Math 213A F23 Homework 8 Solutions

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November 2023

If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Express

$$f(z) = \sum_{n\in\mathbb{Z}} (z-n)^{-4}$$

in terms of trigonometric functions, and use your results to evaluate $\zeta(4) = \sum_1^\infty 1/n^4$.

Solution. We claim that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^4} = \frac{\pi^4}{3} \cdot \csc^2(\pi z) \cdot \left(3\csc^2(\pi z) - 2\right). \tag{1}$$

Assuming this for a moment, we can evaluate $\zeta(4)$ as follows: pull the $1/z^4$ to the right and take the limit as $z \to 0$ to get that

$$2\zeta(4) = \lim_{z\to 0} \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{1}{(z-n)^4} = \lim_{z\to 0} \left(\frac{\pi^4}{3} \cdot \csc^2(\pi z) \cdot \left(3\csc^2(\pi z) - 2\right) - \frac{1}{z^4} \right),$$

and this last limit can be evaluated in several ways (e.g. using L'Hopital's Rule several times; we mention another approach below) to be $\pi^4/45$, and so

$$\zeta(4) = \frac{\pi^4}{90}.$$

Alternatively, you could plug in $z = 1/2$ to (1) to get

$$32 \cdot \zeta(4) \left(1 - \frac{1}{2^4}\right) = \sum_{n\in\mathbb{Z}} \frac{1}{((1/2)-n)^4} = \frac{\pi^4}{3},$$

where in the first step we have used that

$$\zeta(4) = \left(\sum_{n=0}^\infty \frac{1}{2^{4n}}\right)\left(\sum_{n=0}^\infty \frac{1}{(2n+1)^4}\right) = \frac{1}{1-2^{-4}} \left(\sum_{n=0}^\infty \frac{1}{(2n+1)^4}\right).$$

There are several ways to prove (1). Here are a few:

(a) The canonical product formula for $\sin(\pi z)$ looks like

$$\sin(\pi z) = \pi z \prod_{n\in\mathbb{Z}\smallsetminus\{0\}} E_1\left(\frac{z}{n}\right) = \pi z \prod_{n\in\mathbb{Z}\smallsetminus\{0\}} \left(1 - \frac{z}{n}\right) \mathrm{e}^{z/n},$$

where the convergence is uniform on compact subsets of $\mathbb{C}\smallsetminus\mathbb{Z}$. Therefore, logarithmic differentiation is justified and we get that

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{1}{z-n} + \frac{1}{n},$$

where again the sum converges uniformly on compact subsets of $\mathbb{C}\smallsetminus\mathbb{Z}$. Because of this last condition, termwise differentiation is allowed, with the resulting series still converging uniformly on compact subsets of $\mathbb{C}\smallsetminus\mathbb{Z}$; doing this thrice yields (1). The same can also be done using

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^\infty \frac{1}{z-n} + \frac{1}{z+n}.$$

See Remark 1.

(b) For fixed $z \in \mathbb{C}\smallsetminus\mathbb{Z}$, consider the meromorphic function f_z on the plane defined by

$$f_z(w) := \frac{\pi\cot(\pi w)}{(z-w)^4}.$$

This has simple poles at $\mathbb{Z}$, with residue given by

$$\operatorname*{Res}_{w=n} f_z(w) = \frac{1}{(z-n)^4}$$

for $n \in \mathbb{Z}$, along with a quadruple pole at $w = z$ with residue

$$\operatorname*{Res}_{w=z} f_z(w) = -\frac{\pi^4}{3} \cdot \csc^2(\pi z) \cdot \left(3\csc^2(\pi z) - 2\right),$$

as you can check either by repeated differentiation or changing variables to $\zeta = w - z$. For $N \geq 0$, let C_N be the counterclockwise oriented boundary of the square of side $2N+1$ centered at the origin. It follows from the Residue Theorem that for $N \gg 1$ we have

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_N} f_z(w)\,\mathrm{d}w = \sum_{n=-N}^{N} \frac{1}{(z-n)^4} - \frac{\pi^4}{3} \cdot \csc^2(\pi z) \cdot \left(3\csc^2(\pi z) - 2\right). \tag{2}$$

Now we use the following helpful lemma:

