1.13 07/08/24 - Parametric Projective Curves, Pascal's Theorem, and More on Conics

Today, I want to prove Bézout's theorem for conics, and derive some delicious applications. For this, I will need to talk about parametric projective curves.

1.13.1 Parametric Projective Curves and Bézout's Theorem for a Conic

In the affine case, we defined a parametric curve to be an image of $\mathbb{A}^1_k$ under two rational functions. In the projective case, we can always clear denominators and work with $\mathbb{P}^1_k$ instead. This leads us to

Definition 1.13.1. A parametric projective algebraic curve is the image of a map $\Psi : \mathbb{P}^1_k \to \mathbb{P}^2_k$ of the form

$$\Psi[U : V] = [F_1(U,V) : F_2(U,V) : F_3(U,V)],$$

where $F_i(U,V) \in k[U,V]$ for $i = 1,2,3$ are homogeneous polynomials of the same degree, not all zero.

This definition corresponds to the affine one by considering $\mathbb{A}^1_k \subset \mathbb{P}^2_k$ as the set where $V \neq 0$ with coordinate $t = U/V$, in which case the affine part of this parametric projective curve is given by

$$t \mapsto \left(\frac{F_1(t,1)}{F_3(t,1)}, \frac{F_2(t,1)}{F_3(t,1)}\right),$$

which is a parametric affine curve. One can show, either using techniques similar to those in §1.3 or by reducing to the affine case, that a parametric projective algebraic curve is, in fact, a projective algebraic curve (at least when not all F_i are proportional, in which case the image is a single point). I will not do this here, but I encourage you to carry this out yourselves.

Remark 1.13.2. Note that I did not ask for the $F_j(U,V)$ to not have a common root on $\mathbb{P}^1_k$, because if they did, then I would very easily be able to just cancel this common factor from each $F_j(U,V)$. This is a manifestation of the completeness of projective curves–projective curves have no holes, and rational maps out a smooth projective curve always eextends to a regular morphism out of it. As usual, if this doesn't make sense, please ignore it.

Example 1.13.3. The smooth conic defined by $YZ - X^2 \in k[X,Y,Z]_2$ can be parametrized via the map Ψ given as

$$\Psi[U : V] = [UV : U^2 : V^2].$$

This is the projective version of affine parametrization $t \mapsto (t, t^2)$ of the parabola defined by $y - x^2 = 0$. By Corollary 1.12.14, this gives us a parametrization of every smooth conic curve. In particular, every smooth conic curve admits a parametrization.

From this parametrization, we can now prove Bézout's theorem for a conic.

Theorem 1.13.4. If k is an algebraically closed field, $C \subset \mathbb{P}^2_k$ is a conic (i.e. a curve with $\deg C = 2$), and $D \subset \mathbb{P}^2_k$ a curve of degree $d \geq 1$ such that C and D do not share a component, then

$$\sum_{P \in C \cap L} i_P(C,D) = 2d.$$

Of course, $\deg C = 2$ and so $2d = (\deg C)(\deg D)$.

Proof. By Corollary 1.12.14, we can choose coordinates such that C is either the union of the two lines C_X and C_Y, or that $C = C_{YZ-X^2}$; make a change of coordinates so that we are working with this coordinate system. In the first case, neither of the two lines C_X or C_Y can be contained in D, and we are done by additivity of intersection multiplicity and Theorem 1.12.12 (make sure you believe this!). In the second case, C is irreducible and Example 1.13.3 tells us that we can parametrize C as the image of the map

$$[U : V] \mapsto [UV : U^2 : V^2].$$

If F is a minimal polynomial for D, then F is a homogeneous polynomial of degree d, and the intersection points of C and D correspond exactly to $[U_0 : V_0] \in \mathbb{P}^1_k$ such that

$$F(U_0V_0, U_0^2, V_0^2) = 0.$$

Now $F(UV, U^2, V^2) \in k[U, V]_{2d}$ is a homogeneous polynomial of degree $2d$. If it is identically zero, then we conclude that $C \subset D$, contrary to assumption that C and D do not share any components; therefore, this polynomial is not identically zero, and so has exactly $2d$ roots counted with multiplicity, again by Lemma 1.8.3. I will again leave it to the reader to check, perhaps using techniques similar to those from Example 1.9.12, that the intersection multiplicity of C and D at a point $[U_0V_0 : U_0^2 : V_0^2]$ agrees with the multiplicity of $[U_0 : V_0]$ as a root of $F(U_0V_0, U_0^2, V_0^2) = 0$.[33] ∎

We are now ready for some delicious applications!

