigma\sqrt{T},$$

and

$$\Phi(x):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-y^2/2}dy,\qquad x\in\mathbb{R},$$

is the Gaussian cumulative distribution function.

The Black-Scholes formula will be derived explicitly in the subsequent chapters using both PDE and probabilistic methods, cf. Propositions 7.11 and 8.5. It can be regarded as a building block for the pricing of financial derivatives, and its importance is not restricted to the pricing of options on stocks. Indeed, the complexity of the interest rate models makes it in general difficult to obtain closed-form expressions, and in many situations one has to rely on the Black-Scholes framework in order to find pricing formulas, for example in the case of interest rate derivatives as in the Black caplet formula of the BGM model.

Our aim later on will be to price and hedge options directly in continuous-time using stochastic calculus, instead of applying the limit procedure described in the previous section. In addition to the construction of the riskless asset price $(A_t)_{t\in\mathbb{R}_+}$ via (4.38) and (4.39) we now need to construct a mathematical model for the price of the risly asset in continuous time.

The return of the risky asset $S_t^{(1)}$ over the time interval $[t,d+dt]$ will be modeled as

$$\frac{dS_t^{(1)}}{S_t^{(1)}}=\mu dt+\sigma dB_t,$$

where in comparison with (4.40), we add a "small" Gaussian random fluctuation σdB_t which accounts for market volatility. Here, the Brownian increment dB_t is multiplied by the volatility parameter $\sigma>0$. In the next Chapter 5 we will turn to the formal definition of the stochastic process $(B_t)_{t\in\mathbb{R}_+}$ which will be used for the modeling of risky assets in continuous time.

Exercises

Exercise 4.1 (Exercise 3.3 continued). Consider a two-step trinomial market model $\big(S_t^{(1)}\big)_{t=0,1,2}$ with $r = 0$ and three return rates $R_t = -1, 0, 1$. Taking $S_0^{(1)} = 1$, price the European put option with strike price $K = 1$ and maturity $N = 2$ at times $t = 0$ and $t = 1$.

Exercise 4.2 Consider a two-step binomial market model $(S_t)_{t=0,1,2}$ with two return rates $a = 0$, $b = 1$ and $S_0 = 1$, and with the riskless account $A_t = (1+r)^t$ where $r = 0.5$. Price and hedge the *tunnel option* whose payoff C at time $t = 2$ is given by

$$C = \begin{cases} 3 & \text{if } S_2 = 4, \\ \\ 1 & \text{if } S_2 = 2, \\ \\ 3 & \text{if } S_2 = 1. \end{cases}$$

Exercise 4.3 In a two-step trinomial market model $(S_t)_{t=0,1,2}$ with interest rate $r = 0$ and three return rates $R_t = -0.5, 0, 1$, we consider a down-an-out barrier call option with exercise date $N = 2$, strike price K and barrier level B, whose payoff C is given by

$$C = (S_N - K)^+ \mathbb{1}_{\Big\{ \min\limits_{t=1,2,\ldots,N} S_t > B \Big\}} = \begin{cases} (S_N - K)^+ & \text{if } \min\limits_{t=1,2,\ldots,N} S_t > B, \\ \\ 0 & \text{if } \min\limits_{t=1,2,\ldots,N} S_t \leqslant B. \end{cases}$$

a) Show that $\mathbb{P}^*$ given by $r^* = \mathbb{P}^*(R_t = -0.5) := 1/2$, $q^* = \mathbb{P}^*(R_t = 0) := 1/4$, $p^* = \mathbb{P}^*(R_t = 1) := 1/4$ is a risk-neutral probability measure.
b) Taking $S_0 = 1$, compute the possible values of the down-an-out barrier call option payoff C with strike price $K = 1.5$ and barrier level $B = 1$, at maturity $N = 2$.
c) Price the down-an-out barrier call option with exercise date $N = 2$, strike price $K = 1.5$ and barrier level $B = 1$, at time $t = 0$ and $t = 1$.

   Hint: Use the formula

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*[C \mid S_t], \qquad t = 0, 1, \ldots, N,$$

   where N denotes maturity time and C is the option payoff.
d) Is this market complete? Is every contingent claim attainable?

Exercise 4.4 Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a = 0$ and $b = 200\%$, and a riskless asset $A_k = A_0(1+r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$, and $S_0 = A_0 = 1$, under the risk-neutral probabilities $p^* = (r-a)/(b-a) = 1/2$ and $q^* = (b-r)/(b-a) = 1/2$.

a) Draw a binomial tree for the possible values of $(S_k)_{k=0,1,2}$ and compute the values V_k of the hedging portfolio at times $k = 0, 1, 2$ of the European *call* option on S_N with strike price $K = 8$ and maturity $N = 2$.

   Hint: Consider three cases when $k = 2$, and two cases when $k = 1$.

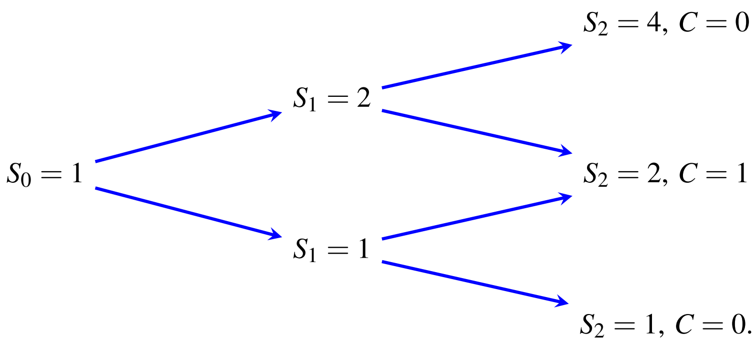
b) Compute the self-financing hedging portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ with values

$$V_0 = \xi_1 S_0 + \eta_1 A_0, \; V_1 = \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \text{ and } V_2 = \xi_2 S_2 + \eta_2 A_2,$$

hedging the European *call* option with strike price $K = 8$ and maturity $N = 2$.

Hint: Consider two separate cases for $k = 2$ and one case for $k = 1$.

Exercise 4.5 We consider a two-step binomial market model $(S_t)_{t=0,1,2}$ with two return rates $a = 0$, $b = 1$, and $S_0 = 1$.



The riskless account is $A_t = \$1$ and the risk-free interest rate is $r = 0$. We consider the *tunnel option* whose payoff C at time $t = 2$ is given by

$$C = \begin{cases} 0 & \text{if } S_2 = 4, \\ \$1 & \text{if } S_2 = 2, \\ 0 & \text{if } S_2 = 1. \end{cases}$$

a) Build a hedging porfolio for the claim C at time $t = 1$ depending on the value of S_1.
b) Price the claim C at time $t = 1$ depending on the value of S_1.
c) Build a hedging porfolio for the claim C at time $t = 0$.
d) Price the claim C at time $t = 0$.
e) Does this model admit an equivalent risk-neutral measure in the sense of Definitions 3.8 and 3.9?
f) Is the model without arbitrage according to Theorem 3.6?

Exercise 4.6 Let $\mathbb{P}^*$ be a risk-neutral probability measure relative to a discrete-time asset price process $(S_n)_{n \in \mathbb{N}}$. Compute the arbitrage-free price $\pi_k(C)$ at time $k = 0, 1, \ldots, N$ of the claim C with maturity time N and affine payoff function

$$C = h(S_N) = \alpha + \beta S_N$$

where $\alpha, \beta \in \mathbb{R}$ are constants, in a discrete-time market with risk free rate r.

Exercise 4.7 Consider a two-step binomial random asset model $(S_k)_{k=0,1,2}$ with possible returns $a = -50\%$ and $b = 150\%$, and a riskless asset $A_k = A_0(1 + r)^k$, $k = 0, 1, 2$ with interest rate $r = 100\%$, and $S_0 = A_0 = 1$, under the risk-neutral probabilities $p^* = (r - a)/(b - a) = 3/4$ and $q^* = (b - r)/(b - a) = 1/4$.

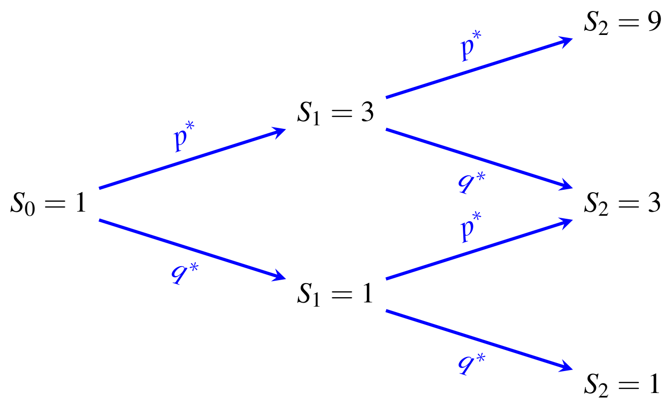
a) Draw a binomial tree for the values of $(S_k)_{k=0,1,2}$.
b) Compute the values V_k at times $k = 0, 1, 2$ of the hedging portfolio of the European *put* option with strike price $K = 5/4$ and maturity $N = 2$ on S_N.

c) Compute the self-financing hedging portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ with values

$$V_0 = \xi_1 S_0 + \eta_1 A_0, \ V_1 = \xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \ \text{ and } \ V_2 = \xi_2 S_2 + \eta_2 A_2,$$

hedging the European *put* option with strike price $K = 5/4$ and maturity $N = 2$.

Exercise 4.8 Consider a two-step binomial model for a stock paying a dividend at the *rate* $\alpha \in (0, 1)$ at times $k = 1$ and $k = 2$, and the following recombining tree represents the *ex-dividend*[*] prices S_k at times $k = 1, 2$, starting from $S_0 = \$1$.




```
install.packages("quantmod")
library(quantmod)
getDividends("Z74.SI",from="2018-01-01",to="2018-12-31",src="yahoo")
getSymbols("Z74.SI",from="2018-11-16",to="2018-12-19",src="yahoo")
T <- chart_theme(); T$col$line.col <- "black"
chart_Series(Op(`Z74.SI`),name="Opening prices (black) - Closing prices (blue)",lty=4,theme=T)
add_TA(Cl(`Z74.SI`),lwd=2,lty=5,legend='Difference',col="blue",on = 1)
```

	Z74.SI.div
2018-07-26	0.107
2018-12-17	0.068
2018-12-18	0.068

[*] "Ex-dividend" means after dividend payment.

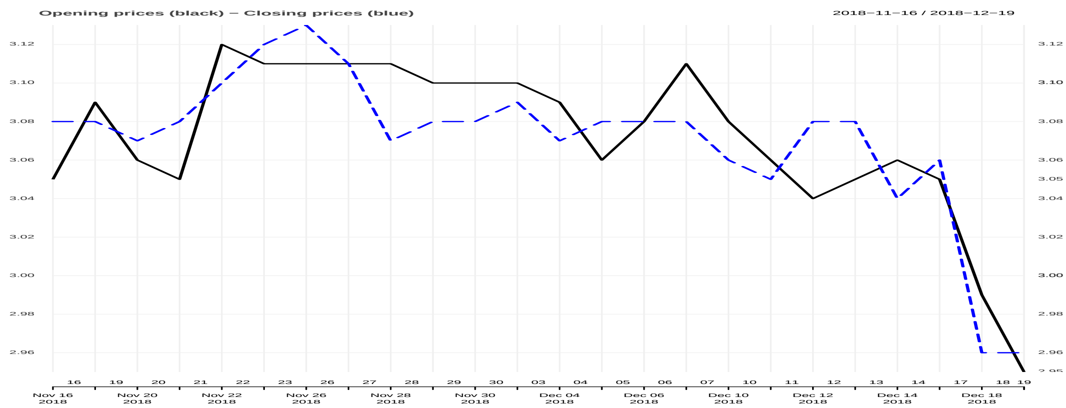


Figure 4.10: SGD0.068 dividend detached on 18 Dec 2018 on Z74.SI.

The difference between the closing price on Dec 17 (\$3.06) and the opening price on Dec 18 (\$2.99) is \$3.06 – \$2.99 = \$0.07. The adjusted price on Dec 17 (\$2.992) is the closing price (\$3.06) minus the dividend (\$0.068).

Z74.SI	Open	High	Low	Close	Volume	Adjusted (ex-dividend)
2018-12-17	3.05	3.08	3.05	**3.06**	17441000	**2.992**
2018-12-18	**2.99**	2.99	2.96	2.96	28456400	2.960

The dividend rate α is given by $\alpha = 0.068/3.06 = 2.22\%$.

We consider a riskless asset $A_k = A_0(1+r)^k$, $k = 0,1,2$ with interest rate $r = 100\%$ and $A_0 = 1$, and two portfolio allocations (ξ_1, η_1) at time $k = 0$ and (ξ_2, η_2) at time $k = 1$, with the values

$$V_1 = \xi_2 S_1 + \eta_2 A_1 \tag{4.42}$$

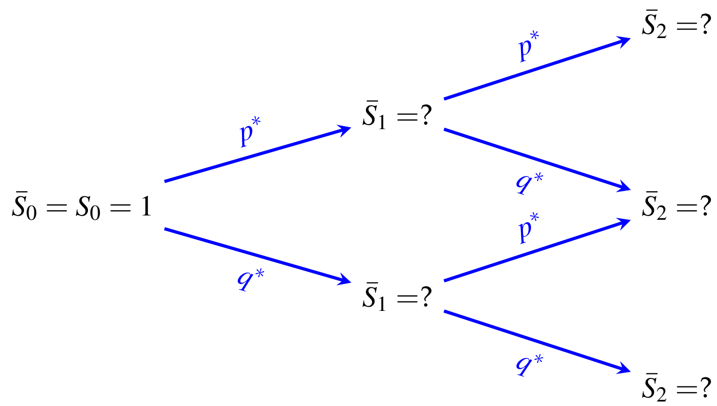
and

$$V_0 = \xi_1 S_0 + \eta_1 A_0. \tag{4.43}$$

We make the following three assumptions:

[A] All dividends are reinvested.
[B] The portfolio strategies are self-financing.
[C] The portfolio value V_2 at time $k = 2$ hedges the European *call* option with payoff $C = (S_T - K)^+$, strike price $K = 8$, and maturity $T = 2$.

a) Using (4.42) and **[A]**, express V_2 in terms of ξ_2, η_2, S_2, A_2 and α.
b) Using (4.43) and **[A]**-**[B]**, express V_1 in terms of ξ_1, η_1, S_1, A_1 and α.
c) Using Assumption **[C]** and the result of Question (a)), compute the portfolio allocation (ξ_2, η_2) in cases $S_1 = 1$ and $S_1 = 3$.
d) Using (4.42) and the portfolio allocation (ξ_2, η_2) obtained in Question (c)), compute the portfolio value V_1 in cases $S_1 = 1$ and $S_1 = 3$.
e) From the results of Questions (b)) and (d)), compute the initial portfolio allocation (ξ_1, η_1).
f) Compute the initial portfolio value V_0 from the result of Question (e)).
g) Knowing that the dividend rate is $\alpha = 25\%$, draw the tree of asset prices $(\bar{S}_k)_{k=1,2}$ *before* the dividend payments.



h) Compute the risk-neutral probabilities p^* and q^* under which the conditional expected return of $(\bar{S}_k)_{k=0,1,2}$ is the risk-free interest rate $r = 100\%$.

i) ✓ Check that the portfolio value V_1 found in Question (d)) satisfies

$$V_1 = \frac{1}{1+r}\mathbb{E}^*\left[(S_2 - K)^+ \mid S_1\right].$$

j) ✓ Check that the portfolio value V_0 found in Question (f)) satisfies

$$V_0 = \frac{1}{(1+r)^2}\mathbb{E}^*\left[(S_2 - K)^+\right] \quad \text{and} \quad V_0 = \frac{1}{1+r}\mathbb{E}^*\left[V_1\right].$$

Exercise 4.9 Analysis of a binary option trading website.

a) In a one-step model with risky asset prices S_0, S_1 at times $t = 0$ and $t = 1$, compute the price at time $t = 0$ of the binary call option with payoff

$$C = \mathbb{1}_{[K,\infty)}(S_1) = \begin{cases} \$1 & \text{if } S_1 \geqslant K, \\ 0 & \text{if } S_1 < K, \end{cases}$$

in terms of the probability $p^* = \mathbb{P}^*(S_1 \geqslant K)$ and of the risk-free interest rate r.

b) Compute the two potential net returns obtained by purchasing one binary call option.

c) Compute the corresponding expected return.

d) A website proposes to pay a return of 86% in case the binary call option matures "in the money", *i.e.* when $S_1 \geqslant K$. Compute the corresponding expected return. What do you conclude?

Exercise 4.10 A *put spread collar* option requires its holder to *sell* an asset at the price $f(S)$ when its market price is at the level S, where $f(S)$ is the function plotted in Figure 4.11, with $K_1 := 80$, $K_2 := 90$, and $K_3 := 110$.

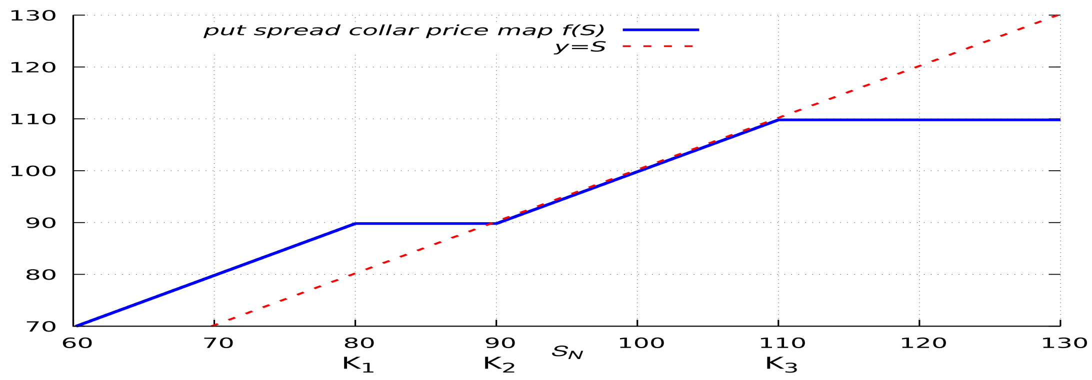


Figure 4.11: Put spread collar price map.

a) Draw the *payoff function* of the put spread collar as a function of the underlying asset price at maturity. See *e.g.* http://optioncreator.com/.
b) Show that this put spread collar option can be realized by purchasing and/or issuing standard European call and put options with strike prices to be specified.

Hints: Recall that an option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\mathbb{E}^*[\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 4.11 A *call spread collar* option requires its holder to *buy* an asset at the price $f(S)$ when its market price is at the level S, where $f(S)$ is the function plotted in Figure 4.11, with $K_1 := 80$, $K_2 := 100$, and $K_3 := 110$.

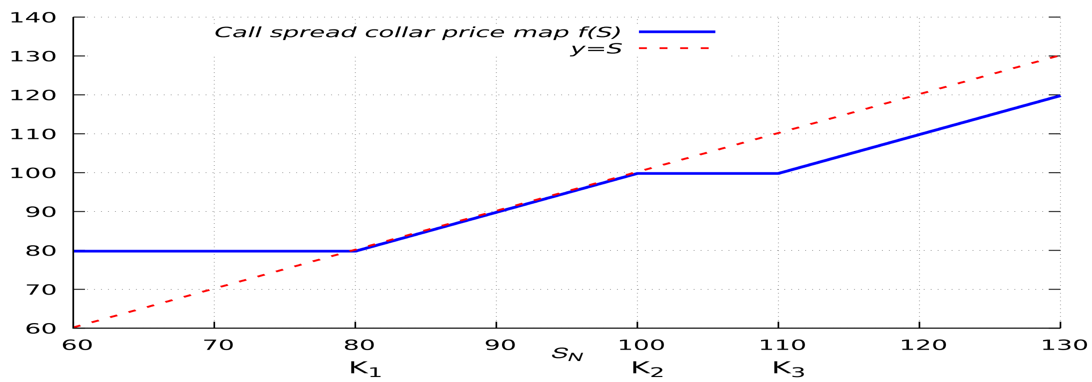


Figure 4.12: Call spread collar price map.

a) Draw the *payoff function* of the call spread collar as a function of the underlying asset price at maturity. See *e.g.* http://optioncreator.com/.
b) Show that this call spread collar option can be realized by purchasing and/or issuing standard European call and put options with strike prices to be specified.

Hints: Recall that an option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\mathbb{E}^*[\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 4.12 Consider an asset price $(S_n)_{n=0,1,\ldots,N}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, with respect to the filtration $(\mathcal{F}_n)_{n=0,1,\ldots,N}$. Given the (convex) function $\phi(x) := (x-K)^+$, show that the price of an Asian option with payoff

$$\phi\left(\frac{S_1+\cdots+S_N}{N}\right)$$

and maturity $N \geqslant 1$ is always lower than the price of the corresponding European *call* option, *i.e.* show that

$$\mathbb{E}^*\left[\phi\left(\frac{S_1+S_2+\cdots+S_N}{N}\right)\right] \leqslant \mathbb{E}^*[\phi(S_N)].$$

Hint: Use in the following order:

(i) the convexity inequality $\phi(x_1/N+\cdots+x_N/N) \leqslant \phi(x_1)/N+\cdots+\phi(x_N)/N$,

(ii) the martingale property $S_k = \mathbb{E}^*[S_N \mid \mathscr{F}_k]$, $k=1,2,\ldots,N$.

(iii) Jensen's inequality

$$\phi(\mathbb{E}^*[S_N \mid \mathscr{F}_k]) \leqslant \mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_k], \qquad k=1,2,\ldots,N,$$

(iv) the tower property $\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_k]] = \mathbb{E}^*[\phi(S_N)]$ of conditional expectations, $k = 1,2,\ldots,N$.

Exercise 4.13 (Exercise 3.5 continued).

a) We consider a forward contract on S_N with strike price K and payoff

$$C := S_N - K.$$

Find a portfolio allocation (η_N, ξ_N) with value

$$V_N = \eta_N \pi_N + \xi_N S_N$$

at time N, such that

$$V_N = C, \tag{4.44}$$

by writing Condition (4.44) as a 2×2 system of equations.

b) Find a portfolio allocation (η_{N-1}, ξ_{N-1}) with value

$$V_{N-1} = \eta_{N-1}\pi_{N-1} + \xi_{N-1}S_{N-1}$$

at time $N-1$, and verifying the self-financing condition

$$V_{N-1} = \eta_N \pi_{N-1} + \xi_N S_{N-1}.$$

Next, at all times $t = 1,2,\ldots,N-1$, find a portfolio allocation (η_t, ξ_t) with value $V_t = \eta_t \pi_t + \xi_t S_t$ verifying (4.44) and the self-financing condition

$$V_t = \eta_{t+1}\pi_t + \xi_{t+1}S_t,$$

where η_t, *resp.* ξ_t, represents the quantity of the riskless, *resp.* risky, asset in the portfolio over the time period $[t-1,t]$, $t=1,2,\ldots,N-1$.

c) Compute the arbitrage-free price $\pi_t(C) = V_t$ of the forward contract C, at time $t = 0,1,\ldots,N$.

d) Check that the arbitrage-free price $\pi_t(C)$ satisfies the relation

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}}\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad t=0,1,\ldots,N.$$

Exercise 4.14 Power option. Let $(S_n)_{n\in\mathbb{N}}$ denote a binomial price process with returns -50% and $+50\%$, and let the riskless asset be valued $A_k = \$1$, $k \in \mathbb{N}$. We consider a power option with payoff $C := (S_N)^2$, and a *predictable* self-financing portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\ldots,N}$ with value

$$V_k = \xi_k S_k + \eta_k A_0, \qquad k=1,2,\ldots,N.$$

a) Find the portfolio allocation (ξ_N, η_N) that matches the payoff $C = (S_N)^2$ at time N, *i.e.* that satisfies

$$V_N = (S_N)^2.$$

Hint: We have $\eta_N = -3(S_{N-1})^2/4$.

b) Compute the portfolio value under the risk-neutral probability measure $p^* = 1/2$

$$V_{N-1} = \mathbb{E}^*[C \mid \mathscr{F}_{N-1}].$$

c) Find the portfolio allocation (η_{N-1}, ξ_{N-1}) at time $N-1$ from the relation

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0.$$

Hint: We have $\eta_{N-1} = -15(S_{N-2})^2/16$.

d) Check that the portfolio satisfies the self-financing condition

$$V_{N-1} = \xi_{N-1} S_{N-1} + \eta_{N-1} A_0 = \xi_N S_{N-1} + \eta_N A_0.$$

Exercise 4.15 Consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$. The price S_t^0 of the riskless asset evolves as $S_t^0 = \pi^0 (1+r)^t$, $t = 0, 1, \ldots, N$. The *return* of the risky asset, defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N,$$

is random and allowed to take only two values a and b, with $-1 < a < r < b$.

The discounted asset price is given by $\widetilde{S}_t := S_t/(1+r)^t$, $t = 0, 1, \ldots, N$.

a) Show that this model admits a unique risk-neutral probability measure $\mathbb{P}^*$ and explicitly compute $\mathbb{P}^*(R_t = a)$ and $\mathbb{P}(R_t = b)$ for all $t = 1, 2, \ldots, N$, with $a = 2\%$, $b = 7\%$, $r = 5\%$.

b) Does there exist arbitrage opportunities in this model? Explain why.

c) Is this market model complete? Explain why.

d) Consider a contingent claim with payoff*

$$C = (S_N)^2.$$

Compute the discounted arbitrage-free price $\widetilde{V}_t$, $t = 0, 1, \ldots, N$, of a self-financing portfolio hedging the claim payoff C, *i.e.* such that

$$\widetilde{V}_N = \widetilde{C} = \frac{(S_N)^2}{(1+r)^N}.$$

e) Compute the portfolio strategy

$$\left(\overline{\xi}_t\right)_{t=1,2,\ldots,N} = \left(\xi_t^0, \xi_t^1\right)_{t=1,2,\ldots,N}$$

associated to $\widetilde{V}_t$, *i.e.* such that

$$\widetilde{V}_t = \overline{\xi}_t \cdot \overline{X}_t = \xi_t^0 X_t^0 + \xi_t^1 X_t^1, \qquad t = 1, 2, \ldots, N.$$

f) Check that the above portfolio strategy is self-financing, *i.e.*

$$\overline{\xi}_t \cdot \overline{S}_t = \overline{\xi}_{t+1} \cdot \overline{S}_t, \qquad t = 1, 2, \ldots, N-1.$$

*This is the payoff of a power call option with strike price $K = 0$.

Exercise 4.16 We consider the discrete-time Cox-Ross-Rubinstein model with $N+1$ time instants $t = 0, 1, \ldots, N$.

The price π_t of the riskless asset evolves as $\pi_t = \pi_0(1+r)^t$, $t = 0, 1, \ldots, N$. The evolution of S_{t-1} to S_t is given by

$$S_t = \begin{cases} (1+b)S_{t-1} & \text{if } R_t = b, \\ \\ (1+a)S_{t-1} & \text{if } R_t = a, \end{cases}$$

with $-1 < a < r < b$. The *return* of the risky asset is defined as

$$R_t := \frac{S_t - S_{t-1}}{S_{t-1}}, \qquad t = 1, 2, \ldots, N.$$

Let ξ_t, *resp.* η_t, denote the (possibly fractional) quantities of the risky, *resp.* riskless, asset held over the time period $[t-1, t]$ in the portfolio with value

$$V_t = \xi_t S_t + \eta_t \pi_t, \qquad t = 0, 1, \ldots, N. \tag{4.45}$$

a) Show that

$$V_t = (1+R_t)\xi_t S_{t-1} + (1+r)\eta_t \pi_{t-1}, \qquad t = 1, 2, \ldots, N. \tag{4.46}$$

b) Show that under the probability $\mathbb{P}^*$ defined by

$$\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) = \frac{b-r}{b-a}, \qquad \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) = \frac{r-a}{b-a},$$

where $\mathcal{F}_{t-1}$ represents the information generated by $\{R_1, R_2, \ldots, R_{t-1}\}$, we have

$$\mathbb{E}^*[R_t \mid \mathcal{F}_{t-1}] = r.$$

c) Under the self-financing condition

$$V_{t-1} = \xi_t S_{t-1} + \eta_t \pi_{t-1}, \qquad t = 1, 2, \ldots, N, \tag{4.47}$$

recover the martingale property

$$V_{t-1} = \frac{1}{1+r}\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}],$$

using the result of Question (a)).

d) Let $a = 5\%$, $b = 25\%$ and $r = 15\%$. Assume that the value V_t at time t of the portfolio is \$3 if $R_t = a$ and \$8 if $R_t = b$, and compute the value V_{t-1} of the portfolio at time $t-1$.

5. Brownian Motion and Stochastic Calculus

Brownian motion is a continuous-time stochastic process having stationary and independent Gaussian distributed increments, and continuous paths. This chapter presents the constructions of Brownian motion and its associated Itô stochastic integral, which will be used for the random modeling of asset and portfolio prices in continuous time.

5.1 Brownian Motion

We start by recalling the definition of Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ of Brownian motion can be constructed on the space $\Omega = \mathscr{C}_0(\mathbb{R}_+)$ of continuous real-valued functions on $\mathbb{R}_+$ started at 0.

Definition 5.1 The standard Brownian motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ such that

1. $B_0 = 0$ almost surely,

2. The sample trajectories $t \mapsto B_t$ are continuous, with probability 1.

3. For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are mutually independent random variables.

4. For any given times $0 \leqslant s < t$, $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t-s$.

In particular, for $t \in \mathbb{R}_+$, the random variable $B_t \simeq \mathcal{N}(0,t)$ has a Gaussian distribution with mean zero and variance $t > 0$. Existence of a stochastic process satisfying the conditions of Definition 5.1 will be covered in Section 5.2.

In Figure 5.1 we draw three sample paths of a standard Brownian motion obtained by computer simulation using (5.3). Note that there is no point in "computing" the value of B_t as it is a *random variable* for all $t > 0$. However, we can generate samples of B_t, which are distributed according to the centered Gaussian distribution with variance t.

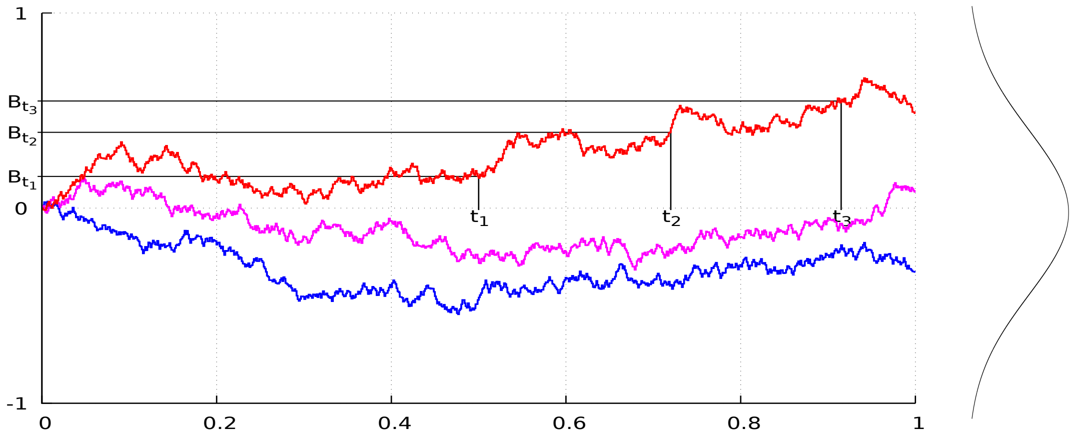


Figure 5.1: Sample paths of a one-dimensional Brownian motion.

In particular, Property *4* in Definition 5.1 implies

$$\mathbb{E}[B_t - B_s] = 0 \quad \text{and} \quad \mathrm{Var}[B_t - B_s] = t - s, \qquad 0 \leqslant s \leqslant t,$$

and we have

$$\begin{aligned}
\mathrm{Cov}(B_s, B_t) &= \mathbb{E}[B_s B_t] \\
&= \mathbb{E}[B_s(B_t - B_s + B_s)] \\
&= \mathbb{E}\left[B_s(B_t - B_s) + (B_s)^2\right] \\
&= \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}\left[(B_s)^2\right] \\
&= \mathbb{E}[B_s]\,\mathbb{E}[B_t - B_s] + \mathbb{E}\left[(B_s)^2\right]
\end{aligned}$$

$$= \text{Var}[B_s]$$
$$= s, \qquad 0 \leqslant s \leqslant t,$$

hence

$$\text{Cov}(B_s, B_t) = \mathbb{E}[B_s B_t] = \min(s,t), \qquad s,t \in \mathbb{R}_+, \tag{5.1}$$

cf. also Exercise 5.2-(5.1). The following graphs present two examples of possible modeling of random data using Brownian motion.

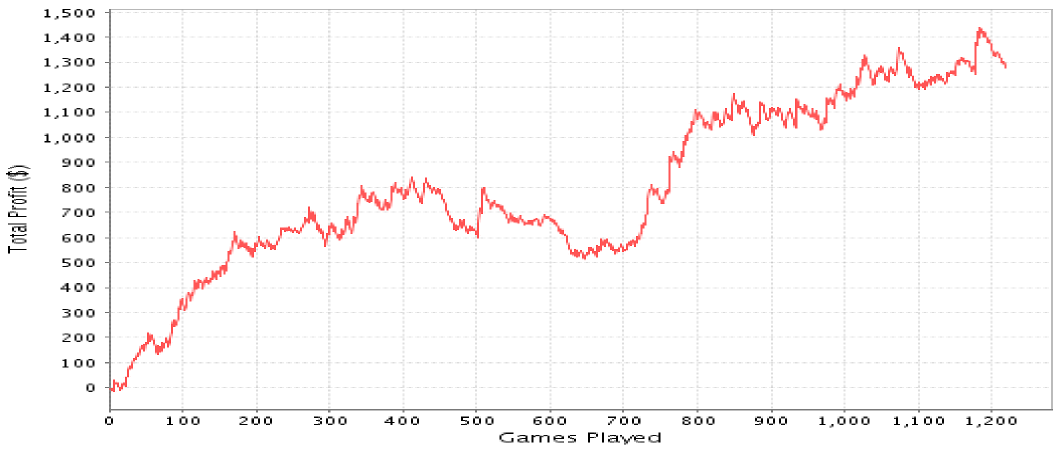


Figure 5.2: Evolution of the fortune of a poker player *vs* number of games played.

Figure 5.3: Web traffic ranking.

In the sequel, we denote by $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ the filtration generated by the Brownian paths up to time t, defined as

$$\mathscr{F}_t := \sigma(B_s \,:\, 0 \leqslant s \leqslant t), \qquad t \geqslant 0. \tag{5.2}$$

Property 3 in Definition 5.1 shows that $B_t - B_s$ is independent of all Brownian increments taken before time s, *i.e.*

$$(B_t - B_s) \perp\!\!\!\perp (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}),$$

$0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant s \leqslant t$, hence $B_t - B_s$ is also independent of the whole Brownian history up to time s, hence $B_t - B_s$ is in fact independent of $\mathscr{F}_s$, $s \geqslant 0$. As in Example 2 page 2 we have the following result.

Proposition 5.1 Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a continuous-time *martingale*.

Proof. We have

$$\begin{aligned}
\mathbb{E}[B_t \mid \mathscr{F}_s] &= \mathbb{E}[B_t - B_s + B_s \mid \mathscr{F}_s] \\
&= \mathbb{E}[B_t - B_s \mid \mathscr{F}_s] + \mathbb{E}[B_s \mid \mathscr{F}_s] \\
&= \mathbb{E}[B_t - B_s] + B_s \\
&= B_s, \qquad 0 \leqslant s \leqslant t,
\end{aligned}$$

because it has centered and independent increments, cf. Section 8.1. □

The n-dimensional Brownian motion can be constructed as $(B_t^1, B_t^2, \ldots, B_t^n)_{t\in\mathbb{R}_+}$ where $(B_t^1)_{t\in\mathbb{R}_+}$, $(B_t^2)_{t\in\mathbb{R}_+}$, $\ldots$,$(B_t^n)_{t\in\mathbb{R}_+}$ are independent copies of $(B_t)_{t\in\mathbb{R}_+}$. Next, we turn to simulations of 2 dimensional and 3 dimensional Brownian motions in Figures 5.4 and 5.5. Recall that the movement of pollen particles originally observed by R. Brown in 1827 was indeed 2-dimensional.

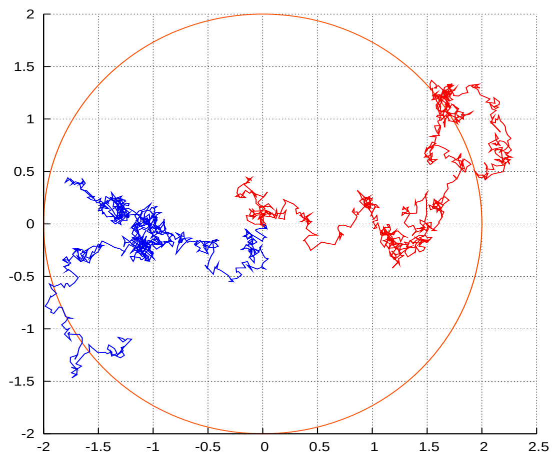


Figure 5.4: Two sample paths of a two-dimensional Brownian motion.

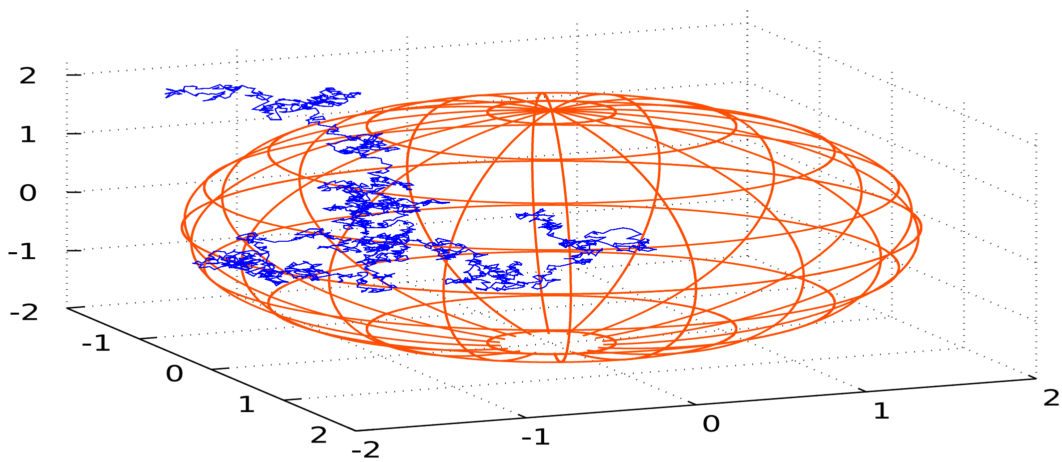


Figure 5.5: Sample path of a three-dimensional Brownian motion.

Figure 5.6 presents an illustration of the scaling property of Brownian motion.

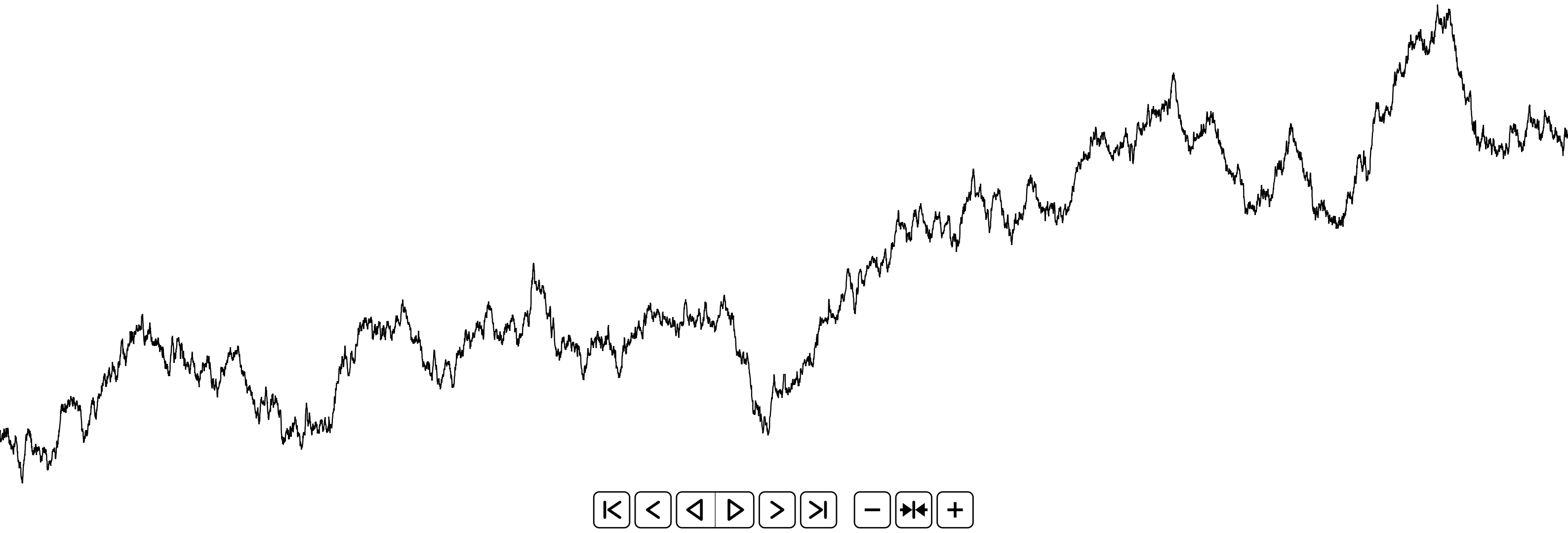


Figure 5.6: Scaling property of Brownian motion.[*]

5.2 Three Constructions of Brownian Motion

We refer to Chapter 1 of Revuz and Yor, 1994 and to Theorem 10.28 in Folland, 1999 for the proof of existence of Brownian motion as a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ satisfying the above Conditions *1-4*.

Brownian motion as a random walk

For convenience we will informally regard Brownian motion as a random walk over infinitesimal time intervals of length Δt, whose increments

$$\Delta B_t := B_{t+\Delta t} - B_t \simeq \mathcal{N}(0, \Delta t)$$

over the time interval $[t, t+\Delta t]$ will be approximated by the Bernoulli random variable

$$\Delta B_t = \pm\sqrt{\Delta t} \tag{5.3}$$

with equal probabilities $(1/2, 1/2)$. Figure 5.7 presents a simulation of Brownian motion as a random walk with $\Delta t = 0.1$.

[*] The animation works in Acrobat Reader on the entire pdf file.

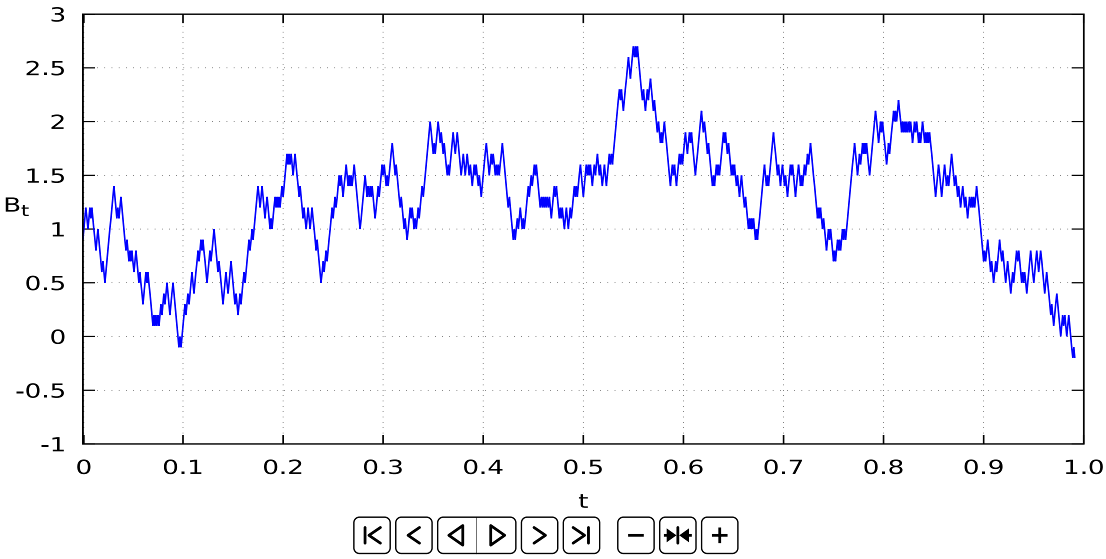


Figure 5.7: Construction of Brownian motion as a random walk.*

R The choice of the square root in (5.3) is in fact not fortuitous. Indeed, any choice of $\pm(\Delta t)^{\alpha}$ with a power $\alpha > 1/2$ would lead to explosion of the process as dt tends to zero, whereas a power $\alpha \in (0, 1/2)$ would lead to a vanishing process.

Note that we have

$$\mathbb{E}[\Delta B_t] = \frac{1}{2}\sqrt{\Delta t} - \frac{1}{2}\sqrt{\Delta t} = 0,$$

and

$$\text{Var}[\Delta B_t] = \mathbb{E}\left[(\Delta B_t)^2\right] = \frac{1}{2}\Delta t + \frac{1}{2}\Delta t = \Delta t.$$

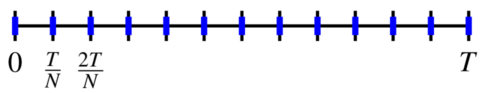
According to this representation, the paths of Brownian motion are not differentiable, although they are continuous by Property 2, as we have

$$\frac{dB_t}{dt} \simeq \frac{\pm\sqrt{dt}}{dt} = \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty. \tag{5.4}$$

In order to recover the Gaussian distribution property of the random variable B_T, we can split the time interval $[0, T]$ into N subintervals

$$\left(\frac{k-1}{N}T, \frac{k}{N}T\right], \qquad k = 1, 2, \ldots, N,$$

of same length $\Delta t = T/N$, with N "large".



Defining the Bernoulli random variable X_k as

$$X_k := \pm\sqrt{T}$$

*The animation works in Acrobat Reader on the entire pdf file.

with equal probabilities $(1/2,1/2)$, we have $\mathrm{Var}(X_k)=T$ and

$$\Delta B_t := \frac{X_k}{\sqrt{N}} = \pm\sqrt{\Delta t}$$

is the increment of B_t over $((k-1)\Delta t, k\Delta t]$, and we get

$$B_T \simeq \sum_{0<t<T} \Delta B_t \simeq \frac{X_1+X_2+\cdots+X_N}{\sqrt{N}}.$$

Hence by the central limit theorem we recover the fact that B_T has the centered Gaussian distribution $\mathcal{N}(0,T)$ with variance T, cf. point 4 of the above Definition 5.1 of Brownian motion, and the illustration given in Figure 5.8.

```
N=1000; t <- 0:N; dt <- 1.0/N; nsim <- 10 # using Bernoulli samples
X <- matrix((dt)^0.5*(rbinom( nsim * N, 1, 0.5)-0.5)*2, nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", ylab = "", type = "l", ylim = c(-2, 2), col = 0)
for (i in 1:nsim){lines(t, X[i, ], xlab = "time", type = "l", ylim = c(-2, 2), col = i)}
# using Gaussian samples
nsim=100;X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)));H<-hist(X[,N])
layout(matrix(c(1,2), nrow =1, byrow = TRUE));par(mar=c(2,2,2,0), oma = c(2, 2, 2, 2))
plot(t*dt, X[1, ], xlab = "time", ylab = "", type = "l", ylim = c(-2, 2), col = 0)
for (i in 1:nsim){lines(t*dt, X[i, ], xlab = "time", type = "l", ylim = c(-2, 2), col = i)}
for (i in 1:nsim){points(N, X[i,N], pch=1, lwd = 5, col = i)}
x <- seq(-2,2, length=100);px <- dnorm(x);par(mar = c(2,2,2,2))
plot(NULL , xlab="", ylab="", xlim = c(0, max(px,H$density)), ylim = c(-2,2),axes=F)
rect(0, H$breaks[1:(length(H$breaks) - 1)], col=rainbow(20,start=0.08,end=0.6), H$density,
      H$breaks[2:length(H$breaks)])
lines(px,x, type="l", lty=1, col="black",lwd=2,xlab="",ylab="", main="")
```

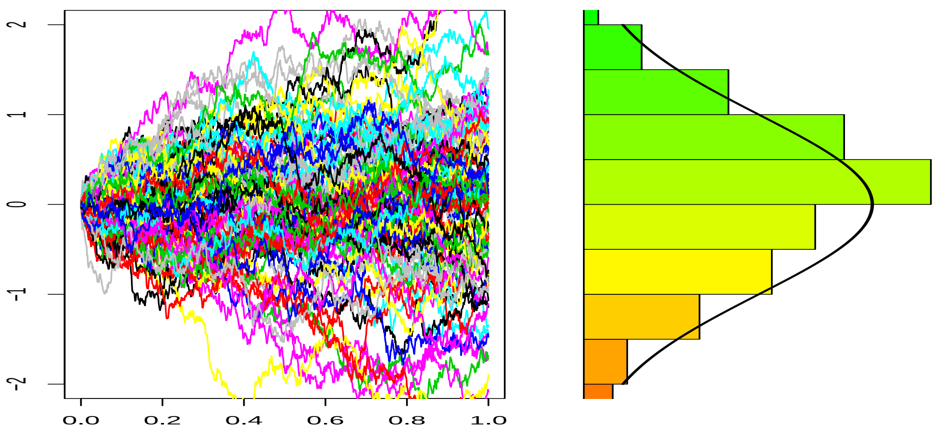


Figure 5.8: Statistics of one-dimensional Brownian paths *vs* Gaussian distribution.

Indeed, the central limit theorem states that given any sequence $(X_k)_{k\geqslant 1}$ of independent identically distributed centered random variables with variance $\sigma^2=\mathrm{Var}[X_k]=T$, the normalized sum

$$\frac{X_1+X_2+\cdots+X_N}{\sqrt{N}}$$

converges (in distribution) to the centered Gaussian random variable $\mathcal{N}(0,\sigma^2)$ with variance σ^2 as N goes to infinity. As a consequence, ΔB_t could in fact be replaced by any centered random variable with variance Δt in the above description.

Lévy's construction of Brownian motion

Figure 5.9 represents the construction of Brownian motion by successive linear interpolations.

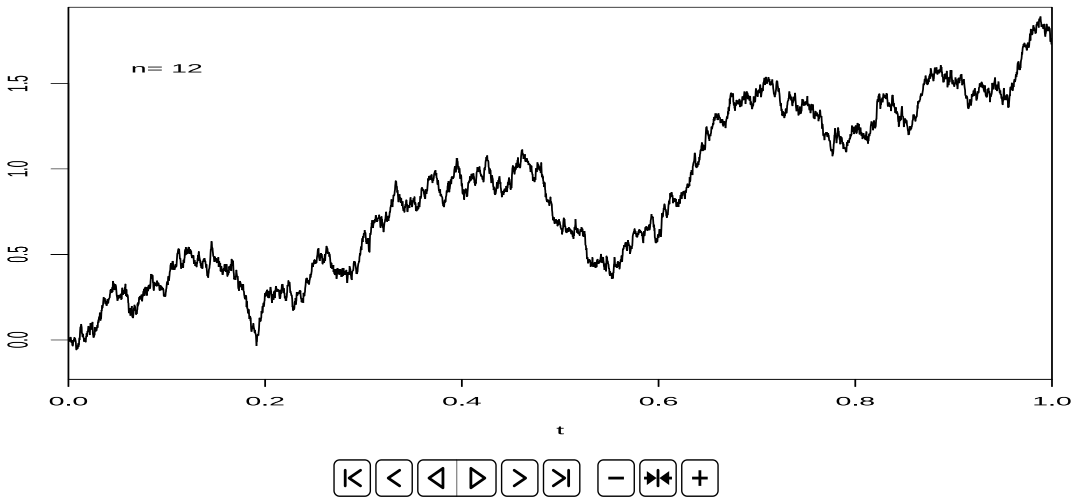


Figure 5.9: Lévy's construction of Brownian motion.[*]

The following R code is used to generate Figure 5.9.[†]

```
alpha=1/2;t <- 0:1;dt <- 1;z=rnorm(1,mean=0,sd=dt^alpha)
plot(t*dt, c(0, z), xlab = "t", ylab = "", main = "", type = "l", xaxs="i")
k=0;while (k<10) {readline("Press <return> to continue")
m <- (z+c(0,head(z,-1)))/2;y <- rnorm(length(t)-1,mean=0,sd=(dt/4)^alpha)
x <- m+y;x <- c(matrix(c(x,z), 2, byrow = T));n=2*length(t)-2;t <- 0:n
plot(t*dt/2, c(0, x), xlab = "t", ylab = "", main = "", type = "l", xaxs="i");z=x;dt=dt/2}
```

Construction by series expansions

Brownian motion on $[0,T]$ can also be constructed by Fourier synthesis via the Paley-Wiener series expansion

$$B_t = \sum_{n\geqslant 1} X_n f_n(t) = \frac{\sqrt{2T}}{\pi} \sum_{n\geqslant 1} X_n \frac{\sin\left((n-1/2)\pi t/T\right)}{(n-1/2)}, \qquad t \in [0,T],$$

where $(X_n)_{n\geqslant 1}$ is a sequence of independent $\mathcal{N}(0,1)$ standard Gaussian random variables, as illustrated in Figure 5.10.[‡]

[*]The animation works in Acrobat Reader on the entire pdf file.
[†]Download the corresponding **R code** or the **IPython notebook** that can be run **here**.
[‡]Download the corresponding **IPython notebook** that can be run **here**.

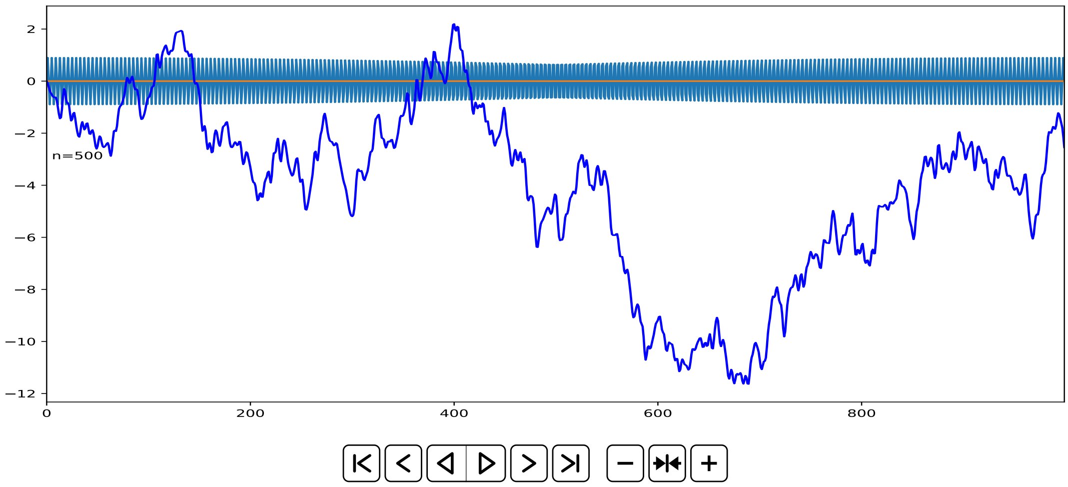


Figure 5.10: Construction of Brownian motion by series expansions.*

5.3 Wiener Stochastic Integral

In this section, we construct the Wiener stochastic integral of square-integrable deterministic functions of time with respect to Brownian motion.

Recall that the price S_t of risky assets was originally modeled in Bachelier, 1900 as $S_t := \sigma B_t$, where σ is a volatility parameter. The stochastic integral

$$\int_0^T f(t) dS_t = \sigma \int_0^T f(t) dB_t$$

can be used to represent the value of a portfolio as a sum of profits and losses $f(t)dS_t$ where dS_t represents the stock price variation and $f(t)$ is the quantity invested in the asset S_t over the short time interval $[t, t+dt]$.

A naive definition of the stochastic integral with respect to Brownian motion would consist in letting

$$\int_0^T f(t) dB_t := \int_0^T f(t) \frac{dB_t}{dt} dt,$$

and evaluating the above integral with respect to dt. However this definition fails because the paths of Brownian motion are not differentiable, cf. (5.4). Next we present Itô's construction of the stochastic integral with respect to Brownian motion. Stochastic integrals will be first constructed as integrals of simple step functions of the form

$$f(t) = \sum_{i=1}^{n} a_i \mathbb{1}_{(t_{i-1}, t_i]}(t), \qquad t \in [0, T], \tag{5.5}$$

i.e. the function f takes the value a_i on the interval $(t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, with $0 \leqslant t_0 < \cdots < t_n \leqslant T$, as illustrated in Figure 5.11.

*The animation works in Acrobat Reader on the entire pdf file.

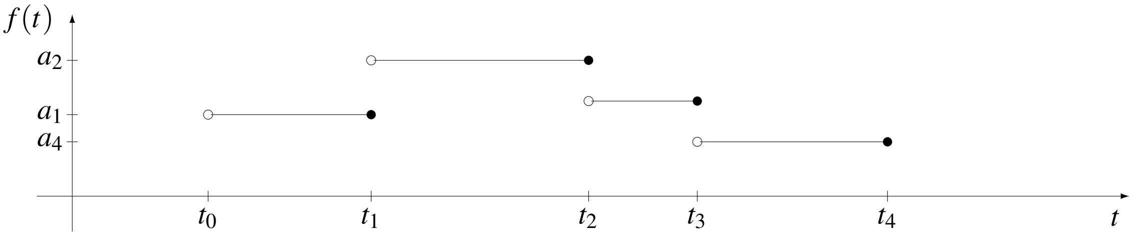


Figure 5.11: Step function $t \mapsto f(t)$.

```
ti<-c(0,2,4.5,7,9)
ai<-c(0,3,1,2,1,0)
plot(stepfun(ti,ai),xlim = c(0,10),do.points = F,main="")
```

Recall that the classical integral of f given in (5.5) is interpreted as the area under the curve f, and computed as

$$\int_0^T f(t)dt = \sum_{i=1}^n a_i(t_i - t_{i-1}).$$

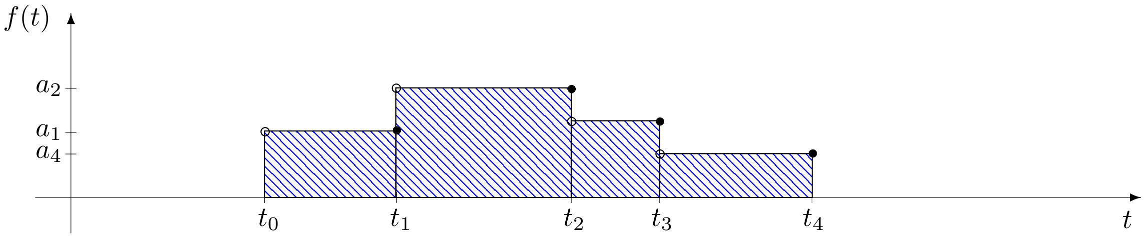


Figure 5.12: Area under the step function $t \mapsto f(t)$.

In the next Definition 5.2 we use such step functions for the construction of the stochastic integral with respect to Brownian motion. The stochastic integral (5.6) for step functions will be interpreted as the sum of profits and losses $a_i(B_{t_i} - B_{t_{i-1}})$, $i = 1,2,\ldots,n$, in a portfolio holding a quantity a_i of a risky asset whose price variation is $B_{t_i} - B_{t_{i-1}}$ at time $i = 1,2,\ldots,n$.

Definition 5.2 The stochastic integral with respect to Brownian motion $(B_t)_{t\in[0,T]}$ of the simple step function f of the form (5.5) is defined by

$$\int_0^T f(t)dB_t := \sum_{i=1}^n a_i(B_{t_i} - B_{t_{i-1}}). \tag{5.6}$$

In the next Lemma 5.2 we determine the probability distribution of $\int_0^T f(t)dB_t$ and we show that it is independent of the particular representation (5.5) chosen for $f(t)$.

Lemma 5.2 Let f be a simple step function f of the form (5.5). The stochastic integral $\int_0^T f(t)dB_t$ defined in (5.6) has the centered Gaussian distribution

$$\int_0^T f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^T |f(t)|^2 dt\right)$$

with mean $\mathbb{E}\left[\int_0^T f(t)dB_t\right] = 0$ and variance given by the *Itô isometry*

$$\operatorname{Var}\left[\int_0^T f(t)dB_t\right] = \mathbb{E}\left[\left(\int_0^T f(t)dB_t\right)^2\right] = \int_0^T |f(t)|^2 dt. \tag{5.7}$$

Proof. Recall that if $X_1, X_2, \ldots, X_n$ are independent Gaussian random variables with probability distributions $\mathcal{N}(m_1, \sigma_1^2), \ldots, \mathcal{N}(m_n, \sigma_n^2)$ then the sum $X_1 + \cdots + X_n$ is a Gaussian random variable with distribution

$$\mathcal{N}\left(m_1 + \cdots + m_n, \sigma_1^2 + \cdots + \sigma_n^2\right).$$

As a consequence, the stochastic integral

$$\int_0^T f(t)dB_t = \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})$$

of the step function

$$f(t) = \sum_{k=1}^n a_k \mathbb{1}_{(t_{k-1}, t_k]}(t), \qquad t \in [0, T],$$

has the centered Gaussian distribution with mean 0 and variance

$$\begin{aligned}
\operatorname{Var}\left[\int_0^T f(t)dB_t\right] &= \operatorname{Var}\left[\sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})\right] \\
&= \sum_{k=1}^n \operatorname{Var}[a_k (B_{t_k} - B_{t_{k-1}})] \\
&= \sum_{k=1}^n |a_k|^2 \operatorname{Var}[B_{t_k} - B_{t_{k-1}}] \\
&= \sum_{k=1}^n (t_k - t_{k-1}) |a_k|^2 \\
&= \sum_{k=1}^n |a_k|^2 \int_{t_{k-1}}^{t_k} dt \\
&= \int_0^T \sum_{k=1}^n |a_k|^2 \mathbb{1}_{(t_{k-1}, t_k]}(t) dt \\
&= \int_0^T |f(t)|^2 dt,
\end{aligned}$$

since the simple function

$$f^2(t) = \sum_{i=1}^n a_i^2 \mathbb{1}_{(t_{i-1}, t_i]}(t), \qquad t \in [0, T],$$

takes the value a_i^2 on the interval $(t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, as can be checked from the following Figure 5.13.

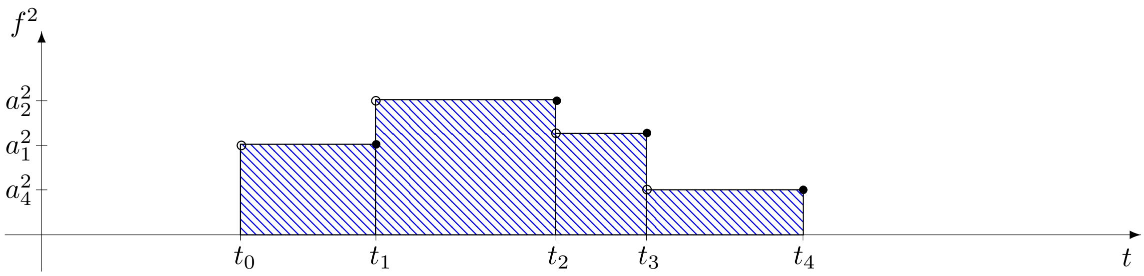


Figure 5.13: Squared step function $t \mapsto f^2(t)$.

□

In the sequel, we will make a repeated use of the space $L^2([0,T])$ of *square-integrable functions*.

Definition 5.3 Let $L^2([0,T])$ denote the space of (measurable) functions $f : [0,T] \longrightarrow \mathbb{R}$ such that

$$\|f\|_{L^2([0,T])} := \sqrt{\int_0^T |f(t)|^2 dt} < \infty, \quad f \in L^2([0,T]). \tag{5.8}$$

For example, the function $f(t) := t^\alpha$, $t \in (0,T]$, belongs to $L^2([0,T])$ if and only if $\alpha > -1/2$, as we have

$$\int_0^T f^2(t)dt = \int_0^T t^{2\alpha} dt = \begin{cases} +\infty & \text{if } \alpha \leqslant -1/2, \\ \left[\dfrac{t^{1+2\alpha}}{1+2\alpha}\right]_{t=0}^{t=T} = \dfrac{T^{1+2\alpha}}{1+2\alpha} < \infty & \text{if } \alpha > -1/2. \end{cases}$$

The norm $\|\cdot\|_{L^2([0,T])}$ on $L^2([0,T])$ induces a *distance* between two functions f and g in $L^2([0,T])$, defined as

$$\|f-g\|_{L^2([0,T])} := \sqrt{\int_0^T |f(t)-g(t)|^2 dt} < \infty,$$

cf. *e.g.* Chapter 3 of Rudin, 1974 for details.

Definition 5.4 *Convergence in* $L^2([0,T])$. We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $L^2([0,T])$ converges in $L^2([0,T])$ to another function $f \in L^2([0,T])$ if

$$\lim_{n\to\infty} \|f-f_n\|_{L^2([0,T])} = \lim_{n\to\infty} \sqrt{\int_0^T |f(t)-f_n(t)|^2 dt} = 0.$$

```
f = function(x){exp(sin(x*1.8*pi))}
for (i in 3:9){n=2^i;x<-cumsum(c(0,rep(1,n)))/n;
z<-c(NA,head(x,-1))
y<-c(f(x)-pmax(f(x)-f(z),0),f(1))
t=seq(0,1,0.01);
plot(f,from=0,to=1,ylim=c(0.3,2.9),type="l",lwd=3,col="red",main="",xaxs="i",yaxs="i")
lines(stepfun(x,y),do.points=F,lwd=2,col="blue",main="");
readline("Press <return> to continue");}
```

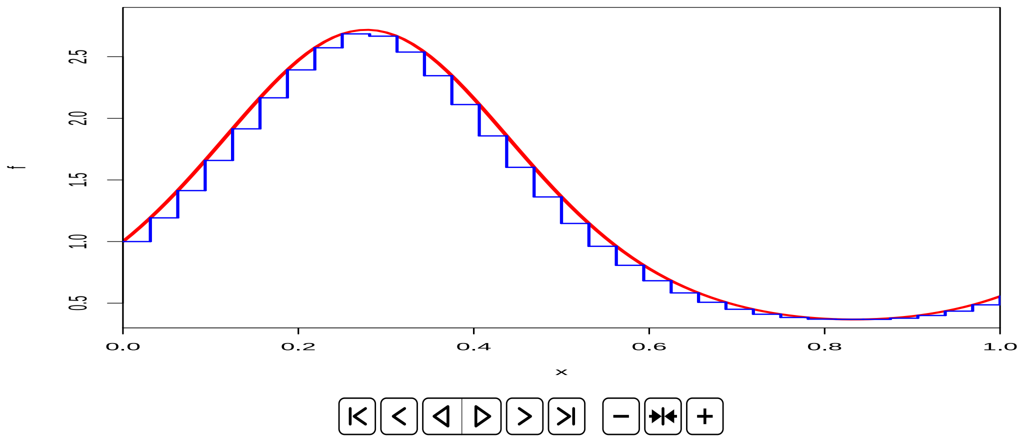


Figure 5.14: Step function approximation.*

By *e.g.* Theorem 3.13 in Rudin, 1974 or Proposition 2.4 page 63 of Hirsch and Lacombe, 1999, we have the following result which states that the set of simple step functions f of the form (5.5) is a linear space which is dense in $L^2([0,T])$ for the norm (5.8), as stated in the next proposition.

Proposition 5.3 For any function $f \in L^2([0,T])$ satisfying (5.8) there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f in $L^2([0,T])$ in the sense that

$$\lim_{n\to\infty} \|f - f_n\|_{L^2([0,T])} = \lim_{n\to\infty} \sqrt{\int_0^T |f(t) - f_n(t)|^2 dt} = 0.$$

In order to extend the definition (5.6) of the stochastic integral $\int_0^T f(t)dB_t$ to any function $f \in L^2([0,T])$, *i.e.* to $f : [0,T] \longrightarrow \mathbb{R}$ measurable such that

$$\int_0^T |f(t)|^2 dt < \infty, \tag{5.9}$$

we will make use of the space $L^2(\Omega)$ of *square-integrable random variables*.

Definition 5.5 Let $L^2(\Omega)$ denote the space of random variables $F : \Omega \longrightarrow \mathbb{R}$ such that

$$\|F\|_{L^2(\Omega\times[0,T])} := \sqrt{\mathbb{E}\left[F^2\right]} < \infty.$$

The norm $\|\cdot\|_{L^2(\Omega)}$ on $L^2(\Omega)$ induces the *distance*

$$\|F - G\|_{L^2(\Omega)} := \sqrt{\mathbb{E}\left[(F-G)^2\right]} < \infty,$$

between the square-integrable random variables F and G in $L^2(\Omega)$.

Definition 5.6 *Convergence in* $L^2(\Omega)$. We say that a sequence $(F_n)_{n\in\mathbb{N}}$ of random variables in $L^2(\Omega)$ converges in $L^2(\Omega)$ to another random variable $F \in L^2(\Omega)$ if

$$\lim_{n\to\infty} \|F - F_n\|_{L^2(\Omega)} = \lim_{n\to\infty} \sqrt{\mathbb{E}\left[(F - F_n)^2\right]} = 0.$$

*The animation works in Acrobat Reader on the entire pdf file.

The next proposition allows us to extend Lemma 5.2 from simple step functions to square-integrable functions in $L^2([0,T])$.

Proposition 5.4 The definition (5.6) of the stochastic integral $\int_0^T f(t)dB_t$ can be extended to any function $f \in L^2([0,T])$. In this case, $\int_0^T f(t)dB_t$ has the centered Gaussian distribution

$$\int_0^T f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^T |f(t)|^2 dt\right)$$

with mean $\mathbb{E}\left[\int_0^T f(t)dB_t\right] = 0$ and variance given by the *Itô isometry*

$$\text{Var}\left[\int_0^T f(t)dB_t\right] = \mathbb{E}\left[\left(\int_0^T f(t)dB_t\right)^2\right] = \int_0^T |f(t)|^2 dt. \tag{5.10}$$

Proof. The extension of the stochastic integral to all functions satisfying (5.9) is obtained by a denseness and Cauchy[*] sequence argument, based on the isometry relation (5.10).

i) Given f a function satisfying (5.9), consider a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f in $L^2([0,T])$, *i.e.*

$$\lim_{n\to\infty} \|f - f_n\|_{L^2([0,T])} = \lim_{n\to\infty} \sqrt{\int_0^T |f(t) - f_n(t)|^2 dt} = 0$$

as in Proposition 5.3.

ii) By the isometry (5.10) and the triangle inequality[†] we have

$$\begin{aligned}
&\left\| \int_0^T f_k(t)dB_t - \int_0^T f_n(t)dB_t \right\|_{L^2(\Omega)} \\
&= \sqrt{\mathbb{E}\left[\left(\int_0^T f_k(t)dB_t - \int_0^T f_n(t)dB_t\right)^2\right]} \\
&= \sqrt{\mathbb{E}\left[\left(\int_0^T (f_k(t) - f_n(t))dB_t\right)^2\right]} \\
&= \|f_k - f_n\|_{L^2([0,T])} \\
&\leqslant \|f_k - f\|_{L^2([0,T])} + \|f - f_n\|_{L^2([0,T])},
\end{aligned}$$

which tends to 0 as k and n tend to infinity, hence $\left(\int_0^T f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$ by for the $L^2(\Omega)$-norm.

iii) Since the sequence $\left(\int_0^T f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is Cauchy and the space $L^2(\Omega)$ is *complete*, cf. *e.g.* Theorem 3.11 in Rudin, 1974 or Chapter 4 of Dudley, 2002, we conclude that $\left(\int_0^T f_n(t)dB_t\right)_{n\in\mathbb{N}}$ *converges* for the L^2-norm to a limit in $L^2(\Omega)$. In this case we let

$$\int_0^T f(t)dB_t := \lim_{n\to\infty} \int_0^T f_n(t)dB_t,$$

which also satisfies (5.10) from (5.7) From (5.10) we can check that the limit is independent of the approximating sequence $(f_n)_{n\in\mathbb{N}}$.

[*] See MH3100 Real Analysis I.

[†] The triangle inequality $\|f_k - f_n\|_{L^2([0,T])} \leqslant \|f_k - f\|_{L^2([0,T])} + \|f - f_n\|_{L^2([0,T])}$ follows from the Minkowski inequality.

iv) Finally, from the convergence of Gaussian characteristic functions

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\alpha\int_0^T f(t)dB_t\right)\right] &= \mathbb{E}\left[\lim_{n\to\infty}\exp\left(i\alpha\int_0^T f_n(t)dB_t\right)\right] \\
&= \lim_{n\to\infty}\mathbb{E}\left[\exp\left(i\alpha\int_0^T f_n(t)dB_t\right)\right] \\
&= \lim_{n\to\infty}\exp\left(-\frac{\alpha^2}{2}\int_0^T |f_n(t)|^2dt\right) \\
&= \exp\left(-\frac{\alpha^2}{2}\int_0^T |f(t)|^2dt\right),
\end{aligned}$$

$f\in L^2([0,T])$, $\alpha\in\mathbb{R}$, we check that $\int_0^T f(t)dB_t$ has the centered Gaussian distribution

$$\int_0^T f(t)dB_t \simeq \mathcal{N}\left(0,\int_0^T |f(t)|^2dt\right),$$

see Theorem 11.5.

□

The next corollary is obtained by bilinearity from the Itô isometry (5.10).

Corollary 5.5 The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ satisfies the isometry

$$\mathbb{E}\left[\int_0^T f(t)dB_t\int_0^T g(t)dB_t\right] = \int_0^T f(t)g(t)dt,$$

for all square-integrable deterministic functions $f,g\in L^2([0,T])$.

Proof. Applying the Itô isometry (5.10) to the processes $f+g$ and $f-g$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T f(t)dB_t\int_0^T g(t)dB_t\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\int_0^T f(t)dB_t+\int_0^T g(t)dB_t\right)^2-\left(\int_0^T f(t)dB_t-\int_0^T g(t)dB_t\right)^2\right] \\
&= \frac{1}{4}\mathbb{E}\left[\left(\int_0^T (f(t)-g(t))dB_t\right)^2\right]-\frac{1}{4}\mathbb{E}\left[\left(\int_0^T (f(t)-g(t))dB_t\right)^2\right] \\
&= \frac{1}{4}\int_0^T (f(t)+g(t))^2dt-\frac{1}{4}\int_0^T (f(t)-g(t))^2dt \\
&= \frac{1}{4}\int_0^T \left((f(t)+g(t))^2-(f(t)-g(t))^2\right)dt \\
&= \int_0^T f(t)g(t)dt.
\end{aligned}$$

□

For example, $\int_0^T \mathrm{e}^{-t}dB_t$ has the centered Gaussian distribution with variance

$$\int_0^T \mathrm{e}^{-2t}dt = \left[-\frac{1}{2}\mathrm{e}^{-2t}\right]_{t=0}^{t=T} = \frac{1}{2}\left(1-\mathrm{e}^{-2T}\right).$$

R The Wiener stochastic integral $\int_0^T f(s)dB_s$ is a Gaussian random variable which cannot be "computed" in the way standard integral are computed via the use of primitives. However, when $f \in L^2([0,T])$ is in $\mathscr{C}^1([0,T])$,* we have the integration by parts relation

$$\int_0^T f(t)dB_t = f(T)B_T - \int_0^T B_t f'(t)dt. \tag{5.11}$$

When $f \in L^2(\mathbb{R}_+)$ is in $\mathscr{C}^1(\mathbb{R}_+)$ we also have following formula

$$\int_0^\infty f(t)dB_t = -\int_0^\infty B_t f'(t)dt, \tag{5.12}$$

provided that $\lim_{t\to\infty} t|f(t)|^2 = 0$ and $f \in L^2(\mathbb{R}_+)$, cf. *e.g.* Remark 2.5.9 in Privault, 2009.

5.4 Itô Stochastic Integral

In this section we extend the Wiener stochastic integral from deterministic functions in $L^2([0,T])$ to random square-integrable (random) *adapted* processes. For this, we will need the notion of *measurability*.

The extension of the stochastic integral to adapted random processes is actually necessary in order to compute a portfolio value when the portfolio process is no longer deterministic. This happens in particular when one needs to update the portfolio allocation based on random events occurring on the market.

A random variable F is said to be $\mathscr{F}_t$*-measurable* if the knowledge of F depends only on the information known up to time t. As an example, if t =today,

- the date of the past course exam is $\mathscr{F}_t$-measurable, because it belongs to the past.
- the date of the next Chinese new year, although it refers to a future event, is also $\mathscr{F}_t$-measurable because it is known at time t.
- the date of the next typhoon is not $\mathscr{F}_t$-measurable since it is not known at time t.
- the maturity date T of the European option is $\mathscr{F}_t$-measurable for all $t \in [0,T]$, because it has been determined at time 0.
- the exercise date τ of an American option after time t is not $\mathscr{F}_t$-measurable because it refers to a future random event.

In the next definition, $(\mathscr{F}_t)_{t\in[0,T]}$ denotes the information flow defined in (5.2), *i.e.*

$$\mathscr{F}_t := \sigma(B_s \ : \ 0 \leqslant s \leqslant t), \qquad t \geqslant 0.$$

Definition 5.7 A stochastic process $(X_t)_{t\in[0,T]}$ is said to be $(\mathscr{F}_t)_{t\in[0,T]}$-adapted if X_t is $\mathscr{F}_t$-measurable for all $t \in [0,T]$.

For example,

- $(B_t)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-*adapted* process,

*This means that f is continuously differentiable on $[0,T]$.

- $(B_{t+1})_{t\in\mathbb{R}_+}$ is *not* an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted process,

- $(B_{t/2})_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-*adapted* process,

- $(B_{\sqrt{t}})_{t\in\mathbb{R}_+}$ is *not* an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted process,

- $\big(\mathrm{Max}_{s\in[0,t]} B_s\big)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-*adapted* process,

- $\left(\int_0^t B_s ds\right)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-*adapted* process,

- $\left(\int_0^t f(s)dB_s\right)_{t\in[0,T]}$ is an $(\mathscr{F}_t)_{t\in[0,T]}$-*adapted* process when $f\in L^2([0,T])$.

In other words, a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ is $(\mathscr{F}_t)_{t\in[0,T]}$-adapted if the value of X_t at time t depends only on information known up to time t. Note that the value of X_t may still depend on "known" future data, for example a fixed future date in the calendar, such as a maturity time $T>t$, as long as its value is known at time t.

The next Figure 5.15 shows an adapted portfolio strategy on two assets, constructed from a sign-switching signal based on spread data, see § 1.5 in Privault, 2020 and this **R code**.

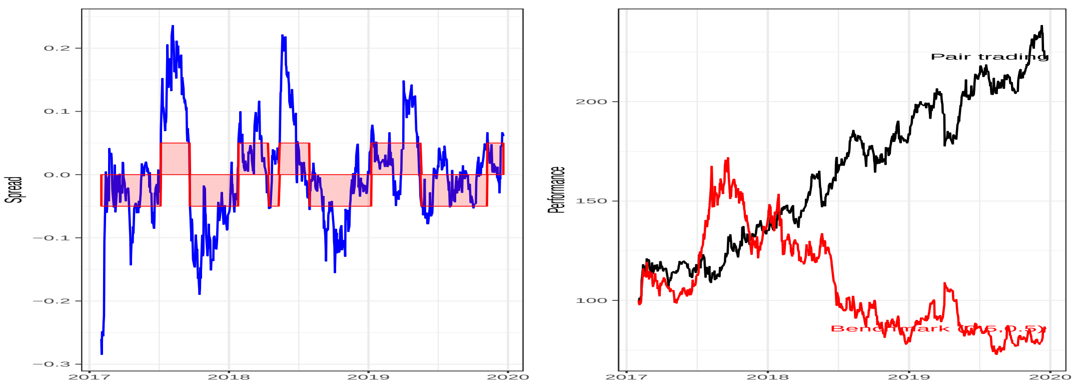


Figure 5.15: Adapted pair trading portfolio strategy.

The stochastic integral of adapted processes is first constructed as integrals of simple predictable processes.

Definition 5.8 A simple predictable processes is a stochastic process $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t := \sum_{i=1}^n F_i \mathbb{1}_{(t_{i-1},t_i]}(t), \qquad t\in\mathbb{R}_+, \tag{5.13}$$

where F_i is an $\mathscr{F}_{t_{i-1}}$-measurable random variable for $i=1,2,\ldots,n$, and $0=t_0<t_1<\cdots<t_{n-1}<t_n=T$.

For example, a natural approximation of $(B_t)_{t\in\mathbb{R}_+}$ by a simple predictable process can be constructed as

$$u_t = \sum_{i=1}^n F_i \mathbb{1}_{(t_{i-1},t_i]}(t) := \sum_{i=1}^n B_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(t), \qquad t\in\mathbb{R}_+, \tag{5.14}$$

since $F_i := B_{t_{i-1}}$ is $\mathscr{F}_{t_{i-1}}$-measurable for $i=1,2,\ldots,n$. The notion of simple predictable process makes full sense in the context of portfolio investment, in which F_i will represent an investment allocation decided at time t_{i-1} and to remain unchanged over the time interval $(t_{i-1}, t_i]$.

By convention, $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is denoted in the sequel by $u_t(\omega)$, $t \in \mathbb{R}_+$, $\omega \in \Omega$, and the random outcome ω is often dropped for convenience of notation.

Definition 5.9 The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of any simple predictable process $(u_t)_{t\in\mathbb{R}_+}$ of the form (5.13) is defined by

$$\int_0^T u_t dB_t := \sum_{i=1}^n F_i (B_{t_i} - B_{t_{i-1}}), \tag{5.15}$$

with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.

The use of predictability in the definition (5.15) is essential from a financial point of view, as F_i will represent a portfolio allocation made at time t_{i-1} and kept constant over the trading interval $[t_{i-1}, t_i]$, while $B_{t_i} - B_{t_{i-1}}$ represents a change in the underlying asset price over $[t_{i-1}, t_i]$. See also the related discussion on self-financing portfolios in Section 6.3 and Lemma 6.7 on the use of stochastic integrals to represent the value of a portfolio.

Definition 5.10 Let $L^2(\Omega \times [0,T])$ denote the space of stochastic processes

$$\begin{aligned} u : \Omega \times [0,T] &\longrightarrow \mathbb{R} \\ (\omega, t) &\longmapsto u_t(\omega) \end{aligned}$$

such that

$$\|u\|_{L^2(\Omega\times[0,T])} := \sqrt{\mathbb{E}\left[\int_0^T |u_t|^2 dt\right]} < \infty, \qquad u \in L^2(\Omega \times [0,T]).$$

The norm $\|\cdot\|_{L^2(\Omega\times[0,T])}$ on $L^2(\Omega \times [0,T])$ induces a *distance* between two stochastic processes u and v in $L^2(\Omega \times [0,T])$, defined as

$$\|u - v\|_{L^2(\Omega\times[0,T])} = \sqrt{\mathbb{E}\left[\int_0^T |u_t - v_t|^2 dt\right]}.$$

Definition 5.11 *Convergence in* $L^2(\Omega \times [0,T])$. We say that a sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ of processes in $L^2(\Omega \times [0,T])$ converges in $L^2(\Omega \times [0,T])$ to another process $u \in L^2(\Omega \times [0,T])$ if

$$\lim_{n\to\infty} \|u - u^{(n)}\|_{L^2(\Omega\times[0,T])} = \lim_{n\to\infty} \sqrt{\mathbb{E}\left[\int_0^T |u_t - u_t^{(n)}|^2 dt\right]} = 0.$$

By Lemma 1.1 of Ikeda and Watanabe, 1989, pages 22 and 46, or Proposition 2.5.3 in Privault, 2009, the set of simple predictable processes forms a linear space which is dense in the subspace $L^2_{\mathrm{ad}}(\Omega \times \mathbb{R}_+)$ made of square-integrable adapted processes in $L^2(\Omega \times \mathbb{R}_+)$, as stated in the next proposition.

Proposition 5.6 Given u a square-integrable adapted process there exists a sequence $(u^{(n)})_{n\in\mathbb{N}}$

of simple predictable processes converging to u in $L^2(\Omega \times \mathbb{R}_+)$, *i.e.*

$$\lim_{n\to\infty} \|u - u^{(n)}\|_{L^2(\Omega\times[0,T])} = \lim_{n\to\infty} \sqrt{\mathbb{E}\left[\int_0^T |u_t - u_t^{(n)}|^2 dt\right]} = 0.$$

The next Proposition 5.7 extends the construction of the stochastic integral from simple predictable processes to square-integrable $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes $(u_t)_{t\in\mathbb{R}_+}$ for which the value of u_t at time t can only depend on information contained in the Brownian path up to time t.

This restriction means that the Itô integrand u_t cannot depend on future information, for example a portfolio strategy that would allow the trader to "buy at the lowest" and "sell at the highest" is excluded as it would require knowledge of future market data. Note that the difference between Relation (5.16) below and Relation (5.10) is the presence of an expectation on the right hand side.

Proposition 5.7 The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ extends to all adapted processes $(u_t)_{t\in\mathbb{R}_+}$ such that

$$\|u\|^2_{L^2(\Omega\times[0,T])} := \mathbb{E}\left[\int_0^T |u_t|^2 dt\right] < \infty,$$

with the Itô isometry

$$\left\|\int_0^T u_t dB_t\right\|^2_{L^2(\Omega)} := \mathbb{E}\left[\left(\int_0^T u_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T |u_t|^2 dt\right]. \tag{5.16}$$

In addition, the Itô integral of an adapted process $(u_t)_{t\in\mathbb{R}_+}$ is always a centered random variable:

$$\mathbb{E}\left[\int_0^T u_t dB_t\right] = 0. \tag{5.17}$$

Proof. We start by showing that the Itô isometry (5.16) holds for the simple predictable process u of the form (5.13). We have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T u_t dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n F_i(B_{t_i} - B_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^n F_i(B_{t_i} - B_{t_{i-1}})\right)\left(\sum_{j=1}^n F_j(B_{t_j} - B_{t_{j-1}})\right)\right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^n F_iF_j(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2 (B_{t_i} - B_{t_{i-1}})^2\right] \\
&\quad +2\,\mathbb{E}\left[\sum_{1\leqslant i<j\leqslant n} F_iF_j(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2 (B_{t_i} - B_{t_{i-1}})^2\right]
\end{aligned}$$

$$
\begin{aligned}
& +2\sum_{1\leqslant i<j\leqslant n}\mathbb{E}\left[F_iF_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})\right]\\
= & \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|F_i|^2(B_{t_i}-B_{t_{i-1}})^2|\mathscr{F}_{t_{i-1}}]]\\
& +2\sum_{1\leqslant i<j\leqslant n}\mathbb{E}[\mathbb{E}[F_iF_j(B_{t_i}-B_{t_{i-1}})(B_{t_j}-B_{t_{j-1}})|\mathscr{F}_{t_{j-1}}]]\\
= & \sum_{i=1}^n \mathbb{E}[|F_i|^2\,\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2|\mathscr{F}_{t_{i-1}}]]\\
& +2\sum_{1\leqslant i<j\leqslant n}\mathbb{E}\Big[F_iF_j(B_{t_i}-B_{t_{i-1}})\underbrace{\mathbb{E}[(B_{t_j}-B_{t_{j-1}})|\mathscr{F}_{t_{j-1}}]}_{=0}\Big]\\
= & \sum_{i=1}^n \mathbb{E}\left[|F_i|^2\,\mathbb{E}\left[(B_{t_i}-B_{t_{i-1}})^2\right]\right]\\
& +2\sum_{1\leqslant i<j\leqslant n}\mathbb{E}[F_iF_j(B_{t_i}-B_{t_{i-1}})\underbrace{\mathbb{E}[B_{t_j}-B_{t_{j-1}}]}_{=0}]\\
= & \sum_{i=1}^n \mathbb{E}[|F_i|^2(t_i-t_{i-1})]\\
= & \ \mathbb{E}\left[\sum_{i=1}^n |F_i|^2(t_i-t_{i-1})\right]\\
= & \ \mathbb{E}\left[\int_0^T |u_t|^2dt\right],
\end{aligned}
$$

where we applied the "tower property" (11.38) of conditional expectations and the facts that $B_{t_i}-B_{t_{i-1}}$ is independent of $\mathscr{F}_{t_{i-1}}$ with

$$
\mathbb{E}[B_{t_i}-B_{t_{i-1}}]=0,\quad \mathbb{E}\left[(B_{t_i}-B_{t_{i-1}})^2\right]=t_i-t_{i-1},\quad i=1,2,\ldots,n.
$$

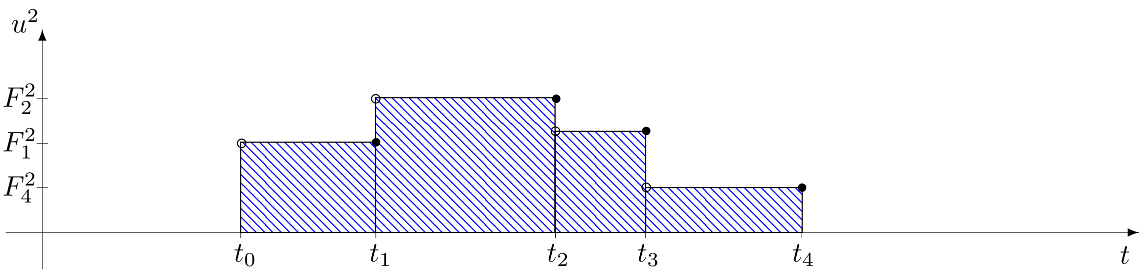


Figure 5.16: Squared simple predictable process $t\mapsto u^2(t)$.

The extension of the stochastic integral to square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$ is obtained by a denseness and Cauchy sequence argument using the isometry (5.16), in the same way as in the proof of Proposition 5.4.

i) By Proposition 5.6 given $u\in L^2(\Omega\times[0,T])$ a square-integrable adapted process there exists a sequence $(u^{(n)})_{n\in\mathbb{N}}$ of simple predictable processes such that

$$
\lim_{n\to\infty}\|u-u^{(n)}\|_{L^2(\Omega\times[0,T])}=\lim_{n\to\infty}\sqrt{\mathbb{E}\left[\int_0^T\left|u_t-u_t^{(n)}\right|^2dt\right]}=0.
$$

ii) Since the sequence $(u^{(n)})_{n\in\mathbb{N}}$ converges it is a Cauchy sequence in $L^2(\Omega\times\mathbb{R}_+)$, hence by the Itô isometry (5.16), the sequence $\left(\int_0^T u_t^{(n)}dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, therefore it admits a limit in the complete space $L^2(\Omega)$. In this case we let

$$\int_0^T u_t dB_t := \lim_{n\to\infty} \int_0^T u_t^{(n)} dB_t$$

and the limit is unique from (5.16) and satisfies (5.16).

iii) The fact that the random variable $\int_0^T u_t dB_t$ is *centered* can be proved first on simple predictable process u of the form (5.13) as

$$\begin{aligned}\mathbb{E}\left[\int_0^T u_t dB_t\right] &= \mathbb{E}\left[\sum_{i=1}^n F_i(B_{t_i} - B_{t_{i-1}})\right] \\ &= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[F_i(B_{t_i} - B_{t_{i-1}}) \mid \mathscr{F}_{t_{i-1}}]] \\ &= \sum_{i=1}^n \mathbb{E}[F_i \mathbb{E}[B_{t_i} - B_{t_{i-1}} \mid \mathscr{F}_{t_{i-1}}]] \\ &= \sum_{i=1}^n \mathbb{E}[F_i \mathbb{E}[B_{t_i} - B_{t_{i-1}}]] \\ &= 0,\end{aligned}$$

and this identity extends as above from simple predictable processes to adapted processes $(u_t)_{t\in\mathbb{R}_+}$ in $L^2(\Omega\times\mathbb{R}_+)$.

□

As an application of the Itô isometry (5.16), we note in particular the identity

$$\mathbb{E}\left[\left(\int_0^T B_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T |B_t|^2 dt\right] = \int_0^T \mathbb{E}\left[|B_t|^2\right] dt = \int_0^T t dt = \frac{T^2}{2},$$

with

$$\int_0^T B_t dB_t \overset{L^2(\Omega)}{=} \lim_{n\to\infty} \sum_{i=1}^n B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

from (5.14).

The next corollary is obtained by bilinearity from the Itô isometry (5.16) by the same argument as in Corollary 5.5.

Corollary 5.8 The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ satisfies the isometry

$$\mathbb{E}\left[\int_0^T u_t dB_t \int_0^T v_t dB_t\right] = \mathbb{E}\left[\int_0^T u_t v_t dt\right],$$

for all square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$, $(v_t)_{t\in\mathbb{R}_+}$.

Proof. Applying the Itô isometry (5.16) to the processes $u+v$ and $u-v$ we have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T u_t dB_t \int_0^T v_t dB_t\right] \\ &= \frac{1}{4}\left(\mathbb{E}\left[\left(\int_0^T u_t dB_t + \int_0^T v_t dB_t\right)^2 - \left(\int_0^T u_t dB_t - \int_0^T v_t dB_t\right)^2\right]\right) \\ &= \frac{1}{4}\left(\mathbb{E}\left[\left(\int_0^T (u_t - v_t) dB_t\right)^2\right] - \mathbb{E}\left[\left(\int_0^T (u_t - v_t) dB_t\right)^2\right]\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\mathbb{E}\left[\int_0^T (u_t+v_t)^2 dt\right] - \mathbb{E}\left[\int_0^T (u_t-v_t)^2 dt\right]\right) \\
&= \frac{1}{4}\mathbb{E}\left[\int_0^T \left((u_t+v_t)^2-(u_t-v_t)^2\right) dt\right] \\
&= \mathbb{E}\left[\int_0^T u_t v_t dt\right].
\end{aligned}
$$

□

In addition, when the integrand $(u_t)_{t\in\mathbb{R}_+}$ is not a deterministic function of time, the random variable $\int_0^T u_t dB_t$ no longer has a Gaussian distribution, except in some exceptional cases.