Lemma 0.0.1. There is a universal $C > 0$ such that for all $N \geq 0$ and $z \in C_N$, we have

$$|\pi \cot(\pi z)| \leq C.$$

Proof. Note that for $z \in \mathbb{C} \smallsetminus \mathbb{R}$ that

$$|\cot(\pi z)| = \left| \mathrm{i} \frac{\mathrm{e}^{2\pi \mathrm{i} z} + 1}{\mathrm{e}^{2\pi \mathrm{i} z} - 1} \right| \leq \frac{1 + \mathrm{e}^{-2\pi \operatorname{Im} z}}{|1 - \mathrm{e}^{-2\pi \operatorname{Im} z}|}$$

and

$$\sup_{|\operatorname{Im} z| \geq 1/2} \frac{1 + \mathrm{e}^{-2\pi \operatorname{Im} z}}{|1 - \mathrm{e}^{-2\pi \operatorname{Im} z}|} = \frac{1 + \mathrm{e}^{-\pi}}{1 - \mathrm{e}^{-\pi}} = \coth\left(\frac{\pi}{2}\right),$$

so by $\mathbb{Z}$-periodicity, we can take

$$C := \max\left\{ \pi \coth\left(\frac{\pi}{2}\right), \sup_{y \in [-1,1]} \left| \pi \cot\left(\pi \cdot \frac{1 + \mathrm{i}y}{2}\right) \right| \right\}.$$

$\blacksquare$

Since for $N > 2|z|$, we have

$$\inf_{w \in C_N} |z - w| \geq \left(N + \frac{1}{2}\right) - \frac{N}{2} = \frac{N+1}{2}$$

it follows from Lemma 0.0.1 that

$$\left| \frac{1}{2\pi \mathrm{i}} \oint_{C_N} f_z(w)\,\mathrm{d}w \right| \leq \frac{1}{2\pi} \cdot 4(2N+1) \sup_{w \in C_N} |f_z(w)| \leq \frac{1}{2\pi} \cdot 4(2N+1) \cdot C \cdot \frac{2}{(N+1)^4}$$

and hence taking $N \to \infty$ in (2) gives the result (noting also that the sum is absolutely convergent for any $z \in \mathbb{C} \smallsetminus \mathbb{Z}$).

(c) Since

$$\sin(z) = z\left(1 - \frac{z^2}{6} + \mathcal{O}(z^4)\right)$$

as $z \to 0$, it follows that as $z \to 0$, we have

$$\pi^2 \csc^2(\pi z) = \frac{1}{z^2}\left(1 + \frac{\pi^2}{3} z^2 + \mathcal{O}(z^4)\right) = \text{ and}$$

$$\pi^4 \csc^4(\pi z) = \frac{1}{z^4}\left(1 + \frac{2\pi^2}{3} z^2 + \mathcal{O}(z^4)\right) = \frac{1}{z^4} + \frac{2\pi^2}{3} \cdot \frac{1}{z^2} + \mathcal{O}(1),$$

where the $\mathcal{O}(1)$ just means that the difference is holomorphic in a neighborhood of 0. It follows that

$$\pi^4 \csc^4(\pi z) - \left(\frac{2\pi^2}{3}\right) \pi^2 \csc^2(\pi z) = \frac{1}{z^4} + \mathcal{O}(1). \tag{3}$$

Note that the left hand side of (3) is the same as the right hand side of (1). The sum defining the left hand side of (1) converges uniformly on compact subsets of $\mathbb{C} \smallsetminus \mathbb{Z}$, and is $\mathbb{Z}$-periodic; by (3), the two sides of (1) have the same polar parts at $z = 0$, and hence, by $\mathbb{Z}$-periodicity, at $z = n$ for every $n \in \mathbb{Z}$. It follows that the difference is a $\mathbb{Z}$-periodic entire function. Finally, one can check that the limit as $|\operatorname{Im} z| \to \infty$ of either side is 0, and hence Liouville's Theorem implies (1).

■

Remark 1. Even though $\pi \cot(\pi z)$ has simple poles at $z \in \mathbb{Z}$ with residue 1 each, you cannot write

$$\pi \cot(\pi z) = \sum_{n \in \mathbb{Z}} \frac{1}{z - n};$$

this latter sum is not convergent in the traditional sense, but only in the sense of principal values (i.e. as $\lim_{N\to\infty} \sum_{-N}^{N} (z - n)^{-1}$). Also, whenever working with infinite series, you need to explicitly justify why termwise operations are allowed; in the above case, the keywords are "uniform convergence on compact subsets of $\mathbb{C} \smallsetminus \mathbb{Z}$".