1.13.2 Pascal's Theorem, Pappus's Theorem, Brocard's Theorem, etc.

Theorem 1.13.5 (Pascal). Let k be an algebraically closed field, $C \subset \mathbb{P}^2_k$ a smooth conic and $P_1, \ldots, P_6$ distinct points on C. For $i = 1, \ldots, 6$, let L_i be the line joining P_i and P_{i+1} (where $P_7 := P_1$), and for $j = 1, 2, 3$, let $Q_j := L_j \cap L_{j+3}$. Then the points $Q_1, Q_2, Q_3 \in \mathbb{P}^2_k$ are collinear, i.e. there is a line $L_0 \subset \mathbb{P}^2_k$ such that $Q_j \in L_0$ for $j = 1, 2, 3$.

Let's first make a few observations about the statement:

(a) The lines L_i are all distinct: if $L_i = L_{i'}$ for some $i \neq i'$, then this line intersects C in at least 3 distinct points, and is hence contained in C (by either Theorem 1.12.12 or 1.13.4); this would mean that C is reducible and hence (by Corollary 1.12.14 if needed) not smooth. In particular, by Proposition 1.11.3, the points Q_1, Q_2, Q_3 are uniquely determined.

(b) Each P_i lies on exactly two lines $L_{i'}$, namely L_{i-1} and L_i, and, in particular, these lines have indices that differ by 1 (modulo 6); conversely, each L_i contains exactly two points P_i and P_{i+1} of C, because again if it contained a third point of C, it would be contained in C entirely.

(c) We have $P_i \neq Q_j$ for all i, j. Indeed, let us take the indices i, j modulo 6; then $P_i = Q_j$ cannot happen because this implies that $P_i \in L_j \cap L_{j+3}$, violating the observation (b).

(d) Finally, we have $Q_1, Q_2, Q_3 \notin C$. Indeed, if some $Q_j \in C$, then $Q_j \in L_j \cap C = \{P_j, P_{j+1}\}$ implies that $Q_j = P_i$ for some i, j, violating (c).

(e) In fact, although we will not need this for the proof, all the 9 points P_i, Q_j are distinct:

Let's now proceed to the proof, which is rather simple given the tools we have.

[33] I'm being lazy partly because, in the proof of Pascal's Theorem (Theorem 1.13.5) below, we will only need the result that C and D intersect in at most $2d$ points unless they share a component, and this we have already proven. Also, we shall do a full proof of the general Bézout Theorem very soon.

Proof. For $i = 1, \ldots, 6$, let $\ell_i \in k[X,Y,Z]_1$ be a homogeneous linear polynomial vanishing on L_i. Consider the family D_Λ of cubic curves parametrized by $\Lambda = [\lambda : \mu] \in \mathbb{P}^1_k$, where D_Λ is defined by the vanishing of the polynomial