Definite stochastic integral

The definite stochastic integral of an adapted process $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ over an interval $[a,b] \subset [0,T]$ is defined as

$$\int_a^b u_t dB_t := \int_0^T \mathbb{1}_{[a,b]}(t) u_t dB_t,$$

with in particular

$$\int_a^b dB_t = \int_0^T \mathbb{1}_{[a,b]}(t) dB_t = B_b - B_a, \quad 0 \leqslant a \leqslant b,$$

We also have the Chasles relation

$$\int_a^c u_t dB_t = \int_a^b u_t dB_t + \int_b^c u_t dB_t, \qquad 0 \leqslant a \leqslant b \leqslant c,$$

and the stochastic integral has the following linearity property:

$$\int_0^T (u_t+v_t) dB_t = \int_0^T u_t dB_t + \int_0^T v_t dB_t, \qquad u,v \in L^2(\mathbb{R}_+).$$

5.5 Stochastic Calculus

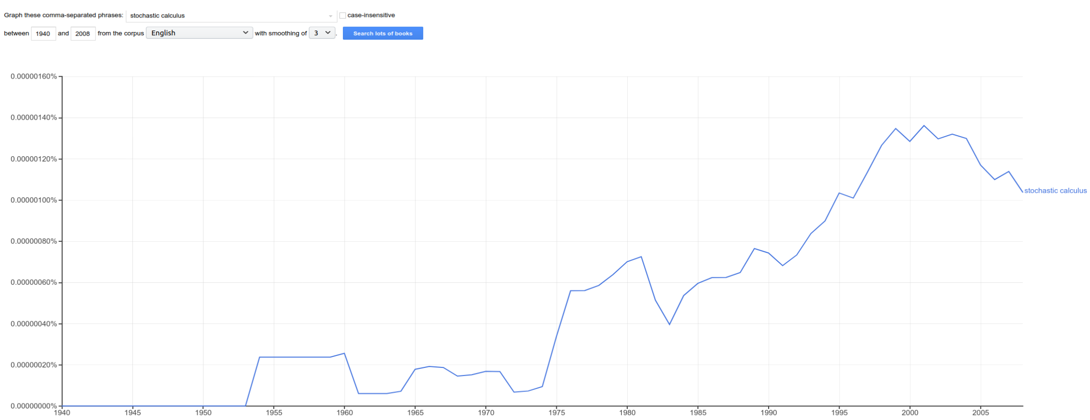


Figure 5.17: NGram Viewer output for the term "stochastic calculus".

Stochastic modeling of asset returns

In the sequel we will define the return at time $t \in \mathbb{R}_+$ of the risky asset $(S_t)_{t\in\mathbb{R}_+}$ as

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad \text{or} \quad \frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{5.18}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. Using the relation

$$X_T = X_0 + \int_0^T dX_t, \qquad T > 0,$$

which holds for any process $(X_t)_{t\in\mathbb{R}_+}$, Equation (5.18) can be rewritten in integral form as

$$S_T = S_0 + \int_0^T dS_t = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t, \tag{5.19}$$

hence the need to define an integral with respect to dB_t, in addition to the usual integral with respect to dt. Note that in view of the definition (5.15), this is a continuous-time extension of the notion portfolio value based on a predictable portfolio strategy.

In Proposition 5.7 we have defined the stochastic integral of square-integrable processes with respect to Brownian motion, thus we have made sense of the equation (5.19) where $(S_t)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in[0,T]}$-adapted process, which can be rewritten in differential notation as in (5.18).

This model will be used to represent the random price S_t of a risky asset at time t. Here the return dS_t/S_t of the asset is made of two components: a constant return μdt and a random return σdB_t parametrized by the coefficient σ, called the volatility.

Our goal is now to solve Equation (5.18) and for this we will need to introduce Itô's calculus in Section 5.5 after a review of classical deterministic calculus.

Deterministic calculus

The *fundamental theorem of calculus* states that for any continuously differentiable (deterministic) function f we have the integral relation

$$f(x) = f(0) + \int_0^x f'(y)dy.$$

In differential notation this relation is written as the first order expansion

$$df(x) = f'(x)dx, \tag{5.20}$$

where dx is "infinitesimally small". Higher-order expansions can be obtained from *Taylor's formula*, which, letting

$$\Delta f(x) := f(x+\Delta x) - f(x),$$

states that

$$\Delta f(x) = f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + \frac{1}{3!}f'''(x)(\Delta x)^3 + \frac{1}{4!}f^{(4)}(x)(\Delta x)^4 + \cdots.$$

Note that Relation (5.20), *i.e.* $df(x) = f'(x)dx$, can be obtained by neglecting all terms of order higher than one in Taylor's formula, since $(\Delta x)^n << \Delta x$, $n \geqslant 2$, as Δx becomes "infinitesimally small".

Stochastic calculus

Let us now apply Taylor's formula to Brownian motion, taking

$$\Delta B_t = B_{t+\Delta t} - B_t \simeq \pm\sqrt{\Delta t},$$

and letting

$$\Delta f(B_t) := f(B_{t+\Delta t}) - f(B_t),$$

we have

$$\Delta f(B_t)$$
$$= f'(B_t)\Delta B_t + \frac{1}{2}f''(B_t)(\Delta B_t)^2 + \frac{1}{3!}f'''(B_t)(\Delta B_t)^3 + \frac{1}{4!}f^{(4)}(B_t)(\Delta B_t)^4 + \cdots.$$

From the construction of Brownian motion by its small increments $\Delta B_t = \pm\sqrt{\Delta t}$, it turns out that the terms in $(\Delta t)^2$ and $\Delta t \Delta B_t \simeq \pm(\Delta t)^{3/2}$ can be neglected in Taylor's formula at the first order of approximation in Δt. However, the term of order two

$$(\Delta B_t)^2 = (\pm\sqrt{\Delta t})^2 = \Delta t$$

can no longer be neglected in front of Δt itself.

Basic Itô formula

For $f \in \mathscr{C}^2(\mathbb{R})$, Taylor's formula written at the second order for Brownian motion reads

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt, \tag{5.21}$$

for "infinitesimally small" dt. Note that writing this formula as

$$\frac{df(B_t)}{dt} = f'(B_t)\frac{dB_t}{dt} + \frac{1}{2}f''(B_t)$$

does not make sense because the pathwise derivative

$$\frac{dB_t}{dt} \simeq \pm\frac{\sqrt{dt}}{dt} \simeq \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty$$

of B_t with respect to t does not exist. Integrating (5.21) on both sides and using the relation

$$f(B_t) - f(B_0) = \int_0^t df(B_s)$$

we get the integral form of Itô's formula for Brownian motion, *i.e.*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Itô processes

We now turn to the general expression of Itô's formula, which is stated for Itô processes.

Definition 5.12 An Itô process is a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ that can be written as

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+, \tag{5.22}$$

or in differential notation

$$dX_t = v_t dt + u_t dB_t,$$

where $(u_t)_{t\in\mathbb{R}_+}$ and $(v_t)_{t\in\mathbb{R}_+}$ are square-integrable adapted processes.

Given $(t,x) \mapsto f(t,x)$ a smooth function of two variables on $\mathbb{R}_+ \times \mathbb{R}$, from now on we let $\frac{\partial f}{\partial t}$ denote partial differentiation with respect to the *first* (time) variable in $f(t,x)$, while $\frac{\partial f}{\partial x}$ denotes partial differentiation with respect to the *second* (price) variable in $f(t,x)$.

Theorem 5.9 (Itô formula for Itô processes). For any Itô process $(X_t)_{t\in\mathbb{R}_+}$ of the form (5.22) and any $f \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ we have

$$
\begin{aligned}
& f(t,X_t) \\
& = f(0,X_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s \\
& \quad + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds. \qquad (5.23)
\end{aligned}
$$

Proof. The proof of the Itô formula can be outlined as follows in the case where $(X_t)_{t\in\mathbb{R}_+} = (B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. We refer to Theorem II-32, page 79 of Protter, 2004 for the general case.

Let $\{0 = t_0^n \leqslant t_1^n \leqslant \cdots \leqslant t_n^n = t\}$, $n \geqslant 1$, be a refining sequence of partitions of $[0,t]$ tending to the identity. We have the telescoping identity

$$
f(B_t) - f(B_0) = \sum_{k=1}^n \big(f(B_{t_i^n}) - f(B_{t_{i-1}^n})\big),
$$

and from Taylor's formula

$$
f(y) - f(x) = (y-x)\frac{\partial f}{\partial x}(x) + \frac{1}{2}(y-x)^2\frac{\partial^2 f}{\partial x^2}(x) + R(x,y),
$$

where the remainder $R(x,y)$ satisfies $R(x,y) \leqslant o(|y-x|^2)$, we get

$$
\begin{aligned}
f(B_t) - f(B_0) &= \sum_{k=1}^n (B_{t_i^n} - B_{t_{i-1}^n})\frac{\partial f}{\partial x}(B_{t_{i-1}^n}) + \frac{1}{2}|B_{t_i^n} - B_{t_{i-1}^n}|^2 \frac{\partial^2 f}{\partial x^2}(B_{t_{i-1}^n}) \\
&\quad + \sum_{k=1}^n R(B_{t_i^n}, B_{t_{i-1}^n}).
\end{aligned}
$$

It remains to show that as n tends to infinity the above converges to

$$
f(B_t) - f(B_0) = \int_0^t \frac{\partial f}{\partial x}(B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(B_s)ds.
$$

□

From the relation

$$
\int_0^t df(s,X_s) = f(t,X_t) - f(0,X_0),
$$

we can rewrite (5.23) as

$$
\begin{aligned}
\int_0^t df(s,X_s) &= \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s \\
&\quad + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds,
\end{aligned}
$$

which allows us to rewrite (5.23) in differential notation, as

$$
\begin{aligned}
& df(t,X_t) && (5.24)\\
&= \frac{\partial f}{\partial t}(t,X_t)dt + v_t\frac{\partial f}{\partial x}(t,X_t)dt + u_t\frac{\partial f}{\partial x}(t,X_t)dB_t + \frac{1}{2}|u_t|^2\frac{\partial^2 f}{\partial x^2}(t,X_t)dt.
\end{aligned}
$$

In case the function $x \mapsto f(x)$ does not depend on the time variable t we get

$$
df(X_t) = u_t\frac{\partial f}{\partial x}(X_t)dB_t + v_t\frac{\partial f}{\partial x}(X_t)dt + \frac{1}{2}|u_t|^2\frac{\partial^2 f}{\partial x^2}(X_t)dt.
$$

Taking $u_t = 1$, $v_t = 0$ and $X_0 = 0$ in (5.22) yields $X_t = B_t$, in which case the Itô formula (5.23) reads

$$
f(t,B_t) = f(0,B_0) + \int_0^t \frac{\partial f}{\partial s}(s,B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds,
$$

i.e. in differential notation:

$$
df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt. \tag{5.25}
$$

Itô multiplication table

Next, consider two Itô processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written in *integral form* as

$$
X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,
$$

and

$$
Y_t = Y_0 + \int_0^t b_s ds + \int_0^t a_s dB_s, \qquad t \in \mathbb{R}_+,
$$

or in *differential notation* as

$$
dX_t = v_t dt + u_t dB_t, \quad \text{and} \quad dY_t = b_t dt + a_t dB_t, \qquad t \in \mathbb{R}_+.
$$

Bivariate Itô formula

The Itô formula can also be written for functions $f \in \mathscr{C}^{1,2,2}(\mathbb{R}_+ \times \mathbb{R}^2)$ we have of two state variables as

$$
\begin{aligned}
df(t,X_t,Y_t) = {} & \frac{\partial f}{\partial t}(t,X_t,Y_t)dt + \frac{\partial f}{\partial x}(t,X_t,Y_t)dX_t + \frac{1}{2}|u_t|^2\frac{\partial^2 f}{\partial x^2}(t,X_t,Y_t)dt \\
& + \frac{\partial f}{\partial y}(t,X_t,Y_t)dY_t + \frac{1}{2}|a_t|^2\frac{\partial^2 f}{\partial y^2}(t,X_t,Y_t)dt + u_t a_t\frac{\partial^2 f}{\partial x \partial y}(t,X_t,Y_t)dt,
\end{aligned} \tag{5.26}
$$

which can be used to show that

$$
d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \bullet dY_t
$$

where the product $dX_t \bullet dY_t$ is computed according to the *Itô rule*

$$
dt \bullet dt = 0, \qquad dt \bullet dB_t = 0, \qquad dB_t \bullet dB_t = dt, \tag{5.27}
$$

which can be encoded in the following Itô multiplication table:

$\cdot$	dt	dB_t
dt	0	0
dB_t	0	dt

Table 5.1: Itô multiplication table.

It follows from the Itô Table 5.1 that

$$\begin{aligned} dX_t \cdot dY_t &= (v_t dt + u_t dB_t) \cdot (b_t dt + a_t dB_t) \\ &= b_t v_t (dt)^2 + b_t u_t dt dB_t + a_t v_t dt dB_t + a_t u_t (dB_t)^2 \\ &= a_t u_t dt. \end{aligned}$$

Hence we also have

$$\begin{aligned} (dX_t)^2 &= (v_t dt + u_t dB_t)^2 \\ &= (v_t)^2 (dt)^2 + (u_t)^2 (dB_t)^2 + 2 u_t v_t (dt \cdot dB_t) \\ &= (u_t)^2 dt, \end{aligned}$$

according to the Itô Table 5.1. Consequently, (5.24) can be rewritten as

$$\boxed{df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2,}$$

and the Itô formula for functions $f \in \mathscr{C}^{1,2,2}(\mathbb{R}_+ \times \mathbb{R}^2)$ of two state variables can be rewritten as

$$\boxed{\begin{aligned} df(t,X_t,Y_t) = \frac{\partial f}{\partial t}(t,X_t,Y_t)dt + \frac{\partial f}{\partial x}(t,X_t,Y_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t,Y_t)(dX_t)^2 \\ + \frac{\partial f}{\partial y}(t,X_t,Y_t)dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t,X_t,Y_t)(dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(t,X_t,Y_t)(dX_t \cdot dY_t). \end{aligned}}$$

Examples

Applying Itô's formula (5.25) to B_t^2 with

$$B_t^2 = f(t,B_t) \quad \text{and} \quad f(t,x) = x^2,$$

and

$$\frac{\partial f}{\partial t}(t,x) = 0, \qquad \frac{\partial f}{\partial x}(t,x) = 2x, \qquad \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 1,$$

we find

$$\begin{aligned} d(B_t^2) &= df(B_t) \\ &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt \\ &= 2B_t dB_t + dt. \end{aligned}$$

Note that from the Itô Table 5.1 we could also write directly

$$d(B_t^2) = B_t dB_t + B_t dB_t + (dB_t)^2 = 2B_t dB_t + dt.$$

Next, by integration in $t \in [0,T]$ we find

$$B_T^2 = B_0 + 2\int_0^T B_s dB_s + \int_0^T dt = 2\int_0^T B_s dB_s + T, \tag{5.28}$$

hence the relation

$$\boxed{\int_0^T B_s dB_s = \frac{1}{2}\left(B_T^2 - T\right).}$$

Similarly, we have

$$\begin{cases} d(B_t^3) = 3B_t^2 dB_t + 3B_t dt, \\ \\ d\big(\sin B_t\big) = \cos(B_t)dB_t - \dfrac{1}{2}\sin(B_t)dt, \\ \\ d\,\mathrm{e}^{B_t} = \mathrm{e}^{B_t}dB_t + \dfrac{1}{2}\mathrm{e}^{B_t}dt, \\ \\ d\log B_t = \dfrac{1}{B_t}dB_t - \dfrac{1}{2B_t^2}dt, \\ \\ d\,\mathrm{e}^{tB_t} = B_t\,\mathrm{e}^{tB_t}dt + \dfrac{t^2}{2}\mathrm{e}^{tB_t}dt + t\,\mathrm{e}^{tB_t}dB_t, \end{cases}$$

etc.

Notation

We close this section with some comments on the practice of Itô's calculus. In certain finance textbooks, Itô's formula for *e.g.* geometric Brownian motion $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

can be found written in the notation

$$\begin{aligned} f(T,S_T) \quad &= \quad f(0,X_0) + \sigma\int_0^T S_t\frac{\partial f}{\partial S_t}(t,S_t)dB_t + \mu\int_0^T S_t\frac{\partial f}{\partial S_t}(t,S_t)dt \\ &\qquad + \int_0^T \frac{\partial f}{\partial t}(t,S_t)dt + \frac{1}{2}\sigma^2\int_0^T S_t^2\frac{\partial^2 f}{\partial S_t^2}(t,S_t)dt, \end{aligned}$$

or

$$df(S_t) = \sigma S_t\frac{\partial f}{\partial S_t}(S_t)dB_t + \mu S_t\frac{\partial f}{\partial S_t}(S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2}(S_t)dt.$$

The notation $\dfrac{\partial f}{\partial S_t}(S_t)$ can in fact be easily misused in combination with the fundamental theorem of classical calculus, and potentially leads to the <u>*wrong*</u> identity

$$\cancel{df(S_t) = \frac{\partial f}{\partial S_t}(S_t)dS_t}.$$

Similarly, writing

$$df(B_t) = \frac{\partial f}{\partial x}(B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t)dt$$

is consistent, while writing

$$df(B_t) = \frac{\partial f(B_t)}{\partial B_t}dB_t + \frac{1}{2}\frac{\partial^2 f(B_t)}{\partial B_t^2}dt$$

is potentially a source of confusion. Note also that the right hand side of the Itô formula uses *partial derivatives* while its left hand side is a *total derivative*.

Stochastic differential equations

In addition to geometric Brownian motion there exists a large family of stochastic differential equations that can be studied, although most of the time they cannot be explicitly solved. Let now

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^n$$

where $\mathbb{R}^d \otimes \mathbb{R}^n$ denotes the space of $d \times n$ matrices, and

$$b : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

satisfy the global Lipschitz condition

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + \|b(t,x) - b(t,y)\|^2 \leqslant K^2 \|x-y\|^2,$$

$t \in \mathbb{R}_+$, $x,y \in \mathbb{R}^n$. Then there exists a unique strong solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s, \quad t \in \mathbb{R}_+, \tag{5.29}$$

i.e., in differential notation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a d-dimensional Brownian motion, see *e.g.* Protter, 2004, Theorem V-7. In addition, the solution process $(X_t)_{t\in\mathbb{R}_+}$ of (5.29) has the *Markov property*, see § V-6 of Protter, 2004.

The term $\sigma(s,X_s)$ in (5.29) will be interpreted later on in Chapter 9 as a *local volatility* component.

Stochastic differential equations can be used to model the behaviour of a variety of quantities, such as

- stock prices,
- interest rates,
- exchange rates,
- weather factors,
- electricity/energy demand,
- commodity (e.g. oil) prices, etc.

Next, we consider several examples of stochastic differential equations that can be solved explicitly using Itô's calculus, in addition to geometric Brownian motion. See *e.g.* § II-4.4 of Kloeden and Platen, 1999 for more examples of explicitly solvable stochastic differential equations.

Examples of stochastic differential equations

1. Consider the stochastic differential equation

$$dX_t = -\alpha X_t dt + \sigma dB_t, \quad X_0 = x_0, \tag{5.30}$$

with $\alpha > 0$ and $\sigma > 0$.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N;alpha=5; sigma=0.4;
Z <- rnorm(N,mean=0,sd=sqrt(dt));X <- rep(0,N);X[1]=0.5
for (j in 2:N){X[j]=X[j-1]-alpha*X[j-1]*dt+sigma*Z[j]}
plot(t, X, xlab = "t", ylab = "", type = "l", ylim = c(-0.5,1), col = "blue")
abline(h=0)
```

We look for a solution of the form

$$X_t = a(t)Y_t = a(t)\left(x_0 + \int_0^t b(s)dB_s\right)$$

where $a(\cdot)$ and $b(\cdot)$ are deterministic functions of time. After applying Theorem 5.9 to the Itô process $x_0 + \int_0^t b(s)dB_s$ of the form (5.22) with $u_t = b(t)$ and $v(t) = 0$, and to the function $f(t,x) = a(t)x$, we find

$$\begin{aligned} dX_t &= d(a(t)Y_t) \\ &= Y_t a'(t)dt + a(t)dY_t \\ &= Y_t a'(t)dt + a(t)b(t)dB_t. \end{aligned} \tag{5.31}$$

By identification of (5.30) with (5.31), we get

$$\begin{cases} a'(t) = -\alpha a(t) \\ \\ a(t)b(t) = \sigma, \end{cases}$$

hence $a(t) = a(0)\,\mathrm{e}^{-\alpha t} = \mathrm{e}^{-\alpha t}$ and $b(t) = \sigma / a(t) = \sigma\,\mathrm{e}^{\alpha t}$, which shows that

$$X_t = x_0\,\mathrm{e}^{-\alpha t} + \sigma \int_0^t \mathrm{e}^{-(t-s)\alpha} dB_s, \qquad t \in \mathbb{R}_+, \tag{5.32}$$

Using integration by parts, we can also write

$$X_t = x_0\,\mathrm{e}^{-\alpha t} + \sigma B_t - \sigma\alpha \int_0^t \mathrm{e}^{-(t-s)\alpha} B_s ds, \qquad t \in \mathbb{R}_+, \tag{5.33}$$

Remark: the solution of the equation (5.30) *cannot* be written as a function $f(t,B_t)$ of t and B_t as in the proof of Proposition 6.8.

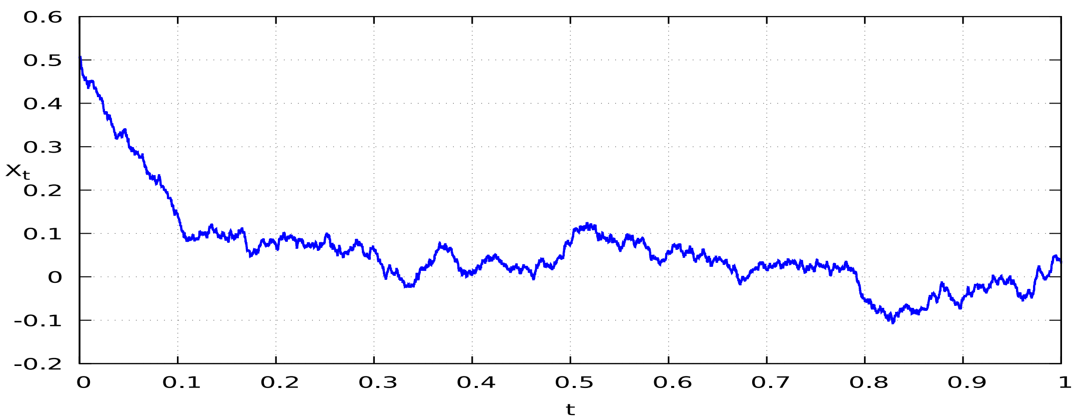


Figure 5.18: Simulated path of (5.30) with $\alpha = 10$ and $\sigma = 0.2$.

2. Consider the stochastic differential equation

$$dX_t = tX_t dt + \mathrm{e}^{t^2/2} dB_t, \qquad X_0 = x_0. \tag{5.34}$$

```
N=10000; T<-2.0; t <- 0:(N-1); dt <- T/N;
Z <- rnorm(N,mean=0,sd= sqrt(dt));X <- rep(0,N);X[1]=0.5
for (j in 2:N){X[j]=X[j-1]+j*X[j-1]*dt*dt+exp(j*dt*j*dt/2)*Z[j]}
plot(t, X, xlab = "t", ylab = "", type = "l", ylim = c(-0.5,10), col = "blue")
abline(h=0)
```

Looking for a solution of the form $X_t = a(t)\left(X_0 + \int_0^t b(s)dB_s\right)$, where $a(\cdot)$ and $b(\cdot)$ are deterministic functions of time, we get $a'(t)/a(t) = t$ and $a(t)b(t) = \mathrm{e}^{t^2/2}$, hence $a(t) = \mathrm{e}^{t^2/2}$ and $b(t) = 1$, which yields $X_t = \mathrm{e}^{t^2/2}(X_0 + B_t)$, $t \in \mathbb{R}_+$.

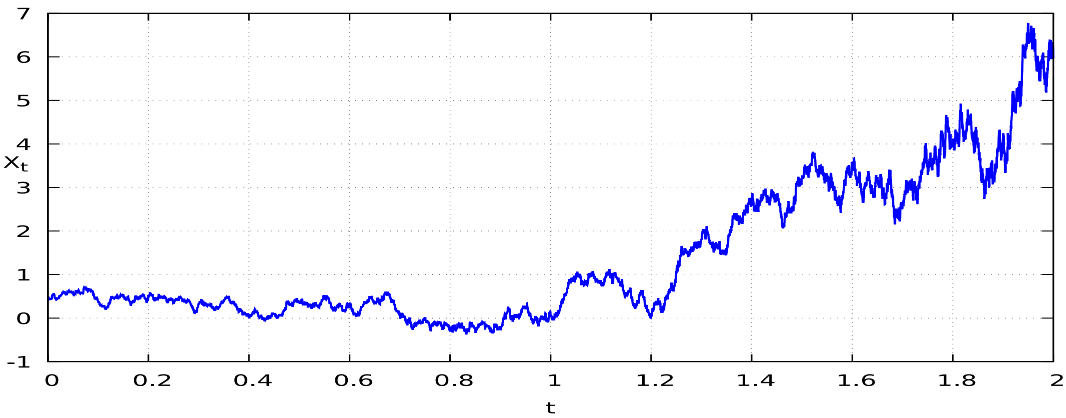


Figure 5.19: Simulated path of (5.34).

3. Consider the stochastic differential equation

$$dY_t = (2\mu Y_t + \sigma^2)dt + 2\sigma\sqrt{Y_t}dB_t, \tag{5.35}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N;mu=-5;sigma=1;
Z <- rnorm(N,mean=0,sd=sqrt(dt));Y <- rep(0,N);Y[1]=0.5
for (j in 2:N){ Y[j]=max(0,Y[j-1]+(2*mu*Y[j-1]+sigma*sigma)*dt+2*sigma*sqrt(Y[j-1])*Z[j])}
plot(t, Y, xlab = "t", ylab = "", type = "l", ylim = c(-0.1,1), col = "blue")
abline(h=0)
```

Letting $X_t = \sqrt{Y_t}$, we find that $dX_t = \mu X_t dt + \sigma dB_t$, hence

$$Y_t = (X_t)^2 = \left(\mathrm{e}^{\mu t}\sqrt{Y_0} + \sigma \int_0^t \mathrm{e}^{\mu(t-s)}dB_s\right)^2.$$

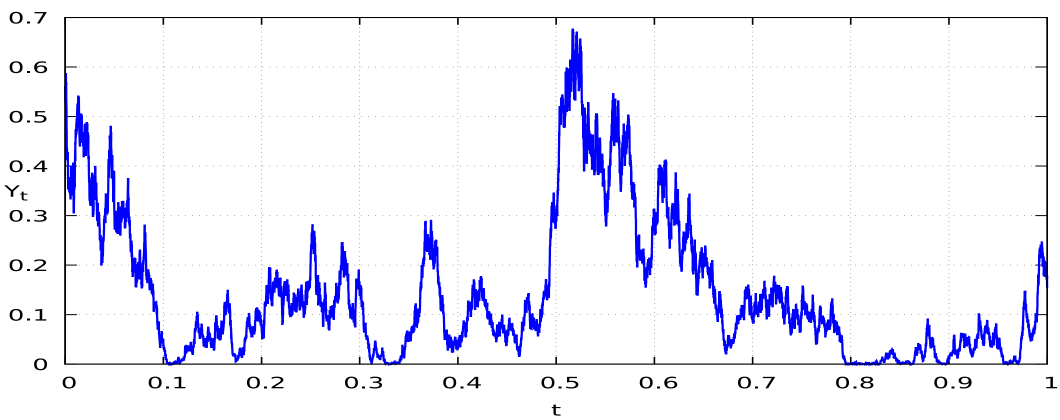


Figure 5.20: Simulated path of (5.35) with $\mu = 5$ and $\sigma = 1$.

Exercises

Exercise 5.1 Compute $\mathbb{E}[B_t B_s]$ in terms of $s,t \in \mathbb{R}_+$.

Exercise 5.2 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion. Let $c > 0$. Among the following processes, tell which is a standard Brownian motion and which is not. Justify your answer.

a) $(X_t)_{t\in\mathbb{R}_+} := \big(B_{c+t} - B_c\big)_{t\in\mathbb{R}_+}$,

b) $(X_t)_{t\in\mathbb{R}_+} := \big(cB_{t/c^2}\big)_{t\in\mathbb{R}_+}$,

c) $(X_t)_{t\in\mathbb{R}_+} := \big(B_{ct^2}\big)_{t\in\mathbb{R}_+}$,

d) $(X_t)_{t\in\mathbb{R}_+} := \big(B_t + B_{t/2}\big)_{t\in\mathbb{R}_+}$.

Exercise 5.3 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion. Compute the stochastic integrals

$$\int_0^T 2dB_t \qquad \text{and} \qquad \int_0^T \big(2\times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)\big)dB_t$$

and determine their probability distributions (including mean and variance).

Exercise 5.4 Determine the probability distribution (including mean and variance) of the stochastic integral $\displaystyle\int_0^{2\pi} \sin(t)\, dB_t$.

Exercise 5.5 Let $T > 0$. Show that for $f : [0,T] \mapsto \mathbb{R}$ a differentiable function such that $f(T) = 0$, we have

$$\int_0^T f(t)dB_t = -\int_0^T f'(t)B_t dt.$$

Hint: Apply Itô's calculus to $t \mapsto f(t)B_t$.

Exercise 5.6

a) Find the probability distribution of the stochastic integral $\int_0^1 t^2 dB_t$ with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

b) Find the probability distribution of the stochastic integral $\int_0^1 t^{-1/2} dB_t$ with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

Exercise 5.7 Given $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $n \geqslant 1$, let the random variable X_n be defined as

$$X_n := \int_0^{2\pi} \sin(nt)dB_t, \qquad n \geqslant 1.$$

a) Give the probability distribution of X_n for all $n \geqslant 1$.

b) Show that $(X_n)_{n\geqslant 1}$ is a sequence of pairwise independent and identically distributed random variables.

Hint: We have $\sin a \sin b = \dfrac{1}{2}\big(\cos(a-b) - \cos(a+b)\big)$, $a,b \in \mathbb{R}$.

Exercise 5.8 Apply the Itô formula to the process $X_t := \sin^2(B_t)$, $t \in \mathbb{R}_+$.

Exercise 5.9 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion.

a) Using the Itô isometry and the known relations

$$B_T = \int_0^T dB_t \quad \text{and} \quad B_T^2 = T + 2\int_0^T B_t dB_t,$$

compute the third and fourth moments $\mathbb{E}[B_T^3]$ and $\mathbb{E}[B_T^4]$.

b) Give the third and fourth moments of the centered normal distribution with variance σ^2.

Exercise 5.10 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{5.36}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with $r \in \mathbb{R}$ and $\sigma > 0$.

a) Find the stochastic differential equation satisfied by the power $(S_t^p)_{t\in\mathbb{R}_+}$ of order $p \in \mathbb{R}$ of $(S_t)_{t\in\mathbb{R}_+}$.
b) Construct a probability measure under which the discounted process $\left(\mathrm{e}^{-rt}S_t^p\right)_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 5.11 Given $T > 0$, find the stochastic integral decomposition of $(B_T)^3$ as

$$(B_T)^3 = C + \int_0^T \zeta_{t,T} dB_t \tag{5.37}$$

where $C \in \mathbb{R}$ is a constant and $(\zeta_{t,T})_{t\in[0,T]}$ is an adapted process to be determined.

Exercise 5.12 Let $f \in L^2([0,T])$, and consider a standard Brownian motion $(B_t)_{t\in[0,T]}$.

a) Compute the conditional expectation

$$\mathbb{E}\left[\mathrm{e}^{\int_0^T f(s)dB_s} \big| \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

where $(\mathscr{F}_t)_{t\in[0,T]}$ denotes the filtration generated by $(B_t)_{t\in[0,T]}$.

b) Using the result of Question (a)), show that the process

$$t \longmapsto \exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right), \qquad t \in [0,T],$$

is an $(\mathscr{F}_t)$-martingale, where $(\mathscr{F}_t)_{t\in[0,T]}$ denotes the filtration generated by $(B_t)_{t\in[0,T]}$.

c) Show that the process $\left(\mathrm{e}^{\sigma B_t - \sigma t^2}\right)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)$-martingale for any $\sigma \in \mathbb{R}$.

Exercise 5.13 Compute the expected value

$$\mathbb{E}\left[\exp\left(\beta\int_0^T B_t dB_t\right)\right]$$

for all $\beta < 1/T$. *Hint*: Expand $(B_T)^2$ using the Itô formula as in (5.28).

Exercise 5.14

a) Solve the stochastic differential equation

$$dX_t = -bX_t dt + \sigma\,\mathrm{e}^{-bt}dB_t, \qquad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\sigma, b \in \mathbb{R}$.

b) Solve the stochastic differential equation

$$dX_t = -bX_t dt + \sigma\,\mathrm{e}^{-at}dB_t, \qquad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $a, b, \sigma \in \mathbb{R}$ are positive constants.

Exercise 5.15 Given $T > 0$, let $(X_t^T)_{t\in[0,T)}$ denote the solution of the stochastic differential equation

$$dX_t^T = \sigma dB_t - \frac{X_t^T}{T-t}dt, \qquad t \in [0,T), \tag{5.38}$$

under the initial condition $X_0^T = 0$ and $\sigma > 0$.

a) Show that

$$X_t^T = (T-t)\sigma \int_0^t \frac{1}{T-s} dB_s, \qquad t \in [0,T).$$

Hint: Start by computing $d(X_t^T/(T-t))$ using the Itô formula.

b) Show that $\mathbb{E}[X_t^T] = 0$ for all $t \in [0,T)$.

c) Show that $\operatorname{Var}[X_t^T] = \sigma^2 t(T-t)/T$ for all $t \in [0,T)$.

d) Show that $\lim_{t\to T} X_t^T = 0$ in $L^2(\Omega)$. The process $(X_t^T)_{t\in[0,T]}$ is called a *Brownian bridge*.

Exercise 5.16 Exponential Vašíček, 1977 model (1). Consider a Vasicek process $(r_t)_{t\in\mathbb{R}_+}$ solution of the stochastic differential equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \qquad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\sigma, a, b > 0$ are positive constants. Show that the exponential $X_t := \mathrm{e}^{r_t}$ satisfies a stochastic differential equation of the form

$$dX_t = X_t(\tilde{a} - \tilde{b} f(X_t))dt + \sigma g(X_t) dB_t,$$

where the coefficients $\tilde{a}$ and $\tilde{b}$ and the functions $f(x)$ and $g(x)$ are to be determined.

Exercise 5.17 Exponential Vasicek model (2). Consider a short-term rate interest rate process $(r_t)_{t\in\mathbb{R}_+}$ in the exponential Vasicek model:

$$dr_t = r_t(\eta - a\log r_t)dt + \sigma r_t dB_t, \tag{5.39}$$

where η, a, σ are positive parameters and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Find the solution $(Z_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dZ_t = -aZ_t dt + \sigma dB_t$$

as a function of the initial condition Z_0, where a and σ are positive parameters.

b) Find the solution $(Y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dY_t = (\theta - aY_t)dt + \sigma dB_t \tag{5.40}$$

as a function of the initial condition Y_0. *Hint*: Let $Z_t := Y_t - \theta/a$.

c) Let $X_t = \mathrm{e}^{Y_t}$, $t \in \mathbb{R}_+$. Determine the stochastic differential equation satisfied by $(X_t)_{t\in\mathbb{R}_+}$.

d) Find the solution $(r_t)_{t\in\mathbb{R}_+}$ of (5.39) in terms of the initial condition r_0.

e) Compute the conditional mean* $\mathbb{E}[r_t|\mathscr{F}_u]$.

f) Compute the conditional variance

$$\operatorname{Var}[r_t|\mathscr{F}_u] := \mathbb{E}[r_t^2|\mathscr{F}_u] - (\mathbb{E}[r_t|\mathscr{F}_u])^2$$

of r_t, $0 \leqslant u \leqslant t$, where $(\mathscr{F}_u)_{u\in\mathbb{R}_+}$ denotes the filtration generated by the Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

*One may use the Gaussian moment generating function $\mathbb{E}[\mathrm{e}^X] = \mathrm{e}^{\alpha^2/2}$ for $X \simeq \mathscr{N}(0,\alpha^2)$.

g) Compute the asymptotic mean and variance $\lim_{t\to\infty} \mathbb{E}[r_t]$ and $\lim_{t\to\infty} \mathrm{Var}[r_t]$.

Exercise 5.18 Cox-Ingersoll-Ross (CIR) model. Consider the equation

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t \tag{5.41}$$

modeling the variations of a short-term interest rate process r_t, where α, β, σ and r_0 are positive parameters and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Write down the equation (5.41) in integral form.

b) Let $u(t) = \mathbb{E}[r_t]$. Show, using the integral form of (5.41), that $u(t)$ satisfies the differential equation

$$u'(t) = \alpha - \beta u(t),$$

and compute $\mathbb{E}[r_t]$ for all $t \in \mathbb{R}_+$.

c) By an application of Itô's formula to r_t^2, show that

$$dr_t^2 = r_t(2\alpha + \sigma^2 - 2\beta r_t)dt + 2\sigma r_t^{3/2}dB_t. \tag{5.42}$$

d) Using the integral form of (5.42), find a differential equation satisfied by $v(t) := \mathbb{E}[r_t^2]$ and compute $\mathbb{E}[r_t^2]$ for all $t \in \mathbb{R}_+$.

e) Show that

$$\mathrm{Var}[r_t] = r_0\frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - \mathrm{e}^{-\beta t}\right)^2, \quad t \in \mathbb{R}_+.$$

6. Continuous-Time Market Model

The continuous-time market model allows for the incorporation of portfolio re-allocation algorithms in a stochastic dynamic programming setting. This chapter starts with a review of the concepts of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also state and solve the equation satisfied by geometric Brownian motion, which will be used for the modeling of continuous asset price processes.

6.1 Asset Price Modeling

The prices at time $t \in \mathbb{R}_+0$ of $d+1$ assets numbered n^o $0,1,\ldots,d$ is denoted by the *random* vector

$$\bar{S}_t = \big(S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}\big)$$

which forms a stochastic process $(\bar{S}_t)_{t\in\mathbb{R}_+}$. As in discrete time, the asset n^o 0 is a riskless asset (of savings account type) yielding an interest rate r, *i.e.* we have

$$S_t^{(0)} = S_0^{(0)} \mathrm{e}^{rt}, \qquad t \in \mathbb{R}_+.$$

Definition 6.1 *Discounting*. Let

$$\overline{X}_t := \left(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\right), \qquad t \in \mathbb{R},$$

denote the vector of discounted asset prices, defined as:

$$\widetilde{S}_t^{(k)} = \mathrm{e}^{-rt} S_t^{(k)}, \qquad t \in \mathbb{R}, \quad k = 0, 1, \ldots, d.$$

We can also write

$$\overline{X}_t := \mathrm{e}^{-rt} \overline{S}_t, \qquad t \in \mathbb{R}_+.$$

The concept of discounting is illustrated in the following figures.

My portfolio S_t grew by $b = 5\%$ this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My portfolio S_t grew by $b = 5\%$ this year.

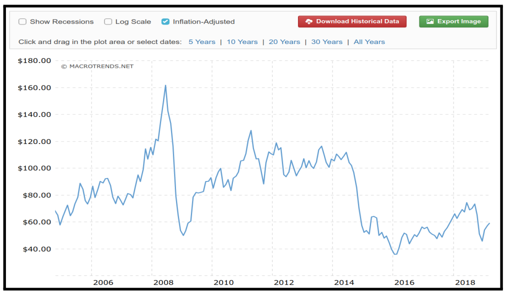
The risk-free or inflation rate is $r = 10\%$.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

(a) *Without* inflation adjustment.



(b) *With* inflation adjustment.

Figure 6.2: Why apply discounting?

Definition 6.2 A portfolio strategy is a stochastic process $(\overline{\xi}_t)_{t \in \mathbb{R}_+} \subset \mathbb{R}^{d+1}$, where $\xi_t^{(k)}$ denotes the (possibly fractional) quantity of asset n^o k held at time $t \in \mathbb{R}_+$.

The *value* at time $t \geqslant 0$ of the portfolio strategy $(\overline{\xi}_t)_{t \in \mathbb{R}_+} \subset \mathbb{R}^{d+1}$ is defined by

$$V_t := \overline{\xi}_t \cdot \overline{S}_t \qquad t \in \mathbb{R}_+.$$

The *discounted* value at time 0 of the portfolio is defined by

$$\widetilde{V}_t := \mathrm{e}^{-rt} V_t, \qquad t \in \mathbb{R}_+.$$

For $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
\widetilde{V}_t &= \mathrm{e}^{-rt} \overline{\xi}_t \cdot \overline{S}_t \\
&= \mathrm{e}^{-rt} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\
&= \sum_{k=0}^{d} \xi_t^{(k)} \widetilde{S}_t^{(k)}
\end{aligned}$$

$$= \overline{\xi}_t \cdot \overline{X}_t, \qquad t \in \mathbb{R}_+.$$

The effect of discounting from time t to time 0 is to divide prices by e^{rt}, making all prices comparable at time 0.

6.2 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the one-step and discrete-time models. In the sequel we will only consider *admissible* portfolio strategies whose total value V_t remains nonnegative for all times $t \in [0, T]$.

Definition 6.3 A portfolio strategy $\left(\xi_t^{(k)}\right)_{t\in[0,T],k=0,1,\ldots,d}$ with value

$$V_t = \overline{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+,$$

constitutes an arbitrage opportunity if all *three* following conditions are satisfied:

i) $V_0 \leqslant 0$ at time $t = 0$, *[start from a zero-cost portfolio or in debt]*
ii) $V_T \geqslant 0$ at time $t = T$, *[finish with a nonnegative amount]*
iii) $\mathbb{P}(V_T > 0) > 0$ at time $t = T$. *[profit made with nonzero probability]*

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure $\mathbb{P}^*$, the return of the risky asset over the time interval $[u, t]$ equals the return of the riskless asset given by

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Recall that the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$\mathscr{F}_t = \sigma(B_u : 0 \leqslant u \leqslant t), \qquad t \in \mathbb{R}_+.$$

Definition 6.4 A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*\left[S_t^{(k)} \mid \mathscr{F}_u\right] = e^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d. \tag{6.1}$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

As in the discrete-time case, $\mathbb{P}^\sharp$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^\sharp\left[S_t^{(k)} \mid \mathscr{F}_u\right] > e^{(t-u)r} S_u, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d,$$

meaning that by taking risks in buying $S_t^{(i)}$, one could make an expected return higher than that of the riskless asset

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\flat$ satisfies

$$\mathbb{E}^\flat\left[S_t^{(k)} \mid \mathscr{F}_u\right] < e^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1, 2, \ldots, d.$$

From the relation

$$S_t^{(0)} = \mathrm{e}^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t,$$

we interpret (6.1) by saying that the expected return of the risky asset $S_t^{(k)}$ under $\mathbb{P}^*$ equals the return of the riskless asset $S_t^{(0)}$, $k = 1,2,\ldots,d$. Recall that the discounted (in \$ at time 0) price $\widetilde{S}_t^{(k)}$ of the risky asset n^o k is defined by

$$\widetilde{S}_t^{(k)} := \mathrm{e}^{-rt} S_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)} / S_0^{(0)}}, \qquad t \in \mathbb{R}_+, \quad k = 0,1,\ldots,d.$$

Definition 6.5 A continuous-time process $(Z_t)_{t\in\mathbb{R}_+}$ of integrable random variables is a martingale under $\mathbb{P}$ and with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[Z_t \mid \mathscr{F}_s] = Z_s, \qquad 0 \leqslant s \leqslant t.$$

Note that when $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale, Z_t is in particular $\mathscr{F}_t$-measurable at all times $t \in \mathbb{R}_+$.

In continuous-time finance, the martingale property can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

As in the discrete-time case, the notion of martingale can be used to characterize risk-neutral probability measures as in the next proposition.

Proposition 6.1 The probability measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted risky asset price process $(\widetilde{S}_t^{(k)})_{t\in\mathbb{R}_+}$ is a martingale *under* $\mathbb{P}^*$, $k = 1,2,\ldots,d$.

Proof. If $\mathbb{P}^*$ is a risk-neutral probability measure, we have

$$\begin{aligned}
\mathbb{E}^*\left[\widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathrm{e}^{-rt} S_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{-rt}\,\mathbb{E}^*\left[S_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{-rt}\,\mathrm{e}^{(t-u)r} S_u^{(k)} \\
&= \mathrm{e}^{-ru} S_u^{(k)} \\
&= \widetilde{S}_u^{(k)}, \qquad 0 \leqslant u \leqslant t,
\end{aligned}$$

hence $\left(\widetilde{S}_t^{(k)}\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$, $k = 1,2,\ldots,d$. Conversely, if $\left(\widetilde{S}_t^{(k)}\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$ then

$$\begin{aligned}
\mathbb{E}^*\left[S_t^{(k)} \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathrm{e}^{rt} \widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{rt}\,\mathbb{E}^*\left[\widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{rt}\,\widetilde{S}_u^{(k)} \\
&= \mathrm{e}^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1,2,\ldots,d,
\end{aligned}$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral according to Definition 6.4. □

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 6.6 A probability measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{P}^*(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}. \tag{6.2}$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 6.2 A market is *without* arbitrage opportunity if and only if it admits at least one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. See Harrison and Pliska, 1981 and Chapter VII-4a of Shiryaev, 1999. □

6.3 Self-Financing Portfolio Strategies

Let $\xi_t^{(i)}$ denote the (possibly fractional) quantity invested at time t over the time interval $[t, t+dt)$, in the asset $S_t^{(k)}$, $k = 0, 1, \ldots, d$, and let

$$\bar{\xi}_t = \left(\xi_t^{(k)}\right)_{k=0,1,\ldots,d}, \qquad \bar{S}_t = \left(S_t^{(k)}\right)_{k=0,1,\ldots,d}, \quad t \in \mathbb{R}_+,$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+. \tag{6.3}$$

Our description of portfolio strategies proceeds in four equivalent formulations (6.4), (6.5) (6.7) and (6.8), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\bar{\xi}_t)_{t \in \mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from $\bar{\xi}_t$ to $\bar{\xi}_{t+dt}$, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t+dt}^{(k)} = \sum_{k=0}^{d} \xi_{t+dt}^{(k)} S_{t+dt}^{(k)} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}, \tag{6.4}$$

which is the continuous-time analog of the self-financing condition already encountered in the discrete setting of Chapter 3, see Definition 3.2. A major difference with the discrete-time case of Definition 3.2, however, is that the continuous-time differentials dS_t and $d\xi_t$ do not make pathwise sense as continuous-time stochastic integrals are defined by L^2 limits, cf. Proposition 5.7, or by convergence in probability.

Portfolio value	$\bar{\xi}_t \cdot \bar{S}_t$ $\longrightarrow$	$\bar{\xi}_t \cdot \bar{S}_{t+dt}$	$=$	$\bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}$ $\longrightarrow$	$\bar{\xi}_{t+dt} \cdot \bar{S}_{t+2dt}$
Asset value	S_t	S_{t+dt}		S_{t+dt}	S_{t+2dt}
Time scale	t	$t+dt$		$t+dt$	$t+2dt$
Portfolio allocation	ξ_t	ξ_t		ξ_{t+dt}	ξ_{t+dt}

Figure 6.3: Illustration of the self-financing condition (6.4).

Equivalently, Condition (6.4) can be rewritten as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} d\xi_t^{(k)} = 0, \tag{6.5}$$

where

$$d\xi_t^{(k)} := \xi_{t+dt}^{(k)} - \xi_t^{(k)}, \qquad k = 0, 1, \ldots, d,$$

denote the respective changes in portfolio allocations. In other words, (6.5) rewrites as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} \big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) = 0. \tag{6.6}$$

Condition (6.6) can be rewritten as

$$\sum_{k=0}^{d} S_t^{(k)} \big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) + \sum_{k=0}^{d} \big(S_{t+dt}^{(k)} - S_t^{(k)}\big) \big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) = 0,$$

which shows that (6.4) and (6.5) are equivalent to

$$\bar{S}_t \cdot d\bar{\xi}_t + d\bar{S}_t \cdot d\bar{\xi}_t = \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)} = 0 \tag{6.7}$$

in differential notation.

Self-financing portfolio differential

In practice, the self-financing portfolio property will be characterized by the following proposition, which states that the value of a self-financing portfolio can be written as the sum of its trading Profits and Losses (P/L).

Proposition 6.3 A portfolio strategy $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ with value

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+,$$

is self-financing according to (6.4) if and only if the relation

$$dV_t = \sum_{k=0}^{d} \underbrace{\xi_t^{(k)} dS_t^{(k)}}_{\text{P/L for asset } n^o\ k} \tag{6.8}$$

holds.

Proof. By Itô's calculus we have

$$dV_t = \sum_{k=0}^{d} \xi_t^{(k)} dS_t^{(k)} + \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)},$$

which shows that (6.7) is equivalent to (6.8). □

Market Completeness

Definition 6.7 A contingent claim with payoff C is said to be attainable if there exists a (self-financing) portfolio strategy $\left(\xi_t^{(k)}\right)_{t\in[0,T],k=0,1,\ldots,d}$ such that at the maturity time T the equality

$$V_T = \bar{\xi}_T \cdot \bar{S}_T = \sum_{k=0}^{d} \xi_T^{(k)} S_T^{(k)} = C$$

holds (almost surely) between random variables.

When a claim with payoff C is attainable, its price at time t will be given by the value V_t of a self-financing portfolio hedging C.

Definition 6.8 A market model is said to be *complete* if every contingent claim is attainable.

The next result is the continuous-time statement of the second fundamental theorem of asset pricing.

Theorem 6.4 A market model without arbitrage opportunities is complete if and only if it admits only one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. See Harrison and Pliska, 1981 and Chapter VII-4a of Shiryaev, 1999. □

In the Black and Scholes, 1973 model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete. From now on, we work with $d = 1$, *i.e.* with a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$.

The riskless asset price process $(A_t)_{t\in\mathbb{R}_+}$ admits the following equivalent constructions:

$$\frac{A_{t+dt} - A_t}{A_t} = rdt, \qquad \frac{dA_t}{A_t} = rdt, \qquad A_t' = rA_t, \quad t \in \mathbb{R}_+,$$

with the solution

$$A_t = A_0 \,\mathrm{e}^{rt}, \qquad t \in \mathbb{R}_+, \tag{6.9}$$

where $r > 0$ is the risk-free interest rate.*

Self-financing portfolio strategies

Let ξ_t and η_t denote the (possibly fractional) quantities invested at time t over the time interval $[t, t+dt)$, respectively in the assets S_t and A_t, and let

$$\bar{\xi}_t = (\eta_t, \xi_t), \qquad \bar{S}_t = (A_t, S_t), \qquad t \in \mathbb{R}_+,$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

Our description of portfolio strategies proceeds in four equivalent formulations presented below in Equations (6.10), (6.11), (6.13) and (6.14), which correspond to different interpretations of the self-financing condition.

*"Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", Kenneth E. Boulding, Boulding, 1973, page 248.

Self-financing portfolio update

The portfolio strategy $(\eta_t, \xi_t)_{t \in \mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}. \tag{6.10}$$

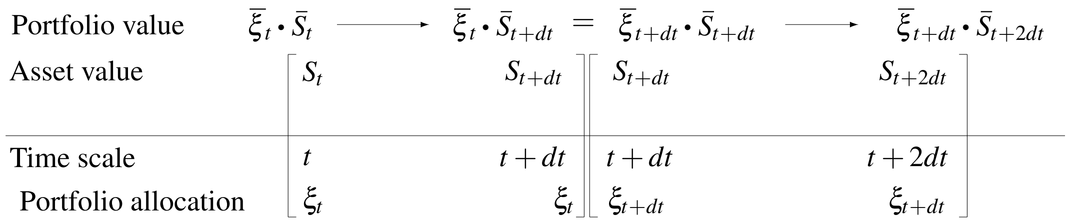


Figure 6.4: Illustration of the self-financing condition (6.10).

Equivalently, Condition (6.10) can be rewritten as

$$A_{t+dt} d\eta_t + S_{t+dt} d\xi_t = 0, \tag{6.11}$$

where

$$d\eta_t := \eta_{t+dt} - \eta_t \quad \text{and} \quad d\xi_t := \xi_{t+dt} - \xi_t$$

denote the respective changes in portfolio allocations. In other words, we have

$$A_{t+dt}(\eta_t - \eta_{t+dt}) = S_{t+dt}(\xi_{t+dt} - \xi_t). \tag{6.12}$$

In other words, when one sells a (possibly fractional) quantity $\eta_t - \eta_{t+dt} > 0$ of the riskless asset valued A_{t+dt} at the end of the time interval $[t, t+dt]$ for the total amount $A_{t+dt}(\eta_t - \eta_{t+dt})$, one should entirely spend this income to buy the corresponding quantity $\xi_{t+dt} - \xi_t > 0$ of the risky asset for the same amount $S_{t+dt}(\xi_{t+dt} - \xi_t) > 0$.

Similarly, if one sells a quantity $-d\xi_t > 0$ of the risky asset S_{t+dt} between the time intervals $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $-S_{t+dt} d\xi_t$, one should entirely use this income to buy a quantity $d\eta_t > 0$ of the riskless asset for an amount $A_{t+dt} d\eta_t > 0$, *i.e.*

$$A_{t+dt} d\eta_t = -S_{t+dt} d\xi_t.$$

Condition (6.12) can also be rewritten as

$$\begin{aligned} S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) \\ + (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = 0, \end{aligned}$$

which shows that (6.10) and (6.11) are equivalent to

$$S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t = 0 \tag{6.13}$$

in differential notation, with

$$dA_t \cdot d\eta_t \simeq (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = rA_t(dt \cdot d\eta_t) = 0$$

in the sense of the Itô calculus by the Itô Table 5.1. This yields the following proposition, which is also consequence of Proposition 6.3.

Proposition 6.5 A portfolio allocation $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+,$$

is self-financing according to (6.10) if and only if the relation

$$dV_t = \underbrace{\eta_t dA_t}_{\text{risk-free P/L}} + \underbrace{\xi_t dS_t}_{\text{risky P/L}} \tag{6.14}$$

holds.

Proof. By Itô's calculus we have

$$dV_t = [\eta_t dA_t + \xi_t dS_t] + [S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t],$$

which shows that (6.13) is equivalent to (6.14). □

Let

$$\widetilde{V}_t := \mathrm{e}^{-rt} V_t \qquad \text{and} \qquad \widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

respectively denote the discounted portfolio value and discounted risky asset price at time $t \geqslant 0$.

Geometric Brownian motion

The risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ will be modeled using a geometric Brownian motion defined from the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \tag{6.15}$$

see Section 6.4.

```r
N=2000; t <- 0:N; dt <- 1.0/N; mu=0.5;sigma=0.2; nsim <- 10
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
for (i in 1:nsim){X[i,] <- exp(mu*t*dt+sigma*X[i,]-sigma*sigma*t*dt/2)}
plot(t*dt, rep(0, N+1), xlab = "time", ylab = "Geometric Brownian motion", lwd=2, ylim =
      c(min(X),max(X)), type = "l", col = 0)
for (i in 1:nsim){lines(t*dt, X[i, ], lwd=2, type = "l", col = i)}
```

By Proposition 6.8 below, we have

$$S_t \quad = \quad S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \in \mathbb{R}_+.$$

The next Figure 6.5 presents sample paths of geometric Brownian motion.

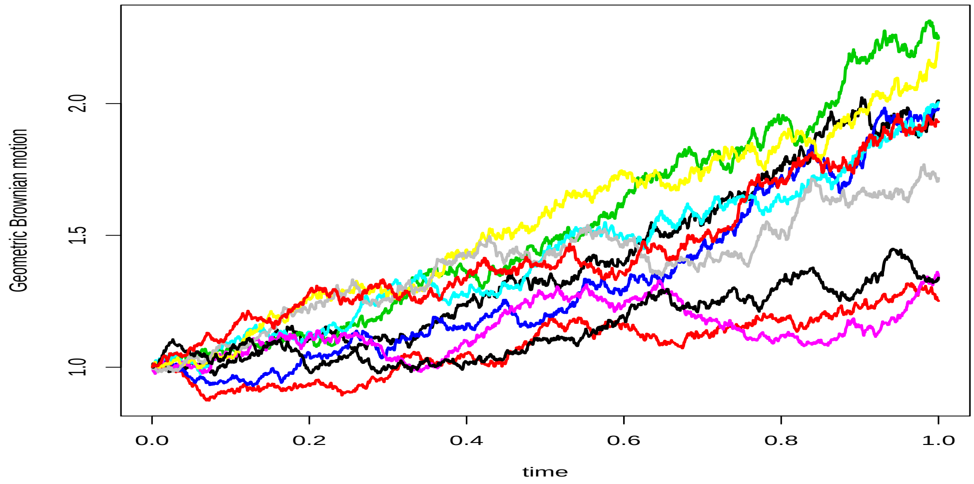


Figure 6.5: Sample paths of geometric Brownian motion.

Lemma 6.6 *Discounting lemma*. Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ be as in (6.15), *i.e.*

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+.$$

Then the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.$$

Proof. We have

$$\begin{aligned}
d\widetilde{S}_t &= d(\mathrm{e}^{-rt} S_t) \\
&= S_t d(\mathrm{e}^{-rt}) + \mathrm{e}^{-rt} dS_t + (d\,\mathrm{e}^{-rt}) \bullet dS_t \\
&= -r\mathrm{e}^{-rt} S_t dt + \mathrm{e}^{-rt} dS_t + (-r\mathrm{e}^{-rt} S_t dt) \bullet dS_t \\
&= -r\mathrm{e}^{-rt} S_t dt + \mu\,\mathrm{e}^{-rt} S_t dt + \sigma\,\mathrm{e}^{-rt} S_t dB_t \\
&= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.
\end{aligned}$$

□

In the next Lemma 6.7, which is the continuous-time analog of Lemma 4.1, we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted trading profits and losses (number of risky assets ξ_t times discounted price variation $d\widetilde{S}_t$).

Note that in Equation (6.16) below, no profit or loss arises from trading the discounted riskless asset $\widetilde{A}_t := \mathrm{e}^{-rt} A_t = A_0$, because its price remains constant over time.

Lemma 6.7 Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

The following statements are equivalent:

(i) the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic integral sum

$$\widetilde{V}_t = \widetilde{V}_0 + \underbrace{\int_0^t \xi_u d\widetilde{S}_u}_{\text{discounted P/L}}, \qquad t \in \mathbb{R}_+, \tag{6.16}$$

of discounted profits and losses.

Proof. Assuming that (i) holds, the self-financing condition and (6.9)-(6.15) show that

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

where we used $V_t = \eta_t A_t + \xi_t S_t$, hence

$$\mathrm{e}^{-rt} dV_t = r\,\mathrm{e}^{-rt} V_t dt + (\mu - r)\,\mathrm{e}^{-rt} \xi_t S_t dt + \sigma\,\mathrm{e}^{-rt} \xi_t S_t dB_t, \quad t \in \mathbb{R}_+,$$

and

$$\begin{aligned} d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt} V_t\right) \\ &= -r\,\mathrm{e}^{-rt} V_t dt + \mathrm{e}^{-rt} dV_t \\ &= (\mu - r)\xi_t\,\mathrm{e}^{-rt} S_t dt + \sigma \xi_t\,\mathrm{e}^{-rt} S_t dB_t \\ &= (\mu - r)\xi_t \widetilde{S}_t dt + \sigma \xi_t \widetilde{S}_t dB_t \\ &= \xi_t d\widetilde{S}_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

i.e. (6.16) holds by integrating on both sides as

$$\widetilde{V}_t - \widetilde{V}_0 = \int_0^t d\widetilde{V}_u = \int_0^t \xi_u d\widetilde{S}_u, \qquad t \in \mathbb{R}_+.$$

(ii) Conversely, if (6.16) is satisfied we have

$$\begin{aligned} dV_t &= d(\mathrm{e}^{rt} \widetilde{V}_t) \\ &= r\,\mathrm{e}^{rt} \widetilde{V}_t dt + \mathrm{e}^{rt} d\widetilde{V}_t \\ &= r\,\mathrm{e}^{rt} \widetilde{V}_t dt + \mathrm{e}^{rt} \xi_t d\widetilde{S}_t \\ &= rV_t dt + \mathrm{e}^{rt} \xi_t d\widetilde{S}_t \\ &= rV_t dt + \mathrm{e}^{rt} \xi_t \widetilde{S}_t ((\mu - r)dt + \sigma dB_t) \\ &= rV_t dt + \xi_t S_t ((\mu - r)dt + \sigma dB_t) \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= \eta_t dA_t + \xi_t dS_t, \end{aligned}$$

hence the portfolio is self-financing according to Definition 6.3. □

As a consequence of Relation (6.16), the problem of hedging a claim payoff C with maturity T also reduces to that of finding the process $(\xi_t)_{t\in[0,T]}$ appearing in the decomposition of the discounted claim payoff $\widetilde{C} = \mathrm{e}^{-rT} C$ as a stochastic integral:

$$\widetilde{C} = \widetilde{V}_T = \widetilde{V}_0 + \int_0^T \xi_t d\widetilde{S}_t,$$

see Section 8.5 on hedging by the martingale method.

Example. *Power options in the Bachelier model.*

In the Bachelier, 1900 model, the underlying asset price can be modeled by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, and may therefore become negative.* The claim payoff $C = (B_T)^2$ of a power option with at maturity $T > 0$ admits the stochastic integral decomposition

$$(B_T)^2 = T + 2\int_0^T B_t dB_t$$

which shows that the claim can be hedged using $\xi_t = 2B_t$ units of the underlying asset at time $t \in [0,T]$, see Exercise 7.1.

Similarly, in the case of power claim payoff $C = (B_T)^3$ we have

$$(B_T)^3 = 3\int_0^T \big(T - t + (B_t)^2\big) dB_t,$$

cf. Exercise 5.11.

Note that according to (6.16), the (non-discounted) self-financing portfolio value V_t can be written as

$$V_t = \mathrm{e}^{rt} V_0 + (\mu - r)\int_0^t \mathrm{e}^{(t-u)r}\xi_u S_u du + \sigma\int_0^t \mathrm{e}^{(t-u)r}\xi_u S_u dB_u, \quad t \in \mathbb{R}_+. \tag{6.17}$$

6.4 Geometric Brownian Motion

In this section we solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

which is used to model the S_t the risky asset price at time t, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in *integral form* as

$$S_t = S_0 + \mu\int_0^t S_s ds + \sigma\int_0^t S_s dB_s, \qquad t \in \mathbb{R}_+. \tag{6.18}$$

It can be solved by applying Itô's formula to the Itô process $(S_t)_{t\in\mathbb{R}_+}$ with $v_t = \mu S_t$ and $u_t = \sigma S_t$, and by taking

$$f(S_t) = \log S_t, \qquad \text{with} \qquad f(x) := \log x.$$

This yields the log-return dynamics

$$\begin{aligned} d\log S_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)dS_t \bullet dS_t \\ &= \mu S_t f'(S_t)dt + \sigma S_t f'(S_t)dB_t + \frac{\sigma^2}{2}S_t^2 f''(S_t)dt \\ &= \mu dt + \sigma dB_t - \frac{\sigma^2}{2}dt, \end{aligned}$$

hence

$$\begin{aligned} \log S_t - \log S_0 &= \int_0^t d\log S_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right)\int_0^t ds + \sigma\int_0^t dB_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t, \end{aligned}$$

*Negative oil prices have been observed in May 2020 when the prices of oil futures contracts fell below zero.

and

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right), \qquad t \in \mathbb{R}_+.$$

The next Figure 6.6 presents an illustration of the geometric Brownian process of Proposition 6.8.

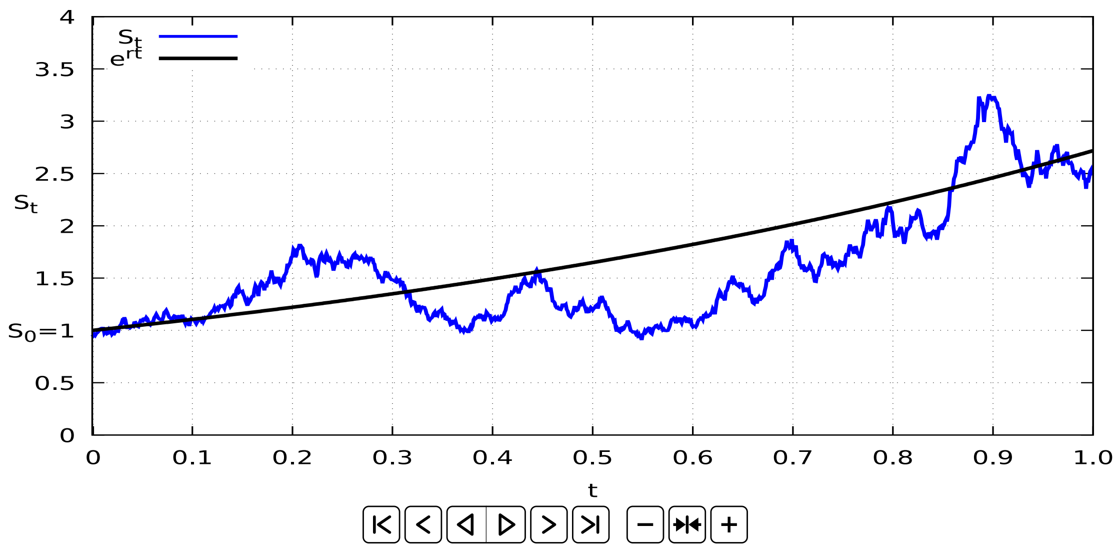


Figure 6.6: Geometric Brownian motion started at $S_0 = 1$, with $\mu = r = 1$ and $\sigma^2 = 0.5$.*

```
N=1000; t <- 0:N; dt <- 1.0/N; sigma=0.6; mu=0.001
Z <- rnorm(N,mean=0,sd=sqrt(dt));
plot(t*dt, exp(mu*t), xlab = "time", ylab = "Geometric Brownian motion", type = "l", ylim = c(0, 4),
    col = 1,lwd=3)
lines(t*dt, exp(sigma*c(0,cumsum(Z))+mu*t-sigma*sigma*t*dt/2),xlab = "time",type = "l",ylim = c(0,
    4), col = 4)
```

The above calculation provides a proof for the next proposition.

Proposition 6.8 The solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{6.19}$$

is given by

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right), \qquad t \in \mathbb{R}_+.$$

Proof. Let us provide an alternative proof by searching for a solution of the form

$$S_t = f(t, B_t)$$

where $f(t,x)$ is a function to be determined. By Itô's formula (5.25) we have

$$dS_t = df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt.$$

*The animation works in Acrobat Reader on the entire pdf file.

Comparing this expression to (6.19) and identifying the terms in dB_t we get

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,B_t) = \sigma S_t, \\ \\ \dfrac{\partial f}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,B_t) = \mu S_t. \end{cases}$$

Using the relation $S_t = f(t,B_t)$, these two equations rewrite as

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,B_t) = \sigma f(t,B_t), \\ \\ \dfrac{\partial f}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,B_t) = \mu f(t,B_t). \end{cases}$$

Since B_t is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,x) = \sigma f(t,x), & \text{(6.22a)} \\ \\ \dfrac{\partial f}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,x) = \mu f(t,x). & \text{(6.22b)} \end{cases}$$

To find the solution $f(t,x) = f(t,0)\,\mathrm{e}^{\sigma x}$ of (6.22a) we let $g(t,x) = \log f(t,x)$ and rewrite (6.22a) as

$$\frac{\partial g}{\partial x}(t,x) = \frac{\partial}{\partial x}\log f(t,x) = \frac{1}{f(t,x)}\frac{\partial f}{\partial x}(t,x) = \sigma,$$

i.e.

$$\frac{\partial g}{\partial x}(t,x) = \sigma,$$

which is solved as

$$g(t,x) = g(t,0) + \sigma x,$$

hence

$$f(t,x) = \mathrm{e}^{g(t,0)}\,\mathrm{e}^{\sigma x} = f(t,0)\,\mathrm{e}^{\sigma x}.$$

Plugging back this expression into the second equation (6.22b) yields

$$\mathrm{e}^{\sigma x}\frac{\partial f}{\partial t}(t,0) + \frac{1}{2}\sigma^2\,\mathrm{e}^{\sigma x} f(t,0) = \mu f(t,0)\,\mathrm{e}^{\sigma x},$$

i.e.

$$\frac{\partial f}{\partial t}(t,0) = \left(\mu - \frac{\sigma^2}{2}\right) f(t,0).$$

In other words, we have $\dfrac{\partial g}{\partial t}(t,0) = \mu - \sigma^2/2$, which yields

$$g(t,0) = g(0,0) + \left(\mu - \frac{\sigma^2}{2}\right) t,$$

i.e.

$$\begin{aligned} f(t,x) &= e^{g(t,x)} = e^{g(t,0)+\sigma x} \\ &= e^{g(0,0)+\sigma x+\left(\mu-\sigma^2/2\right)t} \\ &= f(0,0)\, e^{\sigma x+\left(\mu-\sigma^2/2\right)t}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

We conclude that

$$S_t = f(t,B_t) = f(0,0)\, e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t},$$

and the solution to (6.19) is given by

$$S_t = S_0\, e^{\sigma B_t+(\mu-\sigma^2/2)t}, \qquad t \in \mathbb{R}_+.$$

□

Conversely, taking $S_t = f(t,B_t)$ with $f(t,x) = S_0\, e^{\mu t+\sigma x-\sigma^2 t/2}$ we may apply Itô's formula to check that

$$\begin{aligned} dS_t &= df(t,B_t) \\ &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt \\ &= \left(\mu-\sigma^2/2\right) S_0\, e^{\mu t+\sigma B_t-\sigma^2 t/2}dt + \sigma S_0\, e^{\mu t+\sigma B_t-\sigma^2 t/2}dB_t \\ &\quad +\frac{1}{2}\sigma^2 S_0\, e^{\mu t+\sigma B_t-\sigma^2 t/2}dt \\ &= \mu S_0\, e^{\mu t+\sigma B_t-\sigma^2 t/2}dt + \sigma S_0\, e^{\mu t+\sigma B_t-\sigma^2 t/2}dB_t \\ &= \mu S_t dt + \sigma S_t dB_t. \end{aligned}$$

Exercises

Exercise 6.1 Show that at any time $T > 0$, the random variable $S_T := S_0\, e^{\sigma B_T+(\mu-\sigma^2/2)T}$ has the *lognormal distribution* with probability density function

$$\boxed{x \longmapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}}\, e^{-(-(\mu-\sigma^2/2)T+\log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,}$$

with log-variance σ^2 and log-mean $(\mu-\sigma^2/2)T+\log S_0$, see Figures 4.9 and 6.7.

```
N=1000; t <- 0:N; dt <- 1.0/N; nsim <- 100 # using Bernoulli samples
sigma=0.2;r=0.5;a=(1+r*dt)*(1-sigma*sqrt(dt))-1;b=(1+r*dt)*(1+sigma*sqrt(dt))-1
X <- matrix(a+(b-a)*rbinom( nsim * N, 1, 0.5), nsim, N)
X <- cbind(rep(0, nsim), t(apply((1+X), 1, cumprod)))
X[,1]=1;H<-hist(X[,N]);layout(matrix(c(1,2), nrow =1, byrow = TRUE))
par(mar=c(2,2,2,0), oma = c(2, 2, 2, 2))
plot(t, X[1, ], xlab = "time", ylab = "", type = "l", ylim=c(0.8,3), col = 0)
for (i in 1:nsim){lines(t, X[i, ], xlab = "time", type = "l", col = i)}
lines((1+r*dt)^t, type="l", lty=1, col="black",lwd=3,xlab="",ylab="", main="")
for (i in 1:nsim){points(N, X[i,N], pch=1, lwd = 5, col = i)}
x <- seq(0.01,3, length=100)
px <- exp(-(-(r-sigma^2/2)+log(x))^2/2/sigma^2)/x/sigma/sqrt(2*pi)
par(mar = c(2,2,2,2))
plot(NULL , xlab="", ylab="", xlim = c(0, max(px,H$density)),ylim=c(0.8,3),axes=F)
rect(0, H$breaks[1:(length(H$breaks) - 1)], col=rainbow(20,start=0.08,end=0.6), H$density,
     H$breaks[2:length(H$breaks)])
lines(px,x, type="l", lty=1, col="black",lwd=2,xlab="",ylab="", main="")
```

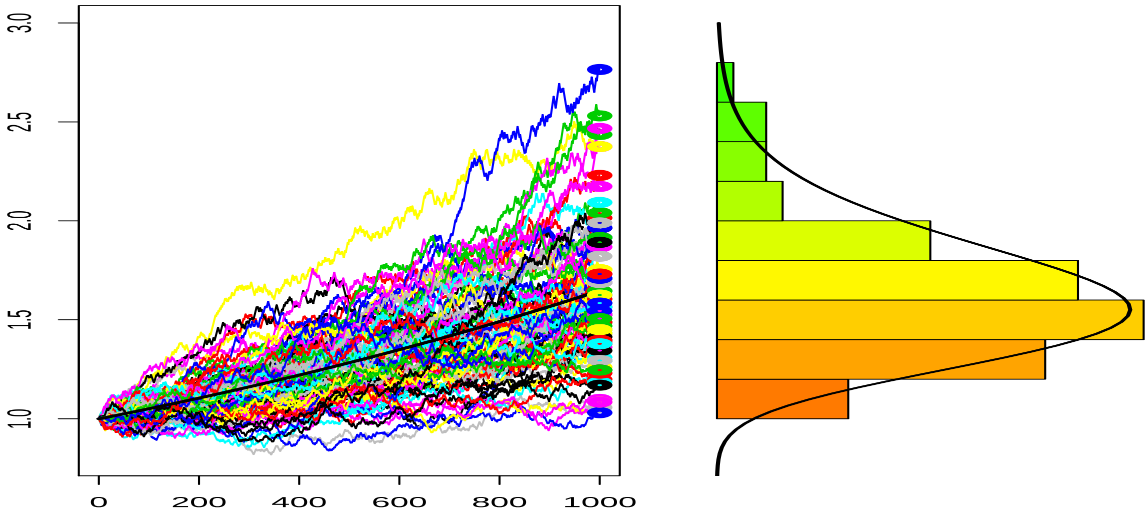


Figure 6.7: Statistics of geometric Brownian paths *vs* lognormal distribution.

Exercise 6.2

a) Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+, \tag{6.23}$$

where $r, \sigma \in \mathbb{R}$ are constants and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Compute $d \log S_t$ using the Itô formula.

b) Solve the ordinary differential equation $df(t) = cf(t)dt$ and the stochastic differential equation (6.23).

c) Compute the lognormal mean and variance

$$\mathbb{E}[S_t] = S_0 \, \mathrm{e}^{rt} \quad \text{and} \quad \mathrm{Var}[S_t] = S_0^2 \, \mathrm{e}^{2rt} (\mathrm{e}^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+,$$

using the Gaussian moment generating function (MGF) formula (11.46).

d) Recover the lognormal mean and variance of Question (c)) using stochatic calculus.

Exercise 6.3 Assume that $(B_t)_{t\in\mathbb{R}_+}$ and $(W_t)_{t\in\mathbb{R}_+}$ are standard Brownian motions, correlated according to the Itô rule $dW_t \bullet dB_t = \rho dt$ for $\rho \in [-1,2]$, and consider the solution $(Y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation $dY_t = \mu Y_t dt + \eta Y_t dW_t$, $t \in \mathbb{R}_+$, where $\mu, \eta \in \mathbb{R}$ are constants. Compute $df(S_t, Y_t)$, for f a $\mathscr{C}^2$ function on $\mathbb{R}^2$ using the bivariate Itô formula (5.26).

Exercise 6.4 Consider the asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Find the stochastic integral decomposition of the random variable S_T, *i.e.*, find the *constant* $C(S_0, r, T) \in \mathbb{R}$ and the process $(\zeta_{t,T})_{t\in[0,T]}$ such that

$$S_T = C(S_0, r, T) + \int_0^T \zeta_{t,T} dB_t. \tag{6.24}$$

Exercise 6.5 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ and the process $(S_t)_{t\in\mathbb{R}_+}$ defined by

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_s + \int_0^t u_s ds \right), \qquad t \in \mathbb{R}_+,$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ and $(u_t)_{t\in\mathbb{R}_+}$ are $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes.

a) Compute dS_t using Itô calculus.
b) Show that S_t satisfies a stochastic differential equation to be determined.

Exercise 6.6 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, and let $\sigma > 0$.
a) Compute the mean and variance of the random variable S_t defined as

$$S_t := 1 + \sigma \int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s, \qquad t \in \mathbb{R}_+. \tag{6.25}$$

b) Express $d\log(S_t)$ using (6.25) and the Itô formula.
c) Show that $S_t = e^{\sigma B_t - \sigma^2 t/2}$ for $t \in \mathbb{R}_+$.

Exercise 6.7 We consider a leveraged fund with factor $\beta : 1$ on an index $(S_t)_{t\in\mathbb{R}_+}$ modeled as the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+,$$

under the risk-neutral probability measure $\mathbb{P}^*$.
a) Find the portfolio allocation (ξ_t, η_t) of the leveraged fund value

$$F_t = \xi_t S_t + \eta_t A_t, \qquad t \in \mathbb{R}_+,$$

where $A_t := A_0 \mathrm{e}^{rt}$ represents the risk-free money market account price.
b) Find the stochastic differential equation satisfied by $(F_t)_{t\in\mathbb{R}_+}$ under the self-financing condition $dF_t = \xi_t dS_t + \eta_t dA_t$.
c) Find the relation between the fund value F_t and the index S_t by solving the stochastic differential equation obtained for F_t in Question (b)). For simplicity we take $F_0 := S_0^\beta$.

Exercise 6.8 Solve the stochastic differential equation

$$dX_t = h(t)X_t dt + \sigma X_t dB_t,$$

where $\sigma > 0$ and $h(t)$ is a deterministic, integrable function of $t \in \mathbb{R}_+$.

Hint: Look for a solution of the form $X_t = f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$, where $f(t)$ is a function to be determined, $t \in \mathbb{R}_+$.

Exercise 6.9 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.
a) Consider the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds, \tag{6.26}$$

where $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$.

Compute $S_t := \mathrm{e}^{X_t}$ by the Itô formula (6.26) applied to $f(x) = \mathrm{e}^x$ and $X_t = \sigma B_t + \nu t$, $\sigma > 0$, $\nu \in \mathbb{R}$.
b) Let $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

c) Given $\sigma > 0$, let $X_t := (B_T - B_t)\sigma$, and compute $\mathrm{Var}[X_t]$, $t \in [0, T]$.

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0\,\mathrm{e}^{\sigma B_t+\nu t}$, $t\in\mathbb{R}_+$. Using the result of Exercise A.2, show that the conditional probability that $S_T > K$ given $S_t = x$ can be computed as

$$\mathbb{P}(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K)+(T-t)\nu}{\sigma\sqrt{T-t}}\right), \qquad t\in[0,T).$$

Hint: Use the time splitting decomposition

$$S_T = S_t\frac{S_T}{S_t} = S_t\,\mathrm{e}^{(B_T-B_t)\sigma+(T-t)\nu}, \qquad t\in[0,T].$$

7. Black-Scholes Pricing and Hedging

The Black and Scholes, 1973 PDE is a Partial Differential Equation which is used for the pricing of vanilla options under absence of arbitrage and self-financing portfolio assumptions. In this chapter we derive the Black-Scholes PDE and present its solution by the the heat kernel method, with application to the pricing and hedging of European call and put options.

7.1 The Black-Scholes PDE

In this chapter we work in a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{A_{t+dt}-A_t}{A_t}=rdt, \qquad \frac{dA_t}{A_t}=rdt, \qquad \frac{dA_t}{dt}=rA_t, \quad t\geqslant 0,$$

with

$$A_t=A_0\,\mathrm{e}^{rt}, \qquad t\geqslant 0,$$

and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ modeled using a geometric Brownian motion defined from the equation

$$\frac{dS_t}{S_t}=\mu dt+\sigma dB_t, \quad t\geqslant 0, \tag{7.1}$$

with solution

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right), \qquad t \geqslant 0,$$

cf. Proposition 6.8.

```
install.packages("quantmod")
library(quantmod)
# getSymbols("0005.HK",from="2016-02-15",to=Sys.Date(),src="yahoo")
getSymbols("0005.HK",from="2016-02-15",to="2017-05-11",src="yahoo")
stock=Ad(`0005.HK`)
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(stock, theme = myTheme)
add_TA(stock, on=1, col="blue", legend=NULL,lwd=1.6)
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

The next Figure 7.1 presents a graph of underlying asset price market data, which is compared to the geometric Brownian motion simulations of Figures 6.5 and 6.6.

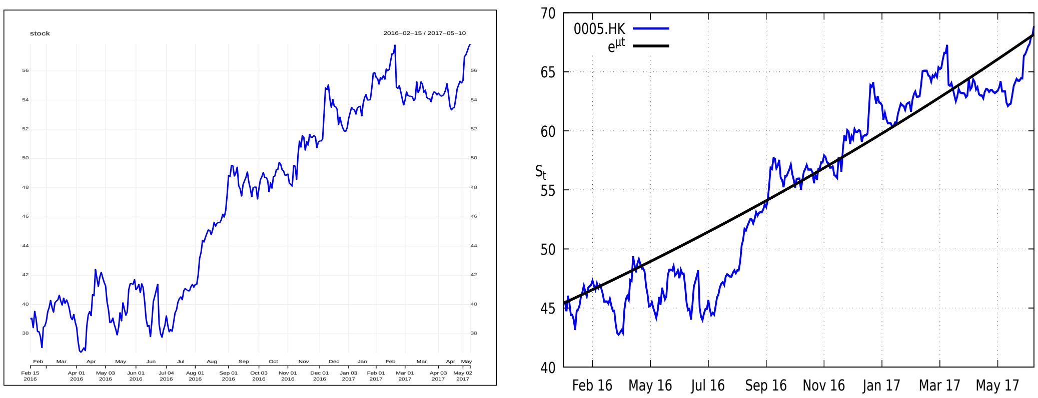


Figure 7.1: Graph of underlying market prices.

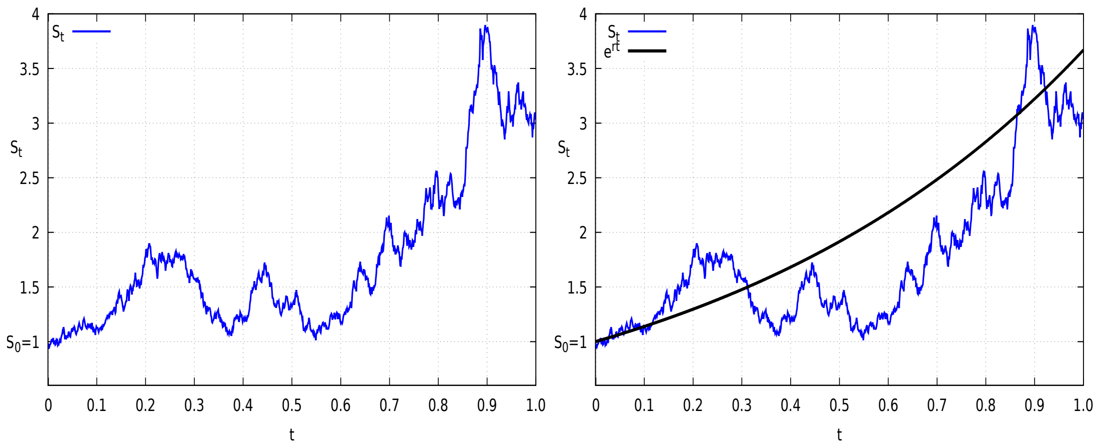


Figure 7.2: Graph of simulated geometric Brownian motion.

In the sequel, we start by deriving the Black and Scholes, 1973 Partial Differential Equation (PDE) for the value of a self-financing portfolio. Note that the drift parameter μ in (7.1) is absent in the PDE (7.2), and it does not appear as well in the Black and Scholes, 1973 formula (7.10).

Proposition 7.1 Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that

(i) the porfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t := \eta_t A_t + \xi_t S_t$, takes the form

$$V_t = g(t, S_t), \qquad t \in \mathbb{R}_+,$$

for some function $g \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ of t and S_t.

Then, the function $g(t,x)$ satisfies the Black and Scholes, 1973 PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \tag{7.2}$$

and $\xi_t = \xi_t(S_t)$ is given by the partial derivative

$$\xi_t = \xi_t(S_t) = \frac{\partial g}{\partial x}(t, S_t), \qquad t \in \mathbb{R}_+. \tag{7.3}$$

Proof. (i) First, we note that the self-financing condition (6.8) in Proposition 6.3 implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t \\ &= rg(t, S_t)dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \end{aligned} \tag{7.4}$$

$t \in \mathbb{R}_+$. We now rewrite (6.18) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

as in (5.22), by taking

$$u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \qquad t \in \mathbb{R}_+.$$

(ii) By (5.24), the application of Itô's formula Theorem 5.9 to $V_t = g(t, S_t)$ leads to

$$\begin{aligned} dV_t &= dg(t, S_t) \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \frac{\partial g}{\partial x}(t, S_t)dS_t + \frac{1}{2}(dS_t)^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + v_t \frac{\partial g}{\partial x}(t, S_t)dt + u_t \frac{\partial g}{\partial x}(t, S_t)dB_t + \frac{1}{2}|u_t|^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t)dB_t. \end{aligned} \tag{7.5}$$

By respective identification of the terms in dB_t and dt in (7.4) and (7.5) we get

$$\begin{cases} rg(t, S_t)dt + (\mu - r)\xi_t S_t dt = \dfrac{\partial g}{\partial t}(t, S_t)dt + \mu S_t \dfrac{\partial g}{\partial x}(t, S_t)dt + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t, S_t)dt, \\ \\ \xi_t S_t \sigma dB_t = S_t \sigma \dfrac{\partial g}{\partial x}(t, S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t,S_t) = \dfrac{\partial g}{\partial t}(t,S_t) + rS_t\dfrac{\partial g}{\partial x}(t,S_t) + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T, \end{cases} \tag{7.6}$$

which yields (7.2) after substituting S_t with $x > 0$. □

The derivative giving ξ_t in (7.3) is called the Delta of the option price, see Proposition 7.4 below. The amount invested on the riskless asset is

$$\eta_t A_t = V_t - \xi_t S_t = g(t,S_t) - S_t \frac{\partial g}{\partial x}(t,S_t),$$

and η_t is given by

$$\begin{aligned} \eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{1}{A_t}\left(g(t,S_t) - S_t\frac{\partial g}{\partial x}(t,S_t)\right) \\ &= \frac{1}{A_0\,\mathrm{e}^{rt}}\left(g(t,S_t) - S_t\frac{\partial g}{\partial x}(t,S_t)\right). \end{aligned}$$

In the next Proposition 7.2 we add a terminal condition $g(T,x) = f(x)$ to the Black-Scholes PDE (7.2) in order to price a claim payoff C of the form $C = h(S_T)$. As in the discrete-time case, the arbitrage-free price $\pi_t(C)$ at time $t \in [0,T]$ of the claim payoff C is defined to be the value V_t of the self-financing portfolio hedging C.

Proposition 7.2 The arbitrage-free price $\pi_t(C)$ at time $t \in [0,T]$ of the (vanilla) option with payoff $C = h(S_T)$ is given by $\pi_t(C) = g(t,S_t)$ and the hedging allocation ξ_t is given by the partial derivative (7.3), where the function $g(t,x)$ is solution of the following Black-Scholes PDE:

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = h(x), \quad x > 0. \end{cases} \tag{7.7}$$

Proof. Proposition 7.1 shows that the solution $g(t,x)$ of (7.2), $g \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$, represents the value $V_t = \eta_t A_t + \xi_t S_t = g(t,S_t)$, $t \in \mathbb{R}_+$, of a self-financing portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$. By Definition 4.1, $\pi_t(C) := V_t = g(t,S_t)$ is the arbitrage-free price at time $t \in [0,T]$ of the vanilla option with payoff $C = h(S_T)$. □

The absence of the drift parameter μ from the PDE (7.7) can be understood in the next forward contract example, in which the claim payoff can be hedged by leveraging on the value S_t of the underlying asset, independently of the trend parameter μ.

Example - forward contracts

When $C = S_T - K$ is the (linear) payoff function of a long forward contract, *i.e.* $h(x) = x - K$, the Black-Scholes PDE (7.7) admits the easy solution

$$g(t,x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T, \tag{7.8}$$

showing that the price at time t of the forward contract with payoff $C = S_T - K$ is

$$S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T.$$

In addition, the Delta of the option price is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \qquad 0 \leqslant t \leqslant T,$$

which leads to a static "hedge and forget" strategy, cf. Exercise 7.7. The forward contract can be realized by the option issuer as follows:

a) At time t, receive the option premium $V_t := S_t - \mathrm{e}^{-(T-t)r}K$ from the option buyer.
b) Borrow $\mathrm{e}^{-(T-t)r}K$ from the bank, to be refunded at maturity.
c) Buy the risky asset using the amount $S_t - \mathrm{e}^{-(T-t)r}K + \mathrm{e}^{-(T-t)r}K = S_t$.
d) Hold the risky asset until maturity (do nothing, constant portfolio strategy).
e) At maturity T, hand in the asset to the option holder, who will pay the amount K in return.
f) Use the amount $K = \mathrm{e}^{(T-t)r}\,\mathrm{e}^{-(T-t)r}K$ to refund the lender of $\mathrm{e}^{-(T-t)r}K$ borrowed at time t.

Another way to compute the option premium V_t is to state that the amount $V_t - S_t$ has to be borrowed at time t in order to purchase the asset, and that the asset price K received at maturity T should be used to refund the loan, which yields

$$(V_t - S_t)\,\mathrm{e}^{-(T-t)r} = K, \quad 0 \leqslant t \leqslant T.$$

Forward contracts can be used for physical delivery, *e.g.* for live cattle. In the case of European options, the basic "hedge and forget" constant strategy

$$\xi_t = 1, \quad \eta_t = \eta_0, \quad 0 \leqslant t \leqslant T,$$

will hedge the option only if

$$S_T + \eta_0 A_T \geqslant (S_T - K)^+,$$

i.e. if $-\eta_0 A_T \leqslant K \leqslant S_T$.

Future contracts

For a future contract expiring at time T we take $K = S_0\,\mathrm{e}^{rT}$ and the contract is usually quoted at time t in terms of the forward price

$$\mathrm{e}^{(T-t)r}\big(S_t - K\,\mathrm{e}^{-(T-t)r}\big) = \mathrm{e}^{(T-t)r}S_t - K = \mathrm{e}^{(T-t)r}S_t - S_0\,\mathrm{e}^{rT},$$

discounted at time T, or simply using $\mathrm{e}^{(T-t)r}S_t$. Future contracts are *non-deliverable* forward contracts which are "marked to market" at each time step via a cash flow exchange between the two parties, ensuring that the absolute difference $|\,\mathrm{e}^{(T-t)r}S_t - K|$ is being credited to the buyer's account if $\mathrm{e}^{(T-t)r}S_t > K$, or to the seller's account if $\mathrm{e}^{(T-t)r}S_t < K$.

7.2 European Call Options

Recall that in the case of the European call option with strike price K the payoff function is given by $h(x) = (x-K)^+$ and the Black-Scholes PDE (7.7) reads

$$\begin{cases} rg_c(t,x) = \dfrac{\partial g_c}{\partial t}(t,x) + rx\dfrac{\partial g_c}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2 g_c}{\partial x^2}(t,x) \\ g_c(T,x) = (x-K)^+. \end{cases} \quad (7.9)$$

The next proposition will be proved in Sections 7.5 and 7.6, see Proposition 7.11.

Proposition 7.3 The solution of the PDE (7.9) is given by the *Black-Scholes* formula for call options

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad (7.10)$$

with

$$d_+(T-t) := \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \quad (7.11)$$

$$d_-(T-t) := \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \qquad 0 \leqslant t < T. \quad (7.12)$$

We note the relation

$$d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t}, \qquad 0 \leqslant t < T. \quad (7.13)$$

Here, "log" denotes the *natural logarithm* "ln" and

$$\Phi(x) := \mathbb{P}(X \leqslant x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian Cumulative Distribution Function (CDF) of a standard normal random variable $X \simeq \mathcal{N}(0,1)$, with the relation

$$\Phi(-x) = 1 - \Phi(x), \qquad x \in \mathbb{R}. \quad (7.14)$$

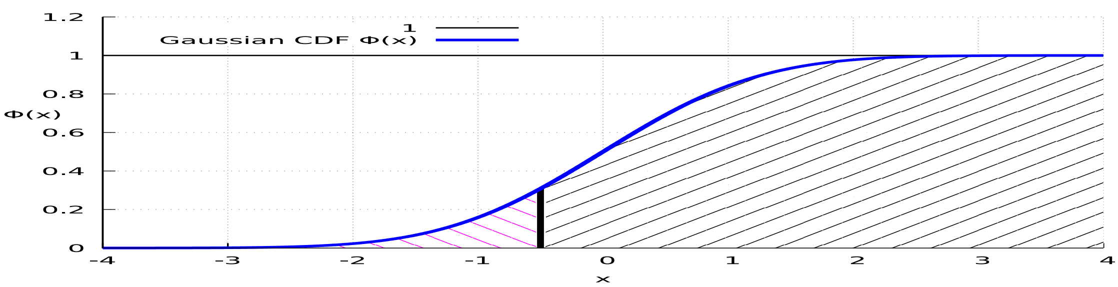


Figure 7.3: Graph of the Gaussian Cumulative Distribution Function (CDF).

In other words, the European call option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$\begin{aligned} g_c(t,S_t) &= \mathrm{Bl}(K,S_t,\sigma,r,T-t) \\ &= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad 0 \leqslant t \leqslant T. \end{aligned}$$

The following R script is an implementation of the Black-Scholes formula for European call options in R.*

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
BSCall}
```

In comparison with the discrete-time Cox-Ross-Rubinstein (CRR) model of Section 3.6, the interest in the formula (7.10) is to provide an analytical solution that can be evaluated in a single step, which is computationally much more efficient.