Q2. Prove:

$$\sum_{n=1}^{\infty} \frac{\coth(n\pi)}{n^7} = \frac{19\pi^7}{56700}.$$

You may use that

$$\operatorname*{Res}_{z=0} \frac{\cot(\pi z)\coth(\pi z)}{z^7} = \frac{-19\pi^6}{14175}.$$

Solution. For $N \geq 0$, let C_N be as above, i.e. the counterclockwise oriented boundary of the square of side $2N+1$ centered at the origin, and look at

$$\frac{1}{2\pi \mathrm{i}} \cdot \oint_{C_N} f(z)\,\mathrm{d}z, \text{ where } f(z) := \frac{\pi \cot(\pi z)\coth(\pi z)}{z^7}.$$

From $\pi\coth(\pi z) = \mathrm{i}\pi\cot(\mathrm{i}\pi z)$, it follows that $\pi\coth(\pi z)$ has simple poles at $\mathrm{i}\mathbb{Z}$ of residue 1 each. Therefore, it follows from the Residue Theorem that

$$\frac{1}{2\pi \mathrm{i}} \cdot \oint_{C_N} f(z)\,\mathrm{d}z = \operatorname*{Res}_{z=0} f(z) + \sum_{\substack{n=-N\\ n\neq 0}}^{N} \operatorname*{Res}_{z=n} f(z) + \sum_{\substack{n=-N\\ n\neq 0}}^{N} f(z).$$

It is given to us that

$$\operatorname*{Res}_{z=0} f(z) = \frac{-19\pi^7}{14175}.$$

Now

$$\operatorname*{Res}_{z=n} f(z) = \frac{\coth(\pi n)}{n^7}$$

and also

$$\operatorname*{Res}_{z=\mathrm{i}n} f(z) = \frac{\cot(\pi \cdot \mathrm{i}n)}{\mathrm{i}^7 n^7} = \frac{\coth(\pi n)}{n^7},$$

so using that these are odd functions, we conclude that

$$\frac{1}{2\pi \mathrm{i}} \cdot \oint_{C_N} f(z)\,\mathrm{d}z = \frac{-19\pi^7}{14175} + 4\sum_{n=1}^{N} \frac{\coth(\pi n)}{n^7}. \tag{4}$$

From Lemma 0.0.1, $\coth(\pi z) = \mathrm{i}\pi\cot(\mathrm{i}\pi z)$, and the invariance of C_N under $z \mapsto \mathrm{i}z$, it follows that

$$\left|\frac{1}{2\pi \mathrm{i}} \cdot \oint_{C_N} f(z)\,\mathrm{d}z\right| \leq \frac{1}{2\pi} \cdot 4(2N+1) \cdot \frac{C^2}{\pi} \cdot \frac{2^7}{(2N+1)^7}$$

and hence taking $\lim_{N\to\infty}$ in Eq. (4) gives us the required result. ■

Remark 2. Some solutions used that $f(z)$ is odd to conclude that $\oint_{C_N} f(z)\,\mathrm{d}z = 0$ for all $N \geq 0$. This is incorrect, because the line element $\mathrm{d}z$ "is also odd", so that the integrals over the top and bottom edges are the same and don't cancel out (and similarly for the right and left edges). Some solutions used the above construction but looked at C_N (or some variant thereof) for $N \in [0,\infty)$ as opposed to $N \in \mathbb{Z}_{\geq 0}$; this is incorrect–no bound of the sort in Lemma 0.0.1 exists in general for C_N for $N \in [0,\infty)$, and indeed the sequence of expanding contours C_N must be chosen to avoid the poles of the integrand.

Q3. Show that for any $\alpha > 0$ we have

$$\int_0^\infty \exp(-x^\alpha)\,\mathrm{d}x = \alpha^{-1}\Gamma(\alpha^{-1}).$$

Solution. In the integral

$$\int_0^\infty \exp(-x^\alpha)\,\mathrm{d}x,$$

use the u-substitution $u = x^\alpha$ so that $\mathrm{d}u = \alpha x^{\alpha-1}\,\mathrm{d}x$ and hence

$$\mathrm{d}x = \alpha^{-1} u^{\alpha^{-1}} \frac{\mathrm{d}u}{u}.$$

This gives us

$$\int_0^\infty \exp(-x^\alpha)\,\mathrm{d}x = \alpha^{-1} \int_0^\infty u^{\alpha^{-1}} \exp(-u)\frac{\mathrm{d}u}{u} = \alpha^{-1}\Gamma(\alpha^{-1}),$$

where in the last step we have used that for any s with $\operatorname{Re} s > 0$, we have by definition

$$\Gamma(s) = \int_0^\infty u^s \exp(-u)\frac{\mathrm{d}u}{u}.$$

■

Q4. Show that the entire function $1/\Gamma(z)$ has order one, but there is no constant $C > 0$ such that $1/\Gamma(z) = \mathcal{O}(\exp C|z|)$.

