Math 213A F23 Homework 12 Solutions

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If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Prove the identity

$$60 \sum_{\lambda \in \mathbb{Z}[\mathrm{i}]}' \lambda^{-4} = 4 \left(\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3 - x}} \right)^4 .$$

Solution. Consider the square lattice $\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$. This has $g_3 = 0$ and so the corresponding Weierstraß equation is

$$y^2 = 4x^3 - g_2 x = 4x(x^2 - e_1^2),$$

from which we get that $e_1 = \sqrt{g_2}/2$ (here we have used that for a rectangular lattice e_1, e_2, e_3 are all real and that the sum of the roots is zero). From the formulae for the period integrals, we know that

$$\frac{1}{2} = \int_{e_1}^\infty \frac{\mathrm{d}x}{\sqrt{4x^3 - g_2 x}}.$$

From the substitution $x = e_1 u$, we get

$$\frac{1}{2} = \int_1^\infty \frac{e_1 \mathrm{d}u}{\sqrt{4e_1^3(u^3 - u)}} = \frac{1}{2\sqrt{e_1}} \int_1^\infty \frac{\mathrm{d}u}{\sqrt{u^3 - u}}.$$

Changing the variable back to x and using that $g_2 = 4e_1^2$, we conclude that

$$60 \sum_{\lambda \in \mathbb{Z}[\mathrm{i}]}' \lambda^{-4} = g_2 = 4e_1^2 = 4 \left(\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3 - x}} \right)^4$$

as needed. ■

Remark 1. This last integral can be evaluated easily with the help of basic properties of the Beta function $\mathrm{B}(x, y)$. Indeed, the substitution $x = t^{-1/2}$ gives us

$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3 - x}} = \frac{1}{2} \int_0^1 t^{-3/4}(1-t)^{-1/2} \mathrm{d}t = \frac{1}{2} \mathrm{B}\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{1}{2\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2,$$

where in the last step we have used the Euler Reflection Formula to simplify a little. This gives us the fascinating little formula

$$\sum_{\lambda \in \mathbb{Z}[\mathrm{i}]}' \lambda^{-4} = \frac{1}{960\pi^2} \Gamma\left(\frac{1}{4}\right)^8 \approx 3.1512\ldots.$$

Q2. Let $V \subset \mathbb{P}^2$ be a smooth cubic curve, and let $F \subset V$ be its 9 flexes.

(a) Show that any line that passes through 2 points of F passes through a third.
(b) Show that there are no 9 points in $\mathbb{R}^2$ with this property, unless they all lie on the same line.

Solution.

(a) Picking any $\mathcal{O} \in F$ as the basepoint allows us to give V the structure of a group with the group law defined by the property that

$$P, Q, R \in V \text{ satisfy } P + Q + R = 0 \text{ iff } P, Q, R \text{ are collinear,}$$

where this statement is to be interpreted with multiplicities.[1] In particular, the 3-torsion points correspond to the flexes F. Since the 3-torsion points clearly form a subgroup of V, it follows from this defining property that any line passing through two points of F passes through a third.

(b) Suppose $F \subset \mathbb{R}^2$ is a collection of 9 points such that any line through 2 points of F passes through the third. If the convex hull of F is not a line segement, then it is either a triangle or a quadrilateral, since there must be a point on the interior of each edge.

(i) If it were a quadrilateral, say $ABCD$, with points X, Y, Z, W on the interiors of edges AB, BC, CD, DA respectively, then by A and C lie on the opposite side of the line XY than B and so by convexity D and hence Z and W must lie on the same side as A as well; in particular, XY cannot contain any of A, B, C, D, Z, or W, and hence must contain the 9th point, say P. But now the same is true of YZ, ZW, and WX, and this is a contradiction. Therefore, the convex hull cannot be a quadrilateral.

(ii) If it were a triangle, say ABC, then not all the points could be on the boundary of this triangle, so there is at least one side with at most one point on the interior, say D on BC. Let X, Y, Z be points on AB, AD and AC respectively, and P, Q be points on XD and ZD respectively. Then AP and AQ are lines which have only two points on them each, and this is a contradiction.

Therefore, the convex hull of F is a line segment as needed. See Figure 1.

