FE6516

Stochastic Calculus in Finance II

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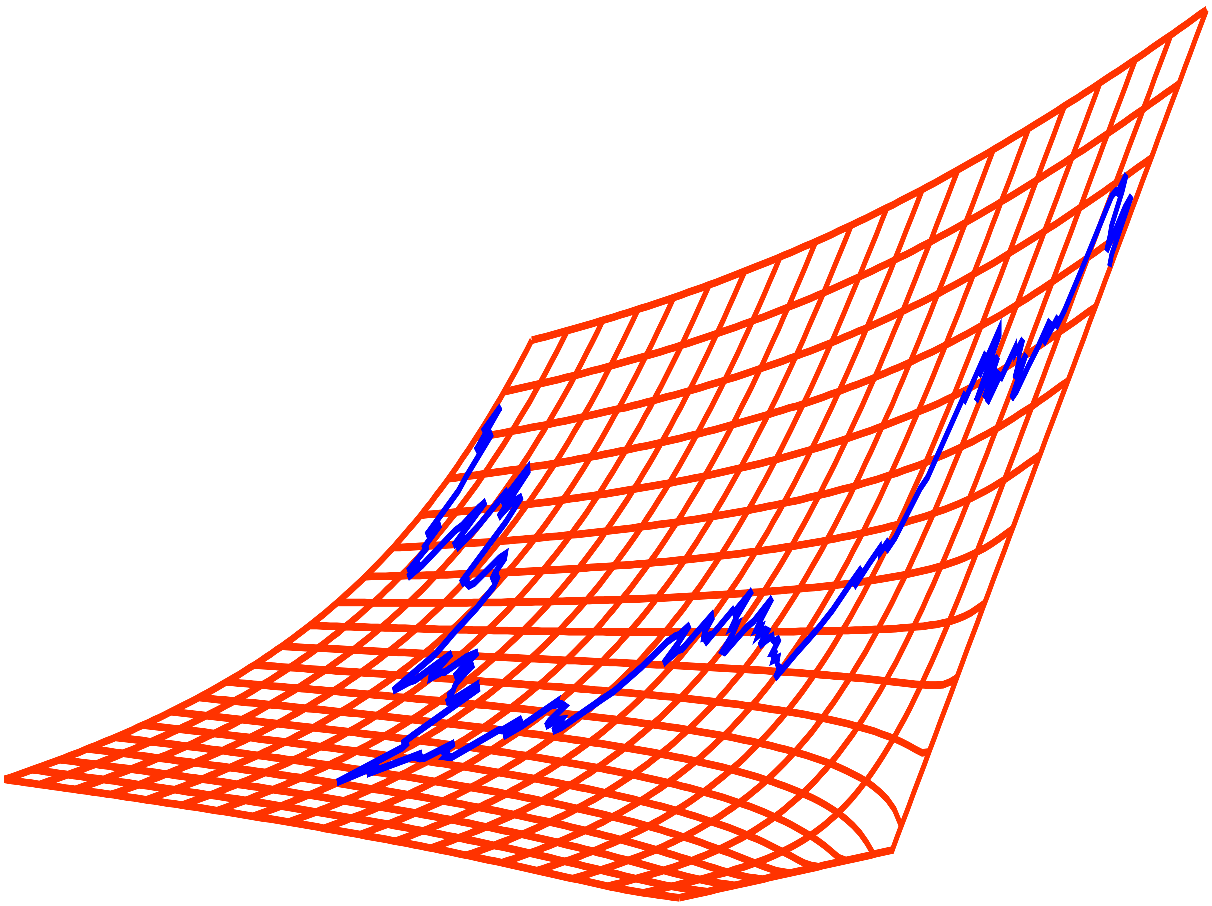

January 20, 2021 https://personal.ntu.edu.sg/nprivault/indext.html

This course provides complements in stochastic calculus for the pricing and hedging of financial derivatives. Stochastic asset price modeling using Brownian motion is reviewed in Section 1, and the Black-Scholes model is presented from the angle of partial differential equation (PDE) methods in Chapter 2, with the derivation of the Black-Scholes formula by transforming the Black-Scholes PDE into the standard heat equation, which is then solved by a heat kernel argument.

The martingale approach to pricing and hedging is then presented in Chapter 3, and complements the PDE approach of Chapter 2 by recovering the Black-Scholes formula via a probabilistic argument. A presentation of volatility estimation tools including historical and implied volatilities is given in Section 2.5 with a comparison of the prices obtained by the Black-Scholes formula with actual option price market data. Optimal stopping and exercise, with application to the pricing of American options, are considered in Chapter 5, following the presentation of background material on filtrations and stopping times in Chapter 4.

Stochastic calculus with jumps is dealt with in Chapter 6 and is restricted to compound Poisson processes, which only have a finite number of jumps on any bounded interval. Those processes are used for option pricing and hedging in jump models in Chapter 7, in which we mostly focus on risk minimizing strategies as markets with jumps are generally incomplete. The text is completed with an appendix containing the needed probabilistic background.

The pdf file contains internal and external links, and 108 figures, including 17 animated figures, *e.g.* Figures 1.6, 2.5, 5.2, 6.12, 6.14, 6.15, 1 embedded video in Figure 2.20, and 2 interacting 3D graphs in Figures 2.4 and 2.11, that may require using Acrobat Reader for viewing on the complete pdf file. It also includes 5 Python codes on pages 24, 56, 40, 125 and 26 R codes on pages 13, 9, 24, 27, 37, 31, 54, 56, 67, 40, 41, 140, 143, 154, 155.

The document also contains 81 solved exercises and includes 5 Python codes on pages 24, 56 and 40, and 26 R codes on pages 13, 9, 24, 27, 37, 31, 54, 56, 67, 40, 41. Clicking on an exercise number inside the solution section will send to the original problem text inside the file. Conversely, clicking on a problem number sends the reader to the corresponding solution, however this feature should not be misused. The cover graph represents the time evolution of the HSBC stock price from January to September 2009, plotted on the price surface of a European *put option* on that asset, expiring on October 05, 2009, cf. § 2.1.

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*Animated figures (work in Acrobat reader).

1. Continuous-Time Market Model

The continuous-time market model allows for the incorporation of portfolio re-allocation algorithms in a stochastic dynamic programming setting. This chapter starts with a review of the concepts of assets, self-financing portfolios, risk-neutral probability measures, and arbitrage in continuous time. We also state and solve the equation satisfied by geometric Brownian motion, which will be used for the modeling of continuous asset price processes.

1.1 Asset Price Modeling

The prices at time $t \in \mathbb{R}_+0$ of $d+1$ assets numbered n^o $0,1,\ldots,d$ is denoted by the *random* vector

$$\bar{S}_t = \left(S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)}\right)$$

which forms a stochastic process $(\bar{S}_t)_{t\in\mathbb{R}_+}$. As in discrete time, the asset n^o 0 is a riskless asset (of savings account type) yielding an interest rate r, *i.e.* we have

$$S_t^{(0)} = S_0^{(0)} \,\mathrm{e}^{rt}, \qquad t \in \mathbb{R}_+.$$

Definition 1.1 *Discounting*. Let

$$\overline{X}_t := \left(\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)}\right), \qquad t \in \mathbb{R},$$

denote the vector of discounted asset prices, defined as:

$$\widetilde{S}_t^{(k)} = \mathrm{e}^{-rt} S_t^{(k)}, \qquad t \in \mathbb{R}, \quad k = 0, 1, \ldots, d.$$

We can also write

$$\overline{X}_t := \mathrm{e}^{-rt} \overline{S}_t, \qquad t \in \mathbb{R}_+.$$

The concept of discounting is illustrated in the following figures.

My portfolio S_t grew by $b = 5\%$ this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My portfolio S_t grew by $b = 5\%$ this year.

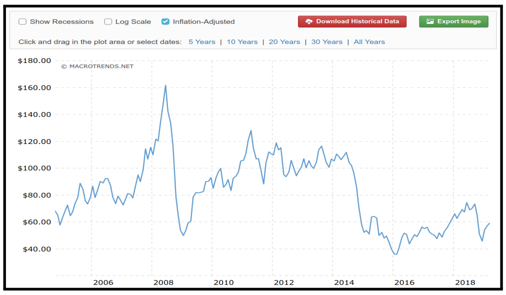
The risk-free or inflation rate is $r = 10\%$.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

(a) *Without* inflation adjustment.



(b) *With* inflation adjustment.

Figure 1.2: Why apply discounting?

Definition 1.2 A portfolio strategy is a stochastic process $(\overline{\xi}_t)_{t\in\mathbb{R}_+} \subset \mathbb{R}^{d+1}$, where $\xi_t^{(k)}$ denotes the (possibly fractional) quantity of asset n^o k held at time $t \in \mathbb{R}_+$.

The *value* at time $t \geqslant 0$ of the portfolio strategy $(\overline{\xi}_t)_{t\in\mathbb{R}_+} \subset \mathbb{R}^{d+1}$ is defined by

$$V_t := \overline{\xi}_t \cdot \overline{S}_t \qquad t \in \mathbb{R}_+.$$

The *discounted* value at time 0 of the portfolio is defined by

$$\widetilde{V}_t := \mathrm{e}^{-rt} V_t, \qquad t \in \mathbb{R}_+.$$

For $t \in \mathbb{R}_+$, we have

$$\begin{aligned} \widetilde{V}_t &= \mathrm{e}^{-rt} \overline{\xi}_t \cdot \overline{S}_t \\ &= \mathrm{e}^{-rt} \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)} \\ &= \sum_{k=0}^{d} \xi_t^{(k)} \widetilde{S}_t^{(k)} \end{aligned}$$

$$= \quad \overline{\xi}_t \cdot \overline{X}_t, \qquad t \in \mathbb{R}_+.$$

The effect of discounting from time t to time 0 is to divide prices by e^{rt}, making all prices comparable at time 0.

1.2 Arbitrage and Risk-Neutral Measures

In continuous-time, the definition of arbitrage follows the lines of its analogs in the one-step and discrete-time models. In the sequel we will only consider *admissible* portfolio strategies whose total value V_t remains nonnegative for all times $t \in [0,T]$.

Definition 1.3 A portfolio strategy $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ with value

$$V_t = \overline{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+,$$

constitutes an arbitrage opportunity if all *three* following conditions are satisfied:

i) $V_0 \leqslant 0$ at time $t = 0$, *[start from a zero-cost portfolio or in debt]*

ii) $V_T \geqslant 0$ at time $t = T$, *[finish with a nonnegative amount]*

iii) $\mathbb{P}(V_T > 0) > 0$ at time $t = T$. *[profit made with nonzero probability]*

Roughly speaking, (ii) means that the investor wants no loss, (iii) means that he wishes to sometimes make a strictly positive gain, and (i) means that he starts with zero capital or even with a debt.

Next, we turn to the definition of risk-neutral probability measures (or martingale measures) in continuous time, which states that under a risk-neutral probability measure $\mathbb{P}^*$, the return of the risky asset over the time interval $[u,t]$ equals the return of the riskless asset given by

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Recall that the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is generated by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$\mathscr{F}_t = \sigma(B_u \ : \ 0 \leqslant u \leqslant t), \qquad t \in \mathbb{R}_+.$$

Definition 1.4 A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*\left[S_t^{(k)} \,\middle|\, \mathscr{F}_u\right] = e^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1,2,\ldots,d. \tag{1.1}$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

As in the discrete-time case, $\mathbb{P}^\sharp$ would be called a risk premium measure if it satisfied

$$\mathbb{E}^\sharp\left[S_t^{(k)} \,\middle|\, \mathscr{F}_u\right] > e^{(t-u)r} S_u, \qquad 0 \leqslant u \leqslant t, \quad k = 1,2,\ldots,d,$$

meaning that by taking risks in buying $S_t^{(i)}$, one could make an expected return higher than that of the riskless asset

$$S_t^{(0)} = e^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t.$$

Similarly, a negative risk premium measure $\mathbb{P}^\flat$ satisfies

$$\mathbb{E}^\flat\left[S_t^{(k)} \,\middle|\, \mathscr{F}_u\right] < e^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1,2,\ldots,d.$$

From the relation

$$S_t^{(0)} = \mathrm{e}^{(t-u)r} S_u^{(0)}, \qquad 0 \leqslant u \leqslant t,$$

we interpret (1.1) by saying that the expected return of the risky asset $S_t^{(k)}$ under $\mathbb{P}^*$ equals the return of the riskless asset $S_t^{(0)}$, $k = 1,2,\ldots,d$. Recall that the discounted (in \$ at time 0) price $\widetilde{S}_t^{(k)}$ of the risky asset n^o k is defined by

$$\widetilde{S}_t^{(k)} := \mathrm{e}^{-rt} S_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)} / S_0^{(0)}}, \qquad t \in \mathbb{R}_+, \quad k = 0,1,\ldots,d,$$

i.e. $S_t^{(0)} / S_0^{(0)}$ plays the role of a *numéraire*.

Definition 1.5 A continuous-time process $(Z_t)_{t\in\mathbb{R}_+}$ of integrable random variables is a martingale under $\mathbb{P}$ and with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[Z_t \mid \mathscr{F}_s] = Z_s, \qquad 0 \leqslant s \leqslant t.$$

Note that when $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale, Z_t is in particular $\mathscr{F}_t$-measurable at all times $t \in \mathbb{R}_+$.

In continuous-time finance, the martingale property can be used to characterize risk-neutral probability measures, for the derivation of pricing partial differential equations (PDEs), and for the computation of conditional expectations.

As in the discrete-time case, the notion of martingale can be used to characterize risk-neutral probability measures as in the next proposition.

Proposition 1.1 The probability measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted risky asset price process $(\widetilde{S}_t^{(k)})_{t\in\mathbb{R}_+}$ is a martingale *under* $\mathbb{P}^*$, $k = 1,2,\ldots,d$.

Proof. If $\mathbb{P}^*$ is a risk-neutral probability measure, we have

$$\begin{aligned}
\mathbb{E}^*\left[\widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathrm{e}^{-rt} S_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{-rt}\, \mathbb{E}^*\left[S_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{-rt}\, \mathrm{e}^{(t-u)r} S_u^{(k)} \\
&= \mathrm{e}^{-ru} S_u^{(k)} \\
&= \widetilde{S}_u^{(k)}, \qquad 0 \leqslant u \leqslant t,
\end{aligned}$$

hence $\left(\widetilde{S}_t^{(k)}\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$, $k = 1,2,\ldots,d$. Conversely, if $\left(\widetilde{S}_t^{(k)}\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$ then

$$\begin{aligned}
\mathbb{E}^*\left[S_t^{(k)} \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathrm{e}^{rt} \widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{rt}\, \mathbb{E}^*\left[\widetilde{S}_t^{(k)} \mid \mathscr{F}_u\right] \\
&= \mathrm{e}^{rt} \widetilde{S}_u^{(k)} \\
&= \mathrm{e}^{(t-u)r} S_u^{(k)}, \qquad 0 \leqslant u \leqslant t, \quad k = 1,2,\ldots,d,
\end{aligned}$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral according to Definition 1.4. □

In the sequel we will only consider probability measures $\mathbb{P}^*$ that are *equivalent* to $\mathbb{P}$, in the sense that they share the same events of zero probability.

Definition 1.6 A probability measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{P}^*(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}. \tag{1.2}$$

Next, we note that the first fundamental theorem of asset pricing also holds in continuous time, and can be used to check for the existence of arbitrage opportunities.

Theorem 1.2 A market is *without* arbitrage opportunity if and only if it admits at least one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. See Harrison and Pliska, 1981 and Chapter VII-4a of Shiryaev, 1999. □

1.3 Self-Financing Portfolio Strategies

Let $\xi_t^{(i)}$ denote the (possibly fractional) quantity invested at time t over the time interval $[t, t+dt)$, in the asset $S_t^{(k)}$, $k = 0, 1, \ldots, d$, and let

$$\bar{\xi}_t = \left(\xi_t^{(k)}\right)_{k=0,1,\ldots,d}, \qquad \bar{S}_t = \left(S_t^{(k)}\right)_{k=0,1,\ldots,d}, \quad t \in \mathbb{R}_+,$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time t is given by

$$V_t = \bar{\xi}_t \bullet \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+. \tag{1.3}$$

Our description of portfolio strategies proceeds in four equivalent formulations (1.4), (1.5) (1.7) and (1.8), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\bar{\xi}_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from $\bar{\xi}_t$ to $\bar{\xi}_{t+dt}$, *i.e.*

$$\bar{\xi}_t \bullet \bar{S}_{t+dt} = \sum_{k=0}^{d} \xi_t^{(k)} S_{t+dt}^{(k)} = \sum_{k=0}^{d} \xi_{t+dt}^{(k)} S_{t+dt}^{(k)} = \bar{\xi}_{t+dt} \bullet \bar{S}_{t+dt}. \tag{1.4}$$

A major difference with the discrete-time case, however, is that the continuous-time differentials dS_t and $d\xi_t$ do not make pathwise sense as continuous-time stochastic integrals are defined by L^2 limits, or by convergence in probability.

Portfolio value	$\bar{\xi}_t \bullet \bar{S}_t \longrightarrow$	$\bar{\xi}_t \bullet \bar{S}_{t+dt}$	$= \bar{\xi}_{t+dt} \bullet \bar{S}_{t+dt} \longrightarrow$	$\bar{\xi}_{t+dt} \bullet \bar{S}_{t+2dt}$
Asset value	S_t	S_{t+dt}	S_{t+dt}	S_{t+2dt}
Time scale	t	$t+dt$	$t+dt$	$t+2dt$
Portfolio allocation	ξ_t	ξ_t	ξ_{t+dt}	ξ_{t+dt}

Figure 1.3: Illustration of the self-financing condition (1.4).

Equivalently, Condition (1.4) can be rewritten as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} d\xi_t^{(k)} = 0, \tag{1.5}$$

where

$$d\xi_t^{(k)} := \xi_{t+dt}^{(k)} - \xi_t^{(k)}, \qquad k = 0, 1, \ldots, d,$$

denote the respective changes in portfolio allocations. In other words, (1.5) rewrites as

$$\sum_{k=0}^{d} S_{t+dt}^{(k)} \big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) = 0. \tag{1.6}$$

Condition (1.6) can be rewritten as

$$\sum_{k=0}^{d} S_t^{(k)} \big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) + \sum_{k=0}^{d} \big(S_{t+dt}^{(k)} - S_t^{(k)}\big)\big(\xi_{t+dt}^{(k)} - \xi_t^{(k)}\big) = 0,$$

which shows that (1.4) and (1.5) are equivalent to

$$\bar{S}_t \cdot d\bar{\xi}_t + d\bar{S}_t \cdot d\bar{\xi}_t = \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)} = 0 \tag{1.7}$$

in differential notation.

Self-financing portfolio differential

In practice, the self-financing portfolio property will be characterized by the following proposition, which states that the value of a self-financing portfolio can be written as the sum of its trading Profits and Losses (P/L).

Proposition 1.3 A portfolio strategy $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ with value

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \sum_{k=0}^{d} \xi_t^{(k)} S_t^{(k)}, \qquad t \in \mathbb{R}_+,$$

is self-financing according to (1.4) if and only if the relation

$$dV_t = \sum_{k=0}^{d} \underbrace{\xi_t^{(k)} dS_t^{(k)}}_{\text{P/L for asset } n^o\ k} \tag{1.8}$$

holds.

Proof. By Itô's calculus we have

$$dV_t = \sum_{k=0}^{d} \xi_t^{(k)} dS_t^{(k)} + \sum_{k=0}^{d} S_t^{(k)} d\xi_t^{(k)} + \sum_{k=0}^{d} dS_t^{(k)} \cdot d\xi_t^{(k)},$$

which shows that (1.7) is equivalent to (1.8). □

Market Completeness

Definition 1.7 A contingent claim with payoff C is said to be attainable if there exists a (self-financing) portfolio strategy $\big(\xi_t^{(k)}\big)_{t\in[0,T],k=0,1,\ldots,d}$ such that at the maturity time T the equality

$$V_T = \bar{\xi}_T \cdot \bar{S}_T = \sum_{k=0}^{d} \xi_T^{(k)} S_T^{(k)} = C$$

holds (almost surely) between random variables.

When a claim with payoff C is attainable, its price at time t will be given by the value V_t of a self-financing portfolio hedging C.

Definition 1.8 A market model is said to be *complete* if every contingent claim is attainable.

The next result is the continuous-time statement of the second fundamental theorem of asset pricing.

Theorem 1.4 A market model without arbitrage opportunities is complete if and only if it admits only one *equivalent* risk-neutral probability measure $\mathbb{P}^*$.

Proof. See Harrison and Pliska, 1981 and Chapter VII-4a of Shiryaev, 1999. □

In the Black and Scholes, 1973 model, one can show the existence of a unique risk-neutral probability measure, hence the model is without arbitrage and complete.

1.4 Black-Scholes Market Model

From now one we work with $d = 1$, *i.e.* with a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$.

The riskless asset price process $(A_t)_{t\in\mathbb{R}_+}$ admits the following equivalent constructions:

$$\frac{A_{t+dt} - A_t}{A_t} = rdt, \qquad \frac{dA_t}{A_t} = rdt, \qquad A'_t = rA_t, \quad t \in \mathbb{R}_+,$$

with the solution

$$A_t = A_0 \, \mathrm{e}^{rt}, \qquad t \in \mathbb{R}_+, \tag{1.9}$$

where $r > 0$ is the risk-free interest rate.*

Self-financing portfolio strategies

Let ξ_t and η_t denote the (possibly fractional) quantities invested at time t over the time interval $[t, t+dt)$, respectively in the assets S_t and A_t, and let

$$\bar{\xi}_t = (\eta_t, \xi_t), \qquad \bar{S}_t = (A_t, S_t), \qquad t \in \mathbb{R}_+,$$

denote the associated portfolio value and asset price processes. The portfolio value V_t at time t is given by

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

Our description of portfolio strategies proceeds in four equivalent formulations presented below in Equations (1.10), (1.11), (1.13) and (1.14), which correspond to different interpretations of the self-financing condition.

Self-financing portfolio update

The portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, *i.e.*

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = \eta_t A_{t+dt} + \xi_t S_{t+dt} = \eta_{t+dt} A_{t+dt} + \xi_{t+dt} S_{t+dt} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}. \tag{1.10}$$

*"Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", Kenneth E. Boulding, Boulding, 1973, page 248.

Portfolio value	$\bar{\xi}_t \cdot \bar{S}_t$	$\longrightarrow$	$\bar{\xi}_t \cdot \bar{S}_{t+dt}$	$=$ $\bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}$	$\longrightarrow$	$\bar{\xi}_{t+dt} \cdot \bar{S}_{t+2dt}$
Asset value	S_t		S_{t+dt}	S_{t+dt}		S_{t+2dt}
Time scale	t		$t+dt$	$t+dt$		$t+2dt$
Portfolio allocation	ξ_t		ξ_t	ξ_{t+dt}		ξ_{t+dt}

Figure 1.4: Illustration of the self-financing condition (1.10).

Equivalently, Condition (1.10) can be rewritten as

$$A_{t+dt}d\eta_t + S_{t+dt}d\xi_t = 0, \tag{1.11}$$

where

$$d\eta_t := \eta_{t+dt} - \eta_t \quad \text{and} \quad d\xi_t := \xi_{t+dt} - \xi_t$$

denote the respective changes in portfolio allocations. In other words, we have

$$A_{t+dt}(\eta_t - \eta_{t+dt}) = S_{t+dt}(\xi_{t+dt} - \xi_t). \tag{1.12}$$

In other words, when one sells a (possibly fractional) quantity $\eta_t - \eta_{t+dt} > 0$ of the riskless asset valued A_{t+dt} at the end of the time interval $[t, t+dt]$ for the total amount $A_{t+dt}(\eta_t - \eta_{t+dt})$, one should entirely spend this income to buy the corresponding quantity $\xi_{t+dt} - \xi_t > 0$ of the risky asset for the same amount $S_{t+dt}(\xi_{t+dt} - \xi_t) > 0$.

Similarly, if one sells a quantity $-d\xi_t > 0$ of the risky asset S_{t+dt} between the time intervals $[t, t+dt]$ and $[t+dt, t+2dt]$ for a total amount $-S_{t+dt}d\xi_t$, one should entirely use this income to buy a quantity $d\eta_t > 0$ of the riskless asset for an amount $A_{t+dt}d\eta_t > 0$, *i.e.*

$$A_{t+dt}d\eta_t = -S_{t+dt}d\xi_t.$$

Condition (1.12) can also be rewritten as

$$S_t(\xi_{t+dt} - \xi_t) + A_t(\eta_{t+dt} - \eta_t) + (S_{t+dt} - S_t)(\xi_{t+dt} - \xi_t) \\ + (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = 0,$$

which shows that (1.10) and (1.11) are equivalent to

$$S_t d\xi_t + A_t d\eta_t + dS_t \cdot d\xi_t + dA_t \cdot d\eta_t = 0 \tag{1.13}$$

in differential notation, with

$$dA_t \cdot d\eta_t \simeq (A_{t+dt} - A_t) \cdot (\eta_{t+dt} - \eta_t) = rA_t(dt \cdot d\eta_t) = 0$$

in the sense of the Itô calculus by the Itô table. This yields the following proposition, which is also consequence of Proposition 1.3.

Proposition 1.5 A portfolio allocation $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+,$$

is self-financing according to (1.10) if and only if the relation

$$dV_t = \underbrace{\eta_t dA_t}_{\text{risk-free P/L}} + \underbrace{\xi_t dS_t}_{\text{risky P/L}} \tag{1.14}$$

holds.

Proof. By Itô's calculus we have

$$dV_t = [\eta_t dA_t + \xi_t dS_t] + [S_t d\xi_t + A_t d\eta_t + dS_t \bullet d\xi_t + dA_t \bullet d\eta_t],$$

which shows that (1.13) is equivalent to (1.14). □

Let

$$\widetilde{V}_t := \mathrm{e}^{-rt} V_t \qquad \text{and} \qquad \widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

respectively denote the discounted portfolio value and discounted risky asset price at time $t \geqslant 0$.

Geometric Brownian motion

The risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ will be modeled using a geometric Brownian motion defined from the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \in \mathbb{R}_+, \tag{1.15}$$

see Section 1.5.

```
N=2000; t <- 0:N; dt <- 1.0/N; mu=0.5;sigma=0.2; nsim <- 10
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
for (i in 1:nsim){X[i,] <- exp(mu*t*dt+sigma*X[i,]-sigma*sigma*t*dt/2)}
plot(t*dt, rep(0, N+1), xlab = "time", ylab = "Geometric Brownian motion", lwd=2, ylim =
      c(min(X),max(X)), type = "l", col = 0)
for (i in 1:nsim){lines(t*dt, X[i, ], lwd=2, type = "l", col = i)}
```

By Proposition 1.8 below, we have

$$S_t \quad = \quad S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \in \mathbb{R}_+.$$

The next Figure 1.5 presents sample paths of geometric Brownian motion.

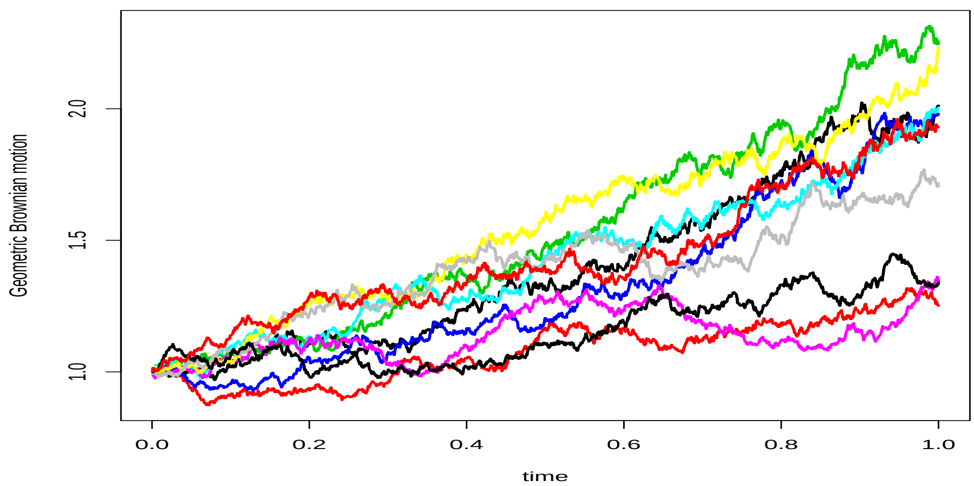


Figure 1.5: Sample paths of geometric Brownian motion.

Lemma 1.6 *Discounting lemma*. Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ be as in (1.15), *i.e.*

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+.$$

Then the discounted asset price process $\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+}$ satisfies the equation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.$$

Proof. We have

$$\begin{aligned}
d\widetilde{S}_t &= d(\mathrm{e}^{-rt}S_t) \\
&= S_t d(\mathrm{e}^{-rt}) + \mathrm{e}^{-rt}dS_t + (d\,\mathrm{e}^{-rt}) \bullet dS_t \\
&= -r\mathrm{e}^{-rt}S_t dt + \mathrm{e}^{-rt}dS_t + (-r\mathrm{e}^{-rt}S_t dt) \bullet dS_t \\
&= -r\mathrm{e}^{-rt}S_t dt + \mu\,\mathrm{e}^{-rt}S_t dt + \sigma\,\mathrm{e}^{-rt}S_t dB_t \\
&= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t.
\end{aligned}$$

□

In the next Lemma 1.7 we show that when a portfolio is self-financing, its discounted value is a gain process given by the sum over time of discounted trading profits and losses (number of risky assets ξ_t times discounted price variation $d\widetilde{S}_t$).

Note that in Equation (1.16) below, no profit or loss arises from trading the discounted riskless asset $\widetilde{A}_t := \mathrm{e}^{-rt}A_t = A_0$, because its price remains constant over time.

Lemma 1.7 Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+.$$

The following statements are equivalent:

(i) the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic integral sum

$$\widetilde{V}_t = \widetilde{V}_0 + \underbrace{\int_0^t \xi_u d\widetilde{S}_u}_{\text{discounted P/L}}, \qquad t \in \mathbb{R}_+, \tag{1.16}$$

of discounted profits and losses.

Proof. Assuming that (i) holds, the self-financing condition and (1.9)-(1.15) show that

$$\begin{aligned}
dV_t &= \eta_t dA_t + \xi_t dS_t \\
&= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

where we used $V_t = \eta_t A_t + \xi_t S_t$, hence

$$\mathrm{e}^{-rt}dV_t = r\mathrm{e}^{-rt}V_t dt + (\mu - r)\mathrm{e}^{-rt}\xi_t S_t dt + \sigma\,\mathrm{e}^{-rt}\xi_t S_t dB_t, \quad t \in \mathbb{R}_+,$$

and

$$\begin{aligned}
d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt}V_t\right) \\
&= -r\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}dV_t
\end{aligned}$$

$$\begin{aligned}
&= (\mu - r)\xi_t \,\mathrm{e}^{-rt} S_t dt + \sigma \xi_t \,\mathrm{e}^{-rt} S_t dB_t \\
&= (\mu - r)\xi_t \widetilde{S}_t dt + \sigma \xi_t \widetilde{S}_t dB_t \\
&= \xi_t d\widetilde{S}_t, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

i.e. (1.16) holds by integrating on both sides as

$$\widetilde{V}_t - \widetilde{V}_0 = \int_0^t d\widetilde{V}_u = \int_0^t \xi_u d\widetilde{S}_u, \qquad t \in \mathbb{R}_+.$$

(ii) Conversely, if (1.16) is satisfied we have

$$\begin{aligned}
dV_t &= d(\mathrm{e}^{rt}\widetilde{V}_t) \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt} d\widetilde{V}_t \\
&= r\,\mathrm{e}^{rt}\widetilde{V}_t dt + \mathrm{e}^{rt} \xi_t d\widetilde{S}_t \\
&= rV_t dt + \mathrm{e}^{rt} \xi_t d\widetilde{S}_t \\
&= rV_t dt + \mathrm{e}^{rt} \xi_t \widetilde{S}_t ((\mu - r)dt + \sigma dB_t) \\
&= rV_t dt + \xi_t S_t ((\mu - r)dt + \sigma dB_t) \\
&= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t \\
&= \eta_t dA_t + \xi_t dS_t,
\end{aligned}$$

hence the portfolio is self-financing according to Definition 1.3. □

As a consequence of Relation (1.16), the problem of hedging a claim payoff C with maturity T also reduces to that of finding the process $(\xi_t)_{t\in[0,T]}$ appearing in the decomposition of the discounted claim payoff $\widetilde{C} = \mathrm{e}^{-rT} C$ as a stochastic integral:

$$\widetilde{C} = \widetilde{V}_T = \widetilde{V}_0 + \int_0^T \xi_t d\widetilde{S}_t,$$

see Section 3.5 on hedging by the martingale method.

Example. *Power options in the Bachelier model.*
In the Bachelier, 1900 model, the underlying asset price can be modeled by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, and may therefore become negative.* The claim payoff $C = (B_T)^2$ of a power option with at maturity $T > 0$ admits the stochastic integral decomposition

$$(B_T)^2 = T + 2\int_0^T B_t dB_t$$

which shows that the claim can be hedged using $\xi_t = 2B_t$ units of the underlying asset at time $t \in [0, T]$, see Exercise 2.1.

Similarly, in the case of power claim payoff $C = (B_T)^3$ we have

$$(B_T)^3 = 3\int_0^T (T - t + (B_t)^2) dB_t.$$

Note that according to (1.16), the (non-discounted) self-financing portfolio value V_t can be written as

$$V_t = \mathrm{e}^{rt} V_0 + (\mu - r)\int_0^t \mathrm{e}^{(t-u)r} \xi_u S_u du + \sigma \int_0^t \mathrm{e}^{(t-u)r} \xi_u S_u dB_u, \quad t \in \mathbb{R}_+. \tag{1.17}$$

*Negative oil prices have been observed in May 2020 when the prices of oil futures contracts fell below zero.

1.5 Geometric Brownian Motion

In this section we solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

which is used to model the S_t the risky asset price at time t, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in *integral form* as

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s, \qquad t \in \mathbb{R}_+. \tag{1.18}$$

It can be solved by applying Itô's formula to the Itô process $(S_t)_{t\in\mathbb{R}_+}$ with $v_t = \mu S_t$ and $u_t = \sigma S_t$, and by taking

$$f(S_t) = \log S_t, \qquad \text{with} \qquad f(x) := \log x.$$

This yields the log-return dynamics

$$\begin{aligned} d\log S_t &= f'(S_t)dS_t + \frac{1}{2} f''(S_t)dS_t \bullet dS_t \\ &= \mu S_t f'(S_t)dt + \sigma S_t f'(S_t)dB_t + \frac{\sigma^2}{2} S_t^2 f''(S_t)dt \\ &= \mu dt + \sigma dB_t - \frac{\sigma^2}{2} dt, \end{aligned}$$

hence

$$\begin{aligned} \log S_t - \log S_0 &= \int_0^t d\log S_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right) \int_0^t ds + \sigma \int_0^t dB_s \\ &= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t, \end{aligned}$$

and

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right), \qquad t \in \mathbb{R}_+.$$

The next Figure 1.6 presents an illustration of the geometric Brownian process of Proposition 1.8.

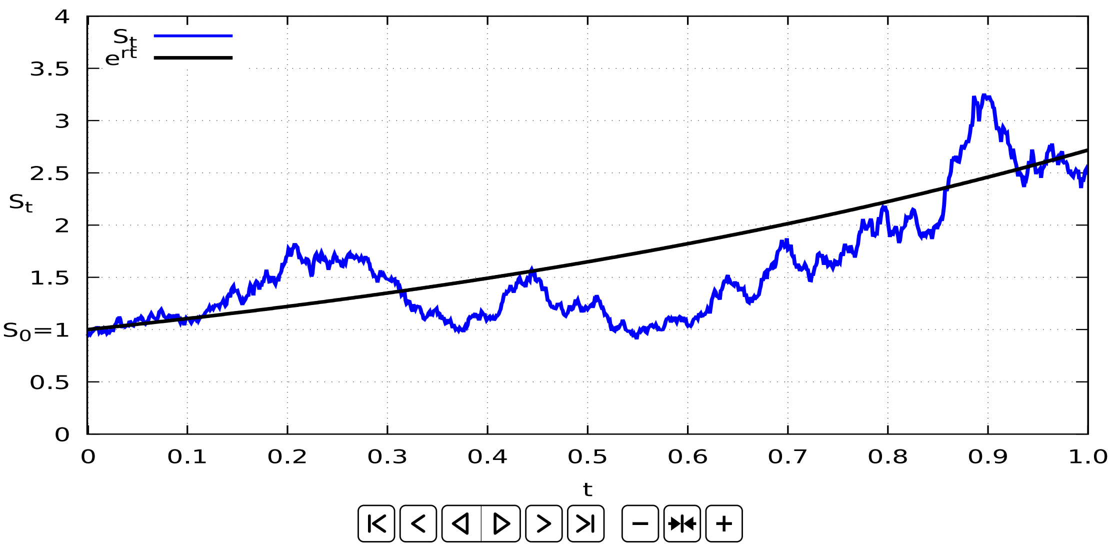


Figure 1.6: Geometric Brownian motion started at $S_0 = 1$, with $\mu = r = 1$ and $\sigma^2 = 0.5$.*

```
N=1000; t <- 0:N; dt <- 1.0/N; sigma=0.6; mu=0.001
Z <- rnorm(N,mean=0,sd=sqrt(dt));
plot(t*dt, exp(mu*t), xlab = "time", ylab = "Geometric Brownian motion", type = "l", ylim = c(0, 4),
      col = 1,lwd=3)
lines(t*dt, exp(sigma*c(0,cumsum(Z))+mu*t-sigma*sigma*t*dt/2),xlab = "time",type = "l",ylim = c(0,
      4), col = 4)
```

The above calculation provides a proof for the next proposition.

Proposition 1.8 The solution of the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1.19}$$

is given by

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \in \mathbb{R}_+.$$

Proof. Let us provide an alternative proof by searching for a solution of the form

$$S_t = f(t, B_t)$$

where $f(t,x)$ is a function to be determined. By Itô's formula, we have

$$dS_t = df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt.$$

Comparing this expression to (1.19) and identifying the terms in dB_t we get

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,B_t) = \sigma S_t, \\ \\ \dfrac{\partial f}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,B_t) = \mu S_t. \end{cases}$$

Using the relation $S_t = f(t,B_t)$, these two equations rewrite as

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,B_t) = \sigma f(t,B_t), \\ \\ \dfrac{\partial f}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,B_t) = \mu f(t,B_t). \end{cases}$$

Since B_t is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

$$\begin{cases} \dfrac{\partial f}{\partial x}(t,x) = \sigma f(t,x), & (1.22a) \\ \\ \dfrac{\partial f}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2}(t,x) = \mu f(t,x). & (1.22b) \end{cases}$$

[*]The animation works in Acrobat Reader on the entire pdf file.

To find the solution $f(t,x)=f(t,0)\,\mathrm{e}^{\sigma x}$ of (1.22a) we let $g(t,x)=\log f(t,x)$ and rewrite (1.22a) as

$$\frac{\partial g}{\partial x}(t,x)=\frac{\partial}{\partial x}\log f(t,x)=\frac{1}{f(t,x)}\frac{\partial f}{\partial x}(t,x)=\sigma,$$

i.e.

$$\frac{\partial g}{\partial x}(t,x)=\sigma,$$

which is solved as

$$g(t,x)=g(t,0)+\sigma x,$$

hence

$$f(t,x)=\mathrm{e}^{g(t,0)}\,\mathrm{e}^{\sigma x}=f(t,0)\,\mathrm{e}^{\sigma x}.$$

Plugging back this expression into the second equation (1.22b) yields

$$\mathrm{e}^{\sigma x}\frac{\partial f}{\partial t}(t,0)+\frac{1}{2}\sigma^2\,\mathrm{e}^{\sigma x}f(t,0)=\mu f(t,0)\,\mathrm{e}^{\sigma x},$$

i.e.

$$\frac{\partial f}{\partial t}(t,0)=\left(\mu-\frac{\sigma^2}{2}\right)f(t,0).$$

In other words, we have $\dfrac{\partial g}{\partial t}(t,0)=\mu-\sigma^2/2$, which yields

$$g(t,0)=g(0,0)+\left(\mu-\frac{\sigma^2}{2}\right)t,$$

i.e.

$$\begin{aligned}f(t,x)&=\mathrm{e}^{g(t,x)}=\mathrm{e}^{g(t,0)+\sigma x}\\&=\mathrm{e}^{g(0,0)+\sigma x+\left(\mu-\sigma^2/2\right)t}\\&=f(0,0)\,\mathrm{e}^{\sigma x+\left(\mu-\sigma^2/2\right)t},\qquad t\in\mathbb{R}_+.\end{aligned}$$

We conclude that

$$S_t=f(t,B_t)=f(0,0)\,\mathrm{e}^{\sigma B_t+\left(\mu-\sigma^2/2\right)t},$$

and the solution to (1.19) is given by

$$S_t=S_0\,\mathrm{e}^{\sigma B_t+(\mu-\sigma^2/2)t},\qquad t\in\mathbb{R}_+.$$

□

Conversely, taking $S_t=f(t,B_t)$ with $f(t,x)=S_0\,\mathrm{e}^{\mu t+\sigma x-\sigma^2t/2}$ we may apply Itô's formula to check that

$$\begin{aligned}dS_t&=df(t,B_t)\\&=\frac{\partial f}{\partial t}(t,B_t)dt+\frac{\partial f}{\partial x}(t,B_t)dB_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt\\&=\left(\mu-\sigma^2/2\right)S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2t/2}dt+\sigma S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2t/2}dB_t\\&\quad+\frac{1}{2}\sigma^2S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2t/2}dt\\&=\mu S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2t/2}dt+\sigma S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2t/2}dB_t\\&=\mu S_tdt+\sigma S_tdB_t.\end{aligned}$$

Exercises

Exercise 1.1 Show that at any time $T > 0$, the random variable $S_T := S_0 \mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T}$ has the *lognormal distribution* with probability density function

$$\boxed{x \longmapsto f(x) = \frac{1}{x\sigma\sqrt{2\pi T}} \mathrm{e}^{-(-(\mu-\sigma^2/2)T + \log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,}$$

with log-variance σ^2 and log-mean $(\mu - \sigma^2/2)T + \log S_0$, see Figure1.7.

```
N=1000; t <- 0:N; dt <- 1.0/N; nsim <- 100 # using Bernoulli samples
sigma=0.2;r=0.5;a=(1+r*dt)*(1-sigma*sqrt(dt))-1;b=(1+r*dt)*(1+sigma*sqrt(dt))-1
X <- matrix(a+(b-a)*rbinom( nsim * N, 1, 0.5), nsim, N)
X <- cbind(rep(0, nsim), t(apply((1+X), 1, cumprod)))
X[,1]=1;H<-hist(X[,N]);layout(matrix(c(1,2), nrow =1, byrow = TRUE))
par(mar=c(2,2,2,0), oma = c(2, 2, 2, 2))
plot(t, X[1, ], xlab = "time", ylab = "", type = "l", ylim=c(0.8,3), col = 0)
for (i in 1:nsim){lines(t, X[i, ], xlab = "time", type = "l", col = i)}
lines((1+r*dt)^t, type="l", lty=1, col="black",lwd=3,xlab="",ylab="", main="")
for (i in 1:nsim){points(N, X[i,N], pch=1, lwd = 5, col = i)}
x <- seq(0.01,3, length=100)
px <- exp(-(-(r-sigma^2/2)+log(x))^2/2/sigma^2)/x/sigma/sqrt(2*pi)
par(mar = c(2,2,2,2))
plot(NULL , xlab="", ylab="", xlim = c(0, max(px,H$density)),ylim=c(0.8,3),axes=F)
rect(0, H$breaks[1:(length(H$breaks) - 1)], col=rainbow(20,start=0.08,end=0.6), H$density,
     H$breaks[2:length(H$breaks)])
lines(px,x, type="l", lty=1, col="black",lwd=2,xlab="",ylab="", main="")
```

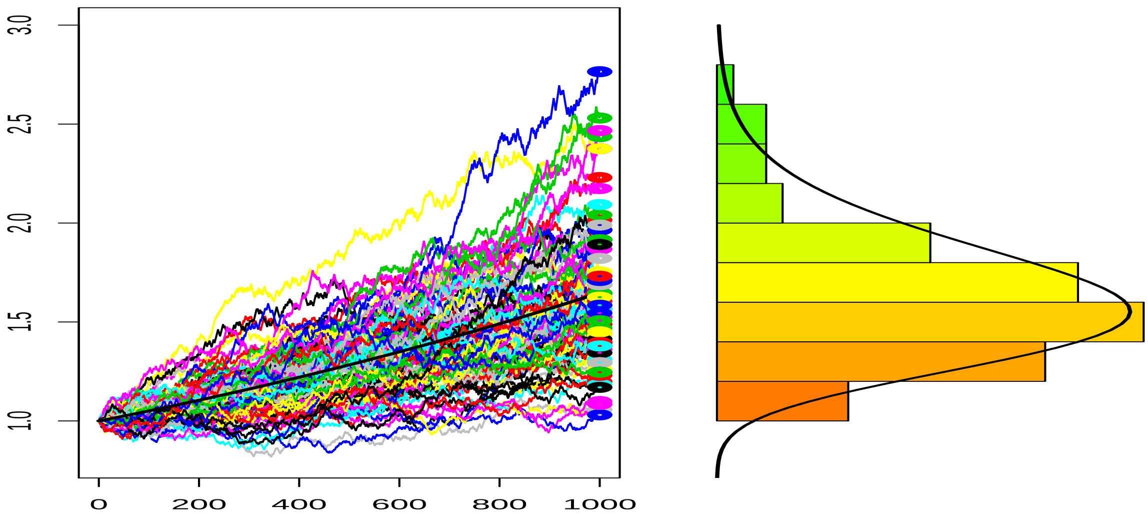


Figure 1.7: Statistics of geometric Brownian paths *vs* lognormal distribution.

Exercise 1.2

a) Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+, \tag{1.23}$$

where $r, \sigma \in \mathbb{R}$ are constants and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Compute $d\log S_t$ using the Itô formula.

b) Solve the ordinary differential equation $df(t) = cf(t)dt$ and the stochastic differential equation (1.23).

c) Compute the lognormal mean and variance

$$\mathbb{E}[S_t] = S_0\,\mathrm{e}^{rt} \quad \text{and} \quad \mathrm{Var}[S_t] = S_0^2\,\mathrm{e}^{2rt}(\mathrm{e}^{\sigma^2 t} - 1), \quad t \in \mathbb{R}_+,$$

using the Gaussian moment generating function (MGF) formula.

d) Recover the lognormal mean and variance of Question (c)) using stochastic calculus.

Exercise 1.3 Assume that $(B_t)_{t\in\mathbb{R}_+}$ and $(W_t)_{t\in\mathbb{R}_+}$ are standard Brownian motions, correlated according to the Itô rule $dW_t \bullet dB_t = \rho dt$ for $\rho \in [-1,2]$, and consider the solution $(Y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation $dY_t = \mu Y_t dt + \eta Y_t dW_t$, $t \in \mathbb{R}_+$, where $\mu, \eta \in \mathbb{R}$ are constants. Compute $df(S_t, Y_t)$, for f a $\mathscr{C}^2$ function on $\mathbb{R}^2$ using the bivariate Itô formula.

Exercise 1.4 Consider the asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Find the stochastic integral decomposition of the random variable S_T, *i.e.*, find the *constant* $C(S_0, r, T) \in \mathbb{R}$ and the process $(\zeta_{t,T})_{t\in[0,T]}$ such that

$$S_T = C(S_0, r, T) + \int_0^T \zeta_{t,T} dB_t. \tag{1.24}$$

Exercise 1.5 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ and the process $(S_t)_{t\in\mathbb{R}_+}$ defined by

$$S_t = S_0 \exp\left(\int_0^t \sigma_s dB_s + \int_0^t u_s ds\right), \qquad t \in \mathbb{R}_+,$$

where $(\sigma_t)_{t\in\mathbb{R}_+}$ and $(u_t)_{t\in\mathbb{R}_+}$ are $(\mathscr{F}_t)_{t\in[0,T]}$-adapted processes.

a) Compute dS_t using Itô calculus.

b) Show that S_t satisfies a stochastic differential equation to be determined.

Exercise 1.6 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, and let $\sigma > 0$.

a) Compute the mean and variance of the random variable S_t defined as

$$S_t := 1 + \sigma \int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s, \qquad t \in \mathbb{R}_+. \tag{1.25}$$

b) Express $d\log(S_t)$ using (1.25) and the Itô formula.

c) Show that $S_t = e^{\sigma B_t - \sigma^2 t/2}$ for $t \in \mathbb{R}_+$.

Exercise 1.7 We consider a leveraged fund with factor $\beta : 1$ on an index $(S_t)_{t\in\mathbb{R}_+}$ modeled as the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+,$$

under the risk-neutral probability measure $\mathbb{P}^*$.

a) Find the portfolio allocation (ξ_t, η_t) of the leveraged fund value

$$F_t = \xi_t S_t + \eta_t A_t, \qquad t \in \mathbb{R}_+,$$

where $A_t := A_0\,\mathrm{e}^{rt}$ represents the risk-free money market account price.

b) Find the stochastic differential equation satisfied by $(F_t)_{t\in\mathbb{R}_+}$ under the self-financing condition $dF_t = \xi_t dS_t + \eta_t dA_t$.

c) Find the relation between the fund value F_t and the index S_t by solving the stochastic differential equation obtained for F_t in Question (b)). For simplicity we take $F_0 := S_0^\beta$.

Exercise 1.8 Solve the stochastic differential equation

$$dX_t = h(t)X_t dt + \sigma X_t dB_t,$$

where $\sigma > 0$ and $h(t)$ is a deterministic, integrable function of $t \in \mathbb{R}_+$.

Hint: Look for a solution of the form $X_t = f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$, where $f(t)$ is a function to be determined, $t \in \mathbb{R}_+$.

Exercise 1.9 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

a) Consider the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s)dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s)ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s)ds, \qquad (1.26)$$

where $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$.

Compute $S_t := \mathrm{e}^{X_t}$ by the Itô formula (1.26) applied to $f(x) = \mathrm{e}^x$ and $X_t = \sigma B_t + \nu t$, $\sigma > 0$, $\nu \in \mathbb{R}$.

b) Let $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

c) Given $\sigma > 0$, let $X_t := (B_T - B_t)\sigma$, and compute $\mathrm{Var}[X_t]$, $t \in [0,T]$.

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0\,\mathrm{e}^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Show that the conditional probability that $S_T > K$ given $S_t = x$ can be computed as

$$\mathbb{P}(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K) + (T-t)\nu}{\sigma\sqrt{T-t}}\right), \qquad t \in [0,T).$$

Hint: Use the time splitting decomposition

$$S_T = S_t \frac{S_T}{S_t} = S_t\,\mathrm{e}^{(B_T - B_t)\sigma + (T-t)\nu}, \qquad t \in [0,T].$$

2. Black-Scholes Pricing and Hedging

The Black and Scholes, 1973 PDE is a Partial Differential Equation which is used for the pricing of vanilla options under absence of arbitrage and self-financing portfolio assumptions. In this chapter we derive the Black-Scholes PDE and present its solution by the the heat kernel method, with application to the pricing and hedging of European call and put options.

2.1 The Black-Scholes PDE

In this chapter we work in a market based on a riskless asset with price $(A_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{A_{t+dt}-A_t}{A_t} = rdt, \qquad \frac{dA_t}{A_t} = rdt, \qquad \frac{dA_t}{dt} = rA_t, \quad t \geqslant 0,$$

with

$$A_t = A_0 \,\mathrm{e}^{rt}, \qquad t \geqslant 0,$$

and a risky asset with price $(S_t)_{t\in\mathbb{R}_+}$ modeled using a geometric Brownian motion defined from the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t \geqslant 0, \tag{2.1}$$

with solution

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \qquad t \geqslant 0,$$

cf. Proposition 1.8.

```
install.packages("quantmod")
library(quantmod)
# getSymbols("0005.HK",from="2016-02-15",to=Sys.Date(),src="yahoo")
getSymbols("0005.HK",from="2016-02-15",to="2017-05-11",src="yahoo")
stock=Ad(`0005.HK`)
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(stock, theme = myTheme)
add_TA(stock, on=1, col="blue", legend=NULL,lwd=1.6)
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

The next Figure 2.1 presents a graph of underlying asset price market data, which is compared to the geometric Brownian motion simulations of Figures 1.5 and 1.6.

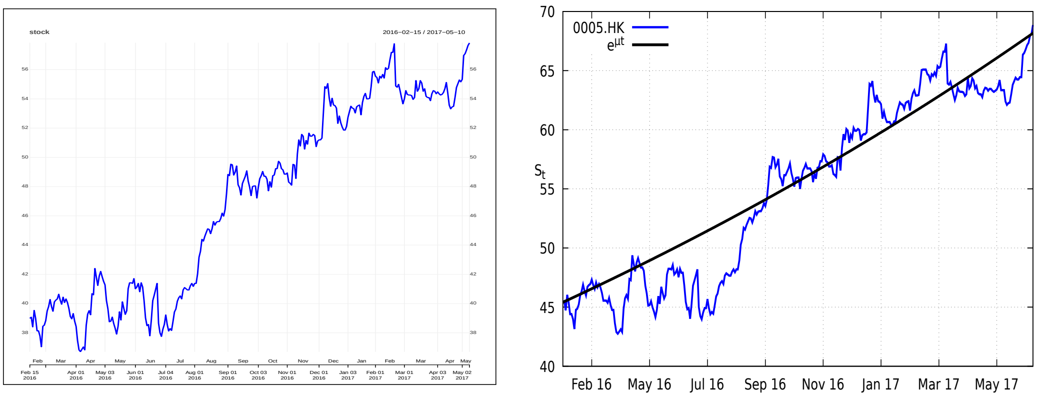


Figure 2.1: Graph of underlying market prices.

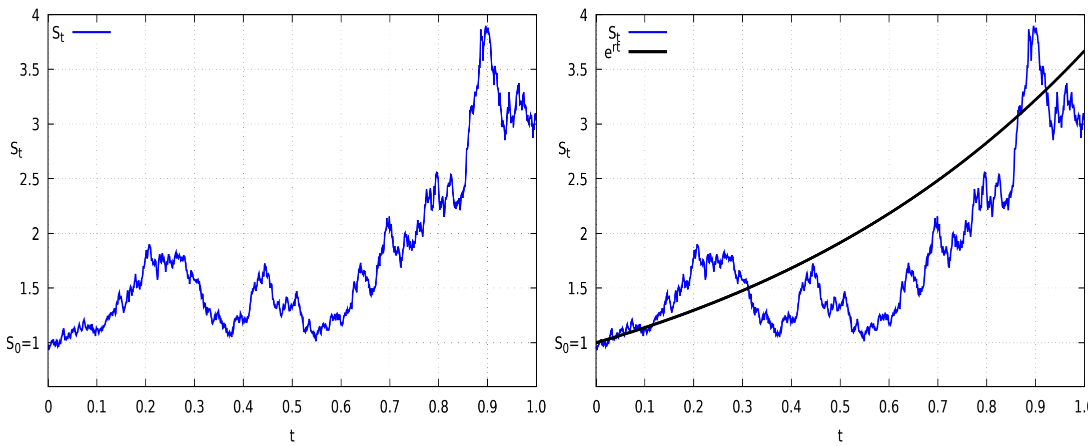


Figure 2.2: Graph of simulated geometric Brownian motion.

In the sequel, we start by deriving the Black and Scholes, 1973 Partial Differential Equation (PDE) for the value of a self-financing portfolio. Note that the drift parameter μ in (2.1) is absent in the PDE (2.2), and it does not appear as well in the Black and Scholes, 1973 formula (2.10).

Proposition 2.1 Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy such that
(i) the porfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is self-financing,

(ii) the portfolio value $V_t := \eta_t A_t + \xi_t S_t$, takes the form

$$V_t = g(t, S_t), \qquad t \in \mathbb{R}_+,$$

for some function $g \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$ of t and S_t.
Then, the function $g(t,x)$ satisfies the Black and Scholes, 1973 PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \tag{2.2}$$

and $\xi_t = \xi_t(S_t)$ is given by the partial derivative

$$\xi_t = \xi_t(S_t) = \frac{\partial g}{\partial x}(t, S_t), \qquad t \in \mathbb{R}_+. \tag{2.3}$$

Proof. (i) First, we note that the self-financing condition (1.8) in Proposition 1.3 implies

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rg(t,S_t)dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \end{aligned} \tag{2.4}$$

$t \in \mathbb{R}_+$. We now rewrite (1.18) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

by taking

$$u_t = \sigma S_t, \quad \text{and} \quad v_t = \mu S_t, \qquad t \in \mathbb{R}_+.$$

(ii) The application of Itô's formula to $V_t = g(t, S_t)$ leads to

$$\begin{aligned} dV_t &= dg(t,S_t) \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}(dS_t)^2\frac{\partial^2 g}{\partial x^2}(t,S_t) \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + v_t\frac{\partial g}{\partial x}(t,S_t)dt + u_t\frac{\partial g}{\partial x}(t,S_t)dB_t + \frac{1}{2}|u_t|^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \mu S_t\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma S_t\frac{\partial g}{\partial x}(t,S_t)dB_t. \end{aligned} \tag{2.5}$$

By respective identification of the terms in dB_t and dt in (2.4) and (2.5) we get

$$\begin{cases} rg(t,S_t)dt + (\mu - r)\xi_t S_t dt = \dfrac{\partial g}{\partial t}(t,S_t)dt + \mu S_t\dfrac{\partial g}{\partial x}(t,S_t)dt + \dfrac{1}{2}\sigma^2 S_t^2\dfrac{\partial^2 g}{\partial x^2}(t,S_t)dt, \\ \xi_t S_t \sigma dB_t = S_t\sigma\dfrac{\partial g}{\partial x}(t,S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t,S_t) = \dfrac{\partial g}{\partial t}(t,S_t) + rS_t\dfrac{\partial g}{\partial x}(t,S_t) + \dfrac{1}{2}\sigma^2 S_t^2 \dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T, \end{cases} \tag{2.6}$$

which yields (2.2) after substituting S_t with $x > 0$. $\square$

The derivative giving ξ_t in (2.3) is called the Delta of the option price, see Proposition 2.4 below. The amount invested on the riskless asset is

$$\eta_t A_t = V_t - \xi_t S_t = g(t,S_t) - S_t \frac{\partial g}{\partial x}(t,S_t),$$

and η_t is given by

$$\begin{aligned} \eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{1}{A_t}\left(g(t,S_t) - S_t\frac{\partial g}{\partial x}(t,S_t)\right) \\ &= \frac{1}{A_0\,\mathrm{e}^{rt}}\left(g(t,S_t) - S_t\frac{\partial g}{\partial x}(t,S_t)\right). \end{aligned}$$

In the next Proposition 2.2 we add a terminal condition $g(T,x) = f(x)$ to the Black-Scholes PDE (2.2) in order to price a claim payoff C of the form $C = h(S_T)$. As in the discrete-time case, the arbitrage price $\pi_t(C)$ at time $t \in [0,T]$ of the claim payoff C is defined to be the value V_t of the self-financing portfolio hedging C.

Proposition 2.2 The arbitrage price $\pi_t(C)$ at time $t \in [0,T]$ of the (vanilla) option with payoff $C = h(S_T)$ is given by $\pi_t(C) = g(t,S_t)$ and the hedging allocation ξ_t is given by the partial derivative (2.3), where the function $g(t,x)$ is solution of the following Black-Scholes PDE:

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = h(x), \quad x > 0. \end{cases} \tag{2.7}$$

Proof. Proposition 2.1 shows that the solution $g(t,x)$ of (2.2), $g \in \mathscr{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}_+)$, represents the value $V_t = \eta_t A_t + \xi_t S_t = g(t,S_t)$, $t \in \mathbb{R}_+$, of a self-financing portfolio strategy $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$. By definition, $\pi_t(C) := V_t = g(t,S_t)$ is the arbitrage price at time $t \in [0,T]$ of the vanilla option with payoff $C = h(S_T)$. $\square$

The absence of the drift parameter μ from the PDE (2.7) can be understood in the next forward contract example, in which the claim payoff can be hedged by leveraging on the value S_t of the underlying asset, independently of the trend parameter μ.

Example - forward contracts

When $C = S_T - K$ is the (linear) payoff function of a long forward contract, *i.e.* $h(x) = x - K$, the Black-Scholes PDE (2.7) admits the easy solution

$$g(t,x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T, \tag{2.8}$$

showing that the price at time t of the forward contract with payoff $C = S_T - K$ is

$$S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad 0 \leqslant t \leqslant T.$$

In addition, the Delta of the option price is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1, \qquad 0 \leqslant t \leqslant T,$$

which leads to a static "hedge and forget" strategy, cf. Exercise 2.6. The forward contract can be realized by the option issuer as follows:

a) At time t, receive the option premium $V_t := S_t - \mathrm{e}^{-(T-t)r}K$ from the option buyer.
b) Borrow $\mathrm{e}^{-(T-t)r}K$ from the bank, to be refunded at maturity.
c) Buy the risky asset using the amount $S_t - \mathrm{e}^{-(T-t)r}K + \mathrm{e}^{-(T-t)r}K = S_t$.
d) Hold the risky asset until maturity (do nothing, constant portfolio strategy).
e) At maturity T, hand in the asset to the option holder, who will pay the amount K in return.
f) Use the amount $K = \mathrm{e}^{(T-t)r}\mathrm{e}^{-(T-t)r}K$ to refund the lender of $\mathrm{e}^{-(T-t)r}K$ borrowed at time t.

Another way to compute the option premium V_t is to state that the amount $V_t - S_t$ has to be borrowed at time t in order to purchase the asset, and that the asset price K received at maturity T should be used to refund the loan, which yields

$$(V_t - S_t)\,\mathrm{e}^{-(T-t)r} = K, \quad 0 \leqslant t \leqslant T.$$

Forward contracts can be used for physical delivery, *e.g.* for live cattle. In the case of European options, the basic "hedge and forget" constant strategy

$$\xi_t = 1, \quad \eta_t = \eta_0, \quad 0 \leqslant t \leqslant T,$$

will hedge the option only if

$$S_T + \eta_0 A_T \geqslant (S_T - K)^+,$$

i.e. if $-\eta_0 A_T \leqslant K \leqslant S_T$.

Future contracts

For a future contract expiring at time T we take $K = S_0\,\mathrm{e}^{rT}$ and the contract is usually quoted at time t in terms of the forward price

$$\mathrm{e}^{(T-t)r}\big(S_t - K\,\mathrm{e}^{-(T-t)r}\big) = \mathrm{e}^{(T-t)r}S_t - K = \mathrm{e}^{(T-t)r}S_t - S_0\,\mathrm{e}^{rT},$$

discounted at time T, or simply using $\mathrm{e}^{(T-t)r}S_t$. Future contracts are *non-deliverable* forward contracts which are "marked to market" at each time step via a cash flow exchange between the two parties, ensuring that the absolute difference $|\mathrm{e}^{(T-t)r}S_t - K|$ is being credited to the buyer's account if $\mathrm{e}^{(T-t)r}S_t > K$, or to the seller's account if $\mathrm{e}^{(T-t)r}S_t < K$.

2.2 European Call Options

Recall that in the case of the European call option with strike price K the payoff function is given by $h(x) = (x-K)^+$ and the Black-Scholes PDE (2.7) reads

$$\begin{cases} rg_{\mathrm{c}}(t,x) = \dfrac{\partial g_{\mathrm{c}}}{\partial t}(t,x) + rx\dfrac{\partial g_{\mathrm{c}}}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2 g_{\mathrm{c}}}{\partial x^2}(t,x) \\[2ex] g_{\mathrm{c}}(T,x) = (x-K)^+. \end{cases} \tag{2.9}$$

The next proposition will be proved in Section2.6, see Proposition 2.11.

Proposition 2.3 The solution of the PDE (2.9) is given by the *Black-Scholes* formula for call options

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad (2.10)$$

with

$$d_+(T-t) := \frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \quad (2.11)$$

$$d_-(T-t) := \frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \qquad 0 \leqslant t < T. \quad (2.12)$$

We note the relation

$$d_+(T-t) = d_-(T-t) + |\sigma|\sqrt{T-t}, \qquad 0 \leqslant t < T. \quad (2.13)$$

Here, "log" denotes the *natural logarithm* "ln" and

$$\Phi(x) := \mathbb{P}(X \leqslant x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian Cumulative Distribution Function (CDF) of a standard normal random variable $X \simeq \mathcal{N}(0,1)$, with the relation

$$\Phi(-x) = 1 - \Phi(x), \qquad x \in \mathbb{R}. \quad (2.14)$$

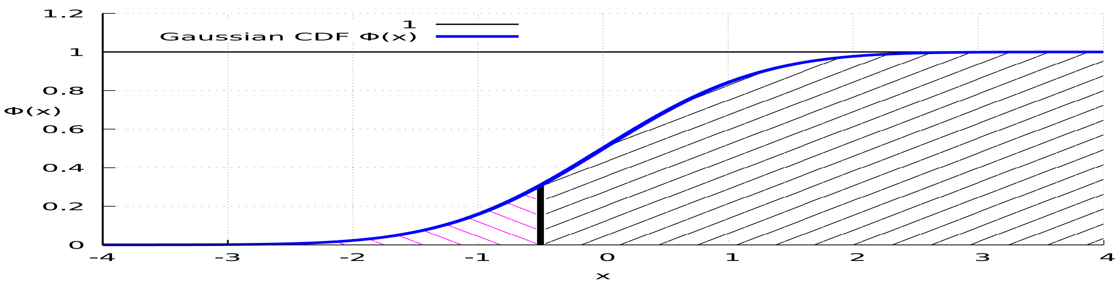


Figure 2.3: Graph of the Gaussian Cumulative Distribution Function (CDF).

In other words, the European call option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$\begin{aligned} g_c(t,S_t) &= \mathrm{Bl}(K,S_t,\sigma,r,T-t) \\ &= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad 0 \leqslant t \leqslant T. \end{aligned}$$

The following R script is an implementation of the Black-Scholes formula for European call options in R.*

*Download the corresponding **IPython notebook** that can be run **here**.

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2)
BSCall}
```

In comparison with the discrete-time Cox-Ross-Rubinstein (CRR) model, the interest in the formula (2.10) is to provide an analytical solution that can be evaluated in a single step, which is computationally much more efficient.

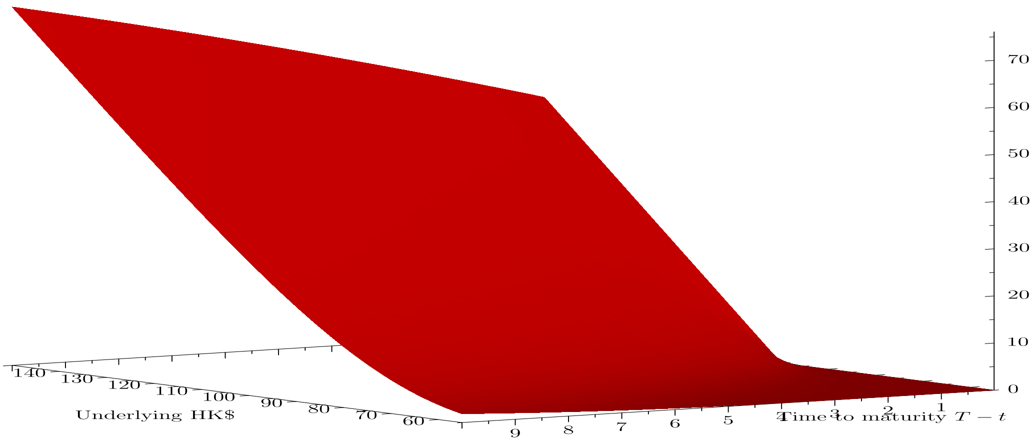


Figure 2.4: Graph of the Black-Scholes call price map with strike price $K = 100$.*

Figure 2.4 presents an interactive graph of the Black-Scholes call price map, *i.e.* the solution

$$(t,x) \longmapsto g_c(t,x) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$$

of the Black-Scholes PDE (2.7) for a call option.

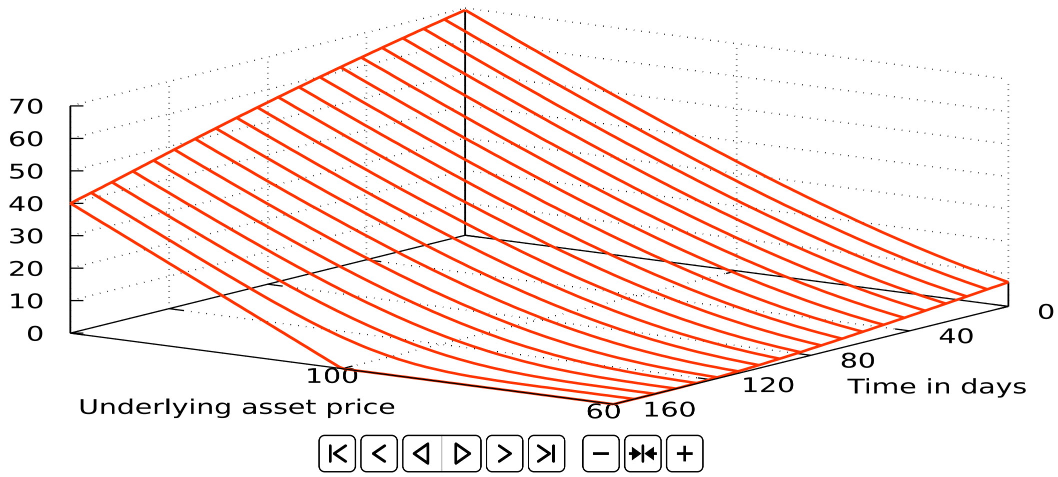


Figure 2.5: Time-dependent solution of the Black-Scholes PDE (call option).†

The next proposition is proved by a direct differentiation of the Black-Scholes function, and will be recovered later using a probabilistic argument in Proposition 3.12 below.

*Right-click on the figure for interaction and "Full Screen Multimedia" view.
†The animation works in Acrobat Reader on the entire pdf file.

Proposition 2.4 The Black-Scholes Delta of the European call option is given by

$$\xi_t = \xi_t(S_t) = \frac{\partial g_c}{\partial x}(t, S_t) = \Phi\big(d_+(T-t)\big) \in [0,1], \tag{2.15}$$

where $d_+(T-t)$ is given by (2.11).

Proof. From Relation (2.13), we note that the standard normal probability density function

$$\varphi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R},$$

satisfies

$$\begin{aligned}
\varphi(d_+(T-t)) &= \varphi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}} + |\sigma|\sqrt{T-t}\right)^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_-(T-t))^2 - (T-t)r - \log\frac{x}{K}\right) \\
&= \frac{K}{x\sqrt{2\pi}} \mathrm{e}^{-(T-t)r} \exp\left(-\frac{1}{2}(d_-(T-t))^2\right) \\
&= \frac{K}{x} \mathrm{e}^{-(T-t)r} \varphi(d_-(T-t)),
\end{aligned}$$

hence by (2.10) we have

$$\begin{aligned}
\frac{\partial g_c}{\partial x}(t,x) &= \frac{\partial}{\partial x}\left(x\Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\left(\Phi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\right) \\
&= \Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad + x\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad + \frac{x}{|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&\quad - \frac{K\mathrm{e}^{-(T-t)r}}{|\sigma|\sqrt{T-t}}\varphi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\
&= \Phi(d_+(T-t)) + \frac{x}{|\sigma|\sqrt{T-t}}\varphi(d_+(T-t)) - \frac{K\mathrm{e}^{-(T-t)r}}{|\sigma|\sqrt{T-t}}\varphi(d_-(T-t)) \\
&= \Phi(d_+(T-t)).
\end{aligned} \tag{2.16}$$

□

As a consequence of Proposition 2.4, the Black-Scholes call price splits into a risky component $S_t\Phi\big(d_+(T-t)\big)$ and a riskless component $-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)$, as follows:

$$g_c(t,S_t) = \underbrace{S_t\Phi\big(d_+(T-t)\big)}_{\text{risky investment (held)}} \underbrace{-K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big)}_{\text{risk-free investment (borrowed)}}, \quad 0 \leqslant t \leqslant T.$$

See Exercise 2.4 for a computation of the boundary values of $g_c(t,x)$, $t\in[0,T)$, $x>0$. The following R script is an implementation of the Black-Scholes Delta for European call options in R.

```
Delta <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
Delta = pnorm(d1);Delta}
```

In Figure 2.6 we plot the Delta of the European call option as a function of the underlying asset price and of the time remaining until maturity.

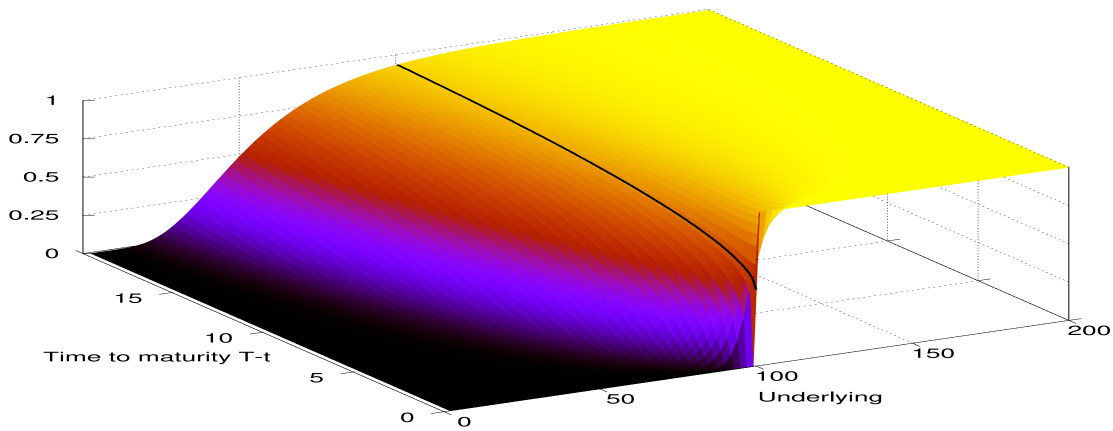


Figure 2.6: Delta of a European call option with strike price $K=100$, $r=3\%$, $\sigma=10\%$.

The *Gamma* of the European call option is defined as the first derivative of Delta, or second derivative of the option price, with respect to the underlying asset price. This gives

$$\begin{aligned}
\gamma_t &= \frac{1}{S_t|\sigma|\sqrt{T-t}}\Phi'\big(d_+(T-t)\big) \\
&= \frac{1}{S_t|\sigma|\sqrt{2(T-t)\pi}}\exp\left(-\frac{1}{2}\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\right) \\
&\geqslant 0.
\end{aligned}$$

In particular, a positive value of γ_t implies that the Delta $\xi_t = \xi_t(S_t)$ should increase when the underlying asset price S_t increases. In other words, the position ξ_t in the underlying asset should be increased by additional purchases if the underlying asset price S_t increases.

In Figure 2.7 we plot the (truncated) value of the Gamma of a European call option as a function of the underlying asset price and of time to maturity.

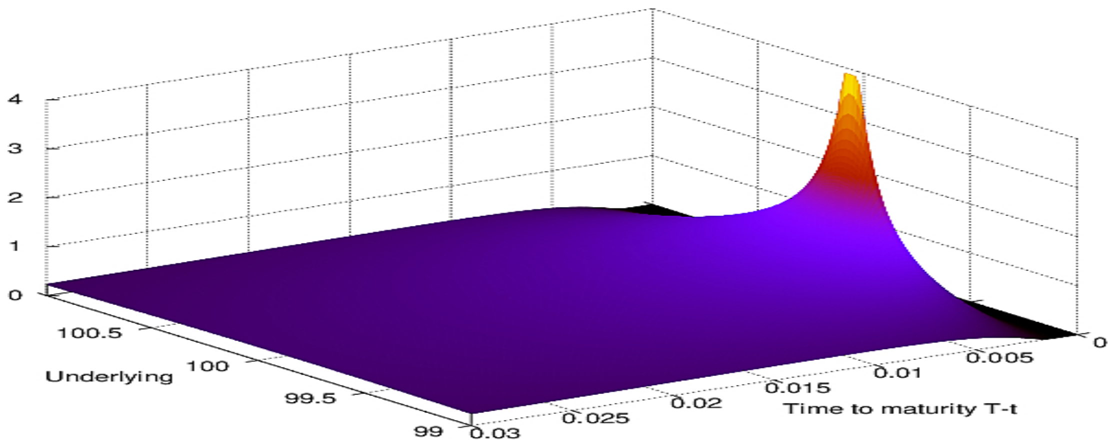


Figure 2.7: Gamma of a European call option with strike price $K = 100$.

As Gamma is always nonnegative, the Black-Scholes hedging strategy is to keep buying the risky underlying asset when its price increases, and to sell it when its price decreases, as can be checked from Figure 2.7.

Numerical example - hedging a call option

In Figure 2.8 we consider the historical stock price of HSBC Holdings (0005.HK) over one year:

Figure 2.8: Graph of the stock price of HSBC Holdings.

Consider the call option issued by Societe Generale on 31 December 2008 with strike price K=\$63.704, maturity T = October 05, 2009, and an entitlement ratio of 100, meaning that one option contract is divided into 100 *warrants*. The next graph gives the time evolution of the Black-Scholes portfolio value

$$t \longmapsto g_c(t, S_t)$$

driven by the market price $t \longmapsto S_t$ of the risky underlying asset as given in Figure 2.8, in which the number of days is counted from the origin and not from maturity.

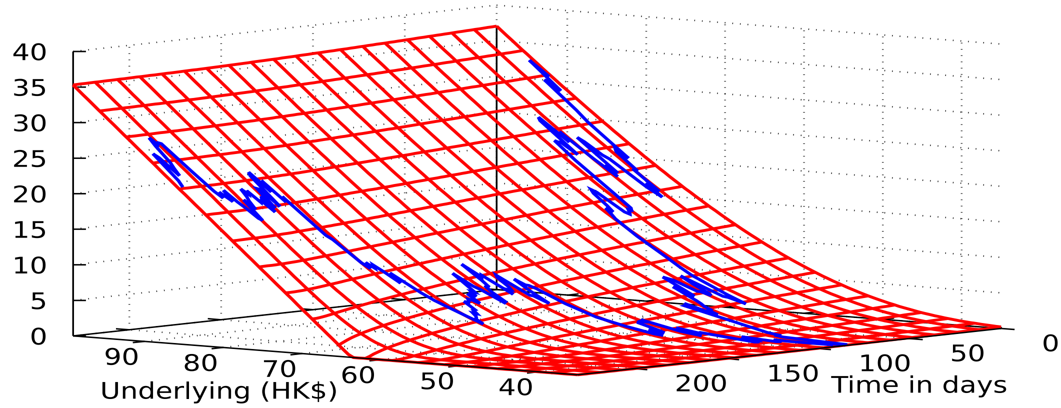


Figure 2.9: Path of the Black-Scholes price for a call option on HSBC.

As a consequence of Proposition 2.4, in the Black-Scholes call option hedging model, the amount invested in the risky asset is

$$\begin{aligned} S_t\xi_t &= S_t\Phi\big(d_+(T-t)\big) \\ &= S_t\Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\geqslant 0, \end{aligned}$$

which is always nonnegative, *i.e.* there is no short selling, and the amount invested on the riskless asset is

$$\begin{aligned} \eta_t A_t &= -K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\ &= -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right) \\ &\leqslant 0, \end{aligned}$$

which is always nonpositive, *i.e.* we are constantly borrowing money on the riskless asset, as noted in Figure 2.10.

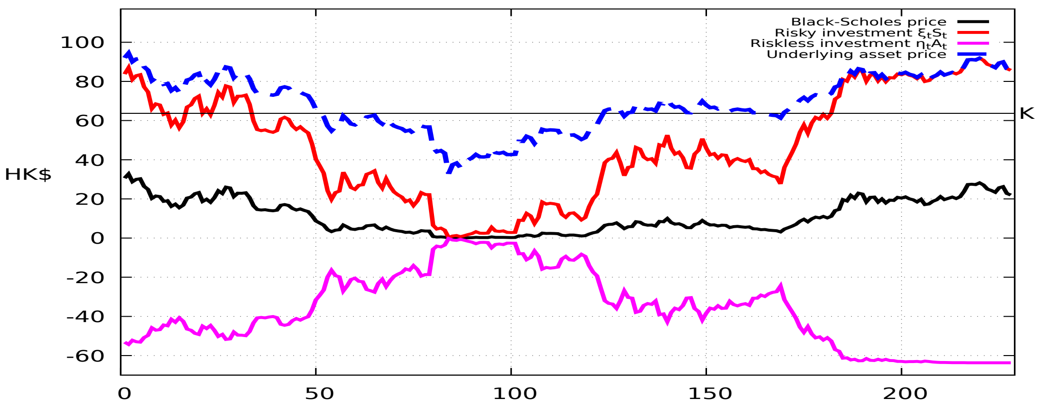


Figure 2.10: Time evolution of a hedging portfolio for a call option on HSBC.

A comparison of Figure 2.10 with market data can be found in Figures 2.28 and 2.29 below.

Cash settlement. In the case of a cash settlement, the option issuer will satisfy the option contract by selling $\xi_T = 1$ stock at the price $S_T = \$83$, refund the $K = \$63$ risk-free investment, and hand in the remaining amount $C = (S_T - K)^+ = 83 - 63 = \$20$ to the option holder.

Physical delivery. In the case of physical delivery of the underlying asset, the option issuer will deliver $\xi_T = 1$ stock to the option holder in exchange for $K = \$63$, which will be used together with the portfolio value to refund the risk-free loan.

2.3 European Put Options

Similarly, in the case of the European put option with strike price K the payoff function is given by $h(x) = (K - x)^+$ and the Black-Scholes PDE (2.7) reads

$$\begin{cases} rg_{\mathrm{p}}(t,x) = \dfrac{\partial g_{\mathrm{p}}}{\partial t}(t,x) + rx\dfrac{\partial g_{\mathrm{p}}}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g_{\mathrm{p}}}{\partial x^2}(t,x), \\ \\ g_{\mathrm{p}}(T,x) = (K - x)^+, \end{cases} \tag{2.17}$$

The next proposition can be proved as in Section2.6, see Proposition 2.11.

Proposition 2.5 The solution of the PDE (2.17) is given by the *Black-Scholes* formula for put options

$$g_{\mathrm{p}}(t,x) = K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big), \tag{2.18}$$

with

$$d_+(T-t) = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{2.19}$$

$$d_-(T-t) = \frac{\log(x/K) + (r - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \tag{2.20}$$

as illustrated in Figure 2.11.

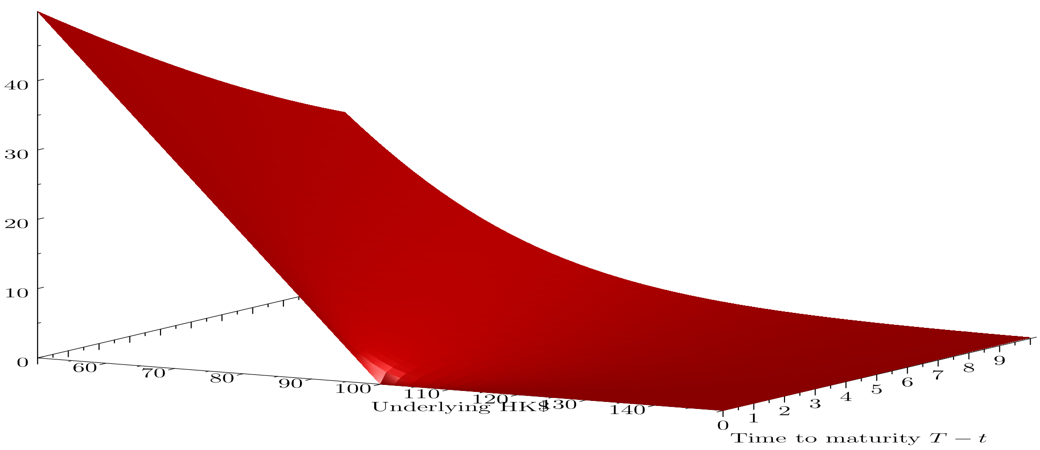


Figure 2.11: Graph of the Black-Scholes put price function with strike price $K = 100$.*

In other words, the European put option with strike price K and maturity T is priced at time $t \in [0,T]$ as

$$g_p(t,S_t) = K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T.$$

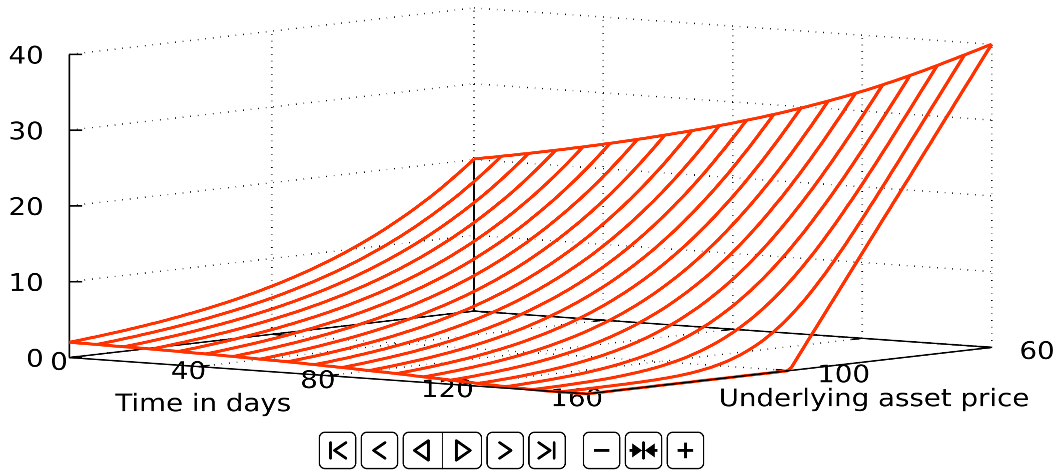


Figure 2.12: Time-dependent solution of the Black-Scholes PDE (put option).*

The following R script is an implementation of the Black-Scholes formula for European put options in R.

```
BSPut <- function(S, K, r, T, sigma)
{d1 = (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T))
d2 = d1 - sigma * sqrt(T)
BSPut = K*exp(-r*T) * pnorm(-d2) - S*pnorm(-d1)
BSPut}
```

Call-put parity

Proposition 2.6 Call-put parity. We have the call-put parity relation

$$g_c(t,S_t) - g_p(t,S_t) = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T, \tag{2.21}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t - K\,\mathrm{e}^{-(T-t)r}$.

Proof. The call-put parity (2.21) is a consequence of the relation

$$x - K = (x-K)^+ - (K-x)^+$$

satisfied by the terminal call and put payoff functions in the Black-Scholes PDE (2.7). It can also be verified directly from (2.10) and (2.18) as

$$\begin{aligned} g_c(t,x) - g_p(t,x) &= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \\ &\quad -\Big(K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)\Big) \\ &= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) \end{aligned}$$

*Right-click on the figure for interaction and "Full Screen Multimedia" view.
*The animation works in Acrobat Reader on the entire pdf file.

$$
\begin{aligned}
& -K\,\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(d_-(T-t)\big)\big)+x\big(1-\Phi\big(d_+(T-t)\big)\big) \\
= \; & x-K.
\end{aligned}
$$

□

The *Delta* of the Black-Scholes put option can be obtained by differentiation of the call-put parity relation (2.21) and Proposition 2.4.

Proposition 2.7 The *Delta* of the Black-Scholes put option is given by

$$\boxed{\xi_t=-\big(1-\Phi\big(d_+(T-t)\big)\big)=-\Phi\big(-d_+(T-t)\big)\in[-1,0],\quad 0\leqslant t\leqslant T.}$$

Proof. By the call-put parity relation (2.21) and Proposition 2.4, we have

$$
\begin{aligned}
\frac{\partial g_{\mathrm{p}}}{\partial x}(t,S_t) &= \frac{\partial g_{\mathrm{c}}}{\partial x}(t,S_t)-1 \\
&= \Phi(d_+(T-t))-1 \\
&= -\Phi(-d_+(T-t)), \qquad 0\leqslant t\leqslant T,
\end{aligned}
$$

where we applied (2.14). □

As a consequence of Proposition 2.7 the Black-Scholes put price splits into a risky component $-S_t\Phi\big(-d_+(T-t)\big)$ and a riskless component $K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)$, as follows:

$$g_{\mathrm{p}}(t,S_t)=\underbrace{K\,\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)}_{\text{risk-free investment (savings)}}-\underbrace{S_t\Phi\big(-d_+(T-t)\big)}_{\text{risky investment (short)}},\quad 0\leqslant t\leqslant T.$$

In Figure 2.13 we plot the Delta of the European put option as a function of the underlying asset price and of the time remaining until maturity.

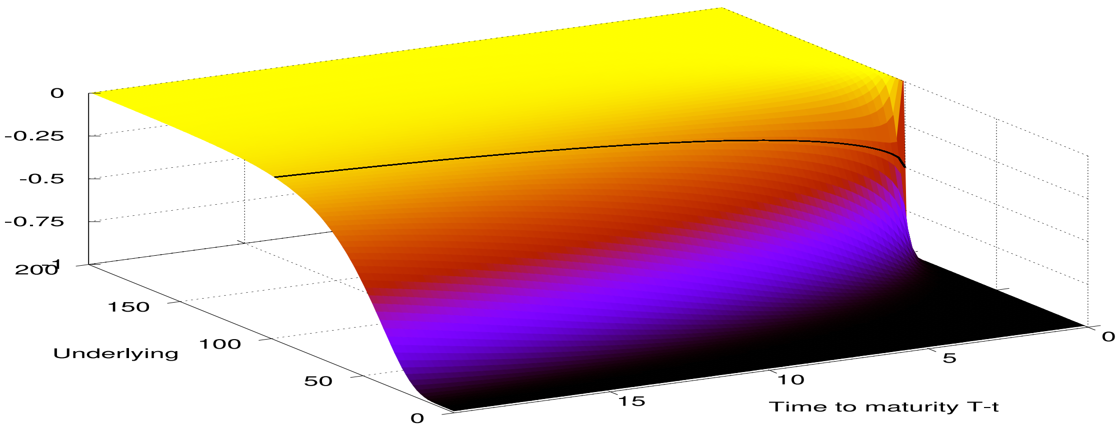


Figure 2.13: Delta of a European put option with strike price $K = 100$, $r = 3\%$, $\sigma = 10\%$.

Numerical example - hedging a put option

For one more example, we consider a put option issued by BNP Paribas on 04 November 2008 with strike price K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593. In the next Figure 2.14, the number of days is counted from the origin, not from maturity.

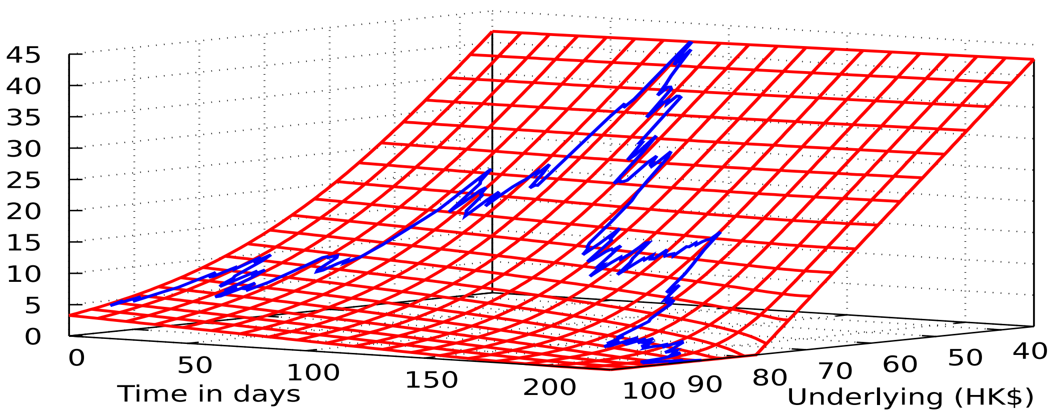


Figure 2.14: Path of the Black-Scholes price for a put option on HSBC.

As a consequence of Proposition 2.7, the amount invested on the risky asset for the hedging of a put option is

$$
\begin{aligned}
-S_t\Phi\big(-d_+(T-t)\big) &= -S_t\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
&\leqslant 0,
\end{aligned}
$$

i.e. there is always short selling, and the amount invested on the riskless asset priced $A_t = \mathrm{e}^{rt}$, $t \in [0,T]$, is

$$
\begin{aligned}
\eta_t A_t &= K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big)\\
&= K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\
&\geqslant 0,
\end{aligned}
$$

which is always nonnegative, *i.e.* we are constantly saving money on the riskless asset, as noted in Figure 2.15.

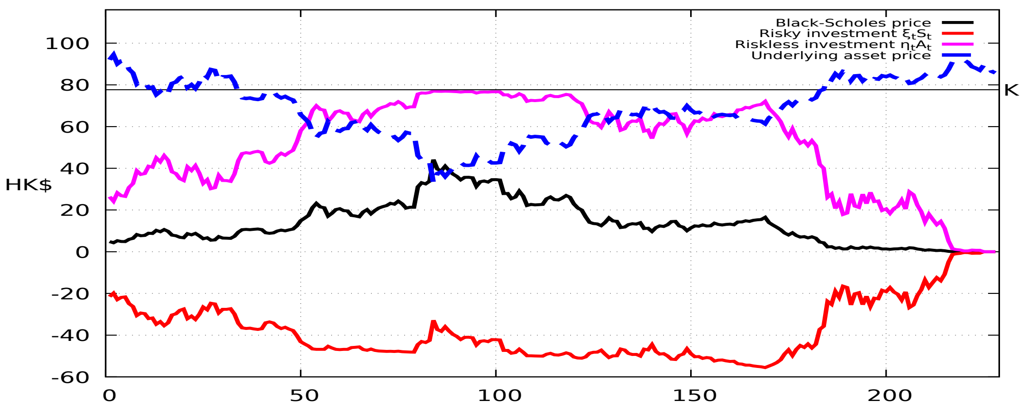


Figure 2.15: Time evolution of the hedging portfolio for a put option on HSBC.

In the above example the put option finished out of the money (OTM), so that no cash settlement or physical delivery occurs. A comparison of Figure 2.10 with market data can be found in Figures 2.30 and 2.31 below.

