Math 213A F23 Homework 6 Solutions

Dhruv Goel

October 2023

If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address. One other comment I want to make this time: please make sure to acknowledge collaborators on your problem sets, so that I do not get too suspicious of answers that look exactly identical.

Q1. Prove that any proper holomorphic map $f : \Delta \to \Delta$ of degree two can be written in the form $f(z) = A(S(B(z))$, with $A, B \in \operatorname{Aut}\Delta$ and $S(z) = z^2$.

Proof. Suppose f is such a map. We proceed in two steps.

(a) The function f has a critical point. If it did not, it would be a local homeomorphism (by the Inverse Function Theorem), and hence by Ex. 4 of the Forward, would be a topological covering map of degree 2; but Δ is a simply connected manifold, and so cannot be $2:1$ covered by another connected manifold.[1] One could also deduce this result by computing the derivative of a Blaschke product of order two (which can be shown to have one critical point inside the disk and one outside), or by proceeding similarly by using the result of Q3 below.

(b) Now, we finish the proof. Let z_0 be some critical point of f. Since $\operatorname{Aut}\Delta$ acts transitively on Δ[2], we can find $U, V \in \operatorname{Aut}\Delta$ such that $V(0) = z_0$ and $U(f(z_0)) = 0$; then the function $g(z) := U(f(V(z))$ is another proper holomorphic map $\Delta \to \Delta$ of degree two such that $g'(0) = 0$. By our classification of proper holomorphic self-maps of the disk as finite Blaschke products, we can write

$$g(z) = \gamma z^m \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z}$$

for some $m, n \geq 0, \alpha_j \in \Delta$ and $\gamma \in S^1$; in this case, we have $\deg g = m + n$. In our case, since $g'(0) = 0$, we must have $m \geq 2$. But then

$$2 = \deg g = m + n \geq 2 + n \geq 2$$

implies that $m = 2$ and $n = 0$. Therefore, $g(z) = \gamma z^2$ for some $\gamma \in S^1$, i.e. $g = R_\gamma(S(z))$ where $R_\gamma : \Delta \to \Delta$ is the map $R_\gamma(z) = \gamma z$. It follows that taking $A := U^{-1} \circ R_\gamma$ and $B := V^{-1}$ suffices. ∎

Aliter. A few students submitted the following proof, which was too brilliant to not include. We may write f as a degree two Blaschke product as before; the only case for which the result is not obvious is when we have

$$f(z) = \gamma \cdot \frac{z - \alpha}{1 - \overline{\alpha} z} \cdot \frac{z - \beta}{1 - \overline{\beta} z}$$

for $\alpha \neq \beta \in \Delta$. The idea is that it suffices to produce a $B \in \operatorname{Aut}\Delta$ such that $B(\alpha) = -B(\beta)$; then we may take $A \in \operatorname{Aut}\Delta$ taking $S(B(\alpha)) = S(B(\beta))$ to 0. Then the composite $A(S(B(z))$ is a proper holomorphic self-map of degree 2 of Δ with zeroes exactly at α and β, and hence by the general theory of Blaschke products is obtainable from $f(z)$ by a rotation, which can be absorbed into A, finishing the proof.

It remains to construct such a B. For this, we can proceed as follows: take the hyperbolic "midpoint" of α and β, i.e. take the unique point μ on the geodesic between α and β whose hyperbolic distance to α is the same as its hyperbolic distance to β. (Why does such a point exist? How do you locate it geometrically? Can you write down an analytic formula for it? It's the unique critical point!) Finally, take $B(z) := (z - \mu)/(1 - \overline{\mu} z)$. This takes $\mu \mapsto 0$ and the geodesic between α and β to a geodesic through 0, i.e. a straight line, with $B(\alpha)$ and $B(\beta)$ on either side of 0 at an equal distance from 0. ∎

Remark 1. A theorem of Heins says that given any collection of points in Δ, there is a unique Blaschke product with these as critical points, up to postcomposition with an automorphism of Δ. see [1] and [2].

[1] One can take this line of reasoning further to deduce that f must have a unique critical point: since f has degree two, if it had exactly $n \geq 1$ distinct critical values, then it would also have exactly n distinct critical points. (Why? What possible ramification types are allowed?) This would then give us a $2:1$ cover of Δ with n points removed (let's call this an n-punctured disk) by a space which is homeomorphic to it. This is possible only if $n = 1$; indeed, an n-punctured disk deformation retracts onto a bouquet of n circles and so has Euler characteristic $1 - n$. If $X \to Y$ is a covering map of nice spaces (e.g. paracompact Hausdorff spaces homotopy equivalent to finite CW complexes, etc.) of degree $d \geq 1$, then the Euler characteristics of X and Y are related by $\chi(X) = d\chi(Y)$. In particular, in the above scenario, we would end up with the equation

$$1 - n = 2(1 - n),$$

which implies $n = 1$.

[2] For example, the Blaschke product $(z - \alpha)/(1 - \overline{\alpha} z)$ sends any $\alpha \in \Delta$ to 0.

Q2. Find the hyperbolic distance between i and i + 1 in $\mathbb{H}$.