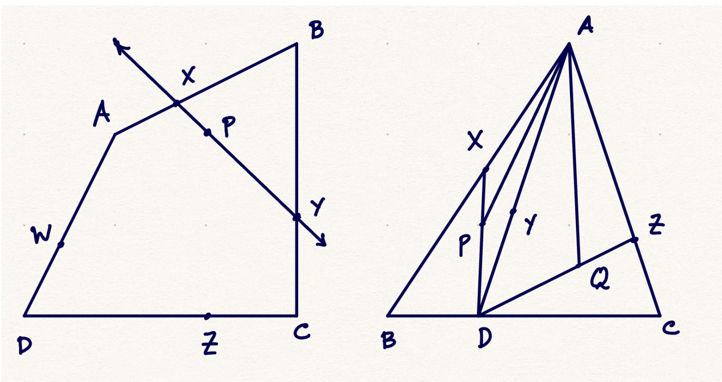

Figure 1: The result of (b).

Remark 2. The result in (b) (for arbitrary n, not just $n = 9$) is called the Sylvester-Gallai Theorem and has many different proofs, a few of them to be found on the Wikipedia webpage about it.

[1] This follows from our uniformization for smooth plane cubics, Corollary 5.30 in the version of the notes dates 11/23/23.

Q3. State and prove a 'double angle' formula for the Weierstraß $\wp$-function. That is, find a rational function $f(z)$ (that may depend on (g_2, g_3)) such that $\wp(2z) = f(\wp(z))$.

Solution. The rational function

$$f(z) = \frac{1}{16} \cdot \frac{(12z^2 - g_2)^2}{4z^3 - g_2 z - g_3} - 2z$$

has this property. This can be shown in several ways: one is to take the limit $w \to z$ in the addition formula

$$\wp(z + w) + \wp(z) + \wp(w) = \frac{1}{4} \left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2,$$

use L'Hopital's rule, and use the equations

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$
$$\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$$

to elimiate $\wp'(z)$ and $\wp''(z)$. Alternatively, you could compute the equation of the tangent to the curve

$$y^2 = 4x^3 - g_2 x - g_3$$

at the point $(\wp(z), \wp'(z))$, and solve for the x-coordinate of the other intersection point of this line with the curve, which would be $\wp(2z)$. ■

Q4. Let $X = \mathbb{C}/\Lambda$ be a complex torus, and define a map $F : X \to X$ by $[F(z)] = [2z]$. Show that F has a dense orbit on X, i.e. that there exists a $p \in X$ such that $\overline{\{F^n(p) : n > 0\}} = X$, where $F^n(p) = F(F(\cdots(F(p))))$ taken n times. Then prove the rational function $f(z)$ of the double angle formula has a dense orbit on $\hat{\mathbb{C}}$.

Proof. It suffices to show the result for $\Lambda = \mathbb{Z}^2$ (why?). For each $k \geq 1$, let C_k be the list of all pairs $(a_j^k, b_j^k)_{j=1,\ldots,2^{2k}}$ of binary strings of length k, enumerated in any order. Construct the point p by asking the binary expansion of its real and imaginary parts to be given by

$$\begin{aligned}\operatorname{Re} p &= 0.a_1^1 a_2^1 a_3^1 a_4^1 a_1^2 \cdots a_{16}^2 a_1^3 \cdots a_{64}^3 \cdots \\ \operatorname{Im} p &= 0.b_1^1 b_2^1 b_3^1 b_4^1 b_1^2 \cdots b_{16}^2 b_1^3 \cdots b_{64}^3 \cdots ,\end{aligned}$$

where juxtaposition represents concatenation. Since F, i.e. multiplication by 2 mod 1, represents a bitwise shift to the left by one bit (omitting the leftmost digit) in this notation, it follows immediately from the construction that p has a dense orbit under F. For the second, it suffices to note that $\wp(p) \in \hat{\mathbb{C}}$ has dense orbit under f; indeed, this follows from $\wp(2z) = f(\wp(z))$ and that $\wp : X \to \hat{\mathbb{C}}$ is a surjective continuous map (how?). ∎

Q5. Suppose $(x, y) = (\wp(z), \wp'(z))$ satisfies $y^2 = 4x^3 + ax + b$ with $a, b \in \mathbb{R}$ and the polynomial $4x^3 + ax + b = 0$ has only one real root. What can you say about the shape of the lattice Λ used to define $\wp(z)$?

Solution. A lattice $\Lambda \subset \mathbb{C}$ is said to be rectangular if it is of the form $\Lambda = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ with $\alpha \in \mathbb{R}_+$ and $\beta \in i\mathbb{R}_+$, and is said to be rhombic if it is of the form $\Lambda = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ with $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{H}$ such that $\beta + \overline{\beta} = \alpha$ (or equivalently $\mathrm{Re}(\beta/\alpha) \equiv 1/2 \pmod 1$); see Figure 2. The point of this problem is to illustrate that a lattice $\Lambda \subset \mathbb{C}$ has invariants $g_2, g_3 \in \mathbb{R}$ iff Λ is either rectangular or rhombic, and the rectangular case corresponds to three real roots and the rhombic to one real root of the polynomial $4x^3 - g_2x - g_3$.