2.4 Market Terms and Data

The following Table 2.1 provides a summary of formulas for the computation of Black-Scholes sensitivities, also called *Greeks*.*

		Call option	Put option
Option price	$g(t,S_t)$	$S_t\Phi(d_+(T-t)) - K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$	$K\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)) - S_t\Phi(-d_+(T-t))$
Delta (Δ)	$\frac{\partial g}{\partial x}(t,S_t)$	$\Phi(d_+(T-t)) \geqslant 0$	$-\Phi(-d_+(T-t)) \leqslant 0$
Gamma (Γ)	$\frac{\partial^2 g}{\partial x^2}(t,S_t)$	$\frac{\Phi'(d_+(T-t))}{S_t\lvert\sigma\rvert\sqrt{T-t}} \geqslant 0$	
Vega	$\frac{\partial g}{\partial \sigma}(t,S_t)$	$S_t\sqrt{T-t}\Phi'(d_+(T-t)) \geqslant 0$	
Theta (Θ)	$\frac{\partial g}{\partial t}(t,S_t)$	$-\frac{S_t\lvert\sigma\rvert\Phi'(d_+(T-t))}{2\sqrt{T-t}} - rK\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \leqslant 0$	$-\frac{S_t\lvert\sigma\rvert\Phi'(d_+(T-t))}{2\sqrt{T-t}} + rK\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))$
Rho (ρ)	$\frac{\partial g}{\partial r}(t,S_t)$	$K(T-t)\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))$	$-K(T-t)\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t))$

Table 2.1: Black-Scholes Greeks (Wikipedia).

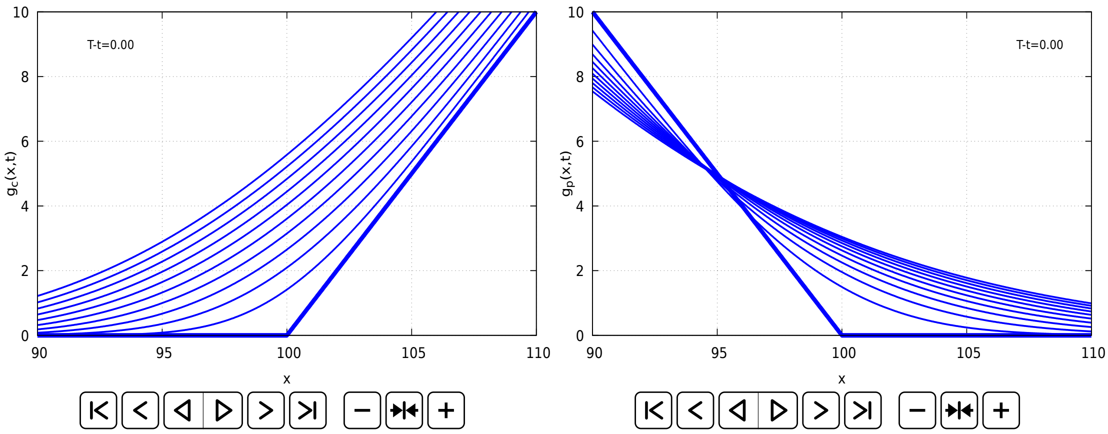
From Table 2.1 we can conclude that call option prices are increasing functions of the underlying asset price S_t, of the interest rate r, and of the volatility parameter σ. Similarly, put option prices are decreasing functions of the underlying asset price S_t, of the interest rate r, and increasing functions of the volatility parameter σ.

Parameter	Variation of call option prices	Variation of put option prices
Underlying S_t	increasing $\nearrow$	decreasing $\searrow$
Volatility σ	increasing $\nearrow$	increasing $\nearrow$
Time t	decreasing $\searrow$	depends on the underlying price level
Interest rate r	increasing $\nearrow$	decreasing $\searrow$

Table 2.2: Variations of Black-Scholes prices.

The change of sign of the sensitivity Theta (Θ) with respect to time t can be verified in the following Figure 2.16.

*"Every class feels like attending a Greek lesson" (AY2018-2019 student feedback).



(a) Black-Scholes call price maps. (b) Black-Scholes put price maps

Figure 2.16: Time-dependent solutions of the Black-Scholes PDE.*

Intrinsic value. The *intrinsic value* at time $t \in [0,T]$ of the option with payoff $C = h\big(S_T^{(1)}\big)$ is given by the immediate exercise payoff $h\big(S_t^{(1)}\big)$. The *extrinsic value* at time $t \in [0,T]$ of the option is the remaining difference $\pi_t(C) - h\big(S_t^{(1)}\big)$ between the option price $\pi_t(C)$ and the immediate exercise payoff $h\big(S_t^{(1)}\big)$. In general, the option price $\pi_t(C)$ decomposes as

$$\pi_t(C) = \underbrace{h\big(S_t^{(1)}\big)}_{\text{intrinsic value}} + \underbrace{\pi_t(C) - h\big(S_t^{(1)}\big)}_{\text{extrinsic value}}, \qquad 0 \leqslant t \leqslant T].$$

Gearing. The *gearing* at time $t \in [0,T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$\mathrm{G}_t := \frac{S_t}{\pi_t(C)} = \frac{S_t}{g(t,S_t)}, \qquad 0 \leqslant t \leqslant T.$$

Effective gearing. The *effective gearing* at time $t \in [0,T]$ of the option with payoff $C = h(S_T)$ is defined as the ratio

$$\begin{aligned} \mathrm{EG}_t &:= \mathrm{G}_t \xi_t \\ &= \frac{\xi_t S_t}{\pi_t(C)} \\ &= \frac{S_t}{\pi_t(C)} \frac{\partial g}{\partial x}(t,S_t) \\ &= \frac{S_t}{g(t,S_t)} \frac{\partial g}{\partial x}(t,S_t) \\ &= S_t \frac{\partial}{\partial x} \log g(t,S_t), \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

The effective gearing

$$\mathrm{EG}_t = \frac{\xi_t S_t}{\pi_t(C)}$$

can be interpreted as the *hedge ratio*, *i.e.* the percentage of the portfolio which is invested on the risky asset. When written as

$$\frac{\Delta g(t,S_t)}{g(t,S_t)} = \mathrm{EG}_t \times \frac{\Delta S_t}{S_t},$$

*The animation works in Acrobat Reader on the entire pdf file.

the effective gearing gives the relative variation, or percentage change, $\Delta g(t,S_t)/g(t,S_t)$ of the option price $g(t,S_t)$ from the relative variation $\Delta S_t/S_t$ in the underlying asset price.

The ratio $\mathrm{EG}_t = S_t \partial \log g(t,S_t)/\partial x$ can also be interpreted as an *elasticity coefficient*.

Break-even price. The *break-even* price BEP_t of the underlying asset is the value of S for which the intrinsic option value $h(S)$ equals the option price $\pi_t(C)$ at time $t \in [0,T]$. For European call options it is given by

$$\mathrm{BEP}_t := K + \pi_t(C) = K + g(t,S_t), \qquad t = 0,1,\ldots,N.$$

whereas for European put options it is given by

$$\mathrm{BEP}_t := K - \pi_t(C) = K - g(t,S_t), \qquad 0 \leqslant t \leqslant T.$$

Premium. The option *premium* OP_t can be defined as the variation required from the underlying asset price in order to reach the break-even price, *i.e.* we have

$$\mathrm{OP}_t := \frac{\mathrm{BEP}_t - S_t}{S_t} = \frac{K + g(t,S_t) - S_t}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European call options, and

$$\mathrm{OP}_t := \frac{S_t - \mathrm{BEP}_t}{S_t} = \frac{S_t + g(t,S_t) - K}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

for European put options, see Figure 2.17 below. The term "premium" is sometimes also used to denote the arbitrage price $g(t,S_t)$ of the option.

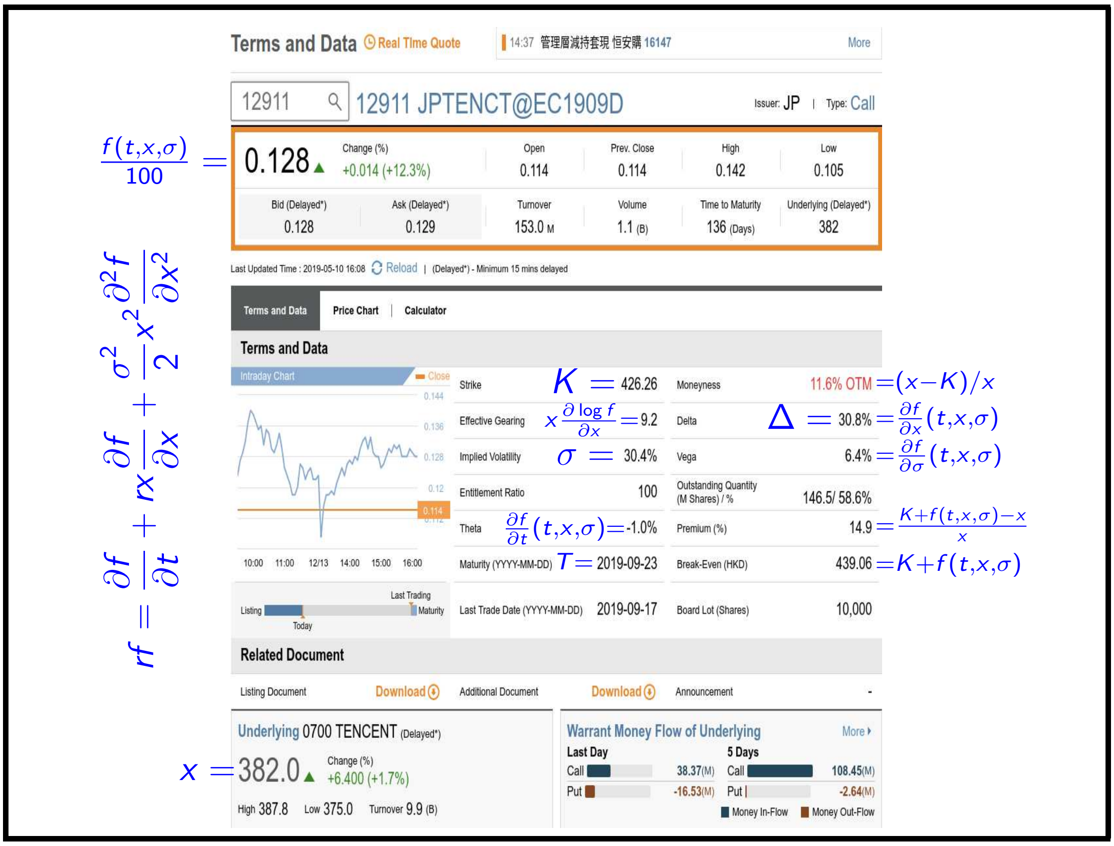


Figure 2.17: Warrant terms and data.

The R package bizdays (requires to install QuantLib) can be used to compute calendar time *vs* business time to maturity

```
install.packages("bizdays")
library(bizdays)
load_quantlib_calendars('HongKong', from='2018-01-01', to='2018-12-31')
load_quantlib_calendars('Singapore', from='2018-01-01', to='2018-12-31')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/HongKong')
bizdays('2018-03-10', '2018-04-03', 'QuantLib/Singapore')
```

2.5 Volatility Estimation

Historical Volatility

We consider the problem of estimating the parameters μ and σ from market data in the stock price model

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{2.22}$$

Historical trend estimation

By discretization of (2.22) along a family $t_0, t_1, \ldots, t_N$ of observation times as

$$\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} = (t_{k+1} - t_k)\mu + (B_{t_{k+1}} - B_{t_k})\sigma, \quad k = 0, 1, \ldots, N-1, \tag{2.23}$$

a natural estimator for the trend parameter μ can be constructed as

$$\widehat{\mu}_N := \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k}\left(\frac{S^M_{t_{k+1}} - S^M_{t_k}}{S^M_{t_k}}\right), \tag{2.24}$$

where $(S^M_{t_{k+1}} - S^M_{t_k})/S^M_{t_k}$, $k = 0, 1, \ldots, N-1$ denotes market returns observed at discrete times $t_0, t_1, \ldots, t_N$ on the market.

Historical log-return estimation

Alternatively, observe that, replacing (2.24) by the log-returns

$$\log\left(1 + \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}}\right) = \log\frac{S_{t_{k+1}}}{S_{t_k}} = \log S_{t_{k+1}} - \log S_{t_k} \simeq \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}},$$

with $t_{k+1} - t_k = T/N$, $k = 0, 1, \ldots, N-1$, one can replace (2.24) with the simpler telescoping estimate*

$$\frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k}(\log S_{t_{k+1}} - \log S_{t_k}) = \frac{1}{T}\log\frac{S_T}{S_0}.$$

Historical volatility estimation

The volatility parameter σ can be estimated by writing, from (2.23),

$$\sigma^2 \sum_{k=0}^{N-1} (B_{t_{k+1}} - B_{t_k})^2 = \sum_{k=0}^{N-1}\left(\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} - (t_{k+1} - t_k)\mu\right)^2,$$

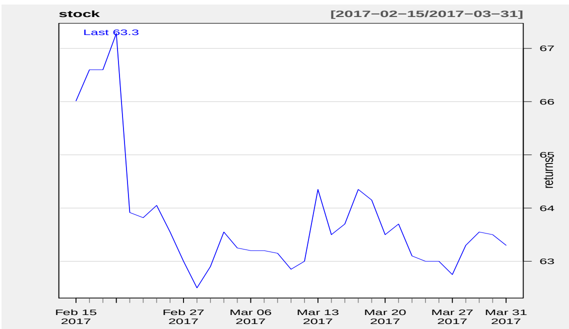
which yields the (unbiased) realized variance estimator

$$\widehat{\sigma}^2_N := \frac{1}{N-1}\sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k}\left(\frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}} - (t_{k+1} - t_k)\widehat{\mu}_N\right)^2.$$

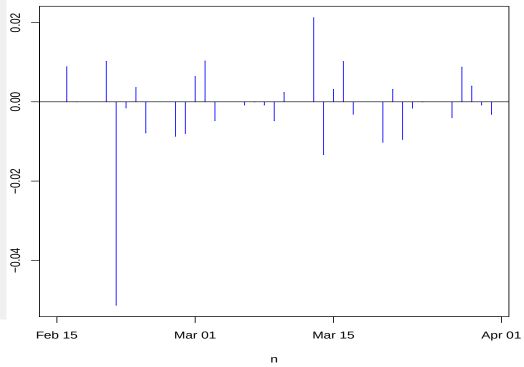
*Note that strictly speaking, the Itô formula reads $d\log S_t = dS_t/S_t - (dS_t)^2/(2S_t^2)$.

```
library(quantmod)
getSymbols("0005.HK",from="2017-02-15",to=Sys.Date(),src="yahoo")
stock=Ad(`0005.HK`)
chartSeries(stock,up.col="blue",theme="white")
```

```
stock=Ad(`0005.HK`);returns=(stock-lag(stock))/stock
returns=diff(log(stock));times=index(returns);returns <- as.vector(returns)
n = sum(is.na(returns))+sum(!is.na(returns))
plot(times,returns,pch=19,cex=0.05,col="blue", ylab="returns", xlab="n", main = '')
segments(x0 = times, x1 = times, cex=0.05,y0 = 0, y1 = returns,col="blue")
abline(seq(1,n),0,FALSE);dt=1.0/365;mu=mean(returns,na.rm=TRUE)/dt
sigma=sd(returns,na.rm=TRUE)/sqrt(dt);mu;sigma
```



(a) Underlying asset price.



(b) Log returns.

Figure 2.18: Graph of underlying asset price *vs* log returns.

```
library(PerformanceAnalytics)
library(quantmod)
returns <- exp(CalculateReturns(stock,method="compound")) - 1; returns[1,] <- 0
histvol <- rollapply(returns, width = 30, FUN=sd.annualized)
myTheme <- chart_theme(); myTheme$col$line.col <- "blue"
chart_Series(stock,name="0005.HK",theme=myTheme)
add_TA(histvol, name="Historical Volatility")
```

The next Figure 2.19 presents a historical volatility graph with a 30 days rolling window.

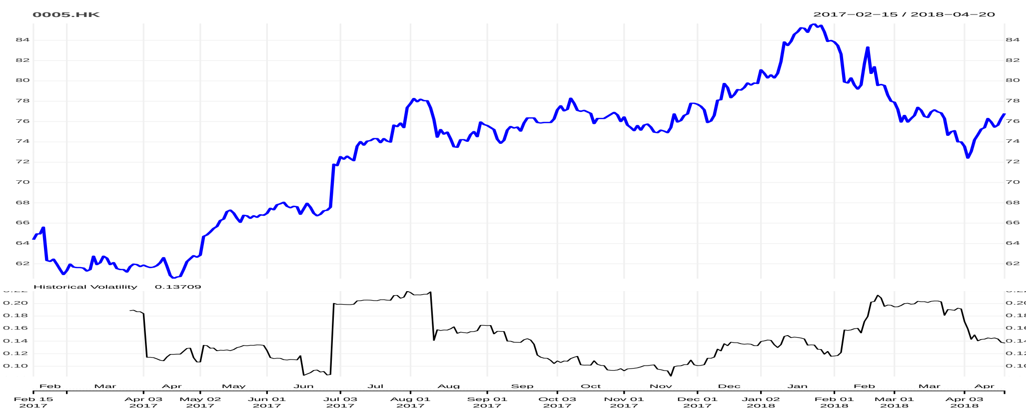


Figure 2.19: Historical volatility graph.

Parameter estimation based on historical data usually requires a lot of samples and it can only be valid on a given time interval, or as a moving average. Moreover, it can only rely on past data, which may not reflect future data.

Figure 2.20: "The fugazi: it's a wazy, it's a woozie. It's fairy dust."*

Implied Volatility

Recall that when $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE is given by

$$\mathrm{Bl}(t,x,K,\sigma,r,T) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) = \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\[2ex] d_-(T-t) = \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \end{cases}$$

In contrast with the historical volatility, the computation of the implied volatility can be done at a fixed time and requires much less data. Equating the Black-Scholes formula

$$\mathrm{Bl}(t,S_t,K,\sigma,r,T) = M \tag{2.25}$$

to the observed value M of a given market price allows one to infer a value of σ when t, S_t, r, T are known, as in *e.g.* Figure 2.36.

*Scorsese, 2013 Click on the figure to play the video (works in Acrobat Reader on the entire pdf file).

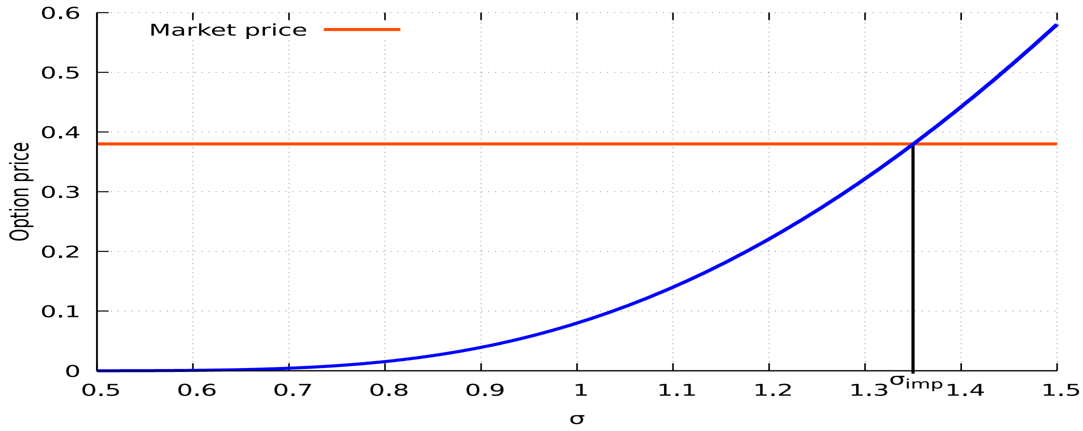


Figure 2.21: Option price as a function of the volatility σ.

This value of σ is called the implied volatility, and it is denoted here by $\sigma_{\text{imp}}(K,T)$, cf. *e.g.* Exercise 2.5. Various algorithms can be implemented to solve (2.25) numerically for $\sigma_{\text{imp}}(K,T)$, such as the bisection method and the Newton-Raphson method.*

```
BS <- function(S, K, T, r, sig){d1 <- (log(S/K) + (r + sig^2/2)*T) / (sig*sqrt(T))
d2 <- d1 - sig*sqrt(T);return(S*pnorm(d1) - K*exp(-r*T)*pnorm(d2))}
implied.vol <- function(S, K, T, r, market){
sig <- 0.20;sig.up <- 10;sig.down <- 0.0001;count <- 0;err <- BS(S, K, T, r, sig) - market
while(abs(err) > 0.00001 && count<1000){
if(err < 0){sig.down <- sig;sig <- (sig.up + sig)/2} else{sig.up <- sig;sig <- (sig.down + sig)/2}
err <- BS(S, K, T, r, sig) - market;count <- count + 1};if(count==1000){return(NA)}else{return(sig)}}
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02; implied.vol(S, K, T, r, market)
BS(S, K, T, r, implied.vol(S, K, T, r, market))
```

The implied volatility value can be used as an alternative way to quote the option price, based on the knowledge of the remaining parameters (such as underlying asset price, time to maturity, interest rate, and strike price). For example, market option price data provided by the Hong Kong stock exchange includes implied volatility computed by inverting the Black-Scholes formula, cf. Figure S.1.

```
library(fOptions)
market = 0.83;K = 62.8;T = 7 / 365.0;S = 63.4;r = 0.02
sig=GBSVolatility(market,"c",S,K,T,r,r,1e-4,maxiter = 10000)
BS(S, K, T, r, sig)
```

*Download the corresponding **R code** or the **IPython notebook** that can be run **here**.

Option chain data in R

```r
install.packages("quantmod")
library(quantmod)
getSymbols("^GSPC", src = "yahoo", from = as.Date("2018-01-01"), to = as.
Date("2018-03-013"))
head(GSPC)
# Only the front-month expiry
GSPC.OPT <- getOptionChain("^GSPC")
# All expiries
GSPC.OPTS <- getOptionChain("^GSPC", NULL)
# All 2018 to 2020 expiries
GSPC.OPTS <- getOptionChain("^GSPC", "2018/2020")
# Only the front-month expiry
AAPL.OPT <- getOptionChain("AAPL")
# All expiries
AAPL.OPTS <- getOptionChain("AAPL", NULL)
# All 2018 to 2020 expiries
AAPL.OPTS <- getOptionChain("AAPL", "2018/2020")
```

Exporting option price data

```r
write.table(AAPL.OPT$puts, file = "AAPLputs")
write.csv(AAPL.OPT$puts, file = "AAPLputs.csv")
install.packages("xlsx")
library(xlsx)
write.xlsx(AAPL.OPTS$Jun.19.2020$puts, file = "AAPL.OPTS$Jun.19.2020$puts.xlsx")
```

Volatility smiles

Given two European call options with strike prices K_1, resp. K_2, maturities T_1, resp. T_2, and prices C_1, resp. C_2, on the same stock S, this procedure should yield two estimates $\sigma_{\rm imp}(K_1,T_1)$ and $\sigma_{\rm imp}(K_2,T_2)$ of implied volatilites according to the following equations.

$$\begin{cases} \mathrm{Bl}(t,S_t,K_1,\sigma_{\rm imp}(K_1,T_1),r,T_1) = M_1, & (2.26a) \\ \\ \mathrm{Bl}(t,S_t,K_2,\sigma_{\rm imp}(K_2,T_2),r,T_2) = M_2, & (2.26b) \end{cases}$$

Clearly, there is no reason a priori for the implied volatilites $\sigma_{\rm imp}(K_1,T_1)$, $\sigma_{\rm imp}(K_2,T_2)$ solutions of (2.26a)-(2.26b) to coincide across different strike prices and different maturities. However, in the standard Black-Scholes model the value of the parameter σ should be unique for a given stock S. This contradiction between a model and market data is a reason for the development of more sophisticated stochastic volatility models.

```r
install.packages("jsonlite")
install.packages("lubridate")
library(jsonlite);library(lubridate);library(quantmod)
# Maturity to be updated as needed
maturity <- as.Date("2022-06-17", format="%Y-%m-%d")
CHAIN <- getOptionChain("AAPL",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
T <- as.numeric((maturity - today)/365);r = 0.02;ImpVol<-1:1
getSymbols("AAPL",from=today-1,to=today,src="yahoo");S=as.numeric(Ad(`AAPL`))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S,CHAIN$calls$Strike[i],T,r,
    CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
    volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
    data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=2)
```

```
# Maturity to be updated as needed
maturity <- as.Date("2019-12-20", format="%Y-%m-%d")
CHAIN <- getOptionChain("^GSPC",maturity)
# Last trading day (may require update)
today <- as.Date(Sys.Date(), format="%Y-%m-%d")
T <- as.numeric((maturity - today)/365);r = 0.02;ImpVol<-1:1
getSymbols("^GSPC",from=today-1,to=today,src="yahoo");S=as.numeric(Ad(`GSPC`))
for (i in 1:length(CHAIN$calls$Strike)){ImpVol[i]<-implied.vol(S, CHAIN$calls$Strike[i], T, r,
       CHAIN$calls$Last[i])}
plot(CHAIN$calls$Strike[!is.na(ImpVol)], ImpVol[!is.na(ImpVol)], xlab = "Strike price", ylab = "Implied
       volatility", lwd =3, type = "l", col = "blue")
fit4 <- lm(ImpVol[!is.na(ImpVol)]~poly(CHAIN$calls$Strike[!is.na(ImpVol)],4,raw=TRUE))
lines(CHAIN$calls$Strike[!is.na(ImpVol)], predict(fit4,
       data.frame(x=CHAIN$calls$Strike[!is.na(ImpVol)])), col="red",lwd=3)
```

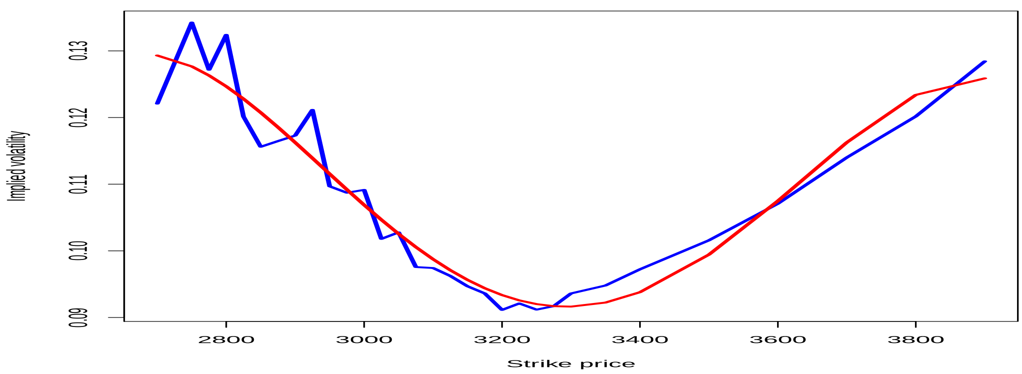


Figure 2.22: S&P500 option prices plotted against strike prices.

When reading option prices on the volatility scale, the smile phenomenon shows that the Black-Scholes formula tends to underprice extreme events for which the underlying asset price S_T is far away from the strike price K. In that sense, the Black-Scholes model, which is based on the Gaussian distribution tails, tends to underestimate the probability of extreme events.

Plotting the different values of the implied volatility σ as a function of K and T will yield a three-dimensional plot called the volatility surface.*

Figure 2.23 presents an estimated implied volatility surface for Asian options on light sweet crude oil futures traded on the New York Mercantile Exchange (NYMEX), based on contract specifications and market data obtained from the Chicago Mercantile Exchange.

*Download the corresponding **IPython notebook** that can be run **here** (© Qu Mengyuan).

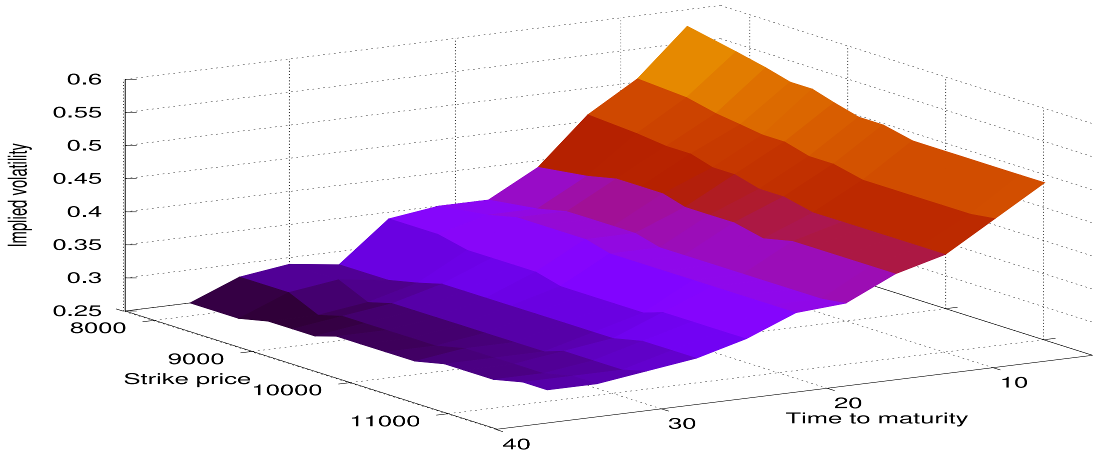


Figure 2.23: Implied volatility surface of Asian options on light sweet crude oil futures.*

As observed in Figure 2.23, the volatility surface can exhibit a *smile* phenomenon, in which implied volatility is higher at a given end (or at both ends) of the range of strike price values.

Black-Scholes Formula *vs* Market Data

On July 28, 2009 a call warrant has been issued by Merrill Lynch on the stock price S of Cheung Kong Holdings (0001.HK) with strike price K=\$109.99, Maturity T = December 13, 2010, and entitlement ratio 100.

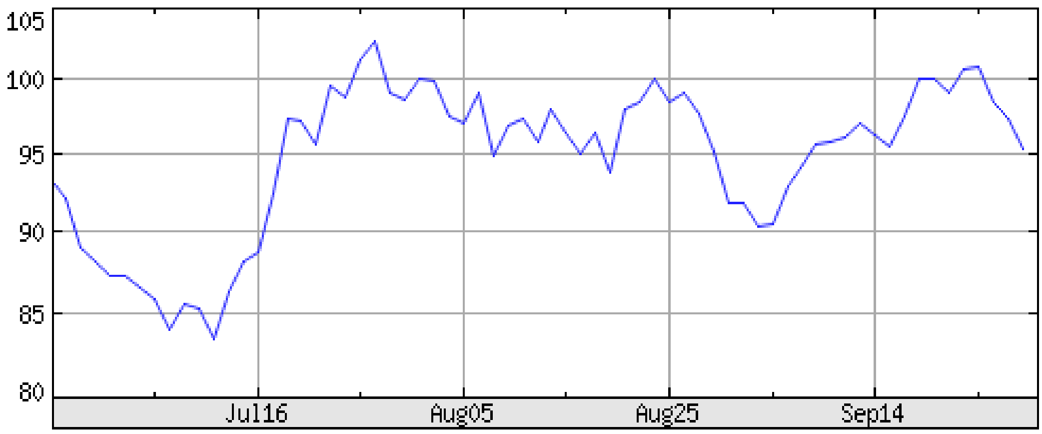


Figure 2.24: Graph of the (market) stock price of Cheung Kong Holdings.

The market price of the option (17838.HK) on September 28 was \$12.30, as obtained from `https://www.hkex.com.hk/eng/dwrc/search/listsearch.asp`.

The next graph in Figure 2.25 shows the evolution of the market price of the option over time. One sees that the option price is much more volatile than the underlying asset price.
In Figure 2.26 we have fitted the path

$$t \longmapsto g_c(t, S_t)$$

of the Black-Scholes price to the data of Figure 2.25 using the market stock price data of Figure 2.24, by varying the values of the volatility σ.

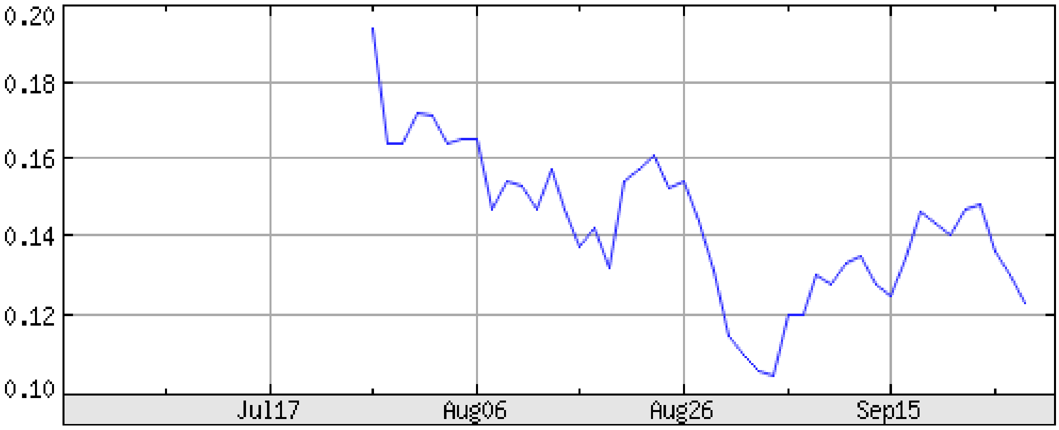


Figure 2.25: Graph of the (market) call option price on Cheung Kong Holdings.

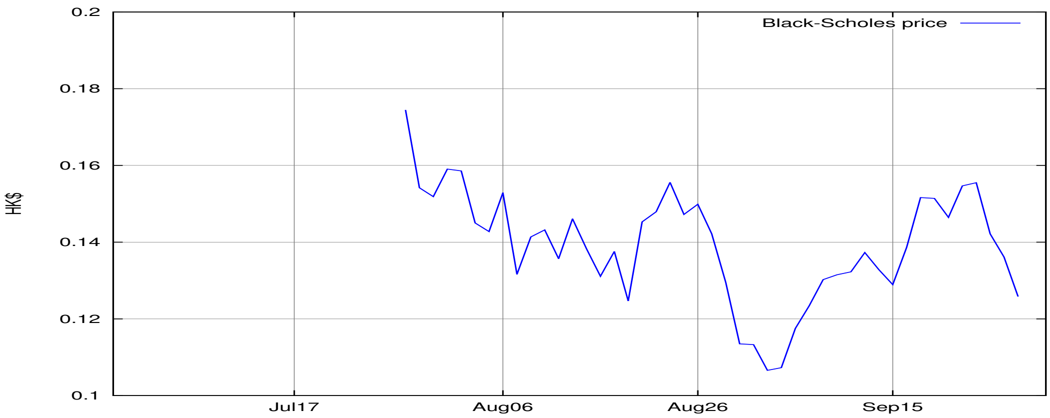


Figure 2.26: Graph of the Black-Scholes call option price on Cheung Kong Holdings.

Another example

Let us consider the stock price of HSBC Holdings (0005.HK) over one year:

Figure 2.27: Graph of the (market) stock price of HSBC Holdings.

Next, we consider the graph of the price of the call option issued by Societe Generale on 31 December 2008 with strike price K=$63.704, maturity T = October 05, 2009, and entitlement ratio 100.

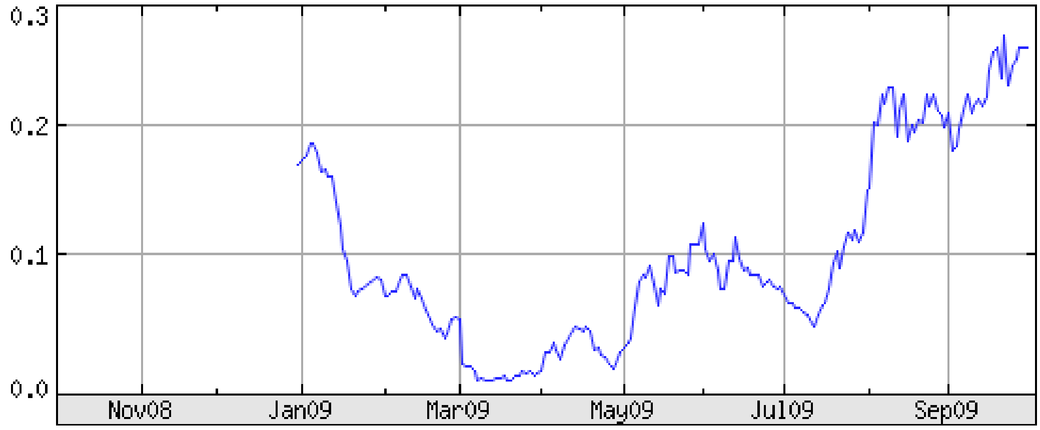


Figure 2.28: Graph of the (market) call option price on HSBC Holdings.

As above, in Figure 2.29 we have fitted the path $t \longmapsto g_c(t, S_t)$ of the Black-Scholes option price to the data of Figure 2.28 using the stock price data of Figure 2.27.

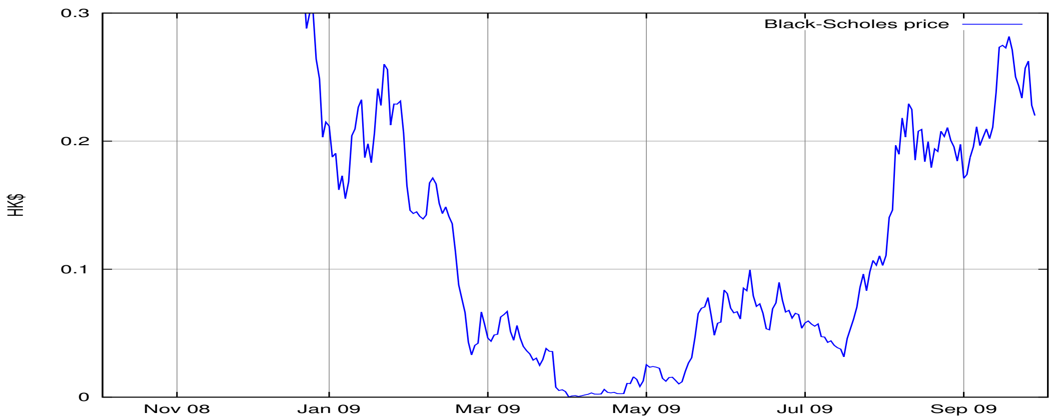


Figure 2.29: Graph of the Black-Scholes call option price on HSBC Holdings.

In this case the option is *in the money* at maturity. We can also check that the option is worth $100 \times 0.2650 = \$26.650$ at that time, which, according to absence of arbitrage, is very close to the actual value \$90 - \$63.703 = \$26.296 of its payoff.

For one more example, consider the graph of the price of a put option issued by BNP Paribas on 04 November 2008 on the underlying asset HSBC, with strike price K=\$77.667, maturity T = October 05, 2009, and entitlement ratio 92.593.
One checks easily that at maturity, the price of the put option is worth \$0.01 (a market price cannot be lower), which almost equals the option payoff \$0, by absence of arbitrage opportunities.
Figure 2.31 is a fit of the Black-Scholes put price graph

$$t \longmapsto g_p(t, S_t)$$

to Figure 2.30 as a function of the stock price data of Figure 2.29. Note that the Black-Scholes price at maturity is strictly equal to 0 while the corresponding market price cannot be lower than one cent.
The normalized market data graph in Figure 2.32 shows how the option price can track the values of the underlying asset price. Note that the range of values $[26.55, 26.90]$ for the underlying asset price

Figure 2.30: Graph of the (market) put option price on HSBC Holdings.

Figure 2.31: Graph of the Black-Scholes put option price on HSBC Holdings.

corresponds to $[0.675, 0.715]$ for the option price, meaning 1.36% *vs* 5.9% in percentage. This is a European call option on the ALSTOM underlying asset with strike price $K = €20$, maturity March 20, 2015, and entitlement ratio 10.

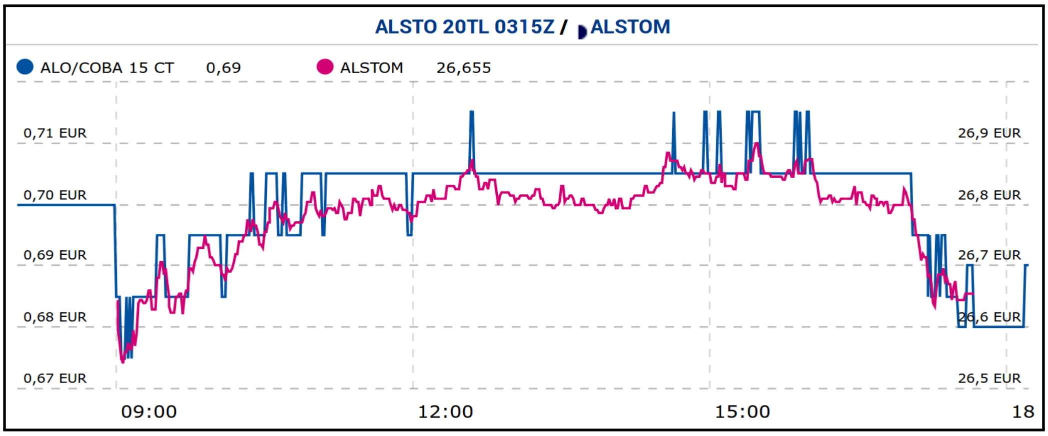


Figure 2.32: Call option price *vs* underlying asset price.

2.6 Solution of the Black-Scholes PDE

The Heat Equation

In the next proposition we notice that the solution $f(t,x)$ of the Black-Scholes PDE (2.7) can be transformed into a solution $g(t,y)$ of the simpler *heat equation* by a change of variable and a time inversion $t \longmapsto T-t$ on the interval $[0,T]$, so that the terminal condition at time T in the Black-Scholes equation (2.27) becomes an initial condition at time $t=0$ in the heat equation (2.30). See also here for a related discussion on changes of variables for the Black-Scholes PDE.

Proposition 2.8 Assume that $f(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} rf(t,x) = \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2}(t,x), \\ \\ f(T,x) = (x-K)^+, \end{cases} \tag{2.27}$$

with terminal condition $h(x) = (x-K)^+$, $x>0$. Then the function $g(t,y)$ defined by

$$g(t,y) = \mathrm{e}^{rt} f\big(T-t, \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}\big) \tag{2.28}$$

solves the heat equation (2.30) with initial condition

$$\psi(y) := h\big(\mathrm{e}^{|\sigma|y}\big), \qquad y \in \mathbb{R}, \tag{2.29}$$

i.e. we have

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = h\big(\mathrm{e}^{|\sigma|y}\big). \end{cases} \tag{2.30}$$

Proposition 2.8 will be proved in Section 2.6. It will allow us to solve the Black-Scholes PDE (2.27) based on the solution of the heat equation (2.30) with initial condition $\psi(y) = h\big(\mathrm{e}^{|\sigma|y}\big)$, $y \in \mathbb{R}$, by inversion of Relation (2.28) with $s = T-t$, $x = \mathrm{e}^{|\sigma|y+(\sigma^2/2-r)t}$, *i.e.*

$$f(s,x) = \mathrm{e}^{-(T-s)r} g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right).$$

Next, we focus on the *heat equation*

$$\frac{\partial \varphi}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) \tag{2.31}$$

which is used to model the diffusion of heat over time through solids. Here, the data of $g(x,t)$ represents the temperature measured at time t and point x. We refer the reader to Widder, 1975 for a complete treatment of this topic.

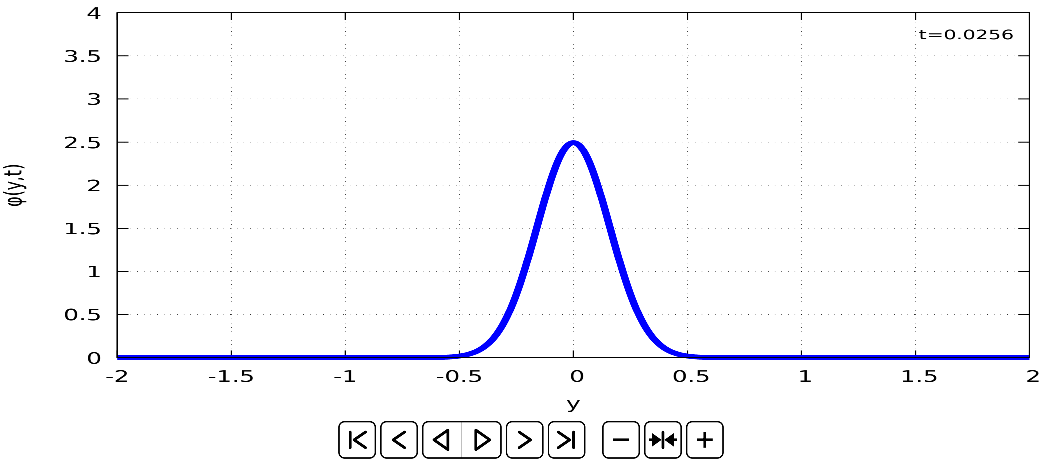


Figure 2.33: Time-dependent solution of the heat equation.*

Proposition 2.9 The fundamental solution of the heat equation (2.31) is given by the Gaussian probability density function

$$\varphi(t,y) := \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-y^2/(2t)}, \qquad y \in \mathbb{R},$$

with variance $t > 0$.

Proof. The proof is done by a direct calculation, as follows:

$$\begin{aligned}
\frac{\partial \varphi}{\partial t}(t,y) &= \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right) \\
&= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t^{3/2}\sqrt{2\pi}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}} \\
&= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y) &= -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{y}{t}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}}\right) \\
&= -\frac{\mathrm{e}^{-y^2/(2t)}}{2t\sqrt{2\pi t}} + \frac{y^2}{2t^2}\frac{\mathrm{e}^{-y^2/(2t)}}{\sqrt{2\pi t}} \\
&= \left(-\frac{1}{2t} + \frac{y^2}{2t^2}\right)\varphi(t,y), \qquad t > 0, \ y \in \mathbb{R}.
\end{aligned}$$

□

In Section 2.6 the heat equation (2.31) will be shown to be equivalent to the Black-Scholes PDE after a change of variables. In particular this will lead to the explicit solution of the Black-Scholes PDE.

*The animation works in Acrobat Reader on the entire pdf file.

Proposition 2.10 The heat equation

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,y) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial y^2}(t,y) \\ \\ g(0,y) = \psi(y) \end{cases} \tag{2.32}$$

with continuous initial condition

$$g(0,y) = \psi(y)$$

has the solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z)\, e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}}, \qquad y \in \mathbb{R}, \quad t > 0. \tag{2.33}$$

Proof. We have

$$\begin{aligned}
\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi(z)\, e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t}\left(\frac{e^{-(y-z)^2/(2t)}}{\sqrt{2\pi t}} \right) dz \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \left(\frac{(y-z)^2}{t^2} - \frac{1}{t} \right) e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial z^2} e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} \psi(z)\, e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}$$

On the other hand, it can be checked that at time $t = 0$ we have

$$\begin{aligned}
\lim_{t \to 0} \int_{-\infty}^{\infty} \psi(z)\, e^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} &= \lim_{t \to 0} \int_{-\infty}^{\infty} \psi(y+z)\, e^{-z^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\
&= \psi(y), \qquad y \in \mathbb{R}.
\end{aligned}$$

□

The next Figure 2.34 shows the evolution of $g(t,x)$ with initial condition based on the European call payoff function $h(x) = (x-K)^+$, *i.e.*

$$g(0,y) = \psi(y) = h\left(e^{|\sigma|y}\right) = \left(e^{|\sigma|y} - K\right)^+, \qquad y \in \mathbb{R}.$$

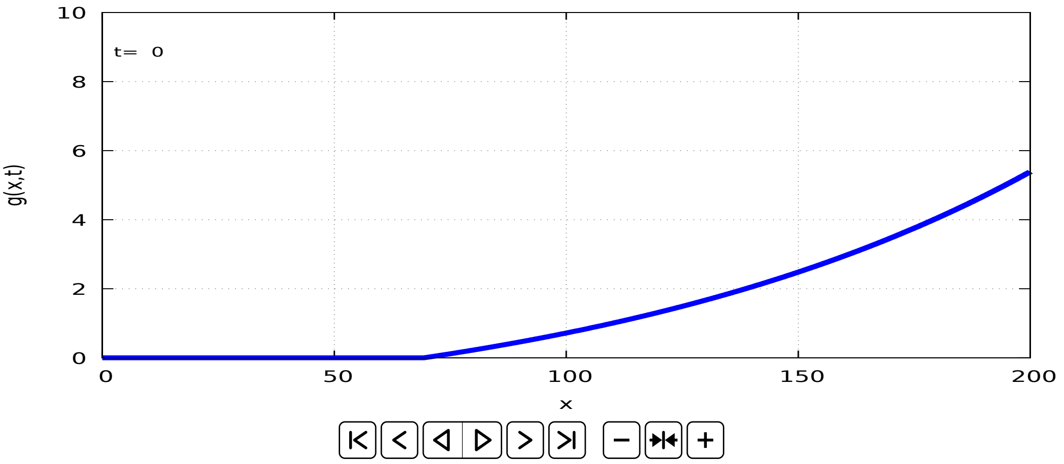


Figure 2.34: Time-dependent solution of the heat equation.*

Let us provide a second proof of Proposition 2.10, this time using Brownian motion and stochastic calculus.

Proof of Proposition 2.10. First, note that under the change of variable $x = z - y$ we have

$$\begin{aligned} g(t,y) &= \int_{-\infty}^{\infty} \psi(z)\, \mathrm{e}^{-(y-z)^2/(2t)} \frac{dz}{\sqrt{2\pi t}} \\ &= \int_{-\infty}^{\infty} \psi(y+x)\, \mathrm{e}^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} \\ &= \mathbb{E}[\psi(y+B_t)] \\ &= \mathbb{E}[\psi(y-B_t)], \end{aligned}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $B_t \simeq \mathcal{N}(0,t)$, $t \in \mathbb{R}_+$. Applying Itô's formula and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we find

$$\begin{aligned} g(t,y) &= \mathbb{E}[\psi(y-B_t)] \\ &= \psi(y) - \mathbb{E}\left[\int_0^t \psi'(y-B_s)dB_s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \psi''(y-B_s)ds\right] \\ &= \psi(y) + \frac{1}{2}\int_0^t \mathbb{E}[\psi''(y-B_s)]\,ds \\ &= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\mathbb{E}[\psi(y-B_s)]\,ds \\ &= \psi(y) + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial y^2}(s,y)ds. \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\mathbb{E}[\psi(y-B_t)] \\ &= \frac{1}{2}\frac{\partial^2}{\partial y^2}\mathbb{E}[\psi(y-B_t)] \\ &= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y). \end{aligned}$$

*The animation works in Acrobat Reader on the entire pdf file.

Regarding the initial condition, we check that

$$g(0,y) = \mathbb{E}[\psi(y - B_0)] = \mathbb{E}[\psi(y)] = \psi(y).$$

□

The expression $g(t,y) = \mathbb{E}[\psi(y - B_t)]$ provides a probabilistic interpretation of the heat diffusion phenomenon based on Brownian motion. Namely, when $\psi_\varepsilon(y) := \mathbb{1}_{[-\varepsilon,\varepsilon]}(y)$, we find that

$$\begin{aligned} g_\varepsilon(t,y) &= \mathbb{E}[\psi_\varepsilon(y - B_t)] \\ &= \mathbb{E}[\mathbb{1}_{[-\varepsilon,\varepsilon]}(y - B_t)] \\ &= \mathbb{P}\big(y - B_t \in [-\varepsilon,\varepsilon]\big) \\ &= \mathbb{P}\big(y - \varepsilon \leqslant B_t \leqslant y + \varepsilon\big) \end{aligned}$$

represents the probability of finding B_t within a neighborhood $[y-\varepsilon, y+\varepsilon]$ of the point $y \in \mathbb{R}$.

Black-Scholes PDE

We now solve the Black-Scholes PDE by the kernel method and a change of variables. This solution method uses the change of variables (2.28) of Proposition 2.8 and a time inversion from which the terminal condition at time T in the Black-Scholes equation becomes an initial condition at time $t = 0$ in the heat equation.

Next, we state the proof Proposition 2.8.

Proof. Letting $s = T - t$ and $x = \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}$ and using Relation (2.28), *i.e.*

$$g(t,y) = \mathrm{e}^{rt} f\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big),$$

we have

$$\begin{aligned} \frac{\partial g}{\partial t}(t,y) &= r\mathrm{e}^{rt} f\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &\quad + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &= r\mathrm{e}^{rt} f(T - t, x) - \mathrm{e}^{rt}\frac{\partial f}{\partial s}(T - t, x) + \left(\frac{\sigma^2}{2} - r\right) \mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x) \\ &= \frac{1}{2}\mathrm{e}^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T - t, x) + \frac{\sigma^2}{2}\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x), \end{aligned} \tag{2.34}$$

where on the last step we used the Black-Scholes PDE. On the other hand we have

$$\frac{\partial g}{\partial y}(t,y) = |\sigma|\, \mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big)$$

and

$$\begin{aligned} \frac{1}{2}\frac{\partial g^2}{\partial y^2}(t,y) &= \frac{\sigma^2}{2}\mathrm{e}^{rt}\, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\frac{\partial f}{\partial x}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &\quad + \frac{\sigma^2}{2}\mathrm{e}^{rt}\, \mathrm{e}^{2|\sigma|y + 2(\sigma^2/2 - r)t}\frac{\partial^2 f}{\partial x^2}\big(T - t, \mathrm{e}^{|\sigma|y + (\sigma^2/2 - r)t}\big) \\ &= \frac{\sigma^2}{2}\mathrm{e}^{rt} x \frac{\partial f}{\partial x}(T - t, x) + \frac{\sigma^2}{2}\mathrm{e}^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T - t, x). \end{aligned} \tag{2.35}$$

We conclude by comparing (2.34) with (2.35), which shows that $g(t,x)$ solves the heat equation (2.32) with initial condition

$$g(0,y) = f\big(T, \mathrm{e}^{|\sigma| y}\big) = h\big(\mathrm{e}^{|\sigma| y}\big).$$

□

In the next proposition, we derive the Black-Scholes formula (2.10) by solving the PDE (2.27). The Black-Scholes formula will also be recovered by a probabilistic argument via the computation of an expected value in Proposition 3.5.

Proposition 2.11 When $h(x) = (x-K)^+$, the solution of the Black-Scholes PDE (2.27) is given by

$$f(t,x) = x\Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \qquad x>0,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

and

$$\begin{cases} d_+(T-t) := \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \\[2ex] d_-(T-t) := \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}, \end{cases}$$

$x>0$, $t \in [0,T)$.

Proof. By inversion of Relation (2.28) with $s = T-t$ and $x = \mathrm{e}^{|\sigma| y + (\sigma^2/2-r)t}$, we get

$$f(s,x) = \mathrm{e}^{-(T-s)r} g\left(T-s, \frac{-(\sigma^2/2-r)(T-s)+\log x}{|\sigma|}\right)$$

and

$$h(x) = \psi\left(\frac{\log x}{|\sigma|}\right), \quad x>0, \quad \text{or} \quad \psi(y) = h\big(\mathrm{e}^{|\sigma| y}\big), \quad y \in \mathbb{R}.$$

Hence, using the solution (2.33) and Relation (2.29), we get

$$\begin{aligned}
f(t,x) &= \mathrm{e}^{-(T-t)r} g\left(T-t, \frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|}\right) \\
&= \mathrm{e}^{-(T-t)r} \int_{-\infty}^{\infty} \psi\left(\frac{-(\sigma^2/2-r)(T-t)+\log x}{|\sigma|} + z\right) \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r} \int_{-\infty}^{\infty} h\big(x\mathrm{e}^{|\sigma| z-(\sigma^2/2-r)(T-t)}\big) \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r} \int_{-\infty}^{\infty} \big(x\mathrm{e}^{|\sigma| z-(\sigma^2/2-r)(T-t)} - K\big)^+ \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= \mathrm{e}^{-(T-t)r} \\
&\quad \times \int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{|\sigma|}}^{\infty} \big(x\mathrm{e}^{|\sigma| z-(\sigma^2/2-r)(T-t)} - K\big) \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\mathrm{e}^{-(T-t)r} \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{|\sigma| z-(\sigma^2/2-r)(T-t)} \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}} \\
&\quad -K\mathrm{e}^{-(T-t)r} \int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/(2(T-t))} \frac{dz}{\sqrt{2(T-t)\pi}}
\end{aligned}$$

$$
\begin{aligned}
&= x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{|\sigma|z-(T-t)\sigma^2/2-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-(z-(T-t)|\sigma|)^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\int_{-d_-(T-t)\sqrt{T-t}-(T-t)|\sigma|}^{\infty} \mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/(2(T-t))}\frac{dz}{\sqrt{2(T-t)\pi}} \\
&= x\int_{-d_-(T-t)-|\sigma|\sqrt{T-t}}^{\infty} \mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}} - K\,\mathrm{e}^{-(T-t)r}\int_{-d_-(T-t)}^{\infty} \mathrm{e}^{-z^2/2}\frac{dz}{\sqrt{2\pi}} \\
&= x\big(1-\Phi\big(-d_+(T-t)\big)\big) - K\,\mathrm{e}^{-(T-t)r}\big(1-\Phi\big(-d_-(T-t)\big)\big) \\
&= x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),
\end{aligned}
$$

where we used the relation (2.14), *i.e.*

$$1-\Phi(a) = \Phi(-a), \qquad a \in \mathbb{R}.$$

□

Exercises

Exercise 2.1 Bachelier, 1900 model. Consider a market made of a riskless asset valued $A_t = A_0$ with zero interest rate, $t \in \mathbb{R}_+$, and a risky asset whose price S_t is modeled by a standard Brownian motion as $S_t = B_t$, $t \in \mathbb{R}_+$.

a) Show that the price $g(t,B_t)$ of the option with payoff $C = B_T^2$ satisfies the heat equation

$$\frac{\partial \varphi}{\partial t}(t,y) = -\frac{1}{2}\frac{\partial^2 \varphi}{\partial y^2}(t,y)$$

with terminal condition $g(T,x) = x^2$.

b) Find the function $g(t,x)$ by solving the PDE of Question (a)).

Hint: Try a solution of the form $g(t,x) = x^2 + f(t)$.

See Exercises 2.10, 3.12 and 3.13 for extensions to nonzero interest rates.

Exercise 2.2 Consider a risky asset price $(S_t)_{t\in\mathbb{R}}$ modeled in the Cox, Ingersoll, and Ross, 1985 (CIR) model as

$$dS_t = \beta(\alpha - S_t)dt + \sigma\sqrt{S_t}dB_t, \qquad \alpha,\beta,\sigma > 0, \tag{2.36}$$

and let $(\eta_t,\xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy whose value $V_t := \eta_t A_t + \xi_t S_t$, takes the form $V_t = g(t,S_t)$, $t \in \mathbb{R}_+$. Figure 2.35 presents a random simulation of the solution to (2.36) with $\alpha = 0.025$, $\beta = 1$, and $\sigma = 1.3$.

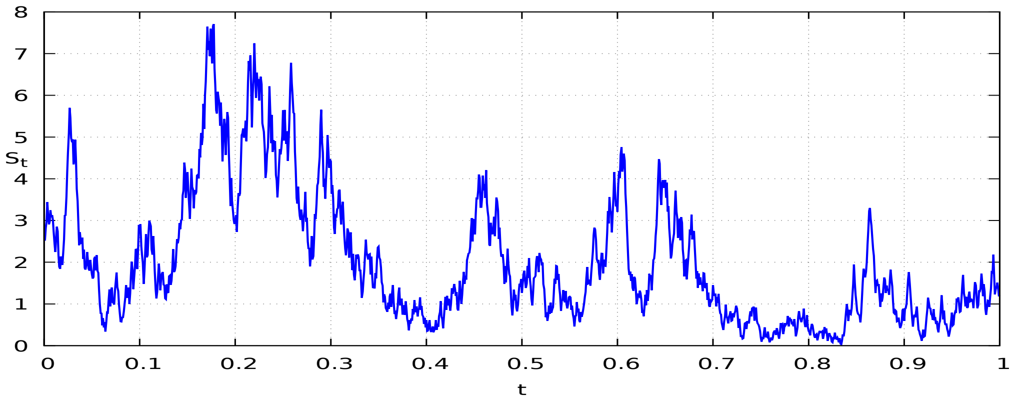


Figure 2.35: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N;a=0.025; b=2; sigma=0.05;
X <- rnorm(N,mean=0,sd=sqrt(dt));R <- rep(0,N);R[1]=0.01
for (j in 2:N){R[j]=max(0,R[j-1]+(a-b*R[j-1])*dt+sigma*sqrt(R[j-1])*X[j])}
plot(t, R, xlab = "t", ylab = "", type = "l", ylim = c(0,0.02), col = "blue")
```

Based on the self-financing condition written as

$$\begin{aligned} dV_t &= rV_t dt - r\xi_t S_t dt + \xi_t dS_t \\ &= rV_t dt - r\xi_t S_t dt + \beta(\alpha - S_t)\xi_t dt + \sigma\xi_t \sqrt{S_t} dB_t, \quad t \in \mathbb{R}_+, \end{aligned} \tag{2.37}$$

derive the PDE satisfied by the function $g(t,x)$ using the Itô formula.

Exercise 2.3 Black-Scholes PDE with dividends. Consider a riskless asset with price $A_t = A_0 e^{rt}$, $t \in \mathbb{R}_+$, and an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion and $\delta > 0$ is a continuous-time dividend rate. By absence of arbitrage, the payment of a dividend entails a drop in the stock price by the same amount occuring generally on the *ex-dividend date*, on which the purchase of the security no longer entitles the investor to the dividend amount. The list of investors entitled to dividend payment is consolidated on the *date of record*, and payment is made on the *payable date*.

```
library(quantmod)
getSymbols("0005.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
getDividends("0005.HK",from="2010-01-01",to=Sys.Date(),src="yahoo")
```

a) Assuming that the portfolio with value $V_t = \xi_t S_t + \eta_t A_t$ at time t is self-financing and that dividends are continuously reinvested, write down the portfolio variation dV_t.
b) Assuming that the portfolio value V_t takes the form $V_t = g(t, S_t)$ at time t, derive the Black-Scholes PDE for the function $g(t,x)$ with its terminal condition.
c) Compute the price at time $t \in [0,T]$ of the European call option with strike price K by solving the corresponding Black-Scholes PDE.
d) Compute the Delta of the option.

Exercise 2.4

a) Check that the Black-Scholes formula (2.10) for European call options

$$g_c(t,x) = x\Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

satisfies the following boundary conditions:

i) at $x = 0$, $g_c(t,0) = 0$,

ii) at maturity $t = T$,

$$g_c(T,x) = (x-K)^+ = \begin{cases} x-K, & x > K \\ 0, & x \leqslant K, \end{cases}$$

iii) as time to maturity tends to infinity,

$$\lim_{T\to\infty} \mathrm{Bl}(K,x,\sigma,r,T-t) = x, \qquad t \in \mathbb{R}_+.$$

b) Check that the Black-Scholes formula (2.18) for European put options

$$g_p(t,x) = K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big)$$

satisfies the following boundary conditions:

i) at $x = 0$, $g_p(t,0) = K\mathrm{e}^{-(T-t)r}$,

ii) as x tends to infinity, $g_p(t,\infty) = 0$ for all $t \in [0,T)$,

iii) at maturity $t = T$,

$$g_p(T,x) = (K-x)^+ = \begin{cases} 0, & x > K \\ K-x, & x \leqslant K, \end{cases}$$

iv) as time to maturity tends to infinity,

$$\lim_{T\to\infty} \mathrm{Bl}_p(K,S_t,\sigma,r,T-t) = 0, \qquad t \in \mathbb{R}_+.$$

Exercise 2.5 On December 18, 2007, a call warrant has been issued by Fortis Bank on the stock price S of the MTR Corporation with maturity $T = 23/12/2008$, strike price $K =$ HK\$ 36.08 and entitlement ratio=10. Recall that in the Black-Scholes model, the price at time t of the European claim on the underlying asset priced S_t with strike price K, maturity T, interest rate r and volatility $\sigma > 0$ is given by the Black-Scholes formula as

$$f(t,S_t) = S_t\Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

where

$$\begin{cases} d_-(T-t) = \dfrac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}, \\[2ex] d_+(T-t) = d_-(T-t)+|\sigma|\sqrt{T-t} = \dfrac{(r+\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}. \end{cases}$$

Recall that by Proposition 2.4 we have

$$\frac{\partial f}{\partial x}(t,S_t) = \Phi\big(d_+(T-t)\big), \qquad 0 \leqslant t \leqslant T.$$

a) Using the values of the Gaussian cumulative distribution function, compute the Black-Scholes price of the corresponding call option at time t =November 07, 2008 with $S_t =$ HK\$ 17.200, assuming a volatility $\sigma = 90\% = 0.90$ and an *annual* risk-free interest rate $r = 4.377\% = 0.04377$,

b) Still using the Gaussian cumulative distribution function, compute the quantity of the risky asset required in your portfolio at time t =November 07, 2008 in order to hedge one such option at maturity $T = 23/12/2008$.

c) Figure 1 represents the Black-Scholes price of the call option as a function of $\sigma \in [0.5, 1.5] = [50\%, 150\%]$.

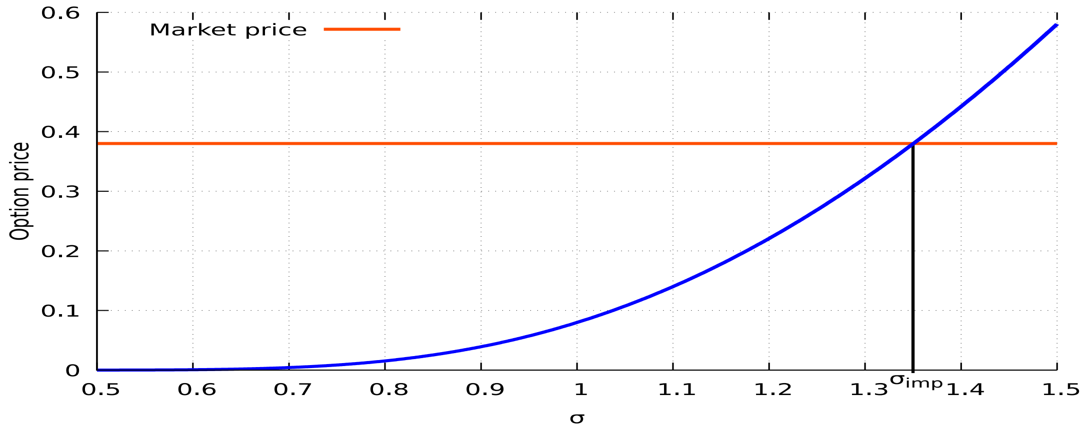


Figure 2.36: Option price as a function of the volatility $\sigma > 0$.

```
BSCall <- function(S, K, r, T, sigma)
{d1 <- (log(S/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));d2 <- d1 - sigma * sqrt(T)
BSCall = S*pnorm(d1) - K*exp(-r*T)*pnorm(d2);BSCall}
sigma <- seq(0.5,1.5, length=100);
plot(sigma,BSCall(17.2,36.08,0.04377,46/365,sigma) , type="l",lty=1, xlab="Sigma",
      ylab="Black-Scholes Call Price", ylim = c(0,0.6),col="blue",lwd=3);grid()
abline(h=0.23,col="red",lwd=3)
```

Knowing that the closing price of the warrant on November 07, 2008 was HK$ 0.023, which value can you infer for the implied volatility σ at this date?*

Exercise 2.6 Forward contracts. Recall that the price $\pi_t(C)$ of a claim payoff $C = h(S_T)$ of maturity T can be written as $\pi_t(C) = g(t, S_t)$, where the function $g(t,x)$ satisfies the *Black-Scholes PDE*

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ \\ g(T,x) = h(x), \end{cases} \qquad (1)$$

with terminal condition $g(T,x) = h(x)$, $x > 0$.

a) Assume that C is a forward contract with payoff

$$C = S_T - K,$$

at time T. Find the function $h(x)$ in (1).

b) Find the solution $g(t,x)$ of the above PDE and compute the price $\pi_t(C)$ at time $t \in [0,T]$. *Hint*: search for a solution of the form $g(t,x) = x - \alpha(t)$ where $\alpha(t)$ is a function of t to be determined.

c) Compute the quantity

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t)$$

of risky assets in a self-financing portfolio hedging C.

*Download the corresponding **R code** or the **IPython notebook** that can be run **here**.

d) Repeat the above questions with the terminal condition $g(T,x) = x$.

Exercise 2.7
a) Solve the Black-Scholes PDE

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(t,x) \tag{2.38}$$

with terminal condition $g(T,x) = 1$, $x > 0$.

Hint: Try a solution of the form $g(t,x) = f(t)$ and find $f(t)$.

b) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = e^{rt}$ in the portfolio with value

$$V_t = g(t,S_t) = \xi_t S_t + \eta_t A_t$$

hedging the contract with payoff \$1 at maturity.

Exercise 2.8 Log-contracts.
a) Solve the PDE

$$0 = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t) \tag{2.39}$$

with the terminal condition $g(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $g(x,t) = f(t) + \log x$, and find $f(t)$.

b) Solve the Black-Scholes PDE

$$rh(x,t) = \frac{\partial h}{\partial t}(x,t) + rx\frac{\partial h}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 h}{\partial x^2}(x,t) \tag{2.40}$$

with the terminal condition $h(x,T) := \log x$, $x > 0$.

Hint: Try a solution of the form $h(x,t) = u(t)g(x,t)$, and find $u(t)$.

c) Find the respective quantities ξ_t and η_t of the risky asset S_t and riskless asset $A_t = e^{rt}$ in the portfolio with value

$$V_t = g(S_t,t) = \xi_t S_t + \eta_t A_t$$

hedging a log-contract with payoff $\log S_T$ at maturity.

Exercise 2.9 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. The binary (or digital) *call* option is a contract with maturity T, strike price K, and payoff

$$C_d := \mathbb{1}_{[K,\infty)}(S_T) = \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K. \end{cases}$$

a) Derive the Black-Schole PDE satisfied by the pricing function $C_d(t,S_t)$ of the binary call option, together with its terminal condition.

b) Show that the solution $C_d(t,x)$ of the Black-Scholes PDE of Question (a)) is given by

$$\begin{aligned} C_d(t,x) &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{|\sigma|\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

where

$$d_-(T-t) := \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{|\sigma|\sqrt{T-t}}, \qquad 0 \leqslant t < T.$$

Exercise 2.10

a) Bachelier, 1900 model. Solve the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t \tag{2.41}$$

in terms of $\alpha, \sigma \in \mathbb{R}$, and the initial condition S_0.

b) Write down the Bachelier PDE satisfied by the function $C(t,x)$, where $C(t,S_t)$ is the price at time $t \in [0,T]$ of the contingent claim with payoff $\phi(S_T) = \exp(S_T)$, and identify the process Delta $(\xi_t)_{t\in[0,T]}$ that hedges this claim.

c) Solve the Black-Scholes PDE of Question (b)) with the terminal condition $\phi(x) = \mathrm{e}^x$, $x \in \mathbb{R}$.

Hint: Search for a solution of the form

$$C(t,x) = \exp\left(-(T-t)r + xh(t) + \frac{\sigma^2}{4r}(h^2(t)-1)\right), \tag{2.42}$$

where $h(t)$ is a function to be determined, with $h(T) = 1$.

d) Compute the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

Exercise 2.11

a) Show that for every fixed value of S, the function

$$d \longmapsto h(S,d) := S\Phi\big(d+|\sigma|\sqrt{T}\big) - K\,\mathrm{e}^{-rT}\Phi(d),$$

reaches its maximum at $d_*(S) := \dfrac{\log(S/K)+(r-\sigma^2/2)T}{|\sigma|\sqrt{T}}$.

b) By the differentiation rule

$$\frac{d}{dS}h(S,d_*(S)) = \frac{\partial h}{\partial S}(S,d_*(S)) + d'_*(S)\frac{\partial h}{\partial d}(S,d_*(S)),$$

recover the value of the Black-Scholes Delta.

Exercise 2.12 Compute the Black-Scholes Vega by differentiation of the Black-Scholes function

$$g_c(t,x) = \mathrm{Bl}(K,x,\sigma,r,T-t) = x\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big),$$

with respect to the volatility parameter σ, knowing that

$$\begin{aligned} -\frac{1}{2}\big(d_-(T-t)\big)^2 &= -\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2 \\ &= -\frac{1}{2}\big(d_+(T-t)\big)^2 + (T-t)r + \log\frac{x}{K}. \end{aligned} \tag{2.43}$$

Exercise 2.13 Consider the backward induction relation

$$\widetilde{v}(t,x) = (1-p_N^*)\widetilde{v}(t+1,x(1+a_N)) + p_N^*\widetilde{v}(t+1,x(1+b_N)),$$

using the renormalizations $r_N := rT/N$ and

$$a_N := (1+r_N)(1-|\sigma|\sqrt{T/N}) - 1, \quad b_N := (1+r_N)(1+|\sigma|\sqrt{T/N}) - 1,$$

$N \geqslant 1$, with

$$p_N^* = \frac{r_N - a_N}{b_N - a_N} \quad \text{and} \quad p_N^* = \frac{b_N - r_N}{b_N - a_N}.$$

a) Show that the Black-Scholes PDE (2.2) of Proposition 2.1 can be recovered when the number N of time steps tends to infinity.

b) Show that the expression of the Delta $\xi_t = \dfrac{\partial g_c}{\partial x}(t,S_t)$ can be similarly recovered from the finite difference relation

$$\xi_t^{(1)}(S_{t-1}) = \frac{v(t,(1+b_N)S_{t-1}) - v(t,(1+a_N)S_{t-1})}{S_{t-1}(b_N - a_N)}$$

as N tends to infinity.

3. Martingale Approach to Pricing and Hedging

In the *martingale approach* to the pricing and hedging of financial derivatives, option prices are expressed as the expected values of discounted option payoffs. This approach relies on the construction of risk-neutral probability measures by the Girsanov theorem, and the associated hedging portfolios are obtained via stochastic integral representations.

3.1 Martingale Property of the Itô Integral

Recall (Definition 1.5) that an integrable process $(X_t)_{t\in\mathbb{R}_+}$ is said to be a *martingale* with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}[X_t \mid \mathscr{F}_s] = X_s, \qquad 0 \leqslant s \leqslant t.$$

Examples of martingales (i)

1. Given $F \in L^2(\Omega)$ a square-integrable random variable and $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ a filtration, the process $(X_t)_{t\in\mathbb{R}_+}$ defined by

$$X_t := \mathbb{E}[F \mid \mathscr{F}_t], \qquad t \in \mathbb{R}_+,$$

is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-martingale under $\mathbb{P}$, as follows from the tower property of conditional expectations:

$$\mathbb{E}[X_t \mid \mathscr{F}_s] = \mathbb{E}[\mathbb{E}[F \mid \mathscr{F}_t] \mid \mathscr{F}_s] = \mathbb{E}[F \mid \mathscr{F}_s] = X_s, \quad 0 \leqslant s \leqslant t. \tag{3.1}$$

2. Any integrable stochastic process $(X_t)_{t\in\mathbb{R}_+}$ whose increments $(X_{t_1}-X_{t_0},\ldots,X_{t_n}-X_{t_{n-1}})$ are independent and centered under $\mathbb{P}$ (*i.e.* $\mathbb{E}[X_t]=0$, $t\in\mathbb{R}_+$) is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(X_t)_{t\in\mathbb{R}_+}$, as we have
$$\begin{aligned}\mathbb{E}[X_t \mid \mathscr{F}_s] &= \mathbb{E}[X_t-X_s+X_s \mid \mathscr{F}_s]\\ &= \mathbb{E}[X_t-X_s \mid \mathscr{F}_s]+\mathbb{E}[X_s \mid \mathscr{F}_s]\\ &= \mathbb{E}[X_t-X_s]+X_s\\ &= X_s, \qquad 0\leqslant s\leqslant t. \qquad (3.2)\end{aligned}$$
In particular, the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale because it has centered and independent increments. This fact is also consequence of Proposition 3.1 below as B_t can be written as
$$B_t=\int_0^t dB_s, \qquad t\in\mathbb{R}_+.$$

The following result shows that the Itô integral yields a martingale with respect to the Brownian filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

Proposition 3.1 The stochastic integral process $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ of a square-integrable adapted process $u\in L^2_{\mathrm{ad}}(\Omega\times\mathbb{R}_+)$ is a martingale, *i.e.*:
$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \,\middle|\, \mathscr{F}_s\right]=\int_0^s u_\tau dB_\tau, \qquad 0\leqslant s\leqslant t. \qquad (3.3)$$

In particular, $\int_0^t u_s dB_s$ is $\mathscr{F}_t$-measurable, $t\in\mathbb{R}_+$, and since $\mathscr{F}_0=\{\emptyset,\Omega\}$, Relation (3.3) applied with $t=0$ recovers the fact that the Itô integral is a centered random variable:
$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right]=\mathbb{E}\left[\int_0^\infty u_s dB_s \,\middle|\, \mathscr{F}_0\right]=\int_0^0 u_s dB_s=0.$$

Proof. The statement is first proved in case $(u_t)_{t\in\mathbb{R}_+}$ is a simple predictable process, and then extended to the general case, cf. *e.g.* Proposition 2.5.7 in Privault, 2009. For example, for u a predictable step process of the form
$$u_s:=F\mathbb{1}_{[a,b]}(s)=\begin{cases}F & \text{if } s\in[a,b],\\ 0 & \text{if } s\notin[a,b],\end{cases}$$
with F an $\mathscr{F}_a$-measurable random variable and $t\in[a,b]$, we have
$$\begin{aligned}\mathbb{E}\left[\int_0^\infty u_s dB_s \,\middle|\, \mathscr{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F\mathbb{1}_{[a,b]}(s)dB_s \,\middle|\, \mathscr{F}_t\right]\\ &= \mathbb{E}[F(B_b-B_a) \mid \mathscr{F}_t]\\ &= F\,\mathbb{E}[B_b-B_a \mid \mathscr{F}_t]\\ &= F(B_t-B_a)\\ &= \int_a^t u_s dB_s\\ &= \int_0^t u_s dB_s, \qquad a\leqslant t\leqslant b.\end{aligned}$$
On the other hand, when $t\in[0,a]$ we have
$$\int_0^t u_s dB_s=0,$$

and we check that

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathscr{F}_t\right] &= \mathbb{E}\left[\int_0^\infty F \mathbb{1}_{[a,b]}(s) dB_s \,\Big|\, \mathscr{F}_t\right] \\
&= \mathbb{E}[F(B_b - B_a) \mid \mathscr{F}_t] \\
&= \mathbb{E}[\mathbb{E}[F(B_b - B_a) \mid \mathscr{F}_a] \mid \mathscr{F}_t] \\
&= \mathbb{E}[F\,\mathbb{E}[B_b - B_a \mid \mathscr{F}_a] \mid \mathscr{F}_t] \\
&= 0, \qquad 0 \leqslant t \leqslant a,
\end{aligned}$$

where we used the tower property of conditional expectations and the fact that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale:

$$\mathbb{E}[B_b - B_a \mid \mathscr{F}_a] = \mathbb{E}[B_b \mid \mathscr{F}_a] - B_a = B_a - B_a = 0.$$

The extension from simple processes to square-integrable processes in $L^2_{\mathrm{ad}}(\Omega \times \mathbb{R}_+)$ can be proved as follows. Given $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ be a sequence of simple predictable processes converging to u in $L^2(\Omega \times [0,T])$ cf. Lemma 1.1 of Ikeda and S. Watanabe, 1989, pages 22 and 46, by Fatou's Lemma and Jensen's inequality we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathscr{F}_t\right]\right)^2\right] \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^{(n)} dB_s - \mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathscr{F}_t\right]\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s \,\Big|\, \mathscr{F}_t\right]\right)^2\right] \\
&\leqslant \liminf_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^{(n)} dB_s - \int_0^\infty u_s dB_s\right)^2 \,\Big|\, \mathscr{F}_t\right]\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^{(n)} - u_s) dB_s\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^{(n)} - u_s|^2 ds\right] \\
&= \liminf_{n\to\infty} \|u^{(n)} - u\|^2_{L^2(\Omega\times[0,T])} \\
&= 0,
\end{aligned}$$

where we used the Itô isometry. We conclude that

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \,\Big|\, \mathscr{F}_t\right] = \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

for $u \in L^2_{\mathrm{ad}}(\Omega \times \mathbb{R}_+)$ a square-integrable adapted process, which leads to (3.3) after applying this identity to the process $(\mathbb{1}_{[0,t]} u_s)_{s\in\mathbb{R}_+}$, *i.e.*,

$$\begin{aligned}
\mathbb{E}\left[\int_0^t u_\tau dB_\tau \,\Big|\, \mathscr{F}_s\right] &= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau \,\Big|\, \mathscr{F}_s\right] \\
&= \int_0^s \mathbb{1}_{[0,t]}(\tau) u_\tau dB_\tau \\
&= \int_0^s u_\tau dB_\tau, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

□

Examples of martingales (ii)

1. The driftless geometric Brownian motion

$$X_t := X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$$

is a martingale. Indeed, we have

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathscr{F}_s] &= \mathbb{E}\left[X_0 \,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \,\mathbb{E}\left[\mathrm{e}^{\sigma B_t} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma + \sigma B_s} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma} \mid \mathscr{F}_s\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathbb{E}\left[\mathrm{e}^{(B_t - B_s)\sigma}\right] \\
&= X_0 \,\mathrm{e}^{-\sigma^2 t/2 + \sigma B_s} \,\mathrm{e}^{\sigma^2 (t-s)/2} \\
&= X_0 \,\mathrm{e}^{\sigma B_s - \sigma^2 s/2} \\
&= X_s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

This fact can also be recovered from Proposition 3.1 since $(X_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$dX_t = \sigma X_t dB_t,$$

which shows that X_t can be written using the Brownian stochastic integral

$$X_t = X_0 + \sigma \int_0^t X_u dB_u, \qquad t \in \mathbb{R}_+.$$

2. Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \qquad t \in \mathbb{R}_+. \tag{3.4}$$

By the *Discounting Lemma 1.6*, the discounted asset price process $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$, $t \in \mathbb{R}_+$, satisfies the stochastic differential equation

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t).$$

The discounted asset price

$$\widetilde{S}_t = \mathrm{e}^{-rt} S_t = S_0 \,\mathrm{e}^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}$ when $\mu = r$. The case $\mu \neq r$ will be treated in Section 3.3 using risk-neutral probability measures and the Girsanov Theorem 3.2, see (3.14) below.

3. The discounted value

$$\widetilde{V}_t = \mathrm{e}^{-rt} V_t, \qquad t \geqslant 0,$$

of a self-financing portfolio is given by

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \geqslant 0,$$

cf. Lemma 1.7 is a martingale when $\mu = r$ by Proposition 3.1 because

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u dB_u, \qquad t \geqslant 0,,$$

since we have

$$d\widetilde{S}_t = \widetilde{S}_t((\mu - r)dt + \sigma dB_t) = \sigma \widetilde{S}_t dB_t$$

by the *Discounting Lemma 1.6*. Since the Black-Scholes theory is in fact valid for any value of the parameter μ we will look forward to including the case $\mu \neq r$ in the sequel.

3.2 Risk-neutral Probability Measures

Recall that by definition, a risk-neutral measure is a probability measure $\mathbb{P}^*$ under which the discounted asset price process

$$\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} := \big(\mathrm{e}^{-rt}S_t\big)_{t\in\mathbb{R}_+}$$

is a *martingale*, cf. Proposition 1.1.

Consider an asset price process $(S_t)_{t\in\mathbb{R}_+}$ given by the stochastic differential equation (3.4). Note that when $\mu = r$, the discounted asset price process $\big(\widetilde{S}_t\big)_{t\in\mathbb{R}_+} = \big(S_0\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}\big)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^* = \mathbb{P}$, which is a risk-neutral probability measure.

In this section, we address the construction of a risk-neutral probability measure $\mathbb{P}^*$ in the general case $\mu \neq r$ using the Girsanov Theorem 3.2 below. Note that the relation

$$d\widetilde{S}_t = \widetilde{S}_t\big((\mu - r)dt + \sigma dB_t\big)$$

where $\mu - r$ is the risk premium, can be rewritten as

$$d\widetilde{S}_t = \sigma\widetilde{S}_t d\widehat{B}_t,$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a drifted Brownian motion given by

$$\widehat{B}_t := \frac{\mu - r}{\sigma}t + B_t, \qquad t \in \mathbb{R}_+,$$

where the drift coefficient $\nu := (\mu - r)/\sigma$ is the "Market Price of Risk" (MPoR). It represents the difference between the return μ expected when investing in the risky asset S_t, and the risk-free interest rate r, measured in units of volatility σ.

Therefore, the search for a risk-neutral probability measure can be replaced by the search for a probability measure $\mathbb{P}^*$ under which $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Let us come back to the informal interpretation of Brownian motion via its infinitesimal increments:

$$\Delta B_t = \pm\sqrt{\Delta t},$$

with

$$\mathbb{P}\big(\Delta B_t = +\sqrt{\Delta t}\big) = \mathbb{P}\big(\Delta B_t = -\sqrt{\Delta t}\big) = \frac{1}{2}.$$

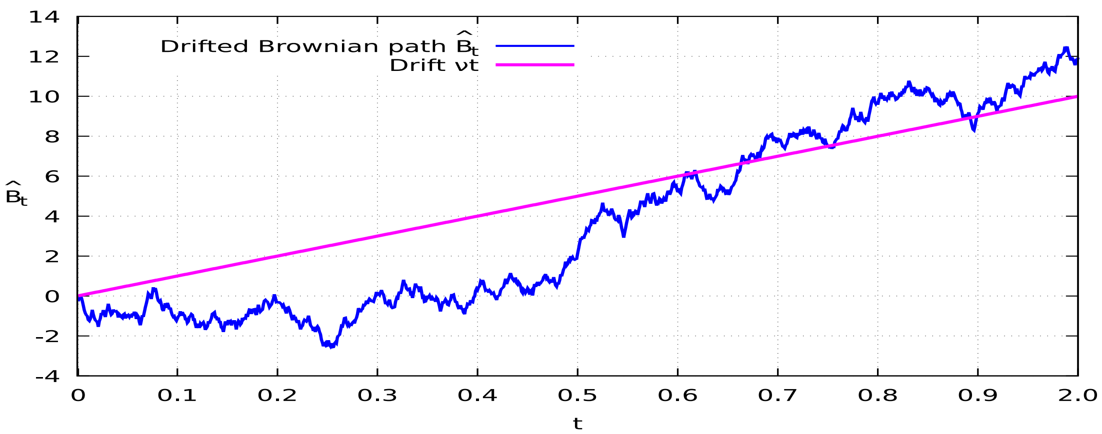


Figure 3.1: Drifted Brownian path $(\widehat{B}_t)_{t\in\mathbb{R}_+}$.

Clearly, given $\nu \in \mathbb{R}$, the drifted process

$$\widehat{B}_t := \nu t + B_t, \qquad t \in \mathbb{R}_+,$$

is no longer a standard Brownian motion because it is not centered:

$$\mathbb{E}\left[\widehat{B}_t\right] = \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

cf. Figure 3.1. This identity can be formulated in terms of infinitesimal increments as

$$\mathbb{E}[\nu \Delta t + \Delta B_t] = \frac{1}{2}(\nu \Delta t + \sqrt{\Delta t}) + \frac{1}{2}(\nu \Delta t - \sqrt{\Delta t}) = \nu \Delta t \neq 0.$$

In order to make $\nu t + B_t$ a centered process (*i.e.* a standard Brownian motion, since $\nu t + B_t$ conserves all the other properties (i)-(iii) in the definition of Brownian motion, one may change the probabilities of ups and downs, which have been fixed so far equal to $1/2$.

That is, the problem is now to find two numbers $p^*, q^* \in [0,1]$ such that

$$\begin{cases} p^*(\nu \Delta t + \sqrt{\Delta t}) + q^*(\nu \Delta t - \sqrt{\Delta t}) = 0 \\ \\ p^* + q^* = 1. \end{cases}$$

The solution to this problem is given by

$$p^* = \frac{1}{2}(1 - \nu\sqrt{\Delta t}) \quad \text{and} \quad q^* = \frac{1}{2}(1 + \nu\sqrt{\Delta t}). \tag{3.5}$$

Coming back to Brownian motion considered as a discrete random walk with independent increments $\pm\sqrt{\Delta t}$, we try to construct a new probability measure denoted $\mathbb{P}^*$, under which the drifted process $\widehat{B}_t := \nu t + B_t$ will be a standard Brownian motion. This probability measure will be defined through its Radon-Nikodym density

$$\begin{aligned} \frac{d\mathbb{P}^*}{d\mathbb{P}} &:= \frac{\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})} \\ &= \frac{\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t})} \\ &= \frac{1}{(1/2)^N}\mathbb{P}^*(\Delta B_{t_1} = \varepsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \varepsilon_N\sqrt{\Delta t}), \end{aligned} \tag{3.6}$$

$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1, 1\}$, with respect to the historical probability measure $\mathbb{P}$, obtained by taking the product of the above probabilities divided by the reference probability $1/2^N$ corresponding to the symmetric random walk.

Interpreting $N = T/\Delta t$ as an (infinitely large) number of discrete time steps and under the identification $[0,T] \simeq \{0 = t_0, t_1, \ldots, t_N = T\}$, this Radon-Nikodym density (3.6) can be rewritten as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) \tag{3.7}$$

where 2^N becomes a normalization factor. Using the expansion

$$\log\left(1 \pm \nu\sqrt{\Delta t}\right) = \pm\nu\sqrt{\Delta t} - \frac{1}{2}(\pm\nu\sqrt{\Delta t})^2 + o(\Delta t)$$

$$= \pm\nu\sqrt{\Delta t} - \frac{\nu^2}{2}\Delta t + o(\Delta t),$$

for small values of Δt, this Radon-Nikodym density can be informally shown to converge as follows as N tends to infinity, *i.e.* as the time step $\Delta t = T/N$ tends to zero:

$$\begin{aligned}
2^N \prod_{0<t<T}\left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) &= \prod_{0<t<T}\left(1 \mp \nu\sqrt{\Delta t}\right) \\
&= \exp\left(\log \prod_{0<t<T}\left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{\Delta t} - \frac{1}{2}\sum_{0<t<T}(\mp\nu\sqrt{\Delta t})^2\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \pm\sqrt{\Delta t} - \frac{\nu^2}{2}\sum_{0<t<T}\Delta t\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \Delta B_t - \frac{\nu^2}{2}\sum_{0<t<T}\Delta t\right) \\
&= \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right),
\end{aligned}$$

based on the identifications

$$B_T \simeq \sum_{0<t<T} \pm\sqrt{\Delta t} \quad \text{and} \quad T \simeq \sum_{0<t<T} \Delta t.$$

Informally, the drifted process $(\widehat{B}_t)_{t\in[0,T]} = (\nu t + B_t)_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right).$$

The following R code is rescaling probabilities as in (3.5) based on the value of the drift μ.

