## 2.1 Exercise Sheet 1

### 2.1.1 Numerical and Exploration

**Exercise 2.1.1.** For an ordered pair $(a, b)$ of rational numbers, consider the polynomial

$$f_{a,b}(x, y) := ax^2 + by^2 - 1 \in \mathbb{Q}[x, y].$$

Let $C(a, b) = C_{f_{a,b}} \subset \mathbb{A}^2_{\mathbb{Q}}$ be the rational affine plane algebraic curve defined by $f_{a,b}$.

(a) Show that $C(2/5, 1/5) = \emptyset$.
(b) Characterize all primes $p$ such that $C(1/p, 1/p) = \emptyset$.
(c) Characterize all pairs $(a, b)$ such that $C(a, b) = \emptyset$.

**Exercise 2.1.2.**

(a) Play around with graphs of real affine plane algebraic curves (RAPACs) on, say, Desmos or WolframAlpha. What is the coolest thing you can get a graph to do (cross itself thrice, look like a heart, etc.)?
(b) How many pieces (i.e. connected components) can a RAPAC of degree $d = 2$ have? How about $d = 3$? What about $d \in \{4, 5, 6, 7\}$?
(c) What can you say in general? Can you come up with upper or lower bounds for the number of pieces?
(d) Does the number of pieces depend on the nesting relations[1] between them? Does it depend on (or dictate) their shapes (e.g. convexity)?[2]

**Exercise 2.1.3.**

(a) Let $P \subset \mathbb{A}^2_{\mathbb{R}}$ be the polar curve implicitly defined by the equation

$$r^3 + r\cos\theta - \sin 4\theta = 0.$$

Find a nonconstant polynomial $f(x, y) \in \mathbb{R}[x, y]$ such that the curve $C_f \subset \mathbb{A}^2_{\mathbb{R}}$ defined by $f$ contains $P$, i.e. satisfies $P \subset C_f$.[3]
(b) What is the degree of your $f$? What is the smallest possible degree of such an $f$?
(c) By your choice of $f$, we have the containment $P \subset C_f$. Is $P$ all of $C_f$? If so, can you explain why (perhaps by retracing steps)? If not, how would you describe the extraneous components of $C_f \smallsetminus P$? Could you have predicted them? Can you pick an $f$ that provably minimizes the number of extraneous components?
(d) Repeat the same analysis as in (a) through (c) for other such implicitly defined polar curves of your own devising.
(e) Can you perform the same analysis as above for the Archimedean spiral, which is the polar curve implicitly defined by the equation $r = \theta$?

Draw pictures, or get a computer to draw them for you, but beware–is your software doing exactly what you think it is? For instance, plotting the equation

$$(x^2 + y^2)^{3/2} + x - \sin\left(4\arctan\left(\frac{y}{x}\right)\right) = 0$$

in Desmos does not give you the correct picture for $P$. (Why?)

[1] What does that mean? What are those?

[2] Here's a harder result to whet your appetite: if $d = 4$ and there is a nested pair of closed ovals, then the inner oval must be convex and there cannot be more components, although there may be up to 4 non-convex components in general. You may not be able to prove this now, but you should be able to solve this problem by the end of the course.

[3] I like to use the symbol $\subset$ to mean "is contained in or equal to". Others prefer the symbol $\subseteq$ to denote the same thing. I will use the symbol $\subsetneq$ when I want to exclude the possibility of equality.

**Exercise 2.1.4.** Consider the surface defined by the equation $z^3+xz-y=0$, pictured in Figure 2.1. The orthogonal projection of this surface to the $xy$-plane outlines a cuspidal curve.

(a) Find the equation describing this cuspidal curve, and prove the assertion made above.
(b) How does all of this relate to the Cardano formula for the solution to the cubic equation?

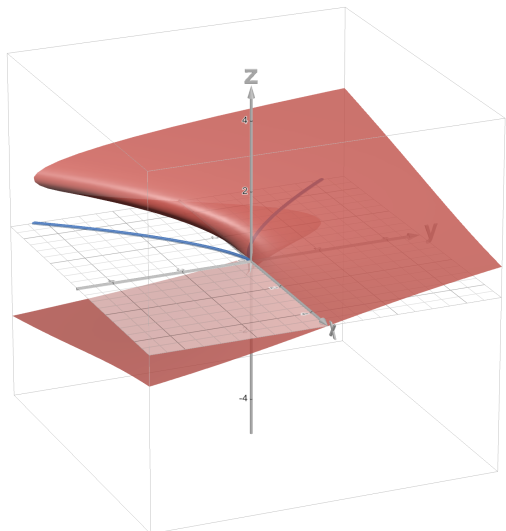


Figure 2.1: The surface $z^3+xz-y=0$ when orthogonally projected onto the $xy$-plane outlines a cuspidal curve. Picture made with Desmos 3D.