Solution (1). The geodesic between i and i + 1 in $\mathbb{H}$ is the part of the semicircle of radius $\sqrt{5}/2$ centered at 1/2 and can be parametrized as $t \mapsto \gamma(t) := (1 + \sqrt{5}e^{it})/2$ for $t \in [\theta, \pi - \theta]$, where $\theta := \arctan 2$. Recalling that the metric on $\mathbb{H}$ is given by $\rho_{\mathbb{H}}(z)|dz| = |dz|/\operatorname{Im} z$, we compute the length of this path as

$$\int_{\theta}^{\pi-\theta} \frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)} dt = \int_{\theta}^{\pi-\theta} \frac{\sqrt{5}/2}{(\sqrt{5}/2)\sin t} dt = \int_{\theta}^{\pi-\theta} \csc t \, dt = [\ln(\csc t - \cot t)]_{\theta}^{\pi-\theta}.$$

Now note that $\csc(\theta) = \csc(\pi - \theta) = \sqrt{5}/2$ whereas $\cot(\theta) = -\cot(\pi - \theta) = 1/2$. Therefore, this value is

$$\ln\left(\frac{\sqrt{5}+1}{2}\right) - \ln\left(\frac{\sqrt{5}-1}{2}\right) = 2\ln\varphi = 0.9624\ldots,$$

where $\varphi := (1 + \sqrt{5})/2$ is the golden ratio. Note that this is less than 1, which is the length of the path that goes horizontally between i and i + 1, as it should be (why?). ■

Solution (2). Consider a Möbius transformation $T : \mathbb{H} \to \Delta$ that takes $i \mapsto 0$ and i + 1 to some point on the real axis, e.g.

$$T(z) := \frac{\sqrt{5}}{1 - 2i} \cdot \frac{z - i}{z + i}.$$

For this T, we have $T(i + 1) = 1/\sqrt{5}$. Since T is a Möbius transformation, it is an isometry for the hyperbolic metrics on $\mathbb{H}$ and Δ, it suffices to compute the distance between 0 and $1/\sqrt{5}$ for the hyperbolic metric $\rho_{\Delta}(z)|dz| = |dz|/(1 - |z|^2)$ on Δ. The geodesic between these points is simply the part of the real axis that lies between them, and we can parametrize this as $t \mapsto \gamma(t) = t$ for $t \in [0, 1/\sqrt{5}]$. Therefore, the length of this path can be computed as

$$\int_0^{1/\sqrt{5}} \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt = \int_0^{1/\sqrt{5}} \frac{1}{1 - t^2} dt = \left[\ln\left(\frac{1+t}{1-t}\right)\right]_0^{1/\sqrt{5}} = \ln\left(\frac{1 + (1/\sqrt{5})}{1 - (1/\sqrt{5})}\right) = 2\ln\varphi.$$ ■

Solution (3). As in the previous solution, we reduce this to computing the hyperbolic distance between 0 and $1/\sqrt{5}$ in Δ. We showed in class that the hyperbolic distance $d_{\Delta}(z, w)$ between two points $z, w \in \Delta$ can be computed as

$$\cosh\left(\frac{d_{\Delta}(z, w)}{2}\right) = \frac{|\langle \tilde{z}, \tilde{w}\rangle_{1,1}|}{\|\tilde{z}\|_{1,1} \, \|\tilde{w}\|_{1,1}},$$

where $\tilde{z}$ (resp. $\tilde{w}$) is a lift of $z \in \Delta \subset \hat{\mathbb{C}} = \mathbb{CP}^1$ (resp. of w) to $\mathbb{C}^2$,

$$\left\langle \begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \begin{bmatrix}\gamma\\ \delta\end{bmatrix} \right\rangle_{1,1} := \alpha\overline{\gamma} - \beta\overline{\delta}$$

is the Hermitian form of type (1, 1), and $\|v\|_{1,1} := \sqrt{|\langle v, v\rangle_{1,1}|}$ for $v \in \mathbb{C}^2$. We pick the "obvious representatives" $\tilde{z} := \begin{bmatrix}1\\ z\end{bmatrix}$ and $\tilde{w} := \begin{bmatrix}1\\ w\end{bmatrix}$ and use this to compute

$$\langle \tilde{z}, \tilde{w}\rangle_{1,1} = 1 - z\overline{w} \text{ and } \|\tilde{z}\|_{1,1} = \sqrt{1 - |z|^2}, \|\tilde{w}\|_{1,1} = \sqrt{1 - |w|^2}.$$

It follows that for $z, w \in \Delta$ we have

$$\cosh\left(\frac{d_{\Delta}(z, w)}{2}\right) = \frac{|1 - z\overline{w}|}{\sqrt{(1 - |z|^2)(1 - |w|^2)}}.$$

Taking $z = 0$ and $w = 1/\sqrt{5}$ gives us that if $d := d_{\Delta}(0, 1/\sqrt{5})$, then

$$\cosh(d/2) = (1 - (1/5))^{-1/2} = \sqrt{5}/2.$$

Since $\cosh^{-1}(x) = \ln\big(x + \sqrt{x^2 - 1}\big)$ for $x > 1$, it follows that

$$d = 2\ln\left(\frac{\sqrt{5}}{2} + \sqrt{\frac{5}{4} - 1}\right) = 2\ln\varphi.$$

■

Q3. Prove that any proper holomorphic map $f : \mathbb{H} \to \mathbb{H}$ can be written in the form

$$f(z) = a_0 z + b_0 + \sum_1^n \frac{a_i}{b_i - z},$$

with $a_i \geq 0$ and $b_i \in \mathbb{R}$.

Proof. We proceed in several steps:

(a) First, we claim that f extends to a rational function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This can be done in several ways, perhaps the simplest of which is to use the biholomorphism $\Delta \to \mathbb{H}$ to transport the result of Corollary 2.30 (in the version of the notes dated 10/25/23) from Δ to $\mathbb{H}$. One could also apply a version of the Schwarz reflection principle (how?).