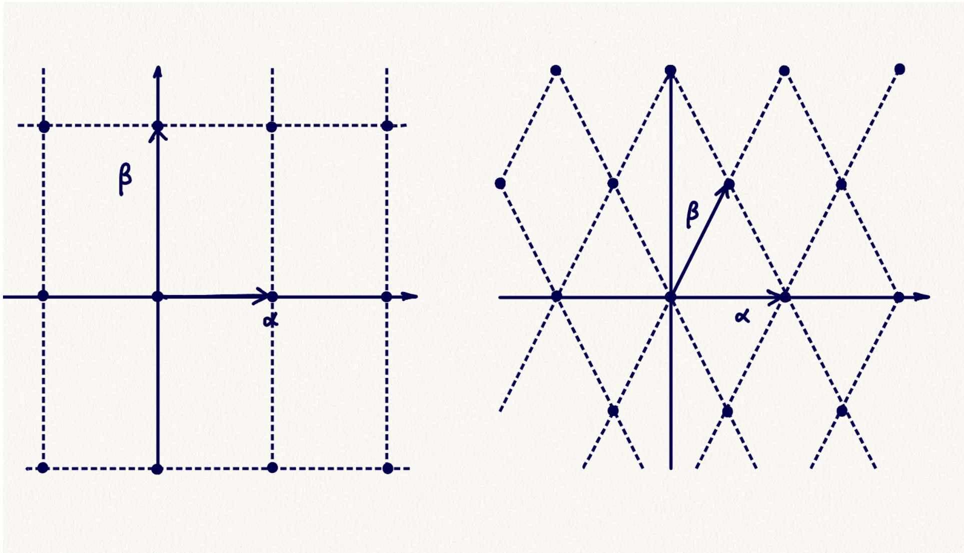

Figure 2: Rectangular and Rhombic Lattices. Note that in the rhombic case, the fundamental parallelogram with vertices $1, \alpha, \beta, \alpha + \beta$ is not necessarily a rhombus; rather, the parallelogram with vertices $1, \beta, \overline{\beta}, \alpha$ is.

(a) First, we have:

Theorem 0.0.1. Let $\Lambda \subset \mathbb{C}$ be a (full rank) lattice. Let $\wp(z), g_k$ and G_k have their usual meanings. Then the following are equivalent:

(i) We have $g_k \in \mathbb{R}$ for $k = 2, 3$.

(ii) We have $G_k \in \mathbb{R}$ for all $k \geq 2$.

(iii) The $\wp$-function takes real values on real inputs, i.e. $\wp(z) = \overline{\wp(\overline{z})}$.

(iv) Λ is conjugation invariant, i.e. $\Lambda = \overline{\Lambda}$.

(v) Λ is either rectangular or rhombic.

Proof. The implication (i) $\Rightarrow$ (ii) follows from $G_k \in \mathbb{Q}[g_2, g_3]$ for all $k \geq 2$; this follows from the recurrence relations that are obtained by using

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{n+1}z^{2n}, \tag{1}$$

and the differential equation

$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2. \tag{2}$$

The implication (ii) $\Rightarrow$ (iii) follows again from (1), and (iii) $\Rightarrow$ (iv) follows from noting that Λ can be recovered from $\wp$ as the set of poles (or periods) of $\wp$. The chain of implications

(v) $\Rightarrow$ (iv) $\Rightarrow$ (i) is clear. To show (iv) $\Rightarrow$ (v), let $\lambda \in \Lambda$ be a vector of minimal nonzero length; then $\overline{\lambda} \in \Lambda$ is a vector of the the same length. If λ and $\overline{\lambda}$ are linearly independent, then they form a minimal system and hence basis of Λ by a previous homework problem, and so Λ is a rhombic lattice: we may take $\alpha = \lambda + \overline{\lambda}$ and β to be whichever one of λ and $\overline{\lambda}$ lies in $\mathbb{H}$ (note that in this case λ is neither real nor purely imaginary). Otherwise, λ and $\overline{\lambda}$ are linearly dependent, and this can only happen if $\overline{\lambda} = \pm\lambda$, i.e. λ is either real or purely imaginary. Suppose first that λ is real, and let α be whichever of $\pm\lambda$ is positive. Let β be a vector of minimal length in Λ which is linearly independent from α; replace β by $-\beta$ if needed to ensure $\beta \in \mathbb{H}$. Then α, β form a minimal system and hence basis of Λ. Now since $\overline{\beta} \in \Lambda$ as well, there are $s, t \in \mathbb{Z}$ such that $\overline{\beta} = s\alpha + t\beta$. Then

$$\beta = s\alpha + t\overline{\beta} = s(1+t)\alpha + t^2\beta, \text{ so that } s(1+t) = t^2 - 1 = 0.$$