**Exercise 2.1.5.** Can you find a way to use the conchoid of Nichomedes (Example 1.2.4) to trisect a given angle? You may suppose that you know how to construct a conchoid with any given parameters. (Hint: see Figure 2.2.) How far can you take this–what else can you do with conchoids of different parameters? Once you've done that, use the cissoid of Diocles to give a compass and ruler (and cissoid) construction of $\sqrt[3]{2}$. Why do these constructions not contradict results from Galois theory?

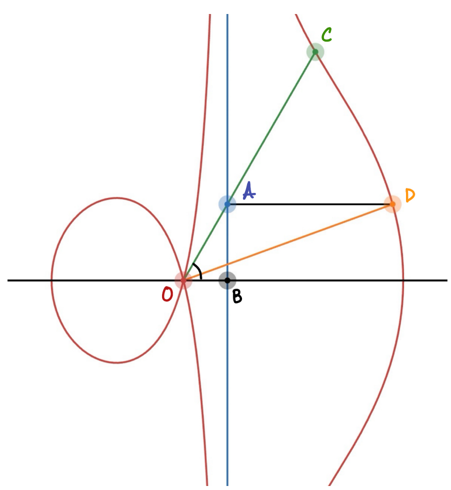


Figure 2.2: The Conchoid of Nichomedes and Angle Trisection. Picture made with Desmos and edited in Notability.

**Exercise 2.1.6.** Show that over $k = \mathbb{C}$, every affine conic section, i.e. plane curve of degree 2 defined by a polynomial of the form

$$f(x, y) = ax^2 + 2hxy + by^2 + 2ex + 2fy + c \in \mathbb{C}[x, y]$$

for some $a, b, c, e, f, h \in \mathbb{C}$, not all zero, can be brought by an affine change of coordinates into one of the following forms:

(a) an ellipse/circle/hyperbola defined by $x^2 + y^2 = 1$,
(b) a parabola defined by $y = x^2$, or
(c) a pair of lines defined by $xy = 0$, or
(d) a double line defined by $x^2 = 0$.

Note that the equivalence of the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ in $\mathbb{A}^2_{\mathbb{C}}$ uses that $\mathbb{C}$ contains a square root of $-1$ (how?). Can you come up with a similar classification over $k = \mathbb{R}$? What about other fields like $k = \mathbb{F}_q$?

### 2.1.2 PODASIPs

Prove or disprove and salvage if possible the following statements.

**Exercise 2.1.7.** Let $k$ be a field, $C \subset \mathbb{A}^2_k$ be an algebraic curve, and $\ell \subset \mathbb{A}^2_k$ be a line. Then the intersection $C \cap \ell \subset \mathbb{A}^2_k$ of $C$ and $\ell$ is finite.

**Exercise 2.1.8.** Given any field $k$ and function $f : k \to k$, we define its **graph** to be the subset

$$\Gamma_f := \mathbb{V}(y - f(x)) = \{(x, f(x)) : x \in k\} \subset \mathbb{A}^2_k.$$

(a) When $k = \mathbb{R}$ and $f(x) = \sin x$, the graph $\Gamma_f \subset \mathbb{A}^2_{\mathbb{R}}$ is an algebraic curve.
(b) When $k = \mathbb{R}$ and $f(x) = \mathrm{e}^x$, the graph $\Gamma_f \subset \mathbb{A}^2_{\mathbb{R}}$ is an algebraic curve.
(c) In the setting of (b), every line $\ell \subset \mathbb{A}^2_{\mathbb{R}}$ meets $\Gamma_f$ in at most two points.
(d) When $k = \mathbb{C}$ and $f(x) = \mathrm{e}^x$, the graph $\Gamma_f \subset \mathbb{A}^2_{\mathbb{C}}$ is an algebraic curve.

[Possible Hints: For (a), see Exercise 2.1.7. For (b), the exponential function grows *very fast*, so that your solution to (a) may not work for (b) thanks to (c). You may either use this growth to your advantage, or you may first solve (d) and use a little bit of complex analysis.]

**Exercise 2.1.9** (Apparently Transcendental Curves).

(a) The curve $C_1 \subset \mathbb{A}^2_{\mathbb{R}}$ given parametrically as

$$C_1 = \{(\mathrm{e}^{2t} + \mathrm{e}^t + 1, \mathrm{e}^{3t} - 2) : t \in \mathbb{R}\}$$

is an algebraic curve.

(b) The curve $C_2 \subset \mathbb{A}^2_{\mathbb{R}}$ defined by the vanishing of the function $f$ defined by

$$f(x, y) = x^2 + y^2 + \sin^2(x + y)$$

is an algebraic curve.

These examples are a little silly, but they illustrate important points (what?). Can we improve our definition of a plane algebraic curve to avoid such silliness?

**Exercise 2.1.10.** Given any $g(r, c, s) \in \mathbb{R}[r, c, s]$, there is a unique polynomial $f(x, y) \in \mathbb{R}[x, y]$ such that the polar algebraic curve $P_g$ implicitly defined by $g$ (see §1.2.2) is contained in the algebraic curve $C_f$ defined by $f$, i.e. satisfies $P_g \subset C_f$.