(b) Next, we claim that the poles of f must lie on $\partial\mathbb{H} = \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$, i.e. cannot lie in the lower half plane $\overline{\mathbb{H}}$. One way to do this is to argue as in (a) that a Blaschke product preserves $\Delta, \partial\Delta = S^1$ and $\hat{\mathbb{C}} \setminus \overline{\Delta}$. Another way to do this is to use the Schwarz Reflection Principle: if we can show that $f(\mathbb{RP}^1) \subset \mathbb{RP}^1$, then we can conclude that $f(z)$ and $\overline{f(\overline{z})}$ are rational maps of $\hat{\mathbb{C}}$ that agree on $\mathbb{RP}^1 \subset \mathbb{CP}^1$, and hence must be the same by the identity principle; in particular, since f has no poles in $\mathbb{H}$, it can't have any in $\overline{\mathbb{H}}$ either. We will actually show $f(\mathbb{R}) \subset \mathbb{RP}^1$; this suffices. For this, let $z \in \mathbb{R}$; if this is a pole, then we are done, so suppose it is not. Pick a sequence $z_n \in \mathbb{H}$ converging to z. By properness of f, the sequence $f(z_n)$ escapes every compact set in $\mathbb{H}$, but has bounded imaginary part, and the only way this can happen is that $\operatorname{Im} f(z) = \lim_{n\to\infty} \operatorname{Im} f(z_n) = 0$.

(c) Next, we claim that the poles of f must all be simple, and further the residue at any pole in $\mathbb{C}$ must be a nonnegative real number. First, suppose we're looking at a pole of f at $b \in \mathbb{R}$. Then we can separate the polar part of f to write it as

$$f(z) = p(z) + \frac{c_1}{z-b} + \cdots + \frac{c_k}{(z-b)^k}$$

for some $k \geq 1, c_i \in \mathbb{C}, p(z) \in \mathbb{C}(z)$ with $p(b) \neq \infty$ and $c_k \neq 0$. Then as $z \to b$ in $\mathbb{H}$, the dominating term is $c_k(z-b)^{-k}$; if $k \geq 2$, then irrespective of the argument of c_k, this dominating term (and hence f) will take values in $\overline{\mathbb{H}}$ even when the domain is restricted to $\mathbb{H}$, since the power $z \mapsto z^k$ stretches angles by a factor of k. Therefore, $k = 1$. Further, we must have $c_1 \in (-\infty, 0)$; this is similar: firstly, c_1 has to be real by $f(z) = \overline{f(\overline{z})}$ and the fact that $p(z)$ and $\overline{p(\overline{z})}$ are both bounded in a neighborhood of $b \in \mathbb{R}$, and, secondly, if $c_1 > 0$, then the dominating term $c_1(z-b)^{-1}$ would again take values in $\overline{\mathbb{H}}$. Now, look at $b = \infty$; this may or may not be a pole, but by an argument similar to the one above, must be at most a simple pole, and in fact, the polar part must look like $a_0 z$ for some $a_0 \in [0, \infty)$.

(d) Now we are almost done: let $b_1, \ldots, b_n \in \mathbb{R}$ be the poles of f on the real axis, and for $1 \leq i \leq n$, let $a_i := -\operatorname{Res}_{z=b_i} f(z)$, which we showed above satisfies $a_i \geq 0$. Let the polar part of $f(z)$ near infinity look like $a_0 z$ with $a_0 \in [0, \infty)$. Then the difference

$$f(z) - \left(a_0 z + \sum_1^n \frac{a_i}{b_i - z}\right)$$

is a bounded entire function and hence constant, say $b_0 \in \mathbb{C}$. It follows finally from $f(z) = \overline{f(\overline{z})}$ (or from $f(0) \in \mathbb{R}$) that $b_0 \in \mathbb{R}$ as well, finishing the proof.

■

Remark 2. The essential idea here is that there are two apparently distinct ways to escape every compact set in H, namely to have $\operatorname{Im} z \to 0$ or $\operatorname{Im} z \to \infty$, but, in reality, these are the same, because both amount to saying $z \to \partial\mathbb{H} = \mathbb{RP}^1$. While Blaschke products give a multiplicative representation of proper self-maps of Δ, this problem gives us an additive representation of proper self-maps of $\mathbb{H}$.

Q4. Let $T(a,b,c) \subset \mathbb{H}$ be a hyperbolic triangle with interior angles a, b, c. Prove that

$$\text{area } T(a,b,c) = \pi - a - b - c$$

as follows.

(a) Note that $T(0,0,0)$ is an 'ideal triangle' with all vertices at infinity. Show all ideal triangles have area π.

(b) Show geometrically that $A(a) = \text{area } T(\pi - a, 0, 0)$ satisfies $A(a + a') = A(a) + A(a')$, and conclude that $A(a) = a$.

(c) Extend the sides of $T(a,b,c)$ to rays as [suggested] and use the corresponding vertices at infinity to relate the area of $T(a,b,c)$ to the area of $T(\pi - a, 0, 0)$, $T(\pi - b, 0, 0)$ and $T(\pi - c, 0, 0)$.

Solution.

