1.12 07/05/24 - Projective Changes of Coordinates, Multiplicity and Smoothness, Classification of Projective Conics

1.12.1 Projective Changes of Coordinates

We defined affine changes of coordinates by setting x and y to be linear polynomials in x' and y' subject to a nondegeneracy condition. We want to mimic the situation in the projective case: we want to set X, Y, Z to be three homogeneous linear polynomials $L, M, N \in k[X', Y', Z']$, but now we need to ensure nondegeneracy as well. If L, M, N were concurrent in $\mathbb{P}^2$, then this point of concurrency would be mapped to $[0 : 0 : 0]$, which doesn't make any sense; therefore, we need to at least ask that L, M, N be nonconcurrent. It turns out that in the projective case, this condition is also sufficient. The discussion in §1.11.1 gives us a direct condition to check to ensure nonconcurrency, and leads us to

Definition 1.12.1. A projective change of coordinates is a transformation

$$\Phi : \mathbb{P}^2_k(X', Y', Z') \to \mathbb{P}^2_k(X, Y, Z)$$

of the form

$$[X : Y : Z] = \Phi[X' : Y' : Z'] = [AX' + BY' + CZ' : DX' + EY' + FZ' : GX' + HY' + IZ']$$

for some $A, B, C, D, E, F, G, H, I \in k$ such that

$$\det \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \neq 0.$$

Again, the nondegeneracy condition on the determinant ensures that the transformation is both well-defined and, in fact, invertible: this is because the transformation is given before homogenization (i.e. quotienting by the equivalence relation of scaling) by the map

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix},$$

so if this transformation matrix has nonzero determinant, then by Cramer's rule it is an invertible matrix, and we can recover $[X' : Y' : Z']$ from $[X : Y : Z]$ using

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

Since, of course, two such transformation matrices define the same transformation if they differ by scalar multiples, the group of all projective changes of coordinates is the group $\mathrm{PGL}_3\, k \subset \mathbb{P}^8_k$ of all 3×3 matrices in k with nonzero determinant up to simultaneously scaling by a nonzero scalar, i.e. $\mathrm{GL}_3\, k$ subject to the equivalence relation $M \sim \lambda M$ for all $M \in \mathrm{GL}_3\, k$ and $\lambda \in k^\times$. This scaling invariance implies that, unlike the affine case, a projective change of coordinates Φ does not quite give us a pullback map on the homogeneous polynomial ring $\Phi^* : k[X, Y, Z] \to k[X', Y', Z']$, but we can always choose such a pullback map which does what we want[27]: such a pullback map would then necessarily be an isomorphism, and any two such maps would be related by a nonscalar scalar.

[27] What do we want?

The key fact to note here is that projective changes of coordinates respect incidence. This is captured by

Lemma 1.12.2. Let $\Phi : \mathbb{P}^2_k(X', Y', Z') \to \mathbb{P}^2_k(X, Y, Z)$ be a projective change of coordinates. Then three points $P_1, P_2, P_3 \in \mathbb{P}^2_k(X', Y', Z')$ are collinear iff $\Phi(P_1), \Phi(P_2)$ and $\Phi(P_3)$ are.

Proof. Write $P_i = [X'_i : Y'_i : Z'_i]$ for $i = 1, 2, 3$. Using Proposition 1.11.4 and the fact that determinants are multiplicative and invariant under taking transposes, we conclude that

$$\begin{aligned} P_1, P_2, P_3 \text{ are collinear } &\Leftrightarrow \det \begin{bmatrix} X'_1 & X'_2 & X'_3 \\ Y'_1 & Y'_2 & Y'_3 \\ Z'_1 & Z'_2 & Z_3 \end{bmatrix} = 0 \\ &\Leftrightarrow \det \left(\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X'_1 & X'_2 & X'_3 \\ Y'_1 & Y'_2 & Y'_3 \\ Z'_1 & Z'_2 & Z_3 \end{bmatrix} \right) = 0 \\ &\Leftrightarrow \Phi(P_1), \Phi(P_2), \Phi(P_3) \text{ are collinear.} \end{aligned}$$

■

Lemma 1.12.2 and Proposition 1.11.3 tell us that projective changes of coordinates preserve all incidence geometry of $\mathbb{P}^2_k$: they take lines to lines, and incidences of points on lines to incidence of points on lines, concurrency of lines to concurrency of lines, etc.