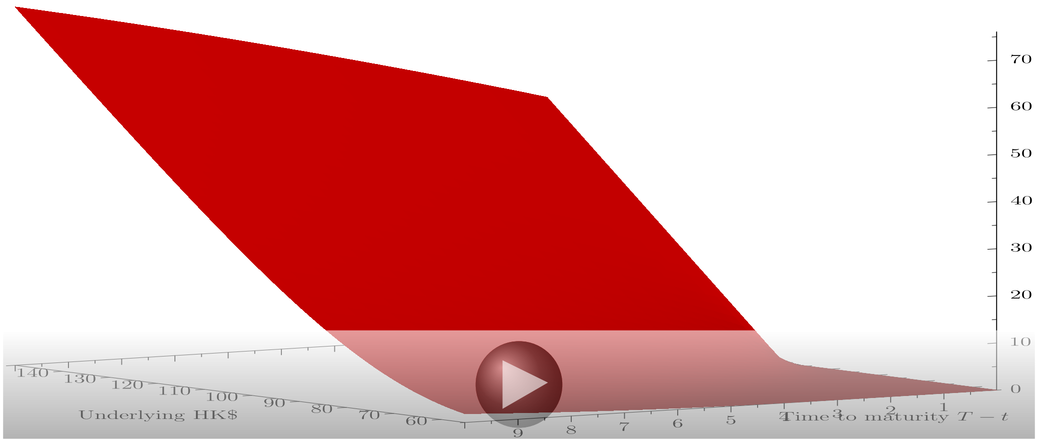


Figure 7.4: Graph of the Black-Scholes call price map with strike price $K = 100$.†

Figure 7.4 presents an interactive graph of the Black-Scholes call price map, *i.e.* the solution

$$(t,x) \longmapsto g_c(t,x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$$

of the Black-Scholes PDE (7.7) for a call option.

*Download the corresponding **IPython notebook** that can be run **here**.
†Right-click on the figure for interaction and "Full Screen Multimedia" view.

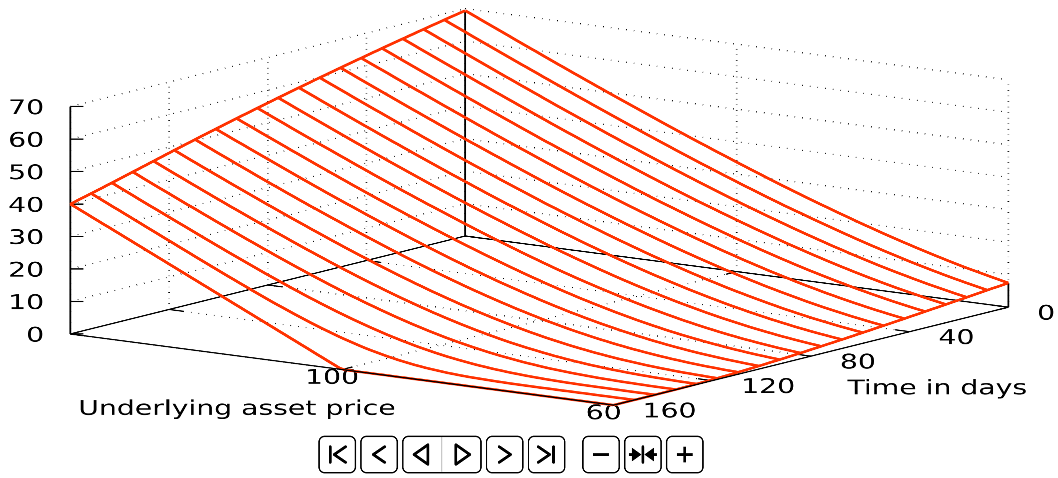


Figure 7.5: Time-dependent solution of the Black-Scholes PDE (call option).*

The next proposition is proved by a direct differentiation of the Black-Scholes function, and will be recovered later using a probabilistic argument in Proposition 8.12 below.

Proposition 7.4 The Black-Scholes Delta of the European call option is given by

$$\xi_t = \xi_t(S_t) = \frac{\partial g_c}{\partial x}(t, S_t) = \Phi\big(d_+(T-t)\big) \in [0,1], \tag{7.15}$$

where $d_+(T-t)$ is given by (7.11).

Proof. From Relation (7.13), we note that the standard normal probability density function

$$\varphi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R},$$

satisfies

$$\begin{aligned}
\varphi(d_+(T-t)) &= \varphi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} + |\sigma|\sqrt{T-t}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_-(T-t))^2 - (T-t)r - \log\frac{x}{K}\right) \\
&= \frac{K}{x\sqrt{2\pi}} \mathrm{e}^{-(T-t)r} \exp\left(-\frac{1}{2}(d_-(T-t))^2\right) \\
&= \frac{K}{x} \mathrm{e}^{-(T-t)r} \varphi(d_-(T-t)),
\end{aligned}$$

hence by (7.10) we have

$$\frac{\partial g_c}{\partial x}(t,x) = \frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right) \tag{7.16}$$

*The animation works in Acrobat Reader on the entire pdf file.

$$
\begin{aligned}
& -K\,\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\left(\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right)\\
= {} & \Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
& +x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
& -K\,\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
= {} & \Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
& +\frac{x}{|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
& -\frac{K\,\mathrm{e}^{-(T-t)r}}{|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
= {} & \Phi(d_+(T-t))+\frac{x}{|\sigma|\sqrt{T-t}}\varphi(d_+(T-t))-\frac{K\,\mathrm{e}^{-(T-t)r}}{|\sigma|\sqrt{T-t}}\varphi(d_-(T-t))\\
= {} & \Phi(d_+(T-t)).
\end{aligned}
$$

□

As a consequence of Proposition 7.4, the Black-Scholes call price splits into a risky component $S_t\Phi\big(d_+(T-t)\big)$ and a riskless component $-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$, as follows:

$$
g_c(t,S_t)=\underbrace{S_t\Phi\big(d_+(T-t)\big)}_{\text{risky investment (held)}}\ \underbrace{-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)}_{\text{risk-free investment (borrowed)}},\qquad 0\leqslant t\leqslant T.
$$

See Exercise 7.4 for a computation of the boundary values of $g_c(t,x)$, $t\in[0,T)$, $x>0$. The following R script is an implementation of the Black-Scholes Delta for European call options in R.

```
Delta <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
Delta = pnorm(d1);Delta}
```

In Figure 7.6 we plot the Delta of the European call option as a function of the underlying asset price and of the time remaining until maturity.

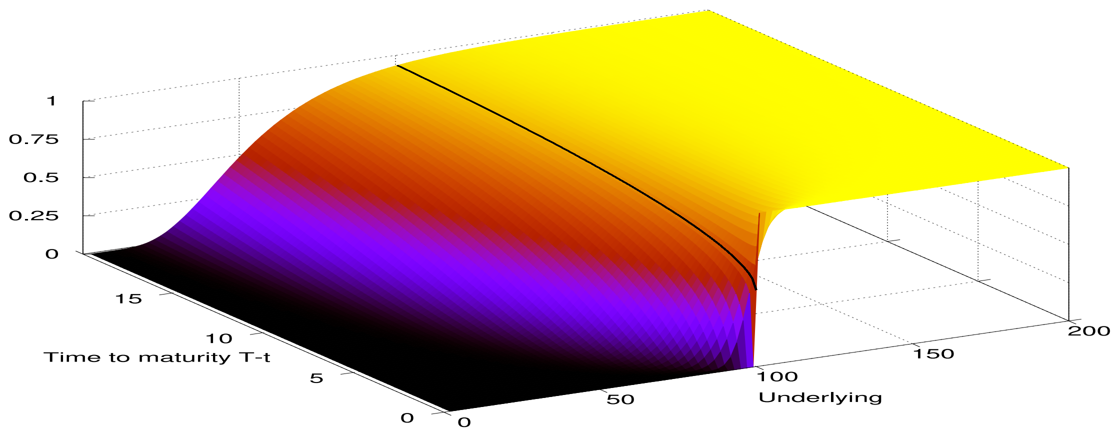


Figure 7.6: Delta of a European call option with strike price $K=100$, $r=3\%$, $\sigma=10\%$.

The *Gamma* of the European call option is defined as the first derivative of Delta, or second derivative of the option price, with respect to the underlying asset price. This gives

$$\begin{aligned}
\gamma_t &= \frac{1}{S_t|\sigma|\sqrt{T-t}}\Phi'\big(d_+(T-t)\big) \\
&= \frac{1}{S_t|\sigma|\sqrt{2(T-t)\pi}}\exp\left(-\frac{1}{2}\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right) \\
&\geqslant 0.
\end{aligned}$$

In particular, a positive value of γ_t implies that the Delta $\xi_t = \xi_t(S_t)$ should increase when the underlying asset price S_t increases. In other words, the position ξ_t in the underlying asset should be increased by additional purchases if the underlying asset price S_t increases.

In Figure 7.7 we plot the (truncated) value of the Gamma of a European call option as a function of the underlying asset price and of time to maturity.

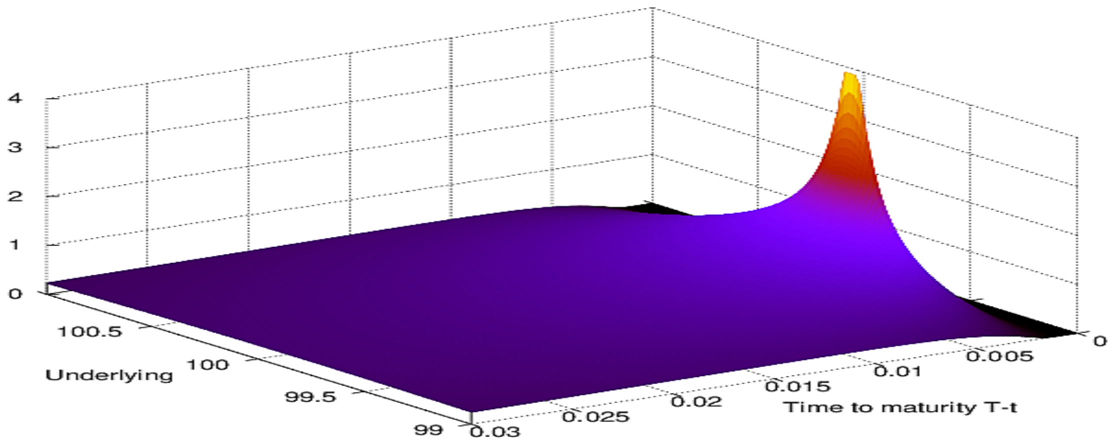


Figure 7.7: Gamma of a European call option with strike price $K = 100$.

As Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the risky underlying asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 7.7.

Numerical example - hedging a call option

In Figure 7.8 we consider the historical stock price of HSBC Holdings (0005.HK) over one year:

Consider the call option issued by Societe Generale on 31 December 2008 with strike price K=\$63.704, maturity T = October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 *warrants*, cf. page 7. The next graph gives the time evolution of the Black-Scholes portfolio value

$$t \longmapsto g_c(t, S_t)$$

driven by the market price $t \longmapsto S_t$ of the risky underlying asset as given in Figure 7.8, in which the number of days is counted from the origin and not from maturity.

Figure 7.8: Graph of the stock price of HSBC Holdings.

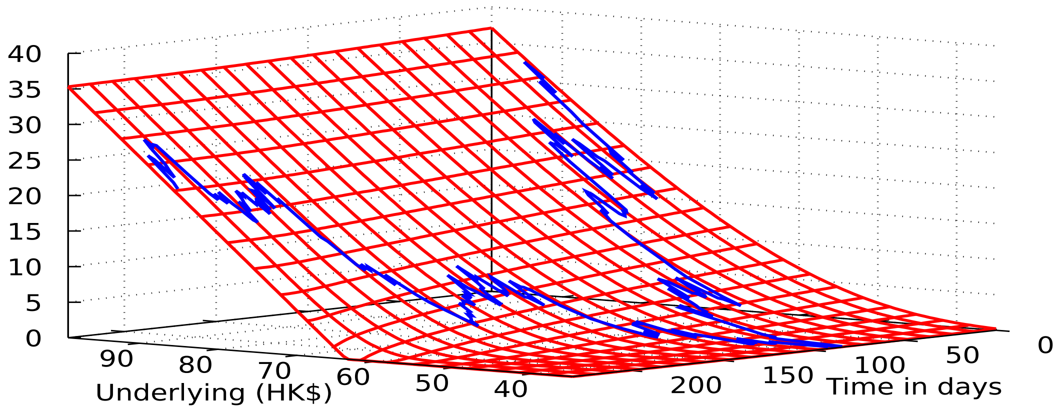


Figure 7.9: Path of the Black-Scholes price for a call option on HSBC.

As a consequence of Proposition 7.4, in the Black-Scholes call option hedging model, the amount invested in the risky asset is

$$\begin{aligned} S_t\xi_t &= S_t\Phi\big(d_+(T-t)\big) \\ &= S_t\Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\geqslant 0, \end{aligned}$$

which is always nonnegative, *i.e.* there is no short selling, and the amount invested on the riskless asset is

$$\begin{aligned} \eta_t A_t &= -K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\ &= -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\leqslant 0, \end{aligned}$$

which is always nonpositive, *i.e.* we are constantly borrowing money on the riskless asset, as noted in Figure 7.10.

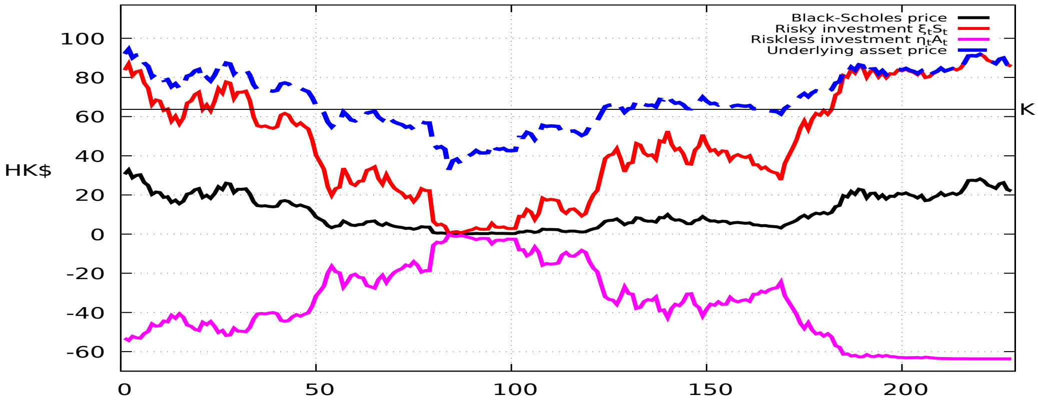


Figure 7.10: Time evolution of a hedging portfolio for a call option on HSBC.

A comparison of Figure 7.10 with market data can be found in Figures 9.11 and 9.12 below.

Cash settlement. In the case of a cash settlement, the option issuer will satisfy the option contract by selling $\xi_T = 1$ stock at the price $S_T = \$83$, refund the $K = \$63$ risk-free investment, and hand in the remaining amount $C = (S_T - K)^+ = 83 - 63 = \$20$ to the option holder.

Physical delivery. In the case of physical delivery of the underlying asset, the option issuer will deliver $\xi_T = 1$ stock to the option holder in exchange for $K = \$63$, which will be used together with the portfolio value to refund the risk-free loan.

7.3 European Put Options

Similarly, in the case of the European put option with strike price K the payoff function is given by $h(x) = (K - x)^+$ and the Black-Scholes PDE (7.7) reads

$$\begin{cases} rg_{\rm p}(t,x) = \dfrac{\partial g_{\rm p}}{\partial t}(t,x) + rx\dfrac{\partial g_{\rm p}}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g_{\rm p}}{\partial x^2}(t,x), \\ \\ g_{\rm p}(T,x) = (K-x)^+, \end{cases} \tag{7.17}$$

The next proposition can be proved as in Sections 7.5 and 7.6, see Proposition 7.11.

Proposition 7.5 The solution of the PDE (7.17) is given by the *Black-Scholes* formula for put options

$$g_p(t,x) = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big), \tag{7.18}$$

with

$$d_+(T-t) = \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{7.19}$$

$$d_-(T-t) = \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{7.20}$$

as illustrated in Figure 7.11.

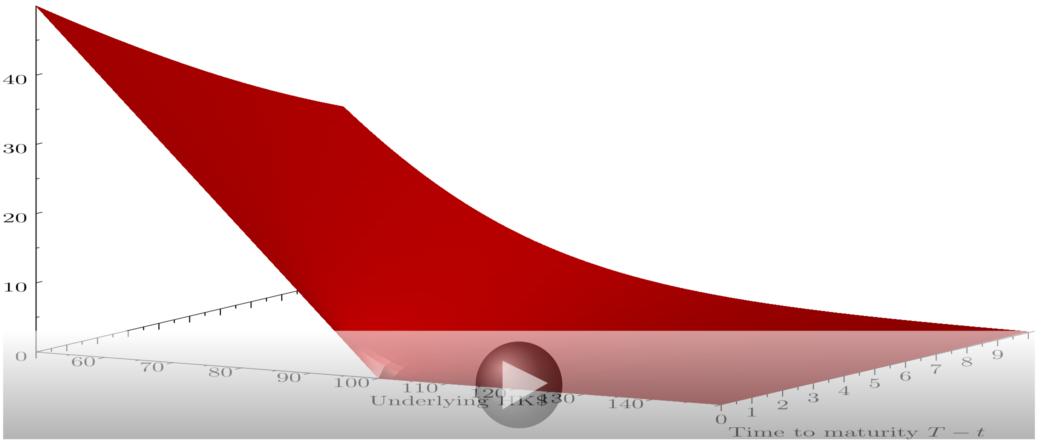


Figure 7.11: Graph of the Black-Scholes put price function with strike price $K = 100$.*

In other words, the European put option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$g_p(t,S_t) = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T.$$

*Right-click on the figure for interaction and “Full Screen Multimedia” view.

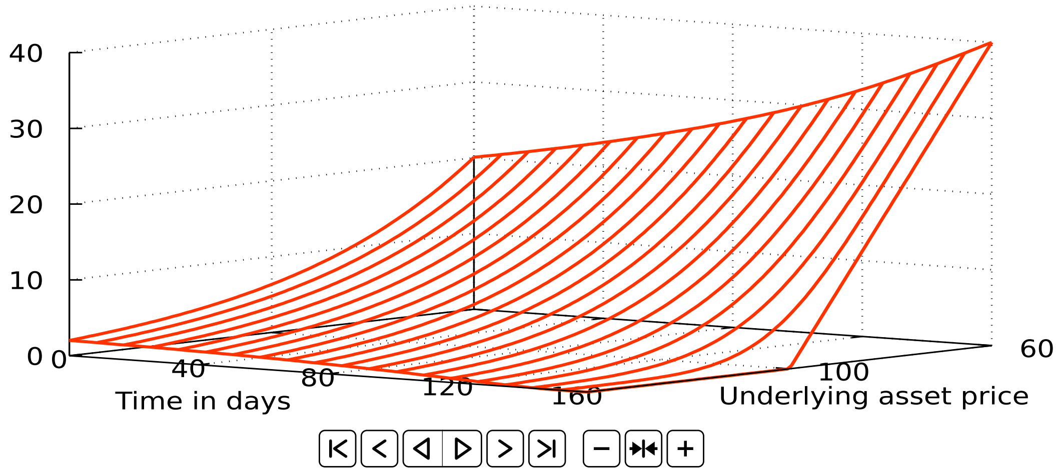


Figure 7.12: Time-dependent solution of the Black-Scholes PDE (put option).*

The following R script is an implementation of the Black-Scholes formula for European put options in R.

```
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 = d1 - sigma * sqrt(T)
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1)
BSPut}
```

Call-put parity

Proposition 7.6 Call-put parity. We have the call-put parity relation

$$g_{\mathrm{c}}(t,S_t) - g_{\mathrm{p}}(t,S_t) = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T, \tag{7.21}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K\,\mathrm{e}^{-(T-t)r}$.

Proof. The call-put parity (7.21) is a consequence of the relation

$$x - K = (x-K)^+ - (K-x)^+$$

satisfied by the terminal call and put payoff functions in the Black-Scholes PDE (7.7). It can also be verified directly from (7.10) and (7.18) as

$$\begin{aligned}
g_{\mathrm{c}}(t,x) - g_{\mathrm{p}}(t,x) &= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\
&\quad -\big(K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)\big) \\
&= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(d_-(T-t)\big)\big) + x\big(1-\Phi\big(d_+(T-t)\big)\big) \\
&= x - K.
\end{aligned}$$

□

The *Delta* of the Black-Scholes put option can be obtained by differentiation of the call-put parity relation (7.21) and Proposition 7.4.

*The animation works in Acrobat Reader on the entire pdf file.

Proposition 7.7 The *Delta* of the Black-Scholes put option is given by

$$\boxed{\xi_t = -(1-\Phi(d_+(T-t))) = -\Phi\big(-d_+(T-t)\big) \in [-1,0], \quad 0 \leqslant t \leqslant T.}$$

Proof. By the call-put parity relation (7.21) and Proposition 7.4, we have

$$\begin{aligned}\frac{\partial g_{\rm p}}{\partial x}(t,S_t) &= \frac{\partial g_{\rm c}}{\partial x}(t,S_t) - 1 \\ &= \Phi(d_+(T-t)) - 1 \\ &= -\Phi(-d_+(T-t)), \qquad 0 \leqslant t \leqslant T,\end{aligned}$$

where we applied (7.14). □

As a consequence of Proposition 7.7 the Black-Scholes put price splits into a risky component $-S_t\Phi\big(-d_+(T-t)\big)$ and a riskless component $K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)$, as follows:

$$g_{\rm p}(t,S_t) = \underbrace{K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)}_{\text{risk-free investment (savings)}} - \underbrace{S_t\Phi\big(-d_+(T-t)\big)}_{\text{risky investment (short)}}, \quad 0 \leqslant t \leqslant T.$$

In Figure 7.13 we plot the Delta of the European put option as a function of the underlying asset price and of the time remaining until maturity.

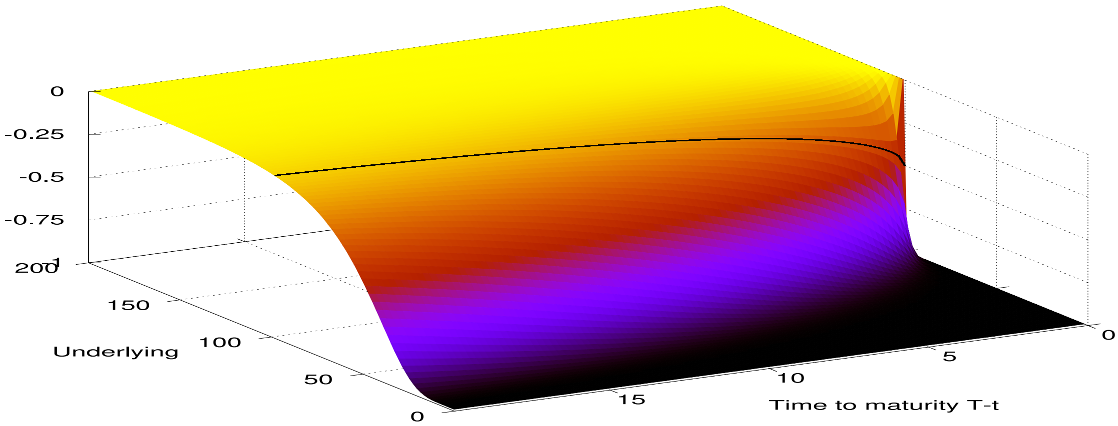


Figure 7.13: Delta of a European put option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

Numerical example - hedging a put option

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price K=$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593, cf. page 7. In the next Figure 7.14, the number of days is counted from the origin, not from maturity.

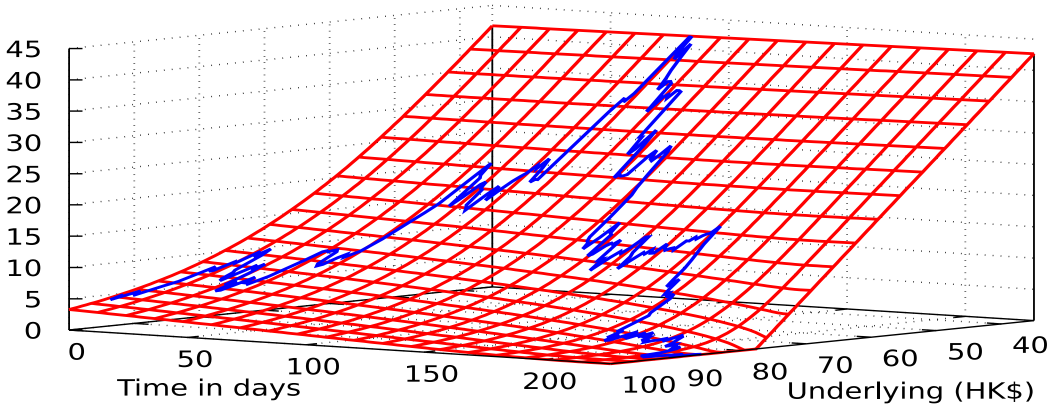


Figure 7.14: Path of the Black-Scholes price for a put option on HSBC.

As a consequence of Proposition 7.7, the amount invested on the risky asset for the hedging of a put option is

$$\begin{aligned} -S_t\Phi\big(-d_+(T-t)\big) &= -S_t\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\leqslant 0, \end{aligned}$$

i.e. there is always short selling, and the amount invested on the riskless asset priced $A_t = \mathrm{e}^{rt}$, $t \in [0,T]$, is

$$\begin{aligned} \eta_t A_t &= K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) \\ &= K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\geqslant 0, \end{aligned}$$

which is always nonnegative, *i.e.* we are constantly saving money on the riskless asset, as noted in Figure 7.15.

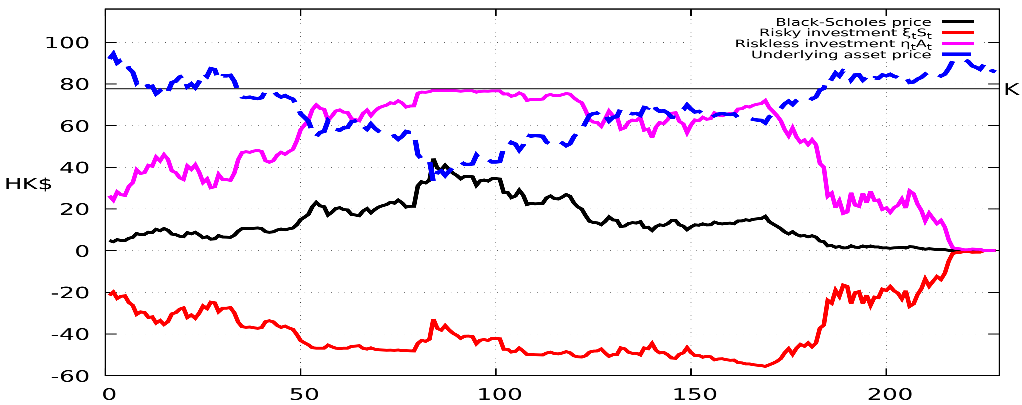


Figure 7.15: Time evolution of the hedging portfolio for a put option on HSBC.

In the above example the put option finished out of the money (OTM), so that no cash settlement or physical delivery occurs. A comparison of Figure 7.10 with market data can be found in Figures 9.13 and 9.14 below.

7.4 Market Terms and Data

The following Table 7.1 provides a summary of formulas for the computation of Black-Scholes sensitivities, also called *Greeks*.*

		Call option	Put option
Option price	$g(t,S_t)$	$S_t\Phi(d_+(T-t))-K\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$	$K\,\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))-S_t\Phi(-d_+(T-t))$
Delta (Δ)	$\frac{\partial g}{\partial x}(t,S_t)$	$\Phi(d_+(T-t))\geqslant 0$	$-\Phi(-d_+(T-t))\leqslant 0$
Gamma (Γ)	$\frac{\partial^2 g}{\partial x^2}(t,S_t)$	$\frac{\Phi'(d_+(T-t))}{S_t\lvert\sigma\rvert\sqrt{T-t}}\geqslant 0$	
Vega	$\frac{\partial g}{\partial \sigma}(t,S_t)$	$S_t\sqrt{T-t}\Phi'(d_+(T-t))\geqslant 0$	
Theta (Θ)	$\frac{\partial g}{\partial t}(t,S_t)$	$-\frac{S_t\lvert\sigma\rvert\Phi'(d_+(T-t))}{2\sqrt{T-t}}-rK\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))\leqslant 0$	$-\frac{S_t\lvert\sigma\rvert\Phi'(d_+(T-t))}{2\sqrt{T-t}}+rK\,\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))$
Rho (ρ)	$\frac{\partial g}{\partial r}(t,S_t)$	$K(T-t)\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$	$-K(T-t)\,\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))$

Table 7.1: Black-Scholes Greeks (Wikipedia).

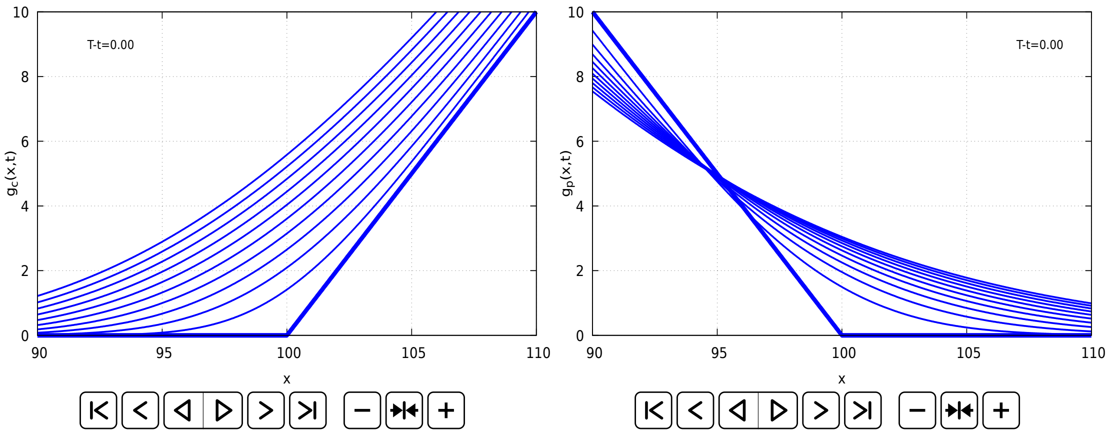
From Table 7.1 we can conclude that call option prices are increasing functions of the underlying asset price S_t, of the interest rate r, and of the volatility parameter σ. Similarly, put option prices are decreasing functions of the underlying asset price S_t, of the interest rate r, and increasing functions of the volatility parameter σ.

Parameter	Variation of call option prices	Variation of put option prices
Underlying S_t	increasing $\nearrow$	decreasing $\searrow$
Volatility σ	increasing $\nearrow$	increasing $\nearrow$
Time t	decreasing $\searrow$	depends on the underlying price level
Interest rate r	increasing $\nearrow$	decreasing $\searrow$

Table 7.2: Variations of Black-Scholes prices.

The change of sign of the sensitivity Theta (Θ) with respect to time t can be verified in the following Figure 7.16.

*"Every class feels like attending a Greek lesson" (AY2018-2019 student feedback).



(a) Black-Scholes call price maps.

(b) Black-Scholes put price maps

Figure 7.16: Time-dependent solutions of the Black-Scholes PDE.*

Intrinsic value. The *intrinsic value* at time $t \in [0,T]$ of the option with payoff $C = h\big(S_T^{(1)}\big)$ is given by the immediate exercise payoff $h\big(S_t^{(1)}\big)$. The *extrinsic value* at time $t \in [0,T]$ of the option is the remaining difference $\pi_t(C) - h\big(S_t^{(1)}\big)$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h\big(S_t^{(1)}\big)$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = \underbrace{h\big(S_t^{(1)}\big)}_{\text{intrinsic value}} + \underbrace{\pi_t(C) - h\big(S_t^{(1)}\big)}_{\text{extrinsic value}}, \qquad 0 \leqslant t \leqslant T].$$

Gearing. The *gearing* at time $t \in [0,T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$\mathrm{G}_t := \frac{S_t}{\pi_t(C)} = \frac{S_t}{g(t,S_t)}, \qquad 0 \leqslant t \leqslant T.$$

Effective gearing. The *effective gearing* at time $t \in [0,T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$\begin{aligned} \mathrm{EG}_t &:= \mathrm{G}_t \xi_t \\ &= \frac{\xi_t S_t}{\pi_t(C)} \\ &= \frac{S_t}{\pi_t(C)} \frac{\partial g}{\partial x}(t,S_t) \\ &= \frac{S_t}{g(t,S_t)} \frac{\partial g}{\partial x}(t,S_t) \\ &= S_t \frac{\partial}{\partial x} \log g(t,S_t), \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

The effective gearing

$$\mathrm{EG}_t = \frac{\xi_t S_t}{\pi_t(C)}$$

can be interpreted as the *hedge ratio*, *i.e.* the percentage of the portfolio which is invested on the risky asset. When written as

$$\frac{\Delta g(t,S_t)}{g(t,S_t)} = \mathrm{EG}_t \times \frac{\Delta S_t}{S_t},$$

*The animation works in Acrobat Reader on the entire pdf file.

the effective gearing gives the relative variation, or percentage change, $\Delta g(t,S_t)/g(t,S_t)$ of the option price $g(t,S_t)$ from the relative variation $\Delta S_t/S_t$ in the underlying asset price.

The ratio $\mathrm{EG}_t = S_t \partial \log g(t,S_t)/\partial x$ can also be interpreted as an *elasticity coefficient*.

Break-even price. The *break-even* price BEP_t of the underlying asset is the value of S for which the intrinsic option value $h(S)$ equals the option price $\pi_t(C)$ at time $t \in [0,T]$. For European call options it is given by

$$\mathrm{BEP}_t := K + \pi_t(C) = K + g(t,S_t), \qquad t = 0,1,\ldots,N.$$

whereas for European put options it is given by

$$\mathrm{BEP}_t := K - \pi_t(C) = K - g(t,S_t), \qquad 0 \leqslant t \leqslant T.$$

Premium. The option *premium* OP_t can be defined as the variation required from the underlying asset price in order to reach the break-even price, *i.e.* we have

$$\mathrm{OP}_t := \frac{\mathrm{BEP}_t - S_t}{S_t} = \frac{K + g(t,S_t) - S_t}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European call options, and

$$\mathrm{OP}_t := \frac{S_t - \mathrm{BEP}_t}{S_t} = \frac{S_t + g(t,S_t) - K}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European put options, see Figure 7.17 below. The term "premium" is sometimes also used to denote the arbitrage-free price $g(t,S_t)$ of the option.

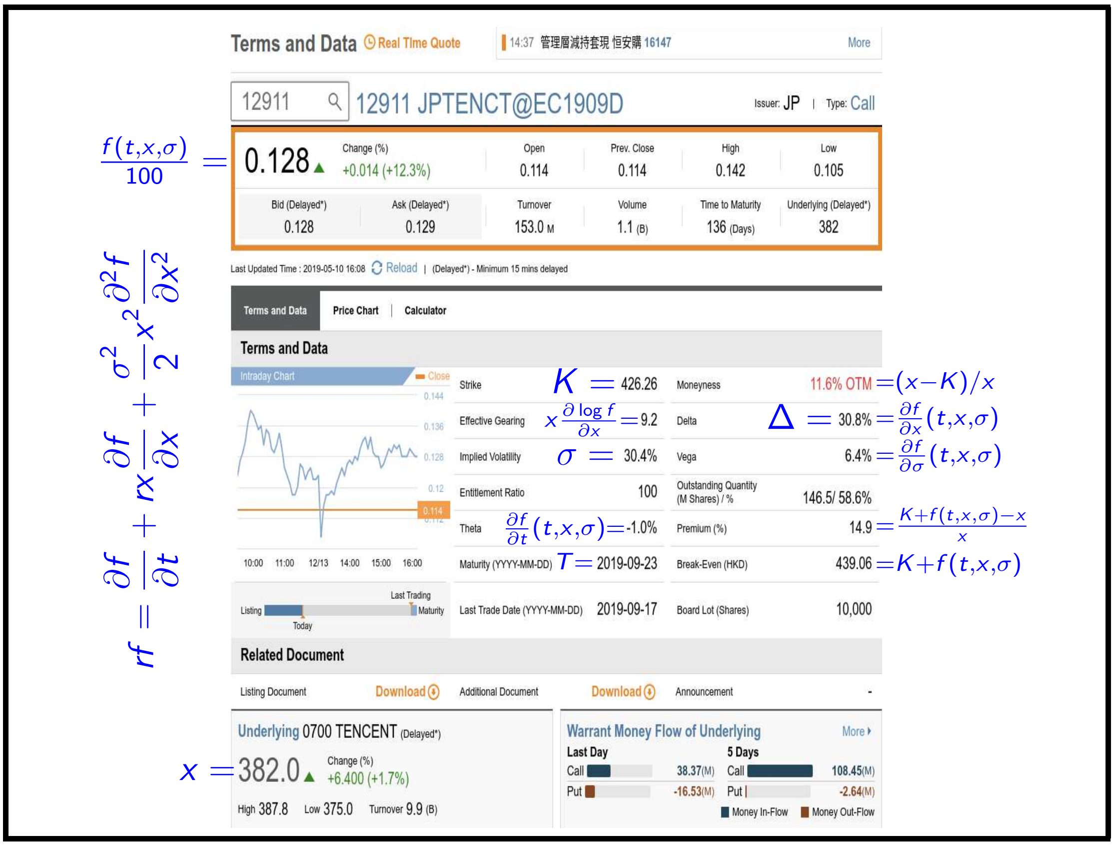


Figure 7.17: Warrant terms and data.

The R package bizdays (requires to install QuantLib) can be used to compute calendar time *vs* business time to maturity

```
install.packages("bizdays")
library(bizdays)
load_quantlib_calendars('HongKong', from='2018-01-01', to='2018-12-31')
load_quantlib_calendars('Singapore', from='2018-01-01', to='2018-12-31')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/HongKong')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/Singapore')
```

7.5 The Heat Equation

In the next proposition we notice that the solution $f(t,x)$ of the Black-Scholes PDE (7.7) can be transformed into a solution $g(t,y)$ of the simpler *heat equation* by a change of variable and a time inversion $t \longmapsto T-t$ on the interval $[0,T]$, so that the terminal condition at time T in the Black-Scholes equation (7.22) becomes an initial condition at time $t=0$ in the heat equation (7.25). See also here for a related discussion on changes of variables for the Black-Scholes PDE.

Proposition 7.8 Assume that $f(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} rf(t,x) = \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2}(t,x), \\ \\ f(T,x) = (x-K)^+, \end{cases} \tag{7.22}$$

with terminal condition $h(x) = (x-K)^+$, $x>0$. Then the function $g(t,y)$ defined by

$$g(t,y) = \mathrm{e}^{rt} f\big(T-t, \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}\big) \tag{7.23}$$

solves the heat equation (7.25) with initial condition

$$\psi(y) := h\big(\mathrm{e}^{|\sigma|y}\big), \qquad y \in \mathbb{R}, \tag{7.24}$$

i.e. we have

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = h\big(\mathrm{e}^{|\sigma|y}\big). \end{cases} \tag{7.25}$$

Proposition 7.8 will be proved in Section 7.6. It will allow us to solve the Black-Scholes PDE (7.22) based on the solution of the heat equation (7.25) with initial condition $\psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big)$, $y \in \mathbb{R}$, by inversion of Relation (7.23) with $s = T-t$, $x = \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}$, *i.e.*

$$f(s,x) = \mathrm{e}^{-(T-s)r} g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right).$$

Next, we focus on the *heat equation*

$$\frac{\partial \varphi}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) \tag{7.26}$$

which is used to model the diffusion of heat over time through solids. Here, the data of $g(x,t)$ represents the temperature measured at time t and point x. We refer the reader to Widder, 1975 for a complete treatment of this topic.

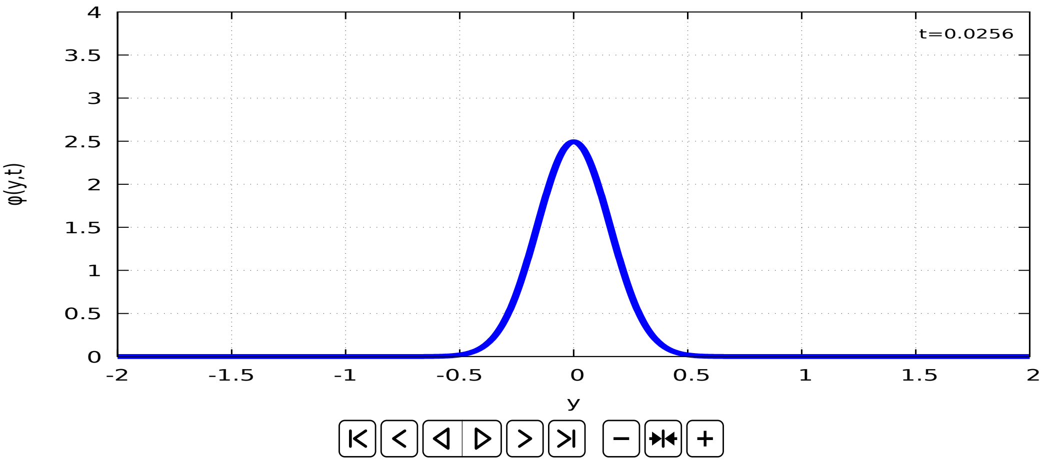


Figure 7.18: Time-dependent solution of the heat equation.*

Proposition 7.9 The fundamental solution of the heat equation (7.26) is given by the Gaussian probability density function

$$\varphi(t,y) := \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-y^2/(2t)}, \qquad y \in \mathbb{R},$$

with variance $t > 0$.

Proof. The proof is done by a direct calculation, as follows:

$$\begin{aligned}
\frac{\partial \varphi}{\partial t}(t,y) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right) \\
&= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t^{3/2}\sqrt{2\pi}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}} \\
&= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) &= -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{y}{t}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right) \\
&= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t\sqrt{2\pi t}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}} \\
&= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y), \qquad t > 0, \ y \in \mathbb{R}.
\end{aligned}$$

□

In Section 7.6 the heat equation (7.26) will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

*The animation works in Acrobat Reader on the entire pdf file.

Proposition 7.10 The heat equation

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = \psi(y) \end{cases} \tag{7.27}$$

with continuous initial condition

$$g(0,y) = \psi(y)$$

has the solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}, \qquad y \in \mathbb{R}, \quad t > 0. \tag{7.28}$$

Proof. We have

$$\begin{aligned}
\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-(y-z)^2/(2t)}}{\sqrt{2\pi t}}\right) dz \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \left(\frac{(y-z)^2}{t^2} - \frac{1}{t}\right) \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial z^2} \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}$$

On the other hand, it can be checked that at time $t = 0$ we have

$$\begin{aligned}
\lim_{t \to 0}\int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} &= \lim_{t \to 0}\int_{-\infty}^{\infty} \psi(y+z)\, \mathrm{e}^{-z^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \psi(y), \qquad y \in \mathbb{R}.
\end{aligned}$$

□

The next Figure 7.19 shows the evolution of $g(t,x)$ with initial condition based on the European call payoff function $h(x) = (x-K)^+$, *i.e.*

$$g(0,y) = \psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big) = \big(\mathrm{e}^{|\sigma|y} - K\big)^+, \qquad y \in \mathbb{R}.$$

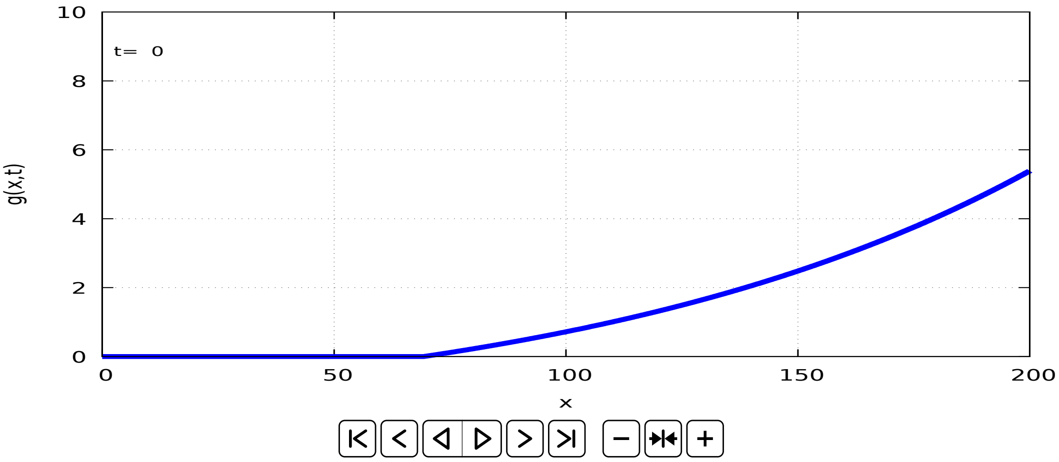


Figure 7.19: Time-dependent solution of the heat equation.*

Let us provide a second proof of Proposition 7.10, this time using Brownian motion and stochastic calculus.

Proof of Proposition 7.10. First, note that under the change of variable $x = z - y$ we have

$$\begin{aligned} g(t,y) &= \int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{\infty} \psi(y+x)\, \mathrm{e}^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} \\ &= \mathbb{E}[\psi(y+B_t)] \\ &= \mathbb{E}[\psi(y-B_t)], \end{aligned}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $B_t \simeq \mathcal{N}(0,t)$, $t \in \mathbb{R}_+$. Applying Itô's formula and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, see Relation (5.17) in Proposition 5.7, we find

$$\begin{aligned} g(t,y) &= \mathbb{E}[\psi(y-B_t)] \\ &= \psi(y) - \mathbb{E}\left[\int_0^t \psi'(y-B_s)dB_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \psi''(y-B_s)ds\right] \\ &= \psi(y) + \frac{1}{2}\int_0^t \mathbb{E}[\psi''(y-B_s)]\,ds \\ &= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\mathbb{E}[\psi(y-B_s)]\,ds \\ &= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial y^2}(s,y)ds. \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\mathbb{E}[\psi(y-B_t)] \\ &= \frac{1}{2}\frac{\partial^2}{\partial y^2}\mathbb{E}[\psi(y-B_t)] \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y). \end{aligned}$$

*The animation works in Acrobat Reader on the entire pdf file.

Regarding the initial condition, we check that

$$g(0,y) = \mathbb{E}[\psi(y - B_0)] = \mathbb{E}[\psi(y)] = \psi(y).$$

$\square$

The expression $g(t,y) = \mathbb{E}[\psi(y - B_t)]$ provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion. Namely, when $\psi_\varepsilon(y) := \mathbb{1}_{[-\varepsilon,\varepsilon]}(y)$, we find that

$$\begin{aligned} g_\varepsilon(t,y) &= \mathbb{E}[\psi_\varepsilon(y - B_t)] \\ &= \mathbb{E}[\mathbb{1}_{[-\varepsilon,\varepsilon]}(y - B_t)] \\ &= \mathbb{P}\big(y - B_t \in [-\varepsilon,\varepsilon]\big) \\ &= \mathbb{P}\big(y - \varepsilon \leqslant B_t \leqslant y + \varepsilon\big) \end{aligned}$$

represents the probability of finding B_t within a neighborhood $[y-\varepsilon, y+\varepsilon]$ of the point $y \in \mathbb{R}$.

7.6 Solution of the Black-Scholes PDE

In this section we solve the Black-Scholes PDE by the kernel method of Section 7.5 and a change of variables. This solution method uses the change of variables (7.23) of Proposition 7.8 and a time inversion from which the terminal condition at time T in the Black-Scholes equation becomes an initial condition at time $t = 0$ in the heat equation.

Next, we state the proof Proposition 7.8.

Proof. Letting $s = T - t$ and $x = \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}$ and using Relation (7.23), *i.e.*

$$g(t,y) = \mathrm{e}^{rt} f\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big),$$

we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= r\mathrm{e}^{rt} f\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &\quad + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &= r\mathrm{e}^{rt} f(T - t, x) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}(T - t, x) + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x) \\ &= \frac{1}{2}\mathrm{e}^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T - t, x) + \frac{\sigma^2}{2}\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x), \end{aligned} \tag{7.29}$$

where on the last step we used the Black-Scholes PDE. On the other hand we have

$$\frac{\partial g}{\partial y}(t,y) = |\sigma|\,\mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big)$$

and

$$\begin{aligned} \frac{1}{2}\frac{\partial g^2}{\partial y^2}(t,y) &= \frac{\sigma^2}{2}\mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &\quad + \frac{\sigma^2}{2}\mathrm{e}^{rt}\, \mathrm{e}^{2|\sigma|y + 2(\sigma^2/2 - r)t}\frac{\partial^2 f}{\partial x^2}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \end{aligned}$$

$$= \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T-t,x) + \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T-t,x). \tag{7.30}$$

We conclude by comparing (7.29) with (7.30), which shows that $g(t,x)$ solves the heat equation (7.27) with initial condition

$$g(0,y) = f\left(T, e^{|\sigma|y}\right) = h\left(e^{|\sigma|y}\right).$$

□

In the next proposition, we derive the Black-Scholes formula (7.10) by solving the PDE (7.22). The Black-Scholes formula will also be recovered by a probabilistic argument via the computation of an expected value in Proposition 8.5.

Proposition 7.11 When $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE (7.22) is given by

$$f(t,x) = x\Phi\big(d_+(T-t)\big) - K e^{-(T-t)r}\Phi\big(d_-(T-t)\big), \qquad x > 0,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) := \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \\ \\ d_-(T-t) := \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \end{cases}$$

$x > 0$, $t \in [0,T)$.

Proof. By inversion of Relation (7.23) with $s = T-t$ and $x = e^{|\sigma|y + (\sigma^2/2-r)t}$, we get

$$f(s,x) = e^{-(T-s)r} g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right)$$

and

$$h(x) = \psi\left(\frac{\log x}{|\sigma|}\right), \quad x > 0, \quad \text{or} \quad \psi(y) = h\left(e^{|\sigma|y}\right), \quad y \in \mathbb{R}.$$

Hence, using the solution (7.28) and Relation (7.24), we get

$$\begin{aligned}
f(t,x) &= e^{-(T-t)r} g\left(T-t, \frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|}\right) \\
&= e^{-(T-t)r} \int_{-\infty}^{\infty} \psi\left(\frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|} + z\right) e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= e^{-(T-t)r} \int_{-\infty}^{\infty} h\left(x e^{|\sigma|z-(\sigma^2/2-r)(T-t)}\right) e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= e^{-(T-t)r} \int_{-\infty}^{\infty} \left(x e^{|\sigma|z-(\sigma^2/2-r)(T-t)} - K\right)^+ e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= e^{-(T-t)r} \\
&\quad \times \int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{|\sigma|}}^{\infty} \left(x e^{|\sigma|z-(\sigma^2/2-r)(T-t)} - K\right) e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x e^{-(T-t)r} \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} e^{|\sigma|z-(\sigma^2/2-r)(T-t)} e^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\end{aligned}$$

$$
\begin{aligned}
& -K\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
= \ & x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{|\sigma|z-(T-t)\sigma^2/2-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
& -K\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
= \ & x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-(z-(T-t)|\sigma|)^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
& -K\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
= \ & x\int_{-d_-(T-t)\sqrt{T-t}-(T-t)|\sigma|}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
& -K\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
= \ & x\int_{-d_-(T-t)-|\sigma|\sqrt{T-t}}^{\infty}\mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}}-K\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)}^{\infty}\mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}} \\
= \ & x\big(1-\Phi\big(-d_+(T-t)\big)\big)-K\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(-d_-(T-t)\big)\big) \\
= \ & x\Phi\big(d_+(T-t)\big)-K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),
\end{aligned}
$$

where we used the relation (7.14), *i.e.*

$$1-\Phi(a)=\Phi(-a), \qquad a\in\mathbb{R}.$$

□

Exercises

Exercise 7.1 Bachelier, 1900 model. Consider a market made of a riskless asset valued $A_t = A_0$ with zero interest rate, $t\in\mathbb{R}_+$, and a risky asset whose price S_t is modeled by a standard Brownian motion as $S_t = B_t$, $t\in\mathbb{R}_+$.

a) Show that the price $g(t,B_t)$ of the option with payoff $C = B_T^2$ satisfies the heat equation

$$\frac{\partial\varphi}{\partial t}(t,y) = -\frac{1}{2}\frac{\partial^2\varphi}{\partial y^2}(t,y)$$

with terminal condition $g(T,x) = x^2$.

b) Find the function $g(t,x)$ by solving the PDE of Question (a)).

Hint: Try a solution of the form $g(t,x) = x^2 + f(t)$.

See Exercises 7.11, 8.13 and 8.14 for extensions to nonzero interest rates.

Exercise 7.2 Consider a risky asset price $(S_t)_{t\in\mathbb{R}}$ modeled in the Cox, Ingersoll, and Ross, 1985 (CIR) model as

$$dS_t = \beta(\alpha - S_t)dt + \sigma\sqrt{S_t}dB_t, \qquad \alpha,\beta,\sigma>0, \tag{7.31}$$

and let $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, takes the form $V_t = g(t,S_t)$, $t\in\mathbb{R}_+$. Figure 7.20 presents a random simulation of the solution to (7.31) with $\alpha = 0.025$, $\beta = 1$, and $\sigma = 1.3$.

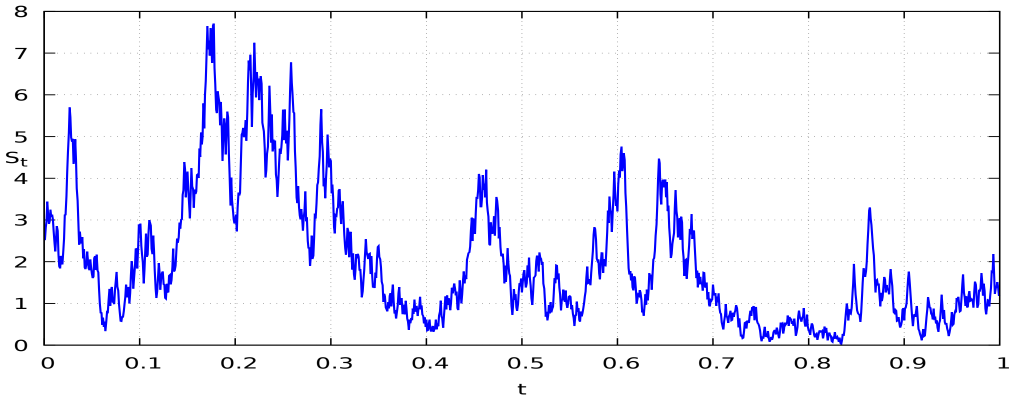


Figure 7.20: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N;a=0.025; b=2; sigma=0.05;
X <- rnorm(N,mean=0,sd=sqrt(dt));R <- rep(0,N);R[1]=0.01
for (j in 2:N){R[j]=max(0,R[j-1]+(a-b*R[j-1])*dt+sigma*sqrt(R[j-1])*X[j])}
plot(t, R, xlab = "t", ylab = "", type = "l", ylim = c(0,0.02), col = "blue")
```

Based on the self-financing condition written as

$$\begin{aligned} dV_t &= rV_t dt - r\xi_t S_t dt + \xi_t dS_t \\ &= rV_t dt - r\xi_t S_t dt + \beta(\alpha - S_t)\xi_t dt + \sigma \xi_t \sqrt{S_t} dB_t, \quad t \in \mathbb{R}_+, \end{aligned} \tag{7.32}$$

derive the PDE satisfied by the function $g(t,x)$ using the Itô formula.

Exercise 7.3 Black-Scholes PDE with dividends. Consider a riskless asset with price $A_t = A_0 \mathrm{e}^{rt}$, $t \in \mathbb{R}_+$, and an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails a drop in the stock price by the same amount occuring generally on the *ex-dividend date*, on which the purchase of the security no longer entitles the investor to the dividend amount. The list of investors entitled to dividend payment is consolidated on the *date of record*, and payment is made on the *payable date*.

```
library(quantmod)
getSymbols("0005.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
getDividends("0005.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
```

a) Assuming that the portfolio with value $V_t = \xi_t S_t + \eta_t A_t$ at time t is self-financing and that dividends are continuously reinvested, write down the portfolio variation dV_t.
b) Assuming that the portfolio value V_t takes the form $V_t = g(t, S_t)$ at time t, derive the Black-Scholes PDE for the function $g(t,x)$ with its terminal condition.
c) Compute the price at time $t \in [0,T]$ of the European call option with strike price K by solving the corresponding Black-Scholes PDE.
d) Compute the Delta of the option.

Exercise 7.4

a) Check that the Black-Scholes formula (7.10) for European call options

$$g_c(t,x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

satisfies the following boundary conditions:
   i) at $x=0$, $g_c(t,0)=0$,
   ii) at maturity $t=T$,

$$g_c(T,x) \quad = \quad (x-K)^+ = \begin{cases} x-K, & x>K \\ \\ 0, & x \leqslant K, \end{cases}$$

   iii) as time to maturity tends to infinity,

$$\lim_{T\to\infty} \mathrm{Bl}(K,x,\sigma,r,T-t) = x, \qquad t \in \mathbb{R}_+.$$

b) Check that the Black-Scholes formula (7.18) for European put options

$$g_p(t,x) = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)$$

satisfies the following boundary conditions:
   i) at $x=0$, $g_p(t,0)=K\,\mathrm{e}^{-(T-t)r}$,
   ii) as x tends to infinity, $g_p(t,\infty)=0$ for all $t\in[0,T)$,
   iii) at maturity $t=T$,

$$g_p(T,x) \quad = \quad (K-x)^+ = \begin{cases} 0, & x>K \\ \\ K-x, & x \leqslant K, \end{cases}$$

   iv) as time to maturity tends to infinity,

$$\lim_{T\to\infty} \mathrm{Bl}_p(K,S_t,\sigma,r,T-t) = 0, \qquad t \in \mathbb{R}_+.$$

Exercise 7.5 Power option. (Exercise 4.14 continued).

a) Solve the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{7.33}$$

with terminal condition $g(x,T)=x^2$, $x>0$, $t\in[0,T]$.

Hint: Try a solution of the form $g(x,t)=x^2 f(t)$, and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t=\mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t, \qquad 0 \leqslant t \leqslant T,$$

hedging the contract with payoff S_T^2 at maturity.

Exercise 7.6 On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price S of the MTR Corporation with maturity $T=23/12/2008$, strike price $K=$ HK\$ 36.08 and entitlement ratio=10. Recall that in the Black-Scholes model, the price at time t of the European claim on the underlying asset priced S_t with strike price K, maturity T, interest rate r and volatility $\sigma>0$ is given by the Black-Scholes formula as

$$f(t,S_t) = S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\begin{cases} d_-(T-t) = \dfrac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}, \\ \\ d_+(T-t) = d_-(T-t)+|\sigma|\sqrt{T-t} = \dfrac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}. \end{cases}$$

Recall that by Proposition 7.4 we have

$$\frac{\partial f}{\partial x}(t,S_t) = \Phi\big(d_+(T-t)\big), \qquad 0 \leqslant t \leqslant T.$$

a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time t =November 07, 2008 with S_t = HK\$ 17.200, assuming a volatility $\sigma = 90\% = 0.90$ and an *annual* risk-free interest rate $r = 4.377\% = 0.04377$,

b) Still using the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time t =November 07, 2008 in order to hedge one such option at maturity $T = 23/12/2008$.

c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

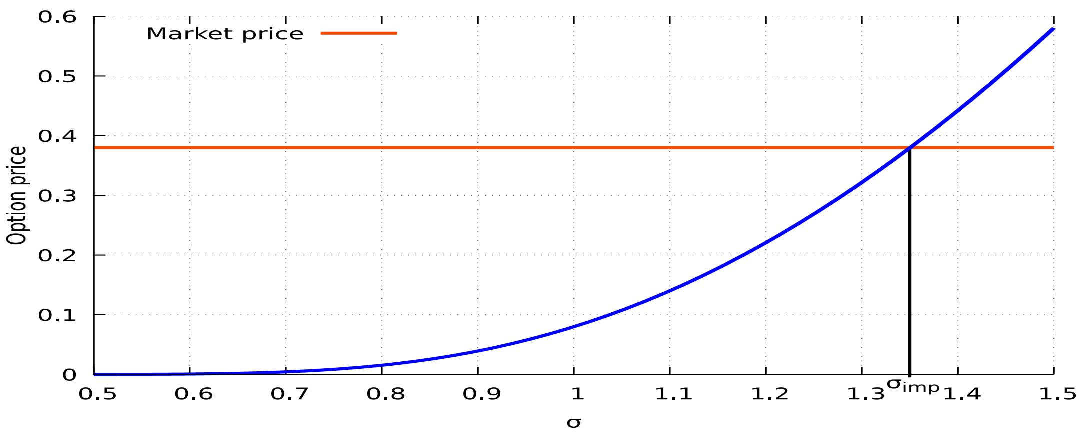


Figure 7.21: Option price as a function of the volatility $\sigma > 0$.

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2);BSCall}
sigma <- seq(0.5,1.5, length=100);
plot(sigma,BSCall(17.2,36.08,0.04377,46/365,sigma) , type="l",lty=1, xlab="Sigma",
      ylab="Black-Scholes Call Price", ylim = c(0,0.6),col="blue",lwd=3);grid()
abline(h=0.23,col="red",lwd=3)
```

Knowing that the closing price of the warrant on November 07, 2008 was HK\$ 0.023, which value can you infer for the implied volatility σ at this date?*

Exercise 7.7 Forward contracts. Recall that the price $\pi_t(C)$ of a claim payoff $C = h(S_T)$ of maturity T can be written as $\pi_t(C) = g(t,S_t)$, where the function $g(t,x)$ satisfies the *Black-Scholes*

*Download the corresponding **R code** or the **IPython notebook** that can be run **here**.

PDE

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = h(x), \end{cases} \qquad (1)$$

with terminal condition $g(T,x) = h(x)$, $x > 0$.

a) Assume that C is a forward contract with payoff

$$C = S_T - K,$$

at time T. Find the function $h(x)$ in (1).

b) Find the solution $g(t,x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0,T]$. *Hint*: search for a solution of the form $g(t,x) = x - \alpha(t)$ where $\alpha(t)$ is a function of t to be determined.

c) Compute the quantity

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t)$$

of risky assets in a self-financing portfolio hedging C.

d) Repeat the above questions with the terminal condition $g(T,x) = x$.

Exercise 7.8

a) Solve the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(t,x) \tag{7.34}$$

with terminal condition $g(T,x) = 1$, $x > 0$.

Hint: Try a solution of the form $g(t,x) = f(t)$ and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = \mathrm{e}^{rt}$ in the portfolio with value

$$V_t = g(t,S_t) = \xi_t S_t + \eta_t A_t$$

hedging the contract with payoff \$1 at maturity.

Exercise 7.9 Log-contracts.

a) Solve the PDE

$$0 = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{7.35}$$

with the terminal condition $g(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $g(x,t) = f(t) + \log x$, and find $f(t)$.

b) Solve the Black-Scholes PDE

$$rh(x,t) = \frac{\partial h}{\partial t}(x,t) + rx\frac{\partial h}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 h}{\partial x^2}(x,t) \tag{7.36}$$

with the terminal condition $h(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $h(x,t) = u(t)g(x,t)$, and find $u(t)$.

c) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = e^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t$$

hedging a log-contract with payoff $\log S_T$ at maturity.

Exercise 7.10 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. The binary (or digital) *call* option is a contract with maturity T, strike price K, and payoff

$$C_d := \mathbb{1}_{[K,\infty)}(S_T) = \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K. \end{cases}$$

a) Derive the Black-Schole PDE satisfied by the pricing function $C_d(t,S_t)$ of the binary call option, together with its terminal condition.
b) Show that the solution $C_d(t,x)$ of the Black-Scholes PDE of Question (a)) is given by

$$\begin{aligned} C_d(t,x) &= e^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{|\sigma|\sqrt{T-t}}\right) \\ &= e^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

where

$$d_-(T-t) := \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}, \qquad 0 \leqslant t < T.$$

Exercise 7.11

a) Bachelier, 1900 model. Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t \tag{7.37}$$

in terms of $\alpha, \sigma \in \mathbb{R}$, and the initial condition S_0.
b) Write down the Bachelier PDE satisfied by the function $C(t,x)$, where $C(t,S_t)$ is the price at time $t \in [0,T]$ of the contingent claim with payoff $\phi(S_T) = \exp(S_T)$, and identify the process Delta $(\xi_t)_{t\in[0,T]}$ that hedges this claim.
c) Solve the Black-Scholes PDE of Question (b)) with the terminal condition $\phi(x) = e^x$, $x \in \mathbb{R}$.

Hint: Search for a solution of the form

$$C(t,x) = \exp\left(-(T-t)r + xh(t) + \frac{\sigma^2}{4r}(h^2(t)-1)\right), \tag{7.38}$$

where $h(t)$ is a function to be determined, with $h(T) = 1$.
d) Compute the portfolio strategy $(\xi_t,\eta_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

Exercise 7.12

a) Show that for every fixed value of S, the function

$$d \longmapsto h(S,d) := S\Phi\big(d+|\sigma|\sqrt{T}\big) - K e^{-rT}\Phi(d),$$

reaches its maximum at $d_*(S) := \dfrac{\log(S/K)+(r-\sigma^2/2)T}{|\sigma|\sqrt{T}}$.

b) By the differentiation rule

$$\frac{d}{dS}h(S,d_*(S)) = \frac{\partial h}{\partial S}(S,d_*(S)) + d_*'(S)\frac{\partial h}{\partial d}(S,d_*(S)),$$

recover the value of the Black-Scholes Delta.

Exercise 7.13 Compute the Black-Scholes Vega by differentiation of the Black-Scholes function

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

with respect to the volatility parameter σ, knowing that

$$\begin{aligned} -\frac{1}{2}\big(d_-(T-t)\big)^2 &= -\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2 \\ &= -\frac{1}{2}\big(d_+(T-t)\big)^2 + (T-t)r + \log\frac{x}{K}. \end{aligned} \tag{7.39}$$

Exercise 7.14 Consider the backward induction relation (4.13), *i.e.*

$$\widetilde{v}(t,x) = (1-p_N^*)\widetilde{v}(t+1,x(1+a_N)) + p_N^*\widetilde{v}(t+1,x(1+b_N)),$$

using the renormalizations $r_N := rT/N$ and

$$a_N := (1+r_N)(1-|\sigma|\sqrt{T/N})-1, \quad b_N := (1+r_N)(1+|\sigma|\sqrt{T/N})-1,$$

of Section 4.6, $N \geqslant 1$, with

$$p_N^* = \frac{r_N-a_N}{b_N-a_N} \quad \text{and} \quad p_N^* = \frac{b_N-r_N}{b_N-a_N}.$$

a) Show that the Black-Scholes PDE (7.2) of Proposition 7.1 can be recovered from the induction relation (4.13) when the number N of time steps tends to infinity.

b) Show that the expression of the Delta $\xi_t = \dfrac{\partial g_c}{\partial x}(t,S_t)$ can be similarly recovered from the finite difference relation (4.19), *i.e.*

$$\xi_t^{(1)}(S_{t-1}) = \frac{v(t,(1+b_N)S_{t-1}) - v(t,(1+a_N)S_{t-1})}{S_{t-1}(b_N-a_N)}$$

as N tends to infinity.

8. Martingale Approach to Pricing and Hedging

In the *martingale approach* to the pricing and hedging of financial derivatives, option prices are expressed as the expected values of discounted option payoffs. This approach relies on the construction of risk-neutral probability measures by the Girsanov theorem, and the associated hedging portfolios are obtained via stochastic integral representations.

8.1 Martingale Property of the Itô Integral

Recall (Definition 6.5) that an integrable process $(X_t)_{t\in\mathbb{R}_+}$ is said to be a *martingale* with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[X_t \mid \mathscr{F}_s] = X_s, \qquad 0 \leqslant s \leqslant t.$$

Examples of martingales (i)

1. Given $F \in L^2(\Omega)$ a square-integrable random variable and $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ a filtration, the process $(X_t)_{t\in\mathbb{R}_+}$ defined by

$$X_t := \mathbb{E}[F \mid \mathscr{F}_t], \qquad t \in \mathbb{R}_+,$$

is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-martingale under $\mathbb{P}$, as follows from the tower property (11.38) of conditional expectations:

$$\mathbb{E}[X_t \mid \mathscr{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathscr{F}_t] \mid \mathscr{F}_s] = \mathbb{E}[F \mid \mathscr{F}_s] = X_s, \quad 0 \leqslant s \leqslant t. \tag{8.1}$$

2. Any integrable stochastic process $(X_t)_{t\in\mathbb{R}_+}$ whose increments $(X_{t_1}-X_{t_0},\ldots,X_{t_n}-X_{t_{n-1}})$ are independent and centered under $\mathbb{P}$ (*i.e.* $\mathbb{E}[X_t]=0$, $t\in\mathbb{R}_+$) is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(X_t)_{t\in\mathbb{R}_+}$, as we have

$$\begin{aligned}\mathbb{E}[X_t \mid \mathscr{F}_s] &= \mathbb{E}[X_t-X_s+X_s \mid \mathscr{F}_s]\\ &= \mathbb{E}[X_t-X_s \mid \mathscr{F}_s]+\mathbb{E}[X_s \mid \mathscr{F}_s]\\ &= \mathbb{E}[X_t-X_s]+X_s\\ &= X_s, \qquad 0\leqslant s\leqslant t. \end{aligned} \tag{8.2}$$

In particular, the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale because it has centered and independent increments. This fact is also consequence of Proposition 8.1 below as B_t can be written as

$$B_t=\int_0^t dB_s, \qquad t\in\mathbb{R}_+.$$

The following result shows that the Itô integral yields a martingale with respect to the Brownian filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. It is the continuous-time analog of the discrete-time Theorem 3.4.

Proposition 8.1 The stochastic integral process $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ of a square-integrable adapted process $u\in L^2_{\rm ad}(\Omega\times\mathbb{R}_+)$ is a martingale, *i.e.*:

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \,\middle|\, \mathscr{F}_s\right]=\int_0^s u_\tau dB_\tau, \qquad 0\leqslant s\leqslant t. \tag{8.3}$$

In particular, $\int_0^t u_s dB_s$ is $\mathscr{F}_t$-measurable, $t\in\mathbb{R}_+$, and since $\mathscr{F}_0=\{\emptyset,\Omega\}$, Relation (8.3) applied with $t=0$ recovers the fact that the Itô integral is a centered random variable:

$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right]=\mathbb{E}\left[\int_0^\infty u_s dB_s \,\middle|\, \mathscr{F}_0\right]=\int_0^0 u_s dB_s=0.$$

Proof. The statement is first proved in case $(u_t)_{t\in\mathbb{R}_+}$ is a simple predictable process, and then extended to the general case, cf. *e.g.* Proposition 2.5.7 in Privault, 2009. For example, for u a predictable step process of the form

$$u_s:=F\mathbb{1}_{[a,b]}(s)=\begin{cases} F & \text{if } s\in[a,b],\\ 0 & \text{if } s\notin[a,b],\end{cases}$$

with F an $\mathscr{F}_a$-measurable random variable and $t\in[a,b]$, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty u_s dB_s \,\middle|\, \mathscr{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbb{1}_{[a,b]}(s) dB_s \,\middle|\, \mathscr{F}_t\right]\\ &= \mathbb{E}[F(B_b-B_a)\mid\mathscr{F}_t]\\ &= F\,\mathbb{E}[B_b-B_a\mid\mathscr{F}_t]\\ &= F(B_t-B_a)\\ &= \int_a^t u_s dB_s\\ &= \int_0^t u_s dB_s, \qquad a\leqslant t\leqslant b.\end{aligned}$$

On the other hand, when $t\in[0,a]$ we have

$$\int_0^t u_s dB_s=0,$$

and we check that

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F \mathbb{1}_{[a,b]}(s) dB_s \,\Big|\, \mathcal{F}_t\right] \\
&= \mathbb{E}[F(B_b - B_a) \mid \mathcal{F}_t] \\
&= \mathbb{E}[\mathbb{E}[F(B_b - B_a) \mid \mathcal{F}_a] \mid \mathcal{F}_t] \\
&= \mathbb{E}[F\,\mathbb{E}[B_b - B_a \mid \mathcal{F}_a] \mid \mathcal{F}_t] \\
&= 0, \qquad 0 \leqslant t \leqslant a,
\end{aligned}$$

where we used the tower property (11.38) of conditional expectations and the fact that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale:

$$\mathbb{E}[B_b - B_a \mid \mathcal{F}_a] = \mathbb{E}[B_b \mid \mathcal{F}_a] - B_a = B_a - B_a = 0.$$

The extension from simple processes to square-integrable processes in $L^2_{\text{ad}}(\Omega \times \mathbb{R}_+)$ can be proved as in Proposition 5.7. Indeed, given $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ be a sequence of simple predictable processes converging to u in $L^2(\Omega \times [0,T])$ cf. Lemma 1.1 of Ikeda and Watanabe, 1989, pages 22 and 46, by Fatou's Lemma 11.2 and Jensen's inequality we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right] \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^{(n)} dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right]\right)^2\right] \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s\right)^2 \,\Big|\, \mathcal{F}_t\right]\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^{(n)} - u_s) dB_s\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^{(n)} - u_s|^2 ds\right] \\
&= \liminf_{n\to\infty} \|u^{(n)} - u\|^2_{L^2(\Omega\times[0,T])} \\
&= 0,
\end{aligned}$$

where we used the Itô isometry (5.16). We conclude that

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathcal{F}_t\right] = \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

for $u \in L^2_{\text{ad}}(\Omega \times \mathbb{R}_+)$ a square-integrable adapted process, which leads to (8.3) after applying this identity to the process $(\mathbb{1}_{[0,t]} u_s)_{s\in\mathbb{R}_+}$, *i.e.*,

$$\begin{aligned}
\mathbb{E}\left[\int_0^t u_\tau dB_\tau \,\Big|\, \mathcal{F}_s\right] &= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau \,\Big|\, \mathcal{F}_s\right] \\
&= \int_0^s \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau \\
&= \int_0^s u_\tau dB_\tau, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

□

Examples of martingales (ii)

1. The driftless geometric Brownian motion

$$X_t := X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$$

is a martingale. Indeed, we have

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathscr{F}_s] &= \mathbb{E}\left[X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \,\mathbb{E}\left[\mathrm{e}^{\sigma B_t} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma + \sigma B_s} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma}\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathrm{e}^{\sigma^2 (t-s)/2} \\
&= X_0 \,\mathrm{e}^{\sigma B_s - \sigma^2 s/2} \\
&= X_s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

This fact can also be recovered from Proposition 8.1 since $(X_t)_{t \in \mathbb{R}_+}$ satisfies the equation

$$dX_t = \sigma X_t dB_t,$$

which shows that X_t can be written using the Brownian stochastic integral

$$X_t = X_0 + \sigma \int_0^t X_u dB_u, \qquad t \in \mathbb{R}_+.$$

2. Consider an asset price process $(S_t)_{t \in \mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+. \tag{8.4}$$

By the *Discounting Lemma* 6.6, the discounted asset price process $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$, $t \in \mathbb{R}_+$, satisfies the stochastic differential equation

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t).$$

The discounted asset price

$$\widetilde{S}_t = \mathrm{e}^{-rt} S_t = S_0 \,\mathrm{e}^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}$ when $\mu = r$. The case $\mu \neq r$ will be treated in Section 8.3 using risk-neutral probability measures and the Girsanov Theorem 8.2, see (8.14) below.

3. The discounted value

$$\widetilde{V}_t = \mathrm{e}^{-rt} V_t, \qquad t \geqslant 0,$$

of a self-financing portfolio is given by

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \geqslant 0,$$

cf. Lemma 6.7 is a martingale when $\mu = r$ by Proposition 8.1 because

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u dB_u, \qquad t \geqslant 0,,$$

since we have

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t) = \sigma \widetilde{S}_t dB_t$$

by the *Discounting Lemma* 6.6. Since the Black-Scholes theory is in fact valid for any value of the parameter μ we will look forward to including the case $\mu \neq r$ in the sequel.

8.2 Risk-neutral Probability Measures

Recall that by definition, a risk-neutral measure is a probability measure $\mathbb{P}^*$ under which the discounted asset price process

$$\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} := \big(\mathrm{e}^{-rt}S_t\big)_{t\in\mathbb{R}_+}$$

is a *martingale*, cf. Proposition 6.1.

Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation (8.4). Note that when $\mu = r$, the discounted asset price process $\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} = \big(S_0\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}\big)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^* = \mathbb{P}$, which is a risk-neutral probability measure.

In this section, we address the construction of a risk-neutral probability measure $\mathbb{P}^*$ in the general case $\mu \neq r$ using the Girsanov Theorem 8.2 below. Note that the relation

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t)$$

where $\mu - r$ is the risk premium, can be rewritten as

$$d\widetilde{S}_t = \sigma\widetilde{S}_t d\widehat{B}_t,$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a drifted Brownian motion given by

$$\widehat{B}_t := \frac{\mu - r}{\sigma}t + B_t, \qquad t \in \mathbb{R}_+,$$

where the drift coefficient $\nu := (\mu - r)/\sigma$ is the "Market Price of Risk" (MPoR). It represents the difference between the return μ expected when investing in the risky asset S_t, and the risk-free interest rate r, measured in units of volatility σ.

Therefore, the search for a risk-neutral probability measure can be replaced by the search for a probability measure $\mathbb{P}^*$ under which $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm\sqrt{\Delta t},$$

with

$$\mathbb{P}\big(\Delta B_t = +\sqrt{\Delta t}\big) = \mathbb{P}\big(\Delta B_t = -\sqrt{\Delta t}\big) = \frac{1}{2}.$$

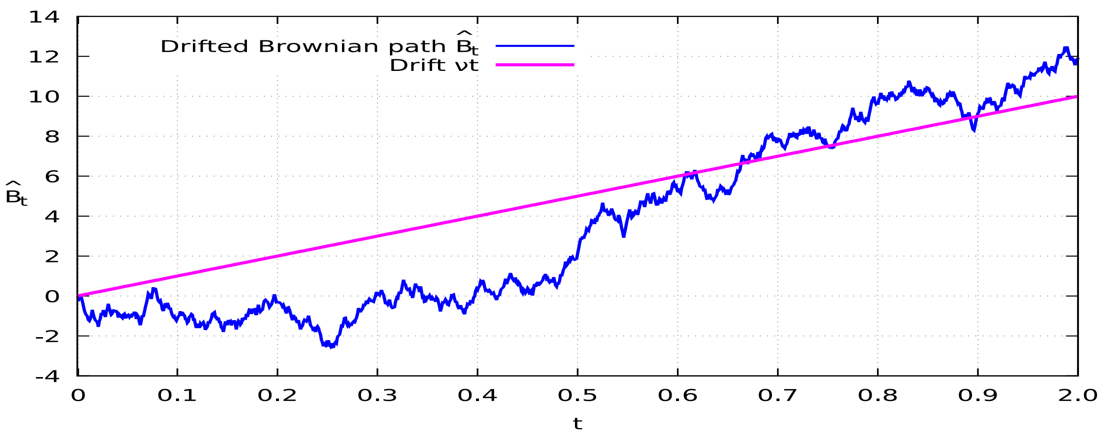


Figure 8.1: Drifted Brownian path $(\widehat{B}_t)_{t\in\mathbb{R}_+}$.

Clearly, given $\nu \in \mathbb{R}$, the drifted process

$$\widehat{B}_t := \nu t + B_t, \qquad t \in \mathbb{R}_+,$$

is no longer a standard Brownian motion because it is not centered:

$$\mathbb{E}\big[\widehat{B}_t\big] = \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

cf. Figure 8.1. This identity can be formulated in terms of infinitesimal increments as

$$\mathbb{E}[\nu \Delta t + \Delta B_t] = \frac{1}{2}(\nu \Delta t + \sqrt{\Delta t}) + \frac{1}{2}(\nu \Delta t - \sqrt{\Delta t}) = \nu \Delta t \neq 0.$$

In order to make $\nu t + B_t$ a centered process (*i.e.* a standard Brownian motion, since $\nu t + B_t$ conserves all the other properties (i)-(iii) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to $1/2$.

That is, the problem is now to find two numbers $p^*, q^* \in [0,1]$ such that

$$\begin{cases} p^*(\nu \Delta t + \sqrt{\Delta t}) + q^*(\nu \Delta t - \sqrt{\Delta t}) = 0 \\ \\ p^* + q^* = 1. \end{cases}$$

The solution to this problem is given by

$$p^* = \frac{1}{2}(1 - \nu\sqrt{\Delta t}) \quad \text{and} \quad q^* = \frac{1}{2}(1 + \nu\sqrt{\Delta t}). \tag{8.5}$$

Coming back to Brownian motion considered as a discrete random walk with independent increments $\pm\sqrt{\Delta t}$, we try to construct a new probability measure denoted $\mathbb{P}^*$, under which the drifted process $\widehat{B}_t := \nu t + B_t$ will be a standard Brownian motion. This probability measure will be defined through its Radon-Nikodym density

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} &:= \frac{\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})} \\ &= \frac{\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})} \\ &= \frac{1}{(1/2)^N}\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t}), \end{aligned} \tag{8.6}$$

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1, 1\}$, with respect to the historical probability measure $\mathbb{P}$, obtained by taking the product of the above probabilities divided by the reference probability $1/2^N$ corresponding to the symmetric random walk.

Interpreting $N = T/\Delta t$ as an (infinitely large) number of discrete time steps and under the identification $[0,T] \simeq \{0 = t_0, t_1, \ldots, t_N = T\}$, this Radon-Nikodym density (8.6) can be rewritten as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) \tag{8.7}$$

where 2^N becomes a normalization factor. Using the expansion

$$\log\left(1 \pm \nu\sqrt{\Delta t}\right) = \pm\nu\sqrt{\Delta t} - \frac{1}{2}(\pm\nu\sqrt{\Delta t})^2 + o(\Delta t)$$

$$= \pm\nu\sqrt{\Delta t} - \frac{\nu^2}{2}\Delta t + o(\Delta t),$$

for small values of Δt, this Radon-Nikodym density can be informally shown to converge as follows as N tends to infinity, *i.e.* as the time step $\Delta t = T/N$ tends to zero:

$$\begin{aligned}
2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) &= \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right) \\
&= \exp\left(\log \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{\Delta t} - \frac{1}{2}\sum_{0<t<T} (\mp\nu\sqrt{\Delta t})^2\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \pm\sqrt{\Delta t} - \frac{\nu^2}{2}\sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \Delta B_t - \frac{\nu^2}{2}\sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right),
\end{aligned}$$

based on the identifications

$$B_T \simeq \sum_{0<t<T} \pm\sqrt{\Delta t} \quad \text{and} \quad T \simeq \sum_{0<t<T} \Delta t.$$

Informally, the drifted process $(\widehat{B}_t)_{t\in[0,T]} = (\nu t + B_t)_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right).$$

The following R code is rescaling probabilities as in (8.5) based on the value of the drift μ.

```
N=1000; t <- 0:N; dt <- 1.0/N; nu=3; p=0.5*(1-nu*(dt)^0.5); nsim <- 10
X <- matrix((dt)^0.5*(rbinom( nsim * N, 1, p)-0.5)*2, nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", type = "l", ylim = c(-2*N*dt, 2*N*dt), col = 0)
for (i in 1:nsim){lines(t,t*nu*dt+X[i,],xlab="time",type="l",ylim=c(-2*N*dt,2*N*dt),col=i)}
```

The discretized illustration in Figure 8.2 displays the drifted Brownian motion $\widehat{B}_t := \nu t + B_t$ under the shifted probability measure $\mathbb{P}^*$ in (8.7) using the above R code with $N = 100$. The code makes big transitions less frequent than small transitions, resulting into a standard, centered Brownian motion under $\mathbb{P}^*$.

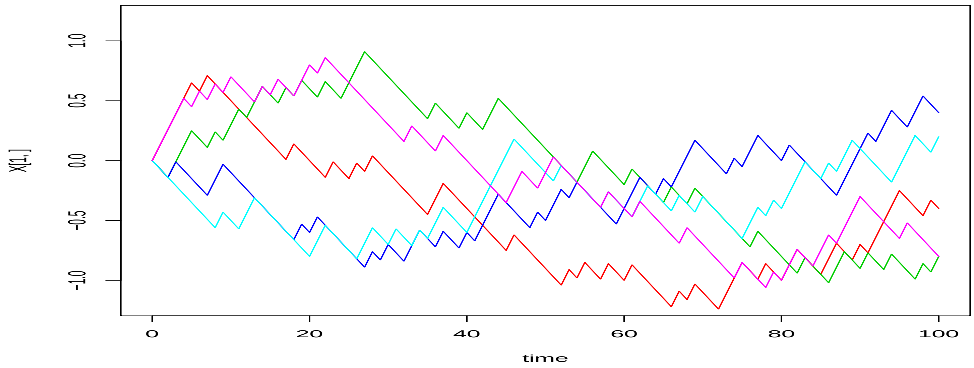


Figure 8.2: Drifted Brownian motion paths under a shifted Girsanov measure.

8.3 Change of Measure and the Girsanov Theorem

In this section we restate the Girsanov Theorem in a more rigorous way, using changes of probability measures.

Definition 8.1 We say that a probability measure $\mathbb{Q}$ is absolutely continuous with respect to another probability measure $\mathbb{P}$ if there exists a nonnegative random variable $F : \Omega \longrightarrow \mathbb{R}$ such that $\mathbb{E}[F] = 1$, and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = F, \qquad \textit{i.e.} \qquad d\mathbb{Q} = F d\mathbb{P}. \tag{8.8}$$

In this case, F is called the Radon-Nikodym *density* of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Relation (8.8) is equivalent to the relation

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[G] &= \int_\Omega G(\omega) d\mathbb{Q}(\omega) \\
&= \int_\Omega G(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega G(\omega) F(\omega) d\mathbb{P}(\omega) \\
&= \mathbb{E}[FG],
\end{aligned}$$

for any integrable random variable G.

The Girsanov Theorem can actually be extended to shifts by adapted processes $(\psi_t)_{t\in[0,T]}$ as follows, cf. *e.g.* Theorem III-42, page 141 of Protter, 2004. Recall also that here, $\Omega = \mathscr{C}_0([0,T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0,T]$ starting at 0 in $t = 0$. The Girsanov Theorem 8.2 will be used in Section 8.4 for the construction of a unique risk-neutral probability measure $\mathbb{P}^*$, showing absence of arbitrage and completeness in the Black-Scholes market, see Theorems 6.2 and 6.4.

Theorem 8.2 Let $(\psi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty, \tag{8.9}$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right).$$

Then

$$\widehat{B}_t := B_t + \int_0^t \psi_s ds, \qquad 0 \leqslant t \leqslant T,$$

is a standard Brownian motion under $\mathbb{Q}$.

In the case of the simple shift

$$\widehat{B}_t := B_t + \nu t, \qquad 0 \leqslant t \leqslant T,$$

by a drift νt with constant $\nu \in \mathbb{R}$, the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard (centered) Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right) d\mathbb{P}(\omega).$$

For example, the fact that $\widehat{B}_T$ has a centered Gaussian distribution under $\mathbb{Q}$ can be recovered as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[f(\widehat{B}_T)\right] &= \mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] \\
&= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(\nu T + B_T) \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P} \\
&= \int_{-\infty}^{\infty} f(\nu T + y) \exp\left(-\nu y - \frac{1}{2}\nu^2 T\right) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(\nu T + x) e^{-(\nu T + x)^2/(2T)} \frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(y) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)],
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] &= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(B_T) d\mathbb{P} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)],
\end{aligned} \tag{8.10}$$

showing that, under $\mathbb{Q}$, $\nu T + B_T$ has the centered $\mathscr{N}(0,T)$ Gaussian distribution with variance T. For example, taking $f(x) = x$, Relation (8.10) recovers the fact that $\widehat{B}_T$ is a centered random variable under $\mathbb{Q}$, *i.e.*

$$\mathbb{E}_{\mathbb{Q}}\left[\widehat{B}_T\right] = \mathbb{E}_{\mathbb{Q}}[\nu T + B_T)] = \mathbb{E}_{\mathbb{P}}[B_T] = 0.$$

The Girsanov Theorem 8.2 also allows us to extend (8.10) as

$$\mathbb{E}[F] = \mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s ds\right) \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right)\right], \tag{8.11}$$

for all random variables $F \in L^1(\Omega)$.

When applied to the (constant) market price of risk (or Sharpe ratio)

$$\psi_t := \frac{\mu - r}{\sigma},$$

the Girsanov Theorem 8.2 shows that

$$\widehat{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad 0 \leqslant t \leqslant T, \tag{8.12}$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T \right). \tag{8.13}$$

Hence by Proposition 8.1 the discounted price process $(\widetilde{S}_t)_{t \in \mathbb{R}_+}$ solution of

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t = \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \tag{8.14}$$

is a martingale under $\mathbb{P}^*$, therefore $\mathbb{P}^*$ is a risk-neutral probability measure, and we obviously have $\mathbb{P} = \mathbb{P}^*$ when $\mu = r$.

In the sequel, we consider probability measures $\mathbb{Q}$ that are *equivalent* to $\mathbb{P}$ in the sense that they share the same events of zero probability.

Definition 8.2 A probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{Q}(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}.$$

Note that when $\mathbb{Q}$ is defined by (8.8), it is *equivalent* to $\mathbb{P}$ if and only if $F > 0$ with $\mathbb{P}$-probability one.

8.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that according to the first fundamental theorem of asset pricing Theorem 6.2, a continuous market is without arbitrage opportunities if and only if there exists (at least) an equivalent risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$\widetilde{S}_t := e^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be *complete*.

The equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+,$$

satisfied by the price process $(S_t)_{t \in \mathbb{R}_+}$ can be rewritten using (8.12) as

$$\frac{dS_t}{S_t} = r dt + \sigma d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \tag{8.15}$$

with the solution

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t - \sigma^2 t/2} = S_0 \mathrm{e}^{rt + \sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

By the discounting Lemma 6.6, we have

$$\begin{aligned} d\widetilde{S}_t &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t \\ &= \sigma \widetilde{S}_t \left(\frac{\mu - r}{\sigma} dt + dB_t \right) \\ &= \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{8.16}$$

hence the discounted price process

$$\begin{aligned} \widetilde{S}_t &:= \mathrm{e}^{-rt} S_t \\ &= S_0 \mathrm{e}^{(\mu - r)t + \sigma B_t - \sigma^2 t/2} \\ &= S_0 \mathrm{e}^{\sigma \widehat{B}_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \end{aligned}$$

is a martingale under the probability measure $\mathbb{P}^*$ defined by (8.13). We note that $\mathbb{P}^*$ is a risk-neutral probability measure equivalent to $\mathbb{P}$, also called martingale measure, whose existence and uniqueness ensure absence of arbitrage and completeness according to Theorems 6.2 and 6.4.

Therefore, by Lemma 6.7 the discounted value $\widetilde{V}_t$ of a self-financing portfolio can be written as

$$\begin{aligned} \widetilde{V}_t &= \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u \\ &= \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+, \end{aligned}$$

and by Proposition 8.1 it becomes a martingale under $\mathbb{P}^*$.

As in Chapter 4, the value V_t at time t of a self-financing portfolio strategy $(\xi_t)_{t \in [0,T]}$ hedging an attainable claim payoff C will be called an *arbitrage-free price* of the claim payoff C at time t and denoted by $\pi_t(C)$, $t \in [0,T]$. Arbitrage-free prices can be used to ensure that financial derivatives are "marked" at their fair value ("mark to market").

Theorem 8.3 Let $(\xi_t, \eta_t)_{t \in [0,T]}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0,T],$$

and let C be a contingent claim payoff, such that
(i) $(\xi_t, \eta_t)_{t \in [0,T]}$ is a self-financing portfolio, and

(ii) $(\xi_t, \eta_t)_{t \in [0,T]}$ hedges the claim payoff C, *i.e.* we have $V_T = C$.
Then the arbitrage-free price of the claim payoff C is given by the portfolio value

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r} \mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{8.17}$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral probability measure $\mathbb{P}^*$.

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t \in \mathbb{R}_+}$ is self-financing, by Lemma 6.7 and (8.16) the discounted portfolio value $\widetilde{V}_t = \mathrm{e}^{-rt} V_t$ satisfies

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 8.1, hence

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\big[\widetilde{V}_T \mid \mathscr{F}_t\big] \\ &= \mathrm{e}^{-rT}\,\mathbb{E}^*[V_T \mid \mathscr{F}_t] \\ &= \mathrm{e}^{-rT}\,\mathbb{E}^*[C \mid \mathscr{F}_t],\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

□

Black-Scholes PDE for vanilla options by the martingale method

The martingale method can be used to recover the Black-Scholes PDE of Proposition 7.1. As the process $(S_t)_{t\in\mathbb{R}_+}$ has the Markov property, see Section 5.5, § V-6 of Protter, 2004 and Definition 8.3 below, the value

$$\begin{aligned}V_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid \mathscr{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t], \quad 0 \leqslant t \leqslant T,\end{aligned}$$

of the portfolio at time $t \in [0,T]$ can be written from (8.17) as a function

$$V_t = C(t,S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t] \tag{8.18}$$

of t and S_t, $0 \leqslant t \leqslant T$.