(a) Note that an angle of zero is only possible at the boundary: this uses that the geodesics in $\mathbb{H}$ are given by semicircles[3] orthogonal to the boundary $\partial\mathbb{H} = \mathbb{RP}^1$, that the metric on $\mathbb{H}$ is conformal (so hyperbolic angles correspond to Euclidean angles), and that two such semicircles cannot be tangent to each other at a point in $\mathbb{H}$ (why?). Next, note that all ideal triangles are congruent to each other; this uses that $\operatorname{Aut}\mathbb{H} = \mathrm{PSL}_2\,\mathbb{R}$ acts by isometries on $\mathbb{H}$ and uniquely triply transitively on $\mathbb{RP}^1 \subset \mathbb{CP}^1$, and hence any collection of three vertices on $\partial\mathbb{H}$ can be taken to any other.[4] Therefore, it suffices to compute the area of one specific ideal triangle, namely with vertices at $1, -1$, and ∞; let's call this T. This can be done explicitly using the metric $\rho_{\mathrm{H}}(z)|\mathrm{d}z| = |\mathrm{d}z|/(\operatorname{Im} z)$ to get

$$\iint_T \frac{1}{y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^{1} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\,\mathrm{d}y \right) \mathrm{d}x = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin(x)\Big|_{-1}^{1} = \pi.$$

Therefore, we have shown that all ideal triangles have hyperbolic area π.

(b) Let $A : (0, \pi] \to [0, \infty)$ be the function defined above as $A(a) := \operatorname{area} T(\pi - a, 0, 0)$. First, we need to show that this is well-defined: that any two triangles with angles $\pi - a, 0, 0$ are congruent. To do this, we use the action of $\operatorname{Aut}\mathbb{H} = \mathrm{PSL}_2\,\mathbb{R}$ to send the vertices of any such triangle with zero vertex angles to $0, \infty$; then it is geometrically clear (see Figure 1) that for a fixed a, the locus of all possible positions of the third vertex consists of two half-rays starting at the origin that are symmetric about the y-axis, and hence any point on this locus can be taken to any another by a composition of scaling by some $\lambda > 0$ (which is an isometry of $\mathbb{H}$) and a reflection across of the y-axis (which is not holomorphic, but antiholomorphic involution and hence an isometry of $\mathbb{H}$).

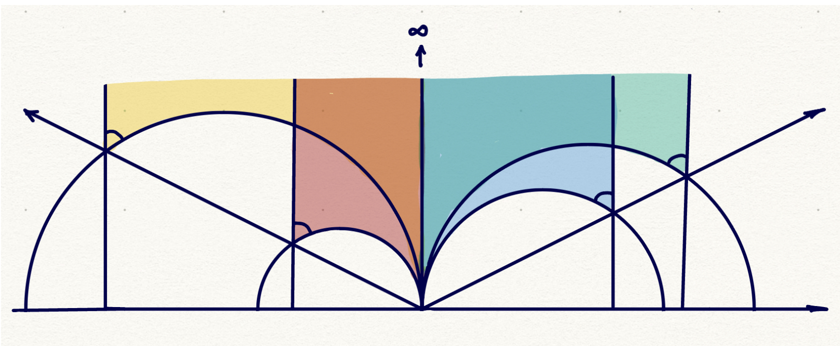

Figure 1: Any two triangles of the form $T(\pi - a, 0, 0)$ are congruent.

[3] This includes half-lines as well, by our convention that "circles" always means "generalized circles".

[4] One way to show this last fact is to the use that $\mathrm{PSL}_2\,\mathbb{C}$ acts uniquely triply transitively on $\mathbb{CP}^1$: given two triples of three distinct points on $\mathbb{RP}^1$, find the unique Möbius transformation $f(z)$ in $\mathrm{PSL}_2\,\mathbb{C}$ that takes one to the other; then uniqueness tells us that $f(z) = \overline{f(\overline{z})}$, so that $f(z)$ is actually in $\mathrm{PSL}_2\,\mathbb{R}$. This last statement is a special instance of the more general fact:

In hyperbolic geometry, similar triangles are congruent.

Note that this is not true in Euclidean geometry. We won't use this directly, and so I refer you to [3, Chapter 11] for a proof of this result and more on hyperbolic geometry.

Therefore, we have a well-defined function $A : (0, \pi] \to [0, \infty)$. It is also clear that A is continuous. Therefore, given that $A(\pi) = \pi$ from the first part, to show that $A(a) = a$, it suffices to show that

$$A(a + a') = A(a) + A(a') \tag{1}$$

whenever $0 < a, a' < a + a' \leq \pi$, by standard properties of Cauchy's functional equation (1).[5] To show (1) geometrically, consider the diagram in Figure 2.

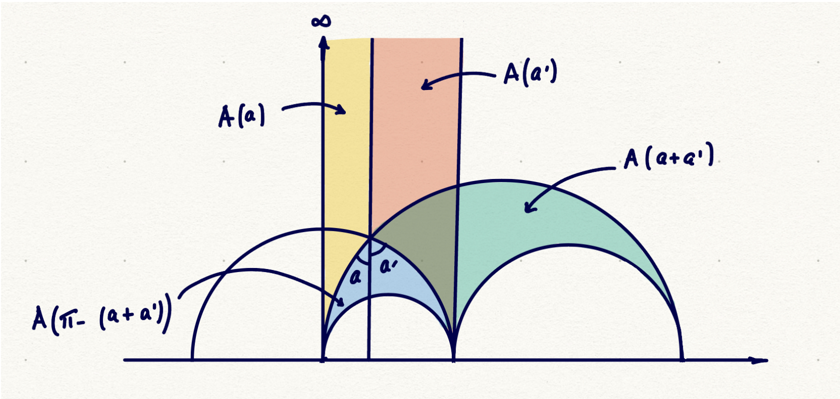

Figure 2: A geometric proof of the functional equation (1).

