MathFinance 345/Stat390
Homework 3 Corrected
Due October 17

1. **Martingale transforms.** Let $(Z_n)_{n\geq 0}$ be a martingale relative to a filtration $(\mathcal{F}_n)_{n\geq 0}$, and let $(Y_n)_{n\geq 1}$ be a predictable sequence. (This just means that, for every $n \geq 1$, the random variable Y_n is measurable relative to $\mathcal{F}_{n-1}$.) The *martingale transform* $(Y \cdot Z)_n$ is defined as follows:

$$(Y \cdot Z)_n = Z_0 + \sum_{j=1}^{n} Y_j (Z_j - Z_{j-1}). \tag{1}$$

Show that if the random variables Y_n are bounded, then the martingale transform $(Y \cdot Z)_n$ is a martingale relative to $(\mathcal{F}_n)_{n\geq 0}$.

2. **Optional Stopping.** Let $(Z_n)_{n\geq 0}$ be a martingale relative to a filtration $(\mathcal{F}_n)_{n\geq 0}$, and let τ be a stopping time.

(a) Show that the sequence $(Z_{\tau\wedge n})_{n\geq 0}$ is a martingale transform. (HINT: You will have to guess what the random variables Y_n must be. Then show that the sequence Y_n is predictable.)

(b) Use the result of (a) to deduce Doob's Optional Stopping Formula.

3. **Bonds.** A (zero–coupon) *bond* with maturity M is a contract that pays the owner \$1 at the maturity date M. Consider a $T-$period market in which bonds of all maturities $M = 1, 2, \dots, T$ are traded; denote by $B_{t,M} = B_{t,M}(\omega)$ the price (in dollars) at time $t \leq M$ of one maturity–M bond.

(a) What is the riskless asset in this market?

(b) A *coupon–bearing* bond is a contract that pays the owner an amount C_t dollars at time t, for $t = 1, 2 \dots, T$. Find a formula for the price at time $t = 0$ of such a contract.

(c) Let π be an equilibrium distribution for the market, and let $E = E_\pi$ denote the expectation operator for the distribution π. For each $t = 0, 1, 2, \dots, T$, define

$$Z_t = \frac{1}{B_{t,T} \prod_{j=0}^{t-1} B_{j,j+1}}$$

Prove that the sequence Z_t is a martingale relative to the usual filtration. (NOTE: For this problem you may use the fact that, for any asset A the *discounted* share price S_t^* at time t satisfies the equation $S_t^* = E(S_{t+1}^* \mid F_t)$, where $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration.)

(d) Is there a self-financing portfolio (in zero-coupon BONDS with different maturities) whose value process is $(Z_t)_{0\leq t\leq T}$? If so, describe it; if not, explain why.

4. **Gambler's Ruin, Revisited.** The game is exactly the same as described in section 8 of the lecture notes *except* that the coin is unfair: the probability that it comes up H is $p > 1/2$, and the probability that it comes up T is $q = 1 - p < 1/2$. The problem is, once

again, to determine the probability that SLIM exhausts his initial fortune \$$B$ before FATS exhausts his initial fortune \$$A$.

(a) Define X_i and S_n as in section 8: thus, S_n is the cumulative change in FATS' fortune after n plays. Show that

$$Z_n := \left(\frac{q}{p}\right)^{S_n}$$

is a martingale relative to the usual filtration.

(b) Define τ to be the minimal integer $n > 0$ such that $S_n = +B$ or $S_n = -A$; thus, τ is the (random) number of plays that the game lasts. Use the Optional Stopping Formula to show that

$$E(Z_\tau) = 1, \tag{3}$$

where Z_n is the martingale introduced in part (a). (HINT: The Optional Stopping Formula implies that $EZ_{\tau\wedge n} = 1$ for every $n = 1, 2, \dots$. Show that as $n \to \infty$ the difference $EZ_\tau - EZ_{\tau\wedge n}$ converges to 0.)

(c) Use the equality in equation (3) to solve for the probability that FATS wins the game.

(d) Show that $S_n - n(p - q)$ is a martingale. Use the Optional Stopping Formula, with the stopping time τ, to show that

$$ES_\tau = (p - q)E\tau. \tag{4}$$

(HINT: The Optional Stopping Formula implies that $ES_{\tau\wedge n} = (p - q)E(\tau \wedge n)$ for each integer $n = 1, 2, \dots$. Use the monotone convergence theorem of measure theory, together with an argument like the one you used in (c), to deduce (4).)

(e) Use the identity (4) and the result of part (c) to solve for $E\tau$ (the expected duration of the game).