

MM5016 Laboration 8 Error analysis

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1 Problems

Uppgift 1.1 (1). The two quantities x and y have a relative error of ϵ_x respectively ϵ_y . Calculate the error in the following cases:

1. $x - 2y$
2. $4x - 5y$
3. $3xy$
4. x^3/y^5
5. \sqrt{x}
6. $\sqrt{x^3}$
7. $\sqrt{2xy}$
8. $2\sqrt{\frac{x}{3y}}$
9. $2\sqrt{\frac{x^3}{3y}}$
10. $\frac{x}{y} + \frac{y}{x}$
11. $x^2 + y^2$

Lemma 1.1 (error of scaled approximate value). Let x be a value and \tilde{x} and approximation with relative error ϵ_x , and λ a scalar. Then the relative error of $\lambda\tilde{x}$ is $\epsilon_{\lambda x} = \epsilon_x$. This is because $\lambda\tilde{x} = \lambda x(1 + \epsilon_{\lambda x})$.

Lemma 1.2 (error of exponent). Let x be a value and \tilde{x} and approximation with relative error ϵ_x , and a an integer. Then the relative error of \tilde{x}^a is $\epsilon_{x^a} = a\epsilon_x$. This is because $\epsilon_{x^a} = \epsilon_{xx^{a-1}} = \epsilon_x + \epsilon_{x^{a-1}}$.

Lösning 1.1.1 (1). (1)

$$\begin{aligned}\epsilon_{x-2y} &= \frac{x}{x - (2y)}\epsilon_x - \frac{(2y)}{x - (2y)}\epsilon_{2y} \\ &= \frac{x}{x - 2y}\epsilon_x - \frac{2y}{x - 2y}\epsilon_y.\end{aligned}$$

(2)

$$\begin{aligned}\epsilon_{4x-5y} &= \frac{4x}{4x - 5y}\epsilon_{4x} - \frac{5y}{4x - 5y}\epsilon_{5y} \\ &= \frac{4x}{4x - 5y}\epsilon_x - \frac{5y}{4x - 5y}\epsilon_y.\end{aligned}$$

(3)

$$\epsilon_{3xy} = \epsilon_{xy} = \epsilon_x + \epsilon_y.$$

(4)

$$\epsilon_{\frac{x^3}{y^5}} = \epsilon_{x^3} - \epsilon_{y^5} = 3\epsilon_x - 5\epsilon_y.$$

(5)

$$\begin{aligned}\epsilon_x &= \epsilon_{\sqrt{x^2}} = 2 \cdot \epsilon_{\sqrt{x}} \\ \implies \epsilon_{\sqrt{x}} &= \epsilon_x/2.\end{aligned}$$

(6)

$$\epsilon_{\sqrt{x^3}} = \frac{\epsilon_{x^3}}{2} = \frac{3}{2}\epsilon_x.$$

(7)

$$\begin{aligned}\epsilon_{\sqrt{2xy}} &= \epsilon_{\sqrt{2} \cdot \sqrt{x} \cdot \sqrt{y}} \\ &= \epsilon_{\sqrt{x} \cdot \sqrt{y}} = \epsilon_{\sqrt{x}} + \epsilon_{\sqrt{y}} \\ &= \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2}.\end{aligned}$$

(7)

$$\epsilon_{2\sqrt{\frac{x^3}{3y}}} = \epsilon_{\sqrt{\frac{x^3}{3y}}} = \frac{\epsilon_{\frac{x^3}{3y}}}{2} = \frac{\epsilon_{x^3} - \epsilon_{3y}}{2} = \frac{3\epsilon_x - \epsilon_y}{2}.$$

