MM5016 Laboration 8 Error analysis

Contents

1 Problems

Uppgift 1.1 (1). The two quantities x and y have a relative error of ϵ_x respectively ϵ_y. Calculate the error in the following cases:

1. $x - 2y$
2. $4x - 5y$
3. $3xy$
4. x^3/y^5
5. $\sqrt{x}$
6. $\sqrt{x^3}$
7. $\sqrt{2xy}$
8. $2\sqrt{\frac{x}{3y}}$
9. $2\sqrt{\frac{x^3}{3y}}$
10. $\frac{x}{y} + \frac{y}{x}$
11. $x^2 + y^2$

Lemma 1.1 (error of scaled approximate value)**.** Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and λ a scalar. Then the relative error of $\lambda\tilde{x}$ is $\epsilon_{\lambda x} = \epsilon_x$. This is because $\lambda\tilde{x} = \lambda x(1 + \epsilon_{\lambda x})$.

Lemma 1.2 (error of exponent)**.** Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and a an integer. Then the relative error of $\tilde{x}^a$ is $\epsilon_{x^a} = a\epsilon_x$. This is because $\epsilon_{x^a} = \epsilon_{xx^{a-1}} = \epsilon_x + \epsilon_{x^{a-1}}$.

Lösning 1.1.1 (1)**.** (1)

$$\begin{aligned}\epsilon_{x-2y} &= \frac{x}{x-(2y)}\epsilon_x - \frac{(2y)}{x-(2y)}\epsilon_{2y}\\ &= \frac{x}{x-2y}\epsilon_x - \frac{2y}{x-2y}\epsilon_y.\end{aligned}$$

(2)

$$\begin{aligned}\epsilon_{4x-5y} &= \frac{4x}{4x-5y}\epsilon_{4x} - \frac{5y}{4x-5y}\epsilon_{5y}\\ &= \frac{4x}{4x-5y}\epsilon_x - \frac{5y}{4x-5y}\epsilon_y.\end{aligned}$$

(3)

$$\epsilon_{3xy} = \epsilon_{xy} = \epsilon_x + \epsilon_y.$$

(4)

$$\epsilon_{\frac{x^3}{y^5}} = \epsilon_{x^3} - \epsilon_{y^5} = 3\epsilon_x - 5\epsilon_y.$$

(5)

$$\begin{aligned}\epsilon_x &= \epsilon_{\sqrt{x}^2} = 2\cdot\epsilon_{\sqrt{x}}\\ \implies \epsilon_{\sqrt{x}} &= \epsilon_x/2.\end{aligned}$$

(6)

$$\epsilon_{\sqrt{x^3}} = \frac{\epsilon_{x^3}}{2} = \frac{3}{2}\epsilon_x.$$

(7)

$$\begin{aligned}\epsilon_{\sqrt{2xy}} &= \epsilon_{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{y}}\\ &= \epsilon_{\sqrt{x}\cdot\sqrt{y}} = \epsilon_{\sqrt{x}} + \epsilon_{\sqrt{y}}\\ &= \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.\end{aligned}$$

(7)

$$\epsilon_{2\sqrt{\frac{x^3}{3y}}} = \epsilon_{\sqrt{\frac{x^3}{3y}}} = \frac{\epsilon_{\frac{x^3}{3y}}}{2} = \frac{\epsilon_{x^3} - \epsilon_{3y}}{2} = \frac{3\epsilon_x - \epsilon_y}{2}.$$

(8)

$$\begin{aligned}\epsilon_{\frac{x}{y}+\frac{y}{x}} = \frac{\frac{x}{y}}{\frac{x}{y}+\frac{y}{x}}\epsilon_{\frac{x}{y}} + \frac{\frac{y}{z}}{\frac{x}{y}+\frac{y}{x}}\epsilon_{\frac{y}{z}} &= \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x}{yz}+\frac{y}{xz}}\epsilon_{\frac{y}{z}}\\ = \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x^2+y^2}{xyz}}\epsilon_{\frac{y}{z}}\\ = \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2+y^2}\epsilon_{\frac{y}{z}}\\ = \frac{x^2}{x^2+y^2}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2+y^2}\epsilon_{\frac{y}{z}}\\ = \frac{x}{x^2+y^2}(\epsilon_{\frac{x}{y}} + y^2z\epsilon_{\frac{y}{z}})\\ = \frac{x}{x^2+y^2}(\epsilon_x - \epsilon_y + y^2z(\epsilon_y - \epsilon_z)).\end{aligned}$$

(8)

$$\begin{aligned}\epsilon_{x^2+y^2} &= \frac{x^2}{x^2+y^2}\epsilon_{x^2} + \frac{y^2}{x^2+y^2}\epsilon_{y^2}\\ &= \frac{x^2}{x^2+y^2}2\epsilon_x + \frac{y^2}{x^2+y^2}2\epsilon_y.\end{aligned}$$

Uppgift 1.2 (2). The radius of a sphere is $R = (22.2 \pm 0.1)cm$. The radius of the base of a cylinder is $r = (12.0 \pm 1.2)cm$, and its height is $h = (24.4 \pm 1.1)cm$. What is the total volum occupied by the sphere and by the cylinder (including the error)?

Lösning 1.2.1 (2). Let the subscript t denote the true value.

$$\begin{aligned}
V_{total} &= \frac{4\pi}{3}(R+\delta R)^3 + \pi(r+\delta r)^2 h \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2 + \delta R^3) + \pi(r^2 + 2\delta r + \delta^2)(h+\delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 + 2\delta r)(h+\delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 h + r^2\delta h + 2\delta h r + 2\delta r\delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R) + \pi(r^2 h + r^2\delta h + 2r\delta h).
\end{aligned}$$

To calculate the minimum volume we let the absolute be so much negative as possible and the reverse for the maximum, if we plug in these values we get:

$$55668.1 \leq V_{total} \leq 58068.$$