Stochastic Calculus

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Tonight —

- Foreign Exchange & Exchange Rate Fluctuations
- Linear Stochastic Differential Equations
- Cameron-Martin-Girsanov Formula

Foreign Exchange

- Stochastic Models for Exchange Rates
- Interest Rates and Exchange Rates
- Options on Currency Exchange

Basic Principles

- Share price processes of tradeable assets are martingales under any risk-neutral probability measure.
- Risk-neutrality of a probability measure depends on the numeraire.
- Currencies are not tradeable assets!
- Money market shares are!

Exchange Rate Model

Let Y_t denote the exchange rate at time t between US Dollars \$ and UK Pounds Sterling $\pounds$, i.e., the number of pounds that one dollar will buy. A simple model:

$$dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t$$

where W_t is a standard Wiener process under the risk neutral measure for $\pounds$ investors, and μ and σ are constants.

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where W_t is a standard Wiener process under the risk neutral measure for $\pounds$ investors, and μ and σ are constants.

In a more realistic model, the drift and/or diffusion coefficients might be time-varying but deterministic:

$$dY_t = \mu_t Y_t \, dt + \sigma_t Y_t \, dW_t$$

Itô Processes

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The local quadratic variation of the Itô process Z_t is defined by

$$d[Z, Z]_t = B_t^2\,dt$$

Itô’s Formula

If Z_t is an Itô process, and if $f(x)$ is a smooth function, then $f(Z_t)$ is also an Itô process whose Itô SDE is

$$df(Z_t) = f'(Z_t)\, dZ_t + \frac{1}{2} f''(Z_t)\, d[Z, Z]_t$$

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Itô's formula has a number of important generalizations. Here is one which is sometimes useful in solving SDEs with time-dependent coefficients: If $u(x,t)$ is a smooth function of two variables, then

$$du(Z_t,t) = u_t\,dt + u_x\,dZ_t + \frac{1}{2}u_{xx}\,d[Z,Z]_t$$

Solving the SDE

The idea is to guess a solution by applying the Itô formula to the right process. Assume that under the probability measure P the exchange rate Y_t satisfies

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$$d\log(Y_t) = \mu\, dt + \sigma\, dW_t - (\sigma^2/2)\, dt$$

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Since μ and σ are constants, this is easily integrated to give the general solution to the SDE:

$$\boxed{Y_t = Y_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t\}}$$

Time-Dependent SDEs

A similar strategy works for equations with time-dependent coefficients, for example:

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and so

$$\boxed{Y_t = Y_0 \exp\{(\bar{\mu}_t - \sigma^2/2)t + \sigma W_t\}}$$

where

$$\bar{\mu}_t = \frac{1}{t} \int_0^t \mu_s \, ds$$

Interest Rates

Assume that for each of the two currencies US Dollar and UK Pound Sterling there is a riskless Money Market. Let A_t and B_t be the "share prices" of US Money Market and UK Money Market, respectively, and for simplicity assume that the time-zero share prices are both 1.

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Assume that the riskless rates of return r_A, r_B in the two currencies are constant, but not necessarily equal. Then

$$A_t = \exp\{r_A t\} \quad \text{dollars}$$

$$B_t = \exp\{r_B t\} \quad \text{pounds}$$

Exchange and Interest Rates

The asset US Money Market is riskless to a Dollar investor, but not to a Pound Sterling investor. Evaluated in Pounds Sterling, the share price of the US Money Market asset is

$$A_t Y_t = Y_0 \exp\{r_A t + \mu t - \sigma^2 t/2 + \sigma W_t\}$$

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Theorem: $\mu = r_B - r_A$.

Proof

Since US Money Market is a tradeable asset, its share price Y_0 at time $t = 0$ must be the expected value of its discounted share price $A_t Y_t$ (in $\pounds$) at time t, where

- the discount rate is r_B, and

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Currency Options

Consider an option Call that gives the owner the right to buy $\$1$ for $\pounds K$ at time T. What is the arbitrage price at time 0?

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Solution: The option is identical to a call on $e^{-r_A T}$ shares of the US Money Market. To a $\pounds$ investor, the US Money Market is a risky asset with price process $e^{-r_A t} Y_t$. Thus, the call option may be priced using the Black-Sholes Formula.

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Exercise: Do it! While you're at it, show how to hedge the option.

Risk-Neutral Measure for $

Theorem: Let Q_A be the risk-neutral probability measure for the US Dollar investor, and Q_B the risk-neutral measure for the UK Pound Sterling investor. Unless $\sigma = 0$ (that is, unless the exchange rate is purely deterministic), it must be the case that

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This is a special case of a more general phenomenon:

Numeraire Change

Suppose that a market has tradeable assets A, B with share price processes S_t^A and S_t^B (evaluated in a common numeraire C). Let Q^A and Q^B be risk-neutral measures for numeraires A, B, respectively.