Solution. Recall from class that

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z} G(z)$$

where

$$G(z) := \prod_1^\infty \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n}.$$

Now $G(z)$ is the canonical product associated to the sequence $(-n)_{n\geq 1}$, which has critical exponent 1, so that by Theorem 3.14 in the notes dated 11/07/23, it follows that $G(z)$ has order 1. Clearly the order of $z \mapsto z\mathrm{e}^{\gamma z}$ is also 1, and hence it follows from the fact that for entire functions f and g we have $\rho(fg) \leq \max\{\rho(f), \rho(g)\}$ that

$$\rho(1/\Gamma) \leq 1.$$

One the other hand, to argue that $\rho(1/\Gamma) \geq 1$, you could either use Corollary 3.13, or Theorem 3.15, or argue as follows. The Euler Reflection Formula (Euler's Supplement) states that for $z \in \mathbb{C}$ we have

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Taking $z = (1/2) - n$ for $n \in \mathbb{Z}_{\geq 0}$ gives us that

$$\log \left| \frac{1}{\Gamma((1/2) - n)} \right| = -\log \pi + \log \Gamma\left(n + \frac{1}{2}\right) \sim \left(n + \frac{1}{2}\right) \log\left(n + \frac{1}{2}\right),$$

where in the last step we have used Stirling's approximation, with $\sim$ meaning that the quotient of the two sides has limit 1 as $n \to \infty$. In particular, this last term is clearly $\mathcal{O}(n^{1+\varepsilon})$ for every $\varepsilon > 0$, but not $\mathcal{O}(n)$; this proves also that $1/\Gamma(z)$ is not $\mathcal{O}(\exp C|z|)$ for any $C > 0$, and also that $\rho(1/\Gamma) \geq 1$. ∎

Remark 3. A few solutions argued that the product of two entire functions of order 1 is of order 1; this is incorrect, and a counterexample is $\mathrm{e}^z \cdot \mathrm{e}^{-z} = 1$. Such a sort of cancellation cannot happen for functions of the form $\mathrm{e}^{Q(z)} \prod_1^\infty E_p(z/a_n)$, where $Q(z)$ is a polynomial of degree p and the second term is a canonical product of order p, but this is a nontrivial result–this involves carefully bounding the canonical product away from disks around the zeroes a_n. See [1, Chapter 5, §5.2] for details.

Q5. Evaluate $\Gamma(1/3)\Gamma(2/3)$. Then find a formula for $\Gamma(3z)$ in terms of Γ at $z, z+1/3, z+2/3$.

Solution. Euler's Reflection Formula says

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

evaluating which at $z = 1/3$ yields

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}}. \tag{5}$$

As in class, we look at the function

$$F(z) := \frac{1}{2\pi\sqrt{3}} \cdot 3^z \cdot \Gamma\left(\frac{z}{3}\right)\Gamma\left(\frac{z+1}{3}\right)\Gamma\left(\frac{z+2}{3}\right).$$

Then, it follows immediately that $F(z+1) = zF(z)$ for all z, and $F(1) = 1$ (using (5) . Since $\Gamma(z)$ is bounded on $\operatorname{Re} z \in [1/3, 4/3]$, it follows (check!) that $F(z)$ is bounded on $\operatorname{Re} z \in [1,2]$ (here we are using also that $\sup_{\operatorname{Re} z\in[1,2]} |3^z| = 9$). Therefore, it follows from Wielandt's Theorem (Theorem 3.22 in the notes dated 11/07/23) that $F(z) = \Gamma(z)$; replacing z by $3z$ yields the triplication formula for $\Gamma(z)$, i.e.

$$\Gamma(3z) = \frac{1}{2\pi\sqrt{3}} \cdot 3^{3z} \cdot \Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right).$$

■

Remark 4. The same argument shows more generally that for any $n \geq 1$, we have the multiplication formula

$$\Gamma(nz) = \frac{1}{(2\pi)^{(n-1)/2}\sqrt{n}} \cdot n^{nz} \cdot \prod_{j=0}^{n-1} \Gamma\left(z + \frac{j}{n}\right),$$

with the analog of (5) being

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}. \tag{6}$$

To prove (6), use again the Reflection Formula: if A denotes the left hand side of (6), then

$$A^2 = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(\frac{n-k}{n}\right) = \prod_{k=1}^{n-1} \frac{\pi}{\sin(\pi k/n)} = \pi^{n-1}\prod_{k=1}^{n-1} \csc\left(\frac{\pi k}{n}\right) = \pi^{n-1} \cdot \frac{2^{n-1}}{n},$$

where in the last step we have used

$$\prod_{k=1}^{n-1}\left[2\sin\left(\frac{\pi k}{n}\right)\right] = \left|\prod_{k=1}^{n}(1-\mathrm{e}^{2\pi \mathrm{i}k/n})\right| = \left(1+z+\cdots+z^{n-1}\right)\Big|_{z=1} = n.$$