Proposition 8.4 Assume that ϕ is a Lipschitz payoff function, and that

$$(S_t)_{t\in\mathbb{R}_+} = \big(S_0\,\mathrm{e}^{\sigma B_t + (r-\sigma^2)t/2}\big)_{t\in\mathbb{R}_+}$$

is a geometric Brownian motion. Then the function $C(t,x)$ defined in (8.18) is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R}_+)$ and solves the Black-Scholes PDE

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 C}{\partial x^2}(t,x) \\[2ex] C(T,x) = \phi(x), \quad x > 0. \end{cases}$$

Proof. It can be checked by integrations by parts that the function $C(t,x)$ defined by

$$C(t,S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(xS_T/S_t)]_{x=S_t},$$

$0 \leqslant t \leqslant T$, is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R}_+)$ when ϕ is a Lipschitz function, from the properties of the lognormal distribution of S_T. We note that by (5.24), the application of Itô's formula Theorem 5.9 to $V_t = C(t,S_t)$ and (8.15) leads to

$$\begin{aligned} d(\,\mathrm{e}^{-rt}C(t,S_t)) &= -r\,\mathrm{e}^{-rt}C(t,S_t)dt\,\mathrm{e}^{-rt}dC(t,S_t) \\ &= -r\,\mathrm{e}^{-rt}C(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt \\ &\quad + \mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dS_t + \frac{1}{2}\,\mathrm{e}^{-rt}(dS_t)^2\frac{\partial^2 C}{\partial x^2}(t,S_t) \\ &= -r\,\mathrm{e}^{-rt}C(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt \end{aligned}$$

$$
\begin{aligned}
&+v_t\,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dt+u_t\,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)d\widehat{B}_t+\frac{1}{2}\,\mathrm{e}^{-rt}|u_t|^2\frac{\partial^2 C}{\partial x^2}(t,S_t)dt\\
= \ & -r\,\mathrm{e}^{-rt}C(t,S_t)dt+\mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt\\
&+rS_t\,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dt+\frac{1}{2}\,\mathrm{e}^{-rt}\sigma^2S_t^2\frac{\partial^2 C}{\partial x^2}(t,S_t)dt+\sigma\,\mathrm{e}^{-rt}S_t\frac{\partial C}{\partial x}(t,S_t)d\widehat{B}_t.
\end{aligned}
$$

By Lemma 6.7 and Proposition 8.1, the discounted price $\widetilde{V}_t=\mathrm{e}^{-rt}C(t,S_t)$ of a self-financing hedging portfolio is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, therefore from *e.g.* Corollary II-6-1, page 72 of Protter, 2004, all terms in dt should vanish in the above expression of

$$d(\mathrm{e}^{-rt}g(t,S_t))=-r\,\mathrm{e}^{-rt}g(t,S_t)dt+\mathrm{e}^{-rt}dg(t,S_t),$$

which shows that

$$-rC(t,S_t)+\frac{\partial C}{\partial t}(t,S_t)+rS_t\frac{\partial C}{\partial x}(t,S_t)+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial x^2}(t,S_t)=0,$$

and leads to the Black-Scholes PDE

$$rC(t,x)=\frac{\partial C}{\partial t}(t,x)+rx\frac{\partial C}{\partial x}(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 C}{\partial x^2}(t,x),\qquad x>0.$$

□

Forward contracts

The long forward contract with payoff $C=S_T-K$ is priced as

$$
\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T-K\mid\mathscr{F}_t]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T\mid\mathscr{F}_t]-K\,\mathrm{e}^{-(T-t)r}\\
&= S_t-K\,\mathrm{e}^{-(T-t)r},\quad 0\leqslant t\leqslant T,
\end{aligned}
$$

which recovers the Black-Scholes PDE solution (7.8), *i.e.*

$$g(t,x)=x-K\,\mathrm{e}^{-(T-t)r},\qquad x>0,\quad t\in[0,T].$$

European call options

In the case of European call options with payoff function $\phi(x)=(x-K)^+$ we recover the Black-Scholes formula (7.10), cf. Proposition 7.11, by a probabilistic argument.

Proposition 8.5 The price at time $t\in[0,T]$ of the European call option with strike price K and maturity T is given by

$$
\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+\mid\mathscr{F}_t] \qquad (8.19)\\
&= S_t\Phi\big(d_+(T-t)\big)-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),\quad 0\leqslant t\leqslant T,
\end{aligned}
$$

with

$$
\begin{cases}
d_+(T-t):=\dfrac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},\\[2ex]
d_-(T-t):=\dfrac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},\quad 0\leqslant t<T,
\end{cases}
$$

where "log" denotes the *natural logarithm* "ln" and Φ is the standard Gaussian Cumulative Distribution Function.

Proof. The proof of Proposition 8.5 is a consequence of (8.17) and Lemma 8.6 below. Using the relation

$$S_T = S_t \,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T,$$

by Theorem 8.3 the value at time $t \in [0,T]$ of the portfolio hedging C is given by

$$\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+ \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_t\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K)^+ \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(x\,\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(\mathrm{e}^{m(x)+X}-K)^+\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := (\widehat{B}_T-\widehat{B}_t)\sigma \simeq \mathscr{N}(0,(T-t)\sigma^2)$$

is a centered Gaussian random variable with variance

$$\mathrm{Var}[X] = \mathrm{Var}\left[(\widehat{B}_T-\widehat{B}_t)\sigma\right] = \sigma^2\mathrm{Var}\left[\widehat{B}_T-\widehat{B}_t\right] = (T-t)\sigma^2$$

under $\mathbb{P}^*$. Hence by Lemma 8.6 below we have

$$\begin{aligned}
C(t,S_t) &= V_t \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(\mathrm{e}^{m(x)+X}-K)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathrm{e}^{m(S_t)+\sigma^2(T-t)/2}\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\
&= S_t\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) - K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\
&= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),
\end{aligned}$$

$0 \leqslant t \leqslant T$. □

Relation (8.19) can also be written as

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+ \mid \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+ \mid S_t\right] \qquad (8.20)\\
= S_t\Phi&\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
-K\,\mathrm{e}^{-(T-t)r}\Phi&\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Lemma 8.6 Let $X \simeq \mathscr{N}(0,v^2)$ be a centered Gaussian random variable with variance $v^2 > 0$. We have

$$\mathbb{E}\left[(\mathrm{e}^{m+X}-K)^+\right] = \mathrm{e}^{m+v^2/2}\Phi(v+(m-\log K)/v) - K\Phi((m-\log K)/v).$$

Proof. We have

$$
\begin{aligned}
\mathbb{E}\left[(\mathrm{e}^{m+X}-K)^+\right] &= \frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}(\mathrm{e}^{m+x}-K)^+\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}(\mathrm{e}^{m+x}-K)\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{e^m}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{x-x^2/(2v^2)}dx - \frac{K}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-(v^2-x)^2/(2v^2)}dx - \frac{K}{\sqrt{2\pi}}\int_{(-m+\log K)/v}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-v^2-m+\log K}^{\infty}\mathrm{e}^{-y^2/(2v^2)}dy - K\Phi((m-\log K)/v) \\
&= \mathrm{e}^{m+v^2/2}\Phi(v+(m-\log K)/v) - K\Phi((m-\log K)/v).
\end{aligned}
$$

□

Call-put parity

Let

$$P(t,S_t) := \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+ \mid \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

denote the price of the put option with strike price K and maturity T.

Proposition 8.7 Call-put parity. We have the relation

$$C(t,S_t) - P(t,S_t) = S_t - \mathrm{e}^{-(T-t)r}K \tag{8.21}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K\,\mathrm{e}^{-(T-t)r}$.

Proof. From Theorem 8.3, we have

$$
\begin{aligned}
&C(t,S_t) - P(t,S_t) \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+ \mid \mathscr{F}_t] - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K-S_T)^+ \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+ - (K-S_T)^+ \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T-K \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid \mathscr{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\
&= S_t - \mathrm{e}^{-(T-t)r}K, \qquad 0 \leqslant t \leqslant T,
\end{aligned}
$$

as we have $\mathbb{E}^*[S_T \mid \mathscr{F}_t] = \mathrm{e}^{-(T-t)r}S_t$, $t \in [0,T]$, under the risk-neutral probability measure $\mathbb{P}^*$.

□

European put options

Using the *call-put parity* Relation (8.21) we can recover the European put option price (7.10) from the European call option price (7.10)-(8.19).

Proposition 8.8 The price at time $t \in [0,T]$ of the European put option with strike price K and maturity T is given by

$$\begin{aligned} P(t,S_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathscr{F}_t] \\ &= K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T, \end{aligned}$$

with

$$\begin{cases} d_+(T-t) := \dfrac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) := \dfrac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T, \end{cases}$$

where "log" denotes the *natural logarithm* "ln" and Φ is the standard Gaussian Cumulative Distribution Function.

Proof. By the *call-put parity* (8.21), we have

$$\begin{aligned} P(t,S_t) &= C(t,S_t) - S_t + \mathrm{e}^{-(T-t)r}K \\ &= S_t\Phi\big(d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K - S_t - \mathrm{e}^{-(T-t)r}K\Phi\big(d_-(T-t)\big) \\ &= -S_t\big(1-\Phi\big(d_+(T-t)\big)\big) + \mathrm{e}^{-(T-t)r}K\big(1-\Phi\big(d_-(T-t)\big)\big) \\ &= -S_t\Phi\big(-d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K\Phi\big(-d_-(T-t)\big). \end{aligned}$$

□

8.5 Hedging by the Martingale Method

Hedging exotic options

In the next Proposition 8.9 we compute a self-financing hedging strategy leading to an arbitrary square-integrable random claim payoff $C \in L^2(\Omega)$ of an exotic option admitting a stochastic integral decomposition of the form

$$C = \mathbb{E}^*[C] + \int_0^T \zeta_t d\widehat{B}_t, \tag{8.22}$$

where $(\zeta_t)_{t\in[0,t]}$ is a square-integrable adapted process. Consequently, the mathematical problem of finding the stochastic integral decomposition (8.22) of a given random variable has important applications in finance. The process $(\zeta_t)_{t\in[0,T]}$ can be computed using the Malliavin gradient on the Wiener space, see e.g. Nunno, Øksendal, and Proske, 2009 or § 8.2 of Privault, 2009.

Simple examples of stochastic integral decompositions include the relations

$$(B_T)^2 = T + 2\int_0^T B_t dB_t,$$

cf. Exercise 8.1, and

$$(B_T)^3 = 3\int_0^T (T-t+B_t^2)dB_t,$$

see Exercise 5.11. In the sequel, recall that the risky asset follows the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0,$$

and by (8.14), the discounted asset price $\widetilde{S}_t := \mathrm{e}^{-rt}S_t$

$$d\widetilde{S}_t = \sigma\widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \qquad \widetilde{S}_0 = S_0 > 0, \tag{8.23}$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. The following proposition applies to arbitrary square-integrable payoff functions, in particular it covers exotic and path-dependent options.

Proposition 8.9 Consider a random claim payoff $C \in L^2(\Omega)$ and the process $(\zeta_t)_{t\in[0,T]}$ given by (8.22), and let

$$\xi_t = \frac{\mathrm{e}^{-(T-t)r}}{\sigma S_t}\zeta_t, \tag{8.24}$$

$$\eta_t = \frac{\mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathscr{F}_t] - \xi_t S_t}{A_t}, \qquad 0 \leqslant t \leqslant T. \tag{8.25}$$

Then the portfolio allocation $(\xi_t,\eta_t)_{t\in[0,T]}$ is self-financing, and letting

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T, \tag{8.26}$$

we have

$$V_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{8.27}$$

In particular we have

$$V_T = C, \tag{8.28}$$

i.e. the portfolio allocation $(\xi_t,\eta_t)_{t\in[0,T]}$ yields a hedging strategy leading to the claim payoff C at maturity, after starting from the initial value

$$V_0 = \mathrm{e}^{-rT}\mathbb{E}^*[C].$$

Proof. Relation (8.27) follows from (8.25) and (8.26), and it implies

$$V_0 = \mathrm{e}^{-rT}\mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0$$

at $t = 0$, and (8.28) at $t = T$. It remains to show that the portfolio strategy $(\xi_t,\eta_t)_{t\in[0,T]}$ is self-financing. By (8.22) and Proposition 8.1 we have

$$\begin{aligned}
V_t &= \eta_t A_t + \xi_t S_t \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{E}^*[C] + \int_0^T \zeta_u d\widehat{B}_u \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\left(\mathbb{E}^*[C] + \int_0^t \zeta_u d\widehat{B}_u\right) \\
&= \mathrm{e}^{rt}V_0 + \mathrm{e}^{-(T-t)r}\int_0^t \zeta_u d\widehat{B}_u \\
&= \mathrm{e}^{rt}V_0 + \sigma\int_0^t \xi_u S_u \mathrm{e}^{(t-u)r} d\widehat{B}_u \\
&= \mathrm{e}^{rt}V_0 + \sigma\,\mathrm{e}^{rt}\int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u.
\end{aligned}$$

By (8.23) this shows that the portfolio strategy $(\xi_t,\eta_t)_{t\in[0,T]}$ given by (8.24)-(8.25) and its discounted portfolio value $\widetilde{V}_t := \mathrm{e}^{-rt}V_t$ satisfy

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad 0 \leqslant t \leqslant T,$$

which implies that $(\xi_t,\eta_t)_{t\in[0,T]}$ is self-financing by Lemma 6.7. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}^*[C]\,\mathrm{e}^{-rT}.$$

In addition, since there exists a hedging strategy leading to

$$\widetilde{V}_T = \mathrm{e}^{-rT}C,$$

then $(\widetilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale, with

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{V}_T \mid \mathscr{F}_t\big] = \mathrm{e}^{-rT}\,\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T,$$

and initial value

$$\widetilde{V}_0 = \mathbb{E}^*\big[\widetilde{V}_T\big] = \mathrm{e}^{-rT}\,\mathbb{E}^*[C].$$

Hedging vanilla options

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t\in[0,T]}$ appearing in (8.22). This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property, see *e.g.* Protter, 2001. The next lemma allows us to compute the process $(\zeta_t)_{t\in[0,T]}$ in case the payoff C is of the form $C = \phi(S_T)$ for some function ϕ.

Lemma 8.10 Assume that ϕ is a Lipschitz payoff function. Then the function $C(t,x)$ defined by

$$C(t,S_t) = \mathbb{E}^*[\phi(S_T) \mid S_t]$$

is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R})$, and the stochastic integral decomposition

$$\phi(S_T) = \mathbb{E}^*\big[\phi(S_T)\big] + \int_0^T \zeta_t d\widehat{B}_t \tag{8.29}$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{8.30}$$

In addition, the hedging strategy $(\xi_t)_{t\in[0,T]}$ satisfies

$$\xi_t = \mathrm{e}^{-(T-t)r}\frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{8.31}$$

Proof. It can be checked as in the proof of Proposition 8.4 the function $C(t,x)$ is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R})$. Therefore, we can apply the Itô formula to the process

$$t \longmapsto C(t,S_t) = \mathbb{E}^*[\phi(S_T) \mid \mathscr{F}_t],$$

which is a martingale from the tower property (11.38) of conditional expectations as in (8.38). From the fact that the finite variation term in the Itô formula vanishes when $(C(t,S_t))_{t\in[0,T]}$ is a martingale, (see *e.g.* Corollary II-6-1 page 72 of Protter, 2004), we obtain:

$$C(t,S_t) = C(0,S_0) + \sigma\int_0^t S_u \frac{\partial C}{\partial x}(u,S_u)d\widehat{B}_u, \quad 0 \leqslant t \leqslant T, \tag{8.32}$$

with $C(0,S_0) = \mathbb{E}^*\big[\phi(S_T)\big]$. Letting $t = T$, we obtain (8.30) by uniqueness of the stochastic integral decomposition (8.29) of $C = \phi(S_T)$. Finally, (8.31) follows from (8.24) and (8.30). $\square$

By (8.39) we also have

$$\begin{aligned}\zeta_t &= \sigma S_t \frac{\partial}{\partial x}\mathbb{E}^*[\phi(S_T) \mid S_t = x]_{x=S_t}\\ &= \sigma S_t \frac{\partial}{\partial x}\mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,\end{aligned}$$

hence

$$\begin{aligned}\xi_t &= \frac{1}{\sigma S_t}\mathrm{e}^{-(T-t)r}\zeta_t \\ &= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,\end{aligned} \tag{8.33}$$

which recovers the formula (7.3) for the Delta of a vanilla option. As a consequence we have $\xi_t \geqslant 0$ and there is no short selling when the payoff function ϕ is non-decreasing.

In the case of European options, the process ζ can be computed via the next proposition which follows from Lemma 8.10 and the relation

$$C(t,x) = \mathbb{E}^*\left[f(S^x_{t,T})\right], \qquad 0 \leqslant t \leqslant T,\ x > 0.$$

Corollary 8.11 Assume that $C = (S_T - K)^+$. Then, for $0 \leqslant t \leqslant T$ we have

$$\zeta_t = \sigma S_t\, \mathbb{E}^*\left[\frac{S_T}{S_t}\mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T, \tag{8.34}$$

and

$$\xi_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{S_T}{S_t}\mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T. \tag{8.35}$$

By evaluating the expectation (8.34) in Corollary 8.11 we can recover the formula (7.15) in Proposition 7.4 for the Delta of the European call option in the Black-Scholes model. In that sense, the next proposition provides another proof of the result of Proposition 7.4.

Proposition 8.12 The Delta of the European call option with payoff function $f(x) = (x-K)^+$ is given by

$$\xi_t = \Phi\big(d_+(T-t)\big) = \Phi\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T.$$

Proof. By Proposition 8.9 and Corollary 8.11, we have

$$\begin{aligned}\xi_t &= \frac{1}{\sigma S_t}\mathrm{e}^{-(T-t)r}\zeta_t \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{S_T}{S_t}\mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r} \\ &\quad \times \mathbb{E}^*\left[\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\mathbb{1}_{[K,\infty)}\big(x\,\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\big)\right]_{x=S_t}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2(T-t)\pi}} \int_{(T-t)\sigma/2-(T-t)r/\sigma+\sigma^{-1}\log(K/S_t)}^{\infty} e^{\sigma y-(T-t)\sigma^2/2-y^2/(2(T-t))} dy \\
&= \frac{1}{\sqrt{2(T-t)\pi}} \int_{-d_-(T-t)/\sqrt{T-t}}^{\infty} e^{-(y-(T-t)\sigma)^2/(2(T-t))} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_-(T-t)}^{\infty} e^{-(y-(T-t)\sigma)^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_+(T-t)}^{\infty} e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+(T-t)} e^{-y^2/2} dy \\
&= \Phi\big(d_+(T-t)\big).
\end{aligned}
$$

□

Proposition 8.12, combined with Proposition 8.5, shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

$$
\xi_t = \Phi\big(d_+(T-t)\big) = \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{8.36}
$$

of the risky asset, and to borrow a quantity

$$
-\eta_t = K\,e^{-rT}\Phi\left(\frac{\log(S_t/K)+(r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{8.37}
$$

of the riskless (savings) account.

As noted above, the result of Proposition 8.12 recovers (7.16) which is obtained by a direct differentiation of the Black-Scholes function as in (7.3) or (8.33).

Markovian semi-group

Definition 8.3 The Markov semi-group $(P_t)_{0\leqslant t\leqslant T}$ associated to $(S_t)_{t\in[0,T]}$ is the mapping P_t defined on functions $f \in \mathcal{C}_b^2(\mathbb{R})$ as

$$
P_t f(x) := \mathbb{E}^*[f(S_t) \mid S_0 = x], \quad t \in \mathbb{R}_+.
$$

By the Markov property and time homogeneity of $(S_t)_{t\in[0,T]}$ we also have

$$
P_t f(S_u) := \mathbb{E}^*[f(S_{t+u}) \mid \mathcal{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t,u \in \mathbb{R}_+,
$$

and the semi-group $(P_t)_{0\leqslant t\leqslant T}$ satisfies the composition property

$$
P_s P_t = P_t P_s = P_{s+t} = P_{t+s}, \qquad s,t \in \mathbb{R}_+,
$$

as we have, using the Markov property and the tower property (11.38) of conditional expectations as in (8.38),

$$
\begin{aligned}
P_s P_t f(x) &= \mathbb{E}^*\left[P_t f(S_s) \mid S_0 = x\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_t) \mid S_0 = y\right]_{y=S_s} \mid S_0 = x\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_{t+s}) \mid S_s = y\right]_{y=S_s} \mid S_0 = x\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_{t+s}) \mid \mathcal{F}_s\right] \mid S_0 = x\right] \\
&= \mathbb{E}^*\left[f(S_{t+s}) \mid \mathcal{F}_s \mid S_0 = x\right]
\end{aligned}
$$

$$= \quad P_{t+s}f(x), \qquad s,t \geqslant 0.$$

Similarly we can show that the process $(P_{T-t}f(S_t))_{t\in[0,T]}$ is an $\mathscr{F}_t$-martingale as in Example (8.1), *i.e.*:

$$\begin{aligned}
\mathbb{E}^*\left[P_{T-t}f(S_t) \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathbb{E}^*[f(S_T) \mid \mathscr{F}_t] \mid \mathscr{F}_u\right] \\
&= \mathbb{E}^*\left[f(S_T) \mid \mathscr{F}_u\right] \\
&= P_{T-u}f(S_u), \qquad 0 \leqslant u \leqslant t \leqslant T,
\end{aligned} \tag{8.38}$$

and we have

$$P_{t-u}f(x) = \mathbb{E}^*\left[f(S_t) \mid S_u = x\right] = \mathbb{E}^*\left[f\left(x\frac{S_t}{S_u}\right)\right], \quad 0 \leqslant u \leqslant t. \tag{8.39}$$

Exercises

Exercise 8.1 (Exercise 7.1 continued). Consider a market made of a riskless asset priced $A_t = A_0$ with zero interest rate, $t \in \mathbb{R}_+$, and a risky asset whose price modeled by a standard Brownian motion as $S_t = B_t$, $t \in \mathbb{R}_+$. Price the vanilla option with payoff $C = (B_T)^2$, and deduce the solution of the Black-Scholes PDE of Exercise 7.1.

Exercise 8.2 Given the price process $(S_t)_{t\in\mathbb{R}_+}$ defined as

$$S_t := S_0\,\mathrm{e}^{\sigma B_t + (r-\sigma^2/2)t}, \qquad t \in \mathbb{R}_+,$$

price the option with payoff function $\phi(S_T)$ by writing $\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi(S_T)\right]$ as an integral.

Exercise 8.3 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$ in a market with interest rate $r = 0$, and let $\phi(x) = (x-K)^+$ be the (convex) European call payoff function.

Show that, for any two maturities $T_1 < T_2$ and $p,q \in [0,1]$ such that $p+q = 1$, the price of the option on average with payoff $\phi(pS_{T_1} + qS_{T_2})$ is upper bounded by the price of the European call option with maturity T_2, *i.e.* show that

$$\mathbb{E}^*\left[\phi(pS_{T_1} + qS_{T_2})\right] \leqslant \mathbb{E}^*\left[\phi(S_{T_2})\right].$$

Hints:

i) For ϕ a convex function we have $\phi(px+qy) \leqslant p\phi(x) + q\phi(y)$ for any $x,y \in \mathbb{R}$ and $p,q \in [0,1]$ such that $p+q = 1$.
ii) Any convex function $\phi(S_t)$ of a martingale S_t is a *sub*martingale.

Exercise 8.4 Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ and a risk-neutral measure $\mathbb{P}^*$.

a) Does the European *call* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ increase or decrease with the strike price K? Justify your answer.
b) Does the European *put* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(K-S_T)^+]$ increase or decrease with the strike price K? Justify your answer.

Exercise 8.5 Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$.

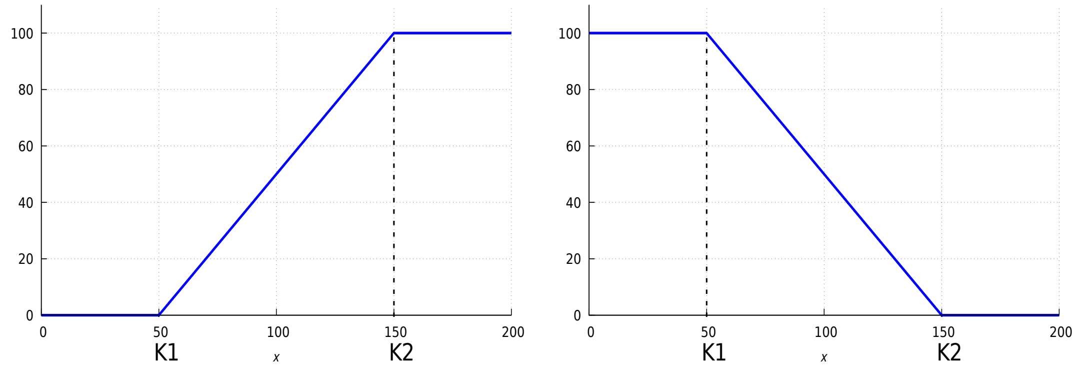
a) Show that the price at time t of the European call option with strike price K and maturity T is lower bounded by the positive part $\left(S_t - K\,\mathrm{e}^{-(T-t)r}\right)^+$ of the corresponding forward contract price, *i.e.*

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+ \mid \mathscr{F}_t\right] \geqslant \left(S_t - K\,\mathrm{e}^{-(T-t)r}\right)^+, \quad 0 \leqslant t \leqslant T.$$

b) Show that the price at time t of the European put option with strike price K and maturity T is lower bounded by $K\,\mathrm{e}^{-(T-t)r} - S_t$, *i.e.*

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+ \mid \mathscr{F}_t\right] \geqslant \left(K\,\mathrm{e}^{-(T-t)r} - S_t\right)^+, \quad 0 \leqslant t \leqslant T.$$

Exercise 8.6 The following two graphs describe the payoff functions ϕ of *bull spread* and *bear spread* options with payoff $\phi(S_N)$ on an underlying asset priced S_N at maturity time N.



(i) Bull spread payoff. (ii) Bear spread payoff.

Figure 8.3: Payoff functions of bull spread and bear spread options.

a) Show that in each case (i) and (ii) the corresponding option can be realized by purchasing and/or short selling standard European call and put options with strike prices to be specified.
b) Price the bull spread option in cases (i) and (ii).

Hint: An option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\,\mathbb{E}^*\left[\phi(S_N)\right]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 8.7 Butterfly options. A butterfly option is designed to deliver a limited payoff when the future volatility of the underlying asset is expected to be low. The payoff function of a butterfly option is plotted in Figure 8.4, with $K_1 := 50$ and $K_2 := 150$.

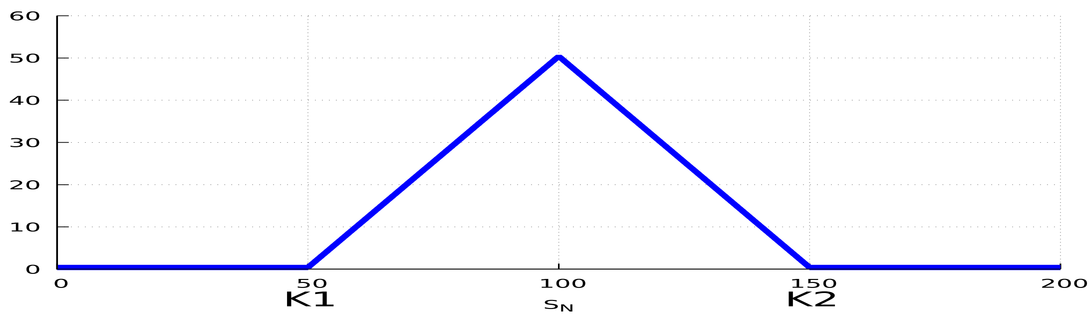


Figure 8.4: Butterfly payoff function.

a) Show that the butterfly option can be realized by purchasing and/or issuing standard European call or put options with strike prices to be specified.

b) Does the hedging strategy of the butterfly option involve holding or shorting the underlying stock?

Hints: Recall that an option with payoff $\phi(S_N)$ is priced in discrete time as $(1+r)^{-N}\mathbb{E}^*[\phi(S_N)]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 8.8 Forward contracts revisited. Consider a risky asset whose price S_t is given by $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Consider a forward contract with maturity T and payoff $S_T - \kappa$.

a) Compute the price C_t of this claim at any time $t \in [0,T]$.

b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 8.9 Option pricing with dividends (Exercise 7.3 continued). Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ paying dividends at the continuous-time rate $\delta > 0$, and modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Show that as in Lemma 6.7, if $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+,$$

where the dividend yield δS_t per share is continuously reinvested in the portfolio, then the discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic integral

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \in \mathbb{R}_+,$$

b) Show that, as in Theorem 8.3, if $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, *i.e.* if $V_T = C$, then the arbitrage-free price of the claim payoff C is given by

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T,$$

where $\widehat{\mathbb{E}}$ denotes expectation under a suitably chosen risk-neutral probability measure $\widehat{\mathbb{P}}$.

c) Compute the price at time $t \in [0,T]$ of a European call option in a market with dividend rate δ by the martingale method.

Exercise 8.10 Forward start options (Rubinstein, 1991). A *forward start* European call option is an option whose holder receives at time T_1 (*e.g.* your birthday) the value of a standard European call option *at the money* and with maturity $T_2 > T_1$. Price this birthday present at any time $t \in [0,T_1]$, *i.e.* compute the price

$$\mathrm{e}^{-(T_1-t)r}\mathbb{E}^*\left[\mathrm{e}^{-(T_2-T_1)r}\mathbb{E}^*\left[(S_{T_2}-S_{T_1})^+ \mid \mathscr{F}_{T_1}\right] \mid \mathscr{F}_t\right]$$

at time $t \in [0,T_1]$, of the *forward start* European call option using the Black-Scholes formula

$$\begin{aligned}\mathrm{Bl}(K,x,\sigma,r,T-t) &= x\Phi\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\ &\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right),\end{aligned}$$

$0 \leqslant t < T$.

Exercise 8.11 Log-contracts. (Exercise 7.9 continued). Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset valued $A_t = A_0 \, \mathrm{e}^{rt}$, $t \in [0,T]$, with $r > 0$. Compute the arbitrage-free price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\log S_T \mid \mathscr{F}_t \right],$$

at time $t \in [0,T]$, of the log-contract with payoff $\log S_T$.

Exercise 8.12 Power option. (Exercise 7.5 continued). Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset valued $A_t = A_0 \, \mathrm{e}^{rt}$, $t \in [0,T]$, with $r > 0$. In this problem, $(\eta_t, \xi_t)_{t\in[0,T]}$ denotes a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T.$$

a) Compute the arbitrage-free price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[|S_T|^2 \mid \mathscr{F}_t \right],$$

at time $t \in [0,T]$, of the power option with payoff $C = |S_T|^2$.

b) Compute a self-financing hedging strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ hedging the claim payoff $|S_T|^2$.

Exercise 8.13 Bachelier, 1900 model (Exercise 7.11 continued).

a) Consider the solution $(S_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t.$$

For which value α_M of α is the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, a martingale under $\mathbb{P}$?

b) For each value of α, build a probability measure $\mathbb{P}_\alpha$ under which the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, is a martingale.

c) Compute the arbitrage-free price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha \left[\mathrm{e}^{S_T} \mid \mathscr{F}_t \right]$$

at time $t \in [0,T]$ of the contingent claim with payoff $\exp(S_T)$, and recover the result of Exercise 7.11.

d) Explicitly compute the portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

e) Check that this strategy is self-financing.

Exercise 8.14 Compute the arbitrage-free price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha \left[(S_T)^2 \mid \mathscr{F}_t \right]$$

at time $t \in [0,T]$ of the power option with payoff $(S_T)^2$ in the framework of the Bachelier, 1900 model of Exercise 8.13.

Exercise 8.15 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Recall that for $f \in \mathscr{C}^2(\mathbb{R}_+\times\mathbb{R})$, Itô's formula for $(B_t)_{t\in\mathbb{R}_+}$ reads

$$\begin{aligned} f(t,B_t) &= f(0,B_0)+\int_0^t \frac{\partial f}{\partial s}(s,B_s)ds \\ &\quad +\int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s+\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds. \end{aligned}$$

a) Let $r \in \mathbb{R}$, $\sigma > 0$, $f(x,t) = \mathrm{e}^{rt+\sigma x-\sigma^2 t/2}$, and $S_t = f(t,B_t)$. Compute $df(t,B_t)$ by Itô's formula, and show that S_t solves the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where $r > 0$ and $\sigma > 0$.

b) Show that

$$\mathbb{E}\left[\mathrm{e}^{\sigma B_T} \mid \mathscr{F}_t\right] = \mathrm{e}^{\sigma B_t+(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T.$$

Hint: Use the independence of increments of $(B_t)_{t\in[0,T]}$ in the time splitting decomposition

$$B_T = (B_t - B_0) + (B_T - B_t),$$

and the Gaussian moment generating function $\mathbb{E}\left[\mathrm{e}^{\alpha X}\right] = \mathrm{e}^{\alpha^2\eta^2/2}$ when $X \simeq \mathscr{N}(0,\eta^2)$.

c) Show that the process $(S_t)_{t\in\mathbb{R}_+}$ satisfies

$$\mathbb{E}\left[S_T \mid \mathscr{F}_t\right] = \mathrm{e}^{(T-t)r}S_t, \qquad 0 \leqslant t \leqslant T.$$

d) Let $C = S_T - K$ denote the payoff of a forward contract with exercise price K and maturity T. Compute the discounted expected payoff

$$V_t := \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[C \mid \mathscr{F}_t\right].$$

e) Find a self-financing portfolio strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ such that

$$V_t = \xi_t S_t + \eta_t A_t, \qquad 0 \leqslant t \leqslant T,$$

where $A_t = A_0\,\mathrm{e}^{rt}$ is the price of a riskless asset with fixed interest rate $r > 0$. Show that it recovers the result of Exercise 7.7-(c)).

f) Show that the portfolio allocation $(\xi_t,\eta_t)_{t\in[0,T]}$ found in Question (e)) *hedges* the payoff $C = S_T - K$ at time T, *i.e.* show that $V_T = C$.

Exercise 8.16 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. A binary (or digital) *call*, *resp. put*, option is a contract with maturity T, strike price K, and payoff

$$C_d := \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K, \end{cases} \qquad \textit{resp.} \quad P_d := \begin{cases} \$1 & \text{if } S_T \leqslant K, \\ 0 & \text{if } S_T > K. \end{cases}$$

Recall that the prices $\pi_t(C_d)$ and $\pi_t(P_d)$ at time t of the binary call and put options are given by the discounted expected payoffs

$$\pi_t(C_d) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[C_d \mid \mathscr{F}_t\right] \quad \text{and} \quad \pi_t(P_d) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[P_d \mid \mathscr{F}_t\right]. \tag{8.40}$$

a) Show that the payoffs C_d and P_d can be rewritten as

$$C_d = \mathbb{1}_{[K,\infty)}(S_T) \qquad \text{and} \qquad P_d = \mathbb{1}_{[0,K]}(S_T).$$

b) Using Relation (8.40), Question (a)), and the relation

$$\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \mid S_t = x\right] = \mathbb{P}^*(S_T \geqslant K \mid S_t = x),$$

show that the price $\pi_t(C_d)$ is given by

$$\pi_t(C_d) = C_d(t, S_t),$$

where $C_d(t,x)$ is the function defined by

$$C_d(t,x) := \mathrm{e}^{-(T-t)r}\mathbb{P}^*(S_T \geqslant K \mid S_t = x).$$

c) Using the results of Exercise 6.9-(d)) and of Question (b)), show that the price $\pi_t(C_d)$ of the binary call option is given by

$$\begin{aligned} C_d(t,x) &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

where

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) Assume that the binary option holder is entitled to receive a "return amount" $\alpha \in [0,1]$ in case the underlying asset price ends out of the money at maturity. Compute the price at time $t \in [0,T]$ of this modified contract.

e) Using Relation (8.40) and Question (a)), prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = \mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T. \tag{8.41}$$

If needed, you may use the fact that $\mathbb{P}^*(S_T = K) = 0$.

f) Using the results of Questions (e)) and (c)), show that the price $\pi_t(P_d)$ of the binary put option is given by

$$\pi_t(P_d) = \mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big).$$

g) Using the result of Question (c)), compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t, S_t)$$

of the binary call option. Does the Black-Scholes hedging strategy of such a call option involve short selling? Why?

h) Using the result of Question (f)), compute the Delta

$$\xi_t := \frac{\partial P_d}{\partial x}(t, S_t)$$

of the binary put option. Does the Black-Scholes hedging strategy of such a put option involve short selling? Why?

Exercise 8.17 Computation of Greeks. Consider an underlying asset whose price $(S_t)_{t\in\mathbb{R}_+}$ is given by a stochastic differential equation of the form

$$dS_t = rS_t dt + \sigma(S_t)dW_t,$$

where $\sigma(x)$ is a Lipschitz coefficient, and an option with payoff function ϕ and price

$$C(x,T) = \mathrm{e}^{-rT}\,\mathbb{E}\big[\phi(S_T)\,\big|\,S_0 = x\big],$$

where $\phi(x)$ is a twice continuously differentiable ($\mathscr{C}^2$) function, with $S_0 = x$. Using the Itô formula, show that the sensitivity

$$\mathrm{Theta}_T = \frac{\partial}{\partial T}\big(\mathrm{e}^{-rT}\,\mathbb{E}\big[\phi(S_T)\,\big|\,S_0 = x\big]\big)$$

of the option price with respect to maturity T can be expressed as

$$\begin{aligned}\mathrm{Theta}_T &= -r\,\mathrm{e}^{-rT}\,\mathbb{E}\big[\phi(S_T)\,\big|\,S_0 = x\big] + r\,\mathrm{e}^{-rT}\,\mathbb{E}\big[S_t\phi'(S_T)\,\big|\,S_0 = x\big]\\ &\quad + \frac{1}{2}\mathrm{e}^{-rT}\,\mathbb{E}\big[\phi''(S_T)\sigma^2(S_T)\,\big|\,S_0 = x\big].\end{aligned}$$

9. Volatility Estimation

Volatility estimation methods include historical, implied and local volatility, and the VIX® volatility index. This chapter presents such estimation methods, together with examples of how the Black-Scholes formula can be fitted to market data. While the market parameters r, t, S_t, T, and K used in Black-Scholes option pricing can be easily obtained from market terms and data, the estimation of volatility parameters can be a more complex task.

9.1 Historical Volatility

We consider the problem of estimating the parameters μ and σ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{9.1}$$

Historical trend estimation

By discretization of (9.1) along a family $t_0, t_1, \ldots, t_N$ of observation times as

$$\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = (t_{k+1} - t_k)\mu + (B_{t_{k+1}} - B_{t_k})\sigma, \quad k = 0, 1, \ldots, N-1, \tag{9.2}$$

a natural estimator for the trend parameter μ can be constructed as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S^M_{t_{k+1}}-S^M_{t_k}}{S^M_{t_k}}\right), \tag{9.3}$$

where $(S^M_{t_{k+1}}-S^M_{t_k})/S^M_{t_k}$, $k=0,1,\ldots,N-1$ denotes market returns observed at discrete times $t_0,t_1,\ldots,t_N$ on the market.

Historical log-return estimation

Alternatively, observe that, replacing (9.3) by the log-returns

$$\log\left(1+\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}\right) = \log\frac{S_{t_{k+1}}}{S_{t_k}} = \log S_{t_{k+1}} - \log S_{t_k} \simeq \frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}},$$

with $t_{k+1}-t_k = T/N$, $k=0,1,\ldots,N-1$, one can replace (9.3) with the simpler telescoping estimate*

$$\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}(\log S_{t_{k+1}}-\log S_{t_k}) = \frac{1}{T}\log\frac{S_T}{S_0}.$$

Historical volatility estimation

The volatility parameter σ can be estimated by writing, from (9.2),

$$\sigma^2\sum_{k=0}^{N-1}(B_{t_{k+1}}-B_{t_k})^2 = \sum_{k=0}^{N-1}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\mu\right)^2,$$

which yields the (unbiased) realized variance estimator

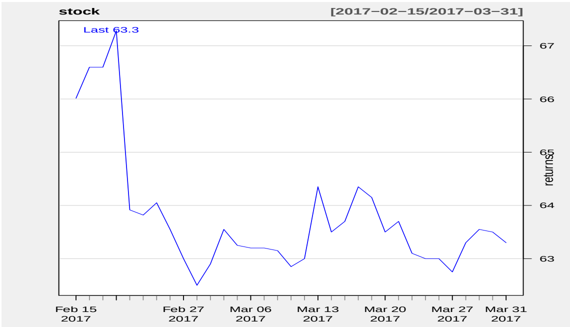
$$\widehat{\sigma}^2_N := \frac{1}{N-1}\sum_{k=0}^{N-1}\frac{1}{t_{k+1}-t_k}\left(\frac{S_{t_{k+1}}-S_{t_k}}{S_{t_k}}-(t_{k+1}-t_k)\widehat{\mu}_N\right)^2.$$

```
library(quantmod)
getSymbols("0005.HK",from="2017-02-15",to=Sys.Date(),src="yahoo")
stock=Ad(`0005.HK`)
chartSeries(stock,up.col="blue",theme="white")
```

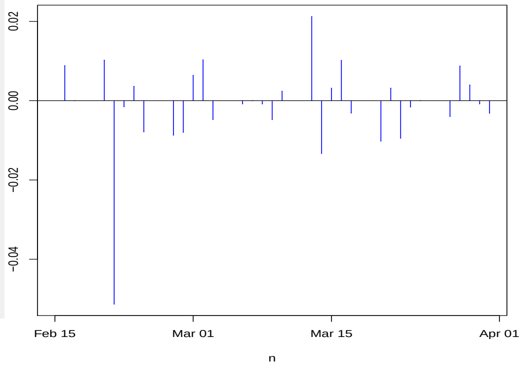
```
stock=Ad(`0005.HK`);returns=(stock-lag(stock))/stock
returns=diff(log(stock));times=index(returns);returns <- as.vector(returns)
n = sum(is.na(returns))+sum(!is.na(returns))
plot(times,returns,pch=19,cex=0.05,col="blue", ylab="returns", xlab="n", main = '')
segments(x0 = times, x1 = times, cex=0.05,y0 = 0, y1 = returns,col="blue")
abline(seq(1,n),0,FALSE);dt=1.0/365;mu=mean(returns,na.rm=TRUE)/dt
sigma=sd(returns,na.rm=TRUE)/sqrt(dt);mu;sigma
```

```
library(PerformanceAnalytics)
library(quantmod)
returns <- exp(CalculateReturns(stock,method="compound")) - 1; returns[1,] <- 0
histvol <- rollapply(returns, width = 30, FUN=sd.annualized)
myTheme <- chart_theme(); myTheme$col$line.col <- "blue"
chart_Series(stock,name="0005.HK",theme=myTheme)
add_TA(histvol, name="Historical Volatility")
```

*Note that strictly speaking, the Itô formula reads $d\log S_t = dS_t/S_t-(dS_t)^2/(2S_t^2)$.



(a) Underlying asset price.



(b) Log returns.

Figure 9.1: Graph of underlying asset price *vs* log returns.

The next Figure 9.2 presents a historical volatility graph with a 30 days rolling window.

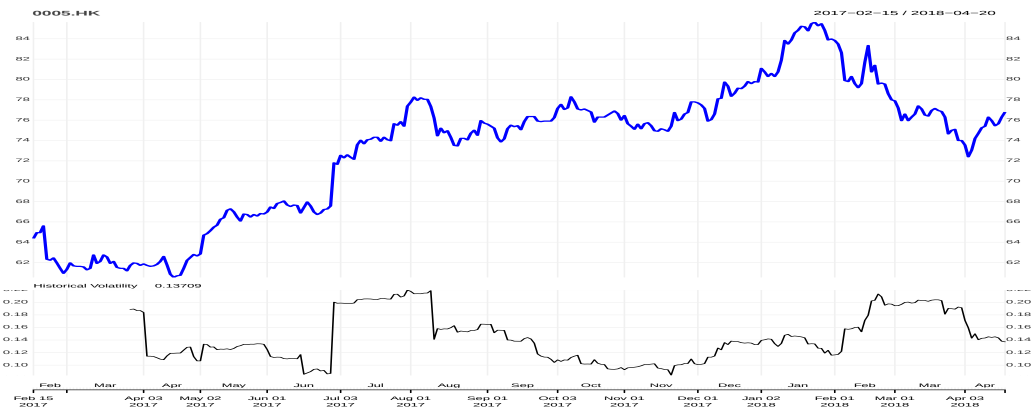


Figure 9.2: Historical volatility graph.

Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

Figure 9.3: "The fugazi: it's a wazy, it's a woozie. It's fairy dust."*

9.2 Implied Volatility

Recall that when $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE is given by

$$\mathrm{Bl}(t,x,K,\sigma,r,T) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) = \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) = \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \end{cases}$$

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data. Equating the Black-Scholes formula

$$\mathrm{Bl}(t,S_t,K,\sigma,r,T) = M \tag{9.4}$$

to the observed value M of a given market price allows one to infer a value of σ when t, S_t, r, T are known, as in *e.g.* Figure 7.21.

*Scorsese, 2013 Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

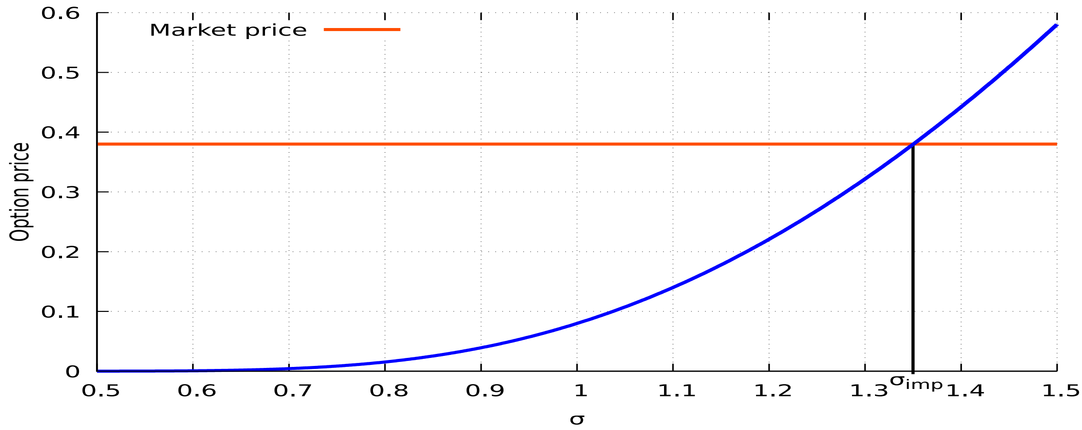


Figure 9.4: Option price as a function of the volatility σ.

This value of σ is called the implied volatility, and it is denoted here by $\sigma_{\rm imp}(K,T)$, cf. *e.g.* Exercise 7.6. Various algorithms can be implemented to solve (9.4) numerically for $\sigma_{\rm imp}(K,T)$, such as the bisection method and the Newton-Raphson method.*

```
BS <- function(S, K, T, r, sig){d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
d2 <- d1 - sig*sqrt(T);return(S*pnorm(d1) - K*exp(-r*T)*pnorm(d2))}
implied.vol <- function(S, K, T, r, market){
sig <- 0.20;sig.up <- 10;sig.down <- 0.0001;count <- 0;err <- BS(S, K, T, r, sig) - market
while(abs(err) > 0.00001 && count<1000){
if(err < 0){sig.down <- sig;sig <- (sig.up + sig)/2} else{sig.up <- sig;sig <- (sig.down + sig)/2}
err <- BS(S, K, T, r, sig) - market;count <- count + 1};if(count==1000){return(NA)}else{return(sig)}}
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02; implied.vol(S, K, T, r, market)
BS(S, K, T, r, implied.vol(S, K, T, r, market))
```

The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, market option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, cf. Figure S.13.

```
library(fOptions)
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02
sig=GBSVolatility(market,"c",S,K,T,r,r,1e-4,maxiter = 10000)
BS(S, K, T, r, sig)
```

*Download the corresponding **R code** or the **IPython notebook** that can be run **here**.

Option chain data in R

```
install.packages("quantmod")
library(quantmod)
getSymbols("^GSPC", src = "yahoo", from = as.Date("2018-01-01"), to = as.
Date("2018-03-013"))
head(GSPC)
# Only the front-month expiry
GSPC.OPT <- getOptionChain("^GSPC")
# All expiries
GSPC.OPTS <- getOptionChain("^GSPC", NULL)
# All 2018 to 2020 expiries
GSPC.OPTS <- getOptionChain("^GSPC", "2018/2020")
# Only the front-month expiry
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2018 to 2020 expiries
AAPL.OPTS <- getOptionChain("AAPL", "2018/2020")
```

Exporting option price data

```
write.table(AAPL.OPT$puts, file = "AAPLputs")
write.csv(AAPL.OPT$puts, file = "AAPLputs.csv")
install.packages("xlsx")
library(xlsx)
write.xlsx(AAPL.OPTS$Jun.19.2020$puts, file = "AAPL.OPTS$Jun.19.2020$puts.xlsx")
```

Volatility smiles

Given two European call options with strike prices K_1, resp. K_2, maturities T_1, resp. T_2, and prices C_1, resp. C_2, on the same stock S, this procedure should yield two estimates $\sigma_{\text{imp}}(K_1, T_1)$ and $\sigma_{\text{imp}}(K_2, T_2)$ of implied volatilites according to the following equations.

$$\begin{cases} \text{Bl}(t, S_t, K_1, \sigma_{\text{imp}}(K_1, T_1), r, T_1) = M_1, & (9.5a) \\ \\ \text{Bl}(t, S_t, K_2, \sigma_{\text{imp}}(K_2, T_2), r, T_2) = M_2, & (9.5b) \end{cases}$$

Clearly, there is no reason a priori for the implied volatilites $\sigma_{\text{imp}}(K_1, T_1)$, $\sigma_{\text{imp}}(K_2, T_2)$ solutions of (9.5a)-(9.5b) to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter σ should be unique for a given stock S. This contradiction between a model and market data is a reason for the development of more sophisticated stochastic volatility models.

```
install.packages("jsonlite")
install.packages("lubridate")
library(jsonlite);library(lubridate);library(quantmod)
# Maturity to be updated as needed
maturity <- as.Date("2022-06-17", format="%Y-%m-%d")
CHAIN <- getOptionChain("AAPL",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
T <- as.numeric((maturity - today)/365);r = 0.02;ImpVol<-1:1
getSymbols("AAPL",from=today-1,to=today,src="yahoo");S=as.numeric(Ad(`AAPL`))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S,CHAIN$calls$Strike[i],T,r,
      CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
      volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
      data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=2)
```

```
# Maturity to be updated as needed
maturity <- as.Date("2019-12-20", format="%Y-%m-%d")
CHAIN <- getOptionChain("^GSPC",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
T <- as.numeric((maturity - today)/365);r = 0.02;ImpVol<-1:1
getSymbols("^GSPC",from=today-1,to=today,src="yahoo");S=as.numeric(Ad(`GSPC`))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S, CHAIN$calls$Strike[i], T, r,
      CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
      volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
      data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=3)
```

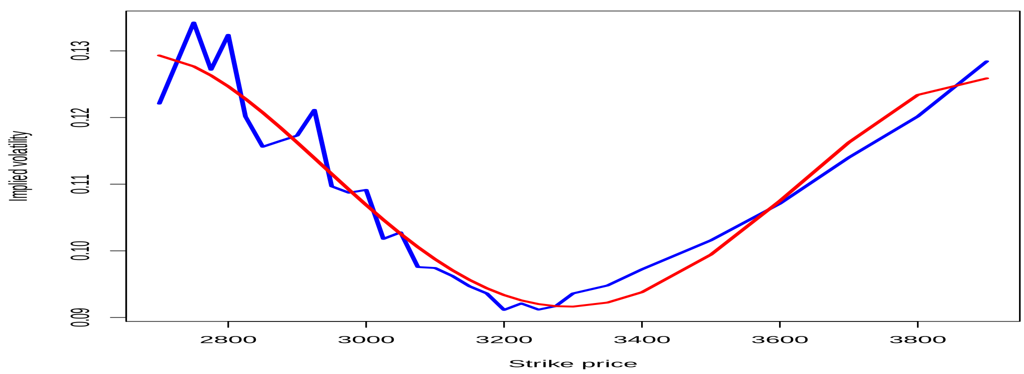


Figure 9.5: S&P500 option prices plotted against strike prices.

When reading option prices on the volatility scale, the smile phenomenon shows that the Black-Scholes formula tends to underprice extreme events for which the underlying asset price S_T is far away from the strike price K. In that sense, the Black-Scholes model, which is based on the Gaussian distribution tails, tends to underestimate the probability of extreme events.

Plotting the different values of the implied volatility σ as a function of K and T will yield a three-dimensional plot called the volatility surface.*

Figure 9.6 presents an estimated implied volatility surface for Asian options on light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange.

*Download the corresponding **IPython notebook** that can be run **here** (© Qu Mengyuan).

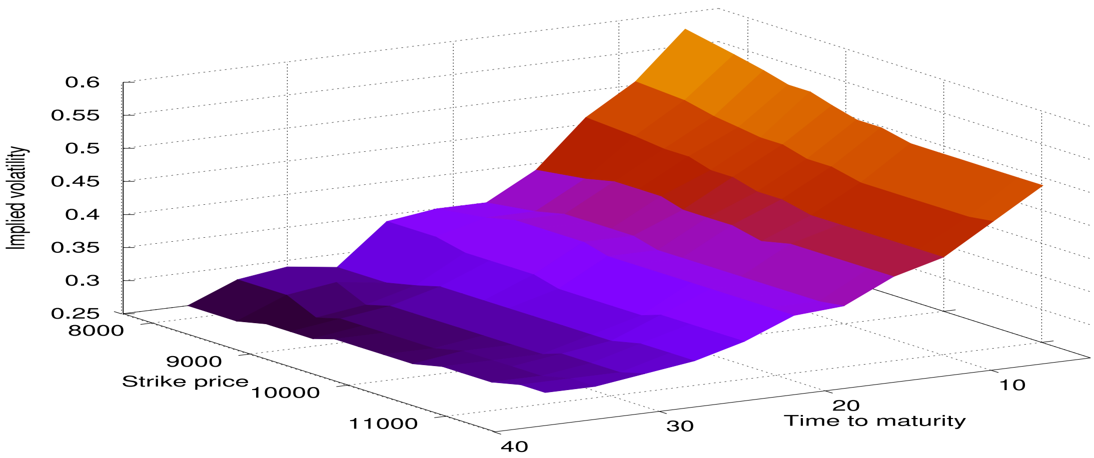


Figure 9.6: Implied volatility surface of Asian options on light sweet crude oil futures.*

As observed in Figure 9.6, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

Black-Scholes Formula *vs* Market Data

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price S of Cheung Kong Holdings (0001.HK) with strike price K=\$109.99, Maturity T = December 13, 2010, and entitlement ratio 100.

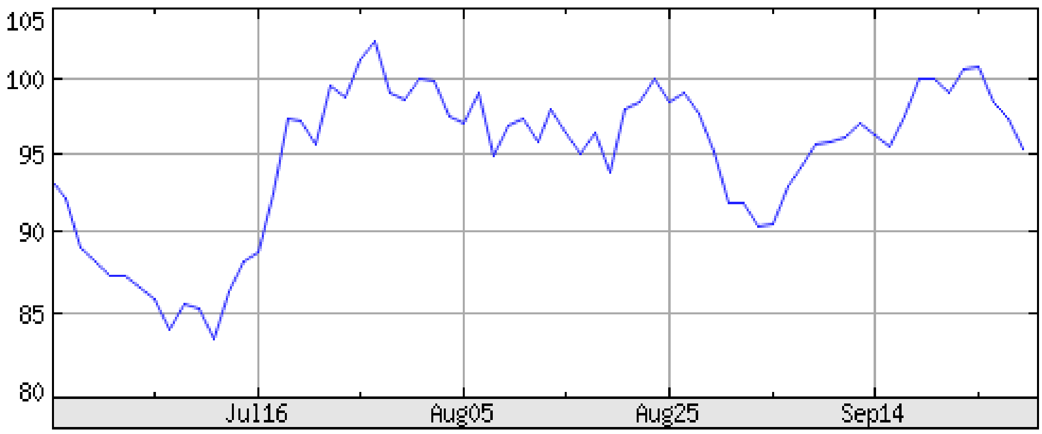


Figure 9.7: Graph of the (market) stock price of Cheung Kong Holdings.

The market price of the option (17838.HK) on September 28 was \$12.30, as obtained from https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp.

The next graph in Figure 9.8 shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying asset price.
In Figure 9.9 we have fitted the path

$$t \longmapsto g_c(t, S_t)$$

of the Black-Scholes price to the data of Figure 9.8 using the market stock price data of Figure 9.7, by varying the values of the volatility σ.

*© Tan Yu Jia.

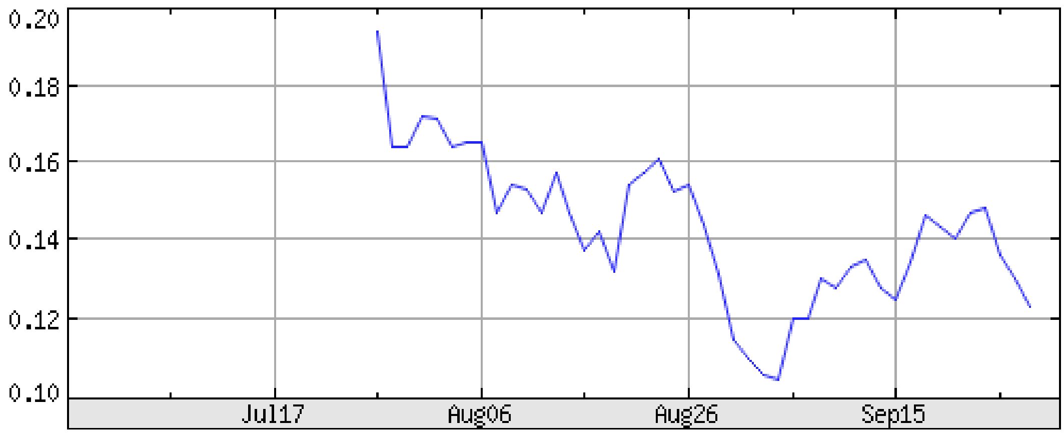


Figure 9.8: Graph of the (market) call option price on Cheung Kong Holdings.

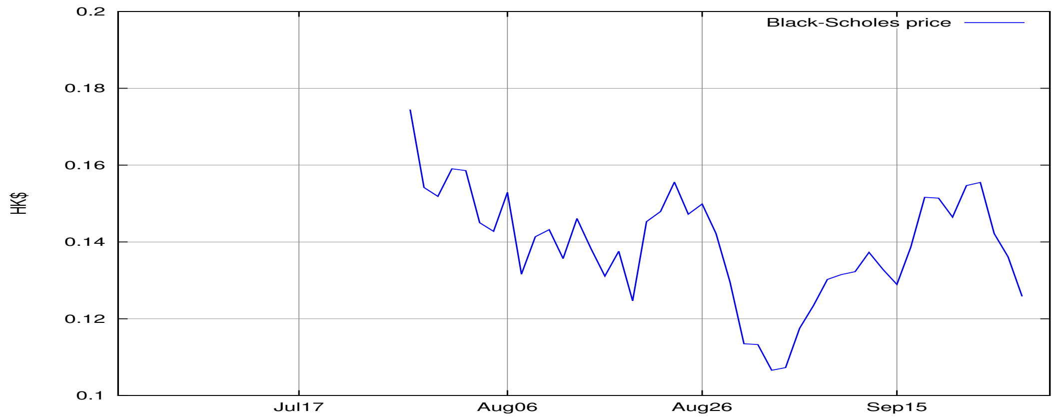


Figure 9.9: Graph of the Black-Scholes call option price on Cheung Kong Holdings.

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

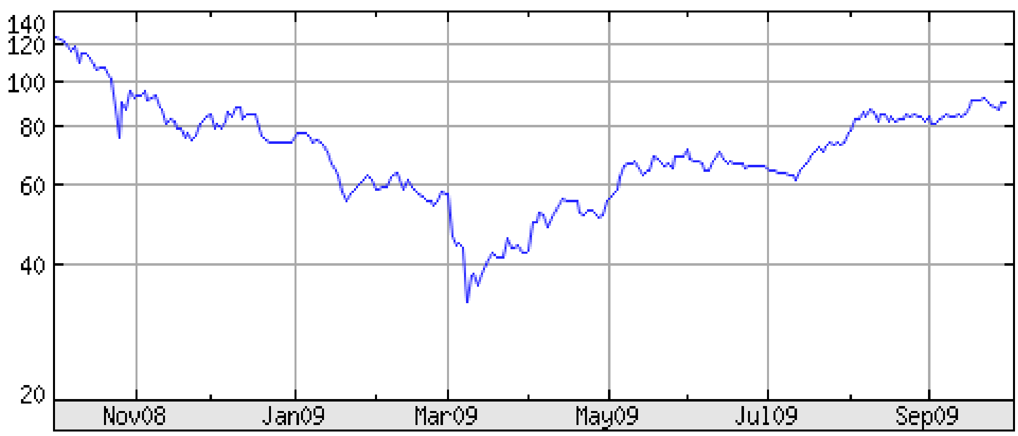


Figure 9.10: Graph of the (market) stock price of HSBC Holdings.

Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price K=$63.704, maturity T = October 05, 2009, and entitlement ratio 100, cf. page 7.

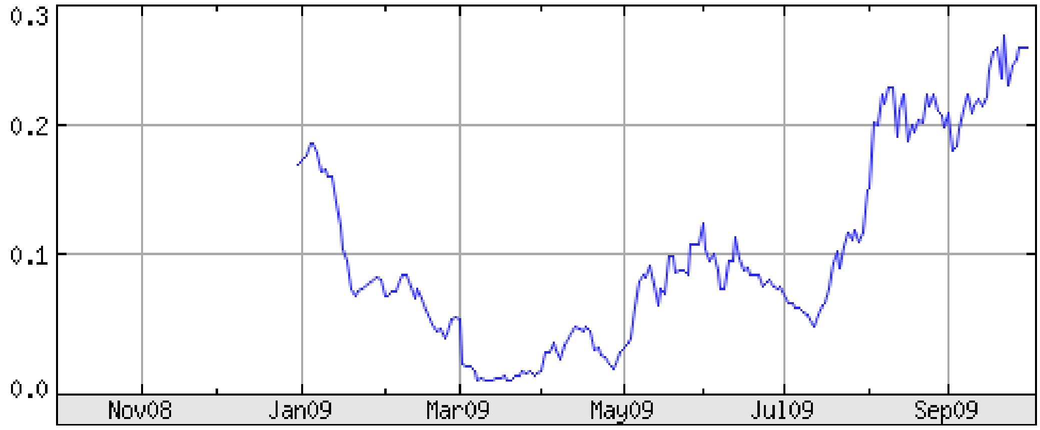


Figure 9.11: Graph of the (market) call option price on HSBC Holdings.

As above, in Figure 9.12 we have fitted the path $t \longmapsto g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 9.11 using the stock price data of Figure 9.10.

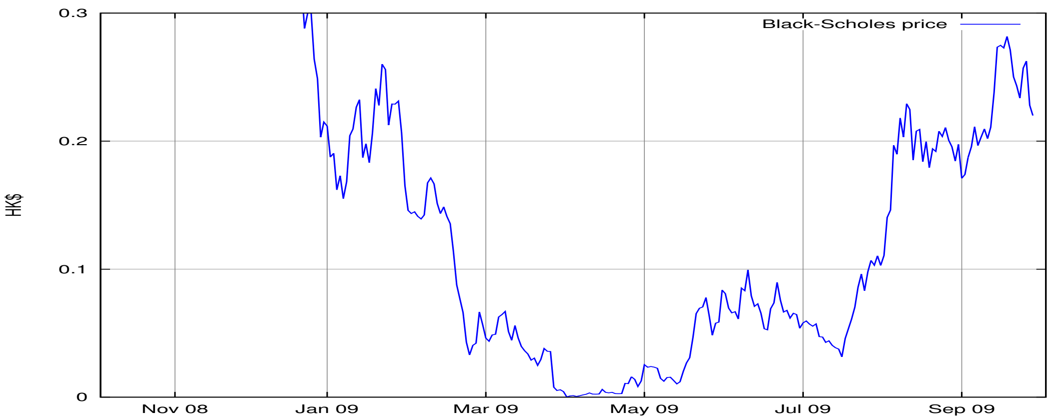


Figure 9.12: Graph of the Black-Scholes call option price on HSBC Holdings.

In this case the option is *in the money* at maturity. We can also check that the option is worth $100 \times 0.2650 = \$26.650$ at that time, which, according to absence of arbitrage, is very close to the actual value \$90 - \$63.703 = \$26.296 of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying asset HSBC, with strike price K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593.
One checks easily that at maturity, the price of the put option is worth \$0.01 (a market price cannot be lower), which almost equals the option payoff \$0, by absence of arbitrage opportunities. Figure 9.14 is a fit of the Black-Scholes put price graph

$$t \longmapsto g_p(t, S_t)$$

to Figure 9.13 as a function of the stock price data of Figure 9.12. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.
The normalized market data graph in Figure 9.15 shows how the option price can track the values of the underlying asset price. Note that the range of values $[26.55, 26.90]$ for the underlying asset price

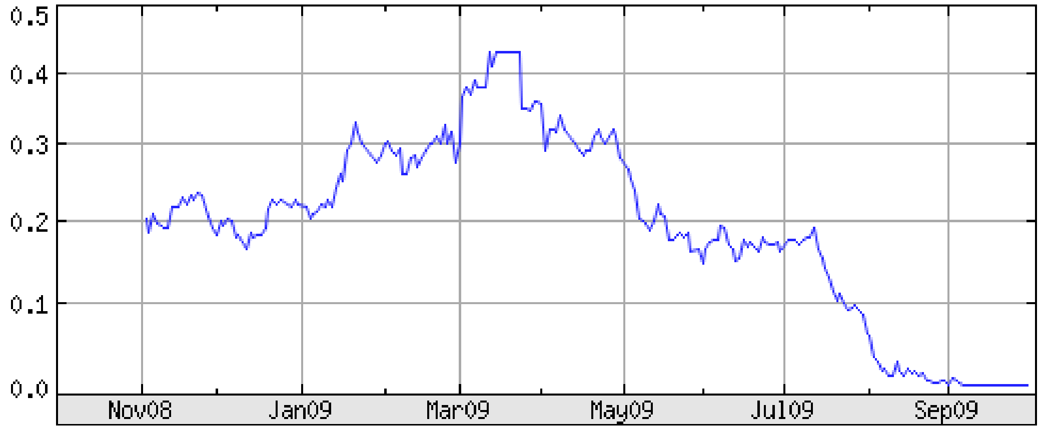


Figure 9.13: Graph of the (market) put option price on HSBC Holdings.

Figure 9.14: Graph of the Black-Scholes put option price on HSBC Holdings.

corresponds to $[0.675, 0.715]$ for the option price, meaning 1.36% *vs* 5.9% in percentage. This is a European call option on the ALSTOM underlying asset with strike price $K = €20$, maturity March 20, 2015, and entitlement ratio 10.

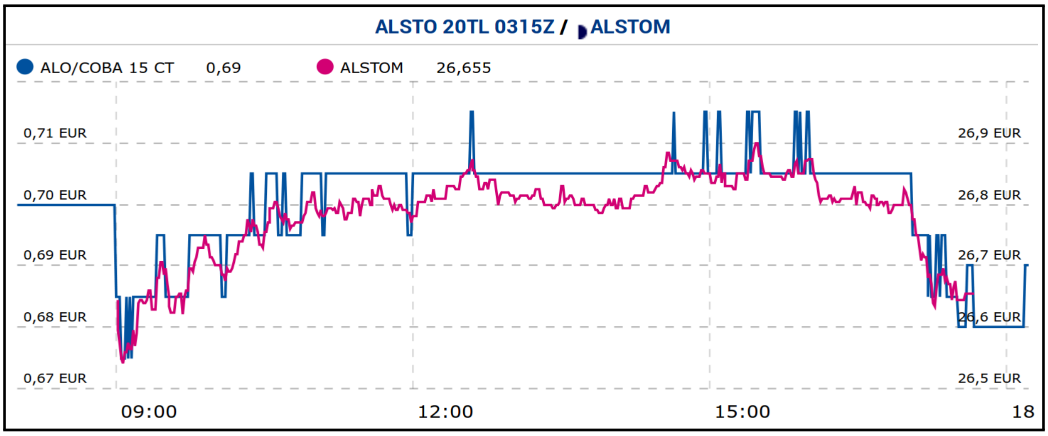


Figure 9.15: Call option price *vs* underlying asset price.

9.3 Local Volatility

As the constant volatility assumption in the Black-Scholes model does not appear to be satisfactory due to the existence of volatility smiles, it can make more sense to consider models of the form

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t$$

where σ_t is a random process. Such models are called stochastic volatility models.

A particular class of stochastic volatility models can be written as

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dB_t \tag{9.6}$$

where $\sigma(t,x)$ is a deterministic function of time t and of the underlying asset price x. Such models are called local volatility models. The corresponding Black-Scholes PDE for the option prices

$$g(t,x,K) := \mathrm{e}^{-(T-t)r}\mathbb{E}[(S_T - K)^+ \mid S_t = x], \tag{9.7}$$

where $(S_t)_{t\in\mathbb{R}_+}$ is defined by (9.6), can be written as

$$\begin{cases} rg(t,x,K) = \dfrac{\partial g}{\partial t}(t,x,K) + rx\dfrac{\partial g}{\partial x}(t,x,K) + \dfrac{1}{2}x^2\sigma^2(t,x)\dfrac{\partial^2 g}{\partial x^2}(t,x,K), \\ \\ g(T,x,K) = (x-K)^+, \end{cases} \tag{9.8}$$

with terminal condition $g(T,x,K) = (x-K)^+$, *i.e.* we consider European call options.

Lemma 9.1 (Relation (1) in Breeden and Litzenberger, 1978). Consider a family $(C^M(T,K))_{T,K>0}$ of market call option prices with maturities T and strike prices K given at time 0. Then the probability density function $\varphi_t(y)$ of S_t, $t \in [0,T]$, is given by

$$\boxed{\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K), \qquad K > 0. \tag{9.9}}$$

Proof. Assume that the market option prices $C^M(T,K)$ match the Black-Scholes prices $\mathrm{e}^{-rT}\mathbb{E}[(S_T-K)^+]$, $K>0$. Letting $\varphi_T(y)$ denote the probability density function of S_T, Condition (9.12) can be written at time $t=0$ as

$$\begin{aligned} C^M(T,K) &= \mathrm{e}^{-rT}\mathbb{E}[(S_T-K)^+] \\ &= \mathrm{e}^{-rT}\int_0^\infty (y-K)^+\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty (y-K)\varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy \\ &= \mathrm{e}^{-rT}\int_K^\infty y\varphi_T(y)dy - K\,\mathrm{e}^{-rT}\mathbb{P}(S_T \geqslant K). \end{aligned} \tag{9.10}$$

By differentiation of (9.10) with respect to K, one gets

$$\begin{aligned} \frac{\partial C^M}{\partial K}(T,K) &= -\mathrm{e}^{-rT}K\varphi_T(K) - \mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy + \mathrm{e}^{-rT}K\varphi_T(K) \\ &= -\mathrm{e}^{-rT}\int_K^\infty \varphi_T(y)dy, \end{aligned}$$

which yields (9.9) by twice differentiation of $C^M(T,K)$ with respect to K. $\square$

In order to implement a stochastic volatility model such as (9.6), it is important to first calibrate the local volatility function $\sigma(t,x)$ to market data.

In principle, the Black-Scholes PDE could allow one to recover the value of $\sigma(t,x)$ as a function of the option price $g(t,x,K)$, as

$$\sigma(t,x) = \sqrt{\frac{2rg(t,x,K) - 2\frac{\partial g}{\partial t}(t,x,K) - 2rx\frac{\partial g}{\partial x}(t,x,K)}{x^2\frac{\partial^2 g}{\partial x^2}(t,x,K)}}, \qquad x,t>0,$$

however, this formula requires the knowledge of the option price for different values of the underlying asset price x, in addition to the knowledge of the strike price K.

The Dupire, 1994 formula brings a solution to the local volatility calibration problem by providing an estimator of $\sigma(t,x)$ as a function $\sigma(t,K)$ based on the values of the strike price K.

Proposition 9.2 (Dupire, 1994, Derman and Kani, 1994) Consider a family $(C^M(T,K))_{T,K>0}$ of market call option prices with maturities T and strike prices K given at time 0 with $S_0=x$, and define the volatility function $\sigma(t,y)$ by

$$\boxed{\sigma(t,y) := \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y) + 2ry\frac{\partial C^M}{\partial y}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \frac{\sqrt{\frac{\partial C^M}{\partial t}(t,y) + ry\frac{\partial C^M}{\partial y}(t,y)}}{y\,\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}},} \tag{9.11}$$

where $\varphi_t(y)$ denotes the probability density function of S_t, $t \in [0,T]$. Then the prices generated from the Black-Scholes PDE (9.8) will be compatible with the market option prices $C^M(T,K)$ in the sense that

$$C^M(T,K) = \mathrm{e}^{-rT}\,\mathbb{E}[(S_T-K)^+], \qquad K > 0. \tag{9.12}$$

Proof. For any sufficiently smooth function $f \in \mathscr{C}_0^\infty(\mathbb{R})$, with $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$, using the Itô formula we have

$$\begin{aligned}
\mathbb{E}[f(S_T)] &= \mathbb{E}\left[f(S_0) + r\int_0^T S_t f'(S_t)dt + \int_0^T S_t f'(S_t)\sigma(t,S_t)dB_t \right.\\
&\quad \left. + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + \mathbb{E}\left[r\int_0^T S_t f'(S_t)dt + \frac{1}{2}\int_0^T S_t^2 f''(S_t)\sigma^2(t,S_t)dt\right]\\
&= f(S_0) + r\int_0^T \mathbb{E}[S_t f'(S_t)]dt + \frac{1}{2}\int_0^T \mathbb{E}[S_t^2 f''(S_t)\sigma^2(t,S_t)]dt\\
&= f(S_0) + r\int_{-\infty}^\infty y f'(y)\int_0^T \varphi_t(y)dtdy\\
&\quad + \frac{1}{2}\int_{-\infty}^\infty y^2 f''(y)\int_0^T \sigma^2(t,y)\varphi_t(y)dtdy,
\end{aligned}$$

hence, after differentiating both sides of the equality with respect to T,

$$\int_{-\infty}^\infty f(y)\frac{\partial\varphi_T}{\partial T}(y)dy = r\int_{-\infty}^\infty yf'(y)\varphi_T(y)dy + \frac{1}{2}\int_{-\infty}^\infty y^2 f''(y)\sigma^2(T,y)\varphi_T(y)dy.$$

Integrating by parts in the above relation yields

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{\partial\varphi_T}{\partial T}(y)f(y)dy\\
&= -r\int_{-\infty}^\infty f(y)\frac{\partial}{\partial y}(y\varphi_T(y))dy + \frac{1}{2}\int_{-\infty}^\infty f(y)\frac{\partial^2}{\partial y^2}(y^2\sigma^2(T,y)\varphi_T(y))dy,
\end{aligned}$$

for all smooth functions $f(y)$ with compact support, hence

$$\frac{\partial\varphi_T}{\partial T}(y) = -r\frac{\partial}{\partial y}(y\varphi_T(y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(y^2\sigma^2(T,y)\varphi_T(y)), \qquad y \in \mathbb{R}.$$

Making use of Relation (9.9) in Lemma 9.1, we have

$$\frac{\partial\varphi_T}{\partial T}(K) = r\,\mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial K^2}(T,K) + \mathrm{e}^{rT}\frac{\partial^3 C^M}{\partial T\partial K^2}(T,K),$$

hence we get

$$\begin{aligned}
&-r\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{\partial^3 C^M}{\partial T\partial y^2}(T,y)\\
&= r\frac{\partial}{\partial y}\left(y\frac{\partial^2 C^M}{\partial y^2}(T,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right), \qquad y \in \mathbb{R}.
\end{aligned}$$

After a first integration with respect to y under the boundary condition $\lim_{y\to+\infty} C^M(T,y) = 0$, we obtain

$$-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)$$

$$= \quad ry\frac{\partial^2 C^M}{\partial y^2}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),$$

i.e.

$$\begin{aligned}
&-r\frac{\partial C^M}{\partial y}(T,y) - \frac{\partial}{\partial T}\frac{\partial C^M}{\partial y}(T,y)\\
&= \quad r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - r\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right),
\end{aligned}$$

or

$$-\frac{\partial}{\partial y}\frac{\partial C^M}{\partial T}(T,y) = r\frac{\partial}{\partial y}\left(y\frac{\partial C^M}{\partial y}(T,y)\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y)\right).$$

Integrating one more time with respect to y yields

$$-\frac{\partial C^M}{\partial T}(T,y) = ry\frac{\partial C^M}{\partial y}(T,y) - \frac{1}{2}y^2\sigma^2(T,y)\frac{\partial^2 C^M}{\partial y^2}(T,y), \qquad y \in \mathbb{R},$$

which conducts to (9.11) and is called the Dupire, 1994 PDE. □

Partial derivatives in time can be approximated using *forward* finite difference approximations as

$$\frac{\partial C}{\partial t}(t_i,y) \simeq \frac{C(t_{i+1},y_j) - C(t_i,y_j)}{\Delta t}, \tag{9.13}$$

or, using *backward* finite difference approximations, as

$$\frac{\partial C}{\partial t}(t_i,y) \simeq \frac{C(t_i,y_j) - C(t_{i-1},y_j)}{\Delta t}. \tag{9.14}$$

First order spatial derivatives can be approximated as

$$\frac{\partial C}{\partial y}(t,y_j) \simeq \frac{C(t,y_j) - C(t_i,y_{j-1})}{\Delta y}, \quad \frac{\partial C}{\partial y}(t,y_{j+1}) \simeq \frac{C(t,y_{j+1}) - C(t_i,y_j)}{\Delta y}. \tag{9.15}$$

Reusing (9.15), second order spatial derivatives can be similarly approximated as

$$\begin{aligned}
\frac{\partial^2 C}{\partial y^2}(t,y_j) \quad &\simeq \quad \frac{1}{\Delta y}\left(\frac{\partial C}{\partial y}(t,y_{j+1}) - \frac{\partial C}{\partial y}(t,y_j)\right)\\
&\simeq \quad \frac{C(t_i,y_{j+1}) + C(t_i,y_{j-1}) - 2C(t_i,y_j)}{(\Delta y)^2}.
\end{aligned} \tag{9.16}$$

Figure 9.16* presents an estimation of local volatility by the finite differences (9.13)-(9.16), based on Boeing (NYSE:BA) option price data.
See Achdou and Pironneau, 2005 and in particular Figure 8.1 therein for numerical methods applied to local volatility estimation using spline functions instead of the discretization (9.13)-(9.16).

The attached **R code** (© Abhishek Vijaykumar) plots a local volatility estimate for a given stock.

Based on (9.11), the local volatility $\sigma(t,y)$ can also be estimated by computing $C^M(T,y)$ from the Black-Scholes formula, based on a value of the implied volatility σ.

*© Yu Zhi Yu.

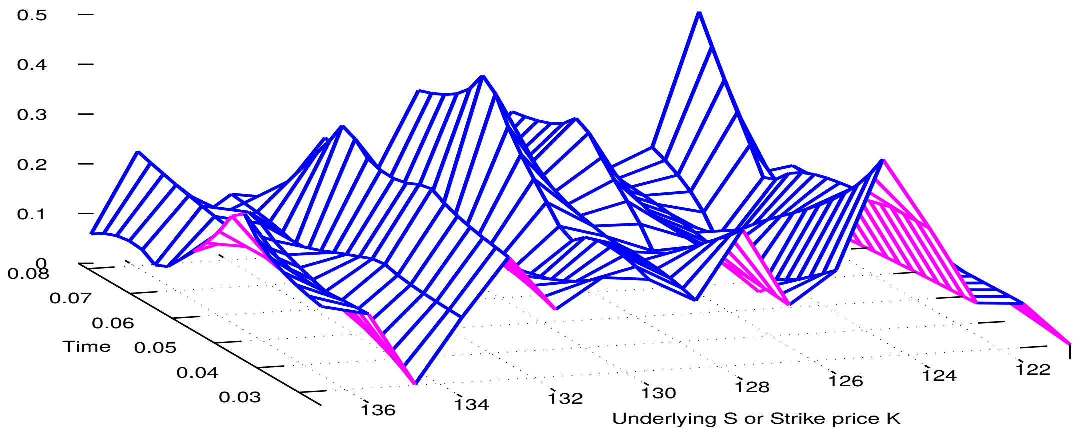


Figure 9.16: Local volatility estimated from Boeing Co. option price data.

Local volatility from put option prices

Note that by the call-put parity relation

$$C^M(T,y) = P^M(T,y) + x - y\mathrm{e}^{-rT}, \qquad y,T > 0,$$

where $S_0 =$, cf. (7.21), we have

$$\begin{cases} \dfrac{\partial C^M}{\partial T}(T,y) = ry\mathrm{e}^{-rT} + \dfrac{\partial P^M}{\partial T}(T,y), \\ \\ \dfrac{\partial P^M}{\partial y}(t,y) = \mathrm{e}^{-rT} + \dfrac{\partial C^M}{\partial y}(t,y), \end{cases}$$

and

$$\frac{\partial C^M}{\partial T}(T,y) + ry\frac{\partial C^M}{\partial y}(T,y) = \frac{\partial P^M}{\partial T}(T,y) + ry\frac{\partial P^M}{\partial y}(T,y).$$

Consequently, the local volatility in Proposition 9.2 can be rewritten in terms of market put option prices as

$$\sigma(t,y) := \sqrt{\frac{2\dfrac{\partial P^M}{\partial t}(t,y) + 2ry\dfrac{\partial P^M}{\partial y}(t,y)}{y^2\dfrac{\partial^2 P^M}{\partial y^2}(t,y)}} = \frac{\sqrt{\dfrac{\partial P^M}{\partial t}(t,y) + ry\dfrac{\partial P^M}{\partial y}(t,y)}}{y\mathrm{e}^{-rT/2}\sqrt{\varphi_t(y)/2}},$$

which is formally identical to (9.11) after replacing market call option prices $C^M(T,K)$ with market put option prices $P^M(T,K)$. In addition, we have the relation

$$\varphi_T(K) = \mathrm{e}^{rT}\frac{\partial^2 C^M}{\partial y^2}(T,y) = \mathrm{e}^{rT}\frac{\partial^2 P^M}{\partial y^2}(T,y) \tag{9.17}$$

between the probability density function φ_T of S_T and the call/put option pricing functions $C^M(T,y)$, $P^M(T,y)$.

9.4 The VIX® Index

Other ways to estimate market volatility include the CBOE Volatility Index® (VIX®) for the S&P 500 stock index, cf. *e.g.* § 3.1.1 of Papanicolaou and Sircar, 2014. Let the asset price process $(S_t)_{t\in\mathbb{R}_+}$ be given as

$$dS_t = rS_t dt + \sigma_t S_t dB_t$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ is a stochastic volatility process which is *independent* of the Brownian motion $(B_t)_{t\in\mathbb{R}_+}$.

The next Proposition 9.3, cf. Friz and Gatheral, 2005, shows that the VIX® Volatility Index defined as

$$\mathrm{VIX}_t := \sqrt{\frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\int_0^{F_{t,t+\tau}} \frac{P(t,t+\tau,K)}{K^2}dK + \int_{F_{t,t+\tau}}^{\infty} \frac{C(t,t+\tau,K)}{K^2}dK\right)} \tag{9.18}$$

at time $t>0$ can be interpreted as an average of future volatility values. Here, $\tau=$30 days,

$$F_{t,t+\tau} := \mathbb{E}^*[S_{t+\tau} \mid \mathscr{F}_t] = \mathrm{e}^{r\tau}S_t$$

represents the future price on $S_{t+\tau}$, and $P(t,t+\tau,K)$, $C(t,t+\tau,K)$ are out-of-the-money put and call option prices with strike price K and maturity $t+\tau$.

Proposition 9.3 The value of the VIX® Volatility Index at time $t \geqslant 0$ is given from the averaged realized variance option as

$$\mathrm{VIX}_t := \sqrt{\frac{1}{\tau}\mathbb{E}^*\left[\int_t^{t+\tau}\sigma_u^2 du\right]}.$$

Proof. We take $t=0$ for simplicity. For any $F,S>0$, we note the relationships

$$\begin{aligned}
\int_0^F (K-S)^+\frac{dK}{K^2} &= \mathbb{1}_{\{S\leqslant F\}}\int_S^F (K-S)\frac{dK}{K^2}\\
&= \mathbb{1}_{\{S\leqslant F\}}\left(\int_S^F\frac{dK}{K} - S\int_S^F\frac{dK}{K^2}\right)\\
&= \mathbb{1}_{\{S\leqslant F\}}\left(\frac{S}{F}-1+\log\frac{F}{S}\right),
\end{aligned}$$

and

$$\begin{aligned}
\int_F^\infty (S-K)^+\frac{dK}{K^2} &= \mathbb{1}_{\{S\geqslant F\}}\int_F^S (S-K)\frac{dK}{K^2}\\
&= \mathbb{1}_{\{S\geqslant F\}}\left(S\int_F^S\frac{dK}{K^2} - \int_F^S\frac{dK}{K}\right)\\
&= \mathbb{1}_{\{S\geqslant F\}}\left(\frac{S}{F}-1+\log\frac{F}{S}\right).
\end{aligned}$$

Hence, taking $F := F_{0,\tau} = \mathrm{e}^{r\tau}S_0$ and $S := S_\tau$, we have

$$\int_0^{F_{0,\tau}}(K-S_\tau)^+\frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty (S_\tau-K)^+\frac{dK}{K^2} = \left(\frac{S_\tau}{F_{0,\tau}}-1+\log\frac{F_{0,\tau}}{S_\tau}\right). \tag{9.19}$$

Next, taking expectations under $\mathbb{P}^*$ on both sides of (9.19), we find

$$\begin{aligned}
\mathrm{VIX}_0^2 &= \frac{2\,\mathrm{e}^{r\tau}}{\tau}\left(\int_0^{F_{0,\tau}}\frac{P(0,\tau,K)}{K^2}dK + \int_{F_{0,\tau}}^\infty\frac{C(0,\tau,K)}{K^2}dK\right)\\
&= \frac{2}{\tau}\int_0^{F_{0,\tau}}\mathbb{E}^*[(K-S_\tau)^+]\frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty \mathbb{E}^*[(S_\tau-K)^+]\frac{dK}{K^2}\\
&= \frac{2}{\tau}\mathbb{E}^*\left[\int_0^{F_{0,\tau}}(K-S_\tau)^+\frac{dK}{K^2} + \int_{F_{0,\tau}}^\infty(S_\tau-K)^+\frac{dK}{K^2}\right]\\
&= \frac{2}{\tau}\mathbb{E}^*\left[\frac{S_\tau}{F_{0,\tau}}-1+\log\frac{F_{0,\tau}}{S_\tau}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\tau}\mathbf{E}^*\left[\log\frac{F_{0,\tau}}{S_\tau}\right] \\
&= \frac{1}{\tau}\mathbf{E}^*\left[\int_0^\tau \sigma_t^2 dt\right].
\end{aligned}
$$

□

The following R code allows us to estimate the VIX® index based on the discretization of (9.18) and market option prices on the S&P 500.

```
library(quantmod)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
getSymbols("^GSPC", src = "yahoo", from = today)
S0 = as.vector(Ad(GSPC)[1])
GSPC.OPTS <- getOptionChain("^GSPC", "2020")
GSPC.OPTS <- getOptionChain("^GSPC", today+30)
Call <- as.data.frame(GSPC.OPTS$calls)
Put <- as.data.frame(GSPC.OPTS$puts)
K0 = max(Put[Put$Strike<S0,]$Strike, Call[Call$Strike<S0,]$Strike)
Call_OTM <- Call[Call$Strike>=K0,];Call_OTM$dif = c(S0-K0, diff(Call_OTM$Strike))
Put_OTM <-Put[Put$Strike<=K0,];Put_OTM$dif = c(diff(Put_OTM$Strike),S0-K0)
T = 30/365;r=0.02
VIX_imp = 100*sqrt((2*exp(r*T)/T)*(sum(Call_OTM$Last/(Call_OTM$Strike^2)
    *Call_OTM$dif)+sum(Put_OTM$Last/(Put_OTM$Strike^2)*Put_OTM$dif)))
getSymbols("^VIX", src = "yahoo", from = today)
VIX_market = as.vector(Ad(VIX)[1])
c("Estimated VIX"= VIX_imp, "market VIX"=VIX_market)
VIX.OPTS <- getOptionChain("^VIX")
```

The following R code is fetching VIX® index data using the quantmod R package.

```
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo")
myTheme <- chart_theme()
myTheme$col$line.col <- "blue"
chart_Series(Ad(`GSPC`),name="S&P500",theme=myTheme)
add_TA(Ad(`VIX`), name="VIX")
```

The impact of various events, such as the June 23, 2016 "Brexit" referendum, can be observed on the VIX® index in Figure 9.17.

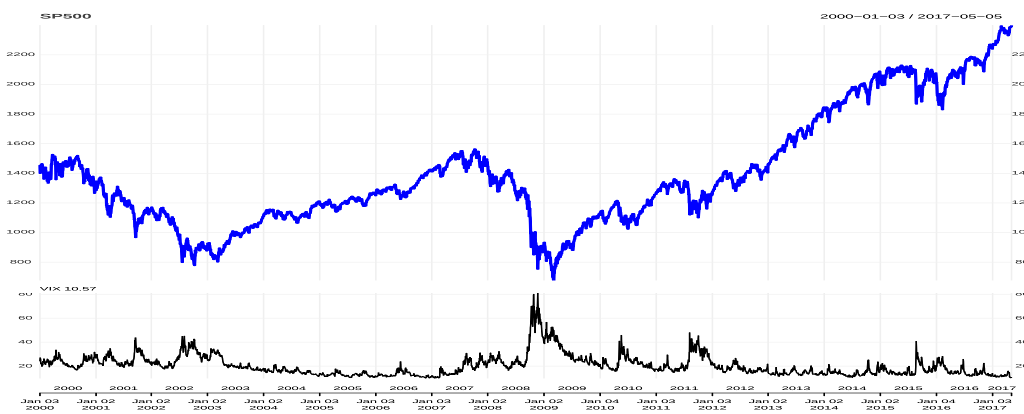


Figure 9.17: VIX® Index *vs* the S&P 500.

```r
library(PerformanceAnalytics)
library(quantmod)
getSymbols("^GSPC",from="2000-01-01",to=Sys.Date(),src="yahoo")
getSymbols("^VIX",from="2000-01-01",to=Sys.Date(),src="yahoo");SP500=Ad(`GSPC`)
SP500.rtn <- exp(CalculateReturns(SP500,method="compound")) - 1;SP500.rtn[1,] <- 0
histvol <- rollapply(SP500.rtn, width = 30, FUN=sd.annualized)
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(SP500,name="SP500",theme=myTheme)
add_TA(histvol, name="Historical Volatility");add_TA(Ad(`VIX`), name="VIX")
```

Figure 9.18 compares the VIX® index estimate to the historical volatility of Section 9.1.

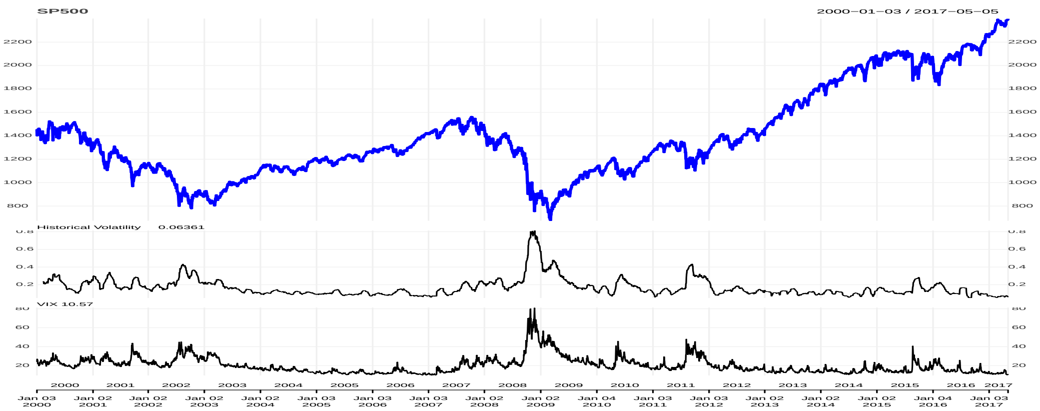
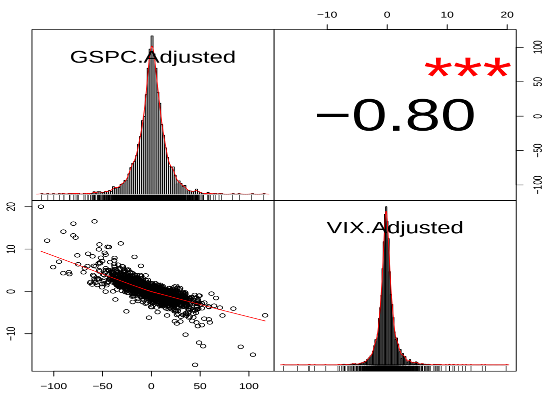
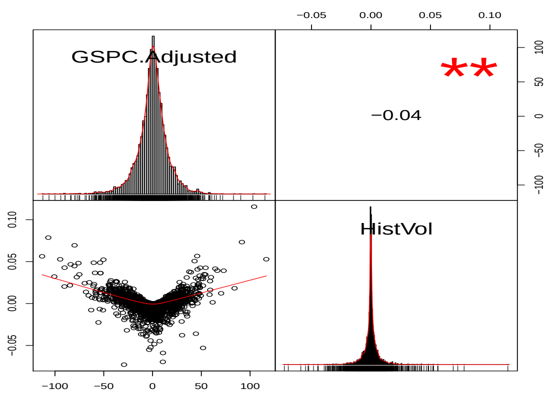


Figure 9.18: VIX® Index *vs* historical volatility for the year 2011.