If $t = 1$, then $s = 0$ and $\overline{\beta} = \beta$, which is not possible since $\beta \notin \mathbb{R}$, so $t = -1$ and hence $\beta + \overline{\beta} = s\alpha$ for some $s \in \mathbb{Z}$. By replacing β by $-\overline{\beta}$, we may assume that $s \geq 0$. If $s \geq 2$, then $|\beta - \alpha| < |\beta|$ since $|s-2| < s$ (check!), and hence by minimality we must have $s \in \{0, 1\}$. If $s = 0$, then $\beta \in \mathrm{i}\mathbb{R}_+$ and Λ is rectangular, and if $s = 1$, then $\beta + \overline{\beta} = \alpha$ and Λ is rhombic. Finally, if λ is purely imaginary, then the same analysis can be applied to $\mathrm{i}\Lambda$, noting that Λ is rectangular (resp. rhombic) iff $\mathrm{i}\Lambda$ is (details left to the reader). ■

(b) Finally, we have:

Theorem 0.0.2. For a lattice $\Lambda \subset \mathbb{C}$ satisfying the equivalent conditions of Theorem 0.0.1, Λ is rectangular iff the polynomial $4x^3 - g_2x - g_3$ has three real roots and rhombic iff it has only one real root.

Proof. It was shown in class that if Λ is rectangular, then $4x^3 - g_2x - g_3$ has three real roots; the key step is to note that in this case $\wp(z)$ is real iff z lies on one of the horizontal or vertical lines through $(1/2)\Lambda$.[2] Now suppose that Λ is rhombic, and write $\Lambda = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ as above; by a scaling argument (recall that $\alpha \in \mathbb{R}_+$) it suffices to deal with $\alpha = 1$. Then $\tau := \beta/\alpha \in \mathbb{H}$ satisfies $\operatorname{Re}\tau = 1/2$. The key claim here is that the J function takes real values on the line $\operatorname{Re}\tau = 1/2$ and satisfies $J(\tau) \leq 1$ here, with equality iff $\tau = (1+\mathrm{i})/2$.[3] If for some τ with $\operatorname{Re}\tau = 1/2$, the modular discriminant $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$ is positive, then $g_2(\tau)^3 \geq g_2(\tau)^3 - 27g_3(\tau)^2 = \Delta(\tau) > 0$ implies that $J(\tau) = g_2(\tau)^3/\Delta(\tau) \geq 1$, which is a contradiction unless $\tau = (1+\mathrm{i})/2$; therefore, for $\tau \neq (1+\mathrm{i})/2$, we must have $\Delta(\tau) < 0$, finishing the proof since $\Delta(\tau)$ is the discriminant of the cubic $4x^3 - g_2(\tau)x - g_3(\tau)$. It only remains to analyze $\tau = (1+\mathrm{i})/2$, and this can be done in many ways: for instance, Δ is a continuous function and nonzero on $\mathbb{H}$, so that if $\Delta(\tau) < 0$ for all τ with $\operatorname{Re}\tau = 1/2$ except possibly $\tau = (1+\mathrm{i})/2$, then necessarily also $\Delta((1+\mathrm{i})/2) < 0$.[4] ■

[2] This already solves the problem as written: namely if Λ is a lattice with $g_2, g_3 \in \mathbb{R}$ such that $4x^3 - g_2x - g_3$ has one real root, then Λ must be rhombic. The rest of the proof shows that every rhombic lattice has this property.

[3] This can be shown by studying the mapping properties of the J-function more closely; see [1, §2.7] for instance. Note that we know what the values of J on the semicircle $|\tau| = 1$ are: as τ goes from 1 to ζ_6 to i to ζ_3 to -1, the values of $J(\tau)$ move along the real axis from ∞ to 0 to 1 to 0 to ∞ respectively, with $J(\tau) = 1$ iff $\tau = \mathrm{i}$. Now the Möbius transformation $\tau \mapsto \tau/(\tau+1)$ is in the modular group $\mathrm{PSL}_2\,\mathbb{Z}$, and takes the semicircle $|\tau| = 1$ to the ray $\operatorname{Re}\tau = 1/2$; therefore, by $\mathrm{PSL}_2\,\mathbb{Z}$-invariance of J, as τ moves along $\operatorname{Re}\tau = 1/2$ from 1/2 to $(\sqrt{3}+\mathrm{i})/(2\sqrt{3})$ to $(1+\mathrm{i})/2$ to ζ_6 to $\mathrm{i}\infty$, the value of $J(\tau)$ moves along the real axis $(-\infty, 1]$ from ∞ to 0 to 1 to 0 to ∞ respectively, with $J(\tau) = 1$ iff $\tau = (1+\mathrm{i})/2$.

