MM5016 Laboration 8 Error analysis

Contents

1 Problems

Uppgift 1.1 (1)**.** The two quantities x and y have a relative error of ϵ_x respectively ϵ_y. Calculate the error in the following cases:

1. $x - 2y$
2. $4x - 5y$
3. $3xy$
4. x^3/y^5
5. $\sqrt{x}$
6. $\sqrt{x^3}$
7. $\sqrt{2xy}$
8. $2\sqrt{\frac{x}{3y}}$
9. $2\sqrt{\frac{x^3}{3y}}$
10. $\frac{x}{y} + \frac{y}{x}$
11. $x^2 + y^2$

Lemma 1.1 (error of scaled approximate value)**.** Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and λ a scalar. Then the relative error of $\lambda\tilde{x}$ is $\epsilon_{\lambda x} = \epsilon_x$. This is because $\lambda\tilde{x} = \lambda x(1 + \epsilon_{\lambda x})$.

Lemma 1.2 (error of exponent)**.** Let x be a value and $\tilde{x}$ and approximation with relative error ϵ_x, and a an integer. Then the relative error of $\tilde{x}^a$ is $\epsilon_{x^a} = a\epsilon_x$. This is because $\epsilon_{x^a} = \epsilon_{xx^{a-1}} = \epsilon_x + \epsilon_{x^{a-1}}$.

Lösning 1.1.1 (1)**.** (1)

$$\begin{aligned}\epsilon_{x-2y} &= \frac{x}{x-(2y)}\epsilon_x - \frac{(2y)}{x-(2y)}\epsilon_{2y} \\ &= \frac{x}{x-2y}\epsilon_x - \frac{2y}{x-2y}\epsilon_y.\end{aligned}$$

(2)

$$\begin{aligned}\epsilon_{4x-5y} &= \frac{4x}{4x-5y}\epsilon_{4x} - \frac{5y}{4x-5y}\epsilon_{5y} \\ &= \frac{4x}{4x-5y}\epsilon_x - \frac{5y}{4x-5y}\epsilon_y.\end{aligned}$$

(3)

$$\epsilon_{3xy} = \epsilon_{xy} = \epsilon_x + \epsilon_y.$$

(4)

$$\epsilon_{\frac{x^3}{y^5}} = \epsilon_{x^3} - \epsilon_{y^5} = 3\epsilon_x - 5\epsilon_y.$$

(5)

$$\begin{aligned}\epsilon_x &= \epsilon_{\sqrt{x}^2} = 2 \cdot \epsilon_{\sqrt{x}} \\ \implies \epsilon_{\sqrt{x}} &= \epsilon_x/2.\end{aligned}$$

(6)

$$\epsilon_{\sqrt{x^3}} = \frac{\epsilon_{x^3}}{2} = \frac{3}{2}\epsilon_x.$$

(7)

$$\begin{aligned}\epsilon_{\sqrt{2xy}} &= \epsilon_{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{y}} \\ &= \epsilon_{\sqrt{x}\cdot\sqrt{y}} = \epsilon_{\sqrt{x}} + \epsilon_{\sqrt{y}} \\ &= \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.\end{aligned}$$

(7)

$$\epsilon_{2\sqrt{\frac{x^3}{3y}}} = \epsilon_{\sqrt{\frac{x^3}{3y}}} = \frac{\epsilon_{\frac{x^3}{3y}}}{2} = \frac{\epsilon_{x^3} - \epsilon_{3y}}{2} = \frac{3\epsilon_x - \epsilon_y}{2}.$$

(8)

$$\begin{aligned}\epsilon_{\frac{x}{y}+\frac{y}{x}} &= \frac{\frac{x}{y}}{\frac{x}{y}+\frac{y}{x}}\epsilon_{\frac{x}{y}} + \frac{\frac{y}{z}}{\frac{x}{y}+\frac{y}{x}}\epsilon_{\frac{y}{z}} = \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x}{yz}+\frac{y}{xz}}\epsilon_{\frac{y}{z}} \\ &= \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{y}{\frac{x^2+y^2}{xyz}}\epsilon_{\frac{y}{z}} \\ &= \frac{x}{x+\frac{y^2}{x}}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2+y^2}\epsilon_{\frac{y}{z}} \\ &= \frac{x^2}{x^2+y^2}\epsilon_{\frac{x}{y}} + \frac{xy^2z}{x^2+y^2}\epsilon_{\frac{y}{z}} \\ &= \frac{x}{x^2+y^2}(\epsilon_{\frac{x}{y}} + y^2z\epsilon_{\frac{y}{z}}) \\ &= \frac{x}{x^2+y^2}(\epsilon_x - \epsilon_y + y^2z(\epsilon_y - \epsilon_z)).\end{aligned}$$