(8)

$$\begin{aligned}\epsilon_{\frac{x}{y} + \frac{y}{x}} &= \frac{\frac{x}{y}}{\frac{x}{y} + \frac{y}{x}} \epsilon_{\frac{x}{y}} + \frac{\frac{y}{x}}{\frac{x}{y} + \frac{y}{x}} \epsilon_{\frac{y}{x}} = \frac{x}{x + \frac{y^2}{x}} \epsilon_{\frac{x}{y}} + \frac{y}{\frac{x^2}{y} + y} \epsilon_{\frac{y}{x}} \\ &= \frac{x}{x + \frac{y^2}{x}} \epsilon_{\frac{x}{y}} + \frac{y}{\frac{x^2 + y^2}{x}} \epsilon_{\frac{y}{x}} \\ &= \frac{x}{x + \frac{y^2}{x}} \epsilon_{\frac{x}{y}} + \frac{xy^2}{x^2 + y^2} \epsilon_{\frac{y}{x}} \\ &= \frac{x^2}{x^2 + y^2} \epsilon_{\frac{x}{y}} + \frac{xy^2}{x^2 + y^2} \epsilon_{\frac{y}{x}} \\ &= \frac{x}{x^2 + y^2} (\epsilon_{\frac{x}{y}} + y^2 \epsilon_{\frac{y}{x}}) \\ &= \frac{x}{x^2 + y^2} (\epsilon_x - \epsilon_y + y^2 (\epsilon_y - \epsilon_z)).\end{aligned}$$

(8)

$$\begin{aligned}\epsilon_{x^2 + y^2} &= \frac{x^2}{x^2 + y^2} \epsilon_{x^2} + \frac{y^2}{x^2 + y^2} \epsilon_{y^2} \\ &= \frac{x^2}{x^2 + y^2} 2\epsilon_x + \frac{y^2}{x^2 + y^2} 2\epsilon_y.\end{aligned}$$

Uppgift 1.2 (2). The radius of a sphere is $R = (22.2 \pm 0.1)cm$. The radius of the base of a cylinder is $r = (12.0 \pm 1.2)cm$, and its height is $h = (24.4 \pm 1.1)cm$. What is the total volum occupied by the sphere and by the cylinder (including the error)?

Lösning 1.2.1 (2). Let the subscript t denote the true value.

$$\begin{aligned}V_{total} &= \frac{4\pi}{3}(R + \delta R)^3 + \pi(r + \delta r)^2 h \\&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2 + \delta R^3) + \pi(r^2 + 2\delta r + \delta^2)(h + \delta h) \\&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 + 2\delta r)(h + \delta h) \\&= \frac{4\pi}{3}(R^3 + 3R^2\delta R + 3R\delta R^2) + \pi(r^2 h + r^2 \delta h + 2\delta h r + 2\delta r \delta h) \\&= \frac{4\pi}{3}(R^3 + 3R^2\delta R) + \pi(r^2 h + r^2 \delta h + 2r\delta h).\end{aligned}$$

To calculate the minimum volume we let the absolute be so much negative as possible and the reverse for the maximum, if we plug in these values we get:

$$55668.1 \leq V_{total} \leq 58068.$$

Uppgift 1.3 (3). Suppose

$$F = \frac{P\pi a^4}{8lw}.$$

Write down an expression for the maximum percentage error in F . Suppose further that $\frac{\delta P}{P} = 0.5\%$, $\frac{\delta a}{a} = 0.5\%$, $\frac{\delta l}{l} = 0.1\%$, and $\frac{\delta w}{w} = 1\%$, calculate the percent error in the result.

Lösning 1.3.1 (3).

$$\begin{aligned}\epsilon_F &= \epsilon_{\frac{P\pi a^4}{8lw}} = \epsilon_{\frac{Pa^4}{lw}} = \epsilon_{Pa^4} - \epsilon_{lw} = (\epsilon_P + \epsilon_{a^4}) - (\epsilon_l + \epsilon_w) \\&= \epsilon_P + 4\epsilon_a - \epsilon_l - \epsilon_w \\&= 0.5\% + 4 \cdot 0.5\% - 0.1\% - 0.1\% \\&= 2.3\%.\end{aligned}$$

Uppgift 1.4 (4). Solve the system

$$\begin{aligned}5x_1 + 7x_2 &= b_1 \\7x_1 + 10x_2 &= b_2,\end{aligned}$$

using Gaussian elimination method to obtain the solution x_1 when $b_T = (b_1, b_2) = (0.7, 1)$. Also solve the above system with $b_A = (b_1, b_2) = (0.69, 1.01)$ using the same method to obtain the solution x_2 . Show that