In this figure, we have extended the sides of the triangles to obtain ideal triangles. The sum of the yellow, red, and blue areas is given by

$$A(a) + A(a') + A(\pi - (a + a')) = A(\pi) = \pi. \tag{2}$$

The sum of the green and blue areas is also given by

$$A(b) + A(\pi - b) = \pi, \tag{3}$$

where in our case $b := a + a'$. Putting Equations (2) and (3) together gives the result.

(c) We draw the suggested diagram in our upper half plane model as follows (see Figure 3).

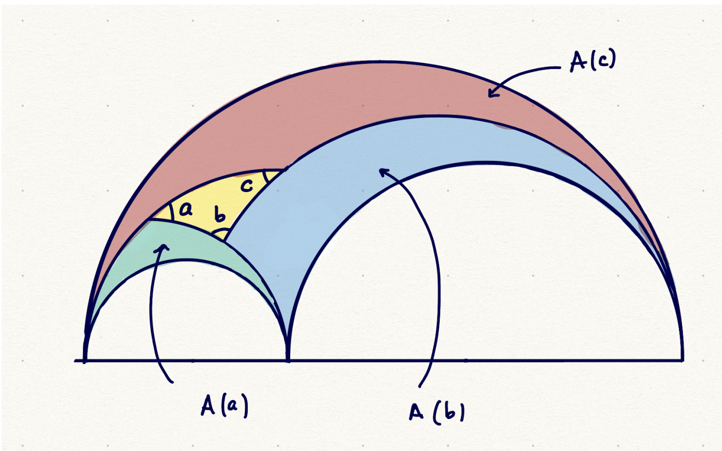

Figure 3: Extending sides of an arbitrary triangle.

It follows from (a) and (b) and summing the areas of the triangles as shown that

$$T(a, b, c) + A(a) + A(b) + A(c) = \pi,$$

which gives the required result. The above figure assumes that none of the geodesics that make up the sides contain the point ∞; we can reduce to this case either by using an isometry or by drawing a picture similar to the one above, which I leave to the diligent reader.

■

[5] One uses this to conclude that $A(q\pi) = q\pi$ for all $q \in (0, 1] \cap \mathbb{Q}$ or for all $q \in \{r/2^n : n \geq 1, 0 < r < 2^n \text{ odd}\}$, and then uses density of the chosen set in $(0, \pi]$ along with continuity.

Q5. Let $f : \Delta \to \mathbb{C}$ be an analytic function such that $\operatorname{Re} f(z) \geq 0$. Show that [for all $z \in \Delta$, we have]

$$|f(z)| \leq |f(0)| \cdot \frac{1+|z|}{1-|z|}.$$

Proof. If f is constant, there is nothing to show. Else, by the Open Mapping Principle,

$$f(\Delta) \subset \operatorname{Int}\{z : \operatorname{Re} z \geq 0\} = \{z : \operatorname{Re} z > 0\} = -\mathrm{i}\mathbb{H}.$$

For any $\lambda \in -\mathrm{i}\mathbb{H}$, the map $\varphi_\lambda : -\mathrm{i}\mathbb{H} \to \Delta$ given by

$$\varphi_\lambda(w) = \frac{w-\lambda}{w+\overline{\lambda}}$$

is a Möbius transformation that takes $\lambda \mapsto 0$, and all such Möbius transformations are obtainable from φ_λ by rotations. Applying this to the point $\lambda = f(0)$, we let $g := \varphi_{f(0)} \circ f : \Delta \to \Delta$; this is given by

$$g(z) = \frac{f(z) - f(0)}{f(z) + \overline{f(0)}}$$

and satisfies $g(0) = 0$. By the Schwarz Lemma, we have $|g(z)| \leq |z|$ for all z, which says that

$$\left| \frac{f(z) - f(0)}{f(z) + \overline{f(0)}} \right| \leq |z| \tag{4}$$

for all $z \in \Delta$. By two applications of the triangle inequality, we have also for all $z \in \Delta$ that

$$\frac{|f(z)| - |f(0)|}{|f(z)| + |\overline{f(0)}|} \leq \left| \frac{f(z) - f(0)}{f(z) + \overline{f(0)}} \right|. \tag{5}$$

The result follows from Inequalities (4) and (5), along with $|\overline{f(0)}| = |f(0)|$ and the following lemma, the proof of which is clear, applied to $u = |f(z)|/|f(0)|$ and $v = |z|$.

Lemma 0.0.1. Let u, v be real numbers with $u > 0$ and $v < 1$. Then

$$\frac{u-1}{u+1} \leq v \Rightarrow u \leq \frac{1+v}{1-v}.$$

∎

Q6. Pick a basis for the Lie algebra of $\mathrm{SL}_2\,\mathbb{C}$ and show how each basis element can be canonically interpreted as a holomorphic vector field on $\hat{\mathbb{C}} = \mathbb{CP}^1$. Check that the Lie bracket corresponds to the [negative of the] bracket of vector fields.