$$\lambda \ell_1 \ell_3 \ell_5 + \mu \ell_2 \ell_4 \ell_6.$$

Note that each curve D_Λ in this family passes through all the P_i's and Q_j's, and we have that $D_{[1:0]} = L_1 \cup L_3 \cup L_5$ and $D_{[0:1]} = L_2 \cup L_4 \cup L_6$.[34] Now pick a point $R \in C \smallsetminus \{P_1, \ldots, P_6\}$, which exists because C is infinite (Proposition 1.11.13). From observation (b) above, we conclude that $R \notin D_{[1:0]} \cup D_{[0:1]}$, from which it follows that there is a unique $\Lambda_0 \in \mathbb{P}^1_k$ such that $R \in D_{\Lambda_0}$.[35] Let $D = D_{\Lambda_0}$. Since D is a cubic curve and C and D intersect in at least 7 points, it follows from Theorem 1.13.4 that C and D share a component. Since C is irreducible (Corollary 1.12.14) and $\deg D = 3$, this can only happen if $C \subset D$ and $D = C \cup L_0$ for some line $L_0 \subset \mathbb{P}^2_k$. But now, D contains Q_1, Q_2, Q_3 (because each D_Λ does), while C does not contain Q_1, Q_2, Q_3 (this was observation (c) above), and hence $Q_1, Q_2, Q_3 \in L_0$. ■

See Figure 1.8 for a visual demonstration of the proof technique.

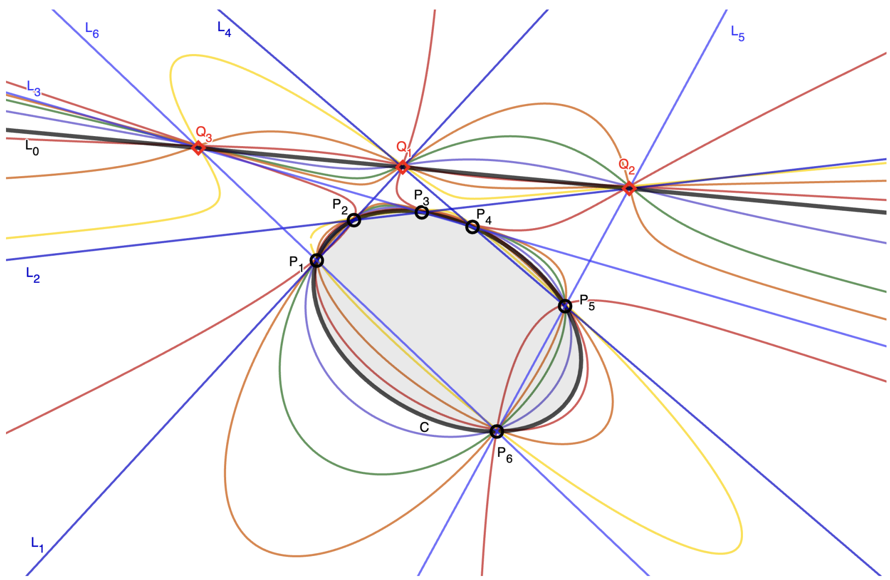

Figure 1.8: Pascal's Theorem. The conic (here ellipse) C and the line L_0 are in thick black style. The various colorful curves represent various members of the one parameter family D_Λ, one member of which is also $C \cup L_0$. Picture made with Geogebra.

Remark 1.13.6. Note that the actual statement of Theorem 1.13.5 does not use an ordering whatsoever on the points $P_1, \ldots, P_6$–indeed, for general fields, it does not even make sense to order points of a conic. In particular, if we start with a collection of 6 distinct unordered points on a conic C, then they can be connected into a hexagon in 60 different ways, and resulting in 60 different instances of Pascal's Theorem and 60 different "Pascal" lines; this configuration of 60 lines associated to 6 points on a hexagon is often called the Hexagrammum Mysticum. Finally, although we have proven the theorem over algebraically closed fields, it follows also immediately

[34] If you were not convinced of this already, then this observation tells us that every curve D_Λ has degree 3: indeed, if it did not, then some D_Λ would be either a line or a union of two lines, neither of which can contain all the P_i's, since no three of them are collinear (why?).