```
N=1000; t <- 0:N; dt <- 1.0/N; nu=3; p=0.5*(1-nu*(dt)^0.5); nsim <- 10
X <- matrix((dt)^0.5*(rbinom( nsim * N, 1, p)-0.5)*2, nsim, N)
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", type = "l", ylim = c(-2*N*dt, 2*N*dt), col = 0)
for (i in 1:nsim){lines(t,t*nu*dt+X[i,],xlab="time",type="l",ylim=c(-2*N*dt,2*N*dt),col=i)}
```

The discretized illustration in Figure 3.2 displays the drifted Brownian motion $\widehat{B}_t := \nu t + B_t$ under the shifted probability measure $\mathbb{P}^*$ in (3.7) using the above R code with $N = 100$. The code makes big transitions less frequent than small transitions, resulting into a standard, centered Brownian motion under $\mathbb{P}^*$.

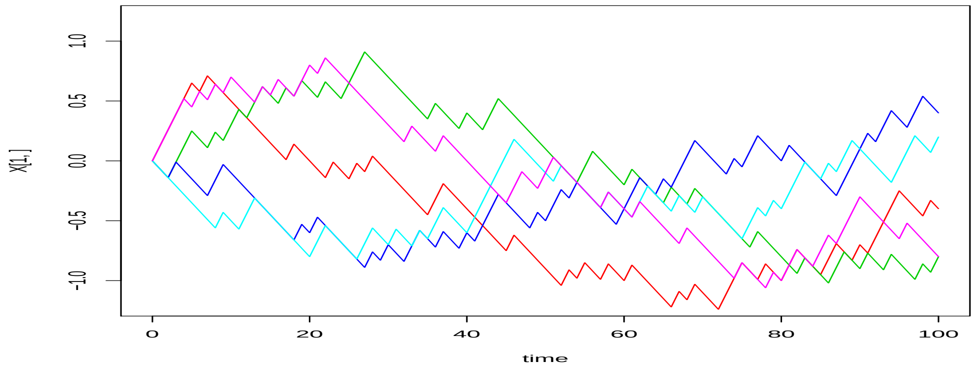


Figure 3.2: Drifted Brownian motion paths under a shifted Girsanov measure.

3.3 Change of Measure and the Girsanov Theorem

In this section we restate the Girsanov Theorem in a more rigorous way, using changes of probability measures.

Definition 3.1 We say that a probability measure $\mathbb{Q}$ is absolutely continuous with respect to another probability measure $\mathbb{P}$ if there exists a nonnegative random variable $F : \Omega \longrightarrow \mathbb{R}$ such that $\mathbb{E}[F] = 1$, and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = F, \qquad \textit{i.e.} \qquad d\mathbb{Q} = F d\mathbb{P}. \tag{3.8}$$

In this case, F is called the Radon-Nikodym *density* of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Relation (3.8) is equivalent to the relation

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[G] &= \int_\Omega G(\omega) d\mathbb{Q}(\omega) \\
&= \int_\Omega G(\omega) \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega G(\omega) F(\omega) d\mathbb{P}(\omega) \\
&= \mathbb{E}[FG],
\end{aligned}$$

for any integrable random variable G.

The Girsanov Theorem can actually be extended to shifts by adapted processes $(\psi_t)_{t\in[0,T]}$ as follows, cf. *e.g.* Theorem III-42, page 141 of Protter, 2004. An extension of the Girsanov Theorem to jump processes will be covered in Section 6.5. Recall also that here, $\Omega = \mathscr{C}_0([0,T])$ is the Wiener space and $\omega \in \Omega$ is a continuous function on $[0,T]$ starting at 0 in $t = 0$. The Girsanov Theorem 3.2 will be used in Section 3.4 for the construction of a unique risk-neutral probability measure $\mathbb{P}^*$, showing absence of arbitrage and completeness in the Black-Scholes market, see Theorems 1.2 and 1.4.

Theorem 3.2 Let $(\psi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty, \tag{3.9}$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right).$$

Then

$$\widehat{B}_t := B_t + \int_0^t \psi_s ds, \qquad 0 \leqslant t \leqslant T,$$

is a standard Brownian motion under $\mathbb{Q}$.

In the case of the simple shift

$$\widehat{B}_t := B_t + \nu t, \qquad 0 \leqslant t \leqslant T,$$

by a drift νt with constant $\nu \in \mathbb{R}$, the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard (centered) Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{\nu^2}{2}T\right) d\mathbb{P}(\omega).$$

For example, the fact that $\widehat{B}_T$ has a centered Gaussian distribution under $\mathbb{Q}$ can be recovered as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[f(\widehat{B}_T)\right] &= \mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] \\
&= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(\nu T + B_T) \exp\left(-\nu B_T - \frac{1}{2}\nu^2 T\right) d\mathbb{P} \\
&= \int_{-\infty}^{\infty} f(\nu T + y) \exp\left(-\nu y - \frac{1}{2}\nu^2 T\right) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(\nu T + x) e^{-(\nu T + x)^2/(2T)} \frac{dx}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{\infty} f(y) e^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)],
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[f(\nu T + B_T)] &= \int_\Omega f(\nu T + B_T) d\mathbb{Q} \\
&= \int_\Omega f(B_T) d\mathbb{P} \\
&= \mathbb{E}_{\mathbb{P}}[f(B_T)],
\end{aligned} \tag{3.10}$$

showing that, under $\mathbb{Q}$, $\nu T + B_T$ has the centered $\mathcal{N}(0,T)$ Gaussian distribution with variance T. For example, taking $f(x) = x$, Relation (3.10) recovers the fact that $\widehat{B}_T$ is a centered random variable under $\mathbb{Q}$, *i.e.*

$$\mathbb{E}_{\mathbb{Q}}\left[\widehat{B}_T\right] = \mathbb{E}_{\mathbb{Q}}[\nu T + B_T)] = \mathbb{E}_{\mathbb{P}}[B_T] = 0.$$

The Girsanov Theorem 3.2 also allows us to extend (3.10) as

$$\mathbb{E}[F] = \mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s ds\right) \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right)\right], \tag{3.11}$$

for all random variables $F \in L^1(\Omega)$.

When applied to the (constant) market price of risk (or Sharpe ratio)

$$\psi_t := \frac{\mu - r}{\sigma},$$

the Girsanov Theorem 3.2 shows that

$$\widehat{B}_t := \frac{\mu - r}{\sigma} t + B_t, \qquad 0 \leqslant t \leqslant T, \tag{3.12}$$

is a standard Brownian motion under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T \right). \tag{3.13}$$

Hence by Proposition 3.1 the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ solution of

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t = \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \tag{3.14}$$

is a martingale under $\mathbb{P}^*$, therefore $\mathbb{P}^*$ is a risk-neutral probability measure, and we obviously have $\mathbb{P} = \mathbb{P}^*$ when $\mu = r$.

In the sequel, we consider probability measures $\mathbb{Q}$ that are *equivalent* to $\mathbb{P}$ in the sense that they share the same events of zero probability.

Definition 3.2 A probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when

$$\mathbb{Q}(A) = 0 \quad \text{if and only if} \quad \mathbb{P}(A) = 0, \quad \text{for all} \quad A \in \mathscr{F}.$$

Note that when $\mathbb{Q}$ is defined by (3.8), it is *equivalent* to $\mathbb{P}$ if and only if $F > 0$ with $\mathbb{P}$-probability one.

3.4 Pricing by the Martingale Method

In this section we give the expression of the Black-Scholes price using expectations of discounted payoffs.

Recall that according to the first fundamental theorem of asset pricing Theorem 1.2, a continuous market is without arbitrage opportunities if and only if there exists (at least) an equivalent risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$\widetilde{S}_t := e^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be *complete*.

The equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+,$$

satisfied by the price process $(S_t)_{t\in\mathbb{R}_+}$ can be rewritten using (3.12) as

$$\frac{dS_t}{S_t} = r dt + \sigma d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \tag{3.15}$$

with the solution

$$S_t = S_0\,\mathrm{e}^{\mu t+\sigma B_t-\sigma^2 t/2} = S_0\,\mathrm{e}^{rt+\sigma \widehat{B}_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

By the discounting Lemma 1.6, we have

$$\begin{aligned} d\widetilde{S}_t &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t \\ &= \sigma \widetilde{S}_t \left(\frac{\mu - r}{\sigma} dt + dB_t \right) \\ &= \sigma \widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{3.16}$$

hence the discounted price process

$$\begin{aligned} \widetilde{S}_t &:= \mathrm{e}^{-rt} S_t \\ &= S_0\,\mathrm{e}^{(\mu-r)t+\sigma B_t-\sigma^2 t/2} \\ &= S_0\,\mathrm{e}^{\sigma \widehat{B}_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \end{aligned}$$

is a martingale under the probability measure $\mathbb{P}^*$ defined by (3.13). We note that $\mathbb{P}^*$ is a risk-neutral probability measure equivalent to $\mathbb{P}$, also called martingale measure, whose existence and uniqueness ensure absence of arbitrage and completeness according to Theorems 1.2 and 1.4.

Therefore, by Lemma 1.7 the discounted value $\widetilde{V}_t$ of a self-financing portfolio can be written as

$$\begin{aligned} \widetilde{V}_t &= \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u \\ &= \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+, \end{aligned}$$

and by Proposition 3.1 it becomes a martingale under $\mathbb{P}^*$.

The value V_t at time t of a self-financing portfolio strategy $(\xi_t)_{t\in[0,T]}$ hedging an attainable claim payoff C will be called an *arbitrage price* of the claim payoff C at time t and denoted by $\pi_t(C)$, $t \in [0,T]$. Arbitrage prices can be used to ensure that financial derivatives are "marked" at their fair value ("mark to market").

Theorem 3.3 Let $(\xi_t, \eta_t)_{t\in[0,T]}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in [0,T],$$

and let C be a contingent claim payoff, such that

(i) $(\xi_t, \eta_t)_{t\in[0,T]}$ is a self-financing portfolio, and

(ii) $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, *i.e.* we have $V_T = C$.

Then the arbitrage price of the claim payoff C is given by the portfolio value

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{3.17}$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral probability measure $\mathbb{P}^*$.

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing, by Lemma 1.7 and (3.16) the discounted portfolio value $\widetilde{V}_t = \mathrm{e}^{-rt} V_t$ satisfies

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 3.1, hence

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\big[\widetilde{V}_T \mid \mathscr{F}_t\big] \\ &= \mathrm{e}^{-rT}\,\mathbb{E}^*[V_T \mid \mathscr{F}_t] \\ &= \mathrm{e}^{-rT}\,\mathbb{E}^*[C \mid \mathscr{F}_t],\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

□

Black-Scholes PDE for vanilla options by the martingale method

The martingale method can be used to recover the Black-Scholes PDE of Proposition 2.1. As the process $(S_t)_{t\in\mathbb{R}_+}$ has the Markov property, see § V-6 of Protter, 2004 and Definition 3.3 below, the value

$$\begin{aligned}V_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid \mathscr{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t], \quad 0 \leqslant t \leqslant T,\end{aligned}$$

of the portfolio at time $t \in [0,T]$ can be written from (3.17) as a function

$$V_t = C(t,S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t] \tag{3.18}$$

of t and S_t, $0 \leqslant t \leqslant T$.

Proposition 3.4 Assume that ϕ is a Lipschitz payoff function, and that

$$(S_t)_{t\in\mathbb{R}_+} = \big(S_0\,\mathrm{e}^{\sigma B_t + (r-\sigma^2)t/2}\big)_{t\in\mathbb{R}_+}$$

is a geometric Brownian motion. Then the function $C(t,x)$ defined in (3.18) is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R}_+)$ and solves the Black-Scholes PDE

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x) + \dfrac{1}{2}x^2\sigma^2\dfrac{\partial^2 C}{\partial x^2}(t,x) \\[2ex] C(T,x) = \phi(x), \quad x > 0. \end{cases}$$

Proof. It can be checked by integrations by parts that the function $C(t,x)$ defined by

$$C(t,S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T) \mid S_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(xS_T/S_t)]_{x=S_t},$$

$0 \leqslant t \leqslant T$, is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R}_+)$ when ϕ is a Lipschitz function, from the properties of the lognormal distribution of S_T. We note that the application of Itô's formula to $V_t = C(t,S_t)$ and (3.15) leads to

$$\begin{aligned} d(\mathrm{e}^{-rt}C(t,S_t)) &= -r\mathrm{e}^{-rt}C(t,S_t)dt\,\mathrm{e}^{-rt}dC(t,S_t) \\ &= -r\mathrm{e}^{-rt}C(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt \\ &\quad + \mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dS_t + \frac{1}{2}\mathrm{e}^{-rt}(dS_t)^2\frac{\partial^2 C}{\partial x^2}(t,S_t) \\ &= -r\mathrm{e}^{-rt}C(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt \end{aligned}$$

$$
\begin{aligned}
& +v_t \,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dt + u_t \,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)d\widehat{B}_t + \frac{1}{2}\,\mathrm{e}^{-rt}|u_t|^2\frac{\partial^2 C}{\partial x^2}(t,S_t)dt \\
= \; & -r\,\mathrm{e}^{-rt}C(t,S_t)dt + \mathrm{e}^{-rt}\frac{\partial C}{\partial t}(t,S_t)dt \\
& +rS_t\,\mathrm{e}^{-rt}\frac{\partial C}{\partial x}(t,S_t)dt + \frac{1}{2}\,\mathrm{e}^{-rt}\sigma^2 S_t^2\frac{\partial^2 C}{\partial x^2}(t,S_t)dt + \sigma\,\mathrm{e}^{-rt}S_t\frac{\partial C}{\partial x}(t,S_t)d\widehat{B}_t.
\end{aligned}
$$

By Lemma 1.7 and Proposition 3.1, the discounted price $\widetilde{V}_t = \mathrm{e}^{-rt}C(t,S_t)$ of a self-financing hedging portfolio is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, therefore from *e.g.* Corollary II-6-1, page 72 of Protter, 2004, all terms in dt should vanish in the above expression of

$$
d(\mathrm{e}^{-rt}g(t,S_t)) = -r\,\mathrm{e}^{-rt}g(t,S_t)dt + \mathrm{e}^{-rt}dg(t,S_t),
$$

which shows that

$$
-rC(t,S_t) + \frac{\partial C}{\partial t}(t,S_t) + rS_t\frac{\partial C}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial x^2}(t,S_t) = 0,
$$

and leads to the Black-Scholes PDE

$$
rC(t,x) = \frac{\partial C}{\partial t}(t,x) + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 C}{\partial x^2}(t,x), \qquad x > 0.
$$

$\square$

Forward contracts

The long forward contract with payoff $C = S_T - K$ is priced as

$$
\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\
&= S_t - K\,\mathrm{e}^{-(T-t)r}, \quad 0 \leqslant t \leqslant T,
\end{aligned}
$$

which recovers the Black-Scholes PDE solution (2.8), *i.e.*

$$
g(t,x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad x > 0, \quad t \in [0,T].
$$

European call options

In the case of European call options with payoff function $\phi(x) = (x-K)^+$ we recover the Black-Scholes formula (2.10), cf. Proposition 2.11, by a probabilistic argument.

Proposition 3.5 The price at time $t \in [0,T]$ of the European call option with strike price K and maturity T is given by

$$
\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] \qquad (3.19)\\
&= S_t\Phi\big(d_+(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \quad 0 \leqslant t \leqslant T,
\end{aligned}
$$

with

$$
\begin{cases}
d_+(T-t) := \dfrac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\[2ex]
d_-(T-t) := \dfrac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T,
\end{cases}
$$

where "log" denotes the *natural logarithm* "ln" and Φ is the standard Gaussian Cumulative Distribution Function.

Proof. The proof of Proposition 3.5 is a consequence of (3.17) and Lemma 3.6 below. Using the relation

$$S_T = S_t \, \mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T,$$

by Theorem 3.3 the value at time $t \in [0,T]$ of the portfolio hedging C is given by

$$\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\mathbf{E}^*[C \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(S_T-K)^+ \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(S_t \, \mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K)^+ \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(x \, \mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(\mathrm{e}^{m(x)+X}-K)^+\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := (\widehat{B}_T - \widehat{B}_t)\sigma \simeq \mathscr{N}(0,(T-t)\sigma^2)$$

is a centered Gaussian random variable with variance

$$\mathrm{Var}[X] = \mathrm{Var}\left[(\widehat{B}_T-\widehat{B}_t)\sigma\right] = \sigma^2\mathrm{Var}\left[\widehat{B}_T-\widehat{B}_t\right] = (T-t)\sigma^2$$

under $\mathbb{P}^*$. Hence by Lemma 3.6 below we have

$$\begin{aligned}
C(t,S_t) &= V_t \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(\mathrm{e}^{m(x)+X}-K)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\mathrm{e}^{m(S_t)+\sigma^2(T-t)/2}\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) \\
&\quad -K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\
&= S_t\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) - K\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\
&= S_t\Phi(d_+(T-t)) - K\,\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)),
\end{aligned}$$

$0 \leqslant t \leqslant T$. □

Relation (3.19) can also be written as

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(S_T-K)^+ \mid \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r}\mathbf{E}^*\left[(S_T-K)^+ \mid S_t\right] \qquad (3.20)\\
= S_t\Phi&\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
-K\,\mathrm{e}^{-(T-t)r}\Phi&\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Lemma 3.6 Let $X \simeq \mathscr{N}(0,v^2)$ be a centered Gaussian random variable with variance $v^2 > 0$. We have

$$\mathbf{E}\left[(\mathrm{e}^{m+X}-K)^+\right] = \mathrm{e}^{m+v^2/2}\Phi(v+(m-\log K)/v) - K\Phi((m-\log K)/v).$$

Proof. We have

$$
\begin{aligned}
\mathbb{E}\left[(\mathrm{e}^{m+X}-K)^+\right] &= \frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}(\mathrm{e}^{m+x}-K)^+\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}(\mathrm{e}^{m+x}-K)\,\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{e^m}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{x-x^2/(2v^2)}dx-\frac{K}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-x^2/(2v^2)}dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-(v^2-x)^2/(2v^2)}dx-\frac{K}{\sqrt{2\pi}}\int_{(-m+\log K)/v}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}}\int_{-v^2-m+\log K}^{\infty}\mathrm{e}^{-y^2/(2v^2)}dy-K\Phi((m-\log K)/v) \\
&= \mathrm{e}^{m+v^2/2}\Phi(v+(m-\log K)/v)-K\Phi((m-\log K)/v).
\end{aligned}
$$

□

Call-put parity

Let

$$P(t,S_t):=\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mid\mathcal{F}_t\right],\qquad 0\leqslant t\leqslant T,$$

denote the price of the put option with strike price K and maturity T.

Proposition 3.7 Call-put parity. We have the relation

$$C(t,S_t)-P(t,S_t)=S_t-\mathrm{e}^{-(T-t)r}K \tag{3.21}$$

between the Black-Scholes prices of call and put options, in terms of the forward contract price $S_t-K\,\mathrm{e}^{-(T-t)r}$.

Proof. From Theorem 3.3, we have

$$
\begin{aligned}
&C(t,S_t)-P(t,S_t) \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+\mid\mathcal{F}_t]-\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K-S_T)^+\mid\mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+-(K-S_T)^+\mid\mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T-K\mid\mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T\mid\mathcal{F}_t]-K\,\mathrm{e}^{-(T-t)r} \\
&= S_t-\mathrm{e}^{-(T-t)r}K,\qquad 0\leqslant t\leqslant T,
\end{aligned}
$$

as we have $\mathbb{E}^*[S_T\mid\mathcal{F}_t]=\mathrm{e}^{-(T-t)r}S_t$, $t\in[0,T]$, under the risk-neutral probability measure $\mathbb{P}^*$.

□

European put options

Using the *call-put parity* Relation (3.21) we can recover the European put option price (2.10) from the European call option price (2.10)-(3.19).

Proposition 3.8 The price at time $t\in[0,T]$ of the European put option with strike price K and maturity T is given by

$$\begin{aligned}P(t,S_t) &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathscr{F}_t] \\ &= K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - S_t\Phi\big(-d_+(T-t)\big), \quad 0 \leqslant t \leqslant T,\end{aligned}$$

with

$$\begin{cases} d_+(T-t) := \dfrac{\log(x/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \\ d_-(T-t) := \dfrac{\log(x/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad 0 \leqslant t < T, \end{cases}$$

where "log" denotes the *natural logarithm* "ln" and Φ is the standard Gaussian Cumulative Distribution Function.

Proof. By the *call-put parity* (3.21), we have

$$\begin{aligned}P(t,S_t) &= C(t,S_t) - S_t + \mathrm{e}^{-(T-t)r}K \\ &= S_t\Phi\big(d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K - S_t - \mathrm{e}^{-(T-t)r}K\Phi\big(d_-(T-t)\big) \\ &= -S_t\big(1-\Phi\big(d_+(T-t)\big)\big) + \mathrm{e}^{-(T-t)r}K\big(1-\Phi\big(d_-(T-t)\big)\big) \\ &= -S_t\Phi\big(-d_+(T-t)\big) + \mathrm{e}^{-(T-t)r}K\Phi\big(-d_-(T-t)\big).\end{aligned}$$

□

3.5 Hedging by the Martingale Method

Hedging exotic options

In the next Proposition 3.9 we compute a self-financing hedging strategy leading to an arbitrary square-integrable random claim payoff $C \in L^2(\Omega)$ of an exotic option admitting a stochastic integral decomposition of the form

$$C = \mathbb{E}^*[C] + \int_0^T \zeta_t d\widehat{B}_t, \tag{3.22}$$

where $(\zeta_t)_{t\in[0,t]}$ is a square-integrable adapted process. Consequently, the mathematical problem of finding the stochastic integral decomposition (3.22) of a given random variable has important applications in finance. The process $(\zeta_t)_{t\in[0,T]}$ can be computed using the Malliavin gradient on the Wiener space, see e.g. Nunno, Øksendal, and Proske, 2009 or § 8.2 of Privault, 2009.

Simple examples of stochastic integral decompositions include the relations

$$(B_T)^2 = T + 2\int_0^T B_t dB_t,$$

cf. Exercise 3.1, and

$$(B_T)^3 = 3\int_0^T (T-t+B_t^2)dB_t.$$

In the sequel, recall that the risky asset follows the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0,$$

and by (3.14), the discounted asset price $\widetilde{S}_t := \mathrm{e}^{-rt}S_t$

$$d\widetilde{S}_t = \sigma\widetilde{S}_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+, \qquad \widetilde{S}_0 = S_0 > 0, \tag{3.23}$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. The following proposition applies to arbitrary square-integrable payoff functions, in particular it covers exotic and path-dependent options.

Proposition 3.9 Consider a random claim payoff $C \in L^2(\Omega)$ and the process $(\zeta_t)_{t\in[0,T]}$ given by (3.22), and let

$$\xi_t = \frac{\mathrm{e}^{-(T-t)r}}{\sigma S_t}\zeta_t, \tag{3.24}$$

$$\eta_t = \frac{\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathscr{F}_t] - \xi_t S_t}{A_t}, \qquad 0 \leqslant t \leqslant T. \tag{3.25}$$

Then the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing, and letting

$$V_t = \eta_t A_t + \xi_t S_t, \qquad 0 \leqslant t \leqslant T, \tag{3.26}$$

we have

$$V_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{3.27}$$

In particular we have

$$V_T = C, \tag{3.28}$$

i.e. the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ yields a hedging strategy leading to the claim payoff C at maturity, after starting from the initial value

$$V_0 = \mathrm{e}^{-rT}\,\mathbb{E}^*[C].$$

Proof. Relation (3.27) follows from (3.25) and (3.26), and it implies

$$V_0 = \mathrm{e}^{-rT}\,\mathbb{E}^*[C] = \eta_0 A_0 + \xi_0 S_0$$

at $t = 0$, and (3.28) at $t = T$. It remains to show that the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing. By (3.22) and Proposition 3.1 we have

$$\begin{aligned}
V_t &= \eta_t A_t + \xi_t S_t \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathbb{E}^*[C] + \int_0^T \zeta_u d\widehat{B}_u \;\Big|\; \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\left(\mathbb{E}^*[C] + \int_0^t \zeta_u d\widehat{B}_u\right) \\
&= \mathrm{e}^{rt} V_0 + \mathrm{e}^{-(T-t)r}\int_0^t \zeta_u d\widehat{B}_u \\
&= \mathrm{e}^{rt} V_0 + \sigma \int_0^t \xi_u S_u \mathrm{e}^{(t-u)r} d\widehat{B}_u \\
&= \mathrm{e}^{rt} V_0 + \sigma\,\mathrm{e}^{rt} \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u.
\end{aligned}$$

By (3.23) this shows that the portfolio strategy $(\xi_t, \eta_t)_{t\in[0,T]}$ given by (3.24)-(3.25) and its discounted portfolio value $\widetilde{V}_t := \mathrm{e}^{-rt} V_t$ satisfy

$$\widetilde{V}_t = V_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad 0 \leqslant t \leqslant T,$$

which implies that $(\xi_t, \eta_t)_{t\in[0,T]}$ is self-financing by Lemma 1.7. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}^*[C]\,\mathrm{e}^{-rT}.$$

In addition, since there exists a hedging strategy leading to

$$\widetilde{V}_T = \mathrm{e}^{-rT} C,$$

then $(\widetilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale, with

$$\widetilde{V}_t = \mathbb{E}^*\big[\widetilde{V}_T \mid \mathscr{F}_t\big] = \mathrm{e}^{-rT}\,\mathbb{E}^*[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T,$$

and initial value

$$\widetilde{V}_0 = \mathbb{E}^*\big[\widetilde{V}_T\big] = \mathrm{e}^{-rT}\,\mathbb{E}^*[C].$$

Hedging vanilla options

In practice, the hedging problem can now be reduced to the computation of the process $(\zeta_t)_{t\in[0,T]}$ appearing in (3.22). This computation, called Delta hedging, can be performed by the application of the Itô formula and the Markov property, see *e.g.* Protter, 2001. The next lemma allows us to compute the process $(\zeta_t)_{t\in[0,T]}$ in case the payoff C is of the form $C = \phi(S_T)$ for some function ϕ.

Lemma 3.10 Assume that ϕ is a Lipschitz payoff function. Then the function $C(t,x)$ defined by

$$C(t,S_t) = \mathbb{E}^*[\phi(S_T) \mid S_t]$$

is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R})$, and the stochastic integral decomposition

$$\phi(S_T) = \mathbb{E}^*\big[\phi(S_T)\big] + \int_0^T \zeta_t\, d\widehat{B}_t \tag{3.29}$$

is given by

$$\zeta_t = \sigma S_t \frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{3.30}$$

In addition, the hedging strategy $(\xi_t)_{t\in[0,T]}$ satisfies

$$\xi_t = \mathrm{e}^{-(T-t)r}\frac{\partial C}{\partial x}(t,S_t), \qquad 0 \leqslant t \leqslant T. \tag{3.31}$$

Proof. It can be checked as in the proof of Proposition 3.4 the function $C(t,x)$ is in $\mathscr{C}^{1,2}([0,T)\times\mathbb{R})$. Therefore, we can apply the Itô formula to the process

$$t \longmapsto C(t,S_t) = \mathbb{E}^*[\phi(S_T) \mid \mathscr{F}_t],$$

which is a martingale from the tower property of conditional expectations as in (3.38). From the fact that the finite variation term in the Itô formula vanishes when $(C(t,S_t))_{t\in[0,T]}$ is a martingale, (see *e.g.* Corollary II-6-1 page 72 of Protter, 2004), we obtain:

$$C(t,S_t) = C(0,S_0) + \sigma \int_0^t S_u \frac{\partial C}{\partial x}(u,S_u) d\widehat{B}_u, \quad 0 \leqslant t \leqslant T, \tag{3.32}$$

with $C(0,S_0) = \mathbb{E}^*\left[\phi(S_T)\right]$. Letting $t = T$, we obtain (3.30) by uniqueness of the stochastic integral decomposition (3.29) of $C = \phi(S_T)$. Finally, (3.31) follows from (3.24) and (3.30). $\square$

By (3.39) we also have

$$\begin{aligned}
\zeta_t &= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^*[\phi(S_T) \mid S_t = x]_{x=S_t} \\
&= \sigma S_t \frac{\partial}{\partial x} \mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

hence

$$\begin{aligned}
\xi_t &= \frac{1}{\sigma S_t} \mathrm{e}^{-(T-t)r} \zeta_t \qquad (3.33)\\
&= \mathrm{e}^{-(T-t)r} \frac{\partial}{\partial x} \mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

which recovers the formula (2.3) for the Delta of a vanilla option. As a consequence we have $\xi_t \geqslant 0$ and there is no short selling when the payoff function ϕ is non-decreasing.

In the case of European options, the process ζ can be computed via the next proposition which follows from Lemma 3.10 and the relation

$$C(t,x) = \mathbb{E}^*\left[f(S^x_{t,T})\right], \qquad 0 \leqslant t \leqslant T, \ x > 0.$$

Corollary 3.11 Assume that $C = (S_T - K)^+$. Then, for $0 \leqslant t \leqslant T$ we have

$$\zeta_t = \sigma S_t \mathbb{E}^*\left[\frac{S_T}{S_t} \mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T, \tag{3.34}$$

and

$$\xi_t = \mathrm{e}^{-(T-t)r} \mathbb{E}^*\left[\frac{S_T}{S_t} \mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}, \qquad 0 \leqslant t \leqslant T. \tag{3.35}$$

By evaluating the expectation (3.34) in Corollary 3.11 we can recover the formula (2.15) in Proposition 2.4 for the Delta of the European call option in the Black-Scholes model. In that sense, the next proposition provides another proof of the result of Proposition 2.4.

Proposition 3.12 The Delta of the European call option with payoff function $f(x) = (x-K)^+$ is given by

$$\xi_t = \Phi(d_+(T-t)) = \Phi\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \quad 0 \leqslant t \leqslant T.$$

Proof. By Proposition 3.9 and Corollary 3.11, we have

$$\xi_t = \frac{1}{\sigma S_t} \mathrm{e}^{-(T-t)r} \zeta_t$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\frac{S_T}{S_t}\mathbb{1}_{[K,\infty)}\left(x\frac{S_T}{S_t}\right)\right]_{x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\\
&\quad\times\mathbb{E}^*\left[\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\mathbb{1}_{[K,\infty)}\big(x\,\mathrm{e}^{(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2+(T-t)r}\big)\right]_{x=S_t}\\
&= \frac{1}{\sqrt{2(T-t)\pi}}\int_{(T-t)\sigma/2-(T-t)r/\sigma+\sigma^{-1}\log(K/S_t)}^{\infty}\mathrm{e}^{\sigma y-(T-t)\sigma^2/2-y^2/(2(T-t))}dy\\
&= \frac{1}{\sqrt{2(T-t)\pi}}\int_{-d_-(T-t)/\sqrt{T-t}}^{\infty}\mathrm{e}^{-(y-(T-t)\sigma)^2/(2(T-t))}dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_-(T-t)}^{\infty}\mathrm{e}^{-(y-(T-t)\sigma)^2/2}dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_+(T-t)}^{\infty}\mathrm{e}^{-y^2/2}dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+(T-t)}\mathrm{e}^{-y^2/2}dy\\
&= \Phi\big(d_+(T-t)\big).
\end{aligned}
$$

□

Proposition 3.12, combined with Proposition 3.5, shows that the Black-Scholes self-financing hedging strategy is to hold a (possibly fractional) quantity

$$
\xi_t = \Phi\big(d_+(T-t)\big) = \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{3.36}
$$

of the risky asset, and to borrow a quantity

$$
-\eta_t = K\,\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_t/K)+(r-\sigma_t^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \geqslant 0 \tag{3.37}
$$

of the riskless (savings) account.

As noted above, the result of Proposition 3.12 recovers (2.16) which is obtained by a direct differentiation of the Black-Scholes function as in (2.3) or (3.33).

Markovian semi-group

Definition 3.3 The Markov semi-group $(P_t)_{0\leqslant t\leqslant T}$ associated to $(S_t)_{t\in[0,T]}$ is the mapping P_t defined on functions $f\in\mathscr{C}_b^2(\mathbb{R})$ as

$$
P_t f(x) := \mathbb{E}^*[f(S_t) \mid S_0 = x], \quad t\in\mathbb{R}_+.
$$

By the Markov property and time homogeneity of $(S_t)_{t\in[0,T]}$ we also have

$$
P_t f(S_u) := \mathbb{E}^*[f(S_{t+u}) \mid \mathscr{F}_u] = \mathbb{E}^*[f(S_{t+u}) \mid S_u], \quad t,u\in\mathbb{R}_+,
$$

and the semi-group $(P_t)_{0\leqslant t\leqslant T}$ satisfies the composition property

$$
P_sP_t = P_tP_s = P_{s+t} = P_{t+s}, \qquad s,t\in\mathbb{R}_+,
$$

as we have, using the Markov property and the tower property of conditional expectations as in (3.38),

$$
P_sP_tf(x) \quad = \quad \mathbb{E}^*\left[P_tf(S_s) \mid S_0 = x\right]
$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_t)\mid S_0=y\right]_{y=S_s}\mid S_0=x\right]\\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_{t+s})\mid S_s=y\right]_{y=S_s}\mid S_0=x\right]\\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[f(S_{t+s})\mid \mathscr{F}_s\right]\mid S_0=x\right]\\
&= \mathbb{E}^*\left[f(S_{t+s})\mid \mathscr{F}_s \mid S_0=x\right]\\
&= P_{t+s}f(x), \qquad s,t\geqslant 0.
\end{aligned}
$$

Similarly we can show that the process $(P_{T-t}f(S_t))_{t\in[0,T]}$ is an $\mathscr{F}_t$-martingale as in Example (3.1), *i.e.*:

$$
\begin{aligned}
\mathbb{E}^*\left[P_{T-t}f(S_t)\mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathbb{E}^*[f(S_T)\mid \mathscr{F}_t]\mid \mathscr{F}_u\right]\\
&= \mathbb{E}^*\left[f(S_T)\mid \mathscr{F}_u\right]\\
&= P_{T-u}f(S_u), \qquad 0\leqslant u\leqslant t\leqslant T,
\end{aligned}
\tag{3.38}
$$

and we have

$$
P_{t-u}f(x) = \mathbb{E}^*\left[f(S_t)\mid S_u=x\right] = \mathbb{E}^*\left[f\left(x\frac{S_t}{S_u}\right)\right], \quad 0\leqslant u\leqslant t. \tag{3.39}
$$

Exercises

Exercise 3.1 (Exercise 2.1 continued). Consider a market made of a riskless asset priced $A_t = A_0$ with zero interest rate, $t\in\mathbb{R}_+$, and a risky asset whose price modeled by a standard Brownian motion as $S_t = B_t$, $t\in\mathbb{R}_+$. Price the vanilla option with payoff $C=(B_T)^2$, and deduce the solution of the Black-Scholes PDE of Exercise 2.1.

Exercise 3.2 Given the price process $(S_t)_{t\in\mathbb{R}_+}$ defined as

$$
S_t := S_0\,\mathrm{e}^{\sigma B_t+(r-\sigma^2/2)t}, \qquad t\in\mathbb{R}_+,
$$

price the option with payoff function $\phi(S_T)$ by writing $\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\phi(S_T)\right]$ as an integral.

Exercise 3.3 Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ which is a martingale under the risk-neutral probability measure $\mathbb{P}^*$ in a market with interest rate $r=0$, and let $\phi(x)=(x-K)^+$ be the (convex) European call payoff function.

Show that, for any two maturities $T_1<T_2$ and $p,q\in[0,1]$ such that $p+q=1$, the price of the option on average with payoff $\phi(pS_{T_1}+qS_{T_2})$ is upper bounded by the price of the European call option with maturity T_2, *i.e.* show that

$$
\mathbb{E}^*\left[\phi(pS_{T_1}+qS_{T_2})\right]\leqslant \mathbb{E}^*\left[\phi(S_{T_2})\right].
$$

Hints:

i) For ϕ a convex function we have $\phi(px+qy)\leqslant p\phi(x)+q\phi(y)$ for any $x,y\in\mathbb{R}$ and $p,q\in[0,1]$ such that $p+q=1$.
ii) Any convex function $\phi(S_t)$ of a martingale S_t is a *sub*martingale.

Exercise 3.4 Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ and a risk-neutral measure $\mathbb{P}^*$.

a) Does the European *call* option price $C(K):=\mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ increase or decrease with the strike price K? Justify your answer.

b) Does the European *put* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(K-S_T)^+]$ increase or decrease with the strike price K? Justify your answer.

Exercise 3.5 Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$.

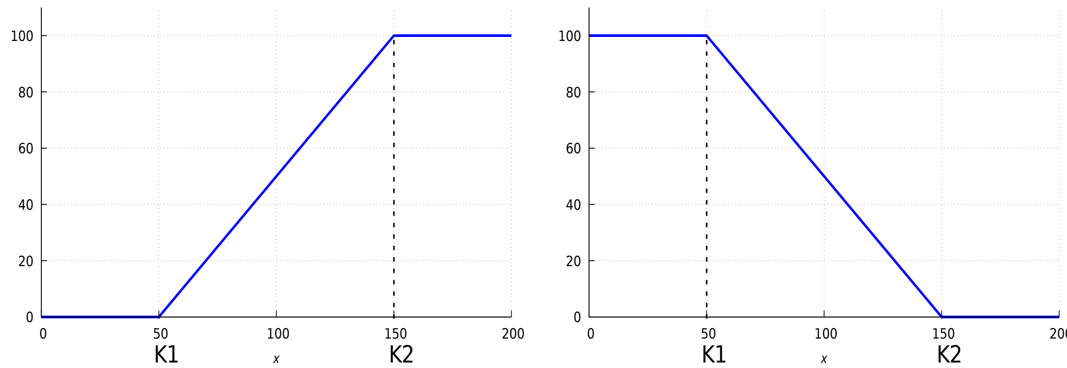
a) Show that the price at time t of the European call option with strike price K and maturity T is lower bounded by the positive part $\left(S_t - K\,\mathrm{e}^{-(T-t)r}\right)^+$ of the corresponding forward contract price, *i.e.*

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+ \mid \mathcal{F}_t\right] \geqslant \left(S_t - K\,\mathrm{e}^{-(T-t)r}\right)^+, \quad 0 \leqslant t \leqslant T.$$

b) Show that the price at time t of the European put option with strike price K and maturity T is lower bounded by $K\,\mathrm{e}^{-(T-t)r} - S_t$, *i.e.*

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+ \mid \mathcal{F}_t\right] \geqslant \left(K\,\mathrm{e}^{-(T-t)r} - S_t\right)^+, \quad 0 \leqslant t \leqslant T.$$

Exercise 3.6 The following two graphs describe the payoff functions ϕ of *bull spread* and *bear spread* options with payoff $\phi(S_N)$ on an underlying asset priced S_N at maturity time N.



(i) Bull spread payoff. (ii) Bear spread payoff.

Figure 3.3: Payoff functions of bull spread and bear spread options.

a) Show that in each case (i) and (ii) the corresponding option can be realized by purchasing and/or short selling standard European call and put options with strike prices to be specified.

b) Price the bull spread option in cases (i) and (ii).

Hint: An option with payoff $\phi(S_N)$ is priced $(1+r)^{-N}\,\mathbb{E}^*\left[\phi(S_N)\right]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 3.7 Butterfly options. A butterfly option is designed to deliver a limited payoff when the future volatility of the underlying asset is expected to be low. The payoff function of a butterfly option is plotted in Figure 3.4 with $K_1 := 50$ and $K_2 := 150$.

a) Show that the butterfly option can be realized by purchasing and/or issuing standard European call or put options with strike prices to be specified.

b) Does the hedging strategy of the butterfly option involve holding or shorting the underlying stock?

Hints: Recall that an option with payoff $\phi(S_N)$ is priced in discrete time as $(1+r)^{-N}\,\mathbb{E}^*\left[\phi(S_N)\right]$ at time 0. The payoff of the European call (resp. put) option with strike price K is $(S_N-K)^+$, resp. $(K-S_N)^+$.

Exercise 3.8 Forward contracts revisited. Consider a risky asset whose price S_t is given by $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. Consider a forward contract with maturity T and payoff $S_T - \kappa$.

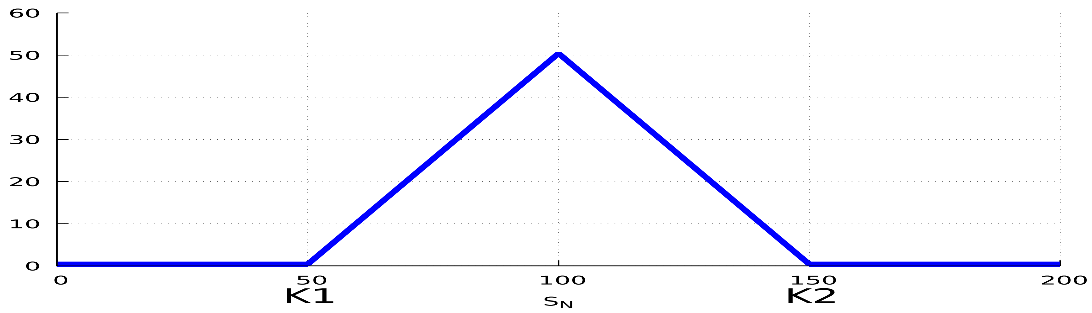


Figure 3.4: Butterfly payoff function.

a) Compute the price C_t of this claim at any time $t \in [0,T]$.

b) Compute a hedging strategy for the option with payoff $S_T - \kappa$.

Exercise 3.9 Option pricing with dividends (Exercise 2.3 continued). Consider an underlying asset price process $(S_t)_{t\in\mathbb{R}_+}$ paying dividends at the continuous-time rate $\delta > 0$, and modeled as

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

a) Show that as in Lemma 1.7, if $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+,$$

where the dividend yield δS_t per share is continuously reinvested in the portfolio, then the discounted portfolio value $\widetilde{V}_t$ can be written as the stochastic integral

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t \xi_u d\widetilde{S}_u, \qquad t \in \mathbb{R}_+,$$

b) Show that, as in Theorem 3.3, if $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, *i.e.* if $V_T = C$, then the arbitrage price of the claim payoff C is given by

$$\pi_t(C) = V_t = \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T,$$

where $\widehat{\mathbb{E}}$ denotes expectation under a suitably chosen risk-neutral probability measure $\widehat{\mathbb{P}}$.

c) Compute the price at time $t \in [0,T]$ of a European call option in a market with dividend rate δ by the martingale method.

Exercise 3.10 Forward start options (Rubinstein, 1991). A *forward start* European call option is an option whose holder receives at time T_1 (*e.g.* your birthday) the value of a standard European call option *at the money* and with maturity $T_2 > T_1$. Price this birthday present at any time $t \in [0,T_1]$, *i.e.* compute the price

$$\mathrm{e}^{-(T_1-t)r}\mathbb{E}^*\left[\mathrm{e}^{-(T_2-T_1)r}\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathcal{F}_{T_1}\right] \mid \mathcal{F}_t\right]$$

at time $t \in [0,T_1]$, of the *forward start* European call option using the Black-Scholes formula

$$\begin{aligned}\mathrm{Bl}(K,x,\sigma,r,T-t) &= x\Phi\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)\\ &\quad -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right),\end{aligned}$$

$0 \leqslant t < T$.

Exercise 3.11 Log-contracts. (Exercise 2.8 continued). Consider the price process $(S_t)_{t\in[0,T]}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$

and a riskless asset valued $A_t = A_0 \, \mathrm{e}^{rt}$, $t \in [0,T]$, with $r > 0$. Compute the arbitrage price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \, \mathbb{E}^* \left[\log S_T \mid \mathscr{F}_t \right],$$

at time $t \in [0,T]$, of the log-contract with payoff $\log S_T$.

Exercise 3.12 Bachelier, 1900 model (Exercise 2.10 continued).

a) Consider the solution $(S_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma dB_t.$$

For which value α_M of α is the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, a martingale under $\mathbb{P}$?

b) For each value of α, build a probability measure $\mathbb{P}_\alpha$ under which the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$, $0 \leqslant t \leqslant T$, is a martingale.

c) Compute the arbitrage price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \, \mathbb{E}_\alpha \left[\mathrm{e}^{S_T} \mid \mathscr{F}_t \right]$$

at time $t \in [0,T]$ of the contingent claim with payoff $\exp(S_T)$, and recover the result of Exercise 2.10.

d) Explicitly compute the portfolio strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ that hedges the contingent claim with payoff $\exp(S_T)$.

e) Check that this strategy is self-financing.

Exercise 3.13 Compute the arbitrage price

$$C(t,S_t) = \mathrm{e}^{-(T-t)r} \, \mathbb{E}_\alpha \left[(S_T)^2 \mid \mathscr{F}_t \right]$$

at time $t \in [0,T]$ of the power option with payoff $(S_T)^2$ in the framework of the Bachelier, 1900 model of Exercise 3.12.

Exercise 3.14 (Exercise 2.2 continued). Price the option with vanilla payoff $C = \phi(S_T)$ using the noncentral Chi square probability density function of the Cox, Ingersoll, and Ross, 1985 (CIR) model.

Exercise 3.15 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Recall that for $f \in \mathscr{C}^2(\mathbb{R}_+ \times \mathbb{R})$, Itô's formula for $(B_t)_{t\in\mathbb{R}_+}$ reads

$$\begin{aligned} f(t,B_t) &= f(0,B_0) + \int_0^t \frac{\partial f}{\partial s}(s,B_s)ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds. \end{aligned}$$

a) Let $r \in \mathbb{R}$, $\sigma > 0$, $f(x,t) = \mathrm{e}^{rt+\sigma x-\sigma^2 t/2}$, and $S_t = f(t,B_t)$. Compute $df(t,B_t)$ by Itô's formula, and show that S_t solves the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where $r > 0$ and $\sigma > 0$.

b) Show that

$$\mathbb{E}\left[\mathrm{e}^{\sigma B_T} \mid \mathscr{F}_t\right] = \mathrm{e}^{\sigma B_t+(T-t)\sigma^2/2}, \qquad 0 \leqslant t \leqslant T.$$

Hint: Use the independence of increments of $(B_t)_{t\in[0,T]}$ in the time splitting decomposition

$$B_T = (B_t - B_0) + (B_T - B_t),$$

and the Gaussian moment generating function $\mathbb{E}\left[\mathrm{e}^{\alpha X}\right] = \mathrm{e}^{\alpha^2\eta^2/2}$ when $X \simeq \mathscr{N}(0,\eta^2)$.

c) Show that the process $(S_t)_{t\in\mathbb{R}_+}$ satisfies

$$\mathbb{E}\left[S_T \mid \mathscr{F}_t\right] = \mathrm{e}^{(T-t)r} S_t, \qquad 0 \leqslant t \leqslant T.$$

d) Let $C = S_T - K$ denote the payoff of a forward contract with exercise price K and maturity T. Compute the discounted expected payoff

$$V_t := \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[C \mid \mathscr{F}_t\right].$$

e) Find a self-financing portfolio strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ such that

$$V_t = \xi_t S_t + \eta_t A_t, \qquad 0 \leqslant t \leqslant T,$$

where $A_t = A_0\,\mathrm{e}^{rt}$ is the price of a riskless asset with fixed interest rate $r > 0$. Show that it recovers the result of Exercise 2.6-(c)).

f) Show that the portfolio allocation $(\xi_t,\eta_t)_{t\in[0,T]}$ found in Question (e)) *hedges* the payoff $C = S_T - K$ at time T, *i.e.* show that $V_T = C$.

Exercise 3.16 Binary options. Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t, \qquad S_0 = 1,$$

under the risk-neutral probability measure $\mathbb{P}^*$. A binary (or digital) *call*, *resp. put*, option is a contract with maturity T, strike price K, and payoff

$$C_d := \begin{cases} \$1 & \text{if } S_T \geqslant K, \\ 0 & \text{if } S_T < K, \end{cases} \qquad resp. \quad P_d := \begin{cases} \$1 & \text{if } S_T \leqslant K, \\ 0 & \text{if } S_T > K. \end{cases}$$

Recall that the prices $\pi_t(C_d)$ and $\pi_t(P_d)$ at time t of the binary call and put options are given by the discounted expected payoffs

$$\pi_t(C_d) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[C_d \mid \mathscr{F}_t\right] \quad \text{and} \quad \pi_t(P_d) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[P_d \mid \mathscr{F}_t\right]. \tag{3.40}$$

a) Show that the payoffs C_d and P_d can be rewritten as

$$C_d = \mathbb{1}_{[K,\infty)}(S_T) \qquad \text{and} \qquad P_d = \mathbb{1}_{[0,K]}(S_T).$$

b) Using Relation (3.40), Question (a)), and the relation

$$\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \mid S_t = x\right] = \mathbb{P}^*(S_T \geqslant K \mid S_t = x),$$

show that the price $\pi_t(C_d)$ is given by

$$\pi_t(C_d) = C_d(t,S_t),$$

where $C_d(t,x)$ is the function defined by

$$C_d(t,x) := \mathrm{e}^{-(T-t)r}\mathbb{P}^*(S_T \geqslant K \mid S_t = x).$$

c) Using the results of Exercise 1.9-(d)) and of Question (b)), show that the price $\pi_t(C_d)$ of the binary call option is given by

$$\begin{aligned} C_d(t,x) &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned}$$

where

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) Assume that the binary option holder is entitled to receive a "return amount" $\alpha \in [0,1]$ in case the underlying asset price ends out of the money at maturity. Compute the price at time $t \in [0,T]$ of this modified contract.

e) Using Relation (3.40) and Question (a)), prove the call-put parity relation

$$\pi_t(C_d) + \pi_t(P_d) = \mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T. \tag{3.41}$$

If needed, you may use the fact that $\mathbb{P}^*(S_T = K) = 0$.

f) Using the results of Questions (e)) and (c)), show that the price $\pi_t(P_d)$ of the binary put option is given by

$$\pi_t(P_d) = \mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big).$$

g) Using the result of Question (c)), compute the Delta

$$\xi_t := \frac{\partial C_d}{\partial x}(t,S_t)$$

of the binary call option. Does the Black-Scholes hedging strategy of such a call option involve short selling? Why?

h) Using the result of Question (f)), compute the Delta

$$\xi_t := \frac{\partial P_d}{\partial x}(t,S_t)$$

of the binary put option. Does the Black-Scholes hedging strategy of such a put option involve short selling? Why?

Exercise 3.17 Computation of Greeks. Consider an underlying asset whose price $(S_t)_{t\in\mathbb{R}_+}$ is given by a stochastic differential equation of the form

$$dS_t = rS_t dt + \sigma(S_t)dW_t,$$

where $\sigma(x)$ is a Lipschitz coefficient, and an option with payoff function ϕ and price

$$C(x,T) = \mathrm{e}^{-rT}\,\mathbb{E}\big[\phi(S_T)\,\big|\,S_0 = x\big],$$

where $\phi(x)$ is a twice continuously differentiable ($\mathscr{C}^2$) function, with $S_0 = x$. Using the Itô formula, show that the sensitivity

$$\mathrm{Theta}_T = \frac{\partial}{\partial T}\left(\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi(S_T) \,\middle|\, S_0 = x \right] \right)$$

of the option price with respect to maturity T can be expressed as

$$\begin{aligned} \mathrm{Theta}_T &= -r\,\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi(S_T) \,\middle|\, S_0 = x \right] + r\,\mathrm{e}^{-rT}\, \mathbb{E}\left[S_t \phi'(S_T) \,\middle|\, S_0 = x \right] \\ &\quad + \frac{1}{2}\,\mathrm{e}^{-rT}\, \mathbb{E}\left[\phi''(S_T)\sigma^2(S_T) \,\middle|\, S_0 = x \right]. \end{aligned}$$

4. Stopping Times and Martingales

Stopping times are random times whose value can be determined by the historical behavior of a stochastic process modeling market data. This chapter presents additional material on optimal stopping and martingales, for use in the pricing and optimal exercise of American options in Chapter 5. Applications are given to hitting probabilities for Brownian motion.

4.1 Filtrations and Information Flow

Let $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ denote the *filtration* generated by a stochastic process $(X_t)_{t\in\mathbb{R}_+}$. In other words, $\mathscr{F}_t$ denotes the collection of all events possibly generated by $\{X_s \,:\, 0 \leqslant s \leqslant t\}$ up to time t. Examples of such events include the event

$$\{X_{t_0} \leqslant a_0,\ X_{t_1} \leqslant a_1,\ \ldots,\ X_{t_n} \leqslant a_n\}$$

for $a_0, a_1, \ldots, a_n$ a given fixed sequence of real numbers and $0 \leqslant t_1 < \cdots < t_n < t$, and $\mathscr{F}_t$ is said to represent the *information* generated by $(X_s)_{s\in[0,t]}$ up to time $t \geqslant 0$.

By construction, $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a *non-decreasing* family of σ-algebras in the sense that we have $\mathscr{F}_s \subset \mathscr{F}_t$ (information known at time s is contained in the information known at time t) when $0 < s < t$.

One refers sometimes to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ as the non-decreasing flow of information generated by $(X_t)_{t\in\mathbb{R}_+}$.

4.2 Submartingales and Supermartingales

Let us recall the definition of *martingale* (cf. Definition 1.5) and introduce in addition the definitions of *super*martingale and *sub*martingale.*

Definition 4.1 An integrable stochastic process $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale (resp. a *super*martingale, resp. a *sub*martingale) with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if it satisfies the property

$$Z_s = \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t,$$

resp.

$$Z_s \geqslant \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t,$$

resp.

$$Z_s \leqslant \mathbb{E}[Z_t \mid \mathcal{F}_s], \qquad 0 \leqslant s \leqslant t.$$

Clearly, a stochastic process $(Z_t)_{t\in\mathbb{R}_+}$ is a martingale if and only if it is both a *super*martingale and a *sub*martingale.

A particular property of martingales is that their expectation is constant over time $t \in \mathbb{R}_+$.

Proposition 4.1 Let $(Z_t)_{t\in\mathbb{R}_+}$ be a martingale. We have

$$\mathbb{E}[Z_t] = \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t.$$

The above proposition follows from the tower property of conditional expectations, which shows that

$$\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t \mid \mathcal{F}_s]] = \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t. \tag{4.1}$$

Similarly, a *super*martingale has a *non-increasing* expectation, while a *sub*martingale has a *non-decreasing* expectation.

Proposition 4.2 Let $(Z_t)_{t\in\mathbb{R}_+}$ be a *super*martingale, resp. a *sub*martingale. Then we have

$$\mathbb{E}[Z_t] \leqslant \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t,$$

resp.

$$\mathbb{E}[Z_t] \geqslant \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t.$$

Proof. As in (4.1) above we have

$$\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t \mid \mathcal{F}_s]] \leqslant \mathbb{E}[Z_s], \qquad 0 \leqslant s \leqslant t.$$

The proof is similar in the *sub*martingale case. □

*"This obviously inappropriate nomenclature was chosen under the malign influence of the noise level of radio's SUPERman program, a favorite supper-time program of Doob's son during the writing of Doob, 1953", cf. Doob, 1984, historical notes, page 808.

Independent increments processes whose increments have negative expectation give examples of *super*martingales. For example, if $(Z_t)_{t\in\mathbb{R}_+}$ is such a stochastic process then we have

$$\begin{aligned}
\mathbb{E}[Z_t \mid \mathscr{F}_s] &= \mathbb{E}[Z_s \mid \mathscr{F}_s] + \mathbb{E}\left[Z_t - Z_s \,\middle|\, \mathscr{F}_s\right] \\
&= \mathbb{E}[Z_s \mid \mathscr{F}_s] + \mathbb{E}[Z_t - Z_s] \\
&\leqslant \mathbb{E}[Z_s \mid \mathscr{F}_s] \\
&= Z_s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

Similarly, a stochastic process with independent increments which have positive expectation will be a *sub*martingale. Brownian motion $B_t + \mu t$ with positive drift $\mu > 0$ is such an example, as in Figure 4.1 below.

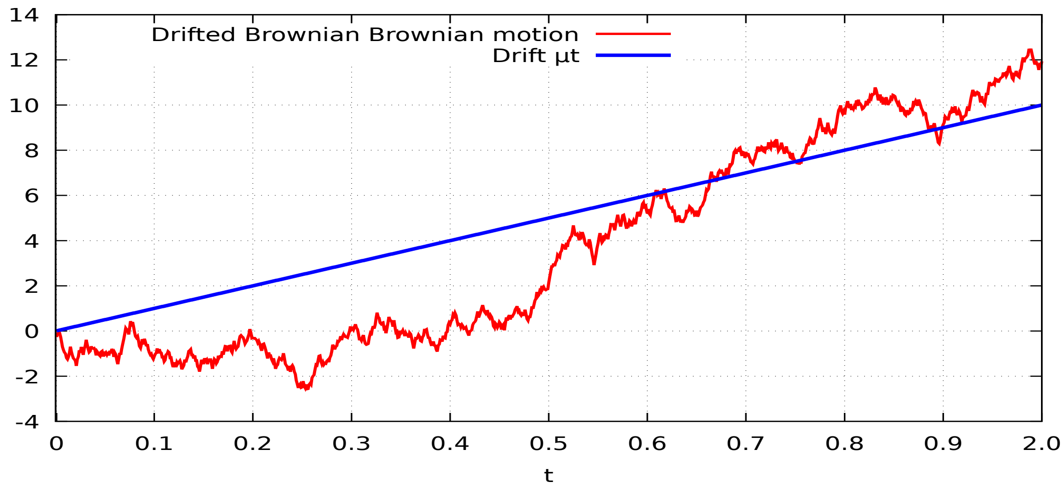


Figure 4.1: Drifted Brownian path.

The following example comes from gambling.

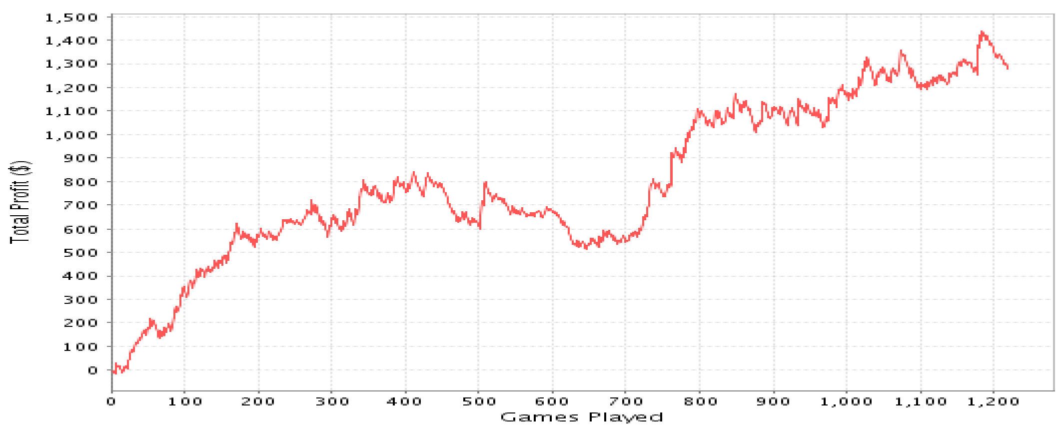


Figure 4.2: Evolution of the fortune of a poker player *vs* number of games played.

A natural way to construct *sub*martingales is to take convex functions of martingales.

Proposition 4.3 a) Given $(M_t)_{t\in\mathbb{R}_+}$ a martingale and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function, the process $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a *sub*martingale.
b) Given $(M_t)_{t\in\mathbb{R}_+}$ a *sub*martingale and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ a *non-decreasing* convex function, the process $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a *sub*martingale.

Proof. By Jensen's inequality we have

$$\phi(\mathbb{E}[M_t \mid \mathscr{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathscr{F}_s], \qquad 0 \leqslant s \leqslant t, \tag{4.2}$$

which shows that

$$\phi(M_s) = \phi(\mathbb{E}[M_t \mid \mathscr{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathscr{F}_s], \qquad 0 \leqslant s \leqslant t.$$

If ϕ is convex non-decreasing and $(M_t)_{\in\mathbb{R}_+}$ is a *sub*martingale, the above rewrites as

$$\phi(M_s) \leqslant \phi(\mathbb{E}[M_t \mid \mathscr{F}_s]) \leqslant \mathbb{E}[\phi(M_t) \mid \mathscr{F}_s], \qquad 0 \leqslant s \leqslant t,$$

showing that $(\phi(M_t))_{t\in\mathbb{R}_+}$ is a *sub*martingale. □

Similarly, $(\phi(M_t))_{t\in\mathbb{R}_+}$ will be a *super*martingale when $(M_t)_{\in\mathbb{R}_+}$ is a martingale and the function ϕ is *concave*.

Other examples of (super, sub)-martingales include geometric Brownian motion

$$S_t = S_0 \, \mathrm{e}^{rt+\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

which is a martingale for $r = 0$, a *super*martingale for $r \leqslant 0$, and a *sub*martingale for $r \geqslant 0$.

4.3 Stopping Times

Next, we turn to the definition of *stopping time*.

Definition 4.2 An $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time is a random variable $\tau : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\{\tau > t\} \in \mathscr{F}_t, \qquad t \in \mathbb{R}_+. \tag{4.3}$$

The meaning of Relation (4.3) is that the knowledge of the event $\{\tau > t\}$ depends only on the information present in $\mathscr{F}_t$ up to time t, *i.e.* on the knowledge of $(X_s)_{0\leqslant s\leqslant t}$.

In other words, an event occurs at a *stopping time* τ if at any time t it can be decided whether the event has already occurred ($\tau \leqslant t$) or not ($\tau > t$) based on the information $\mathscr{F}_t$ generated by $(X_s)_{s\in\mathbb{R}_+}$ up to time t.

For example, the day you bought your first car is a stopping time (one can always answer the question "did I ever buy a car"), whereas the day you will buy your *last* car may not be a stopping time (one may not be able to answer the question "will I ever buy another car").

Proposition 4.4 Every constant time is a stopping time. In addition, if τ and θ are stopping times, then
i) the minimum $\tau \wedge \theta := \min(\tau, \theta)$ of τ and θ is also a stopping time,
ii) the maximum $\tau \vee \theta := \mathrm{Max}(\tau, \theta)$ of τ and θ is also a stopping time.

Proof. Point (i) is easily checked. Regarding (ii), we have

$$\{\tau \wedge \theta > t\} = \{\tau > t \text{ and } \theta > t\} = \{\tau > t\} \cap \{\theta > t\} \in \mathscr{F}_t, \quad t \in \mathbb{R}_+.$$

On the other hand, we have

$$\{\tau \vee \theta \leqslant t\} = \{\tau \leqslant t \text{ and } \theta \leqslant t\} = \{\tau > t\}^c \cap \{\theta > t\}^c \in \mathscr{F}_t, \quad t \in \mathbb{R}_+,$$

which implies

$$\{\tau \vee \theta > t\} = \{\tau \vee \theta \leqslant t\}^c \in \mathscr{F}_t, \qquad t \in \mathbb{R}_+.$$

□

Hitting times

Hitting times provide natural examples of stopping times. The hitting time of level x by the process $(X_t)_{t\in\mathbb{R}_+}$, defined as

$$\tau_x = \inf\{t \in \mathbb{R}_+ \ : \ X_t = x\},$$

is a stopping time,* as we have (here in discrete time)

$$\begin{aligned} \{\tau_x > t\} &= \{X_s \neq x \text{ for all } s \in [0,t]\} \\ &= \{X_0 \neq x\} \cap \{X_1 \neq x\} \cap \cdots \cap \{X_t \neq x\} \in \mathscr{F}_t, \qquad t \in \mathbb{N}. \end{aligned}$$

In gambling, a hitting time can be used as an exit strategy from the game. For example, letting

$$\tau_{x,y} := \inf\{t \in \mathbb{R}_+ \ : \ X_t = x \text{ or } X_t = y\} \tag{4.4}$$

defines a hitting time (hence a stopping time) which allows a gambler to exit the game as soon as losses become equal to $x = -10$, or gains become equal to $y = +100$, whichever comes first. Hitting times can be used to trigger for “buy limit” or “sell stop” orders in finance.

However, not every $\mathbb{R}_+$-valued random variable is a stopping time. For example the random time

$$\tau = \inf\left\{ t \in [0,T] \ : \ X_t = \operatorname*{Sup}_{s\in[0,T]} X_s \right\},$$

which represents the first time the process $(X_t)_{t\in[0,T]}$ reaches its maximum over $[0,T]$, is not a stopping time with respect to the filtration generated by $(X_t)_{t\in[0,T]}$. Indeed, the information known at time $t \in (0,T)$ is not sufficient to determine whether $\{\tau > t\}$.

Stopped process

Given $(Z_t)_{t\in\mathbb{R}_+}$ a stochastic process and $\tau : \Omega \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ a stopping time, the *stopped process* $(Z_{t\wedge\tau})_{t\in\mathbb{R}_+}$ is defined as

$$Z_{t\wedge\tau} = \begin{cases} Z_t & \text{if } t < \tau, \\ \\ Z_\tau & \text{if } t \geqslant \tau, \end{cases}$$

Using indicator functions we may also write

$$Z_{t\wedge\tau} = Z_t \mathbb{1}_{\{t<\tau\}} + Z_\tau \mathbb{1}_{\{t\geqslant\tau\}}, \qquad t \in \mathbb{R}_+.$$

The following Figure 4.3 is an illustration of the path of a stopped process.

*As a convention we let $\tau = +\infty$ in case there exists no $t \geqslant 0$ such that $X_t = x$.

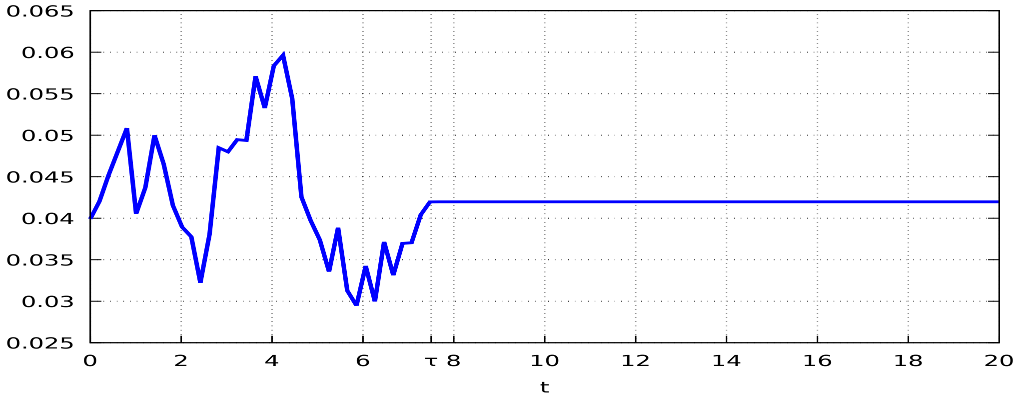


Figure 4.3: Stopped process.

Theorem 4.5 below is called the *Stopping Time* (or *Optional Sampling*, or *Optional Stopping*) Theorem, it is due to the mathematician J.L. Doob (1910-2004). It is also used in Exercise 4.6 below.

Theorem 4.5 Assume that $(M_t)_{t\in\mathbb{R}_+}$ is a martingale with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, and that τ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time. Then the *stopped process* $(M_{t\wedge\tau})_{t\in\mathbb{R}_+}$ is also a martingale with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

Proof. We only give the proof in discrete time by applying a martingale transform argument. Writing the telescoping sum

$$M_n = M_0 + \sum_{l=1}^{n}(M_l - M_{l-1}),$$

we have

$$M_{\tau\wedge n} = M_0 + \sum_{l=1}^{\tau\wedge n}(M_l - M_{l-1}) = M_0 + \sum_{l=1}^{n}\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}),$$

and for $k \leqslant n$,

$$\begin{aligned}
\mathbb{E}[M_{\tau\wedge n} \mid \mathscr{F}_k] &= \mathbb{E}\left[M_0 + \sum_{l=1}^{n}\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \,\middle|\, \mathscr{F}_k\right] \\
&= M_0 + \sum_{l=1}^{n}\mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathscr{F}_k] \\
&= M_0 + \sum_{l=1}^{k}\mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathscr{F}_k] \\
&\quad + \sum_{l=k+1}^{n}\mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}(M_l - M_{l-1}) \mid \mathscr{F}_k] \\
&= M_0 + \sum_{l=1}^{k}(M_l - M_{l-1})\,\mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}} \mid \mathscr{F}_k] \\
&\quad + \sum_{l=k+1}^{n}\mathbb{E}[\mathbb{E}[(M_l - M_{l-1})\mathbb{1}_{\{l\leqslant\tau\}} \mid \mathscr{F}_{l-1}] \mid \mathscr{F}_k] \\
&= M_0 + \sum_{l=1}^{k}(M_l - M_{l-1})\mathbb{1}_{\{l\leqslant\tau\}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=k+1}^{n}\mathbb{E}[\mathbb{1}_{\{l\leqslant\tau\}}\underbrace{\mathbb{E}[(M_l-M_{l-1})\mid\mathscr{F}_{l-1}]}_{=0}\mid\mathscr{F}_k] \\
&= M_0+\sum_{l=1}^{k}(M_l-M_{l-1})\mathbb{1}_{\{l\leqslant\tau\}} \\
&= M_0+\sum_{l=1}^{\tau\wedge k}(M_l-M_{l-1}) \\
&= M_{\tau\wedge k}, \qquad k=0,1,\ldots,n,
\end{aligned}
$$

as by the martingale property of $(M_l)_{l\in\mathbb{N}}$, we have

$$
\begin{aligned}
\mathbb{E}[(M_l-M_{l-1})\mid\mathscr{F}_{l-1}] &= \mathbb{E}[M_l\mid\mathscr{F}_{l-1}]-\mathbb{E}[M_{l-1}\mid\mathscr{F}_{l-1}] \\
&= \mathbb{E}[M_l\mid\mathscr{F}_{l-1}]-M_{l-1} \\
&= 0, \qquad l\geqslant 1.
\end{aligned}
$$

□

Remarks.

a) More generally, if $(M_t)_{t\in\mathbb{R}_+}$ is a *super* (resp. *sub*)-martingale with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, then the *stopped process* $(M_{t\wedge\tau})_{t\in\mathbb{R}_+}$ remains a *super* (resp. *sub*)-martingale with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, see *e.g.* Exercise 4.6 below for the case of *sub*martingales in discrete time.

b) Since by Theorem 4.5 the stopped process $(M_{\tau\wedge t})_{t\in\mathbb{R}_+}$ is a martingale, we find that its expected value $\mathbb{E}[M_{\tau\wedge t}]$ is constant over time $t\in\mathbb{R}_+$ by Proposition 4.1.

As a consequence, if $(M_t)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-martingale and τ is a stopping time bounded by a constant $T>0$, *i.e.* $\tau\leqslant T$ almost surely,* we have

$$
\mathbb{E}[M_\tau]=\mathbb{E}[M_{\tau\wedge T}]=\mathbb{E}[M_{\tau\wedge 0}]=\mathbb{E}[M_0]=\mathbb{E}[M_T]. \tag{4.5}
$$

c) From (4.5), if τ,ν are two stopping times *a.s. bounded* by a constant $T>0$ and $(M_t)_{t\in\mathbb{R}_+}$ is a martingale, we have

$$
\mathbb{E}[M_0]=\mathbb{E}[M_\tau]=\mathbb{E}[M_\nu]=\mathbb{E}[M_T]. \tag{4.6}
$$

d) If τ,ν are two stopping times *a.s. bounded* by a constant $T>0$ and such that $\tau\leqslant\nu$ *a.s.*, then
   (i) when $(M_t)_{t\in\mathbb{R}_+}$ is a *super*martingale, we have

$$
\mathbb{E}[M_0]\geqslant\mathbb{E}[M_\tau]\geqslant\mathbb{E}[M_\nu]\geqslant\mathbb{E}[M_T], \tag{4.7}
$$

   (ii) when $(M_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, we have

$$
\mathbb{E}[M_0]\leqslant\mathbb{E}[M_\tau]\leqslant\mathbb{E}[M_\nu]\leqslant\mathbb{E}[M_T], \tag{4.8}
$$

see Exercise 4.6 below for a proof in discrete time.

*"$\tau\leqslant T$ almost surely" means $\mathbb{P}(\tau\leqslant T)=1$, *i.e.* $\mathbb{P}(\tau>T)=0$.

e) In case τ is finite with probability one (but not bounded) we may also write

$$\mathbb{E}[M_\tau] = \mathbb{E}\left[\lim_{t\to\infty} M_{\tau\wedge t}\right] = \lim_{t\to\infty} \mathbb{E}[M_{\tau\wedge t}] = \mathbb{E}[M_0], \tag{4.9}$$

provided that

$$|M_{\tau\wedge t}| \leqslant C, \qquad a.s., \quad t \in \mathbb{R}_+. \tag{4.10}$$

More generally, (4.9) holds provided that the limit and expectation signs can be exchanged, and this can be done using *e.g.* the *Dominated Convergence Theorem*. In some situations the exchange of limit and expectation signs may not be valid.*

In case $\mathbb{P}(\tau = +\infty) > 0$, (4.9) holds under the above conditions, provided that

$$M_\infty := \lim_{t\to\infty} M_t \tag{4.11}$$

exists with probability one.

Relations (4.7), (4.8) and (4.6) can be extended to unbounded stopping times along the same lines and conditions as (4.9), such as (4.10) applied to both τ and ν. Dealing with unbounded stopping times can be necessary in the case of hitting times.

f) In general, for all *a.s.* finite (bounded or unbounded) stopping times τ it remains true that

$$\mathbb{E}[M_\tau] = \mathbb{E}\left[\lim_{t\to\infty} M_{\tau\wedge t}\right] \leqslant \lim_{t\to\infty} \mathbb{E}\left[M_{\tau\wedge t}\right] \leqslant \lim_{t\to\infty} \mathbb{E}[M_0] = \mathbb{E}[M_0], \tag{4.12}$$

provided that $(M_t)_{t\in\mathbb{R}_+}$ is a nonnegative *super*martingale, where we used Fatou's Lemma.† As in (4.9), the limit (4.11) is required to exist with probability one if $\mathbb{P}(\tau = +\infty) > 0$.

g) As a counterexample to (4.6), the random time

$$\tau := \inf\left\{ t \in [0,T] \ : \ M_t = \operatorname*{Sup}_{s\in[0,T]} M_s \right\},$$

which is not a stopping time, will satisfy

$$\mathbb{E}[M_\tau] > \mathbb{E}[M_T],$$

although $\tau \leqslant T$ almost surely. Similarly,

$$\tau := \inf\left\{ t \in [0,T] \ : \ M_t = \inf_{s\in[0,T]} M_s \right\},$$

is not a stopping time and satisfies

$$\mathbb{E}[M_\tau] < \mathbb{E}[M_T].$$

*Consider for example the sequence $M_n = n\mathbb{1}_{\{X<1/n\}}$, $n \geqslant 1$, where $X \simeq U(0,1]$ is a uniformly distributed random variable on $(0,1]$.

†$\mathbb{E}[\lim_{n\to\infty} F_n] \leqslant \lim_{n\to\infty} \mathbb{E}[F_n]$ for any sequence $(F_n)_{n\in\mathbb{N}}$ of *nonnegative* random variables, provided that the limits exist.

		bounded stopping times $\tau,\ \nu$
$(M_t)_{t\in\mathbb{R}_+}$	*super*martingale	$\mathbb{E}[M_\tau] \geqslant \mathbb{E}[M_\nu]$ if $\tau \leqslant \nu$.
	martingale	$\mathbb{E}[M_\tau] = \mathbb{E}[M_\nu]$.
	*sub*martingale	$\mathbb{E}[M_\tau] \leqslant \mathbb{E}[M_\nu]$ if $\tau \leqslant \nu$.

Table 4.1: Martingales and stopping times.

Martingales and stopping times as gambling strategies

When $(M_t)_{t\in[0,T]}$ is a martingale, *e.g.* a centered random walk with independent increments, the message of the Stopping Time Theorem 4.5 is that the expected gain of the exit strategy $\tau_{x,y}$ of (4.4) remains zero on average since

$$\mathbb{E}\left[M_{\tau_{x,y}}\right] = \mathbb{E}[M_0] = 0,$$

if $M_0 = 0$. Therefore, on average, this exit strategy does not increase the average gain of the player. More precisely we have

$$\begin{aligned} 0 &= M_0 = \mathbb{E}[M_{\tau_{x,y}}] = x\mathbb{P}(M_{\tau_{x,y}} = x) + y\mathbb{P}(M_{\tau_{x,y}} = y) \\ &= -10 \times \mathbb{P}(M_{\tau_{x,y}} = -10) + 100 \times \mathbb{P}(M_{\tau_{x,y}} = 100), \end{aligned}$$

which shows that

$$\mathbb{P}(M_{\tau_{x,y}} = -10) = \frac{10}{11} \quad \text{and} \quad \mathbb{P}(M_{\tau_{x,y}} = 100) = \frac{1}{11},$$

provided that the relation $\mathbb{P}(M_{\tau_{x,y}} = x) + \mathbb{P}(M_{\tau_{x,y}} = y) = 1$ is satisfied, see below for further applications to Brownian motion.

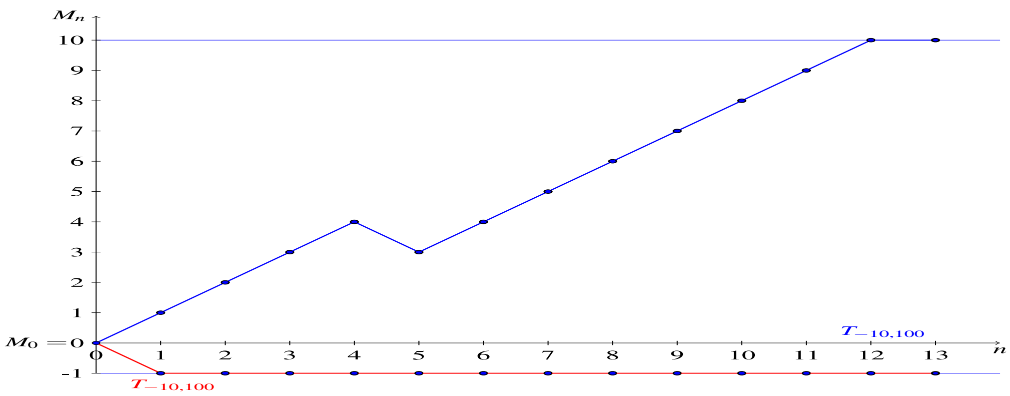


Figure 4.4: Sample paths of a gambling process $(M_n)_{n\in\mathbb{N}}$.

In the next Table 4.1 we summarize some of the results obtained in this section for bounded stopping times.

In the sequel we note that, as an application of the Stopping Time Theorem 4.5, a number of expectations can be computed in a simple and elegant way.

4.4 Application to drifted Brownian motion

Brownian motion hitting a barrier

Given $a,b \in \mathbb{R}$, $a < b$, let the hitting* time $\tau_{a,b} : \Omega \longrightarrow \mathbb{R}_+$ be defined by

$$\tau_{a,b} = \inf\{t \geqslant 0 \ : \ B_t = a \text{ or } B_t = b\},$$

which is the hitting time of the boundary $\{a,b\}$ of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, $a < b \in \mathbb{R}$.

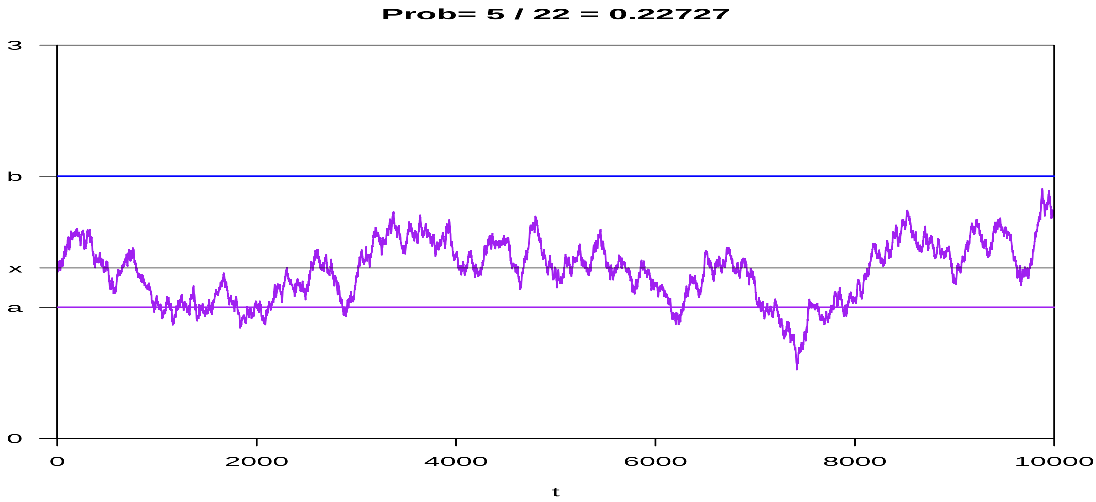


Figure 4.5: Brownian motion hitting a barrier.

Recall that Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a martingale since it has independent increments, and those increments are centered:

$$\mathbb{E}[B_t - B_s] = 0, \qquad 0 \leqslant s \leqslant t.$$

Consequently, $(B_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is still a martingale, and by (4.9) we have

$$\mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = \mathbb{E}[B_0 \mid B_0 = x] = x,$$

as the exchange between limit and expectation in (4.9) can be justified since

$$|B_{t\wedge\tau_{a,b}}| \leqslant \mathrm{Max}(|a|,|b|), \qquad t \in \mathbb{R}_+.$$

Hence we have

$$\begin{cases} x = \mathbb{E}[B_{\tau_{a,b}} \mid B_0 = x] = a \times \mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) + b \times \mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x), \\ \\ \mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1, \end{cases}$$

which yields

$$\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) = \frac{x-a}{b-a}, \qquad a \leqslant x \leqslant b,$$

and also shows that

$$\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) = \frac{b-x}{b-a}, \qquad a \leqslant x \leqslant b.$$

Note that the above result and its proof actually apply to any continuous martingale, and not only to Brownian motion.

*Hitting times are stopping times.

Drifted Brownian motion hitting a barrier

Next, let us turn to the case of drifted Brownian motion

$$X_t = x + B_t + \mu t, \qquad t \in \mathbb{R}_+.$$

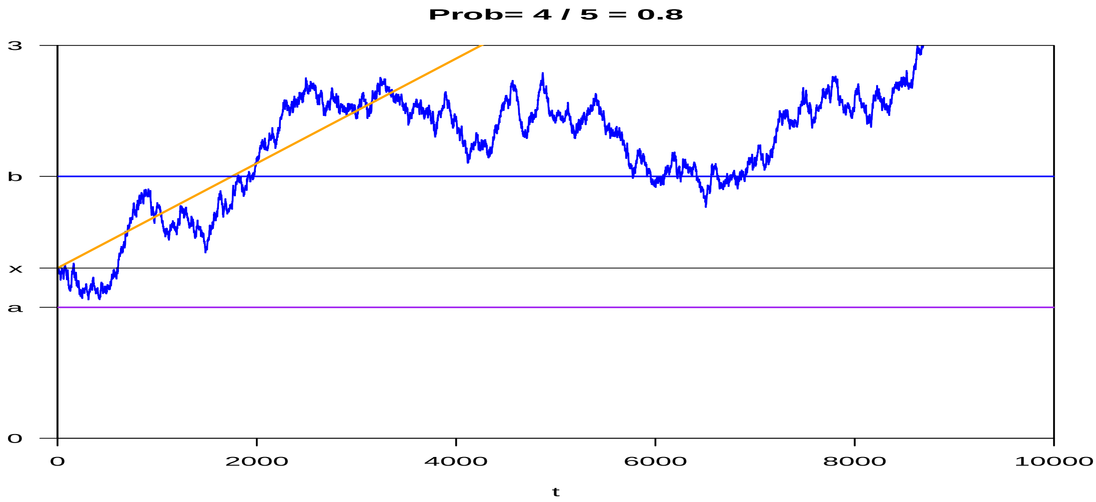


Figure 4.6: Drifted Brownian motion hitting a barrier.

In this case, the process $(X_t)_{t\in\mathbb{R}_+}$ is no longer a martingale and in order to use Theorem 4.5 we need to construct a martingale of a different type. Here we note that the process

$$M_t := \mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

is a martingale with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Indeed, we have

$$\mathbb{E}[M_t \mid \mathscr{F}_s] = \mathbb{E}\left[\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \mid \mathscr{F}_s\right] = \mathrm{e}^{\sigma B_s - \sigma^2 s/2}, \qquad 0 \leqslant s \leqslant t,$$

cf. *e.g.* Example 1 page 64.

By Theorem 4.5 we know that the stopped process $(M_{\tau_{a,b}\wedge t})_{t\in\mathbb{R}_+}$ is a martingale, hence its expected value is constant over time $t \in \mathbb{R}_+$ by Proposition 4.1, and (4.9) yields

$$1 = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau_{a,b}}],$$

as the exchange between limit and expectation in (4.9) can be justified since

$$|M_{t\wedge\tau_{a,b}}| \leqslant \operatorname{Max}\left(\mathrm{e}^{\sigma|a|}, \mathrm{e}^{\sigma|b|}\right), \qquad t \in \mathbb{R}_+.$$

Next, we note that taking $\mu = -\sigma/2$, *i.e.* $\sigma = -2\mu$, we have $M_t = \mathrm{e}^{-\sigma x}\mathrm{e}^{\sigma X_t}$, and

$$\mathrm{e}^{\sigma X_t} = \mathrm{e}^{\sigma x + \sigma B_t + \sigma\mu t} = \mathrm{e}^{\sigma x + \sigma B_t - \sigma^2 t/2} = \mathrm{e}^{\sigma x} M_t,$$

hence

$$\begin{aligned}
1 &= \mathbb{E}[M_{\tau_{a,b}}] \\
&= \mathrm{e}^{-\sigma x}\,\mathbb{E}[\mathrm{e}^{\sigma X_{\tau_{a,b}}}] \\
&= \mathrm{e}^{(a-x)\sigma}\,\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathrm{e}^{(b-x)\sigma}\,\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x),
\end{aligned}$$

under the additional condition

$$\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) + \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = 1.$$

Finally, this gives

$$\begin{cases} \mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) = \dfrac{\mathrm{e}^{\sigma x} - \mathrm{e}^{\sigma b}}{\mathrm{e}^{\sigma a} - \mathrm{e}^{\sigma b}} = \dfrac{\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} & \text{(4.13a)} \\[2ex] \mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) = \dfrac{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}}, & \text{(4.13b)} \end{cases}$$

$a \leqslant x \leqslant b$, see Figure 4.7 for an illustration with $a = 1$, $b = 2$, $x = 1.3$, $\mu = 2.0$, and $\left(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}\right) / \left(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}\right) = 0.7118437$.