Solution. The Lie algebra $\mathfrak{sl}_2\,\mathbb{C}$ of $\mathrm{SL}_2\,\mathbb{C}$ can be realized as the space of 2×2 traceless complex matrices

$$\mathfrak{sl}_2\,\mathbb{C} := \left\{ \begin{bmatrix} a & b, \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{C}, a+d=0 \right\}$$

with the Lie bracket given by the commutator of matrices, $[X,Y] = XY - YX$. Equivalently, it is the Lie algebra on the generators E, F, H subject to the relations

$$[H,E] = 2E, \quad [H,F] = -2F, \text{ and } \quad [E,F] = H, \tag{6}$$

with the correspondence between these given by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We pick E, F, H as our basis of $\mathfrak{sl}_2\,\mathbb{C}$. To interpret an $X \in \mathfrak{sl}_2\,\mathbb{C}$ as a holomorphic vector field on $\hat{\mathbb{C}}$, we:

(a) find some curve representing it, i.e. a holomorphic map $\gamma_X : \Delta \to \mathrm{SL}_2\,\mathbb{C}$ that satisfies $\gamma_X(0) = \mathrm{id}_2$ and $\gamma_X'(0) = X$,

(b) then use the representation $\rho : \mathrm{SL}_2\,\mathbb{C} \to \mathrm{Aut}\,\hat{\mathbb{C}}$ as Möbius transformations to consider the corresponding one-parameter family $t \mapsto (z \mapsto \gamma_{X,z}(t) := \rho(\gamma_X(t))(z))$ of biholomorphisms of $\hat{\mathbb{C}}$,

(c) differentiating which yields the holomorphic vector field $X_z : z \mapsto (\mathrm{d}_0\gamma_{X,z})\,(\partial_t|_{t=0}) = \gamma_{X,z}'(0)\,\partial_z$ on $\hat{\mathbb{C}}$ (where $\partial_z := \partial/\partial z$, and the prime denotes the derivative with respect to t).

In (a), one choice of such curve guaranteed to work is $\gamma_X(t) := \exp(tX)$; this is because

$$\det\exp(tX) = \exp(\mathrm{tr}(tX)) = \exp(t\,\mathrm{tr}(X)) = \exp(0) = 1$$

for all t.[6] Let's carry this out for E, F, H above.

(i) For E, the curve $\gamma_E(t)$ is given by

$$\gamma_E(t) = \exp(tE) = 1 + tE = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

because $E^2 = 0$. The corresponding one-parameter family of biholomorphisms of $\hat{\mathbb{C}}$ is given (in the local coordinate z on $\mathbb{C} \subset \hat{\mathbb{C}}$) by the

$$z \mapsto \gamma_{E,z}(t) := \frac{z+t}{0\cdot z + 1} = z + t.$$

[6]The relation $\det\exp(X) = \exp(\mathrm{tr}\,X)$ for any matrix X can be proved in various ways. One could argue, for instance, that $f(t) := \det(\exp(tX))$ satisfies the differential equation $f'(t) = \mathrm{tr}(X)f(t)$ and $f(0) = 1$, and hence must be $\exp(t\,\mathrm{tr}(X))$, by the uniqueness of solutions to ODE's in $\mathbb{C}$. Alternatively, over $\mathbb{C}$, one could triangularize X and write $X = TYT^{-1}$ where Y is upper triangular. Then $\exp(X) = T\exp(Y)T^{-1}$ and so $\det(X) = \det\bigl(T\exp(Y)T^{-1}\bigr) = \det(\exp(Y))$. But now $\exp(Y)$ is upper triangular as well, and if λ_i are the diagonal entries of Y, then those of $\exp(Y)$ are $\exp(\lambda_i)$. It follows then that

$$\det\exp(X) = \det\exp(Y) = \prod \exp(\lambda_i) = \exp\left(\sum \lambda_i\right) = \exp(\mathrm{tr}\,Y) = \exp(\mathrm{tr}\,X),$$

where in the last step we have used $\mathrm{tr}\,X = \mathrm{tr}\,TYT^{-1} = \mathrm{tr}\,T$. Alternatively, one could proceed via the Jordan decomposition or reduce to diagonalizable matrices by continuity of $\exp$, $\det$ and tr, and the density of diagonalizable matrices. This is part of the more general framework which says that if $G \hookrightarrow \mathrm{GL}_n$ is some matrix Lie group, then the Lie algebra $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ of $\mathfrak{g}$ can be thought to consist of matrices, and then $X \mapsto \exp(tX)$, where the righthand side now denotes the matrix exponential, maps $\mathfrak{g} \to G$; indeed, there is a notion of an exponential map $\exp : \mathfrak{g} \to G$ for arbitrary Lie groups, and the above observation amounts to saying that (a) this construction is functorial, and (b) for the group GL_n, this agrees with the matrix exponential.

Therefore, we have $\gamma'_{E,z}(t) = 1$ for all z, t and the corresponding vector field E_z is given by

$$z \mapsto \gamma'_{E,z}(0)\,\partial_z = \partial_z.^7 \tag{7}$$

(ii) For F, we have

$$\gamma_F(t) = \exp(tF) = 1 + tF = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

Next, we have

$$\gamma_{F,z}(t) = \frac{z}{tz+1}$$

and hence the corresponding vector field F_z is

$$z \mapsto \gamma'_{F,z}(0)\,\partial_z = -z^2\,\partial_z. \tag{8}$$

(iii) For H, we have

$$\gamma_H(t) = \exp(tH) = \begin{bmatrix} \exp(t) & 0 \\ 0 & \exp(-t) \end{bmatrix}.$$