[35] Indeed, if we pick a representative (X_0, Y_0, Z_0) for $R = [X_0 : Y_0 : Z_0]$, then neither of $\ell_1\ell_3\ell_5|_{(X_0,Y_0,Z_0)}$ and $\ell_2\ell_4\ell_6|_{(X_0,Y_0,Z_0)}$ is zero, and this unique Λ_0 is $\Lambda_0 = [-\ell_2\ell_4\ell_6|_{(X_0,Y_0,Z_0)} : \ell_1\ell_3\ell_5|_{(X_0,Y_0,Z_0)}] \in \mathbb{P}^1_k$.

over all fields (e.g. over $k = \mathbb{R}$), thanks to Theorem 1.4.5 and the observation that Proposition 1.11.3 does not use that the base field is algebraically closed, which implies, for instance, that if three points $Q_1, Q_2, Q_3 \in \mathbb{P}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}}$ are collinear in $\mathbb{P}^2_{\mathbb{C}}$, then they are collinear in $\mathbb{P}^2_{\mathbb{R}}$, i.e. the line joining them is real. Over the real numbers, other proofs can also be given; after all, Pascal did not actually have Bézout's Theorem. One approach involves using a variant of the classification of projetive conics over $\mathbb{R}$ (see Remark 1.12.15) to conclude that any smooth conic can be taken by a projective change of coordinates over $\mathbb{R}$ to a circle $X^2 + Y^2 - Z^2 = 0$, and then to use other techniques from Euclidean geometry (e.g. Menelaus's Theorem).

In the proof of Pascal's Theorem, we did not really use that C was a smooth conic other than to rule out certain degenerate cases. Therefore, the same proof technique also yields

Theorem 1.13.7 (Pappus). Let k be any field. Let $L_1, L_2 \subset \mathbb{P}^2_k$ two distinct lines, and $P_1, Q_1, R_1 \in L_1 \smallsetminus L_2$ and $P_2, Q_2, R_2 \in L_2 \smallsetminus L_1$ be distinct points. If

$$\begin{aligned} S_1 &= \overline{Q_1R_2} \cap \overline{Q_2R_1}, \\ S_2 &= \overline{P_1R_2} \cap \overline{{}_2R_1}, \text{ and} \\ S_3 &= \overline{P_1Q_2} \cap \overline{P_2Q_1}. \end{aligned}$$

. Then $S_1, S_2, S_3 \subset \mathbb{P}^2_k$ are collinear.

Proof. By Theorem 1.4.5 and Proposition 1.11.3, we may replace k by an algebraically closed field and still have the same result (check!), and then the same proof technique as in Theorem 1.13.5 works. I leave the verification of the nondegeneracy conditions to the diligent reader. ■

Finally, Pascal's Theorem can also be applied with "multiplicities". The key result needed to do this is

Lemma 1.13.8. Let $C \subset \mathbb{P}^2_k$ be a curve and $P \in C$ be a smooth point. Let F be a minimal polynomial for C, and let $G, H \in k[X, Y, Z]$ be homogeneous polynomials such that $G, H, G + H \neq 0$. Then

$$i_P(F, G + H) \geq \max\{i_P(F, G), i_P(F, H)\}$$

with equality if $i_P(F, G) \neq i_P(F, H)$.

Proof Sketch. This is a local property invariant under changes of coordinates, and so we may work in the affine chart $Z \neq 0$ and assume that $P = (0, 0)$ and that the tangent line to C at P is the x-axis C_y. Let $f = F^{\mathrm{i}}$. The claim is that for any $0 \neq g \in \mathcal{O}_P$, there is a unique integer $n \geq 0$ such that for some unit $u \in \mathcal{O}_P^\times$ we have $g - ux^n \in (f)\mathcal{O}_P$. Uniqueness is clear, because then $i_P(f, g) = i_P(f, ux^n) = n$. For existence, scale f and write it as $f = y + x^n p + y^2 q$ for some $p \in k[x]$ such that $p(0) \neq 0$ and then $y - p(1 + yq)^{-1}x^n \in (f)\mathcal{O}_P$, proving the claim for $g = y$. The statement for $g = x$ is clear, and so is the fact that if such an n exists for g and h, then it does also for $g \cdot h$. Showing the result for the sum $g + h$ when $0 \neq g + h$ is slightly more involved, but in any case if $g \equiv ux^n \pmod{f\mathcal{O}_P}$ and $h \equiv vx^m \pmod{f\mathcal{O}_P}$ for some $n, m \geq 0$ and $u, v \in \mathcal{O}_P^\times$, then $f + g \equiv (ux^{n-m} + v)x^m \pmod{f\mathcal{O}_P}$ showing that the result holds for $f + g$ (as well as the claim in the lemma), unless we have $n = m$ and $u + v = 0$; this case needs some more effort, but is not too difficult. See, for instance, the discussion in the proof of [3, §3.3, Theorem 3(8)]. ■

