MathFinance 345/Stat390
Homework 4
Due October 24

1. **Double or Nothing.** The Optional Stopping Formula is not generally true if $\tau \wedge n$ is replaced by τ. Following is a simple example: Let $\xi_1, \xi_2, \dots$ be an infinite sequence of independent Bernoulli random variables with success parameter 1/2: that is, $P\{\xi_i = 1\} = P\{\xi_i = 0\} = 1/2$. Define a sequence $(Z_n)_{n \geq 0}$ of random variables inductively as follows:

$$\begin{aligned} Z_0 &= 1 \\ Z_n &= 2\xi_n Z_{n-1} \qquad \forall n \geq 1. \end{aligned}$$

(a) Prove that the sequence $(Z_n)_{n \geq 0}$ is a martingale relative to the usual filtration.
(b) Define $\tau = \min\{n : Z_n = 0\}$. Prove that $P\{\tau < \infty\} = 1$, and that τ is a stopping time.
(c) Show that $EZ_0 \neq EZ_\tau$.

2. **Arbitrages in Infinite Period Markets.** The economic assumption that efficient markets should not allow the existence of arbitrages may be reasonable in markets with finitely many trading periods, but not in markets with *infinitely* many trading periods. For example: Consider a homogeneous binary market with a risky asset STOCK and a riskless asset BOND with rate of return $r = 0$. The share price S_t of the asset STOCK evolves as in the $T-$period homogeneous binary market: thus, there are constants $d < 1 < u$ such that, given any partial scenario $\omega = \omega_1\omega_2 \dots \omega_t$ of length t,

$$\begin{aligned} S_{t+1}(\omega_1\omega_2 \dots \omega_t +) &= uS_t(\omega_1\omega_2 \dots \omega_t); \\ S_{t+1}(\omega_1\omega_2 \dots \omega_t -) &= dS_t(\omega_1\omega_2 \dots \omega_t). \end{aligned}$$

For the sake of simplicity, assume that

$$\frac{u-1}{u-d} = \frac{1-d}{u-d} = \frac{1}{2}.$$

Show that this market permits an arbitrage. (HINT: Construct a self-financing portfolio θ whose value V_t^θ evolves as the double-or-nothing martingale Z_n in problem 3. You will, of course, have to give a definition of an "arbitrage" in an infinite period market; your defnition must be reasonable.)

3. **American Put Option:** The American *put* with strike K is a contract that gives the owner the right to *sell* one share of STOCK for $\$K$ at *any* time $t = 1, 2, \dots, T$. Assume that there is a riskless asset BOND whose rate of return is $r > 0$. Give an example to show that, in certain markets and in certain circumstances, it may be better to exercise the put option early.

HINT: Consider a homogeneous 2-period binary market. Show that for some values of u, d, r and K, you get a higher expected payoff by exercising the put at $t = 1$ when the partial scenario is $-$ than waiting until $t = 2$. Note that if $r = 0$, then there is no advantage to early exercise.

4. **First-Passage Time Distribution:** Let $\xi_1, \xi_2, \dots$ be an infinite sequence of independent Bernoulli-$\frac{1}{2}$ random variables, and let $S_n = \sum_{i=1}^n \xi_i$. Define $\tau = \min\{n : S_n = 1\}$ to be the first time that the "random walk" reaches the level 1. (Such random variables play an important role in *barrier options*, about which we shall have more to say later.) The purpose of this exercise is to find the distribution of the random variable τ.

(a) Fix $z > 0$. Show that the sequence of random variables

$$Y_n = z^{S_n} / \varphi(z)^n$$

is a martingale relative to the natural filtration, where $\varphi(z) = (z + z^{-1})/2$.

(b) Show that if $z \geq 1$ then $E(1/\varphi(z)^\tau) = 1/z$. What goes wrong if $z < 1$?

(c) Conclude that for any $0 < \zeta < 1$, $E\zeta^\tau =$ (you figure out what goes here).

(d) Use the fact that $E\zeta^\tau = \sum_{n=1}^\infty \zeta^n P\{\tau = n\}$ and calculus to find a formula for $P\{\tau = n\}$.

Reminder: The midterm exam will be held on October 24 during the regular class period in KENT 107. You may bring one page of notes.