Example 1.12.3. An affine change of coordinates of the form $(x, y) = (ax' + by' + p, cx' + dy' + q)$ is the affine "shadow" of a projective change of coordinates given by the matrix

$$\begin{bmatrix} a & b & p \\ c & d & q \\ 0 & 0 & 1 \end{bmatrix},$$

where the affine and projective nondegeneracy conditions are identical because the determinant of this matrix is $ad - bc$. Note that this "projectivization" of any affine change of coordinates fixes the line at infinity $L_\infty \subset \mathbb{P}^2_k$ as a set (although perhaps not pointwise!), and conversely, any projective change of coordinates that fixes the line at infinity must arise from an affine change of coordinates. Projective changes of coordinates are, however, more powerful, and treat all points (resp. lines) "equally," including points (resp. the line) at infinity.

From the construction itself, it is pretty clear that given any three tuple $L, M, N \in k[X, Y, Z]_1$ of homogeneous linear polynomials which vanish on three nonconcurrent lines, there is a change of coordinates taking the lines given by the vanishing of X, Y, Z to those given by L, M, N; in Exercise 2.4.8 you are invited to make this precise, and to explore whether such a transformation is unique. This incredible flexibility of projective transformations often makes explicit computations with projective curves really easy. Here's some terminology and a proposition we will have repeated ocassion to use.

Definition 1.12.4. A subset $S \subset \mathbb{P}^2_k$ is said to be in general position if no three points in S are collinear. We also say that the points $P_j \in S$ are in general position.

Proposition 1.12.5. Given any two ordered 4-tuples

$$P = (P_1, P_2, P_3, P_4) \text{ and } Q = (Q_1, Q_2, Q_3, Q_4)$$

of points in $\mathbb{P}^2_k$, both in general position, there is a unique projective change of coordinates $\Phi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ taking one to the other, i.e. such that $\Phi(P_i) = Q_i$ for $i = 1, 2, 3, 4$.

Proof. It suffices to show the result when

$$P_1 = E_1 := [1 : 0 : 0], P_2 = E_2 := [0 : 1 : 0], P_3 = E_3 := [0 : 0 : 1] \text{ and } P_4 = E_4 := [1 : 1 : 1],$$

because then we can first uniquely take an arbitrary 4-tuple P to this standard 4-tuple E (because projective changes of curves are invertible), and then further take this standard 4-tuple to an arbitrary collection Q.[28] If we write $Q_i = [X_i : Y_i : Z_i]$, then any Φ taking $E_i \mapsto Q_i$ for $i = 1, 2, 3$ must be given by a matrix of the form

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

for some $\lambda, \mu, \nu \in k^\times$; that this matrix has nonzero determinant uses that Q_1, Q_2, Q_3 are non-collinear. Then the condition $E_4 \mapsto Q_4$ uniquely determines the triple (λ, μ, ν), up to simultaneous scaling, by the requirement that

$$\begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} = t \cdot \begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix}$$

for some $t \in k^\times$, since the matrix on the left is invertible; the fact that the resulting λ, μ, ν from this matrix equation are nonzero then is equivalent to saying that Q_4 does not lie in the lines $\overline{Q_2Q_3}, \overline{Q_1Q_3}$ and $\overline{Q_1Q_2}$ respectively.[29] ∎

1.12.2 Multiplicity, Smoothness, and Intersection Multiplicity

We would like to define the notions of multiplicity, smoothness, tangent lines, and intersection multiplicity in a way that is both invariant under projective changes of coordinates and compatible with dehomogenization. One way to do this is to define these local notions by first changing coordinates so that the point in consideration is $P = [0 : 0 : 1]$, and then use dehomogenization, and then rehomogenize–so for instance, the tangent line to a projective curve at a point would be the projective closure of its affine tangent line in some chart. This approach works, but has the disadvantage that checking invariance under projective changes of coordinates is a much more daunting task than in the affine case. A slightly more elegant approach is given by thinking about local rings.