Next, we have

$$\gamma_{H,z}(t) = \exp(2t)\cdot z$$

and hence the corresponding vector field H_z is

$$z \mapsto \gamma'_{H,z}(0)\,\partial_z = 2z\,\partial_z. \tag{9}$$

Finally, recall that the bracket of two vector fields Z, W on $\mathbb{C}$ is given by

$$[Z, W](f) = Z(Wf) - W(Zf)$$

for holomorphic $f : \mathbb{C} \to \mathbb{C}$. If we can write $Z = p(z)\,\partial_z$ and $W = q(z)\,\partial_z$ for $p, q : \mathbb{C} \to \mathbb{C}$ holomorphic, then we have for holomorphic $f : \mathbb{C} \to \mathbb{C}$ that

$$\begin{aligned} Z(Wf) - W(Zf) &= p(z)\,\partial_z(q(z)\,\partial_z f(z)) - q(z)\,\partial_z(p(z)\,\partial_z f(z)) \\ &= p(z)\,\partial_z(q(z)f'(z)) - q(z)\,\partial_z(p(z)f'(z)) \\ &= p(z)\,(q(z)f''(z) + q'(z)f'(z)) - q(z)\,(p(z)f''(z) + p'(z)f'(z)) \\ &= ((p(z)q'(z) - q(z)p'(z))\partial_z)\,f(z), \end{aligned}$$

so that $[Z, W]$ is given by

$$(p(z)q'(z) - q(z)p'(z))\,\partial_z. \tag{10}$$

Using formula (10), the commutation relations (6), and the explicit formulae (7), (8), and (9), one can check that this construction takes the Lie bracket to the bracket of vector fields, up to a negative sign. Explicitly, we have

$$[H_z, E_z] = [2z\,\partial_z, \partial_z] = (2z \cdot 0 - 1 \cdot 2)\,\partial_z = -2\,\partial_z = -2E_z,$$

and similarly

$$[H_z, F_z] = 2F_z \text{ and } [E_z, F_z] = -H_z.$$

This negative sign is not unimportant and shows up for a subtle reason; one could avoid it by redefining the commutator of the vector fields to be the negative of what we chose above, but that seems arbitrary. See Remark 3 for an explanation of this sign. ■

[7] In the local coordinate $w = 1/z$ in $\hat{\mathbb{C}} \setminus \{0\}$, the one-parameter transformation is given by $w \mapsto \gamma_{E,w}(t) := (w^{-1} + t) = w(1 + tw)^{-1}$, and here $\gamma'_{E,w}(t) = -w^2(1 + tw)^{-2}$, so that the vector field here is given by $w \mapsto \gamma'_{E,w}(0)\,\partial_w = -w^2\,\partial_w$. On the intersection $\mathbb{C}^*$, this agrees with what we computed above because $\mathrm{d}z = -w^{-2}\mathrm{d}w \Rightarrow \partial_z = -w^2\,\partial_w$ because ∂_z (resp. ∂_w) is dual to $\mathrm{d}z$ (resp. $\mathrm{d}w$). Technically, only by specifying the maps $z \mapsto \partial_z$ on $\mathbb{C}$ and $w \mapsto -w^2\partial_w$ on $\hat{\mathbb{C}} \setminus \{0\}$ have we specified a complete vector field on $\hat{\mathbb{C}}$. Similarly, in computing the bracket of vector fields, to be completely rigorous, we must compute it separately on $\mathbb{C}$ and $\hat{\mathbb{C}} \setminus \{0\}$. In this solution, we will work with the local coordinate z on $\mathbb{C} \subset \hat{\mathbb{C}}$ and leave computations around ∞ to the diligent reader/student.

Remark 3. This problem gives us a glimpse into a very general framework that can be summarized by saying that for a closed manifold M[8], we have the canonical isomorphism of Lie algebras

$$\mathrm{Lie}(\mathrm{Aut}(M)) := \mathrm{T}_{\mathrm{id}_M}\,\mathrm{Aut}(M) \cong \mathrm{Vect}(M), \tag{11}$$

where Lie denotes the Lie algebra of the group $\mathrm{Aut}(M)$, and the Lie bracket is given by

$$\mathrm{ad}_X(Y) = -\boldsymbol{\mathcal{L}}_X Y$$

where $\boldsymbol{\mathcal{L}}_X Y = [X, Y]$ is the Lie derivative, i.e. usual Lie bracket, of vector fields on M (this is where the negative sign shows up above). In particular, if you have a Lie group G acting on M, then we get a representing smooth map

$$\rho : G \to \mathrm{Aut}(M),$$

applying the functor Lie (i.e. taking the differential at the identity) to which yields the Lie algebra homomorphism

$$\mathrm{d}_e\rho : \mathfrak{g} \to \mathrm{Vect}(M),$$

which allows us to interpret any vector in the Lie algebra of G as a vector field on M, with the Lie bracket on the left side corresponding to the negative of the usual Lie bracket of vector fields on the right side; this is essentially what we did above, in a much more down-to-earth fashion. Here's how the correspondence in (11) is obtained: given a curve[9] $\gamma : \Delta \to \mathrm{Aut}(M)$ with $\gamma(0) = \mathrm{id}_M$ representing a tangent vector to $\mathrm{Aut}(M)$ at id_M, send it to the vector field

$$z \mapsto (\mathrm{d}_0\gamma_z)\left(\partial_t|_{t=0}\right),$$

where $\gamma_z : \Delta \to M$ is the curve given by $\gamma_z(t) = \gamma(t) \cdot z$. Equivalently, this is the vector field that takes a function f on M to the function $z \mapsto \partial_t|_{t=0} f(\gamma(t)z)$. The exponential map

$$\exp : \mathrm{Vect}(M) \to \mathrm{Aut}(M)$$

satisfies

$$\exp(tX) = \varphi_t^X,$$

where φ_t^X denotes the automorphism given by locally flowing along X for time t.[10] Now, in any Lie group, we may compute the Lie bracket

$$\mathrm{ad}_X(Y) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} \exp(tX) \cdot \exp(sY) \cdot \exp(tX).$$