Remark 1.13.9. The grown-up way to prove (and understand) Lemma 1.13.8 is to say that if $C \subset \mathbb{P}^2_k$ is a curve and $P \in C$ is a smooth point, then the local ring $\mathcal{O}_{C,P}$ of C at P is a discrete valuation ring with uniformizer given by the class of any line not tangent to C at P. I haven't defined what those terms are yet, so do not worry too much about this at the moment.

Given Lemma 1.13.8, however, it is very straightforward to extend the proof of Pascal's Theorem to cases where the points "degenerate". Here's one example of how to do this; you are invited to explore other examples of this sort in Exercise 2.5.3.

Theorem 1.13.10 (Brocard)**.** Let k be an algebraically closed field and $C \subset \mathbb{P}^2_k$ be a smooth conic and $P_1, P_2, P_3, P_4 \in C$ be distinct points. For $i = 1, \dots, 4$, let $T_i := \mathrm{T}_{P_i}C$, and for $1 \le i, j \le 4$, let L_{ij} be the line joining P_i and P_j. Let

$$\begin{aligned} S_1 &= L_{12} \cap L_{34}, \\ S_2 &= L_{23} \cap L_{41}, \\ Q_{13} &= T_1 \cap T_3, \\ Q_{24} &= T_2 \cap T_4. \end{aligned}$$

The points S_1, S_2, Q_{13} and Q_{24} in $\mathbb{P}^2_k$ are collinear.

I will leave to the reader the verification of many implicit claims in the statement of the theorem, e.g. the definition of S_1 uses Proposition 1.11.3 and that $L_{12} \neq L_{34}$. The line joining S_1, S_2, Q_{13} and Q_{24} is called the polar of the last intersection point $S_3 := L_{13} \cap L_{41}$ with respect to the conic C. Again, the ordering of the points P_1, P_2, P_3, P_4 does not matter, and we end up with 3 different such configurations.

Proof. It suffices to show that S_1, S_2 and Q_{13} are collinear, because then S_2, S_1 and Q_{24} are collinear by an application of the proven claim to P_2, P_3, P_4, P_1 in that order. To show the first claim, apply Pascal's Theorem (Theorem 1.13.5) to the "hexagon" $P_1P_1P_2P_3P_3P_4$. To say more, take

$$\begin{aligned} L_1 &= T_1, \\ L_2 &= L_{12}, \\ L_3 &= L_{23}, \\ L_4 &= T_3, \\ L_5 &= L_{34}, \text{ and} \\ L_6 &= L_{41} \end{aligned}$$

in the setup of Theorem 1.8, so that $Q_1 = Q_{13}$, $Q_2 = S_1$ and $Q_3 = S_2$. Take linear polynomials as ℓ_i as before, and again consider the 1-parameter family D_Λ. Again, take a new point R and a unique Λ_0 such that $R \in D$. Since $L_1 \cup L_3 \cup L_5$ and $L_2 \cup L_4 \cup L_6$ each meet C in multiplicity at least 2 at both P_1 and P_3, it follows from Lemma 1.13.8 then every memebr of the family D_Λ meets C both passes through the points $P_1, P_2, P_3, P_4, S_1, S_2$ and Q_{13}, and meets C to multiplicity at least two at both P_1 and P_3. It follows as before from Bézout's Theorem for a conic (Theorem 1.13.4), but this time applied with multiplicities, that $C \subset D$, and the rest of the proof is identical to that of Theorem 1.13.5. ■

See Figure 1.9 for an illustration of Theorems 1.13.7 and 1.13.10.