Recall from Definition 1.10.2 that given a point $P \in \mathbb{A}^2_k$, we define its local ring $\mathcal{O}_{\mathbb{A}^2_k,P} \subset k(x, y)$ to consist of all rational functions on $\mathbb{A}^2_k$ which can be evaluated at P, in which case evaluation at P gives us a ring homomorphism

$$\mathrm{eval}_P : \mathcal{O}_{\mathbb{A}^2_k,P} \to k$$

with kernel

$$I_{\mathbb{A}^2_k,P} := \ker \mathrm{eval}_P$$

[28] Make this statement precise, particularly if it doesn't obviously make sense!

[29] Check this! This uses Cramer's rule.

consisting of all rational functions that vanish at P.[30] If $P = (0,0)$ is the origin, then $I_{\mathbb{A}^2_k,P}$ is an ideal of $\mathcal{O}_{\mathbb{A}^2_k,P}$ generated by x and y. It follows from this that $I^2_{\mathbb{A}^2_k,P}$ is generated by x^2, xy, and y^2, or more generally that for any $n \geq 1$, the ideal $I^n_{\mathbb{A}^2_k,P}$ is generated over $\mathcal{O}_{\mathbb{A}^2_k,P}$ by $x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n$. In particular, we have that

$$\bigcap_{n\geq 0} I^n_{\mathbb{A}^2_k,P} = 0,$$

and hence given any nonzero polynomial $f \in k[x,y] \subset \mathcal{O}_{\mathbb{A}^2_k,P}$, there is a unique largest integer $m \geq 0$ such that $f \in k[x,y] \cap I^m_{\mathbb{A}^2_k,P}$. A moment's reflection shows that this m is nothing but the multiplicity $m_P(f)$ of f at P. This could have been used as an alternative definition of multiplicity, and this notion would then be somewhat visibly invariant under changes of coordinates, since the local ring $\mathcal{O}_{\mathbb{A}^2_k,P}$ and its maximal ideal $I_{\mathbb{A}^2_k,P}$ visibly behave well under changes of coordinates. Further, the above discussion gives us some added flexibility: the same definition applies to any nonzero element of $\mathcal{O}_{\mathbb{A}^2_k,P}$, and so we are now allowed to talk about intersection multiplicities of rational functions that one can evaluate at P. This is a crucial generalization needed to check the invariance of multiplicity under projective changes of coordinates irrespective of the chosen definition. Similarly, the notion of local intersection multiplicity is local: we observed in the proof of existence in Theorem 1.9.9 that $i_P(f,g)$ is just $\dim_k \mathcal{O}_{\mathbb{A}^2_k,P}/(f,g)\mathcal{O}_{\mathbb{A}^2_k,P}$, again dependent only on the local ring.

The above discussion tells us that if we can define projective analogs of the rational function field of $\mathbb{A}^2_k$ and of these local rings in a way that is compatible with taking affine charts, then we would be in good shape to define multiplicity in this case. And indeed, this is possible.

Definition 1.12.6. The rational function field of $\mathbb{P}^2_k$ is the subfield

$$k(\mathbb{P}^2_k) := \left\{ \frac{F}{G} \in k(X,Y,Z) : F, G \text{ are homogeneous of the same degree} \right\} \subset k(X,Y,Z).$$

Given a point $P \in \mathbb{P}^2_k$, we define the local ring of $\mathbb{P}^2_k$ at P to be the ring

$$\mathcal{O}_{\mathbb{P}^2_k,P} := \{ r \in k(\mathbb{P}^2_k) : r = F/G \text{ for some homogeneous } F, G \in k[X,Y,Z] \text{ s.t. } G|_P \neq 0. \}$$