We note that the variations of the stock index are negatively correlated to the variations of the VIX® index, however the same cannot be said of the correlation to the variations of historical volatility.



(a) Underlying *vs* the VIX® index.



(b) Underlying *vs* hist. volatility.

Figure 9.19: Correlation estimates between GSPC and the VIX®.

```r
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),Ad(`VIX`)-lag(Ad(`VIX`))), histogram=TRUE,
      pch="+")
colnames(histvol) <- "HistVol"
chart.Correlation(cbind(Ad(`GSPC`)-lag(Ad(`GSPC`)),histvol-lag(histvol)), histogram=TRUE,
      pch="+")
```

The next Figure 9.20 shortens the time range to year 2011 and shows the increased reactivity of the VIX® index to volatility spikes, in comparison with the moving average of historical volatility.

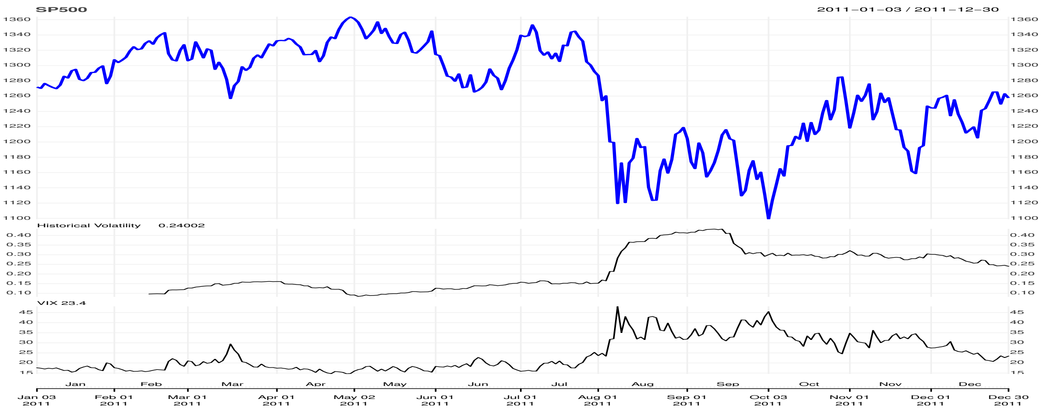


Figure 9.20: VIX® Index *vs* 30 day historical volatility for the S&P 500.

Exercises

Exercise 9.1 Consider the Black-Scholes call pricing formula

$$C(T-t,x,K) = Kf(T-t,x/K)$$

written using the function

$$f(\tau,z) := z\Phi\left(\frac{(r+\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right) - \mathrm{e}^{-r\tau}\Phi\left(\frac{(r-\sigma^2/2)\tau+\log z}{|\sigma|\sqrt{\tau}}\right).$$

a) Compute $\frac{\partial C}{\partial x}$ and $\frac{\partial C}{\partial K}$ using the function f, and find the relation between $\frac{\partial C}{\partial K}(T-t,x,K)$ and $\frac{\partial C}{\partial x}(T-t,x,K)$.

b) Compute $\frac{\partial^2 C}{\partial x^2}$ and $\frac{\partial^2 C}{\partial K^2}$ using the function f, and find the relation between $\frac{\partial C^2}{\partial K^2}(T-t,x,K)$ and $\frac{\partial C^2}{\partial x^2}(T-t,x,K)$.

c) From the Black-Scholes PDE

$$\begin{aligned} rC(T-t,x,K) &= \frac{\partial C}{\partial t}(T-t,x,K) + rx\frac{\partial C}{\partial x}(T-t,x,K) \\ &\quad + \frac{\sigma^2 x^2}{2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K), \end{aligned}$$

recover the Dupire, 1994 PDE for the constant volatility σ.

Exercise 9.2 Let $\sigma_{\mathrm{imp}}(K)$ denote the implied volatility of a call option with strike price K, defined from the relation

$$M_C(K,S,r,\tau) = C(K,S,\sigma_{\mathrm{imp}}(K),r,\tau),$$

where M_C is the market price of the call option, $C(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)$ is the Black-Scholes call pricing function, S is the underlying asset price, τ is the time remaining until maturity, and r is the risk-free interest rate.

a) Compute the partial derivative

$$\frac{\partial M_C}{\partial K}(K,S,r,\tau).$$

using the functions C and σ_{imp}.

b) Knowing that market call option prices $M_C(K,S,r,\tau)$ are *decreasing* in the strike prices K, find an upper bound for the slope $\sigma'_{\text{imp}}(K)$ of the implied volatility curve.

c) Similarly, knowing that the market *put* option prices $M_P(K,S,r,\tau)$ are *increasing* in the strike prices K, find a lower bound for the slope $\sigma'_{\text{imp}}(K)$ of the implied volatility curve.

Exercise 9.3 (Hagan et al., 2002) Consider the European option priced as $e^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ in a local volatility model $dS_t = \sigma_{\text{loc}}(S_t)S_t dB_t$. The implied volatility $\sigma_{\text{imp}}(K,S_0)$, computed from the equation

$$\text{Bl}\big(S_0,K,T,\sigma_{\text{imp}}(K,S_0),r\big) = \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+],$$

is known to admit the approximation

$$\sigma_{\text{imp}}(K,S_0) \simeq \sigma_{\text{loc}}\left(\frac{K+S_0}{2}\right).$$

a) Taking a local volatility of the form $\sigma_{\text{loc}}(x) := \sigma_0 + \beta(x-S_0)^2$, estimate the implied volatility $\sigma_{\text{imp}}(K,S)$ when the underlying asset price is at the level S.

b) Express the Delta of the Black Scholes call option price given by

$$\text{Bl}\big(S,K,T,\sigma_{\text{imp}}(K,S),r\big),$$

using the standard Black-Scholes Delta and the Black-Scholes Vega.

10. Basic Numerical Methods

Numerical methods in finance include finite difference methods, and statistical and Monte Carlo methods for computation of option prices and hedging strategies. This chapter is a basic introduction to finite difference methods for the resolution of PDEs and stochastic differential equations. We cover the explicit and implicit finite difference schemes for the heat equations and the Black-Scholes PDE, as well as the Euler and Milshtein schemes for stochastic differential equations.

10.1 Discretized Heat Equation

Consider the heat equation

$$\frac{\partial \phi}{\partial t}(t,x) = \frac{\partial^2 \phi}{\partial x^2}(t,x) \tag{10.1}$$

with initial condition

$$\phi(0,x) = f(x)$$

on a compact time-space interval $[0,T] \times [0,X]$.

The intervals $[0,T]$ and $[0,X]$ are respectively discretized according to $\{t_0 = 0, t_1, \ldots, t_N = T\}$ and $\{x_0 = 0, t_1, \ldots, t_M = X\}$ with $\Delta t = T/N$ and $\Delta x = X/M$, from which we construct a grid

$$(t_i, x_j) = (i\Delta t, j\Delta x), \qquad i = 0, \ldots, N, \quad j = 0, \ldots, M,$$

on $[0,T]\times[0,X]$.

Our goal is to solve the heat equation (10.1) with *initial* condition $\phi(0,x)$ and lateral boundary conditions $\phi(t,0)$, $\phi(t,X)$, via a discrete approximation

$$(\phi(t_i,x_j))_{0\leqslant i\leqslant N,\ 0\leqslant j\leqslant M}$$

of the solution to (10.1), by evaluating derivatives using finite differences.

Explicit scheme

Using the *forward* time difference approximation

$$\frac{\partial\phi}{\partial t}(t_i,x)\simeq\frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t}$$

of the time derivative, and the related space difference approximations

$$\frac{\partial\phi}{\partial x}(t,x_j)\simeq\frac{\phi(t,x_j)-\phi(t_i,x_{j-1})}{\Delta x},\qquad\frac{\partial\phi}{\partial x}(t,x_{j+1})\simeq\frac{\phi(t,x_{j+1})-\phi(t_i,x_j))}{\Delta x}$$

and

$$\frac{\partial^2\phi}{\partial x^2}(t,x_j)\simeq\frac{1}{\Delta x}\left(\frac{\partial\phi}{\partial x}(t,x_{j+1})-\frac{\partial\phi}{\partial x}(t,x_j)\right)=\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}$$

of the time and space derivatives, we discretize (10.1) as

$$\frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t}=\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}. \tag{10.2}$$

Letting $\rho=(\Delta t)/(\Delta x)^2$, this yields

$$\phi(t_{i+1},x_j)=\rho\phi(t_i,x_{j+1})+(1-2\rho)\phi(t_i,x_j)+\rho\phi(t_i,x_{j-1}),$$

$1\leqslant j\leqslant M-1$, $1\leqslant i\leqslant N$, *i.e.*

$$\Phi_{i+1}=A\Phi_i+\rho\begin{bmatrix}\phi(t_i,x_0)\\0\\\vdots\\0\\\phi(t_i,x_M)\end{bmatrix},\qquad i=0,1,\ldots,N-1, \tag{10.3}$$

with

$$\Phi_i=\begin{bmatrix}\phi(t_i,x_1)\\\vdots\\\phi(t_i,x_{M-1})\end{bmatrix},\qquad i=0,1,\ldots,N,$$

and

$$A=\begin{bmatrix}1-2\rho&\rho&0&\cdots&0&0&0\\\rho&1-2\rho&\rho&\cdots&0&0&0\\0&\rho&1-2\rho&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&\cdots&1-2\rho&\rho&0\\0&0&0&\cdots&\rho&1-2\rho&\rho\\0&0&0&\cdots&0&\rho&1-2\rho\end{bmatrix}.$$

The vector

$$\begin{bmatrix} \phi(t_i, x_0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, x_M) \end{bmatrix} = \begin{bmatrix} \phi(t_i, 0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, X) \end{bmatrix}, \qquad i = 0, 1, \ldots, N,$$

in (10.3) can be given by the lateral boundary conditions $\phi(t,0)$ and $\phi(t,X)$. From those boundary conditions and the initial data of

$$\Phi_0 = \begin{bmatrix} \phi(0, x_0) \\ \phi(0, x_1) \\ \vdots \\ \phi(0, x_{M-1}) \\ \phi(0, x_M) \end{bmatrix}$$

we can apply (10.3) in order to solve (10.2) recursively for $\Phi_1, \Phi_2, \Phi_3, \ldots$

Implicit scheme

Using the *backward* time difference approximation

$$\frac{\partial \phi}{\partial t}(t_i, x) \simeq \frac{\phi(t_i, x_j) - \phi(t_{i-1}, x_j)}{\Delta t}$$

of the time derivative, we discretize (10.1) as

$$\frac{\phi(t_i, x_j) - \phi(t_{i-1}, x_j)}{\Delta t} = \frac{\phi(t_i, x_{j+1}) + \phi(t_i, x_{j-1}) - 2\phi(t_i, x_j)}{(\Delta x)^2} \tag{10.4}$$

and letting $\rho = (\Delta t)/(\Delta x)^2$ we get

$$\phi(t_{i-1}, x_j) = -\rho \phi(t_i, x_{j+1}) + (1 + 2\rho)\phi(t_i, x_j) - \rho \phi(t_i, x_{j-1}),$$

$1 \leqslant j \leqslant M-1$, $1 \leqslant i \leqslant N$, *i.e.*

$$\Phi_{i-1} = B\Phi_i + \rho \begin{bmatrix} \phi(t_i, x_0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, x_M) \end{bmatrix}, \qquad i = 1, 2, \ldots, N,$$

with

$$B = \begin{bmatrix} 1+2\rho & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+2\rho & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+2\rho & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\rho & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+2\rho & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1+2\rho \end{bmatrix}.$$

By inversion of the matrix B, Φ_i is given in terms of Φ_{i-1} as

$$\Phi_i = B^{-1}\Phi_{i-1} - \rho B^{-1} \begin{bmatrix} \phi(t_i, x_0) \\ 0 \\ \vdots \\ 0 \\ \phi(t_i, x_M) \end{bmatrix}, \qquad i = 1, \ldots, N,$$

which also allows for a recursive solution of (10.4).

10.2 Discretized Black-Scholes PDE

Consider the Black-Scholes PDE

$$r\phi(t,x)=\frac{\partial \phi}{\partial t}(t,x)+rx\frac{\partial \phi}{\partial x}(t,x)+\frac{1}{2}x^2\sigma^2\frac{\partial^2 \phi}{\partial x^2}(t,x), \tag{10.5}$$

under the terminal condition $\phi(T,x)=(x-K)^+$, *resp.* $\phi(T,x)=(K-x)^+$, for a European call, *resp.* put, option. The constant volatility coefficient σ may also be replaced with a function $\sigma(t,x)$ of the underlying asset price, in the case local volatility models.

Note that in the solution of the Black-Scholes PDE, time is run *backwards* as we start from a terminal condition $\phi(T,x)$ at time T. Thus here the explicit scheme uses *backward* differences while the implicit scheme uses *forward* differences.

Explicit scheme

Using here the *backward* time difference approximation

$$\frac{\partial \phi}{\partial t}(t_i,x)\simeq\frac{\phi(t_i,x_j)-\phi(t_{i-1},x_j)}{\Delta t}$$

of the time derivative, we discretize (10.5) as

$$\begin{aligned} r\phi(t_i,x_j) &= \frac{\phi(t_i,x_j)-\phi(t_{i-1},x_j)}{\Delta t}+rx_j\frac{\phi(t_i,x_{j+1})-\phi(t_i,x_{j-1})}{2\Delta x} \\ &\quad +\frac{1}{2}x_j^2\sigma^2\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}, \end{aligned} \tag{10.6}$$

$1\leqslant j\leqslant M-1$, $0\leqslant i\leqslant N-1$, *i.e.*

$$\begin{aligned} \phi(t_{i-1},x_j) &= \frac{1}{2}(\sigma^2 j^2-rj)\phi(t_i,x_{j-1})\Delta t+\phi(t_i,x_j)(1-(\sigma^2 j^2+r)\Delta t) \\ &\quad +\frac{1}{2}(\sigma^2 j^2+rj)\phi(t_i,x_{j+1})\Delta t, \end{aligned}$$

$1\leqslant j\leqslant M-1$, where the lateral boundary conditions $\phi(t_i,0)$ and $\phi(t_i,x_M)$ are (approximately) given as follows.

European call options. We take

$$\phi(t_i,x_0)=0, \quad \text{and} \quad \phi(t_i,x_M)\simeq\left(x_M-K\,\mathrm{e}^{-r(T-t_i)}\right)^+=x_M-K\,\mathrm{e}^{-r(T-t_i)},$$

$i=0,1,\ldots,N$, provided that x_M is sufficiently large.

European put options. We take

$$\phi(t_i,x_0)=\left(K\,\mathrm{e}^{-(T-t_i)r}-x_0\right)^+=K\,\mathrm{e}^{-(T-t_i)r}, \quad \text{and} \quad \phi(t_i,x_M)=0,$$

$i=0,1,\ldots,N$, with here $x_0=0$.

Given a terminal condition of the form

$$\phi(T,x_j)=(x_j-K)^+, \text{ resp. } \phi(T,x_j)=(K-x_j)^+, \quad j=1,\ldots,M-1,$$

this allows us to solve (10.6) successively for

$$\phi(t_{N-1},x_j),\ \phi(t_{N-2},x_j),\ \phi(t_{N-3},x_j),\ldots,\phi(t_1,x_j),\ \phi(t_0,x_j).$$

The explicit finite difference method is nevertheless known to have a divergent behaviour as time is run backwards, as illustrated in Figure 10.1.

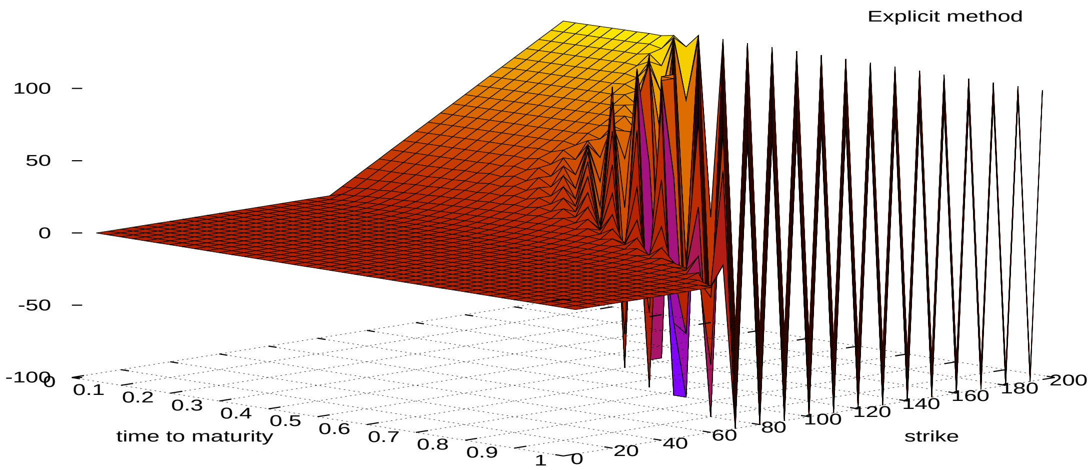


Figure 10.1: Divergence of the explicit finite difference method.

Implicit scheme

Using the *forward* time difference approximation

$$\frac{\partial \phi}{\partial t}(t_i,x) \simeq \frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t}$$

of the time derivative, we discretize (10.5) as

$$\begin{aligned} r\phi(t_i,x_j) &= \frac{\phi(t_{i+1},x_j)-\phi(t_i,x_j)}{\Delta t}+rx_j\frac{\phi(t_i,x_{j+1})-\phi(t_i,x_{j-1})}{\Delta x} \\ &\quad +\frac{1}{2}x_j^2\sigma^2\frac{\phi(t_i,x_{j+1})+\phi(t_i,x_{j-1})-2\phi(t_i,x_j)}{(\Delta x)^2}, \end{aligned} \tag{10.7}$$

$1 \leqslant j \leqslant M-1$, $0 \leqslant i \leqslant N-1$, *i.e.*

$$\begin{aligned} \phi(t_{i+1},x_j) &= -\frac{1}{2}(\sigma^2 j^2-rj)\phi(t_i,x_{j-1})\Delta t+\phi(t_i,x_j)(1+(\sigma^2 j^2+r)\Delta t) \\ &\quad -\frac{1}{2}(\sigma^2 j^2+rj)\phi(t_i,x_{j+1})\Delta t, \end{aligned}$$

$1 \leqslant j \leqslant M-1$, *i.e.*

$$\Phi_{i+1}=B\Phi_i+\begin{bmatrix} \frac{1}{2}\left(r-\sigma^2\right)\phi(t_i,x_0)\Delta t \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}\left(r(M-1)+(M-1)^2\sigma^2\right)\phi(t_i,x_M)\Delta t \end{bmatrix},$$

$i=0,1,\ldots,N-1$, with

$$B_{j,j-1}=\frac{1}{2}\left(rj-\sigma^2 j^2\right)\Delta t, \quad B_{j,j}=1+\sigma^2 j^2\Delta t+r\Delta t,$$

and

$$B_{j,j+1} = -\frac{1}{2}\left(rj+\sigma^2 j^2\right)\Delta t,$$

for $j = 1,2,\ldots,M-1$, and $B(i,j) = 0$ otherwise.

By inversion of the matrix B, Φ_i is given in terms of Φ_{i+1} as

$$\Phi_i = B^{-1}\Phi_{i+1} - B^{-1}\begin{bmatrix} \frac{1}{2}\left(r-\sigma^2\right)\phi(t_i,x_0)\Delta t \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{2}\left(r(M-1)+(M-1)^2\sigma^2\right)\phi(t_i,x_M)\Delta t \end{bmatrix},$$

$i = 0,1,\ldots,N-1$, where the lateral boundary conditions $\phi(t_i,x_0)$ and $\phi(t_i,x_M)$ can be provided as in the case of the explicit scheme, allowing us to solve (10.7) recursively for $\phi(t_{N-1},x_j), \phi(t_{N-2},x_j), \phi(t_{N-3},x_j), \ldots$

Remark. Note that for all $j = 1,2,\ldots,M-1$ we have

$$B_{j,j-1} + B_{j,j} + B_{j,j+1} = 1 + r\Delta t,$$

hence when the terminal condition is a constant $\phi(T,x) = c > 0$ we get

$$\phi(t_i,x) = c(1+r\Delta t)^{-(N-i)} = c\left(1+r\frac{T}{N}\right)^{-(N-i)}, \qquad i = 0,\ldots,N.$$

In particular, when the number N of discretization steps tends to infinity, denoting by $[x]$ the integer part of $x \in \mathbb{R}$ we find

$$\begin{aligned} \phi(s,x) &= \lim_{N\to\infty} \phi\left(t_{[Ns/T]},x\right) \\ &= c\lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^{-(N-[Ns/T])} \\ &= c\lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^{-[N(T-s)/T]} \\ &= c\lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^{-(T-s)/T} \\ &= c\,\mathrm{e}^{-r(T-s)}, \end{aligned}$$

for all $s \in [0,T]$, as expected.

The implicit finite difference method is known to be more stable than the explicit scheme, as illustrated in Figure 10.2, in which the discretization parameters have been taken to be the same as in Figure 10.1.

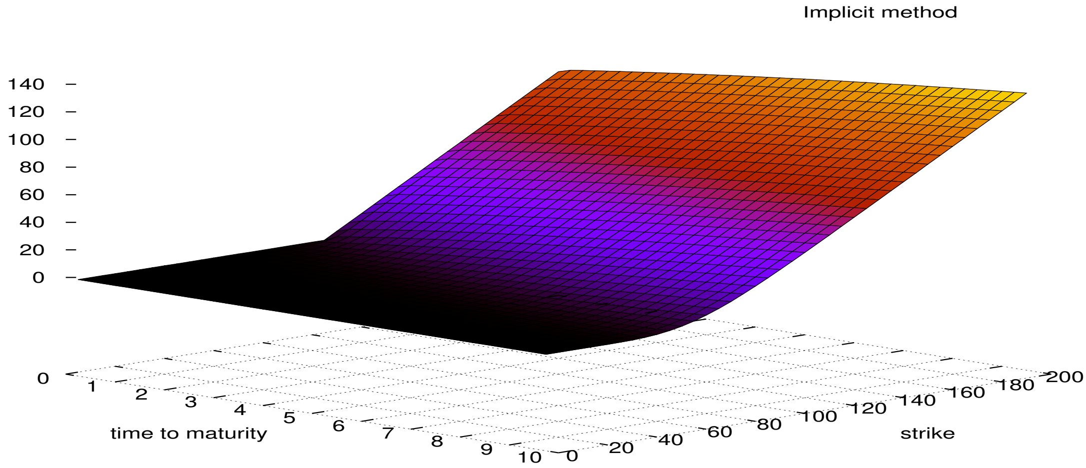


Figure 10.2: Stability of the implicit finite difference method.

10.3 Euler Discretization

In order to apply the Monte Carlo method in option pricing, we need to generate a sequence $(\widehat{X}_1, \ldots, \widehat{X}_N)$ of sample values of a random variable X, such that the empirical mean

$$\mathbb{E}[\phi(X)] \simeq \frac{\phi(\widehat{X}_1) + \cdots + \phi(\widehat{X}_N)}{N}$$

can be used according to the strong law of large number for the evaluation of the expected value $\mathbb{E}[\phi(X)]$. Despite its apparent simplicity, the Monte Carlo method can converge slowly. The optimization of Monte Carlo algorithms and of random number generators have been the object of numerous studies which are outside the scope of this text, cf. *e.g.* Glasserman, 2004, R. Korn, E. Korn, and Kroisandt, 2010.

Random samples for the solution of a stochastic differential equation of the form

$$dX_t = b(X_t)dt + a(X_t)dW_t \tag{10.8}$$

can be generated by discretization. More precisely, the Euler discretization scheme for the stochastic differential equation (10.8) is given by

$$\begin{aligned} \widehat{X}^N_{t_{k+1}} &= \widehat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + \int_{t_k}^{t_{k+1}} a(X_s)dW_s \\ &\simeq \widehat{X}^N_{t_k} + b(\widehat{X}^N_{t_k})(t_{k+1} - t_k) + a(\widehat{X}^N_{t_k})(W_{t_{k+1}} - W_{t_k}), \end{aligned}$$

where $W_{t_{k+1}} - W_{t_k} \simeq \mathcal{N}(0, t_{k+1} - W_{t_k})$, $k = 0, 1, \ldots, N-1$.

In particular, when X_t is the geometric Brownian motion given by

$$dX_t = rX_t dt + \sigma X_t dW_t$$

we get

$$\widehat{X}^N_{t_{k+1}} = \widehat{X}^N_{t_k} + r\widehat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma\widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}),$$

which can be computed as

$$\widehat{X}^N_{t_k} = \widehat{X}^N_{t_0} \prod_{i=1}^{k} \big(1 + r(t_i - t_{i-1}) + (W_{t_i} - W_{t_{i-1}})\sigma\big), \quad k = 0, 1, \ldots, N.$$

10.4 Milshtein Discretization

In the Milshtein scheme we use (10.8) to expand $a(X_s)$ as

$$\begin{aligned} a(X_s) &\simeq a(X_{t_k}) + a'(X_{t_k})(X_s - X_{t_k}) \\ &\simeq a(X_{t_k}) + a'(X_{t_k})\big(b(X_{t_k})(s-t_k) + a(X_{t_k})(W_s - W_{t_k})\big), \end{aligned}$$

$0 \leqslant t_k < s$. As a consequence, we get

$$\begin{aligned} \widehat{X}^N_{t_{k+1}} &= \widehat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + \int_{t_k}^{t_{k+1}} a(X_s)dW_s \\ &\simeq \widehat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ &\quad + a'(X_{t_k})b(X_{t_k}) \int_{t_k}^{t_{k+1}} (s-t_k)dW_s \\ &\quad + a'(X_{t_k})a(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s \\ &\simeq \widehat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ &\quad + a'(X_{t_k})a(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s. \end{aligned}$$

Next, using Itô's formula we note that

$$(W_{t_{k+1}} - W_{t_k})^2 = 2\int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s + \int_{t_k}^{t_{k+1}} ds,$$

hence

$$\int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s = \frac{1}{2}\big((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\big),$$

and

$$\begin{aligned} \widehat{X}^N_{t_{k+1}} &\simeq \widehat{X}^N_{t_k} + \int_{t_k}^{t_{k+1}} b(X_s)ds + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ &\quad + \frac{1}{2}a'(X_{t_k})a(X_{t_k})\big((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\big) \\ &\simeq \widehat{X}^N_{t_k} + b(X_{t_k})(t_{k+1} - t_k) + a(X_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ &\quad + \frac{1}{2}a'(X_{t_k})a(X_{t_k})\big((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\big). \end{aligned}$$

As a consequence the Milshtein scheme is written as

$$\begin{aligned} \widehat{X}^N_{t_{k+1}} &\simeq \widehat{X}^N_{t_k} + b(\widehat{X}^N_{t_k})(t_{k+1} - t_k) + a(\widehat{X}^N_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ &\quad + \frac{1}{2}a'(\widehat{X}^N_{t_k})a(\widehat{X}^N_{t_k})\big((W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)\big), \end{aligned}$$

i.e. in the Milshtein scheme we take into account the "small" difference

$$(W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)$$

existing between $(\Delta W_t)^2$ and Δt. Taking $(\Delta W_t)^2$ equal to Δt brings us back to the Euler scheme.

When X_t is the geometric Brownian motion given by

$$dX_t = rX_t dt + \sigma X_t dW_t$$

we get

$$\widehat{X}^N_{t_{k+1}} = \widehat{X}^N_{t_k} + (r - \sigma^2/2)\widehat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma \widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}) + \frac{1}{2}\sigma^2 \widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k})^2,$$

which can be computed as

$$\widehat{X}^N_{t_k} = \widehat{X}^N_{t_0} \prod_{i=1}^{k} \left(1 + (r - \sigma^2/2)(t_i - t_{i-1}) + (W_{t_i} - W_{t_{i-1}})\sigma + \frac{1}{2}(W_{t_i} - W_{t_{i-1}})^2 \sigma^2 \right).$$

Appendix: Background on Probability Theory

In this appendix we review a number of basic probabilistic tools that are needed in option pricing and hedging. We refer to Jacod and Protter, 2000, Devore, 2003, Pitman, 1999 for more information on the needed probability background.

11.1 Probability Sample Space and Events

We will need the following notation coming from set theory. Given A and B to abstract sets, "$A \subset B$" means that A is contained in B, and in this case, $B \setminus A$ denotes the set of elements of B which do not belong to A. The property that the element ω belongs to the set A is denoted by "$\omega \in A$", and given two sets A and Ω such that $A \subset \Omega$, we let $A^c = \Omega \setminus A$ denote the *complement* of A in Ω. The finite set made of n elements $\omega_1, \ldots, \omega_n$ is denoted by $\{\omega_1, \ldots, \omega_n\}$, and we will usually distinguish between the element ω and its associated singleton set $\{\omega\}$.

A probability sample space is an abstract set Ω that contains the possible outcomes of a random experiment.

Examples

i) Coin tossing: $\Omega = \{H, T\}$.

ii) Rolling one die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

iii) Picking one card at random in a pack of 52: $\Omega = \{1,2,3,\ldots,52\}$.

iv) An integer-valued random outcome: $\Omega = \mathbb{N} = \{0,1,2,\ldots\}$.

In this case the outcome $\omega \in \mathbb{N}$ can be the random number of trials needed until some event occurs.

v) A nonnegative, real-valued outcome: $\Omega = \mathbb{R}_+$.

In this case the outcome $\omega \in \mathbb{R}_+$ may represent the (nonnegative) value of a continuous random time.

vi) A random continuous parameter (such as time, weather, price or wealth, temperature, ...): $\Omega = \mathbb{R}$.

vii) Random choice of a continuous path in the space $\Omega = \mathscr{C}(\mathbb{R}_+)$ of all continuous functions on $\mathbb{R}_+$.

In this case, $\omega \in \Omega$ is a function $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and a typical example is the graph $t \longmapsto \omega(t)$ of a stock price over time.

Product spaces:

Probability sample spaces can be built as product spaces and used for the modeling of repeated random experiments.

i) Rolling two dice: $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.

In this case a typical element of Ω is written as $\omega = (k,l)$ with $k,l \in \{1,2,3,4,5,6\}$.

ii) A finite number n of real-valued samples: $\Omega = \mathbb{R}^n$.

In this case the outcome ω is a vector $\omega = (x_1,\ldots,x_n) \in \mathbb{R}^n$ with n components.
Note that to some extent, the more complex Ω is, the better it fits a practical and useful situation, *e.g.* $\Omega = \{H,T\}$ corresponds to a simple coin tossing experiment while $\Omega = \mathscr{C}(\mathbb{R}_+)$ the space of continuous functions on $\mathbb{R}_+$ can be applied to the modeling of stock markets. On the other hand, in many cases and especially in the most complex situations, we will *not* attempt to specify Ω explicitly.

Events

An event is a collection of outcomes, which is represented by a subset of Ω. The collections $\mathscr{G}$ of events that we will consider are called σ-algebras, according to the following definition.

Definition 11.1 A collection $\mathscr{G}$ of events is a σ-algebra provided that it satisfies the following conditions:

(i) $\emptyset \in \mathscr{G}$,

(ii) For all countable sequences $(A_n)_{n\geqslant 1}$ such that $A_n \in \mathscr{G}$, $n \geqslant 1$, we have $\bigcup_{n\geqslant 1} A_n \in \mathscr{G}$,

(iii) $A \in \mathscr{G} \Longrightarrow (\Omega \setminus A) \in \mathscr{G}$,

where $\Omega \setminus A := \{\omega \in \Omega \ : \ \omega \notin A\}$.

Note that Properties (ii) and (iii) above also imply

$$\bigcap_{n\geqslant 1} A_n = \left(\bigcup_{n\geqslant 1} A_n^c\right)^c \in \mathscr{G}, \tag{11.9}$$

for all countable sequences $A_n \in \mathscr{G}$, $n \geqslant 1$.

The collection of all events in Ω will often be denoted by $\mathscr{F}$. The empty set $\emptyset$ and the full space Ω are considered as events but they are of less importance because Ω corresponds to "any outcome may occur" while $\emptyset$ corresponds to an absence of outcome, or no experiment.

In the context of stochastic processes, two σ-algebras $\mathscr{F}$ and $\mathscr{G}$ such that $\mathscr{F} \subset \mathscr{G}$ will refer to two different amounts of information, the amount of information associated to $\mathscr{F}$ being here lower than the one associated to $\mathscr{G}$.

The formalism of σ-algebras helps in describing events in a short and precise way.

Examples

i) Let $\Omega = \{1,2,3,4,5,6\}$.

The event $A = \{2,4,6\}$ corresponds to

"the result of the experiment is an even number".

ii) Taking again $\Omega = \{1,2,3,4,5,6\}$,

$$\mathscr{F} := \{\Omega, \emptyset, \{2,4,6\}, \{1,3,5\}\}$$

defines a σ-algebra on Ω which corresponds to the knowledge of parity of an integer picked at random from 1 to 6.

Note that in the set-theoretic notation, an event A is a subset of Ω, *i.e.* $A \subset \Omega$, while it is an element of $\mathscr{F}$, *i.e.* $A \in \mathscr{F}$. For example, we have $\Omega \supset \{2,4,6\} \in \mathscr{F}$, while $\{\{2,4,6\},\{1,3,5\}\} \subset \mathscr{F}$.

iii) Taking

$$\mathscr{G} := \{\Omega, \emptyset, \{2,4,6\}, \{2,4\}, \{6\}, \{1,2,3,4,5\}, \{1,3,5,6\}, \{1,3,5\}\} \supset \mathscr{F},$$

defines a σ-algebra on Ω which is bigger than $\mathscr{F}$, and corresponds to the parity information contained in $\mathscr{F}$, completed by the knowledge of whether the outcome is equal to 6 or not.

iv) Take

$$\Omega = \{H,T\} \times \{H,T\} = \{(H,H), (H,T), (T,H), (T,T)\}.$$

In this case, the collection $\mathscr{F}$ of all possible events is given by

$$\begin{aligned}\mathscr{F} = \{&\emptyset, \{(H,H)\}, \{(T,T)\}, \{(H,T)\}, \{(T,H)\},\\ &\{(T,T),(H,H)\}, \{(H,T),(T,H)\}, \{(H,T),(T,T)\},\\ &\{(T,H),(T,T)\}, \{(H,T),(H,H)\}, \{(T,H),(H,H)\},\\ &\{(H,H),(T,T),(T,H)\}, \{(H,H),(T,T),(H,T)\},\\ &\{(H,T),(T,H),(H,H)\}, \{(H,T),(T,H),(T,T)\}, \Omega\}.\end{aligned} \tag{11.10}$$

Note that the set $\mathscr{F}$ of all events considered in (11.10) above has altogether

$1 = \binom{n}{0}$ event of cardinality 0,

$4 = \binom{n}{1}$ events of cardinality 1,

$6 = \binom{n}{2}$ events of cardinality 2,

$4 = \binom{n}{3}$ events of cardinality 3,

$1 = \binom{n}{4}$ event of cardinality 4,

with $\underline{n = 4}$, for a total of

$$16 = 2^n = \sum_{k=0}^{4} \binom{4}{k} = 1+4+6+4+1$$

events. The collection of events

$$\mathscr{G} := \{\emptyset, \{(T,T),(H,H)\}, \{(H,T),(T,H)\}, \Omega\}$$

defines a sub σ-algebra of $\mathscr{F}$, associated to the information "the results of two coin tossings are different".

Exercise: Write down the set of all events on $\Omega = \{H,T\}$.

Note also that (H,T) is different from (T,H), whereas $\{(H,T),(T,H)\}$ is equal to $\{(T,H),(H,T)\}$.

In addition, we will distinguish between the outcome $\omega \in \Omega$ and its associated event $\{\omega\} \in \mathscr{F}$, which satisfies $\{\omega\} \subset \Omega$.

11.2 Probability Measures

Definition 11.2 A *probability measure* is a mapping $\mathbb{P} : \mathscr{F} \longrightarrow [0,1]$ that assigns a probability $\mathbb{P}(A) \in [0,1]$ to any event $A \in \mathscr{F}$, with the properties

a) $\mathbb{P}(\Omega) = 1$, and

b) $\mathbb{P}\left(\bigcup_{n\geqslant 1} A_n\right) = \sum_{n\geqslant 1} \mathbb{P}(A_n)$, whenever $A_k \cap A_l = \emptyset$, $k \neq l$.

Property (b)) above is named the *law of total probability*. It states in particular that we have

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)$$

when the subsets $A_1, \ldots, A_n$ of Ω are disjoints, and

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \tag{11.11}$$

if $A \cap B = \emptyset$. We also have the *complement rule*

$$\mathbb{P}(A^c) = \mathbb{P}(\Omega \setminus A) = \mathbb{P}(\Omega) - \mathbb{P}(A) = 1 - \mathbb{P}(A).$$

When A and B are not necessarily disjoint we can write

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

The triple

$$(\Omega, \mathscr{F}, \mathbb{P}) \tag{11.12}$$

is called a *probability space*, and was introduced by A.N. Kolmogorov (1903-1987). This setting is generally referred to as the *Kolmogorov framework*.

A property or event is said to hold $\mathbb{P}$-almost surely (also written $\mathbb{P}$-*a.s.*) if it holds with probability equal to one.

Example

1. Take

$$\Omega = \{(T,T),(H,H),(H,T),(T,H)\}$$

and

$$\mathscr{F} = \{\emptyset, \{(T,T),(H,H)\}, \{(H,T),(T,H)\}, \Omega\}.$$

The *uniform* probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ is given by setting

$$\mathbb{P}(\{(T,T),(H,H)\}) := \frac{1}{2} \quad \text{and} \quad \mathbb{P}(\{(H,T),(T,H)\}) := \frac{1}{2}.$$

In addition, we have the following convergence properties.

1. Let $(A_n)_{n\in\mathbb{N}}$ be a *non-decreasing* sequence of events, *i.e.* $A_n \subset A_{n+1}$, $n \in \mathbb{N}$. Then we have

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n). \tag{11.13}$$

2. Let $(A_n)_{n\in\mathbb{N}}$ be a *non-increasing* sequence of events, *i.e.* $A_{n+1} \subset A_n$, $n \in \mathbb{N}$. Then we have

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n). \tag{11.14}$$

Theorem 11.1 Borel-Cantelli Lemma. Let $(A_n)_{n\geqslant 1}$ denote a sequence of events on $(\Omega, \mathscr{F}, \mathbb{P})$, such that

$$\sum_{n\geqslant 1} \mathbb{P}(A_n) < \infty.$$

Then we have

$$\mathbb{P}\left(\bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k\right) = 0,$$

i.e. the probability that A_n occurs infinitely many times occur is zero.

Proposition 11.2 (Fatou's lemma) Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of *nonnegative* random variable. Then we have

$$\mathbb{E}\left[\liminf_{n\to\infty} F_n\right] \leqslant \liminf_{n\to\infty} \mathbb{E}[F_n].$$

In particular, Fatou's lemma shows that if in addition the sequence $(F_n)_{n\in\mathbb{N}}$ converges with probability one and the sequence $(\mathbb{E}[F_n])_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ then we have

$$\mathbb{E}\Big[\lim_{n\to\infty} F_n\Big] \leqslant \lim_{n\to\infty} \mathbb{E}[F_n].$$

11.3 Conditional Probabilities and Independence

We start with an example.

Consider a population $\Omega = M \cup W$ made of a set M of men and a set W of women. Here the σ-algebra $\mathscr{F} = \{\Omega, \emptyset, W, M\}$ corresponds to the information given by gender. After polling the population, *e.g.* for a market survey, it turns out that a proportion $p \in [0,1]$ of the population declares to like apples, while a proportion $1-p$ declares to dislike apples. Let $A \subset \Omega$ denote the subset of individuals who like apples, while $A^c \subset \Omega$ denotes the subset individuals who dislike apples, with

$$p = \mathbb{P}(A) \quad \text{and} \quad 1-p = \mathbb{P}(A^c),$$

e.g. $p = 60\%$ of the population likes apples. It may be interesting to get a more precise information and to determine

- the relative proportion $\dfrac{\mathbb{P}(A\cap W)}{\mathbb{P}(W)}$ of women who like apples, and
- the relative proportion $\dfrac{\mathbb{P}(A\cap M)}{\mathbb{P}(M)}$ of men who like apples.

Here, $\mathbb{P}(A\cap W)/\mathbb{P}(W)$ represents the probability that a randomly chosen woman in W likes apples, and $\mathbb{P}(A\cap M)/\mathbb{P}(M)$ represents the probability that a randomly chosen man in M likes apples. Those two ratios are interpreted as conditional probabilities, for example $\mathbb{P}(A\cap M)/\mathbb{P}(M)$ denotes the probability that an individual likes apples *given that* he is a man.

For another example, suppose that the population Ω is split as $\Omega = Y \cup O$ into a set Y of "young" people and another set O of "old" people, and denote by $A \subset \Omega$ the set of people who voted for candidate A in an election. Here it can be of interest to find out the relative proportion

$$\mathbb{P}(A \mid Y) = \frac{\mathbb{P}(Y\cap A)}{\mathbb{P}(Y)}$$

of young people who voted for candidate A.

More generally, given any two events $A, B \subset \Omega$ with $\mathbb{P}(B) \neq 0$, we call

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

the probability of A *given* B, or *conditionally to* B.

R We note that if $\mathbb{P}(B) = 1$ we have $\mathbb{P}(A\cap B^c) \leqslant \mathbb{P}(B^c) = 0$, hence $\mathbb{P}(A\cap B^c) = 0$, which implies

$$\mathbb{P}(A) = \mathbb{P}(A\cap B) + \mathbb{P}(A\cap B^c) = \mathbb{P}(A\cap B),$$

and $\mathbb{P}(A \mid B) = \mathbb{P}(A)$.

We also recall the following property:

$$\begin{aligned}\mathbb{P}\left(B \cap \bigcup_{n\geqslant 1} A_n\right) &= \sum_{n\geqslant 1} \mathbb{P}(B\cap A_n)\\ &= \sum_{n\geqslant 1} \mathbb{P}(B \mid A_n)\mathbb{P}(A_n)\\ &= \sum_{n\geqslant 1} \mathbb{P}(A_n \mid B)\mathbb{P}(B),\end{aligned}$$

for any family of disjoint events $(A_n)_{n\geqslant 1}$ with $A_i \cap A_j = \emptyset$, $i \neq j$, and $\mathbb{P}(B) > 0$, $n \geqslant 1$. This also shows that conditional probability measures are probability measures, in the sense that whenever $\mathbb{P}(B) > 0$ we have

a) $\mathbb{P}(\Omega \mid B) = 1$, and

b) $\mathbb{P}\left(\bigcup_{n\geqslant 1} A_n \,\middle|\, B\right) = \sum_{n\geqslant 1} \mathbb{P}(A_n \mid B)$, whenever $A_k \cap A_l = \emptyset$, $k \neq l$.

In particular, if $\bigcup_{n\geqslant 1} A_n = \Omega$, $(A_n)_{n\geqslant 1}$ becomes a *partition* of Ω and we get the *law of total probability*

$$\mathbb{P}(B) = \sum_{n\geqslant 1} \mathbb{P}(B\cap A_n) = \sum_{n\geqslant 1} \mathbb{P}(A_n \mid B)\mathbb{P}(B) = \sum_{n\geqslant 1} \mathbb{P}(B \mid A_n)\mathbb{P}(A_n), \tag{11.15}$$

provided that $A_i \cap A_j = \emptyset$, $i \neq j$, and $\mathbb{P}(B) > 0$, $n \geqslant 1$. However, in general we have

$$\mathbb{P}\left(A \,\middle|\, \bigcup_{n\geqslant 1} B_n\right) \neq \sum_{n\geqslant 1} \mathbb{P}(A \mid B_n),$$

even when $B_k \cap B_l = \emptyset$, $k \neq l$. Indeed, taking for example $A = \Omega = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$ and $\mathbb{P}(B_1) = \mathbb{P}(B_2) = 1/2$, we have

$$1 = \mathbb{P}(\Omega \mid B_1 \cup B_2) \neq \mathbb{P}(\Omega \mid B_1) + \mathbb{P}(\Omega \mid B_2) = 2.$$

Independent events

Two events A and B such that $\mathbb{P}(A), \mathbb{P}(B) > 0$ are said to be *independent* if

$$\mathbb{P}(A \mid B) = \mathbb{P}(A),$$

which is equivalent to

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

In this case we find

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

11.4 Random Variables

A real-valued random variable is a mapping

$$\begin{aligned} X \ : \ &\Omega \longrightarrow \mathbb{R}\\ &\omega \longmapsto X(\omega)\end{aligned}$$

from a probability sample space Ω into the state space $\mathbb{R}$. Given

$$X : \Omega \longrightarrow \mathbb{R}$$

a random variable and A a (measurable)* subset of $\mathbb{R}$, we denote by $\{X \in A\}$ the event

$$\{X \in A\} := \{\omega \in \Omega \, : \, X(\omega) \in A\}.$$

Examples

i) Let $\Omega := \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$, and consider the mapping

$$\begin{aligned} X \ &: \ \Omega \longrightarrow \mathbb{R} \\ &(k,l) \longmapsto k+l. \end{aligned}$$

Then X is a random variable giving the sum of the two numbers appearing on each die.

ii) the time needed everyday to travel from home to work or school is a random variable, as the precise value of this time may change from day to day under unexpected circumstances.

iii) the price of a risky asset is modeled using a random variable.

In the sequel we will often use the notion of indicator function $\mathbb{1}_A$ of an event $A \subset \Omega$.

Definition 11.3 For any $A \subset \Omega$, the indicator function $\mathbb{1}_A$ is the random variable

$$\begin{aligned} \mathbb{1}_A \ &: \ \Omega \longrightarrow \{0,1\} \\ &\omega \longmapsto \mathbb{1}_A(\omega) \end{aligned}$$

defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Indicator functions satisfy the property

$$\mathbb{1}_{A\cap B}(\omega) = \mathbb{1}_A(\omega)\mathbb{1}_B(\omega), \tag{11.16}$$

since

$$\begin{aligned} \omega \in A \cap B &\iff \{\omega \in A \text{ and } \omega \in B\} \\ &\iff \{\mathbb{1}_A(\omega) = 1 \text{ and } \mathbb{1}_B(\omega) = 1\} \\ &\iff \mathbb{1}_A(\omega)\mathbb{1}_B(\omega) = 1. \end{aligned}$$

We also have

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B,$$

and

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B, \tag{11.17}$$

if $A \cap B = \emptyset$.

For example, if $\Omega = \mathbb{N}$ and $A = \{k\}$, for all $l \in \mathbb{N}$ we have

$$\mathbb{1}_{\{k\}}(l) = \begin{cases} 1 & \text{if } k = l, \\ \\ 0 & \text{if } k \neq l. \end{cases}$$

Given X a random variable, we also let

$$\mathbb{1}_{\{X=n\}} = \begin{cases} 1 & \text{if } X = n, \\ \\ 0 & \text{if } X \neq n, \end{cases}$$

*Measurability of subsets of $\mathbb{R}$ refers to *Borel measurability*, a concept which will not be defined in this text.

and

$$\mathbb{1}_{\{X<n\}} = \begin{cases} 1 & \text{if } X < n, \\ 0 & \text{if } X \geqslant n. \end{cases}$$

11.5 Probability Distributions

The probability distribution of a random variable $X : \Omega \longrightarrow \mathbb{R}$ is the collection

$$\{\mathbb{P}(X \in A) \ : \ A \text{ is a measurable subset of } \mathbb{R}\}.$$

As the collection of *measurable* subsets of $\mathbb{R}$ coincides with the σ-algebra generated by the intervals in $\mathbb{R}$, the distribution of X can be reduced to the knowledge of either

$$\{\mathbb{P}(a < X \leqslant b) = \mathbb{P}(X \leqslant b) - \mathbb{P}(X \leqslant a) \ : \ a < b \in \mathbb{R}\},$$

or

$$\{\mathbb{P}(X \leqslant a) \ : \ a \in \mathbb{R}\}, \quad \text{or} \quad \{\mathbb{P}(X \geqslant a) \ : \ a \in \mathbb{R}\},$$

see *e.g.* Corollary 3.8 in Çınlar, 2011.

Two random variables X and Y are said to be independent under the probability $\mathbb{P}$ if their probability distributions satisfy

$$\mathbb{P}(X \in A\,,\, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

for all (measurable) subsets A and B of $\mathbb{R}$.

Distributions admitting a density

We say that the distribution of X admits a probability density distribution function $\varphi_X : \mathbb{R} \longrightarrow \mathbb{R}_+$ if, for all $a \leqslant b$, the probability $\mathbb{P}(a \leqslant X \leqslant b)$ can be written as

$$\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b \varphi_X(x)dx.$$

We also say that the distribution of X is absolutely continuous, or that X is an absolutely continuous random variable. This, however, does *not* imply that the density function $\varphi_X : \mathbb{R} \longrightarrow \mathbb{R}_+$ is continuous.

In particular, we always have

$$\int_{-\infty}^{\infty} \varphi_X(x)dx = \mathbb{P}(-\infty \leqslant X \leqslant \infty) = 1$$

for all probability density functions $\varphi_X : \mathbb{R} \longrightarrow \mathbb{R}_+$.

R Note that if the distribution of X admits a probability density function φ_X, then for all $a \in \mathbb{R}$ we have

$$\mathbb{P}(X = a) = \int_a^a \varphi_X(x)dx = 0, \tag{11.18}$$

and this is not a contradiction.

In particular, Remark 11.5 shows that

$$\mathbb{P}(a \leqslant X \leqslant b) = \mathbb{P}(X = a) + \mathbb{P}(a < X \leqslant b) = \mathbb{P}(a < X \leqslant b) = \mathbb{P}(a < X < b),$$

for $a \leqslant b$. Property (11.18) appears for example in the framework of lottery games with a large number of participants, in which a given number "a" selected in advance has a very low (almost zero) probability to be chosen.

The probability density function φ_X can be recovered from the cumulative distribution functions

$$x \longmapsto F_X(x) := \mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} \varphi_X(s)ds,$$

and

$$x \longmapsto 1 - F_X(x) = \mathbb{P}(X \geqslant x) = \int_{x}^{\infty} \varphi_X(s)ds,$$

as

$$\varphi_X(x) = \frac{\partial F_X}{\partial x}(x) = \frac{\partial}{\partial x}\int_{-\infty}^{x} \varphi_X(s)ds = -\frac{\partial}{\partial x}\int_{x}^{\infty} \varphi_X(s)ds, \qquad x \in \mathbb{R}.$$

Examples

i) The *uniform* distribution on an interval.

The probability density function of the uniform distribution on the interval $[a,b]$, $a < b$, is given by

$$\varphi(x) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(x), \qquad x \in \mathbb{R}.$$

ii) The *Gaussian* distribution.

The probability density function of the standard normal distribution is given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R}.$$

More generally, the probability density function of the Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is given by

$$\varphi(x) := \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}, \qquad x \in \mathbb{R}.$$

In this case, we write $X \simeq \mathcal{N}(\mu,\sigma^2)$.

iii) The *exponential* distribution.

The probability density function of the exponential distribution with parameter $\lambda > 0$ is given by

$$\varphi(x) := \lambda\mathbb{1}_{[0,\infty)}(x)\,\mathrm{e}^{-\lambda x} = \begin{cases} \lambda\,\mathrm{e}^{-\lambda x}, & x \geqslant 0 \\ 0, & x < 0. \end{cases} \tag{11.19}$$

We also have

$$\mathbb{P}(X > t) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+. \tag{11.20}$$

iv) The *gamma* distribution.

The probability density function of the gamma distribution is given by

$$\varphi(x) := \frac{a^\lambda}{\Gamma(\lambda)} \mathbb{1}_{[0,\infty)}(x) x^{\lambda-1} \mathrm{e}^{-ax} = \begin{cases} \dfrac{a^\lambda}{\Gamma(\lambda)} x^{\lambda-1} \mathrm{e}^{-ax}, & x \geqslant 0 \\ 0, & x < 0, \end{cases}$$

where $a > 0$ and $\lambda > 0$ are parameters, and

$$\Gamma(\lambda) := \int_0^\infty x^{\lambda-1} \mathrm{e}^{-x} dx, \qquad \lambda > 0,$$

is the gamma function.

v) The *Cauchy* distribution.

The probability density function of the Cauchy distribution is given by

$$\varphi(x) := \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbb{R}.$$

vi) The *lognormal* distribution.

The probability density function of the lognormal distribution is given by

$$\varphi(x) := \mathbb{1}_{[0,\infty)}(x) \frac{1}{x\sigma\sqrt{2\pi}} \mathrm{e}^{-(\mu-\log x)^2/(2\sigma^2)} = \begin{cases} \dfrac{1}{x\sigma\sqrt{2\pi}} \mathrm{e}^{-(\mu-\log x)^2/(2\sigma^2)}, & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

Exercise: For each of the above probability density functions φ, check that the condition

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1$$

is satisfied.

Joint densities

Given two absolutely continuous random variables $X : \Omega \longrightarrow \mathbb{R}$ and $Y : \Omega \longrightarrow \mathbb{R}$ we can form the $\mathbb{R}^2$-valued random variable (X,Y) defined by

$$\begin{aligned} (X,Y) \ : \ & \Omega \longrightarrow \mathbb{R}^2 \\ & \omega \longmapsto (X(\omega), Y(\omega)). \end{aligned}$$

We say that (X,Y) admits a joint probability density

$$\varphi_{(X,Y)} : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$$

when

$$\mathbb{P}((X,Y) \in A \times B) = \int_B \int_A \varphi_{(X,Y)}(x,y) dx dy$$

for all *measurable* subsets A, B of $\mathbb{R}$, cf. Figure 11.3.

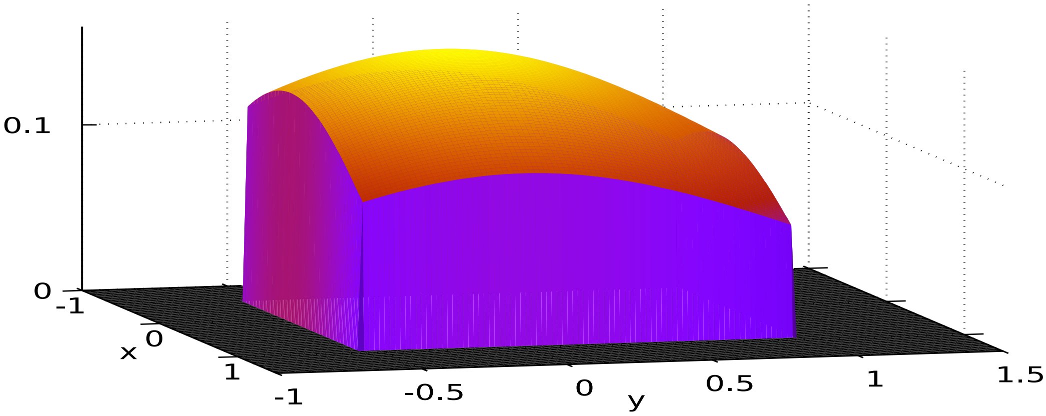


Figure 11.3: Probability $\mathbb{P}((X,Y) \in [-0.5,1] \times [-0.5,1])$ computed as a volume integral.

The probability density function $\varphi_{(X,Y)}$ can be recovered from the joint cumulative distribution function

$$(x,y) \longmapsto F_{(X,Y)}(x,y) := \mathbb{P}(X \leqslant x \text{ and } Y \leqslant y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi_{(X,Y)}(s,t)dsdt,$$

and

$$(x,y) \longmapsto \mathbb{P}(X \geqslant x \text{ and } Y \geqslant y) = \int_{x}^{\infty} \int_{y}^{\infty} \varphi_{(X,Y)}(s,t)dsdt,$$

as

$$\begin{aligned} \varphi_{(X,Y)}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{(X,Y)}(x,y) & (11.21) \\ &= \frac{\partial^2}{\partial x \partial y} \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi_{(X,Y)}(s,t)dsdt & (11.22) \\ &= \frac{\partial^2}{\partial x \partial y} \int_{x}^{\infty} \int_{y}^{\infty} \varphi_{(X,Y)}(s,t)dsdt, \end{aligned}$$

$x,y \in \mathbb{R}$.

The probability densities $\varphi_X : \mathbb{R} \longrightarrow \mathbb{R}_+$ and $\varphi_Y : \mathbb{R} \longrightarrow \mathbb{R}_+$ of $X : \Omega \longrightarrow \mathbb{R}$ and $Y : \Omega \longrightarrow \mathbb{R}$ are called the marginal densities of (X,Y), and are given by

$$\varphi_X(x) = \int_{-\infty}^{\infty} \varphi_{(X,Y)}(x,y)dy, \qquad x \in \mathbb{R}, \tag{11.23}$$

and

$$\varphi_Y(y) = \int_{-\infty}^{\infty} \varphi_{(X,Y)}(x,y)dx, \qquad y \in \mathbb{R}.$$

The conditional probability density $\varphi_{X|Y=y} : \mathbb{R} \longrightarrow \mathbb{R}_+$ of X given $Y = y$ is defined by

$$\varphi_{X|Y=y}(x) := \frac{\varphi_{(X,Y)}(x,y)}{\varphi_Y(y)}, \qquad x,y \in \mathbb{R}, \tag{11.24}$$

provided that $\varphi_Y(y) > 0$. In particular, X and Y are independent if and only if $\varphi_{X|Y=y}(x) = \varphi_X(x)$, $x,y \in \mathbb{R}$, *i.e.*,

$$\varphi_{(X,Y)}(x,y) = \varphi_X(x)\varphi_Y(y), \qquad x,y \in \mathbb{R}.$$

Example

1. If $X_1,\ldots,X_n$ are independent exponentially distributed random variables with parameters $\lambda_1,\ldots,\lambda_n$ we have
$$\begin{aligned}\mathbb{P}(\min(X_1,\ldots,X_n)>t) &= \mathbb{P}(X_1>t,\ldots,X_n>t)\\ &= \mathbb{P}(X_1>t)\cdots\mathbb{P}(X_n>t)\\ &= \mathrm{e}^{-(\lambda_1+\cdots+\lambda_n)t}, \qquad t\in\mathbb{R}_+, \end{aligned} \tag{11.25}$$
hence $\min(X_1,\ldots,X_n)$ is an exponentially distributed random variable with parameter $\lambda_1+\cdots+\lambda_n$.

   From the joint probability density function of (X_1,X_2) given by
$$\varphi_{(X_1,X_2)}(x,y)=\varphi_{X_1}(x)\varphi_{X_2}(y)=\lambda_1\lambda_2\,\mathrm{e}^{-\lambda_1x-\lambda_2y}, \qquad x,y\geqslant 0,$$
   we can write
$$\begin{aligned}\mathbb{P}(X_1<X_2) &= \mathbb{P}(X_1\leqslant X_2)\\ &= \int_0^\infty\int_0^y \varphi_{(X_1,X_2)}(x,y)dxdy\\ &= \lambda_1\lambda_2\int_0^\infty\int_0^y \mathrm{e}^{-\lambda_1x-\lambda_2y}dxdy\\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}, \end{aligned} \tag{11.26}$$
   and we note that
$$\mathbb{P}(X_1=X_2)=\lambda_1\lambda_2\int_{\{(x,y)\in\mathbb{R}_+^2\ :\ x=y\}}\mathrm{e}^{-\lambda_1x-\lambda_2y}dxdy=0.$$

Discrete distributions

We only consider integer-valued random variables, *i.e.* the distribution of X is given by the values of $\mathbb{P}(X=k)$, $k\in\mathbb{N}$.

Examples

i) The *Bernoulli* distribution.

   We have
$$\mathbb{P}(X=1)=p \qquad \text{and} \qquad \mathbb{P}(X=0)=1-p, \tag{11.27}$$
   where $p\in[0,1]$ is a parameter.

   Note that any Bernoulli random variable $X:\Omega\longrightarrow\{0,1\}$ can be written as the indicator function
$$X=\mathbb{1}_A$$
   on Ω with $A=\{X=1\}=\{\omega\in\Omega\ :\ X(\omega)=1\}$.

ii) The *binomial* distribution.

   We have
$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}, \qquad k=0,1,\ldots,n,$$
   where $n\geqslant 1$ and $p\in[0,1]$ are parameters.

iii) The *geometric* distribution.

In this case, we have

$$\mathbb{P}(X=k) = (1-p)p^k, \qquad k \in \mathbb{N}, \tag{11.28}$$

where $p \in (0,1)$ is a parameter. For example, if $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent Bernoulli random variables with distribution (11.27), then the random variable,*

$$T_0 := \inf\{k \in \mathbb{N} \ : \ X_k = 0\}$$

can denote the duration of a game until the time that the wealth X_k of a player reaches 0. The random variable T_0 has the geometric distribution (11.28) with parameter $p \in (0,1)$.

iv) The *negative binomial* (or *Pascal*) distribution.

We have

$$\mathbb{P}(X=k) = \binom{k+r-1}{r-1}(1-p)^r p^k, \qquad k \in \mathbb{N}, \tag{11.29}$$

where $p \in (0,1)$ and $r \geqslant 1$ are parameters. Note that the sum of $r \geqslant 1$ independent geometric random variables with parameter p has a negative binomial distribution with parameter (r,p). In particular, the negative binomial distribution recovers the geometric distribution when $r=1$.

v) The *Poisson* distribution.

We have

$$\mathbb{P}(X=k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \qquad k \in \mathbb{N},$$

where $\lambda > 0$ is a parameter.

The probability that a discrete nonnegative random variable $X : \Omega \longrightarrow \mathbb{N} \cup \{+\infty\}$ is finite is given by

$$\mathbb{P}(X<\infty) = \sum_{k\geqslant 0} \mathbb{P}(X=k), \tag{11.30}$$

and we have

$$1 = \mathbb{P}(X=\infty) + \mathbb{P}(X<\infty) = \mathbb{P}(X=\infty) + \sum_{k\geqslant 0}\mathbb{P}(X=k).$$

R The distribution of a discrete random variable cannot admit a probability density. If this were the case, by Remark 11.5 we would have $\mathbb{P}(X=k)=0$ for all $k \in \mathbb{N}$ and

$$1 = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(X \in \mathbb{N}) = \sum_{k\geqslant 0}\mathbb{P}(X=k) = 0,$$

which is a contradiction.

Given two discrete random variables X and Y, the conditional distribution of X given $Y=k$ is given by

$$\mathbb{P}(X=n \mid Y=k) = \frac{\mathbb{P}(X=n \text{ and } Y=k)}{\mathbb{P}(Y=k)}, \qquad n \in \mathbb{N},$$

provided that $\mathbb{P}(Y=k) > 0$, $k \in \mathbb{N}$.

*The notation "inf" stands for "infimum", meaning the smallest $n \geqslant 0$ such that $X_n = 0$, if such an n exists.

11.6 Expectation of Random Variables

The expectation, or expected value, of a random variable X is the mean, or average value, of X. In practice, expectations can be even more useful than probabilities. For example, knowing that a given equipment (such as a bridge) has a failure probability of 1.78493 out of a billion can be of less practical use than knowing the expected lifetime (*e.g.* 200000 years) of that equipment.

For example, the time $T(\omega)$ to travel from home to work/school can be a random variable with a new outcome and value every day, however we usually refer to its expectation $\mathbb{E}[T]$ rather than to its sample values that may change from day to day.

Expected value of a Bernoulli random variable

Any Bernoulli random variable $X : \Omega \longrightarrow \{0,1\}$ can be written as the indicator function $X := \mathbb{1}_A$ where A is the event $A = \{X = 1\}$, and the parameter $p \in [0,1]$ of X is given by

$$p = \mathbb{P}(X = 1) = \mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A] = \mathbb{E}[X].$$

The expectation of a Bernoulli random variable with parameter p is defined as

$$\boxed{\mathbb{E}[\mathbb{1}_A] := 1 \times \mathbb{P}(A) + 0 \times \mathbb{P}(A^c) = \mathbb{P}(A).} \qquad (11.31)$$

Expected value of a discrete random variable

Next, let $X : \Omega \longrightarrow \mathbb{N}$ be a discrete random variable. The expectation $\mathbb{E}[X]$ of X is defined as the sum

$$\mathbb{E}[X] = \sum_{k \geqslant 0} k \mathbb{P}(X = k), \qquad (11.32)$$

in which the possible values $k \in \mathbb{N}$ of X are weighted by their probabilities. More generally we have

$$\mathbb{E}[\phi(X)] = \sum_{k \geqslant 0} \phi(k) \mathbb{P}(X = k),$$

for all sufficiently summable functions $\phi : \mathbb{N} \longrightarrow \mathbb{R}$.

The expectation of the indicator function $X = \mathbb{1}_A = \mathbb{1}_{\{X=1\}}$ can be recovered from (11.32) as

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{1}_A] = 0 \times \mathbb{P}(\Omega \setminus A) + 1 \times \mathbb{P}(A) = 0 \times \mathbb{P}(\Omega \setminus A) + 1 \times \mathbb{P}(A) = \mathbb{P}(A).$$

Note that the expectation is a linear operation, *i.e.* we have

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y], \qquad a, b \in \mathbb{R}, \qquad (11.33)$$

provided that

$$\mathbb{E}[|X|] + \mathbb{E}[|Y|] < \infty.$$

Examples

i) Expected value of a Poisson random variable with parameter $\lambda > 0$:

$$\mathbb{E}[X] = \sum_{k \geqslant 0} k \mathbb{P}(X = k) = \mathrm{e}^{-\lambda} \sum_{k \geqslant 1} k \frac{\lambda^k}{k!} = \lambda \, \mathrm{e}^{-\lambda} \sum_{k \geqslant 0} \frac{\lambda^k}{k!} = \lambda. \qquad (11.34)$$

ii) Estimating the expected value of a Poisson random variable using R:

Taking $\lambda := 2$, we can use the following R code:

```
poisson_samples <- rpois(100000, lambda = 2)
poisson_samples
mean(poisson_samples)
```

Given $X : \Omega \longrightarrow \mathbb{N} \cup \{+\infty\}$ a discrete nonnegative random variable X, we have

$$\mathbb{P}(X < \infty) = \sum_{k \geqslant 0} \mathbb{P}(X = k),$$

and

$$1 = \mathbb{P}(X = \infty) + \mathbb{P}(X < \infty) = \mathbb{P}(X = \infty) + \sum_{k \geqslant 0} \mathbb{P}(X = k),$$

and in general

$$\mathbb{E}[X] = +\infty \times \mathbb{P}(X = \infty) + \sum_{k \geqslant 0} k \mathbb{P}(X = k).$$

In particular, $\mathbb{P}(X = \infty) > 0$ implies $\mathbb{E}[X] = \infty$, and the finiteness $\mathbb{E}[X] < \infty$ condition implies $\mathbb{P}(X < \infty) = 1$, however the converse is *not true*.

Examples

a) Assume that X has the geometric distribution

$$\mathbb{P}(X = k) := \frac{1}{2^{k+1}}, \qquad k \geqslant 0, \tag{11.35}$$

with parameter $p = 1/2$, and

$$\mathbb{E}[X] = \sum_{k \geqslant 0} \frac{k}{2^{k+1}} = \frac{1}{4} \sum_{k \geqslant 1} \frac{k}{2^{k-1}} = \frac{1}{4} \frac{1}{(1 - 1/2)^2} = 1 < \infty.$$

Letting $\phi(X) := 2^X$, we have

$$\mathbb{P}(\phi(X) < \infty) = \mathbb{P}(X < \infty) = \sum_{k \geqslant 0} \frac{1}{2^{k+1}} = 1,$$

and

$$\mathbb{E}[\phi(X)] = \sum_{k \geqslant 0} \phi(k) \mathbb{P}(X = k) = \sum_{k \geqslant 0} \frac{2^k}{2^{k+1}} = \sum_{k \geqslant 0} \frac{1}{2} = +\infty,$$

hence the expectation $\mathbb{E}[\phi(X)]$ is <u>infinite</u> although $\phi(X)$ is *finite* with probability one.*

b) The uniform random variable U on $[0, 1]$ satisfies $\mathbb{E}[U] = 1/2 < \infty$ and

$$\mathbb{P}(1/U < \infty) = \mathbb{P}(U > 0) = \mathbb{P}(U \in (0, 1]) = 1,$$

however we have

$$\mathbb{E}[1/U] = \int_0^1 \frac{dx}{x} = +\infty,$$

and $\mathbb{P}(1/U = +\infty) = \mathbb{P}(U = 0) = 0$.

c) If the random variable X has an exponential distribution with parameter $\mu > 0$ we have

$$\mathbb{E}\left[e^{\lambda X}\right] = \mu \int_0^\infty e^{\lambda x} e^{-\mu x} dx = \begin{cases} \dfrac{\mu}{\mu - \lambda} < \infty & \text{if } \mu > \lambda, \\ \\ +\infty, & \text{if } \mu \leqslant \lambda. \end{cases}$$

*This is the St. Petersburg paradox.

Conditional expectation

The notion of expectation takes its full meaning under conditioning. For example, the expected return of a random asset usually depends on information such as economic data, location, etc. In this case, replacing the expectation by a conditional expectation will provide a better estimate of the expected value.

For instance, life expectancy is a natural example of a conditional expectation since it typically depends on location, gender, and other parameters.

The conditional expectation of a finite discrete random variable $X : \Omega \longrightarrow \mathbb{N}$ given an event A is defined by

$$\mathbb{E}[X \mid A] = \sum_{k\geqslant 0} k\mathbb{P}(X = k \mid A) = \sum_{k\geqslant 0} k\frac{\mathbb{P}(X = k \text{ and } A)}{\mathbb{P}(A)}.$$

Lemma 11.3 Given an event A such that $\mathbb{P}(A) > 0$, we have

$$\mathbb{E}[X \mid A] = \frac{1}{\mathbb{P}(A)}\mathbb{E}\left[X\mathbb{1}_A\right]. \tag{11.36}$$

Proof. The proof is done only for $X : \Omega \longrightarrow \mathbb{N}$ a discrete random variable, however (11.36) is valid for general real-valued random variables. By Relation (11.16) we have

$$\begin{aligned}
\mathbb{E}[X \mid A] &= \sum_{k\geqslant 0} k\mathbb{P}(X = k \mid A) \\
&= \frac{1}{\mathbb{P}(A)}\sum_{k\geqslant 0} k\mathbb{P}(\{X = k\} \cap A) = \frac{1}{\mathbb{P}(A)}\sum_{k\geqslant 0} k\,\mathbb{E}\left[\mathbb{1}_{\{X=k\}\cap A}\right] \\
&= \frac{1}{\mathbb{P}(A)}\sum_{k\geqslant 0} k\,\mathbb{E}\left[\mathbb{1}_{\{X=k\}}\mathbb{1}_A\right] = \frac{1}{\mathbb{P}(A)}\mathbb{E}\left[\mathbb{1}_A\sum_{k\geqslant 0} k\mathbb{1}_{\{X=k\}}\right] \\
&= \frac{1}{\mathbb{P}(A)}\mathbb{E}\left[\mathbb{1}_A X\right],
\end{aligned}$$

where we used the relation

$$X = \sum_{k\geqslant 0} k\mathbb{1}_{\{X=k\}}$$

which holds since X takes only integer values. □

Example

i) For example, consider $\Omega = \{1, 3, -1, -2, 5, 7\}$ with the non-uniform probability measure given by

$$\mathbb{P}(\{-1\}) = \mathbb{P}(\{-2\}) = \mathbb{P}(\{1\}) = \mathbb{P}(\{3\}) = \frac{1}{7},\ \mathbb{P}(\{5\}) = \frac{2}{7},\ \mathbb{P}(\{7\}) = \frac{1}{7},$$

and the random variable

$$X : \Omega \longrightarrow \mathbb{Z}$$

given by

$$X(k) = k, \qquad k = 1, 3, -1, -2, 5, 7.$$

Here, $\mathbb{E}[X \mid X > 1]$ denotes the expected value of X given

$$A = \{X > 1\} = \{3, 5, 7\} \subset \Omega,$$

i.e. the mean value of X given that X is strictly positive. This conditional expectation can be computed as

$$\begin{aligned}
&\mathbb{E}[X \mid X > 1] \\
&= 3 \times \mathbb{P}(X = 3 \mid X > 1) + 5 \times \mathbb{P}(X = 5 \mid X > 1) + 7 \times \mathbb{P}(X = 7 \mid X > 1) \\
&= \frac{3 + 2 \times 5 + 7}{4} \\
&= \frac{3 + 5 + 5 + 7}{7 \times 4/7} \\
&= \frac{1}{\mathbb{P}(X > 1)} \mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right],
\end{aligned}$$

where $\mathbb{P}(X > 1) = 4/7$ and the truncated expectation $\mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right]$ is given by $\mathbb{E}\left[X \mathbb{1}_{\{X>1\}}\right] = (3 + 2 \times 5 + 7)/7$.

ii) Estimating a conditional expectation using *R*:

```
geo_samples <- rgeom(100000, prob = 1/4)
mean(geo_samples)
mean(geo_samples[geo_samples<10])
```

Taking $p := 3/4$, we have

$$\mathbb{E}[X] = (1-p) \sum_{k \geqslant 1} k p^k = \frac{p}{1-p} = 3,$$

and

$$\begin{aligned}
\mathbb{E}[X \mid X < 10] &= \frac{1}{\mathbb{P}(X < 10)} \mathbb{E}\left[X \mathbb{1}_{\{X<10\}}\right] \\
&= \frac{1}{\mathbb{P}(X < 10)} \sum_{k=0}^{9} k \mathbb{P}(X = k) \\
&= \frac{1}{\sum_{k=0}^{9} p^k} \sum_{k=1}^{9} k p^k \\
&= \frac{p(1-p)}{1-p^{10}} \frac{\partial}{\partial p} \sum_{k=0}^{9} p^k \\
&= \frac{p(1-p)}{1-p^{10}} \frac{\partial}{\partial p} \left(\frac{1-p^{10}}{1-p} \right) \\
&= \frac{p(1-p^{10} - 10(1-p)p^9)}{(1-p)(1-p^{10})} \\
&\simeq 2.4032603455.
\end{aligned}$$

If the random variable $X : \Omega \longrightarrow \mathbb{N}$ is independent* of the event A, we have

$$\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[X] \, \mathbb{E}[\mathbb{1}_A] = \mathbb{E}[X] \mathbb{P}(A),$$

and we naturally find

$$\mathbb{E}[X \mid A] = \mathbb{E}[X]. \tag{11.37}$$

*$i.e.$, $\mathbb{P}(\{X = k\} \cap A) = \mathbb{P}(\{X = k\}) \mathbb{P}(A)$ for all $k \in \mathbb{N}$.

Taking $X = \mathbb{1}_A$ with

$$\begin{aligned} \mathbb{1}_A : \quad & \Omega \longrightarrow \{0,1\} \\ & \omega \longmapsto \mathbb{1}_A := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A, \end{cases} \end{aligned}$$

shows that, in particular,

$$\begin{aligned} \mathbb{E}[\mathbb{1}_A \mid A] &= 0 \times \mathbb{P}(X = 0 \mid A) + 1 \times \mathbb{P}(X = 1 \mid A) \\ &= \mathbb{P}(X = 1 \mid A) \\ &= \mathbb{P}(A \mid A) \\ &= 1. \end{aligned}$$

One can also define the conditional expectation of X given $A = \{Y = k\}$, as

$$\mathbb{E}[X \mid Y = k] = \sum_{n \geqslant 0} n \mathbb{P}(X = n \mid Y = k),$$

where $Y : \Omega \longrightarrow \mathbb{N}$ is a discrete random variable.

Proposition 11.4 Given X a discrete random variable such that $\mathbb{E}[|X|] < \infty$, we have the relation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]], \tag{11.38}$$

which is sometimes referred to as the *tower property*.

Proof. We have

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X \mid Y]] &= \sum_{k \geqslant 0} \mathbb{E}[X \mid Y = k] \mathbb{P}(Y = k) \\ &= \sum_{k \geqslant 0} \sum_{n \geqslant 0} n \mathbb{P}(X = n \mid Y = k) \mathbb{P}(Y = k) \\ &= \sum_{n \geqslant 0} n \sum_{k \geqslant 0} \mathbb{P}(X = n \text{ and } Y = k) \\ &= \sum_{n \geqslant 0} n \mathbb{P}(X = n) = \mathbb{E}[X], \end{aligned}$$

where we used the marginal distribution

$$\mathbb{P}(X = n) = \sum_{k \geqslant 0} \mathbb{P}(X = n \text{ and } Y = k), \qquad n \in \mathbb{N},$$

that follows from the *law of total probability* (11.15) with $A_k = \{Y = k\}$, $k \geqslant 0$. □

Taking

$$Y = \sum_{k \geqslant 0} k \mathbb{1}_{A_k},$$

with $A_k := \{Y = k\}$, $k \in \mathbb{N}$, from (11.38) we also get the *law of total expectation*

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid Y]] \\ &= \sum_{k \geqslant 0} \mathbb{E}[X \mid Y = k] \mathbb{P}(Y = k) \\ &= \sum_{k \geqslant 0} \mathbb{E}[X \mid A_k] \mathbb{P}(A_k). \end{aligned} \tag{11.39}$$

Example

Life expectancy in Singapore is $\mathbb{E}[T]=80$ years overall, where T denotes the lifetime of a given individual chosen at random. Let $G\in\{m,w\}$ denote the gender of that individual. The statistics show that

$$\mathbb{E}[T\mid G=m]=78 \quad \text{and} \quad \mathbb{E}[T\mid G=w]=81.9,$$

and we have

$$\begin{aligned} 80 &= \mathbb{E}[T] \\ &= \mathbb{E}[\mathbb{E}[T|G]] \\ &= \mathbb{P}(G=w)\,\mathbb{E}[T\mid G=w]+\mathbb{P}(G=m)\,\mathbb{E}[T\mid G=m] \\ &= 81.9\times\mathbb{P}(G=w)+78\times\mathbb{P}(G=m) \\ &= 81.9\times(1-\mathbb{P}(G=m))+78\times\mathbb{P}(G=m), \end{aligned}$$

showing that

$$80=81.9\times(1-\mathbb{P}(G=m))+78\times\mathbb{P}(G=m),$$

i.e.

$$\mathbb{P}(G=m)=\frac{81.9-80}{81.9-78}=\frac{1.9}{3.9}=0.487.$$

Variance

The variance of a random variable X is defined in general by

$$\operatorname{Var}[X]:=\mathbb{E}\left[X^2\right]-(\mathbb{E}[X])^2,$$

provided that $\mathbb{E}\left[|X|^2\right]<\infty$. If $(X_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables we have

$$\begin{aligned} \operatorname{Var}\left[\sum_{k=1}^n X_k\right] &= \mathbb{E}\left[\left(\sum_{k=1}^n X_k\right)^2\right]-\left(\mathbb{E}\left[\sum_{k=1}^n X_k\right]\right)^2 \\ &= \mathbb{E}\left[\sum_{k=1}^n X_k\sum_{l=1}^n X_l\right]-\mathbb{E}\left[\sum_{k=1}^n X_k\right]\mathbb{E}\left[\sum_{l=1}^n X_l\right] \\ &= \mathbb{E}\left[\sum_{k=1}^n\sum_{l=1}^n X_kX_l\right]-\sum_{k=1}^n\sum_{l=1}^n\mathbb{E}[X_k]\,\mathbb{E}[X_l] \\ &= \sum_{k=1}^n\mathbb{E}\left[X_k^2\right]+\sum_{1\leqslant k\neq l\leqslant n}\mathbb{E}[X_kX_l]-\sum_{k=1}^n(\mathbb{E}[X_k])^2-\sum_{1\leqslant k\neq l\leqslant n}\mathbb{E}[X_k]\,\mathbb{E}[X_l] \\ &= \sum_{k=1}^n\left(\mathbb{E}\left[X_k^2\right]-(\mathbb{E}[X_k])^2\right) \\ &= \sum_{k=1}^n\operatorname{Var}[X_k]. \end{aligned} \tag{11.40}$$

Random sums

In the sequel we consider $Y:\Omega\longrightarrow\mathbb{N}$ an *a.s.* finite, integer-valued random variable, *i.e.* we have $\mathbb{P}(Y<\infty)=1$ and $\mathbb{P}(Y=\infty)=0$. Based on the tower property of conditional expectations (11.38) or ordinary conditioning, the expectation of a random sum $\sum_{k=1}^{Y}X_k$, where $(X_k)_{k\in\mathbb{N}}$ is a sequence of random variables, can be computed from the *tower property* (11.38) or from the *law of total expectation* (11.39) as

$$\mathbb{E}\left[\sum_{k=1}^{Y}X_k\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{Y}X_k\,\middle|\,Y\right]\right]$$

$$= \sum_{n\geqslant 0} \mathbb{E}\left[\sum_{k=1}^{Y} X_k \,\middle|\, Y=n\right] \mathbb{P}(Y=n)$$
$$= \sum_{n\geqslant 0} \mathbb{E}\left[\sum_{k=1}^{n} X_k \,\middle|\, Y=n\right] \mathbb{P}(Y=n),$$

and if the sequence $(X_k)_{k\in\mathbb{N}}$ is (mutually) independent of Y, this yields

$$\mathbb{E}\left[\sum_{k=1}^{Y} X_k\right] = \sum_{n\geqslant 0} \mathbb{E}\left[\sum_{k=1}^{n} X_k\right] \mathbb{P}(Y=n) = \sum_{n\geqslant 0} \mathbb{P}(Y=n) \sum_{k=1}^{n} \mathbb{E}[X_k].$$

Random products

Similarly, for a random product we will have, using the independence of Y with $(X_k)_{k\in\mathbb{N}}$,

$$\mathbb{E}\left[\prod_{k=1}^{Y} X_k\right] = \sum_{n\geqslant 0} \mathbb{E}\left[\prod_{k=1}^{n} X_k\right] \mathbb{P}(Y=n) = \sum_{n\geqslant 0} \mathbb{P}(Y=n) \prod_{k=1}^{n} \mathbb{E}[X_k], \qquad (11.41)$$

where the last equality requires the (mutual) independence of the random variables in the sequence $(X_k)_{k\geqslant 1}$.

Distributions admitting a density

Given a random variable X whose distribution admits a probability density $\varphi_X : \mathbb{R} \longrightarrow \mathbb{R}_+$ we have

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\varphi_X(x)dx,$$

and more generally,

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x)\varphi_X(x)dx, \qquad (11.42)$$

for all sufficiently integrable function ϕ on $\mathbb{R}$. For example, if X has a standard normal distribution we have

$$\mathbb{E}[\phi(X)] = \int_{-\infty}^{\infty} \phi(x)\,\mathrm{e}^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.$$

In case X has a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ we get

$$\mathbb{E}[\phi(X)] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \phi(x)\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}dx. \qquad (11.43)$$

Exercise: In case $X \simeq \mathcal{N}(\mu,\sigma^2)$ has a Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, check that

$$\mu = \mathbb{E}[X] \qquad \text{and} \qquad \sigma^2 = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.$$

When $(X,Y) : \Omega \longrightarrow \mathbb{R}^2$ is a $\mathbb{R}^2$-valued couple of random variables whose distribution admits a probability density $\varphi_{X,Y} : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ we have

$$\mathbb{E}[\phi(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\varphi_{X,Y}(x,y)dxdy,$$

for all sufficiently integrable function ϕ on $\mathbb{R}^2$.

The expectation of an absolutely continuous random variable satisfies the same linearity property (11.33) as in the discrete case.

The conditional expectation of an absolutely continuous random variable can be defined as

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x\varphi_{X|Y=y}(x)dx$$

where the conditional probability density $\varphi_{X|Y=y}(x)$ is defined in (11.24), with the relation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] \tag{11.44}$$

which is called the *tower property* and holds as in the discrete case, since

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid Y]] &= \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y]\varphi_Y(y)dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\varphi_{X|Y=y}(x)\varphi_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} x\int_{-\infty}^{\infty} \varphi_{(X,Y)}(x,y)dydx \\
&= \int_{-\infty}^{\infty} x\varphi_X(x)dx = \mathbb{E}[X],
\end{aligned}$$

where we used Relation (11.23) between the probability density of (X,Y) and its marginal X.

For example, an exponentially distributed random variable X with probability density function (11.19) has the expected value

$$\mathbb{E}[X] = \lambda \int_0^{\infty} x\mathrm{e}^{-\lambda x}dx = \frac{1}{\lambda}.$$

Moment Generating Functions

Characteristic functions

The characteristic function of a random variable X is the function

$$\Psi_X : \mathbb{R} \longrightarrow \mathbb{C}$$

defined by

$$\Psi_X(t) = \mathbb{E}\left[\mathrm{e}^{itX}\right], \qquad t \in \mathbb{R}.$$

The characteristic function Ψ_X of a random variable X with probability density function $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\Psi_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{ixt}\varphi(x)dx, \qquad t \in \mathbb{R}.$$

On the other hand, if $X : \Omega \longrightarrow \mathbb{N}$ is a discrete random variable we have

$$\Psi_X(t) = \sum_{n\geqslant 0} \mathrm{e}^{itn}\mathbb{P}(X = n), \qquad t \in \mathbb{R}.$$

One of the main applications of characteristic functions is to provide a characterization of probability distributions, as in the following theorem.

Theorem 11.5 Two random variables $X : \Omega \longrightarrow \mathbb{R}$ and $Y : \Omega \longrightarrow \mathbb{R}$ have same distribution if and only if

$$\Psi_X(t) = \Psi_Y(t), \qquad t \in \mathbb{R}.$$

Theorem 11.5 is used to identify or to determine the probability distribution of a random variable X, by comparison with the characteristic function Ψ_Y of a random variable Y whose distribution is known.

The characteristic function of a random vector (X,Y) is the function $\Psi_{X,Y} : \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined by

$$\Psi_{X,Y}(s,t) = \mathbb{E}\left[e^{isX+itY}\right], \qquad s,t \in \mathbb{R}.$$

Theorem 11.6 The random variables $X : \Omega \longrightarrow \mathbb{R}$ and $Y : \Omega \longrightarrow \mathbb{R}$ are *independent* if and only if

$$\Psi_{X,Y}(s,t) = \Psi_X(s)\Psi_Y(t), \qquad s,t \in \mathbb{R}.$$

A random variable X has a Gaussian distribution with mean μ and variance σ^2 if and only if its characteristic function satisfies

$$\mathbb{E}\left[e^{i\alpha X}\right] = e^{i\alpha\mu - \alpha^2\sigma^2/2}, \qquad \alpha \in \mathbb{R}. \tag{11.45}$$

In terms of moment generating functions (MGF) we have, replacing $i\alpha$ by α,

$$\mathbb{E}\left[e^{\alpha X}\right] = e^{\alpha\mu + \alpha^2\sigma^2/2}, \qquad \alpha \in \mathbb{R}. \tag{11.46}$$

From Theorems 11.5 and 11.6 we deduce the following proposition.

Proposition 11.7 Let $X \simeq \mathcal{N}(\mu, \sigma_X^2)$ and $Y \simeq \mathcal{N}(\nu, \sigma_Y^2)$ be independent Gaussian random variables. Then $X+Y$ also has a Gaussian distribution

$$X + Y \simeq \mathcal{N}(\mu + \nu, \sigma_X^2 + \sigma_Y^2).$$

Proof. Since X and Y are independent, by Theorem 11.6 the characteristic function Ψ_{X+Y} of $X+Y$ is given by

$$\begin{aligned}
\Phi_{X+Y}(t) &= \Phi_X(t)\Phi_Y(t) \\
&= e^{it\mu - t^2\sigma_X^2/2} e^{it\nu - t^2\sigma_Y^2/2} \\
&= e^{it(\mu+\nu) - t^2(\sigma_X^2+\sigma_Y^2)/2}, \qquad t \in \mathbb{R},
\end{aligned}$$

where we used (11.45). Consequently, the characteristic function of $X+Y$ is that of a Gaussian random variable with mean $\mu + \nu$ and variance $\sigma_X^2 + \sigma_Y^2$ and we conclude by Theorem 11.5. □

Moment generating functions

The moment generating function of a random variable X is the function $\Phi_X : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\Phi_X(t) = \mathbb{E}\left[e^{tX}\right], \qquad t \in \mathbb{R},$$

provided that the expectation is finite. In particular, we have

$$\mathbb{E}[X^n] = \frac{\partial^n}{\partial t}\Phi_X(0), \qquad n \geqslant 1,$$

provided that $\mathbb{E}[|X|^n] < \infty$, and

$$\Phi_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{n\geqslant 0} \frac{t^n}{n!}\mathbb{E}[X^n],$$

provided that $\mathbb{E}\left[e^{t|X|}\right] < \infty$, $t \in \mathbb{R}$, and for this reason the moment generating function G_X characterizes the *moments* $\mathbb{E}[X^n]$ of $X : \Omega \longrightarrow \mathbb{N}$, $n \geqslant 0$.

The moment generating function Φ_X of a random variable X with probability density function $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{xt}\varphi(x)dx, \qquad t \in \mathbb{R}.$$

Note that in probability, the moment generating function is written as a *bilateral* transform defined using an integral from $-\infty$ to $+\infty$.

11.7 Conditional Expectation

The construction of conditional expectations of the form $\mathbb{E}[X \mid Y]$ given above for discrete and absolutely continuous random variables can be generalized to σ-algebras.

Definition 11.4 Given $\mathscr{F}$ a σ-algebra on Ω, a random variable $F : \Omega \longrightarrow \mathbb{R}$ is said to be $\mathscr{F}$-measurable if

$$\{F \leqslant x\} := \{\omega \in \Omega \ : \ F(\omega) \leqslant x\} \in \mathscr{F},$$

for all $x \in \mathbb{R}$.

Intuitively, when F is $\mathscr{F}$-measurable, the knowledge of the values of F depends only on the information contained in $\mathscr{F}$. For example, when $\mathscr{F} = \sigma(A_1,\ldots,A_n)$ where $(A_n)_{n\geqslant 1}$ is a *partition* of Ω with $\bigcup_{n\geqslant 1} A_n = \Omega$, any $\mathscr{F}$-measurable random variable F can be written as

$$F(\omega) = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}(\omega), \qquad \omega \in \Omega,$$

for some $c_1,\ldots,c_n \in \mathbb{R}$.

Definition 11.5 Given $(\Omega,\mathscr{F},\mathbb{P})$ a probability space we let $L^2(\Omega,\mathscr{F})$ denote the space of $\mathscr{F}$-measurable and square-integrable random variables, *i.e.*

$$L^2(\Omega,\mathscr{F}) := \left\{F : \Omega \longrightarrow \mathbb{R} \ : \ \mathbb{E}\left[|F|^2\right] < \infty\right\}.$$

More generally, for $p \geqslant 1$ one can define the space $L^p(\Omega,\mathscr{F})$ of $\mathscr{F}$-measurable and p-integrable random variables as

$$L^p(\Omega,\mathscr{F}) := \left\{F : \Omega \longrightarrow \mathbb{R} \ : \ \mathbb{E}[|F|^p] < \infty\right\}.$$

We define a *scalar product* $\langle\cdot,\cdot\rangle_{L^2(\Omega,\mathscr{F})}$ between elements of $L^2(\Omega,\mathscr{F})$, as

$$\langle F,G\rangle_{L^2(\Omega,\mathscr{F})} := \mathbb{E}[FG], \quad F,G \in L^2(\Omega,\mathscr{F}).$$

This scalar product is associated to the norm $\|\cdot\|_{L^2(\Omega)}$ by the relation

$$\|F\|_{L^2(\Omega)} = \sqrt{\mathbb{E}\left[F^2\right]} = \sqrt{\langle F,F\rangle_{L^2(\Omega,\mathscr{F})}}, \qquad F \in L^2(\Omega,\mathscr{F}).$$

The norm $\|\cdot\|_{L^2(\Omega)}$ also defines the *mean-square* distance

$$\|F-G\|_{L^2(\Omega)} = \sqrt{\mathbb{E}\left[(F-G)^2\right]}$$

between random variables $F, G \in L^2(\Omega, \mathscr{F})$, and it induces a notion of *orthogonality*, namely F is *orthogonal* to G in $L^2(\Omega, \mathscr{F})$ if and only if

$$\langle F, G \rangle_{L^2(\Omega,\mathscr{F})} = 0.$$

Proposition 11.8 The ordinary expectation $\mathbb{E}[F]$ achieves the minimum distance

$$\|F - \mathbb{E}[F]\|^2_{L^2(\Omega)} = \min_{c\in\mathbb{R}} \|F - c\|^2_{L^2(\Omega)}. \tag{11.47}$$

Proof. It suffices to differentiate

$$\frac{\partial}{\partial c}\mathbb{E}\left[(F-c)^2\right] = 2\,\mathbb{E}[F-c] = 0,$$

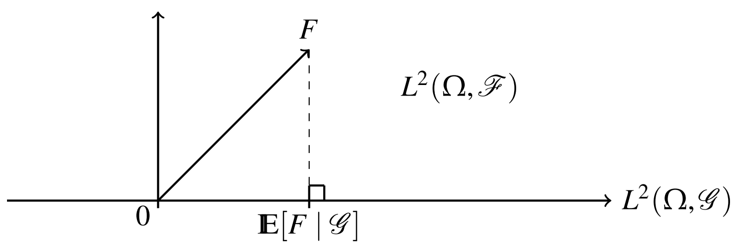
showing that the minimum in (11.47) is reached when $\mathbb{E}[F-c] = 0$, *i.e.* $c = \mathbb{E}[F]$. □

Similarly to Proposition 11.8, the conditional expectation will be defined by a distance minimizing procedure.

Definition 11.6 Given $\mathscr{G} \subset \mathscr{F}$ a sub σ-algebra of $\mathscr{F}$ and $F \in L^2(\Omega, \mathscr{F})$, the conditional expectation of F given $\mathscr{G}$, and denoted

$$\mathbb{E}[F \mid \mathscr{G}],$$

is defined as the *orthogonal projection* of F onto $L^2(\Omega, \mathscr{G})$.