```
nsim <- 1000;a=1;b=2;x=1.3;mu=2.0;N=10001; T<-2.0; t <- 0:(N-1); dt <- T/N;prob=0.0;
for (i in 1:nsim){signal=0;colour="blue";Z <- rnorm(N,mean=0,sd= sqrt(dt));
X <- c(1,N);X[1]=x;for (j in 2:N){X[j]=X[j-1]+Z[j]+mu*dt
if (X[j]<=a && signal==0) {signal=-1;colour="purple"}
if (X[j]>=b && signal==0) {signal=1;colour="blue";prob=prob+1}}
plot(t, X, xlab = "t", ylab = "", type = "l", ylim = c(-0,3), col =
      colour,main=paste("Prob=",prob,"/",i,"=",round(prob/i, digits=5)), xaxs="i", yaxs="i", yaxt="n")
lines(t, x+mu*t*dt, type = "l", col = "orange",lwd=2);yticks<-c(0,a,x,b,3)
axis(side=2, at=yticks,labels = c(0,"a","x","b",3), las = 2)
abline(h=x);abline(h=a,col="purple",lwd=2);abline(h=b,col="blue",lwd=2)# Sys.sleep(0.5)
readline(prompt = "Pause. Press <Enter> to continue...")}
(exp(-2*mu*a)-exp(-2*mu*x))/(exp(-2*mu*a)-exp(-2*mu*b))
```

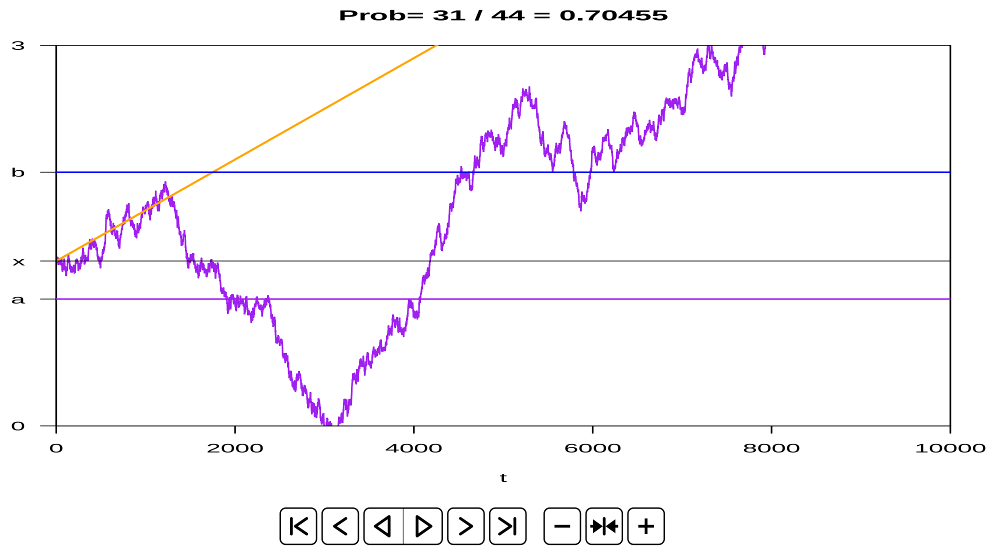


Figure 4.7: Hitting probabilities of drifted Brownian motion.*

Letting b tend to infinity in the above equalities shows that the probability $\mathbb{P}(\tau_a = +\infty)$ of escape to infinity of Brownian motion started from $x \in [a, \infty)$ is equal to

$$\mathbb{P}(\tau_a = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = 1 - \mathrm{e}^{-2\mu(x-a)}, & \mu \geqslant 0, \\[1ex] 0, \quad \mu \leqslant 0, \end{cases}$$

*The animation works in Acrobat Reader on the entire pdf file.

(4.14)

i.e.

$$\mathbb{P}(\tau_a < +\infty) = \begin{cases} \mathbb{P}(X_{\tau_{a,\infty}} = a \mid X_0 = x) = \mathrm{e}^{-2\mu(x-a)}, & \mu \geqslant 0, \\ 1, \quad \mu \leqslant 0. \end{cases} \tag{4.15}$$

Similarly for $x \in (-\infty, b]$, letting a tend to infinity we have

$$\mathbb{P}(\tau_b = +\infty) = \begin{cases} 1 - \mathbb{P}(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = 1 - \mathrm{e}^{-2\mu(x-b)}, & \mu \leqslant 0, \\ 0, \quad \mu \geqslant 0, \end{cases} \tag{4.16}$$

i.e.

$$\mathbb{P}(\tau_b < +\infty) = \begin{cases} \mathbb{P}(X_{\tau_{-\infty,b}} = b \mid X_0 = x) = \mathrm{e}^{-2\mu(x-b)}, & \mu \leqslant 0, \\ 1, \quad \mu \geqslant 0. \end{cases} \tag{4.17}$$

Mean hitting times for Brownian motion

The martingale method also allows us to compute the expectation $\mathbb{E}[B_{\tau_{a,b}}]$, after rechecking that

$$B_t^2 - t = 2\int_0^t B_s dB_S, \qquad t \in \mathbb{R}_+,$$

is also a martingale. Indeed, we have

$$\begin{aligned} \mathbb{E}[B_t^2 - t \mid \mathscr{F}_s] &= \mathbb{E}[(B_s + (B_t - B_s))^2 - t \mid \mathscr{F}_s] \\ &= \mathbb{E}[B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s) - t \mid \mathscr{F}_s] \\ &= \mathbb{E}[B_s^2 - s \mid \mathscr{F}_s] - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathscr{F}_s] + 2\,\mathbb{E}[B_s(B_t - B_s) \mid \mathscr{F}_s] \\ &= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2 \mid \mathscr{F}_s] + 2B_s\,\mathbb{E}[B_t - B_s \mid \mathscr{F}_s] \\ &= B_s^2 - s - (t-s) + \mathbb{E}[(B_t - B_s)^2] + 2B_s\,\mathbb{E}[B_t - B_s] \\ &= B_s^2 - s, \qquad 0 \leqslant s \leqslant t. \end{aligned}$$

Consequently the stopped process $(B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t)_{t\in\mathbb{R}_+}$ is still a martingale by Theorem 4.5 hence the expectation $\mathbb{E}[B^2_{\tau_{a,b}\wedge t} - \tau_{a,b} \wedge t]$ is constant over time $t \in \mathbb{R}_+$, hence by (4.9) we get*

$$\begin{aligned} x^2 &= \mathbb{E}[B_0^2 - 0 \mid B_0 = x] \\ &= \mathbb{E}[B^2_{\tau_{a,b}} - \tau_{a,b} \mid B_0 = x] \\ &= \mathbb{E}[B^2_{\tau_{a,b}} \mid B_0 = x] - \mathbb{E}[\tau_{a,b} \mid B_0 = x] \\ &= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - \mathbb{E}[\tau_{a,b} \mid B_0 = x], \end{aligned}$$

i.e.

$$\begin{aligned} \mathbb{E}[\tau_{a,b} \mid B_0 = x] &= b^2\mathbb{P}(B_{\tau_{a,b}} = b \mid B_0 = x) + a^2\mathbb{P}(B_{\tau_{a,b}} = a \mid B_0 = x) - x^2 \\ &= b^2\frac{x-a}{b-a} + a^2\frac{b-x}{b-a} - x^2 \\ &= (x-a)(b-x), \qquad a \leqslant x \leqslant b. \end{aligned}$$

*Here we note that it can be showed that $\mathbb{E}[\tau_{a,b}] < \infty$ in order to apply (4.9).

Mean hitting time for drifted Brownian motion

Finally we show how to recover the value of the mean hitting time $\mathbb{E}[\tau_{a,b} \mid X_0 = x]$ of drifted Brownian motion $X_t = x + B_t + \mu t$. As above, the process $X_t - \mu t$ is a martingale the stopped process $(X_{\tau_{a,b}\wedge t} - \mu(\tau_{a,b}\wedge t))_{t\in\mathbb{R}_+}$ is still a martingale by Theorem 4.5. Hence the expectation $\mathbb{E}[X_{\tau_{a,b}\wedge t} - \mu(\tau_{a,b}\wedge t)]$ is constant over time $t \in \mathbb{R}_+$.

Since the stopped process $\big(X_{\tau_{a,b}\wedge t} - \mu(\tau_{a,b}\wedge t)\big)_{t\in\mathbb{R}_+}$ is a martingale, we have

$$x = \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x],$$

which gives

$$\begin{aligned} x &= \mathbb{E}[X_{\tau_{a,b}} - \mu\tau_{a,b} \mid X_0 = x] \\ &= \mathbb{E}[X_{\tau_{a,b}} \mid X_0 = x] - \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x] \\ &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x], \end{aligned}$$

i.e. by (4.13a),

$$\begin{aligned} \mu\,\mathbb{E}[\tau_{a,b} \mid X_0 = x] &= b\mathbb{P}(X_{\tau_{a,b}} = b \mid X_0 = x) + a\mathbb{P}(X_{\tau_{a,b}} = a \mid X_0 = x) - x \\ &= b\frac{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} + a\frac{\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}} - x \\ &= \frac{b(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}) + a(\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}) - x(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})}{\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b}}, \end{aligned}$$

hence

$$\mathbb{E}[\tau_{a,b} \mid X_0 = x] = \frac{b(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu x}) + a(\mathrm{e}^{-2\mu x} - \mathrm{e}^{-2\mu b}) - x(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})}{\mu(\mathrm{e}^{-2\mu a} - \mathrm{e}^{-2\mu b})},$$

$a \leqslant x \leqslant b$.

Table 4.2 presents a summary of the families of martingales used in this chapter.

Problem \ Probabilities	Non drifted	Drifted
Hitting probability $\mathbb{P}(X_{\tau_{a,b}} = a, b)$	B_t	$\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$
Mean hitting time $\mathbb{E}[\tau_{a,b}]$	$B_t^2 - t$	$B_t + \mu t$

Table 4.2: List of martingales.

Exercises

Exercise 4.1 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0, *i.e.* $B_0 = 0$.

a) Is the process $t \longmapsto (2 - B_t)^+$ a *sub*martingale, a martingale or a *super*martingale?
b) Is the process $(\mathrm{e}^{B_t})_{t\in\mathbb{R}_+}$ a *sub*martingale, a martingale, or a *super*martingale?

c) Consider the random time ν defined by

$$\nu := \inf\{t \in \mathbb{R}_+ \ : \ B_t = B_{2t}\},$$

which represents the first intersection time of the curves $(B_t)_{t\in\mathbb{R}_+}$ and $(B_{2t})_{t\in\mathbb{R}_+}$.

Is ν a stopping time?

d) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \mathrm{e}^{B_t - t/2} = \alpha + \beta t\right\},$$

which represents the first time geometric Brownian motion $\mathrm{e}^{B_t - t/2}$ crosses the straight line $t \longmapsto \alpha + \beta t$. Is τ a stopping time?

e) If τ is a stopping time, compute $\mathbb{E}[\tau]$ by the Doob Stopping Time Theorem 4.5 in each of the following two cases:

i) $\alpha > 1$ and $\beta < 0$,

ii) $\alpha < 1$ and $\beta > 0$.

Exercise 4.2 Stopping times. Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0.

a) Consider the random time ν defined by

$$\nu := \inf\{t \in \mathbb{R}_+ \ : \ B_t = B_1\},$$

which represents the first time Brownian motion B_t hits the level B_1. Is ν a stopping time?

b) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ \mathrm{e}^{B_t} = \alpha\,\mathrm{e}^{-t/2}\right\},$$

which represents the first time the exponential of Brownian motion B_t crosses the path of $t \longmapsto \alpha\,\mathrm{e}^{-t/2}$, where $\alpha > 1$.

Is τ a stopping time? If τ is a stopping time, compute $\mathbb{E}\left[\mathrm{e}^{-\tau}\right]$ by applying the Stopping Time Theorem 4.5.

c) Consider the random time τ defined by

$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ B_t^2 = 1 + \alpha t\right\},$$

which represents the first time the process $(B_t^2)_{t\in\mathbb{R}_+}$ crosses the straight line $t \longmapsto 1 + \alpha t$, with $\alpha < 1$.

Is τ a stopping time? If τ is a stopping time, compute $\mathbb{E}[\tau]$ by the Doob Stopping Time Theorem 4.5.

Exercise 4.3 Consider a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ started at $B_0 = 0$, and let

$$\tau_L = \inf\{t \in \mathbb{R}_+ \ : \ B_t = L\}$$

denote the first hitting time of the level $L > 0$ by $(B_t)_{t\in\mathbb{R}_+}$.

a) Compute the Laplace transform $\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right]$ of τ_L for all $r \geqslant 0$.

Hint: Use the Stopping Time Theorem 4.5 and the fact that $\left(\mathrm{e}^{\sqrt{2r}B_t - rt}\right)_{t\in\mathbb{R}_+}$ is a martingale when $r > 0$.

b) Find the optimal level stopping strategy depending on the value of $r > 0$ for the maximization problem
$$\operatorname*{Sup}_{L>0} \mathbb{E}\left[\mathrm{e}^{-r\tau_L} B_{\tau_L}\right].$$

Exercise 4.4 Consider $(B_t)_{t\in\mathbb{R}_+}$ a Brownian motion started at $B_0 = x \in [a,b]$ with $a < b$, and let
$$\tau := \inf\left\{t \in \mathbb{R}_+ \ : \ B_t = a \ \text{ or } \ B_t = b\right\}$$
denote the first exit time of the interval $[a,b]$. Show that the solution $f(x)$ of the differential equation $f''(x) = -2$ with $f(a) = f(b) = 0$ satisfies $f(x) = \mathbb{E}[\tau \mid B_0 = x]$.

Hint: Consider the process $X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)ds$, and apply the Doob Stopping Time Theorem 4.5.

Exercise 4.5 Consider a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ started at $B_0 = 0$, and let
$$\tau := \inf\{t \in \mathbb{R}_+ \ : \ B_t = \alpha + \beta t\}$$
denote the first hitting time of the straight line $t \longmapsto \alpha + \beta t$ by $(B_t)_{t\in\mathbb{R}_+}$. Compute the Laplace transform $\mathbb{E}[\mathrm{e}^{-r\tau}]$ of τ for all $r > 0$.

Exercise 4.6 (Doob-Meyer decomposition in discrete time). Let $(M_n)_{n\in\mathbb{N}}$ be a discrete-time *sub*martingale with respect to a filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$, with $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$.

a) Show that there exists two processes $(N_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ such that
   i) $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$,
   ii) $(A_n)_{n\in\mathbb{N}}$ is non-decreasing, *i.e.* $A_n \leqslant A_{n+1}$ *a.s.*, $n \in \mathbb{N}$,
   iii) $(A_n)_{n\in\mathbb{N}}$ is predictable in the sense that A_n is $\mathscr{F}_{n-1}$-measurable, $n \in \mathbb{N}$, and
   iv) $M_n = N_n + A_n$, $n \in \mathbb{N}$.

   Hint: Let $A_0 := 0$,
$$A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n], \quad n \geqslant 0,$$
   and define $(N_n)_{n\in\mathbb{N}}$ in such a way that it satisfies the four required properties.

b) Show that for all bounded stopping times σ and τ such that $\sigma \leqslant \tau$ *a.s.*, we have
$$\mathbb{E}[M_\sigma] \leqslant \mathbb{E}[M_\tau].$$
   Hint: Use the Stopping Time Theorem 4.5 for martingales and (4.6).

5. American Options

American options are financial derivatives that can be exercised at any time before maturity, in contrast to European options which have fixed maturities. The prices of American options are evaluated as an optimization problem in which one has to find the optimal time to exercise in order to maximize the claim option payoff.

5.1 Perpetual American Options

The price of an American put option with finite expiration time $T > 0$ and strike price K can be expressed as the expected value of its discounted payoff:

$$f(t,S_t) = \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(K-S_\tau)^+ \,\middle|\, S_t \right],$$

under the risk-neutral probability measure $\mathbb{P}^*$, where the supremum is taken over stopping times between t and a fixed maturity T. Similarly, the price of a finite expiration American call option with strike price K is expressed as

$$f(t,S_t) = \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau-K)^+ \,\middle|\, S_t \right].$$

In this section we take $T = +\infty$, in which case we refer to these options as *perpetual* options, and the corresponding put and call options are respectively priced as

$$f(t,S_t) = \underset{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K-S_\tau)^+ \,\middle|\, S_t \right],$$

and

$$f(t,S_t) = \underset{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(S_\tau-K)^+ \,\middle|\, S_t \right].$$

Two-choice optimal stopping at a fixed price level for perpetual put options

In this section we consider the pricing of perpetual put options. Given $L \in (0,K)$ a fixed price, consider the following choices for the exercise of a *put* option with strike price K:

1. If $S_t \leqslant L$, then exercise at time t.

2. Otherwise if $S_t > L$, wait until the first hitting time

$$\tau_L := \inf\{u \geqslant t \ : \ S_u \leqslant L\} \tag{5.1}$$

   of the level $L > 0$, and exercise the option at time τ_L if $\tau_L < \infty$.

Note that by definition of (5.1) we have $\tau_L = t$ if $S_t \leqslant L$.

In case $S_t \leqslant L$, the payoff will be

$$(K - S_t)^+ = K - S_t$$

since $K > L \geqslant S_t$, however in this case one would buy the option at price $K - S_t$ only to exercise it immediately for the same amount.

In case $S_t > L$, as $r > 0$ the price of the option is given by

$$\begin{aligned}
f_L(t,S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L-t)r}(K-S_{\tau_L})^+ \,\middle|\, S_t \right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L-t)r}(K-S_{\tau_L})^+ \mathbb{1}_{\{\tau_L<\infty\}} \,\middle|\, S_t \right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L-t)r}(K-L)^+ \mathbb{1}_{\{\tau_L<\infty\}} \,\middle|\, S_t \right] \\
&= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-(\tau_L-t)r} \,\middle|\, S_t \right].
\end{aligned} \tag{5.2}$$

We note that the starting date t does not matter when pricing perpetual options, which have an infinite time horizon. Hence, $f_L(t,x) = f_L(x)$, $x > 0$, does not depend on $t \in \mathbb{R}_+$, and the pricing of the perpetual put option can be performed at $t = 0$. Recall that the underlying asset price is written as

$$S_t = S_0\,\mathrm{e}^{rt+\sigma\widehat{B}_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \tag{5.3}$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, r is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

Lemma 5.1 Assume that $r > 0$. We have

$$\boxed{\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \,\middle|\, S_t = x \right] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}, \qquad x \geqslant L.} \tag{5.4}$$

Proof. We take $t = 0$ without loss of generality. We note that from (5.3), for all $\lambda \in \mathbb{R}$ we have

$$S_t^\lambda = \mathrm{e}^{\lambda r t + \lambda\sigma\widehat{B}_t - \lambda\sigma^2 t/2},$$

the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := S_0^\lambda \,\mathrm{e}^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2} = S_t^\lambda \,\mathrm{e}^{-(\lambda r - \lambda(1-\lambda)\sigma^2/2)t}, \qquad t \in \mathbb{R}_+, \tag{5.5}$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$. Choosing $\lambda \in \mathbb{R}$ such that

$$r = r\lambda - \lambda(1-\lambda)\frac{\sigma^2}{2}, \tag{5.6}$$

we have

$$Z_t^{(\lambda)} = S_t^\lambda \,\mathrm{e}^{-rt}, \qquad t \in \mathbb{R}_+.$$

The equation (5.6) rewrites as

$$0 = \lambda^2\frac{\sigma^2}{2} + \lambda\left(r - \frac{\sigma^2}{2}\right) - r = \frac{\sigma^2}{2}\left(\lambda + \frac{2r}{\sigma^2}\right)(\lambda - 1), \tag{5.7}$$

with solutions

$$\lambda_+ = 1 \quad \text{and} \quad \lambda_- = -\frac{2r}{\sigma^2}.$$

Choosing the negative solution* $\underline{\lambda_- = -2r/\sigma^2 < 0}$, we have

$$0 \leqslant Z_t^{(\lambda_-)} = \mathrm{e}^{-rt}(S_t)^{\lambda_-} \leqslant \mathrm{e}^{-rt}L^{\lambda_-} \leqslant L^{\lambda_-}, \qquad 0 \leqslant t < \tau_L, \tag{5.8}$$

since $r > 0$, hence $\lim_{t\to\infty} Z_t^{(\lambda_-)} = 0$ and $\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)} = 0$ on $\{\tau_L < \infty\}$. Therefore, since $r > 0$ we have

$$\begin{aligned}
L^{\lambda_-}\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] &= L^{\lambda_-}\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}}\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\left(S_{\tau_L}\right)^{\lambda_-}\mathbb{1}_{\{\tau_L<\infty\}}\right] \\
&= \mathbb{E}^*\left[Z_{\tau_L}^{(\lambda_-)}\mathbb{1}_{\{\tau_L<\infty\}}\right] \\
&= \mathbb{E}^*\left[\mathbb{1}_{\{\tau_L<\infty\}}\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)}\right] \\
&= \mathbb{E}^*\left[\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_-)}\right] &(5.9)\\
&= \lim_{t\to\infty}\mathbb{E}^*\left[Z_{\tau_L\wedge t}^{(\lambda_-)}\right] &(5.10)\\
&= \lim_{t\to\infty}\mathbb{E}^*\left[Z_0^{(\lambda_-)}\right] \\
&= (S_0)^{\lambda_-},
\end{aligned}$$

where by (5.8) we used the dominated convergence theorem from (5.9) to (5.10), hence we find

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \,\middle|\, S_0 = x\right] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}, \qquad x \geqslant L.$$

□

Next, we apply Lemma 5.1 in order to price the perpetual American put option.

*Note that $\mathbb{P}(\tau_L = \infty) > 0$ since $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, cf. (4.14), and the bound (5.8) does not hold for the positive solution $\lambda_+ = 1$.

Proposition 5.2 Assume that $r > 0$. We have

$$f_L(x) = \mathbb{E}^*\left[e^{-(\tau_L - t)r}(K - S_{\tau_L})^+ \,\middle|\, S_t = x\right] = \begin{cases} K - x, & 0 < x \leqslant L, \\ (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L. \end{cases}$$

Proof. We take $t = 0$ without loss of generality.
$i)$ The result is obvious for $S_0 = x \leqslant L$ since in this case we have $\tau_L = t = 0$ and $S_{\tau_L} = S_0 = x$, so that we only focus on the case $x \geqslant L$.
$ii)$ Next, we consider the case $\underline{S_0 = x > L}$. We have

$$\begin{aligned} \mathbb{E}^*\left[e^{-r\tau_L}(K - S_{\tau_L})^+ \,\middle|\, S_0 = x\right] &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau_L < \infty\}} e^{-r\tau_L}(K - S_{\tau_L})^+ \,\middle|\, S_0 = x\right] \\ &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau_L < \infty\}} e^{-r\tau_L}(K - L) \,\middle|\, S_0 = x\right] \\ &= (K-L)\,\mathbb{E}^*\left[e^{-r\tau_L} \,\middle|\, S_0 = x\right], \end{aligned} \tag{5.11}$$

and we conclude by the expression of $\mathbb{E}^*\left[e^{-r\tau_L} \,\middle|\, S_0 = x\right]$ given in Lemma 5.1. □

We note that taking $L = K$ would yield a payoff always equal to 0 for the option holder, hence the value of L should be strictly lower than K. On the other hand, if $L = 0$ the value of τ_L will be infinite almost surely, hence the option price will be 0 when $r \geqslant 0$ from (5.2). Therefore there should be an optimal value L^*, which should be strictly comprised between 0 and K.

Figure 5.1 shows for $K = 100$ that there exists an optimal value $L^* = 85.71$ which maximizes the option price for all values of the underlying asset price.

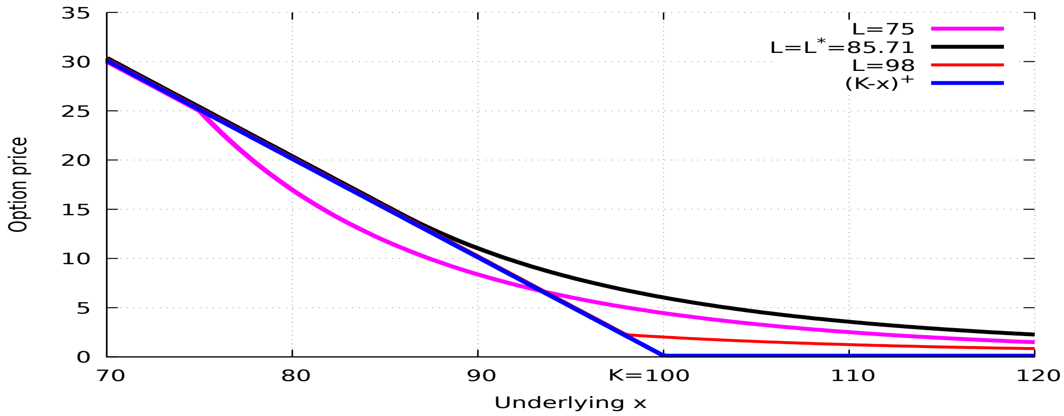


Figure 5.1: American put by exercising at τ_L for different values of L and $K = 100$.

In order to compute L^* we observe that, geometrically, the slope of $f_L(x)$ at $x = L^*$ is equal to -1, *i.e.*

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(K - L^*)\frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -1,$$

i.e.

$$\frac{2r}{\sigma^2}(K - L^*) = L^*,$$

or

$$\boxed{L^* = \frac{2r}{2r+\sigma^2}K < K.} \quad (5.12)$$

We note that L^* tends to zero as σ becomes large or r becomes small, and that L^* tends to K when σ becomes small.

The same conclusion can be reached from the vanishing of the derivative of $L \longmapsto f_L(x)$:

$$\frac{\partial f_L(x)}{\partial L} = -\left(\frac{x}{L}\right)^{-2r/\sigma^2} + \frac{2r}{\sigma^2}\frac{K-L}{L}\left(\frac{x}{L}\right)^{-2r/\sigma^2} = 0,$$

cf. page 351 of Shreve, 2004. The next Figure 5.2 is a 2-dimensional animation that also shows the optimal value L^* of L.

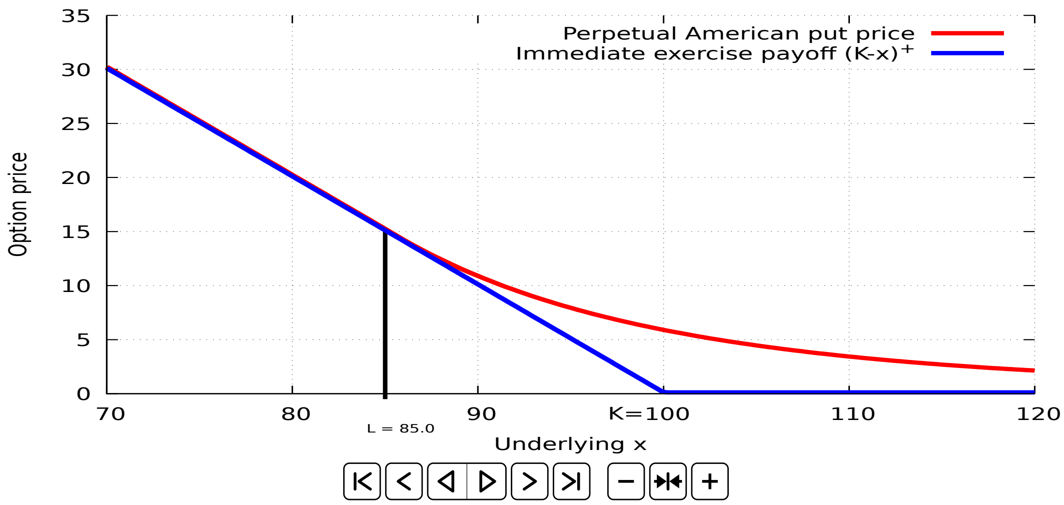


Figure 5.2: Animated graph of American put prices depending on L with $K = 100$.*

The next Figure 5.3 gives another view of the put option prices according to different values of L, with the optimal value $L^* = 85.71$.

*The animation works in Acrobat Reader on the entire pdf file.

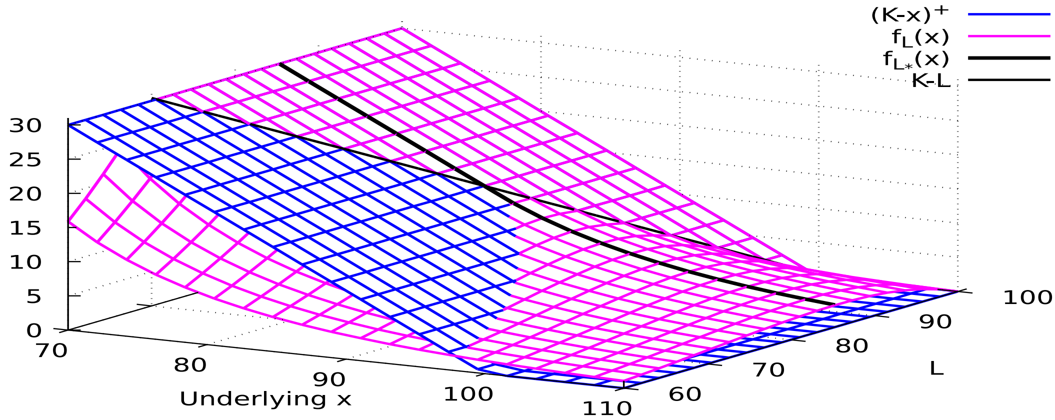


Figure 5.3: Option price as a function of L and of the underlying asset price.

In Figure 5.4, which is based on the stock price of HSBC Holdings (0005.HK) over year 2009 as in Figures 2.8-2.15, the optimal exercise strategy for an American put option with strike price K=\$77.67 would have been to exercise whenever the underlying asset price goes above $L^* = \$62$, *i.e.* at approximately 54 days, for a payoff of \$25.67. Exercising after a longer time, *e.g.* 85 days, could yield an even higher payoff of over \$65, however this choice is not made because decisions are taken based on existing (past) information, and optimization is in expected value (or average) over all possible future paths.

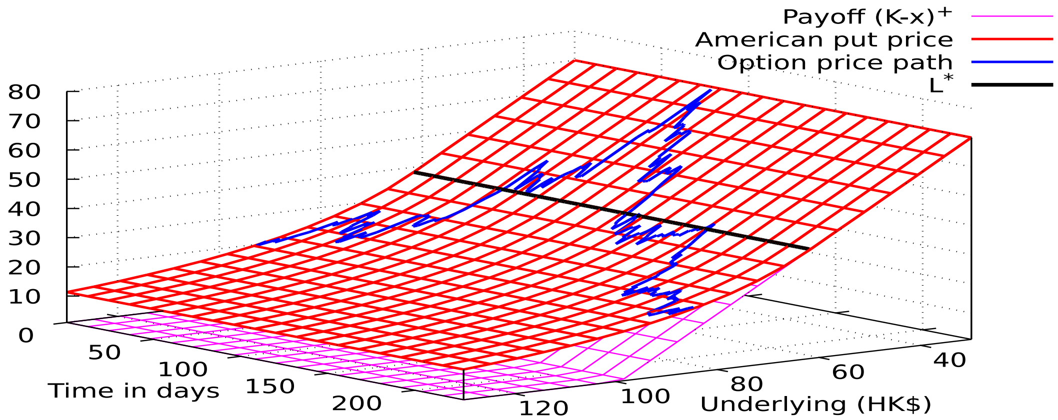


Figure 5.4: Path of the American put option price on the HSBC stock.

See Exercise 5.5 for the pricing of perpetual American put options with dividends.

5.2 PDE approach

Exercise. Check by hand calculations that the function f_{L^*} defined as

$$f_{L^*}(x) := \begin{cases} K - x, & 0 < x \leqslant L^* = \dfrac{2r}{2r+\sigma^2}K, \\ \\ \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{x}{K}\right)^{-2r/\sigma^2}, & x \geqslant L^* = \dfrac{2r}{2r+\sigma^2}K, \end{cases} \tag{5.13}$$

satisfies the Partial Differential Equation (PDE)

$$
\begin{aligned}
-rf_{L^*}(x)+rxf'_{L^*}(x)+\frac{1}{2}\sigma^2x^2f''_{L^*}(x) &= \mathbb{1}_{\{x\leqslant L^*\}} \\
&= \begin{cases} -rK<0, & 0<x\leqslant L^*<K, \qquad \text{[Exercise now]} \\ 0, & x>L^*. \qquad \text{[Wait]} \end{cases}
\end{aligned}
\tag{5.14}
$$

in addition to the conditions

$$
\begin{cases} f_{L^*}(x)=K-x, & 0<x\leqslant L^*<K, \qquad \text{[Exercise now]} \\ \\ f_{L^*}(x)>(K-x)^+, & x>L^*, \qquad \text{[Wait]} \end{cases}
$$

see (5.13).

The above statements can be summarized in the following proposition.

Proposition 5.3 The function f_{L^*} satisfies the following set of differential inequalities, or variational differential equation:

$$
\begin{cases} f_{L^*}(x)\geqslant (K-x)^+, & (5.15a) \\ rxf'_{L^*}(x)+\dfrac{\sigma^2}{2}x^2f''_{L^*}(x)\leqslant rf_{L^*}(x), & (5.15b) \\ \left(rf_{L^*}(x)-rxf'_{L^*}(x)-\dfrac{\sigma^2}{2}x^2f''_{L^*}(x)\right)\left(f_{L^*}(x)-(K-x)^+\right)=0. & (5.15c) \end{cases}
$$

The equation (5.15c) admits an interpretation in terms of absence of arbitrage, as shown below. By (5.14) and Itô's formula applied to

$$
dS_t = rS_tdt+\sigma S_td\widehat{B}_t,
$$

the discounted portfolio value process

$$
\widetilde{f}_{L^*}(S_t) = \mathrm{e}^{-rt}f_{L^*}(S_t), \qquad t\in\mathbb{R}_+,
$$

satisfies

$$
\begin{aligned}
& d\left(\widetilde{f}_{L^*}(S_t)\right) \\
&= \left(-rf_{L^*}(S_t)+rS_tf'_{L^*}(S_t)+\frac{1}{2}\sigma^2S_t^2f''_{L^*}(S_t)\right)\mathrm{e}^{-rt}dt+\mathrm{e}^{-rt}\sigma S_tf'_{L^*}(S_t)d\widehat{B}_t \\
&= -\mathbb{1}_{\{S_t\leqslant L^*\}}rK\,\mathrm{e}^{-rt}dt+\mathrm{e}^{-rt}\sigma S_tf'_{L^*}(S_t)d\widehat{B}_t \\
&= -\mathbb{1}_{\{f_{L^*}(S_t)=(K-S_t)^+\}}rK\,\mathrm{e}^{-rt}dt+\mathrm{e}^{-rt}\sigma S_tf'_{L^*}(S_t)d\widehat{B}_t,
\end{aligned}
\tag{5.16}
$$

hence we have the relation

$$
\begin{aligned}
& \widetilde{f}_{L^*}(S_T)-\widetilde{f}_{L^*}(S_t) \\
&= -rK\int_t^T \mathbb{1}_{\{f_{L^*}(S_u)\leqslant(K-S_u)^+\}}\mathrm{e}^{-ru}du+\int_t^T \mathrm{e}^{-ru}\sigma S_uf'_{L^*}(S_u)d\widehat{B}_u,
\end{aligned}
$$

which implies

$$
\mathbf{E}^*\left[\widetilde{f}_{L^*}(S_T)-\widetilde{f}_{L^*}(S_t)\mid\mathscr{F}_t\right] = \mathbf{E}^*\left[\widetilde{f}_{L^*}(S_T)\mid\mathscr{F}_t\right]-\widetilde{f}_{L^*}(S_t)
$$

$$= \mathbb{E}^*\left[-rK\int_t^T \mathbb{1}_{\{f_{L^*}(S_u)\leqslant (K-S_u)^+\}}\,\mathrm{e}^{-ru}du + \int_t^T \mathrm{e}^{-ru}\sigma S_u f'_{L^*}(S_u)d\widehat{B}_u \,\Big|\, \mathscr{F}_t\right]$$

$$= -\mathbb{E}^*\left[rK\int_t^T \mathbb{1}_{\{f_{L^*}(S_u)\leqslant (K-S_u)^+\}}\,\mathrm{e}^{-ru}du \,\Big|\, \mathscr{F}_t\right],$$

hence the following decomposition of the perpetual American put price into the sum of a European put price and an early exercise premium:

$$\begin{aligned}
&\widetilde{f}_{L^*}(S_t)\\
&= \mathbb{E}^*\big[\widetilde{f}_{L^*}(S_T) \mid \mathscr{F}_t\big] + rK\,\mathbb{E}^*\left[\int_t^T \mathbb{1}_{\{f_{L^*}(S_u)\leqslant (K-S_u)^+\}}\,\mathrm{e}^{-ru}du \,\Big|\, \mathscr{F}_t\right]\\
&= \underbrace{\mathrm{e}^{-rT}\,\mathbb{E}^*\big[(K-S_T)^+ \mid \mathscr{F}_t\big]}_{\text{European put price}} + \underbrace{rK\,\mathbb{E}^*\left[\int_t^T \mathbb{1}_{\{S_u\leqslant L^*\}}\,\mathrm{e}^{-ru}du \,\Big|\, \mathscr{F}_t\right]}_{\text{Early exercise premium}},
\end{aligned} \tag{5.17}$$

$0 \leqslant t \leqslant T$, see also Theorem 8.4.1 in § 8.4 in Elliott and Kopp, 2005 on early exercise premiums. From (5.16) we also make the following observations.

a) From Equation (5.15c), $\widetilde{f}_{L^*}(S_t)$ is a martingale when

$$f_{L^*}(S_t) > (K-S_t)^+, \quad i.e. \quad S_t > L^*, \quad \text{[Wait]}$$

and in this case the expected rate of return of the hedging portfolio value $f_{L^*}(S_t)$ equals the rate r of the riskless asset, as

$$d\big(\widetilde{f}_{L^*}(S_t)\big) = \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t,$$

or

$$d\big(f_{L^*}(S_t)\big) = d\big(\mathrm{e}^{rt}\widetilde{f}_{L^*}(S_t)\big) = rf_{L^*}(S_t)dt + \sigma S_t f'_{L^*}(S_t)d\widehat{B}_t,$$

and the investor prefers to wait.

b) On the other hand, if

$$f_{L^*}(S_t) = (K-S_t)^+, \quad i.e. \quad 0 < S_t < L^*, \quad \text{[Exercise now]}$$

the return of the hedging portfolio becomes lower than r as $d\big(\widetilde{f}_{L^*}(S_t)\big) = -rK\,\mathrm{e}^{-rt}dt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t$ and

$$\begin{aligned}
d\big(f_{L^*}(S_t)\big) &= d\big(\mathrm{e}^{rt}\widetilde{f}_{L^*}(S_t)\big)\\
&= rf_{L^*}(S_t)dt - rKdt + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t.
\end{aligned}$$

In this case it is not worth waiting as (5.15b)-(5.15c) show that the return of the hedging portfolio is lower than that of the riskless asset, *i.e.*:

$$-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t) = -rK < 0,$$

exercise becomes immediate since the process $\widetilde{f}_{L^*}(S_t)$ becomes a (strict) *super*martingale, and (5.15c) implies $f_{L^*}(x) = (K-x)^+$.

In view of the above derivation, it should make sense to assert that $f_{L^*}(S_t)$ is the price at time t of the perpetual American put option. The next proposition confirms that this is indeed the case, and that the optimal exercise time is $\tau^* = \tau_{L^*}$.

Proposition 5.4 The price of the perpetual American put option is given for all $t \geqslant 0$ by

$$
\begin{aligned}
f_{L^*}(S_t) &= \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K-S_\tau)^+ \,\middle|\, S_t \right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_{L^*}-t)r}(K-S_{\tau_{L^*}})^+ \,\middle|\, S_t \right] \\
&= \begin{cases} K - S_t, & 0 < S_t \leqslant L^*, \\[2ex] \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2r+\sigma^2}{2r}\dfrac{S_t}{K}\right)^{-2r/\sigma^2}, & S_t \geqslant L^*. \end{cases}
\end{aligned}
$$

Proof. i) Since the drift

$$
-rf_{L^*}(S_t) + rS_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t)
$$

in Itô's formula (5.16) is nonpositive by the inequality (5.15b), the discounted portfolio value process

$$
u \longmapsto \mathrm{e}^{-ru} f_{L^*}(S_u), \qquad u \in [t, \infty),
$$

is a *super*martingale. As a consequence, for all (*a.s.* finite) stopping times $\tau \in [t, \infty)$ we have, by (4.12),

$$
\mathrm{e}^{-rt} f_{L^*}(S_t) \geqslant \mathbb{E}^*\left[\mathrm{e}^{-r\tau} f_{L^*}(S_\tau) \,\middle|\, S_t \right] \geqslant \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \,\middle|\, S_t \right],
$$

from (5.15a), which implies

$$
\mathrm{e}^{-rt} f_{L^*}(S_t) \geqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \,\middle|\, S_t \right]. \tag{5.18}
$$

ii) The converse inequality is obvious by Proposition 5.2, as

$$
\begin{aligned}
f_{L^*}(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_{L^*}-t)r}(K-S_{\tau_{L^*}})^+ \,\middle|\, S_t \right] \\
&\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K-S_\tau)^+ \,\middle|\, S_t \right], \qquad (5.19)
\end{aligned}
$$

since τ_{L^*} is a stopping time larger than $t \in \mathbb{R}_+$. The inequalities (5.18) and (5.19) allow us to conclude to the equality

$$
f_{L^*}(S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K-S_\tau)^+ \,\middle|\, S_t \right].
$$

□

Remark. Note that the converse inequality (5.19) can also be obtained from the variational PDE (5.15a)-(5.15c) itself, without relying on Proposition 5.2. For this, taking $\tau = \tau_{L^*}$ we note that the process

$$
u \longmapsto \mathrm{e}^{-ru \wedge \tau_{L^*}} f_{L^*}(S_{u \wedge \tau_{L^*}}), \qquad u \geqslant t,
$$

is not only a *super*martingale, it is also a martingale until exercise at time τ_{L^*} by (5.14) since $S_{u \wedge \tau_{L^*}} \geqslant L^*$, hence we have

$$
\mathrm{e}^{-rt} f_{L^*}(S_t) = \mathbb{E}^*\left[\mathrm{e}^{-(u \wedge \tau_{L^*})r} f_{L^*}(S_{u \wedge \tau_{L^*}}) \,\middle|\, S_t \right], \qquad u \geqslant t,
$$

hence after letting u tend to infinity we obtain

$$\begin{aligned}
\mathrm{e}^{-rt} f_{L^*}(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_{L^*}} f_{L^*}(S_{\tau_{L^*}}) \,\middle|\, S_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_{L^*}} f_{L^*}(L^*) \,\middle|\, S_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_{L^*}} (K - S_{\tau_{L^*}})^+ \,\middle|\, S_t\right] \\
&\leqslant \sup_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_{L^*}} (K - S_{\tau_{L^*}})^+ \,\middle|\, S_t\right],
\end{aligned}$$

which shows that

$$\mathrm{e}^{-rt} f_{L^*}(S_t) \leqslant \sup_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau} (K - S_\tau)^+ \,\middle|\, S_t\right], \qquad t \in \mathbb{R}_+.$$

Two-choice optimal stopping at a fixed price level for perpetual call options

In this section we consider the pricing of perpetual call options. Given $L > K$ a fixed price, consider the following choices for the exercise of a *call* option with strike price K:

1. If $S_t \geqslant L$, then exercise at time t.

2. Otherwise, wait until the first hitting time

$$\tau_L = \inf\{u \geqslant t \ : \ S_u = L\}$$

   and exercise the option and time τ_L.

In case $S_t \geqslant L$, the immediate exercise (or intrinsic) payoff will be

$$(S_t - K)^+ = S_t - K,$$

since $K < L \leqslant S_t$.

In case $S_t < L$, as $r > 0$ the price of the option will be given by

$$\begin{aligned}
f_L(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L - t)r} (S_{\tau_L} - K)^+ \,\middle|\, S_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L - t)r} (S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \,\middle|\, S_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L - t)r} (L - K)^+ \,\middle|\, S_t\right] \\
&= (L - K)\, \mathbb{E}^*\left[\mathrm{e}^{-(\tau_L - t)r} \,\middle|\, S_t\right].
\end{aligned}$$

Proposition 5.5 In case $S_t < L$, as $r > 0$ the price of the option is given by $f_L(S_t)$, where

$$f_L(x) = \begin{cases} x - K, & x \geqslant L > K, \\ (L - K)\dfrac{x}{L}, & 0 < x \leqslant L. \end{cases} \tag{5.20}$$

Proof. We only need to consider the case $S_0 = x < L$. Note that for all $\lambda \in \mathbb{R}$, the process $(Z_t^{(\lambda)})_{t \in \mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := S_t^\lambda \,\mathrm{e}^{-r\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} = S_0^\lambda \,\mathrm{e}^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

defined in (5.5) is a martingale under the risk-neutral probability measure $\widetilde{\mathbb{P}}$. Hence the stopped process $\big(Z^{(\lambda)}_{t\wedge\tau_L}\big)_{t\in\mathbb{R}_+}$ is a martingale and it has constant expectation, *i.e.* we have

$$\mathbb{E}^*\big[Z^{(\lambda)}_{t\wedge\tau_L}\big] = \mathbb{E}^*\big[Z^{(\lambda)}_0\big] = S_0^\lambda, \qquad t\in\mathbb{R}_+. \tag{5.21}$$

Choosing λ such that

$$r = r\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2,$$

i.e.

$$0 = \lambda^2\sigma^2/2 + \lambda(r-\sigma^2/2) - r = \frac{\sigma^2}{2}(\lambda + 2r/\sigma^2)(\lambda - 1),$$

Relation (5.21) rewrites as

$$\mathbb{E}^*\big[(S_{t\wedge\tau_L})^\lambda\, \mathrm{e}^{-(t\wedge\tau_L)r}\big] = S_0^\lambda, \qquad t\in\mathbb{R}_+. \tag{5.22}$$

Choosing the positive solution* $\lambda_+ = 1$ yields the bound

$$0 \leqslant Z^{(\lambda_+)}_t = \mathrm{e}^{-rt}S_t \leqslant S_t \leqslant L, \qquad 0\leqslant t < \tau_L, \tag{5.23}$$

since $S_0 = x < L$. Hence, noting that $\lim_{t\to\infty} Z^{(\lambda_+)}_t = 0$ on $\{\tau_L = +\infty\}$ by letting t go to infinity in (5.22), by (5.23) and the dominated convergence theorem we get, since $r > 0$,

$$\begin{aligned}
L\,\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}\big] &= \mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}S_{\tau_L}\mathbb{1}_{\{\tau_L<\infty\}}\big]\\
&= \mathbb{E}^*\Big[\lim_{t\to\infty}\mathrm{e}^{-(\tau_L\wedge t)r}S_{\tau_L\wedge t}\Big]\\
&= \mathbb{E}^*\Big[\lim_{t\to\infty}Z^{(\lambda_+)}_{\tau_L\wedge t}\Big]\\
&= \lim_{t\to\infty}\mathbb{E}^*\big[Z^{(\lambda_+)}_{\tau_L\wedge t}\big]\\
&= \lim_{t\to\infty}\mathbb{E}^*\big[Z^{(\lambda_+)}_0\big]\\
&= S_0,
\end{aligned}$$

which yields

$$\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}\big] = \frac{S_0}{L}. \tag{5.24}$$

□

One can check from Figures 5.5 and 5.6 that the situation completely differs from the perpetual put option case, as there does not exist an optimal value L^* that would maximize the option price for all values of the underlying asset price.

*We actually have $\mathbb{P}(\tau_L = \infty) = 0$ since $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, cf. (4.16), and the bound (5.23) does not hold for the negative solution $\lambda_- = -2r/\sigma^2$.

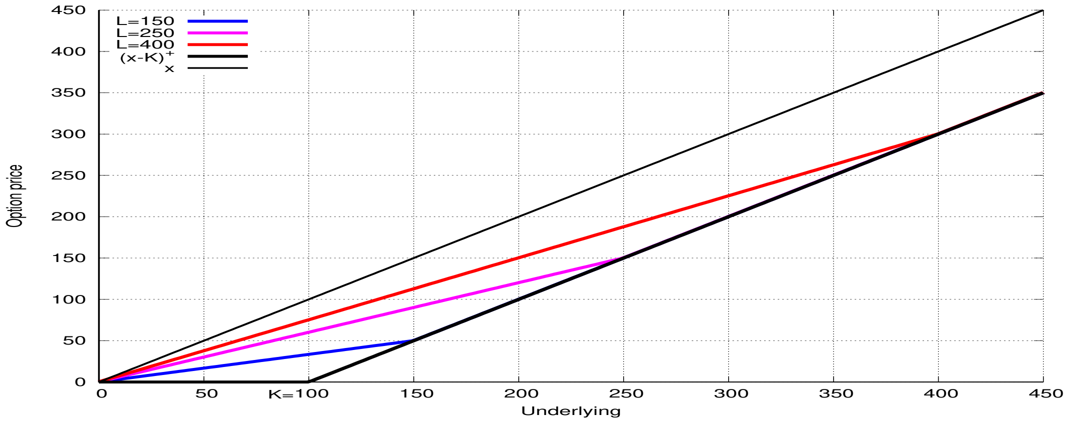


Figure 5.5: American call prices by exercising at τ_L for different values of L and $K = 100$.

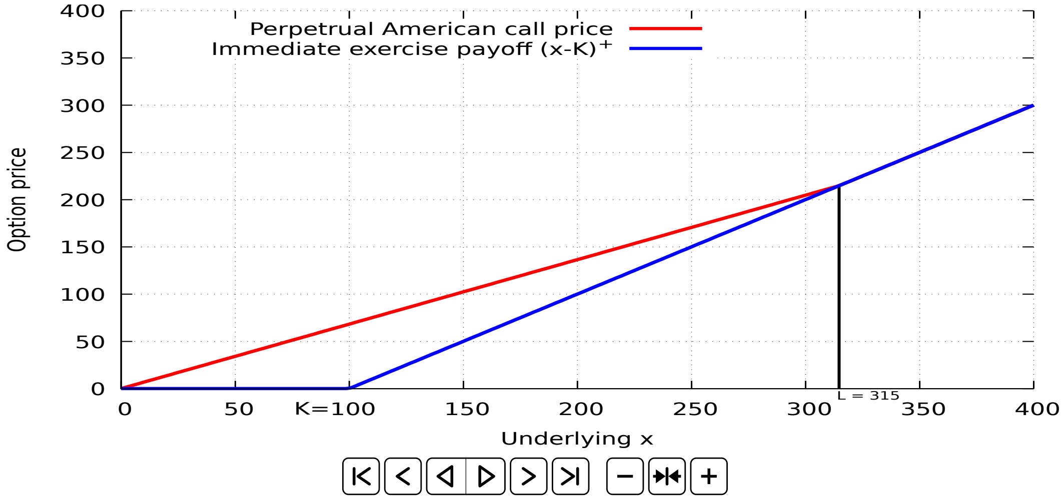


Figure 5.6: Animated graph of American option prices depending on L with $K = 100$.*

The intuition behind this picture is that there is no upper limit above which one should exercise the option, and in order to price the American perpetual call option we have to let L go to infinity, *i.e.* the "optimal" exercise strategy is to wait indefinitely.

*The animation works in Acrobat Reader on the entire pdf file.

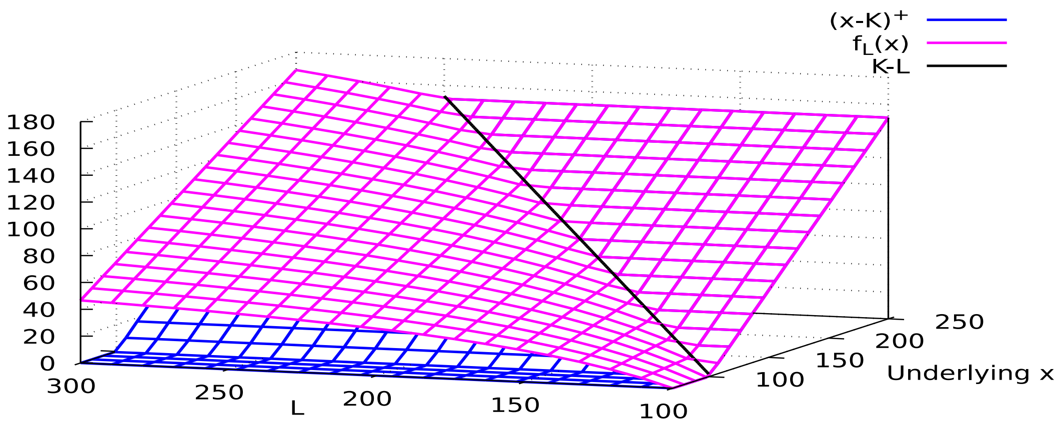


Figure 5.7: American call prices for different values of L.

We check from (5.20) that

$$\lim_{L\to\infty} f_L(x) = x - \lim_{L\to\infty} K\frac{x}{L} = x, \qquad x > 0. \tag{5.25}$$

As a consequence we have the following proposition.

Proposition 5.6 Assume that $r \geqslant 0$. The price of the perpetual American call option is given by

$$\operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right] = S_t, \qquad t \in \mathbb{R}_+. \tag{5.26}$$

Proof. For all $L > K$ we have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[e^{-(\tau_L-t)r}(S_{\tau_L} - K)^+ \,\middle|\, S_t \right] \\ &\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right], \qquad t \in \mathbb{R}_+, \end{aligned}$$

hence from (5.25), taking the limit as $L \to \infty$ yields

$$S_t \leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right]. \tag{5.27}$$

On the other hand, since $u \longmapsto e^{-(u-t)r}S_u$ is a martingale, by (4.12) we have, for all stopping times $\tau \in [t, \infty)$,

$$\mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right] \leqslant \mathbb{E}^*\left[e^{-(\tau-t)r}S_\tau \,\middle|\, S_t \right] \leqslant S_t, \quad t \in \mathbb{R}_+,$$

hence

$$\operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right] \leqslant S_t, \qquad t \in \mathbb{R}_+,$$

which shows (5.26) by (5.27). □

We may also check that since $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, the process $t \longmapsto (e^{-rt}S_t - K)^+$ is a *sub*martingale since the function $x \longmapsto (x-K)^+$ is convex, hence for all bounded stopping times τ such that $t \leqslant \tau$ we have

$$(S_t - K)^+ \leqslant \mathbb{E}^*\left[(e^{-(\tau-t)r}S_\tau - K)^+ \,\middle|\, S_t \right] \leqslant \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau - K)^+ \,\middle|\, S_t \right],$$

$t \in \mathbb{R}_+$, showing that it is always better to wait than to exercise at time t, and the optimal exercise time is $\tau^* = +\infty$. This argument does not apply to American put options.

See Exercise 5.6 for the pricing of perpetual American call options with dividends.

5.3 Finite Expiration American Options

In this section we consider finite expirations American put and call options with strike price K. The prices of such options can be expressed as

$$f(t, S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}}{\operatorname{Sup}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau - t)r} (K - S_\tau)^+ \,\middle|\, S_t \right],$$

and

$$f(t, S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}}{\operatorname{Sup}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau - t)r} (S_\tau - K)^+ \,\middle|\, S_t \right].$$

Two-choice optimal stopping at fixed times with finite expiration

We start by considering the optimal stopping problem in a simplified setting where $\tau \in \{t, T\}$ is allowed at time t to take only *two* values t (which corresponds to immediate exercise) and T (wait until maturity).

Proposition 5.7 For any stopping time $\tau \geqslant t$ we have

$$(x - K)^+ \leqslant \mathbb{E}^*[\mathrm{e}^{-(\tau - t)r} (S_\tau - K)^+ \mid S_t = x], \qquad x, t > 0. \tag{5.28}$$

Proof. Since the function $x \longmapsto x^+ = \operatorname{Max}(x, 0)$ is convex non-decreasing and the process $(\mathrm{e}^{-rt} S_t - \mathrm{e}^{-rt} K)_{t \in \mathbb{R}_+}$ is a *sub*martingale under $\mathbb{P}^*$, Proposition 4.3-(b)) shows that that $t \longmapsto (\mathrm{e}^{-rt} S_t - \mathrm{e}^{-rt} K)^+$ is a *sub*martingale by the Jensen inequality (4.2). Hence, by (4.12) applied to *sub*martingales, for any stopping time τ bounded by $t > 0$ we have

$$\begin{aligned}
(S_t - K)^+ &= \mathrm{e}^{rt} (\mathrm{e}^{-rt} S_t - \mathrm{e}^{-rt} K)^+ \\
&\leqslant \mathrm{e}^{rt} \mathbb{E}^*[(\mathrm{e}^{-r\tau} S_\tau - \mathrm{e}^{-r\tau} K)^+ \mid \mathcal{F}_t] \\
&= \mathbb{E}^*[\mathrm{e}^{-(\tau - t)r} (S_\tau - K)^+ \mid \mathcal{F}_t],
\end{aligned}$$

which yields (5.28). □

In particular, for the deterministic time $\tau := T \geqslant t$ we get

$$(x - K)^+ \leqslant \mathrm{e}^{-(T - t)r} \mathbb{E}^*[(S_T - K)^+ \mid S_t = x], \qquad x, t > 0.$$

as illustrated in Figure 5.8 using the Black-Scholes formula for European call options, see also Figure 2.16. In other words, taking $x = S_t$, the payoff $(S_t - K)^+$ of immediate exercise at time t is always lower than the expected payoff $\mathrm{e}^{-(T - t)r} \mathbb{E}^*[(S_T - K)^+ \mid S_t = x]$ given by exercise at maturity T. As a consequence, the optimal strategy for the investor is to wait until time T to exercise an American call option, rather than exercising earlier at time t.

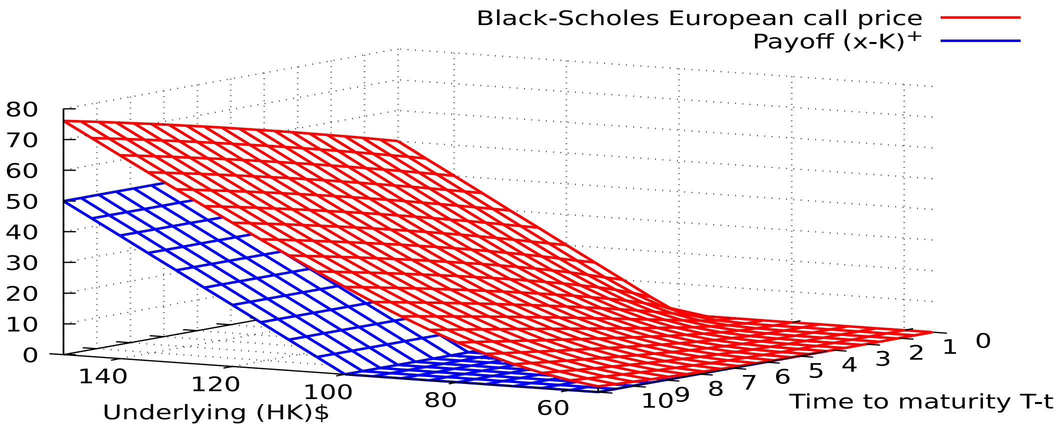


Figure 5.8: Expected Black-Scholes European call option price *vs* $(x,t) \longmapsto (x-K)^+$.

More generally, it can be shown that the price of the American call option equals the price of the corresponding European call option with maturity T, *i.e.*

$$f(t,S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+ \,\middle|\, S_t\right],$$

i.e. T is the optimal exercise date, see Proposition 5.8 below or §14.4 of Steele, 2001 for a proof.

Put options

For put options the situation is entirely different. The Black-Scholes formula for European put options shows that the inequality

$$(K-x)^+ \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid S_t = x],$$

does not always hold, as illustrated in Figure 5.9, see also Figure 2.16.

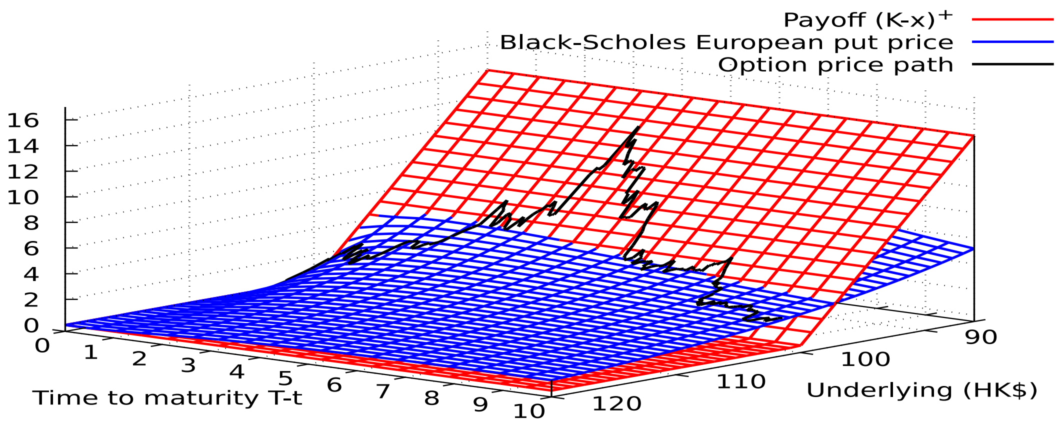


Figure 5.9: Black-Scholes put option price map *vs* $(x,t) \longmapsto (K-x)^+$.

As a consequence, the optimal exercise decision for a put option depends on whether $(K-S_t)^+ \leqslant \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid S_t]$ (in which case one chooses to exercise at time T) or $(K-S_t)^+ > \mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid S_t]$ (in which case one chooses to exercise at time t).

A view from above of the graph of Figure 5.9 shows the existence of an optimal frontier depending on time to maturity and on the price of the underlying asset, instead of being given by a constant level L^* as in Section 5.1, see Figure 5.10.

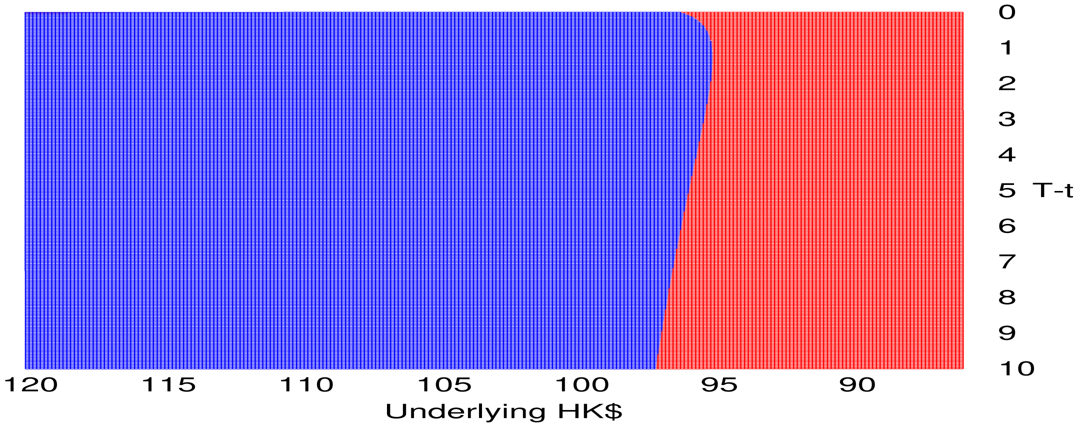


Figure 5.10: Optimal frontier for the exercise of a put option.

At a given time t, one will choose to exercise immediately if $(S_t, T-t)$ belongs to the blue area on the right, and to wait until maturity if $(S_t, T-t)$ belongs to the red area on the left.

When $r = 0$ we have $L^* = 0$, and the next remark shows that in this case it is always better to exercise the finite expiration American put option at maturity T, see also Exercise 5.8.

R Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative convex function such that $\phi(0) = 0$ and assume that r=0. Then the price of the finite expiration American option with payoff function ϕ on the underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ coincides with the corresponding vanilla option price:

$$f(t, S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T\\ \tau \text{ stopping time}}} \mathbb{E}^*\big[\phi(S_\tau) \,\big|\, S_t\big] = \mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big],$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option, and $\tau^* = T$.

Proof. Since the function ϕ is convex and $(S_{t+s})_{s\in[0,T-t]}$ is a martingale under the risk-neutral measure $\mathbb{P}^*$, we know from Proposition 4.3-(a)) that the process $(\phi(S_{t+s}))_{s\in[0,T-t]}$ is a martingale. Therefore, for all (bounded) stopping times τ comprised between t and T we have,

$$\mathbb{E}^*[\phi(S_\tau) \mid \mathcal{F}_t] \leqslant \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

i.e. it is always better to wait until time T than to exercise at time $\tau \in [t, T]$, and this yields

$$\operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T\\ \tau \text{ stopping time}}} \mathbb{E}^*\big[\phi(S_\tau) \,\big|\, S_t\big] \leqslant \mathbb{E}^*\big[\phi(S_T) \,\big|\, S_t\big].$$

Since the constant T is a stopping time, it attains the above supremum. □

5.4 PDE approach

Let us describe the PDE associated to American put options. After discretization $\{0 = t_0 < t_1 < \ldots < t_N = T\}$ of the time interval $[0, T]$, the optimal exercise strategy for the American put option can be described as follow at each time step:

If $f(t, S_t) > (K - S_t)^+$, wait.

If $f(t, S_t) = (K - S_t)^+$, exercise the option at time t.

Note that we cannot have $f(t,S_t) < (K-S_t)^+$.

If $f(t,S_t) > (K-S_t)^+$ the expected return of the hedging portfolio equals that of the riskless asset. This means that $f(t,S_t)$ follows the Black-Scholes PDE

$$rf(t,S_t) = \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t,S_t),$$

whereas if $f(t,S_t) = (K-S_t)^+$ it is not worth waiting as the return of the hedging portfolio is lower than that of the riskless asset:

$$rf(t,S_t) \geqslant \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t,S_t).$$

As a consequence, $f(t,x)$ should solve the following variational PDE, see Theorem 8.5.9 in Elliott and Kopp, 2005 and Jaillet, Lamberton, and Lapeyre, 1990:

$$\begin{cases} f(t,x) \geqslant f(T,x) = (K-x)^+, & \text{(5.29a)} \\[2ex] \dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{\sigma^2}{2}x^2\dfrac{\partial^2 f}{\partial x^2}(t,x) \leqslant rf(t,x), & \text{(5.29b)} \\[2ex] \left(\dfrac{\partial f}{\partial t}(t,x) + rx\dfrac{\partial f}{\partial x}(t,x) + \dfrac{\sigma^2}{2}x^2\dfrac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x)\right) & \\ \qquad\qquad\qquad\qquad \times \left(f(t,x) - (K-x)^+\right) = 0, & \text{(5.29c)} \end{cases}$$

$x > 0$, $0 \leqslant t \leqslant T$, subject to the terminal condition $f(T,x) = (K-x)^+$. In other words, equality holds either in (5.29a) or in (5.29b) due to the presence of the term $(f(t,x) - (K-x)^+)$ in (5.29c).

The optimal exercise strategy consists in exercising the put option as soon as the equality $f(u,S_u) = (K-S_u)^+$ holds, *i.e.* at the time

$$\tau^* = \inf\{u \geqslant t \ : \ f(u,S_u) = (K-S_u)^+\},$$

after which the process $\widetilde{f}_{L^*}(S_t)$ ceases to be a martingale and becomes a (strict) *super*martingale.

A simple procedure to compute numerically the price of an American put option is to use a finite difference scheme while simply enforcing the condition $f(t,x) \geqslant (K-x)^+$ at every iteration by adding the condition

$$f(t_i,x_j) := \mathrm{Max}(f(t_i,x_j),(K-x_j)^+)$$

right after the computation of $f(t_i,x_j)$.

The next Figure 5.11 shows a numerical resolution of the variational PDE (5.29a)-(5.29c) using the above simplified (implicit) finite difference scheme, see also Jacka, 1991 for properties of the optimal boundary function. In comparison with Figure 5.4, one can check that the PDE solution becomes time-dependent in the finite expiration case.

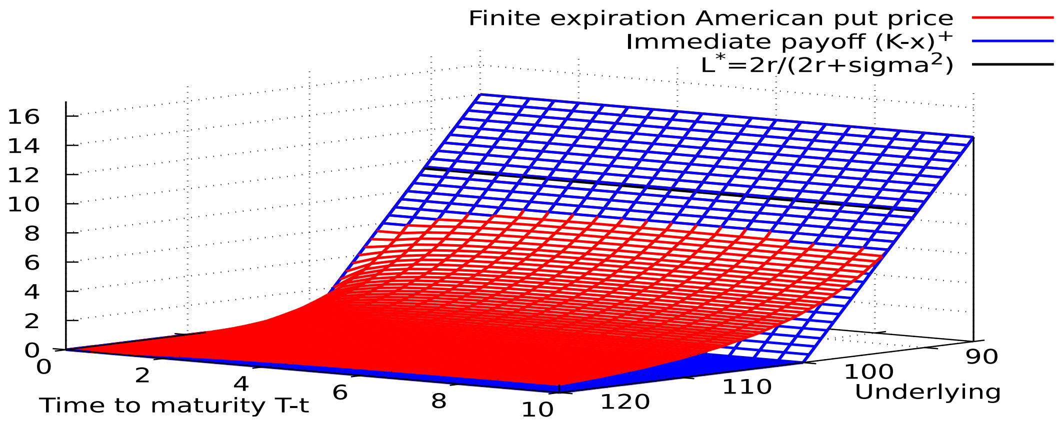


Figure 5.11: PDE estimates of finite expiration American put option prices.

In general, one will choose to exercise the put option when

$$f(t,S_t) = (K - S_t)^+,$$

i.e. within the blue area in Figure (5.11). We check that the optimal threshold $L^* = 90.64$ of the corresponding perpetual put option is within the exercise region, which is consistent since the perpetual optimal strategy should allow one to wait longer than in the finite expiration case.

The numerical computation of the American put option price

$$f(t,S_t) = \underset{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K - S_\tau)^+ \,\middle|\, S_t \right]$$

can also be done by dynamic programming and backward optimization using the Longstaff and Schwartz, 2001 (or Least Square Monte Carlo, LSM) algorithm as in Figure 5.12.

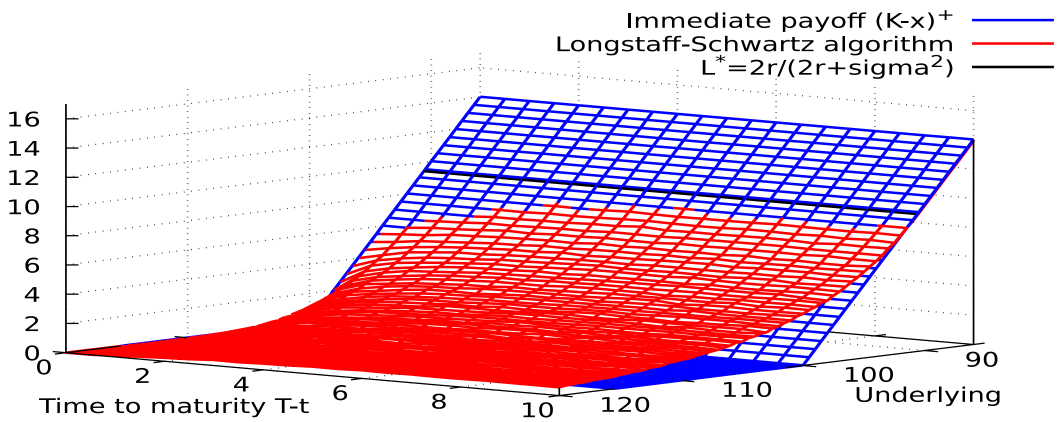


Figure 5.12: Longstaff-Schwartz estimates of finite expiration American put option prices.

In Figure 5.12 above and Figure 5.13 below the optimal threshold of the corresponding perpetual put option is again $L^* = 90.64$ and falls within the exercise region. Also, the optimal threshold is closer to L^* for large time to maturities, which shows that the perpetual option approximates the finite expiration option in that situation. In the next Figure 5.13 we compare the numerical computation of the American put option price by the finite difference and Longstaff-Schwartz methods.

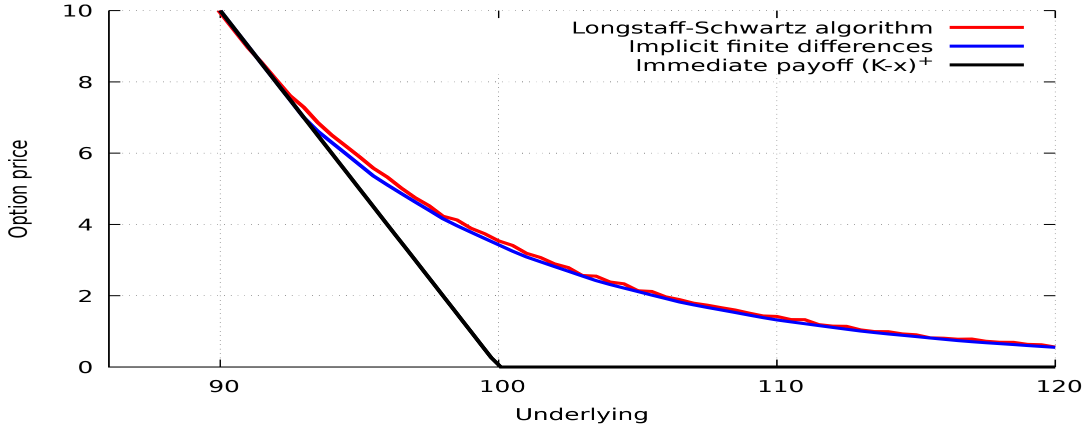


Figure 5.13: Comparison between Longstaff-Schwartz and finite differences.

It turns out that, although both results are very close, the Longstaff-Schwartz method performs better in the critical area close to exercise at it yields the expected continuously differentiable solution, and the simple numerical PDE solution tends to underestimate the optimal threshold. Also, a small error in the values of the solution translates into a large error on the value of the optimal exercise threshold.

The fOptions package in R contains a finite expiration American put option pricer based on the Barone-Adesi and Whaley, 1987 approximation, see Exercise 5.3, however the approximation is valid only for certain parameter ranges. See also Allegretto, Barone-Adesi, and Elliott, 1995 for a related approximation of the early exercise premium (5.17).

The finite expiration American call option

In the next proposition we compute the price of a finite expiration American call option with an arbitrary convex payoff function ϕ.

Proposition 5.8 Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative convex function such that $\phi(0) = 0$. The price of the finite expiration American call option with payoff function ϕ on the underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ is given by

$$f(t,S_t) = \sup_{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}\phi(S_\tau) \,\middle|\, S_t\right] = e^{-(T-t)r}\mathbb{E}^*\left[\phi(S_T) \,\middle|\, S_t\right],$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option, and $\tau^* = T$.

Proof. Since the function ϕ is convex and $\phi(0) = 0$, we have

$$\phi(px) = \phi((1-p)\times 0 + px) \leqslant (1-p)\times\phi(0) + p\phi(x) = p\phi(x), \tag{5.30}$$

for all $p \in [0,1]$ and $x \geqslant 0$. Next, taking $p := e^{-rs}$ in (5.30) we note that

$$\begin{aligned} e^{-rs}\mathbb{E}^*\left[\phi(S_{t+s}) \,\middle|\, \mathcal{F}_t\right] &\geqslant e^{-rs}\phi\left(\mathbb{E}^*\left[S_{t+s} \,\middle|\, \mathcal{F}_t\right]\right) \\ &\geqslant \phi\left(e^{-rs}\mathbb{E}^*\left[S_{t+s} \,\middle|\, \mathcal{F}_t\right]\right) \\ &= \phi(S_t), \end{aligned}$$

where we used Jensen's inequality (4.2) applied to the convex function ϕ, hence the process $s \longmapsto e^{-rs}\phi(S_{t+s})$ is a *sub*martingale. Hence by the optional stopping theorem for *sub*martingales, cf (4.8), for all (bounded) stopping times τ comprised between t and T we have,

$$\mathbb{E}^*[e^{-(\tau-t)r}\phi(S_\tau) \mid \mathcal{F}_t] \leqslant e^{-(T-t)r}\mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t],$$

Option type	Perpetual	Finite expiration
Put option	$\begin{cases} K - S_t, & 0 < S_t \leqslant L^*, \\ (K-L^*)\left(\dfrac{S_t}{L^*}\right)^{-2r/\sigma^2}, & S_t \geqslant L^*. \end{cases}$ $\tau^* = \tau_{L^*}$	Solve the PDE (5.29a)-(5.29c) for $f(t,x)$ or use Longstaff and Schwartz, 2001. $\tau^* = T \wedge \inf\{u \geqslant t : f(u,S_u) = (K-S_u)^+\}$.
Call option	S_t, $\quad \tau^* = +\infty$.	$e^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid S_t]$, $\quad \tau^* = T$.

Table 5.1: Optimal exercise strategies.

i.e. it is always better to wait until time T than to exercise at time $\tau \in [t,T]$, and this yields

$$\operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}\phi(S_\tau) \,\middle|\, S_t\right] \leqslant e^{-(T-t)r}\mathbb{E}^*\left[\phi(S_T) \,\middle|\, S_t\right].$$

The converse inequality

$$e^{-(T-t)r}\mathbb{E}^*\left[\phi(S_T) \,\middle|\, S_t\right] \leqslant \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}\phi(S_\tau) \,\middle|\, S_t\right],$$

is obvious because T is a stopping time. □

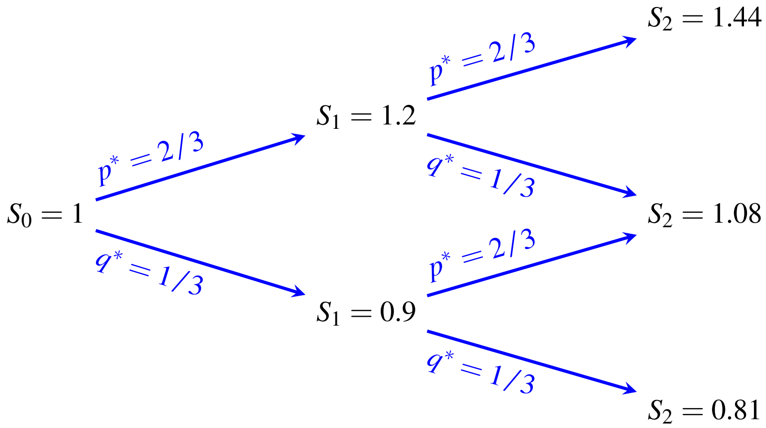
As a consequence of Proposition 5.8 applied to the convex function $\phi(x) = (x-K)^+$, the price of the finite expiration American call option is given by

$$\begin{aligned} f(t,S_t) &= \operatorname*{Sup}_{\substack{t \leqslant \tau \leqslant T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}(S_\tau-K)^+ \,\middle|\, S_t\right] \\ &= e^{-(T-t)r}\mathbb{E}^*\left[(S_T-K)^+ \,\middle|\, S_t\right], \end{aligned}$$

i.e. the optimal strategy is to wait until the maturity time T to exercise the option. In the following Table 5.1 we summarize the optimal exercise strategies for the pricing of American options.

Exercises

Exercise 5.1 Consider a two-step binomial model $(S_k)_{k=0,1,2}$ with interet rate $r = 0\%$ and risk-neutral probabilities (p^*, q^*):



a) At time $t = 1$, would you exercise the American put option with strike price $\underline{K = 1.25}$ if $S_1 = 1.2$? If $S_1 = 0.9$?
b) What would be your strategy at time $t = 0$?*

Exercise 5.2 Let $r > 0$ and $\sigma > 0$.
a) Show that for every $C > 0$, the function $f(x) := Cx^{-2r/\sigma^2}$ solves the differential equation

$$\begin{cases} rf(x) = rxf'(x) + \dfrac{1}{2}\sigma^2 x^2 f''(x), \\ \\ \lim\limits_{x\to\infty} f(x) = 0. \end{cases}$$

b) Show that for every $K > 0$ there exists a unique level $L^* \in (0, K)$ and constant $C > 0$ such that $f(x)$ also solves the smooth fit conditions $f(L^*) = K - L^*$ and $f'(L^*) = -1$.

Exercise 5.3 (Barone-Adesi and Whaley, 1987) We approximate the finite expiration American put option price with strike price K as

$$f(x,T) \simeq \begin{cases} \mathrm{BS_p}(x,T) + \alpha(x/S^*)^{-2r/\sigma^2}, & x > S^*, \quad (5.31) \\ \\ K - x, & x \leqslant S^*, \quad (5.32) \end{cases}$$

where $\alpha > 0$ is a parameter, $S^* > 0$ is called the *critical price*, and $\mathrm{BS}_p(x,T) = \mathrm{e}^{-rT}K\Phi(-d_-(x,T)) - x\Phi(-d_+(x,T))$ is the Black-Scholes *put* pricing function.
a) Find the value α^* of α which achieves a smooth fit (equality of derivatives in x) between (5.31) and (5.32) at $x = S^*$.
b) Derive the equation satisfied by the critical price S^*.

Exercise 5.4 Consider the process $(X_t)_{t\in\mathbb{R}_+}$ given by $X_t := tZ$, $t \in \mathbb{R}_+$, where $Z \in \{0,1\}$ is a Bernoulli random variable with $\mathbb{P}(Z = 1) = \mathbb{P}(Z = 0) = 1/2$. Given $\varepsilon \geqslant 0$, let the random time τ_ε be defined as

$$\tau_\varepsilon := \inf\{t > 0 \ : \ X_t > \varepsilon\},$$

with $\inf\emptyset = +\infty$, and let $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ denote the filtration generated by $(X_t)_{t\in\mathbb{R}_+}$.
a) Give the possible values of τ_ε in $[0,\infty]$ depending on the value of Z.
b) Take $\varepsilon = 0$. Is $\tau_0 := \inf\{t > 0 \ : \ X_t > 0\}$ an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time? *Hint:* Consider the event $\{\tau_0 > 0\}$.

*Download the corresponding discrete-time **IPython notebook** that can be run **here**.

c) Take $\varepsilon > 0$. Is $\tau_\varepsilon := \inf\{t > 0 \ : \ X_t > \varepsilon\}$ an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time? *Hint:* Consider the event $\{\tau_\varepsilon > t\}$ for $t \geqslant 0$.

Exercise 5.5 American put options with dividends, cf. Exercise 8.5 in Shreve, 2004. Consider a dividend-paying asset priced as

$$S_t = S_0 \,\mathrm{e}^{(r-\delta)t+\sigma\widehat{B}_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

where $r > 0$ is the risk-free interest rate, $\delta \geqslant 0$ is a continuous dividend rate, $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, and $\sigma > 0$ is the volatility coefficient. Consider the american put option with payoff

$$(\kappa - S_\tau)^+ = \begin{cases} \kappa - S_\tau & \text{if } S_\tau \leqslant \kappa, \\ 0 & \text{if } S_\tau > \kappa, \end{cases}$$

when exercised at the stopping time $\tau > 0$. Given $L \in (0,\kappa)$ a fixed level, consider the following exercise strategy for the above option:
- If $S_t \leqslant L$, then exercise at time t.
- If $S_t > L$, wait until the hitting time $\tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and exercise the option at time τ_L.

a) Give the intrinsic option value at time $t = 0$ in case $S_0 \leqslant L$.

In the sequel we work with $S_0 = x > L$.

b) Show that for all $\lambda \in \mathbb{R}$ the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined as

$$Z_t^{(\lambda)} := \left(\frac{S_t}{S_0}\right)^{\lambda} \mathrm{e}^{-((r-\delta)\lambda-\lambda(1-\lambda)\sigma^2/2)t}$$

is a martingale under the risk-neutral probability measure $\mathbb{P}^*$.

c) Show that $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ can be rewritten as

$$Z_t^{(\lambda)} = \left(\frac{S_t}{S_0}\right)^{\lambda} \mathrm{e}^{-rt}, \qquad t \in \mathbb{R}_+,$$

for two values $\lambda_- \leqslant 0 \leqslant \lambda_+$ of λ that can be computed explicitly.

d) Choosing the negative solution λ_-, show that

$$0 \leqslant Z_t^{(\lambda_-)} \leqslant \left(\frac{L}{S_0}\right)^{\lambda_-}, \qquad 0 \leqslant t < \tau_L.$$

e) Let τ_L denote the hitting time

$$\tau_L = \inf\{u \in \mathbb{R}_+ \ : \ S_u \leqslant L\}.$$

By application of the Stopping Time Theorem 4.5 to the martingale $(Z_t)_{t\in\mathbb{R}_+}$, show that

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \left(\frac{S_0}{L}\right)^{\lambda_-}, \tag{5.33}$$

with

$$\lambda_- := \frac{-(r-\delta-\sigma^2/2) - \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}. \tag{5.34}$$

f) Show that for all $L \in (0,K)$ we have

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \,\big|\, S_0 = x\right] = \begin{cases} K-x, & 0 < x \leqslant L, \\ \\ (K-L)\left(\dfrac{x}{L}\right)^{\frac{-(r-\delta-\sigma^2/2)-\sqrt{(r-\delta-\sigma^2/2)^2+4r\sigma^2/2}}{\sigma^2}}, & x \geqslant L. \end{cases}$$

g) Show that the value L^* of L that maximizes

$$f_L(x) := \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \,\big|\, S_0 = x\right]$$

for every $x > 0$ is given by

$$L^* = \frac{\lambda_-}{\lambda_- - 1}K.$$

h) Show that

$$f_{L^*}(x) = \begin{cases} K-x, & 0 < x \leqslant L^* = \dfrac{\lambda_-}{\lambda_- - 1}K, \\ \\ \left(\dfrac{1-\lambda_-}{K}\right)^{\lambda_- - 1}\left(\dfrac{x}{-\lambda_-}\right)^{\lambda_-}, & x \geqslant L^* = \dfrac{\lambda_-}{\lambda_- - 1}K, \end{cases}$$

i) Show by hand computation that $f_{L^*}(x)$ satisfies the variational differential equation

$$\begin{cases} f_{L^*}(x) \geqslant (K-x)^+, & (5.35a) \\ \\ (r-\delta)xf'_{L^*}(x) + \dfrac{1}{2}\sigma^2x^2 f''_{L^*}(x) \leqslant rf_{L^*}(x), & (5.35b) \\ \\ \left(rf_{L^*}(x) - (r-\delta)xf'_{L^*}(x) - \dfrac{1}{2}\sigma^2x^2 f''_{L^*}(x)\right) & (5.35c) \\ \qquad \times \left(f_{L^*}(x) - (K-x)^+\right) = 0. \end{cases}$$

j) Using Itô's formula, check that the discounted portfolio value process

$$t \longmapsto \mathrm{e}^{-rt} f_{L^*}(S_t)$$

is a *super*martingale.

k) Show that we have

$$f_{L^*}(S_0) \geqslant \operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \,\big|\, S_0\right].$$

l) Show that the stopped process

$$s \longmapsto \mathrm{e}^{-(s\wedge\tau_{L^*})r} f_{L^*}(S_{s\wedge\tau_{L^*}}), \qquad s \in \mathbb{R}_+,$$

is a martingale, and that

$$f_{L^*}(S_0) \leqslant \operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+\right].$$

m) Fix $t \in \mathbb{R}_+$ and let τ_{L^*} denote the hitting time

$$\tau_{L^*} = \inf\{u \geqslant t \ : \ S_u = L^*\}.$$

Conclude that the price of the perpetual American put option with dividend is given for all $t \in \mathbb{R}_+$ by

$$\begin{aligned} f_{L^*}(S_t) &= \mathbb{E}^*\left[e^{-(\tau_{L^*}-t)r}(K - S_{\tau_{L^*}})^+ \,\middle|\, S_t \right] \\ &= \begin{cases} K - S_t, & 0 < S_t \leqslant \dfrac{\lambda_-}{\lambda_- - 1}K, \\ \left(\dfrac{1-\lambda_-}{K}\right)^{\lambda_- - 1}\left(\dfrac{S_t}{-\lambda_-}\right)^{\lambda_-}, & S_t \geqslant \dfrac{\lambda_-}{\lambda_- - 1}K, \end{cases} \end{aligned}$$

where $\lambda_- < 0$ is given by (5.34), and

$$\tau_{L^*} = \inf\{u \geqslant t \ : \ S_u \leqslant L\}.$$

Exercise 5.6 American call options with dividends, see § 9.3 of Wilmott, 2006. Consider a dividend-paying asset priced as $S_t = S_0 e^{(r-\delta)t + \sigma \widehat{B}_t - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, where $r > 0$ is the risk-free interest rate, $\delta \geqslant 0$ is a continuous dividend rate, and $\sigma > 0$.

a) Show that for all $\lambda \in \mathbb{R}$ the process $Z_t^{(\lambda)} := (S_t)^\lambda e^{-((r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2)t}$ is a martingale under $\mathbb{P}^*$.
b) Show that we have $Z_t^{(\lambda)} = (S_t)^\lambda e^{-rt}$ for two values $\lambda_- \leqslant 0$, $1 \leqslant \lambda_+$ of λ satisfying a certain equation.
c) Show that $0 \leqslant Z_t^{(\lambda_+)} \leqslant L^{\lambda_+}$ for $0 \leqslant t < \tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and compute $\mathbb{E}^*\left[e^{-r\tau_L}(S_{\tau_L} - K)^+ \,\middle|\, S_0 = x \right]$ when $S_0 = x < L$ and $K < L$.

Exercise 5.7 Optimal stopping for exchange options (Gerber and Shiu, 1996). We consider two risky assets S_1 and S_2 modeled by

$$S_1(t) = S_1(0) e^{\sigma_1 W_t + rt - \sigma_2^2 t/2} \quad \text{and} \quad S_2(t) = S_2(0) e^{\sigma_2 W_t + rt - \sigma_2^2 t/2}, \tag{5.36}$$

$t \in \mathbb{R}_+$, with $\underline{\sigma_2 > \sigma_1 \geqslant 0}$, and the perpetual optimal stopping problem

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau}(S_1(\tau) - S_2(\tau))^+],$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$.

a) Find $\alpha > 1$ such that the process

$$Z_t := e^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha}, \qquad t \in \mathbb{R}_+, \tag{5.37}$$

is a martingale.
b) For some fixed $L \geqslant 1$, consider the hitting time

$$\tau_L = \inf\{t \in \mathbb{R}_+ \ : \ S_1(t) \geqslant L S_2(t)\},$$

and show that

$$\mathbb{E}[e^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+] = (L-1)\,\mathbb{E}[e^{-r\tau_L} S_2(\tau_L)].$$

c) By an application of the Stopping Time Theorem 4.5 to the martingale (5.37), show that we have

$$\mathbb{E}[\mathrm{e}^{-r\tau_L}(S_1(\tau_L)-S_2(\tau_L))^+] = \frac{L-1}{L^\alpha}S_1(0)^\alpha S_2(0)^{1-\alpha}.$$

d) Show that the price of the perpetual exchange option is given by

$$\operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}[\mathrm{e}^{-r\tau}(S_1(\tau)-S_2(\tau))^+] = \frac{L^*-1}{(L^*)^\alpha}S_1(0)^\alpha S_2(0)^{1-\alpha},$$

where

$$L^* = \frac{\alpha}{\alpha-1}.$$

e) As an application of Question (d)), compute the perpetual American put option price

$$\operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}[\mathrm{e}^{-r\tau}(\kappa - S_2(\tau))^+]$$

when $r = \sigma_2^2/2$.

Exercise 5.8 Consider an underlying asset whose price is written as

$$S_t = S_0\,\mathrm{e}^{rt+\sigma B_t-\sigma^2 t/2}, \qquad t\in\mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $\sigma>0$ denotes the volatility coefficient, and $\underline{r\in\mathbb{R}}$ is the risk-free interest rate. For any $\lambda\in\mathbb{R}$ we consider the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ defined by

$$\begin{aligned} Z_t^{(\lambda)} &:= \mathrm{e}^{-rt}(S_t)^\lambda \\ &= (S_0)^\lambda\,\mathrm{e}^{\lambda\sigma B_t-\lambda^2\sigma^2 t/2+(\lambda-1)(\lambda+2r/\sigma^2)\sigma^2 t/2}, \quad t\in\mathbb{R}_+. \end{aligned} \tag{5.38}$$

a) Assume that $\underline{r\geqslant -\sigma^2/2}$. Show that, under $\mathbb{P}^*$, the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ is a *super*martingale when $-2r/\sigma^2\leqslant\lambda\leqslant 1$, and that it is a *sub*martingale when $\lambda\in(-\infty,-2r/\sigma^2]\cup[1,\infty)$.
b) Assume that $\underline{r\leqslant -\sigma^2/2}$. Show that, under $\mathbb{P}^*$, the process $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ is a *super*martingale when $1\leqslant\lambda\leqslant -2r/\sigma^2$, and that it is a *sub*martingale when $\lambda\in(-\infty,1]\cup[-2r/\sigma^2,\infty)$.
c) From this question onwards, we assume that $\underline{r<0}$. Given $L>0$, let τ_L denote the hitting time

$$\tau_L = \inf\{u\in\mathbb{R}_+ \ : \ S_u = L\}.$$

By application of the Stopping Time Theorem 4.5 to $\big(Z_t^{(\lambda)}\big)_{t\in\mathbb{R}_+}$ to suitable values of λ, show that

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \leqslant \begin{cases} \left(\dfrac{x}{L}\right)^{\mathrm{Max}(1,-2r/\sigma^2)}, & x\geqslant L, \\[2ex] \left(\dfrac{x}{L}\right)^{\min(1,-2r/\sigma^2)}, & 0<x\leqslant L. \end{cases}$$

d) Deduce an upper bound on the price

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \,\middle|\, S_0 = x\right]$$

of the European put option exercised in finite time under the stopping strategy τ_L when $L\in(0,K)$ and $x\geqslant L$.

e) Show that when $r \leqslant -\sigma^2/2$, the upper bound of Question (d)) increases and tends to $+\infty$ when L decreases to 0.

f) Find an upper bound on the price

$$\mathbb{E}^* \left[\mathrm{e}^{-r\tau_L} (S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x \right]$$

of the European call option exercised in finite time under the stopping strategy τ_L when $L \geqslant K$ and $x \leqslant L$.

g) Show that when $-\sigma^2/2 \leqslant r < 0$, the upper bound of Question (f)) increases in $L \geqslant K$ and tends to S_0 as L increases to $+\infty$.

Exercise 5.9 Perpetual American binary options.

a) Compute the price

$$C_b^{\mathrm{Am}}(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau - t)r} \mathbb{1}_{\{S_\tau \geqslant K\}} \,\middle|\, S_t \right]$$

of the perpetual American binary call option.

b) Compute the price

$$P_b^{\mathrm{Am}}(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau - t)r} \mathbb{1}_{\{S_\tau \leqslant K\}} \,\middle|\, S_t \right]$$

of the perpetual American binary put option.

Exercise 5.10 Finite expiration American binary options. An American binary (or digital) call (resp. put) option with maturity $T > 0$ on an underlying asset process $(S_t)_{t \in \mathbb{R}_+} = (\mathrm{e}^{rt + \sigma B_t - \sigma^2 t/2})_{t \in \mathbb{R}_+}$ can be exercised at any time $t \in [0, T]$, at the choice of the option holder.

The call (resp. put) option exercised at time t yields the payoff $\mathbb{1}_{[K,\infty)}(S_t)$ (resp. $\mathbb{1}_{[0,K]}(S_t)$), and the option holder wants to find an exercise strategy that will maximize his payoff.

a) Consider the following possible situations at time t:
   i) $S_t \geqslant K$,
   ii) $S_t < K$.

   In each case (i) and (ii), tell whether you would choose to exercise the call option immediately, or to wait.

b) Consider the following possible situations at time t:
   i) $S_t > K$,
   ii) $S_t \leqslant K$.

   In each case (i) and (ii), tell whether you would choose to exercise the put option immediately, or to wait.

c) The price $C_d^{\mathrm{Am}}(t, T, S_t)$ of an American binary call option is known to satisfy the Black-Scholes PDE

$$rC_d^{\mathrm{Am}}(t, T, x) = \frac{\partial C_d^{\mathrm{Am}}}{\partial t}(t, T, x) + rx\frac{\partial C_d^{\mathrm{Am}}}{\partial x}(t, T, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C_d^{\mathrm{Am}}}{\partial x^2}(t, T, x).$$

   Based on your answers to Question (a)), how would you set the boundary conditions $C_d^{\mathrm{Am}}(t, T, K)$, $0 \leqslant t < T$, and $C_d^{\mathrm{Am}}(T, T, x)$, $0 \leqslant x < K$?

d) The price $P_d^{\mathrm{Am}}(t, T, S_t)$ of an American binary put option is known to satisfy the same Black-Scholes PDE

$$rP_d^{\mathrm{Am}}(t, T, x) = \frac{\partial P_d^{\mathrm{Am}}}{\partial t}(t, T, x) + rx\frac{\partial P_d^{\mathrm{Am}}}{\partial x}(t, T, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P_d^{\mathrm{Am}}}{\partial x^2}(t, T, x). \tag{5.39}$$

   Based on your answers to Question (b)), how would you set the boundary conditions $P_d^{\mathrm{Am}}(t, T, K)$, $0 \leqslant t < T$, and $P_d^{\mathrm{Am}}(T, T, x)$, $x > K$?

e) Show that the optimal exercise strategy for the American binary call option with strike price K is to exercise as soon as the price of the underlying asset reaches the level K, *i.e.* at time

$$\tau_K := \inf\{u \geqslant t \ : \ S_u = K\},$$

starting from any level $S_t \leqslant K$, and that the price $C_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary call option is given by

$$C_d^{\mathrm{Am}}(t,x) = \mathbb{E}\left[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{\tau_K<T\}} \mid S_t = x\right].$$

f) Show that the price $C_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary call option is equal to

$$C_d^{\mathrm{Am}}(t,T,x) = \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right) + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leqslant x \leqslant K,$$

that this formula is consistent with the answer to Question (c)), and that it recovers the answer to Question (a)) of Exercise 5.9 as T tends to infinity.

g) Show that the optimal exercise strategy for the American binary put option with strike price K is to exercise as soon as the price of the underlying asset reaches the level K, *i.e.* at time

$$\tau_K := \inf\{u \geqslant t \ : \ S_u = K\},$$

starting from any level $S_t \geqslant K$, and that the price $P_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary put option is

$$P_d^{\mathrm{Am}}(t,T,x) = \mathbb{E}[\mathrm{e}^{-(\tau_K-t)r}\mathbb{1}_{\{\tau_K<T\}} \mid S_t = x], \qquad x \geqslant K.$$

h) Show that the price $P_d^{\mathrm{Am}}(t,T,S_t)$ of the American binary put option is equal to

$$P_d^{\mathrm{Am}}(t,T,x) = \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right) + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad x \geqslant K,$$

that this formula is consistent with the answer to Question (d)), and that it recovers the answer to Question (b)) of Exercise 5.9 as T tends to infinity.

i) Does the standard call-put parity relation hold for American binary options?

Exercise 5.11 American forward contracts. Consider $(S_t)_{t\in\mathbb{R}_+}$ an asset price process given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

where $r > 0$ and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$.

a) Compute the price

$$f(t,S_t) = \underset{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(K-S_\tau) \,\big|\, S_t\right],$$

and optimal exercise strategy of a finite expiration American-type short forward contract with strike price K on the underlying asset priced $(S_t)_{t\in\mathbb{R}_+}$, with payoff $K - S_\tau$ when exercised at time $\tau \in [0,T]$.

b) Compute the price

$$f(t,S_t) = \underset{\substack{t\leqslant\tau\leqslant T \\ \tau \text{ stopping time}}}{\mathrm{Sup}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}(S_\tau-K) \,\big|\, S_t\right],$$

and optimal exercise strategy of a finite expiration American-type long forward contract with strike price K on the underlying asset priced $(S_t)_{t\in\mathbb{R}_+}$, with payoff $S_\tau - K$ when exercised at time $\tau \in [0,T]$.

c) How are the answers to Questions (a)) and (b)) modified in the case of perpetual options with $T = +\infty$?

Exercise 5.12 Consider an underlying asset price process written as

$$S_t = S_0 \,\mathrm{e}^{rt+\sigma\widehat{B}_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

where $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, with $\sigma, r > 0$.

a) Show that the processes $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ defined as

$$Y_t := \mathrm{e}^{-rt} S_t^{-2r/\sigma^2} \quad \text{and} \quad Z_t := \mathrm{e}^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

are both martingales under $\mathbb{P}^*$.

b) Let τ_L denote the hitting time

$$\tau_L = \inf\{u \in \mathbb{R}_+ \ : \ S_u = L\}.$$

By application of the Stopping Time Theorem 4.5 to the martingales $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$, show that

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \,\middle|\, S_0 = x\right] = \begin{cases} \dfrac{x}{L}, & 0 < x \leqslant L, \\ \\ \left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L. \end{cases}$$

c) Compute the price $\mathbb{E}^*\big[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})\big]$ of a short forward contract under the exercise strategy τ_L.

d) Show that for every value of $S_0 = x$ there is an optimal value L_x^* of L that maximizes $L \longmapsto \mathbb{E}\big[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})\big]$.

e) Would you use the stopping strategy

$$\tau_{L_x^*} = \inf\{u \in \mathbb{R}_+ \ : \ S_u = L_x^*\}$$

as an optimal exercise strategy for the short forward contract with payoff $K - S_\tau$?

Exercise 5.13 Let $p \geqslant 1$ and consider a power put option with payoff

$$((\kappa - S_\tau)^+)^p = \begin{cases} (\kappa - S_\tau)^p & \text{if } S_\tau \leqslant \kappa, \\ 0 & \text{if } S_\tau > \kappa, \end{cases}$$

exercised at time τ, on an underlying asset whose price S_t is written as

$$S_t = S_0 \,\mathrm{e}^{rt+\sigma B_t-\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$, $r \geqslant 0$ is the risk-free interest rate, and $\sigma > 0$ is the volatility coefficient.

Given $L \in (0, \kappa)$ a fixed price, consider the following choices for the exercise of a *put* option with strike price κ:

i) If $S_t \leqslant L$, then exercise at time t.

ii) Otherwise, wait until the first hitting time $\tau_L := \inf\{u \geqslant t \ : \ S_u = L\}$, and exercise the option at time τ_L.

a) Under the above strategy, what is the option payoff equal to if $S_t \leqslant L$?

b) Show that in case $S_t > L$, the price of the option is equal to

$$f_L(S_t) = (\kappa - L)^p \, \mathbb{E}^* \left[\mathrm{e}^{-(\tau_L - t)r} \,\middle|\, S_t \right].$$

c) Compute the price $f_L(S_t)$ of the option at time t.

Hint: Recall that by (5.4) we have $\mathbb{E}^*[\mathrm{e}^{-(\tau_L - t)r} \mid S_t = x] = (x/L)^{-2r/\sigma^2}$, $x \geqslant L$.

d) Compute the optimal value L^* that maximizes $L \longmapsto f_L(x)$ for all fixed $x > 0$.

Hint: Observe that, geometrically, the slope of $x \longmapsto f_L(x)$ at $x = L^*$ is equal to $-p(\kappa - L^*)^{p-1}$.

e) How would you compute the American put option price

$$f(t, S_t) = \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau - t)r} ((\kappa - S_\tau)^+)^p \,\middle|\, S_t \right] \ ?$$

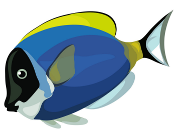
Exercise 5.14 Same questions as in Exercise 5.13, this time for the option with payoff $\kappa - (S_\tau)^p$ exercised at time τ, with $p > 0$.

6. Stochastic Calculus for Jump Processes

Jump processes are stochastic processes whose trajectories have discontinuities called jumps, that can occur at random times. This chapter presents the construction of jump processes with independent increments, such as the Poisson and compound Poisson processes, followed by an introduction to stochastic integrals and stochastic calculus with jumps. We also present the Girsanov Theorem for jump processes, which will be used for the construction of risk-neutral probability measures in Chapter 7 for option pricing and hedging in markets with jumps, in relation with market incompleteness.

6.1 The Poisson Process

The most elementary and useful jump process is the *standard Poisson process* $(N_t)_{t\in\mathbb{R}_+}$ which is a *counting process*, *i.e.* $(N_t)_{t\in\mathbb{R}_+}$ has jumps of size $+1$ only, and its paths are constant in between two jumps. In addition, the standard Poisson process starts at $\underline{N_0 = 0}$.



The Poisson process can be used to model discrete arrival times such as claim dates in insurance, or connection logs.

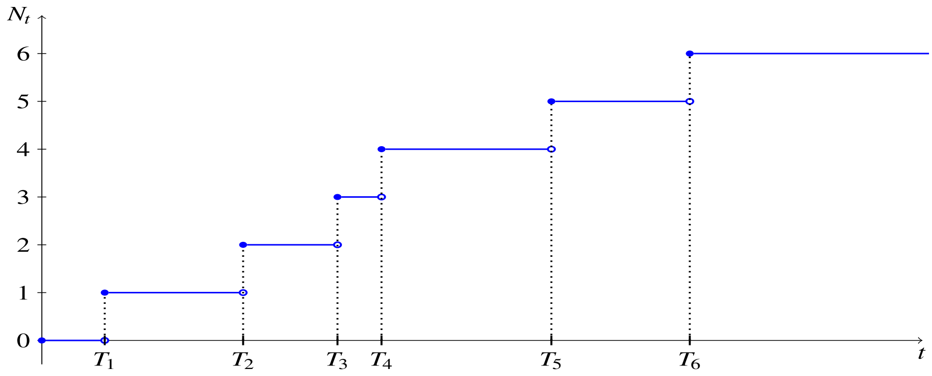


Figure 6.1: Sample path of a Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In other words, the value N_t at time t is given by*

$$N_t = \sum_{k\geqslant 1} \mathbb{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+, \tag{6.1}$$

where

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 & \text{if } t \geqslant T_k, \\ \\ 0 & \text{if } 0 \leqslant t < T_k, \end{cases}$$

$k \geqslant 1$, and $(T_k)_{k\geqslant 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty.$$

In addition, the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is assumed to satisfy the following conditions:

1. Independence of increments: for all $0 \leqslant t_0 < t_1 < \cdots < t_n$ and $n \geqslant 1$ the increments

$$N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}},$$

are mutually independent random variables.

2. Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leqslant s \leqslant t$.

The meaning of the above stationarity condition is that for all fixed $k \in \mathbb{N}$ we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = k) = \mathbb{P}(N_t - N_s = k),$$

for all $h > 0$, *i.e.*, the value of the probability

$$\mathbb{P}(N_{t+h} - N_{s+h} = k)$$

does not depend on $h > 0$, for all fixed $0 \leqslant s \leqslant t$ and $k \in \mathbb{N}$.

Based on the above assumption, given $T > 0$ a time value, a natural question arises:

*The notation N_t is not to be confused with the notation used for numéraire processes.

what is the probability distribution of the random variable N_T?

We already know that N_t takes values in $\mathbb{N}$ and therefore it has a discrete distribution for all $t \in \mathbb{R}_+$.

It is a remarkable fact that the distribution of the increments of $(N_t)_{t\in\mathbb{R}_+}$, can be completely determined from the above conditions, as shown in the following theorem.

As seen in the next result, cf. Theorem 4.1 in Bosq and Nguyen, 1996, the Poisson increment $N_t - N_s$ has the Poisson distribution with parameter $(t-s)\lambda$.

Theorem 6.1 Assume that the counting process $(N_t)_{t\in\mathbb{R}_+}$ satisfies the above independence and stationarity Conditions 1 and 2 on page 136. Then for all fixed $0 \leqslant s \leqslant t$ the increment $N_t - N_s$ follows the Poisson distribution with parameter $(t-s)\lambda$, *i.e.* we have

$$\mathbb{P}(N_t - N_s = k) = \mathrm{e}^{-(t-s)\lambda}\frac{((t-s)\lambda)^k}{k!}, \qquad k \geqslant 0, \tag{6.2}$$

for some constant $\lambda > 0$.

The parameter $\lambda > 0$ is called the <u>intensity</u> of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and it is given by

$$\lambda := \lim_{h\to 0}\frac{1}{h}\mathbb{P}(N_h = 1). \tag{6.3}$$

The proof of the above Theorem 6.1 is technical and not included here, cf. *e.g.* Bosq and Nguyen, 1996 for details, and we could in fact take this distribution property (6.2) as one of the hypotheses that define the Poisson process.

Precisely, we could restate the definition of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ as being a stochastic process defined by (6.1), which is assumed to have independent increments distributed according to the Poisson distribution, in the sense that for all $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n$,

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$((t_1 - t_0)\lambda, \ldots, (t_n - t_{n-1})\lambda).$$

In particular, N_t has the Poisson distribution with parameter λt, *i.e.*,

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!}\mathrm{e}^{-\lambda t}, \qquad t > 0.$$

The *expected value* $\mathbb{E}[N_t]$ and variance of N_t can be computed as

$$\mathbb{E}[N_t] = \mathrm{Var}[N_t] = \lambda t. \tag{6.4}$$

As a consequence, the *dispersion index* of the Poisson process is

$$\frac{\mathrm{Var}[N_t]}{\mathbb{E}[N_t]} = 1, \qquad t \in \mathbb{R}_+. \tag{6.5}$$

Short time behaviour

From (6.3) above we deduce the *short time asymptotics*[*]

$$\begin{cases} \mathbb{P}(N_h = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), & h \to 0, \\ \mathbb{P}(N_h = 1) = h\lambda\, \mathrm{e}^{-h\lambda} \simeq h\lambda, & h \to 0. \end{cases}$$

By stationarity of the Poisson process we also find more generally that

$$\begin{cases} \mathbb{P}(N_{t+h} - N_t = 0) = \mathrm{e}^{-h\lambda} = 1 - h\lambda + o(h), & h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 1) = h\lambda\, \mathrm{e}^{-h\lambda} \simeq h\lambda, & h \to 0, \\ \mathbb{P}(N_{t+h} - N_t = 2) \simeq h^2 \dfrac{\lambda^2}{2} = o(h), & h \to 0, \quad t > 0, \end{cases} \tag{6.6}$$

for all $t > 0$. This means that within a "short" interval $[t, t+h]$ of length h, the increment $N_{t+h} - N_t$ behaves like a Bernoulli random variable with parameter λh. This fact can be used for the random simulation of Poisson process paths.

More generally, for $k \geqslant 1$ we have

$$\mathbb{P}(N_{t+h} - N_t = k) \simeq h^k \frac{\lambda^k}{k!}, \qquad h \to 0, \qquad t > 0.$$

The intensity of the Poisson process can in fact be made time-dependent (*e.g.* by a time change), in which case we have

$$\mathbb{P}(N_t - N_s = k) = \exp\left(-\int_s^t \lambda(u)du\right) \frac{\left(\int_s^t \lambda(u)du\right)^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

Assuming that $\lambda(t)$ is a continuous function of time t we have in particular, as h tends to zero,

$$\mathbb{P}(N_{t+h} - N_t = k) = \begin{cases} \exp\left(-\int_t^{t+h} \lambda(u)du\right) = 1 - \lambda(t)h + o(h), & k = 0, \\ \exp\left(-\int_t^{t+h} \lambda(u)du\right) \int_t^{t+h} \lambda(u)du = \lambda(t)h + o(h), & k = 1, \\ o(h), & k \geqslant 2. \end{cases}$$

The next code and Figure 6.2 present a simulation of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ according to its short time behavior (6.6).