(8)

$$\begin{aligned}\epsilon_{x^2+y^2} &= \frac{x^2}{x^2+y^2}\epsilon_{x^2} + \frac{y^2}{x^2+y^2}\epsilon_{y^2} \\ &= \frac{x^2}{x^2+y^2}2\epsilon_x + \frac{y^2}{x^2+y^2}2\epsilon_y.\end{aligned}$$

Uppgift 1.2 (2). The radius of a sphere is $R = (22.2 \pm 0.1)cm$. The radius of the base of a cylinder is $r = (12.0 \pm 1.2)cm$, and its height is $h = (24.4 \pm 1.1)cm$. What is the total volum occupied by the sphere and by the cylinder (including the error)?

Lösning 1.2.1 (2). Let the subscript t denote the true value.

$$\begin{aligned}V_{total} &= \frac{4\pi}{3}(R+\delta R)^3 + \pi(r+\delta r)^2 h \\ &= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2 + \delta R^3) + \pi(r^2 + 2\delta r + \delta^2)(h+\delta h) \\ &= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 + 2\delta r)(h+\delta h) \\ &= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 h + r^2\delta h + 2\delta h r + 2\delta r\delta h) \\ &= \frac{4\pi}{3}(R^3 + 3R^2\delta R) + \pi(r^2 h + r^2\delta h + 2r\delta h).\end{aligned}$$

To calculate the minimum volume we let the absolute be so much negative as possible and the reverse for the maximum, if we plug in these values we get:

$$55668.1 \le V_{total} \le 58068.$$

Uppgift 1.3 (3). Suppose

$$F = \frac{P\pi a^4}{8lw}.$$

Write down an expression for the maximum percentage error in F. Suppose further that $\frac{\delta P}{P} = 0.5\%$, $\frac{\delta a}{a} = 0.5\%$, $\frac{\delta l}{l} = 0.1\%$, and $\frac{\delta w}{w} = 1\%$, calculate the percent error in the result.

Lösning 1.3.1 (3).

$$\begin{aligned}\epsilon_F &= \epsilon_{\frac{P\pi a^4}{8lw}} = \epsilon_{\frac{Pa^4}{lw}} = \epsilon_{Pa^4} - \epsilon_{lw} = (\epsilon_P + \epsilon_{a^4}) - (\epsilon_l + \epsilon_w) \\ &= \epsilon_P + 4\epsilon_a - \epsilon_l - \epsilon_w \\ &= 0.5\% + 4\cdot 0.5\% - 0.1\% - 0.1\% \\ &= 2.3\%.\end{aligned}$$

Uppgift 1.4 (4). Solve the system

$$\begin{aligned}5x_1 + 7x_2 &= b_1 \\ 7x_1 + 10x_2 &= b_2,\end{aligned}$$

using Gaussian elimination method to obtain the solution x_1 when $b_T = (b_1, b_2) = (0.7, 1)$. Also solve the above system with $b_A = (b_1, b_2) = (0.69, 1.01)$ using the same method to obtain the solution x_2. Show that

Lösning 1.4.1 (4). **Solving for x_1:**

$$\begin{aligned}5x_1 + 7x_2 &= b_1 \\ 7x_1 + 10x_2 &= b_2.\end{aligned}$$

Let us scale each row, so that the factors in x_1 match:

$$\begin{aligned}35x_1 + 49x_2 &= 7b_1 \\ 35x_1 + 50x_2 &= 5b_2.\end{aligned}$$

Let's eliminate:

$$\begin{aligned} 35x_1 + 49x_2 &= 7b_2 \\ x_2 &= 5b_2 - 7b_1. \end{aligned}$$

And let's substitute:

$$\begin{aligned} 35x_1 + 245b_2 - 343b_1 &= 7b_2 \\ x_2 &= 5b_2 - 7b_1. \end{aligned}$$