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Theorem: $Q^A = Q^B$ if and only if S_t^A / S_t^B is a constant random variable. Furthermore, in general, for any finite time T,

$$\left(\frac{dQ^B}{dQ^A}\right)_{\mathcal{F}_T} = \left(\frac{S_T^B}{S_T^A}\right)\left(\frac{S_0^A}{S_0^B}\right)$$

Consequence

In the foreign exchange context, the riskless assets for the two numeraires are US Money Market and UK Money Market, with share prices (in \$)

$$A_t = \exp\{r_A t\}$$
$$B_t = \exp\{r_B t\}/Y_t$$

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Therefore, the likelihood ratio between the risk-neutral measures for $\pounds$ and \$ investors is

$$\left(\frac{dQ^B}{dQ^A}\right)_{\mathcal{F}_T} = \left(\frac{Y_T}{Y_0}\right)^{-1} \exp\{(r_B - r_A)T\}$$

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$$V_0^A = V_0^C / S_0^A = E^A V_t^C / S_t^A$$
$$V_0^B = V_0^C / S_0^B = E^B V_t^C / S_t^B$$

Likelihood Ratio Identity

It follows that for every contingent claim V with share price V_t^C (in numeraire C),

$$S_0^A E^A(V_t^C/S_t^A) = S_0^B E^B(V_t^C/S_t^B)$$

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Apply this to the contingent claim with payoff $V_T^C S_T^B$ at time T to obtain the following identity, valid for all nonnegative random variables V_T^C measurable $\mathcal{F}_T$:

$$E^B V_T^C = E^A V_T^C \left(\frac{S_T^B S_0^A}{S_T^A S_0^B} \right)$$

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This is the defining property of a likelihood ratio.

Exponential Martingales

Let W_t be a standard Wiener process, wth Brownian filtration $\mathcal{F}_t$, and let θ_t be a bounded, adapted process. Define

$$Z_t = \exp\left\{\int_0^t \theta_s \, dW_s - \int_0^t \theta_s^2 \, ds/2\right\}$$

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Fact: Z_t is a positive martingale. **Proof:** Itô!

$$\begin{aligned} dZ_t &= Z_t\theta_t\, dW_t - Z_t\theta_t^2\, dt/2 + Z_t\theta_t^2\, dt/2 \\ &= Z_t\theta_t\, dW_t \qquad \Longrightarrow \end{aligned}$$

$$Z_t = Z_0 + \int_0^t Z_s\theta_s\, dW_s$$

Girsanov's Theorem

Because Z_t is a positive martingale under P with initial value $Z_0 = 1$, for every fixed time T the random variable Z_T is a likelihood ratio: that is,

$$Q(F) := E_P(I_F Z_T)$$

defines a new probability measure on the $\sigma-$algebra $\mathcal{F}_T$ of events F that are observable by time T.

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Theorem: Under the measure Q, the process $\{W_t - \int_0^t \theta_s ds\}_{0 \leq t \leq T}$ is a standard Wiener process.

Exchange Rates

Consider again the \$ and $\pounds$ currencies. Assume that each has a riskless Money Market, and that the rates of return r_A, r_B are constant. Assume that the exchange rate Y_t obeys

$$dY_t = (r_B - r_A)Y_t\,dt + \sigma Y_t\,dW_t$$

where W_t is a standard Wiener process under the risk-neutral probability Q^B for $\pounds$ investors. Thus,

$$Y_t = Y_0 \exp\{(r_B - r_A - \sigma^2/2)t + \sigma W_t\}.$$

Exchange Rates

Since

$$\left(\frac{dQ^A}{dQ^B}\right)_{\mathcal{F}_T} = \left(\frac{Y_T}{Y_0}\right)\exp\{-(r_B - r_A)T\}$$
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Girsanov implies that under Q^A the process W_t is a Wiener process with drift σ. Thus, to the \$ investor, it appears that the exchange rate obeys

$$dY_t = (r_B - r_A - \sigma^2)Y_t\,dt + \sigma Y_t\,d\tilde{W}_t$$

where $\tilde{W}_t$ is a standard Wiener process under Q^A.

Proof of Girsanov 1

The statement that X is a standard Wiener process is an assertion that the increments of X are independent Gaussian random variables with the correct variances. Let's show that under Q, the distribution of $W_T - \Theta_T$ is gaussian with var T (where $\Theta_T = \int_0^T \theta_s \, ds$).

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$$E_Q \exp\{\lambda(W_T - \Theta_T)\} = \exp\{\lambda^2 T/2\}$$

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To evaluate the expectation, change measure:

$$E_Q \exp\{\lambda(W_T - \Theta_T)\} = E_P \exp\{\lambda(W_T - \Theta_T)\} Z_T$$

Proof of Girsanov 2

Objective: Show that $E_P H_T = 1$, where

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Thus, H_t is an exponential martingale under P, and so its expectation is constant over time. A similar calculation establishes the independence of the increments.

Scratch

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