```
lambda = 0.6;T=10;N=1000;dt=T*1.0/N
t=0;s=c();for (k in 1:N) {if (runif(1)<lambda*dt) {s=c(s,t)};t=t+dt}
dev.new(width=T, height=5)
plot(stepfun(s,cumsum(c(0,rep(1,length(s))))),xlim =c(0,T),xlab="t",ylab="Nt",pch=1, cex=0.8,
     col='blue', lwd=2, main="")
```

[*] The notation $f(h) = o(h^k)$ means $\lim_{h\to 0} f(h)/h^k = 0$, and $f(h) \simeq h^k$ means $\lim_{h\to 0} f(h)/h^k = 1$.

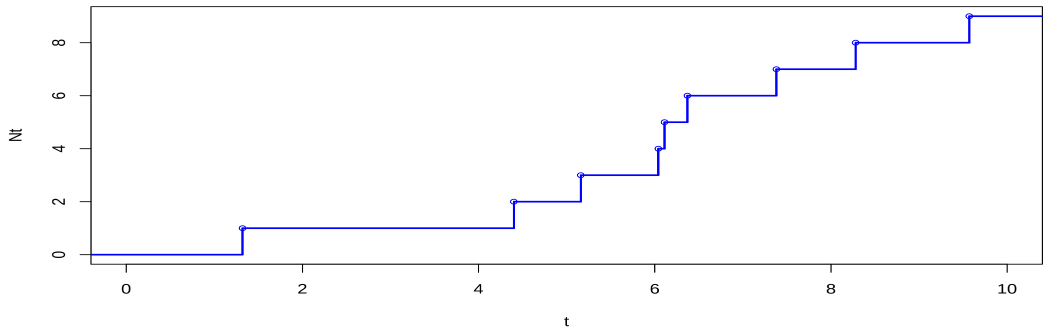


Figure 6.2: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

The intensity process $(\lambda(t))_{t\in\mathbb{R}_+}$ can also be made random, as in the case of Cox processes.

Poisson process jump times

In order to determine the distribution of the first jump time T_1 we note that we have the equivalence

$$\{T_1 > t\} \Longleftrightarrow \{N_t = 0\},$$

which implies

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,$$

i.e., T_1 has an exponential distribution with parameter $\lambda > 0$.

In order to prove the next proposition we note that more generally, we have the equivalence

$$\{T_n > t\} \Longleftrightarrow \{N_t \leqslant n-1\},$$

for all $n \geqslant 1$. This allows us to compute the distribution of the random jump time T_n with its probability density function. It coincides with the *gamma* distribution with integer parameter $n \geqslant 1$, also known as the Erlang distribution in queueing theory.

Proposition 6.2 For all $n \geqslant 1$ the probability distribution of T_n has the gamma probability density function

$$t \longmapsto \lambda^n \mathrm{e}^{-\lambda t} \frac{t^{n-1}}{(n-1)!}$$

on $\mathbb{R}_+$, *i.e.*, for all $t > 0$ the probability $\mathbb{P}(T_n \geqslant t)$ is given by

$$\mathbb{P}(T_n \geqslant t) = \lambda^n \int_t^\infty \mathrm{e}^{-\lambda s} \frac{s^{n-1}}{(n-1)!} ds.$$

Proof. We have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = \mathrm{e}^{-\lambda t}, \qquad t \in \mathbb{R}_+,$$

and by induction, assuming that

$$\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty \mathrm{e}^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \geqslant 2,$$

we obtain

$$\begin{aligned}
\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \geqslant T_{n-1}) + \mathbb{P}(T_{n-1} > t) \\
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= \mathrm{e}^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty \mathrm{e}^{-\lambda s}\frac{(\lambda s)^{n-2}}{(n-2)!}ds \\
&= \lambda \int_t^\infty \mathrm{e}^{-\lambda s}\frac{(\lambda s)^{n-1}}{(n-1)!}ds, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

where we applied an integration by parts to derive the last line. □

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!},$$

i.e., $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geqslant 1$, is the probability density function of the random jump time T_n.

In addition to Proposition 6.2 we could show the following proposition which relies on the *strong Markov property*, see *e.g.* Theorem 6.5.4 of Norris, 1998.

Proposition 6.3 The (random) interjump times

$$\tau_k := T_{k+1} - T_k$$

spent at state $k \in \mathbb{N}$, with $T_0 = 0$, form a sequence of independent identically distributed random variables having the exponential distribution with parameter $\lambda > 0$, *i.e.*,

$$\mathbb{P}(\tau_0 > t_0, \ldots, \tau_n > t_n) = \mathrm{e}^{-(t_0+t_1+\cdots+t_n)\lambda}, \qquad t_0, t_1, \ldots, t_n \in \mathbb{R}_+.$$

As the expectation of the exponentially distributed random variable τ_k with parameter $\lambda > 0$ is given by

$$\mathbb{E}[\tau_k] = \lambda \int_0^\infty x\mathrm{e}^{-\lambda x}dx = \frac{1}{\lambda},$$

we can check that the *nth* jump time $T_n = \tau_0 + \cdots + \tau_{n-1}$ has the mean

$$\mathbb{E}[T_n] = \frac{n}{\lambda}, \qquad n \geqslant 1.$$

Consequently, the higher the intensity $\lambda > 0$ is (*i.e.*, the higher the probability of having a jump within a small interval), the smaller the time spent in each state $k \in \mathbb{N}$ is on average.

In addition, conditionally to $\{N_T = n\}$, the n jump times on $[0, T]$ of the Poisson process $(N_t)_{t \in \mathbb{R}_+}$ are independent uniformly distributed random variables on $[0, T]^n$, cf. *e.g.* § 12.1 of Privault, 2018. This fact can be useful for the random simulation of the Poisson process.

As a consequence of Propositions 6.2 and 6.2, random samples of Poisson process jump times can be generated using the following *R* code.

```
lambda = 0.6;n = 100;T=10;Z<-cumsum(c(0,rep(1,n)))
tau_n <- rexp(n,rate=lambda); Tn <- cumsum(tau_n)
dev.new(width=T, height=5)
plot(stepfun(Tn,Z),xlim =c(0,T),ylim=c(0,8),xlab="t",ylab="Nt",pch=1, cex=0.8, col="blue", lwd=2,
     main="")
```

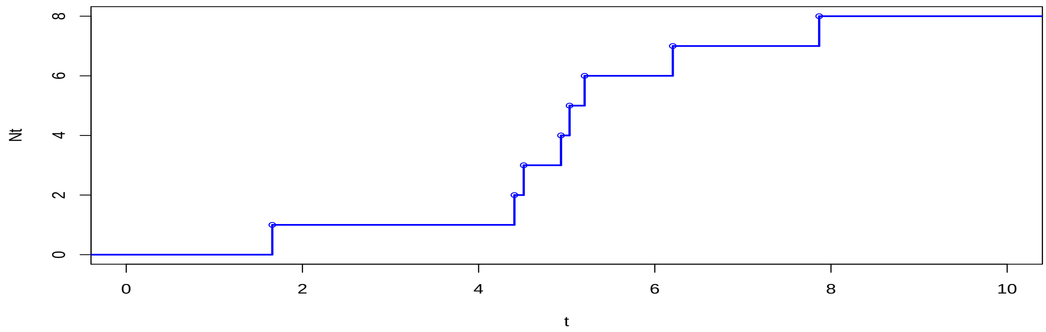


Figure 6.3: Sample path of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Compensated Poisson martingale

From (6.4) above we deduce that

$$\mathbb{E}[N_t - \lambda t] = 0, \tag{6.7}$$

i.e., the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ has *centered increments*.

```
lambda = 0.6;n = 20;Z<-cumsum(c(0,rep(1,n)));
tau_n <- rexp(n,rate=lambda); Tn <- cumsum(tau_n)
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
dev.new(width=T, height=5)
plot(t,N(t)-lambda*t,xlim = c(0,10),ylim =
      c(-2,2),xlab="t",ylab="Nt-t",type="l",lwd=2,col="blue",main="", xaxs = "i", yaxs = "i", xaxs =
      "i", yaxs = "i")
abline(h = 0, col="black", lwd =2)
points(Tn,N(Tn)-lambda*Tn,pch=1,cex=0.8,col="blue",lwd=2)
```

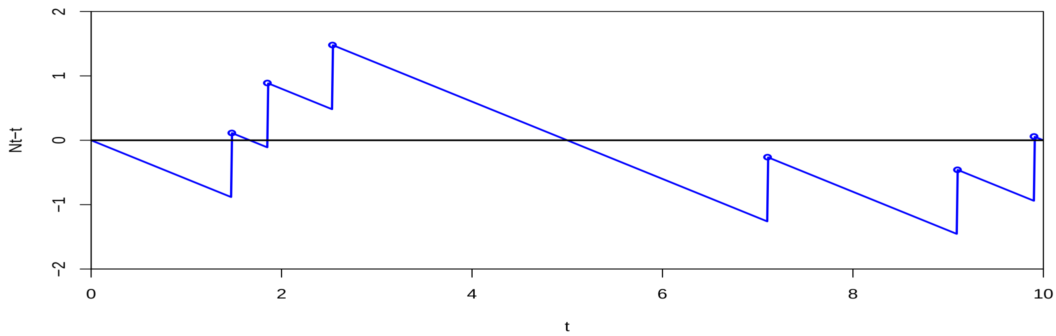


Figure 6.4: Sample path of the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$.

Since in addition $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ also has independent increments, we get the following proposition, cf. e.g. Example 2 page 62. We let

$$\mathscr{F}_t := \sigma\big(N_s \ : \ s\in[0,t]\big), \qquad t\in\mathbb{R}_+,$$

denote the *filtration* generated by the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Proposition 6.4 The compensated Poisson process

$$(N_t - \lambda t)_{t\in\mathbb{R}_+}$$

is a *martingale* with respect $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

Extensions of the Poisson process include Poisson processes with time-dependent intensity, and with random time-dependent intensity (Cox processes). Poisson processes belong to the family of *renewal processes* which are counting processes of the form

$$N_t = \sum_{n\geqslant 1} \mathbb{1}_{[T_n,\infty)}(t), \qquad t\in\mathbb{R}_+,$$

for which $\tau_k := T_{k+1} - T_k$, $k \geqslant 0$, is a sequence of independent identically distributed random variables.

6.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore there is some interest in considering jump processes that can have random jump sizes.

Let $(Z_k)_{k\geqslant 1}$ denote an *i.i.d.* sequence of square-integrable random variables distributed as the common random variable Z with the probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$. We have

$$\mathbb{P}(Z\in[a,b]) = \nu([a,b]) = \int_a^b \nu(dy), \quad -\infty < a \leqslant b < \infty, \quad k\geqslant 1,$$

and when the distribution $\nu(dy)$ admits a probability density $\varphi(y)$ on $\mathbb{R}$, we write $\nu(dy) = \varphi(y)dy$ and

$$\mathbb{P}(Z\in[a,b]) = \nu([a,b]) = \int_a^b \nu(dy) = \int_a^b \varphi(y)dy, \quad -\infty < a \leqslant b < \infty, \quad k\geqslant 1.$$

Definition 6.1 The process $(Y_t)_{t\in\mathbb{R}_+}$ given by the random sum

$$Y_t := Z_1 + Z_2 + \cdots + Z_{N_t} = \sum_{k=1}^{N_t} Z_k, \qquad t\in\mathbb{R}_+, \tag{6.8}$$

is called a compound Poisson process.[a]

[a] We use the convention $\sum_{k=1}^{n} Z_k = 0$ if $n=0$, so that $Y_0 = 0$.

Letting Y_{t^-} denote the left limit

$$Y_{t^-} := \lim_{s\nearrow t} Y_s, \qquad t>0,$$

we note that the jump size

$$\Delta Y_t := Y_t - Y_{t^-}, \qquad t\in\mathbb{R}_+,$$

of $(Y_t)_{t\in\mathbb{R}_+}$ at time t is given by the relation

$$\Delta Y_t = Z_{N_t}\Delta N_t, \qquad t\in\mathbb{R}_+, \tag{6.9}$$

where

$$\Delta N_t := N_t - N_{t^-} \in \{0,1\}, \qquad t\in\mathbb{R}_+,$$

denotes the jump size of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and N_{t^-} is the left limit

$$N_{t^-} := \lim_{s\nearrow t} N_s, \qquad t>0,$$

Example. Assume that jump sizes are Gaussian distributed with mean δ and variance η^2, with

$$\nu(dy) = \frac{1}{\sqrt{2\pi\eta^2}} \mathrm{e}^{-(y-\delta)^2/(2\eta^2)} dy.$$

The next Figure 6.5 represents a sample path of a compound Poisson process, with here $Z_1 = 0.9$, $Z_2 = -0.7$, $Z_3 = 1.4$, $Z_4 = 0.6$, $Z_5 = -2.5$, $Z_6 = 1.5$, $Z_7 = -0.5$, with the relation

$$Y_{T_k} = Y_{T_k^-} + Z_k, \qquad k \geqslant 1.$$

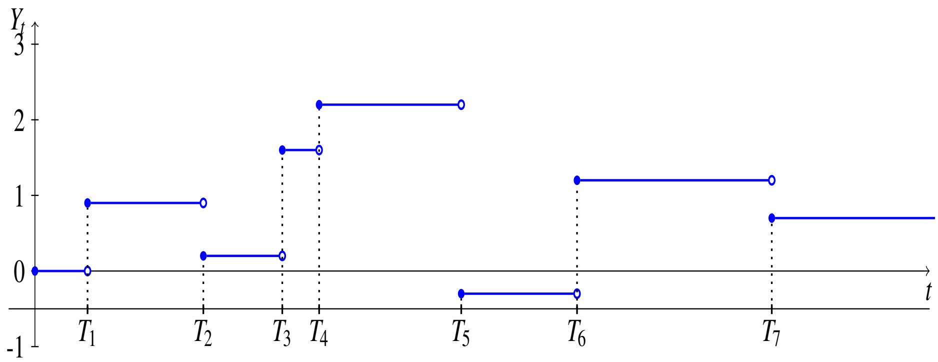


Figure 6.5: Sample path of a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$.

```
N<-50
x<-cumsum(rexp(N,rate=0.5))
y<-cumsum(c(0,rexp(N,rate=0.5)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5",col="blue")
y<-cumsum(c(0,rnorm(N,mean=0,sd=1)))
plot(stepfun(x,y),xlim = c(0,10),do.points = F,main="L=0.5",col="blue")
```

Given that $\{N_T = n\}$, the n jump sizes of $(Y_t)_{t\in\mathbb{R}_+}$ on $[0,T]$ are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. Based on this fact, the next proposition allows us to compute the *Moment Generating Function* (MGF) of the increment $Y_T - Y_t$.

Proposition 6.5 For any $t \in [0,T]$ we have

$$\mathbb{E}\left[\mathrm{e}^{\alpha(Y_T-Y_t)}\right] = \exp\left((T-t)\lambda \int_{-\infty}^{\infty} (\mathrm{e}^{\alpha y}-1)\nu(dy)\right), \quad \alpha\in\mathbb{R}. \tag{6.10}$$

Proof. Since N_t has a Poisson distribution with parameter $t>0$ and is independent of $(Z_k)_{k\geqslant 1}$, for all $\alpha\in\mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\alpha(Y_T-Y_t)}\right] &= \mathbb{E}\left[\exp\left(\alpha \sum_{k=N_t+1}^{N_T} Z_k\right)\right] \\
&= \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{N_T-N_t} Z_k\right)\right] \\
&= \sum_{n\geqslant 0} \mathbb{E}\left[\exp\left(\alpha \sum_{k=1}^{n} Z_k\right) \Big| N_T - N_t = n\right] \mathbb{P}(N_T - N_t = n)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \mathbb{E}\left[\exp\left(\alpha\sum_{k=1}^n Z_k\right)\right] \\
&= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \prod_{k=1}^n \mathbb{E}\left[\mathrm{e}^{\alpha Z_k}\right] \\
&= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{\lambda^n}{n!}(T-t)^n \left(\mathbb{E}\left[\mathrm{e}^{\alpha Z}\right]\right)^n \\
&= \exp\left((T-t)\lambda\left(\mathbb{E}\left[\mathrm{e}^{\alpha Z}\right]-1\right)\right) \\
&= \exp\left((T-t)\lambda\int_{-\infty}^{\infty}\mathrm{e}^{\alpha y}\nu(dy) - (T-t)\lambda\int_{-\infty}^{\infty}\nu(dy)\right) \\
&= \exp\left((T-t)\lambda\int_{-\infty}^{\infty}(\mathrm{e}^{\alpha y}-1)\nu(dy)\right),
\end{aligned}
$$

since the probability distribution $\nu(dy)$ of Z satisfies

$$
\mathbb{E}\left[\mathrm{e}^{\alpha Z}\right] = \int_{-\infty}^{\infty}\mathrm{e}^{\alpha y}\nu(dy) \quad \text{and} \quad \int_{-\infty}^{\infty}\nu(dy) = 1.
$$

□

From the moment generating function (6.10) we can compute the expectation of Y_t for fixed t as the product of the mean number of jump times $\mathbb{E}[N_t] = \lambda t$ and the mean jump size $\mathbb{E}[Z]$, *i.e.*,

$$
\mathbb{E}[Y_t] = \frac{\partial}{\partial\alpha}\mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0} = \lambda t\int_{-\infty}^{\infty} y\nu(dy) = \mathbb{E}[N_t]\,\mathbb{E}[Z] = \lambda t\,\mathbb{E}[Z]. \tag{6.11}
$$

Note that the above identity requires to exchange the differentiation and expectation operators, which is possible when the moment generating function (6.10) takes finite values for all α in a certain neighborhood $(-\varepsilon,\varepsilon)$ of 0.

Relation (6.11) states that the mean value of Y_t is the mean jump size $\mathbb{E}[Z]$ times the mean number of jumps $\mathbb{E}[N_t]$. It can be directly recovered using series summations, as

$$
\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\left[\sum_{k=1}^{N_t} Z_k\right] \\
&= \sum_{n\geqslant 1}\mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\Big|\, N_t = n\right]\mathbb{P}(N_t = n) \\
&= \mathrm{e}^{-\lambda t}\sum_{n\geqslant 1}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Z_k \,\Big|\, N_t = n\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n\geqslant 1}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Z_k\right] \\
&= \lambda t\,\mathrm{e}^{-\lambda t}\,\mathbb{E}[Z]\sum_{n\geqslant 1}\frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= \lambda t\,\mathbb{E}[Z] \\
&= \mathbb{E}[N_t]\,\mathbb{E}[Z].
\end{aligned}
$$

Regarding the variance, we have

$$
\mathbb{E}\left[Y_t^2\right] = \frac{\partial^2}{\partial\alpha^2}\mathbb{E}[\mathrm{e}^{\alpha Y_t}]_{|\alpha=0}
$$

$$
\begin{aligned}
&= \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) + (\lambda t)^2 \left(\int_{-\infty}^{\infty} y \nu(dy) \right)^2 \\
&= \lambda t \, \mathbb{E}\left[Z^2\right] + (\lambda t \, \mathbb{E}[Z])^2,
\end{aligned}
$$

which yields

$$
\mathrm{Var}\,[Y_t] = \lambda t \int_{-\infty}^{\infty} y^2 \nu(dy) = \lambda t \, \mathbb{E}\left[|Z|^2\right] = \mathbb{E}[N_t] \, \mathbb{E}\left[|Z|^2\right]. \tag{6.12}
$$

As a consequence, the *dispersion index* of the compound Poisson process

$$
\frac{\mathrm{Var}\,[Y_t]}{\mathbb{E}[Y_t]} = \frac{\mathbb{E}\left[|Z|^2\right]}{\mathbb{E}[Z]}, \qquad t \in \mathbb{R}_+.
$$

is the dispersion index of the random jump size Z. Proposition 6.5 can be used to show the next result.

Proposition 6.6 The compound Poisson process

$$
Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,
$$

has independent increments, *i.e.* for any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$
Y_{t_1} - Y_{t_0}, \; Y_{t_2} - Y_{t_1}, \ldots, \; Y_{t_n} - Y_{t_{n-1}}
$$

are mutually independent random variables.

Proof. This result relies on the fact that the result of Proposition 6.5 can be extended to sequences $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, as

$$
\begin{aligned}
\mathbb{E}\left[\prod_{k=1}^{n} \mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right] &= \mathbb{E}\left[\exp\left(i \sum_{k=1}^{n} \alpha_k (Y_{t_k} - Y_{t_{k-1}})\right)\right] \\
&= \exp\left(\lambda \sum_{k=1}^{n} (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1) \nu(dy)\right) \qquad (6.13) \\
&= \prod_{k=1}^{n} \exp\left((t_k - t_{k-1}) \lambda \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1) \nu(dy)\right) \\
&= \prod_{k=1}^{n} \mathbb{E}\left[\mathrm{e}^{i\alpha_k (Y_{t_k} - Y_{t_{k-1}})}\right].
\end{aligned}
$$

□

Since the compensated compound Poisson process also has independent and centered increments by (6.7) we have the following counterpart of Proposition 6.4, cf. also Example 2 page 62.

Proposition 6.7 The compensated compound Poisson process

$$
M_t := Y_t - \lambda t \, \mathbb{E}[Z], \qquad t \in \mathbb{R}_+,
$$

is a *martingale*.

By construction, compound Poisson processes only have a *finite* number of jumps on any interval. They belong to the family of *Lévy processes* which may have an infinite number of jumps on any finite time interval, see *e.g.* § 4.4.1 of Cont and Tankov, 2004.

The stochastic integral of a deterministic function $f(t)$ with respect to $(Y_t)_{t\in\mathbb{R}_+}$ is defined as

$$\int_0^T f(t)dY_t = \sum_{k=1}^{N_T} Z_k f(T_k).$$

Relation (6.13) can be used to show that, more generally, the moment generating function of $\int_0^T f(t)dY_t$ is given by

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] &= \exp\left(\lambda \int_0^T \int_{\mathbb{R}} \left(\mathrm{e}^{yf(t)}-1\right)\nu(dy)dt\right)\\ &= \exp\left(\lambda \int_0^T \left(\mathbb{E}\left[\mathrm{e}^{f(t)Z}\right]-1\right)dt\right).\end{aligned}$$

We also have

$$\begin{aligned}\log \mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] &= \lambda \int_0^T \int_{\mathbb{R}} \left(\mathrm{e}^{yf(t)}-1\right)\nu(dy)dt\\ &= \lambda \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^T \int_{\mathbb{R}} y^n f^n(t)\nu(dy)dt\\ &= \lambda \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{E}[Z^n] \int_0^T f^n(t)dt,\end{aligned}$$

hence the *cumulant* of order $n \geqslant 1$ of $\displaystyle\int_0^T f(t)dY_t$ is given by

$$\kappa_n = \lambda\, \mathbb{E}[Z^n] \int_0^T f^n(t)dt,$$

which recovers (6.11) and (6.12) by taking $f(t) = \mathbb{1}_{[0,T]}(t)$ when $n = 1,2$.

6.3 Stochastic Integrals and Itô Formula with Jumps

Based on the relation

$$\Delta Y_t = Z_{N_t}\Delta N_t,$$

we can define the stochastic integral of a stochastic process $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(Y_t)_{t\in\mathbb{R}_+}$ by

$$\boxed{\int_0^T \phi_t dY_t = \int_0^T \phi_t Z_{N_t} dN_t := \sum_{k=1}^{N_T} \phi_{T_k} Z_k. \qquad (6.14)}$$

As a consequence of Proposition 6.5 we can derive the following version of the Lévy-Khintchine formula:

$$\mathbb{E}\left[\exp\left(\int_0^T f(t)dY_t\right)\right] = \exp\left(\lambda \int_0^T \int_{-\infty}^{\infty} \left(\mathrm{e}^{yf(t)}-1\right)\nu(dy)dt\right)$$

for $f : [0,T] \longrightarrow \mathbb{R}$ a bounded deterministic function of time.

Note that the expression (6.14) of $\int_0^T \phi_t dY_t$ has a natural financial interpretation as the value at time T of a portfolio containing a (possibly fractional) quantity ϕ_t of a risky asset at time t, whose price evolves according to random returns Z_k, generating profits/losses $\phi_{T_k} Z_k$ at random times T_k.

In particular, the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ in (6.1) admits the stochastic integral representation

$$Y_t = Y_0 + \sum_{k=1}^{N_t} Z_k = Y_0 + \int_0^t Z_{N_s} dN_s.$$

The next result is also called the smoothing lemma, cf. Theorem 9.2.1 in Brémaud, 1999.

Proposition 6.8 Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a stochastic process *adapted* to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, admitting left limits and such that

$$\mathbb{E}\left[\int_0^T |\phi_t| dt\right] < \infty, \qquad T > 0.$$

The expected value of the compound Poisson stochastic integral can be expressed as

$$\boxed{\mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] = \mathbb{E}\left[\int_0^T \phi_{t^-} Z_{N_t} dN_t\right] = \lambda\, \mathbb{E}[Z]\, \mathbb{E}\left[\int_0^T \phi_{t^-} dt\right],} \tag{6.15}$$

where ϕ_{t^-} denotes the left limit

$$\phi_{t^-} := \lim_{s\nearrow t} \phi_s, \qquad t > 0.$$

Proof. By Proposition 6.7 the compensated compound Poisson process $(Y_t - \lambda t\, \mathbb{E}[Z])_{t\in\mathbb{R}_+}$ is a *martingale*, and as a consequence the stochastic integral process

$$t \longmapsto \int_0^t \phi_{s^-} d(Y_s - \lambda\, \mathbb{E}[Z] ds) = \int_0^t \phi_{s^-} (Z_{N_s} dN_s - \lambda\, \mathbb{E}[Z] ds)$$

is also a martingale, by an argument similar to that in the proof of Proposition 3.1 because the adaptedness of $(\phi_t)_{t\in\mathbb{R}_+}$ to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, makes $(\phi_{t^-})_{t>0}$ *predictable*, *i.e.* adapted with respect to the filtration

$$\mathscr{F}_{t^-} := \sigma(Y_s \ : \ s \in [0,t)), \qquad t > 0.$$

It remains to use the fact that the expectation of a martingale remains constant over time, which shows that

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_0^T \phi_{t^-} (dY_t - \lambda\, \mathbb{E}[Z] dt)\right] \\ &= \mathbb{E}\left[\int_0^T \phi_{t^-} dY_t\right] - \lambda\, \mathbb{E}[Z]\, \mathbb{E}\left[\int_0^T \phi_{t^-} dt\right]. \end{aligned}$$

□

For example, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_{t^-} dN_t = \sum_{k=1}^{N_T} (k-1) = \frac{1}{2} N_T (N_T - 1),$$

hence

$$\begin{aligned}\mathbb{E}\left[\int_0^T N_{t^-} dN_t\right] &= \frac{1}{2}\left(\mathbb{E}\left[N_T^2\right] - \mathbb{E}[N_T]\right)\\ &= \frac{(\lambda T)^2}{2}\\ &= \lambda \int_0^T \lambda t dt\\ &= \lambda \int_0^T \mathbb{E}[N_t] dt,\end{aligned}$$

as in (6.15). Note however that while the identity in expectations (6.15) holds for the left limit ϕ_{t^-}, it need not hold for ϕ_t itself. Indeed, taking $\phi_t = Y_t := N_t$ we have

$$\int_0^T N_t dN_t = \sum_{k=1}^{N_T} k = \frac{1}{2} N_T (N_T + 1),$$

hence

$$\begin{aligned}\mathbb{E}\left[\int_0^T N_t dN_t\right] &= \frac{1}{2}\left(\mathbb{E}\left[N_T^2\right] + \mathbb{E}[N_T]\right)\\ &= \frac{1}{2}\left((\lambda T)^2 + 2\lambda T\right)\\ &= \frac{(\lambda T)^2}{2} + \lambda T\\ &\neq \lambda\,\mathbb{E}\left[\int_0^T N_t dt\right].\end{aligned}$$

Under similar conditions, the compound Poisson compensated stochastic integral can be shown to satisfy the Itô isometry (6.16) in the next proposition.

Proposition 6.9 Let $(\phi_t)_{t\in\mathbb{R}_+}$ be a stochastic process *adapted* to the filtration generated by $(Y_t)_{t\in\mathbb{R}_+}$, admitting left limits and such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < \infty, \qquad T > 0.$$

The expected value of the squared compound Poisson compensated stochastic integral can be computed as

$$\boxed{\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2\right] = \lambda\,\mathbb{E}\left[|Z|^2\right]\mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt\right], \qquad (6.16)}$$

Note that in (6.16), the generic jump size Z is squared but λ is not.

Proof. From the stochastic Fubini-type theorem, we have

$$\begin{aligned}&\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2 & (6.17)\\ &= 2\int_0^T \phi_{t^-}\int_0^{t^-} \phi_{s^-}(dY_s - \lambda\,\mathbb{E}[Z]ds)(dY_t - \lambda\,\mathbb{E}[Z]dt) & (6.18)\end{aligned}$$

$$+\int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t, \tag{6.19}$$

where integration over the diagonal $\{s = t\}$ has been excluded in (6.18) as the inner integral has an upper limit t^- rather than t. Next, taking expectation on both sides of (6.17)-(6.19), we find

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \phi_{t^-}(dY_t - \lambda\,\mathbb{E}[Z]dt)\right)^2\right] &= \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 |Z_{N_t}|^2 dN_t\right] \\
&= \lambda\,\mathbb{E}\left[|Z|^2\right]\mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt\right],
\end{aligned}$$

where we used the vanishing of the expectation of the double stochastic integral:

$$\mathbb{E}\left[\int_0^T \phi_{t^-} \int_0^{t^-} \phi_{s^-}(dY_s - \lambda\,\mathbb{E}[Z]ds)(dY_t - \lambda\,\mathbb{E}[Z]dt)\right] = 0,$$

and the martingale property of the compensated compound Poisson process

$$t \longmapsto \left(\sum_{k=1}^{N_t} |Z_k|^2\right) - \lambda t\,\mathbb{E}\left[Z^2\right], \qquad t \in \mathbb{R}_+,$$

as in the proof of Proposition 6.8. The isometry relation (6.16) can also be proved using simple predictable processes. □

Next, take $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion independent of $(Y_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ a jump-diffusion process of the form

$$X_t := \int_0^t u_s dB_s + \int_0^t v_s ds + Y_t, \qquad t \in \mathbb{R}_+,$$

where $(u_t)_{t\in\mathbb{R}_+}$ is a stochastic process which is adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$, and such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^2 |u_t|^2 dt\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T |\phi_t v_t| dt\right] < \infty, \quad T > 0.$$

We define the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ by

$$\begin{aligned}
\int_0^T \phi_t dX_t &:= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \int_0^T \phi_t dY_t \\
&:= \int_0^T \phi_t u_t dB_t + \int_0^T \phi_t v_t dt + \sum_{k=1}^{N_T} \phi_{T_k} Z_k, \qquad T > 0.
\end{aligned}$$

For the mixed continuous-jump martingale

$$X_t := \int_0^t u_s dB_s + Y_t - \lambda t\,\mathbb{E}[Z], \qquad t \in \mathbb{R}_+,$$

we then have the isometry:

$$\boxed{\mathbb{E}\left[\left(\int_0^T \phi_{t^-} dX_t\right)^2\right] = \mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 |u_t|^2 dt\right] + \lambda\,\mathbb{E}\left[|Z|^2\right]\mathbb{E}\left[\int_0^T |\phi_{t^-}|^2 dt\right].} \tag{6.20}$$

provided that $(\phi_t)_{t\in\mathbb{R}_+}$ is adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$. The isometry formula (6.20) will be used in Section 7.6 for mean-variance hedging in jump-diffusion models.

More generally, when $(X_t)_{t\in\mathbb{R}_+}$ contains an additional drift term,

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \in \mathbb{R}_+,$$

the stochastic integral of $(\phi_t)_{t\in\mathbb{R}_+}$ with respect to $(X_t)_{t\in\mathbb{R}_+}$ is given by

$$\begin{aligned}\int_0^T \phi_s dX_s &:= \int_0^T \phi_s u_s dB_s + \int_0^T \phi_s v_s ds + \int_0^T \eta_s \phi_s dY_s \\ &= \int_0^T \phi_s u_s dB_s + \int_0^T \phi_s v_s ds + \sum_{k=1}^{N_T} \phi_{T_k} \eta_{T_k} Z_k, \qquad T > 0.\end{aligned}$$

Itô Formula with Jumps

The next proposition gives the simplest instance of the Itô formula with jumps, in the case of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity λ.

Proposition 6.10 Itô formula for the standard Poisson process. We have

$$f(N_t) = f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s, \qquad t \in \mathbb{R}_+,$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h\searrow 0} N_{s-h}$.

Proof. We note that

$$N_s = N_{s^-} + 1 \ \text{ if } \ dN_s = 1 \ \text{ and } \ k = N_{T_k} = 1 + N_{T_k^-}, \qquad k \geqslant 1.$$

Hence we have the *telescoping sum*

$$\begin{aligned}f(N_t) &= f(0) + \sum_{k=1}^{N_t} (f(k) - f(k-1)) \\ &= f(0) + \sum_{k=1}^{N_t} (f(N_{T_k}) - f(N_{T_k^-})) \\ &= f(0) + \sum_{k=1}^{N_t} (f(1+N_{T_k^-}) - f(N_{T_k^-})) \\ &= f(0) + \int_0^t (f(1+N_{s^-}) - f(N_{s^-}))dN_s \\ &= f(0) + \int_0^t (f(N_s) - f(N_s - 1))dN_s \\ &= f(0) + \int_0^t (f(N_s) - f(N_{s^-}))dN_s,\end{aligned}$$

where N_{s^-} denotes the left limit $N_{s^-} = \lim_{h\searrow 0} N_{s-h}$. □

The next result deals with the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ in (6.1) via a similar argument.

Proposition 6.11 Itô formula for the compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$. We have the *pathwise*

Itô formula

$$f(Y_t) = f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))dN_s, \qquad t \in \mathbb{R}_+. \tag{6.21}$$

Proof. We have

$$\begin{aligned}
f(Y_t) &= f(0) + \sum_{k=1}^{N_t} (f(Y_{T_k}) - f(Y_{T_k^-})) \\
&= f(0) + \sum_{k=1}^{N_t} (f(Y_{T_k^-} + Z_k) - f(Y_{T_k^-})) \\
&= f(0) + \int_0^t (f(Y_{s^-} + Z_{N_s}) - f(Y_{s^-}))dN_s \\
&= f(0) + \int_0^t (f(Y_s) - f(Y_{s^-}))dN_s, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

□

From the expression

$$Y_t = Y_0 + \sum_{k=1}^{N_t} Z_k = Y_0 + \int_0^t Z_{N_s} dN_s,$$

the Itô formula (6.21) can be decomposed using a compensated Poisson stochastic integral as

$$\begin{aligned}
df(Y_t) &= (f(Y_t) - f(Y_{t^-}))dN_t - \mathbb{E}[(f(y+Z) - f(y)]_{y=Y_{t^-}} dt \\
&\quad + \mathbb{E}[(f(y+Z) - f(y)]_{y=Y_{t^-}} dt,
\end{aligned} \tag{6.22}$$

where $(f(Y_t) - f(Y_{t^-}))dN_t - \mathbb{E}[(f(y+Z_{N_t}) - f(y)]_{y=Y_{t^-}} dt$ is the differential of a martingale by the smoothing lemma Proposition 6.8.

More generally, we have the following result.

Proposition 6.12 For an Itô process of the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dY_s, \qquad t \in \mathbb{R}_+,$$

we have the Itô formula

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t v_s f'(X_s)ds + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds \\
&\quad + \int_0^t (f(X_s) - f(X_{s^-}))dN_s, \qquad t \in \mathbb{R}_+.
\end{aligned} \tag{6.23}$$

Proof. By combining the Itô formula for Brownian motion with the above argument we find

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds + \int_0^t v_s f'(X_s)ds \\
&\quad + \sum_{k=1}^{N_T} (f(X_{T_k^-} + \eta_{T_k} Z_k) - f(X_{T_k^-})) \\
&= f(X_0) + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds + \int_0^t v_s f'(X_s)ds \\
&\quad + \int_0^t (f(X_{s^-} + \eta_s Z_{N_s}) - f(X_{s^-}))dN_s, \quad t \in \mathbb{R}_+,
\end{aligned}$$

which yields (6.23). □

The integral Itô formula (6.23) can be rewritten in differential notation as

$$df(X_t) = v_t f'(X_t)dt + u_t f'(X_t)dB_t + \frac{|u_t|^2}{2} f''(X_t)dt + (f(X_t) - f(X_{t^-}))dN_t, \tag{6.24}$$

$t \in \mathbb{R}_+$.

Itô multiplication table with jumps

For a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ given by

$$X_t = \int_0^t u_s dB_s + \int_0^t v_s ds + \int_0^t \eta_s dN_s, \qquad t \in \mathbb{R}_+,$$

the Itô formula with jumps reads

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s)dB_s \\ &\quad + \int_0^t v_s f'(X_s)ds + \int_0^t (f(X_{s^-} + \eta_s) - f(X_{s^-}))dN_s \\ &= f(0) + \int_0^t u_s f'(X_s)dB_s + \frac{1}{2}\int_0^t |u_s|^2 f''(X_s)dB_s \\ &\quad + \int_0^t v_s f'(X_s)ds + \int_0^t (f(X_s) - f(X_{s^-}))dN_s. \end{aligned}$$

Given two Itô processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written in differential notation as

$$dX_t = u_t dB_t + v_t dt + \eta_t dN_t, \qquad t \in \mathbb{R}_+,$$

and

$$dY_t = a_t dB_t + b_t dt + c_t dN_t, \qquad t \in \mathbb{R}_+,$$

the Itô formula for jump processes can also be written as

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \bullet dY_t$$

where the product $dX_t \bullet dY_t$ is computed according to the following extension of the Itô multiplication table. The relation $dB_t \bullet dN_t = 0$ is due to the fact that $(N_t)_{t\in\mathbb{R}_+}$ has finite variation on any finite interval.

$\bullet$	dt	dB_t	dN_t
dt	0	0	0
dB_t	0	dt	0
dN_t	0	0	dN_t

Table 6.1: Itô multiplication table with jumps.

In other words, we have

$$\begin{aligned} dX_t \bullet dY_t &= (v_t dt + u_t dB_t + \eta_t dN_t)(b_t dt + a_t dB_t + c_t dN_t) \\ &= v_t b_t dt \bullet dt + u_t b_t dB_t \bullet dt + \eta_t b_t dN_t \bullet dt \\ &\quad + v_t a_t dt \bullet dB_t + u_t a_t dB_t \bullet dB_t + \eta_t a_t dN_t \bullet dB_t \end{aligned}$$

$$
\begin{aligned}
& +v_t c_t dt \bullet dN_t + u_t c_t dB_t \bullet dN_t + \eta_t c_t dN_t \bullet dN_t \\
= & +u_t a_t dB_t \bullet dB_t + \eta_t c_t dN_t \bullet dN_t \\
= & \ u_t a_t dt + \eta_t c_t dN_t,
\end{aligned}
$$

since

$$dN_t \bullet dN_t = (dN_t)^2 = dN_t,$$

as $\Delta N_t \in \{0,1\}$. In particular, we have

$$(dX_t)^2 = (v_t dt + u_t dB_t + \eta_t dN_t)^2 = u_t^2 dt + \eta_t^2 dN_t.$$

Jump processes with infinite activity

Given $\eta(s)$, $s \in \mathbb{R}_+$, a deterministic function of time and $(X_t)_{t\in\mathbb{R}_+}$ an Itô process of the form

$$X_t := X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s + \int_0^t \eta(s) dY_t, \qquad t \geqslant 0,$$

the Itô formula with jumps (6.23) can be rewritten as

$$
\begin{aligned}
f(X_t) = {} & f(X_0) + \int_0^t v_s f'(X_s) ds + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds \\
& + \int_0^t (f(X_{s^-} + \eta(s)\Delta Y_s) - f(X_{s^-}))\, dN_s - \lambda \int_0^t \mathbb{E}\left[f(x+\eta(s)Z) - f(x)\right]_{x=X_{s^-}} ds \\
& + \lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_{s^-}))\, \nu(dy) ds, \qquad t \geqslant 0,
\end{aligned}
$$

using the compensated martingale

$$
\begin{aligned}
& \int_0^t (f(X_s) - f(X_{s^-}))\, dN_s - \lambda \int_0^t \mathbb{E}\left[f(x+\eta(s)Z) - f(x)\right]_{x=X_{s^-}} ds \\
& \quad = \int_0^t (f(X_{s^-} + \eta(s)\Delta Y_s) - f(X_{s^-}))\, dN_s \\
& \qquad - \lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_s))\, \nu(dy) ds,
\end{aligned}
$$

with the relation $dX_s = \eta_s \Delta Y_s$. We note that from the relation

$$\mathbb{E}[Z] = \int_{-\infty}^{\infty} y \nu(dy),$$

the above compensator can be written as

$$
\begin{aligned}
& \lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_{s^-}))\, \nu(dy) ds \\
& \quad = \lambda \int_0^t \int_{-\infty}^{\infty} (f(X_{s^-} + \eta(s)y) - f(X_{s^-}) - \eta(s) y f'(X_{s^-}))\, \nu(dy) ds \\
& \qquad + \lambda\, \mathbb{E}[Z] \int_0^t \eta(s) f'(X_{s^-}) ds.
\end{aligned}
\tag{6.25}
$$

The expression (6.25) above is at the basis of the extension of Itô's formula to Lévy processes with an infinite number of jumps on any interval under the conditions

$$\int_{|y|\leqslant 1} y^2 \nu(dy) < \infty \quad \text{and} \quad \nu([-1,1]^c) < \infty,$$

using the bound

$$|f(x+y) - f(x) - y f'(x)| \leqslant C y^2, \qquad y \in [-1,1],$$

that follows from Taylor's theorem for f a $\mathscr{C}^2(\mathbb{R})$ function. This yields

$$f(X_t) = f(X_0) + \int_0^t v_s f'(X_s) ds + \int_0^t u_s f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s)|u_s|^2 ds$$

$$
\begin{aligned}
&+\int_0^t (f(X_{s^-}+\eta(s)\Delta Y_s)-f(X_{s^-}))\,dN_s-\lambda\int_0^t \mathbb{E}\left[f(x+\eta(s)Z)-f(x)\right]_{x=X_{s^-}} ds\\
&+\lambda\int_0^t\int_{-\infty}^{\infty}(f(X_{s^-}+\eta(s)y)-f(X_{s^-})-\eta(s)yf'(X_{s^-}))\,\nu(dy)ds\\
&+\lambda\,\mathbb{E}[Z]\int_0^t \eta(s)f'(X_{s^-})ds, \qquad t\geqslant 0,
\end{aligned}
$$

see *e.g.* Theorem 1.16 in Øksendal and Sulem, 2005 and Theorem 4.4.7 in Applebaum, 2009 in the setting of Poisson random measures. Such processes, also called "infinite activity Lévy processes" are also useful in financial modeling, cf. Cont and Tankov, 2004, and include the gamma process, stable processes, variance gamma processes, inverse Gaussian processes, etc, as in the following illustrations.

1. Gamma process.

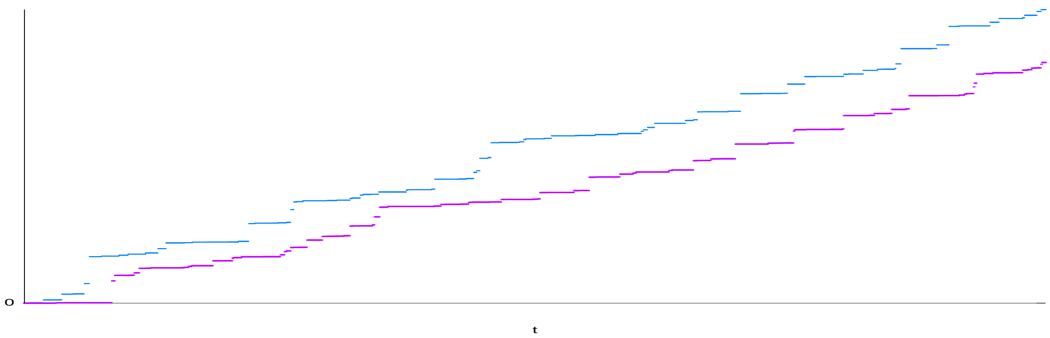


Figure 6.6: Sample trajectories of a gamma process.

The next R code can be used to generate the gamma process paths of Figure 6.6.

```
N=2000; t <- 0:N; dt <- 1.0/N; nsim <- 6; alpha=20.0
X = matrix(0, nsim, N)
for (i in 1:nsim){X[i,]=rgamma(N,alpha*dt);}
X <- cbind(rep(0, nsim), t(apply(X, 1, cumsum)))
plot(t, X[1, ], xlab = "time", type = "l", ylim = c(0, 2*N*alpha*dt), col = 0)
for (i in 1:nsim){points(t, X[i, ], xlab = "time", type = "p", pch=20, cex =0.02, col = i)}
```

2. Variance gamma process.

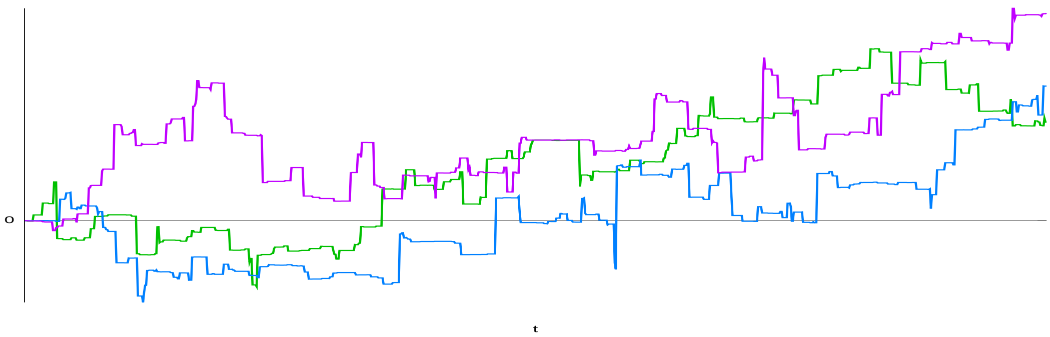


Figure 6.7: Sample trajectories of a variance gamma process.

3. Inverse Gaussian process.

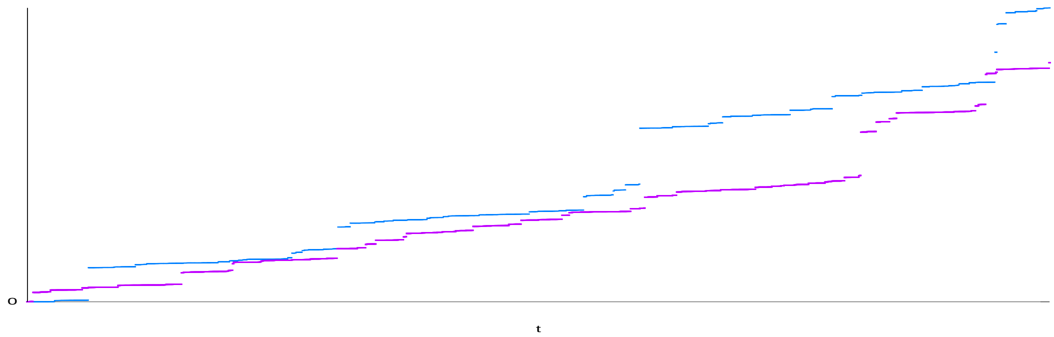


Figure 6.8: Sample trajectories of an inverse Gaussian process.

4. Negative Inverse Gaussian process.

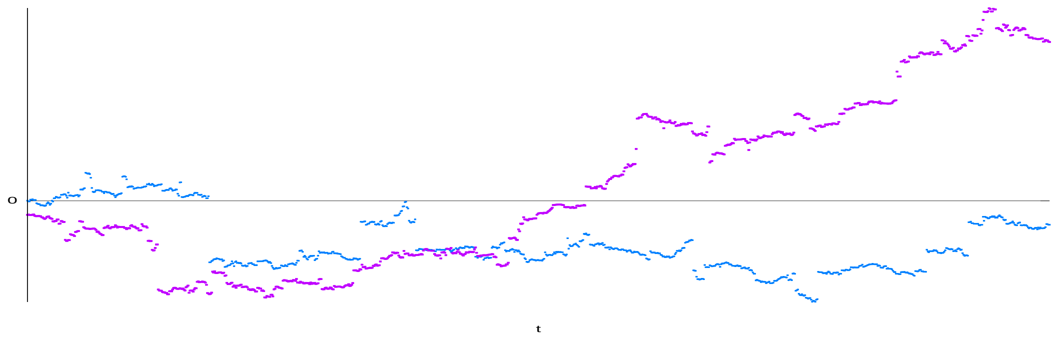


Figure 6.9: Sample trajectories of a negative inverse Gaussian process.

5. Stable process.

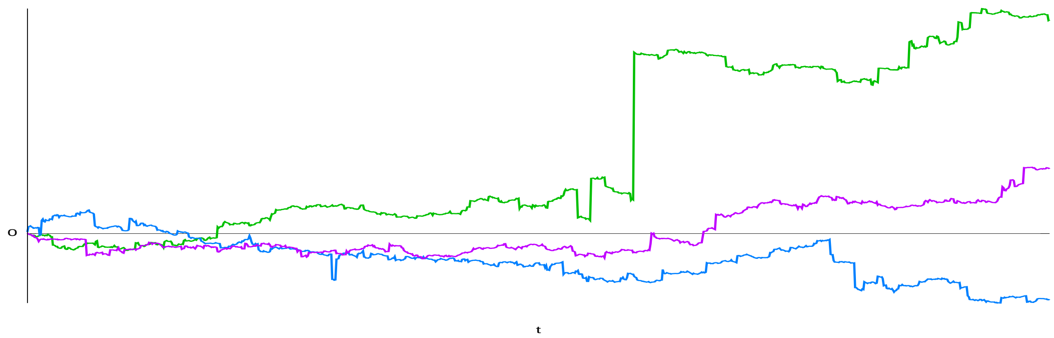


Figure 6.10: Sample trajectories of a stable process.

The above sample paths of a stable process can be compared to the USD/CNY exchange rate over the year 2015, according to the date retrieved using the following code.

```
library(quantmod)
getSymbols("USDCNY=X",from="2015-01-01",to="2015-12-06",src="yahoo")
rate=Ad(`USDCNY=X`)
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
chart_Series(rate, theme = myTheme)
add_TA(rate, on=1, col="blue", legend=NULL,lwd=2)
getSymbols("EURCHF=X",from="2013-12-30",to="2016-01-01",src="yahoo")
rate=Ad(`EURCHF=X`)
```

The adjusted close price Ad() is the closing price after adjustments for applicable splits and dividend distributions.

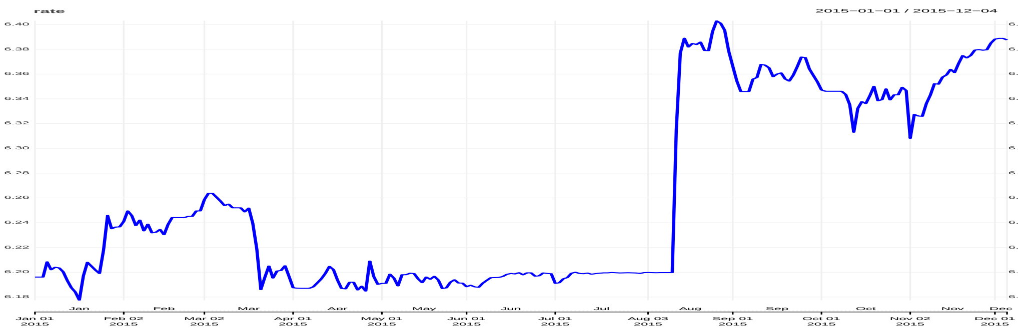


Figure 6.11: USD/CNY Exchange rate data.

6.4 Stochastic Differential Equations with Jumps

In the continuous asset price model, the returns of the riskless asset price process $(A_t)_{t\in\mathbb{R}_+}$ and of the risky asset price process $(S_t)_{t\in\mathbb{R}_+}$ are modeled as

$$\frac{dA_t}{A_t} = rdt \quad \text{and} \quad \frac{dS_t}{S_t} = \mu dt + \sigma dB_t.$$

In this section we are interested in using jump processes in order to model an asset price process $(S_t)_{t\in\mathbb{R}_+}$.

i) Constant market return $\eta > -1$.

In the case of discontinuous asset prices, let us start with the simplest example of a constant market return η written as

$$\eta := \frac{S_t - S_{t^-}}{S_{t^-}}, \tag{6.26}$$

assuming the presence of a jump at time t, *i.e.*, $dN_t = 1$. Using the relation $dS_t := S_t - S_{t^-}$, (6.26) rewrites as

$$\eta dN_t = \frac{S_t - S_{t^-}}{S_{t^-}} = \frac{dS_t}{S_{t^-}}, \tag{6.27}$$

or

$$dS_t = \eta S_{t^-} dN_t, \tag{6.28}$$

which is a stochastic differential equation with respect to the standard Poisson process, with constant volatility $\eta \in \mathbb{R}$. Note that the left limit S_{t^-} in (6.28) occurs naturally from the definition (6.27) of market returns when dividing by the previous index value S_{t^-}.

In the presence of a jump at time t, the equation (6.26) also reads

$$S_t = (1+\eta)S_{t^-}, \qquad dN_t = 1,$$

which can be applied by induction at the successive jump times $T_1, T_2, \ldots, T_{N_t}$ until time t, to derive the solution

$$S_t = S_0(1+\eta)^{N_t}, \qquad t \in \mathbb{R}_+,$$

of (6.28).

The use of the left limit S_{t^-} turns out to be necessary when computing pathwise solutions by solving for S_t from S_{t^-}.

ii) Time-dependent market returns η_t, $t \in \mathbb{R}_+$.

Next, consider the case where η_t is time-dependent, *i.e.*,

$$dS_t = \eta_t S_{t^-} dN_t. \tag{6.29}$$

At each jump time T_k, Relation (6.29) reads

$$dS_{T_k} = S_{T_k} - S_{T_k^-} = \eta_{T_k} S_{T_k^-},$$

i.e.,

$$S_{T_k} = (1+\eta_{T_k})S_{T_k^-},$$

and repeating this argument for all $k = 1, 2, \ldots, N_t$ yields the product solution

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} (1+\eta_{T_k}) \\ &= S_0 \prod_{\substack{\Delta N_s = 1 \\ 0 \leqslant s \leqslant t}} (1+\eta_s) \\ &= S_0 \prod_{0 \leqslant s \leqslant t} (1+\eta_s \Delta N_s), \quad t \in \mathbb{R}_+. \end{aligned}$$

By a similar argument, we obtain the following proposition.

Proposition 6.13 The stochastic differential equation with jumps

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt), \tag{6.30}$$

admits the solution

$$\boxed{S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda \int_0^t \eta_s ds\right) \prod_{k=1}^{N_t} (1+\eta_{T_k}), \qquad t \in \mathbb{R}_+.}$$

Note that the equations

$$dS_t = \mu_t S_{t^-} dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

and

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dN_t - \lambda dt)$$

are equivalent because $S_{t^-} dt = S_t dt$ as the set $\{T_k\}_{k \geqslant 1}$ of jump times has zero measure of length.

A random simulation of the numerical solution of the above equation (6.30) is given in Figure 6.12 for $\eta = 1.29$ and constant $\mu = \mu_t$, $t \in \mathbb{R}_+$.

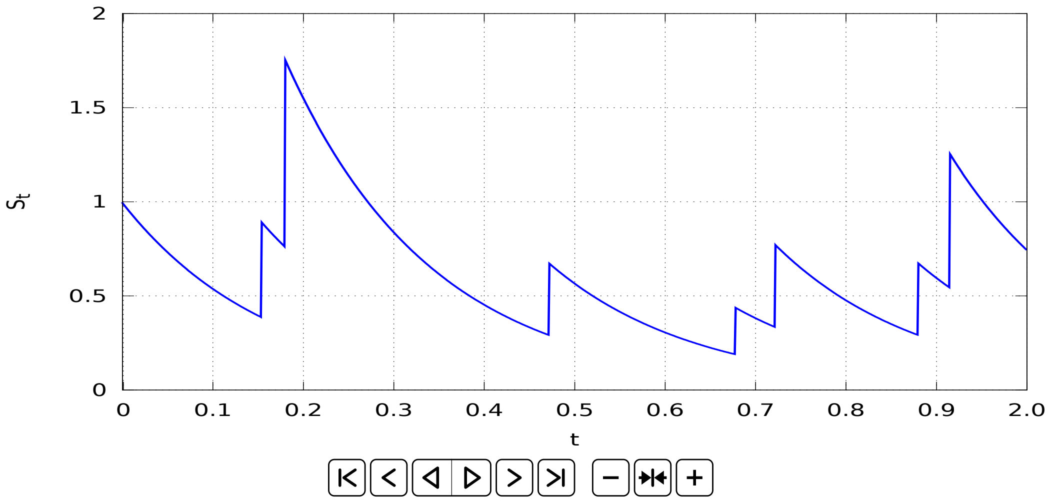


Figure 6.12: Geometric Poisson process.*

The above simulation can be compared to the real sales ranking data of Figure 6.13.

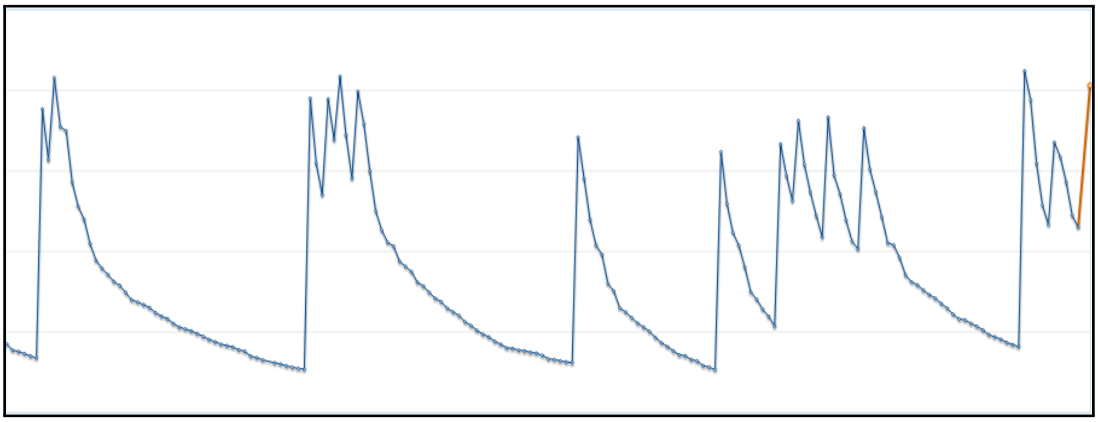


Figure 6.13: Ranking data.

Next, consider the equation

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda\, \mathbb{E}[Z]dt)$$

driven by the compensated compound Poisson process $(Y_t - \lambda\, \mathbb{E}[Z]t)_{t\in\mathbb{R}_+}$, also written as

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(Z_{N_t} dN_t - \lambda\, \mathbb{E}[Z]dt),$$

with solution

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds - \lambda\, \mathbb{E}[Z]\int_0^t \eta_s ds\right)\prod_{k=1}^{N_t}(1+\eta_{T_k} Z_k) \qquad t \in \mathbb{R}_+. \tag{6.31}$$

A random simulation of the geometric compound Poisson process (6.31) is given in Figure 6.14.

*The animation works in Acrobat Reader on the entire pdf file.

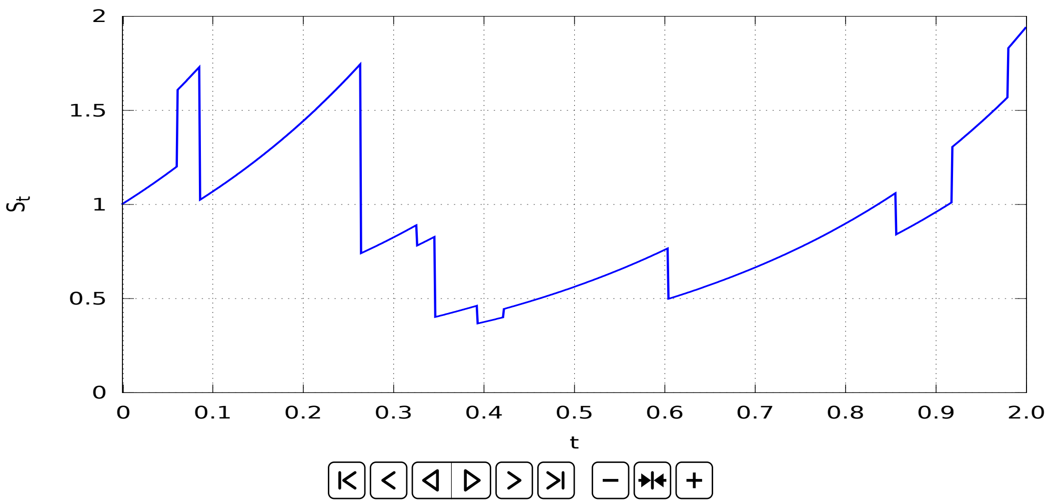


Figure 6.14: Geometric compound Poisson process.*

In the case of a jump-diffusion stochastic differential equation of the form

$$dS_t = \mu_t S_t dt + \eta_t S_{t^-}(dY_t - \lambda \mathbb{E}[Z]dt) + \sigma_t S_t dB_t,$$

we get

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu_s ds - \lambda \mathbb{E}[Z] \int_0^t \eta_s ds + \int_0^t \sigma_s dB_s - \frac{1}{2} \int_0^t |\sigma_s|^2 ds \right) \\ &\quad \times \prod_{k=1}^{N_t} (1 + \eta_{T_k} Z_k), \qquad t \in \mathbb{R}_+. \end{aligned}$$

A random simulation of the geometric Brownian motion with compound Poisson jumps is given in Figure 6.15.

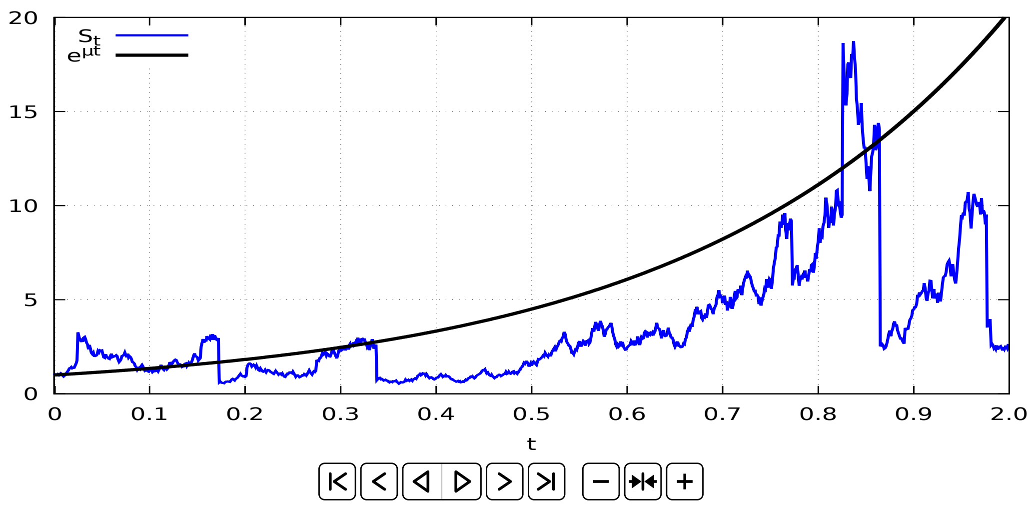


Figure 6.15: Geometric Brownian motion with compound Poisson jumps.†

*The animation works in Acrobat Reader on the entire pdf file.
†The animation works in Acrobat Reader on the entire pdf file.

By rewriting S_t as

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \eta_s (dY_s - \lambda\,\mathbb{E}[Z]ds) + \int_0^t \sigma_s dB_s - \frac{1}{2}\int_0^t |\sigma_s|^2 ds\right) \times \prod_{k=1}^{N_t} \left((1+\eta_{T_k} Z_k)\,\mathrm{e}^{-\eta_{T_k} Z_k}\right),$$

$t \in \mathbb{R}_+$, one can extend this jump model to processes with an infinite number of jumps on any finite time interval, cf. Cont and Tankov, 2004. The next Figure 6.16 shows a number of downward and upward jumps occuring in the SMRT historical share price data, with a typical geometric Brownian behavior in between jumps.

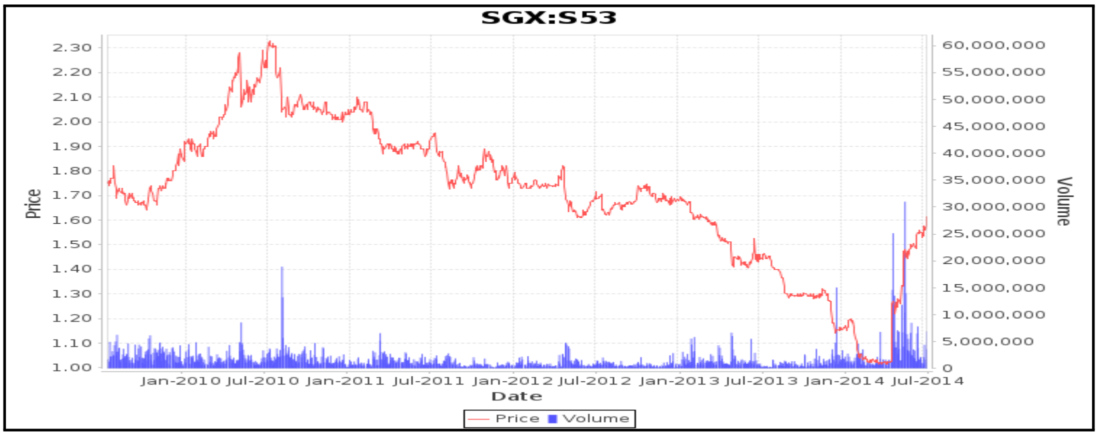


Figure 6.16: SMRT Share price.

6.5 Girsanov Theorem for Jump Processes

Recall that in its simplest form, cf. Section 3.2, the Girsanov Theorem for Brownian motion states the following.

Under the probability measure $\widetilde{\mathbb{P}}_{-\mu}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{-\mu}}{d\mathbb{P}} := \mathrm{e}^{-\mu B_T - \mu^2 T/2},$$

the random variable $B_T + \mu T$ has the centered Gaussian distribution $\mathcal{N}(0,T)$.

This fact follows from the calculation

$$\begin{aligned}
\widetilde{\mathbb{E}}_{-\mu}[f(B_T+\mu T)] &= \mathbb{E}[f(B_T+\mu T)\,\mathrm{e}^{-\mu B_T - \mu^2 T/2}] \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} f(x+\mu T)\,\mathrm{e}^{-\mu x - \mu^2 T/2}\,\mathrm{e}^{-x^2/(2T)}dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} f(x+\mu T)\,\mathrm{e}^{-(x+\mu T)^2/(2T)}dx \\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-y^2/(2T)}dy \\
&= \mathbb{E}[f(B_T)], \qquad (6.32)
\end{aligned}$$

for any bounded measurable function f on $\mathbb{R}$, which shows that $B_T + \mu T$ is a centered Gaussian random variable under $\widetilde{\mathbb{P}}_{-\mu}$.

More generally, the Girsanov Theorem states that $(B_t + \mu t)_{t\in[0,T]}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{-\mu}$.

When Brownian motion is replaced with a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, a spatial shift of the type

$$B_t \longmapsto B_t + \mu t$$

can no longer be used because $N_t + \mu t$ cannot be a Poisson process, whatever the change of probability applied, since by construction, the paths of the standard Poisson process has jumps of unit size and remain constant between jump times.

The correct way to extend the Girsanov Theorem to the Poisson case is to replace the space shift with a shift in the intensity of the Poisson process as in the following statement.

Proposition 6.14 Consider a random variable N_T having the Poisson distribution $\mathscr{P}(\lambda T)$ with parameter λT under $\mathbb{P}_\lambda$. Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T},$$

the random variable N_T has a Poisson distribution with intensity $\tilde{\lambda}T$.

Proof. This follows from the relation

$$\widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_T = k) = \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{k}\mathbb{P}_\lambda(N_T = k) = \mathrm{e}^{-\tilde{\lambda}T}\frac{\tilde{\lambda}^k}{k!}, \quad k \geqslant 0.$$

□

Assume now that $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ under a probability measure $\mathbb{P}_\lambda$. In order to extend (6.32) to the Poisson case we can replace the space shift with a *time contraction* (or dilation)

$$N_t \longmapsto N_{t/(1+c)} \quad \text{or} \quad N_t \longmapsto N_{(1+c)t},$$

by a factor $1+c$, where

$$c := -1 + \frac{\tilde{\lambda}}{\lambda} > -1,$$

or $\tilde{\lambda} = (1+c)\lambda$. We note that

$$\begin{aligned}
\mathbb{P}_\lambda(N_{(1+c)T} = k) &= \frac{(\lambda(1+c)T)^k}{k!}\mathrm{e}^{-\lambda(1+c)T} \\
&= (1+c)^k \mathrm{e}^{-\lambda c T}\mathbb{P}_\lambda(N_T = k) \\
&= \widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_T = k), \quad k \geqslant 0,
\end{aligned}$$

and by analogy with (6.32) we have

$$\mathbb{E}_\lambda\left[f(N_{(1+c)T})\right] = \sum_{k\geqslant 0} f(k)\mathbb{P}_\lambda(N_{(1+c)T} = k) \tag{6.33}$$

$$
\begin{aligned}
&= \mathrm{e}^{-\lambda cT} \sum_{k \geqslant 0} f(k)(1+c)^k \mathbb{P}_\lambda(N_T = k) \\
&= \mathrm{e}^{-\lambda cT} \mathbb{E}\left[f(N_T)(1+c)^{N_T}\right] \\
&= \mathrm{e}^{-\lambda cT} \int_\Omega (1+c)^{N_T} f(N_T) \mathrm{d}\mathbb{P}_\lambda \\
&= \int_\Omega f(N_T) d\widetilde{\mathbb{P}}_{\tilde{\lambda}} \\
&= \widetilde{\mathbb{E}}_{\tilde{\lambda}}[f(N_T)],
\end{aligned}
$$

for any bounded function f on $\mathbb{N}$. In other words, taking $f(x) := \mathbb{1}_{\{x \leqslant n\}}$ we have

$$
\mathbb{P}_\lambda\left(N_{(1+c)T} \leqslant n\right) = \widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_T \leqslant n), \qquad n \in \mathbb{N},
$$

or

$$
\widetilde{\mathbb{P}}_{\tilde{\lambda}}(N_{T/(1+c)} \leqslant n) = \mathbb{P}_\lambda\left(N_T \leqslant n\right), \qquad n \in \mathbb{N}.
$$

As a consequence, we have the following proposition.

Proposition 6.15 Let $\lambda, \tilde{\lambda} > 0$, and set

$$
c := -1 + \frac{\tilde{\lambda}}{\lambda} > -1.
$$

The process $(N_{t/(1+c)})_{t \in \mathbb{R}_+}$ is a standard Poisson process with intensity λ under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$
\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{\mathrm{d}\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T} = \mathrm{e}^{-c\lambda T}(1+c)^{N_T}.
$$

In particular, the compensated Poisson process

$$
N_{t/(1+c)} - \lambda t, \qquad t \in \mathbb{R}_+,
$$

is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$.

Proof. As in (6.33) we have

$$
\mathbb{E}[f(N_T)] = \widetilde{\mathbb{E}}_{\tilde{\lambda}}\left[f(N_{T/(1+c)})\right],
$$

i.e., under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ the distribution of $N_{T/(1+c)}$ is that of a standard Poisson random variable with parameter λT. As a consequence, $(N_{t/(1+c)})_{t \in \mathbb{R}_+}$ is a standard Poisson process with intensity λ under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, and since $(N_{t/(1+c)})_{t \in \mathbb{R}_+}$ has independent increments, the compensated process $(N_{t/(1+c)} - \lambda t)_{t \in \mathbb{R}_+}$ is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ by (3.2). □

Similarly, since

$$
(N_t - (1+c)\lambda t)_{t \in \mathbb{R}_+} = (N_t - \tilde{\lambda} t)_{t \in \mathbb{R}_+}
$$

has independent increments, the compensated Poisson process

$$
N_t - (1+c)\lambda t = N_t - \tilde{\lambda} t
$$

is a martingale under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$. We also have

$$
N_{t/(1+c)} = \sum_{n \geqslant 1} \mathbb{1}_{[T_n, \infty)}\left(\frac{t}{1+c}\right) = \sum_{n \geqslant 1} \mathbb{1}_{[(1+c)T_n, \infty)}(t), \quad t \in \mathbb{R}_+,
$$

which shows that the jump times $((1+c)T_n)_{n\geqslant 1}$ of $(N_{t/(1+c)})_{t\in[0,T]}$ are distributed under $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ as the jump times of a Poisson process with intensity λ.

The next R code shows that the compensated Poisson process $(N_{t/(1+c)}-\lambda t)_{t\in\mathbb{R}_+}$, remains a martingale after the Poisson process interjump times $(\tau_k)_{k\geqslant 1}$ have been generated using exponential random variables with parameter $\tilde{\lambda}>0$.

```
lambda = 0.5;lambdat=2;c=-1+lambdat/lambda;n = 20;Z<-cumsum(c(0,rep(1,n)))
for (k in 1:n){tau_k <- rexp(n,rate=lambdat); Tn <- cumsum(tau_k)}
N <- function(t) {return(stepfun(Tn,Z)(t))};t <- seq(0,10,0.01)
plot(t,N(t/(1+c))-lambda*t,xlim = c(0,10),ylim =
    c(-2,2),xlab="t",ylab="Nt-t",type="l",lwd=2,col="blue",main="", xaxs = "i", yaxs = "i", xaxs =
    "i", yaxs = "i");abline(h = 0, col="black", lwd =2)
points(Tn*(1+c),N(Tn)-lambda*Tn*(1+c),pch=1,cex=0.8,col="blue",lwd=2)
```

When $\mu\neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}=(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ written as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}}=(\mu-r)dt+\sigma(dN_t-\lambda dt) \tag{6.34}$$

is not a martingale under $\mathbb{P}_\lambda$. However, we can rewrite (6.34) as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}}=\sigma\left(dN_t-\left(\lambda-\frac{\mu-r}{\sigma}\right)dt\right)$$

and letting

$$\tilde{\lambda}:=\lambda-\frac{\mu-r}{\sigma}=(1+c)\lambda$$

with

$$c:=-\frac{\mu-r}{\sigma\lambda},$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}}=\sigma\left(dN_t-\tilde{\lambda}dt\right)$$

hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{\mathrm{d}\mathbb{P}_\lambda}:=\mathrm{e}^{-\lambda cT}(1+c)^{N_T}d\mathbb{P}_\lambda=\mathrm{e}^{(\mu-r)/\sigma}\left(1-\frac{\mu-r}{\sigma\lambda}\right)^{N_T}.$$

We note that if

$$\mu-r\leqslant\sigma\lambda$$

then the risk-neutral probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ exists and is unique, therefore by Theorems 1.2 and 1.4 the market is without arbitrage and complete. If $\mu-r>\sigma\lambda$ then the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is always increasing, and arbitrage becomes possible by borrowing from the savings account and investing on the risky underlying asset.

Girsanov Theorem for compound Poisson processes

In the case of compound Poisson processes, the Girsanov Theorem can be extended to variations in jump sizes in addition to time variations, and we have the following more general result.

Theorem 6.16 Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another intensity parameter $\tilde{\lambda} > 0$ and jump size distribution $\tilde{\nu}(dx)$, and let

$$\psi(x) := \frac{\tilde{\lambda}\,\tilde{\nu}(dx)}{\lambda\,\nu(dx)} - 1, \qquad x \in \mathbb{R}. \tag{6.35}$$

Then,

under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\prod_{k=1}^{N_T}(1+\psi(Z_k)),$$

the process

$$Y_t := \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,$$

is a compound Poisson process with

- modified intensity $\tilde{\lambda} > 0$, and

- modified jump size distribution $\tilde{\nu}(dx)$.

Proof. For any bounded measurable function f on $\mathbb{R}$, we extend (6.33) to the following change of variable

$$\begin{aligned}
\mathbf{E}_{\tilde{\lambda},\tilde{\nu}}[f(Y_T)] &= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\,\mathbf{E}_{\lambda,\nu}\left[f(Y_T)\prod_{i=1}^{N_T}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-(\tilde{\lambda}-\lambda)T}\sum_{k\geqslant 0}\mathbf{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k}Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\;\Big|\;N_T=k\right]\mathbb{P}_{\lambda}(N_T=k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\mathbf{E}_{\lambda,\nu}\left[f\left(\sum_{i=1}^{k}Z_i\right)\prod_{i=1}^{k}(1+\psi(Z_i))\right] \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\lambda T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\prod_{i=1}^{k}(1+\psi(z_i))\nu(dz_1)\cdots\nu(dz_k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\tilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\left(\prod_{i=1}^{k}\frac{\tilde{\nu}(dz_i)}{\nu(dz_i)}\right)\nu(dz_1)\cdots\nu(dz_k) \\
&= \mathrm{e}^{-\tilde{\lambda}T}\sum_{k\geqslant 0}\frac{(\tilde{\lambda}T)^k}{k!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+\cdots+z_k)\tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_k).
\end{aligned}$$

This shows that under $\mathbb{P}_{\tilde{\lambda},\tilde{\nu}}$, Y_T has the distribution of a compound Poisson process with intensity $\tilde{\lambda}$ and jump size distribution $\tilde{\nu}$. We refer to Proposition 9.6 of Cont and Tankov, 2004 for the independence of increments of $(Y_t)_{t\in\mathbb{R}_+}$ under $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$. □

Example. In case $\nu \simeq \mathcal{N}(\alpha,\sigma^2)$ and $\tilde{\nu} \simeq \mathcal{N}(\beta,\eta^2)$, we have

$$\nu(dx) = \frac{dx}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(x-\alpha)^2\right), \quad \tilde{\nu}(dx) = \frac{dx}{\sqrt{2\pi\eta^2}}\exp\left(-\frac{1}{2\eta^2}(x-\beta)^2\right),$$

$x \in \mathbb{R}$, hence

$$\frac{\tilde{\nu}(dx)}{\nu(dx)} = \frac{\eta}{\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2 \right), \qquad x \in \mathbb{R},$$

and $\psi(x)$ in (6.35) is given by

$$1 + \psi(x) = \frac{\tilde{\lambda}}{\lambda} \frac{\tilde{\nu}(dx)}{\nu(dx)} = \frac{\tilde{\lambda}\eta}{\lambda\sigma} \exp\left(\frac{1}{2\eta^2}(x-\beta)^2 - \frac{1}{2\sigma^2}(x-\alpha)^2 \right), \quad x \in \mathbb{R}.$$

Note that the compound Poisson process with intensity $\tilde{\lambda} > 0$ and jump size distribution $\tilde{\nu}$ can be built as

$$X_t := \sum_{k=1}^{N_{\tilde{\lambda}t/\lambda}} h(Z_k),$$

provided that $\tilde{\nu}$ is the *pushforward* measure of ν by the function $h : \mathbb{R} \to \mathbb{R}$, *i.e.*,

$$\mathbb{P}(h(Z_k) \in A) = \mathbb{P}(Z_k \in h^{-1}(A)) = \nu(h^{-1}(A)) = \tilde{\nu}(A),$$

for all (measurable) subsets A of $\mathbb{R}$. As a consequence of Theorem 6.16 we have the following proposition.

Proposition 6.17 The compensated process

$$Y_t - \tilde{\lambda} t \, \mathbb{E}_{\tilde{\nu}}[Z]$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} = \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \prod_{k=1}^{N_T} (1 + \psi(Z_k)).$$

Finally, the Girsanov Theorem can be extended to the linear combination of a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and a compound Poisson process $(Y_t)_{t\in\mathbb{R}_+}$ independent of $(B_t)_{t\in\mathbb{R}_+}$, as in the following result which is a particular case of Theorem 33.2 of Sato, 1999.

Theorem 6.18 Let $(Y_t)_{t\geqslant 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\nu(dx)$. Consider another jump size distribution $\tilde{\nu}(dx)$ and intensity parameter $\tilde{\lambda} > 0$, and let

$$\psi(x) := \frac{\tilde{\lambda}}{\lambda} \frac{d\tilde{\nu}}{d\nu}(x) - 1, \qquad x \in \mathbb{R},$$

and let $(u_t)_{t\in\mathbb{R}_+}$ be a bounded adapted process. Then the process

$$\left(B_t + \int_0^t u_s ds + Y_t - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[Z] t \right)_{t\in\mathbb{R}_+}$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ defined by the Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}}{d\widetilde{\mathbb{P}}_{\lambda,\nu}} = \exp\left(-(\tilde{\lambda}-\lambda)T - \int_0^T u_s dB_s - \frac{1}{2}\int_0^T |u_s|^2 ds \right) \prod_{k=1}^{N_T} (1 + \psi(Z_k)). \tag{6.36}$$

As a consequence of Theorem 6.18, if

$$B_t + \int_0^t v_s ds + Y_t \tag{6.37}$$

is not a martingale under $\widetilde{\mathbb{P}}_{\lambda,\nu}$, it will become a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ provided that $u, \tilde{\lambda}$ and $\tilde{\nu}$ are chosen in such a way that

$$v_s = u_s - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z], \qquad s \in \mathbb{R}, \tag{6.38}$$

in which case (6.37) can be rewritten into the martingale decomposition

$$dB_t + u_t dt + dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt,$$

in which both $\left(B_t + \int_0^t u_s ds\right)_{t\in\mathbb{R}_+}$ and $\left(Y_t - \tilde{\lambda} t\,\mathbb{E}_{\tilde{\nu}}[Z]\right)_{t\in\mathbb{R}_+}$ are martingales under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$

The following remarks will be of importance for arbitrage pricing in jump models in Chapter 7.

a) When $\tilde{\lambda} = \lambda = 0$, Theorem 6.18 coincides with the usual Girsanov Theorem for Brownian motion, in which case (6.38) admits only one solution given by $u = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{u,0,0}$.
b) Uniqueness also occurs when $u = 0$ in the absence of Brownian motion with Poisson jumps of fixed size a (*i.e.*, $\tilde{\nu}(dx) = \nu(dx) = \delta_a(dx)$) since in this case (6.38) also admits only one solution $\tilde{\lambda} = v$ and there is uniqueness of $\widetilde{\mathbb{P}}_{0,\tilde{\lambda},\delta_a}$.

When $\mu \neq r$, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ defined by

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = (\mu - r)dt + \sigma dB_t + \eta(dY_t - \lambda t\,\mathbb{E}_\nu[Z])$$

is not martingale under $\mathbb{P}_{\lambda,\nu}$, however we can rewrite the equation as

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma(udt + dB_t) + \eta\left(dY_t - \left(\frac{u\sigma}{\eta} + \lambda\,\mathbb{E}_{\tilde{\nu}}[Z] - \frac{\mu - r}{\eta}\right)dt\right)$$

and choosing u, $\tilde{\nu}$, and $\tilde{\lambda}$ such that

$$\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z] = \frac{u\sigma}{\eta} + \lambda\,\mathbb{E}_{\tilde{\nu}}[Z] - \frac{\mu - r}{\eta}, \tag{6.39}$$

we have

$$\frac{d\widetilde{S}_t}{\widetilde{S}_{t^-}} = \sigma(udt + dB_t) + \eta\left(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt\right)$$

hence the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$, and the market is without arbitrage by Theorem 1.2 and the existence of a risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. However, the market is not complete due to the non uniqueness of solutions $(u, \tilde{\nu}, \tilde{\lambda})$ to (6.39), and Theorem 1.4 does not apply in this situation.

Exercises

Exercise 6.1 Analysis of user login activity to the DBX digibank app showed that the times elapsed between two logons are independent and exponentially distributed with mean $1/\lambda$. Find the CDF of the time $T - T_{N_T}$ elapsed since the last logon before time T, given that the user has logged on at least once.

Hint: The number of logins until time $t > 0$ can be modeled by a standard Poisson process $(N_t)_{t\in[0,T]}$ with intensity λ.

Exercise 6.2 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Solve the stochastic differential equation

$$dS_t = \eta S_{t^-} dN_t - \eta \lambda S_t dt = \eta S_{t^-} (dN_t - \lambda dt).$$

b) Using the first Poisson jump time T_1, solve the stochastic differential equation

$$dS_t = -\lambda \eta S_t dt + dN_t, \qquad t \in (0, T_2).$$

Exercise 6.3 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ over the time intervals $[0, T_1)$, $[T_1, T_2)$, $[T_2, T_3)$, $[T_3, T_4)$, where $X_0 = 1$.
b) Write a differential equation for $f(t) := \mathbb{E}[X_t]$, and solve it for $t \in \mathbb{R}_+$.

Exercise 6.4 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = \sigma X_{t^-} dN_t$ for $(X_t)_{t\in\mathbb{R}_+}$, where $\sigma > 0$ and $X_0 = 1$.
b) Show that the solution $(S_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dS_t = r dt + \sigma S_{t^-} dN_t,$$

is given by $S_t = S_0 X_t + r X_t \int_0^t X_s^{-1} ds$.
c) Compute $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t / X_s]$, $0 \leqslant s \leqslant t$.
d) Compute $\mathbb{E}[S_t]$, $t \in \mathbb{R}_+$.

Exercise 6.5 Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with intensity $\lambda > 0$, started at $N_0 = 0$.

a) Is the process $t \mapsto N_t - 2\lambda t$ a *sub*martingale, a martingale, or a *super*martingale?
b) Let $r > 0$. Solve the stochastic differential equation

$$dS_t = r S_t dt + \sigma S_{t^-} (dN_t - \lambda dt).$$

c) Is the process $t \mapsto S_t$ of Question (b)) a *sub*martingale, a martingale, or a *super*martingale?
d) Compute the price at time 0 of the European call option with strike price $K = S_0 \, \mathrm{e}^{(r-\lambda\sigma)T}$, where $\sigma > 0$.

Exercise 6.6 Affine stochastic differential equation with jumps. Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$.

a) Solve the stochastic differential equation $dX_t = a dN_t + \sigma X_{t^-} dN_t$, where $\sigma > 0$, and $a \in \mathbb{R}$.
b) Compute $\mathbb{E}[X_t]$ for $t \in \mathbb{R}_+$.

Exercise 6.7 Consider the compound Poisson process $Y_t := \sum_{k=1}^{N_t} Z_k$, where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$, and $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of $\mathcal{N}(0,1)$ Gaussian random variables. Solve the stochastic differential equation

$$dS_t = r S_t dt + \eta S_{t^-} dY_t,$$

where $\eta, r \in \mathbb{R}$.

Exercise 6.8 Show, by direct computation or using the moment generating function (6.10), that the variance of the compound Poisson process Y_t with intensity $\lambda > 0$ satisfies

$$\mathrm{Var}\,[Y_t] = \lambda t \, \mathbb{E}\left[|Z|^2\right] = \lambda t \int_{-\infty}^{\infty} x^2 \nu(dx).$$

Exercise 6.9 Consider an exponential compound Poisson process of the form

$$S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t + Y_t}, \qquad t \in \mathbb{R}_+,$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process of the form (6.8).

a) Derive the stochastic differential equation with jumps satisfied by $(S_t)_{t\in\mathbb{R}_+}$.
b) Let $r > 0$. Find a family $(\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}})$ of probability measures under which the discounted asset price $\mathrm{e}^{-rt} S_t$ is a martingale.

Exercise 6.10 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + Z_{N_t} S_{t^-} dN_t, \tag{6.40}$$

where $(Z_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of random variables of the form

$$Z_k = \mathrm{e}^{X_k} - 1, \quad \text{where} \quad X_k \simeq \mathcal{N}(0, \sigma^2), \quad k \geqslant 1.$$

a) Solve the equation (6.40).
b) We assume that μ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ and r?
c) Under the relation of Question (b)), compute the price at time t of the European call option on S_T with strike price κ and maturity T, using a series expansion of Black-Scholes functions.

Exercise 6.11 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be the mean-reverting process defined by the stochastic differential equation

$$dS_t = -\alpha S_t dt + \sigma (dN_t - \beta dt), \tag{6.41}$$

where $S_0 > 0$ and $\alpha, \beta > 0$.

a) Solve the equation (6.41) for S_t.
b) Compute $f(t) := \mathbb{E}[S_t]$ for all $t \in \mathbb{R}_+$.
c) Under which condition on α, β, σ and λ does the process S_t become a *sub*martingale?
d) Propose a method for the calculation of expectations of the form $\mathbb{E}[\phi(S_T)]$ where ϕ is a payoff function.

Exercise 6.12 Let $(N_t)_{t\in[0,T]}$ be a standard Poisson process started at $N_0 = 0$, with intensity $\lambda > 0$ under the probability measure $\mathbb{P}_\lambda$, and consider the compound Poisson process $(Y_t)_{t\in[0,T]}$ with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$.

a) Under the probability measure $\mathbb{P}_\lambda$, the process $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z])$ is a:

submartingale |	martingale |	supermartingale |

b) Consider the process $(S_t)_{t\in[0,T]}$ given by

$$dS_t = \mu S_t dt + \sigma S_{t^-} dY_t.$$

Find $\tilde{\lambda}$ such that the discounted process $(\widetilde{S}_t)_{t\in[0,T]} := (e^{-rt} S_t)_{t\in[0,T]}$ is a martingale under the probability measure $\mathbb{P}_{\tilde{\lambda}}$ defined by the Radon-Nikodym density

$$\frac{d\mathbb{P}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_\lambda$.

c) Price the forward contract with payoff $S_T - \kappa$.

Exercise 6.13 Consider $(Y_t)_{t\in\mathbb{R}_+}$ a compound Poisson process written as

$$Y_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+,$$

where $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ and $(Z_k)_{k\geqslant 1}$ is an *i.i.d* family of random variables with probability distribution $\nu(dx)$ on $\mathbb{R}$, under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + S_{t^-} dY_t. \tag{6.42}$$

a) Solve the equation (6.42).
b) We assume that μ, $\nu(dx)$ and the risk-free interest rate $r > 0$ are chosen such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$. What relation does this impose on μ, $\nu(dx)$ and r?
c) Under the relation of Question (b)), compute the price at time t of the European call option on S_T with strike price κ and maturity T, using a series expansion of integrals.

Exercise 6.14 Consider a standard Poisson process $(N_t)_{t\in[0,T]}$ with intensity $\lambda > 0$ and a standard Brownian motion $(B_t)_{t\in[0,T]}$ independent of $(N_t)_{t\in[0,T]}$ under the probability measure $\mathbb{P}_\lambda$. Let also $(Y_t)_{t\in[0,T]}$ be a compound Poisson process with *i.i.d.* jump sizes $(Z_k)_{k\geqslant 1}$ of distribution $\nu(dx)$ under $\mathbb{P}_\lambda$, and consider the jump process $(S_t)_{t\in[0,T]}$ solution of

$$dS_t = rS_t dt + \sigma S_t dB_t + \eta S_{t^-}(dY_t - \tilde{\lambda} t\, \mathbb{E}[Z_1]).$$

with $r, \sigma, \eta, \lambda, \tilde{\lambda} > 0$.

a) Assume that $\tilde{\lambda} = \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

| submartingale | | martingale | | supermartingale | |

b) Assume $\tilde{\lambda} > \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

| submartingale | | martingale | | supermartingale | |

c) Assume $\tilde{\lambda} < \lambda$. Under the probability measure $\mathbb{P}_\lambda$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

| submartingale | | martingale | | supermartingale | |

d) Consider the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$ defined by its Radon-Nikodym density

$$\frac{d\widetilde{\mathbb{P}}_{\tilde{\lambda}}}{d\mathbb{P}_\lambda} := \mathrm{e}^{-(\tilde{\lambda}-\lambda)T} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_T}.$$

with respect to $\mathbb{P}_\lambda$. Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a:

| submartingale | | martingale | | supermartingale | |

7. Pricing and Hedging in Jump Models

This chapter considers the pricing and hedging of financial derivatives using discontinuous processes that can model sharp movements in asset prices. Unlike in the case of continuous asset price modeling, uniqueness of risk-neutral probability measures can be lost and, as a consequence, the computation of perfect replicating hedging strategies may not be possible in general.

7.1 Fitting the Distribution of Market Returns

The modeling of risky asset by stochastic processes with continuous paths, based on Brownian motions, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes, gaps or opening jumps, see *e.g.* Chapter 1 of Cont and Tankov, 2004. Secondly, the modeling of risky asset prices by Brownian motion relies on the use of the Gaussian distribution which tends to underestimate the probabilities of extreme events.

The R package Quantmod is installed through the command:

```
install.packages("quantmod")
```

The following scripts allow us to fetch DJI and STI index data using Quantmod.