Evaluation at P gives us a surjective map

$$\mathrm{eval}_P : \mathcal{O}_{\mathbb{P}^2_k,P} \to k$$

whose kernel we will denote by $I_{\mathbb{P}^2_k,P}$. Finally, given any integer $n \geq 1$ and homogeneous $F_1, \ldots, F_n \in k[X,Y,Z]$, we define the ideal $(F_1, \ldots, F_n)\mathcal{O}_{\mathbb{P}^2_k,P}$ to consist of all linear combinations of the form

$$\sum_{i=1}^{n} \frac{H_i}{G_i} \cdot F_i,$$

where the $H_i, G_i \in k[X,Y,Z]$ are homogeneous such that $\deg H_i = \deg F_i + \deg G_i$ and after cancellation of common factors we have $G_i|_P \neq 0$.

Given this, we are now ready to handle defining multiplicity of a homogeneous polynomial at a point $P \in \mathbb{P}^2_k$ and the intersection multiplicity of two polynomials, etc.

[30] Some textbooks denote this kernel by $\mathfrak{m}_{\mathbb{A}^2_k,P}$ or $\mathfrak{m}_P$ to emphasize that it is a maximal ideal of $\mathcal{O}_{\mathbb{A}^2_k,P}$, but we will not need this idea and I will stick to $I_{\mathbb{A}^2_k,P}$ or I_P.

Definition 1.12.7. Let $P \in \mathbb{P}^2_k$ be a point.

(a) Given a nonzero homogeneous polynomial $F \in k[X,Y,Z]$, we define the multiplicity of F at P to be the largest integer $m \geq 0$ such that

$$(F)\mathcal{O}_{\mathbb{P}^2_k,P} \subset I^m_{\mathbb{P}^2_k,P}.$$

(b) Given a curve $C \subset \mathbb{P}^2_k$ and point $P \in \mathbb{P}^2_k$, we define the multiplicity of C at P to be

$$m_P(C) := m_P(F)$$

where F is any minimal polynomial for C.

(c) Given two nonzero homogeneous polynomials $F, G \in k[X,Y,Z]$, we define the local intersection multiplicity of F and G at P to be

$$i_P(F,G) := \dim_k \mathcal{O}_{\mathbb{P}^2_k,P}/(F,G)\mathcal{O}_{\mathbb{P}^2_k,P}.$$

(d) Given two curves $C, D \subset \mathbb{P}^2_k$ and point $P \in \mathbb{P}^2_k$, we define the intersection multiplicity of C and D at P to be

$$i_P(C,D) := i_P(F,G)$$

where F, G are any minimal polynomials for C and D.

These definitions have the advantage of being visibly invariant under projective changes of coordinates, but we observe also that they are compatible with definitions from the affine case: setting $x := X/Z$ and $y := Y/Z$ gives us an isomorphism

$$k(\mathbb{P}^2_k) \xrightarrow{\sim} k(x,y)$$

with the property that if $P = [x_0 : y_0 : 1] \in \mathbb{A}^2_k \subset \mathbb{P}^2_k$, then this map takes

$$\mathcal{O}_{\mathbb{P}^2_k,[x_0:y_0:1]} \xrightarrow{\sim} \mathcal{O}_{\mathbb{A}^2_k,(x_0,y_0)} \text{ and } I_{\mathbb{P}^2_k,[x_0:y_0:1]} \xrightarrow{\sim} I_{\mathbb{A}^2_k,(x_0,y_0)}.$$

From this isomorphism and our above discussion on multiplicity, it follows immediately that if $P \in \mathbb{A}^2_k \subset \mathbb{P}^2_k$ and $F \in k[X,Y,Z]$ is a nonzero homogeneous polynomial, then

$$m_P(F) = m_P(F^{\mathrm{i}}),$$

and similarly that if $F, G \in k[X,Y,Z]$ are nonzero homogeneous polynomials, then

$$i_P(F,G) = i_P(F^{\mathrm{i}}, G^{\mathrm{i}}).$$

It follows from this that the function i satisfies axioms similar to (1)-(7) and is also completely characterized by them. Henceforth, we will use notions of (intersection) multiplicity for projective curves without further comment.

