MathFinance 345/Stat390
Homework 5
Due November 7

1. **One–Sided Stable-1/2 Law.** For each $a > 0$, let τ_a be the first–passage time to the level a by the Brownian path $W(t)$. Let $T_1, T_2, \dots$ be a sequence of independent, identically distributed random variables each having the same distribution as τ_1.

(a) Show that $ET_1 = \infty$.
(b) Show that τ_n has the same distribution as $\sum_{j=1}^n T_j$.
(c) Show that T_1 has the same distribution as $n^{-2}\sum_{j=1}^n T_j$.

HINT for (b): Use the strong Markov property repeatedly.

2. **First Passage to a Tilted Line.** Define $\tau = \min\{t > 0 : W(t) = a - bt\}$ where $a, b > 0$ are positive constants. Find the Laplace transform and/or the probability density function of τ.

3. **Nondifferentiability of Brownian Paths.**
Prove that, with probability one, $W(t)$ is not differentiable at $t = 0$.

HINT: If $W(t)$ were differentiable at $t = 0$ then for all sufficiently small $\varepsilon > 0$ the graph of $\{W(t)\}_{0\le t\le\varepsilon}$ would have to lie between the lines $w = -t/\varepsilon$ and $w = t/\varepsilon$. But the probability that even the *endpoint* $(\varepsilon, W(\varepsilon))$ lies between these two lines is only

$$P\{|W(\varepsilon)| < \varepsilon\} = \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} e^{-x^2/2}\,dx.$$

NOTE: It is a theorem of DVORETSKY, ERDÖS, and KAKUTANI that, with probability one, $W(t)$ is not differentiable *anywhere*. This is considerably more difficult to prove than nondifferentiability at 0.

4. Verify that the Gauss kernel $p_t(x, y)$ solves the heat equation.

5.* **Embedded Simple Random Walk.** Define stopping times τ_n inductively as follows: let $\tau_0 = 0$, and let

$$\tau_1 = \min\{t : |W(t)| = 1\};$$
$$\tau_{n+1} = \min\{t : |W(t + \tau_n) - W(\tau_n)| = 1\}$$

(a) Use the strong Markov property to show that the sequence $\{W(\tau_n)\}_{n\ge0}$ is a simple random walk.
(b) Use the Optional Sampling Formula to show that $E\tau_1 = 1$.
(c) Use the SLLN to conclude that $\tau_n/n \to 1$.
(d) Use (a), (c), path continuity, and the scaling property of Brownian motion to give a new proof of the DeMoivre–Laplace Central Limit theorem.