As a consequence of the uniqueness of the orthogonal projection onto the subspace $L^2(\Omega, \mathscr{G})$ of $L^2(\Omega, \mathscr{F})$, the conditional expectation $\mathbb{E}[F \mid \mathscr{G}]$ is characterized by the relation

$$\langle G, F - \mathbb{E}[F \mid \mathscr{G}] \rangle_{L^2(\Omega,\mathscr{F})} = 0,$$

which rewrites as

$$\mathbb{E}[G(F - \mathbb{E}[F \mid \mathscr{G}])] = 0,$$

i.e.

$$\mathbb{E}[GF] = \mathbb{E}[G\,\mathbb{E}[F \mid \mathscr{G}]],$$

for all bounded and $\mathscr{G}$-measurable random variables G, where $\langle \cdot, \cdot \rangle_{L^2(\Omega,\mathscr{F})}$ denotes the inner product in $L^2(\Omega, \mathscr{F})$. The next proposition extends Proposition 11.8 as a consequence of Definition 11.6, see Theorem 5.1.4 page 197 of Stroock, 2011 for an extension of the construction of conditional expectation to the space $L^1(\Omega, \mathscr{F})$ of integrable random variable.

Proposition 11.9 The conditional expectation $\mathbb{E}[F \mid \mathscr{G}]$ realizes the minimum in mean-square distance between $F \in L^2(\Omega, \mathscr{F})$ and $L^2(\Omega, \mathscr{G})$, *i.e.* we have

$$\|F - \mathbb{E}[F \mid \mathscr{G}]\|_{L^2(\Omega)} = \inf_{G \in L^2(\Omega, \mathscr{G})} \|F - G\|_{L^2(\Omega)}. \tag{11.48}$$

The following proposition will often be used as a characterization of $\mathbb{E}[F \mid \mathscr{G}]$.

Proposition 11.10 Given $F \in L^2(\Omega, \mathscr{F})$, $X := \mathbb{E}[F \mid \mathscr{G}]$ is the unique random variable X in $L^2(\Omega, \mathscr{G})$ that satisfies the relation

$$\mathbb{E}[GF] = \mathbb{E}[GX] \tag{11.49}$$

for all bounded and $\mathscr{G}$-measurable random variables G.

We note that taking $G = \mathbf{1}$ in (11.49) yields

$$\mathbb{E}[\mathbb{E}[F \mid \mathscr{G}]] = \mathbb{E}[F]. \tag{11.50}$$

In particular, when $\mathscr{G} = \{\emptyset, \Omega\}$ we have $\mathbb{E}[F \mid \mathscr{G}] = \mathbb{E}[F \mid \{\emptyset, \Omega\}]$ and

$$\mathbb{E}[F \mid \{\emptyset, \Omega\}] = \mathbb{E}[\mathbb{E}[F \mid \{\emptyset, \Omega\}]] = \mathbb{E}[F]$$

because $\mathbb{E}[F \mid \{\emptyset, \Omega\}]$ is in $L^2(\Omega, \{\emptyset, \Omega\})$ and is *a.s.* constant. In addition, the conditional expectation operator has the following properties.

i) $\mathbb{E}[FG \mid \mathscr{G}] = G\,\mathbb{E}[F \mid \mathscr{G}]$ if G depends only on the information contained in $\mathscr{G}$.

   Proof. By the characterization (11.49) it suffices to show that

   $$\mathbb{E}[H(GF)] = \mathbb{E}[H(G\,\mathbb{E}[F|\mathscr{G}])], \tag{11.51}$$

   for all bounded and $\mathscr{G}$-measurable random variables H, which implies $\mathbb{E}[FG \mid \mathscr{G}] = G\,\mathbb{E}[F \mid \mathscr{G}]$.

   Relation (11.51) holds from (11.49) because the product HG is $\mathscr{G}$-measurable hence G in (11.49) can be replaced with HG.

ii) $\mathbb{E}[G|\mathscr{G}] = G$ when G depends only on the information contained in $\mathscr{G}$.

   Proof. This is a consequence of point (i) above by taking $F := \mathbf{1}$.

iii) $\mathbb{E}[\mathbb{E}[F|\mathscr{G}] \mid \mathscr{H}] = \mathbb{E}[F|\mathscr{H}]$ if $\mathscr{H} \subset \mathscr{G}$, called the *tower property*.

   Proof. First, we note that by (11.50), (iii) holds when $\mathscr{H} = \{\emptyset, \Omega\}$. Next, by the characterization (11.49) it suffices to show that

   $$\mathbb{E}[H\,\mathbb{E}[F|\mathscr{G}]] = \mathbb{E}[H\,\mathbb{E}[F|\mathscr{H}]], \tag{11.52}$$

   for all bounded and $\mathscr{G}$-measurable random variables H, which will imply (iii) from (11.49).

   In order to prove (11.52) we check that by point (i) above and (11.50) we have

   $$\begin{aligned}\mathbb{E}[H\,\mathbb{E}[F|\mathscr{G}]] &= \mathbb{E}[\mathbb{E}[HF|\mathscr{G}]] = \mathbb{E}[HF] \\ &= \mathbb{E}[\mathbb{E}[HF|\mathscr{H}]] = \mathbb{E}[H\,\mathbb{E}[F|\mathscr{H}]],\end{aligned}$$

   and we conclude by the characterization (11.49).

iv) $\mathbb{E}[F|\mathscr{G}] = \mathbb{E}[F]$ when F "does not depend" on the information contained in $\mathscr{G}$ or, more precisely stated, when the random variable F is *independent* of the σ-algebra $\mathscr{G}$.

Proof. It suffices to note that for all bounded $\mathscr{G}$-measurable G we have

$$\mathbb{E}[FG] = \mathbb{E}[F]\,\mathbb{E}[G] = \mathbb{E}[G\,\mathbb{E}[F]],$$

and we conclude again by (11.49).

v) If G depends only on $\mathscr{G}$ and F is independent of $\mathscr{G}$, then

$$\mathbb{E}[h(F,G)|\mathscr{G}] = \mathbb{E}[h(F,x)]_{x=G}. \tag{11.53}$$

Proof. This relation can be proved using the tower property, by noting that for any $K \in L^2(\Omega,\mathscr{G})$ we have

$$\begin{aligned}
\mathbb{E}[K\,\mathbb{E}[h(x,F)]_{x=G}] &= \mathbb{E}[K\,\mathbb{E}[h(x,F)\mid\mathscr{G}]_{x=G}] \\
&= \mathbb{E}[K\,\mathbb{E}[h(G,F)\mid\mathscr{G}]] \\
&= \mathbb{E}[\mathbb{E}[Kh(G,F)\mid\mathscr{G}]] \\
&= \mathbb{E}[Kh(G,F)],
\end{aligned}$$

which yields (11.53) by the characterization (11.49).

The notion of conditional expectation can be extended from square-integrable random variables in $L^2(\Omega,\mathscr{F})$ to integrable random variables in $L^1(\Omega,\mathscr{F})$, cf. *e.g.* Theorem 5.1 in Kallenberg, 2002.

Proposition 11.11 When the σ-algebra $\mathscr{G} := \sigma(A_1,A_2,\ldots,A_n)$ is generated by n disjoint events $A_1,A_2,\ldots,A_n \in \mathscr{F}$, we have

$$\mathbb{E}[F\mid\mathscr{G}] = \sum_{k=1}^n \mathbb{1}_{A_k}\mathbb{E}[F\mid A_k] = \sum_{k=1}^n \mathbb{1}_{A_k}\frac{\mathbb{E}[F\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.$$

Proof. It suffices to note that the $\mathscr{G}$-measurable random variables can be generated by indicators of the form $\mathbb{1}_{A_l}$, and that

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{A_l}\sum_{k=1}^n \mathbb{1}_{A_k}\frac{\mathbb{E}[F\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}\right] &= \mathbb{E}\left[\mathbb{1}_{A_l}\frac{\mathbb{E}[F\mathbb{1}_{A_l}]}{\mathbb{P}(A_l)}\right] \\
&= \frac{\mathbb{E}[F\mathbb{1}_{A_l}]}{\mathbb{P}(A_l)}\mathbb{E}\left[\mathbb{1}_{A_l}\right] \\
&= \mathbb{E}\left[F\mathbb{1}_{A_l}\right], \qquad l = 1,2,\ldots,n,
\end{aligned}$$

showing (11.49). The relation

$$\mathbb{E}[F\mid A_k] = \frac{\mathbb{E}[F\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}, \qquad k = 1,2,\ldots,n,$$

follows from Lemma 11.3. □

For example, in case $\Omega = \{a,b,c,d\}$ and $\mathscr{G} = \big\{\emptyset,\Omega,\{a,b\},\{c\},\{d\}\big\}$, we have

$$\begin{aligned}
\mathbb{E}[F\mid\mathscr{G}] &= \mathbb{1}_{\{a,b\}}\mathbb{E}[F\mid\{a,b\}] + \mathbb{1}_{\{c\}}\mathbb{E}[F\mid\{c\}] + \mathbb{1}_{\{d\}}\mathbb{E}[F\mid\{d\}] \\
&= \mathbb{1}_{\{a,b\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{a,b\}}\right]}{\mathbb{P}(\{a,b\})} + \mathbb{1}_{\{c\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{c\}}\right]}{\mathbb{P}(\{c\})} + \mathbb{1}_{\{d\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{d\}}\right]}{\mathbb{P}(\{d\})}.
\end{aligned}$$

Regarding conditional probabilities we have similarly, for $A \subset \Omega = \{a,b,c,d\}$,

$$\begin{aligned}
\mathbb{P}(A \mid \mathscr{G}) &= \mathbb{1}_{\{a,b\}} \frac{\mathbb{P}\big(A \cap \{a,b\}\big)}{\mathbb{P}(\{a,b\})} + \mathbb{1}_{\{c\}} \frac{\mathbb{P}\big(A \cap \{c\}\big)}{\mathbb{P}(\{c\})} + \mathbb{1}_{\{d\}} \frac{\mathbb{P}\big(A \cap \{d\}\big)}{\mathbb{P}(\{d\})} \\
&= \mathbb{1}_{\{a,b\}} \mathbb{P}(A \mid \{a,b\}) + \mathbb{1}_{\{c\}} \mathbb{P}(A \mid \{c\}) + \mathbb{1}_{\{d\}} \mathbb{P}(A \mid \{d\}).
\end{aligned}$$

In particular, if $A = \{a\} \subset \Omega = \{a,b,c,d\}$ we find

$$\begin{aligned}
\mathbb{P}(\{a\} \mid \mathscr{G}) &= \mathbb{1}_{\{a,b\}} \mathbb{P}(\{a\} \mid \{a,b\}) \\
&= \mathbb{1}_{\{a,b\}} \frac{\mathbb{P}\big(\{a\} \cap \{a,b\}\big)}{\mathbb{P}(\{a,b\})} \\
&= \mathbb{1}_{\{a,b\}} \frac{\mathbb{P}(\{a\})}{\mathbb{P}(\{a,b\})}.
\end{aligned}$$

In other words, the probability of getting the outcome a is $\mathbb{P}(\{a\})/\mathbb{P}(\{a,b\})$ knowing that the outcome is either a or b, otherwise it is zero.

Exercises

Exercise A.1 Compute the expected value $\mathbb{E}[X]$ of a Poisson random variable X with parameter $\lambda > 0$.

Exercise A.2 Let X denote a centered Gaussian random variable with variance η^2, $\eta > 0$. Show that the probability $P(e^X > c)$ is given by

$$P(e^X > c) = \Phi(-(\log c)/\eta),$$

where $\log = \ln$ denotes the natural logarithm and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

denotes the Gaussian cumulative distribution function.

Exercise A.3 Let $X \simeq \mathscr{N}(\mu, \sigma^2)$ be a Gaussian random variable with parameters $\mu > 0$ and $\sigma^2 > 0$, and density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}.$$

a) Write down $\mathbb{E}[X]$ as an integral and show that

$$\mu = \mathbb{E}[X].$$

b) Write down $\mathbb{E}[X^2]$ as an integral and show that

$$\sigma^2 = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

c) Consider the function $x \mapsto (x-K)^+$ from $\mathbb{R}$ to $\mathbb{R}_+$, defined as

$$(x-K)^+ = \begin{cases} x - K & \text{if } x \geqslant K, \\ 0 & \text{if } x \leqslant K, \end{cases}$$

where $K \in \mathbb{R}$ be a fixed real number. Write down $\mathbb{E}[(X-K)^+]$ as an integral and compute this integral.

Hints: $(x-K)^+$ is zero when $x < K$, and when $\mu = 0$ and $\sigma = 1$ the result is

$$\mathbb{E}[(X-K)^+] = \frac{1}{\sqrt{2\pi}} e^{-\frac{K^2}{2}} - K\Phi(-K),$$

where

$$\Phi(x) := \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R}.$$

d) Write down $\mathbb{E}[e^X]$ as an integral, and compute $\mathbb{E}[e^X]$.

Exercise A.4 Let X be a centered Gaussian random variable with variance $\alpha^2 > 0$ and density $x \mapsto \frac{1}{\sqrt{2\pi\alpha^2}} e^{-x^2/(2\alpha^2)}$ and let $\beta \in \mathbb{R}$.

a) Write down $\mathbb{E}[(\beta - X)^+]$ as an integral. Hint: $(\beta - x)^+$ is zero when $x > \beta$.

b) Compute this integral to show that

$$\mathbb{E}[(\beta - X)^+] = \frac{\alpha}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2\alpha^2}} + \beta\Phi(\beta/\alpha),$$

where

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R}.$$

Exercise A.5 Let X be a centered Gaussian random variable with variance v^2.

a) Compute

$$\mathbb{E}\left[e^{\sigma X} \mathbf{1}_{[K,\infty[}(xe^{\sigma X})\right] = \frac{1}{\sqrt{2\pi v^2}} \int_{\frac{1}{\sigma}\log\frac{K}{x}}^{\infty} e^{\sigma y - y^2/(2v^2)} dy.$$

Hint: use the decomposition

$$\sigma y - \frac{y^2}{v^2} = \frac{v^2\sigma^2}{4} - \left(\frac{y}{v} - \frac{v\sigma}{2}\right)^2.$$

b) Compute

$$\mathbb{E}[(e^{m+X} - K)^+] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} (e^{m+x} - K)^+ e^{-x^2/(2v^2)} dx.$$

Some Useful Identities

Here we present a summary of algebraic identities that are used in this text.

Indicator functions

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \qquad \mathbb{1}_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leqslant x \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

Binomial coefficients

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}, \qquad k = 0, 1, \ldots, n.$$

Exponential series

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in \mathbb{R}. \tag{12.54}$$

Geometric sum

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}, \qquad r \neq 1. \tag{12.55}$$

Geometric series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \qquad -1 < r < 1. \tag{12.56}$$

Differentiation of geometric series

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{\partial}{\partial r} \sum_{k=0}^{\infty} r^k = \frac{\partial}{\partial r} \frac{1}{1-r} = \frac{1}{(1-r)^2}, \qquad -1 < r < 1. \tag{12.57}$$

Binomial identities

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{12.58}$$

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} k \binom{n}{k} a^k b^{n-k} &= \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} a^k b^{n-k} \\
&= \sum_{k=0}^{n-1} \frac{n!}{(n-1-k)!k!} a^{k+1} b^{n-1-k} \\
&= n \sum_{k=0}^{n-1} \binom{n-1}{k} a^{k+1} b^{n-1-k} \\
&= na(a+b)^{n-1}, \qquad n \geqslant 1,
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} k \binom{n}{k} a^k b^{n-k} &= a \frac{\partial}{\partial a} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\
&= a \frac{\partial}{\partial a} (a+b)^n \\
&= na(a+b)^{n-1}, \qquad n \geqslant 1.
\end{aligned}$$

Sums of integers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{12.59}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{12.60}$$

Taylor expansion

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{x^k}{k!}\alpha(\alpha-1)\times\cdots\times(\alpha-(k-1)). \tag{12.61}$$

Differential equation

$$\text{The solution of } f'(t) = cf(t) \text{ is given by } f(t) = f(0)\,\mathrm{e}^{ct}, \quad t \in \mathbb{R}_+. \tag{12.62}$$

Exercise Solutions

Chapter 1 - Discrete-Time Martingales

Exercise 1.1

a) We have

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\sum_{k=1}^{n+1} 2^{k-1} X_k \,\middle|\, \mathcal{F}_n\right] \\ &= \mathbb{E}\left[\sum_{k=1}^{n} 2^{k-1} X_k \,\middle|\, \mathcal{F}_n\right] + \mathbb{E}[2^n X_{n+1} \mid \mathcal{F}_n] \\ &= \sum_{k=1}^{n} 2^{k-1} X_k + 2^n \mathbb{E}[X_{n+1}] \\ &= M_n, \qquad n \geqslant 0.\end{aligned}$$

b) This random time is a hitting time, so it is a stopping time.

c) The strategy of the gambler is to double the stakes each time he loses (double down strategy), and to quit the game as soon as his gains reach \$1.

d) The two possible values of $M_{\tau\wedge n}$ are 1 and

$$-\sum_{k=1}^{n} 2^{k-1} = -\frac{1-2^n}{1-2} = 1-2^n, \qquad n \geqslant 1.$$

We have

$$\mathbb{P}(M_{n\wedge\tau} = 1-2^n) = 2^{-n} \quad \text{and} \quad \mathbb{P}(M_{n\wedge\tau} = 1) = 1-2^{-n}, \qquad n \geqslant 1.$$

e) We have

$$\begin{aligned}\mathbb{E}[M_{n\wedge\tau}] &= (1-2^n)\mathbb{P}(M_{n\wedge\tau} = 1-2^n) + \mathbb{P}(M_{n\wedge\tau} = 1) \\ &= (1-2^n)2^{-n} + (1-2^{-n}) \\ &= 0, \qquad n \geqslant 1.\end{aligned}$$

f) The Stopping Time Theorem 1.3 directly states that

$$\mathbb{E}[M_{n\wedge\tau}] = \mathbb{E}[M_0] = 0.$$

Exercise 1.2 We note that for all $n \geqslant 1$ we have

$$\{\tau = n\} = \{\tau \geqslant n\} \setminus \{\tau > n\} = \{\tau > n-1\} \cap \{\tau > n\}^c \in \mathscr{F}_n,$$

since $\{\tau > n\} \in \mathscr{F}_n$ and $\{\tau > n-1\} \in \mathscr{F}_{n-1} \subset \mathscr{F}_n$. In case $n = 0$, we have

$$\{\tau = 0\} = \{\tau > 0\}^c \in \mathscr{F}_0.$$

Conversely, assuming that (1.9) holds, we have

$$\{\tau > n\} = \{\tau \leqslant n\}^c = \left(\bigcup_{k=0}^{n} \{\tau = k\} \right)^c = \bigcap_{k=0}^{n} \{\tau = k\}^c \in \mathscr{F}_n,$$

since

$$\{\tau = k\} \in \mathscr{F}_k \subset \mathscr{F}_n, \qquad k = 0, 1, \ldots, n.$$

Exercise 1.3 We consider several examples.

i) The stopped martingale $(X_n)_{n\in\mathbb{N}} := (M_{\tau\wedge n})_{n\in\mathbb{N}}$ of Exercise 1.1, for which we have $\mathbb{E}[X_n] = \mathbb{E}[M_{\tau\wedge n}] = 0$, $n \in \mathbb{N}$, by Question (e)), while $\lim_{n\to\infty} X_n = \lim_{n\to\infty} M_{\tau\wedge n} = 1$, *a.s.*, with $\inf_{n\in\mathbb{N}} M_n = -\infty$ and $\operatorname{Sup}_{n\in\mathbb{N}} |M_n| = +\infty$.

ii) The sequence $X_n := n \mathbb{1}_{\{U<1/n\}}$, $n \geqslant 1$, where $U \simeq U(0,1]$ is a uniformly distributed random variable on $(0,1]$. More formally, $(X_n)_{n\geqslant 1}$ can be constructed by *e.g.* taking $\Omega = (0,1)$ with the probability measure $\mathbb{P}$ defined by $\mathbb{P}([a,b]) = b-a$, $0 < a \leqslant b < 1$, and defining the random variable $X_n(\omega) = n \mathbb{1}_{\{0<\omega<1/n\}}$, $\omega \in (0,1)$, in which case we have $\operatorname{Sup}_{n\geqslant 1} |X_n| = +\infty$, $\mathbb{E}[X_n] = n \times (1/n - 0) = 1$, $n \geqslant 1$, and $\lim_{n\to\infty} X_n(\omega) = 0$ for all $\omega \in (0,1)$.

iii) The sequence $X_n := n^2 \mathbb{1}_{\{U_n<1/n^2\}}$, $n \geqslant 1$, where $(U_n)_{n\geqslant 1} \simeq U(0,1]$ is a sequence of uniformly distributed random variables on $(0,1]$. In this case, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

hence by the Borel-Cantelli Lemma, the probability that $X_n > 0$ infinitely many times is zero.

Exercise 1.4

a) From the tower property of conditional expectations (11.38), we have:

$$\mathbb{E}[M_{n+1}] = \mathbb{E}[\mathbb{E}[M_{n+1} \mid \mathscr{F}_n]] \geqslant \mathbb{E}[M_n], \qquad n \geqslant 0.$$

b) If $(Z_n)_{n\in\mathbb{N}}$ is a stochastic process with independent increments and nonnegative expectations, we have

$$\begin{aligned}
\mathbb{E}[Z_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}[Z_n \mid \mathscr{F}_n] + \mathbb{E}[Z_{n+1} - Z_n \mid \mathscr{F}_n] \\
&= \mathbb{E}[Z_n \mid \mathscr{F}_n] + \mathbb{E}[Z_{n+1} - Z_n] \\
&\geqslant \mathbb{E}[Z_n \mid \mathscr{F}_n] = Z_n, \qquad n \geqslant 0.
\end{aligned}$$

c) We let $A_0 := 0$, $A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n]$, $n \geqslant 0$, and

$$N_n := M_n - A_n, \qquad n \in \mathbb{N}. \tag{A.1}$$

(i) For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{E}[N_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}[M_{n+1} - A_{n+1} \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathscr{F}_n] - \mathbb{E}[\mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathscr{F}_n] - \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \\
&= -\mathbb{E}[A_n \mid \mathscr{F}_n] + \mathbb{E}[M_n \mid \mathscr{F}_n] = M_n - A_n = N_n,
\end{aligned}$$

hence $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

(ii) For all $n \in \mathbb{N}$, we have

$$\begin{aligned} A_{n+1} - A_n &= \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \\ &= \mathbb{E}[M_{n+1} \mid \mathscr{F}_n] - \mathbb{E}[M_n \mid \mathscr{F}_n] \\ &= \mathbb{E}[M_{n+1} \mid \mathscr{F}_n] - M_n \geqslant 0, \end{aligned}$$

since $(M_n)_{n\in\mathbb{N}}$ is a submartingale.

(iii) By induction we have $A_{n+1} = A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n]$, $n \in \mathbb{N}$, which is $\mathscr{F}_n$-measurable if A_n is $\mathscr{F}_{n-1}$-measurable, $n \geqslant 1$.

(iv) This property is obtained by construction in (A.1).

d) For all bounded stopping times σ and τ such that $\sigma \leqslant \tau$ *a.s.*, we have $\mathbb{E}[N_\sigma] = \mathbb{E}[N_\tau]$ by (1.5), hence

$$\begin{aligned} \mathbb{E}[M_\sigma] &= \mathbb{E}[N_\sigma] + \mathbb{E}[A_\sigma] \\ &= \mathbb{E}[N_\tau] + \mathbb{E}[A_\sigma] \\ &\leqslant \mathbb{E}[N_\tau] + \mathbb{E}[A_\tau] \\ &= \mathbb{E}[M_\tau], \end{aligned}$$

by (1.5), since $(M_n)_{n\in\mathbb{N}}$ is a martingale and $(A_n)_{n\in\mathbb{N}}$ is non-decreasing.

Exercise 1.5

a) We will show more generally that

$$\phi(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \leqslant p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_n\phi(x_n), \tag{A.2}$$

$x_1, \ldots, x_n \in \mathbb{R}$, for any sequence of coefficients $p_1, p_2, \ldots, p_n \geqslant 0$ such that $p_1 + p_2 + \cdots + p_n = 1$. The inequality (A.2) clearly holds for $n = 1$, and for $n = 2$ it coincides with the convexity property of ϕ, *i.e.*

$$\phi(p_1x_1 + p_2x_2) \leqslant p_1\phi(x_1) + p_2\phi(x_2), \qquad x_1, x_2 \in \mathbb{R}.$$

Assuming that (A.2) holds for some $n \geqslant 1$ and taking $p_1, p_2, \ldots, p_{n+1} \geqslant 0$ such that $p_1 + p_2 + \cdots + p_{n+1} = 1$ and $0 < p_{n+1} < 1$ and applying (1.10) at the second order, we have

$$\begin{aligned} &\phi(p_1x_1 + p_2x_2 + \cdots + p_{n+1}x_{n+1}) \\ &= \phi\left((1 - p_{n+1})\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{1 - p_{n+1}} + p_{n+1}x_{n+1}\right) \\ &\leqslant (1 - p_{n+1})\phi\left(\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{1 - p_{n+1}}\right) + p_{n+1}\phi(x_{n+1}) \\ &\leqslant (1 - p_{n+1})\left(\frac{p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_n\phi(x_n)}{1 - p_{n+1}}\right) + p_{n+1}\phi(x_{n+1}) \\ &= p_1\phi(x_1) + p_2\phi(x_2) + \cdots + p_{n+1}\phi(x_{n+1}), \end{aligned}$$

and we conclude by induction.

b) Taking $p_1 = p_1 = \cdots = p_N = 1/N$, we have

$$\begin{aligned} &\mathbb{E}^*\left[\phi\left(\frac{S_1 + \cdots + S_N}{N}\right)\right] \leqslant \mathbb{E}^*\left[\frac{\phi(S_1) + \cdots + \phi(S_N)}{N}\right] && \text{since } \phi \text{ is convex,} \\ &= \frac{\mathbb{E}^*[\phi(S_1)] + \cdots + \mathbb{E}^*[\phi(S_N)]}{N} \\ &= \frac{\mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathscr{F}_1])] + \cdots + \mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathscr{F}_N])]}{N} && \text{because } (S_n)_{n\in\mathbb{N}} \text{ is a martingale,} \\ &\leqslant \frac{\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_1]] + \cdots + \mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_N]]}{N} && \text{by Jensen's inequality,} \\ &= \frac{\mathbb{E}^*[\phi(S_N)] + \cdots + \mathbb{E}^*[\phi(S_N)]}{N} && \text{by the tower property,} \\ &= \mathbb{E}^*[\phi(S_N)]. \end{aligned}$$

c) We conclude by applying the above to the convex function $x \mapsto (x-K)^+$, see Figure S.1 for an illustration.

```
nSim=99999;p=0.4;q=1-p;n=7;a=q/p;r=1;european=0;asian=0;K=1.5
for (j in 1:nSim){S<-a^cumsum(2*rbinom(n,1,p)-1);color="blue"
A<-sum(c(1,S))/(n+1);if (S[n]>=K) {european=european+S[n]-K}
if (A>=K) {asian=asian+A-K};if (S[n]>A) {color="darkred"} else {color="darkgreen"}
plot(seq(0,n),c(1,S), xlab = "Time", xlim=c(0,n), type='o',ylim = c(0,a^(n-2)), lwd = 3, ylab = "", col =
      color,main=paste("Asian Price=",format(round(asian,2)),"/", j,"=",format(round(asian/j,2)),"European
      Price=",format(round(european,2)), "/",j,"=",format(round(european/j,2))),
      xaxs='i',xaxt='n',yaxt='n', yaxs='i', yaxp = c(0,10,10))
text(3,6,paste("A-Payoff=",format(round(max(A-K,0),2)),", E-Payoff=", format(round(max(S[n]-K,0),2))),
      col=color,cex=2)
axis(1, at=seq(0,n), labels=seq(0,n), las=1)
axis(2, at=c(0,K,A,1,2,3,4,5,6,7,8,9,10), labels=c(0,"K","Average",1,2,3,4,5,6,7,8,9,10), las=2)
lines(seq(0,n),rep(K,n+1),col = "red",lty = 1, lwd = 4);
lines(seq(0,n),rep(A,n+1),col = "darkgreen",lty = 2, lwd = 4);
text(7,K, paste("K"));Sys.sleep(0.1)
if (S[n]>K || A>K) {readline(prompt = "Pause. Press <Enter> to continue...")}}
```

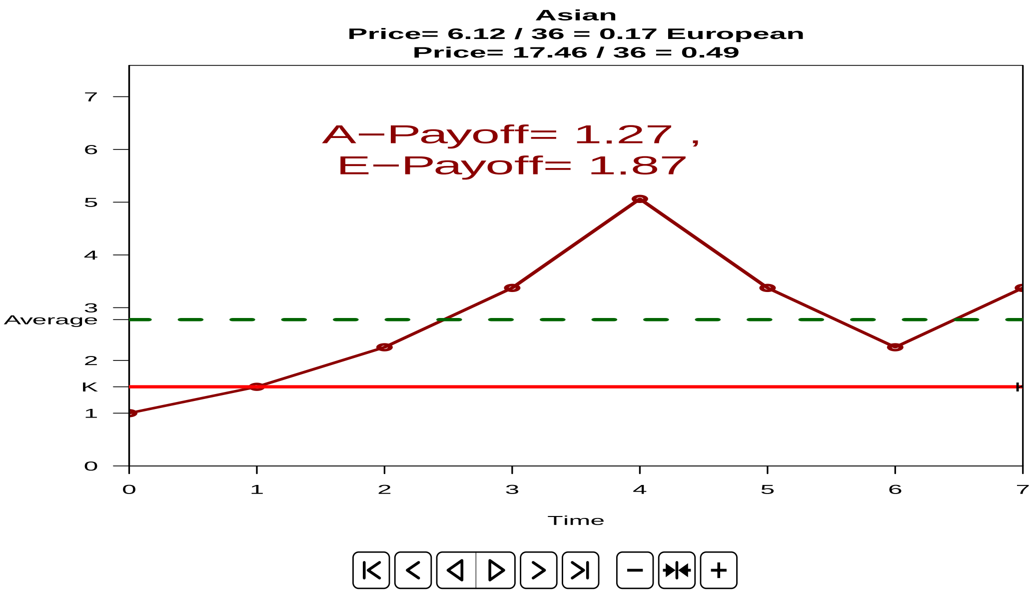


Figure S.1: Asian option price *vs* European option price.[*]

Exercise 1.6

a) We have

$$\mathbb{E}[M_n] \leqslant \mathbb{E}[\mathbb{E}[M_{n+1} \mid \mathscr{F}_n]] = \mathbb{E}[M_{n+1}], \qquad n \in \mathbb{N}.$$

b) Writing

$$S_n - \alpha n = \underbrace{(X_1 + X_2 + \cdots + X_n - np)}_{\text{martingale}} + n(p-\alpha)$$

as the sum of a martingale (a stochastic process with centered independent increments) and $n(p-\alpha)$, we conclude that $(S_n)_{n\in\mathbb{N}}$ is a *sub*martingale if and only if $p \geqslant \alpha$. Indeed, we have

$$\begin{aligned} \mathbb{E}[S_n - \alpha n \mid \mathscr{F}_k] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n - np \mid \mathscr{F}_k] + n(p-\alpha) \\ &= X_1 + X_2 + \cdots + X_k - kp + n(p-\alpha) \\ &= X_1 + X_2 + \cdots + X_k - k\alpha + (n-k)(p-\alpha) \\ &\geqslant S_k - k\alpha, \qquad k = 0, 1, \ldots, n, \end{aligned}$$

[*]The animation works in Acrobat Reader on the entire pdf file.

if and only if $p \geqslant \alpha$.

Exercise 1.7

a) We have

$$\phi(M_k) = \phi\left(\mathbb{E}[M_n \mid \mathscr{F}_k]\right) \leqslant \mathbb{E}[\phi(M_n) \mid \mathscr{F}_k], \qquad k = 0,1,\ldots,n$$

b) We have

$$\phi(M_k) \leqslant \phi\left(\mathbb{E}[M_n \mid \mathscr{F}_k]\right) \leqslant \mathbb{E}[\phi(M_n) \mid \mathscr{F}_k], \qquad k = 0,1,\ldots,n.$$

Problem 1.8

a) We have

$$\{\tau_x > n\} = \bigcap_{k=0}^{n} \{M_k < x\}.$$

On the other hand, for all $k = 0,1,\ldots,n$ we have $\{M_k < x\} \in \mathscr{F}_k \subset \mathscr{F}_n$, hence $\{\tau_x > n\} \in \mathscr{F}_n$ by stability of σ-algebras by intersection, cf. (11.9).

b) We have

$$\begin{aligned} x\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\right) &= x\mathbb{P}(\tau_x \leqslant n) \\ &= x\mathbb{E}[\mathbb{1}_{\{\tau_x \leqslant n\}}] \\ &\leqslant \mathbb{E}\left[M_{\tau_x \wedge n}\mathbb{1}_{\{\tau_x \leqslant n\}}\right] \qquad \text{(A.3)} \\ &\leqslant \mathbb{E}[M_{\tau_x \wedge n}] \qquad \text{(A.4)} \\ &\leqslant \mathbb{E}[M_{\tau_x \wedge 0}] \\ &\leqslant \mathbb{E}[M_0] \\ &= \mathbb{E}[M_n], \qquad \text{(A.5)} \end{aligned}$$

where we used the condition $M_{\tau_x \wedge n} \geqslant 0$ from (A.3) to (A.4), hence

$$\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\right) \leqslant \frac{\mathbb{E}[M_n]}{x}, \qquad x > 0. \tag{A.6}$$

Remark. The nonnegativity of $(M_n)_{n\in\mathbb{N}}$ is used to reach (A.4), and the Doob Stopping Time Theorem 1.3 is used to conclude to (A.5).

c) When $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale, we have

$$\mathbb{E}[M_{\tau_x \wedge n}] \leqslant \mathbb{E}[M_n],$$

by the Doob Stopping Time Theorem 1.3 for *sub*martingales, *cf.* Exercise 1.4-(d)), hence we can still conclude to (A.5) and (A.6) above.

d) Since $x \mapsto x^2$ is a convex function, $((M_n)^2)_{n\in\mathbb{N}}$ is a *sub*martingale by Question (a)), hence by Question (c)) we have

$$\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\right) = \mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} (M_k)^2 \geqslant x^2\right) \leqslant \frac{\mathbb{E}[(M_n)^2]}{x^2}, \qquad x > 0.$$

e) Similarly to Question (d)), $x \mapsto x^p$ is a convex function for all $p \geqslant 1$ hence $((M_n)^2)_{n\in\mathbb{N}}$ is a *sub*martingale by Question (a)), and by Question (c)) we find

$$\mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\right) = \mathbb{P}\left(\operatorname*{Max}_{k=0,1,\ldots,n} (M_k)^p \geqslant x^p\right) \leqslant \frac{\mathbb{E}[(M_n)^p]}{x^p}, \qquad x > 0.$$

f) We note that $(S_n)_{n\in\mathbb{N}}$ is a martingale because it has centered and independent increments, with
$$\mathbb{E}[(S_n)^2] = \mathrm{Var}[S_n] = n\,\mathrm{Var}[Y_1] = n\sigma^2,$$
hence by Question (d)) we have
$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} S_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[(S_n)^2]}{x^2} = \frac{n\sigma^2}{x^2}, \qquad x > 0.$$

g) When $(M_n)_{n\in\mathbb{N}}$ is a (not necessarily nonnegative) *sub*martingale we can modify the answer to Question (b)) using the Doob Stopping Time Theorem 1.3 for *sub*martingales, see Exercise 1.4-(d)), as follows:
$$\begin{aligned} x\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) &= x\mathbb{E}[\mathbb{1}_{\{\tau_x\leqslant n\}}] \\ &\leqslant \mathbb{E}\left[M_{\tau_x\wedge n}\mathbb{1}_{\{\tau_x\leqslant n\}}\right] \\ &\leqslant \mathbb{E}[(M_{\tau_x\wedge n})^+] \\ &\leqslant \mathbb{E}[(M_n)^+], \end{aligned}$$
since $((M_k)^+)_{k\in\mathbb{N}}$ is a *sub*martingale because $x \mapsto x^+$ is a non-decreasing convex function, *cf.* Exercise 1.7-(b)), hence
$$\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) \leqslant \frac{\mathbb{E}[(M_n)^+]}{x}, \qquad x > 0.$$

h) We have
$$\begin{aligned} x\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) &= x\mathbb{P}(\tau_x \leqslant n) \\ &= x\mathbb{E}[\mathbb{1}_{\{\tau_x\leqslant n\}}] \\ &\leqslant \mathbb{E}\left[M_{\tau_x\wedge n}\mathbb{1}_{\{\tau_x\leqslant n\}}\right] \\ &\leqslant \mathbb{E}[M_{\tau_x\wedge n}] \\ &\leqslant \mathbb{E}[M_0]. \end{aligned}$$

i) We have
$$\begin{aligned} x\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} \phi(M_k) \geqslant x\Big) &= x\mathbb{E}\left[\mathbb{1}_{\{\mathrm{Max}_{k=0,1,\ldots,n}\phi(M_k)\geqslant x\}}\right] \\ &= x\mathbb{E}\left[\mathbb{1}_{\{\tau_x^\phi\leqslant n\}}\right] \\ &= \mathbb{E}\left[\phi\left(M_{\tau_x^\phi\wedge n}\right)\mathbb{1}_{\{\tau_x^\phi\leqslant n\}}\right] \\ &= \mathbb{E}\left[\phi(M_{\tau_x^\phi\wedge n})\right] \\ &= \mathbb{E}[\phi(M_n)], \end{aligned}$$
where the last inequality follows from Exercise 1.4-(d)), since both $\tau_x^\phi \wedge n$ and n are stopping times.

j) Consider for example any nonnegative martingale such as $M_n = (p/q)^{S_n}$ where $S_n = X_1 + \cdots + X_n$ and $(X_k)_{k\geqslant 1}$ is a sequence of independent identically distributed Bernoulli random variables with $p = \mathbb{P}(X_k = 1)$ and $q = 1 - p = \mathbb{P}(X_k = -1)$, $k \geqslant 1$. Then $Z_n := \mathrm{e}^{-n}M_n$ will be a *super*martingale, since
$$\begin{aligned} \mathbb{E}[Z_n \mid \mathscr{F}_k] &= \mathrm{e}^{-n}\mathbb{E}[M_n \mid \mathscr{F}_k] \\ &= \mathrm{e}^{-n}M_k \\ &\leqslant \mathrm{e}^{-k}M_k \\ &= Z_k, \qquad k = 0,1,\ldots,n. \end{aligned}$$

Exercise 1.9 We have

$$\begin{aligned}
\mathbb{E}[(M_n)^r] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{rS_n}\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\left(\frac{q}{p}\right)^{r(S_k-S_{k-1})}\right] \\
&= \prod_{k=1}^{n}\mathbb{E}\left[\left(\frac{q}{p}\right)^{r(S_k-S_{k-1})}\right] \\
&= \prod_{k=1}^{n}\left(p\left(\frac{q}{p}\right)^{r}+q\left(\frac{q}{p}\right)^{-r}\right) \\
&= \left(p\left(\frac{q}{p}\right)^{r}+q\left(\frac{p}{q}\right)^{r}\right)^{n} \\
&= \left(\frac{pq^{2r}+qp^{2r}}{(pq)^r}\right)^{n} \\
&= \big(\mathbb{E}[(M_1)^r]\big)^{n}.
\end{aligned}$$

We note that by Jensen's inequality we have

$$\mathbb{E}[M_1^r] \geqslant \big(\mathbb{E}[M_1]\big)^r = 1,$$

and $\mathbb{E}[M_n^r]$ is non-decreasing in n as $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale. We also check that

$$\begin{aligned}
\mathbb{E}[(M_{2n})^r] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{rS_{2n}}\right] \\
&= \sum_{k=-n}^{n}\left(\frac{q}{p}\right)^{2kr}\mathbb{P}(S_{2n}=2k) \\
&= \sum_{k=-n}^{n}\binom{2n}{n+k}\left(\frac{q}{p}\right)^{2kr}p^{n+k}q^{n-k} \\
&= \sum_{k=0}^{2n}\binom{2n}{k}\left(\frac{q}{p}\right)^{2(k-n)r}p^{k}q^{2n-k} \\
&= \left(\frac{q}{p}\right)^{-2nr}q^{2n}\sum_{k=0}^{2n}\binom{2n}{k}\left(\left(\frac{q}{p}\right)^{2r-1}\right)^{k} \\
&= \left(p\left(\frac{q}{p}\right)^{r}+q\left(\frac{p}{q}\right)^{r}\right)^{2n},
\end{aligned}$$

hence by Problem 1.8-(e)), for all $n \geqslant 0$ and $r \geqslant 1$ we have

$$\begin{aligned}
\mathbb{P}\Big(\operatorname*{Max}_{k=0,1,\ldots,n} M_k \geqslant x\Big) &\leqslant \frac{\mathbb{E}[(M_n)^r]}{x^r} \\
&= \frac{(p(q/p)^r+q(p/q)^r)^{2n}}{x^r}, \qquad x>0.
\end{aligned}$$

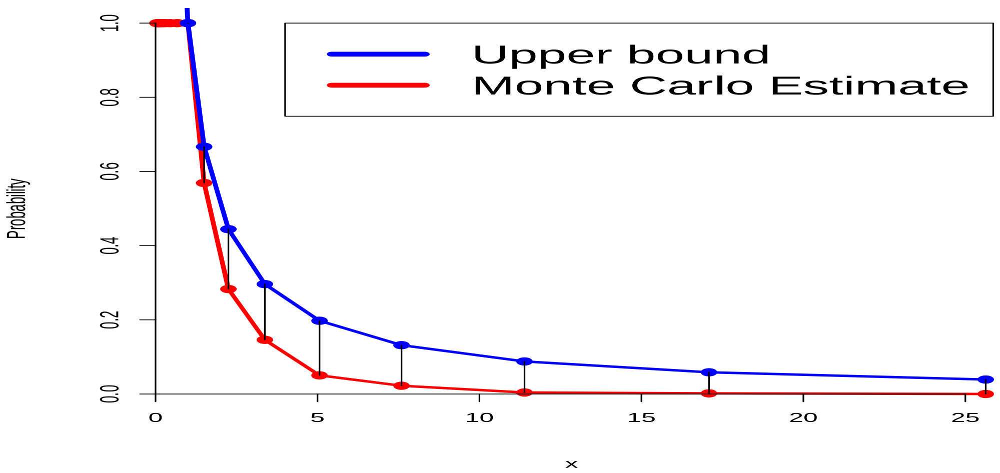


Figure S.2: Supremum deviation probability with $n = 7$ and $p = 0.4$.

```
nSim=99999;p=0.4;q=1-p;n=7;a=q/p;r=1;prob=rep(0,2*n+2)
for (i in (-n):(n+1)){for (j in 1:nSim){
M<-a^cumsum(2*rbinom(n,1,p)-1);color="blue"
if (max(c(1,M))>=a^i) {prob[n+1+i]=prob[n+1+i]+1;color="darkred"}
if ((j%%10000)==0){
plot(seq(0,n),c(1,M), xlab = "Time", xlim=c(0,n), type='o',ylim = c(0,a^(n-1)), lwd = 3,
     ylab = "", col = color, main='',xaxs='i',yaxs='i',yaxp = c(0, 11, 11))
text(3,8,paste("x=",format(round(a^i,2)),",
     ",prob[n+1+i],"/",j,"=",format(prob[n+1+i]/j,digits=4)),col=color,cex=2)
axis(2, at=c(a^i), labels=c("x"), las=2)
lines(seq(0,n),rep(a^i,n+1),col = "black",lty = 2, lwd = 2);
text(-0.1,a^i, paste("x"))
Sys.sleep(0.9)}}
prob[n+1+i]=prob[n+1+i]/nSim}
x=a^seq(-n,n+1)
plot(x,prob,type="o",pch=19,lwd=3,col="red",xlab="x",ylim=c(0,1),
     ylab="Probability",main="",axes=FALSE)
lines(x,(p*a^r+q*(p/q)^r)^(2*n)/x^r,type="o",pch=19,lwd=3,col="blue",main="")
axis(1, pos=0);axis(2, pos=0);
legend(4, 1, legend=c("Upper bound", "Monte Carlo Estimate"), col=c("blue","red"), lty=1,
     lwd=6,cex=2)
for (i in 0:(n+1)){segments(x0=a^i,y0=prob[n+1+i],x1=a^i,(p*a^r+q*a^(-r))^(2*n)
     /a^(i*r),col="black")}
```

Chapter 2

Exercise 2.1 The payoff C is that of a *put* option with strike price $K = \$3$.

Exercise 2.2 Each of the two possible scenarios yields one equation:

$$\begin{cases} 5\alpha + \beta = 0 \\ \\ 2\alpha + \beta = 6, \end{cases} \quad \text{with solution} \quad \begin{cases} \alpha = -2 \\ \\ \beta = +10. \end{cases}$$

The hedging strategy at $t = 0$ is to **shortsell** $-\alpha = +2$ units of the asset S priced $S_0 = 4$, and to put $\beta = \$10$ on the savings account. The price $V_0 = \alpha S_0 + \beta$ of the initial portfolio at time $t = 0$ is

$$V_0 = \alpha S_0 + \beta = -2 \times 4 + 10 = \$2,$$

*The animation works in Acrobat Reader on the entire pdf file.

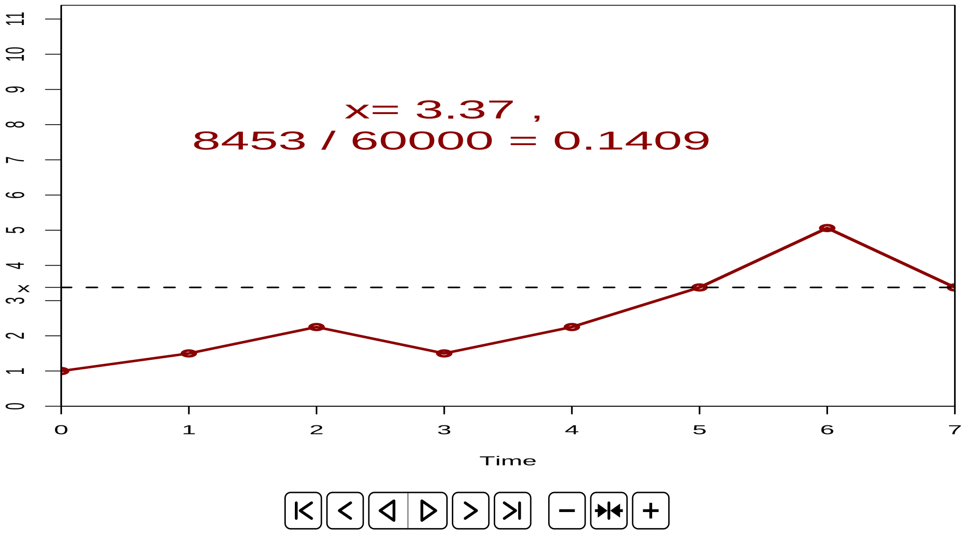


Figure S.3: Martingale supremum as a function of time.*

which yields the price of the claim at time $t = 0$. In order to hedge then option, one should:

i) At time $t = 0$,
   (a) Charge the \$2 option price.
   (b) Shortsell $-\alpha = +2$ units of the stock priced $S_0 = 4$, which yields \$8.
   (c) Put $\beta = \$8 + \$2 = \$10$ on the savings account.

ii) At time $t = 1$,
   (a) If $S_1 = \$5$, spend \$10 from savings to buy back $-\alpha = +2$ stocks.
   (b) If $S_1 = \$2$, spend \$4 from savings to buy back $-\alpha = +2$ stocks, and deliver a \$10 - \$4 = \$6 payoff.

Pricing the option by the expected value $\mathbb{E}^*[C]$ yields the equality

$$\begin{aligned}
\$2 &= \mathbb{E}^*[C] \\
&= 0 \times \mathbb{P}^*(C = 0) + 6 \times \mathbb{P}^*(C = 6) \\
&= 0 \times \mathbb{P}^*(S_1 = 2) + 6 \times \mathbb{P}^*(S_1 = 5) \\
&= 6 \times q^*,
\end{aligned}$$

hence the risk-neutral probability measure $\mathbb{P}^*$ is given by

$$p^* = \mathbb{P}^*(S_1 = 5) = \frac{2}{3} \quad \text{and} \quad q^* = \mathbb{P}^*(S_1 = 2) = \frac{1}{3}.$$

Exercise 2.3

a) i) Does this model allow for arbitrage? | Yes | ✓ | | No | |

ii) If this model allows for arbitrage opportunities, how can they be realized?

| By shortselling | | By borrowing on savings | ✓ | N.A. | |

b) i) Does this model allow for arbitrage? | Yes | | | No | ✓ |

ii) If this model allows for arbitrage opportunities, how can they be realized? $\boxed{\text{By shortselling }|\ \ }$ $\boxed{\text{By borrowing on savings }|\ \ }$ $\boxed{\text{N.A. }|\ \checkmark}$

c) i) Does this model allow for arbitrage? $\boxed{\text{Yes }|\ \checkmark}$ $\boxed{\text{No }|\ \ }$

ii) If this model allows for arbitrage opportunities, how can they be realized? $\boxed{\text{By shortselling }|\ \checkmark}$ $\boxed{\text{By borrowing on savings }|\ \ }$ $\boxed{\text{N.A. }|\ \ }$

Exercise 2.4

a) We need to search for possible risk-neutral probability measure(s) $\mathbb{P}^*$ such that $\mathbb{E}^*\big[S_1^{(1)}\big] = (1+r)S_0^{(1)}$. Letting

$$\begin{cases} p^* = \mathbb{P}^*\big(S_1^{(1)} = S_0^{(1)}(1+a)\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = a\right), \\ \\ \theta^* = \mathbb{P}^*\big(S_1^{(1)} = S_0^{(1)}(1+b)\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = b\right), \\ \\ q^* = \mathbb{P}^*\big(S_1^{(1)} = (1+c)S_0^{(1)}\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = c\right), \end{cases}$$

We have

$$\begin{cases} (1+a)p^*S_0^{(1)} + (1+b)\theta^*S_0^{(1)} + (1+c)q^*S_0^{(1)} = (1+r)S_0^{(1)} \\ \\ p^* + \theta^* + q^* = 1, \end{cases}$$

from which we obtain

$$\begin{cases} p^*a + \theta^*b + q^*c = r, \\ \\ p^* + \theta^* + q^* = 1. \end{cases} \quad \Longrightarrow \quad \begin{cases} p^* = \dfrac{(1-\theta)c + \theta^*b - r}{c-a} \in (0,1), \\ \\ q^* = \dfrac{r - (1-\theta)a - \theta^*b}{c-a} \in (0,1), \end{cases}$$

for any $\theta \in (0,1)$ such that

$$(1-\theta^*)a - \theta^*b < r < (1-\theta^*)c + \theta^*b,$$

or $(1-\theta^*)a < r < (1-\theta^*)c$ in case $b = 0$. Therefore there exists an infinity of risk-neutral probability measures depending on the value of $\theta^* \in (0,1)$, and the market is without arbitrage but not complete.

b) Hedging a claim with possible payoff values C_a, C_b, C_c would require to solve

$$\begin{cases} (1+a)\alpha S_0^{(1)} + (1+r)\beta S_0^{(0)} = C_a \\ \\ (1+b)\alpha S_0^{(1)} + (1+r)\beta S_0^{(0)} = C_b \\ \\ (1+c)\alpha S_0^{(1)} + (1+r)\beta S_0^{(0)} = C_c, \end{cases}$$

for α and β, which is not possible in general due to the existence of three conditions with only two unknowns.

Exercise 2.5

a) The risk-neutral condition $\mathbb{E}^*[R_1] = 0$ reads

$$b\mathbb{P}^*(R_1 = b) + 0 \times \mathbb{P}^*(R_1 = 0) + (-b) \times (R_1 = -b) = bp^* - bq^* = 0,$$

hence

$$p^* = q^* = \frac{1-\theta^*}{2},$$

since $p^* + q^* + \theta^* = 1$.

b) We have

$$\begin{aligned}\mathrm{Var}^*\left[\frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}}\right] &= \mathbb{E}^*\left[R_1^2\right] - (\mathbb{E}^*[R_1])^2 \\ &= \mathbb{E}^*\left[R_1^2\right] \\ &= b^2\mathbb{P}^*(R_1 = b) + 0^2 \times \mathbb{P}^*(R_1 = 0) + (-b)^2 \times (R_1 = -b) \\ &= b^2(p^* + q^*) \\ &= b^2(1-\theta^*) \\ &= \sigma^2,\end{aligned}$$

hence $\theta^* = 1 - \sigma^2/2b^2$, and therefore

$$p^* = q^* = \frac{1-\theta^*}{2} = \frac{\sigma^2}{2b^2},$$

provided that $\sigma^2 \leqslant 2b^2$.

Exercise 2.6

a) i) $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid \checkmark}$

Comment: No loss is possible, while a 100% profit is possible with non-zero probability $1/3$.

ii) $\boxed{\text{Yes} \mid}$ $\boxed{\text{No} \mid \checkmark}$

Comment: The (unique) risk-neutral measure (p^*, q^*) is given by

$$\$2 \times p^* + \$1 \times q^* = \$1 \times (1+r) = \$1 \quad \text{and} \quad p^* + q^* = 1,$$

hence $(p^*, q^*) = (0, 1)$ is not equivalent to $\mathbb{P}$ given by $(p, q) = (1/3, 2/3)$.

b) i) $\boxed{\text{Yes} \mid \checkmark}$ $\boxed{\text{No} \mid}$ Comment: The risk-neutral measure (p^*, θ^*, q^*) is given by the equations

$$\$2 \times p^* + \$1 \times \theta^* + \$0 \times q^* = \$1 \times (1+r) = \$1 \quad \text{and} \quad p^* + \theta^* + q^* = 1, \tag{A.7}$$

which clearly admit solutions, see (iv)) below.

ii) $\boxed{\text{Yes} \mid \checkmark}$ $\boxed{\text{No} \mid}$ Comment: Realizing arbitrage would mean building a portfolio achieving no strictly negative return with probability one, which is impossible since the probability of 100% loss is $\mathbb{P}(S_1 = 0) = 1 - 1/4 - 1/9 = 23/36 > 0$.

iii) Yes | No | ✓

Comment: For example, the claim $\mathbb{1}_{\{S_1>0\}}$ cannot be attained since the there is no portfolio allocation (α,β) satisfying

$$\begin{cases} \$2\times\alpha+\beta=\$1 \\ \$1\times\alpha+\beta=\$1 \\ \$0\times\alpha+\beta=\$0. \end{cases}$$

iv) Yes | No | ✓

Comment: The risk-neutral measure is clearly not unique, as for example

$$(p^*,\theta^*,q^*)=(1/4,1/2,1/4) \quad \text{and} \quad (p^*,\theta^*,q^*)=(1/3,1/3,1/3)$$

are both solutions of (A.7).

Exercise 2.7

a) The possible values of R are a and b.

b) We have

$$\begin{aligned} \mathbb{E}^*[R] &= a\mathbb{P}^*(R=a)+b\mathbb{P}^*(R=b) \\ &= a\frac{b-r}{b-a}+b\frac{r-a}{b-a} \\ &= r. \end{aligned}$$

c) By Theorem 2.1, there do not exist arbitrage opportunities in this market since from Question (b)) there exists a risk-neutral probability measure $\mathbb{P}^*$ whenever $a<r<b$.

d) The risk-neutral probability measure is unique hence the market model is complete by Theorem 2.4.

e) Taking

$$\eta=\frac{\alpha(1+b)-\beta(1+a)}{\pi_1(b-a)} \quad \text{and} \quad \xi=\frac{\beta-\alpha}{S_0(b-a)},$$

we check that

$$\begin{cases} \eta\pi_1+\xi S_0(1+a)=\alpha & \text{if } R=a, \\ \\ \eta\pi_1+\xi S_0(1+b)=\beta & \text{if } R=b, \end{cases}$$

which shows that

$$\eta\pi_1+\xi S_1=C$$

in both cases $R=a$ and $R=b$.

f) We have

$$\begin{aligned} \pi_0(C) &= \eta\pi_0+\xi S_0 \\ &= \frac{\alpha(1+b)-\beta(1+a)}{(1+r)(b-a)}+\frac{\beta-\alpha}{b-a} \\ &= \frac{\alpha(1+b)-\beta(1+a)-(1+r)(\alpha-\beta)}{(1+r)(b-a)} \\ &= \frac{\alpha b-\beta a-r(\alpha-\beta)}{(1+r)(b-a)}. \end{aligned} \tag{A.8}$$

g) We have

$$\begin{aligned} \mathbb{E}^*[C] &= \alpha\mathbb{P}^*(R=a)+\beta\mathbb{P}^*(R=b) \\ &= \alpha\frac{b-r}{b-a}+\beta\frac{r-a}{b-a}. \end{aligned} \tag{A.9}$$

h) Comparing (A.8) and (A.9) above we do obtain

$$\pi_0(C) \quad = \quad \frac{1}{1+r}\mathbb{E}^*[C]$$

i) The initial value $\pi_0(C)$ of the portfolio is interpreted as the arbitrage-free price of the option contract and it equals the expected value of the discounted payoff.

j) We have

$$C = (K - S_1)^+ = (11 - S_1)^+ = \begin{cases} 11 - S_1 & \text{if } K > S_1, \\ 0 & \text{if } K \leqslant S_1. \end{cases}$$

k) We have $S_0 = 1$, $a = 8$, $b = 11$, $\alpha = 2$, $\beta = 0$, hence

$$\xi = \frac{\beta - \alpha}{S_0(b-a)} = \frac{0-2}{11-8} = -\frac{2}{3},$$

and

$$\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_1(b-a)} = \frac{24}{3 \times 1.05}.$$

l) The arbitrage-free price $\pi_0(C)$ of the contingent claim with payoff C is

$$\pi_0(C) = \eta\pi_0 + \xi S_0 = 6.952.$$

Exercise 2.8 Let $a := (152-180)/180 = -7/45$ and $b := (203-180)/180 = 23/180$ denote the potential market returns, with $r = 0.03$. From the strike price K and the risk-neutral probabilities

$$p_r^* = \frac{r-a}{b-a} = 0.6549 \quad \text{and} \quad q_r^* = \frac{b-r}{b-a} = 0.3451,$$

the price of the option at the beginning of the year is given from Proposition 2.5 as the discounted expected value

$$\frac{1}{1+r}\mathbb{E}^*[(K-S_1)^+] = \frac{1}{1+r}\left(p_r^*(K-203)^+ + q_r^*(K-152)^+\right).$$

Equating this price with the intrinsic value $(K-180)^+$ of the put option yields the equation

$$(K-180)^+ = \frac{1}{1+r}\left(p_r^*(K-203)^+ + q_r^*(K-152)^+\right)$$

which requires $K > 180$ (the case $K \leqslant 152$ is not considered because both the option price and option payoff vanish in this case). Hence we consider the equation

$$K - 180 = \frac{1}{1+r}\left(p_r^*(K-203)^+ + q_r^*(K-152)^+\right),$$

with the following cases.

i) If $K \in [180, 203]$ we get

$$(1+r)(K-180) = q_r^*(K-152),$$

hence

$$K = \frac{(1+r)180 - q_r^* 152}{1+r-q_r^*} = \frac{(1+r)180 - q_r^* 152}{p_r^* + r} = 194.11.$$

ii) If $K \geqslant 203$ we find

$$K = \frac{180(1+r) - 203p_r^* - 152q_r^*}{r} < 203,$$

which is out of range and leads to a contradiction.

We note that the above formula

$$K = \frac{(1+r)180 - q_r^* 152}{p_r^* + r} = \frac{28b - 180a + r(180(b-a)+152)}{(b+1-a)r - a}$$

yields a decreasing function $K(r)$ of r in the interval $[0, 100\%]$, although the function is not monotone over $\mathbb{R}_+$.

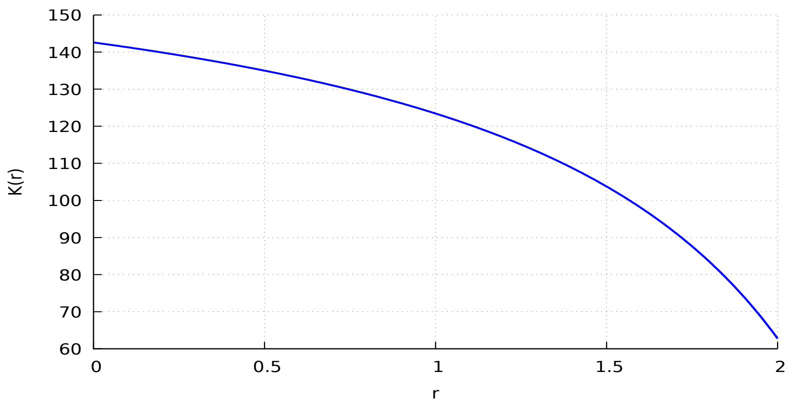


Figure S.4: Strike price as a function of risk-free rate r.

Chapter 3

Exercise 3.1 Let $m := \$2,550$ denote the amount invested each year.

a) By (3.1), the value of the plan after $N = 10$ years becomes

$$m \sum_{k=1}^{N} (1+r)^k = m(1+r)\frac{(1+r)^N - 1}{r},$$

which in turns becomes

$$(1+r)^N m \sum_{k=1}^{N} (1+r)^k = m(1+r)^{N+1}\frac{(1+r)^N - 1}{r},$$

after N additional years without further contributions to the plan. Equating

$$A = 30835 = m(1+r)^{N+1}\frac{(1+r)^N - 1}{r}$$

shows that

$$\frac{(1+r)^{2N+1} - (1+r)^{N+1}}{r} = \frac{A}{m},$$

with $m = 2550$, or

$$\frac{(1+r)^{21} - (1+r)^{11}}{r} = \frac{30835}{2550} \simeq 12.09215,$$

hence $r \simeq 1.23\%$ according to Figure S.5, which is typical of an annual fixed deposit interest rate.

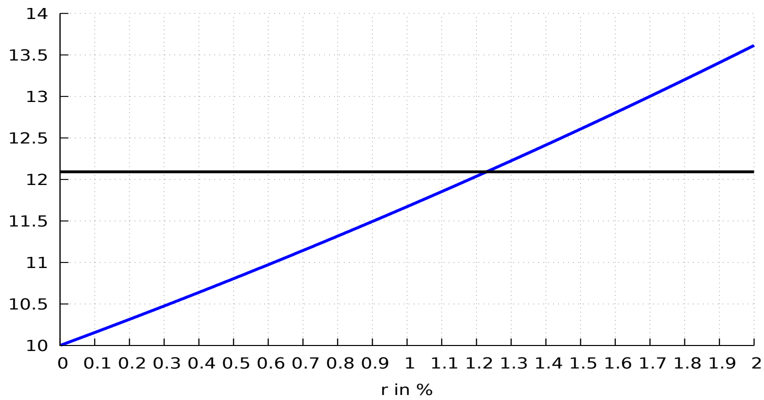


Figure S.5: Graph of $r \longmapsto ((1+r)^{21} - (1+r)^{11})/r$.

In the hypothesis $r = 3.25\%$ we would find

$$A = m(1+r)^{N+1}\frac{(1+r)^N - 1}{r} = 42040.42.$$

b) Taking $N = 10$, $m = 2,550$ and $r = 0.0325$, we find

$$\begin{aligned} A_{2N} : &= m(1+r)^N \sum_{k=1}^{N} (1+r)^{N-k+1} \\ &= m(1+r)^N \sum_{k=1}^{N} (1+r)^k \\ &= m(1+r)^{N+1}\frac{(1+r)^N - 1}{r} \\ &= \$42040.42. \end{aligned}$$

c) In this case, we find

$$A_{2N} = m\sum_{k=1}^{N}(1+r)^{N-k+1} = m(1+r)\frac{(1+r)^N - 1}{r} = \$30532.79.$$

Exercise 3.2

a) Let $m := \$3,581$ denote the amount invested each year. After multiplying (3.1) by $(1+r)^N$ in order to account for the compounded interest from year 11 until year 20, we get the equality

$$A = m(1+r)^{N+1}\frac{(1+r)^N - 1}{r}$$

shows that

$$(1+r)^{21} - (1+r)^{11} = r\frac{50862}{3581} \simeq 14.2033r,$$

showing that $r \simeq 2.28\%$ according to Figure S.6.

b) Taking $N = 10$, $m = 3,581$ and $r = 0.0325$, we find

$$\begin{aligned} A_{2N} : &= m(1+r)^N \sum_{k=1}^{N} (1+r)^{N-k+1} \\ &= m(1+r)^N \sum_{k=1}^{N} (1+r)^k \end{aligned}$$

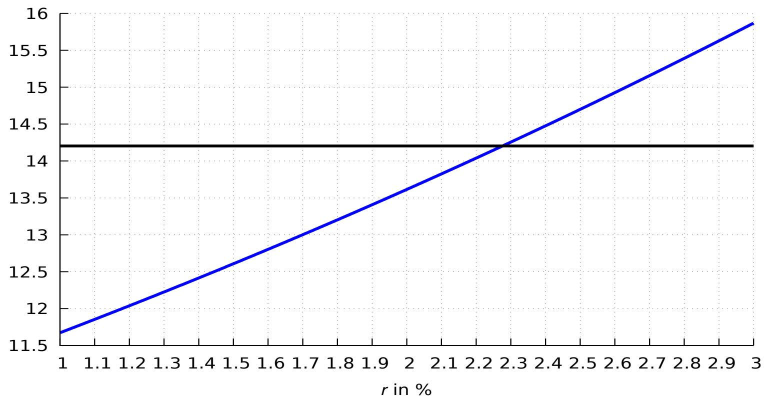


Figure S.6: Graph of $r \longmapsto ((1+r)^{21}-(1+r)^{11})/r$.

$$
\begin{aligned}
&= m(1+r)^{N+1}\frac{(1+r)^N-1}{r} \\
&= \$59037.94.
\end{aligned}
$$

c) In this case, we find

$$
A_{2N} = m\sum_{k=1}^{N}(1+r)^{N-k+1} = m(1+r)\frac{(1+r)^N-1}{r} = \$42877.61.
$$

Exercise 3.3 We check that for any $\mathbb{P}^*$ of the form $\mathbb{P}^*(R_t=-1) := p^*$, $\mathbb{P}^*(R_t=0) := 1-2p^*$, $\mathbb{P}^*(R_t=1) := p^*$, we have

$$
\mathbb{E}^*[S_1] = S_0(2p^*+1-2p^*) = S_0,
$$

and similarly

$$
\mathbb{E}^*[S_2 \mid S_1] = S_1(2p^*+(1-2p^*)) = S_1,
$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral.

Exercise 3.4

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure $\mathbb{P}^*$ which should satisfy

$$
\mathbb{E}^*\left[S^{(1)}_{k+1} \mid \mathscr{F}_k\right] = (1+r)S^{(1)}_k, \qquad k=0,1,\ldots,N-1,
$$

with $r=0$. Rewriting $\mathbb{E}^*\left[S^{(1)}_{k+1} \mid \mathscr{F}_k\right]$ as

$$
\begin{aligned}
&\mathbb{E}^*\left[S^{(1)}_{k+1} \mid \mathscr{F}_k\right] \\
&= (1-b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=a \mid \mathscr{F}_k) + S^{(1)}_k\mathbb{P}^*(R_{k+1}=0 \mid \mathscr{F}_k) \\
&\quad + (1+b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=b \mid \mathscr{F}_k) \\
&= (1-b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=a) + S^{(1)}_k\mathbb{P}^*(R_{k+1}=0) + (1+b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=b),
\end{aligned}
$$

$k=0,1,\ldots,N-1$, it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$
\begin{cases}
(1+b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=b) + S^{(1)}_k\mathbb{P}^*(R_{k+1}=0) + (1-b)S^{(1)}_k\mathbb{P}^*(R_{k+1}=a) = S^{(1)}_k \\
\\
\mathbb{P}^*(R_{k+1}=b) + \mathbb{P}^*(R_{k+1}=0) + \mathbb{P}^*(R_{k+1}=-b) = 1,
\end{cases}
$$

$k = 0, 1, \ldots, N-1$, *i.e.*

$$\begin{cases} b\mathbb{P}^*(R_k = b) - b\mathbb{P}^*(R_k = -b) = 0, \\ \\ \mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = -b) = 1 - \mathbb{P}^*(R_k = 0), \end{cases}$$

$k = 1, 2, \ldots, N$, with solution

$$\mathbb{P}^*(R_k = b) = \mathbb{P}^*(R_k = -b) = \frac{1-\theta^*}{2},$$

$k = 1, 2, \ldots, N$.

b) We have

$$\begin{aligned} &\mathrm{Var}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \,\middle|\, \mathscr{F}_k\right] \\ &= \mathbb{E}^*\left[\left(\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}}\right)^2 \,\middle|\, \mathscr{F}_k\right] - \left(\mathbb{E}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \,\middle|\, \mathscr{F}_k\right]\right)^2 \\ &= \mathbb{E}^*\left[\left(\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}}\right)^2 \,\middle|\, \mathscr{F}_k\right] \\ &= b^2\mathbb{P}_\sigma^*(R_{k+1} = -b \mid \mathscr{F}_k) + b^2\mathbb{P}_\sigma^*(R_{k+1} = b \mid \mathscr{F}_k) \\ &= b^2\frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} + b^2\frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} \\ &= b^2(1-\theta) \\ &= \sigma^2, \end{aligned}$$

$k = 0, 1, \ldots, N-1$, hence

$$\mathbb{P}_\sigma^*(R_k = 0) = \theta = 1 - \frac{\sigma^2}{b^2},$$

and therefore

$$\mathbb{P}_\sigma^*(R_k = b) = \mathbb{P}_\sigma^*(R_k = -b) = \frac{1 - \mathbb{P}_\sigma^*(R_k = 0)}{2} = \frac{\sigma^2}{2b^2},$$

$k = 0, 1, \ldots, N-1$, under the condition $0 < \sigma^2 < b^2$.

Exercise 3.5

a) The possible values of R_t are a and b.

b) We have

$$\begin{aligned} \mathbb{E}^*[R_{t+1} \mid \mathscr{F}_t] &= a\mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t) + b\mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t) \\ &= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} = r. \end{aligned}$$

c) Letting $p^* = (r-a)/(b-a)$ and $q^* = (b-r)/(b-a)$ we have

$$\begin{aligned} \mathbb{E}^*[S_{t+k} \mid \mathscr{F}_t] &= \sum_{i=0}^{k} (p^*)^i (q^*)^{k-i} \binom{k}{i} (1+b)^i (1+a)^{k-i} S_t \\ &= S_t \sum_{i=0}^{k} \binom{k}{i} (p^*(1+b))^i (q^*(1+a))^{k-i} \\ &= S_t \left(p^*(1+b) + q^*(1+a)\right)^k \\ &= S_t \left(\frac{r-a}{b-a}(1+b) + \frac{b-r}{b-a}(1+a)\right)^k \end{aligned}$$

$$= (1+r)^k S_t.$$

Assuming that the formula holds for $k=1$, its extension to $k \geqslant 2$ can also be proved recursively from the "tower property" (11.38) of conditional expectations, as follows:

$$\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathscr{F}_t] &= \mathbb{E}^*[\mathbb{E}^*[S_{t+k} \mid \mathscr{F}_{t+k-1}] \mid \mathscr{F}_t] \\
&= (1+r)\,\mathbb{E}^*[S_{t+k-1} \mid \mathscr{F}_t] \\
&= (1+r)\,\mathbb{E}^*[\mathbb{E}^*[S_{t+k-1} \mid \mathscr{F}_{t+k-2}] \mid \mathscr{F}_t] \\
&= (1+r)^2\,\mathbb{E}^*[S_{t+k-2} \mid \mathscr{F}_t] \\
&= (1+r)^2\,\mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} \mid \mathscr{F}_{t+k-3}] \mid \mathscr{F}_t] \\
&= (1+r)^3\,\mathbb{E}^*[S_{t+k-3} \mid \mathscr{F}_t] \\
&= \cdots \\
&= (1+r)^{k-2}\,\mathbb{E}^*[S_{t+2} \mid \mathscr{F}_t] \\
&= (1+r)^{k-2}\,\mathbb{E}^*[\mathbb{E}^*[S_{t+2} \mid \mathscr{F}_{t+1}] \mid \mathscr{F}_t] \\
&= (1+r)^{k-1}\,\mathbb{E}^*[S_{t+1} \mid \mathscr{F}_t] \\
&= (1+r)^k S_t.
\end{aligned}$$

Exercise 3.6

a) We check that

$$\begin{aligned}
\mathbb{E}^*[R_{t+1} \mid \mathscr{F}_t] &= a\mathbb{P}^*(R_{t+1} = a \mid \mathscr{F}_t) + b\mathbb{P}^*(R_{t+1} = b \mid \mathscr{F}_t) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} = r.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}^*[(S_N)^\beta] &= S_0\,\mathbb{E}^*\left[\left(\prod_{k=1}^N (1+R_k)\right)^\beta\right] \\
&= S_0\,\mathbb{E}^*\left[\prod_{k=1}^N (1+R_k)^\beta\right] \\
&= S_0 \prod_{k=1}^N \mathbb{E}^*\left[(1+R_k)^\beta\right],
\end{aligned}$$

after using the independence of the returns $(R_k)_{k=1,2,\ldots,N}$, with

$$\mathbb{E}^*\left[(1+R_k)^\beta\right] = (1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}, \quad k = 0,1,\ldots,N,$$

hence we find

$$\mathbb{E}^*[(S_N)^\beta] = S_0^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}\right)^N.$$

c) We have

$$\begin{aligned}
&\mathbb{P}^*\Big(S_t \geqslant \alpha\pi_t \text{ for some } t \in \{0,1,\ldots,N\}\Big) = \mathbb{P}^*\left(\operatorname*{Max}_{t=0,1,\ldots,n} \frac{S_t}{\pi_t} \geqslant x\right) \\
&\leqslant \frac{\mathbb{E}[(M_N)^\beta]}{x^\beta} \\
&\leqslant \left(\frac{S_0}{(1+r)x\pi_0}\right)^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}\right)^N,
\end{aligned}$$

since the discounted price process

$$(M_t)_{t=0,1,\ldots,N} := \left(\frac{S_t}{\pi_t}\right)_{t=0,1,\ldots,N}$$

is a nonnegative martingale.

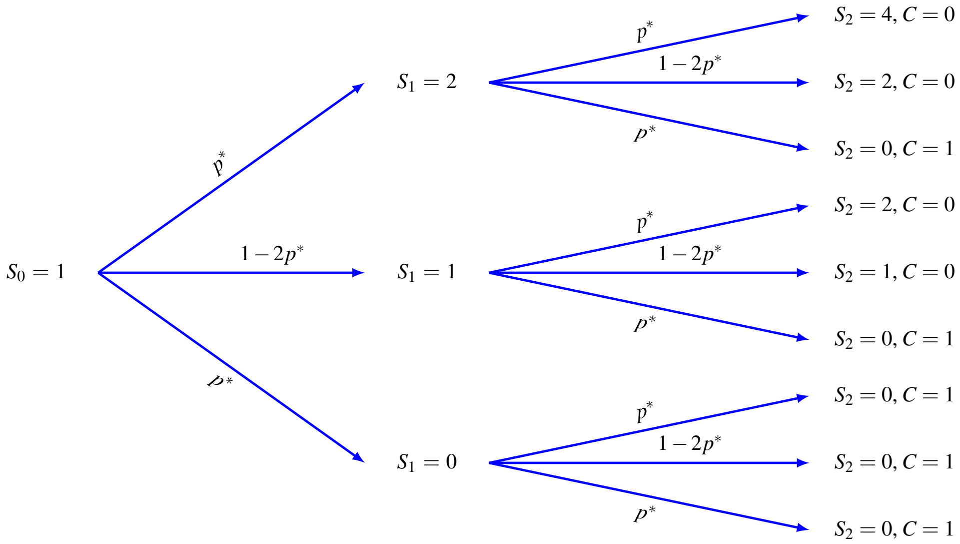
d) We have

$$\begin{aligned}\mathbb{P}^*\Big(\operatorname*{Max}_{t=0,1,\ldots,n} S_t \geqslant x\Big) &\leqslant \frac{\mathbb{E}\left[(M_N)^\beta\right]}{x^\beta}\\ &\leqslant \left(\frac{S_0}{x}\right)^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}\right)^N,\end{aligned}$$

since the price process $(M_t)_{t=0,1,\ldots,N} := (S_t)_{t=0,1,\ldots,N}$ is a nonnegative *sub*martingale.

Chapter 4

Exercise 4.1 (Exercise 3.3 continued). We consider the following trinomial tree.



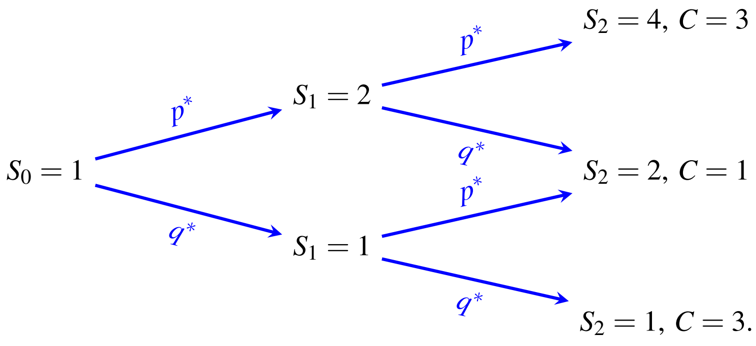
At time $t = 0$, we find

$$\begin{aligned}\pi_0(C) &= \frac{1}{(1+r)^2}\mathbb{E}^*[(K-S_2)^+]\\ &= p^*(p^*+(1-2p^*)+p^*)+(1-2p^*)p^*+(p^*)^2\\ &= p^*+(1-2p^*)p^*+(p^*)^2\\ &= 2p^*-(p^*)^2.\end{aligned}$$

At time $t = 1$, we find

$$\begin{aligned}\pi_1(C) &= \frac{1}{1+r}\mathbb{E}^*[(K-S_2)^+ \mid S_1]\\ &= \begin{cases} p^* & \text{if } S_1 = 2S_0,\\ \\ p^* & \text{if } S_1 = S_0,\\ \\ 1 & \text{if } S_1 = 0.\end{cases}\end{aligned}$$

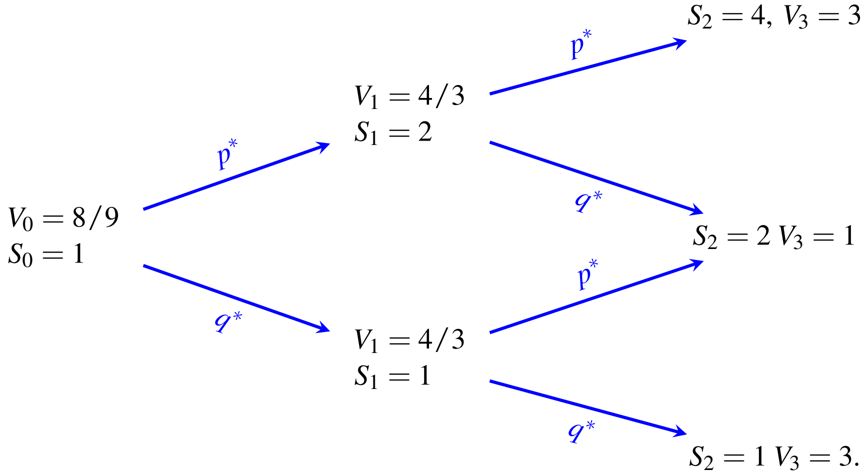
Exercise 4.2 We have $p^* = (r-a)/(b-a) = 1/2$ and $q^* = (b-r)/(b-a) = 1/2$, and the following underlying asset price tree:



We first price, and then hedge. At time $t = 1$, by Theorem 4.2 we have

$$\pi_1(C) = V_1 = \begin{cases} \dfrac{3p^* + q^*}{1+r} = \dfrac{4}{3} & \text{if } S_1 = 2 \\ \\ \dfrac{p^* + 3q^*}{1+r} = \dfrac{4}{3} & \text{if } S_1 = 1, \end{cases} \qquad \text{and} \quad V_0 = \frac{4}{3}\frac{p^* + q^*}{1+r} = \frac{8}{9}.$$

This leads to the following option pricing tree:



Regarding hedging, if $S_1 = 2$ the condition $\bar{\xi}_2 \cdot \bar{S}_2 = 0$ reads

$$S_1 = 2 \Longrightarrow \begin{cases} 4\xi_2 + \eta_2(1+r)^2 = 3 \\ \\ 2\xi_2 + \eta_2(1+r)^2 = 1, \end{cases}$$

hence $(\xi_2, \eta_2) = (1, -4/9)$. On the other hand, if $S_1 = 1$ we have

$$S_1 = 1 \Longrightarrow \begin{cases} 2\xi_2 + \eta_2(1+r)^2 = 1 \\ \\ \xi_2 + \eta_2(1+r)^2 = 3, \end{cases}$$

hence $(\xi_2, \eta_2) = (-2, 20/9)$. Finally, at time $t = 0$ with $S_0 = 1$ we have

$$\begin{cases} 2\xi_1 + \eta_1(1+r) = \dfrac{4}{3} \\ \\ \xi_1 + \eta_1(1+r) = \dfrac{4}{3}, \end{cases}$$

hence $(\xi_1, \eta_1) = (0, 8/9)$. The results can be summarized in the following table:

<table>
<tr><td rowspan="6">$S_0 = 1$
$V_0 = 8/9$
$\xi_1 = 0$
$\eta_1 = 8/9$</td><td rowspan="3">$S_1 = 2$, $V_1 = 4/3$
$\xi_2 = 1$, $\eta_2 = -4/9$</td><td>$S_2 = 4$
$V_2 = 3$</td></tr>
<tr><td rowspan="2">$S_2 = 1$
$V_2 = 3$</td></tr>
<tr></tr>
<tr><td rowspan="3">$S_1 = 1$, $V_1 = 4/3$
$\xi_2 = -2$, $\eta_2 = 20/9$</td></tr>
<tr><td rowspan="2">$S_2 = 1$
$V_2 = 3$</td></tr>
<tr></tr>
</table>

Table 13.1: CRR pricing and hedging table.

In addition, it can be checked that the portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ is self-financing as we have

$$\begin{aligned} \xi_1 S_1 + \eta_1 A_1 &= \frac{8}{9} \times \frac{3}{2} \\ &= \begin{cases} 2 - \dfrac{4}{9} \times \dfrac{3}{2} \\ \\ -2 + \dfrac{20}{9} \times \dfrac{3}{2} \end{cases} \\ &= \xi_2 S_1 + \eta_2 A_1. \end{aligned}$$

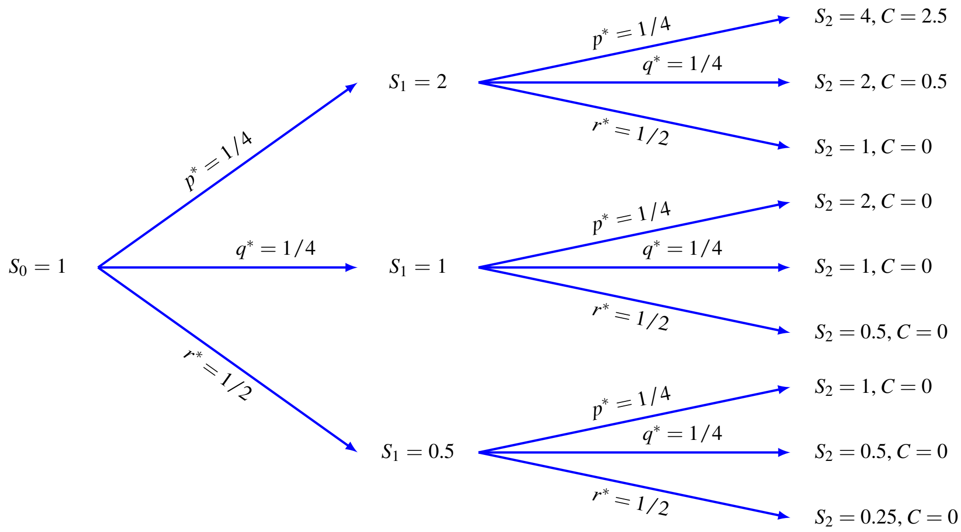
Exercise 4.3

a) We have

$$\begin{aligned} \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}^*[S_{t+1} \mid S_t] \\ &= \frac{S_t}{2} \mathbb{P}^*(R_t = -0.5) + S_t \mathbb{P}^*(R_t = 0) + 2 S_t \mathbb{P}^*(R_t = 1) \\ &= S_t \left(\frac{r^*}{2} + q^* + 2p^* \right) \\ &= S_t, \qquad t = 0, 1, \end{aligned}$$

with $r = 0$.

b) We have the following graph:



c) The down-an-out barrier call option is priced at time $t = 0$ as

$$V_0 = \mathbb{E}^*[C] = 2.5 \times (p^*)^2 + 0.5 \times p^* q^* = \frac{3}{16}.$$

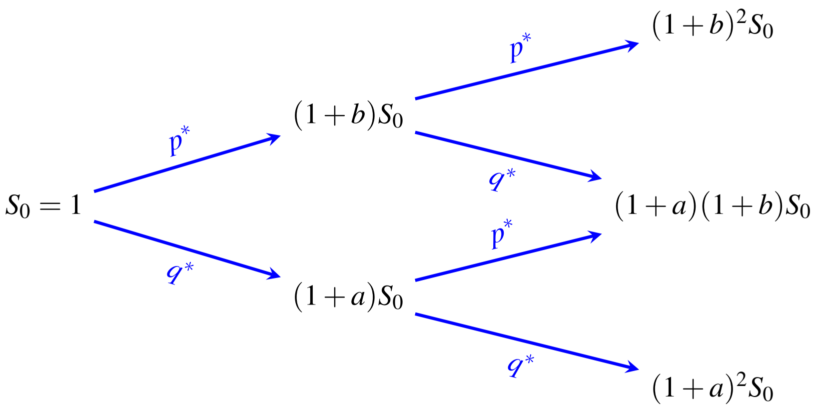
At time $t = 1$ we have

$$V_1 = 2.5 \times p^* + 0.5 \times q^* = 2.5 \times \frac{1}{4} + 0.5 \times \frac{1}{4} = \frac{3}{4}$$

if $S_1 = 2$, and $V_1 = 0$ in both cases $S_1 = 1$ and $S_1 = 0.5$.

d) This market is not complete, and not every contingent claim is attainable, because the risk-neutral probability measure $\mathbb{P}^*$ is not unique, for example $(r^*, q^*, p^*) = (1/4, 5/8, 1/8)$ and $(r^*, q^*, p^*) = (1/2, 1/4, 1/4)$ are both risk-neutral probability measures.

Exercise 4.4 The CRR model can be described by the following binomial tree.



a) By the formulas

$$\begin{aligned} V_1 &= \frac{1}{1+r} \mathbb{E}^*[V_2 \mid \mathscr{F}_1] = \frac{1}{1+r} \mathbb{E}^*[V_2 \mid S_1] \\ &= \frac{S_0(1+b)^2 - 8}{1+r} \mathbb{P}^*(S_2 = S_0(1+b)^2 \mid S_1) \\ &= p^* \frac{(S_0(1+b)^2 - 8)}{1+r} \mathbb{1}_{\{S_1 = S_0(1+b)\}}, \end{aligned}$$

and

$$\begin{aligned} V_0 &= \frac{1}{1+r} \mathbb{E}^*[V_1 \mid \mathscr{F}_0] \\ &= \frac{1}{1+r} \left(p^* \frac{(S_0(1+b)^2 - 8)}{1+r} \times \mathbb{P}^*(S_1 = S_0(1+b)) + 0 \times \mathbb{P}^*(S_1 = S_0(1+a)) \right) \\ &= (p^*)^2 \frac{(S_0(1+b)^2 - 8)}{(1+r)^2}, \end{aligned}$$

we find the table

<table>
<tr><td rowspan="3">$S_0 = 1$
$V_0 = 1/16$</td><td rowspan="2">$S_1 = 3$, $V_1 = 1/4$</td><td>$S_2 = 9$
$V_2 = 1$</td></tr>
<tr><td>$S_2 = 3$
$V_2 = 0$</td></tr>
<tr><td>$S_1 = 1$, $V_1 = 0$</td><td>$S_2 = 1$
$V_2 = 0$</td></tr>
</table>

Table 13.2: CRR pricing tree.

Note that we could also directly compute V_0 from

$$V_0 = \frac{1}{(1+r)^2} \mathbb{E}^*[V_2 \mid \mathscr{F}_0].$$

b) When $S_1 = S_0(1+b)$, the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = S_0(1+b)^2 - 8 \\ \\ \xi_2 S_0(1+b)(1+a) + \eta_2 A_0(1+r)^2 = 0, \end{cases}$$

which yields

$$\xi_2 = \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_2 = -\frac{(S_0(1+b)^2 - 8)(1+a)}{(b-a)A_0(1+r)^2}. \tag{A.10}$$

On the other hand, when $S_1 = S_0(1+a)$ the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+a)^2 + \eta_2 A_0(1+r)^2 = 0 \\ \\ \xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = 0, \end{cases}$$

which has the unique solution $(\xi_2, \eta_2) = (0, 0)$. Next, the equation $\xi_1 S_1 + \eta_1 A_1 = V_1$ reads

$$\begin{cases} \xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \dfrac{p^*(S_0(1+b)^2 - 8)}{1+r}, \\ \\ \xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0 \end{cases}$$

which yields

$$\xi_1 = p^* \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+r)} \quad \text{and} \quad \eta_1 = -p^* \frac{(1+a)(S_0(1+b)^2 - 8)}{(b-a)A_0(1+r)^2}. \tag{A.11}$$

This can be summarized in the following table:

<table>
<tr><td rowspan="6">$S_0 = 1$
$V_0 = 1/16$
$\xi_1 = 1/8$
$\eta_1 = -1/16$</td><td rowspan="3">$S_1 = 3$, $V_1 = 1/4$

$\xi_2 = 1/6$, $\eta_2 = -1/8$</td><td rowspan="2">$S_2 = 9$
$V_2 = 1$</td></tr>
<tr></tr>
<tr><td rowspan="2">$S_2 = 3$
$V_2 = 0$</td></tr>
<tr><td rowspan="3">$S_1 = 1$, $V_1 = 0$

$\xi_2 = 0$, $\eta_2 = 0$</td></tr>
<tr><td rowspan="2">$S_2 = 1$
$V_2 = 0$</td></tr>
<tr></tr>
</table>

Table 13.3: CRR pricing and hedging tree.

When $S_1 = S_0(1+a)$ at time $t = 1$ the option price is $V_1 = 0$ and the hedging strategy is to cut all positions: $\xi_2 = \eta_2 = 0$. On the other hand, if $S_1 = S_0(1+b)$ then there is a chance of being in the money at maturity and we need to increase our position in the underlying asset from $\xi_1 = 1/8$ to $\xi_2 = 1/6$.

Note that the self-financing condition

$$\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \tag{A.12}$$

is verified. For example when $S_1 = S_0(1+a)$ we have

$$\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = 0 \times S_1 + 0 \times A_1 = 0,$$

while when $S_1 = S_0(1+b)$ we find

$$\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = \frac{1}{6} \times S_1 - \frac{1}{8} \times A_1 = \frac{1}{4}.$$

On the other hand, we can also use the self-financing condition (A.12) to recover (A.11) by rewriting the system of equations as

$$\begin{cases} \xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \xi_2 S_0(1+b) + \eta_2 A_0(1+r) \\ \\ \xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0, \end{cases}$$

with (ξ_2, η_2) given by (A.10), which recovers

$$V_1 = \xi_1 S_1 + \eta_1 A_1 = \begin{cases} \dfrac{3}{8} - \dfrac{2}{16} = \dfrac{1}{4} & \text{if } S_1 = 3, \\ \\ \dfrac{1}{8} - \dfrac{2}{16} = 0 & \text{if } S_1 = 1. \end{cases}$$

Exercise 4.5

a) We build a portfolio based on α units of stock and $\$\beta$ in cash. When $S_1 = 2$, we should have

$$\begin{cases} 4\alpha_2 + \beta_2 = 0 \\ 2\alpha_2 + \beta_2 = 1, \end{cases}$$

hence $(\alpha_2, \beta_2) = (-1/2, 2)$. On the other hand, when $S_1 = 1$ we should have

$$\begin{cases} 2\alpha_1 + \beta_1 = 1 \\ \alpha_1 + \beta_1 = 0, \end{cases}$$

hence $(\alpha_1, \beta_1) = (1, -1)$.

b) When $S_1 = 2$, the price of the claim at $t = 1$ is

$$\alpha_2 S_2 + \beta_2 = 2\alpha_2 + \beta_2 = 1.$$

When $S_1 = 1$, the price of the claim at $t = 1$ is $\alpha_1 S_1 + \beta_1 = \alpha_1 + \beta_1 = 0$.

c) We build a portfolio based on α_0 units of stock and $\$\beta_0$ in cash. At time $t = 1$, we should have

$$\begin{cases} 2\alpha_0 + \beta_2 = 1 \\ \alpha_0 + \beta_2 = 0, \end{cases}$$

hence $(\alpha_0, \beta_0) = (1, -1)$.

d) The price of the claim C at time $t = 0$ is $\alpha_0 S_0 + \beta_0 = \alpha_0 + \beta_0 = 0$.

e) The probabilities $(p^*, q^*) = ((r-a)/(b-a), (b-r)/(b-a)) = (0, 1)$ are clearly risk-neutral in the sense of Definition 3.8, as they yield

$$\mathbb{E}^*[S_2 \mid S_1] = S_1 \qquad \text{and} \qquad \mathbb{E}^*[S_1 \mid S_0] = S_0.$$

with the risk-free rate $r = 0$. However, this does not form a risk-neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 3.9 when $\mathbb{P}(S_2 \neq S_1) > 0$ and/or $\mathbb{P}(S_1 \neq S_0) > 0$.

f) According to Theorem 3.6 this model allows for arbitrage opportunities as the unique available risk-neutral probability measure $\mathbb{P}^*$ may not be equivalent to the historical probability measure $\mathbb{P}$. Here, arbitrage opportunities can be easily implemented by purchasing the option at the price 0 of part (d)) while receiving a strictly positive payoff at maturity. More generally, arbitrage opportunities follow from the fact that the underlying price may increase with nonzero probability, without decreasing.