Remark 1.12.8. One can reasonably ask: which subring of $k(x,y)$ does $\mathcal{O}_{\mathbb{P}^2_k,P}$ map to when $P \in L_\infty$? The answer is pretty fun to work out and straightforward: if $P = [1:0:0]$, then $\mathcal{O}_{\mathbb{P}^2_k,P} \subset k(x,y)$ corresponds to the ring

$$k\left[\frac{y}{x}, \frac{1}{x}\right]_{(y/x,1/x)} \subset k(x,y)$$

which is the localization of the polynomial ring $k[y/x, 1/x]$ at the maximal ideal $(y/x, 1/x)$. If you do not know what this remark means, you can safely ignore it.

1.12.3 Projective Jacobi Criterion

One useful result that we would like to have in our toolkit is a projective analog of Theorem 1.8.8. For this, we will need

Lemma 1.12.9 (Euler)**.** Suppose $F \in k[X, Y, Z]$ is a homogeneous polynomial of degree $d \geq 0$. If $\partial_X F$ (resp. $\partial_Y F$, $\partial_Z F$) denotes the formal partial derivative of F with respect to X (resp. Y, Z), then

$$X \cdot \partial_X F + Y \cdot \partial_Y F + Z \cdot \partial_Z F = d \cdot F.$$

Proof. Both sides of the equation are k-linear in F, so it suffices to show the result for a monomial of the form $F = X^a Y^b Z^c$, where $a + b + c = d$; but then the statement is clear. ∎

Of course, there is nothing special about the polynomial ring in three variables, and a similar result holds in any number of variables. Lemma 1.12.9 tells us also that if $\operatorname{ch} k \nmid d$ (in particular always in characteristic zero), then the conditions $\partial_X F|_P = \partial_Y F|_P = \partial_Z F|_P = 0$ also imply $F|_P = 0$. We are now ready to prove

Theorem 1.12.10 (Projective Jacobi Criterion)**.** Suppose we are given a curve $C \subset \mathbb{P}^2_k$ and a point $P \in \mathbb{P}^2_k$. Let $F \in k[X, Y, Z]$ be a minimal polynomial for C. Then

(a) $P \in C$ iff $F|_P = 0$, and in this case
(b) P is a singular point of C iff

$$\partial_X F|_P = \partial_Y F|_P = \partial_Z F|_P = 0.$$

(c) If $P \in C$ is a smooth point, then the tangent line $\mathrm{T}_P C$ is defined by the vanishing of

$$\partial_X F|_P \cdot X + \partial_Y F|_P \cdot Y + \partial_Z F|_P \cdot Z = 0,$$

where in these evaluations we use the same representative (X_0, Y_0, Z_0) for the point $P = [X_0 : Y_0 : Z_0]$.

Proof. The statement in (a) is clear. As in the proof of Theorem 1.8.8, all parts are invariant under projective coordinate changes,[31] so it suffices to do the case $P = [0 : 0 : 1]$, and so we may work in the affine chart $\mathbb{A}^2_k$. For (b), we note that P is a singular point for C iff it is a singular point for C°, which by Theorem 1.11.21 has minimal polynomial F^{i}. Theorem 1.8.8 tells us that this happens iff

$$\partial_x F^{\mathrm{i}}|_P = \partial_y F^{\mathrm{i}}|_P = 0.$$

But now we observe that

$$\begin{aligned} \partial_x F^{\mathrm{i}} &= (\partial_X F)^{\mathrm{i}}, \\ \partial_y F^{\mathrm{i}} &= (\partial_Y F)^{\mathrm{i}}, \text{ and} \\ \partial_Z F|_P &= d \cdot F|_P, \end{aligned}$$

where in the last equality we are using Lemma 1.12.9. It follows that if (a) holds, then

$$\partial_x F^{\mathrm{i}}|_P = \partial_y F^{\mathrm{i}}|_P = 0 \Leftrightarrow \partial_X F|_P = \partial_Y F|_P = \partial_Z F|_P = 0,$$

