2.4 Exercise Sheet 4

2.4.1 Numerical and Exploration

Exercise 2.4.1. What can you say about $\mathrm{Der}_{\mathbb{Q}}\left(\mathbb{Q}[\sqrt{-3}]\right)$? $\mathrm{Der}_{\mathbb{Q}}(\mathbb{Q}[\sqrt{7}, \cos(2\pi/5)])$? $\mathrm{Der}_{\mathbb{R}}(\mathbb{C})$? $\mathrm{Der}_{\mathbb{Q}}(\mathbb{Q}[\pi])$? $\mathrm{Der}_{\mathbb{Q}}(\mathbb{C})$? $\mathrm{Der}_{\mathbb{F}_p(t)}(\mathbb{F}_p(t)[s]/(s^p - t))$? What about $\mathrm{Der}_k K$, where k is any field and $K = \mathrm{Frac}\, k[x, y]/(f)$ for $f = y, y - x^2, y^2 - x^3, y^2 - x^3 + x$? Make (and prove) conjectures.

Exercise 2.4.2. (Adapted from [3, Exercise 5.2].) Define what it means for a projective plane curve to be irreducible. For each of the following polynomials F, identify whether the projective curve $C_F \subset \mathbb{P}^2_k$ is irreducible, find all the multiple points, their multiplicities, and tangent lines at the multiple poins.

(a) $XY^4 + YZ^4 + ZX^4$.
(b) $X^2Y^3 + Y^2Z^3 + Z^2X^3$.
(c) $Y^2Z - X(X - Z)(X - \lambda Z)$ for $\lambda \in k$.
(d) $X^n + Y^n + Z^n$ for $n \geq 1$.

What is the relationship between the irreducibility of F and that of C_F? Do your answers depend on the characteristic of the base field?

Exercise 2.4.3. (Adapted from [3, Exercise 5.3].) Find all points of intersection of the following pairs of curves, and the intersection numbers at these points.

(a) $X^2 + Y^2 - Z^2$ and Z.
(b) $(X^2 + Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$.
(c) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$.
(d) $(X^2 + Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2 + Y^2)^3 - 4X^2Y^2Z^2$.

Do your answers depend on the base field?

Exercise 2.4.4 (Singular Plane Cubics). Let $F \in k[X, Y, Z]$ be an irreducible homogeneous cubic polynomial, and suppose that $C = C_F$ has a cusp at a point $P \in C$ (see Exercise 2.3.5).

(a) Show that there is a projective change of coordinates such that $P = [0 : 0 : 1]$ and $\mathrm{T}_P C$ is defined by $Y = 0$. Show that in these coordinates,
$$F = Y^2Z - AX^3 - BX^2Y - CXY^2 - DY^3$$
for some $A, B, C, D \in k$ with $A \neq 0$, up to scaling F by a nonzero scalar.
(b) Find a projective change of coordinates to make $C = D = 0$. In other words, find a projective change of coordinates $\phi : \mathbb{P}^2_k(X_1, Y_1, Z_1) \to \mathbb{P}^2_k(X, Y, Z)$ such that we have $\phi^* F = Y_1 Z_1^2 - AX_1^3 - BX_1^2 Y$.
(c) Now suppose that k is algebraically closed (or even that $k^\times = (k^\times)^3$, i.e. that every nonzero element is a cube) and also that $\mathrm{ch}\, k \neq 3$. Find a projective change of coordinates to make $A = 1$ and $B = 0$. Conclude that when k satisfies the above hypotheses (e.g. $k = \mathbb{C}$ or $k = \mathbb{F}_5$), there is a unique cuspidal plane cubic up to projective changes of coordinates, and this has no other singularities. What happens when these hypotheses on k are not satisfied?
(d) Similarly, show that under suitable hypotheses on k, there is a unique nodal plane cubic up to projective changes of coordinates, and this has no other singularities. Explore what happens when these hypothesis on k do not apply.
(e) Give at least two proofs of the following fact: under suitable hypothesis on the base field k, any irreducible projective plane cubic is either nonsingular, or has at most one singular point of multiplicity at most 2, which must be either a node or a cusp. (Hint: For one, use (c) and (d). For the other, use the correct salvage of Exercise 2.4.9 below.)

(f) What can you say about irreducible singular plane quartic curves? Can you come up with a similar clasification? What about singular plane quintic curves? Can you explore and make some general conjectures?

Exercise 2.4.5 (Hessian)**.** (Adapted from [4, Exercise 3.29].) Let $F \in k[X, Y, Z]$ be a homogeneous polynomial. We define the Hessian polynomial of F to be

$$\mathrm{Hess}(F) := \det \begin{bmatrix} \partial^2 F/\partial X^2 & \partial^2 F/\partial X \partial Y & \partial^2 F/\partial X \partial Z \\ \partial^2 F/\partial X \partial Y & \partial^2 F/\partial Y^2 & \partial^2 F/\partial Y \partial Z \\ \partial^2 F/\partial X \partial Z & \partial^2 F/\partial Y \partial Z & \partial^2 F/\partial Z^2 \end{bmatrix}.$$

(a) Show that if $\phi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ is a projective change of coordinates, then we have that $\mathrm{Hess}(\phi^* F) = C \cdot \phi^*(\mathrm{Hess}(F))$ for some nonzero constant C. What is C in terms of F and ϕ?

(b) Compute the Hessian for

$$F_\lambda := Y^2 Z - X(X - Z)(X - \lambda Z),$$

where $\lambda \in k$, and describe the intersection $C_{F_\lambda} \cap C_{\mathrm{Hess}(F_\lambda)}$? (If the general case is too hard, can you do this for some special values of λ?)

(c) Show that if $\mathrm{ch}\, k \neq 2, 3$, if F is irreducible of $\deg F \geq 2$ and if $P \in C_F$ is a smooth point of C_F, then $P \in C_F \cap C_{\mathrm{Hess}(F)}$ iff $i_P(C_F, T_P C_F) \geq 3$. Such a point is called an inflection point of C_F.

(d) How many inflection points can a smooth curve of degree 2 have? What about 3? 4? 5? Find patterns and make some conjectures.

2.4.2 PODASIPs

Prove or disprove and salvage if possible the following statements.

Exercise 2.4.6. If k is any field and $f \in k[t]$ a nonconstant polynomial, then $\partial_t f \neq 0$.

Exercise 2.4.7. If k is any infinite field and $C \subset \mathbb{P}^2_k$ a projective plane curve, then C is infinite.

Exercise 2.4.8. Given any two ordered sets of nonconcurrent lines (L_1, L_2, L_3) and (L_1', L_2', L_3') in $\mathbb{P}^2_k$, there is a unique projective change of coordinates $\phi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ such that $\phi(L_i) = L_i'$ for $i = 1, 2, 3$.

Exercise 2.4.9 (Bézout's Theorem for a Line)**.** If k is any field and $C \subset \mathbb{P}^2_k$ a projective curve of degree $n \geq 1$ with minimal polynomial $F \in k[X, Y, Z]_n$, then for any line $C_L \subset \mathbb{P}^2_k$ where $L \in k[X, Y, Z]_1$, we have

$$\sum_{P \in C_F \cap L} i_P(F, L) = n.$$

Exercise 2.4.10. If $F \in k[X, Y, Z]$ is a nonconstant homogenous polynomial, the the projective curve C_F defined by F is irreducible iff F is.