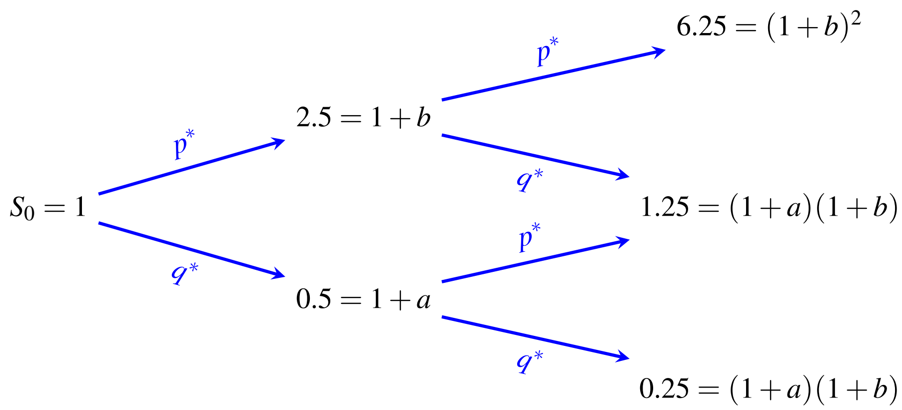
Exercise 4.6 We have

$$\begin{aligned}\pi_k(C) &= \frac{1}{(1+r)^{N-k}}\mathbb{E}^*[h(S_N) \mid \mathscr{F}_k] \\ &= \frac{1}{(1+r)^{N-k}}\mathbb{E}^*[a+bS_N \mid \mathscr{F}_k] \\ &= \frac{a}{(1+r)^{N-k}} + \frac{b}{(1+r)^{N-k}}\mathbb{E}^*[S_N \mid \mathscr{F}_k] \\ &= \frac{a(1+r)^k}{(1+r)^N} + bS_k, \qquad k = 0,1,\ldots,N.\end{aligned}$$

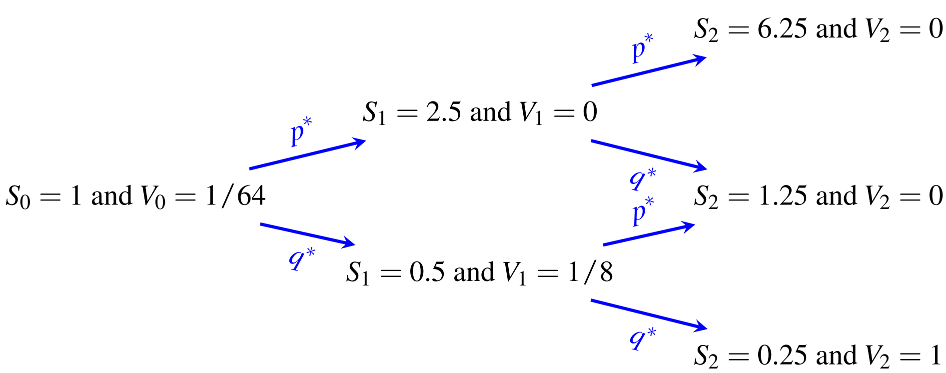
The hedging portfolio strategy is to hold β units of the underlying asset priced S_k and $a/(1+r)^N$ units of the riskless asset priced $(1+r)^k$ at time $k = 0,1,\ldots,N$.

Exercise 4.7

a) Taking $q^* = 1 - p^* = 1/4$, we find the binary tree



b) We find the binary tree



and the table

$S_0 = 1$ $V_0 = 1/64$	$S_1 = 2.5$, $V_1 = 0$	$S_2 = 6.25$ $V_2 = 0$
		$S_2 = 1.25$ $V_2 = 0$
	$S_1 = 0.5$, $V_1 = 1/8$	$S_2 = 0.25$ $V_2 = 1$

Table 13.4: CRR pricing tree.

c) Here we compute the hedging strategy from the option prices. When $S_1 = S_0(1+b)$ we clearly have $\xi_2 = \eta_2 = 0$. When $S_1 = S_0(1+a)$, the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+a)^2 + \eta_2(1+r)^2 = S_0(K-(1+a)(1+b)) \\ \\ \xi_2 S_0(1+b)(1+a) + \eta_2(1+r)^2 = 0 \end{cases}$$

hence

$$\xi_2 = -\frac{(K - S_0(1+a)(1+b))}{S_0(b-a)(1+a)} \quad \text{and} \quad \eta_2 = \frac{(K - S_0(1+a)(1+b))(1+b)}{S_0(b-a)(1+r)^2}.$$

Next, at time $t = 1$ the equation $\xi_1 S_1 + \eta_1 A_1 = V_1$ reads

$$\begin{cases} \xi_1 S_0(1+a) + \eta_1(1+r) = S_0 \dfrac{q^*(K-(1+a)(1+b))}{1+r}, \\ \\ \xi_1 S_0(1+b) + \eta_1(1+r) = 0 \end{cases}$$

which yields

$$\xi_1 = -\frac{q^*(K - S_0(1+a)(1+b))}{S_0(b-a)(1+r)} \quad \text{and} \quad \eta_1 = \frac{q^*(K - S_0(1+a)(1+b))(1+b)}{S_0(b-a)(1+r)^2}.$$

This can be summarized in the following table:

$S_0 = 1$ $V_0 = 1/64$ $\xi_1 = -1/16$ $\eta_1 = 5/64$	$S_1 = 2.5$, $V_1 = 0$ $\xi_2 = 0$, $\eta_2 = 0$	$S_2 = 6.25$ $V_2 = 0$
		$S_2 = 1.25$ $V_2 = 0$
	$S_1 = 0.5$, $V_1 = 1/8$ $\xi_2 = -1$, $\eta_2 = 5/16$	$S_2 = 0.25$ $V_2 = 1$

Table 13.5: CRR pricing and hedging tree.

If $S_1 = S_0(1+a)$ then there is a chance of being in the money at maturity and we need to short sell further by decreasing ξ_1 from $\xi_1 = -1/16$ to $\xi_2 = -1$. Note that the self-financing condition

$$\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1$$

is satisfied.

Exercise 4.8

a) Denoting $\widehat{S}_2$ the asset price at time 2 before the dividend is paid at the rate α, we find that the ex-dividend asset price S_2 after dividend payment is

$$S_2 = \widehat{S}_2 - \alpha \widehat{S}_2,$$

hence

$$\begin{aligned} V_2 &= \xi_2 S_2 + \eta_2 A_2 + \alpha \xi_2 \widehat{S}_2 \\ &= \xi_2 S_2 + \eta_2 A_2 + \alpha \xi_2 \frac{S_2}{1-\alpha} \\ &= \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2. \end{aligned}$$

b) Denoting $\widehat{S}_1$ the asset price at time 1 before the dividend is paid at the rate α, we find that the ex-dividend asset price S_1 after dividend payment is

$$S_1 = \widehat{S}_1 - \alpha \widehat{S}_1,$$

hence

$$\begin{aligned} V_1 &= \xi_1 S_1 + \eta_1 A_1 + \alpha \xi_1 \widehat{S}_1 \\ &= \xi_1 S_1 + \eta_1 A_1 + \alpha \xi_1 \frac{S_1}{1-\alpha} \\ &= \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1. \end{aligned}$$

c) If $S_1 = 3$ we have

$$V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases} \dfrac{9\xi_2}{1-\alpha} + \eta_2 2^2 = \$1 & \text{if } S_2 = 9, \\ \\ \dfrac{3\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3, \end{cases}$$

hence $(\xi_2, \eta_2) = ((1-\alpha)/6, -1/8)$.

If $S_1 = 1$ we have

$$V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases} \dfrac{3\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3, \\ \\ \dfrac{\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 1, \end{cases}$$

hence $(\xi_2, \eta_2) = (0, 0)$.

d) We have

$$\begin{cases} V_1 = \xi_2 S_1 + 2\eta_2 = 3 \times \dfrac{1-\alpha}{6} - 2 \times \dfrac{1}{8} = \dfrac{1-2\alpha}{4} & \text{if } S_1 = 3, \\ \\ V_1 = \xi_2 S_1 + 2\eta_2 = 0 \times 1 + 0 \times 2 = 0 & \text{if } S_1 = 1. \end{cases}$$

e) We have

$$V_1 = \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1 = \begin{cases} \dfrac{3\xi_1}{1-\alpha} + 2\eta_1 = \dfrac{1-2\alpha}{4} & \text{if } S_1 = 3, \\ \\ \dfrac{\xi_1}{1-\alpha} + 2\eta_1 = 0 & \text{if } S_1 = 1, \end{cases}$$

hence $(\xi_1, \eta_1) = ((\alpha-1)(2\alpha-1)/8, (2\alpha-1)/16)$.

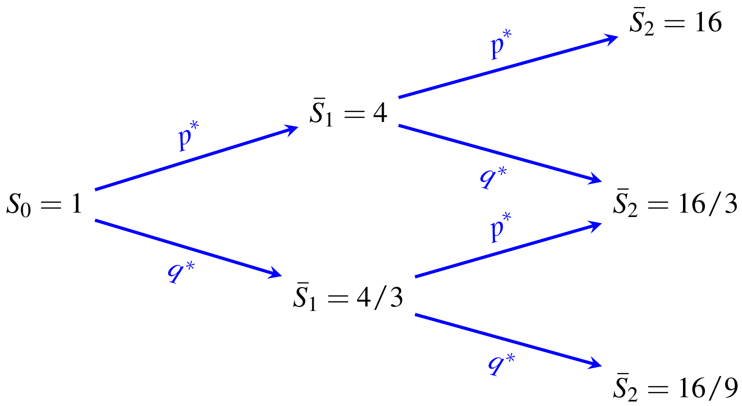
f) At time $k = 0$ we have

$$V_0 = \xi_1 S_0 + \eta_1 = \frac{(\alpha-1)(2\alpha-1)}{8} + \frac{2\alpha-1}{16} = \frac{(2\alpha-1)^2}{16}.$$

g) Multiplying the prices $(S_k)_{k=1,2}$ of the original tree by

$$\left(\frac{1}{1-\alpha}\right)^k = \left(\frac{1}{1-1/4}\right)^k = \left(\frac{4}{3}\right)^k,$$

we find the prices $(\bar{S}_k)_{k=1,2} = (S_k/(1-\alpha)^k)_{k=1,2}$ as in the following tree:



h) The market returns found in Question (g)) are $\bar{a} = 1/3$ and $\bar{b} = 3$, with $r = 1\%$. Therefore we have

$$p^* = \frac{r-a}{b-a} = \frac{1-1/3}{3-1/3} = \frac{1}{4} \quad \text{and} \quad q^* = \frac{3-1}{3-1/3} = \frac{b-r}{b-a} = \frac{3}{4}.$$

i) If $S_1 = 3$ we have

$$\frac{1}{1+r}\mathbb{E}^*\left[(S_2-K)^+ \mid \bar{S}_1 = 3\right] = \$1 \times \frac{p^*}{2} = \frac{1}{8},$$

which coincides with

$$V_1 = \xi_2 S_1 + 2\eta_2 = \frac{3}{8} - \frac{2}{8} = \frac{1}{8}.$$

If $S_1 = 1$ we have

$$\frac{1}{1+r}\mathbb{E}^*\left[(S_2-K)^+ \mid \bar{S}_1 = 1\right] = 0,$$

which coincides with

$$V_1 = \xi_2 S_1 + 2\eta_2 = 0.$$

j) At time $k = 0$ we have

$$\frac{1}{(1+r)^2}\mathbb{E}^*\left[(S_2-K)^+\right] = \frac{(p^*)^2}{(1+r)^2} = \frac{1}{64},$$

which coincides with

$$V_0 = \xi_1 S_0 + \eta_1 = \frac{3}{64} - \frac{1}{32} = \frac{1}{64}.$$

We also have

$$\frac{1}{1+r}\mathbb{E}^*\left[V_1\right] = \frac{p^*}{1+r} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}.$$

Exercise 4.9

a) Taking the risk-free interest rate r equal to zero, the binary call option can be priced as

$$\mathbb{E}^*[C] = \mathbb{E}^*[\mathbb{1}_{[K,\infty)}(S_N)] = \mathbb{P}^*(S_N \geqslant K) =: p^*$$

under the risk-neutral probability measure $\mathbb{P}^*$.

b) Investing $\$p^*$ by purchasing one binary call option yields a potential net return of

$$\begin{cases} \dfrac{\$1 - p^*}{p^*} = \dfrac{\$1}{p^*} - 1 & \text{if } S_N \geqslant K, \\[2ex] \dfrac{\$0 - p^*}{p^*} = -100\% & \text{if } S_N < K. \end{cases}$$

c) The corresponding expected return is

$$p^* \times \left(\frac{1}{p^*} - 1\right) + (1 - p^*) \times (-1) = 0.$$

d) The corresponding expected return is

$$p^* \times 0.86 + (1 - p^*) \times (-1) = p^* \times 1.86 - 1,$$

which will be *negative* if

$$p^* < \frac{1}{1.86} \simeq 0.538.$$

That means, the expected gain can be negative even if

$$0.538 > p^* = \mathbb{P}^*(S_N \geqslant K) > 0.5.$$

Similarly, the expected gain

$$(1 - p^*) \times 0.86 + p^* \times (-1) = 0.86 - p^* \times 1.86,$$

on binary put options will be *negative* if $1 - p^* > 1/1.86$, i.e. if

$$p^* > \frac{0.86}{1.86} \simeq 0.462.$$

That means, the expected gain can be negative even if $1 - 0.462 > \mathbb{P}^*(S_N < K) > 0.5$. In conclusion, the average gains of both call and put options will be negative if $p^* \in (0.462, 0.538)$.

Note that the average of call and put option gains will still be negative, as

$$\frac{p^* \times 1.86 - 1}{2} + \frac{0.86 - p^* \times 1.86}{2} = \frac{0.86 - 1}{2} < 0.$$

Exercise 4.10

a) Based on the price map of the put spread collar option:

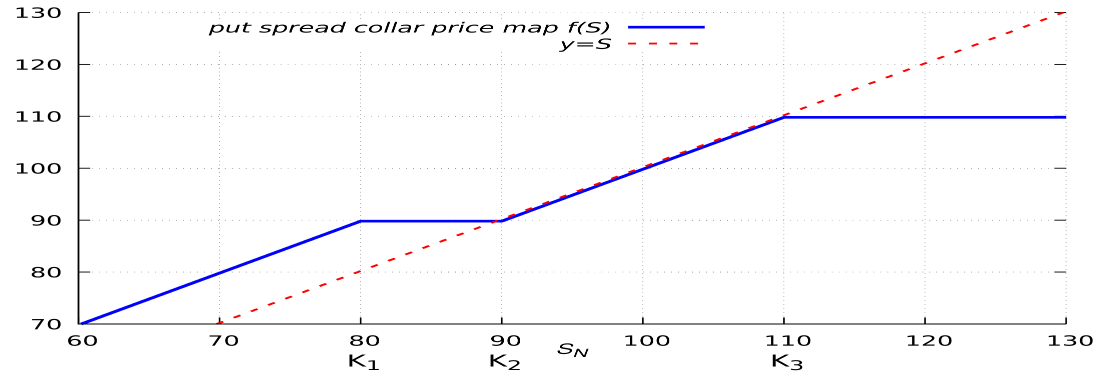


Figure S.7: Put spread collar price map.

we deduce the following payoff function graph of the put spread collar option in the next Figure S.8.

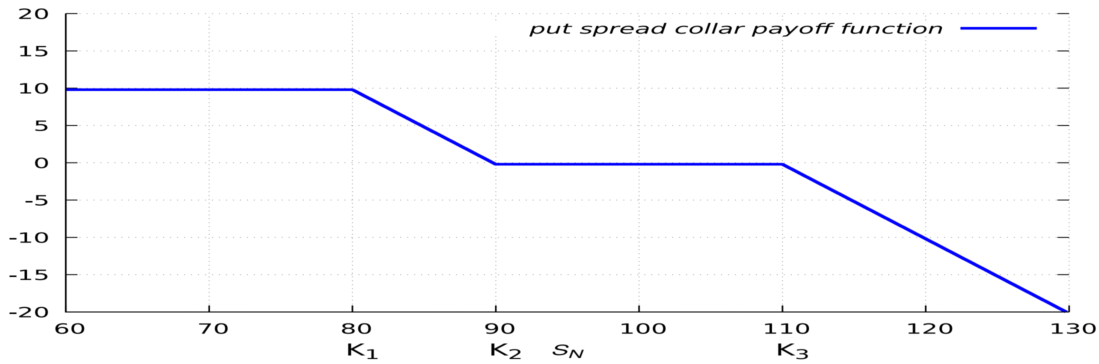


Figure S.8: Put spread collar payoff function.

b) The payoff function can be written as

$$
\begin{aligned}
&-(K_1-x)^+ + (K_2-x)^+ - (x-K_3)^+ \\
&\quad = \quad -(80-x)^+ + (90-x)^+ - (x-110)^+,
\end{aligned}
$$

see also http://optioncreator.com/stp7xy2.

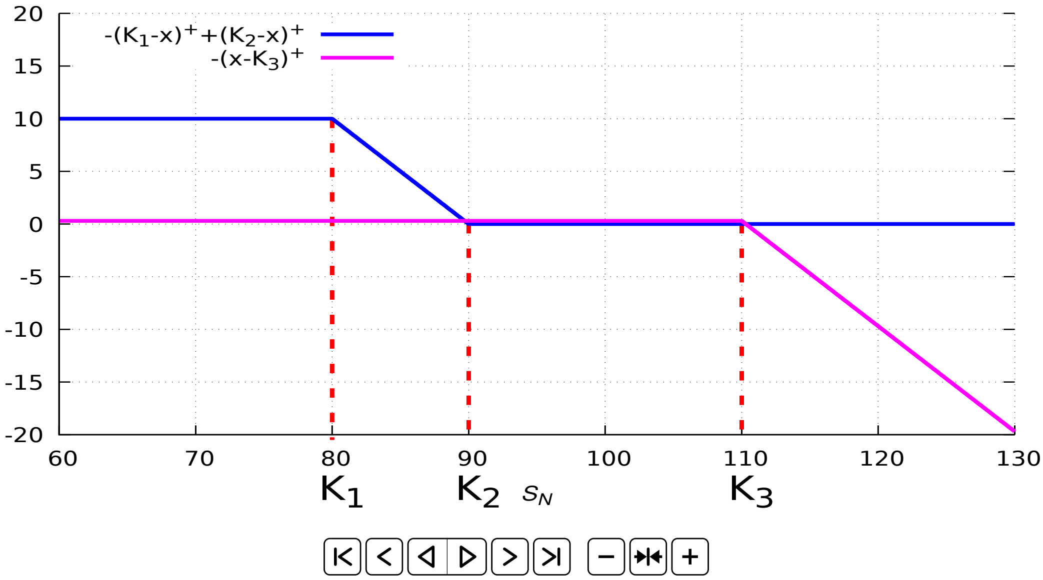


Figure S.9: Put spread collar payoff as a combination of call and put option payoffs.*

Hence this collar option payoff can be realized by

1. issuing (or selling) one *put option* with strike price $K_1 = 80$, and
2. purchasing one *put option* with strike price $K_2 = 90$, and
3. issuing (or selling) one *call option* with strike price $K_3 = 110$.

Exercise 4.11

a) Based on the price map of the call spread collar option:

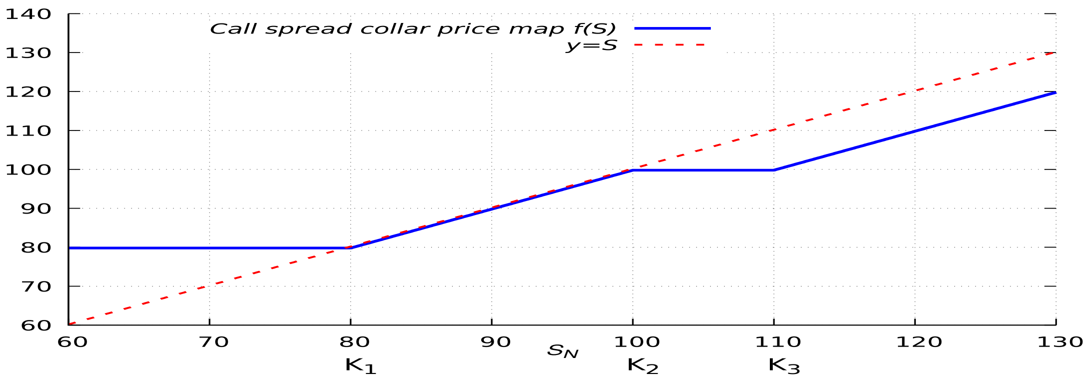


Figure S.10: Call spread collar price map.

we deduce the following payoff function graph of the call spread collar option in the next Figure S.11.

b) The payoff function can be written as

$$-(K_1 - x)^+ + (x - K_2)^+ - (x - K_3)^+$$
$$= \quad -(80 - x)^+ + (x - 100)^+ - (x - 110)^+,$$

see also http://optioncreator.com/st3e4cz.

*The animation works in Acrobat Reader on the entire pdf file.

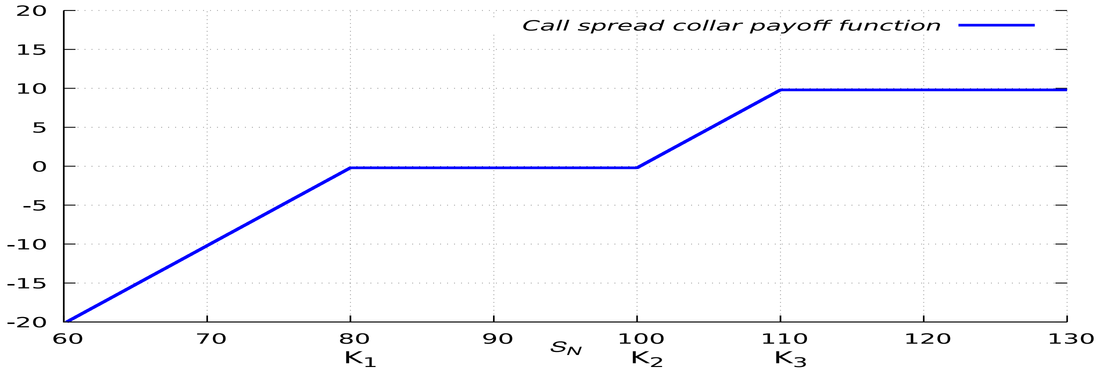


Figure S.11: Call spread collar payoff function.

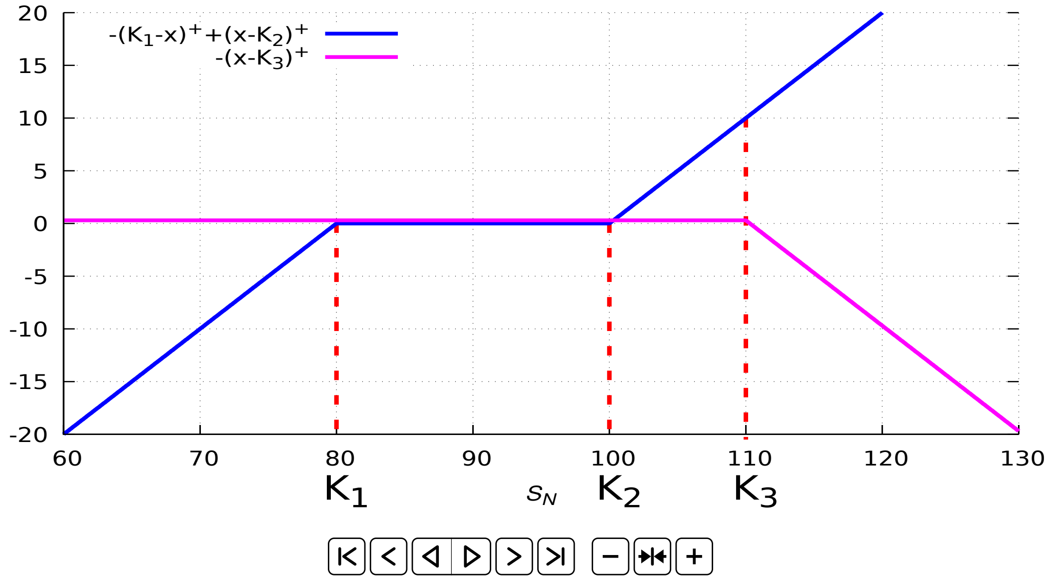


Figure S.12: Call spread collar payoff as a combination of call and put option payoffs.*

Hence this collar option payoff can be realized by

1. issuing (or selling) one *put option* with strike price $K_1 = 80$, and
2. purchasing one *call option* with strike price $K_2 = 100$, and
3. issuing (or selling) one *call option* with strike price $K_3 = 110$.

Exercise 4.12 We have

*The animation works in Acrobat Reader on the entire pdf file.

$$\begin{aligned}
\mathbb{E}^*\left[\phi\left(\frac{S_1+\cdots+S_N}{N}\right)\right] &\leqslant \mathbb{E}^*\left[\frac{\phi(S_1)+\cdots+\phi(S_N)}{N}\right] && \text{since } \phi \text{ is convex,}\\
&= \frac{\mathbb{E}^*[\phi(S_1)]+\cdots+\mathbb{E}^*[\phi(S_N)]}{N}\\
&= \frac{\mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathscr{F}_1])]+\cdots+\mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathscr{F}_N])]}{N} && \text{because } (S_n)_{n\in\mathbb{N}} \text{ is a martingale,}\\
&\leqslant \frac{\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_1]]+\cdots+\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_N]]}{N} && \text{by Jensen's inequality,}\\
&= \frac{\mathbb{E}^*[\phi(S_N)]+\cdots+\mathbb{E}^*[\phi(S_N)]}{N} && \text{by the tower property.}\\
&= \mathbb{E}^*[\phi(S_N)].
\end{aligned}$$

The above argument is implicitly using the fact that a convex function $\phi(S_n)$ of a martingale $(S_n)_{n\in\mathbb{N}}$ is itself a *sub*martingale, as

$$\phi(S_k) = \phi(\mathbb{E}^*[S_N \mid \mathscr{F}_k]) \leqslant \mathbb{E}^*[\phi(S_N) \mid \mathscr{F}_k], \quad k = 1,2,\ldots,N.$$

Exercise 4.13 (Exercise 3.5 continued).

a) The condition $V_N = C$ reads

$$\begin{cases} \eta_N\pi_N + \xi_N(1+a)S_{N-1} = (1+a)S_{N-1} - K \\ \\ \eta_N\pi_N + \xi_N(1+b)S_{N-1} = (1+b)S_{N-1} - K, \end{cases}$$

from which we deduce the (static) hedging strategy $\xi_N = 1$ and $\eta_N = -K(1+r)^{-N}/\pi_0$.

b) We have

$$\begin{cases} \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+a)S_{N-2} = \eta_N\pi_{N-1} + \xi_N(1+a)S_{N-2} \\ \\ \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+b)S_{N-2} = \eta_N\pi_{N-1} + \xi_N(1+b)S_{N-2}, \end{cases}$$

which yields $\xi_{N-1} = \xi_N = 1$ and $\eta_{N-1} = \eta_N = -K(1+r)^{-N}/\pi_0$. Similarly, solving the self-financing condition

$$\begin{cases} \eta_t\pi_t + \xi_t(1+a)S_{t-1} = \eta_{t+1}\pi_t + \xi_{t+1}(1+a)S_{t-1} \\ \\ \eta_t\pi_t + \xi_t(1+b)S_{t-1} = \eta_{t+1}\pi_t + \xi_{t+1}(1+b)S_{t-1} \end{cases}$$

at time t yields

$$\xi_t = 1 \quad \text{and} \quad \eta_t = -(1+r)^{-N}\frac{K}{\pi_0}, \qquad t = 1,2,\ldots,N.$$

c) We have

$$\begin{aligned}
\pi_t(C) &= V_t\\
&= \eta_t\pi_t + \xi_t S_t\\
&= S_t - K(1+r)^{-N}\frac{\pi_t}{\pi_0}\\
&= S_t - K(1+r)^{-(N-t)}.
\end{aligned}$$

d) For all $t = 0, 1, \ldots, N$ we have

$$\begin{aligned}(1+r)^{-(N-t)} \mathbb{E}^*[C \mid \mathscr{F}_t] &= (1+r)^{-(N-t)} \mathbb{E}^*[S_N - K \mid \mathscr{F}_t],\\ &= (1+r)^{-(N-t)} \mathbb{E}^*[S_N \mid \mathscr{F}_t] - (1+r)^{-(N-t)} \mathbb{E}^*[K \mid \mathscr{F}_t]\\ &= (1+r)^{-(N-t)}(1+r)^{N-t} S_t - K(1+r)^{-(N-t)}\\ &= S_t - K(1+r)^{-(N-t)}\\ &= V_t = \pi_t(C).\end{aligned}$$

For a future contract expiring at time N we take $K = S_0(1+r)^N$ and the contract is usually quoted at time t using the forward price $(1+r)^{N-t}(S_t - K(1+r)^{N-t}) = (1+r)^{N-t}S_t - K = (1+r)^{N-t}S_t - S_0(1+r)^N$, or simply using $(1+r)^{N-t}S_t$. Future contracts are "marked to market" at each time step $t = 1, 2, \ldots, N$ via a positive or negative cash flow exchange $(1+r)^{N-t}S_t - (1+r)^{N-t+1}S_{t-1}$ from the seller to the buyer, ensuring that the absolute difference $|(1+r)^{N-t}S_t - K|$ has been credited to the buyer's account if it is positive, or to the seller's account if it is negative.

Exercise 4.14

a) We write

$$V_N = \begin{cases} \xi_N S_{N-1}(1+1/2) + \eta_N = (S_{N-1}(1+1/2))^2 \\ \\ \xi_N S_{N-1}(1-1/2) + \eta_N = (S_{N-1}(1-1/2))^2, \end{cases}$$

which yields

$$\begin{cases} \xi_N = 2S_{N-1} \\ \\ \eta_N = -3(S_{N-1})^2/4. \end{cases}$$

b) We have

$$\begin{aligned}\mathbb{E}^*[(S_N)^2 \mid \mathscr{F}_{N-1}] &= p^*(S_{N-1})^2(1+1/2)^2 + (1-p^*)(S_{N-1})^2(1-1/2)^2\\ &= \frac{1}{2}(S_{N-1})^2\left((1+1/2)^2 + (1-1/2)^2\right)\\ &= 5(S_{N-1})^2/4.\end{aligned}$$

c) We have

$$\begin{aligned}\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 &= \begin{cases} \xi_{N-1}S_{N-2}(1+1/2) + \eta_{N-1} \\ \\ \xi_{N-1}S_{N-2}(1-1/2) + \eta_{N-1} \end{cases}\\ &= V_{N-1}\\ &= 5(S_{N-1})^2/4\\ &= \begin{cases} 5(S_{N-2}(1+1/2))^2/4 \\ \\ 5(S_{N-2}(1-1/2))^2/4, \end{cases}\end{aligned}$$

hence

$$\begin{cases} \xi_{N-1} = 5S_{N-2}/2 \\ \\ \eta_{N-1} = -15(S_{N-2})^2/16. \end{cases}$$

d) We have

$$\begin{aligned}\xi_{N-1}S_{N-1} + \eta_{N-1}A_0 &= 5S_{N-2}S_{N-1}/2 - 15(S_{N-2})^2/16\\ &= \begin{cases} 5(S_{N-2})^2(1+1/2)/2 - 15(S_{N-2})^2/16 \\ \\ 5(S_{N-2})^2(1-1/2)/2 - 15(S_{N-2})^2/16 \end{cases}\end{aligned}$$

$$= \begin{cases} 15(S_{N-2})^2/4 - 15(S_{N-2})^2/16 \\ \\ 5(S_{N-2})^2 - 15(S_{N-2})^2/16 \end{cases}$$

$$= \begin{cases} 45(S_{N-2})^2/16 \\ \\ 5(S_{N-2})^2/16, \end{cases}$$

and on the other hand,

$$\begin{aligned} \xi_N S_{N-1} + \eta_N A_0 &= 2(S_{N-1})^2 - 3(S_{N-1})^2/4 \\ &= \begin{cases} 2(S_{N-2})^2(1+1/2)^2 - 3(S_{N-2})^2(1+1/2)^2/4 \\ \\ 2(S_{N-2})^2(1-1/2)^2 - 3(S_{N-2})^2(1-1/2)^2/4 \end{cases} \\ &= \begin{cases} 45(S_{N-2})^2/16 \\ \\ 5(S_{N-2})^2/16. \end{cases} \end{aligned}$$

Remark: We could also determine (ξ_{N-1}, η_{N-1}) as in Proposition 4.6, from (ξ_N, η_N) and the self-financing condition

$$\xi_{N-1} S_{N-1} + \eta_{N-1} A_0 = \xi_N S_{N-1} + \eta_N A_{N-1},$$

as

$$\begin{aligned} \xi_{N-1} S_{N-1} + \eta_{N-1} A_0 &= \begin{cases} \xi_{N-1} S_{N-2}(1+1/2) + \eta_{N-1} \\ \\ \xi_{N-1} S_{N-2}(1-1/2) + \eta_{N-1} \end{cases} \\ &= \xi_N S_{N-1} + \eta_N A_0 \\ &= 2(S_{N-1})^2 - 3(S_{N-1})^2/4 \\ &= \begin{cases} 2(S_{N-2})^2(1+1/2)^2 - 3(S_{N-2})^2(1+1/2)^2/4 \\ \\ 2(S_{N-2})^2(1-1/2)^2 - 3(S_{N-2})^2(1-1/2)^2/4, \end{cases} \end{aligned}$$

which recovers $\xi_{N-1} = 5S_{N-2}/2$ and $\eta_{N-1} = -15(S_{N-2})^2/16$.

Exercise 4.15

a) By Theorem 3.8 this model admits a unique risk-neutral probability measure $\mathbb{P}^*$ because $a < r < b$, and from (3.16) we have

$$\mathbb{P}^*(R_t = a) = \frac{b-r}{b-a} = \frac{0.07-0.05}{0.07-0.02},$$

and

$$\mathbb{P}(R_t = b) = \frac{r-a}{b-a} = \frac{0.05-0.02}{0.07-0.02},$$

$t = 1, 2, \ldots, N$.

b) There are no arbitrage opportunities in this model, due to the existence of a risk-neutral probability measure.

c) This market model is complete because the risk-neutral probability measure is unique.

d) We have

$$C = (S_N)^2,$$

hence

$$\widetilde{C} = \frac{(S_N)^2}{(1+r)^N} = h(X_N),$$

with
$$h(x) = x^2(1+r)^N.$$

Now we have
$$\widetilde{V}_t = \tilde{v}(t, X_t),$$

where the function $v(t,x)$ is given from Proposition 4.4 by

$$\begin{aligned}
\tilde{v}(t,x) = & \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
& \times (p^*)^k (q^*)^{N-t-k} h\left(x\left(\frac{1+b}{1+r}\right)^k \left(\frac{1+a}{1+r}\right)^{N-t-k}\right) \\
= \; & x^2(1+r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
& \times (p^*)^k (q^*)^{N-t-k} \left(\frac{1+b}{1+r}\right)^{2k} \left(\frac{1+a}{1+r}\right)^{2(N-t-k)} \\
= \; & x^2(1+r)^N \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
& \times \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2}\right)^k \left(\frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t-k} \\
= \; & x^2(1+r)^N \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} + \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t} \\
= \; & \frac{x^2\left((r-a)(1+b)^2 + (b-r)(1+a)^2\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
= \; & \frac{x^2\left((r-a)(1+2b+b^2) + (b-r)(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
= \; & \frac{x^2\left(r(1+2b+b^2) - a(1+2b+b^2) + b(1+2a+a^2) - r(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
= \; & x^2 \frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-2t}}.
\end{aligned}$$

e) We have

$$\begin{aligned}
\xi_t^1 & = \frac{v\left(t, \frac{1+b}{1+r}X_{t-1}\right) - v\left(t, \frac{1+a}{1+r}X_{t-1}\right)}{X_{t-1}(b-a)/(1+r)} \\
& = X_{t-1} \frac{\left(\frac{1+b}{1+r}\right)^2 - \left(\frac{1+a}{1+r}\right)^2}{(b-a)/(1+r)} \frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-2t}} \\
& = S_{t-1}(a+b+2)\frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-t}}, \qquad t = 1,2,\ldots,N,
\end{aligned}$$

representing the quantity of the risky asset to be present in the portfolio at time t. On the other hand we have

$$\begin{aligned}
\xi_t^0 & = \frac{V_t - \xi_t^1 X_t}{X_t^0} \\
& = \frac{V_t - \xi_t^1 X_t}{\pi_0} \\
& = X_t(1+r(a+b+2)-ab)^{N-t} \frac{X_t - X_{t-1}(a+b+2)/(1+r)}{\pi_0(1+r)^{N-2t}}
\end{aligned}$$

$$= S_t(1+r(a+b+2)-ab)^{N-t}\frac{S_t - S_{t-1}(a+b+2)}{\pi_0(1+r)^N}$$
$$= -(S_{t-1})^2(1+r(a+b+2)-ab)^{N-t}\frac{(1+a)(1+b)}{\pi_0(1+r)^N},$$

$t = 1,2,\ldots,N.$

f) Let us check that the portfolio is self-financing. We have

$$\begin{aligned}
\bar{\xi}_{t+1}\cdot\bar{S}_t &= \xi^0_{t+1}S^0_t + \xi^1_{t+1}S^1_t \\
&= -(S_t)^2(1+r(a+b+2)-ab)^{N-t-1}\frac{(1+a)(1+b)}{\pi_0(1+r)^N}S^0_t \\
&\quad +(S_t)^2(a+b+2)\frac{(1+r(a+b+2)-ab)^{N-t-1}}{(1+r)^{N-t-1}} \\
&= (S_t)^2\frac{(1+r(a+b+2)-ab)^{N-t-1}}{(1+r)^{N-t}} \\
&\quad \times((a+b+2)(1+r)-(1+a)(1+b)) \\
&= \frac{1}{(1+r)^{N-3t}}(X_t)^2(1+r(a+b+2)-ab)^{N-t} \\
&= (1+r)^tV_t \\
&= \bar{\xi}_t\cdot\bar{S}_t, \qquad t=1,2,\ldots,N.
\end{aligned}$$

Exercise 4.16

a) We have

$$\begin{aligned}
V_t &= \xi_tS_t + \eta_t\pi_t \\
&= \xi_t(1+R_t)S_{t-1} + \eta_t(1+r)\pi_{t-1}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}^*[R_t|\mathcal{F}_{t-1}] &= a\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + b\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} \\
&= b\frac{r}{b-a} - a\frac{r}{b-a} \\
&= r.
\end{aligned}$$

c) By the result of Question (a)), we have

$$\begin{aligned}
\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] &= \mathbb{E}^*[\xi_t(1+R_t)S_{t-1} \mid \mathcal{F}_{t-1}] + \mathbb{E}^*[\eta_t(1+r)\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= \xi_tS_{t-1}\,\mathbb{E}^*[1+R_t \mid \mathcal{F}_{t-1}] + (1+r)\,\mathbb{E}^*[\eta_t\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= (1+r)\xi_tS_{t-1} + (1+r)\eta_t\pi_{t-1} \\
&= (1+r)\xi_{t-1}S_{t-1} + (1+r)\eta_{t-1}\pi_{t-1} \\
&= (1+r)V_{t-1},
\end{aligned}$$

where we used the self-financing condition.

d) We have

$$\begin{aligned}
V_{t-1} &= \frac{1}{1+r}\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \\
&= \frac{3}{1+r}\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1+r}\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= \frac{1}{1+0.15}\left(3\frac{0.25-0.15}{0.25-0.05} + 8\frac{0.15-0.05}{0.25-0.05}\right) \\
&= \frac{1}{1.15}\left(\frac{3}{2}+\frac{8}{2}\right) \\
&= 4.78.
\end{aligned}$$

Chapter 5

Exercise 5.1 If $0 \leqslant s \leqslant t$, using the facts that $\mathbb{E}[B_t] = 0$ and $\mathbb{E}\left[B_t^2\right] = 0$, $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
\mathbb{E}[B_t B_s] &= \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}\left[B_s^2\right] \\
&= \mathbb{E}[(B_t - B_s)]\,\mathbb{E}[B_s] + \mathbb{E}\left[B_s^2\right] \\
&= 0 + s \\
&= s,
\end{aligned}$$

and similarly we obtain $\mathbb{E}[B_t B_s] = t$ when $0 \leqslant t \leqslant s$, hence in general we have

$$\mathbb{E}[B_t B_s] = \min(s,t), \qquad s,t \in \mathbb{R}_+.$$

Exercise 5.2 We need to check whether the four properties of the definition of Brownian motion are satisfied.

a) Checking Conditions 1-2-3 are easily satisfied using the time shift $t \longmapsto c+t$. As for Condition 4, $B_{c+t} - B_{c+s}$ clearly has the centered Gaussian distribution with variance $c+t-(c-s) = t-s$. We conclude that $(B_{c+t} - B_c)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

b) Similarly, checking Conditions 1-2-3 does not pose any particular problem using the time change $t \longmapsto t/c^2$. As for Condition 4, $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with

$$\begin{aligned}
\mathrm{Var}\left(c\left(B_{t/c^2} - B_{s/c^2}\right)\right) &= c^2 \mathrm{Var}\left(B_{t/c^2} - B_{s/c^2}\right) \\
&= (t-s)c^2/c^2 \\
&= t-s.
\end{aligned}$$

As a consequence, $\left(B_{t/c^2}\right)_{t \in \mathbb{R}_+}$ is a standard Brownian motion.

c) We note that B_{ct^2} is a centered Gaussian random variable with variance ct^2 - not t, hence $(B_{ct^2})_{t \in \mathbb{R}_+}$ is not a standard Brownian motion.

d) This process does not have independent increments, hence it cannot be a Brownian motion. For example, by (5.1) we have

$$\begin{aligned}
&\mathbb{E}\left[\left(B_t + B_{t/2} - \left(B_s + B_{s/2}\right)\right)\left(B_s + B_{s/2}\right)\right] \\
&= \mathbb{E}\left[B_t B_s + B_t B_{s/2} + B_{t/2} B_s + B_{t/2} B_{s/2}\right] \\
&\quad - \mathbb{E}\left[B_s B_s + B_s B_{s/2} + B_{s/2} B_s + B_{s/2} B_{s/2}\right] \\
&= s + \frac{s}{2} + s + \frac{s}{2} - s - \frac{s}{2} - \frac{s}{2} - \frac{s}{2} \\
&= \frac{s}{2},
\end{aligned}$$

which differs from 0, hence the two increments are not independent. Indeed, independence of $B_t + B_{t/2} - \left(B_s + B_{s/2}\right)$ and $B_s + B_{s/2}$ would yield

$$\begin{aligned}
&\mathbb{E}\left[\left(B_t + B_{t/2} - \left(B_s + B_{s/2}\right)\right)\left(B_s + B_{s/2}\right)\right] \\
&= \mathbb{E}\left[B_t + B_{t/2} - \left(B_s + B_{s/2}\right)\right]\mathbb{E}\left[\left(B_s + B_{s/2}\right)\right] \\
&= 0.
\end{aligned}$$

Exercise 5.3 We have

$$\int_0^T 2dB_t = 2(B_T - B_0) = 2B_T,$$

which has a Gaussian distribution with mean 0 and variance $4T$. On the other hand,

$$\int_0^T \left(2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)\right)dB_t = 2(B_{T/2} - B_0) + (B_T - B_{T/2})$$

$$= B_T + B_{T/2},$$

which has a Gaussian distribution with mean 0 and variance

$$\begin{aligned}
\mathrm{Var}[B_T + B_{T/2}] &= \mathrm{Var}[(B_T - B_{T/2}) + 2B_{T/2}] \\
&= \mathrm{Var}[B_T - B_{T/2}] + 4\,\mathrm{Var}[B_{T/2}] \\
&= \frac{T}{2} + \frac{4T}{2} \\
&= \frac{5T}{2}.
\end{aligned}$$

Equivalently, using the Itô isometry (5.7), we have

$$\begin{aligned}
&\mathrm{Var}\left[\left(\int_0^T (2\times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t))dB_t\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^T (2\times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t))dB_t\right)\right] \\
&= \int_0^T \big(2\times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)\big)dt \\
&= 4\int_0^{T/2} dt + \int_{T/2}^T dt \\
&= \frac{5T}{2}.
\end{aligned}$$

Exercise 5.4 By Proposition 5.4, the stochastic integral $\int_0^{2\pi} \sin(t)\,dB_t$ has a Gaussian distribution with mean 0 and variance

$$\int_0^{2\pi} \sin^2(t)dt = \frac{1}{2}\int_0^{2\pi}(1-\cos(2t))dt = \pi.$$

Exercise 5.5 By the Itô formula (5.27), we have

$$\begin{aligned}
d(f(t)B_t) &= f(t)dB_t + B_t df(t) + df(t)\bullet dB_t \\
&= f(t)dB_t + B_t f'(t)dt + f'(t)dt \bullet dB_t \\
&= f(t)dB_t + B_t f'(t)dt,
\end{aligned}$$

and by integration on both sides we get

$$\begin{aligned}
\int_0^T f(t)dB_t + \int_0^T B_t f'(t)dt &= \int_0^T d(f(t)B_t) \\
&= f(T)B_T - f(0)B_0 \\
&= 0,
\end{aligned}$$

since $f(T) = 0$ and $B_0 = 0$, hence the conclusion. Note that this result can also be obtained by integration by parts.

Exercise 5.6

a) The stochastic integral $\int_0^1 t^2 dB_t$ is a centered Gaussian random variable with variance

$$\mathbb{E}\left[\left(\int_0^1 t^2 dB_t\right)^2\right] = \int_0^1 t^4 dt = \frac{1}{5}.$$

b) The stochastic integral $\int_0^1 t^{-1/2}dB_t$ has the variance

$$\mathbb{E}\left[\left(\int_0^1 t^{-1/2}dB_t\right)^2\right] = \int_0^1 \frac{1}{t}dt = +\infty.$$

In fact, the stochastic integral $\int_0^1 t^{-1/2}dB_t$ does not exist as a random variable in $L^2(\Omega)$ because the function $t \mapsto t^{-1/2}$ is not in $L^2([0,1])$.

Remark. Writing Relation (5.3.7) page 190 with $f(t) = t^{-1/2}$ gives

$$\int_0^1 t^{-1/2}dB_t = \frac{B_T}{\sqrt{T}} + \frac{1}{2}\int_0^1 t^{-3/2}B_t dt,$$

however this is only a formal statement as f is not in $\mathscr{C}^1([0,1])$. Informally, we can check that the term $\displaystyle\int_0^T t^{-3/2}B_t dt$ has the infinite variance

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T B_t f'(t)dt\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T B_t f'(t)dt\right)\left(\int_0^T B_s f'(s)ds\right)\right] \\
&= \mathbb{E}\left[\int_0^T\int_0^T B_s B_t f'(s)f'(t)dsdt\right] \\
&= \int_0^T\int_0^T f'(s)f'(t)\,\mathbb{E}[B_s B_t]dsdt \\
&= \frac{1}{4}\int_0^T\int_0^T s^{-3/2}t^{-3/2}\min(s,t)dsdt \\
&= \frac{1}{4}\int_0^T t^{-3/2}\int_0^t s^{-5/2}dsdt + \frac{1}{4}\int_0^T t^{-5/2}\int_t^T s^{-3/2}dsdt \\
&= \frac{1}{4}\int_0^T t^{-3/2}\int_0^t s^{-5/2}dsdt + \frac{1}{4}\int_0^T t^{-5/2}\int_t^T s^{-3/2}dsdt \\
&= +\infty,
\end{aligned}$$

where we used Relation (5.1) or the result of Exercise 5.1

Exercise 5.7

a) By Proposition 5.4, the probability distribution of X_n is Gaussian with mean zero and variance

$$\begin{aligned}
\mathrm{Var}[X_n] &= \mathbb{E}\left[\left(\int_0^{2\pi}\sin(nt)dB_t\right)^2\right] \\
&= \int_0^{2\pi}\sin^2(nt)dt \\
&= \frac{1}{2}\int_0^{2\pi}\cos(0)dt - \frac{1}{2}\int_0^{2\pi}\cos(2nt)dt \\
&= \pi, \qquad n \geqslant 1.
\end{aligned}$$

b) The random variables $(X_n)_{n\geqslant 1}$ have same Gaussian distribution, and they are pairwise independent as from Corollary 5.5 we have

$$\begin{aligned}
\mathbb{E}[X_n X_m] &= \mathbb{E}\left[\int_0^{2\pi}\sin(nt)dB_t\int_0^{2\pi}\sin(mt)dB_t\right] \\
&= \int_0^{2\pi}\sin(nt)\sin(mt)dt \\
&= \frac{1}{2}\int_0^{2\pi}\cos((n-m)t)dt - \frac{1}{2}\int_0^{2\pi}\cos((n+m)t)dt \\
&= 0
\end{aligned}$$

and the vector (X_n, X_m) is jointly Gaussian, for $n,m \geqslant 1$ such that $n \neq m$. Note that this condition implies independence only when the random variables have a Gaussian distribution.

Exercise 5.8 We have $X_t = f(B_t)$ with $f(x) = \sin^2 x$, $f'(x) = 2\sin x \cos x = \sin(2x)$, and $f''(x) = 2\cos(2x)$, hence

$$\begin{aligned} dX_t &= d\sin^2(B_t) \\ &= df(B_t) \\ &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \\ &= \sin(2B_t)dB_t + \cos(2B_t)dt. \end{aligned}$$

Exercise 5.9

a) We have

$$\begin{aligned} \mathbb{E}\left[B_T^3\right] &= \mathbb{E}\left[\int_0^T dB_t \left(T + 2\int_0^T B_t dB_t\right)\right] \\ &= T\,\mathbb{E}\left[\int_0^T dB_t\right] + 2\,\mathbb{E}\left[\int_0^T dB_t \int_0^T B_t dB_t\right] \\ &= 2\,\mathbb{E}\left[\int_0^T B_t dt\right] \\ &= 2\int_0^T \mathbb{E}[B_t]dt \\ &= 0. \end{aligned}$$

We also have

$$\begin{aligned} \mathbb{E}\left[B_T^4\right] &= \mathbb{E}\left[\left(T + 2\int_0^T B_t dB_t\right)^2\right] \\ &= \mathbb{E}\left[T^2 + 2T\int_0^T B_t dB_t + 4\left(\int_0^T B_t dB_t\right)^2\right] \\ &= T^2 + 2T\,\mathbb{E}\left[\int_0^T B_t dB_t\right] + 4\,\mathbb{E}\left[\left(\int_0^T B_t dB_t\right)^2\right] \\ &= T^2 + 4\,\mathbb{E}\left[\int_0^T |B_t|^2 dt\right] \\ &= T^2 + 4\int_0^T \mathbb{E}\left[|B_t|^2\right]dt \\ &= T^2 + 4\int_0^T t dt \\ &= T^2 + 4\frac{T^2}{2} \\ &= 3T^2. \end{aligned}$$

b) If $X \simeq \mathcal{N}(0, \sigma^2)$, we have $X \simeq B_T$ with $\sigma^2 = T$, hence the answer to Question (a)) yields

$$\mathbb{E}\left[X^3\right] = 0 \quad \text{and} \quad \mathbb{E}\left[X^4\right] = 3\sigma^4.$$

We note that those moments can be recovered directly from the Gaussian probability density function as

$$\mathbb{E}\left[X^3\right] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^3\, \mathrm{e}^{-x^2/(2\sigma^2)}dx = 0$$

and

$$\mathbb{E}\left[X^4\right] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^4\, \mathrm{e}^{-x^2/(2\sigma^2)}dx = 3\sigma^4.$$

Exercise 5.10

a) By the Itô formula, we have

$$
\begin{aligned}
dS_t^p &= pS_t^{p-1}dS_t^p + \frac{p(p-1)}{2}S_t^{p-2}dS_t \cdot dS_t \\
&= pS_t^{p-1}(rS_t dt + \sigma S_t dB_t) + \frac{p(p-1)}{2}S_t^{p-2}(rS_t dt + \sigma S_t dB_t)\cdot(rS_t dt + \sigma S_t dB_t) \\
&= prS_t^p dt + \sigma p S_t^p dB_t + \sigma^2 \frac{p(p-1)}{2}S_t^p dt \\
&= \left(pr + \sigma^2\frac{p(p-1)}{2}\right)S_t^p dt + \sigma p S_t^p dB_t.
\end{aligned}
$$

b) By the Girsanov Theorem, letting

$$
\nu := \frac{1}{p\sigma}\left((p-1)r + \sigma^2\frac{p(p-1)}{2}\right),
$$

the drifted process

$$
\widehat{B}_t := B_t + \nu t, \qquad 0 \leqslant t \leqslant T,
$$

is a standard (centered) Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$
d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right) d\mathbb{P}(\omega).
$$

Therefore, the differential of $(S_t^p)_{t\in\mathbb{R}_+}$ can be written as

$$
\begin{aligned}
dS_t^p &= \left(pr + \sigma^2\frac{p(p-1)}{2}\right)S_t^p dt + \sigma p S_t^p dB_t \\
&= (r + p\sigma\nu)S_t^p dt + \sigma p S_t^p dB_t \\
&= rS_t^p dt + \sigma p S_t^p(dB_t + \nu dt) \\
&= rS_t^p dt + \sigma p S_t^p d\widehat{B}_t,
\end{aligned}
$$

hence the discounted process $\widetilde{S}_t := e^{-rt}S_t^p$ satisfies $d\widetilde{S}_t = \sigma p \widetilde{S}_t d\widehat{B}_t$, and $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is a martingale under the probability measure $\mathbb{Q}$.

Exercise 5.11 Taking expectation on both sides of (5.37) shows that

$$
\begin{aligned}
0 &= \mathbb{E}\left[(B_T)^3\right] \\
&= \mathbb{E}\left[C + \int_0^T \zeta_{t,T} dB_t\right] \\
&= C + \mathbb{E}\left[\int_0^T \zeta_{t,T} dB_t\right] \\
&= 0
\end{aligned}
$$

by (5.17), hence $C = 0$. Next, applying Itô's formula to the function $f(x) = x^3$ shows that

$$
\begin{aligned}
(B_T)^3 &= f(B_T) \\
&= f(B_0) + \int_0^T f'(B_t)dB_t + \frac{1}{2}\int_0^T f''(B_t)dt \\
&= 3\int_0^T B_t^2 dB_t + 3\int_0^T B_t dt.
\end{aligned}
$$

By the integration by parts formula (5.11) applied to $f(t) = t$, we find

$$
\int_0^T B_t dt = TB_T - \int_0^T t dB_t = \int_0^T (T-t)dB_t,
$$

hence

$$\begin{aligned}(B_T)^3 &= 3\int_0^T B_t^2 dB_t + 3\left(TB_T - \int_0^T t dB_t\right)\\ &= 3\int_0^T (T-t+B_t^2) dB_t,\end{aligned}$$

and we find $\zeta_{t,T} = 3(T-t+B_t^2)$, $t \in [0,T]$. This type of stochastic integral decomposition can be used for option hedging, cf. Section 8.5.

Exercise 5.12

a) We have

$$\begin{aligned}E\left[e^{\int_0^T f(s)dB_s} \Big| \mathscr{F}_t\right] &= e^{\int_0^t f(s)dB_s} E\left[e^{\int_t^T f(s)dB_s} \Big| \mathscr{F}_t\right]\\ &= e^{\int_0^t f(s)dB_s} E\left[e^{\int_t^T f(s)dB_s}\right]\\ &= \exp\left(\int_0^t f(s)dB_s + \frac{1}{2}\int_t^T |f(s)|^2 ds\right), \qquad \text{(A.13)}\end{aligned}$$

$0 \leqslant t \leqslant T$, where we used the Gaussian moment generating function $\mathbb{E}\left[e^X\right] = e^{\sigma^2/2}$ for $X \simeq \mathscr{N}(0,\sigma^2)$ and the fact that $\int_t^T f(s)dB_s \simeq \mathscr{N}\left(0, \int_t^T f^2(s)ds\right)$ by Proposition 5.4.

b) We have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right) \Big| \mathscr{F}_u\right]\\ &= \exp\left(-\frac{1}{2}\int_0^t f^2(s)ds\right)\mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s\right) \Big| \mathscr{F}_u\right]\\ &= \exp\left(-\frac{1}{2}\int_0^t f^2(s)ds\right)\mathbb{E}\left[\exp\left(\int_0^u f(s)dB_s + \int_u^t f(s)dB_s\right) \Big| \mathscr{F}_u\right]\\ &= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right)\mathbb{E}\left[\exp\left(\int_u^t f(s)dB_s\right) \Big| \mathscr{F}_u\right]\\ &= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right)\mathbb{E}\left[\exp\left(\int_u^t f(s)dB_s\right)\right]\\ &= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds + \frac{1}{2}\int_u^t f^2(s)ds\right)\\ &= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^u f^2(s)ds\right), \qquad 0 \leqslant u \leqslant t.\end{aligned}$$

This result can also be obtained by directly applying (A.13).

c) We apply the conclusion of Question (b)) to the constant function $f(t) := \sigma$, $t \in \mathbb{R}_+$.

Exercise 5.13 We have

$$\begin{aligned}E\left[\exp\left(\beta\int_0^T B_t dB_t\right)\right] &= E\left[\exp\left(\beta(B_T^2 - T)/2\right)\right]\\ &= e^{-\beta T/2} E\left[e^{\beta(B_T)^2/2}\right]\\ &= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} e^{\beta x^2/2} e^{-x^2/(2T)} dx\\ &= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} e^{(\beta - 1/T)x^2/2} dx\\ &= \frac{e^{-\beta T/2}}{\sqrt{1-\beta T}}\int_{-\infty}^{\infty} \frac{e^{-x^2/(2/(1/T-\beta))}}{\sqrt{2\pi/(1/T-\beta)}} dx\end{aligned}$$

$$= \frac{e^{-\beta T/2}}{\sqrt{1-\beta T}},$$

for all $\beta < 1/T$, where we applied Relation (11.43) to $\phi(x) = e^{\beta x^2/2}$, knowing that $B_T \simeq \mathcal{N}(0,T)$.

Exercise 5.14

a) Letting $Y_t = e^{bt}X_t$, we have

$$\begin{aligned} dY_t &= d(e^{bt}X_t) \\ &= b e^{bt}X_t dt + e^{bt}dX_t \\ &= b e^{bt}X_t dt + e^{bt}(-bX_t dt + \sigma e^{-bt}dB_t) \\ &= \sigma dB_t, \end{aligned}$$

hence

$$Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t dB_s = Y_0 + \sigma B_t,$$

and

$$X_t = e^{-bt}Y_t = e^{-bt}Y_0 + \sigma e^{-bt}B_t = e^{-bt}X_0 + \sigma e^{-bt}B_t.$$

b) Letting $Y_t = e^{bt}X_t$, we have

$$\begin{aligned} dY_t &= d(e^{bt}X_t) \\ &= b e^{bt}X_t dt + e^{bt}dX_t \\ &= b e^{bt}X_t dt + e^{bt}(-bX_t dt + \sigma e^{-at}dB_t) \\ &= \sigma e^{(b-a)t}dB_t, \end{aligned}$$

hence we can solve for Y_t by integrating on both sides as

$$Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t e^{(b-a)s}dB_s, \qquad t \in \mathbb{R}_+.$$

This yields the solution

$$X_t = e^{-bt}Y_t = e^{-bt}X_0 + \sigma e^{-bt}\int_0^t e^{(b-a)s}dB_s, \qquad t \in \mathbb{R}_+.$$

Comments:

(i) This type of computation appears anywhere *discounting* by the factor e^{-bt} is involved.

(ii) The stochastic integral $\int_0^t e^{(b-a)s}dB_s$ cannot be computed in closed form. It is a centered Gaussian random variable with variance

$$\int_0^t e^{2(b-a)s}ds = \frac{e^{2(b-a)t}-1}{2(b-a)}$$

if $b \neq a$, and variance t if $a = b$.

Exercise 5.15

a) Note that the stochastic integral

$$\int_0^T \frac{1}{T-s}dB_s$$

is not defined in $L^2(\Omega)$ as the function $s \longmapsto 1/(T-s)$ is not in $L^2([0,T])$, and by the Itô isometry we have

$$\mathbb{E}\left[\left(\int_0^T \frac{1}{T-s}dB_s\right)^2\right] = \int_0^T \frac{1}{(T-s)^2}ds = \left[\frac{1}{T-s}\right]_0^\infty = +\infty.$$

By (5.38) we have

$$d\left(\frac{X_t^T}{T-t}\right) = \frac{dX_t^T}{T-t} + \frac{X_t^T}{(T-t)^2}dt = \sigma\frac{dB_t}{T-t},$$

hence, by integration over the time interval $[0,t]$ and using the initial condition $X_0 = 0$, we find

$$\frac{X_t^T}{T-t} = \frac{X_0^T}{T} + \int_0^t d\left(\frac{X_s^T}{T-s}\right) = \sigma\int_0^t \frac{dB_s 1}{T-s}, \qquad 0 \leqslant t < T.$$

b) By (5.17), we have

$$\mathbb{E}[X_t^T] = (T-t)\sigma\,\mathbb{E}\left[\int_0^t \frac{1}{T-s}dB_s\right] = 0, \qquad 0 \leqslant t < T.$$

c) By the Itô isometry, we have

$$\begin{aligned}
\mathrm{Var}[X_t^T] &= (T-t)^2\sigma^2\,\mathrm{Var}\left[\int_0^t \frac{1}{T-s}dB_s\right] \\
&= (T-t)^2\sigma^2\,\mathbb{E}\left[\left(\int_0^t \frac{1}{T-s}dB_s\right)^2\right] \\
&= (T-t)^2\sigma^2\int_0^t \frac{1}{(T-s)^2}ds \\
&= (T-t)^2\sigma^2\left(\frac{1}{T-t} - \frac{1}{T}\right) \\
&= \sigma^2\left(1 - \frac{t}{T}\right), \qquad 0 \leqslant t < T.
\end{aligned}$$

d) We have

$$\lim_{t\to T}\|X_t^T\|_{L^2(\Omega)} = \lim_{t\to T}\mathrm{Var}[X_t^T] = \lim_{t\to T}\sigma^2\left(1-\frac{t}{T}\right) = 0.$$

Exercise 5.16 Exponential Vašíček, 1977 model (1). Applying the Itô formula to $X_t = \mathrm{e}^{r_t} = f(r_t)$ with $f(x) = \mathrm{e}^x$, we have

$$\begin{aligned}
dX_t &= d\,\mathrm{e}^{r_t} \\
&= \mathrm{e}^{r_t}dr_t + \frac{1}{2}\mathrm{e}^{r_t}|dr_t|^2 \\
&= \mathrm{e}^{r_t}((a-br_t)dt + \sigma dB_t) + \frac{1}{2}\mathrm{e}^{r_t}((a-br_t)dt + \sigma dB_t)^2 \\
&= \mathrm{e}^{r_t}((a-br_t)dt + \sigma dB_t) + \frac{\sigma^2}{2}\mathrm{e}^{r_t}dt \\
&= X_t\left(a + \frac{\sigma^2}{2} - b\log(X_t)\right)dt + \sigma X_t dB_t \\
&= X_t(\tilde{a} - \tilde{b}f(X_t))dt + \sigma g(X_t)dB_t,
\end{aligned}$$

hence

$$\tilde{a} = a + \frac{\sigma^2}{2} \qquad \text{and} \qquad \tilde{b} = b$$

the functions $f(x)$ and $g(x)$ are given by $f(x) = \log x$ and $g(x) = x$. Note that this stochastic differential equation is that of the exponential Vasicek model.

Exercise 5.17 Exponential Vasicek model (2).

a) We have $Z_t = \mathrm{e}^{-at} Z_0 + \sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s$.

b) We have $Y_t = \mathrm{e}^{-at} Y_0 + \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s$.

c) We have $dX_t = X_t \left(\theta + \frac{\sigma^2}{2} - a \log X_t \right) dt + \sigma X_t dB_t$.

d) We have $r_t = \exp\left(\mathrm{e}^{-at} \log r_0 + \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s \right)$.

e) Using the Gaussian moment generating function identity $\mathbb{E}[\mathrm{e}^X] = \mathrm{e}^{\alpha^2/2}$ for $X \simeq \mathcal{N}(0,\alpha^2)$, we have

$$
\begin{aligned}
\mathbb{E}[r_t \mid \mathcal{F}_u] &= \mathbb{E}\left[\exp\left(\mathrm{e}^{-at} \log r_0 + \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \mathrm{e}^{\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\left(1-\mathrm{e}^{-at}\right) + \sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s} \mathbb{E}\left[\exp\left(\sigma \int_u^t \mathrm{e}^{-(t-s)a} dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \mathrm{e}^{\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\left(1-\mathrm{e}^{-at}\right) + \sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s} \mathbb{E}\left[\exp\left(\sigma \int_u^t \mathrm{e}^{-(t-s)a} dB_s \right) \right] \\
&= \exp\left(\mathrm{e}^{-at} \log r_0 + \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s + \frac{\sigma^2}{2} \int_u^t \mathrm{e}^{-2(t-s)a} ds \right) \\
&= \exp\left(\mathrm{e}^{-at} \log r_0 + \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s + \frac{\sigma^2}{4a}(1-\mathrm{e}^{-2(t-u)a}) \right) \\
&= \exp\left(\mathrm{e}^{-(t-u)a} \left(\mathrm{e}^{-au} \log r_0 + \frac{\theta}{a}(1-\mathrm{e}^{-au}) + \sigma \int_0^u \mathrm{e}^{-(u-s)a} dB_s \right) \right. \\
&\quad \left. + \frac{\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{4a}(1-\mathrm{e}^{-2(t-u)a}) \right) \\
&= \exp\left(\mathrm{e}^{-(t-u)a} \log r_u + \frac{\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{4a}(1-\mathrm{e}^{-2(t-u)a}) \right) \\
&= r_u^{\mathrm{e}^{-(t-u)a}} \exp\left(\frac{\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{4a}(1-\mathrm{e}^{-2(t-u)a}) \right).
\end{aligned}
$$

In particular, for $u = 0$ we find

$$
\mathbb{E}[r_t] = r_0^{\mathrm{e}^{-at}} \exp\left(+ \frac{\theta}{a}(1-\mathrm{e}^{-at}) + \frac{\sigma^2}{4a}(1-\mathrm{e}^{-2at}) \right).
$$

f) Similarly, we have

$$
\begin{aligned}
\mathbb{E}[r_t^2 \mid \mathcal{F}_u] &= \mathbb{E}\left[\exp\left(2\,\mathrm{e}^{-at} \log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-at}) + 2\sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \mathrm{e}^{2\,\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}\left(1-\mathrm{e}^{-at}\right) + 2\sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s} \mathbb{E}\left[\exp\left(2\sigma \int_u^t \mathrm{e}^{-(t-s)a} dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \mathrm{e}^{2\,\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}\left(1-\mathrm{e}^{-at}\right) + 2\sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s} \mathbb{E}\left[\exp\left(2\sigma \int_u^t \mathrm{e}^{-(t-s)a} dB_s \right) \right] \\
&= \exp\left(2\,\mathrm{e}^{-at} \log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-at}) + 2\sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s + 2\sigma^2 \int_u^t \mathrm{e}^{-2(t-s)a} ds \right) \\
&= \exp\left(2\,\mathrm{e}^{-at} \log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-at}) + 2\sigma \int_0^u \mathrm{e}^{-(t-s)a} dB_s + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a}) \right) \\
&= \exp\left(2\,\mathrm{e}^{-(t-u)a} \left(2\,\mathrm{e}^{-au} \log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-au}) + 2\sigma \int_0^u \mathrm{e}^{-(u-s)a} dB_s \right) \right. \\
&\quad \left. + \frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a}) \right)
\end{aligned}
$$

$$= \exp\left(2\,e^{-(t-u)a}\log r_u + \frac{2\theta}{a}(1-e^{-(t-u)a}) + \frac{\sigma^2}{a}(1-e^{-2(t-u)a})\right)$$
$$= r_u^{2\,e^{-(t-u)a}} \exp\left(\frac{2\theta}{a}(1-e^{-(t-u)a}) + \frac{\sigma^2}{a}(1-e^{-2(t-u)a})\right),$$

hence

$$\begin{aligned}
\mathrm{Var}[r_t \mid \mathscr{F}_u] &= \mathbb{E}[r_t^2 \mid \mathscr{F}_u] - (\mathbb{E}[r_t \mid \mathscr{F}_u])^2 \\
&= r_u^{2\,e^{-(t-u)a}} \exp\left(\frac{2\theta}{a}(1-e^{-(t-u)a}) + \frac{\sigma^2}{a}(1-e^{-2(t-u)a})\right) \\
&\quad - r_u^{2\,e^{-(t-u)a}} \exp\left(\frac{2\theta}{a}(1-e^{-(t-u)a}) + \frac{\sigma^2}{2a}(1-e^{-2(t-u)a})\right) \\
&= r_u^{2\,e^{-(t-u)a}} \exp\left(\frac{2\theta}{a}(1-e^{-(t-u)a}) + \frac{\sigma^2}{a}(1-e^{-2(t-u)a})\right) \\
&\quad \times \left(1 - \exp\left(-\frac{\sigma^2}{2a}(1-e^{-2(t-u)a})\right)\right).
\end{aligned}$$

g) We find $\lim_{t\to\infty} \mathbb{E}[r_t] = r_0 \exp\left(\frac{\theta}{a} + \frac{\sigma^2}{4a}\right)$ and

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{Var}[r_t] &= \exp\left(\frac{2\theta}{a} + \frac{\sigma^2}{a}\right)\left(1 - \exp\left(-\frac{\sigma^2}{2a}\right)\right) \\
&= \exp\left(\frac{2\theta}{a}\right)\left(\exp\left(\frac{\sigma^2}{a}\right) - 1\right).
\end{aligned}$$

Exercise 5.18 Cox-Ingersoll-Ross (CIR) model.

a) We have

$$r_t = r_0 + \int_0^t (\alpha - \beta r_s)ds + \sigma \int_0^t \sqrt{r_s}\, dB_s, \qquad t \geqslant 0. \tag{A.14}$$

b) Taking expectations on both sides of (A.14) and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we find

$$\begin{aligned}
u(t) &= \mathbb{E}[r_t] \\
&= \mathbb{E}\left[r_0 + \int_0^t (\alpha - \beta r_s)ds + \sigma \int_0^t \sqrt{r_s}\, dB_s\right] \\
&= \mathbb{E}\left[r_0 + \int_0^t (\alpha - \beta r_s)ds\right] \\
&= r_0 + \mathbb{E}\left[\int_0^t (\alpha - \beta r_s)ds\right] \\
&= r_0 + \int_0^t (\alpha - \beta\,\mathbb{E}[r_s])ds \\
&= r_0 + \int_0^t (\alpha - \beta u(s))ds,
\end{aligned}$$

which yields the differential equation $u'(t) = \alpha - \beta u(t)$. Letting $w(t) : e^{\beta t} u(t)$ we have

$$w'(t) = \beta\, e^{\beta t} u(t) + e^{\beta t} u'(t) = \alpha\, e^{\beta t},$$

hence

$$\begin{aligned}
\mathbb{E}[r_t] &= u(t) \\
&= e^{-\beta t} w(t) \\
&= e^{-\beta t}\left(w(0) + \alpha \int_0^t e^{\beta s} ds\right) \\
&= e^{-\beta t}\left(u(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1)\right)
\end{aligned}$$

$$= \quad e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}), \qquad t \geqslant 0. \tag{A.15}$$

c) By applying Itô's formula to

$$r_t^2 = f\left(r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s\right),$$

with $f(x) = x^2$, we find

$$d(r_t)^2 = r_t(\sigma^2 + 2\alpha - 2\beta r_t)dt + 2\sigma r_t^{3/2} dB_t$$

or, in integral form,

$$r_t^2 = r_0^2 + \int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s)ds + 2\sigma \int_0^t r_s^{3/2} dB_s, \qquad t \geqslant 0. \tag{A.16}$$

d) Taking again the expectation on both sides of (A.16), we find

$$\begin{aligned}
v(t) &= \mathbb{E}[r_t^2] \\
&= \mathbb{E}\left[r_0^2 + \int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s)ds + 2\sigma \int_0^t r_s^{3/2} dB_s\right] \\
&= r_0^2 + \mathbb{E}\left[\int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s)ds\right] \\
&= r_0^2 + \int_0^t (\sigma^2 \mathbb{E}[r_s] + 2\alpha \mathbb{E}[r_s] - 2\beta \mathbb{E}[r_s^2])ds \\
&= v(0) + \int_0^t (\sigma^2 u(s) + 2\alpha u(s) - 2\beta v(s))ds,
\end{aligned}$$

and after differentiation with respect to t this yields the differential equation

$$v'(t) = (\sigma^2 + 2\alpha)u(t) - 2\beta v(t), \qquad t \geqslant 0.$$

By (A.15) we find

$$v'(t) = (\sigma^2 + 2\alpha)\left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right) e^{-\beta t}\right) - 2\beta v(t), \qquad t \geqslant 0.$$

Looking for a solution of the form

$$v(t) = c_0 + c_1 e^{-\beta t} + c_2 e^{-2\beta t}, \qquad t \geqslant 0,$$

we find

$$\begin{aligned}
v'(t) &= -\beta c_1 e^{-\beta t} \\
&= (\sigma^2 + 2\alpha)\left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right) e^{-\beta t}\right) - 2\beta(c_0 + c_1 e^{-\beta t}) \\
&= \frac{\alpha}{\beta}(\sigma^2 + 2\alpha) + (\sigma^2 + 2\alpha)\left(r_0 - \frac{\alpha}{\beta}\right) e^{-\beta t} - 2\beta c_0 - 2\beta c_1 e^{-\beta t},
\end{aligned}$$

$t \in \mathbb{R}_+$, hence

$$\begin{cases}
0 = \dfrac{\alpha}{\beta}(\sigma^2 + 2\alpha) - 2\beta c_0, \\[2ex]
-\beta c_1 = (\sigma^2 + 2\alpha)\left(r_0 - \dfrac{\alpha}{\beta}\right) - 2\beta c_1,
\end{cases}$$

and

$$c_0 = \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha), \qquad c_1 = \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right),$$

with

$$r_0^2 = v(0) = c_0 + c_1 + c_2,$$

which yields

$$\begin{aligned} c_2 &= r_0^2 - c_0 - c_1 \\ &= r_0^2 - \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) - \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \\ &= r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[r_t^2] &= v(t) \\ &= c_0 + c_1 \mathrm{e}^{-\beta t} + c_2 \mathrm{e}^{-2\beta t} \\ &= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \mathrm{e}^{-\beta t} \\ &\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right) \mathrm{e}^{-2\beta t}, \qquad t \geqslant 0. \end{aligned}$$

e) We have

$$\begin{aligned} \mathrm{Var}[r_t] &= \mathbb{E}[r_t^2] - (\mathbb{E}[r_t])^2 \\ &= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \mathrm{e}^{-\beta t} \\ &\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right) \mathrm{e}^{-2\beta t} \\ &\quad - \left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right) \mathrm{e}^{-\beta t}\right)^2 \\ &= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \mathrm{e}^{-\beta t} \\ &\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right) \mathrm{e}^{-2\beta t} \\ &\quad - \left(\frac{\alpha}{\beta}\right)^2 - \frac{2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \mathrm{e}^{-\beta t} - \left(r_0^2 - 2r_0\frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta}\right)^2\right) \mathrm{e}^{-2\beta t} \\ &= r_0 \frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2} - \frac{\alpha\sigma^2}{\beta^2} \mathrm{e}^{-\beta t} + \frac{\alpha\sigma^2}{2\beta^2} \mathrm{e}^{-2\beta t} \\ &= r_0 \frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - \mathrm{e}^{-\beta t}\right)^2, \quad t \geqslant 0. \end{aligned}$$

Chapter 6

Exercise 6.1 For all $x \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{P}(S_T \leqslant x) &= \mathbb{P}\left(S_0 \mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} \leqslant x\right) \\ &= \mathbb{P}\left(\sigma B_T + \left(\mu - \frac{\sigma^2}{2}\right) T \leqslant \log \frac{x}{S_0}\right) \\ &= \mathbb{P}\left(B_T \leqslant \frac{1}{\sigma}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\ &= \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/\sigma} \mathrm{e}^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\ &= \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right), \end{aligned}$$

where

$$\Phi(x) := \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi T}}, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian cumulative distribution function. After differentiation with respect to x we find the lognormal probability density function

$$\begin{aligned}
f(x) &= \frac{d\mathbb{P}(S_T \leqslant x)}{dx} \\
&= \frac{\partial}{\partial x} \int_{-\infty}^{(\log(x/S_0)-(\mu-\sigma^2/2)T)/\sigma} e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \frac{\partial}{\partial x} \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T\right)\right) \\
&= \frac{1}{x\sigma} \varphi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T\right)\right) \\
&= \frac{1}{x\sigma\sqrt{2\pi T}} e^{-(-(\mu-\sigma^2/2)T+\log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,
\end{aligned}$$

where

$$\varphi(y) = \Phi'(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \qquad y \in \mathbb{R},$$

denotes the standard Gaussian probability density function.

Exercise 6.2

a) We have

$$d\log S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2 = rdt + \sigma dB_t - \frac{\sigma^2}{2} dt, \quad t \in \mathbb{R}_+.$$

b) We have $f(t) = f(0)\,e^{ct}$ (continuous-time interest rate compounding), and

$$S_t = S_0\, e^{\sigma B_t - \sigma^2 t/2 + rt}, \qquad t \in \mathbb{R}_+,$$

(geometric Brownian motion).

c) Those quantities can be directly computed from the expression of S_t as a function of the $\mathcal{N}(0,t)$ random variable B_t. Namely, we have

$$\begin{aligned}
\mathbb{E}[S_t] &= \mathbb{E}\left[S_0\, e^{\sigma B_t - \sigma^2 t/2 + rt}\right] \\
&= S_0\, e^{-\sigma^2 t/2 + rt}\, \mathbb{E}\left[e^{\sigma B_t}\right] \\
&= S_0\, e^{rt},
\end{aligned}$$

where we used the Gaussian moment generating function (MGF) formula (11.46)

$$\mathbb{E}\left[e^{\sigma B_t}\right] = e^{\sigma^2 t/2}$$

for the normal random variable $B_t \simeq \mathcal{N}(0,t)$, $t > 0$. Similarly, we have

$$\begin{aligned}
\mathbb{E}[S_t^2] &= \mathbb{E}\left[S_0^2\, e^{2\sigma B_t - \sigma^2 t + 2rt}\right] \\
&= S_0^2\, e^{-\sigma^2 t + 2rt}\, \mathbb{E}\left[e^{2\sigma B_t}\right] \\
&= S_0^2\, e^{\sigma^2 t + 2rt}, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

d) We note that from the stochastic differential equation

$$S_t = S_0 + r\int_0^t S_s ds + \sigma \int_0^t S_s dB_s,$$

the function $u(t) := \mathbb{E}[S_t]$ satisfies the ordinary differential equation $u'(t) = ru(t)$ with $u(0) = S_0$ and solution $u(t) = \mathbb{E}[S_t] = S_0\,\mathrm{e}^{rt}$. On the other hand, by the Itô formula we have

$$dS_t^2 = 2S_t dS_t + (dS_t)^2 = 2rS_t^2 dt + \sigma^2 S_t^2 dt + 2\sigma S_t dB_t,$$

hence letting $v(t) = \mathbb{E}\left[S_t^2\right]$ and taking expectations on both sides of

$$S_t^2 = S_0^2 + 2r\int_0^t S_u^2 du + \sigma^2 \int_0^t S_u^2 du + 2\sigma \int_0^t S_u dB_u,$$

we find

$$\begin{aligned} v(t) &= \mathbb{E}\left[S_t^2\right] \\ &= S_0^2 + (2r+\sigma^2)\,\mathbb{E}\left[\int_0^t S_u^2 du\right] + 2\sigma\,\mathbb{E}\left[\int_0^t S_u dB_u\right] \\ &= S_0^2 + (2r+\sigma^2)\int_0^t \mathbb{E}\left[S_u^2\right] du \\ &= S_0^2 + (2r+\sigma^2)\int_0^t v(u) du, \end{aligned}$$

hence $v(t) := \mathbb{E}\left[S_t^2\right]$ satisfies the ordinary differential equation

$$v'(t) = (\sigma^2 + 2r)v(t),$$

with $v(0) = S_0^2$ and solution

$$v(t) = \mathbb{E}\left[S_t^2\right] = S_0^2\,\mathrm{e}^{(\sigma^2+2r)t},$$

which recovers

$$\begin{aligned} \mathrm{Var}[S_t] &= \mathbb{E}\left[S_t^2\right] - (\mathbb{E}[S_t])^2 \\ &= v(t) - u^2(t) \\ &= S_0^2\,\mathrm{e}^{(\sigma^2+2r)t} - S_0^2\,\mathrm{e}^{2rt} \\ &= S_0^2\,\mathrm{e}^{2rt}(\mathrm{e}^{\sigma^2 t} - 1), \qquad t \in \mathbb{R}_+. \end{aligned}$$

Exercise 6.3 Using the bivariate Itô formula (5.26) in two variables, we find

$$\begin{aligned} df(S_t,Y_t) &= \frac{\partial f}{\partial x}(S_t,Y_t)dS_t + \frac{\partial f}{\partial y}(S_t,Y_t)dY_t \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S_t,Y_t)(dS_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(S_t,Y_t)(dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(S_t,Y_t)dS_t \bullet dY_t \\ &= \frac{\partial f}{\partial x}(S_t,Y_t)(rS_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial y}(S_t,Y_t)(\mu Y_t dt + \eta Y_t dW_t) \\ &\quad + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial x^2}(S_t,Y_t)dt + \frac{\eta^2 Y_t^2}{2}\frac{\partial^2 f}{\partial y^2}(S_t,Y_t)dt + \rho\sigma\eta S_t Y_t \frac{\partial^2 f}{\partial x \partial y}(S_t,Y_t)dt. \end{aligned}$$

Exercise 6.4 Taking expectations on both sides of (6.24) shows that

$$\mathbb{E}[S_T] = C(S_0,r,T) + \mathbb{E}\left[\int_0^T \zeta_{t,T} dB_t\right] = C(S_0,r,T),$$

hence

$$\begin{aligned} C(S_0,r,T) &= \mathbb{E}[S_T] \\ &= \mathbb{E}[S_0\,\mathrm{e}^{\mu T + \sigma B_T - \sigma^2 T/2}] \end{aligned}$$

$$
\begin{aligned}
&= S_0 \,\mathrm{e}^{\mu T - \sigma^2 T/2} \,\mathbb{E}[\mathrm{e}^{\sigma B_T}] \\
&= S_0 \,\mathrm{e}^{\mu T - \sigma^2 T/2 + \sigma^2 T/2} \\
&= S_0 \,\mathrm{e}^{\mu T},
\end{aligned}
$$

where we used the moment generating function

$$
\mathbb{E}[\mathrm{e}^{\sigma B_T}] = \mathrm{e}^{\sigma^2 T/2}
$$

of the Gaussian random variable $B_T \simeq \mathcal{N}(0,T)$. On the other hand, the discounted asset price $X_t := \mathrm{e}^{-rt} S_t$ satisfies $dX_t = \sigma X_t dB_t$, which shows that

$$
X_T = X_0 + \sigma \int_0^T X_t dB_t.
$$

Multiplying both sides by e^{rT} shows that

$$
S_T = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{rT} X_t dB_t = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{(T-t)r} S_t dB_t,
$$

which recovers the relation $C(S_0,r,T) = S_0 \,\mathrm{e}^{rT}$, and shows that $\zeta_{t,T} = \sigma \,\mathrm{e}^{(T-t)r} S_t$, $t \in [0,T]$.

Exercise 6.5

a) We have $S_t = f(X_t)$, $t \in \mathbb{R}_+$, where $f(x) = S_0 \,\mathrm{e}^x$ and $(X_t)_{t\in\mathbb{R}_+}$ is the Itô process given by

$$
X_t := \int_0^t \sigma_s dB_s + \int_0^t u_s ds, \qquad t \in \mathbb{R}_+,
$$

or in differential form

$$
dX_t := \sigma_t dB_t + u_t dt, \qquad t \in \mathbb{R}_+,
$$

hence

$$
\begin{aligned}
dS_t &= df(X_t) \\
&= f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2 \\
&= u_t f'(X_t) dt + \sigma_t f'(X_t) dB_t + \frac{1}{2}\sigma_t^2 f''(X_t) dt \\
&= S_0 u_t \,\mathrm{e}^{X_t} dt + S_0 \sigma_t \,\mathrm{e}^{X_t} dB_t + \frac{1}{2} S_0 \sigma_t^2 \,\mathrm{e}^{X_t} dt \\
&= u_t S_t dt + \sigma_t S_t dB_t + \frac{1}{2}\sigma_t^2 S_t dt.
\end{aligned}
$$

b) The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$
dS_t = \left(u_t + \frac{1}{2}\sigma_t^2 \right) S_t dt + \sigma_t S_t dB_t.
$$

Exercise 6.6

a) We have $\mathbb{E}[S_t] = 1$ because the expected value of the Itô stochastic integral is zero. Regarding the variance, using the Itô isometry (5.7) we have

$$
\begin{aligned}
\mathrm{Var}[S_t] &= \sigma^2 \,\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s \right)^2 \right] \\
&= \sigma^2 \,\mathbb{E}\left[\int_0^t \left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2} \right)^2 ds \right] \\
&= \sigma^2 \int_0^t \mathbb{E}\left[\left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2} \right)^2 \right] ds
\end{aligned}
$$

$$\begin{aligned}
&= \sigma^2 \int_0^t \mathbb{E}\left[e^{2\sigma B_s - \sigma^2 s} \right] ds \\
&= \sigma^2 \int_0^t e^{-\sigma^2 s} \mathbb{E}\left[e^{2\sigma B_s} \right] ds \\
&= \sigma^2 \int_0^t e^{-\sigma^2 s} e^{2\sigma^2 s} ds \\
&= \sigma^2 \int_0^t e^{\sigma^2 s} ds \\
&= e^{\sigma^2 t} - 1.
\end{aligned}$$

b) Taking $f(x) = \log x$, we have

$$\begin{aligned}
d\log(S_t) &= df(S_t) \\
&= \sigma f'(S_t) dS_t + \frac{1}{2}\sigma^2 f''(S_t)(dS_t)^2 \\
&= \sigma f'(S_t) e^{\sigma B_t - \sigma^2 t/2} dB_t + \frac{1}{2}\sigma^2 f''(S_t) e^{2\sigma B_t - \sigma^2 t} dt \\
&= \frac{\sigma}{S_t} e^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} e^{2\sigma B_t - \sigma^2 t} dt. \qquad \text{(A.17)}
\end{aligned}$$

c) We check that when $S_t = e^{\sigma B_t - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, we have

$$\log S_t = \sigma B_t - \sigma^2 t/2, \quad \text{and} \quad d\log S_t = \sigma dB_t - \frac{\sigma^2}{2} dt.$$

On the other hand, we also find

$$\sigma dB_t - \frac{\sigma^2}{2} dt = \frac{\sigma}{S_t} e^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} e^{2\sigma B_t - \sigma^2 t} dt,$$

showing by (A.17) that the equation

$$d\log S_t = \frac{\sigma}{S_t} e^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} e^{2\sigma B_t - \sigma^2 t} dt$$

is satisfied. By uniqueness of solutions, we conclude that $S_t := e^{\sigma B_t - \sigma^2 t/2}$ solves

$$S_t = 1 + \sigma \int_0^t e^{\sigma B_s - \sigma^2 s/2} dB_s, \qquad t \in \mathbb{R}_+.$$

Exercise 6.7

a) Leveraging with a factor $\beta : 1$ means that we invest the amount $\xi_t S_t = \beta F_t$ on the risky asset priced S_t. In this case, the fund value decomposes into the portfolio

$$F_t = \xi_t S_t + \eta_t A_t = \beta \frac{F_t}{S_t} S_t - (\beta - 1)\frac{F_t}{A_t} A_t, \qquad t \in \mathbb{R}_+,$$

with $\xi_t = \beta F_t / S_t$ and $\eta_t = -(\beta - 1)F_t / A_t$, $t \in \mathbb{R}_+$.

b) We have

$$\begin{aligned}
dF_t &= \xi_t dS_t + \eta_t dA_t \\
&= \beta \frac{F_t}{S_t} dS_t - (\beta - 1)\frac{F_t}{A_t} dA_t \\
&= \beta \frac{F_t}{S_t} dS_t - (\beta - 1) r F_t dt \\
&= \beta F_t (r dt + \sigma dB_t) - (\beta - 1) r F_t dt \\
&= r F_t dt + \beta \sigma F_t dB_t, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

c) We have

$$\begin{aligned}
F_t &= F_0\,\mathrm{e}^{\beta\sigma B_t+rt-\beta^2\sigma^2t/2}\\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt/\beta-\beta\sigma^2t/2}\right)^\beta\\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt-\sigma^2t/2-(1-1/\beta)rt-(\beta-1)\sigma^2t/2}\right)^\beta\\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt-\sigma^2t/2}\right)^\beta\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2t/2}\\
&= \left(S_0\,\mathrm{e}^{\sigma B_t+rt-\sigma^2t/2}\right)^\beta\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2t/2}\\
&= S_t^\beta\,\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2t/2}, \qquad t\in\mathbb{R}_+.
\end{aligned}$$

Exercise 6.8 Letting $X_t := f(t)\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}$ and noting the relation

$$d\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}\sigma f(t)\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}dB_t, \qquad t\in\mathbb{R}_+,$$

see Proposition 6.8 with $\mu=0$, we have

$$\begin{aligned}
dX_t &= \mathrm{e}^{\sigma B_t-\sigma^2t/2}f'(t)dt+f(t)d\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}\\
&= \mathrm{e}^{\sigma B_t-\sigma^2t/2}f'(t)dt+\sigma f(t)\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}dB_t\\
&= \frac{f'(t)}{f(t)}X_tdt+\sigma X_tdB_t\\
&= h(t)X_tdt+\sigma X_tdB_t,
\end{aligned}$$

hence

$$\frac{d}{dt}\log f(t)=\frac{f'(t)}{f(t)}=h(t),$$

which shows that

$$\log f(t)=\log f(0)+\int_0^t h(s)ds,$$

and

$$\begin{aligned}
X_t &= f(t)\,\mathrm{e}^{\sigma B_t-\sigma^2t/2}\\
&= f(0)\exp\left(\int_0^t h(s)ds+\sigma B_t-\frac{\sigma^2}{2}t\right)\\
&= X_0\exp\left(\int_0^t h(s)ds+\sigma B_t-\frac{\sigma^2}{2}t\right), \qquad t\in\mathbb{R}_+.
\end{aligned}$$

Exercise 6.9

a) We have

$$\begin{aligned}
S_t &= \mathrm{e}^{X_t}\\
&= \mathrm{e}^{X_0}+\int_0^t u_s\,\mathrm{e}^{X_s}dB_s+\int_0^t v_s\,\mathrm{e}^{X_s}ds+\frac{1}{2}\int_0^t u_s^2\,\mathrm{e}^{X_s}ds\\
&= \mathrm{e}^{X_0}+\sigma\int_0^t \mathrm{e}^{X_s}dB_s+\nu\int_0^t \mathrm{e}^{X_s}ds+\frac{\sigma^2}{2}\int_0^t \mathrm{e}^{X_s}ds\\
&= S_0+\sigma\int_0^t S_sdB_s+\nu\int_0^t S_sds+\frac{\sigma^2}{2}\int_0^t S_sds.
\end{aligned}$$

b) Let $r>0$. The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t=rS_tdt+\sigma S_tdB_t$$

when $r=\nu+\sigma^2/2$.

c) We have

$$\operatorname{Var}[X_t] = \operatorname{Var}[(B_T - B_t)\sigma] = \sigma^2 \operatorname{Var}[B_T - B_t] = (T-t)\sigma^2, \quad t \in [0,T].$$

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 \,\mathrm{e}^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the time splitting decomposition

$$S_T = S_t \frac{S_T}{S_t} = S_t \,\mathrm{e}^{(B_T - B_t)\sigma + \nu\tau},$$

we have

$$\begin{aligned}
\mathbb{P}(S_T > K \mid S_t = x) &= \mathbb{P}(S_t \,\mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K \mid S_t = x) \\
&= \mathbb{P}(x \,\mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K) \\
&= \mathbb{P}(\mathrm{e}^{(B_T-B_t)\sigma} > K \,\mathrm{e}^{-(T-t)\nu}/x) \\
&= \mathbb{P}\left(\frac{B_T - B_t}{\sqrt{T-t}} > \frac{1}{\sigma\sqrt{T-t}} \log\left(K \,\mathrm{e}^{-(T-t)\nu}/x\right)\right) \\
&= 1 - \Phi\left(\frac{\log\left(K \,\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(-\frac{\log\left(K \,\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(\frac{\log(x/K) + \nu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

where $\tau = T - t$.