In our case, this amounts to saying that if X and Y are vector fields, then the Lie bracket in $\mathrm{Lie}(\mathrm{Aut}(M))$ can be computed by saying that $\mathrm{ad}_X(Y)$ is the vector field that takes a function f on M to the function $\mathrm{ad}_X(Y)f$ defined by

$$\left(\mathrm{ad}_X(Y)f\right)(z) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} f\left(\varphi_t^X\left(\varphi_s^Y\left(\varphi_{-t}^X(z)\right)\right)\right).$$

By definition of the flow, we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} f(\varphi_t^X \varphi_s^Y \varphi_{-t}^X z) = Y_{\varphi_{-t}^X(z)}((\varphi_t^X)^* f),$$

[8]Here, M could be a real smooth manifold and $\mathrm{Aut}(M) = \mathrm{Diff}^\infty(M)$ the (infinite-dimensional Lie) group of diffeomorphisms of M, in which case $\mathrm{Vect}(M)$ should be interpreted as the space of smooth vector fields on M; or M could be a symplectic manifold and $\mathrm{Aut}(M)$ the group of its symplectomorphisms, in which case $\mathrm{Vect}(M)$ should be interpreted as the space of symplectic vector fields on M; or M could be a complex manifold and $\mathrm{Aut}(M)$ the group of its biholomorphisms, in which case $\mathrm{Vect}(M)$ should be interpreted as the space of holomorphic vector fields on M; or M could be a proper smooth variety over a field k and $\mathrm{Aut}(M)$ the group of regular automorphisms, in which case $\mathrm{Vect}(M)$ should be interpreted as the space of algebraic vector fields, i.e. sections of $\Omega_{M/k}^\vee$, and so on.

[9]When working in the real setting, we can take $\Delta = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$; when working in the holomorphic setting, we can take $\Delta = \Delta(0, \varepsilon)$ for some $\varepsilon > 0$; when working in the algebro-geometric setting, we can take $\Delta = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ etc.

[10]The compactness of M guarantees the existence of this flow globally, at least for small time t, by the local existence of solutions to ODE's. This flow could, in general, not be defined globally for any t, but is certainly defined for small t locally at any point of M, and the calculus on the infinite-dimensional spaces is set up so that this local existence result suffices in what follows in any case.

and hence it follows by definition of the Lie derivative that

$$\mathrm{ad}_X(Y) = -\mathcal{L}_X Y$$

as needed. For more on Lie groups and Lie algebras (at least finite-dimensional ones), see [4]. The theory of infinite-dimensional Lie groups is substantially harder; see [5] and the references there as a starting point. You were not expected to know this; this is just for your interest.

Remark 4. In fact, the space of holomorphic vector fields on $\hat{\mathbb{C}}$ is 3-dimensional, and spanned by E_z, F_z and H_z. Indeed, any holomorphic vector field on $\hat{\mathbb{C}}$ looks on $\mathbb{C} \subset \hat{\mathbb{C}}$ like $f(z)\,\partial_z$ for some entire function f; changing variables $z = 1/w$, this looks like

$$-w^2 f(w^{-1})\,\partial_w.$$

If $-w^2 f(w^{-1})$ extends to a holomorphic function at $w = 0$, then $f(w^{-1})$ has a pole of order ≤ 2 at $w = 0$. It follows that there are $R, C \gg 0$ such that $|f(z)| \leq C|z|^2$ for all $|z| > R$, which proves that $f(z)$ is a polynomial of degree ≤ 2. This tells us that our original vector field lies in the span of E_z, F_z and H_z. For algebraic geometry enthusiasts, this is a good simple illustration of the GAGA principle–that all of projective complex geometry comes from algebraic geometry: we have shown that the holomorphic tangent bundle $\mathrm{T}\mathbb{CP}^1 \cong \mathcal{O}_{\mathbb{CP}^1}(2)$ and hence $\Gamma(\mathbb{CP}^1, \mathrm{T}\mathbb{CP}^1) = \mathrm{H}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2))$, which has dimension $h^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(2)) = 3$. If you know what Chern classes (and/or numbers) are, this is another good illustration of why $c_1(\mathbb{CP}^1) = 2 = \chi(S^2)$. The above problem gives us an explicit way to identify the adjoint representation of $\mathrm{SL}_2\,\mathbb{C}$ as the action of $\mathrm{SL}_2\,\mathbb{C}$ on $\mathrm{Vect}(\mathbb{CP}^1) = \Gamma(\mathbb{CP}^1, \mathrm{T}\mathbb{CP}^1)$ via Möbius transformations.

References

[1] M. Heins, "On a Class of Conformal Metrics," *Nagoya Mathematical Journal*, vol. 21, pp. 1–60, 1962.

[2] D. Singer, "The Location of Critical Points of Finite Blaschke Products," *Conformal Geometry and Dynamics: An Electronic Journal of the American Mathematical Society*, vol. 10, pp. 117–124.

[3] A. Presley, *Elementary Differential Geometry*. Springer Undergraduate Mathematics Series, Springer, second ed., 2010.

[4] W. Fulton and J. Harris, *Representation Theory: A First Course*. No. 129 in Graduate Texts in Mathematics, Springer, 1991.

[5] R. Schmid, "Infinite-Dimensional Lie Groups and Algebras in Mathematical Physics," *Advances in Mathematical Physics*, vol. 2010, pp. 1–35, 2010.