Simpify:

$$\begin{aligned} x_1 &= 10b_1 - 7b_2 \\ x_2 &= 5b_2 - 7b_1. \end{aligned}$$

We can then plugin the b_T or b_A, to get the desired solution. If we plugin b_T we get the solution (we will now transition from x_1 and x_2 being numbers to vectors, specified by the correction in Discord):

$$x_1 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}.$$

and for b_A we get:

$$x_2 = \begin{pmatrix} \text{-0.17} \\ 0.22 \end{pmatrix}.$$

Proving the inequality: First let us calculate the left hand side:

$$\begin{aligned} &\frac{\|x_1 - x_2\|_2}{\|x_1\|_2} \\ &= \frac{\sqrt{0.17^2 + 0.02^2}}{\sqrt{0.1^2}} \\ &= 10 \cdot \sqrt{0.0293}. \end{aligned}$$

Let us now calculate the right hand side. $\|A\|_2\|A^{-1}\|_2$ is equal to the eigenvalue with greatest magnitude. So let us calculate the eigenvalues. The charactheristic equation for A is

$$\begin{aligned}
& (5-\lambda)(10-\lambda)-49=0 \\
\iff & \lambda^2-15x+1=0 \\
\iff & (\lambda-\frac{15}{2})^2=\frac{15^2}{4}-1 \\
\iff & (\lambda-\frac{15}{2})^2=\frac{15^2-4}{4} \\
\iff & (\lambda-\frac{15}{2})=\pm\frac{\sqrt{15^2-4}}{2} \\
\iff & \lambda=\frac{15}{2}\pm\frac{\sqrt{15^2-4}}{2} \\
\iff & \lambda=\frac{15\pm\sqrt{15^2-4}}{2} \\
\iff & \lambda=\frac{15\pm\sqrt{221}}{2}.
\end{aligned}$$

So the greatest eigenvalue is

$$\frac{15+\sqrt{221}}{2}.$$

Let us now calculate the rest of the right hand side:

$$\begin{aligned}
& \frac{\|b_T-b_A\|_2}{\|b_T\|_2} \\
& =\frac{\sqrt{0.01^2+0.01^2}}{\sqrt{0.7^2+1}} \\
& =\frac{\sqrt{0.0002}}{\sqrt{1.49}}.
\end{aligned}$$

The inequality is thus equivalent to:

$$\begin{aligned}
& 10\cdot\sqrt{0.0293}\leq(223+15\sqrt{221})\frac{\sqrt{0.0002}}{\sqrt{1.49}} \\
\iff & \sqrt{100\cdot 0.0293}\leq(223+15\sqrt{221})\sqrt{\frac{0.0002}{1.49}} \\
\iff & \sqrt{2.93}\leq(223+15\sqrt{221})\sqrt{\frac{1}{10^4}\cdot\frac{2}{1.49}} \\
\iff & \sqrt{2.93}\leq(223+15\sqrt{221})\frac{1}{10^2}\cdot\sqrt{\frac{2}{1.49}} \\
\iff & 10^2\sqrt{2.93}\leq(223+15\sqrt{221})\cdot\sqrt{\frac{2}{1.49}} \\
\iff & \sqrt{293}\leq(223+15\sqrt{221})\cdot\sqrt{\frac{2}{1.49}} \\
\iff & \sqrt{\frac{1.49\cdot 293}{2}}\leq 223+15\sqrt{221} \\
\iff & \sqrt{218.285}\leq 223+15\sqrt{221} \\
& .
\end{aligned}$$

Which is true, and thus the proof is done.

Uppgift 1.5 (5)**.** Show by an example that $\|\cdot\|_M$ defined by $\|A\|_M = \max_{1\leq i,j\leq n} \|a_{ij}\|$ does not define an induced matrix norm.

Lösning 1.5.1 (5)**.** Assume $\|\cdot\|_M$ is a norm. Let

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}.$$

Thus

$$\|A\|\|B\| = 9$$

and

$$\|AB\| = 10.$$

Which means that

$$\|A\|\|B\| < \|AB\|.$$

This is a contradiction for the rules about norms. The assumption was thus wrong, and the proof is complete.

Uppgift 1.6 (6)**.** Show that $\kappa(A) \geq 1$ for any non-singular matrix A.

Lösning 1.6.1 (6)**.** Let us prove it for matrices of dimension two or higher. Let A be a non singular matrix of dimension n. That means there exists n linearly independent eigenvectors, as otherwise the span of A wouldn't be the whole room, and A would be singular. But from linear algebra we know that each distinct eigenvector has it's own eigen value, which are all distinct. Because they are distinct there exists a largest one, let's call it λ.