Remark 1.13.11. Over $k = \mathbb{R}$ or $k = \mathbb{C}$, the proof of these "degenerate" cases can also be given by continuity. Similarly, once you have Pascal's Theorem, you can also derive from it Pappus's

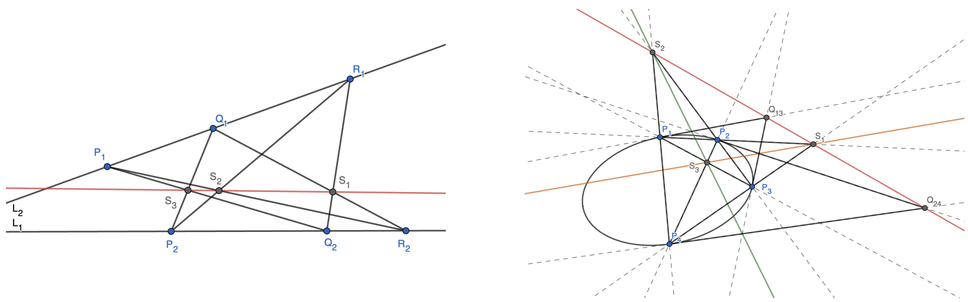

Figure 1.9: Theorems of Pappus and Brocard. Pictures made with Geogebra.

Theorem by continuity (by letting a hyperbola degenerate to a pair of lines). Such proofs are also available over other fields, but only with significantly more sophisticated tools.

1.13.3 More on Conics

Finally, let's talk about how Bézout's Theorem can be used to solve interpolation problems, i.e. problems involving finding curves of certain degrees passing through given points in $\mathbb{P}^2_k$.

Theorem 1.13.12. Let $S \subset \mathbb{P}^2_k$ be a set with 5 elements.

(a) There is a conic $C \subset \mathbb{P}^2_k$ passing through S, i.e. such that $S \subset C$.
(b) If no four distinct points in S are collinear, then such a conic as in (a) is unique.
(c) If no three distinct points in S are collinear (i.e. S is in general position), then the unique conic as in (b) is smooth.

Note that (b) and (c) are the best possible refinements of (a): if four points in S were collinear, then (at least if k is infinite), there would be infinitely many (reducible) conics containing S, and similarly if three points in S are collinear, then there is no hope of a conic containing S being irreducible or equivalently smooth (thanks again to Theorem 1.13.4).

Proof.

(a) Let $S = \{P_1, \ldots, P_5\}$, and pick representatives (X_i, Y_i, Z_i) for $P_i = [X_i : Y_i : Z_i]$ for $i = 1, 2, \ldots, 5$. The vector space of homogeneous quadratic polynomials in
(b) If there are two distinct conics $C, D \subset \mathbb{P}^2_k$ through S, then by Bézout's Theorem (Theorem 1.13.4), C and D must have a common component. Then neither C nor D can be irreducible, and, in fact, we must have $C = L_1 \cup L_2$ and $D = L_2 \cup L_3$ for some distinct lines $L_1, L_2, L_3 \subset \mathbb{P}^2_k$ (check!). In this case, $S \subset C \cap D = L_2 \cup (L_1 \cap L_3)$. Since $L_1 \cap L_3$ is one point, at least four points of S must lie on L_2.
(c) If the unique conic C as in (b) is singular, then it is reducible and hence a union of two lines. By the Pigeonhole Principle, at least three elements of S must lie on a line.

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Remark 1.13.13. You are invited to explore similar interpolation problems in Exercise 2.5.1. In the above result, there is some subtlety involving whether of not we're working over algebraically closed fields; I'll let you work through the details of that. Remark 1.12.15 may be of some help.

We will have just a little more to say about conics in the next few lectures–when we talk about one-parameter families (i.e. pencils) of conics. Next time, we will finally go over two proofs of Bézout's Theorem.