MM5016 Laboration 8 Error analysis

Contents

1 Problems

Uppgift 1.1 (1). The two quantities x and y have a relative error of ϵ_x respectively ϵ_y. Calculate the error in the following cases:

1. $x - 2y$

2. $4x - 5y$

3. $3xy$

4. x^3/y^5

5. $\sqrt{x}$

6. $\sqrt{x^3}$

7. $\sqrt{2xy}$

8. $2\sqrt{\frac{x}{3y}}$

9. $2\sqrt{\frac{x^3}{3y}}$

10. $\frac{x}{y} + \frac{y}{x}$

11. $x^2 + y^2$

Lemma 1.1 (error of scaled approximate value). Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and λ a scalar. Then the relative error of $\lambda\tilde{x}$ is $\epsilon_{\lambda x} = \epsilon_x$. This is because $\lambda\tilde{x} = \lambda x(1 + \epsilon_{\lambda x})$.

Lemma 1.2 (error of exponent). Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and a an integer. Then the relative error of $\tilde{x}^a$ is $\epsilon_{x^a} = a\epsilon_x$. This is because $\epsilon_{x^a} = \epsilon_{xx^{a-1}} = \epsilon_x + \epsilon_{x^{a-1}}$.

Lösning 1.1.1 (1). (1)

$$\epsilon_{x-2y} = \frac{x}{x - (2y)}\epsilon_x - \frac{(2y)}{x - (2y)}\epsilon_{2y}$$
$$= \frac{x}{x - 2y}\epsilon_x - \frac{2y}{x - 2y}\epsilon_y.$$

(2)

$$\epsilon_{4x-5y} = \frac{4x}{4x - 5y}\epsilon_{4x} - \frac{5y}{4x - 5y}\epsilon_{5y}$$
$$= \frac{4x}{4x - 5y}\epsilon_x - \frac{5y}{4x - 5y}\epsilon_y.$$

(3)

$$\epsilon_{3xy} = \epsilon_{xy} = \epsilon_x + \epsilon_y.$$

(4)

$$\epsilon_{\frac{x^3}{y^5}} = \epsilon_{x^3} - \epsilon_{y^5} = 3\epsilon_x - 5\epsilon_y.$$

(5)

$$\epsilon_x = \epsilon_{\sqrt{x^2}} = 2 \cdot \epsilon_{\sqrt{x}}$$
$$\implies \epsilon_{\sqrt{x}} = \epsilon_x/2.$$

(6)

$$\epsilon_{\sqrt{x^3}} = \frac{\epsilon_{x^3}}{2} = \frac{3}{2}\epsilon_x.$$

(7)

$$\epsilon_{\sqrt{2xy}} = \epsilon_{\sqrt{2} \cdot \sqrt{x} \cdot \sqrt{y}}$$
$$= \epsilon_{\sqrt{x} \cdot \sqrt{y}} = \epsilon_{\sqrt{x}} + \epsilon_{\sqrt{y}}$$
$$= \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.$$

(7)

$$\epsilon_{2\sqrt{\frac{x^3}{3y}}} = \epsilon_{\sqrt{\frac{x^3}{3y}}} = \frac{\epsilon_{\frac{x^3}{3y}}}{2} = \frac{\epsilon_{x^3} - \epsilon_{3y}}{2} = \frac{3\epsilon_x - \epsilon_y}{2}.$$

(8)

$$\epsilon_{\frac{x}{y}+\frac{y}{x}} = \frac{\frac{x}{y}}{\frac{x}{y} + \frac{y}{x}}\epsilon_{\frac{x}{y}} + \frac{\frac{y}{z}}{\frac{x}{y} + \frac{y}{x}}\epsilon_{\frac{y}{z}} = \frac{x}{x + \frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x}{yz} + \frac{y}{xz}}\epsilon_{\frac{y}{z}}$$
$$= \frac{x}{x + \frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x^2+y^2}{xyz}}\epsilon_{\frac{y}{z}}$$
$$= \frac{x}{x + \frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2 + y^2}\epsilon_{\frac{y}{z}}$$
$$= \frac{x^2}{x^2 + y^2}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2 + y^2}\epsilon_{\frac{y}{z}}$$
$$= \frac{x}{x^2 + y^2}\left(\epsilon_{\frac{x}{y}} + y^2z\epsilon_{\frac{y}{z}}\right)$$
$$= \frac{x}{x^2 + y^2}\left(\epsilon_x - \epsilon_y + y^2z(\epsilon_y - \epsilon_z)\right).$$

(8)

$$\epsilon_{x^2+y^2} = \frac{x^2}{x^2 + y^2}\epsilon_{x^2} + \frac{y^2}{x^2 + y^2}\epsilon_{y^2}$$
$$= \frac{x^2}{x^2 + y^2}2\epsilon_x + \frac{y^2}{x^2 + y^2}2\epsilon_y.$$

Uppgift 1.2 (2). The radius of a sphere is $R = (22.2 \pm 0.1)cm$. The radius of the base of a cylinder is $r = (12.0 \pm 1.2)cm$, and its height is $h = (24.4 \pm 1.1)cm$. What is the total volum occupied by the sphere and by the cylinder (including the error)?

Lösning 1.2.1 (2). Let the subscript t denote the true value.

$$
\begin{aligned}
V_{total} &= \frac{4\pi}{3}(R + \delta R)^3 + \pi(r + \delta r)^2 h \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2 + \delta R^3) + \pi(r^2 + 2\delta r + \delta^2)(h + \delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 + 2\delta r)(h + \delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 h + r^2\delta h + 2\delta hr + 2\delta r\delta h) \\
&= \frac{4\pi}{3}(R^3 + 3R^2\delta R) + \pi(r^2 h + r^2\delta h + 2r\delta h).
\end{aligned}
$$

To calculate the minimum volume we let the absolute be so much negative as possible and the reverse for the maximum, if we plug in these values we get:

$$55668.1 \le V_{total} \le 58068.$$

Uppgift 1.3 (3). Suppose

$$F = \frac{P\pi a^4}{8lw}.$$

Write down an expression for the maximum percentage error in F. Suppose further that $\frac{\delta P}{P} = 0.5\%$, $\frac{\delta a}{a} = 0.5\%$, $\frac{\delta l}{l} = 0.1\%$, and $\frac{\delta w}{w} = 1\%$, calculate the percent error in the result.

Lösning 1.3.1 (3).

$$
\begin{aligned}
\epsilon_F = \epsilon_{\frac{P\pi a^4}{8lw}} = \epsilon_{\frac{Pa^4}{lw}} &= \epsilon_{Pa^4} - \epsilon_{lw} = \left(\epsilon_P + \epsilon_{a^4}\right) - \left(\epsilon_l + \epsilon_w\right) \\
&= \epsilon_P + 4\epsilon_a - \epsilon_l - \epsilon_w \\
&= 0.5\% + 4 \cdot 0.5\% - 0.1\% - 0.1\% \\
&= 2.3\%.
\end{aligned}
$$

Uppgift 1.4 (4). Solve the system

$$
\begin{aligned}
5x_1 + 7x_2 &= b_1 \\
7x_1 + 10x_2 &= b_2,
\end{aligned}
$$

using Gaussian elimination method to obtain the solution x_1 when $b_T = (b_1, b_2) = (0.7, 1)$. Also solve the above system with $b_A = (b_1, b_2) = (0.69, 1.01)$ using the same method to obtain the solution x_2. Show that