[4] Alternatively, one can note that $(1+\mathrm{i})\mathbb{Z} \oplus \mathbb{Z}\left(\frac{1+\mathrm{i}}{2}\right) = \mathbb{Z}[\mathrm{i}]$, so that $g_2((1+\mathrm{i})/2) = (1+\mathrm{i})^4 g_2(\mathrm{i}) = -4g_2(\mathrm{i})$, whereas $g_3((1+\mathrm{i})/2) = 0$. Therefore, the corresponding polynomial is $4x^3 - g_2((1+\mathrm{i})/2)x = 4x(x^2 + g_2(\mathrm{i}))$, which has one real and two imaginary roots because $g_2(\mathrm{i}) \in (0, \infty)$.

Q6. Let $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ be a lattice in $\mathbb{C}$, with associated Weierstraß $\wp$-function $\wp(z)$. Let

$$\sigma(z) := z \prod_{\Lambda}{}' \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2\right)$$

be the canonical product with zeroes at the points of Λ. Prove that there is a unique odd meromorphic function $\zeta(z)$ on $\mathbb{C}$ such that $\zeta'(z) = -\wp(z)$, and relate $\zeta(z)$ to $\sigma(z)$.

Proof. First let's show uniquess: if ζ_i for $i = 1, 2$ are two such functions, $\zeta_1' = \zeta_2'$ implies that the difference $\zeta_1 - \zeta_2$ is constant; then using oddness of ζ_i, this constant is necessarily zero. Next, to show existence, note that the canonical product defining σ converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$, and so termwise logarithmic differentiation is justified and gives us

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\Lambda}{}' \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right),$$

and the series again converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$. Note also that this function is odd, since $\sigma(z)$ is clearly odd. Differentiating termwise one more time, we get

$$\frac{\sigma'(z)^2 - \sigma(z)\sigma''(z)}{\sigma(z)^2} = -\frac{1}{z^2} + \sum_{\Lambda}{}' \frac{-1}{(z - \lambda)^2} + \frac{1}{\lambda^2} = -\wp(z),$$

so that by uniqueness, we must have

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$$

as needed.

■

Q7. Show that $\zeta(z+\lambda_i) = \eta_i$ for suitable $\eta_i \in \mathbb{C}$. Show that these 'dual periods' satisfy

$$\det \begin{bmatrix} \eta_1 & \eta_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = 2\pi \mathrm{i}.$$

(Hint: translate the period parallelogram so it is centered at $z=0$, where $\zeta(z)$ has a pole, and integrate along its boundary.)

Proof. From

$$\zeta'(z+\lambda_i) = -\wp(z+\lambda_i) = -\wp(z) = \zeta'(z),$$

we conclude that $\zeta(z+\lambda_i) - \zeta(z)$ is constant, say η_i. Now note that $\zeta'(z) = -\wp(z)$ tells us that $\zeta(z)$ has simple poles along Λ with residue 1 and is holomorphic elsewhere. In particular, translating the period parallelogram and integrating $\zeta(z)\mathrm{d}z$ as suggested gives us that

$$2\pi\mathrm{i} = \int_0^{\lambda_1} \zeta(z)\,\mathrm{d}z + \int_{\lambda_1}^{\lambda_1+\lambda_2} \zeta(z)\,\mathrm{d}z + \int_{\lambda_1+\lambda_2}^{\lambda_2} \zeta(z)\,\mathrm{d}z + \int_{\lambda_2}^{0} \zeta(z)\,\mathrm{d}z,$$

where the integrals are taken along the sides of the parallelogram. The first and third integrals combine to give

$$\int_0^{\lambda_1} (\zeta(z) - \zeta(z+\lambda_2))\,\mathrm{d}z = -\lambda_1\eta_2,$$

whereas the first and fourth integrals combine to give

$$\int_0^{\lambda_2} (\zeta(z+\lambda_1) - \zeta(z))\,\mathrm{d}z = \lambda_2\eta_1.$$

Putting these equations together yields the desired identity. ∎

References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*. No. 41 in Graduate Texts in Mathematics, Springer-Verlag, second ed., 1990.