MM5016 Lab Assignment 6 (QR method)

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1 Task 1

Uppgift 1.1 (1). Find a QR decomposition for the matrix based on the Gram-Schmidt method.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Lösning 1.1.1 (1). To QR decompose A we will interpret the columns of A as three column vectors $A = (a_1, a_2, a_3)$ expressed in an orthonormal basis β. By the Gram-Schmidt method we will find an orthonormal basis γ such that if r_1, r_2, r_3 are the respective vectors a_1, a_2, a_3 in the basis γ, then the matrix $R = (r_1, r_2, r_3)$ we be an upper triangular matrix.
By the Gram-Schmidt method we will have an orthogonal basis and add our new vector by removing all components that are unorthogonal relative to the given orthogonal basis. We will get these components by projecting our new vector on the vectors in the orthogonal basis. Then we can add an arbitrary ammount of new vectors by using this defintion recursively. When we're done we will normalise all our vectors in the basis and thus it will be our orthonormal basis γ.
So let v_k denote the orthogonal basis vectors we are looking for that we will in the end normalise to get the orthonormal basis. So we will start by simply letting $v_1 = a_1$ as the set $\{v_1\}$ is orthogonal by definition.
We will get v_2 by removing the v_1 parallel component of a_2:

$$\begin{aligned} v_2 &= a_2 - \mathrm{proj}_{v_1}(a_2) \\ &= a_2 - \frac{\langle v_1, a_2 \rangle}{|v_1|^2} v_1 \\ &= a_2 - v_1 \\ &= a_2 - a_1 \\ &= (1,1,0)^t - (0,1,0)^t \\ &= (1,0,0)^t. \end{aligned}$$

From this we can also conclude that $a_2 = v_1 + v_2$ which we will use when determinening r_2.
We will do the same process with a_3 to obtain v_3, but this time we need to remove both the v_1 and v_2 parallel components:

$$\begin{aligned} v_3 &= a_3 - \mathrm{proj}_{v_1}(a_3) - \mathrm{proj}_{v_2}(a_3) \\ &= a_3 - \frac{\langle v_1, a_3 \rangle}{|v_1|^2} v_1 - \frac{\langle v_2, a_3 \rangle}{|v_2|^2} v_2 \\ &= a_3 - 2v_1 - v_2 \\ &= (1,2,3)^t - 2(0,1,0)^t - (1,0,0)^t \\ &= (0,0,3)^t. \end{aligned}$$

From this we can also conclude that $a_3 = 2v_1 + v_2 + v_3$, which we will use when determining r_3.

So now we have an orthogonal basis

$$\{v_1, v_2, v_3\} = \{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\}.$$

As each vector is has a single coordinate we can easily see that we can normalise it my dividing by v_3 by 3. If we denote our orthonormal basis γ as, $\gamma = \{q_1, q_2, q_3\}$, then $q_1 = v_1$, $q_2 = v_2$ and $q_3 = \frac{v_3}{3}$.
As r_k are a_k in the basis γ, we know have enough information to determine r_k. $a_1 = v_1 = q_1$, so $r_1 = (1, 0, 0)^t$. $a_2 = v_1 + v_2 = q_1 + q_2$, so $r_2 = (1, 1, 0)^t$. And lastly we have that $a_3 = 2v_1 + v_2 + v_3 = 2q_1 + q_2 + 3q_3$, so $r_3 = (2, 1, 3)^t$.
If we had a vector v in the basis γ and we wanted to express it in the basis β, we could write Qv where Q is the matrix $Q = (q_1, q_2, q_3)$. v doesn't need to be a single column vector, it can also be a row vector of column vectors, and by multiplying Q on the left we get a new row vector where each row column vector has changed basis. Thus we have that $A = QR$, where $R = (r_1, r_2, r_3)$.
We can explicitly write out the matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

To control that Q is orthonormal we will verify that $\langle q_i, q_j \rangle = \delta_{ij}$, where w$\delta_{ij}$ is the kronecker delta function. If we define the matrix $M_{ij} = \langle q_i, q_j \rangle$, then the statement that Q is orthonormal is equivalent to saying that $M = I$, where I is the 3×3 identity matrix.
Let us denote the i:th component of a vector v_k in the basis β, as v_k^i. Let us consider the matrix Q^tQ. It is defined as

$$\begin{aligned} (Q^tQ)_{ij} &= \sum_{k=0}^{n} (Q^t)_{ik} Q_{kj} \\ &= \sum_{k=0}^{n} Q_{ki} Q_{kj} \\ &= \sum_{k=0}^{n} q_i^k q_j^k \\ &= \langle q_i, q_j \rangle . \end{aligned}$$

We can now see that $M = Q^tQ$, which means that the statement that $M = I$ is equivalent to that $Q^tQ = I$, which is equivalent to

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And because the last statement is true, that means that our original statement that γ is an orthonormal basis is true.