```
library(quantmod)
getSymbols("^STI",from="1990-01-03",to="2015-01-03",src="yahoo");stock=Ad(`STI`);
getSymbols("^DJI",from="1990-01-03",to=Sys.Date(),src="yahoo");stock=Ad(`DJI`);
stock.rtn=diff(log(stock));returns <- as.vector(stock.rtn)
m=mean(returns,na.rm=TRUE);s=sd(returns,na.rm=TRUE)
times=index(stock.rtn)
n = sum(is.na(returns))+sum(!is.na(returns))
x=seq(1,n);y=rnorm(n,mean=m,sd=s)
plot(times,returns,pch=19,xaxs="i",cex=0.03,col="blue", ylab="", xlab="", main = '')
segments(x0 = times, x1 = times, y0 = 0, y1 = returns,col="blue")
points(times,y,pch=19,cex=0.3,col="red")
abline(h = m+3*s, col="black", lwd =1)
abline(h = m, col="black", lwd =1)
abline(h = m+-3*s, col="black", lwd =1)
length(returns[abs(returns-m)>3*s])/length(stock.rtn)
length(y[abs(y-m)>3*s])/length(y)
2*(1-pnorm(3*s,0,s))
```

The next Figures 7.1-7.5 illustrate the mismatch between the distributional properties of market log returns *vs* Gaussian returns, based on the above code.

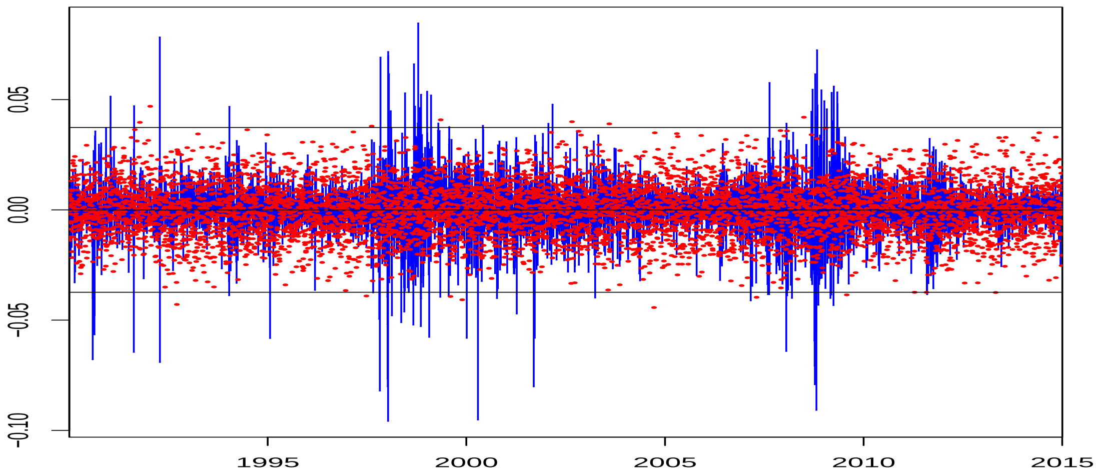


Figure 7.1: Market returns *vs* normalized Gaussian returns.

```
stock.ecdf=ecdf(as.vector(stock.rtn))
x <- seq(-0.25, 0.25, length=100)
px <- pnorm((x-m)/s)
plot(stock.ecdf, xlab = 'Sample Quantiles', col="blue",ylab = '', main = '')
lines(x, px, type="l", lty=2, col="red",xlab="x value",ylab="Probability", main="")
legend("topleft", legend=c("Empirical CDF", "Gaussian CDF"),col=c("blue", "red"), lty=1:2, cex=0.8)
```

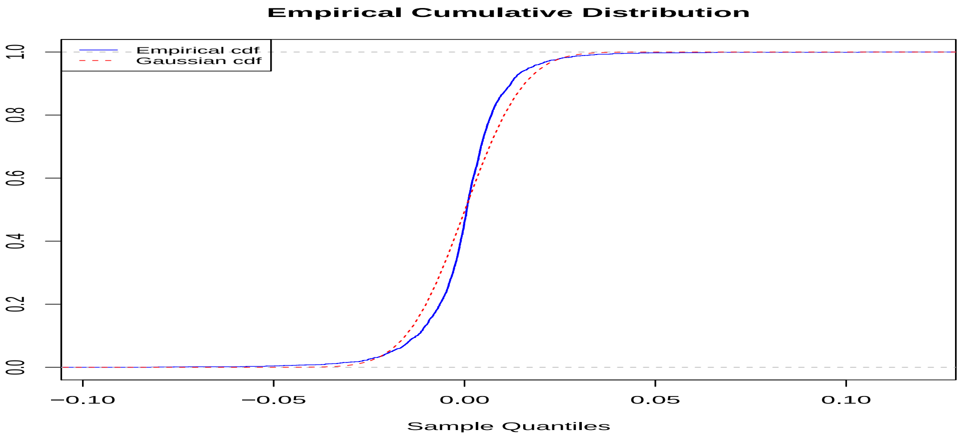


Figure 7.2: Empirical *vs* Gaussian CDF.

The following Quantile-Quantile plot is plotting the normalized empirical quantiles against the standard Gaussian quantiles, and is obtained with the `qqnorm(returns)` command.

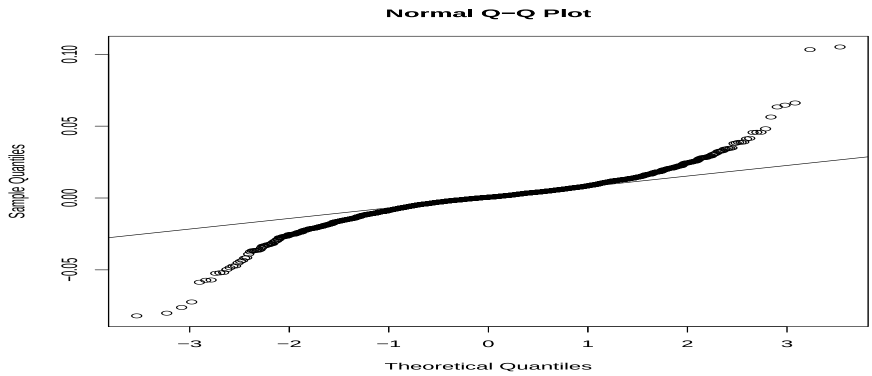


Figure 7.3: Quantile-Quantile plot.

```
ks.test(y,"pnorm",mean=m,sd=s)
ks.test(returns,"pnorm",mean=m,sd=s)
```

The Kolmogorov-Smirnov test clearly rejects the null (normality) hypothesis.

```
One-sample Kolmogorov-Smirnov test

data:  returns
D = 0.075577, p-value < 2.2e-16
alternative hypothesis: two-sided
```

```r
stock.dens=density(returns,na.rm=TRUE)
x <- stock.dens$x
qx <- dnorm(x,mean=m,sd=s)
plot(stock.dens, xlab = 'x', lwd=2, col="red",ylab = '', main = 'Empirical density',panel.first = abline(h
      = 0, col='grey', lwd =0.2))
lines(x, qx, type="l", lty=2, lwd=2, col="blue",xlab="x value",ylab="Density", main="")
legend("topleft", legend=c("Empirical density", "Gaussian density"),col=c("red", "blue"), lty=1:2,
      cex=0.8)
```

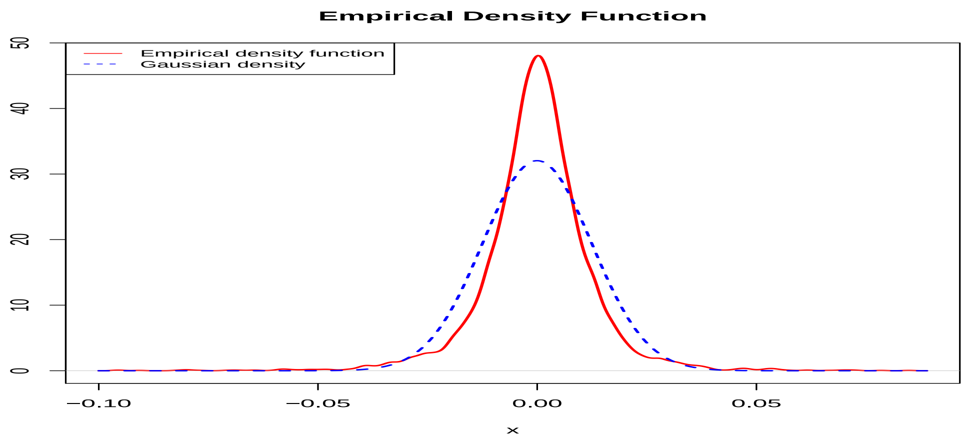


Figure 7.4: Empirical density *vs* normalized Gaussian density.

Power tail distributions

We note that the empirical density has significantly higher kurtosis and non zero skewness in comparison with the Gaussian probability density. On the other hand, power tail probability densities of the form $\varphi(x) \simeq C_\alpha / x^\alpha$, $x \to \infty$, can provide a better fit of empirical probability density functions, as shown in Figure 7.5.

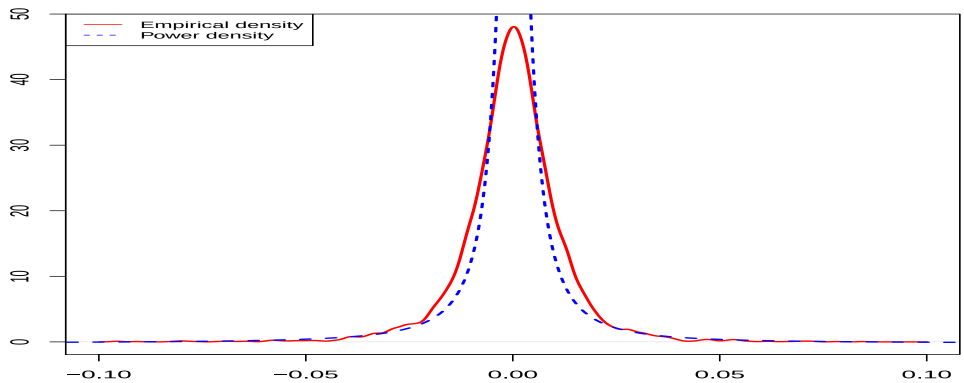


Figure 7.5: Empirical density *vs* power density.

The above fitting of empirical probability density function is using a power probability density function defined by a rational fraction obtained by the following R script.

```
install.packages("pracma")
library(pracma); x <- seq(-0.25, 0.25, length=1000)
stock.dens=density(returns,na.rm=TRUE, from = -0.1, to = 0.1, n = 1000)
a<-rationalfit(stock.dens$x, stock.dens$y, d1=2, d2=2)
plot(stock.dens$x,stock.dens$y, lwd=2, type = "l",xlab = '', col="red",ylab = '', main = '', panel.first =
    abline(h = 0, col='grey', lwd =0.2))
lines(x,(a$p1[3]+a$p1[2]*x+a$p1[1]*x^2)/(a$p2[3]+a$p2[2]*x+a$p2[1]*x^2),
    type="l",lty=2,col="blue",xlab="x value",lwd=2, ylab="Density",main="")
legend("topleft", legend=c("Empirical density", "Power density"),col=c("red", "blue"), lty=1:2, cex=0.8)
```

The output of the rationalfit command is

```
$p1
[1] -0.184717249 -0.001591433  0.001385017

$p2
[1]  1.000000e+00 -6.460948e-04  1.314672e-05
```

which yields a rational fraction of the form

$$\begin{aligned} x &\longmapsto \frac{0.001385017 - 0.001591433 \times x - 0.184717249 \times x^2}{1.314672\ 10^{-5} - 6.460948\ 10^{-4} \times x + x^2} \\ &\simeq -0.184717249 - \frac{0.001591433}{x} + \frac{0.001385017}{x^2}, \end{aligned}$$

which approximates the empirical probability density function of DJI returns in the least squares sense.

A solution to this tail problem is to use stochastic processes with jumps, that will account for sudden variations of the asset prices. On the other hand, such jump models are generally based on the Poisson distribution which has a slower tail decay than the Gaussian distribution. This allows one to assign higher probabilities to extreme events, resulting in a more realistic modeling of asset prices. *Stable distributions* with parameter $\alpha \in (0,2)$ provide typical examples of probability laws with power tails, as their probability density functions behave asymptotically as $x \mapsto C_\alpha / |x|^{1+\alpha}$ when $x \to \pm\infty$, see Figure 6.10 for stable processes.

Edgeworth and Gram-Charlier expansions

Let

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}, \qquad x \in \mathbb{R},$$

denote the standard normal density function, and let

$$\Phi(x) := \int_{-\infty}^{x} \varphi(y) dy, \qquad x \in \mathbb{R},$$

denote the standard normal cumulative distribution function. Let also

$$H_n(x) := \frac{(-1)^n}{\varphi(x)} \frac{\partial^n \varphi}{\partial x^n}(x), \qquad x \in \mathbb{R},$$

denote the Hermite polynomial of degree n, with $H_0(x) = 1$.

Given X a random variable, the sequence $(\kappa_n^X)_{n \geqslant 1}$ of cumulants of X has been introduced in Thiele, 1899. In the sequel we will use the Moment Generating Function (MGF) of the random variable X, defined as

$$\mathscr{M}_X(t) := \mathbb{E}\left[\mathrm{e}^{tX}\right] = 1 + \sum_{n \geqslant 1} \frac{t^n}{n!} \mathbb{E}[X^n], \qquad t \in \mathbb{R}. \tag{7.1}$$

Definition 7.1 The *cumulants* of a random variable X are defined to be the coefficients $(\kappa_n^X)_{n\geqslant 1}$ appearing in the series expansion

$$\log\left(\mathbb{E}\left[\mathrm{e}^{tX}\right]\right) = \log\left(1+\sum_{n\geqslant 1}\frac{t^n}{n!}\mathbb{E}[X^n]\right) = \sum_{n\geqslant 1}\kappa_n^X\frac{t^n}{n!}, \quad t\in\mathbb{R}, \tag{7.2}$$

of the logarithmic moment generating function (log-MGF) of X.

The cumulants of X were originally called "semi-invariants" due to the property $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$, $n\geqslant 1$, when X and Y are independent random variables. Indeed, in this case we have

$$\begin{aligned}
\sum_{n\geqslant 1}\kappa_n^{X+Y}\frac{t^n}{n!} &= \log\left(\mathbb{E}\left[\mathrm{e}^{t(X+Y)}\right]\right) \\
&= \log\left(\mathbb{E}\left[\mathrm{e}^{tX}\right]\mathbb{E}\left[\mathrm{e}^{tY}\right]\right) \\
&= \log\mathbb{E}\left[\mathrm{e}^{tX}\right] + \log\mathbb{E}\left[\mathrm{e}^{tY}\right] \\
&= \sum_{n\geqslant 1}\kappa_n^X\frac{t^n}{n!} + \sum_{n\geqslant 1}\kappa_n^Y\frac{t^n}{n!} \\
&= \sum_{n\geqslant 1}\left(\kappa_n^X+\kappa_n^Y\right)\frac{t^n}{n!}, \qquad t\in\mathbb{R},
\end{aligned}$$

showing that $\kappa_n^{X+Y} = \kappa_n^X + \kappa_n^Y$, $n\geqslant 1$.

a) First moment and cumulant. Taking $n=1$ and $\pi=\{1\}$, we find $\kappa_1^X = \mathbb{E}[X]$.

b) Variance and second cumulant. We have

$$\kappa_2^X = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right],$$

and $\sqrt{\kappa_X^2}$ is the *standard deviation* of X.

c) The third cumulant of X is given as the third central moment

$$\kappa_3^X = \mathbb{E}[(X-\mathbb{E}[X])^3],$$

and the coefficient

$$\frac{\kappa_3^X}{(\kappa_2^X)^{3/2}} = \frac{\mathbb{E}\left[(X-\mathbb{E}[X])^3\right]}{(\mathbb{E}[(X-\mathbb{E}[X])^2])^{3/2}}$$

is the *skewness* of X.

d) Similarly, we have

$$\begin{aligned}
\kappa_4^X &= \mathbb{E}\left[(X-\mathbb{E}[X])^4\right] - 3\left(\kappa_2^X\right)^2 \\
&= \mathbb{E}\left[(X-\mathbb{E}[X])^4\right] - 3\left(\mathbb{E}\left[(X-\mathbb{E}[X])^2\right]\right)^2,
\end{aligned}$$

and the *excess kurtosis* of X is defined as

$$\frac{\kappa_4^X}{(\kappa_2^X)^2} = \frac{\mathbb{E}[(X-\mathbb{E}[X])^4]}{(\mathbb{E}[(X-\mathbb{E}[X])^2])^2} - 3.$$

The next proposition summarizes the Gram-Charlier expansion method to obtain series expansion of a probability density function, see Gram, 1883, Charlier, 1914 and § 17.6 of Cramér, 1946.

Proposition 7.1 (Proposition 2.1 in Tanaka, Yamada, and T. Watanabe, 2010) The Gram-Charlier expansion of the continuous probability density function $\phi_X(x)$ of a random variable X is given by

$$\phi_X(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + \frac{1}{\sqrt{\kappa_2}}\sum_{n=3}^{\infty} c_n H_n\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right),$$

where $c_0 = 1$, $c_1 = c_2 = 0$, and the sequence $(c_n)_{n\geqslant 3}$ is given from the cumulants $(\kappa_n)_{n\geqslant 1}$ of X as

$$c_n = \frac{1}{\kappa_2^{n/2}}\sum_{m=1}^{[n/3]}\sum_{\substack{l_1+\cdots+l_m=n\\ l_1,\ldots,l_m\geqslant 3}}\frac{\kappa_{l_1}\cdots\kappa_{l_m}}{m!l_1!\cdots l_m!}, \qquad n\geqslant 3.$$

The coefficients c_3 and c_4 can be expressed from the skweness $\kappa_3/\kappa_2^{3/2}$ and the excess kurtosis κ_4/κ_2^2 as

$$c_3 = \frac{\kappa_3}{3!\kappa_2^{3/2}} \quad \text{and} \quad c_4 = \frac{\kappa_4}{4!\kappa_2^2}.$$

a) The first-order expansion

$$\phi_X^{(1)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)$$

corresponds to normal moment matching approximation.

b) The third-order expansion is given by

$$\phi_X^{(3)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\right)$$

c) The fourth-order expansion is given by

$$\phi_X^{(4)}(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\left(1 + c_3 H_3\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + c_4 H_4\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\right).$$