Q6. Show that for any polynomial $p(z)$, there exists a meromorphic function $f(z)$ such that

$$f(z+1) = p(z)f(z).$$

Solution. We can always choose $f(z) = 0$, and this is the only such function when $p(z) = 0$; hence assume we are not in this trivial case, and write $p(z) = C\prod_{j=1}^{n}(z - \alpha_j)$ as a product of linear factors, with $n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $C \in \mathbb{C}^*$. Then the function

$$f(z) := C^z \prod_{j=1}^{n} \Gamma(z - \alpha_j)$$

works, where $C^z := \exp(z \log C)$ for any choice of $\log C \in \mathbb{C}$. ■

Remark 5. The idea here is that the function $\delta : \mathcal{M}(\mathbb{C})^* \to \mathcal{M}(\mathbb{C})^*$ given by

$$\delta(f)(z) := \frac{f(z+1)}{f(z)}$$

is a group homomorphism. Therefore, to find an element of $\delta^{-1}(p)$ for a nonzero polynomial $p(z) = C\prod_{j=1}^{n}(z - \alpha_j)$, it suffices to find an element of $\delta^{-1}(C)$ and an element each of $\delta^{-1}(z - \alpha_j)$, and then multiply them together.

Q7. Let $M(x) > 0$ be a continuous function on $\mathbb{R}$.

(a) Prove there exists an entire function with $|f(x)| > M(x)$ for all $x \in \mathbb{R}$.
(b) Prove that there exists an entire function with $0 < |g(x)| < M(x)$ for all $x \in \mathbb{R}$.
(c) Prove that there does not exist an entire function with $|f(z)| > |z|$ for all $z \in \mathbb{C}$.

Solution.

(a) By replacing $M(x)$ by $\max\{M(x), M(-x)\}$, assume without loss of generality that M is even; it then suffices to produce an even entire function f such that $|f(x)| > M(x)$ on $x \in [0, \infty)$. For this, pick any increasing sequence $(a_n)_{n \geq 1}$ of even integers such that for all $n \geq 1$ we have $a_n > n$ and $2^{a_n} > \sup_{t \in [n,n+1]} M(t)$, and let $C > \sup_{t \in [0,1]} M(t)$ be any real constant. Define

$$f(z) := C + \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^{a_n} z^{a_n}.$$

By the Cauchy-Hadamard Formula, the radius of convergence $R \in [0, \infty]$ of the above power series satisfies

$$\frac{1}{R} = \varlimsup_{n \to \infty} \left(\left(\frac{2}{n}\right)^{a_n}\right)^{1/a_n} = \varlimsup_{n \to \infty} \frac{2}{n} = 0,$$

which tells us that this power series defines an entire function. Clearly, f is even and $f(\mathbb{R}) \subset \mathbb{R}^+$. Finally, for any $x \in [0, \infty)$, there is a unique $n \geq 0$ such that $x \in [n, n+1)$; if $n = 0$, then

$$f(x) \geq C > \sup_{t \in [0,1]} M(t) \geq M(x),$$

and if $n \geq 1$, then

$$f(x) \geq 2^{a_n} \left(\frac{x}{n}\right)^{a_n} \geq 2^{a_n} > \sup_{t \in [n,n+1]} M(t) \geq M(x)$$

as needed.

(b) This is an immediate consequence of (a): by our proof of (a), we may pick an entire function $f(z)$ such that $f(\mathbb{R}) \subset \mathbb{R}^+$ and $f(x) > 1/M(x)$ for all $x \in \mathbb{R}$. Set $g(z) := \exp(-f(z))$; then $g(\mathbb{R}) \subset \mathbb{R}^+$ as well, and we have for all $x \in \mathbb{R}$ that $f(x) < \exp(f(x))$ and hence

$$g(x) = \frac{1}{\exp(f(x))} < \frac{1}{f(x)} < M(x)$$

as needed.

(c) Suppose f is such a function. Then $\lim_{z \to \infty} |f(z)| = \infty$ implies that f is proper, and hence a nonconstant polynomial. Next, $|f(z)| > |z| \geq 0$ for all $z \in \mathbb{C}$ implies that $f(z)$ is never zero, but a nonconstant polynomial must be zero somewhere; this is the required contradiction.

$\blacksquare$

References

[1] E. M. Stein and R. Shakarchi, *Complex Analysis.* No. 2 in Princeton Lectures in Analysis, Princeton University Press, 2007.