Chapter 7

Exercise 7.1

a) By the Itô formula, we have

$$dV_t = dg(t, B_t) = \frac{\partial g}{\partial t}(t, B_t)dt + \frac{\partial g}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, B_t)dt. \tag{A.18}$$

Consider a hedging portfolio with value $V_t = \eta_t A_t + \xi_t B_t$, satisfying the self-financing condition

$$dV_t = \eta_t dA_t + \xi_t dB_t = \xi_t dB_t, \quad t \in \mathbb{R}_+. \tag{A.19}$$

By respective identification of the terms in dB_t and dt in (A.18) and (A.19) we get

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t, B_t)dt + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t, B_t)dt, \\[2ex] \xi_t dB_t = \dfrac{\partial g}{\partial x}(t, B_t)dB_t, \end{cases}$$

hence

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t, B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t, B_t), \\[2ex] \xi_t = \dfrac{\partial g}{\partial x}(t, B_t), \end{cases}$$

and

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t,B_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,B_t), \end{cases}$$

hence the function $g(t,x)$ satisfies the heat equation

$$0 = \frac{\partial g}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \tag{A.20}$$

with terminal condition $g(T,x) = x^2$, and ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t,B_t), \qquad t \in \mathbb{R}_+.$$

b) In order to solve (A.20) we substitute a solution of the form $g(t,x) = x^2 + f(t)$ and find $1 + f'(t) = 0$, which yields $f(T-t) = T-t$ and $g(t,x) = x^2 + T - t$, $t \in [0,T]$.

Exercise 7.2 By the Itô formula, we have

$$\begin{aligned} dV_t &= dg(t,S_t) \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma\sqrt{S_t}\frac{\partial g}{\partial x}(t,S_t)dB_t. \end{aligned} \tag{A.21}$$

By respective identification of the terms in dB_t and dt in (7.32) and (A.21) we get

$$\begin{cases} rg(t,S_t)dt + \beta(\alpha - S_t)\xi_t dt - r\xi_t S_t dt \\ \qquad = \dfrac{\partial g}{\partial t}(t,S_t)dt + \beta(\alpha - S_t)\dfrac{\partial g}{\partial x}(t,S_t)dt + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t,S_t)dt, \\ \\ \sigma\xi_t\sqrt{S_t}dB_t = \sigma\sqrt{S_t}\dfrac{\partial g}{\partial x}(t,S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t,S_t) + \beta(\alpha - S_t)\xi_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t,S_t) + \beta(\alpha - S_t)\dfrac{\partial g}{\partial x}(t,S_t) + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t), \end{cases}$$

and

$$\begin{cases} rg(t,S_t) + \beta(\alpha - S_t)\xi_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t,S_t) + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t), \end{cases}$$

hence the function $g(t,x)$ satisfies the PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x \frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0,$$

and ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t), \qquad t \in \mathbb{R}_+.$$

Exercise 7.3

a) Let $V_t := \xi_t S_t + \eta_t A_t$ denote the hedging portfolio value at time $t \in [0,T]$. Since the dividend yield δS_t per share is continuously reinvested in the portfolio, the portfolio change dV_t decomposes as

$$\begin{aligned} dV_t &= \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{dividend payout}} \\ &= r\eta_t A_t dt + \xi_t((\mu-\delta)S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt \\ &= r\eta_t A_t dt + \xi_t(\mu S_t dt + \sigma S_t dB_t) \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \in \mathbb{R}_+. \end{aligned}$$

b) By Itô's formula we have

$$\begin{aligned} dg(t,S_t) &= \frac{\partial g}{\partial t}(t,S_t)dt + (\mu-\delta)S_t\frac{\partial g}{\partial x}(t,S_t)dt \\ &\quad + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma S_t \frac{\partial g}{\partial x}(t,S_t)dB_t, \end{aligned}$$

hence by identification of the terms in dB_t and dt in the expressions of dV_t and $dg(t,S_t)$, we get

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t),$$

and we derive the Black-Scholes PDE with dividend

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + (r-\delta)x\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x). \tag{A.22}$$

c) In order to solve (A.22) we note that, letting $f(t,x) := \mathrm{e}^{(T-t)\delta} g(t,x)$, the PDE (A.22) reads

$$rf(t,x) = \delta f(t,x) + \frac{\partial f}{\partial t}(t,x) + (r-\delta)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x),$$

hence $f(t,x) := \mathrm{e}^{(T-t)\delta} g(t,x)$, satisfies the standard Black-Scholes PDE with interest rate $r-\delta$, *i.e.* we have

$$(r-\delta)f(t,x) = \frac{\partial f}{\partial t}(t,x) + (r-\delta)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x),$$

with same terminal condition $f(T,x) = g(T,x) = (x-K)^+$, hence we have

$$\begin{aligned} f(t,x) &= \mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\ &= x\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(r-\delta)(T-t)}\Phi\big(d_-^\delta(T-t)\big), \end{aligned}$$

where

$$d_\pm^\delta(T-t) := \frac{\log(x/K) + (r-\delta \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Consequently, the pricing function of the European call option with dividend rate δ is

$$g(t,x) = \mathrm{e}^{-(T-t)\delta} f(t,x)$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\
&= x\mathrm{e}^{-(T-t)\delta}\Phi\big(d_+^\delta(T-t)\big)-K\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big), \quad 0\leqslant t\leqslant T.
\end{aligned}
$$

We also have

$$
g(t,x)=\mathrm{Bl}\big(x\mathrm{e}^{-(T-t)\delta},K,\sigma,r,T-t\big), \qquad 0\leqslant t\leqslant T.
$$

d) As in Proposition 7.4, we have

$$
\frac{\partial g}{\partial a}(t,x)=\mathrm{e}^{-(T-t)\delta}\Phi\big(d_+^\delta(T-t)\big), \qquad x>0,\ 0\leqslant t<T.
$$

Exercise 7.4

a) We easily check that $g_{\mathrm{c}}(t,0)=0$, as when $x=0$ we have $d_+(T-t)=d_-(T-t)=-\infty$ for all $t\in[0,T)$. On the other hand, we have

$$
\lim_{t\nearrow T} d_+(T-t)=\lim_{t\nearrow T} d_-(T-t)=\begin{cases} +\infty, & x>K, \\ 0, & x=K, \\ -\infty, & x<K, \end{cases}
$$

which allows us to recover the boundary condition

$$
\begin{aligned}
g_{\mathrm{c}}(T,x) &= \lim_{t\nearrow T} g_{\mathrm{c}}(t,x) \\
&= \left\{\begin{array}{ll} x\Phi(+\infty)-K\Phi(+\infty)=x-K, & x>K \\ \dfrac{x}{2}-\dfrac{K}{2}=0, & x=K \\ x\Phi(-\infty)-K\Phi(-\infty)=0, & x<K \end{array}\right\} = (x-K)^+
\end{aligned}
$$

at $t=T$. Similarly, we can check that

$$
\lim_{T\to\infty} d_-(T-t)=\begin{cases} +\infty, & r>\dfrac{\sigma^2}{2}, \\ 0, & r=\dfrac{\sigma^2}{2}, \\ -\infty, & r<\dfrac{\sigma^2}{2}, \end{cases}
$$

and $\lim_{T\to\infty} d_+(T-t)=+\infty$, hence

$$
\begin{aligned}
&\lim_{T\to\infty}\mathrm{Bl}(K,x,\sigma,r,T-t) \\
&\quad = x\lim_{T\to\infty}\Phi(d_+(T-t))-\lim_{T\to\infty}\big(\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))\big) \\
&\quad = x, \qquad t\in\mathbb{R}_+.
\end{aligned}
$$

b) We check that $g_{\mathrm{p}}(t,0)=K\mathrm{e}^{-(T-t)r}$ and $g_{\mathrm{p}}(t,\infty)=0$ as when $x=0$ we have $d_+(T-t)=d_-(T-t)=-\infty$ and as x tends to infinity we have $d_+(T-t)=d_-(T-t)=+\infty$ for all $t\in[0,T)$. On the other hand, we have

$$
g_{\mathrm{p}}(T,x)=\left\{\begin{array}{ll} K\Phi(+\infty)-x\Phi(+\infty)=K-x, & x<K \\ \dfrac{K}{2}-\dfrac{x}{2}=0, & x=K \\ K\Phi(-\infty)-x\Phi(-\infty)=0, & x>K \end{array}\right\} = (K-x)^+
$$

at $t = T$. Similarly, we can check that

$$\lim_{T\to\infty} d_-(T-t) = \begin{cases} +\infty, & r > \dfrac{\sigma^2}{2}, \\ 0, & r = \dfrac{\sigma^2}{2}, \\ -\infty, & r < \dfrac{\sigma^2}{2}, \end{cases}$$

and $\lim_{T\to\infty} d_+(T-t) = +\infty$, hence

$$\lim_{T\to\infty} \mathrm{Bl_p}(K,x,\sigma,r,T-t) = 0, \qquad t \in \mathbb{R}_+.$$

Exercise 7.5 (Exercise 4.14 continued).

a) Substituting $g(x,t) = x^2 f(t)$ in (7.33), we find $f'(t) = -(r+\sigma^2)f(t)$, hence

$$f(t) = f(0)\,\mathrm{e}^{-(r+\sigma^2)t} = f(T)\,\mathrm{e}^{(r+\sigma^2)(T-t)},$$

hence $g(x,t) = f(T)x^2\,\mathrm{e}^{(r+\sigma^2)(T-t)} = x^2\,\mathrm{e}^{(r+\sigma^2)(T-t)}$ due to the terminal condition $g(x,T) = x^2$.

b) We have $\xi_t = \dfrac{\partial g}{\partial x} g(S_t,t) = 2S_t\,\mathrm{e}^{(r+\sigma^2)(T-t)}$, and

$$\begin{aligned} \eta_t &= \frac{1}{A_t}\left(g(S_t,t) - \xi_t S_t\right) \\ &= \frac{1}{A_0 e^{rt}}\left(S_t^2\,\mathrm{e}^{(r+\sigma^2)(T-t)} - 2S_t^2\,\mathrm{e}^{(r+\sigma^2)(T-t)}\right) \\ &= -\frac{S_t^2}{A_0}\,\mathrm{e}^{(T-2t)r+(T-t)\sigma^2}, \qquad t \in [0,T]. \end{aligned}$$

Exercise 7.6

a) Counting approximately 46 days to maturity, we have

$$\begin{aligned} d_-(T-t) &= \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}} \\ &= \frac{(0.04377-(0.9)^2/2)(46/365)+\log(17.2/36.08)}{0.9\sqrt{46/365}} \\ &= -2.461179058, \end{aligned}$$

and

$$d_+(T-t) = d_-(T-t) + 0.9\sqrt{46/365} = -2.14167602.$$

From the standard Gaussian cumulative distribution table we get

$$\Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098$$

and

$$\Phi(d_-(T-t)) = \Phi(-2.46) = 0.00692406,$$

hence

$$\begin{aligned} f(t,S_t) &= S_t\Phi(d_+(T-t)) - K\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\ &= 17.2\times 0.0161098 - 36.08\times\mathrm{e}^{-0.04377\times 46/365}\times 0.00692406 \\ &= \text{HK\$ } 0.028642744. \end{aligned}$$

For comparison, running the corresponding Black-Scholes R script of Figure 7.21 yields

$$\mathrm{BSCall}(17.2, 36.08, 0.04377, 46/365, 0.9) = 0.02864235.$$

b) We have

$$\eta_t = \frac{\partial f}{\partial x}(t, S_t) = \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098, \tag{A.23}$$

hence one should only hold a fractional quantity equal to 16.10 units in the risky asset in order to hedge 1000 such call options when $\sigma = 0.90$.

c) From the curve it turns out that when $f(t, S_t) = 10 \times 0.023 = \text{HK\$ } 0.23$, the volatility σ is approximately equal to $\sigma = 122\%$.

This approximate value of implied volatility can be found under the column “Implied Volatility (IV.)” on this set of market data from the Hong Kong Stock Exchange:

Updated: 6 November 2008

Basic Data

DW Code	Issuer	UL	Call /Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike	Entitle-ment Ratio^
01897	FB	00066	Call	Standard	18-12-2007	23-12-2008	36.08	10

Market Data

Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Day High ($)	Day Low ($)	Closing Price # ($)	T/O ('000)	UL Price ($)
138,000,000	16.43	0.780	125.375	0.000	0.000	0.023	0	17.200

Figure S.13: Market data for the warrant #01897 on the MTR Corporation.

Remark: a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

Exercise 7.7

a) We find $h(x) = x - K$.

b) Letting $g(t, x)$, the PDE rewrites as

$$(x - \alpha(t))r = -\alpha'(t) + rx,$$

hence $\alpha(t) = \alpha(0)\,\mathrm{e}^{rt}$ and $g(t,x) = x - \alpha(0)\,\mathrm{e}^{rt}$. The final condition

$$g(T, x) = h(x) = x - K$$

yields $\alpha(0) = K\,\mathrm{e}^{-rT}$ and $g(t,x) = x - K\,\mathrm{e}^{-(T-t)r}$.

c) We have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1,$$

hence

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{g(t, S_t) - S_t}{A_t} = \frac{S_t - K\,\mathrm{e}^{-(T-t)r} - S_t}{A_t} = -K\,\mathrm{e}^{-rT}.$$

Note that we could also have directly used the identification

$$V_t = g(S_t,t) = S_t - K\mathrm{e}^{-(T-t)r} = S_t - K\mathrm{e}^{-rT}A_t = \xi_t S_t + \eta_t A_t,$$

which immediately yields $\xi_t = 1$ and $\eta_t = -K\mathrm{e}^{-rT}$.

d) It suffices to take $K = 0$, which shows that $g(t,x) = x$, $\xi_t = 1$ and $\eta_t = 0$.

Exercise 7.8

a) We develop two approaches.

(i) By financial intuition. We need to replicate a fixed amount of \$1 at maturity T, *without risk*. For this there is no need to invest in the stock. Simply invest $g(t,S_t) := \mathrm{e}^{-(T-t)r}$ at time $t \in [0,T]$ and at maturity T you will have $g(T,S_T) = \mathrm{e}^{(T-t)r}g(t,S_t) = \$1$.

(ii) By analysis and the Black-Scholes PDE. Given the hint, we try plugging a solution of the form $\underline{g(t,x) = f(t)}$, *not depending on the variable* x, into the Black-Scholes PDE (7.34). Given that here we have

$$\frac{\partial g}{\partial x}(t,x) = 0, \qquad \frac{\partial^2 g}{\partial x^2}(t,x) = 0, \qquad \text{and} \qquad \frac{\partial g}{\partial t}(t,x) = f'(t),$$

we find that the Black-Scholes PDE reduces to $rf(t) = f'(t)$ with the terminal condition $f(T) = g(T,x) = 1$. This equation has for solution $f(t) = \mathrm{e}^{-(T-t)r}$ and this is also the unique solution $g(t,x) = f(t) = \mathrm{e}^{-(T-t)r}$ of the Black-Scholes PDE (7.34) with terminal condition $g(T,x) = 1$.

b) We develop two approaches.

(i) By financial intuition. Since the terminal payoff \$1 is risk-free we do not need to invest in the risky asset, hence we should keep $\underline{\xi_t = 0}$. Our portfolio value at time t becomes

$$V_t = g(t,S_t) = \mathrm{e}^{-(T-t)r} = \xi_t S_t + \eta_t A_t = \eta_t A_t$$

with $A_t = \mathrm{e}^{rt}$, so that we find $\eta_t = \mathrm{e}^{-rT}$, $t \in [0,T]$. This portfolio strategy remains constant over time, hence it is clearly self-financing.

(ii) By analysis. The Black-Scholes theory of Proposition 7.1 tells us that

$$\xi_t = \frac{\partial g}{\partial x}(t,x) = 0,$$

and

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{V_t}{A_t} = \frac{\mathrm{e}^{-(T-t)r}}{\mathrm{e}^{rt}} = \mathrm{e}^{-rT}.$$

Exercise 7.9 Log-contracts.

a) Substituting the function $g(x,t) := f(t) + \log x$ in the PDE (7.33) we have

$$0 = f'(t) + r - \frac{\sigma^2}{2},$$

hence

$$f(t) = f(0) - \left(r - \frac{\sigma^2}{2}\right)t,$$

with $f(0) = \left(r - \frac{\sigma^2}{2}\right)T$ in order to match the terminal condition $g(x,T) := \log x$, hence we have

$$g(x,t) = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x, \qquad x > 0.$$

b) Substituting the function

$$h(x,t) := u(t)g(x,t) = u(t)\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right)$$

in the PDE (7.35), we find $u'(t) = ru(t)$, hence $u(t) = u(0)\,\mathrm{e}^{rt} = \mathrm{e}^{-(T-t)r}$, with $u(T) = 1$, and we conclude to

$$h(x,t) = u(t)g(x,t) = \mathrm{e}^{-(T-t)r}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right),$$

$x > 0$, $t \in [0,T]$.

c) We have

$$\xi_t = \frac{\partial h}{\partial x}(t,S_t) = \frac{\mathrm{e}^{-(T-t)r}}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

and

$$\begin{aligned}\eta_t &= \frac{1}{A_t}(h(t,S_t) - \xi_t S_t) \\ &= \frac{\mathrm{e}^{-rT}}{A_0}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x - 1\right),\end{aligned}$$

$0 \leqslant t \leqslant T$.

Exercise 7.10 Binary options.

a) From Proposition 7.1, the function $C_d(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 C}{\partial x^2}(t,x), \\ \\ C(T,x) = \mathbb{1}_{[K,\infty)}(x). \end{cases}$$

b) We check by direct differentiation that the Black-Scholes PDE is satisfied by the function $C(t,x)$, together with the terminal condition $C(T,x) = \mathbb{1}_{[K,\infty)}(x)$ as t tends to T.

Exercise 7.11

a) By (5.32) we have

$$S_t = S_0\,\mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s.$$

b) By the self-financing condition (6.8) we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \alpha\xi_t S_t dt + \sigma\xi_t dB_t \\ &= rV_t dt + (\alpha - r)\xi_t S_t dt + \sigma\xi_t dB_t, \end{aligned} \tag{A.24}$$

$t \in \mathbb{R}_+$. Rewriting (7.37) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,$$

with

$$u_t = \sigma, \quad \text{and} \quad v_t = \alpha S_t, \qquad t \in \mathbb{R}_+,$$

the application of Itô's formula Theorem 5.9 to $V_t = C(t,S_t)$ shows that

$$\begin{aligned} dC(t,S_t) = {} & v_t \frac{\partial C}{\partial x}(t,S_t)dt + u_t \frac{\partial C}{\partial x}(t,S_t)dB_t \\ & + \frac{\partial C}{\partial t}(t,S_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt \end{aligned}$$

$$= \frac{\partial C}{\partial t}(t,S_t)dt + \alpha S_t \frac{\partial C}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt + \sigma \frac{\partial C}{\partial x}(t,S_t)dB_t. \tag{A.25}$$

Identifying the terms in dB_t and dt in (A.24) and (A.25) above, we get

$$\begin{cases} rC(t,S_t) = \frac{\partial C}{\partial t}(t,S_t) + rS_t \frac{\partial C}{\partial x}(t,S_t) + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial x^2}(t,S_t), \\ \xi_t = \frac{\partial C}{\partial x}(t,S_t), \end{cases}$$

hence the function $C(t,x)$ satisfies the usual Black-Scholes PDE

$$rC(t,x) = \frac{\partial C}{\partial t}(t,x) + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t,x), \quad x > 0, \quad 0 \leqslant t \leqslant T, \tag{A.26}$$

with the terminal condition $C(T,x) = \mathrm{e}^x$, $x \in \mathbb{R}$.

c) Based on (7.38), we compute

$$\begin{cases} \frac{\partial C}{\partial t}(t,x) = \left(r + xh'(t) + \frac{\sigma^2}{2r}h(t)h'(t) \right) C(t,x), \\ \frac{\partial C}{\partial x}(t,x) = h(t)C(t,x) \\ \frac{\partial^2 C}{\partial x^2}(t,x) = (h(t))^2 C(t,x), \end{cases}$$

hence the substitution of (7.38) into the Black-Scholes PDE (A.26) yields the ordinary differential equation

$$xh'(t) + \frac{\sigma^2}{2r}h'(t)h(t) + rxh(t) + \frac{\sigma^2}{2}(h(t))^2 = 0, \quad x > 0, \quad 0 \leqslant t \leqslant T,$$

which reduces to the ordinary differential equation $h'(t) + rh(t) = 0$ with terminal condition $h(T) = 1$ and solution $h(t) = \mathrm{e}^{(T-t)r}$, $t \in [0,T]$, which yields

$$C(t,x) = \exp\left(-(T-t)r + x\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1) \right).$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t,S_t) = \exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1) \right).$$

Exercise 7.12

a) Noting that $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$, we have the

$$\begin{aligned} \frac{\partial h}{\partial d}(S,d) &= S\varphi(d + \sigma\sqrt{T}) - K\mathrm{e}^{-rT}\varphi(d) \\ &= \frac{S}{\sqrt{2\pi}}\mathrm{e}^{-\left(d+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\,\mathrm{e}^{-d^2/2} \\ &= \frac{S}{\sqrt{2\pi}}\mathrm{e}^{-d^2/2-\sigma\sqrt{T}d-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\,\mathrm{e}^{-d^2/2}, \end{aligned}$$

hence the vanishing of $\frac{\partial h}{\partial d}(S,d_*(S))$ at $d = d_*(S)$ yields

$$\frac{S}{\sqrt{2\pi}}\mathrm{e}^{-d_*^2(S)/2-\sigma\sqrt{T}d_*(S)-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2} = 0,$$

i.e. $d_*(S) = \dfrac{\log(S/K) + rT - \sigma^2 T/2}{\sigma\sqrt{T}}$. We can also check that

$$\begin{aligned}
&\frac{\partial^2 h}{\partial d^2}(S, d_*(S)) = \frac{\partial}{\partial d}\left(\frac{S}{\sqrt{2\pi}}\,\mathrm{e}^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}}\,\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2}\right)\\
&= -\big(d_*(S) + \sigma\sqrt{T}\big)\frac{S}{\sqrt{2\pi}}\,\mathrm{e}^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} + \frac{K}{\sqrt{2\pi}} d_*(S)\,\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2}\\
&= -\big(d_*(S) + \sigma\sqrt{T}\big)\frac{K}{\sqrt{2\pi}}\,\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2} + \frac{K}{\sqrt{2\pi}} d_*(S)\,\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2}\\
&= -\sigma\sqrt{T}\frac{K}{\sqrt{2\pi}}\,\mathrm{e}^{-rT}\,\mathrm{e}^{-d_*^2(S)/2} < 0,
\end{aligned}$$

hence the function $d \longmapsto h(S,d) := S\Phi\big(d + \sigma\sqrt{T}\big) - K\,\mathrm{e}^{-rT}\Phi(d)$ admits a *maximum* at $d = d_*(S)$, and

$$\begin{aligned}
h(S, d_*(S)) &= S\Phi\big(d_*(S) + \sigma\sqrt{T}\big) - K\,\mathrm{e}^{-rT}\Phi(d_*(S))\\
&= S\Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - K\,\mathrm{e}^{-rT}\Phi\left(\frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

is the Black-Scholes call option price.

b) Since $\frac{\partial h}{\partial d}(S, d_*(S)) = 0$, we find

$$\begin{aligned}
\Delta &= \frac{d}{dS}h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d_*'(S)\frac{\partial h}{\partial d}(S, d_*(S))\\
&= \Phi\big(d_*(S) + \sigma\sqrt{T}\big) = \Phi\left(\frac{\log(S/K) + rT + \sigma^2 T/2}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Exercise 7.13 When $\sigma > 0$ we have

$$\begin{aligned}
\frac{\partial g_c}{\partial \sigma} &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}d_+(T-t) - K\,\mathrm{e}^{-(T-t)r}\Phi'\big(d_-(T-t)\big)\frac{\partial}{\partial\sigma}d_-(T-t)\\
&= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}d_+(T-t)\\
&\quad -K\,\mathrm{e}^{-(T-t)r}\Phi'\big(d_+(T-t)\big)\,\mathrm{e}^{(T-t)r+\log(x/K)}\frac{\partial}{\partial\sigma}d_-(T-t)\\
&= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}\big(d_+(T-t) - d_-(T-t)\big)\\
&= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}\big(\sigma\sqrt{T-t}\big)\\
&= x\sqrt{T-t}\,\Phi'\big(d_+(T-t)\big),
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\Phi'\big(d_-(T-t)\big) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(d_-(T-t))^2/2}\\
&= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(d_-(T-t))^2/2+(T-t)r+\log(x/K)}\\
&= \Phi'\big(d_+(T-t)\big)\,\mathrm{e}^{(T-t)r+\log(x/K)}.
\end{aligned}$$

We note that the Black-Scholes European call price is an increasing function of the volatility parameter $\sigma > 0$. Relation (7.39) can be obtained from

$$\begin{aligned}
&\big(d_+(T-t)\big)^2 - \big(d_-(T-t)\big)^2\\
&= \big(d_+(T-t) + d_-(T-t)\big)\big(d_+(T-t) - d_-(T-t)\big)
\end{aligned}$$

$$= \quad 2r(T-t) + 2\log\frac{x}{K}.$$

Exercise 7.14

a) Given that

$$p^* = \frac{r_N - a_N}{b_N - a_N} = \frac{1}{2} \quad \text{and} \quad q^* = \frac{b_N - r_N}{b_N - a_N} = \frac{1}{2},$$

Relation (4.13) reads

$$\begin{aligned}\widetilde{v}(t,x) &= \frac{1}{2}\widetilde{v}(t+T/N, x(1+rT/N)(1-\sigma\sqrt{T/N})) \\ &\quad + \frac{1}{2}\widetilde{v}(t+T/N, x(1+rT/N)(1+\sigma\sqrt{T/N})).\end{aligned}$$

After letting $\Delta T := T/N$ and applying Taylor's formula at the second order we obtain

$$\begin{aligned}0 &= \frac{1}{2}\left(\widetilde{v}\left(t+\Delta T, x\left(1+r\Delta T-\sigma\sqrt{\Delta T}\right)\right) - \widetilde{v}(t,x)\right) \\ &\quad + \frac{1}{2}\left(\widetilde{v}\left(t+\Delta T, x\left(1+r\Delta T+\sigma\sqrt{\Delta T}\right)\right) - \widetilde{v}(t,x)\right) + o(\Delta T) \\ &= \frac{1}{2}\left(\Delta T\frac{\partial \widetilde{v}}{\partial t}(t,x) + x\left(r\Delta T - \sigma\sqrt{\Delta T}\right)\frac{\partial \widetilde{v}}{\partial x}(t,x)\right. \\ &\quad \left. + \frac{x^2}{2}\left(r\Delta T - \sigma\sqrt{\Delta T}\right)^2\frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x) + o(\Delta T)\right) \\ &\quad + \frac{1}{2}\left(\Delta T\frac{\partial \widetilde{v}}{\partial t}(t,x) + x\left(r\Delta T + \sigma\sqrt{\Delta T}\right)\frac{\partial \widetilde{v}}{\partial x}(t,x)\right. \\ &\quad \left. + \frac{x^2}{2}\left(r\Delta T + \sigma\sqrt{\Delta T}\right)^2\frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x) + o(\Delta T)\right) + o(\Delta T) \\ &= \Delta T\frac{\partial \widetilde{v}}{\partial t}(t,x) + rx\Delta T\frac{\partial \widetilde{v}}{\partial x}(t,x) + \frac{x^2}{2}\left(\sigma\sqrt{\Delta T}\right)^2\frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x) + o(\Delta T),\end{aligned}$$

which shows that

$$\frac{\partial \widetilde{v}}{\partial t}(t,x) + rx\frac{\partial \widetilde{v}}{\partial x}(t,x) + x^2\frac{\sigma^2}{2}\frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x) = -\frac{o(\Delta T)}{\Delta T},$$

hence as N tends to infinity (or as ΔT tends to 0) we find*

$$0 = \frac{\partial \widetilde{v}}{\partial t}(t,x) + rx\frac{\partial \widetilde{v}}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 \widetilde{v}}{\partial x^2}(t,x),$$

showing that the function $v(t,x) := \mathrm{e}^{(T-t)r}\widetilde{v}(t,x)$ solves the classical Black-Scholes PDE

$$rv(t,x) = \frac{\partial v}{\partial t}(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x).$$

b) Similarly, we have

$$\begin{aligned}\xi_t^{(1)}(x) &= \frac{v\left(t,(1+b_N)x\right) - v\left(t,(1+a_N)x\right)}{x(b_N-a_N)} \\ &= \frac{v\left(t,(1+r/N)(1+\sigma\sqrt{T/N})x\right) - v\left(t,(1+r/N)(1-\sigma\sqrt{T/N})x\right)}{2x(1+r/N)\sigma\sqrt{T/N}} \\ &\to \frac{\partial v}{\partial x}(t,x),\end{aligned}$$

as N tends to infinity.

*The notation $o(\Delta T)$ denotes any function of ΔT such that $\lim_{\Delta T\to 0} o(\Delta T)/\Delta T = 0$.

Chapter 8

Exercise 8.1 (Exercise 7.1 continued). Since $r=0$ we have $\mathbb{P}=\mathbb{P}^*$ and

$$\begin{aligned}
g(t,B_t) &= \mathbb{E}^*\left[B_T^2 \mid \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[(B_T-B_t+B_t)^2 \mid \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[(B_T-B_t+x)^2\right]_{x=B_t} \\
&= \mathbb{E}^*\left[(B_T-B_t)^2+2x(B_T-B_t)+x^2\right]_{x=B_t} \\
&= \mathbb{E}^*\left[(B_T-B_t)^2\right]+2x\mathbb{E}^*[B_T-B_t]+B_t^2 \\
&= B_t^2+T-t, \qquad 0\leqslant t\leqslant T,
\end{aligned}$$

hence ξ_t is given by the partial derivative

$$\xi_t=\frac{\partial g}{\partial x}(t,B_t)=2B_t, \qquad 0\leqslant t\leqslant T,$$

with

$$\eta_t=\frac{g(t,B_t)-\xi_t B_t}{A_0}=\frac{B_t^2+(T-t)-2B_t^2}{A_0}=\frac{(T-t)-B_t^2}{A_0}, \quad 0\leqslant t\leqslant T.$$

Exercise 8.2 Since $B_T \simeq \mathcal{N}(0,T)$, we have

$$\begin{aligned}
\mathbb{E}[\phi(S_T)] &= \mathbb{E}\left[\phi\left(S_0\,\mathrm{e}^{\sigma B_T+(r-\sigma^2/2)T}\right)\right] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\phi\left(S_0\,\mathrm{e}^{\sigma y+(r-\sigma^2/2)T}\right)\mathrm{e}^{-y^2/(2T)}dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\infty}\phi(x)\,\mathrm{e}^{-((\sigma^2/2-r)T+\log x)^2/(2\sigma^2T)}\frac{dx}{x} \\
&= \int_{-\infty}^{\infty}\phi(x)g(x)dx,
\end{aligned}$$

under the change of variable

$$x=S_0\,\mathrm{e}^{\sigma y+(r-\sigma^2/2)T}, \quad \text{with } dx=\sigma S_0\,\mathrm{e}^{\sigma y+(r-\sigma^2/2)T}dy=\sigma x dy,$$

i.e.

$$y=\frac{(\sigma^2/2-r)T+\log(x/S_0)}{\sigma} \quad \text{and} \quad dy=\frac{dx}{\sigma x},$$

where

$$g(x):=\frac{1}{x\sqrt{2\pi\sigma^2T}}\,\mathrm{e}^{-((\sigma^2/2-r)T+\log(x/S_0))^2/(2\sigma^2T)}$$

is the *lognormal* probability density function with location parameter $(r-\sigma^2/2)T+\log S_0$ and scale parameter $\sigma\sqrt{T}$.

Exercise 8.3 We have

$$\begin{aligned}
&\mathbb{E}^*[\phi(pS_{T_1}+qS_{T_2})]\leqslant\mathbb{E}^*[p\phi(S_{T_1})+q\phi(S_{T_2})] && \text{since } \phi \text{ is convex,}\\
&\quad=p\,\mathbb{E}^*[\phi(S_{T_1})]+q\,\mathbb{E}^*[\phi(S_{T_2})] \\
&\quad=p\,\mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_2}\mid\mathscr{F}_{T_1}])]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{because } (S_t)_{t\in\mathbb{R}_+} \text{ is a martingale,}\\
&\quad\leqslant p\,\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_2})\mid\mathscr{F}_{T_1}]]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by Jensen's inequality,}\\
&\quad=p\,\mathbb{E}^*[\phi(S_{T_2})]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by the tower property,}
\end{aligned}$$

$$= \mathbb{E}^*[\phi(S_{T_2})], \qquad \text{because } p+q=1.$$

Remark: This type of technique can be useful in order to get an upper price estimate from Black-Scholes when the actual option price is difficult to compute: here the closed-form computation would involve a double integration of the form

$$\begin{aligned}
\mathbb{E}^*[\phi(pS_{T_1}+qS_{T_2})] &= \mathbb{E}^*\left[\phi\left(pS_0\,\mathrm{e}^{\sigma B_{T_1}-\sigma^2 T_1/2}+qS_0\,\mathrm{e}^{\sigma B_{T_2}-\sigma^2 T_2/2}\right)\right]\\
&= \mathbb{E}^*\left[\phi\left(S_0\,\mathrm{e}^{\sigma B_{T_1}-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{(B_{T_2}-B_{T_1})\sigma-(T_2-T_1)\sigma^2/2}\right)\right)\right]\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\left(S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2}\right)\right)\\
&\qquad\times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2}\right)-K\right)^+\\
&\qquad\times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \frac{1}{2\pi}\int_{\{(x,y)\in\mathbb{R}^2\,:\,S_0\,\mathrm{e}^{\sigma x}(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2})\geqslant K\,\mathrm{e}^{\sigma^2 T_1/2}\}}\\
&\qquad (S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2})-K)\\
&\qquad\times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \cdots
\end{aligned}$$

Exercise 8.4

a) The European *call* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ decreases with the strike price K, because the option payoff $(S_T-K)^+$ decreases and the expectation operator preserves the ordering of random variables.
b) The European *put* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(K-S_T)^+]$ increases with the strike price K, because the option payoff $(K-S_T)^+$ increases and the expectation operator preserves the ordering of random variables.

Exercise 8.5

a) Using Jensen's inequality and the martingale property of the discounted asset price process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+\mid\mathcal{F}_t] &\geqslant \mathrm{e}^{-(T-t)r}\left(\mathbb{E}^*[S_T-K\mid\mathcal{F}_t]\right)^+\\
&= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{(T-t)r}S_t-K\right)^+\\
&= \left(S_t-K\,\mathrm{e}^{-(T-t)r}\right)^+,\quad 0\leqslant t\leqslant T.
\end{aligned}$$

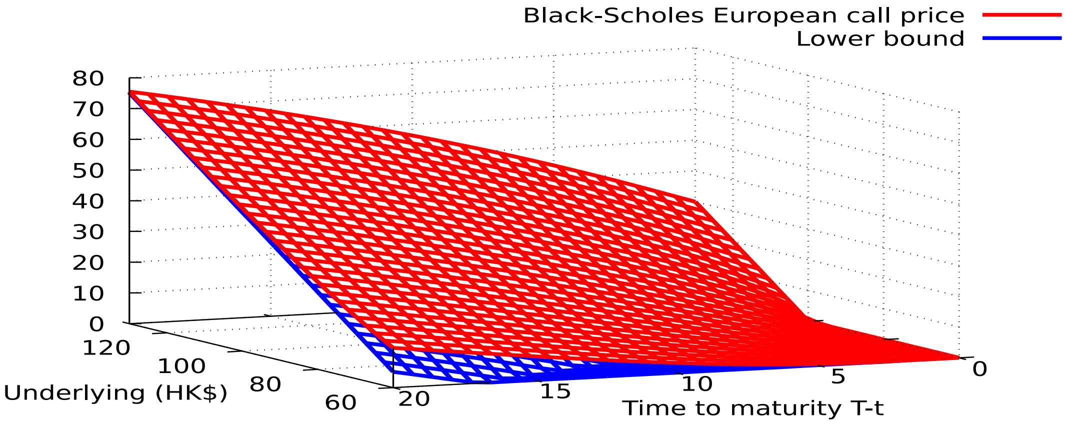


Figure S.14: Lower bound *vs* Black-Scholes call price.

b) Similarly, by Jensen's inequality and the martingale property, we find

$$\begin{aligned} e^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t] &\geqslant e^{-(T-t)r}\big(\mathbb{E}^*[K-S_T \mid \mathcal{F}_t]\big)^+ \\ &= e^{-(T-t)r}\big(K-e^{(T-t)r}S_t\big)^+ \\ &= \big(K e^{-(T-t)r}-S_t\big)^+, \quad 0 \leqslant t \leqslant T. \end{aligned}$$

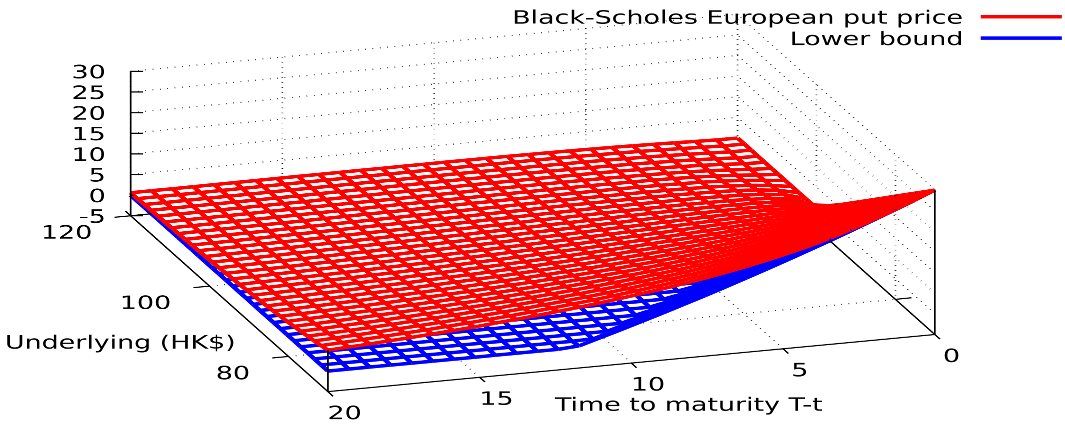


Figure S.15: Lower bound *vs* Black-Scholes put option price.

Exercise 8.6

a) (i) The bull spread option can be realized by purchasing one European call option with strike price K_1 and by short selling (or issuing) one European call option with strike price K_2, because the bull spread payoff function can be written as

$$x \longmapsto (x-K_1)^+ - (x-K_2)^+.$$

see http://optioncreator.com/st3ce7z.

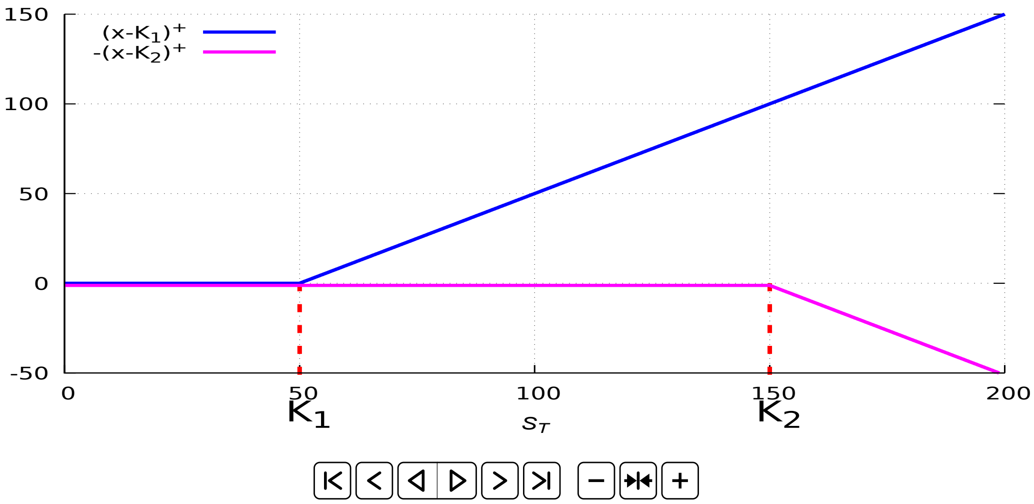


Figure S.16: Bull spread option as a combination of call and put options.*

(ii) The bear spread option can be realized by purchasing one European put option with strike price K_2 and by short selling (or issuing) one European put option with strike price K_1, because the bear spread payoff function can be written as

$$x \longmapsto -(K_1 - x)^+ + (K_2 - x)^+,$$

see http://optioncreator.com/stmomsb.

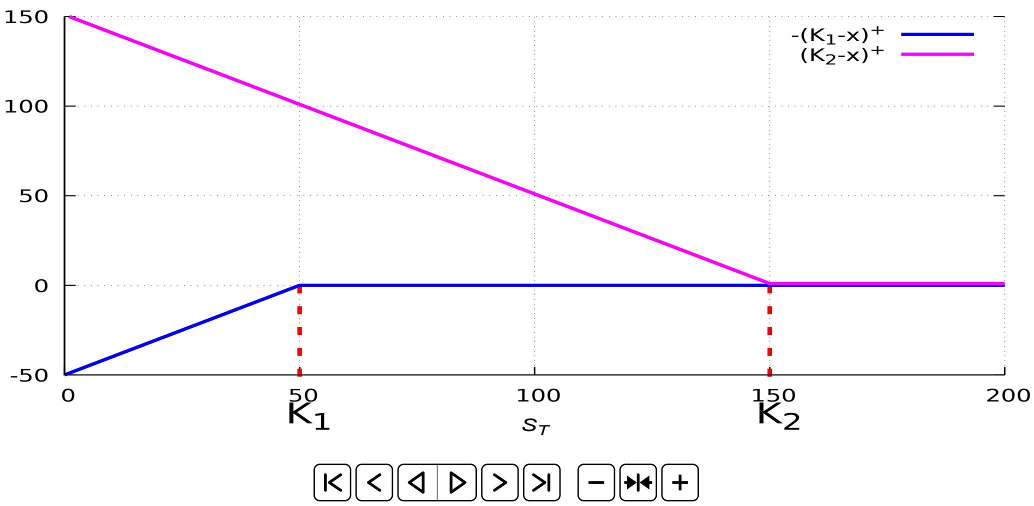


Figure S.17: Bear spread option as a combination of call and put options.†

b) (i) The bull spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\mathrm{Bl}(K_1, S_t, \sigma, r, T - t) - \mathrm{Bl}(K_2, S_t, \sigma, r, T - t).$$

(ii) The bear spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\mathrm{Bl}(K_2, S_t, \sigma, r, T - t) - \mathrm{Bl}(K_1, S_t, \sigma, r, T - t).$$

*The animation works in Acrobat Reader on the entire pdf file.
†The animation works in Acrobat Reader on the entire pdf file.

Exercise 8.7

a) The payoff function can be written as

$$\begin{aligned}(K_1-x)^+ + (K_2-x)^+ - 2(x-(K_1+K_2)/2)^+ \\ = \quad (50-x)^+ + (150-x)^+ - 2(x-100)^+,\end{aligned}$$

see also https://optioncreator.com/stnurzg.

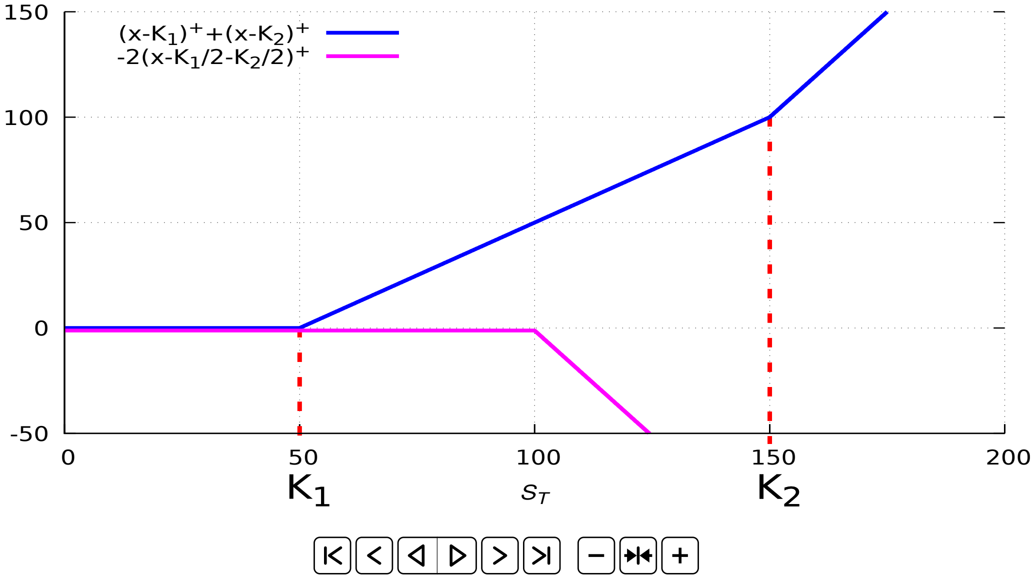


Figure S.18: Butterfly option as a combination of call options.*

Hence the butterfly option can be realized by:

1. purchasing one *call option* with strike price $K_1 = 50$, and
2. purchasing one *call option* with strike price $K_2 = 150$, and
3. issuing (or selling) two *call options* with strike price $(K_1+K_2)/2 = 100$.

b) Denoting by $\phi(x)$ the payoff function, the self-financing replicating portfolio strategy $\left(\xi_t(S_{t-1})\right)_{t=1,2,\ldots,N}$ hedging the contingent claim with payoff $C = \phi(S_N)$ is given by

$$\xi_t(x) = \frac{\mathbb{E}^*\left[\phi\left(x(1+b)\prod_{j=t+1}^N(1+R_j)\right) - \phi\left(x(1+a)\prod_{j=t+1}^N(1+R_j)\right)\right]}{(b-a)(1+r)^{N-t}S_{t-1}}$$

with $x = S_{t-1}$. Therefore, $\xi_t(x)$ will be positive (holding) when $x = S_{t-1}$ is sufficiently below $(K_1+K_2)/2$, and $\xi_t(x)$ will be negative (short selling) when $x = S_{t-1}$ is sufficiently above $(K_1+K_2)/2$.

*The animation works in Acrobat Reader.

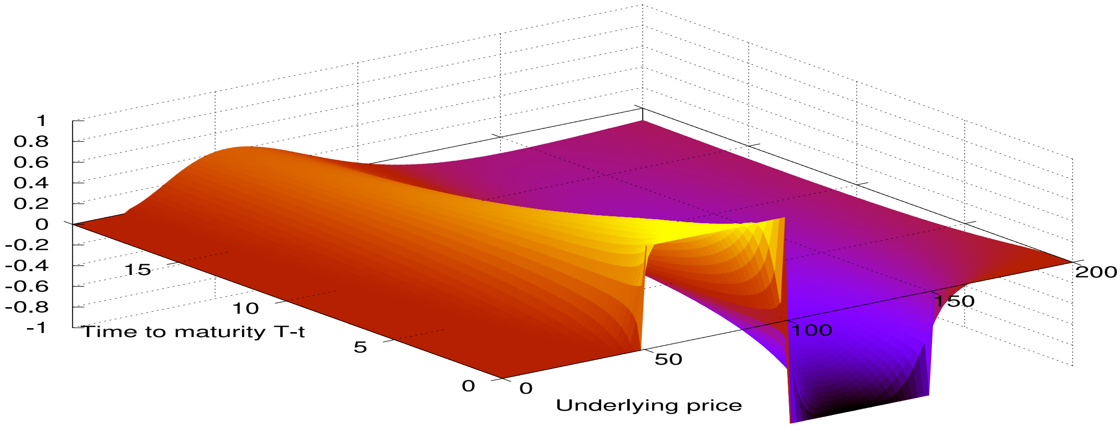


Figure S.19: Delta of a butterfly option with strike prices $K_1 = 50$ and $K_2 = 150$.

Exercise 8.8

a) We have

$$\begin{aligned} C_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\,\mathbb{E}^*[\mathrm{e}^{-rT}S_T \mid \mathcal{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\,\mathrm{e}^{-rt}S_t - K\,\mathrm{e}^{-(T-t)r} \\ &= S_t - K\,\mathrm{e}^{-(T-t)r}. \end{aligned}$$

We can check that the function $g(x,t) = x - K\,\mathrm{e}^{-(T-t)r}$ satisfies the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t)$$

with terminal condition $g(x,T) = x - K$, since $\partial g(x,t)/\partial t = -rK\,\mathrm{e}^{-(T-t)r}$ and $\partial g(x,t)/\partial x = 1$.

b) We simply take $\xi_t = 1$ and $\eta_t = -K\,\mathrm{e}^{-rT}$ in order to have

$$C_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt} = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T.$$

Note again that this hedging strategy is *constant* over time, and the relation $\xi_t = \partial g(S_t,t)/\partial x$ for the option Delta, cf. (A.23), is satisfied.

Exercise 8.9 Option pricing with dividends (Exercise 7.3 continued).

a) Let $\widehat{\mathbb{P}}$ denote the probability measure under which the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ defined by

$$d\widehat{B}_t = \frac{\mu - r}{\sigma}dt + dB_t$$

is a standard Brownian motion. Under absence of arbitrage the asset price process $(S_t)_{t\in\mathbb{R}_+}$ has the dynamics

$$\begin{aligned} dS_t &= (\mu - \delta)S_t dt + \sigma S_t dB_t \\ &= (r - \delta)S_t dt + \sigma S_t d\widehat{B}_t, \end{aligned}$$

and the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ satisfies

$$d\widetilde{S}_t = -\delta\widetilde{S}_t dt + \sigma\widetilde{S}_t d\widehat{B}_t.$$

Assuming that the dividend yield δS_t per share is continuously reinvested in the portfolio, the self-financing portfolio condition

$$dV_t = \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta\xi_t S_t dt}_{\text{dividend payout}}$$

$$\begin{aligned}
&= r\eta_t A_t dt + \xi_t((r-\delta)S_t dt + \sigma S_t d\widehat{B}_t) + \delta\xi_t S_t dt \\
&= r\eta_t A_t dt + \xi_t(rS_t dt + \sigma S_t d\widehat{B}_t) \\
&= rV_t dt + \sigma\xi_t S_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

which yields

$$\begin{aligned}
d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt}V_t\right) \\
&= -r\,\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}dV_t \\
&= \sigma\xi_t\,\mathrm{e}^{-rt}S_t d\widehat{B}_t \\
&= \sigma\xi_t\widetilde{S}_t d\widehat{B}_t \\
&= \xi_t(d\widetilde{S}_t + \delta\widetilde{S}_t dt), \qquad t \in \mathbb{R}_+.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\widetilde{V}_t - \widetilde{V}_0 &= \int_0^t d\widetilde{V}_u \\
&= \sigma\int_0^t \xi_u\widetilde{S}_u d\widehat{B}_u \\
&= \int_0^t \xi_u d\widetilde{S}_u + \delta\int_0^t \widetilde{S}_u du, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

Here, the asset price process $(\mathrm{e}^{\delta t}S_t)_{t\in\mathbb{R}_+}$ with added dividend yield satisfies the equation

$$d(\mathrm{e}^{\delta t}S_t) = r\,\mathrm{e}^{\delta t}S_t dt + \sigma\left(\mathrm{e}^{\delta t}S_t\right)d\widehat{B}_t,$$

and after discount, the process $(\mathrm{e}^{-rt}\,\mathrm{e}^{\delta t}S_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-(r-\delta)t}S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\widehat{\mathbb{P}}$.

b) We have

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma\int_0^t \xi_u\widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale under $\widehat{\mathbb{P}}$ from Proposition 8.1, hence

$$\begin{aligned}
\widetilde{V}_t &= \widehat{\mathbb{E}}\big[\widetilde{V}_T \mid \mathscr{F}_t\big] \\
&= \mathrm{e}^{-rT}\widehat{\mathbb{E}}[V_T \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-rT}\widehat{\mathbb{E}}[C \mid \mathscr{F}_t],
\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

c) After discounting the payoff $(S_T-K)^+$ at the continuously compounded interest rate r, we obtain

$$\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}\big[(S_T-K)^+ \mid \mathscr{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}\big[\big(S_0\,\mathrm{e}^{\sigma\widehat{B}_T+(r-\delta-\sigma^2/2)T}-K\big)^+ \mid \mathscr{F}_t\big] \\
&= \mathrm{e}^{-(T-t)\delta}\big(\mathrm{e}^{-(T-t)(r-\delta)}\widehat{\mathbb{E}}\big[\big(S_0\,\mathrm{e}^{\sigma\widehat{B}_T+(r-\delta-\sigma^2/2)T}-K\big)^+ \mid \mathscr{F}_t\big]\big) \\
&= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\
&= \mathrm{e}^{-(T-t)\delta}\big(S_t\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(T-t)(r-\delta)}\Phi\big(d_-^\delta(T-t)\big)\big) \\
&= \mathrm{e}^{-(T-t)\delta}S_t\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big), \qquad 0 \leqslant t < T,
\end{aligned}$$

where

$$d_+^\delta(T-t) := \frac{\log(S_t/K)+(r-\delta+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}$$

and

$$d_-^\delta(T-t) := \frac{\log(S_t/K)+(r-\delta-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}.$$

We also have

$$g(t,x) = \mathrm{Bl}\big(x\,\mathrm{e}^{-(T-t)\delta}, K, \sigma, r, T-t\big), \qquad 0 \leqslant t \leqslant T.$$

Exercise 8.10 We start by pricing the "inner" at-the-money option with payoff $(S_{T_2} - S_{T_1})^+$ and strike price $K = S_{T_1}$ at time T_1 as

$$\begin{aligned}
&\mathrm{e}^{-(T_2-T_1)r}\,\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathscr{F}_{T_1}\right] \\
&= S_{T_1}\Phi\left(\frac{(r+\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
&\quad - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{(r-\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
&= S_{T_1}\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right),
\end{aligned}$$

where we applied (8.20) with $T = T_2$, $t = T_1$, and $K = S_{T_1}$. As a consequence, the forward start option can be priced as

$$\begin{aligned}
&\mathrm{e}^{-(T_1-t)r}\,\mathbb{E}^*\left[\mathrm{e}^{-(T_2-T_1)r}\,\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathscr{F}_{T_1}\right] \,\Big|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T_1-t)r} \\
&\quad \times \mathbb{E}^*\left[S_{T_1}\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T_1-t)r} \\
&\quad \times \left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right)\mathbb{E}^*[S_{T_1} \mid \mathscr{F}_t] \\
&= S_t\left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right),
\end{aligned}$$

$0 \leqslant t \leqslant T_1$.

Exercise 8.11 (Exercise 7.9 continued). We have

$$\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\log S_T \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\log S_t + (\widehat{B}_T - \widehat{B}_t)\sigma + \left(r - \frac{\sigma^2}{2}\right)(T-t) \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\log S_t + \mathrm{e}^{-(T-t)r}\left(r - \frac{\sigma^2}{2}\right)(T-t),
\end{aligned}$$

$t \in [0,T]$.

Exercise 8.12 (Exercise 7.5 continued).

a) For all $t \in [0,T]$, we have

$$\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}[S_T^2 \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[S_t^2\frac{S_T^2}{S_t^2} \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}S_t^2\,\mathbb{E}\left[\frac{S_T^2}{S_t^2} \mid \mathscr{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r} S_t^2 \,\mathbb{E}\left[\frac{S_T^2}{S_t^2}\right] \\
&= \mathrm{e}^{-(T-t)r} S_t^2 \,\mathbb{E}\left[\mathrm{e}^{2(B_T-B_t)\sigma-(T-t)\sigma^2+2(T-t)r}\right] \\
&= \mathrm{e}^{-(T-t)r} S_t^2 \,\mathrm{e}^{-(T-t)\sigma^2+2(T-t)r}\,\mathbb{E}\left[\mathrm{e}^{2(B_T-B_t)\sigma}\right] \\
&= S_t^2 \,\mathrm{e}^{(r+\sigma^2)(T-t)},
\end{aligned}
$$

where we used the Gaussian moment generating function (MGF) formula (11.46)

$$
\mathbb{E}\left[\mathrm{e}^{2(B_T-B_t)\sigma}\right] = \mathrm{e}^{2(T-t)\sigma^2}
$$

for the normal random variable $B_T - B_t \simeq \mathscr{N}(0, T-t)$, $0 \leqslant t < T$.

b) For all $t \in [0,T]$, we have

$$
\xi_t = \frac{\partial C}{\partial x}(t,x)_{|x=S_t} = 2S_t \,\mathrm{e}^{(r+\sigma^2)(T-t)},
$$

i.e.

$$
\xi_t S_t = 2S_t^2 \,\mathrm{e}^{(r+\sigma^2)(T-t)} = 2C(t,S_t),
$$

and

$$
\begin{aligned}
\eta_t = \frac{C(t,S_t)-\xi_t S_t}{A_t} &= \frac{\mathrm{e}^{-rt}}{A_0}\left(S_t^2 \,\mathrm{e}^{(r+\sigma^2)(T-t)} - 2S_t^2 \,\mathrm{e}^{(r+\sigma^2)(T-t)}\right) \\
&= -\frac{S_t^2}{A_0} \,\mathrm{e}^{\sigma^2(T-t)+(T-2t)r},
\end{aligned}
$$

i.e.

$$
\eta_t A_t = -S_t^2 \frac{A_t}{A_0} \,\mathrm{e}^{\sigma^2(T-t)+(T-2t)r} = -S_t^2 \,\mathrm{e}^{\sigma^2(T-t)+(T-t)r} = -C(t,S_t).
$$

As for the self-financing condition, we have

$$
\begin{aligned}
dC(t,S_t) &= d\left(S_t^2 \,\mathrm{e}^{(r+\sigma^2)(T-t)}\right) \\
&= -(r+\sigma^2)\,\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + \mathrm{e}^{(r+\sigma^2)(T-t)} d\left(S_t^2\right) \\
&= -(r+\sigma^2)\,\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + \mathrm{e}^{(r+\sigma^2)(T-t)}\left(2S_t dS_t + \sigma^2 S_t^2 dt\right) \\
&= -r\,\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t \,\mathrm{e}^{(r+\sigma^2)(T-t)} dS_t,
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_t dS_t + \eta_t dA_t &= 2S_t \,\mathrm{e}^{(r+\sigma^2)(T-t)} dS_t - r\frac{S_t^2}{A_0} \,\mathrm{e}^{\sigma^2(T-t)+(T-2t)r} A_t dt \\
&= 2S_t \,\mathrm{e}^{(r+\sigma^2)(T-t)} dS_t - rS_t^2 \,\mathrm{e}^{\sigma^2(T-t)+(T-t)r} dt,
\end{aligned}
$$

which recovers $dC(t,S_t) = \xi_t dS_t + \eta_t dA_t$, *i.e.* the portfolio strategy is self-financing.

Exercise 8.13 (Exercise 7.11 continued).

a) The discounted process $X_t := \mathrm{e}^{-rt} S_t$ satisfies

$$
dX_t = (\alpha - r) X_t dt + \sigma \,\mathrm{e}^{-rs} dB_s,
$$

which is a martingale when $\alpha = r$ by Proposition 8.1, as in this case it becomes a stochastic integral with respect to a standard Brownian motion. This fact can be recovered by directly computing the conditional expectation $\mathbb{E}[X_t \mid \mathscr{F}_s]$ and showing it is equal to X_s. By (5.32), see Exercise 7.11, we have

$$
S_t = S_0 \,\mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s,
$$

hence

$$X_t = S_0 + \sigma \int_0^t \mathrm{e}^{-rs} dB_s, \qquad t \in \mathbb{R}_+,$$

and

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}\left[S_0 + \sigma \int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= \mathbb{E}[S_0] + \sigma\, \mathbb{E}\left[\int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= S_0 + \sigma\, \mathbb{E}\left[\int_0^s \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] + \sigma\, \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u + \sigma\, \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u \right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u \\
&= X_s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

b) We rewrite the stochastic differential equation satisfied by $(S_t)_{t\in\mathbb{R}_+}$ as

$$dS_t = \alpha S_t dt + \sigma dB_t = r S_t dt + \sigma d\widehat{B}_t,$$

where

$$d\widehat{B}_t := \frac{\alpha - r}{\sigma} S_t dt + dB_t,$$

which allows us to rewrite (5.32), by taking $\alpha := -r$ therein, as

$$S_t = \mathrm{e}^{rt}\left(S_0 + \sigma \int_0^t \mathrm{e}^{-rs} d\widehat{B}_s \right) = S_0\, \mathrm{e}^{rt} + \sigma \int_0^t \mathrm{e}^{(t-s)r} d\widehat{B}_s. \tag{A.27}$$

Taking

$$\psi_t := \frac{\alpha - r}{\sigma} S_t, \qquad 0 \leqslant t \leqslant T,$$

in the Girsanov Theorem 8.2, the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\mathbb{P}_\alpha$ defined by

$$\begin{aligned}
\frac{d\mathbb{P}_\alpha}{d\mathbb{P}} &:= \exp\left(-\int_0^T \psi_t dB_t - \frac{1}{2}\int_0^T \psi_t^2 dt \right) \\
&= \exp\left(-\frac{\alpha - r}{\sigma} \int_0^T S_t dB_t - \frac{1}{2}\left(\frac{\alpha - r}{\sigma} \right)^2 \int_0^T S_t^2 dt \right),
\end{aligned}$$

and $(X_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}_\alpha$.

c) Using (A.27) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
C(t, S_t) &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}_\alpha[\exp(S_T) \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{(T-u)r} d\widehat{B}_u \right) \Big|\, \mathcal{F}_t \right] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma \int_0^t \mathrm{e}^{(T-u)r} d\widehat{B}_u + \sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u \right) \Big|\, \mathcal{F}_t \right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t \right) \mathbb{E}_\alpha\left[\exp\left(\sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u \right) \Big|\, \mathcal{F}_t \right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t \right) \mathbb{E}_\alpha\left[\exp\left(\sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u \right) \right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t \right) \exp\left(\frac{\sigma^2}{2} \int_t^T \mathrm{e}^{2(T-u)r} du \right) \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t + \frac{\sigma^2}{4r}\left(\mathrm{e}^{2(T-t)r} - 1 \right) \right), \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)$$

and

$$\begin{aligned}\eta_t &= \frac{C(t, S_t) - \xi_t S_t}{A_t}\\ &= \frac{\mathrm{e}^{-(T-t)r}}{A_t}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)\\ &\quad -\frac{S_t}{A_t}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right).\end{aligned}$$

e) We have

$$\begin{aligned}dC(t, S_t) &= r\,\mathrm{e}^{-(T-t)r}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt\\ &\quad -rS_t\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt\\ &\quad -\frac{\sigma^2}{2}\mathrm{e}^{(T-t)r}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt\\ &\quad +\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dS_t\\ &\quad +\frac{1}{2}\mathrm{e}^{(T-t)r}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)\sigma^2 dt\\ &= r\,\mathrm{e}^{-(T-t)r}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt\\ &\quad -rS_t\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt + \xi_t dS_t.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\xi_t dS_t + \eta_t dA_t = \xi_t dS_t\\ &\qquad +r\,\mathrm{e}^{-(T-t)r}\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt\\ &\qquad -rS_t\exp\left(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt,\end{aligned}$$

showing that

$$dC(t, S_t) = \xi_t dS_t + \eta_t dA_t,$$

and confirming that the strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing.

Exercise 8.14

a) Using (A.27) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}&C(t, S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha[S_T^2 \mid \mathcal{F}_t]\\ &\quad = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \,\Big|\, \mathcal{F}_t\right]\\ &\quad = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u + \sigma\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \,\Big|\, \mathcal{F}_t\right]\\ &\quad = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \,\Big|\, \mathcal{F}_t\right]\\ &\qquad +2\sigma\,\mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)\mathbb{E}_\alpha\left[\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u \,\Big|\, \mathcal{F}_t\right]\end{aligned}$$

$$
\begin{aligned}
&+\sigma^2 e^{-(T-t)r} \mathbb{E}_\alpha \left[\left(\int_t^T e^{(T-u)r} d\widehat{B}_u \right)^2 \;\Big|\; \mathcal{F}_t \right] \\
&= e^{-(T-t)r} \left(e^{rT} S_0 + \sigma \int_0^t e^{(T-u)r} d\widehat{B}_u \right)^2 + \sigma^2 e^{-(T-t)r} \mathbb{E}_\alpha \left[\left(\int_t^T e^{(T-u)r} d\widehat{B}_u \right)^2 \right] \\
&= e^{-(T-t)r} \left(e^{rT} S_0 + \sigma \int_0^t e^{(T-u)r} d\widehat{B}_u \right)^2 + \sigma^2 e^{-(T-t)r} \int_t^T e^{2(T-u)r} du \\
&= e^{(T-t)r} S_t^2 + \frac{\sigma^2}{2r} \left(e^{(T-t)r} - e^{-(T-t)r} \right) \\
&= e^{(T-t)r} S_t^2 + \sigma^2 \frac{\sinh((T-t)r)}{r}, \qquad 0 \leqslant t \leqslant T.
\end{aligned}
$$

b) We find

$$
\xi_t = \frac{\partial C}{\partial x}(t, S_t) = 2 e^{(T-t)r} S_t, \qquad 0 \leqslant t \leqslant T.
$$

Exercise 8.15

a) We have

$$
\frac{\partial f}{\partial t}(t,x) = (r - \sigma^2/2) f(t,x), \qquad \frac{\partial f}{\partial x}(t,x) = \sigma f(t,x),
$$

and

$$
\frac{\partial^2 f}{\partial x^2}(t,x) = \sigma^2 f(t,x),
$$

hence

$$
\begin{aligned}
dS_t &= df(t, B_t) \\
&= \frac{\partial f}{\partial t}(t, B_t) dt + \frac{\partial f}{\partial x}(t, B_t) dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, B_t) dt \\
&= \left(r - \frac{1}{2}\sigma^2 \right) f(t, B_t) dt + \sigma f(t, B_t) dB_t + \frac{1}{2}\sigma^2 f(t, B_t) dt \\
&= r f(t, B_t) dt + \sigma f(t, B_t) dB_t \\
&= r S_t dt + \sigma S_t dB_t.
\end{aligned}
$$

b) We have

$$
\begin{aligned}
\mathbb{E}\left[e^{\sigma B_T} \mid \mathcal{F}_t \right] &= \mathbb{E}\left[e^{(B_T - B_t + B_t)\sigma} \mid \mathcal{F}_t \right] \\
&= e^{\sigma B_t} \mathbb{E}\left[e^{(B_T - B_t)\sigma} \mid \mathcal{F}_t \right] \\
&= e^{\sigma B_t} \mathbb{E}\left[e^{(B_T - B_t)\sigma} \right] \\
&= e^{\sigma B_t + \sigma^2 (T-t)/2}.
\end{aligned}
$$

c) We have

$$
\begin{aligned}
\mathbb{E}[S_T \mid \mathcal{F}_t] &= \mathbb{E}\left[e^{\sigma B_T + rT - \sigma^2 T/2} \mid \mathcal{F}_t \right] \\
&= e^{rT - \sigma^2 T/2} \mathbb{E}\left[e^{\sigma B_T} \mid \mathcal{F}_t \right] \\
&= e^{rT - \sigma^2 T/2} e^{\sigma B_t + \sigma^2 (T-t)/2} \\
&= e^{rT + \sigma B_t - \sigma^2 t/2} \\
&= e^{(T-t)r + \sigma B_t + rt - \sigma^2 t/2} \\
&= e^{(T-t)r} S_t, \qquad 0 \leqslant t \leqslant T.
\end{aligned}
$$

d) We have

$$
V_t = e^{-(T-t)r} \mathbb{E}[C \mid \mathcal{F}_t]
$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\mathbb{E}[S_T-K \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}[S_T \mid \mathscr{F}_t]-\mathrm{e}^{-(T-t)r}\mathbb{E}[K \mid \mathscr{F}_t] \\
&= S_t-\mathrm{e}^{-(T-t)r}K, \qquad 0 \leqslant t \leqslant T.
\end{aligned}
$$

e) We take $\xi_t = 1$ and $\eta_t = -K\,\mathrm{e}^{-rT}/A_0$, $t \in [0,T]$.

f) We find

$$V_T = \mathbb{E}[C \mid \mathscr{F}_{\mathscr{T}}] = C.$$

Exercise 8.16 Binary options. (Exercise 7.10 continued).

a) By definition of the indicator (or step) functions $\mathbb{1}_{[K,\infty)}$ and $\mathbb{1}_{[0,K]}$ we have

$$\mathbb{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geqslant K, \\ \\ 0 & \text{if } x < K, \end{cases} \qquad \textit{resp.} \quad \mathbb{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \leqslant K, \\ \\ 0 & \text{if } x > K, \end{cases}$$

which shows the claimed result by the definition of C_b and P_b.

b) We have

$$
\begin{aligned}
\pi_t(C_b) &= \mathrm{e}^{-(T-t)r}\mathbb{E}[C_b \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \,\middle|\, S_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t) \\
&= C_b(t,S_t).
\end{aligned}
$$

c) We have $\pi_t(C_b) = C_b(t,S_t)$, where

$$
\begin{aligned}
C_b(t,x) &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T > K \mid S_t = x) \\
&= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&= \mathrm{e}^{-(T-t)r}\Phi\left(d_-(T-t)\right),
\end{aligned}
$$

with

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) The price of this modified contract with payoff

$$C_\alpha = \mathbb{1}_{[K,\infty)}(S_T)+\alpha\mathbb{1}_{[0,K)}(S_T)$$

is given by

$$
\begin{aligned}
\pi_t(C_\alpha) &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T)+\alpha\mathbb{1}_{[0,K)}(S_T) \,\middle|\, S_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t)+\alpha\,\mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \leqslant K \mid S_t) \\
&= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t)+\alpha\,\mathrm{e}^{-(T-t)r}(1-\mathbb{P}(S_T \geqslant K \mid S_t)) \\
&= \alpha\,\mathrm{e}^{-(T-t)r}\mathrm{e}^{-(T-t)r}+(1-\alpha)\mathbb{P}(S_T \geqslant K \mid S_t) \\
&= \alpha\,\mathrm{e}^{-(T-t)r}+(1-\alpha)\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

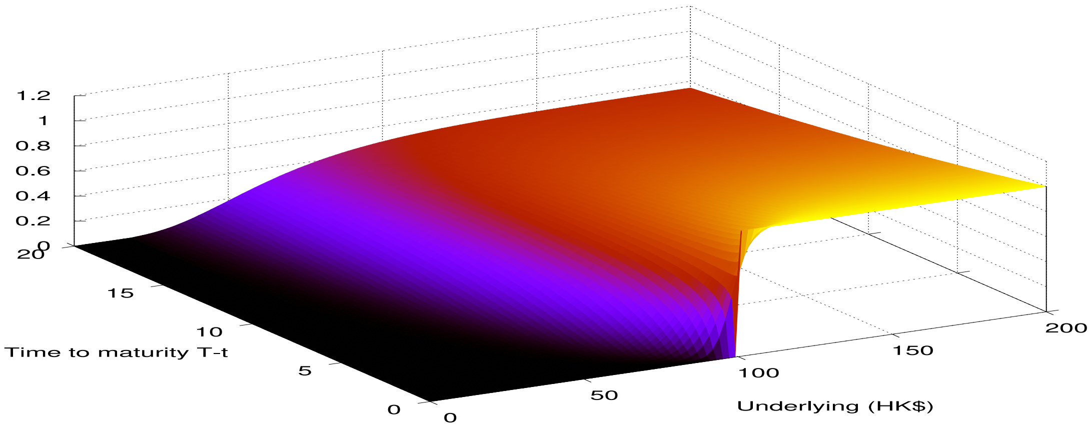


Figure S.20: Price of a binary call option.

e) We note that

$$\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) = \mathbb{1}_{[0,\infty)}(S_T),$$

almost surely since $\mathbb{P}(S_T = K) = 0$, hence

$$\begin{aligned}
\pi_t(C_b) + \pi_t(P_b) &= \mathrm{e}^{-(T-t)r}\mathbb{E}[C_b \mid \mathscr{F}_t] + \mathrm{e}^{-(T-t)r}\mathbb{E}[P_b \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}[C_b + P_b \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[0,\infty)}(S_T) \,\middle|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}[1 \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

f) We have

$$\begin{aligned}
\pi_t(P_b) &= \mathrm{e}^{-(T-t)r} - \pi_t(C_b) \\
&= \mathrm{e}^{-(T-t)r} - \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}}\right) \\
&= \mathrm{e}^{-(T-t)r}(1 - \Phi(d_-(T-t))) \\
&= \mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)).
\end{aligned}$$

g) We have

$$\begin{aligned}
\xi_t &= \frac{\partial C_b}{\partial x}(t, S_t) \\
&= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}}\right)_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{\sigma S_t\sqrt{2(T-t)\pi}}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&> 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a call option does not involve short selling because $\xi_t > 0$ for all t, cf. Figure S.21 which represents the risky investment in the hedging portfolio of a binary call option.

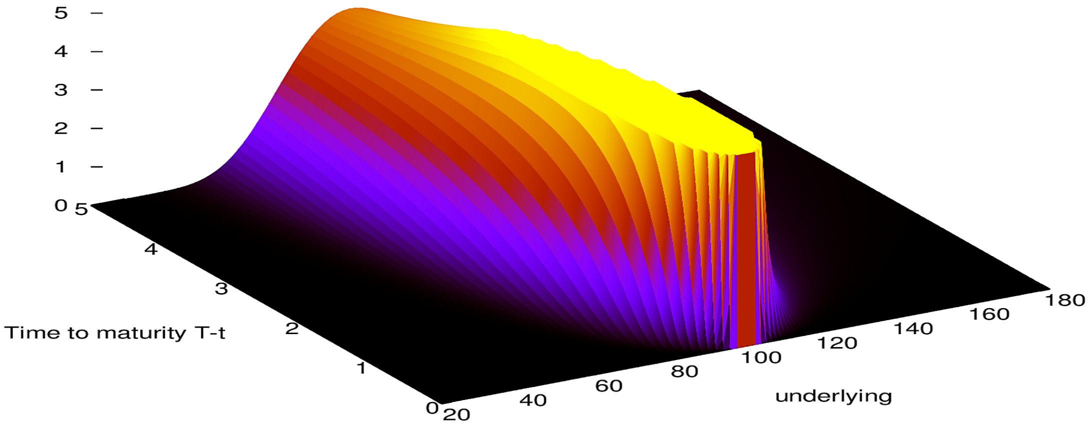


Figure S.21: Risky hedging portfolio value for a binary call option.

Figure S.22 presents the risk-free hedging portfolio value for a binary call option.

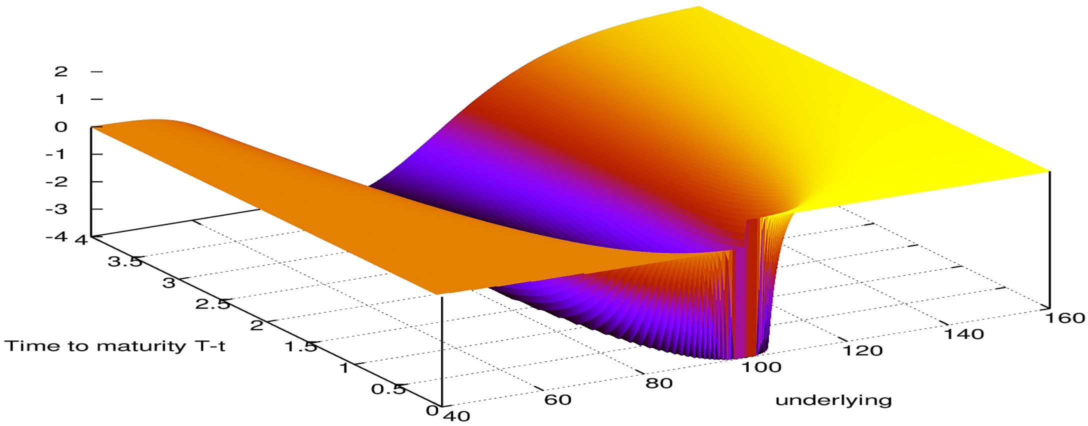


Figure S.22: Risk-free hedging portfolio value for a binary call option.

h) Here we have

$$\begin{aligned}
\xi_t &= \frac{\partial P_b}{\partial x}(t, S_t) \\
&= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(x/K)}{\sigma\sqrt{T-t}}\right)_{x=S_t} \\
&= -\mathrm{e}^{-(T-t)r}\frac{1}{\sigma\sqrt{2(T-t)\pi}S_t}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&< 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a put option does involve short selling because $\xi_t < 0$ for all t.

Exercise 8.17 Using Itô's formula and the fact that the expectation of the stochastic integral with respect to $(W_t)_{t\in\mathbb{R}_+}$ is zero, cf. Relation (5.17), we have

$$\begin{aligned}
C(x,T) &= \mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\,\middle|\,S_0=x\right] \qquad \text{(A.28)} \\
&= \phi(x) - \mathbb{E}\left[\int_0^T r\mathrm{e}^{-rs}\phi(S_t)dt\,\middle|\,S_0=x\right] \\
&\quad + r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dt\,\middle|\,S_0=x\right]
\end{aligned}$$

$$
\begin{aligned}
& +\sigma \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt} S_t \phi'(S_t) dB_t \,\Big|\, S_0 = x\right] \\
& +\frac{1}{2}\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt} \phi''(S_t)\sigma^2(S_t) dt \,\Big|\, S_0 = x\right] \\
= \ & \phi(x) - r\mathbb{E}\left[\int_0^T \mathrm{e}^{-rs} \phi(S_t) dt \,\Big|\, S_0 = x\right] \\
& +r\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt} S_t \phi'(S_t) dt \,\Big|\, S_0 = x\right] \\
& +\frac{1}{2}\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt} \phi''(S_t)\sigma^2(S_t) dt \,\Big|\, S_0 = x\right] \\
= \ & \phi(x) - \int_0^T r\mathrm{e}^{-rt}\, \mathbb{E}\left[\phi(S_t) \,\big|\, S_0 = x\right] dt \\
& +r\int_0^T \mathrm{e}^{-rt}\, \mathbb{E}\left[S_t\phi'(S_t) \,\big|\, S_0 = x\right] dt \\
& +\frac{1}{2}\int_0^T \mathrm{e}^{-rt}\, \mathbb{E}\left[\phi''(S_t)\sigma^2(S_t) \,\big|\, S_0 = x\right] dt,
\end{aligned}
$$

hence by differentiation with respect to T we find

$$
\begin{aligned}
\mathrm{Theta}_T &= \frac{\partial}{\partial T}\left(\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi(S_T) \,\big|\, S_0 = x\right]\right) \\
&= -r\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi(S_T) \big| S_0 = x\right] + r\mathrm{e}^{-rT}\, \mathbb{E}\left[S_T\phi'(S_T) \,\big|\, S_0 = x\right] \\
&\quad +\frac{1}{2}\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi''(S_T)\sigma^2(S_T) \,\big|\, S_0 = x\right].
\end{aligned}
$$

Chapter 9

Exercise 9.1

a) We have $\dfrac{\partial C}{\partial x}(T-t,x,K) = \dfrac{\partial f}{\partial z}\left(T-t,\dfrac{x}{K}\right)$ and

$$
\begin{aligned}
\frac{\partial C}{\partial K}(T-t,x,K) &= f\left(T-t,\frac{x}{K}\right) - \frac{x}{K}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) \\
&= \frac{1}{K}C(T-t,x,K) - \frac{x}{K}\frac{\partial C}{\partial x}(T-t,x,K),
\end{aligned}
$$

hence

$$
\frac{\partial C}{\partial x}(T-t,x,K) = \frac{1}{x}C(T-t,x,K) - \frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K).
$$

b) We have $\dfrac{\partial^2 C}{\partial x^2}(T-t,x,K) = \dfrac{1}{K}\dfrac{\partial^2 f}{\partial z^2}\left(T-t,\dfrac{x}{K}\right)$ and

$$
\begin{aligned}
& \frac{\partial^2 C}{\partial K^2}(T-t,x,K) \\
&= -\frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) + \frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) + \frac{x^2}{K^3}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) \\
&= \frac{x^2}{K^3}\frac{\partial^2 f}{\partial z^2}\left(T-t,\frac{x}{K}\right) \\
&= \frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K),
\end{aligned}
$$

hence

$$
\frac{\partial^2 C}{\partial x^2}(T-t,x,K) = \frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).
$$

c) Noting that

$$\frac{\partial C}{\partial t}(T-t,x,K) = -\frac{\partial C}{\partial T}(T-t,x,K),$$

we can rewrite the Black-Scholes PDE as

$$\begin{aligned} rC(T-t,x,K) &= -\frac{\partial C}{\partial T}(T-t,x,K) \\ &\quad + rx\left(\frac{1}{x}C(T-t,x,K) - \frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K)\right) \\ &\quad + \frac{\sigma^2 x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K), \end{aligned}$$

i.e.

$$\frac{\partial C}{\partial T}(T-t,x,K) = -rK\frac{\partial C}{\partial K}(T-t,x,K) + \frac{\sigma^2 x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).$$

Remarks:

1. Using the Black-Scholes Greek Gamma expression

$$\begin{aligned} \frac{\partial^2 C}{\partial x^2}(T-t,x,K) &= \frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_+(T-t)) \\ &= \frac{1}{\sigma x\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2}, \end{aligned}$$

we can recover the lognormal probability density function $\varphi_T(y)$ of geometric Brownian motion S_T as follows:

$$\begin{aligned} \varphi_T(K) &= \mathrm{e}^{(T-t)r}\frac{\partial^2 C}{\partial K^2}(T-t,x,K) \\ &= \mathrm{e}^{(T-t)r}\frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K) \\ &= \frac{\mathrm{e}^{(T-t)r}x}{\sigma K^2\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2} \\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_-(T-t))^2/2} \\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\exp\left(-\frac{\big((r-\sigma^2/2)(T-t)+\log(x/K)\big)^2}{2(T-t)\sigma^2}\right), \end{aligned}$$

knowing that

$$\begin{aligned} -\frac{1}{2}\big(d_-(T-t)\big)^2 &= -\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2 \\ &= -\frac{1}{2}\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2 + (T-t)r + \log\frac{x}{K} \\ &= -\frac{1}{2}\big(d_+(T-t)\big)^2 + (T-t)r + \log\frac{x}{K}, \end{aligned}$$

which can be obtained from the relation

$$\begin{aligned} &\big(d_+(T-t)\big)^2 - \big(d_-(T-t)\big)^2 \\ &= \big((d_+(T-t)+d_-(T-t)\big)\big((d_+(T-t)-d_-(T-t)\big) \\ &= 2r(T-t) + 2\log\frac{x}{K}. \end{aligned}$$

2. Using the expressions of the Black-Scholes Greeks Delta and Theta we can also recover

$$
\begin{aligned}
& 2\frac{\frac{\partial C}{\partial T}(T-t,x,K)+rK\frac{\partial C}{\partial K}(T-t,x,K)}{K^2\frac{\partial^2 C}{\partial K^2}(T-t,x,K)} \\
= & \ 2\frac{-\frac{\partial C}{\partial t}(T-t,x,K)+rK\left(\frac{1}{K}C(T-t,x,K)-\frac{x}{K}\frac{\partial C}{\partial x}(T-t,x,K)\right)}{x^2\frac{\partial^2 C}{\partial x^2}(T-t,x,K)} \\
= & \ 2\frac{x\sigma\Phi'(d_+(T-t))/(2\sqrt{T-t})+rK\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})} \\
& +2\frac{rC(T-t,x,K)-rx\Phi(d_+(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})} \\
= & \ \sigma^2.
\end{aligned}
$$

Exercise 9.2

a) We have

$$
\frac{\partial M_C}{\partial K}(K,S,r,\tau) = \frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)+\sigma'_{\mathrm{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau).
$$

b) We have

$$
\frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)+\sigma'_{\mathrm{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)\leqslant 0,
$$

which shows that

$$
\sigma'_{\mathrm{imp}}(K)\leqslant -\frac{\frac{\partial C}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}{\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}
$$

c) We have

$$
\frac{\partial P}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)+\sigma'_{\mathrm{imp}}(K)\frac{\partial P}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)\geqslant 0,
$$

which shows that

$$
\sigma'_{\mathrm{imp}}(K)\geqslant -\frac{\frac{\partial P}{\partial K}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}{\frac{\partial P}{\partial \sigma}(K,S,\sigma_{\mathrm{imp}}(K),r,\tau)}
$$

Exercise 9.3

a) We have

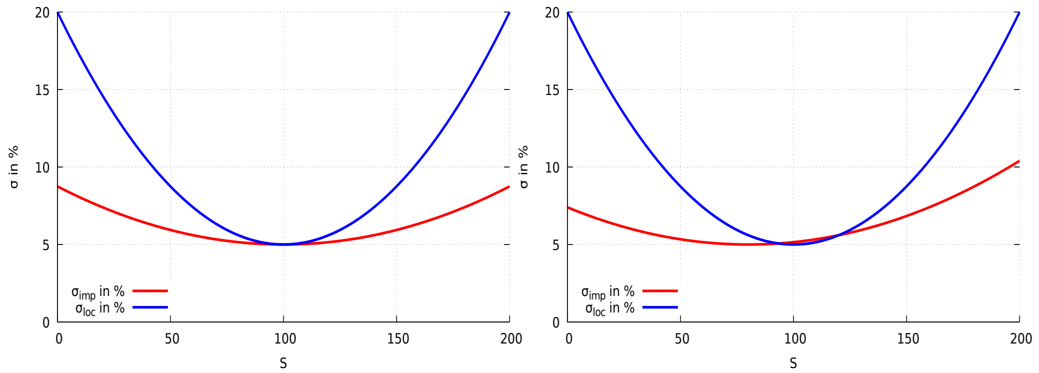
$$
\begin{aligned}
\sigma_{\mathrm{imp}}(K,S) & \simeq \sigma_{\mathrm{loc}}((K+S)/2) \\
& = \sigma_0+\beta((K+S)/2-S_0)^2 \\
& = \sigma_0+\frac{\beta}{4}(K-(2S_0-S))^2.
\end{aligned}
$$

b) We find

$$
\begin{aligned}
\frac{\partial}{\partial S}\big((S,K,T,\sigma_{\mathrm{imp}}(K,S),r)\big) & = \frac{\partial \mathrm{Bl}}{\partial x}\big(x,K,T,\sigma_{\mathrm{imp}}(K,S),r\big)_{x=S} \\
& \quad +\frac{\partial \sigma_{\mathrm{imp}}}{\partial S}\frac{\partial \mathrm{Bl}}{\partial \sigma}\big(x,K,T,\sigma,r\big)_{\sigma=\sigma_{\mathrm{imp}}(K,S)} \\
& = \Delta+\nu\frac{\beta}{2}(K-(2S_0-S)),
\end{aligned}
$$

where

$$
\Delta=\frac{\partial \mathrm{Bl}}{\partial x}\big(x,K,T,\sigma_{\mathrm{imp}}(K,S),r\big)_{x=S}
$$



(a) At the money $K = S_0$. (b) Out of the money $K > S_0$.

Figure S.23: Implied *vs* local volatility.

is the Black-Scholes Delta and

$$v = \frac{\partial \mathrm{Bl}}{\partial \sigma}(S,K,T,\sigma,r)_{\sigma=\sigma_{\mathrm{imp}}(K,S)}$$

is the Black-Scholes Vega, cf. §2.2 of Hagan et al., 2002.

Background on Probability Theory

Exercise A.1

a) We have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k\geqslant 0} k\mathbb{P}(X=k) = \mathrm{e}^{-\lambda}\sum_{k\geqslant 0} k\frac{\lambda^k}{k!} \\ &= \mathrm{e}^{-\lambda}\sum_{k\geqslant 1}\frac{\lambda^k}{(k-1)!} = \lambda\,\mathrm{e}^{-\lambda}\sum_{k\geqslant 0}\frac{\lambda^k}{k!} = \lambda.\end{aligned}$$

b) We have

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{k\geqslant 0} k^2\mathbb{P}(X=k) \\ &= \mathrm{e}^{-\lambda}\sum_{k\geqslant 1} k^2\frac{\lambda^k}{k!} \\ &= \mathrm{e}^{-\lambda}\sum_{k\geqslant 1} k\frac{\lambda^k}{(k-1)!} \\ &= \mathrm{e}^{-\lambda}\sum_{k\geqslant 2}\frac{\lambda^k}{(k-2)!} + \mathrm{e}^{-\lambda}\sum_{k\geqslant 1}\frac{\lambda^k}{(k-1)!} \\ &= \lambda^2\mathrm{e}^{-\lambda}\sum_{k\geqslant 0}\frac{\lambda^k}{k!} + \lambda\,\mathrm{e}^{-\lambda}\sum_{k\geqslant 0}\frac{\lambda^k}{k!} \\ &= \lambda^2 + \lambda,\end{aligned}$$

and

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda = \mathbb{E}[X].$$

Exercise A.2 We have

$$\mathbb{P}(\mathrm{e}^X > c) = \mathbb{P}(X > \log c) = \int_{\log c}^{\infty} \mathrm{e}^{-y^2/(2\eta^2)}\frac{dy}{\sqrt{2\pi\eta^2}}$$

$$= \int_{(\log c)/\eta}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} = 1 - \Phi((\log c)/\eta) = \Phi(-(\log c)/\eta).$$

Exercise A.3

a) Using the change of variable $z = (x-\mu)/\sigma$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \varphi(x) dx &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz.
\end{aligned}$$

Next, using the polar change of coordinates $dxdy = rdrd\theta$, we find*

$$\begin{aligned}
\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz\right)^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y^2/2} dy \int_{-\infty}^{\infty} e^{-z^2/2} dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(y^2+z^2)/2} dydz \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r e^{-r^2/2} drd\theta \\
&= \int_0^{\infty} r e^{-r^2/2} dr \\
&= \lim_{R\to+\infty} \int_0^R r e^{-r^2/2} dy \\
&= -\lim_{R\to+\infty} \left[e^{-r^2/2} \right]_0^R \\
&= \lim_{R\to+\infty} (1 - e^{-R^2/2}) \\
&= 1.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x\varphi(x) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\mu+y) e^{-y^2/(2\sigma^2)} dx \\
&= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy \\
&= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy + \frac{\sigma}{\sqrt{2\pi}} \lim_{A\to+\infty} \int_{-A}^{A} y e^{-y^2/2} dy \\
&= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \\
&= \mu \int_{-\infty}^{\infty} \varphi(y) dy \\
&= \mu \mathbb{P}(X \in \mathbb{R}) \\
&= \mu,
\end{aligned}$$

by symmetry of the function $y \longmapsto y e^{-y^2/2}$ on $\mathbb{R}$.

c) Similarly, after a double integration by parts on $\mathbb{R}$, we find

$$\mathbb{E}[(X - \mathbb{E}[X])] = \int_{-\infty}^{\infty} (x-\mu)^2 \varphi(x) dx$$

*"In a discussion with Grothendieck, Messing mentioned the formula expressing the integral of e^{-x^2} in terms of π, which is proved in every calculus course. Not only did Grothendieck not know the formula, but he thought that he had never seen it in his life". Milne, 2005.

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} y^2 \mathrm{e}^{-(y-\mu)^2/(2\sigma^2)}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\times y\mathrm{e}^{-y^2/2}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/2}dy \\
&= \sigma^2.
\end{aligned}$$

d) We have

$$\begin{aligned}
\mathbb{E}[(X-\mathbb{E}[X])] &= \int_{-\infty}^{\infty}(x-\mu)^2\varphi(x)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y\times y\mathrm{e}^{-y^2/2}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2/2}dy \\
&= \sigma^2.
\end{aligned}$$

e) By a completion of squares argument, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^X] &= \int_{-\infty}^{\infty}\mathrm{e}^x\varphi(x)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{x-(x-\mu)^2/(2\sigma^2)}dx \\
&= \frac{\mathrm{e}^\mu}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{y-y^2/(2\sigma^2)}dy \\
&= \frac{\mathrm{e}^\mu}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{\sigma^2/2+(y-\sigma^2)^2/(2\sigma^2)}dy \\
&= \frac{\mathrm{e}^\mu+\sigma^2/2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \mathrm{e}^{x^2/(2\sigma^2)}dy \\
&= \mathrm{e}^\mu+\sigma^2/2.
\end{aligned}$$

Exercise A.4

a) We have

$$\begin{aligned}
\mathbb{E}[X^+] &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^+ \mathrm{e}^{-x^2/(2/\sigma^2)}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\int_0^{\infty} x\mathrm{e}^{-x^2/2}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-\mathrm{e}^{-x^2/2}\right]_{x=0}^{x=\infty} \\
&= \frac{\sigma}{\sqrt{2\pi}}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}[(X-K)^+] &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}(x-K)^+ \mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_K^{\infty}(x-K)\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_K^{\infty}x\mathrm{e}^{-x^2/(2\sigma^2)}dx - \frac{K}{\sqrt{2\pi\sigma^2}}\int_K^{\infty}\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-\mathrm{e}^{-x^2/(2\sigma^2)}dx\right]_{x=K}^{\infty} - \frac{K}{\sqrt{2\pi}}\int_{-\infty}^{-K/\sigma}\mathrm{e}^{-x^2/2}dx
\end{aligned}$$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-K^2/(2\sigma)^2} - K\Phi(-K/\sigma).$$

c) Similarly, we have

$$\begin{aligned}
\mathbb{E}[(K-X)^+] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (K-x)^+ e^{-x^2/(2\sigma^2)} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{K} (K-x) e^{-x^2/(2\sigma^2)} dx \\
&= \frac{K}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{K} e^{-x^2/(2\sigma^2)} dx - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{K} x e^{-x^2/(2\sigma^2)} dx \\
&= \frac{K}{\sqrt{2\pi}} \int_{-\infty}^{K/\sigma} e^{-x^2/2} dx - \frac{\sigma}{\sqrt{2\pi}} \left[- e^{-x^2/(2\sigma^2)} dx \right]_{-\infty}^{x=K} \\
&= \frac{\sigma}{\sqrt{2\pi}} e^{-K^2/(2\sigma)^2} + K\Phi(K/\sigma).
\end{aligned}$$

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