[31] Check! This is the reason for the symmetric shape of the statement, although we will break the symmetry by invoking the affine Jacobi criterion below.

proving (b). Theorem 1.8.8 also tells us that the affine tangent line to C° at P is

$$\partial_x F^{\mathrm{i}}|_{(0,0)} \cdot x + \partial_y F^{\mathrm{i}}|_{(0,0} \cdot y = 0$$

which has projective closure

$$\partial_X F|_P \cdot X + \partial_Y F|_P \cdot Y + \partial_Z F|_P \cdot Z = 0$$

as needed. ■

One immediate consequence of this criterion is an analog of Theorem 1.9.7; this is

Theorem 1.12.11. If $C \subset \mathbb{P}^2_k$ is any curve, then C has only finitely many singular points.

Proof. Identical to the proof of Theorem 1.9.7, using Theorem 1.12.10 instead of Theorem 1.11.21. We can also reduce to the affine case. I leave the details to the reader. ■

That's more than enough abstract theory for now. Let's return to some concrete examples now.

1.12.4 Bézout's Theorem for a Line, Classification of Projective Conics up to Changes of Coordinates

Let's first prove Bézout's theorem for a line.

Theorem 1.12.12. If k is an algebraically closed field, $C \subset \mathbb{P}^2_k$ is a curve of degree $d \geq 1$, and $L \subset \mathbb{P}^2_k$ is a line such that $L \not\subset C$, then

$$\sum_{P \in C \cap L} i_P(C, L) = d.$$

Of course, $\deg L = 1$, so that $d = (\deg C)(\deg L)$.

Proof. By a projective change of coordinates, we can assume $L = L_\infty$. By Theorem 1.11.21, if F is a minimal polynomial for C, then C° has minimal polynomial $f := F^{\mathrm{i}}$, and $L_\infty \not\subset C$ implies that $Z \nmid F$ and so $\deg f = \deg F = \deg C = d$. If we write $f = f_0 + \cdots + f_d$, where each $f_j \in k[x, y]$ is homogeneous of degree j, then

$$F = f^{\mathrm{h}} = Z^d f_0(X, Y) + \cdots + f_d(X, Y).$$

Then points $P \in C \cap L$ are exactly points of the form $[X_0 : Y_0 : 0]$, where $f_d(X_0, Y_0) = 0$, and there are exactly d such points counted with multiplicity, by Lemma 1.8.3, where the two notions of multiplicity coincide by the computation in Example 1.9.12; I leave the details of this verification to the reader, since we will do the more general case soon. ■

We can now use this to classify all projective conics–at least when the base field k is algebraically closed.

Theorem 1.12.13. If k is algebraically closed and $Q \in k[X, Y, Z]_2$ is a nonzero homogeneous polynomial of degree 2, then there is a projective change of coordinates $\Phi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ and a lift to the homogeneous polynomial ring $\Phi^* : k[X, Y, Z] \to k[X, Y, Z]$ such that $\Phi^* Q$ is either X^2, XY or $YZ - X^2$.

It is also clear that three cases are disjoint, since the corresponding projective curves are not isomorphic. One immediate consequence of this algebraic result is

Corollary 1.12.14 (Classification of Projective Conics)**.** If k is an algebraically closed field and $C \subset \mathbb{P}^2_k$ a conic (i.e. curve of degree 2), then there is a projective change of coordinates $\Phi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ taking C to one, and only one, of the following forms:

(a) $C = C_{XY}$, which is a union of two lines that is singular at $[0 : 0 : 1]$, and
(b) $C = C_{YZ-X^2}$, which is a smooth conic.

In particular, it follows that any irreducible conic is smooth; compare this with the proof of this result from Exercise 2.3.4(b).