```
install.packages("SimMultiCorrData");install.packages("PDQutils")
library(quantmod);library(SimMultiCorrData);library(PDQutils)
plot(stock.dens$x,stock.dens$y, xlab = 'x', type = 'l', lwd=2, col="red",ylab = '', main = 'Empirical
      density',panel.first = abline(h = 0, col='grey', lwd =0.2))
lines(x, qx, type="l", lty=2, lwd=2, col="blue")
m<-calc_moments(returns[!is.na(returns)])
cumulants<-c(m[1],m[2]**2);d2 <- dapx_edgeworth(x, cumulants)
lines(x, d2, type="l", lty=2, lwd=2, col="blue")
cumulants<-c(m[1],m[2]**2,m[3]*m[2]**3);d3 <- dapx_edgeworth(x, cumulants)
lines(x, d3, type="l", lty=2, lwd=2, col="green")
cumulants<-c(m[1],m[2]**2,0.5*m[3]*m[2]**3,0.2*m[4]*m[2]**4)
d4 <- dapx_edgeworth(x, cumulants)
lines(x, d4, type="l", lty=2, lwd=2, col="purple")
legend("topleft", legend=c("Empirical density", "Gaussian density", "Third order Gram-Charlier", "Fourth
      order Gram-Charlier"),col=c("red", "blue", "green", "purple"), lty=1:2,cex=0.8)
```

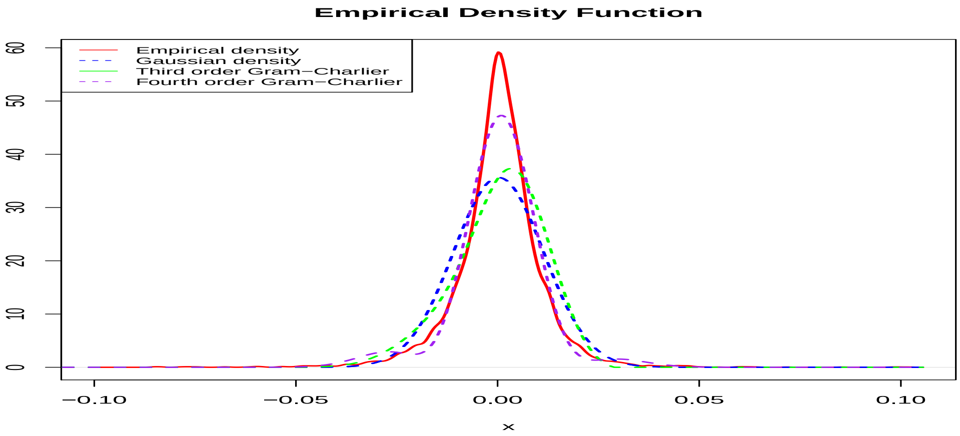


Figure 7.6: Gram-Charlier expansions

7.2 Risk-Neutral Probability Measures

Consider an asset price process modeled by the equation,

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t, \tag{7.3}$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is the compound Poisson process defined in Section 6.2, with jump size distribution $\nu(dx)$ under $\mathbb{P}_\nu$. The equation (7.3) has for solution

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{\sigma^2}{2} t\right) \prod_{k=1}^{N_t} (1 + Z_k), \tag{7.4}$$

$t \in \mathbb{R}_+$. An important issue for non-abitrage pricing is to determine a risk-neutral probability measure (or martingale measure) $\mathbb{P}^*$ under which the discounted asset price process $(\widehat{S}_t)_{t\in\mathbb{R}_+} := (\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale, and this goal can be achieved using the Girsanov Theorem for jump processes, cf. Section 6.5.

The discounted asset price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

satisfies the equation

$$\begin{aligned} d\widetilde{S}_t &= (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dB_t + \mathrm{e}^{-rt} \widetilde{S}_{t^-} dY_t \\ &= (\mu - r + \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] - \sigma u)\widetilde{S}_t dt + \sigma \widetilde{S}_t (dB_t + u dt) + \widetilde{S}_{t^-}(dY_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] dt), \end{aligned}$$

for any $u \in \mathbb{R}$. When the drift parameter u, the intensity $\tilde{\lambda} > 0$ and then jump size distribution $\tilde{\nu}$ are chosen to satisfy the condition

$$\mu - r + \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] - \sigma u = 0 \tag{7.5}$$

with $\sigma u + r - \mu > 0$, then

$$\tilde{\lambda} = \frac{\sigma u + r - \mu}{\mathbb{E}_{\tilde{\nu}}[Z]} > 0,$$

and the Girsanov Theorem 6.18 for jump processes then shows that

$$dB_t + u dt + dY_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[Z] dt$$

is a martingale under the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ defined in Theorem 6.18. As a consequence, the discounted price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ becomes a martingale is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$.

In this setting, the non-uniqueness of the risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ is apparent since additional degrees of freedom are involved in the choices of u, λ and the measure $\tilde{\nu}$, whereas in the continuous case the choice of $u = (\mu - r)/\sigma$ in (3.12) was unique.

7.3 Pricing in Jump Models

Recall that a market is without arbitrage if and only it admits at least one risk-neutral probability measure.

Consider the probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ constructed in Theorem 6.18, under which the discounted asset price process

$$d\widehat{S}_t = \widehat{S}_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\nu}[Z]dt) + \sigma\widehat{S}_t d\widehat{B}_t,$$

is a martingale, and $\widehat{B}_t = B_t + udt$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$.

Then the arbitrage price of a claim with payoff C is given by

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[C \mid \mathscr{F}_t] \tag{7.6}$$

under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$.

Clearly the price (7.6) of C is no longer unique in the presence of jumps due to the infinity of choices satisfying the martingale condition (7.5), and such a market is not complete, except if either $\tilde{\lambda} = \lambda = 0$, or ($\sigma = 0$ and $\tilde{\nu} = \nu = \delta_1$).

Various techniques can be used for the selection of a risk-neutral probability measure, such as the determination of a minimal entropy risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ that minimizes the Kullback-Leibler relative entropy

$$\mathbb{Q} \longmapsto I(\mathbb{Q},\mathbb{P}) := \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

among the probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$.

Pricing vanilla options

The price of a vanilla option with payoff of the form $\phi(S_T)$ on the underlying asset S_T can be written from (7.6) as

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathscr{F}_t], \tag{7.7}$$

where the expectation can be computed as

$$\begin{aligned}
&\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(S_T) \mid \mathscr{F}_t\right] \\
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(S_0\exp\left(\mu T + \sigma B_T - \frac{\sigma^2}{2}T\right)\prod_{k=1}^{N_T}(1+Z_k)\right) \,\Big|\, \mathscr{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(S_t \exp\left((T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2}(T-t)\right)\prod_{k=N_t+1}^{N_T}(1+Z_k)\right) \Big| \mathscr{F}_t\right] \\
&= \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x \exp\left((T-t)\mu + (B_T - B_t)\sigma - \frac{\sigma^2}{2}(T-t)\right)\prod_{k=N_t+1}^{N_T}(1+Z_k)\right)\right]_{x=S_t} \\
&= \sum_{n\geqslant 0} \mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}(N_T - N_t = n) \\
&\quad \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu + (B_T-B_t)\sigma - (T-t)\sigma^2/2}\prod_{k=N_t+1}^{N_T}(1+Z_k)\right) \Big| N_T - N_t = n\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu + (B_T-B_t)\sigma - (T-t)\sigma^2/2}\prod_{k=1}^{n}(1+Z_k)\right)\right]_{x=S_t} \\
&= \mathrm{e}^{-\tilde{\lambda}(T-t)} \sum_{n\geqslant 0} \frac{(\tilde{\lambda}(T-t))^n}{n!} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{\text{n times}} \\
&\quad \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu + (B_T-B_t)\sigma - (T-t)\sigma^2/2}\prod_{k=1}^{n}(1+z_k)\right)\right]_{x=S_t} \tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_n),
\end{aligned}
$$

hence the price of the vanilla option with payoff $\phi(S_T)$ is given by

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathscr{F}_t] \\
&= \frac{1}{\sqrt{2(T-t)\pi}} \mathrm{e}^{-(r+\tilde{\lambda})(T-t)} \sum_{n\geqslant 0} \frac{(\tilde{\lambda}(T-t))^n}{n!} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{\text{n+1 times}} \\
&\quad \phi\left(S_t\,\mathrm{e}^{(T-t)\mu + \sigma x - (T-t)\sigma^2/2}\prod_{k=1}^{n}(1+z_k)\right) \mathrm{e}^{-(x+(T-t)u)^2/(2(T-t))}\tilde{\nu}(dz_1)\cdots\tilde{\nu}(dz_n)dx.
\end{aligned}
$$

7.4 Black-Scholes PDE with Jumps

In this section, we consider the asset price process $(S_t)_{t\in\mathbb{R}_+}$ modeled by the equation (7.3), *i.e.*

$$
dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t, \tag{7.8}
$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a compound Poisson process with jump size distribution $\nu(dx)$. Recall that by the Markov property of $(S_t)_{t\in\mathbb{R}_+}$, the price (7.7) at time t of the option with payoff $\phi(S_T)$ can be written as a function $f(t,S_t)$ of t and S_t, *i.e.*

$$
f(t,S_t) = \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathscr{F}_t] = \mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid S_t], \tag{7.9}
$$

with the terminal condition $f(T,x) = \phi(x)$. In addition, the process

$$
t \longmapsto \mathrm{e}^{(T-t)r} f(t,S_t)
$$

is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ by the same argument as in (3.1).

In the next proposition we derive a Partial Integro-Differential Equation (PIDE) for the function $(t,x) \longmapsto f(t,x)$.

Proposition 7.2 The price $f(t, S_t)$ of the vanilla option with payoff function ϕ in the model (7.8) satisfies the Partial *Integro-Differential* Equation (PIDE)

$$\begin{aligned} rf(t,x) &= \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \tilde{\lambda}\int_{-\infty}^{\infty}\left(f(t,x(1+y)) - f(t,x) - yx\frac{\partial f}{\partial x}(t,x)\right)\tilde{\nu}(dy), \end{aligned} \tag{7.10}$$

under the terminal condition $f(T,x) = \phi(x)$.

Proof. We have

$$dS_t = rS_t dt + \sigma S_t d\widehat{B}_t + S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt), \tag{7.11}$$

where $\widehat{B}_t = B_t + ut$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. Next, by the Itô formula with jumps (6.23), we have

$$\begin{aligned} & df(t,S_t) \\ &= \frac{\partial f}{\partial t}(t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,S_t)dt + \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)dt \\ &\quad - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)dt + (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t \\ &= \sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t + (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t \\ &\quad - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt \\ &\quad + \left(\frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)\right)dt \\ &\quad + \left(\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)\right)dt. \end{aligned}$$

Based on the discounted portfolio value differential

$$\begin{aligned} & d(\mathrm{e}^{-rt}f(t,S_t)) \\ &= \mathrm{e}^{-rt}\sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t \\ &\quad + \mathrm{e}^{-rt}\left(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt\right) \\ &\quad + \mathrm{e}^{-rt}\left(-rf(t,S_t) + \frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)\right)dt \qquad (7.12) \\ &\quad + \mathrm{e}^{-rt}\left(\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)\right)dt, \qquad (7.13) \end{aligned}$$

obtained from the Itô Table 6.1 with jumps, and the facts that

- the Brownian motion $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$,
- by the smoothing lemma Proposition 6.8, the process given by the differential

$$(f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t}dt,$$

is a martingale under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$, see also (6.22),

- the discounted portfolio value process $t \longmapsto e^{-rt} f(t,S_t)$, is also a martingale under the risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$,

we conclude to the vanishing of the terms (7.12)-(7.13) above, *i.e.*

$$-rf(t,S_t)+\frac{\partial f}{\partial t}(t,S_t)+rS_t\frac{\partial f}{\partial x}(t,S_t)+\frac{\sigma^2}{2}S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t)$$
$$+\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z))-f(t,x))]_{x=S_t}-\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]S_t\frac{\partial f}{\partial x}(t,S_t)=0,$$

or

$$\frac{\partial f}{\partial t}(t,x)+rx\frac{\partial f}{\partial x}(t,x)+\frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x)$$
$$+\tilde{\lambda}\int_{-\infty}^{\infty}(f(t,x(1+y))-f(t,x))\tilde{\nu}(dy)-\tilde{\lambda}x\frac{\partial f}{\partial x}(t,x)\int_{-\infty}^{\infty}y\tilde{\nu}(dy)=rf(t,x),$$

which leads to the Partial *Integro-Differential* Equation (7.10). □

A major technical difficulty when solving the PIDE (7.10) numerically is that the operator

$$f\longmapsto\int_{-\infty}^{\infty}\left(f(t,x(1+y))-f(t,x)-yx\frac{\partial f}{\partial x}(t,x)\right)\tilde{\nu}(dy)$$

is *nonlocal*, therefore adding significant difficulties to the application of standard discretization schemes.

In addition, we have shown that the change $df(t,S_t)$ in the portfolio value (7.9) is given by

$$df(t,S_t)=\sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t+rf(t,S_t)dt \tag{7.14}$$
$$+(f(t,S_{t^-}(1+Z_{N_t}))-f(t,S_{t^-}))dN_t-\tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t,x(1+Z))-f(t,x))]_{x=S_t}dt.$$

In the case of Poisson jumps with fixed size a, *i.e.* when $Y_t=aN_t$ and $\nu(dx)=\delta_a(dx)$, the PIDE (7.10) reads

$$\begin{aligned} rf(t,x) &= \frac{\partial f}{\partial t}(t,x)+rx\frac{\partial f}{\partial x}(t,x)+\frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x)\\ &\quad+\tilde{\lambda}\left(f(t,x(1+a))-f(t,x)-ax\frac{\partial f}{\partial x}(t,x)\right),\end{aligned}$$

and we have

$$df(t,S_t)=\sigma S_t\frac{\partial f}{\partial x}(t,S_t)d\widehat{B}_t+rf(t,S_t)dt$$
$$+(f(t,S_{t^-}(1+a))-f(t,S_{t^-}))dN_t-\tilde{\lambda}(f(t,S_t(1+a))-f(t,S_t))dt.$$

7.5 Exponential Lévy Models

Instead of modeling the asset price $(S_t)_{t\in\mathbb{R}_+}$ through a stochastic exponential (7.4) solution of the stochastic differential equation with jumps of the form (7.3), we may consider an exponential price process of the form

$$\begin{aligned} S_t &:= S_0\,e^{\mu t+\sigma B_t+Y_t}\\ &= S_0\exp\left(\mu t+\sigma B_t+\sum_{k=1}^{N_t}Z_k\right)\end{aligned}$$

$$\begin{aligned}
&= S_0 e^{\mu t+\sigma B_t} \prod_{k=1}^{N_t} e^{Z_k} \\
&= S_0 e^{\mu t+\sigma B_t} \prod_{0\leqslant s\leqslant t} e^{\Delta Y_t}, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

from Relation (6.9), *i.e.* $\Delta Y_t = Z_{N_t}\Delta N_t$. The process $(S_t)_{t\in\mathbb{R}_+}$ is equivalently given by the log-return dynamics

$$d\log S_t = \mu dt + \sigma dB_t + dY_t, \qquad t \in \mathbb{R}_+.$$

In the exponential Lévy model we also have

$$S_t = S_0 e^{(\mu+\sigma^2/2)t+\sigma B_t-\sigma^2 t/2+Y_t}$$

and the process S_t satisfies the stochastic differential equation

$$\begin{aligned}
dS_t &= \left(\mu+\frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dB_t + S_{t^-}(e^{\Delta Y_t}-1)dN_t \\
&= \left(\mu+\frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dB_t + S_{t^-}(e^{Z_{N_t}}-1)dN_t,
\end{aligned}$$

hence the process S_t has jumps of size $S_{T_k^-}(e^{Z_k}-1)$, $k \geqslant 1$, and (7.5) reads

$$\mu+\frac{\sigma^2}{2} - r = \sigma u - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[e^Z-1].$$

Under this condition we can choose a risk-neutral probability measure $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ under which $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the expected value

$$e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathscr{F}_t]$$

represents a (non-unique) arbitrage price at time $t \in [0,T]$ for the contingent claim with payoff $\phi(S_T)$.

This arbitrage price can be expressed as

$$\begin{aligned}
&e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(S_T)\middle|\mathscr{F}_t\right] = e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(S_0 e^{\mu T+\sigma B_T+Y_T})\middle|\mathscr{F}_t\right] \\
&= e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(S_t e^{(T-t)\mu+(B_T-B_t)\sigma+Y_T-Y_t})\middle|\mathscr{F}_t\right] \\
&= e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi(x e^{(T-t)\mu+(B_T-B_t)\sigma+Y_T-Y_t})\right]_{x=S_t} \\
&= e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\exp\left((T-t)\mu+(B_T-B_t)\sigma+\sum_{k=N_t+1}^{N_T} Z_k\right)\right)\right]_{x=S_t} \\
&= e^{-(T-t)r-(T-t)\tilde{\lambda}} \\
&\quad\times \sum_{n\geqslant 0}\frac{(\tilde{\lambda}(T-t))^n}{n!}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x e^{(T-t)\mu+(B_T-B_t)\sigma}\exp\left(\sum_{k=1}^{n} Z_k\right)\right)\right]_{x=S_t}.
\end{aligned}$$

The Merton model

We assume that $(Z_k)_{k\geqslant 1}$ is a family of independent identically distributed Gaussian $\mathcal{N}(\delta,\eta^2)$ random variables under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$ with

$$\mu+\frac{\sigma^2}{2} - r = \sigma u - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[e^Z-1] = \sigma u - \tilde{\lambda}(e^{\delta+\eta^2/2}-1),$$

as in (7.5), hence by the Girsanov Theorem 6.18 for jump processes, $B_t + ut + Y_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[\mathrm{e}^Z - 1]t$ is a martingale and $B_t + ut$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_{u,\tilde{\lambda},\tilde{\nu}}$. For simplicity we choose $u = 0$, which yields

$$\mu = r - \frac{\sigma^2}{2} - \tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2} - 1),$$

and we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[\phi(S_T) \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}\left[\phi\left(x\,\mathrm{e}^{(T-t)\mu+(B_T-B_t)\sigma}\exp\left(\sum_{k=1}^n Z_k\right)\right)\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!}\,\mathbb{E}\left[\phi(x\,\mathrm{e}^{(T-t)\mu+n\delta+X_n})\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \int_{-\infty}^{\infty} \phi(S_t\,\mathrm{e}^{(T-t)\mu+n\delta+y}) \frac{\mathrm{e}^{-y^2/(2((T-t)\sigma^2+n\eta^2))}}{\sqrt{4((T-t)\sigma^2+n\eta^2)\pi}}dy,
\end{aligned}$$

where

$$X_n := (B_T - B_t)\sigma + \sum_{k=1}^n (Z_k - \delta) \simeq \mathscr{N}(0, (T-t)\sigma^2 + n\eta^2), \qquad n \geqslant 0,$$

is a centered Gaussian random variable with variance

$$v_n^2 := (T-t)\sigma^2 + \sum_{k=1}^n \mathrm{Var}\,Z_k = (T-t)\sigma^2 + n\eta^2.$$

Hence when $\phi(x) = (x-\kappa)^+$ is the payoff function of a European call option, using the relation

$$\mathrm{Bl}(x,\kappa,v_n^2/\tau,r,\tau) = \mathrm{e}^{-r\tau}\,\mathbb{E}\left[\left(x\mathrm{e}^{X_n - v_n^2/2 + r\tau} - K\right)^+\right]$$

we get

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}}\,\mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T-\kappa)^+ \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!}\,\mathbb{E}\left[\left(x\,\mathrm{e}^{(T-t)\mu+n\delta+X_n} - \kappa\right)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}\left[\left(x\,\mathrm{e}^{(r-\sigma^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1))(T-t)+n\delta+X_n} - \kappa\right)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathbb{E}\left[\left(x\,\mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)+X_n-v^2)n/2+(T-t)r} - \kappa\right)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \\
&\quad \times \mathrm{Bl}(S_t\,\mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)}, \kappa, \sigma^2 + n\eta^2/(T-t), r, T-t).
\end{aligned}$$

We may also write

$$\begin{aligned}
& \mathrm{e}^{-(T-t)r-(T-t)\tilde{\lambda}} \mathbb{E}_{\tilde{\lambda},\tilde{\nu}}[(S_T-\kappa)^+ \mid \mathscr{F}_t] \\
= \ & \mathrm{e}^{-(T-t)\tilde{\lambda}} \sum_{n\geqslant 0} \frac{((T-t)\tilde{\lambda})^n}{n!} \mathrm{e}^{n\delta+n\eta^2/2-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)} \\
& \times \mathrm{Bl}\left(S_t, \kappa \mathrm{e}^{-n\delta-n\eta^2/2+\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1)(T-t)}, \sigma^2+n\eta^2/(T-t), r, T-t\right) \\
= \ & \mathrm{e}^{-\tilde{\lambda}\,\mathrm{e}^{\delta+\eta^2/2}(T-t)} \sum_{n\geqslant 0} \frac{(\tilde{\lambda}\,\mathrm{e}^{\delta+n\eta^2/2}(T-t))^n}{n!} \\
& \times \mathrm{Bl}\left(S_t, \kappa, \sigma^2+n\eta^2/(T-t), r+n\frac{\delta+\eta^2/2}{T-t}-\tilde{\lambda}(\mathrm{e}^{\delta+\eta^2/2}-1), T-t\right).
\end{aligned}$$

7.6 Mean-Variance Hedging with Jumps

Consider a portfolio valued

$$V_t := \eta_t A_t + \xi_t S_t = \eta_t \mathrm{e}^{rt} + \xi_t S_t$$

at time $t \in \mathbb{R}_+$, and satisfying the self-financing condition (1.3), *i.e.*

$$dV_t = \eta_t dA_t + \xi_t dS_t = r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t.$$

Assuming that the portfolio value takes the form $V_t = f(t, S_t)$ at all times $t \in [0, T]$, by (7.11) we have

$$\begin{aligned}
dV_t &= df(t, S_t) \\
&= r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t \\
&= r\eta_t \mathrm{e}^{rt} dt + \xi_t (rS_t dt + \sigma S_t d\widehat{B}_t + S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt)) \\
&= rV_t dt + \sigma \xi_t S_t d\widehat{B}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt) \\
&= rf(t, S_t)dt + \sigma \xi_t S_t d\widehat{B}_t + \xi_t S_{t^-}(dY_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[Z]dt),
\end{aligned} \tag{7.15}$$

has to match

$$\begin{aligned}
& df(t, S_t) = rf(t, S_t)dt + \sigma S_t \frac{\partial f}{\partial x}(t, S_t) d\widehat{B}_t \\
& + (f(t, S_{t^-}(1+Z_{N_t})) - f(t, S_{t^-}))dN_t - \tilde{\lambda}\,\mathbb{E}_{\tilde{\nu}}[(f(t, x(1+Z)) - f(t, x))]_{x=S_t} dt,
\end{aligned} \tag{7.16}$$

which is obtained from (7.14).

In such a situation we say that the claim payoff C can be exactly replicated.

Exact replication is possible in essentially only two situations:

(i) *Continuous market*, $\lambda = \tilde{\lambda} = 0$. In this case we find the usual Black-Scholes Delta:

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t). \tag{7.17}$$

(ii) *Poisson jump market*, $\sigma = 0$ and $Y_t = aN_t$, $\nu(dx) = \delta_a(dx)$. In this case we find

$$\xi_t = \frac{1}{aS_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-})). \tag{7.18}$$

Note that in the limit $a \to 0$ this expression recovers the Black-Scholes Delta formula (7.17).

When Conditions (i) or (ii) above are not satisfied, exact replication is not possible, and this results into an hedging error given from (7.15) and (7.16) by

$$\begin{aligned}
V_T - \phi(S_T) &= V_T - f(T,S_T)\\
&= V_0 + \int_0^T dV_t - f(0,S_0) - \int_0^T df(t,S_t)\\
&= V_0 - f(0,S_0) + \sigma \int_0^T S_t \left(\xi_t - \frac{\partial f}{\partial x}(t,S_t) \right) d\widehat{B}_t\\
&\quad + \int_0^T \xi_t S_{t^-} (Z_{N_t} dN_t - \tilde{\lambda}\, \mathbf{E}_{\tilde{\nu}}[Z] dt)\\
&\quad - \int_0^T (f(t,S_{t^-}(1+Z_{N_t})) - f(t,S_{t^-})) dN_t\\
&\quad + \tilde{\lambda} \int_0^T \mathbf{E}_{\tilde{\nu}}[(f(t,x(1+Z)) - f(t,x))]_{x=S_t} dt.
\end{aligned}$$

Assuming for simplicity that $Y_t = aN_t$, *i.e.* $\nu(dx) = \delta_a(dx)$, we get

$$\begin{aligned}
V_T - f(T,S_T) &= V_0 - f(0,S_0) + \sigma \int_0^T S_{t^-} \left(\xi_t - \frac{\partial f}{\partial x}(t,S_{t^-}) \right) d\widehat{B}_t\\
&\quad - \int_0^T (f(t,S_{t^-}(1+a)) - f(t,S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda} dt),
\end{aligned}$$

hence the mean-square hedging error is given by

$$\begin{aligned}
&\mathbf{E}_{u,\tilde{\lambda}} \left[(V_T - f(T,S_T))^2 \right]\\
&= (V_0 - f(0,S_0))^2 + \sigma^2 \mathbf{E}_{u,\tilde{\lambda}} \left[\left(\int_0^T S_{t^-} \left(\xi_t - \frac{\partial f}{\partial x}(t,S_{t^-}) \right) d\widehat{B}_t \right)^2 \right]\\
&\quad + \mathbf{E}_{u,\tilde{\lambda}} \left[\left(\int_0^T (f(t,S_{t^-}(1+a)) - f(t,S_{t^-}) - a\xi_t S_{t^-})(dN_t - \tilde{\lambda} dt) \right)^2 \right]\\
&= (V_0 - f(0,S_0))^2 + \sigma^2 \mathbf{E}_{u,\tilde{\lambda}} \left[\int_0^T S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t,S_{t^-}) \right)^2 dt \right]\\
&\quad + \tilde{\lambda}\, \mathbf{E}_{u,\tilde{\lambda}} \left[\int_0^T ((f(t,S_{t^-}(1+a)) - f(t,S_{t^-}) - a\xi_t S_{t^-}))^2 dt \right],
\end{aligned}$$

where we applied the Itô isometry (6.20). Clearly, the initial portfolio value V_0 that minimizes the above quantity is

$$V_0 = f(0,S_0) = \mathrm{e}^{-rT} \mathbf{E}_{u,\tilde{\lambda},\tilde{\nu}}[\phi(S_T)].$$

When hedging only the risk generated by the Brownian part we let

$$\xi_t = \frac{\partial f}{\partial x}(t,S_{t^-})$$

as in the Black-Scholes model, and in this case the hedging error due to the presence of jumps becomes

$$\mathbf{E}_{u,\tilde{\lambda}} \left[(V_T - f(T,S_T))^2 \right] = \tilde{\lambda}\, \mathbf{E}_{u,\tilde{\lambda}} \left[\int_0^T ((f(t,S_{t^-}(1+a)) - f(t,S_{t^-}) - a\xi_t S_{t^-}))^2 dt \right].$$

Next, let us find the optimal strategy $(\xi_t)_{t\in\mathbb{R}_+}$ that minimizes the remaining hedging error

$$\mathbf{E}_{u,\tilde{\lambda}} \left[\int_0^T \left(\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t,S_{t^-}) \right)^2 + \tilde{\lambda} ((f(t,S_{t^-}(1+a)) - f(t,S_{t^-}) - a\xi_t S_{t^-}))^2 \right) dt \right].$$

For all $t \in [0,T]$, the almost-sure minimum of

$$\xi_t \longmapsto \sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right)^2 + \tilde{\lambda} \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-}) \right)^2$$

is given by differentiation with respect to ξ_t, as the solution of

$$2\sigma^2 S_{t^-}^2 \left(\xi_t - \frac{\partial f}{\partial x}(t, S_{t^-}) \right) - 2a\tilde{\lambda} S_{t^-} \left((f(t, S_{t^-}(1+a)) - f(t, S_{t^-}) - a\xi_t S_{t^-}) \right) = 0,$$

i.e.

$$\boxed{\xi_t = \frac{\sigma^2}{\sigma^2 + a^2\tilde{\lambda}} \frac{\partial f}{\partial x}(t, S_{t^-}) + \frac{a^2\tilde{\lambda}}{\sigma^2 + a^2\tilde{\lambda}} \times \frac{f(t, S_{t^-}(1+a)) - f(t, S_{t^-})}{aS_{t^-}},} \tag{7.19}$$

$t \in (0,T]$. We note that the optimal strategy (7.19) is a weighted average of the Brownian and jump hedging strategies (7.17) and (7.18) according to the respective variance parameters σ^2 and $a^2\tilde{\lambda}$ of the continuous and jump components.

Clearly, if $a\tilde{\lambda} = 0$ we get

$$\xi_t = \frac{\partial f}{\partial x}(t, S_{t^-}), \qquad t \in [0,T],$$

which is the Black-Scholes perfect replication strategy, and when $\sigma = 0$ we recover

$$\xi_t = \frac{f(t, (1+a)S_{t^-}) - f(t, S_{t^-})}{aS_{t^-}}, \qquad t \in [0,T].$$

which is (7.18). See § 10.4.2 of Cont and Tankov, 2004 for mean-variance hedging in exponential Lévy model, and § 12.6 of Nunno, Øksendal, and Proske, 2009 for mean-variance hedging by the Malliavin calculus.

Note that the fact that perfect replication is not possible in a jump-diffusion model can be interpreted as a more realistic feature of the model, as perfect replication is not possible in the real world.

See Jeanblanc and Privault, 2002 for an example of a complete market model with jumps, in which continuous and jump noise are mutually excluding each other over time.

In the following table we summarize the properties of geometric Brownian motion *vs* jump-diffusion models in terms of asset price and market behaviors.

Exercises

Exercise 7.1 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + \eta S_{t^-}(dN_t - \alpha dt),$$

where $\eta > 0$.

Model / Properties	Geometric Brownian motion	Jump-diffusion model	Real world
Discontinuous asset prices	✗	✓	✓
Fat tailed market returns	✗	✓	✓
Complete market	✓	✗	✗
Unique prices and risk-neutral measure	✓	✗	✗

Table 7.1: Market models and their properties.

a) Find the value of $\alpha \in \mathbb{R}$ such that the discounted process $\left(e^{-rt} S_t\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Compute the price at time $t \in [0,T]$ of a power option with payoff $|S_T|^2$ at maturity T.

Exercise 7.2 Consider a long forward contract with payoff $S_T - K$ on a jump diffusion risky asset $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t^-} dY_t.$$

a) Show that the forward claim admits a unique arbitrage price to be computed in a market with risk-free rate $r > 0$.
b) Show that the forward claim admits an exact replicating portfolio strategy based on the two assets S_t and e^{rt}.
c) Recover portfolio strategy of Question (b)) using the optimal portfolio strategy formula (7.19).

Exercise 7.3 Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$, independent of $(B_t)_{t\in\mathbb{R}_+}$, under a probability measure $\mathbb{P}^*$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = \mu S_t dt + \eta S_{t^-} dN_t + \sigma S_t dB_t. \tag{7.20}$$

a) Solve the equation (7.20).
b) We assume that μ, η and the risk-free rate $r > 0$ are chosen such that the discounted process $\left(e^{-rt} S_t\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$. What relation does this impose on μ, η, λ and r?
c) Under the relation of Question (b)), compute the price at time $t \in [0,T]$ of a European call option on S_T with strike price κ and maturity T, using a series expansion of Black-Scholes functions.

Exercise 7.4 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + Y_{N_t} S_{t^-} dN_t,$$

where $(Y_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of uniformly distributed random variables on $[-1,1]$.
a) Show that the discounted process $\left(e^{-rt} S_t\right)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Compute the price at time 0 of a European call option on S_T with strike price κ and maturity T, using a series of multiple integrals.

Exercise 7.5 Consider a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with intensity $\lambda > 0$ under a probability measure $\mathbb{P}$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + Y_{N_t} S_{t^-} (dN_t - \alpha dt),$$

where $(Y_k)_{k\geqslant 1}$ is an *i.i.d.* sequence of uniformly distributed random variables on $[0,1]$.

a) Find the value of $\alpha \in \mathbb{R}$ such that the discounted process $(\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Compute the price at time $t \in [0,T]$ of the long forward contract with maturity T and payoff $S_T - \kappa$.

Exercise 7.6 Consider $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ under a risk-neutral probability measure $\mathbb{P}^*$. Let $(S_t)_{t\in\mathbb{R}_+}$ be defined by the stochastic differential equation

$$dS_t = rS_t dt + \alpha S_{t^-}(dN_t - \lambda dt), \tag{7.21}$$

where $\alpha > 0$. Consider a portfolio with value

$$V_t = \eta_t \mathrm{e}^{rt} + \xi_t S_t$$

at time $t \in [0,T]$, and satisfying the self-financing condition

$$dV_t = r\eta_t \mathrm{e}^{rt} dt + \xi_t dS_t.$$

We assume that the portfolio hedges a claim payoff $C = \phi(S_T)$, and that the portfolio value can be written as a function $V_t = f(t,S_t)$ of t and S_t for all times $t \in [0,T]$.
a) Solve the stochastic differential equation (7.21).
b) Price the claim $C = \phi(S_T)$ at time $t \in [0,T]$ using a series expansion.
c) Show that under self-financing, the variation dV_t of the portfolio value V_t satisfies

$$dV_t = rf(t,S_t)dt + \alpha\xi_t S_{t^-}(dN_t - \lambda dt). \tag{7.22}$$

d) Show that the claim payoff $C = \phi(S_T)$ can be exactly replicated by the hedging strategy

$$\xi_t = \frac{1}{\alpha S_{t^-}}\big(f(t,S_{t^-}(1+\alpha)) - f(t,S_{t^-})\big).$$

Exercise 7.7 Pricing by the Esscher transform (Gerber and Shiu, 1994). Consider a compound Poisson process $(Y_t)_{t\in[0,T]}$ with $\mathbb{E}\left[\mathrm{e}^{\theta(Y_t-Y_s)}\right] = \mathrm{e}^{(t-s)m(\theta)}$, $0 \leqslant s \leqslant t$, with $m(\theta)$ a function of $\theta \in \mathbb{R}$, and the asset price process $S_t := \mathrm{e}^{rt+Y_t}$, $t \in [0,T]$. Given $\theta \in \mathbb{R}$, let

$$N_t := \frac{\mathrm{e}^{\theta Y_t}}{\mathbb{E}\left[\mathrm{e}^{\theta Y_t}\right]} = \mathrm{e}^{\theta Y_t - tm(\theta)} = S_t^\theta \,\mathrm{e}^{-r\theta t - tm(\theta)},$$

and consider the probability measure $\mathbb{P}^\theta$ defined as

$$\frac{d\mathbb{P}^\theta_{|\mathscr{F}_t}}{d\mathbb{P}_{|\mathscr{F}_t}} := \frac{N_T}{N_t} = \mathrm{e}^{(Y_T-Y_t)\theta-(T-t)m(\theta)}, \qquad 0 \leqslant t \leqslant T.$$

a) Check that $(N_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}$.
b) Find a condition on θ such that the discounted price process $(\mathrm{e}^{-rt}S_t)_{t\in[0,T]} = \left(\mathrm{e}^{Y_t}\right)_{t\in[0,T]}$ is a martingale under $\mathbb{P}^\theta$.
c) Price the European call option with payoff $(S_T - K)^+$ by taking $\mathbb{P}^\theta$ as risk-neutral probability measure.

Exercise Solutions

Chapter 1

Exercise 1.1 For all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{P}(S_T \leqslant x) &= \mathbb{P}\big(S_0 \mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} \leqslant x\big) \\
&= \mathbb{P}\left(\sigma B_T + \left(\mu - \frac{\sigma^2}{2}\right) T \leqslant \log \frac{x}{S_0}\right) \\
&= \mathbb{P}\left(B_T \leqslant \frac{1}{\sigma}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/\sigma} \mathrm{e}^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right),
\end{aligned}$$

where

$$\Phi(x) := \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian cumulative distribution function. After differentiation with respect to x we find the lognormal probability density function

$$\begin{aligned}
f(x) &= \frac{d\mathbb{P}(S_T \leqslant x)}{dx} \\
&= \frac{\partial}{\partial x} \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/\sigma} \mathrm{e}^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \frac{\partial}{\partial x} \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \frac{1}{x\sigma} \varphi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \frac{1}{x\sigma\sqrt{2\pi T}} \mathrm{e}^{-(-(\mu - \sigma^2/2)T + \log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,
\end{aligned}$$

where

$$\varphi(y) = \Phi'(y) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}, \qquad y \in \mathbb{R},$$

denotes the standard Gaussian probability density function.

Exercise 1.2

a) We have

$$d \log S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2 = r dt + \sigma dB_t - \frac{\sigma^2}{2} dt, \quad t \in \mathbb{R}_+.$$

b) We have $f(t) = f(0)\,\mathrm{e}^{ct}$ (continuous-time interest rate compounding), and

$$S_t = S_0\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2 + rt}, \qquad t \in \mathbb{R}_+,$$

(geometric Brownian motion).

c) Those quantities can be directly computed from the expression of S_t as a function of the $\mathcal{N}(0,t)$ random variable B_t. Namely, we have

$$\begin{aligned} \mathbb{E}[S_t] &= \mathbb{E}\left[S_0\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2 + rt}\right] \\ &= S_0\,\mathrm{e}^{-\sigma^2 t/2 + rt}\,\mathbb{E}\left[\mathrm{e}^{\sigma B_t}\right] \\ &= S_0\,\mathrm{e}^{rt}, \end{aligned}$$

where we used the Gaussian moment generating function (MGF) formula

$$\mathbb{E}\left[\mathrm{e}^{\sigma B_t}\right] = \mathrm{e}^{\sigma^2 t/2}$$

for the normal random variable $B_t \simeq \mathcal{N}(0,t)$, $t > 0$. Similarly, we have

$$\begin{aligned} \mathbb{E}[S_t^2] &= \mathbb{E}\left[S_0^2\,\mathrm{e}^{2\sigma B_t - \sigma^2 t + 2rt}\right] \\ &= S_0^2\,\mathrm{e}^{-\sigma^2 t + 2rt}\,\mathbb{E}\left[\mathrm{e}^{2\sigma B_t}\right] \\ &= S_0^2\,\mathrm{e}^{\sigma^2 t + 2rt}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

d) We note that from the stochastic differential equation

$$S_t = S_0 + r\int_0^t S_s ds + \sigma \int_0^t S_s dB_s,$$

the function $u(t) := \mathbb{E}[S_t]$ satisfies the ordinary differential equation $u'(t) = ru(t)$ with $u(0) = S_0$ and solution $u(t) = \mathbb{E}[S_t] = S_0\,\mathrm{e}^{rt}$. On the other hand, by the Itô formula we have

$$dS_t^2 = 2S_t dS_t + (dS_t)^2 = 2rS_t^2 dt + \sigma^2 S_t^2 dt + 2\sigma S_t dB_t,$$

hence letting $v(t) = \mathbb{E}\left[S_t^2\right]$ and taking expectations on both sides of

$$S_t^2 = S_0^2 + 2r\int_0^t S_u^2 du + \sigma^2 \int_0^t S_u^2 du + 2\sigma \int_0^t S_u dB_u,$$

we find

$$\begin{aligned} v(t) &= \mathbb{E}\left[S_t^2\right] \\ &= S_0^2 + (2r + \sigma^2)\,\mathbb{E}\left[\int_0^t S_u^2 du\right] + 2\sigma\,\mathbb{E}\left[\int_0^t S_u dB_u\right] \\ &= S_0^2 + (2r + \sigma^2)\int_0^t \mathbb{E}\left[S_u^2\right] du \\ &= S_0^2 + (2r + \sigma^2)\int_0^t v(u) du, \end{aligned}$$

hence $v(t) := \mathbb{E}\left[S_t^2\right]$ satisfies the ordinary differential equation

$$v'(t) = (\sigma^2 + 2r)v(t),$$

with $v(0) = S_0^2$ and solution

$$v(t) = \mathbb{E}\left[S_t^2\right] = S_0^2 \, \mathrm{e}^{(\sigma^2+2r)t},$$

which recovers

$$\begin{aligned} \mathrm{Var}[S_t] &= \mathbb{E}\left[S_t^2\right] - (\mathbb{E}[S_t])^2 \\ &= v(t) - u^2(t) \\ &= S_0^2 \, \mathrm{e}^{(\sigma^2+2r)t} - S_0^2 \, \mathrm{e}^{2rt} \\ &= S_0^2 \, \mathrm{e}^{2rt}(\mathrm{e}^{\sigma^2 t} - 1), \qquad t \in \mathbb{R}_+. \end{aligned}$$

Exercise 1.3 Using the bivariate Itô formula, we find

$$\begin{aligned} df(S_t, Y_t) &= \frac{\partial f}{\partial x}(S_t, Y_t)dS_t + \frac{\partial f}{\partial y}(S_t, Y_t)dY_t \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S_t, Y_t)(dS_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(S_t, Y_t)(dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t)dS_t \bullet dY_t \\ &= \frac{\partial f}{\partial x}(S_t, Y_t)(rS_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial y}(S_t, Y_t)(\mu Y_t dt + \eta Y_t dW_t) \\ &\quad + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial x^2}(S_t, Y_t)dt + \frac{\eta^2 Y_t^2}{2}\frac{\partial^2 f}{\partial y^2}(S_t, Y_t)dt + \rho\sigma\eta S_t Y_t \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t)dt. \end{aligned}$$

Exercise 1.4 Taking expectations on both sides of (1.24) shows that

$$\mathbb{E}[S_T] = C(S_0, r, T) + \mathbb{E}\left[\int_0^T \zeta_{t,T} \, dB_t\right] = C(S_0, r, T),$$

hence

$$\begin{aligned} C(S_0, r, T) &= \mathbb{E}[S_T] \\ &= \mathbb{E}[S_0 \, \mathrm{e}^{\mu T + \sigma B_T - \sigma^2 T/2}] \\ &= S_0 \, \mathrm{e}^{\mu T - \sigma^2 T/2} \, \mathbb{E}[\mathrm{e}^{\sigma B_T}] \\ &= S_0 \, \mathrm{e}^{\mu T - \sigma^2 T/2 + \sigma^2 T/2} \\ &= S_0 \, \mathrm{e}^{\mu T}, \end{aligned}$$

where we used the moment generating function

$$\mathbb{E}[\mathrm{e}^{\sigma B_T}] = \mathrm{e}^{\sigma^2 T/2}$$

of the Gaussian random variable $B_T \simeq \mathcal{N}(0, T)$. On the other hand, the discounted asset price $X_t := \mathrm{e}^{-rt} S_t$ satisfies $dX_t = \sigma X_t dB_t$, which shows that

$$X_T = X_0 + \sigma \int_0^T X_t \, dB_t.$$

Multiplying both sides by e^{rT} shows that

$$S_T = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{rT} X_t \, dB_t = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{(T-t)r} S_t \, dB_t,$$

which recovers the relation $C(S_0, r, T) = S_0 \, \mathrm{e}^{rT}$, and shows that $\zeta_{t,T} = \sigma \, \mathrm{e}^{(T-t)r} S_t$, $t \in [0, T]$.

Exercise 1.5

a) We have $S_t = f(X_t)$, $t \in \mathbb{R}_+$, where $f(x) = S_0 \, \mathrm{e}^x$ and $(X_t)_{t\in\mathbb{R}_+}$ is the Itô process given by

$$X_t := \int_0^t \sigma_s dB_s + \int_0^t u_s ds, \qquad t \in \mathbb{R}_+,$$

or in differential form

$$dX_t := \sigma_t dB_t + u_t dt, \qquad t \in \mathbb{R}_+,$$

hence

$$\begin{aligned} dS_t &= df(X_t) \\ &= f'(X_t)dX_t + \frac{1}{2} f''(X_t)(dX_t)^2 \\ &= u_t f'(X_t)dt + \sigma_t f'(X_t)dB_t + \frac{1}{2}\sigma_t^2 f''(X_t)dt \\ &= S_0 u_t \, \mathrm{e}^{X_t} dt + S_0 \sigma_t \, \mathrm{e}^{X_t} dB_t + \frac{1}{2} S_0 \sigma_t^2 \, \mathrm{e}^{X_t} dt \\ &= u_t S_t dt + \sigma_t S_t dB_t + \frac{1}{2}\sigma_t^2 S_t dt. \end{aligned}$$

b) The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = \left(u_t + \frac{1}{2}\sigma_t^2\right) S_t dt + \sigma_t S_t dB_t.$$

Exercise 1.6

a) We have $\mathbb{E}[S_t] = 1$ because the expected value of the Itô stochastic integral is zero. Regarding the variance, using the Itô isometry we have

$$\begin{aligned} \mathrm{Var}[S_t] &= \sigma^2 \, \mathbb{E}\left[\left(\int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s\right)^2\right] \\ &= \sigma^2 \, \mathbb{E}\left[\int_0^t \left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2}\right)^2 ds\right] \\ &= \sigma^2 \int_0^t \mathbb{E}\left[\left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2}\right)^2\right] ds \\ &= \sigma^2 \int_0^t \mathbb{E}\left[\mathrm{e}^{2\sigma B_s - \sigma^2 s}\right] ds \\ &= \sigma^2 \int_0^t \mathrm{e}^{-\sigma^2 s} \, \mathbb{E}\left[\mathrm{e}^{2\sigma B_s}\right] ds \\ &= \sigma^2 \int_0^t \mathrm{e}^{-\sigma^2 s} \, \mathrm{e}^{2\sigma^2 s} ds \\ &= \sigma^2 \int_0^t \mathrm{e}^{\sigma^2 s} ds \\ &= \mathrm{e}^{\sigma^2 t} - 1. \end{aligned}$$

b) Taking $f(x) = \log x$, we have

$$\begin{aligned} d\log(S_t) &= df(S_t) \\ &= \sigma f'(S_t)dS_t + \frac{1}{2}\sigma^2 f''(S_t)(dS_t)^2 \\ &= \sigma f'(S_t) \, \mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t + \frac{1}{2}\sigma^2 f''(S_t) \, \mathrm{e}^{2\sigma B_t - \sigma^2 t} dt \\ &= \frac{\sigma}{S_t} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2} \mathrm{e}^{2\sigma B_t - \sigma^2 t} dt. \end{aligned} \tag{A.1}$$

c) We check that when $S_t = e^{\sigma B_t - \sigma^2 t/2}$, $t \in \mathbb{R}_+$, we have

$$\log S_t = \sigma B_t - \sigma^2 t/2, \quad \text{and} \quad d\log S_t = \sigma dB_t - \frac{\sigma^2}{2} dt.$$

On the other hand, we also find

$$\sigma dB_t - \frac{\sigma^2}{2}dt = \frac{\sigma}{S_t}\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}dB_t - \frac{\sigma^2}{2S_t^2}\,\mathrm{e}^{2\sigma B_t - \sigma^2 t}dt,$$

showing by (A.1) that the equation

$$d\log S_t = \frac{\sigma}{S_t}\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}dB_t - \frac{\sigma^2}{2S_t^2}\,\mathrm{e}^{2\sigma B_t - \sigma^2 t}dt$$

is satisfied. By uniqueness of solutions, we conclude that $S_t := e^{\sigma B_t - \sigma^2 t/2}$ solves

$$S_t = 1 + \sigma \int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2}dB_s, \qquad t \in \mathbb{R}_+.$$

Exercise 1.7

a) Leveraging with a factor $\beta : 1$ means that we invest the amount $\xi_t S_t = \beta F_t$ on the risky asset priced S_t. In this case, the fund value decomposes into the portfolio

$$F_t = \xi_t S_t + \eta_t A_t = \beta \frac{F_t}{S_t} S_t - (\beta - 1)\frac{F_t}{A_t} A_t, \qquad t \in \mathbb{R}_+,$$

with $\xi_t = \beta F_t / S_t$ and $\eta_t = -(\beta - 1)F_t / A_t$, $t \in \mathbb{R}_+$.

b) We have

$$\begin{aligned} dF_t &= \xi_t dS_t + \eta_t dA_t \\ &= \beta \frac{F_t}{S_t} dS_t - (\beta - 1)\frac{F_t}{A_t} dA_t \\ &= \beta \frac{F_t}{S_t} dS_t - (\beta - 1) r F_t dt \\ &= \beta F_t (r dt + \sigma dB_t) - (\beta - 1) r F_t dt \\ &= r F_t dt + \beta \sigma F_t dB_t, \qquad t \in \mathbb{R}_+. \end{aligned}$$

c) We have

$$\begin{aligned} F_t &= F_0\,\mathrm{e}^{\beta\sigma B_t + rt - \beta^2\sigma^2 t/2} \\ &= F_0 \left(\mathrm{e}^{\sigma B_t + rt/\beta - \beta\sigma^2 t/2}\right)^{\beta} \\ &= F_0 \left(\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2 - (1-1/\beta)rt - (\beta-1)\sigma^2 t/2}\right)^{\beta} \\ &= F_0 \left(\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}\right)^{\beta}\,\mathrm{e}^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2} \\ &= \left(S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}\right)^{\beta}\,\mathrm{e}^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2} \\ &= S_t^{\beta}\,\mathrm{e}^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

Exercise 1.8 Letting $X_t := f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$ and noting the relation

$$d\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \sigma f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}dB_t, \qquad t \in \mathbb{R}_+,$$

see Proposition 1.8 with $\mu = 0$, we have

$$\begin{aligned} dX_t &= \mathrm{e}^{\sigma B_t - \sigma^2 t/2} f'(t)dt + f(t)d\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \\ &= \mathrm{e}^{\sigma B_t - \sigma^2 t/2} f'(t)dt + \sigma f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}dB_t \\ &= \frac{f'(t)}{f(t)} X_t dt + \sigma X_t dB_t \end{aligned}$$

$$= \quad h(t)X_t dt + \sigma X_t dB_t,$$

hence

$$\frac{d}{dt}\log f(t) = \frac{f'(t)}{f(t)} = h(t),$$

which shows that

$$\log f(t) = \log f(0) + \int_0^t h(s)ds,$$

and

$$\begin{aligned}
X_t &= f(t)\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \\
&= f(0)\exp\left(\int_0^t h(s)ds + \sigma B_t - \frac{\sigma^2}{2}t\right) \\
&= X_0\exp\left(\int_0^t h(s)ds + \sigma B_t - \frac{\sigma^2}{2}t\right), \qquad t\in\mathbb{R}_+.
\end{aligned}$$

Exercise 1.9

a) We have

$$\begin{aligned}
S_t &= \mathrm{e}^{X_t} \\
&= \mathrm{e}^{X_0} + \int_0^t u_s\,\mathrm{e}^{X_s}dB_s + \int_0^t v_s\,\mathrm{e}^{X_s}ds + \frac{1}{2}\int_0^t u_s^2\,\mathrm{e}^{X_s}ds \\
&= \mathrm{e}^{X_0} + \sigma\int_0^t \mathrm{e}^{X_s}dB_s + \nu\int_0^t \mathrm{e}^{X_s}ds + \frac{\sigma^2}{2}\int_0^t \mathrm{e}^{X_s}ds \\
&= S_0 + \sigma\int_0^t S_s dB_s + \nu\int_0^t S_s ds + \frac{\sigma^2}{2}\int_0^t S_s ds.
\end{aligned}$$

b) Let $r>0$. The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t$$

when $r = \nu + \sigma^2/2$.

c) We have

$$\mathrm{Var}[X_t] = \mathrm{Var}[(B_T - B_t)\sigma] = \sigma^2\,\mathrm{Var}[B_T - B_t] = (T-t)\sigma^2, \quad t\in[0,T].$$

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0\,\mathrm{e}^{\sigma B_t + \nu t}$, $t\in\mathbb{R}_+$. Using the time splitting decomposition

$$S_T = S_t\frac{S_T}{S_t} = S_t\,\mathrm{e}^{(B_T - B_t)\sigma + \nu\tau},$$

we have

$$\begin{aligned}
\mathbb{P}(S_T > K \mid S_t = x) &= \mathbb{P}(S_t\,\mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K \mid S_t = x) \\
&= \mathbb{P}(x\,\mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K) \\
&= \mathbb{P}(\mathrm{e}^{(B_T-B_t)\sigma} > K\,\mathrm{e}^{-(T-t)\nu}/x) \\
&= \mathbb{P}\left(\frac{B_T - B_t}{\sqrt{T-t}} > \frac{1}{\sigma\sqrt{T-t}}\log\left(K\,\mathrm{e}^{-(T-t)\nu}/x\right)\right) \\
&= 1 - \Phi\left(\frac{\log\left(K\,\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(-\frac{\log\left(K\,\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(\frac{\log(x/K) + \nu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

where $\tau = T - t$.

Chapter 2

Exercise 2.1

a) By the Itô formula, we have

$$dV_t = dg(t,B_t) = \frac{\partial g}{\partial t}(t,B_t)dt + \frac{\partial g}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)dt. \tag{A.2}$$

Consider a hedging portfolio with value $V_t = \eta_t A_t + \xi_t B_t$, satisfying the self-financing condition

$$dV_t = \eta_t dA_t + \xi_t dB_t = \xi_t dB_t, \quad t \in \mathbb{R}_+. \tag{A.3}$$

By respective identification of the terms in dB_t and dt in (A.2) and (A.3) we get

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t,B_t)dt + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t,B_t)dt, \\ \\ \xi_t dB_t = \dfrac{\partial g}{\partial x}(t,B_t)dB_t, \end{cases}$$

hence

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t,B_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,B_t), \end{cases}$$

and

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t,B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t,B_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t,B_t), \end{cases}$$

hence the function $g(t,x)$ satisfies the heat equation

$$0 = \frac{\partial g}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x), \quad x > 0, \tag{A.4}$$

with terminal condition $g(T,x) = x^2$, and ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t,B_t), \qquad t \in \mathbb{R}_+.$$

b) In order to solve (A.4) we substitute a solution of the form $g(t,x) = x^2 + f(t)$ and find $1 + f'(t) = 0$, which yields $f(T-t) = T-t$ and $g(t,x) = x^2 + T - t$, $t \in [0,T]$.

Exercise 2.2 By the Itô formula, we have

$$\begin{aligned} dV_t &= dg(t,S_t) \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma\sqrt{S_t}\frac{\partial g}{\partial x}(t,S_t)dB_t. \end{aligned} \tag{A.5}$$

By respective identification of the terms in dB_t and dt in (2.37) and (A.5) we get

$$\begin{cases} rg(t,S_t)dt+\beta(\alpha-S_t)\xi_t dt - r\xi_t S_t dt \\ \qquad = \dfrac{\partial g}{\partial t}(t,S_t)dt+\beta(\alpha-S_t)\dfrac{\partial g}{\partial x}(t,S_t)dt+\dfrac{1}{2}\sigma^2 S_t\dfrac{\partial^2 g}{\partial x^2}(t,S_t)dt, \\ \\ \sigma\xi_t\sqrt{S_t}dB_t=\sigma\sqrt{S_t}\dfrac{\partial g}{\partial x}(t,S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t,S_t)+\beta(\alpha-S_t)\xi_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t,S_t)+\beta(\alpha-S_t)\dfrac{\partial g}{\partial x}(t,S_t)+\dfrac{1}{2}\sigma^2 S_t\dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t=\dfrac{\partial g}{\partial x}(t,S_t), \end{cases}$$

and

$$\begin{cases} rg(t,S_t)+\beta(\alpha-S_t)\xi_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t,S_t)+\dfrac{1}{2}\sigma^2 S_t\dfrac{\partial^2 g}{\partial x^2}(t,S_t), \\ \\ \xi_t=\dfrac{\partial g}{\partial x}(t,S_t), \end{cases}$$

hence the function $g(t,x)$ satisfies the PDE

$$rg(t,x)=\frac{\partial g}{\partial t}(t,x)+rx\frac{\partial g}{\partial x}(t,x)+\frac{1}{2}\sigma^2 x\frac{\partial^2 g}{\partial x^2}(t,x), \quad x>0,$$

and ξ_t is given by the partial derivative

$$\xi_t=\frac{\partial g}{\partial x}(t,S_t), \qquad t\in\mathbb{R}_+.$$

Exercise 2.3

a) Let $V_t := \xi_t S_t + \eta_t A_t$ denote the hedging portfolio value at time $t\in[0,T]$. Since the dividend yield δS_t per share is continuously reinvested in the portfolio, the portfolio change dV_t decomposes as

$$\begin{aligned} dV_t &= \underbrace{\eta_t dA_t+\xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta\xi_t S_t dt}_{\text{dividend payout}} \\ &= r\eta_t A_t dt+\xi_t((\mu-\delta)S_t dt+\sigma S_t dB_t)+\delta\xi_t S_t dt \\ &= r\eta_t A_t dt+\xi_t(\mu S_t dt+\sigma S_t dB_t) \\ &= rV_t dt+(\mu-r)\xi_t S_t dt+\sigma\xi_t S_t dB_t, \qquad t\in\mathbb{R}_+. \end{aligned}$$

b) By Itô's formula we have

$$\begin{aligned} dg(t,S_t) &= \frac{\partial g}{\partial t}(t,S_t)dt+(\mu-\delta)S_t\frac{\partial g}{\partial x}(t,S_t)dt \\ &\quad +\frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt+\sigma S_t\frac{\partial g}{\partial x}(t,S_t)dB_t, \end{aligned}$$

hence by identification of the terms in dB_t and dt in the expressions of dV_t and $dg(t,S_t)$, we get

$$\xi_t=\frac{\partial g}{\partial x}(t,S_t),$$

and we derive the Black-Scholes PDE with dividend

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + (r-\delta)x\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 g}{\partial x^2}(t,x). \tag{A.6}$$

c) In order to solve (A.6) we note that, letting $f(t,x) := \mathrm{e}^{(T-t)\delta}g(t,x)$, the PDE (A.6) reads

$$rf(t,x) = \delta f(t,x) + \frac{\partial f}{\partial t}(t,x) + (r-\delta)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x),$$

hence $f(t,x) := \mathrm{e}^{(T-t)\delta}g(t,x)$, satisfies the standard Black-Scholes PDE with interest rate $r-\delta$, *i.e.* we have

$$(r-\delta)f(t,x) = \frac{\partial f}{\partial t}(t,x) + (r-\delta)x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x),$$

with same terminal condition $f(T,x) = g(T,x) = (x-K)^+$, hence we have

$$\begin{aligned} f(t,x) &= \mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\ &= x\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(r-\delta)(T-t)}\Phi\big(d_-^\delta(T-t)\big), \end{aligned}$$

where

$$d_\pm^\delta(T-t) := \frac{\log(x/K) + (r-\delta\pm\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Consequently, the pricing function of the European call option with dividend rate δ is

$$\begin{aligned} g(t,x) &= \mathrm{e}^{-(T-t)\delta}f(t,x) \\ &= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\ &= x\,\mathrm{e}^{-(T-t)\delta}\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big), \quad 0 \leqslant t \leqslant T. \end{aligned}$$

We also have

$$g(t,x) = \mathrm{Bl}\big(x\,\mathrm{e}^{-(T-t)\delta},K,\sigma,r,T-t\big), \qquad 0 \leqslant t \leqslant T.$$

d) As in Proposition 2.4, we have

$$\frac{\partial g}{\partial a}(t,x) = \mathrm{e}^{-(T-t)\delta}\Phi\big(d_+^\delta(T-t)\big), \qquad x>0,\ 0\leqslant t<T.$$

Exercise 2.4

a) We easily check that $g_c(t,0) = 0$, as when $x=0$ we have $d_+(T-t) = d_-(T-t) = -\infty$ for all $t\in[0,T)$. On the other hand, we have

$$\lim_{t\nearrow T} d_+(T-t) = \lim_{t\nearrow T} d_-(T-t) = \begin{cases} +\infty, & x>K, \\ 0, & x=K, \\ -\infty, & x<K, \end{cases}$$

which allows us to recover the boundary condition

$$\begin{aligned} g_c(T,x) &= \lim_{t\nearrow T} g_c(t,x) \\ &= \left\{\begin{array}{ll} x\Phi(+\infty) - K\Phi(+\infty) = x-K, & x>K \\ \dfrac{x}{2} - \dfrac{K}{2} = 0, & x=K \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x<K \end{array}\right\} = (x-K)^+ \end{aligned}$$

at $t = T$. Similarly, we can check that

$$\lim_{T\to\infty} d_-(T-t) = \begin{cases} +\infty, & r > \dfrac{\sigma^2}{2}, \\ 0, & r = \dfrac{\sigma^2}{2}, \\ -\infty, & r < \dfrac{\sigma^2}{2}, \end{cases}$$

and $\lim_{T\to\infty} d_+(T-t) = +\infty$, hence

$$\begin{aligned} & \lim_{T\to\infty} \mathrm{Bl}(K,x,\sigma,r,T-t) \\ & = x \lim_{T\to\infty} \Phi(d_+(T-t)) - \lim_{T\to\infty} \left(\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))\right) \\ & = x, \qquad t \in \mathbb{R}_+. \end{aligned}$$

b) We check that $g_{\mathrm{p}}(t,0) = K\mathrm{e}^{-(T-t)r}$ and $g_{\mathrm{p}}(t,\infty) = 0$ as when $x = 0$ we have $d_+(T-t) = d_-(T-t) = -\infty$ and as x tends to infinity we have $d_+(T-t) = d_-(T-t) = +\infty$ for all $t \in [0,T)$. On the other hand, we have

$$g_{\mathrm{p}}(T,x) = \left\{ \begin{array}{ll} K\Phi(+\infty) - x\Phi(+\infty) = K - x, & x < K \\ \dfrac{K}{2} - \dfrac{x}{2} = 0, & x = K \\ K\Phi(-\infty) - x\Phi(-\infty) = 0, & x > K \end{array} \right\} = (K-x)^+$$

at $t = T$. Similarly, we can check that

$$\lim_{T\to\infty} d_-(T-t) = \begin{cases} +\infty, & r > \dfrac{\sigma^2}{2}, \\ 0, & r = \dfrac{\sigma^2}{2}, \\ -\infty, & r < \dfrac{\sigma^2}{2}, \end{cases}$$

and $\lim_{T\to\infty} d_+(T-t) = +\infty$, hence

$$\lim_{T\to\infty} \mathrm{Bl}_{\mathrm{p}}(K,x,\sigma,r,T-t) = 0, \qquad t \in \mathbb{R}_+.$$

Exercise 2.5

a) Counting approximately 46 days to maturity, we have

$$\begin{aligned} d_-(T-t) & = \frac{(r-\sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \\ & = \frac{(0.04377 - (0.9)^2/2)(46/365) + \log(17.2/36.08)}{0.9\sqrt{46/365}} \\ & = -2.461179058, \end{aligned}$$

and

$$d_+(T-t) = d_-(T-t) + 0.9\sqrt{46/365} = -2.14167602.$$

From the standard Gaussian cumulative distribution table we get

$$\Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098$$

and

$$\Phi(d_-(T-t)) = \Phi(-2.46) = 0.00692406,$$

hence

$$\begin{aligned} f(t,S_t) &= S_t\Phi(d_+(T-t)) - K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\ &= 17.2 \times 0.0161098 - 36.08 \times \mathrm{e}^{-0.04377\times 46/365} \times 0.00692406 \\ &= \text{HK\$ } 0.028642744. \end{aligned}$$

For comparison, running the corresponding Black-Scholes R script of Figure 2.36 yields

$$\mathrm{BSCall}(17.2, 36.08, 0.04377, 46/365, 0.9) = 0.02864235.$$

b) We have

$$\eta_t = \frac{\partial f}{\partial x}(t,S_t) = \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098, \tag{A.7}$$

hence one should only hold a fractional quantity equal to 16.10 units in the risky asset in order to hedge 1000 such call options when $\sigma = 0.90$.

c) From the curve it turns out that when $f(t,S_t) = 10 \times 0.023 = \text{HK\$ } 0.23$, the volatility σ is approximately equal to $\sigma = 122\%$.

This approximate value of implied volatility can be found under the column “Implied Volatility (IV.)” on this set of market data from the Hong Kong Stock Exchange:

Updated: 6 November 2008

Basic Data

DW Code ▽	Issuer △	UL	Call /Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y) △	Strike △	Entitlement Ratio^
01897	FB	00066	Call	Standard	18-12-2007	23-12-2008	36.08	10

Market Data

Total Issue Size	O/S (%) △	Delta (%) △	IV. (%) △	Day High ($)	Day Low ($)	Closing Price # ($)	T/O ('000)	UL Price ($)
138,000,000	16.43	0.780	125.375	0.000	0.000	0.023	0	17.200

Figure S.1: Market data for the warrant #01897 on the MTR Corporation.

Remark: a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

Exercise 2.6

a) We find $h(x) = x - K$.

b) Letting $g(t,x)$, the PDE rewrites as

$$(x-\alpha(t))r=-\alpha'(t)+rx,$$

hence $\alpha(t)=\alpha(0)\,\mathrm{e}^{rt}$ and $g(t,x)=x-\alpha(0)\,\mathrm{e}^{rt}$. The final condition

$$g(T,x)=h(x)=x-K$$

yields $\alpha(0)=K\,\mathrm{e}^{-rT}$ and $g(t,x)=x-K\,\mathrm{e}^{-(T-t)r}$.

c) We have

$$\xi_t=\frac{\partial g}{\partial x}(t,S_t)=1,$$

hence

$$\eta_t=\frac{V_t-\xi_tS_t}{A_t}=\frac{g(t,S_t)-S_t}{A_t}=\frac{S_t-K\,\mathrm{e}^{-(T-t)r}-S_t}{A_t}=-K\,\mathrm{e}^{-rT}.$$

Note that we could also have directly used the identification

$$V_t=g(S_t,t)=S_t-K\,\mathrm{e}^{-(T-t)r}=S_t-K\,\mathrm{e}^{-rT}A_t=\xi_tS_t+\eta_tA_t,$$

which immediately yields $\xi_t=1$ and $\eta_t=-K\,\mathrm{e}^{-rT}$.

d) It suffices to take $K=0$, which shows that $g(t,x)=x$, $\xi_t=1$ and $\eta_t=0$.

Exercise 2.7

a) We develop two approaches.

(i) By financial intuition. We need to replicate a fixed amount of \$1 at maturity T, *without risk*. For this there is no need to invest in the stock. Simply invest $g(t,S_t):=\mathrm{e}^{-(T-t)r}$ at time $t\in[0,T]$ and at maturity T you will have $g(T,S_T)=\mathrm{e}^{(T-t)r}g(t,S_t)=\$1$.

(ii) By analysis and the Black-Scholes PDE. Given the hint, we try plugging a solution of the form $\underline{g(t,x)=f(t)}$, *not depending on the variable* x, into the Black-Scholes PDE (2.38). Given that here we have

$$\frac{\partial g}{\partial x}(t,x)=0,\qquad \frac{\partial^2 g}{\partial x^2}(t,x)=0,\qquad \text{and}\qquad \frac{\partial g}{\partial t}(t,x)=f'(t),$$

we find that the Black-Scholes PDE reduces to $rf(t)=f'(t)$ with the terminal condition $f(T)=g(T,x)=1$. This equation has for solution $f(t)=\mathrm{e}^{-(T-t)r}$ and this is also the unique solution $g(t,x)=f(t)=\mathrm{e}^{-(T-t)r}$ of the Black-Scholes PDE (2.38) with terminal condition $g(T,x)=1$.

b) We develop two approaches.

(i) By financial intuition. Since the terminal payoff \$1 is risk-free we do not need to invest in the risky asset, hence we should keep $\underline{\xi_t=0}$. Our portfolio value at time t becomes

$$V_t=g(t,S_t)=\mathrm{e}^{-(T-t)r}=\xi_tS_t+\eta_tA_t=\eta_tA_t$$

with $A_t=\mathrm{e}^{rt}$, so that we find $\eta_t=\mathrm{e}^{-rT}$, $t\in[0,T]$. This portfolio strategy remains constant over time, hence it is clearly self-financing.

(ii) By analysis. The Black-Scholes theory of Proposition 2.1 tells us that

$$\xi_t=\frac{\partial g}{\partial x}(t,x)=0,$$

and

$$\eta_t=\frac{V_t-\xi_tS_t}{A_t}=\frac{V_t}{A_t}=\frac{\mathrm{e}^{-(T-t)r}}{\mathrm{e}^{rt}}=\mathrm{e}^{-rT}.$$

Exercise 2.8 Log-contracts.

a) Substituting the function $g(x,t) := f(t) + \log x$ in the PDE (2.39), we have

$$0 = f'(t) + r - \frac{\sigma^2}{2},$$

hence

$$f(t) = f(0) - \left(r - \frac{\sigma^2}{2}\right)t,$$

with $f(0) = \left(r - \frac{\sigma^2}{2}\right)T$ in order to match the terminal condition $g(x,T) := \log x$, hence we have

$$g(x,t) = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x, \qquad x > 0.$$

b) Substituting the function

$$h(x,t) := u(t)g(x,t) = u(t)\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right)$$

in the PDE (2.39), we find $u'(t) = ru(t)$, hence $u(t) = u(0)\,\mathrm{e}^{rt} = \mathrm{e}^{-(T-t)r}$, with $u(T) = 1$, and we conclude to

$$h(x,t) = u(t)g(x,t) = \mathrm{e}^{-(T-t)r}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right),$$

$x > 0$, $t \in [0,T]$.

c) We have

$$\xi_t = \frac{\partial h}{\partial x}(t,S_t) = \frac{\mathrm{e}^{-(T-t)r}}{S_t}, \qquad 0 \leqslant t \leqslant T,$$

and

$$\begin{aligned}\eta_t &= \frac{1}{A_t}(h(t,S_t) - \xi_t S_t)\\ &= \frac{\mathrm{e}^{-rT}}{A_0}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x - 1\right),\end{aligned}$$

$0 \leqslant t \leqslant T$.

Exercise 2.9 Binary options.

a) From Proposition 2.1, the function $C_d(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} rC(t,x) = \dfrac{\partial C}{\partial t}(t,x) + rx\dfrac{\partial C}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 C}{\partial x^2}(t,x), \\ \\ C(T,x) = \mathbb{1}_{[K,\infty)}(x). \end{cases}$$

b) We check by direct differentiation that the Black-Scholes PDE is satisfied by the function $C(t,x)$, together with the terminal condition $C(T,x) = \mathbb{1}_{[K,\infty)}(x)$ as t tends to T.

Exercise 2.10

a) We have

$$S_t = S_0\,\mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s.$$

b) By the self-financing condition (1.8) we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \alpha\xi_t S_t dt + \sigma\xi_t dB_t \\ &= rV_t dt + (\alpha - r)\xi_t S_t dt + \sigma\xi_t dB_t, \end{aligned} \tag{A.8}$$

$t \in \mathbb{R}_+$. Rewriting (2.41) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,$$

with

$$u_t = \sigma, \quad \text{and} \quad v_t = \alpha S_t, \qquad t \in \mathbb{R}_+,$$

the application of Itô's formula to $V_t = C(t, S_t)$ shows that

$$\begin{aligned} dC(t,S_t) &= v_t \frac{\partial C}{\partial x}(t,S_t)dt + u_t \frac{\partial C}{\partial x}(t,S_t)dB_t \\ &\quad + \frac{\partial C}{\partial t}(t,S_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt \\ &= \frac{\partial C}{\partial t}(t,S_t)dt + \alpha S_t \frac{\partial C}{\partial x}(t,S_t)dt + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t,S_t)dt + \sigma \frac{\partial C}{\partial x}(t,S_t)dB_t. \end{aligned} \tag{A.9}$$

Identifying the terms in dB_t and dt in (A.8) and (A.9) above, we get

$$\begin{cases} rC(t,S_t) = \dfrac{\partial C}{\partial t}(t,S_t) + rS_t \dfrac{\partial C}{\partial x}(t,S_t) + \dfrac{\sigma^2}{2}\dfrac{\partial^2 C}{\partial x^2}(t,S_t), \\ \xi_t = \dfrac{\partial C}{\partial x}(t,S_t), \end{cases}$$

hence the function $C(t,x)$ satisfies the usual Black-Scholes PDE

$$rC(t,x) = \frac{\partial C}{\partial t}(t,x) + rx\frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t,x), \quad x > 0, \quad 0 \leqslant t \leqslant T, \tag{A.10}$$

with the terminal condition $C(T,x) = \mathrm{e}^x$, $x \in \mathbb{R}$.

c) Based on (2.42), we compute

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,x) = \left(r + xh'(t) + \dfrac{\sigma^2}{2r}h(t)h'(t) \right) C(t,x), \\ \dfrac{\partial C}{\partial x}(t,x) = h(t)C(t,x) \\ \dfrac{\partial^2 C}{\partial x^2}(t,x) = (h(t))^2 C(t,x), \end{cases}$$

hence the substitution of (2.42) into the Black-Scholes PDE (A.10) yields the ordinary differential equation

$$xh'(t) + \frac{\sigma^2}{2r}h'(t)h(t) + rxh(t) + \frac{\sigma^2}{2}(h(t))^2 = 0, \quad x > 0, \quad 0 \leqslant t \leqslant T,$$

which reduces to the ordinary differential equation $h'(t) + rh(t) = 0$ with terminal condition $h(T) = 1$ and solution $h(t) = \mathrm{e}^{(T-t)r}$, $t \in [0,T]$, which yields

$$C(t,x) = \exp\left(-(T-t)r + x\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1) \right).$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t \, e^{(T-t)r} + \frac{\sigma^2}{4r}\left(e^{2(T-t)r} - 1\right)\right).$$

Exercise 2.11

a) Noting that $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2} e^{-x^2/2}$, we have the

$$\begin{aligned} \frac{\partial h}{\partial d}(S,d) &= S\varphi\big(d+\sigma\sqrt{T}\big) - K e^{-rT} \varphi(d) \\ &= \frac{S}{\sqrt{2\pi}} e^{-\left(d+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2} \\ &= \frac{S}{\sqrt{2\pi}} e^{-d^2/2-\sigma\sqrt{T}d-\sigma^2 T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d^2/2}, \end{aligned}$$

hence the vanishing of $\frac{\partial h}{\partial d}(S, d_*(S))$ at $d = d_*(S)$ yields

$$\frac{S}{\sqrt{2\pi}} e^{-d_*^2(S)/2-\sigma\sqrt{T}d_*(S)-\sigma^2 T/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} = 0,$$

i.e. $d_*(S) = \dfrac{\log(S/K) + rT - \sigma^2 T/2}{\sigma\sqrt{T}}$. We can also check that

$$\begin{aligned} \frac{\partial^2 h}{\partial d^2}(S, d_*(S)) &= \frac{\partial}{\partial d}\left(\frac{S}{\sqrt{2\pi}} e^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2}\right) \\ &= -\big(d_*(S)+\sigma\sqrt{T}\big)\frac{S}{\sqrt{2\pi}} e^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*^2(S)/2} \\ &= -\big(d_*(S)+\sigma\sqrt{T}\big)\frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} + \frac{K}{\sqrt{2\pi}} d_*(S) e^{-rT} e^{-d_*^2(S)/2} \\ &= -\sigma\sqrt{T}\frac{K}{\sqrt{2\pi}} e^{-rT} e^{-d_*^2(S)/2} < 0, \end{aligned}$$

hence the function $d \longmapsto h(S,d) := S\Phi\big(d+\sigma\sqrt{T}\big) - K e^{-rT}\Phi(d)$ admits a *maximum* at $d = d_*(S)$, and

$$\begin{aligned} h(S, d_*(S)) &= S\Phi\big(d_*(S)+\sigma\sqrt{T}\big) - K e^{-rT}\Phi(d_*(S)) \\ &= S\Phi\left(\frac{\log(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - K e^{-rT}\Phi\left(\frac{\log(S/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

is the Black-Scholes call option price.

b) Since $\frac{\partial h}{\partial d}(S, d_*(S)) = 0$, we find

$$\begin{aligned} \Delta &= \frac{d}{dS} h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d_*'(S)\frac{\partial h}{\partial d}(S, d_*(S)) \\ &= \Phi\big(d_*(S)+\sigma\sqrt{T}\big) = \Phi\left(\frac{\log(S/K)+rT+\sigma^2 T/2}{\sigma\sqrt{T}}\right). \end{aligned}$$

Exercise 2.12 When $\sigma > 0$ we have

$$\begin{aligned} \frac{\partial g_c}{\partial \sigma} &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma} d_+(T-t) - K e^{-(T-t)r}\Phi'\big(d_-(T-t)\big)\frac{\partial}{\partial\sigma} d_-(T-t) \\ &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma} d_+(T-t) \\ &\quad - K e^{-(T-t)r}\Phi'\big(d_+(T-t)\big) e^{(T-t)r+\log(x/K)}\frac{\partial}{\partial\sigma} d_-(T-t) \\ &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}\big(d_+(T-t) - d_-(T-t)\big) \end{aligned}$$

$$\begin{aligned}
&= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial\sigma}\big(\sigma\sqrt{T-t}\big)\\
&= x\sqrt{T-t}\,\Phi'\big(d_+(T-t)\big),
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\Phi'\big(d_-(T-t)\big) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(d_-(T-t))^2/2}\\
&= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(d_-(T-t))^2/2+(T-t)r+\log(x/K)}\\
&= \Phi'\big(d_+(T-t)\big)\,\mathrm{e}^{(T-t)r+\log(x/K)}.
\end{aligned}$$

We note that the Black-Scholes European call price is an increasing function of the volatility parameter $\sigma > 0$. Relation (2.43) can be obtained from

$$\begin{aligned}
&\big(d_+(T-t)\big)^2 - \big(d_-(T-t)\big)^2\\
&= \big(d_+(T-t)+d_-(T-t)\big)\big(d_+(T-t)-d_-(T-t)\big)\\
&= 2r(T-t)+2\log\frac{x}{K}.
\end{aligned}$$

Exercise 2.13

a) Given that

$$p^* = \frac{r_N-a_N}{b_N-a_N} = \frac{1}{2} \quad\text{and}\quad q^* = \frac{b_N-r_N}{b_N-a_N} = \frac{1}{2},$$

we have

$$\begin{aligned}
\widetilde{v}(t,x) &= \frac{1}{2}\widetilde{v}\big(t+T/N, x(1+rT/N)(1-\sigma\sqrt{T/N})\big)\\
&\quad+\frac{1}{2}\widetilde{v}\big(t+T/N, x(1+rT/N)(1+\sigma\sqrt{T/N})\big).
\end{aligned}$$

After letting $\Delta T := T/N$ and applying Taylor's formula at the second order we obtain

$$\begin{aligned}
0 &= \frac{1}{2}\big(\widetilde{v}\big(t+\Delta T, x\big(1+r\Delta T-\sigma\sqrt{\Delta T}\big)\big)-\widetilde{v}(t,x)\big)\\
&\quad+\frac{1}{2}\big(\widetilde{v}\big(t+\Delta T, x\big(1+r\Delta T+\sigma\sqrt{\Delta T}\big)\big)-\widetilde{v}(t,x)\big)+o\big(\Delta T\big)\\
&= \frac{1}{2}\Big(\Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+x\big(r\Delta T-\sigma\sqrt{\Delta T}\big)\frac{\partial\widetilde{v}}{\partial x}(t,x)\\
&\quad+\frac{x^2}{2}\big(r\Delta T-\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o(\Delta T)\Big)\\
&\quad+\frac{1}{2}\Big(\Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+x\big(r\Delta T+\sigma\sqrt{\Delta T}\big)\frac{\partial\widetilde{v}}{\partial x}(t,x)\\
&\quad+\frac{x^2}{2}\big(r\Delta T+\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o(\Delta T)\Big)+o\big(\Delta T\big)\\
&= \Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\Delta T\frac{\partial\widetilde{v}}{\partial x}(t,x)+\frac{x^2}{2}\big(\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o\big(\Delta T\big),
\end{aligned}$$

which shows that

$$\frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\frac{\partial\widetilde{v}}{\partial x}(t,x)+x^2\frac{\sigma^2}{2}\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x) = -\frac{o\big(\Delta T\big)}{\Delta T},$$

hence as N tends to infinity (or as ΔT tends to 0) we find*

$$0 = \frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\frac{\partial\widetilde{v}}{\partial x}(t,x)+\frac{\sigma^2}{2}x^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x),$$

*The notation $o(\Delta T)$ denotes any function of ΔT such that $\lim_{\Delta T\to 0} o(\Delta T)/\Delta T = 0$.

showing that the function $v(t,x) := \mathrm{e}^{(T-t)r}\widetilde{v}(t,x)$ solves the classical Black-Scholes PDE

$$rv(t,x) = \frac{\partial v}{\partial t}(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x).$$

b) Similarly, we have

$$\begin{aligned}\xi_t^{(1)}(x) &= \frac{v(t,(1+b_N)x) - v(t,(1+a_N)x)}{x(b_N-a_N)} \\ &= \frac{v\left(t,(1+r/N)(1+\sigma\sqrt{T/N})x\right) - v\left(t,(1+r/N)(1-\sigma\sqrt{T/N})x\right)}{2x(1+r/N)\sigma\sqrt{T/N}} \\ &\to \frac{\partial v}{\partial x}(t,x),\end{aligned}$$

as N tends to infinity.

Chapter 3

Exercise 3.1 (Exercise 2.1 continued). Since $r=0$ we have $\mathbb{P} = \mathbb{P}^*$ and

$$\begin{aligned} g(t,B_t) &= \mathbb{E}^*\left[B_T^2 \mid \mathscr{F}_t\right] \\ &= \mathbb{E}^*\left[(B_T - B_t + B_t)^2 \mid \mathscr{F}_t\right] \\ &= \mathbb{E}^*\left[(B_T - B_t + x)^2\right]_{x=B_t} \\ &= \mathbb{E}^*\left[(B_T-B_t)^2 + 2x(B_T-B_t) + x^2\right]_{x=B_t} \\ &= \mathbb{E}^*\left[(B_T-B_t)^2\right] + 2x\,\mathbb{E}^*[B_T-B_t] + B_t^2 \\ &= B_t^2 + T - t, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

hence ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t,B_t) = 2B_t, \qquad 0 \leqslant t \leqslant T,$$

with

$$\eta_t = \frac{g(t,B_t) - \xi_t B_t}{A_0} = \frac{B_t^2 + (T-t) - 2B_t^2}{A_0} = \frac{(T-t) - B_t^2}{A_0}, \quad 0 \leqslant t \leqslant T.$$

Exercise 3.2 Since $B_T \simeq \mathscr{N}(0,T)$, we have

$$\begin{aligned} \mathbb{E}[\phi(S_T)] &= \mathbb{E}\left[\phi\left(S_0\,\mathrm{e}^{\sigma B_T + (r-\sigma^2/2)T}\right)\right] \\ &= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty} \phi\left(S_0\,\mathrm{e}^{\sigma y + (r-\sigma^2/2)T}\right)\mathrm{e}^{-y^2/(2T)}dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\infty}\phi(x)\,\mathrm{e}^{-((\sigma^2/2-r)T+\log x)^2/(2\sigma^2T)}\frac{dx}{x} \\ &= \int_{-\infty}^{\infty}\phi(x)g(x)dx, \end{aligned}$$

under the change of variable

$$x = S_0\,\mathrm{e}^{\sigma y + (r-\sigma^2/2)T}, \quad \text{with } dx = \sigma S_0\,\mathrm{e}^{\sigma y + (r-\sigma^2/2)T}dy = \sigma x dy,$$

i.e.

$$y = \frac{(\sigma^2/2 - r)T + \log(x/S_0)}{\sigma} \quad \text{and} \quad dy = \frac{dx}{\sigma x},$$

where

$$g(x) := \frac{1}{x\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-((\sigma^2/2-r)T+\log(x/S_0))^2/(2\sigma^2 T)}$$

is the *lognormal* probability density function with location parameter $(r-\sigma^2/2)T + \log S_0$ and scale parameter $\sigma\sqrt{T}$.

Exercise 3.3 We have

$$\begin{aligned}
\mathbb{E}^*[\phi(pS_{T_1}+qS_{T_2})] &\leqslant \mathbb{E}^*[p\phi(S_{T_1})+q\phi(S_{T_2})] && \text{since } \phi \text{ is convex,}\\
&= p\,\mathbb{E}^*[\phi(S_{T_1})]+q\,\mathbb{E}^*[\phi(S_{T_2})] \\
&= p\,\mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_2} \mid \mathscr{F}_{T_1}])]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{because } (S_t)_{t\in\mathbb{R}_+} \text{ is a martingale,}\\
&\leqslant p\,\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_2}) \mid \mathscr{F}_{T_1}]]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by Jensen's inequality,}\\
&= p\,\mathbb{E}^*[\phi(S_{T_2})]+q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by the tower property,}\\
&= \mathbb{E}^*[\phi(S_{T_2})], && \text{because } p+q=1.
\end{aligned}$$

Remark: This type of technique can be useful in order to get an upper price estimate from Black-Scholes when the actual option price is difficult to compute: here the closed-form computation would involve a double integration of the form

$$\begin{aligned}
\mathbb{E}^*[\phi(pS_{T_1}+qS_{T_2})] &= \mathbb{E}^*\left[\phi\left(pS_0\,\mathrm{e}^{\sigma B_{T_1}-\sigma^2 T_1/2}+qS_0\,\mathrm{e}^{\sigma B_{T_2}-\sigma^2 T_2/2}\right)\right]\\
&= \mathbb{E}^*\left[\phi\left(S_0\,\mathrm{e}^{\sigma B_{T_1}-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{(B_{T_2}-B_{T_1})\sigma-(T_2-T_1)\sigma^2/2}\right)\right)\right]\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\left(S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2}\right)\right)\\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}\left(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2}\right)-K\right)^+\\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \frac{1}{2\pi}\int_{\{(x,y)\in\mathbb{R}^2\,:\,S_0\,\mathrm{e}^{\sigma x}(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2})\geqslant K\,\mathrm{e}^{\sigma^2 T_1/2}\}}\\
&\qquad (S_0\,\mathrm{e}^{\sigma x-\sigma^2 T_1/2}(p+q\,\mathrm{e}^{\sigma y-(T_2-T_1)\sigma^2/2})-K)\\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1)-y^2(2(T_2-T_1))}\frac{dxdy}{\sqrt{T_1(T_2-T_1)}}\\
&= \cdots
\end{aligned}$$

Exercise 3.4

a) The European *call* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+]$ decreases with the strike price K, because the option payoff $(S_T-K)^+$ decreases and the expectation operator preserves the ordering of random variables.

b) The European *put* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(K-S_T)^+]$ increases with the strike price K, because the option payoff $(K-S_T)^+$ increases and the expectation operator preserves the ordering of random variables.

Exercise 3.5

a) Using Jensen's inequality and the martingale property of the discounted asset price process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] &\geqslant \mathrm{e}^{-(T-t)r}\big(\mathbb{E}^*[S_T-K \mid \mathcal{F}_t]\big)^+ \\ &= \mathrm{e}^{-(T-t)r}\big(\mathrm{e}^{(T-t)r}S_t-K\big)^+ \\ &= \big(S_t-K\,\mathrm{e}^{-(T-t)r}\big)^+, \quad 0\leqslant t\leqslant T.\end{aligned}$$

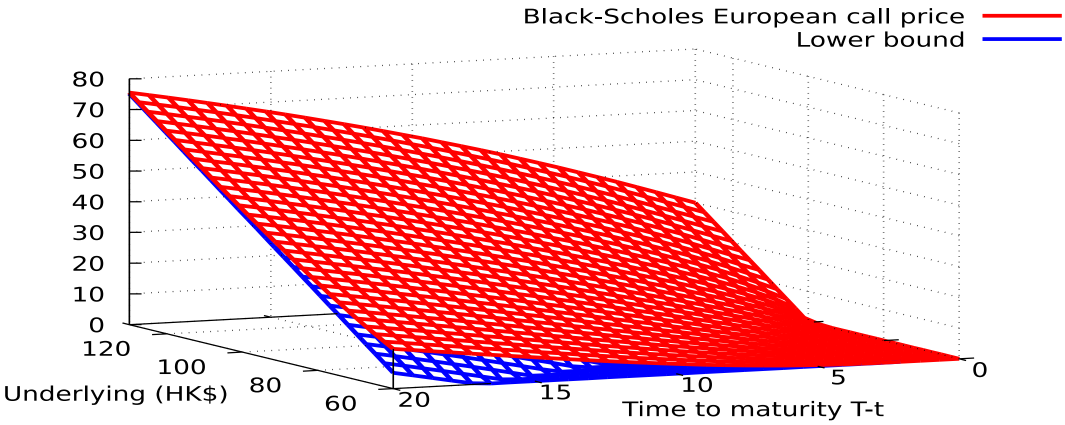


Figure S.2: Lower bound *vs* Black-Scholes call price.

b) Similarly, by Jensen's inequality and the martingale property, we find

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t] &\geqslant \mathrm{e}^{-(T-t)r}\big(\mathbb{E}^*[K-S_T \mid \mathcal{F}_t]\big)^+ \\ &= \mathrm{e}^{-(T-t)r}\big(K-\mathrm{e}^{(T-t)r}S_t\big)^+ \\ &= \big(K\,\mathrm{e}^{-(T-t)r}-S_t\big)^+, \quad 0\leqslant t\leqslant T.\end{aligned}$$

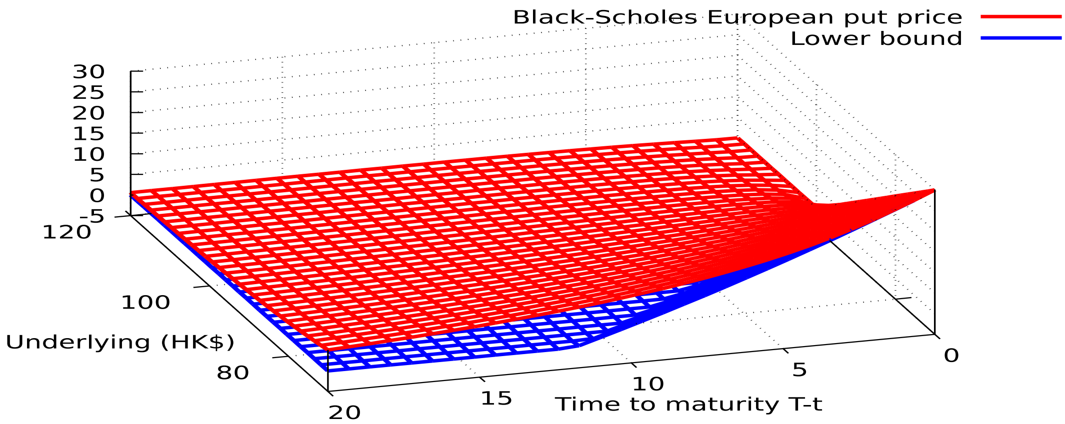


Figure S.3: Lower bound *vs* Black-Scholes put option price.

Exercise 3.6

a) (i) The bull spread option can be realized by purchasing one European call option with strike price K_1 and by short selling (or issuing) one European call option with strike price K_2, because the bull spread payoff function can be written as

$$x \longmapsto (x-K_1)^+ - (x-K_2)^+.$$

see http://optioncreator.com/st3ce7z.

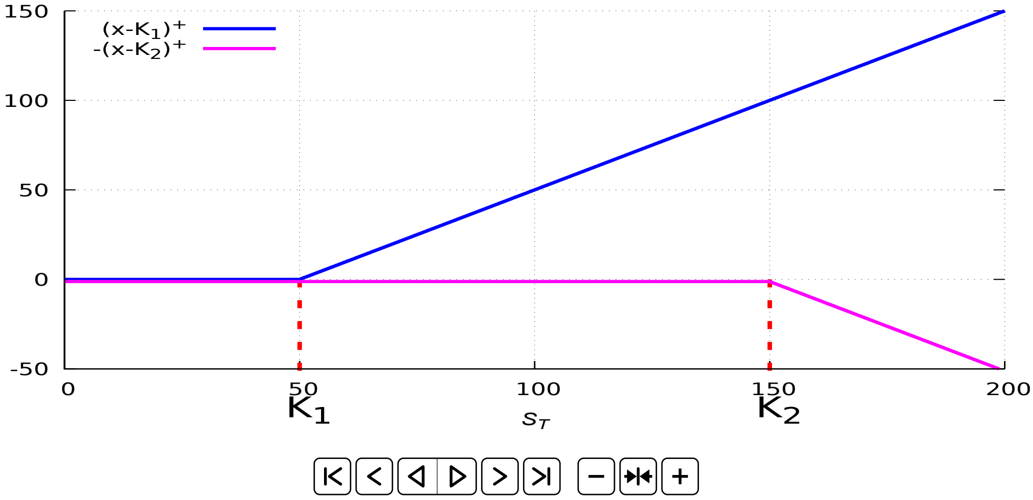


Figure S.4: Bull spread option as a combination of call and put options.*

(ii) The bear spread option can be realized by purchasing one European put option with strike price K_2 and by short selling (or issuing) one European put option with strike price K_1, because the bear spread payoff function can be written as

$$x \longmapsto -(K_1 - x)^+ + (K_2 - x)^+,$$

see http://optioncreator.com/stmomsb.

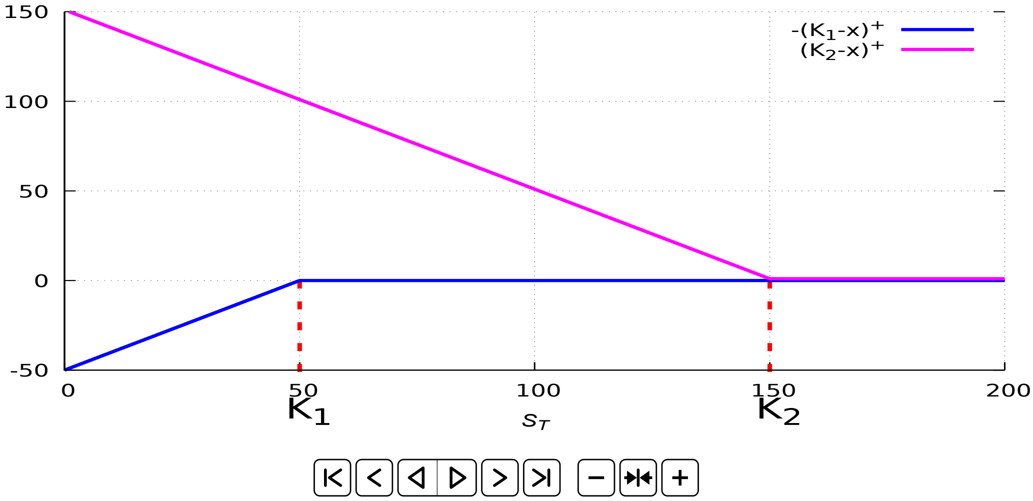


Figure S.5: Bear spread option as a combination of call and put options.†

b) (i) The bull spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\mathrm{Bl}(K_1, S_t, \sigma, r, T - t) - \mathrm{Bl}(K_2, S_t, \sigma, r, T - t).$$

(ii) The bear spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\mathrm{Bl}(K_2, S_t, \sigma, r, T - t) - \mathrm{Bl}(K_1, S_t, \sigma, r, T - t).$$

*The animation works in Acrobat Reader on the entire pdf file.
†The animation works in Acrobat Reader on the entire pdf file.

Exercise 3.7

a) The payoff function can be written as

$$\begin{aligned}(K_1 - x)^+ + (K_2 - x)^+ - 2(x - (K_1 + K_2)/2)^+ \\ = \quad (50 - x)^+ + (150 - x)^+ - 2(x - 100)^+,\end{aligned}$$

see also https://optioncreator.com/stnurzg.

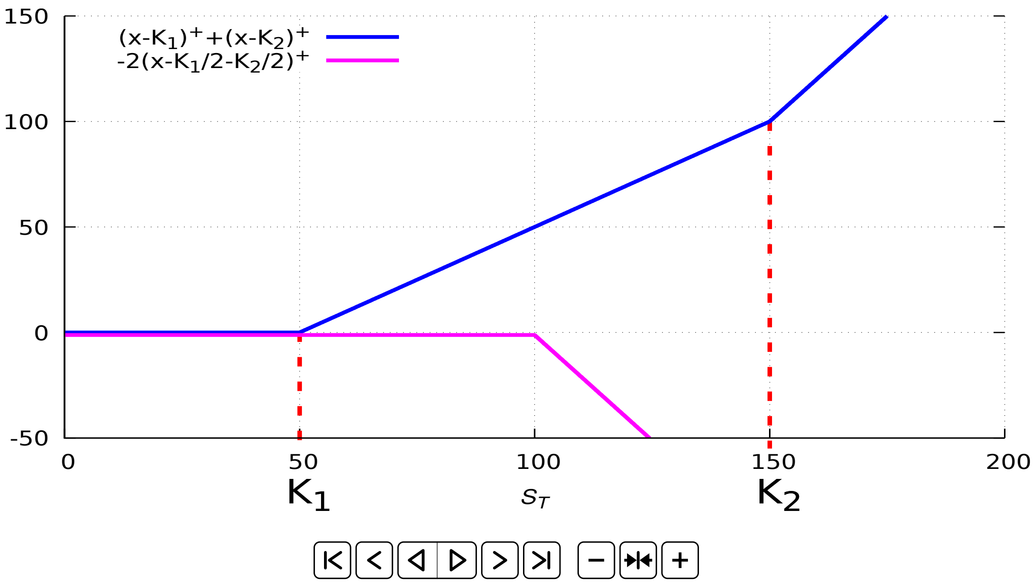


Figure S.6: Butterfly option as a combination of call options.*

Hence the butterfly option can be realized by:

1. purchasing one *call option* with strike price $K_1 = 50$, and
2. purchasing one *call option* with strike price $K_2 = 150$, and
3. issuing (or selling) two *call options* with strike price $(K_1 + K_2)/2 = 100$.

b) Denoting by $\phi(x)$ the payoff function, the self-financing replicating portfolio strategy $(\xi_t(S_{t-1}))_{t=1,2,\ldots,N}$ hedging the contingent claim with payoff $C = \phi(S_N)$ is given by

$$\xi_t(x) = \frac{\mathbb{E}^*\left[\phi\left(x(1+b)\prod_{j=t+1}^N (1+R_j)\right) - \phi\left(x(1+a)\prod_{j=t+1}^N (1+R_j)\right)\right]}{(b-a)(1+r)^{N-t} S_{t-1}}$$

with $x = S_{t-1}$. Therefore, $\xi_t(x)$ will be positive (holding) when $x = S_{t-1}$ is sufficiently below $(K_1 + K_2)/2$, and $\xi_t(x)$ will be negative (short selling) when $x = S_{t-1}$ is sufficiently above $(K_1 + K_2)/2$.

*The animation works in Acrobat Reader.

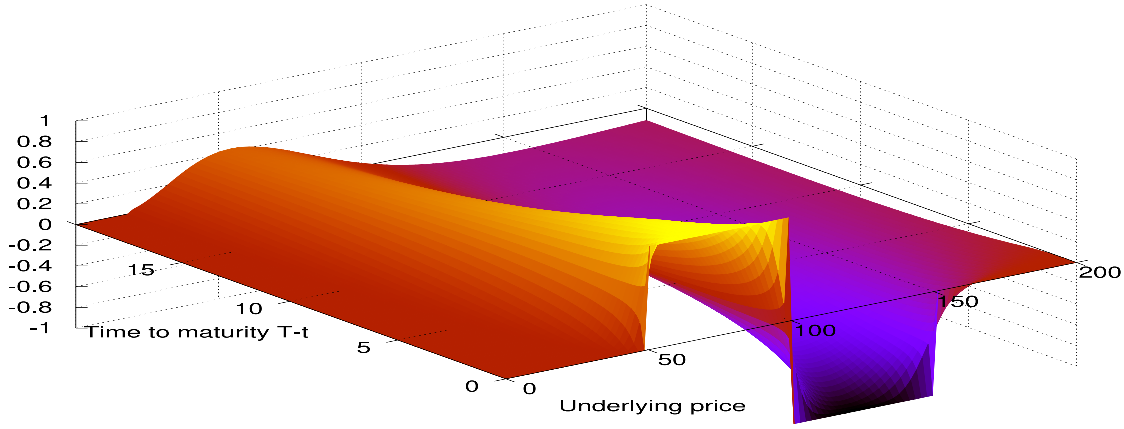


Figure S.7: Delta of a butterfly option with strike prices $K_1 = 50$ and $K_2 = 150$.

Exercise 3.8

a) We have

$$\begin{aligned} C_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T - K \mid \mathscr{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid \mathscr{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\,\mathbb{E}^*[\,\mathrm{e}^{-rT} S_T \mid \mathscr{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\,\mathrm{e}^{-rt} S_t - K\,\mathrm{e}^{-(T-t)r} \\ &= S_t - K\,\mathrm{e}^{-(T-t)r}. \end{aligned}$$

We can check that the function $g(x,t) = x - K\,\mathrm{e}^{-(T-t)r}$ satisfies the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t)$$

with terminal condition $g(x,T) = x - K$, since $\partial g(x,t)/\partial t = -rK\,\mathrm{e}^{-(T-t)r}$ and $\partial g(x,t)/\partial x = 1$.

b) We simply take $\xi_t = 1$ and $\eta_t = -K\,\mathrm{e}^{-rT}$ in order to have

$$C_t = \xi_t S_t + \eta_t\,\mathrm{e}^{rt} = S_t - K\,\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T.$$

Note again that this hedging strategy is *constant* over time, and the relation $\xi_t = \partial g(S_t,t)/\partial x$ for the option Delta, cf. (A.7), is satisfied.

Exercise 3.9 Option pricing with dividends (Exercise 2.3 continued).

a) Let $\widehat{\mathbb{P}}$ denote the probability measure under which the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ defined by

$$d\widehat{B}_t = \frac{\mu - r}{\sigma}dt + dB_t$$

is a standard Brownian motion. Under absence of arbitrage the asset price process $(S_t)_{t\in\mathbb{R}_+}$ has the dynamics

$$\begin{aligned} dS_t &= (\mu - \delta)S_t dt + \sigma S_t dB_t \\ &= (r - \delta)S_t dt + \sigma S_t d\widehat{B}_t, \end{aligned}$$

and the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\,\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ satisfies

$$d\widetilde{S}_t = -\delta\widetilde{S}_t dt + \sigma\widetilde{S}_t d\widehat{B}_t.$$

Assuming that the dividend yield δS_t per share is continuously reinvested in the portfolio, the self-financing portfolio condition

$$dV_t = \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta\xi_t S_t dt}_{\text{dividend payout}}$$

$$\begin{aligned}
&= r\eta_t A_t dt + \xi_t((r-\delta)S_t dt + \sigma S_t d\widehat{B}_t) + \delta\xi_t S_t dt \\
&= r\eta_t A_t dt + \xi_t(rS_t dt + \sigma S_t d\widehat{B}_t) \\
&= rV_t dt + \sigma\xi_t S_t d\widehat{B}_t, \qquad t \in \mathbb{R}_+,
\end{aligned}$$

which yields

$$\begin{aligned}
d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt}V_t\right) \\
&= -r\,\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}dV_t \\
&= \sigma\xi_t\,\mathrm{e}^{-rt}S_t d\widehat{B}_t \\
&= \sigma\xi_t\widetilde{S}_t d\widehat{B}_t \\
&= \xi_t(d\widetilde{S}_t + \delta\widetilde{S}_t dt), \qquad t \in \mathbb{R}_+.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\widetilde{V}_t - \widetilde{V}_0 &= \int_0^t d\widetilde{V}_u \\
&= \sigma\int_0^t \xi_u\widetilde{S}_u d\widehat{B}_u \\
&= \int_0^t \xi_u d\widetilde{S}_u + \delta\int_0^t \widetilde{S}_u du, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

Here, the asset price process $(\mathrm{e}^{\delta t}S_t)_{t\in\mathbb{R}_+}$ with added dividend yield satisfies the equation

$$d(\mathrm{e}^{\delta t}S_t) = r\,\mathrm{e}^{\delta t}S_t dt + \sigma\left(\mathrm{e}^{\delta t}S_t\right)d\widehat{B}_t,$$

and after discount, the process $(\mathrm{e}^{-rt}\,\mathrm{e}^{\delta t}S_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-(r-\delta)t}S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\widehat{\mathbb{P}}$.

b) We have

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma\int_0^t \xi_u\widetilde{S}_u d\widehat{B}_u, \qquad t \in \mathbb{R}_+,$$

which is a martingale under $\widehat{\mathbb{P}}$ from Proposition 3.1, hence

$$\begin{aligned}
\widetilde{V}_t &= \widehat{\mathbb{E}}\big[\widetilde{V}_T \mid \mathscr{F}_t\big] \\
&= \mathrm{e}^{-rT}\widehat{\mathbb{E}}[V_T \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-rT}\widehat{\mathbb{E}}[C \mid \mathscr{F}_t],
\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

c) After discounting the payoff $(S_T-K)^+$ at the continuously compounded interest rate r, we obtain

$$\begin{aligned}
V_t &= \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}\big[(S_T-K)^+ \mid \mathscr{F}_t\big] \\
&= \mathrm{e}^{-(T-t)r}\widehat{\mathbb{E}}\Big[\big(S_0\,\mathrm{e}^{\sigma\widehat{B}_T+(r-\delta-\sigma^2/2)T}-K\big)^+ \mid \mathscr{F}_t\Big] \\
&= \mathrm{e}^{-(T-t)\delta}\Big(\mathrm{e}^{-(T-t)(r-\delta)}\widehat{\mathbb{E}}\Big[\big(S_0\,\mathrm{e}^{\sigma\widehat{B}_T+(r-\delta-\sigma^2/2)T}-K\big)^+ \mid \mathscr{F}_t\Big]\Big) \\
&= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}(K,x,\sigma,r-\delta,T-t) \\
&= \mathrm{e}^{-(T-t)\delta}\Big(S_t\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(T-t)(r-\delta)}\Phi\big(d_-^\delta(T-t)\big)\Big) \\
&= \mathrm{e}^{-(T-t)\delta}S_t\Phi\big(d_+^\delta(T-t)\big) - K\,\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big), \qquad 0 \leqslant t < T,
\end{aligned}$$

where

$$d_+^\delta(T-t) := \frac{\log(S_t/K)+(r-\delta+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}$$

and

$$d_-^\delta(T-t) := \frac{\log(S_t/K)+(r-\delta-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}.$$

We also have

$$g(t,x) = \mathrm{Bl}\big(x\,\mathrm{e}^{-(T-t)\delta}, K, \sigma, r, T-t\big), \qquad 0 \leqslant t \leqslant T.$$

Exercise 3.10 We start by pricing the "inner" at-the-money option with payoff $(S_{T_2} - S_{T_1})^+$ and strike price $K = S_{T_1}$ at time T_1 as

$$\begin{aligned}
&\mathrm{e}^{-(T_2-T_1)r}\,\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathscr{F}_{T_1}\right] \\
&= S_{T_1}\Phi\left(\frac{(r+\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
&\quad - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{(r-\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
&= S_{T_1}\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right),
\end{aligned}$$

where we applied (3.20) with $T = T_2$, $t = T_1$, and $K = S_{T_1}$. As a consequence, the forward start option can be priced as

$$\begin{aligned}
&\mathrm{e}^{-(T_1-t)r}\,\mathbb{E}^*\left[\mathrm{e}^{-(T_2-T_1)r}\,\mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathscr{F}_{T_1}\right] \Big| \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T_1-t)r} \\
&\quad \times \mathbb{E}^*\left[S_{T_1}\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1}\,\mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) \Big| \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T_1-t)r} \\
&\quad \times \left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right)\mathbb{E}^*[S_{T_1} \mid \mathscr{F}_t] \\
&= S_t\left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r}\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right),
\end{aligned}$$

$0 \leqslant t \leqslant T_1$.

Exercise 3.11 (Exercise 2.8 continued). We have

$$\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\log S_T \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\log S_t + (\widehat{B}_T - \widehat{B}_t)\sigma + \left(r - \frac{\sigma^2}{2}\right)(T-t) \mid \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\log S_t + \mathrm{e}^{-(T-t)r}\left(r - \frac{\sigma^2}{2}\right)(T-t),
\end{aligned}$$

$t \in [0,T]$.

Exercise 3.12 (Exercise 2.10 continued).

a) The discounted process $X_t := \mathrm{e}^{-rt}S_t$ satisfies

$$dX_t = (\alpha - r)X_t dt + \sigma\,\mathrm{e}^{-rs}dB_s,$$

which is a martingale when $\alpha = r$ by Proposition 3.1, as in this case it becomes a stochastic integral with respect to a standard Brownian motion. This fact can be recovered by directly computing the conditional expectation $\mathbb{E}[X_t \mid \mathscr{F}_s]$ and showing it is equal to X_s. By Exercise 2.10, we have

$$S_t = S_0 \mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s,$$

hence

$$X_t = S_0 + \sigma \int_0^t \mathrm{e}^{-rs} dB_s, \qquad t \in \mathbb{R}_+,$$

and

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathscr{F}_s] &= \mathbb{E}\left[S_0 + \sigma \int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathscr{F}_s\right] \\
&= \mathbb{E}[S_0] + \sigma \mathbb{E}\left[\int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathscr{F}_s\right] \\
&= S_0 + \sigma \mathbb{E}\left[\int_0^s \mathrm{e}^{-ru} dB_u \,\Big|\, \mathscr{F}_s\right] + \sigma \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathscr{F}_s\right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u + \sigma \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u\right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u \\
&= X_s, \qquad 0 \leqslant s \leqslant t.
\end{aligned}$$

b) We rewrite the stochastic differential equation satisfied by $(S_t)_{t\in\mathbb{R}_+}$ as

$$dS_t = \alpha S_t dt + \sigma dB_t = r S_t dt + \sigma d\widehat{B}_t,$$

where

$$d\widehat{B}_t := \frac{\alpha - r}{\sigma} S_t dt + dB_t,$$

which allows us to write, by taking $\alpha := -r$, as

$$S_t = \mathrm{e}^{rt}\left(S_0 + \sigma \int_0^t \mathrm{e}^{-rs} d\widehat{B}_s\right) = S_0 \mathrm{e}^{rt} + \sigma \int_0^t \mathrm{e}^{(t-s)r} d\widehat{B}_s. \tag{A.11}$$

Taking

$$\psi_t := \frac{\alpha - r}{\sigma} S_t, \qquad 0 \leqslant t \leqslant T,$$

in the Girsanov Theorem 3.2, the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\mathbb{P}_\alpha$ defined by

$$\begin{aligned}
\frac{d\mathbb{P}_\alpha}{d\mathbb{P}} &:= \exp\left(-\int_0^T \psi_t dB_t - \frac{1}{2}\int_0^T \psi_t^2 dt\right) \\
&= \exp\left(-\frac{\alpha - r}{\sigma}\int_0^T S_t dB_t - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \int_0^T S_t^2 dt\right),
\end{aligned}$$

and $(X_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}_\alpha$.

c) Using (A.11) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
C(t, S_t) &= \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha[\exp(S_T) \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r} \mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma \int_0^t \mathrm{e}^{(T-u)r} d\widehat{B}_u + \sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t\right) \mathbb{E}_\alpha\left[\exp\left(\sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t\right) \mathbb{E}_\alpha\left[\exp\left(\sigma \int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right)\right]
\end{aligned}$$

$$= \exp\Big(-(T-t)r + \mathrm{e}^{(T-t)r}S_t\Big)\exp\Big(\frac{\sigma^2}{2}\int_t^T \mathrm{e}^{2(T-u)r}du\Big)$$
$$= \exp\Big(-(T-t)r + \mathrm{e}^{(T-t)r}S_t + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big), \qquad 0 \leqslant t \leqslant T.$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t,S_t) = \exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)$$

and

$$\begin{aligned}
\eta_t &= \frac{C(t,S_t)-\xi_t S_t}{A_t} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{A_t}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big) \\
&\quad -\frac{S_t}{A_t}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big).
\end{aligned}$$

e) We have

$$\begin{aligned}
dC(t,S_t) &= r\,\mathrm{e}^{-(T-t)r}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt \\
&\quad -rS_t\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt \\
&\quad -\frac{\sigma^2}{2}\mathrm{e}^{(T-t)r}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt \\
&\quad +\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dS_t \\
&\quad +\frac{1}{2}\mathrm{e}^{(T-t)r}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)\sigma^2 dt \\
&= r\,\mathrm{e}^{-(T-t)r}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt \\
&\quad -rS_t\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt + \xi_t dS_t.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\xi_t dS_t + \eta_t dA_t &= \xi_t dS_t \\
&\quad + r\,\mathrm{e}^{-(T-t)r}\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt \\
&\quad -rS_t\exp\Big(S_t\,\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\Big)dt,
\end{aligned}$$

showing that

$$dC(t,S_t) = \xi_t dS_t + \eta_t dA_t,$$

and confirming that the strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ is self-financing.

Exercise 3.13

a) Using (A.11) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
C(t,S_t) &= \mathrm{e}^{-(T-t)r}\mathbf{E}_\alpha[S_T^2 \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}_\alpha\Big[\Big(\mathrm{e}^{rT}S_0 + \sigma\int_0^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\Big)^2 \Bigm| \mathscr{F}_t\Big] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}_\alpha\Big[\Big(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u + \sigma\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\Big)^2 \Bigm| \mathscr{F}_t\Big] \\
&= \mathrm{e}^{-(T-t)r}\mathbf{E}_\alpha\Big[\Big(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\Big)^2 \Bigm| \mathscr{F}_t\Big]
\end{aligned}$$

$$+2\sigma \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0+\sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)\mathbb{E}_\alpha\left[\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u \,\Big|\, \mathscr{F}_t\right]$$
$$+\sigma^2 \mathrm{e}^{-(T-t)r}\mathbb{E}_\alpha\left[\left(\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \,\Big|\, \mathscr{F}_t\right]$$
$$= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0+\sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2+\sigma^2 \mathrm{e}^{-(T-t)r}\mathbb{E}_\alpha\left[\left(\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2\right]$$
$$= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0+\sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2+\sigma^2 \mathrm{e}^{-(T-t)r}\int_t^T \mathrm{e}^{2(T-u)r}du$$
$$= \mathrm{e}^{(T-t)r}S_t^2+\frac{\sigma^2}{2r}\left(\mathrm{e}^{(T-t)r}-\mathrm{e}^{-(T-t)r}\right)$$
$$= \mathrm{e}^{(T-t)r}S_t^2+\sigma^2\frac{\sinh((T-t)r)}{r}, \qquad 0\leqslant t\leqslant T.$$

b) We find

$$\xi_t=\frac{\partial C}{\partial x}(t,S_t)=2\,\mathrm{e}^{(T-t)r}S_t, \qquad 0\leqslant t\leqslant T.$$

Exercise 3.14 (Exercise 2.2 continued). If C is a contingent claim payoff of the form $C=\phi(S_T)$ such that $(\xi_t,\eta_t)_{t\in[0,T]}$ hedges the claim payoff C, the arbitrage price of the claim payoff C at time $t\in[0,T]$ is given by

$$\pi_t(X)=V_t=\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[\phi(S_T)\mid\mathscr{F}_t], \qquad 0\leqslant t\leqslant T,$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral measure $\mathbb{P}^*$. Hence, from the noncentral Chi square probability density function

$$f_{T-t}(x)$$
$$=\frac{2\beta}{\sigma^2\left(1-\mathrm{e}^{-\beta(T-t)}\right)}\exp\left(-\frac{2\beta\left(x+r_t\,\mathrm{e}^{-\beta(T-t)}\right)}{\sigma^2\left(1-\mathrm{e}^{-\beta(T-t)}\right)}\right)\left(\frac{x}{r_t\,\mathrm{e}^{-\beta(T-t)}}\right)^{\alpha\beta/\sigma^2-1/2}$$
$$\times I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{r_t x\,\mathrm{e}^{-\beta(T-t)}}}{\sigma^2\left(1-\mathrm{e}^{-\beta(T-t)}\right)}\right),$$

of S_T given S_t, $x>0$, we find

$$g(t,S_t)=\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[\phi(S_T)\mid\mathscr{F}_t]$$
$$=\frac{2\beta\,\mathrm{e}^{-r(T-t)}}{\sigma^2\left(1-\mathrm{e}^{-\beta(T-t)}\right)}\int_0^\infty\phi(x)\left(\frac{x}{S_t\,\mathrm{e}^{-\beta(T-t)}}\right)^{\alpha\beta/\sigma^2-1/2}\mathrm{e}^{-2\frac{\beta(x+S_t\,\mathrm{e}^{-\beta(T-t)})}{\sigma^2(1-\mathrm{e}^{-\beta(T-t)})}}$$
$$\times I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{xS_t\,\mathrm{e}^{-\beta(T-t)}}}{\sigma^2\left(1-\mathrm{e}^{-\beta(T-t)}\right)}\right)dx$$

$0\leqslant t\leqslant T$, under the Feller condition $2\alpha\beta\geqslant\sigma^2$.

Exercise 3.15

a) We have

$$\frac{\partial f}{\partial t}(t,x)=(r-\sigma^2/2)f(t,x), \qquad \frac{\partial f}{\partial x}(t,x)=\sigma f(t,x),$$

and

$$\frac{\partial^2 f}{\partial x^2}(t,x) = \sigma^2 f(t,x),$$

hence

$$\begin{aligned} dS_t &= df(t,B_t) \\ &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) f(t,B_t)dt + \sigma f(t,B_t)dB_t + \frac{1}{2}\sigma^2 f(t,B_t)dt \\ &= rf(t,B_t)dt + \sigma f(t,B_t)dB_t \\ &= rS_t dt + \sigma S_t dB_t. \end{aligned}$$

b) We have

$$\begin{aligned} \mathbb{E}\left[e^{\sigma B_T} \mid \mathscr{F}_t\right] &= \mathbb{E}\left[e^{(B_T - B_t + B_t)\sigma} \mid \mathscr{F}_t\right] \\ &= e^{\sigma B_t}\mathbb{E}\left[e^{(B_T - B_t)\sigma} \mid \mathscr{F}_t\right] \\ &= e^{\sigma B_t}\mathbb{E}\left[e^{(B_T - B_t)\sigma}\right] \\ &= e^{\sigma B_t + \sigma^2(T-t)/2}. \end{aligned}$$

c) We have

$$\begin{aligned} \mathbb{E}[S_T \mid \mathscr{F}_t] &= \mathbb{E}\left[e^{\sigma B_T + rT - \sigma^2 T/2} \mid \mathscr{F}_t\right] \\ &= e^{rT - \sigma^2 T/2}\mathbb{E}\left[e^{\sigma B_T} \mid \mathscr{F}_t\right] \\ &= e^{rT - \sigma^2 T/2} e^{\sigma B_t + \sigma^2(T-t)/2} \\ &= e^{rT + \sigma B_t - \sigma^2 t/2} \\ &= e^{(T-t)r + \sigma B_t + rt - \sigma^2 t/2} \\ &= e^{(T-t)r} S_t, \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

d) We have

$$\begin{aligned} V_t &= e^{-(T-t)r}\mathbb{E}[C \mid \mathscr{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}[S_T - K \mid \mathscr{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}[S_T \mid \mathscr{F}_t] - e^{-(T-t)r}\mathbb{E}[K \mid \mathscr{F}_t] \\ &= S_t - e^{-(T-t)r}K, \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

e) We take $\xi_t = 1$ and $\eta_t = -K e^{-rT}/A_0$, $t \in [0,T]$.

f) We find

$$V_T = \mathbb{E}[C \mid \mathscr{F}_{\mathscr{T}}] = C.$$

Exercise 3.16 Binary options. (Exercise 2.9 continued).

a) By definition of the indicator (or step) functions $\mathbb{1}_{[K,\infty)}$ and $\mathbb{1}_{[0,K]}$ we have

$$\mathbb{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geqslant K, \\ 0 & \text{if } x < K, \end{cases} \qquad \textit{resp.} \quad \mathbb{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \leqslant K, \\ 0 & \text{if } x > K, \end{cases}$$

which shows the claimed result by the definition of C_b and P_b.

b) We have

$$\begin{aligned} \pi_t(C_b) &= e^{-(T-t)r}\mathbb{E}[C_b \mid \mathscr{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \mid S_t\right] \\ &= e^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t) \\ &= C_b(t,S_t). \end{aligned}$$

c) We have $\pi_t(C_b) = C_b(t,S_t)$, where

$$\begin{aligned} C_b(t,x) &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T > K \mid S_t = x) \\ &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)), \end{aligned}$$

with

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) The price of this modified contract with payoff

$$C_\alpha = \mathbb{1}_{[K,\infty)}(S_T) + \alpha\mathbb{1}_{[0,K)}(S_T)$$

is given by

$$\begin{aligned} \pi_t(C_\alpha) &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T)+\alpha\mathbb{1}_{[0,K)}(S_T)\,\middle|\,S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t) + \alpha\,\mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \leqslant K \mid S_t) \\ &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geqslant K \mid S_t) + \alpha\,\mathrm{e}^{-(T-t)r}(1-\mathbb{P}(S_T \geqslant K \mid S_t)) \\ &= \alpha\,\mathrm{e}^{-(T-t)r}\mathrm{e}^{-(T-t)r} + (1-\alpha)\mathbb{P}(S_T \geqslant K \mid S_t) \\ &= \alpha\,\mathrm{e}^{-(T-t)r} + (1-\alpha)\,\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

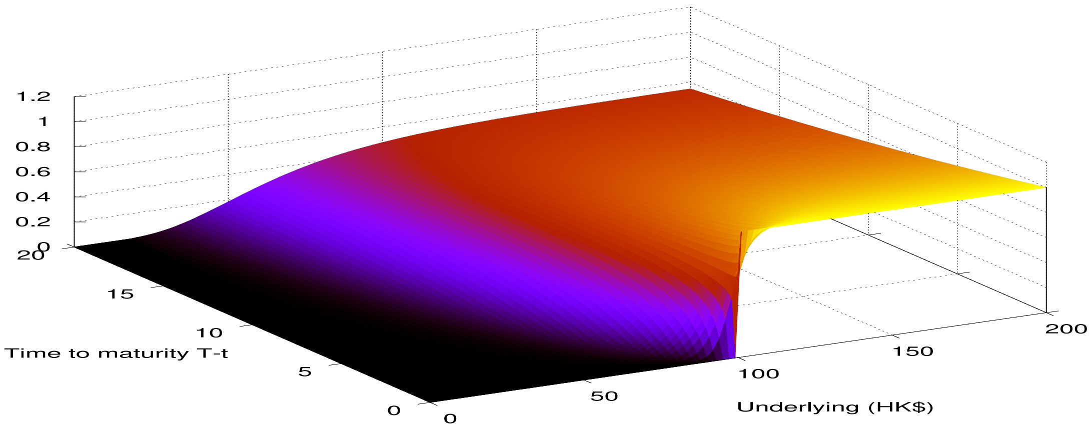


Figure S.8: Price of a binary call option.

e) We note that

$$\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) = \mathbb{1}_{[0,\infty)}(S_T),$$

almost surely since $\mathbb{P}(S_T = K) = 0$, hence

$$\begin{aligned} \pi_t(C_b)+\pi_t(P_b) &= \mathrm{e}^{-(T-t)r}\mathbb{E}[C_b \mid \mathcal{F}_t] + \mathrm{e}^{-(T-t)r}\mathbb{E}[P_b \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}[C_b + P_b \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T)+\mathbb{1}_{[0,K]}(S_T)\,\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}\left[\mathbb{1}_{[0,\infty)}(S_T)\,\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}[1 \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

f) We have

$$\pi_t(P_b) = \mathrm{e}^{-(T-t)r} - \pi_t(C_b)$$

$$= \mathrm{e}^{-(T-t)r} - \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$= \mathrm{e}^{-(T-t)r}(1-\Phi(d_-(T-t)))$$
$$= \mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)).$$

g) We have

$$\begin{aligned}
\xi_t &= \frac{\partial C_b}{\partial x}(t,S_t) \\
&= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(x/K)}{\sigma\sqrt{T-t}}\right)_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{\sigma S_t\sqrt{2(T-t)\pi}}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&> 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a call option does not involve short selling because $\xi_t > 0$ for all t, cf. Figure S.9 which represents the risky investment in the hedging portfolio of a binary call option.

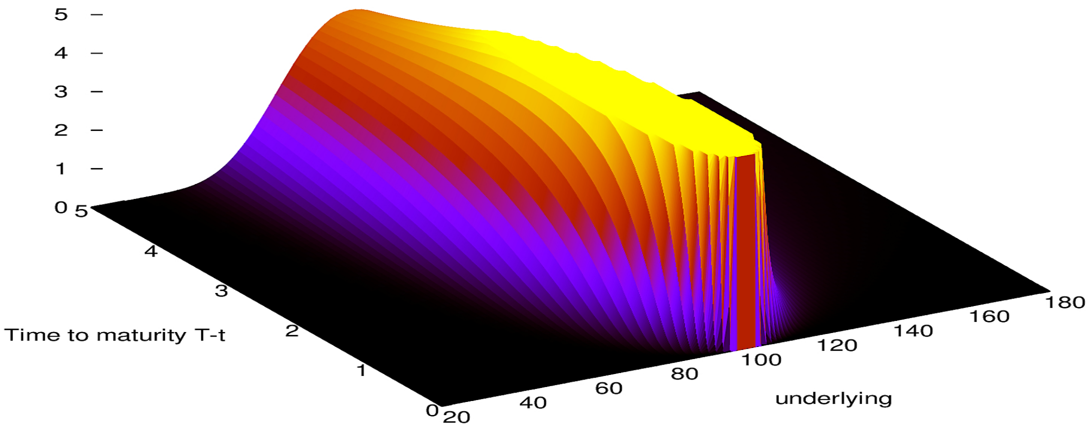


Figure S.9: Risky hedging portfolio value for a binary call option.

Figure S.10 presents the risk-free hedging portfolio value for a binary call option.

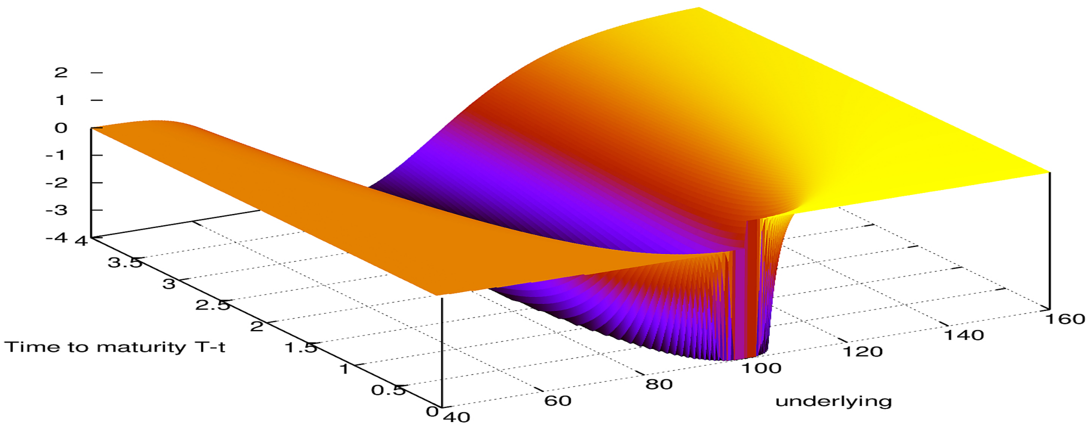


Figure S.10: Risk-free hedging portfolio value for a binary call option.

h) Here we have

$$\begin{aligned}
\xi_t &= \frac{\partial P_b}{\partial x}(t,S_t) \\
&= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(x/K)}{\sigma\sqrt{T-t}}\right)_{x=S_t} \\
&= -\mathrm{e}^{-(T-t)r}\frac{1}{\sigma\sqrt{2(T-t)\pi}S_t}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&< 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a put option does involve short selling because $\xi_t < 0$ for all t.

Exercise 3.17 Using Itô's formula and the fact that the expectation of the stochastic integral with respect to $(W_t)_{t\in\mathbb{R}_+}$ is zero, we have

$$\begin{aligned}
C(x,T) &= \mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\,\middle|\,S_0=x\right] \qquad \text{(A.12)}\\
&= \phi(x)-\mathbb{E}\left[\int_0^T r\,\mathrm{e}^{-rs}\phi(S_t)dt\,\middle|\,S_0=x\right] \\
&\quad +r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dt\,\middle|\,S_0=x\right] \\
&\quad +\sigma\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dB_t\,\middle|\,S_0=x\right] \\
&\quad +\frac{1}{2}\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}\phi''(S_t)\sigma^2(S_t)dt\,\middle|\,S_0=x\right] \\
&= \phi(x)-r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rs}\phi(S_t)dt\,\middle|\,S_0=x\right] \\
&\quad +r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dt\,\middle|\,S_0=x\right] \\
&\quad +\frac{1}{2}\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}\phi''(S_t)\sigma^2(S_t)dt\,\middle|\,S_0=x\right] \\
&= \phi(x)-\int_0^T r\,\mathrm{e}^{-rt}\,\mathbb{E}\left[\phi(S_t)\,\middle|\,S_0=x\right]dt \\
&\quad +r\int_0^T \mathrm{e}^{-rt}\,\mathbb{E}\left[S_t\phi'(S_t)\,\middle|\,S_0=x\right]dt \\
&\quad +\frac{1}{2}\int_0^T \mathrm{e}^{-rt}\,\mathbb{E}\left[\phi''(S_t)\sigma^2(S_t)\,\middle|\,S_0=x\right]dt,
\end{aligned}$$

hence by differentiation with respect to T we find

$$\begin{aligned}
\text{Theta}_T &= \frac{\partial}{\partial T}\left(\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\,\middle|\,S_0=x\right]\right) \\
&= -r\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\middle|\,S_0=x\right]+r\mathrm{e}^{-rT}\,\mathbb{E}\left[S_T\phi'(S_T)\,\middle|\,S_0=x\right] \\
&\quad +\frac{1}{2}\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi''(S_T)\sigma^2(S_T)\,\middle|\,S_0=x\right].
\end{aligned}$$

Chapter 4

Exercise 4.1

a) The process $((2-B_t)^+)_{t\in\mathbb{R}_+}$ is a convex function $x \longmapsto (2-x)^+$ of the Brownian martingale $(B_t)_{t\in\mathbb{R}_+}$, hence it is a *sub*martingale by Proposition 4.3-(a)).

b) The process e^{B_t} can be written as

$$\mathrm{e}^{\sigma B_t - \sigma^2 t/2 + \mu t} = \mathrm{e}^{\mu t}\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

with $\sigma = 1$ and $\mu = \sigma^2/2 > 0$, hence it is a *sub*martingale as the geometric Brownian motion $\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$ is a martingale.

c) When $t > 0$, the question "is $\nu > t$?" cannot be answered at time t without waiting to know the value of B_{2t} at time $2t > t$. Therefore ν is *not* a stopping time.

d) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\leqslant s\leqslant t}$ and of the (deterministic) curve $(\mathrm{e}^{s/2} + \alpha s\,\mathrm{e}^{s/2})_{0\leqslant s\leqslant t}$ up to the time t. Therefore τ is a stopping time.

e) Since τ is a stopping time and $\left(\mathrm{e}^{B_t - t/2}\right)_{t\in\mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 4.5 shows that $\left(\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right)_{t\in\mathbb{R}_+}$ is also a martingale and, in particular, its expected value*

$$\mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_{0\wedge\tau} - (0\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_0 - 0/2}\right] = 1$$

is constantly equal to 1 for all $t \geqslant 0$. This shows that

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] &= \mathbb{E}\left[\lim_{t\to\infty} \mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] \\ &= 1.\end{aligned}$$

Next, we note that $\mathrm{e}^{B_\tau} = (\alpha + \beta\tau)\,\mathrm{e}^{\tau/2}$ at time τ, hence $\alpha + \beta\tau = \mathrm{e}^{B_\tau - \tau/2}$ and

$$\alpha + \beta\,\mathbb{E}[\tau] = \mathbb{E}[\alpha + \beta\tau] = \mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] = 1,$$

i.e. $\mathbb{E}[\tau] = (1-\alpha)/\beta$.

Remark: This argument also recovers $\mathbb{E}[\tau] = 0$ when $\alpha = 1$, however it fails when ($\alpha > 1$ and $\beta > 0$) and when ($\alpha < 1$ and $\beta < 0$), because τ is not *a.s.* finite ($\mathbb{P}(\tau < \infty) < 1$) in those cases.

Exercise 4.2 Stopping times.

a) When $0 \leqslant t < 1$ the question "is $\nu > t$?" cannot be answered at time t without waiting to know the value of B_1 at time 1. Therefore ν is *not* a stopping time.

b) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\leqslant s\leqslant t}$ and of the (deterministic) curve $\left(\alpha\,\mathrm{e}^{-s/2}\right)_{0\leqslant s\leqslant t}$ up to the time t. Therefore τ is a stopping time.
Since τ is a stopping time and $(B_t)_{t\in\mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 4.5 shows that $\left(\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right)_{t\in\mathbb{R}_+}$ is also a martingale and in particular its expected value

$$\mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_{0\wedge\tau} - (0\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_0 - 0/2}\right] = 1$$

is constantly equal to 1 for all t. This shows that

$$\mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] = \mathbb{E}\left[\lim_{t\to\infty} \mathrm{e}^{B_{\tau\wedge t} - (\tau\wedge t)/2}\right] = \lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{B_{\tau\wedge t} - (\tau\wedge t)/2}\right] = 1.$$

Next, we note that we have $\mathrm{e}^{B_\tau} = \alpha\,\mathrm{e}^{-\tau/2}$ at time τ, hence

$$\alpha\,\mathbb{E}\left[\mathrm{e}^{-\tau}\right] = \mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] = 1, \quad i.e. \quad \mathbb{E}\left[\mathrm{e}^{-\tau}\right] = \frac{1}{\alpha} \leqslant 1.$$

Remark: note that this argument fails when $\alpha < 1$ because in that case τ is not *a.s.* finite.

*We let $t \wedge \tau := \min(t, \tau)$.

c) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0 \leqslant s \leqslant t}$ and of the (deterministic) curve $(1+\alpha s)_{0 \leqslant s \leqslant t}$ up to the time t. Therefore τ is a stopping time.

Since τ is a stopping time and $(B_t)_{t \in \mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 4.5 shows that $(B^2_{t\wedge\tau} - (t \wedge \tau))_{t \in \mathbb{R}_+}$ is also a martingale and in particular its expected value

$$\mathbb{E}[B^2_{t\wedge\tau} - (t\wedge\tau)] = \mathbb{E}[B^2_{0\wedge\tau} - (0\wedge\tau)] = \mathbb{E}[B_0^2 - 0] = 0$$

is constantly equal to 0 for all t. This shows that

$$\mathbb{E}[B_\tau^2 - \tau] = \mathbb{E}\left[\lim_{t\to\infty}(B^2_{t\wedge\tau} - (t\wedge\tau))\right] = \lim_{t\to\infty}\mathbb{E}\left[(B^2_{t\wedge\tau} - (t\wedge\tau))\right] = 0.$$

Next, we note that $B_\tau^2 = 1 + \alpha\tau$ at time τ, hence

$$1 + \alpha\,\mathbb{E}[\tau] = \mathbb{E}[1+\alpha\tau] = \mathbb{E}[B_\tau^2] - \mathbb{E}[\tau] = 0,$$

i.e.

$$\mathbb{E}[\tau] = \frac{1}{1-\alpha}.$$

Remark: Note that this argument is valid whenever $\alpha \leqslant 1$ and yields $\mathbb{E}[\tau] = +\infty$ when $\alpha = 1$, however it fails when $\alpha > 1$ because in that case τ is not *a.s.* finite.

Exercise 4.3

a) By the Stopping Time Theorem 4.5, for all $n \geqslant 0$ we have

$$\begin{aligned}
1 &= \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_{\tau_L\wedge n} - r(\tau_L\wedge n)}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_{\tau_L\wedge n} - r(\tau_L\wedge n)}\mathbb{1}_{\{\tau_L<n\}}\right] + \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_{\tau_L\wedge n} - r(\tau_L\wedge n)}\mathbb{1}_{\{\tau_L\geqslant n\}}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_{\tau_L} - r\tau_L}\mathbb{1}_{\{\tau_L<n\}}\right] + \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_n - rn}\mathbb{1}_{\{\tau_L\geqslant n\}}\right] \\
&= \mathrm{e}^{L\sqrt{2r}}\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<n\}}\right] + \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_n - rn}\mathbb{1}_{\{\tau_L\geqslant n\}}\right].
\end{aligned}$$

The first term above converges to

$$\mathrm{e}^{L\sqrt{2r}}\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}}\right] = \mathrm{e}^{L\sqrt{2r}}\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right]$$

as n tends to infinity, by dominated or monotone convergence and the fact that $r > 0$. The second term can be bounded as

$$0 \leqslant \mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_n - rn}\mathbb{1}_{\{\tau_L\geqslant n\}}\right] \leqslant \mathrm{e}^{-rn}\,\mathbb{E}\left[\mathrm{e}^{L\sqrt{2r}}\mathbb{1}_{\{\tau_L\geqslant n\}}\right] \leqslant \mathrm{e}^{-rn}\,\mathrm{e}^{L\sqrt{2r}},$$

which tends to 0 as n tends to infinity because $r > 0$. Therefore we have

$$1 = \lim_{n\to\infty}\mathbb{E}\left[\mathrm{e}^{\sqrt{2r}B_{\tau_L\wedge n} - r(\tau_L\wedge n)}\right] = \mathrm{e}^{L\sqrt{2r}}\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right],$$

which yields $\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right] = \mathrm{e}^{-L\sqrt{2r}}$ for any $r \geqslant 0$. When $r < 0$ we could in fact show that $\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right] = +\infty$.

b) In order to maximize the quantity

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{-r\tau_L}B_{\tau_L}\right] &= \mathbb{E}\left[\mathrm{e}^{-r\tau_L}B_{\tau_L}\mathbb{1}_{\{\tau_L<\infty\}}\right] \\
&= L\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}}\right] \\
&= L\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L}\right]
\end{aligned}$$

$$= \quad L e^{-L\sqrt{2r}},$$

we differentiate

$$\frac{\partial}{\partial L}(L e^{-L\sqrt{2r}}) = e^{-L\sqrt{2r}} - L\sqrt{2r} e^{-L\sqrt{2r}} = 0,$$

which yields the optimal level $L^* = 1/\sqrt{2r}$.

This shows that when the value of r is "large" the better strategy is to opt for a "small gain" at the level $L^* = 1/\sqrt{2r}$ rather than to wait for a longer time.

Exercise 4.4 See *e.g.* Theorem 6.16 page 161 of Klebaner, 2005. By the Itô formula, we have

$$\begin{aligned} X_t &= f(B_t) - \frac{1}{2}\int_0^t f''(B_s)ds \\ &= f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds - \frac{1}{2}\int_0^t f''(B_s)ds \\ &= f(B_0) + \int_0^t f'(B_s)dB_s, \end{aligned}$$

hence the process $(X_t)_{t\in\mathbb{R}_+}$ is a *martingale*. By the Stopping Time Theorem 4.5 we have

$$\begin{aligned} f(x) &= \mathbb{E}[X_0 \mid B_0 = x] \\ &= \mathbb{E}[X_{\tau\wedge t} \mid B_0 = x] \\ &= \mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] - \frac{1}{2}\mathbb{E}\left[\int_0^{\tau\wedge t} f''(B_s)ds \mid B_0 = x\right] \\ &= \mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] + \mathbb{E}[\tau\wedge t \mid B_0 = x], \end{aligned}$$

since $f''(y) = -2$ for all $y \in \mathbb{R}$. We note that, by dominated convergence,

$$\begin{aligned} \mathbb{E}\left[\tau \mid B_0 = x\right] &= \mathbb{E}\left[\lim_{t\to\infty}(\tau\wedge t) \mid B_0 = x\right] \\ &= \lim_{t\to\infty}\mathbb{E}[\tau\wedge t \mid B_0 = x] \\ &\leqslant |f(x)| + \operatorname*{Max}_{y\in[a,b]}|f(y)| \\ &< \infty, \end{aligned}$$

hence $\mathbb{E}[\tau] < \infty$ and therefore $\mathbb{P}(\tau < \infty) = 1$. Next, we have

$$\begin{aligned} f(x) &= \lim_{t\to\infty}\mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] + \lim_{t\to\infty}\mathbb{E}[\tau\wedge t \mid B_0 = x] \\ &= \mathbb{E}\left[\lim_{t\to\infty} f(B_{\tau\wedge t}) \mid B_0 = x\right] + \mathbb{E}[\tau \mid B_0 = x] \\ &= \mathbb{E}[f(B_\tau) \mid B_0 = x] + \mathbb{E}[\tau \mid B_0 = x] \\ &= \mathbb{E}[\tau \mid B_0 = x], \end{aligned}$$

since $f''(x) = -2$ and $f(a) = f(b) = 0$ with $B_\tau \in \{a, b\}$.

Remarks.

i) The above exchanges between $\lim_{t\to\infty}$ and the expectation operator $\mathbb{E}[\,\cdot \mid B_0 = x]$ is justified by the *dominated convergence theorem*, since

$$|f(B_{\tau\wedge t})| \leqslant \operatorname*{Max}_{y\in[a,b]}|f(y)|, \qquad t\in\mathbb{R}_+.$$

ii) The function $f(x)$ can be determined by searching for a quadratic solution of the form $f(x) = \alpha + \beta x + \gamma x^2$, which shows that $f''(x) = 2\gamma = -2$ hence $\gamma = -1$, and

$$\begin{cases} f(a) = \alpha + \beta a - a^2 = 0, \\ \\ f(b) = \alpha + \beta b - b^2 = 0, \end{cases}$$

hence $\alpha = -ab$ and $\beta = a + b$. Therefore, we have

$$\mathbb{E}[\tau \mid B_0 = x] = f(x) = -ab + (a+b)x - x^2 = (x-a)(b-x).$$

Exercise 4.5 We use the Stopping Time Theorem 4.5 and the fact that $\left(e^{\sigma B_t - \sigma^2 t/2}\right)_{t\in\mathbb{R}_+}$ is a martingale for all $\sigma \in \mathbb{R}$. By the stopping time theorem, for all $n \geqslant 0$ we have

$$\begin{aligned} 1 &= \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\right] \\ &= \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\mathbb{1}_{\{\tau\geqslant n\}}\right] \\ &= \mathbb{E}\left[e^{\sigma B_\tau - \sigma^2\tau/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_n - \sigma^2 n/2}\mathbb{1}_{\{\tau\geqslant n\}}\right] \\ &= e^{\sigma\alpha}\,\mathbb{E}\left[e^{\sigma\beta\tau - \sigma^2\tau/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_n - \sigma^2 n/2}\mathbb{1}_{\{\tau\geqslant n\}}\right]. \end{aligned}$$

If $\sigma^2 \geqslant 2\sigma\beta$, the first term above converges to

$$e^{\sigma\alpha}\,\mathbb{E}\left[e^{-(\sigma^2/2-\sigma\beta)\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = e^{\sigma\alpha}\,\mathbb{E}\left[e^{-(\sigma^2/2-\sigma\beta)\tau}\right]$$

as n tends to infinity, by dominated or monotone convergence. In the sequel we use the solutions $\sigma_\pm = \beta \pm \sqrt{\beta^2 + 2r}$ of the equation $r = \sigma^2/2 - \sigma\beta$, and we distinguish two cases.

i) If $\alpha \geqslant 0$, we have $B_n \leqslant \alpha + \beta n$, $n \leqslant \tau$, hence the second term above can be bounded as

$$\begin{aligned} 0 &\leqslant \mathbb{E}\left[e^{\sigma_+ B_n - n\sigma_+^2/2}\mathbb{1}_{\{\tau\geqslant n\}}\right] \\ &\leqslant e^{-n\sigma_+^2/2}\,\mathbb{E}\left[e^{\alpha\sigma_+ + \beta\sigma_+ n}\mathbb{1}_{\{\tau\geqslant n\}}\right] \\ &\leqslant e^{\alpha\sigma_+ - n\sigma_+^2/2 + \beta\sigma_+ n} \\ &= e^{\alpha\sigma_+ - rn}, \end{aligned}$$

which tend to 0 as n tends to infinity. Therefore, we have

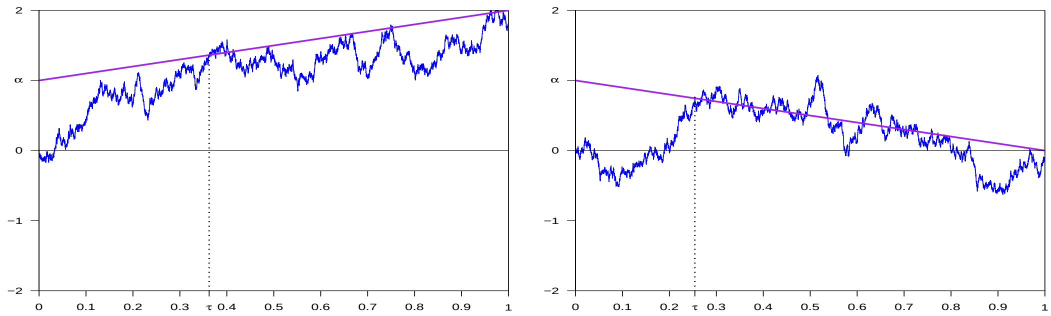
$$1 = \lim_{n\to\infty}\mathbb{E}\left[e^{\sigma_+ B_{\tau\wedge n} - \sigma_+^2(\tau\wedge n)/2}\right] = e^{\alpha\sigma_+}\,\mathbb{E}\left[e^{-(\sigma_+^2/2-\beta\sigma_+)\tau}\right],$$

which yields

$$\begin{aligned} \mathbb{E}\left[e^{-r\tau}\right] &= \mathbb{E}\left[e^{-(\sigma_+^2/2-\beta\sigma_+)\tau}\right] \\ &= e^{-\alpha\sigma_+} \\ &= e^{-\alpha\beta - \alpha\sqrt{\beta^2+2r}} \\ &= e^{-\alpha\beta - |\alpha|\sqrt{\beta^2+2r}}, \end{aligned}$$

with

$$\mathbb{P}(\tau < +\infty) = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}] = \lim_{r\to 0}\mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}e^{-r\tau}\right] = \begin{cases} e^{-2\alpha\beta} & \text{if } \beta \geqslant 0, \\ 1 & \text{if } \beta \leqslant 0. \end{cases}$$



(a) $\alpha = 1$, $\beta = 0.5$ (b) $\alpha = 1$, $\beta = -0.5$

Figure S.11: Hitting times of a straight line started at $\alpha < 0$.

ii) If $\alpha \leqslant 0$, we have $B_n \geqslant \alpha + \beta n$, $n \leqslant \tau$, hence the second term above can be bounded as

$$\begin{aligned} 0 &\leqslant \mathbb{E}\left[\mathrm{e}^{\sigma_- B_n - n\sigma_-^2/2} \mathbb{1}_{\{\tau \geqslant n\}} \right] \\ &\leqslant \mathrm{e}^{-n\sigma_-^2/2} \mathbb{E}\left[\mathrm{e}^{\alpha\sigma_- + \beta n \sigma_-} \mathbb{1}_{\{\tau \geqslant n\}} \right] \leqslant \mathrm{e}^{\alpha\sigma_- - n\sigma_-^2/2 + \beta n \sigma_-} \\ &= \mathrm{e}^{\alpha\sigma_- - rn}, \end{aligned}$$

which tends to 0 as n tends to infinity. Therefore, we have

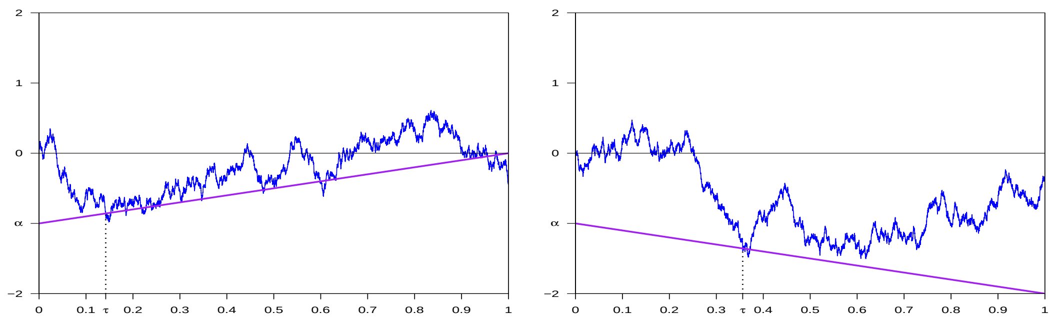
$$1 = \lim_{n \to \infty} \mathbb{E}\left[\mathrm{e}^{\sigma_- B_{\tau \wedge n} - \sigma_-^2 (\tau \wedge n)/2} \right] = \mathrm{e}^{\alpha\sigma_-} \mathbb{E}\left[\mathrm{e}^{-(\sigma_-^2/2 - \beta\sigma_-)\tau} \right],$$

which yields

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-r\tau} \right] &= \mathbb{E}\left[\mathrm{e}^{-(\sigma_-^2/2 - \beta\sigma_-)\tau} \right] = \mathrm{e}^{-\alpha\sigma_-} \\ &= \mathrm{e}^{-\alpha\beta + \alpha\sqrt{\beta^2 + 2r}} \\ &= \mathrm{e}^{-\alpha\beta - |\alpha|\sqrt{\beta^2 + 2r}}, \end{aligned}$$

with

$$\mathbb{P}(\tau < +\infty) = \mathbb{E}[\mathbb{1}_{\{\tau < \infty\}}] = \lim_{r \to 0} \mathbb{E}\left[\mathbb{1}_{\{\tau < \infty\}} \mathrm{e}^{-r\tau} \right] = \begin{cases} 1 & \text{if } \beta \geqslant 0, \\ \mathrm{e}^{-2\alpha\beta} & \text{if } \beta \leqslant 0. \end{cases}$$



(a) $\alpha = -1$, $\beta = 0.5$ (b) $\alpha = -1$, $\beta = -0.5$

Figure S.12: Hitting times of a straight line started at $\alpha < 0$.

Exercise 4.6

a) Letting $A_0 := 0$,

$$A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n], \quad n \geqslant 0,$$

and

$$N_n := M_n - A_n, \qquad n \in \mathbb{N}, \tag{A.13}$$

we have

(i) for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}[N_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}[M_{n+1} - A_{n+1} \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathscr{F}_n] - \mathbb{E}[\mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathscr{F}_n] - \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \\
&= -\mathbb{E}[A_n \mid \mathscr{F}_n] + \mathbb{E}[M_n \mid \mathscr{F}_n] \\
&= M_n - A_n \\
&= N_n,
\end{aligned}$$

hence $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

(ii) We have

$$\begin{aligned}
A_{n+1} - A_n &= \mathbb{E}[M_{n+1} - M_n \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} \mid \mathscr{F}_n] - \mathbb{E}[M_n \mid \mathscr{F}_n] \\
&= \mathbb{E}[M_{n+1} \mid \mathscr{F}_n] - M_n \geqslant 0, \qquad n \in \mathbb{N},
\end{aligned}$$

since $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale.

(iii) By induction we have

$$A_n = A_{n-1} + \mathbb{E}[M_n - M_{n-1} \mid \mathscr{F}_{n-1}], \quad n \geqslant 1,$$

which is $\mathscr{F}_{n-1}$-measurable provided that A_n is $\mathscr{F}_{n-1}$-measurable, $n \geqslant 1$.

(iv) This property is obtained by construction in (A.13).

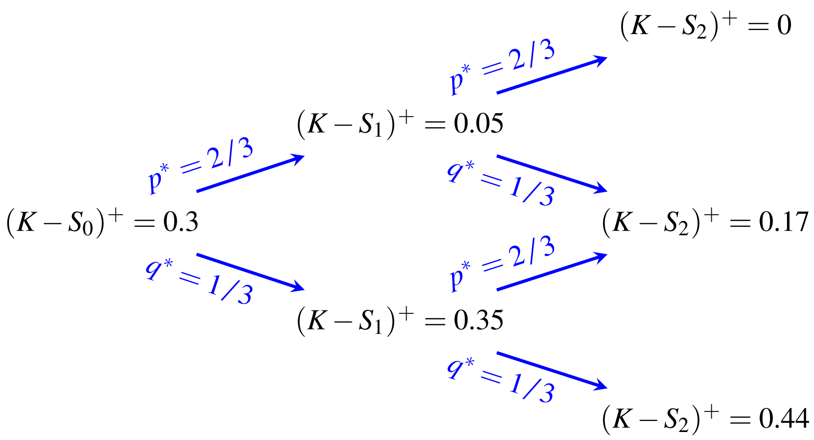
b) For all bounded stopping times σ and τ such that $\sigma \leqslant \tau$ *a.s.*, we have

$$\begin{aligned}
\mathbb{E}[M_\sigma] &= \mathbb{E}[N_\sigma] + \mathbb{E}[A_\sigma] \\
&\leqslant \mathbb{E}[N_\sigma] + \mathbb{E}[A_\tau] \\
&= \mathbb{E}[N_\tau] + \mathbb{E}[A_\tau] \\
&= \mathbb{E}[M_\tau],
\end{aligned}$$

by (4.6), since $(M_n)_{n\in\mathbb{N}}$ is a martingale and $(A_n)_{n\in\mathbb{N}}$ is non-decreasing.

Chapter 5

Exercise 5.1 The option payoffs at immediate exercise are given as follows:



On the other hand, the expected payoffs are given by:

$$
\begin{array}{lll}
 & & (K-S_2)^+ = 0 \\
\mathbb{E}^*[(K-S_2)^+ \mid S_1 = 1.2] = 0.17/3 & \nearrow & \\
 & \searrow & \\
 & & (K-S_2)^+ = 0.17 \\
 & \nearrow & \\
\mathbb{E}^*[(K-S_2)^+ \mid S_1 = 0.9] = 0.26 & & \\
 & \searrow & \\
 & & (K-S_2)^+ = 0.44
\end{array}
$$

Consequently, at time $t=1$ we would exercise immediately if $S_1 = 0.9$, and wait if $S_1 = 1.2$. At time $t=0$ with $S_0 = 1$ the initial value of the option is $(0.34/3+0.35)/3 = 1.39/9 \simeq 0.154 < 0.25$ so we would exercise immediately as well.

Exercise 5.2

a) Taking $f(x) := Cx^{-2r/\sigma^2}$, we have

$$
\begin{aligned}
rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) &= -C\frac{2r^2}{\sigma^2}x^{-2r/\sigma^2} + Cr\left(1+\frac{2r}{\sigma^2}\right)x^{-2r/\sigma^2} \\
&= Crx^{-2r/\sigma^2} \\
&= rf(x),
\end{aligned}
$$

and the condition $\lim_{x\to\infty} f(x) = 0$ is satisfied since $r > 0$.

b) The conditions $f(L^*) = K - L^*$ and $f'(L^*) = -1$ read

$$
\begin{cases}
C(L^*)^{-2r/\sigma^2} = K - L^*, \\
-\dfrac{2r}{\sigma^2}C(L^*)^{-1-2r/\sigma^2} = -1,
\end{cases}
$$

i.e.

$$
\begin{cases}
C(L^*)^{-2r/\sigma^2} = K - L^* \\
\dfrac{2r}{\sigma^2}(K - L^*) = L^*,
\end{cases}
$$

hence

$$
\begin{cases}
L^* = \dfrac{2rK}{2r+\sigma^2} \\
C = \dfrac{K\sigma^2}{2r+\sigma^2}\left(\dfrac{2rK}{2r+\sigma^2}\right)^{2r/\sigma^2} = \dfrac{\sigma^2}{2r}\left(\dfrac{2rK}{2r+\sigma^2}\right)^{1+2r/\sigma^2}.
\end{cases}
$$

Exercise 5.3

a) Given the value

$$
\frac{\partial}{\partial x}\mathrm{BS_p}(x,T) = -\Phi(-d_+(x,T))
$$

of the Delta of the Black-Scholes put option see Proposition 2.7, the smooth fit condition states that at $x = S^*$, the left derivative of (5.31), which is

$$
\frac{\partial}{\partial x}\mathrm{BS_p}(x,T) + \alpha\frac{\partial}{\partial x}(x/S^*)^{-2r/\sigma^2} = -\Phi(-d_+(x,T)) + \alpha\frac{(S^*)^{2r/\sigma^2}}{x^{1+2r/\sigma^2}},
$$

$x > S^*$, should match the right derivative of (5.32), which is -1, hence

$$-1 = -\Phi(-d_+(S^*,T)) - \frac{2r\alpha}{\sigma^2}(S^*)^{-1},$$

which yields

$$\alpha^* = \frac{\sigma^2 S^*}{2r}(1-\Phi(-d_+(S^*,T))) = \frac{\sigma^2 S^*}{2r}\Phi(d_+(S^*,T)),$$

and

$$f(x,T) \simeq \begin{cases} \mathrm{BS_p}(x,T) + \dfrac{\sigma^2(S^*)^{1+2r/\sigma^2}}{2rx^{2r/\sigma^2}}\Phi(d_+(S^*,T)), & x > S^*, \\ K - x, & x \leqslant S^*. \end{cases}$$

Note that at maturity ($T = 0$ here) we have $d_+(S^*,0) = -\infty$ since $S^* < K$, hence $\Phi(d_+(S^*,0)) = 0$ and $f(x,0) = K - x$ as expected.

b) Equating (5.31) to (5.32) at $x = S^*$ yields the equation

$$K - S^* = \mathrm{BS_p}(x,T) + \alpha^*,$$

i.e.

$$1 = \mathrm{e}^{-rT}\Phi(-d_-(S^*,T)) + \frac{S^*}{K}\left(1 + \frac{\sigma^2}{2r}\right)\Phi(d_+(S^*,T)),$$

which can be used to determine the value of S^*, and then the corresponding value of α. The proposed strategy is to exercise the put option as soon as the underlying asset price reaches the critical level S^*.

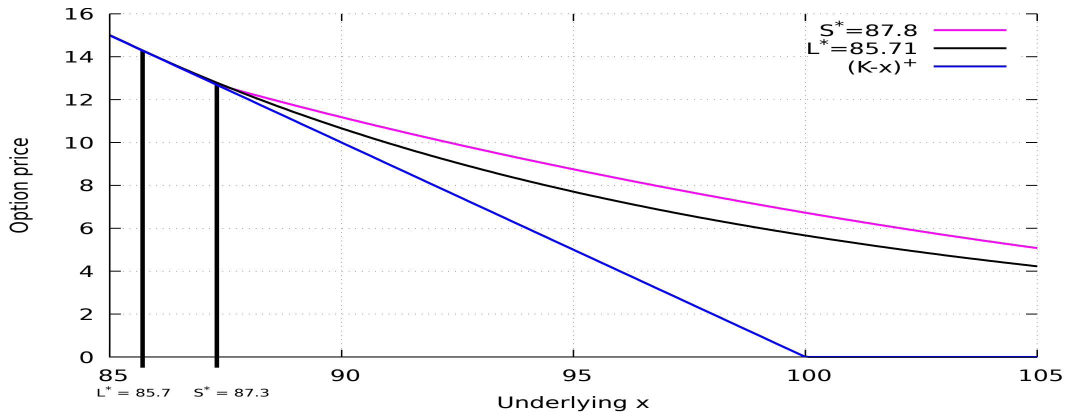


Figure S.13: Perpetual *vs* finite expiration American put option price.

The plot in Figure S.13 yields a finite expiration critical price $S^* = 87.3$ which is expectedly higher than the perpetual critical price $L^* = 85.71$, with $K = 100$, $\sigma = 10\%$, and $r = 3\%$. The perpetual price, however, appears higher than the finite expiration price.

In Figure S.14 we plot the graph of a Barone-Adesi and Whaley, 1987 approximation, together with the European put option price, using the R fOptions package. Note that this approximation is valid only for certain parameter ranges.

