MathFinance 345/Stat390
Homework 7
Due November 21

Problem 1: Consider a contingent claim that pays S_T^n at time T, where n is a positive integer.

(A) Show that the value of this contingent claim at time $t \leq T$ is

$$h(t, T) S_t^n$$

for some function h of (t, T). HINT: Use the fact that the process S_t is a geometric Brownian motion. You should not need the Itô formula.

(B) Derive an ordinary differential equation for $h(t, T)$ in the variable t, and solve it. HINT: Your ordinary differential equation should be first-order, and it should involve only the short rate r_t.

In Problems 2 and 3, let $C(S_t, t) = C(S_t, t; K, T)$ be the price at time t of a European call option on the tradable asset (S_t) with strike price K and exercise time T. Assume that the riskless rate of return r is constant and nonnegative, and that the share price process S_t of the underlying asset STOCK follows the stochastic differential equation

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t.$$

Problem 2:

(A) Show that the price function C satisfies the following symmetry properties:for any positive constant a,

$$C(S, t; K, T) = C(S, 0; K, T - t) \tag{1}$$

$$C(aS, t; aK, T) = aC(S, t; K, T) \qquad \forall\, a > 0. \tag{2}$$

(B) Use the result of part (A) to derive an identity relating the partial derivatives C_S and C_K.

(C) Find a PDE in the variables K, T for the function $C(x, 0; K, T)$. (The equation should involve first and second partial derivatives.)

Problem 3. Denote by

$$C^*(S_t, t) = e^{-rt} C(S_t, t)$$

the *discounted* value of the option.

(A) Show that, for each fixed $t \leq T$,

$$\lim_{x \to \infty} (C(x,t) - e^{-rT+rt}x) = -K.$$

(B) Show that, for each fixed $x > 0$, the function $C^*(x,t)$ is decreasing in t and converges to $e^{-rT}(x-K)_+$ as $t \to T$.

(C) Show that $0 \leq C_x(x,t) \leq 1$ for all $x > 0$ and all $0 \leq t \leq T$. Also, verify that

$$\begin{aligned} \lim_{t \to T} C_x(x,t) &= 1 \qquad \text{if } x > K \text{ and} \\ \lim_{t \to T} C_x(x,t) &= 0 \qquad \text{if } x < K. \end{aligned}$$

Discuss the implications for hedging.

Note: In problems 2 and 3, you avoid the use of the Black–Scholes formula wherever possible. Instead, use the representation of the call option price as a (conditional) expectation.