MathFinance 345/Stat390
Homework 1
Due October 3, 2001

In the following problems, all markets are assumed to be arbitrage–free, and to contain a riskless asset (called BOND), with riskless rate of return r.

1. **A Swap Contract:** The contract calls for the following:
    (a) The buyer X pays the seller Y an amount q to enter into the contract at time 0.
    (b) The seller agrees to exchange 1 share of asset A for 1 share of asset B at time 1.

The share prices of assets A and B at times $t = 0$ and $t = 1$ are S_t^A and S_t^B, respectively. As in all such problems, the share prices of the underlying assets at the termination time $t = 1$ are subject to uncertainty. Assume that there is a riskless asset MONEYMARKET with rate of return r, as in Problem 1. Determine the fair market value q of the contract in two ways:

   (a) by an arbitrage argument; and
   (b) using the Fundamental Theorem.

2. **Put Options:** A (European) *put* on an asset STOCK is a contract that gives the owner the right to sell 1 share of STOCK at time $t = 1$ for an amount K fixed at time $t = 0$ (called the *strike*). Consider a two–scenario market in which the share values of STOCK at time $t = 1$ in the two scenarios ω_1, ω_2 are $d_1 < d_2$. Let S_0 be the share price of STOCK at $t = 0$ and r be the riskless rate of return.

   (a) Find a formula for the market price of a put with strike K in terms of S_0, r, d_1, d_2.
   (b) Find a replicating portfolio for the put in the assets BOND and STOCK.

3. **An Incomplete Market:** Consider a market with two freely traded assets, BOND and STOCK, and three scenarios $\omega_1, \omega_2, \omega_3$. Assume that the $t = 1$ share price of STOCK in scenario ω_i is d_i, and that $d_1 < d_2 < d_3$. Let r be the riskless rate of return, and S_0 the share price of STOCK at $t = 0$.

   (a) Show that this market is incomplete.
   (b) Exhibit a derivative security for which there is *no* replicating portfolio in the assets BOND and STOCK.
   (c) Show that the $t = 0$ market price of the derivative security you found in part (b) is not uniquely determined. (That is, show that there are equilibrium measures for the market that give different prices for the derivative security.)
   (d) Show that the set of possible market prices of the derivative security in (b) is an interval of real numbers.

4. **Markets with Infinitely Many Scenarios.** Does the Fundamental Theorem of Arbitrage Pricing remain valid when the set of scenarios is infinite? The answer, unfortunately, is NO: it is not always valid. Here is an example: Consider a market with 3 traded assets

A^1, A^2, and B, where B is riskless, with rate of return 0. Let S_t^1 and S_t^2 denote the share prices of assets A^1 and A^2 at time $t = 0, 1$. Assume that

$$\Omega = \{(a_1, a_2) \ : \ a_1 > 0, \ a_2 > 0, \ \text{and} \ a_1 + a_2 > 1\} \cup \{(0, 1)\};$$
$$\mathcal{F} = \{\text{Borel subsets of } \Omega\};$$
$$A_1^1((a_1, a_2) = a_1 \ \text{and} \ A_0^1 = 0;$$
$$A_1^2((a_1, a_2)) = a_2 \ \text{and} \ A_0^2 = 2.$$

(a) Prove that there are no arbitrages.

(b) Prove that there is no equilibrium measure.