```r
r=0.1;sig=0.15;T=0.5;K=100;library(ragtop)
library(fOptions);payoff <- function(x){return(max(K-x,0))};vpayoff <- Vectorize(payoff)
par(new=TRUE)
curve(vpayoff, from=85, to=120, xlab="", lwd = 3, ylim=c(0,10),ylab="",col="red")
par(new=TRUE)
curve(blackscholes(callput=-1, x, K, r, T, sig, 0)$Price, from=85, to=120, xlab="", lwd = 3,
      ylim=c(0,10),ylab="",col="orange")
par(new=TRUE)
curve(BAWAmericanApproxOption("p",x,K,T,r,b=0,sig,title = NULL, description = NULL)@price,
      from=85, to=120 , xlab="Underlying asset price", lwd = 3,ylim=c(0,10),ylab="",col="blue")
grid (lty = 5);legend(105,9.5,legend=c("Approximation","European payoff","Black-Scholes
      put"),col=c("blue","red","orange"),lty=1:1, cex=1.)
```

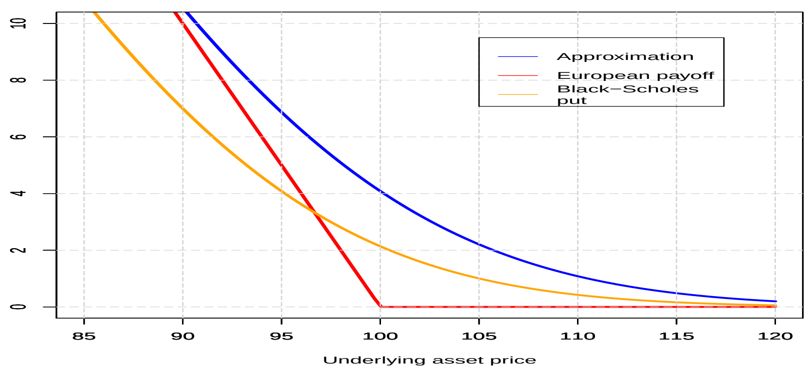


Figure S.14: American put price approximation.

Exercise 5.4

a) We have

$$\tau_\varepsilon = \begin{cases} \varepsilon & \text{if } Z = 1, \\ +\infty & \text{if } Z = 0. \end{cases}$$

b) First, we note that

$$\mathscr{F}_t = \begin{cases} \{\emptyset, \Omega\} & \text{if } t = 0, \\ \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\} & \text{if } t > 0. \end{cases}$$

Next, we have

$$\{\tau_\varepsilon > 0\} = \{Z = 0\},$$

hence

$$\{\tau_\varepsilon > 0\} \notin \mathscr{F}_0 = \{\emptyset, \Omega\},$$

and therefore τ_0 *is not* an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

c) i) For $t = 0$ we have $\{\tau_\varepsilon > 0\} = \{Z = 0\} \cup \{Z = 1\} = \Omega$, hence

$$\{\tau_\varepsilon > 0\} \in \mathscr{F}_0 = \{\emptyset, \Omega\}.$$

ii) For $0 < t < \varepsilon$ we have $\{\tau_\varepsilon > t\} = \Omega$, hence

$$\{\tau_\varepsilon > t\} \in \mathscr{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

iii) For $t > \varepsilon$ we have $\{\tau_\varepsilon > t\} = \{Z = 0\}$, hence

$$\{\tau_\varepsilon > t\} \in \mathscr{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Therefore τ_ε *is* an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time when $\varepsilon > 0$.

Note that here the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is not right-continuous, as

$$\{\emptyset, \Omega\} = \mathscr{F}_0 \neq \mathscr{F}_{0^+} := \bigcap_{t>0} \mathscr{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Exercise 5.5

a) This intrinsic payoff is $\kappa - S_0$.

b) We note that the process $(Z_t)_{t\in\mathbb{R}_+}$ defined as

$$\begin{aligned} Z_t &:= \left(\frac{S_t}{S_0}\right)^\lambda \mathrm{e}^{-(r-\delta)\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} \\ &= \left(\mathrm{e}^{(r-\delta)t + \sigma\widehat{B}_t - \sigma^2 t/2}\right)^\lambda \mathrm{e}^{-(r-\delta)\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} \\ &= \mathrm{e}^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2}, \qquad t \in \mathbb{R}_+, \end{aligned}$$

is a geometric Brownian motion without drift, hence a martingale, under the risk-neutral probability measure $\mathbb{P}^*$.

c) The parameter λ should satisfy the equation

$$r = (r-\delta)\lambda - \frac{\sigma^2}{2}\lambda(1-\lambda),$$

i.e.

$$\lambda^2\sigma^2/2 + \lambda(r - \delta - \sigma^2/2) - r = 0.$$

This equation admits two solutions

$$\lambda_\pm = \frac{-(r-\delta-\sigma^2/2) \pm \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2},$$

d) We have

$$\begin{aligned} 0 \leqslant Z_t^{(\lambda_-)} &= \left(\frac{S_t}{S_0}\right)^{\lambda_-} \mathrm{e}^{-rt} \\ &\leqslant \left(\frac{S_t}{S_0}\right)^{\lambda_-} \\ &\leqslant \left(\frac{L}{S_0}\right)^{\lambda_-}, \qquad 0 \leqslant t < \tau_L, \end{aligned}$$

since $\lambda_- < 0$ and $S_t > L$ for $t \in [0, \tau_L)$.

e) By the Stopping Time Theorem 4.5 we have

$$\mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1,$$

which rewrites as

$$\mathbb{E}^*\left[\left(\frac{S_{\tau_L}}{S_0}\right)^\lambda \mathrm{e}^{-((r-\delta)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

or, given the relation $S_{\tau_L} = L$,

$$\left(\frac{L}{S_0}\right)^{\lambda} \mathbb{E}^*\left[\mathrm{e}^{-((r-\delta)\lambda-\lambda\sigma^2/2+\lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

i.e.

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \left(\frac{S_0}{L}\right)^{\lambda},$$

provided that we choose λ such that

$$-((r-\delta)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2) = -r, \tag{A.14}$$

i.e.

$$\lambda = \frac{-(r-\delta-\sigma^2/2) \pm \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2},$$

and we choose the negative solution

$$\lambda := \frac{-(r-\delta-\sigma^2/2) - \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}$$

since $S_0/L = x/L > 1$ and the expectation $\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] < 1$ is lower than 1 as $r \geqslant 0$.

f) This follows from (5.8) and the fact that $r > 0$. Using the fact that $S_{\tau_L} = L < K$ when $\tau_L < \infty$, we find

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \,\Big|\, S_0 = x\right] &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-L) \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \,\Big|\, S_0 = x\right].
\end{aligned}$$

Next, noting that $\tau_L = 0$ if $S_0 \leqslant L$, for all $L \in (0, K)$ we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \mid S_0 = x\right] \\
&= \begin{cases} K - x, & 0 < x \leqslant L, \\ \mathbb{E}\left[\mathrm{e}^{-r\tau_L}(K-L)^+ \mid S_0 = x\right], & x \geqslant L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leqslant L, \\ (K-L)\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right], & x \geqslant L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leqslant L, \\ (K-L)\left(\dfrac{x}{L}\right)^{\frac{-(r-a-\sigma^2/2)-\sqrt{(r-a-\sigma^2/2)^2+4r\sigma^2/2}}{\sigma^2}}, & x \geqslant L. \end{cases}
\end{aligned}$$

g) In order to compute L^* we observe that, geometrically, the slope of $x \longmapsto f_L(x) = (K - L)(x/L)^{\lambda_-}$ at $x = L^*$ is equal to -1, *i.e.*

$$f'_{L^*}(L^*) = \lambda_-(K-L^*)\frac{(L^*)^{\lambda_- -1}}{(L^*)^{\lambda_-}} = -1,$$

hence

$$\lambda_-(K-L^*) = L^*, \quad \text{or} \quad L^* = \frac{\lambda_-}{\lambda_- - 1}K < K. \tag{A.15}$$

Equivalently we may recover the value of L^* from the optimality condition

$$\frac{\partial f_L(x)}{\partial L} = -\left(\frac{x}{L}\right)^{\lambda_-} - \lambda_- x(K-L)\left(\frac{x}{L}\right)^{\lambda_-+1} = 0,$$

at $L = L^*$, hence

$$-\left(\frac{x}{L}\right)^{\lambda_-} - \lambda_-(K-L)x^{\lambda_-}L^{-\lambda_--1} = 0,$$

hence

$$L^* = \frac{\lambda_-}{1-\lambda_-}K = \frac{1}{1-1/\lambda_-}K,$$

and

$$\operatorname*{Sup}_{L\in(0,K)} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \mid S_0 = x\right] = -\frac{1}{\lambda_-}\left(\frac{K}{1-1/\lambda_-}\right)^{1-\lambda_-} x^{\lambda_-}.$$

h) For $x \geqslant L$ we have

$$\begin{aligned}
f_{L^*}(x) &= (K-L^*)\left(\frac{x}{L^*}\right)^{\lambda_-} \\
&= \left(K - \frac{\lambda_-}{\lambda_- - 1}K\right)\left(\frac{x}{\frac{\lambda_-}{\lambda_--1}K}\right)^{\lambda_-} \\
&= \left(-\frac{K}{\lambda_- - 1}\right)\left(\frac{x(\lambda_- - 1)}{\lambda_- K}\right)^{\lambda_-} \\
&= \left(-\frac{K}{\lambda_- - 1}\right)\left(\frac{x}{-\lambda_-}\right)^{\lambda_-}\left(\frac{\lambda_- - 1}{-K}\right)^{\lambda_-} \\
&= \left(\frac{x}{-\lambda_-}\right)^{\lambda_-}\left(\frac{\lambda_- - 1}{-K}\right)^{\lambda_- -1} \\
&= \left(\frac{x}{K}\right)^{\lambda_-}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-}\frac{K}{1-\lambda_-}.
\end{aligned} \tag{A.16}$$

i) Let us check that the relation

$$f_{L^*}(x) \geqslant (K-x)^+ \tag{A.17}$$

holds. For all $x \leqslant K$ we have

$$\begin{aligned}
f_{L^*}(x) - (K-x) &= \left(\frac{x}{K}\right)^{\lambda_-}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-}\frac{K}{1-\lambda_-} + x - K \\
&= K\left(\left(\frac{x}{K}\right)^{\lambda_-}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-}\frac{1}{1-\lambda_-} + \frac{x}{K} - 1\right).
\end{aligned}$$

Hence it suffices to take $K = 1$ and to show that for all

$$L^* = \frac{\lambda_-}{\lambda_- - 1} \leqslant x \leqslant 1$$

we have

$$f_{L^*}(x) - (1-x) = \frac{x^{\lambda}_-}{1-\lambda_-}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} + x - 1 \geqslant 0.$$

Equality to 0 holds for $x = \lambda_-/(\lambda_- - 1)$. By differentiation of this relation we get

$$f'_{L^*}(x) - (1-x)' = \lambda_- x^{\lambda_- - 1}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-}\frac{1}{1-\lambda_-} + 1$$

$$\begin{aligned} &= x^{\lambda_- -1}\left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_- -1} + 1 \\ &\geqslant 0, \end{aligned}$$

hence the function $f_{L^*}(x) - (1-x)$ is non-decreasing and the inequality holds throughout the interval $[\lambda_-/(\lambda_- - 1), K]$.

On the other hand, using (A.14) it can be checked by hand that f_{L^*} given by (A.16) satisfies the equality

$$(r-\delta)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = rf_{L^*}(x) \tag{A.18}$$

for $x \geqslant L^* = \dfrac{\lambda_-}{\lambda_- - 1}K$. In case

$$0 \leqslant x \leqslant L^* = \frac{\lambda_-}{\lambda_- - 1}K < K,$$

we have

$$f_{L^*}(x) = K - x = (K-x)^+,$$

hence the relation

$$\left(rf_{L^*}(x) - (r-\delta)xf'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x)\right)(f_{L^*}(x) - (K-x)^+) = 0$$

always holds. On the other hand, in that case we also have

$$(r-\delta)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = -(r-\delta)x,$$

and to conclude we need to show that

$$(r-\delta)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leqslant rf_{L^*}(x) = r(K-x), \tag{A.19}$$

which is true if

$$\delta x \leqslant rK.$$

Indeed, by (A.14) we have

$$\begin{aligned} (r-\delta)\lambda_- &= r + \lambda_-(\lambda_- - 1)\sigma^2/2 \\ &\geqslant r, \end{aligned}$$

hence

$$\delta\frac{\lambda_-}{\lambda_- - 1} \leqslant r,$$

since $\lambda_- < 0$, which yields

$$\delta x \leqslant \delta L^* \leqslant \delta\frac{\lambda_-}{\lambda_- - 1}K \leqslant rK.$$

j) By Itô's formula and the relation

$$dS_t = (r-\delta)S_t dt + \sigma S_t d\widehat{B}_t$$

we have

$$d\big(\widetilde{f}_{L^*}(S_t)\big) = -r\mathrm{e}^{-rt}f_{L^*}(S_t)dt + \mathrm{e}^{-rt}df_{L^*}(S_t)$$

$$
\begin{aligned}
&= -re^{-rt} f_{L^*}(S_t)dt + e^{-rt} f'_{L^*}(S_t)dS_t + \frac{\sigma^2}{2} e^{-rt} S_t^2 f''_{L^*}(S_t) \\
&= e^{-rt}\left(-rf_{L^*}(S_t) + (r-\delta)S_t f'_{L^*}(S_t) + \frac{\sigma^2}{2} S_t^2 f''_{L^*}(S_t)\right)dt \\
&\quad + e^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t,
\end{aligned}
$$

and from Equations (A.18) and (A.19) we have

$$
(r-\delta)x f'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leqslant r f_{L^*}(x),
$$

hence

$$
t \longmapsto e^{-rt} f_{L^*}(S_t)
$$

is a *super*martingale.

k) By the *super*martingale property of

$$
t \longmapsto e^{-rt} f_{L^*}(S_t),
$$

for all stopping times τ we have

$$
f_{L^*}(S_0) \geqslant \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(S_\tau) \mid S_0\right] \geqslant \mathbb{E}^*\left[e^{-r\tau}(K-S_\tau)^+ \mid S_0\right],
$$

by (A.17), hence

$$
f_{L^*}(S_0) \geqslant \operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}^*\left[e^{-r\tau}(K-S_\tau)^+ \mid S_0\right]. \tag{A.20}
$$

l) The stopped process

$$
t \longmapsto e^{-rt\wedge\tau_{L^*}} f_{L^*}(S_{t\wedge\tau_{L^*}})
$$

is a martingale since it has vanishing drift up to time τ_{L^*} by (A.18), and it is constant after time τ_{L^*}, hence by the Stopping Time Theorem 4.5 we find

$$
\begin{aligned}
f_{L^*}(S_0) &= \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(S_{\tau_{L^*}}) \,\Big|\, S_0\right] \\
&= \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(L^*) \,\Big|\, S_0\right] \\
&= \mathbb{E}^*\left[e^{-r\tau}(K-S_{\tau_{L^*}})^+ \,\Big|\, S_0\right] \\
&\leqslant \operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}^*\left[e^{-r\tau}(K-S_\tau)^+ \,\Big|\, S_0\right].
\end{aligned}
$$

m) By combining the above results and conditioning at time t instead of time 0 we deduce that

$$
\begin{aligned}
f_{L^*}(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_{L^*}-t)}(K-S_{\tau_{L^*}})^+ \,\Big|\, S_t\right] \\
&= \begin{cases}
K - S_t, & 0 < S_t \leqslant \dfrac{\lambda_-}{\lambda_- - 1}K, \\[2ex]
\left(\dfrac{\lambda_- - 1}{-K}\right)^{\lambda_- - 1}\left(\dfrac{-S_t}{\lambda_-}\right)^{\lambda_-}, & S_t \geqslant \dfrac{\lambda_-}{\lambda_- - 1}K,
\end{cases}
\end{aligned}
$$

for all $t \in \mathbb{R}_+$, where

$$
\tau_{L^*} = \inf\{u \geqslant t \;:\; S_u \leqslant L\}.
$$

We note that the perpetual put option price does not depend on the value of $t \geqslant 0$.

Exercise 5.6

a) We have

$$Z_t^{(\lambda)} = (S_t)^\lambda \, \mathrm{e}^{-t((r-\delta)\lambda-\lambda(1-\lambda)\sigma^2/2)} = (S_0)^\lambda \, \mathrm{e}^{\lambda\sigma\widehat{B}_t-\lambda^2\sigma^2 t/2},$$

which is a driftless geometric Brownian motion, and therefore a martingale under $\mathbb{P}^*$.

b) The condition is $r = (r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2$, with solutions

$$\lambda_- = \frac{\delta - r + \sigma^2/2 - \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \leqslant 0,$$

$$\lambda_+ = \frac{\delta - r + \sigma^2/2 + \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \geqslant 1.$$

c) Due to the inequality

$$0 \leqslant Z_t^{(\lambda_+)} = (S_t)^{\lambda_+} \, \mathrm{e}^{-rt} \leqslant L^{\lambda_+},$$

which holds because $\lambda_+ > 0$ and $S_t \leqslant L$, $0 \leqslant t < \tau_L$, we note that since $\lim_{t\to\infty} Z_t^{(\lambda_+)} = 0$, we have

$$\begin{aligned} L^{\lambda_+} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] &= \mathbb{E}^*\left[\left(S_{\tau_L}\right)^{\lambda_+} \mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L<\infty\}}\right] = \mathbb{E}^*\left[Z_{\tau_L}^{(\lambda_+)} \mathbb{1}_{\{\tau_L<\infty\}}\right] \\ &= \mathbb{E}^*\left[\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_+)}\right] = \lim_{t\to\infty} \mathbb{E}^*\left[Z_{\tau_L\wedge t}^{(\lambda_+)}\right] \\ &= \mathbb{E}^*\left[Z_0^{(\lambda_+)}\right] = (S_0)^{\lambda_+}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L}-K)^+ \mid S_0 = x\right] &= \mathbb{E}^*\left[(S_{\tau_L}-K)\,\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ &= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ &= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \\ &= (L-K)\left(\frac{x}{L}\right)^{\lambda_+}, \end{aligned}$$

when $S_0 = x > L$. In order to maximize

$$\operatorname*{Sup}_{L\in(0,K)} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \mid S_0 = x\right],$$

we differentiate $L \longmapsto (L-K)\,(x/L)^{\lambda_+}$ with respect to L, to find

$$\left(\frac{x}{L}\right)^{\lambda+} - \lambda_+(L-K)x^{\lambda_+}L^{-\lambda_+-1} = 0,$$

hence

$$L_\delta^* = \frac{\lambda_+}{\lambda_+-1}K = \frac{K}{1-1/\lambda_+},$$

and

$$\operatorname*{Sup}_{L\in(0,K)} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \mid S_0 = x\right] = \frac{1}{\lambda_+}\left(\frac{K}{1-1/\lambda_+}\right)^{1-\lambda_+} x^{\lambda_+}.$$

We note that as $\delta \searrow 0$ we have $\lambda_+ \searrow 1$ and $L_\delta^* \nearrow \infty$, and since

$$\left(\frac{K}{1-1/\lambda_+}\right)^{1-\lambda_+} = \exp\left((\lambda_+-1)\log\frac{\lambda_+-1}{\lambda_+K}\right) \to 1,$$

we find that the perpetual American call option price without dividend ($\delta = 0$) is $S_0 = x$.

Exercise 5.7

a) By the definition (5.36) of $S_1(t)$ and $S_2(t)$ we have

$$\begin{aligned} Z_t &= \mathrm{e}^{-rt} S_2(t) \left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \\ &= \mathrm{e}^{-rt} S_1(t)^\alpha S_2(t)^{1-\alpha} \\ &= S_1(0)^\alpha S_2(0)^{1-\alpha} \, \mathrm{e}^{(\alpha\sigma_1+(1-\alpha)\sigma_2)W_t - \sigma_2^2 t/2}, \end{aligned}$$

which is a martingale when

$$\sigma_2^2 = (\alpha\sigma_1 + (1-\alpha)\sigma_2)^2,$$

i.e.

$$\alpha\sigma_1 + (1-\alpha)\sigma_2 = \pm\sigma_2,$$

which yields either $\alpha = 0$ or

$$\alpha = \frac{2\sigma_2}{\sigma_2 - \sigma_1} > 1,$$

since $0 \leqslant \sigma_1 < \sigma_2$.

b) We have

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\right] &= \mathbb{E}\left[\mathrm{e}^{-r\tau_L}(LS_2(\tau_L) - S_2(\tau_L))^+\right] \\ &= (L-1)^+ \, \mathbb{E}\left[\mathrm{e}^{-r\tau_L} S_2(\tau_L)\right]. \end{aligned} \tag{A.21}$$

c) Since $\tau_L \wedge t$ is a bounded stopping time we can write

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[\mathrm{e}^{-r(\tau_L\wedge t)} S_2(\tau_L \wedge t)\left(\frac{S_1(\tau_L\wedge t)}{S_2(\tau_L\wedge t)}\right)^\alpha\right] \tag{A.22}$$

$$= \mathbb{E}\left[\mathrm{e}^{-r\tau_L} S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha \mathbb{1}_{\{\tau_L<t\}}\right] + \mathbb{E}\left[\mathrm{e}^{-rt} S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}}\right]$$

We have

$$\mathrm{e}^{-rt} S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}} \leqslant \mathrm{e}^{-rt} S_2(t) L^\alpha \mathbb{1}_{\{\tau_L>t\}} \leqslant \mathrm{e}^{-rt} S_2(t) L^\alpha,$$

hence by a uniform integrability argument,

$$\lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{-rt} S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}}\right] = 0,$$

and letting t go to infinity in (A.22) shows that

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[\mathrm{e}^{-r\tau_L} S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha\right] = L^\alpha \, \mathbb{E}\left[\mathrm{e}^{-r\tau_L} S_2(\tau_L)\right],$$

since $S_1(\tau_L)/S_2(\tau_L) = L/L = 1$. The conclusion

$$\mathbb{E}\left[\mathrm{e}^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\right] = (L-1)^+ L^{-\alpha} S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha \tag{A.23}$$

then follows by an application of (A.21).

d) In order to maximize (A.23) as a function of L we consider the derivative

$$\frac{\partial}{\partial L}\frac{L-1}{L^\alpha} = \frac{1}{L^\alpha} - \alpha(L-1)L^{-\alpha-1} = 0,$$

which vanishes for

$$L^* = \frac{\alpha}{\alpha-1},$$

and we substitute L in (A.23) with the value of L^*.

e) In addition to $r = \sigma_2^2/2$ it is sufficient to let $S_1(0) = \kappa$ and $\sigma_1 = 0$ which yields $\alpha = 2$, $L^* = 2$, and we find

$$\operatorname*{Sup}_{\tau \text{ stopping time}} \mathbb{E}\left[\mathrm{e}^{-r\tau}(\kappa - S_2(\tau))^+\right] = \frac{1}{S_2(0)}\left(\frac{\kappa}{2}\right)^2,$$

which coincides with the result of Proposition 5.4.

Exercise 5.8

a) It suffices to check the sign of the quantity

$$(\lambda - 1)(\lambda + 2r/\sigma^2), \tag{A.24}$$

in (5.38), which is positive when $\lambda \in (-\infty, -2r/\sigma^2] \cup [1, \infty)$, and negative when $-2r/\sigma^2 \leqslant \lambda \leqslant 1$.

b) The sign of (A.24) is positive when $\lambda \in (-\infty, 1] \cup [-2r/\sigma^2, \infty)$, and negative when $1 \leqslant \lambda \leqslant -2r/\sigma^2$.

c) By the Stopping Time Theorem 4.5, for any $n \geqslant 0$ we have

$$\begin{aligned} x^\lambda &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L\wedge n)} Z^{(\lambda)}_{\tau_L\wedge n} \mid S_0 = x\right] \\ &= \mathbb{E}^*\left[Z^{(\lambda)}_{\tau_L} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right] + \mathrm{e}^{-rn}\,\mathbb{E}^*\left[Z^{(\lambda)}_n \mathbb{1}_{\{\tau_L > n\}} \mid S_0 = x\right] \\ &\geqslant \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L})^\lambda \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right] \\ &= L^\lambda\, \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right]. \end{aligned}$$

By the results of Questions (a))-(b)), the process $\left(Z^{(\lambda)}_t\right)_{t\in\mathbb{R}_+}$ is a martingale when $\lambda \in \{1, -2r\sigma^2/2\}$. Next, letting n to infinity, by monotone convergence we find

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \leqslant \left(\frac{x}{L}\right)^\lambda \leqslant \begin{cases} \left(\dfrac{x}{L}\right)^{\operatorname{Max}(1,-2r/\sigma^2)}, & x \geqslant L, \\[2ex] \left(\dfrac{x}{L}\right)^{\min(1,-2r/\sigma^2)}, & 0 < x \leqslant L. \end{cases}$$

d) We note that $\mathbb{P}^*(\tau_L < \infty) = 1$ by (4.14), hence if $-\sigma^2/2 \leqslant r < 0$ we have

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ &= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \leqslant \begin{cases} (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L, \\[2ex] (K-L)\dfrac{x}{L}, & 0 < x \leqslant L. \end{cases} \end{aligned}$$

Similarly, if $r \leqslant -\sigma^2/2$ we have

$$\begin{aligned} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] &= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \\ &\leqslant \begin{cases} (K-L)\dfrac{x}{L}, & x \geqslant L, \\[2ex] (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & 0 < x \leqslant L. \end{cases} \end{aligned}$$

e) This follows by noting that $(K-L)(x/L) = (K/L - 1)x$ increases to ∞ when L tends to zero.

f) If $-\sigma^2/2 \leqslant r < 0$ we have

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L} - K)^+ \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ &= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \end{aligned}$$

$$\leqslant \begin{cases} (L-K)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L, \\ \\ (L-K)\dfrac{x}{L}, & 0 < x \leqslant L. \end{cases}$$

If $r \leqslant -\sigma^2/2$ we have

$$\begin{aligned} & \mathbb{E}^*\left[e^{-r\tau_L}(S_{\tau_L}-K)^+\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ = & \ (L-K)\,\mathbb{E}^*\left[e^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0 = x\right] \\ \leqslant & \begin{cases} (L-K)\dfrac{x}{L}, & x \geqslant L, \\ \\ (L-K)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & 0 < x \leqslant L. \end{cases} \end{aligned}$$

g) This follows by noting that for fixed $x > 0$, the quantity $(L-K)x/L = (1-K/L)x$ increases to x when L tends to infinity.

Exercise 5.9 Perpetual American binary options.

a) Similarly, for $x \geqslant K$, immediate exercise is the optimal strategy and we have $C_b^{\mathrm{Am}}(t,x) = 1$. When $x < K$ the optimal exercise level of the perpetual American binary call option is $L^* = K$ with the optimal exercice time τ_K, and by *e.g.* (4.4.22) page 135 we have

$$\begin{aligned} C_b^{\mathrm{Am}}(t,x) &= \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}\mathbb{1}_{\{S_\tau \geqslant K\}} \,\middle|\, S_t = x\right] \\ &= \mathbb{E}^*\left[e^{-(\tau_K-t)r} \,\middle|\, S_t = x\right] \\ &= \frac{x}{K}, \qquad x < K. \end{aligned}$$

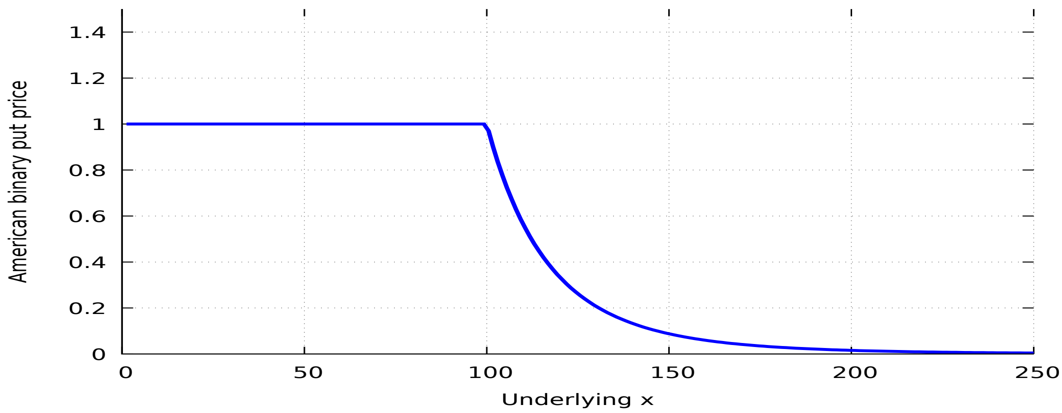


Figure S.15: Perpetual American binary put price map with $K = 100$.

b) For $x \leqslant K$, immediate exercise is the optimal strategy and we have $P_b^{\mathrm{Am}}(t,x) = 1$. When $x > K$ the optimal exercise level of the perpetual American binary put option is $L^* = K$ with the optimal exercice time τ_K, and by *e.g.* (4.4.11) page 125 we have

$$\begin{aligned} P_b^{\mathrm{Am}}(t,x) &= \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-(\tau-t)r}\mathbb{1}_{\{S_\tau \leqslant K\}} \,\middle|\, S_t = x\right] \\ &= \mathbb{E}^*\left[e^{-(\tau_K-t)r} \,\middle|\, S_t = x\right] \\ &= \left(\frac{x}{K}\right)^{-2r/\sigma^2}, \qquad x > K. \end{aligned}$$

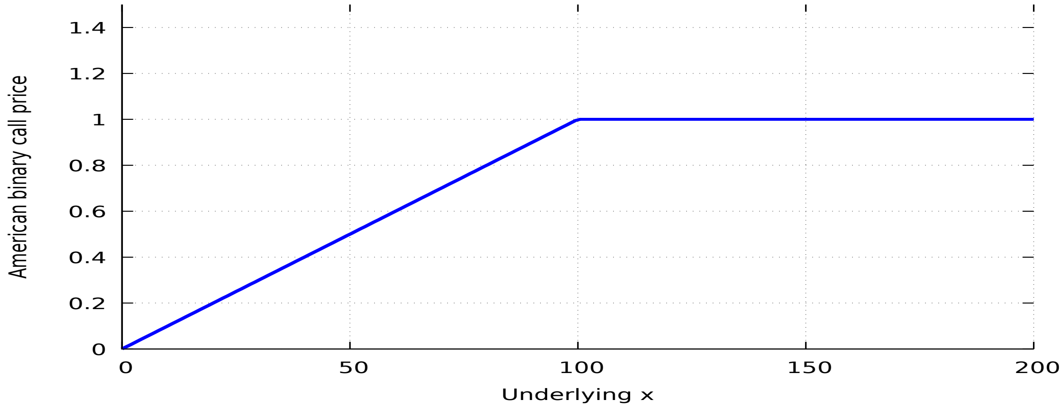


Figure S.16: Perpetual American binary call price map with $K = 100$.

Exercise 5.10 Finite expiration American binary options.

a) The optimal strategy is as follows:
   (i) if $S_t \geqslant K$, then exercise immediately.
   (ii) if $S_t < K$, then wait.

b) The optimal strategy is as follows:
   (i) if $S_t > K$, then wait.
   (ii) if $S_t \leqslant K$, exercise immediately.

c) Based on the answers to Question (a)) we set

$$C_d^{\mathrm{Am}}(t,T,K) = 1, \qquad 0 \leqslant t < T,$$

and

$$C_d^{\mathrm{Am}}(T,T,x) = 0, \qquad 0 \leqslant x < K.$$

d) Based on the answers to Question (b)), we set

$$P_d^{\mathrm{Am}}(t,T,K) = 1, \qquad 0 \leqslant t < T,$$

and

$$P_d^{\mathrm{Am}}(T,T,x) = 0, \qquad x > K.$$

e) Starting from $S_t \leqslant K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $\mathrm{e}^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}}$.

f) The first hitting time τ_a of the level a by a μ-drifted Brownian motion $(W_u + \mu u)_{u \in \mathbb{R}_+}$ satisfies

$$\mathbb{P}(\tau_a \leqslant u) = \Phi\left(\frac{a - \mu u}{\sqrt{u}}\right) - \mathrm{e}^{2\mu a} \Phi\left(\frac{-a - \mu u}{\sqrt{u}}\right), \qquad u > 0,$$

and by differentiation with respect to u this yields the probability density function

$$f_{\tau_a}(u) = \frac{\partial}{\partial u} \mathbb{P}(\tau_a \leqslant u) = \frac{a}{\sqrt{2\pi u^3}} \mathrm{e}^{-(a-\mu u)^2/(2u)} \mathbb{1}_{[0,\infty)}(u)$$

of the first hitting time of level a by Brownian motion with drift μ. Given the relation

$$S_u = S_t \, \mathrm{e}^{\sigma W_{u-t} - (u-t)\sigma^2/2 + (u-t)r} = S_t \, \mathrm{e}^{(W_{u-t} + (u-t)\mu)\sigma}, \quad u \geqslant t,$$

with $\mu = r/\sigma - \sigma/2$, we find that $(S_u)_{u\in[t,\infty)}$ hits the level K at a time $\tau_K = t + \tau_a$, such that

$$S_{\tau_K} = S_t \, \mathrm{e}^{\sigma W_{\tau_a} - \sigma^2 \tau_a/2 + r\tau_a} = S_t \, \mathrm{e}^{(W_{\tau_a} + \mu\tau_a)\sigma} = K,$$

i.e.

$$a = W_{\tau_a} + \mu\tau_a = \frac{1}{\sigma} \log \frac{K}{S_t}.$$

Therefore, the probability density function of the first hitting time τ_K of level K after time t by $(S_u)_{u\in[t,\infty)}$ is given by

$$s \longmapsto \frac{a}{\sqrt{2\pi(s-t)^3}} \mathrm{e}^{-(a-(s-t)\mu)^2/(2(s-t))}, \qquad s > t,$$

with

$$\mu := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) \quad \text{and} \quad a := \frac{1}{\sigma}\log\frac{K}{x},$$

given that $S_t = x$. Hence, for $x \in (0, K)$ we have

$$\begin{aligned}
C_d^{\mathrm{Am}}(t,T,x) &= \mathbb{E}\left[\mathrm{e}^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}} \mid S_t = x\right] \\
&= \int_t^T \mathrm{e}^{-(s-t)r} \frac{a}{\sqrt{2\pi(s-t)^3}} \mathrm{e}^{-(a-(s-t)\mu)^2/(2(s-t))} ds \\
&= \int_0^{T-t} \mathrm{e}^{-rs} \frac{a}{\sqrt{2\pi s^3}} \mathrm{e}^{-(a-\mu s)^2/(2s)} ds \\
&= \int_0^{T-t} \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}} \exp\left(-rs - \frac{1}{2\sigma^2 s}\left(-\left(r - \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right) ds \\
&= \left(\frac{K}{x}\right)^{(r/\sigma^2 - 1/2) \pm (r/\sigma^2 + 1/2)} \\
&\quad \times \int_0^{T-t} \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}} \exp\left(-\frac{1}{2\sigma^2 s}\left(\pm\left(r + \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right) ds \\
&= \frac{1}{\sqrt{2\pi}} \frac{x}{K} \int_{y_-}^{\infty} \mathrm{e}^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \left(\frac{K}{x}\right)^{2r/\sigma^2} \int_{y_+}^{\infty} \mathrm{e}^{-y^2/2} dy \\
&= \frac{x}{K} \Phi\left(\frac{(r + \sigma^2/2)(T-t) + \log(x/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi\left(\frac{-(r + \sigma^2/2)(T-t) + \log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad 0 < x < K,
\end{aligned}$$

where

$$y_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\pm\left(r + \frac{\sigma^2}{2}\right)(T-t) + \log\frac{K}{x}\right),$$

and we used the decomposition

$$\log\frac{K}{x} = \frac{1}{2}\left(\left(r + \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right) + \frac{1}{2}\left(-\left(r + \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right).$$

We check that

$$C_d^{\mathrm{Am}}(T,T,K) = \Phi(0) + \Phi(0) = 1,$$

and

$$C_d^{\mathrm{Am}}(T,T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \qquad x < K,$$

since $t = T$, which is consistent with the answers to Question (c)).

In addition, as T tends to infinity we have

$$\begin{aligned}
\lim_{T\to\infty} C_d^{\mathrm{Am}}(t,T,x) &= \frac{x}{K}\lim_{T\to\infty}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)\\
&\quad +\left(\frac{x}{K}\right)^{-2r/\sigma^2}\lim_{T\to\infty}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)\\
&= \frac{x}{K}, \qquad 0<x<K,
\end{aligned}$$

which is consistent with the answer to Question (a)) of Exercise 5.9.

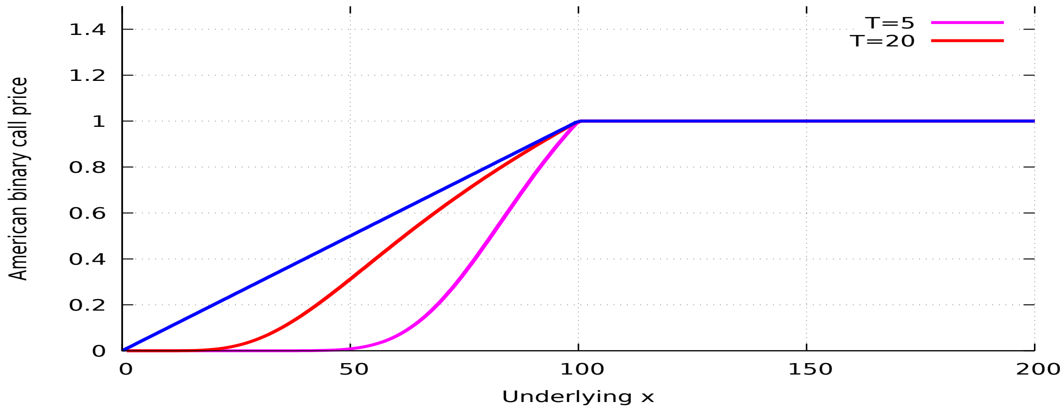


Figure S.17: Finite expiration American binary call price map with $K = 100$.

g) Starting from $S_t \geqslant K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $\mathrm{e}^{-r(\tau_K-t)}\mathbb{1}_{\{\tau_K<T\}}$.

h) Using the notation and answer to Question (f)), for $x > K$ we find

$$\begin{aligned}
P_d^{\mathrm{Am}}(t,T,x) &= \mathbb{E}\left[\mathrm{e}^{-r(\tau_K-t)}\mathbb{1}_{\{\tau_K<T\}} \mid S_t = x\right]\\
&= \int_0^{T-t}\mathrm{e}^{-rs}\frac{a}{\sqrt{2\pi s^3}}\mathrm{e}^{-(a-\mu s)^2/(2s)}ds\\
&= \int_0^{T-t}\frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-rs-\frac{1}{2\sigma^2 s}\left(\left(r-\frac{\sigma^2}{2}\right)s+\log\frac{x}{K}\right)^2\right)ds\\
&= \left(\frac{K}{x}\right)^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\pm\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)}\\
&\quad\times\int_0^{T-t}\frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-\frac{1}{2\sigma^2 s}\left(\mp\left(r+\frac{\sigma^2}{2}\right)s+\log\frac{x}{K}\right)^2\right)ds\\
&= \frac{1}{\sqrt{2\pi}}\frac{x}{K}\int_{y_-}^{\infty}\mathrm{e}^{-y^2/2}dy+\frac{1}{\sqrt{2\pi}}\left(\frac{x}{K}\right)^{2r/\sigma^2}\int_{y_+}^{\infty}\mathrm{e}^{-y^2/2}dy\\
&= \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)\\
&\quad+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad x>K,
\end{aligned}$$

with

$$y_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\mp\left(r+\frac{\sigma^2}{2}\right)(T-t)+\log\frac{x}{K}\right).$$

We check that

$$P_d^{\mathrm{Am}}(T,T,K) = \Phi(0) + \Phi(0) = 1,$$

and

$$P_d^{\mathrm{Am}}(T,T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \qquad 0 < x < K,$$

since $t = T$, which is consistent with the answers to Question (c)).

In addition, as T tends to infinity we have

$$\begin{aligned}\lim_{T\to\infty} P_d^{\mathrm{Am}}(t,T,x) &= \frac{x}{K}\lim_{T\to\infty}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\lim_{T\to\infty}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &= \left(\frac{x}{K}\right)^{-2r/\sigma^2}, \qquad x > K,\end{aligned}$$

which is consistent with the answer to Question (b)) of Exercise 5.9

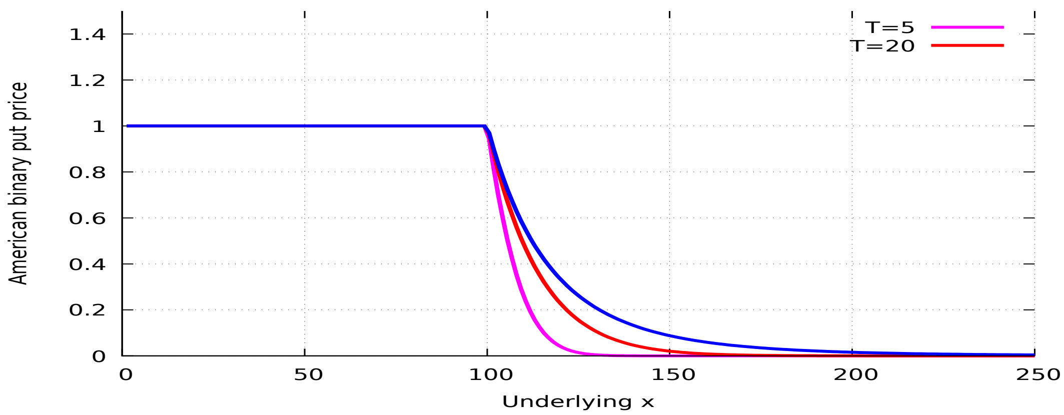


Figure S.18: Finite expiration American binary put price map with $K = 100$.

i) The call-put parity does not hold for American binary options since for $x \in (0,K)$ we have

$$\begin{aligned}C_d^{\mathrm{Am}}(t,T,x) + P_d^{\mathrm{Am}}(t,T,x) &= 1 + \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right),\end{aligned}$$

while for $x > K$ we find

$$\begin{aligned}C_d^{\mathrm{Am}}(t,T,x) + P_d^{\mathrm{Am}}(t,T,x) &= 1 + \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

Exercise 5.11 American forward Contracts.

a) For all (bounded) stopping times $\tau \in [t,T]$, since the discounted asset price process $(\widetilde{S}_u)_{u\in[t,\infty)} := (\mathrm{e}^{-(u-t)r}S_u)_{u\in[t,\infty)}$ is a martingale and the stopped process $(\widetilde{S}_{\tau\wedge u})_{u\in[t,\infty)} = (\mathrm{e}^{-(\tau\wedge u-t)r}S_{\tau\wedge u})_{u\in[t,\infty)}$ is also a martingale by the Stopping Time Theorem 4.5, we have

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau)\;\middle|\;\mathscr{F}_t\right] &= K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathscr{F}_t\right] - \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau\;\middle|\;\mathscr{F}_t\right]\\ &= K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathscr{F}_t\right] - \mathbb{E}^*\left[\widetilde{S}_{\tau\wedge T}\;\middle|\;\mathscr{F}_t\right]\end{aligned}$$

$$
\begin{aligned}
&= K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] - \widetilde{S}_{\tau\wedge t} \\
&= K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] - \widetilde{S}_t \\
&= K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] - S_t,
\end{aligned}
$$

and the above quantity is clearly maximized by taking $\tau = t$. Hence we have

$$
f(t,S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau) \mid \mathscr{F}_t\right] = K - S_t,
$$

and the optimal strategy is to exercise immediately (or avoiding to buy the option) at time t.

b) Similarly to the above, we have

$$
\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau \mid \mathscr{F}_t\right] - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] \\
&= S_t - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right],
\end{aligned}
$$

since $\tau \in [t,T]$ is bounded and $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale. As the above quantity is clearly maximized by taking $\tau = T$, we have

$$
f(t,S_t) = \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] = S_t - \mathrm{e}^{-(T-t)r}K,
$$

and the optimal strategy is to wait until the maturity time T in order to exercise.

c) Regarding the perpetual American long forward contract, since the discounted asset price process $(\widetilde{S}_u)_{u\in[t,\infty)} := (\mathrm{e}^{-(u-t)r}S_u)_{u\in[t,\infty)}$ is a martingale, by the Stopping Time Theorem 4.5, for all stopping times $\tau \geqslant t$ we have*,

$$
\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] = \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau \mid \mathscr{F}_t\right] - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\widetilde{S}_\tau \mid \mathscr{F}_t\right] - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] \\
&= \widetilde{S}_t - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] \\
&= S_t - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)} \mid \mathscr{F}_t\right] \\
&\leqslant S_t, \qquad t \geqslant 0.
\end{aligned}
$$

On the other hand, for all fixed $T > 0$ we have

$$
\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r(T-t)}(S_T-K) \mid \mathscr{F}_t\right] &= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[S_T \mid \mathscr{F}_t] - \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[K \mid \mathscr{F}_t] \\
&= S_t - \mathrm{e}^{-r(T-t)}K, \qquad t \in [0,T],
\end{aligned}
$$

hence

$$
(S_t - \mathrm{e}^{-r(T-t)}K) \leqslant \operatorname*{Sup}_{\substack{t\leqslant\tau\leqslant T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] \leqslant S_t, \quad T \geqslant t,
$$

and letting $T \to \infty$ we get

$$
\begin{aligned}
S_t &= \lim_{T\to\infty}(S_t - \mathrm{e}^{-r(T-t)}K) \\
&\leqslant \operatorname*{Sup}_{\substack{\tau\geqslant t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] \\
&\leqslant S_t,
\end{aligned}
$$

hence we have

$$
f(t,S_t) = \operatorname*{Sup}_{\substack{\tau\geqslant t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K) \mid \mathscr{F}_t\right] = S_t,
$$

*using Fatou's Lemma.

and the optimal strategy $\tau^* = +\infty$ is to wait indefinitely.

Regarding the perpetual American short forward contract, we have

$$\begin{aligned} f(t,S_t) &= \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau) \mid \mathscr{F}_t \right] \\ &\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau)^+ \mid \mathscr{F}_t \right] \\ &= f_{L^*}(S_t), \end{aligned} \tag{A.25}$$

with

$$L^* = \frac{2r}{2r+\sigma^2}K < K$$

as defined in (5.12). On the other hand, for $\tau = \tau_{L^*}$ we have

$$(K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)^*$$

since $0 < L^* = 2Kr/(2r+\sigma^2) < K$, hence

$$\begin{aligned} f_{L^*}(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_{L^*}-t)}(K-S_{\tau_{L^*}})^+ \mid \mathscr{F}_t \right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_{L^*}-t)}(K-S_{\tau_{L^*}}) \mid \mathscr{F}_t \right] \\ &\leqslant \operatorname*{Sup}_{\substack{\tau \geqslant t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau) \mid \mathscr{F}_t \right] \\ &= f(t,S_t), \end{aligned}$$

which, together with (A.25), shows that

$$f(t,S_t) = f_{L^*}(S_t),$$

i.e. the perpetual American short forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 5.12

a) We have

$$\begin{aligned} Y_t &= \mathrm{e}^{-rt}\left(S_0\,\mathrm{e}^{rt+\sigma\widehat{B}_t-\sigma^2t/2}\right)^{-2r/\sigma^2} \\ &= S_0^{-2r/\sigma^2}\,\mathrm{e}^{-rt-2r^2t/\sigma^2+2r\widehat{B}_t/\sigma+rt} \\ &= S_0^{-2r/\sigma^2}\,\mathrm{e}^{2r\widehat{B}_t/\sigma-(2r/\sigma)^2t/2}, \qquad t \in \mathbb{R}_+, \end{aligned}$$

and

$$Z_t = \mathrm{e}^{-rt}S_t = S_0\,\mathrm{e}^{\sigma\widehat{B}_t-\sigma^2t/2}, \qquad t \in \mathbb{R}_+,$$

which are both martingales under $\mathbb{P}^*$ because they are standard geometric Brownian motions with respective volatilities σ and $2r/\sigma$.

b) Since $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ are both martingales and τ_L is a stopping time, we have

$$\begin{aligned} S_0^{-2r/\sigma^2} &= \mathbb{E}^*[Y_0] \\ &= \mathbb{E}^*[Y_{\tau_L}] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}S_{\tau_L}^{-2r/\sigma^2}\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}L^{-2r/\sigma^2}\right] \\ &= L^{-2r/\sigma^2}\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right], \end{aligned}$$

hence

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}$$

if $S_0 = x \geqslant L$ (note that in this case $Y_{\tau_L \wedge t}$ remains bounded by L^{-2r/σ^2}), and

$$S_0 = \mathbb{E}^*[Z_0] = \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} S_{\tau_L}\right] = \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} L\right] = L\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right],$$

hence

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \frac{x}{L}$$

if $S_0 = x \leqslant L$. Note that in this case $Z_{\tau_L \wedge t}$ remains bounded by L.

c) We find

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L}) \mid S_0 = x\right] &= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \\
&= \begin{cases} x\dfrac{K-L}{L}, & 0 < x \leqslant L, \\[2ex] (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L. \end{cases}
\end{aligned} \tag{A.26}$$

d) By differentiating

$$\begin{aligned}
&\frac{\partial}{\partial L}\mathbb{E}\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L}) \mid S_0 = x\right] \\
&= \begin{cases} \left(\dfrac{x}{L}\right)^{-2r/\sigma^2}\left(\dfrac{2r}{\sigma^2}\left(\dfrac{K}{L}-1\right)-1\right), & 0 < L < x, \\[2ex] -\dfrac{Kx}{L^2}, & L > x, \end{cases}
\end{aligned}$$

and check that the minimum occurs for $L^* = x$.

e) The value $L^* = x$ shows that the optimal strategy for the American *finite expiration* short forward contract is to exercize immediately starting from $S_0 = x$, which is consistent with the result of Exercise 5.11-(a)), since given any stopping time τ upper bounded by T we have

$$\mathbb{E}\left[\mathrm{e}^{-r\tau}(K - S_\tau)\right] = K\,\mathbb{E}\left[\mathrm{e}^{-r\tau}\right] - \mathbb{E}\left[\mathrm{e}^{-r\tau}S_\tau\right] = K\,\mathbb{E}\left[\mathrm{e}^{-r\tau}\right] - S_0 \leqslant K - S_0.$$

Exercise 5.13

a) The option payoff equals $(\kappa - S_t)^p$ if $S_t \leqslant L$.

b) We have

$$\begin{aligned}
f_L(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}((\kappa - S_{\tau_L})^+)^p \mid \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}((\kappa - L)^+)^p \mid \mathscr{F}_t\right] \\
&= (\kappa - L)^p\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)} \mid \mathscr{F}_t\right].
\end{aligned}$$

c) We have

$$\begin{aligned}
f_L(x) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}(\kappa - S_{\tau_L})^+ \mid \mathscr{F}_t = x\right] \\
&= \begin{cases} (\kappa - x)^p, & 0 < x \leqslant L, \\[2ex] (\kappa - L)^p\left(\dfrac{L}{x}\right)^{2r/\sigma^2}, & x \geqslant L. \end{cases}
\end{aligned} \tag{A.27}$$

d) By the differentiation $\dfrac{d}{dx}(\kappa - x)^p = -p(\kappa - x)^{p-1}$ we find

$$\frac{\partial f_L(x)}{\partial L} = \frac{2r}{\sigma^2 L}(\kappa - L)^p\left(\frac{L}{x}\right)^{2r/\sigma^2} - p(\kappa - L)^{p-1}\left(\frac{L}{x}\right)^{2r/\sigma^2},$$

hence the condition $\dfrac{\partial f'_{L^*}(x)}{\partial L}_{|x=L^*} = 0$ reads

$$\frac{2r}{\sigma^2 L^*}(\kappa - L^*) - p = 0, \quad \text{or} \quad L^* = \frac{2r}{2r + p\sigma^2}\kappa < \kappa.$$

e) By (A.27) the price can be computed as

$$f(t, S_t) = f_{L^*}(S_t) = \begin{cases} (\kappa - S_t)^p, & 0 < S_t \leqslant L^*, \\ \\ \left(\dfrac{p\sigma^2\kappa}{2r + p\sigma^2}\right)^p \left(\dfrac{2r + p\sigma^2}{2r}\dfrac{S_t}{\kappa}\right)^{-2r/\sigma^2}, & S_t \geqslant L^*, \end{cases}$$

using (4.12) as in the proof of Proposition 5.4, since the process

$$u \longmapsto \mathrm{e}^{-ru} f_{L^*}(S_u), \qquad u \geqslant t,$$

is a nonnegative *super*martingale.

Exercise 5.14

a) The option payoff is $\kappa - (S_t)^p$.

b) We have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p) \,\Big|\, \mathscr{F}_t\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}(\kappa - L^p) \,\Big|\, \mathscr{F}_t\right] \\ &= (\kappa - L^p)\mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)} \,\Big|\, \mathscr{F}_t\right]. \end{aligned}$$

c) We have

$$\begin{aligned} f_L(x) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p) \,\Big|\, S_t = x\right] \\ &= \begin{cases} \kappa - x^p, & 0 < x \leqslant L, \\ \\ (\kappa - L^p)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geqslant L. \end{cases} \end{aligned}$$

d) We have

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(\kappa - (L^*)^p)\frac{(L^*)^{-2r/\sigma^2 - 1}}{(L^*)^{-2r/\sigma^2}} = -p(L^*)^{p-1},$$

i.e.

$$\frac{2r}{\sigma^2}(\kappa - (L^*)^p) = p(L^*)^p,$$

or

$$L^* = \left(\frac{2r\kappa}{2r + p\sigma^2}\right)^{1/p} < (\kappa)^{1/p}. \tag{A.28}$$

<u>Remark</u>: We may also compute L^* by maximizing $L \longmapsto f_L(x)$ for all fixed x. The derivative $\partial f_L(x)/\partial L$ can be computed as

$$\begin{aligned} \frac{\partial f_L(x)}{\partial L} &= \frac{\partial}{\partial L}\left((\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2}\right) \\ &= -pL^{p-1}\left(\frac{L}{x}\right)^{2r/\sigma^2} + \frac{2r}{\sigma^2}L^{-1}(\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2}, \end{aligned}$$

and equating $\partial f_L(x)/\partial L$ to 0 at $L = L^*$ yields

$$-p(L^*)^{p-1} + \frac{2r}{\sigma^2}(L^*)^{-1}(\kappa - (L^*)^p) = 0,$$

which recovers (A.28).

e) We have

$$\begin{aligned} f_{L^*}(S_t) &= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leqslant L^*, \\ \\ (\kappa - (L^*)^p)\dfrac{(S_t)^{-2r/\sigma^2}}{(L^*)^{-2r/\sigma^2}}, & S_t \geqslant L^* \end{cases} \\ &= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leqslant L^*, \\ \\ \dfrac{\sigma^2}{2r} p (S_t)^{-2r/\sigma^2}(L^*)^{p+2r/\sigma^2}, & S_t \geqslant L^*, \end{cases} \\ &= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leqslant L^*, \\ \\ \dfrac{p\sigma^2\kappa}{2r+p\sigma^2}\left(\dfrac{2r+p\sigma^2}{2r}\dfrac{S_t^p}{\kappa}\right)^{-2r/(p\sigma^2)} < \kappa, & S_t \geqslant L^*, \end{cases} \end{aligned}$$

however we cannot conclude as in Exercise 5.13-(e)) since the process

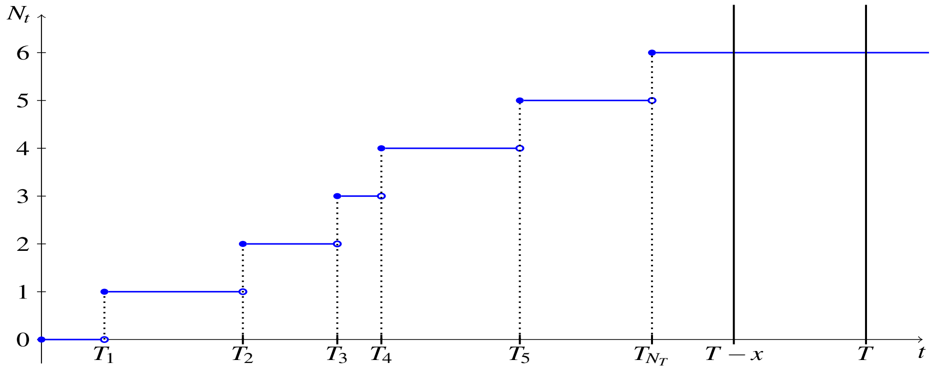
$$u \longmapsto \mathrm{e}^{-ru} f_{L^*}(S_u), \qquad u \geqslant t,$$

does not remain nonnegative when $p > 1$, so that (4.12) cannot be applied as in the proof of Proposition 5.4.

Chapter 6

Exercise 6.1 For any $x \in [0, T]$, we have

$$\begin{aligned} \mathbb{P}(T - T_{N_T} > x \mid N_T \geqslant 1) &= \frac{\mathbb{P}(T - T_{N_T} > x \text{ and } N_T \geqslant 1)}{\mathbb{P}(N_T \geqslant 1)} \\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0 \text{ and } N_T \geqslant 1)}{\mathbb{P}(N_T \geqslant 1)} \\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0 \text{ and } N_{T-x} \geqslant 1)}{\mathbb{P}(N_T \geqslant 1)} \\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0)\mathbb{P}(N_{T-x} \geqslant 1)}{\mathbb{P}(N_T \geqslant 1)} \\ &= \frac{\mathrm{e}^{-(T-(T-x))\lambda}(1 - \mathrm{e}^{-(T-x)\lambda})}{1 - \mathrm{e}^{-\lambda T}} \\ &= \frac{\mathrm{e}^{-(T-(T-x))\lambda} - \mathrm{e}^{-\lambda T}}{1 - \mathrm{e}^{-\lambda T}} \\ &= \frac{\mathrm{e}^{-\lambda x} - \mathrm{e}^{-\lambda T}}{1 - \mathrm{e}^{-\lambda T}}, \qquad 0 \leqslant t \leqslant T. \end{aligned}$$



We note that

$$\mathbb{P}(T - T_{N_T} > 0 \mid N_T \geqslant 1) = 1 \text{ and } \mathbb{P}(T - T_{N_T} > T \mid N_T \geqslant 1) = 0.$$

Exercise 6.2

a) When $t \in [0, T_1)$, the equation reads

$$dS_t = -\eta\lambda S_{t^-} dt = -\eta\lambda S_t dt,$$

which is solved as $S_t = S_0 \, \mathrm{e}^{-\eta\lambda t}$, $0 \leqslant t < T_1$. Next, at the first jump time $\underline{t = T_1}$ we have

$$\Delta S_t := S_t - S_{t^-} = \eta S_{t^-} dN_t = \eta S_{t^-},$$

which yields $S_t = (1+\eta)S_{t^-}$, hence $S_{T_1} = (1+\eta)S_{T_1^-} = S_0(1+\eta)\,\mathrm{e}^{-\eta\lambda T_1}$. Repeating this procedure over the N_t jump times contained in the interval $[0,t]$ we get

$$S_t = S_0(1+\eta)^{N_t}\,\mathrm{e}^{-\lambda\eta t}, \qquad t \in \mathbb{R}_+.$$

b) When $t \in [0, T_1)$ the equation reads

$$dS_t = -\eta\lambda S_{t^-} dt = -\eta\lambda S_t dt,$$

which is solved as $S_t = S_0 \, \mathrm{e}^{-\eta\lambda t}$, $0 \leqslant t < T_1$. Next, at the first jump time $\underline{t = T_1}$ we have

$$dS_t = S_t - S_{t^-} = dN_t = 1,$$

which yields $S_t = 1 + S_{t^-}$, hence $S_{T_1} = 1 + S_{T_1^-} = 1 + S_0\,\mathrm{e}^{-\eta\lambda T_1}$, and for $t \in [T_1, T_2)$ we will find

$$S_t = (1 + S_0\,\mathrm{e}^{-\eta\lambda T_1})\,\mathrm{e}^{-(t-T_1)\eta\lambda}, \qquad t \in [T_1, T_2).$$

More generally, the equation can be solved by letting $Y_t := \mathrm{e}^{\eta\lambda t} S_t$ and noting that $(Y_t)_{t\in\mathbb{R}_+}$ satisfies $dY_t = \mathrm{e}^{\lambda\eta t} dN_t$, which has the solution

$$Y_t = Y_0 + \int_0^t \mathrm{e}^{\eta\lambda s} dN_s, \qquad t \in \mathbb{R}_+,$$

hence in general we have

$$S_t = \mathrm{e}^{-\eta\lambda t} S_0 + \int_0^t \mathrm{e}^{-(t-s)\eta\lambda} dN_s, \qquad t \in \mathbb{R}_+,$$

Exercise 6.3

a) We have

$$X_t = \begin{cases} X_0 \, \mathrm{e}^{\alpha t}, & 0 \leqslant t < T_1, \\ \left(X_0 \, \mathrm{e}^{\alpha T_1} + \sigma \right) \mathrm{e}^{(t-T_1)\alpha} = X_0 \, \mathrm{e}^{\alpha t} + \sigma \, \mathrm{e}^{(t-T_1)\alpha}, & T_1 \leqslant t < T_2, \\ \left(\left(X_0 \, \mathrm{e}^{\alpha T_1} + \sigma \right) \mathrm{e}^{(T_2-T_1)\alpha} + \sigma \right) \mathrm{e}^{(t-T_2)\alpha} & \\ \qquad = X_0 \, \mathrm{e}^{\alpha t} + \sigma \, \mathrm{e}^{(t-T_1)\alpha} + \sigma \, \mathrm{e}^{(t-T_2)\alpha}, & T_2 \leqslant t < T_3, \end{cases}$$

and more generally the solution $(X_t)_{t \in \mathbb{R}_+}$ can be written as

$$X_t = X_0 \, \mathrm{e}^{\alpha t} + \sigma \sum_{k=1}^{N_t} \mathrm{e}^{(t-T_k)\alpha} = X_0 \, \mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dN_s, \qquad t \in \mathbb{R}_+. \tag{A.29}$$

b) Letting $f(t) := \mathbb{E}[X_t]$ and taking expectation on both sides of the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ we find

$$df(t) = \alpha f(t) dt + \sigma \lambda dt,$$

or

$$f'(t) = \alpha f(t) + \sigma \lambda.$$

Letting $g(t) = f(t) \, \mathrm{e}^{-\alpha t}$ we check that

$$g'(t) = \sigma \lambda \, \mathrm{e}^{-\alpha t},$$

hence

$$g(t) = g(0) + \int_0^t g'(s) ds = g(0) + \sigma \lambda \int_0^t \mathrm{e}^{-\alpha s} ds = f(0) + \sigma \frac{\lambda}{\alpha} (1 - \mathrm{e}^{-\alpha t}),$$

and

$$\begin{aligned} f(t) &= \mathbb{E}[X_t] \\ &= g(t) \, \mathrm{e}^{\alpha t} \\ &= f(0) \, \mathrm{e}^{\alpha t} + \sigma \frac{\lambda}{\alpha} (\mathrm{e}^{\alpha t} - 1) \\ &= X_0 \, \mathrm{e}^{\alpha t} + \sigma \frac{\lambda}{\alpha} (\mathrm{e}^{\alpha t} - 1), \qquad t \in \mathbb{R}_+. \end{aligned}$$

We could also take the expectation on both sides of (A.29) and directly find

$$f(t) = \mathbb{E}[X_t] = X_0 \, \mathrm{e}^{\alpha t} + \sigma \lambda \int_0^t \mathrm{e}^{(t-s)\alpha} ds = X_0 \, \mathrm{e}^{\alpha t} + \sigma \frac{\lambda}{\alpha} (\mathrm{e}^{\alpha t} - 1), \qquad t \in \mathbb{R}_+.$$

Exercise 6.4

a) We have $X_t = X_0 \prod_{k=1}^{N_t} (1 + \sigma) = X_0 (1 + \sigma)^{N_t} = (1 + \sigma)^{N_t}$, $t \in \mathbb{R}_+$.

b) By stochastic calculus and using the relation $dX_t = \sigma X_{t^-} dN_t$, we have

$$\begin{aligned} dS_t &= d\left(S_0 X_t + r X_t \int_0^t X_s^{-1} ds \right) = S_0 dX_t + r d\left(X_t \int_0^t X_s^{-1} ds \right) \\ &= S_0 dX_t + r X_t d\left(\int_0^t X_s^{-1} ds \right) + r \left(\int_0^t X_s^{-1} ds \right) dX_t + r dX_t \bullet d\left(\int_0^t X_s^{-1} ds \right) \\ &= S_0 dX_t + r X_t X_t^{-1} dt + r \left(\int_0^t X_s^{-1} ds \right) dX_t + r dX_t \bullet (X_t^{-1} dt) \\ &= S_0 dX_t + r dt + r \left(\int_0^t X_s^{-1} ds \right) dX_t = r dt + \left(S_0 + r \int_0^t X_s^{-1} ds \right) dX_t \\ &= r dt + \sigma \left(S_0 X_{t^-} + r X_{t^-} \int_0^t X_s^{-1} ds \right) dN_t = r dt + \sigma S_{t^-} dN_t. \end{aligned}$$

c) We have

$$\begin{aligned}\mathbb{E}[X_t/X_s] &= \mathbb{E}[(1+\sigma)^{N_t-N_s}]\\ &= \sum_{k\geqslant 0}(1+\sigma)^k\mathbb{P}(N_t-N_s=k)\\ &= \mathrm{e}^{-(t-s)\lambda}\sum_{k\geqslant 0}(1+\sigma)^k\frac{((t-s)\lambda)^k}{k!} = \mathrm{e}^{-(t-s)\lambda}\sum_{k\geqslant 0}\frac{((t-s)(1+\sigma)\lambda)^k}{k!}\\ &= \mathrm{e}^{-(t-s)\lambda}\mathrm{e}^{(t-s)(1+\sigma)\lambda} = \mathrm{e}^{(t-s)\lambda\sigma}, \qquad 0\leqslant s\leqslant t.\end{aligned}$$

Remarks: We could also let $f(t)=\mathbb{E}[X_t]$ and take expectation in the equation $dX_t=\sigma X_{t^-}dN_t$ to get $f'(t)=\sigma\lambda f(t)dt$ and $f(t)=\mathbb{E}[X_t]=f(0)\mathrm{e}^{\lambda\sigma t}=\mathrm{e}^{\lambda\sigma t}$. Note that the relation $\mathbb{E}[X_t/X_s]=\mathbb{E}[X_t]/\mathbb{E}[X_s]$, which happens to be true here, is *wrong* in general.

d) We have

$$\begin{aligned}\mathbb{E}[S_t] &= \mathbb{E}\left[S_0X_t+rX_t\int_0^t X_s^{-1}ds\right] = S_0\mathbb{E}[X_t]+r\int_0^t\mathbb{E}[X_t/X_s]ds\\ &= S_0\mathrm{e}^{\lambda\sigma t}+r\int_0^t\mathrm{e}^{(t-s)\lambda\sigma}ds = S_0\mathrm{e}^{\lambda\sigma t}+r\int_0^t\mathrm{e}^{\lambda\sigma s}ds\\ &= S_0\mathrm{e}^{\lambda\sigma t}+\frac{(\mathrm{e}^{\lambda\sigma t}-1)r}{\lambda\sigma}, \qquad t\in\mathbb{R}_+.\end{aligned}$$

Exercise 6.5

a) Since $\mathbb{E}[N_t]=\lambda t$, the expectation $\mathbb{E}[N_t-2\lambda t]=-\lambda t$ is a decreasing function of $t\in\mathbb{R}_+$, and $(N_t-2\lambda t)_{t\in\mathbb{R}_+}$ is a *super*martingale.

b) We have

$$S_t=S_0\mathrm{e}^{rt-\lambda\sigma t}(1+\sigma)^{N_t}, \qquad t\in\mathbb{R}_+.$$

c) The stochastic differential equation

$$dS_t=rS_tdt+\sigma S_{t^-}(dN_t-\lambda dt)$$

contains a martingale component $(dN_t-\lambda dt)$ and a positive drift rS_tdt, therefore $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale.

d) Given that $\sigma>0$ we have $((1+\sigma)^k-1)^+=(1+\sigma)^k-1$, hence

$$\begin{aligned}\mathrm{e}^{-rT}\mathbb{E}^*[(S_T-K)^+] &= \mathrm{e}^{-rT}\mathbb{E}^*[(S_0\mathrm{e}^{(r-\sigma\lambda)T}(1+\sigma)^{N_T}-K)^+]\\ &= \mathrm{e}^{-rT}\mathbb{E}^*[(S_0\mathrm{e}^{(r-\sigma\lambda)T}(1+\sigma)^{N_T}-S_0\mathrm{e}^{(r-\lambda\sigma)T})^+]\\ &= S_0\mathrm{e}^{-\sigma\lambda T}\mathbb{E}^*[((1+\sigma)^{N_T}-1)^+]\\ &= S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geqslant 0}((1+\sigma)^k-1)^+\mathbb{P}(N_T=k)\\ &= S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geqslant 0}((1+\sigma)^k-1)\mathbb{P}(N_T=k)\\ &= S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geqslant 0}(1+\sigma)^k\mathbb{P}(N_T=k)-S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geqslant 0}\mathbb{P}(N_T=k)\\ &= S_0\mathrm{e}^{-\sigma\lambda T-\lambda T}\sum_{k\geqslant 0}\frac{(T(1+\sigma)\lambda)^k}{k!}-S_0\mathrm{e}^{-\sigma\lambda T}\\ &= S_0(1-\mathrm{e}^{-\sigma\lambda T}),\end{aligned}$$

where we applied the exponential identity

$$\mathrm{e}^x=\sum_{k\geqslant 0}\frac{x^k}{k!}$$

to $x:=T(1+\sigma)\lambda$.

Exercise 6.6

a) For all $k = 1, 2, \ldots, N_t$ we have

$$X_{T_k} - X_{T_k^-} = a + \sigma X_{T_k^-},$$

hence

$$X_{T_k} = a + (1+\sigma) X_{T_k^-},$$

and continuing by induction, we obtain

$$\begin{aligned} X_{T_k} &= a + (1+\sigma)a + \cdots + (1+\sigma)^{k-1} a + X_0 (1+\sigma)^k \\ &= a \frac{(1+\sigma)^k - 1}{\sigma} + X_0 (1+\sigma)^k, \end{aligned}$$

which shows that

$$\begin{aligned} X_t &= X_{T_{N_t}} \\ &= X_0 (1+\sigma)^{N_t} + a \frac{(1+\sigma)^{N_t} - 1}{\sigma} \\ &= (1+\sigma)^{N_t} \left(X_0 + \frac{a}{\sigma} \right) - \frac{a}{\sigma}, \quad t \in \mathbb{R}_+. \end{aligned}$$

This result can also be obtained by noting that

$$X_{T_k} + \frac{a}{\sigma} = (1+\sigma) \left(X_{T_k^-} + \frac{a}{\sigma} \right), \qquad k = 1, 2, \ldots, N_t.$$

b) We have

$$\mathbb{E}[(1+\sigma)^{N_t}] = \mathrm{e}^{-\lambda t} \sum_{n \geqslant 0} (1+\sigma)^k \frac{(\lambda t)^k}{k!} = \mathrm{e}^{\sigma \lambda t}, \qquad t \in \mathbb{R}_+,$$

hence

$$\mathbb{E}[X_t] = X_0 \, \mathrm{e}^{\lambda \sigma t} + a \frac{\mathrm{e}^{\lambda \sigma t} - 1}{\sigma} = \mathrm{e}^{\lambda \sigma t} \left(X_0 + \frac{a}{\sigma} \right) - \frac{a}{\sigma}, \qquad t \in \mathbb{R}_+.$$

Exercise 6.7 We have $S_t = S_0 \, \mathrm{e}^{rt} \prod_{k=1}^{N_t} (1 + \eta Z_k)$, $t \in \mathbb{R}_+$.

Exercise 6.8 We have

$$\begin{aligned} \mathrm{Var}\,[Y_T] &= \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T] \right)^2 \right] \\ &= \sum_{n \geqslant 0} \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \lambda t \, \mathbb{E}[Z] \right)^2 \Big| N_T = k \right] \mathbb{P}(N_T = k) \\ &= \mathrm{e}^{-\lambda t} \sum_{n \geqslant 0} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k - \lambda t \, \mathbb{E}[Z] \right)^2 \right] \\ &= \mathrm{e}^{-\lambda t} \sum_{n \geqslant 0} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k \right)^2 - 2\lambda t \, \mathbb{E}[Z] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z])^2 \right] \\ &= \mathrm{e}^{-\lambda t} \sum_{n \geqslant 0} \frac{\lambda^n t^n}{n!} \\ &\quad \times \mathbb{E}\left[2 \sum_{1 \leqslant k < l \leqslant n} Z_k Z_l + \sum_{k=1}^{n} |Z_k|^2 - 2\lambda t \, \mathbb{E}[Z] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z])^2 \right] \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-\lambda t} \sum_{n \geqslant 0} \frac{\lambda^n t^n}{n!} \\
&\quad \times (n(n-1)(\mathbb{E}[Z])^2 + n\mathbb{E}\left[|Z|^2\right] - 2n\lambda t(\mathbb{E}[Z])^2 + \lambda^2 t^2 (\mathbb{E}[Z])^2) \\
&= \mathrm{e}^{-\lambda t} (\mathbb{E}[Z])^2 \sum_{n \geqslant 2} \frac{\lambda^n t^n}{(n-2)!} + \mathrm{e}^{-\lambda t} \mathbb{E}\left[|Z|^2\right] \sum_{n \geqslant 1} \frac{\lambda^n t^n}{(n-1)!} \\
&\quad -2\,\mathrm{e}^{-\lambda t} \lambda t (\mathbb{E}[Z])^2 \sum_{n \geqslant 1} \frac{\lambda^n t^n}{(n-1)!} + \lambda^2 t^2 (\mathbb{E}[Z])^2) \\
&= \lambda t \, \mathbb{E}\left[|Z|^2\right],
\end{aligned}
$$

or, using the *moment generating function* of Proposition 6.5,

$$
\begin{aligned}
\operatorname{Var}[Y_T] &= \mathbb{E}\left[|Y_T|^2\right] - (\mathbb{E}[Y_T])^2 \\
&= \frac{\partial^2}{\partial \alpha^2} \mathbb{E}[\mathrm{e}^{\alpha Y_T}]_{|\alpha=0} - \lambda^2 t^2 (\mathbb{E}[Z])^2 \\
&= \lambda t \int_{-\infty}^{\infty} |y|^2 \mu(dy) = \lambda t \, \mathbb{E}\left[|Z|^2\right].
\end{aligned}
$$

Exercise 6.9

a) Applying the Itô formula (6.24) to the function $f(x) = \mathrm{e}^x$ and to the process $X_t = \mu t + \sigma W_t + Y_t$, we find

$$
\begin{aligned}
dS_t &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_t - S_{t^-}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 \,\mathrm{e}^{\mu t + \sigma W_t + Y_t} - S_0 \,\mathrm{e}^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 \,\mathrm{e}^{\mu t + \sigma W_t + Y_{t^-} + Z_{N_t}} - \mathrm{e}^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + S_{t^-} (\mathrm{e}^{Z_{N_t}} - 1) dN_t,
\end{aligned}
$$

hence the jumps of S_t are given by the sequence $(\mathrm{e}^{Z_k} - 1)_{k \geqslant 1}$.

b) The discounted process $\mathrm{e}^{-rt} S_t$ satisfies

$$
d(\mathrm{e}^{-rt} S_t) = \mathrm{e}^{-rt} \left(\mu - r + \frac{1}{2}\sigma^2\right) S_t dt + \sigma \,\mathrm{e}^{-rt} S_t dW_t + \mathrm{e}^{-rt} S_{t^-} (\mathrm{e}^{Z_{N_t}} - 1) dN_t.
$$

Hence by the Girsanov Theorem 6.18, choosing u, $\tilde{\lambda}$, $\tilde{\nu}$ such that

$$
\mu - r + \frac{1}{2}\sigma^2 = \sigma u - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[\mathrm{e}^Z - 1],
$$

shows that

$$
d(\mathrm{e}^{-rt} S_t) = \sigma \,\mathrm{e}^{-rt} S_t (dW_t + u dt) + \mathrm{e}^{-rt} S_{t^-} ((\mathrm{e}^{Z_{N_t}} - 1) dN_t - \tilde{\lambda} \, \mathbb{E}_{\tilde{\nu}}[\mathrm{e}^Z - 1] dt)
$$

is a martingale under $(\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}})$.

Exercise 6.10

a) We have

$$
S_t = S_0 \,\mathrm{e}^{\mu t} \prod_{k=1}^{N_t} (1 + Z_k) = S_0 \exp\left(\mu t + \sum_{k=1}^{N_t} X_k\right), \qquad t \in \mathbb{R}_+.
$$

b) We have the discounted asset price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t = S_0 \exp\left((\mu - r)t + \sum_{k=1}^{N_t} X_k\right), \qquad t \in \mathbb{R}_+,$$

satisfies the stochastic differential equation

$$\begin{aligned} d\widetilde{S}_t &= (\mu - r)\widetilde{S}_t dt + X_{N_t}\widetilde{S}_{t^-} dN_t \\ &= (\mu - r + \lambda\,\mathbb{E}[Z])\widetilde{S}_t dt + \big(X_{N_t} - \lambda\,\mathbb{E}[Z]\big)\widetilde{S}_{t^-} dN_t \qquad t \in \mathbb{R}_+, \end{aligned}$$

hence it is a martingale if

$$0 = \mu - r + \lambda\,\mathbb{E}[Z] = \mu - r + \lambda\,\mathbb{E}[\mathrm{e}^{X_k} - 1] = \mu - r + (\mathrm{e}^{\sigma^2/2} - 1)\lambda.$$

c) We have

$$\begin{aligned} &\mathrm{e}^{-(T-t)r}\,\mathbb{E}[(S_T - \kappa)^+ \mid S_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_0 \exp\left(\mu T + \sum_{k=1}^{N_T} X_k\right) - \kappa\right)^+ \Bigg| S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_t \exp\left(\mu(T-t) + \sum_{k=N_t+1}^{N_T} X_k\right) - \kappa\right)^+ \Bigg| S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(x \exp\left(\mu(T-t) + \sum_{k=N_t+1}^{N_T} X_k\right) - \kappa\right)^+\right]_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(x \exp\left(\mu(T-t) + \sum_{k=1}^{N_T-N_t} X_k\right) - \kappa\right)^+\right]_{x=S_t} \\ &= \mathrm{e}^{-(T-t)r} \sum_{n\geqslant 0} \mathbb{E}\left[\left(x\,\mathrm{e}^{\mu(T-t)+\sum_{k=1}^n X_k} - \kappa\right)^+\right]_{x=S_t} \mathbb{P}(N_T - N_t = n) \\ &= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geqslant 0} \mathbb{E}\left[\left(x\,\mathrm{e}^{\mu(T-t)+\sum_{k=1}^n X_k} - \kappa\right)^+\right]_{x=S_t} \frac{((T-t)\lambda)^n}{n!} \\ &= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \frac{((T-t)\lambda)^n}{n!} \mathrm{Bl}(S_t\,\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}, r, n\sigma^2/(T-t), \kappa, T-t) \\ &= \mathrm{e}^{-(T-t)\lambda} \sum_{n\geqslant 0} \left(S_t\,\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}\Phi(d_+) - \kappa\,\mathrm{e}^{-(T-t)r}\Phi(d_-)\right) \frac{((T-t)\lambda)^n}{n!}, \end{aligned}$$

with

$$\begin{aligned} d_+ &= \frac{\log(S_t\,\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + (T-t)r + n\sigma^2/2}{\sigma\sqrt{n}} \\ &= \frac{\log(S_t/\kappa) + \mu(T-t) + n\sigma^2}{\sigma\sqrt{n}}, \\ d_- &= \frac{\log(S_t\,\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + (T-t)r - n\sigma^2/2}{\sigma\sqrt{n}} \\ &= \frac{\log(S_t/\kappa) + \mu(T-t)}{\sigma\sqrt{n}}, \end{aligned}$$

and $\mu = r + (1 - \mathrm{e}^{\sigma^2/2})\lambda$.

Exercise 6.11

a) We have

$$d(\mathrm{e}^{\alpha t} S_t) = \sigma\,\mathrm{e}^{\alpha t}(dN_t - \beta dt),$$

hence

$$e^{\alpha t} S_t = S_0 + \sigma \int_0^t e^{\alpha s} (dN_s - \beta d),$$

and

$$S_t = S_0 e^{-\alpha t} + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds), \quad t \in \mathbb{R}_+. \tag{A.30}$$

b) We have

$$\begin{aligned} f(t) &= \mathbb{E}[S_t] \\ &= S_0 e^{-\alpha t} + \sigma \mathbb{E}\left[\int_0^t e^{-(t-s)\alpha} dN_s\right] - \beta\sigma \int_0^t e^{-(t-s)\alpha} ds \\ &= S_0 e^{-\alpha t} + \lambda\sigma \int_0^t e^{-(t-s)\alpha} ds - \beta\sigma \int_0^t e^{-(t-s)\alpha} ds \\ &= S_0 e^{-\alpha t} + (\lambda - \beta)\sigma \frac{1 - e^{-\alpha t}}{\alpha} \\ &= \sigma \frac{\lambda - \beta}{\alpha} + \left(S_0 + \sigma \frac{\beta - \lambda}{\alpha}\right) e^{-\alpha t}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

c) By rewriting (A.30) as

$$\begin{aligned} S_t &= S_0 - \alpha S_0 \int_0^t e^{-(t-s)\alpha} ds + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \beta ds) \\ &= S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - (\beta + \alpha S_0/\sigma) ds) \\ &= S_0 + \sigma \int_0^t e^{-(t-s)\alpha} (\lambda - \beta - \alpha S_0/\sigma) ds) + \sigma \int_0^t e^{-(t-s)\alpha} (dN_s - \lambda ds), \end{aligned}$$

$t \in \mathbb{R}_+$, we check that the process $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, provided that $\lambda - \beta - \alpha S_0/\sigma \geqslant 0$, *i.e.* $S_0 + (\beta - \lambda)\sigma/\alpha \leqslant 0$. We also check that this condition makes the expectation $f(t) = \mathbb{E}[S_t]$ decreasing in Question (b)).

d) Since, given that $N_T = n$ the jump times $(T_1, T_2, \ldots, T_n)$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are independent uniformly distributed random variables over $[0, T]$, hence we can write

$$\begin{aligned} &\mathbb{E}[\phi(S_T)] = \mathbb{E}\left[\phi\left(S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds),\right)\right] \\ &= \sum_{n\geqslant 0} \mathbb{P}(N_T = n) \\ &\quad \times \mathbb{E}\left[\phi\left(S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha} (dN_s - \beta ds)\right) \Big| N_T = n\right] \\ &= e^{-\lambda T} \sum_{n\geqslant 0} \frac{(\lambda T)^n}{n!} \\ &\quad \times \mathbb{E}\left[\phi\left(S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-T_k)\alpha} - \sigma\beta \int_0^T e^{-(T-s)\alpha} ds\right) \Big| N_T = n\right] \\ &= e^{-\lambda T} \sum_{n\geqslant 0} \frac{\lambda^n}{n!} \\ &\quad \times \int_0^T \cdots \int_0^T \phi\left(S_0 e^{-\alpha T} + \sigma \sum_{k=1}^n e^{-(T-s_k)\alpha} - \sigma\beta \frac{1 - e^{-\alpha T}}{\alpha}\right) ds_1 \cdots ds_n, \end{aligned}$$

$T \geqslant 0$.

Exercise 6.12

a) From the decomposition $Y_t - \lambda t(t + \mathbb{E}[Z]) = Y_t - \lambda\, \mathbb{E}[Z] t - \lambda t^2$ as the sum of a martingale and a decreasing function, we conclude that $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z])$ is a *super*martingale.

b) Writing

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_{t^-} dY_t \\ &= r S_t dt + \sigma S_{t^-}\left(dY_t - \frac{r-\mu}{\sigma} dt\right) \\ &= r S_t dt + \sigma S_{t^-}\left(dY_t - \tilde{\lambda}\,\mathbb{E}[Z] dt\right), \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

we conclude that $(S_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$ provided that

$$\frac{\mu - r}{\sigma} = -\tilde{\lambda}\,\mathbb{E}[Z] dt,$$

i.e.

$$\tilde{\lambda} = \frac{r-\mu}{\sigma\,\mathbb{E}[Z]}.$$

We note that $\tilde{\lambda} < 0$ if $\mu < r$, hence in this case there is no risk-neutral probability measure and the market admits arbitrage opportunities as the risky asset always overperforms the risk-free interest rate r.

c) We have

$$\begin{aligned} \mathrm{e}^{-(T-t)r}\,\mathbb{E}_{\tilde{\lambda}}[S_T - \kappa \mid \mathscr{F}_t] &= \mathrm{e}^{rt}\,\mathbb{E}_{\tilde{\lambda}}[\mathrm{e}^{-rT} S_T \mid \mathscr{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\ &= S_t - K\,\mathrm{e}^{-(T-t)r}, \end{aligned}$$

since $(S_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$.

Exercise 6.13

a) We have

$$S_t = S_0\,\mathrm{e}^{\mu t}\prod_{k=1}^{N_t}(1+Z_k), \qquad t \in \mathbb{R}_+.$$

b) Letting $X_k = \log(1+Z_k)$, $k \geqslant 1$, we find that

$$\mathrm{e}^{-rt} S_t = S_0 \exp\left((\mu - r)t + \sum_{k=1}^{N_t} X_k\right), \qquad t \in \mathbb{R}_+,$$

and (6.42) can be rewritten for the discounted price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \in \mathbb{R}_+,$$

as

$$d\widetilde{S}_t = (\mu - r + \lambda\,\mathbb{E}[Z])\widetilde{S}_t dt + \widetilde{S}_{t^-}(dY_t + \lambda\,\mathbb{E}[Z]),$$

which becomes a martingale if

$$0 = \mu - r + \lambda\,\mathbb{E}[Z] = \mu - r + \lambda\int_{-\infty}^{\infty} z\nu(dz).$$

c) We have

$$\begin{aligned} &\mathrm{e}^{-(T-t)r}\,\mathbb{E}[(S_T-\kappa)^+ \mid S_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_0\,\mathrm{e}^{\mu T}\prod_{k=1}^{N_T} Z_k - \kappa\right)^+ \Bigg| S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\sum_{n\geqslant 0}\mathbb{E}\left[\left(S_t\,\mathrm{e}^{\mu(T-t)}\prod_{k=N_t+1}^{N_T} Z_k - \kappa\right)^+ \Bigg| S_t\right]\mathbb{P}(N_T - N_t = n) \end{aligned}$$

$$= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geqslant 0} \mathbb{E}\left[\left(S_t\, \mathrm{e}^{\mu(T-t)} \prod_{k=N_t+1}^{N_T} Z_k - \kappa\right)^+ \Bigg| S_t\right] \frac{((T-t)\lambda)^n}{n!}$$
$$= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geqslant 0} \frac{((T-t)\lambda)^n}{n!}$$
$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(S_t\, \mathrm{e}^{\mu(T-t)} \prod_{k=1}^{n} z_k - \kappa\right)^+ \nu(dz_1)\cdots\nu(dz_n).$$

Exercise 6.14

a) The discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a martingale, hence it is both a *sub*martingale and a *super*martingale.

b) The discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a *super*martingale.

c) The discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a *sub*martingale.

d) Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, the discounted price process $(\mathrm{e}^{-rt} S_t)_{t\in[0,T]}$ is a martingale, hence it is both a *sub*martingale and a *super*martingale.

Chapter 7

Exercise 7.1

a) We have $\mathbb{E}[N_t - \alpha t] = \mathbb{E}[N_t] - \alpha t = \lambda t - \alpha t$, hence $N_t - \alpha t$ is a martingale if and only if $\alpha = \lambda$. Given that
$$d(\mathrm{e}^{-rt} S_t) = \eta\, \mathrm{e}^{-rt} S_{t^-}(dN_t - \alpha dt),$$
we conclude that the discounted price process $\mathrm{e}^{-rt} S_t$ is a martingale if and only if $\underline{\alpha = \lambda}$.

b) Since we are pricing under the risk-neutral probability measure we take $\alpha = \lambda$. Next, we note that
$$S_T = S_0\, \mathrm{e}^{(r-\eta\lambda)T}(1+\eta)^{N_T} = S_t\, \mathrm{e}^{(r-\eta\lambda)(T-t)}(1+\eta)^{N_T - N_t}, \qquad 0 \leqslant t \leqslant T,$$
hence the price at time t of the option is
$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}[|S_T|^2 \mid \mathscr{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}[|S_t|^2 \mathrm{e}^{2(r-\eta\lambda)(T-t)}(1+\eta)^{2(N_T-N_t)} \mid \mathscr{F}_t] \\
&= |S_t|^2 \mathrm{e}^{(r-2\eta\lambda)(T-t)} \mathbb{E}[(1+\eta)^{2(N_T-N_t)} \mid \mathscr{F}_t] \\
&= |S_t|^2 \mathrm{e}^{(r-2\eta\lambda)(T-t)} \mathbb{E}[(1+\eta)^{2(N_T-N_t)}] \\
&= |S_t|^2 \mathrm{e}^{(r-2\eta\lambda)(T-t)} \sum_{n\geqslant 0} (1+\eta)^{2n} \mathbb{P}(N_T - N_t = n) \\
&= |S_t|^2 \mathrm{e}^{(r-2\eta\lambda-\lambda)(T-t)} \sum_{n\geqslant 0} (1+\eta)^{2n} \frac{(\lambda(T-t))^n}{n!} \\
&= |S_t|^2 \mathrm{e}^{(r-2\eta\lambda-\lambda)(T-t)+(1+\eta)^2\lambda(T-t)} \\
&= |S_t|^2 \mathrm{e}^{(r+\eta^2\lambda)(T-t)}, \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Exercise 7.2

a) Regardless of the choice of a particular risk-neutral probability measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$, we have
$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[S_T - K \mid \mathscr{F}_t] &= \mathrm{e}^{rt}\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[\mathrm{e}^{-rT} S_T \mid \mathscr{F}_t] - K\,\mathrm{e}^{-(T-t)r} \\
&= \mathrm{e}^{rt}\,\mathrm{e}^{-rt} S_t - K\,\mathrm{e}^{-(T-t)r} \\
&= S_t - K\,\mathrm{e}^{-(T-t)r} \\
&= f(t, S_t),
\end{aligned}$$

for

$$f(t,x) = x - K\,\mathrm{e}^{-(T-t)r}, \qquad t,x > 0.$$

b) Clearly, holding one unit of the risky asset and shorting a (possibly fractional) quantity $K\,\mathrm{e}^{-rT}$ of the riskless asset will hedge the payoff $S_T - K$, and this (static) hedging strategy is self-financing because it is constant in time.

c) Since $\dfrac{\partial f}{\partial x}(t,x) = 1$ we have

$$\begin{aligned}
\xi_t &= \frac{\sigma^2 \dfrac{\partial f}{\partial x}(t,S_{t^-}) + \dfrac{a\tilde{\lambda}}{S_{t^-}}(f(t,S_{t^-}(1+a)) - f(t,S_{t^-}))}{\sigma^2 + a^2\tilde{\lambda}} \\
&= \frac{\sigma^2 + \dfrac{a\tilde{\lambda}}{S_{t^-}}(S_{t^-}(1+a) - S_{t^-})}{\sigma^2 + a^2\tilde{\lambda}} \\
&= 1, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

which coincides with the result of Question (b)).

Exercise 7.3

a) We have

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t}.$$

b) We have

$$\widetilde{S}_t = S_0 \exp\left((\mu - r)t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t},$$

and

$$d\widetilde{S}_t = (\mu - r + \lambda\eta)\widetilde{S}_t dt + \eta \widetilde{S}_{t^-}(dN_t - \lambda dt) + \sigma \widetilde{S}_t dW_t,$$

hence we need to take

$$\mu - r + \lambda\eta = 0,$$

since the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale.

c) We have

$$\begin{aligned}
&\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[(S_T - \kappa)^+ \mid S_t] \\
&= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\left(S_0 \exp\left(\mu T + \sigma B_T - \frac{1}{2}\sigma^2 T\right)(1+\eta)^{N_T} - \kappa\right)^+ \,\Big|\, S_t\right] \\
&= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\left(S_t\,\mathrm{e}^{\mu(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^{N_T-N_t} - \kappa\right)^+ \,\Big|\, S_t\right] \\
&= \mathrm{e}^{-r(T-t)} \sum_{n\geqslant 0} \mathbb{P}(N_T - N_t = n) \\
&\quad \times \mathbb{E}^*\left[\left(S_t\,\mathrm{e}^{\mu(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^n - \kappa\right)^+ \,\Big|\, S_t\right] \\
&= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geqslant 0} \frac{(\lambda(T-t))^n}{n!} \\
&\quad \times \mathbb{E}^*\left[\left(S_t\,\mathrm{e}^{(r-\lambda\eta)(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^n - \kappa\right)^+ \,\Big|\, S_t\right] \\
&= \mathrm{e}^{-\lambda(T-t)} \sum_{n\geqslant 0} \mathrm{Bl}(S_t\,\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n, r, \sigma^2, T-t, \kappa)\frac{(\lambda(T-t))^n}{n!} \\
&= \mathrm{e}^{-\lambda(T-t)} \sum_{n\geqslant 0} \left(S_t\,\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n \Phi(d_+) - \kappa\,\mathrm{e}^{-r(T-t)}\Phi(d_-)\right)\frac{(\lambda(T-t))^n}{n!},
\end{aligned}$$

with

$$d_+ = \frac{\log(S_t\,\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\log(S_t(1+\eta)^n/\kappa)+(r-\lambda\eta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_- = \frac{\log(S_t\,\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\log(S_t(1+\eta)^n/\kappa)+(r-\lambda\eta-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Exercise 7.4

a) The discounted process $\widetilde{S}_t = \mathrm{e}^{-rt}S_t$ satisfies the equation

$$d\widetilde{S}_t = Y_{N_t}\widetilde{S}_{t^-}dN_t,,$$

and it is a martingale since the compound Poisson process $Y_{N_t}dN_t$ is centered with independent increments as $\mathbb{E}[Y_1]=0$.

b) We have

$$S_T = S_0\,\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k),$$

hence

$$\mathrm{e}^{-rT}\,\mathbb{E}\left[(S_T-\kappa)^+\right] = \mathrm{e}^{-rT}\,\mathbb{E}\left[\left(S_0\,\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k)-\kappa\right)^+\right]$$

$$= \mathrm{e}^{-rT}\sum_{n\geqslant 0}\mathbb{E}\left[\left(S_0\,\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k)-\kappa\right)^+ \,\Big|\, N_T=n\right]\mathbb{P}(N_T=n)$$

$$= \mathrm{e}^{-rT-\lambda T}\sum_{k\geqslant 0}\mathbb{E}\left[\left(S_0\,\mathrm{e}^{rT}\prod_{k=1}^{n}(1+Y_k)-\kappa\right)^+\right]\frac{(\lambda T)^n}{n!}$$

$$= \mathrm{e}^{-rT-\lambda T}\sum_{k\geqslant 0}\frac{(\lambda T)^n}{2^n n!}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(S_0\,\mathrm{e}^{rT}\prod_{k=1}^{n}(1+y_k)-\kappa\right)^+ dy_1\cdots dy_n.$$

Exercise 7.5

a) We find $\alpha=\lambda$ where λ is the intensity of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

b) We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}[S_T-\kappa \mid \mathscr{F}_t] = \mathrm{e}^{rt}\,\mathbb{E}[\mathrm{e}^{-rT}S_T \mid \mathscr{F}_t] - \mathrm{e}^{-(T-t)r}\,\mathbb{E}[\kappa \mid \mathscr{F}_t]$$

$$= \mathrm{e}^{rt}\,\mathbb{E}[\mathrm{e}^{-rt}S_t \mid \mathscr{F}_t] - \mathrm{e}^{-(T-t)r}\kappa$$

$$= S_t - \mathrm{e}^{-(T-t)r}\kappa,$$

since the process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 7.6

a) We have

$$S_t = S_0\,\mathrm{e}^{(r-\lambda)t}(1+\alpha)^{N_t}, \qquad t\in\mathbb{R}_+.$$

b) We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T)\mid\mathscr{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(xS_T/S_t)]_{x=S_t}$$

$$= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi(x\,\mathrm{e}^{(r-\lambda)(T-t)}(1+\alpha)^{N_T-N_t})\right]_{x=S_t}$$

$$= \mathrm{e}^{-(r+\lambda)(T-t)}\sum_{k=0}^{\infty}\frac{((T-t)\lambda)^k}{k!}\phi\left(S_t\,\mathrm{e}^{(r-\lambda)(T-t)}(1+\alpha)^k\right), \quad 0\leqslant t\leqslant T.$$

c) We have

$$\begin{aligned} dV_t &= r\eta_t\,\mathrm{e}^{rt}dt + \xi_t dS_t \\ &= r\eta_t\,\mathrm{e}^{rt}dt + \xi_t(rS_t dt + \alpha S_{t^-}(dN_t - \lambda dt)) \\ &= rV_t dt + \alpha\xi_t S_{t^-}(dN_t - \lambda dt) \\ &= rf(t,S_t)dt + \alpha\xi_t S_{t^-}(dN_t - \lambda dt). \end{aligned} \tag{A.31}$$

d) We apply the Itô formula with jumps and make use of the martingale property of $t \longmapsto \mathrm{e}^{-rt}f(t,S_t)$ to get the expression

$$df(t,S_t) = rf(t,S_t)dt$$
$$+(f(t,S_{t^-}(1+\alpha)) - f(t,S_{t^-}))dN_t - \lambda(f(t,S_t(1+\alpha)) - f(t,S_t))dt.$$

Finally, we identify the terms in the above formula with those appearing in (A.31).

Exercise 7.7

a) We have

$$\begin{aligned} \mathbb{E}[N_t \mid \mathscr{F}_s] &= \mathrm{e}^{\theta Y_s - sm(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)} \mid \mathscr{F}_s\right] \\ &= \mathrm{e}^{\theta Y_s - sm(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)}\right] = N_s, \quad 0 \leqslant s \leqslant t. \end{aligned}$$

b) We have

$$\begin{aligned} \mathbb{E}^{\theta}\left[\mathrm{e}^{-rt}S_t \mid \mathscr{F}_s\right] &= \mathbb{E}\left[\mathrm{e}^{Y_t}\frac{N_T}{N_s} \,\Big|\, \mathscr{F}_s\right] \\ &= \mathbb{E}\left[\mathrm{e}^{Y_t}\frac{N_t}{N_s} \,\Big|\, \mathscr{F}_s\right] \\ &= \mathrm{e}^{Y_s}\,\mathbb{E}\left[\mathrm{e}^{Y_t-Y_s}\,\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)} \mid \mathscr{F}_s\right] \\ &= \mathrm{e}^{Y_s}\,\mathrm{e}^{-(t-s)m(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(1+\theta)(Y_t-Y_s)}\right] \\ &= \mathrm{e}^{Y_s}\,\mathrm{e}^{-(t-s)m(\theta)}\,\mathrm{e}^{(t-s)m(\theta+1)}, \end{aligned}$$

hence we should have $m(\theta) = m(\theta+1)$. For example, when $(Y_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is a compensated Poisson process we have $m(\theta) = \mathrm{e}^{\theta} - \theta - 1$ and the condition reads $\mathrm{e}^{\theta} + 1 = \mathrm{e}^{\theta+1}$, *i.e.* $\theta = -\log(\mathrm{e}-1)$.

c) We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^{\theta}[(S_T-K)^+ \mid \mathscr{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T-K)^+\frac{N_T}{N_t} \,\Big|\, \mathscr{F}_t\right]$$
$$= \mathrm{e}^{-(T-t)((1+r)\theta+m(\theta))}\,\mathbb{E}\left[(S_T-K)^+\left(\frac{S_T}{S_t}\right)^{\theta} \,\Big|\, \mathscr{F}_t\right].$$

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