Proof of Theorem 1.12.13. Either $Q = \ell^2$ for some $L \in k[X, Y, Z]_1$, in which case we can take ℓ to X via some Φ^*; or $Q = \ell_1\ell_2$ for some distinct irreducibles $\ell_1, \ell_2 \in k[X, Y, Z]_1$, in which case we can take $\ell_1 \mapsto X$ and $\ell_2 \mapsto Y$ by a simple application of Proposition 1.12.5; or Q is irreducible. Consider the curve C defined by Q; then C is also irreducible. By Proposition 1.11.13, C is infinite, but by Theorem 1.12.11, C has only finitely many singular points; in particular, all but finitely many points on C are smooth.

Let $P_1, P_2 \in C$ be any two distinct smooth points, and let $L_i = \mathrm{T}_{P_i} C$ for $i = 1, 2$ be the tangent lines at those points. We claim that $P_1 \notin L_2$ (and so, by symmetry, we have $P_2 \notin L_1$), and in particular $L_1 \neq L_2$. Indeed, if $P_1 \in L_2$, then we get that

$$\sum_{P \in C \cap L_2} i_P(C, L_2) \geq i_{P_1}(C, L_2) + i_{P_2}(C, L_2) \geq 1 + 2 = 3,$$

where $i_{P_2}(C, L_2) \geq 2$ because L_2 is tangent to C at P_2 (check!). This, combined with Theorem 1.12.12 tells us that $L_2 \subset C$, which by 1.11.17(b) implies that $L_2 = C$, contradicting the fact that $\deg C = 2$.[32] Since $L_1 \neq L_2$, we conclude from Proposition 1.11.3 that L_1 and L_2 intersect in a unique point, say P_3. Since $P_1 \notin L_2$, it follows that $P_1 \neq P_3$; similarly, $P_2 \neq P_3$. In fact, it follows that P_1, P_2, P_3 are not collinear: if they were collinear, then Proposition 1.11.3 would tell us that the line containing them would have to be both L_1 and L_2, contradicting $L_1 \neq L_2$.

It then follows from 1.12.5 that there is a projective change of coordinates Φ taking $P_1 \mapsto [0 : 0 : 1]$, $P_2 \mapsto [0 : 1 : 0]$ and $P_3 \mapsto [1 : 0 : 0]$. In this coordinate system, L_1 is the line $Y = 0$, and L_2 is the line $Z = 0$. For this Φ and any choice of Φ^*, if we write

$$\Phi^* Q = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2,$$

then $f = 0$ (because $P_1 \in C$), $c = 0$ (because $P_2 \in C$), $d = 0$ (because $L_1 = C_Y$) and $b = 0$ (because $L_2 = C_Z$). In particular, we will have

$$\Phi^* Q = aX^2 + eYZ.$$

Now neither a nor e is zero, because otherwise Q would be reducible. Then we may scale Φ^* by $-a^{-1}$ and further change coordinates so Y is replaced by $-ae^{-1}Y$ to bring Q into the form $YZ - X^2$. ∎

[32] The fact that $P_1 \notin L_2$ uses crucially that C is a conic–for instance, a tangent to a cubic or higher degree curve meets the curve in at least one other point in general.

Remark 1.12.15. A careful analysis of the proof shows that we did not really use, in the last case, that k is algebraically closed, but only that C has at least two points. The above proof can be upgraded, with some care, to also obtain a classification over other fields: namely that if k is *any* field and $Q \in k[X, Y, Z]_2$ is a homogeneous irreducible element of degree 2 such that C_Q has at least two points, then, after a suitable change of coordinates, $Q = YZ - X^2$. This is, in fact, the best we can do in general: if $k = \mathbb{R}$, then the possibilities for Q include, in addition to $X^2, XY, YZ - X^2$, also the "conics" $X^2 + Y^2$ (which defines one point) and $X^2 + Y^2 + Z^2$ (which defines the empty set). The classification of projective conics over an arbitrary field is closely related to the classification of binary quadratic forms in 3 variables over that field. See, for instance, [5, §1.6] for another perspective on this result via this approach.

Next time, we will start by discussing very cool applications of these results–including Pascal's Theorem!