MathFinance 345/Stat390
Homework 6
Due November 12

1. Use the Itô formula to show that for any integer $k \geq 2$,

$$EW(t)^k = \frac{1}{2}k(k-1)\int_0^t EW(s)^{k-2}\, ds,$$

and use this to evaluate the first 240 moments of the normal distribution.

2. Use the Itô formula to show that, for any *nonrandom*, continuously differentiable function $f(t)$,

$$\int_0^t f(s)\, dW_s = f(t)W(t) - \int_0^t f'(s)W_s ds.$$

3. Use the Itô formula to show that, for any $t > 0$ and any scalar θ,

$$Ee^{\theta W(t)} = \frac{\theta^2}{2}\int_0^t Ee^{\theta W(s)}\, ds.$$

Now take the derivative of each side to get a differential equation for $Ee^{\theta W(t)}$, and then solve this differential equation to get a formula for $Ee^{\theta W(t)}$. Hint: For any locally $\mathcal{H}^2$ integrand, $E\int \alpha s\, dW_s = 0$.

4. Let $u(x,t)$ be a bounded, continuous solution to the *heat equation* $u_{xx}/2 = u_t$ whose first and second partial derivatives are bounded, and which satisfies the initial condition $u(x,0) = f(x)$. Use the Itô formula (what else?) to show that for every $x \in \mathbb{R}$ and every $t > 0$,

$$u(x,t) = Ef(x + W(t)).$$

Conclude that for any initial condition $u(x,0) = f(x)$, there can be at most one continuous bounded solution to the heat equation with bounded first and second partial derivatives.

5. **Stock Price Processes.** Let $\{S_t\}_{t\geq 0}$ be the price process of a Stock in a market with a riskless asset Bond whose price at time t is $B_t = e^{rt}$. Suppose that $\{S_t\}$ obeys the SDE

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t)$$

for some parameters μ and $\sigma > 0$. Show that $\mu = r$.