Math 213A F23 Homework 13 Solutions

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November 2023

If you find any errors, or have any comments or questions, please send them to me via e-mail at this e-mail address.

Q1. Prove that there is no nonconstant analytic function $f : \Delta \to \Delta$ with zeroes at the points $z_n = 1 - 1/(n+1), n = 1, 2, 3 \ldots$. (Hint: consider $f(0)/B_n(0)$, where $B_n(z)$ is a Blaschke product with zeroes at $z_1, \ldots, z_n$.)

Proof. Suppose $f : \Delta \to \Delta$ is a nonconstant analytic function such that $f(z_n) = 0$ for all $n \geq 1$. If f has a zero of order $m \geq 0$ at $z = 0$, then we may write $f(z) = z^m f_1(z)$ for some function f_1 holomorphic on Δ with $f_1(0) \neq 0$; then, by a repeated application of the Schwarz Lemma, f_1 still maps Δ to Δ, has zeroes at the z_n, and satisfies $f_1(0) \neq 0$, so replacing f by f_1, we may assume that $f(0) \neq 0$.

As suggested, let

$$B_n(z) := \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z}_k z}.$$

Since each Blaschke factor is an automorphism of Δ preserving S^1, it follows that $B_n : \Delta \to \Delta$ is proper: that

$$\lim_{r \to 1^-} \inf_{z \in S^1(r)} |B_n(z)| = 1.$$

Since $|f(z)| \leq 1$ on Δ, it follows from the Maximum Principle applied to the holomorphic function $f(z)/B_n(z)$ on Δ (how?), that for all $z \in \Delta$,

$$\left| \frac{f(z)}{B_n(z)} \right| \leq 1,$$

i.e.

$$|f(z)| \leq |B_n(z)|.$$

Plugging in $z = 0$ gives us

$$|f(0)| \leq (-1)^n \frac{1}{n+1},$$

and since this is true for each $n \geq 1$, we cocnlude that $f(0) = 0$, which is a contradiction. ■

Remark 1. Alternatively, one can show using Jensen's Formula that a sequence of points (z_n) in Δ is the set of zeroes of some nonconstant bounded holomorphic function on Δ iff $\sum_1^\infty (1 - |z_n|) < \infty$ (see [1, §15.21-23]), so the above problem is equivalent to the divergence of the harmonic series.

Q2. State and prove a necessary and sufficient condition for a meromorphic 1-form $\omega = \omega(z)\,\mathrm{d}z$ on $\mathbb{C}$ to be the logarithmic derivative, $\omega = \mathrm{d}\log f = f'(z)/f(z)\,\mathrm{d}z$ of a meromorphic function $f(z)$.

Solution. A meromorphic 1-form is the logarithmic derivative of a meromorphic function iff it has only simple poles with integer residues. Clearly, this is necessary because if $f(z)$ is meromorphic in a neighborhood of $\alpha \in \mathbb{C}$, then the function $f'(z)/f(z)$ has polar part

$$\frac{\operatorname{ord}(f, \alpha)}{z - \alpha} + \mathcal{O}(1)$$

at α. Conversely, if ω has only simple poles with integer residues, then picking any base point $p \in \mathbb{C}$ which is not a pole of ω, we may define

$$f(z) := \exp\left(\int_p^z \omega\right),$$

where the integral is taken along any path avoiding the poles of ω. Since the residues of ω are integers, it follows that the integral $\int_p^z \omega$ is independent of the path up to $2\pi\mathrm{i}\mathbb{Z}$, i.e. well-defined as an element of $\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$, and hence $f(z)$ is a well-defined holomorphic function away from the poles of ω. It remains to show that f extends to a meromorphic function around any simple pole α of ω. It suffices to show this when $\alpha = 0$, so suppose we can write

$$\omega(z) = \frac{m}{z} + g(z)$$

for some $g(z)$ holomorphic in a neighborhood of 0 and $m \in \mathbb{Z}$. Then it follows for any $p \neq 0$ sufficiently close to zero, we have

$$\begin{aligned} f(z) &= \exp\left(\int_p^z \left(\frac{m}{w} + g(w)\right) \mathrm{d}w\right) \\ &= \exp\left(m \log\left(\frac{z}{p}\right) + \int_p^z g(w)\,\mathrm{d}w\right) \\ &= \left(\frac{z}{p}\right)^m \exp\left(\int_p^z g(w)\,\mathrm{d}w\right), \end{aligned}$$

where log denotes some local branch (the choice of which is irrelevant because $m \in \mathbb{Z}$). This tells us that $f(z)$ extends to a meromorphic function in a neighborhood of 0, completing the proof. ■

Remark 2. The condition that ω have at most simple poles is necessary. For instance, the above recipe yields for $\omega(z) = z^{-2}$ the function $f(z) = \exp\bigl(-z^{-1}\bigr)$ (up to constants), which has an essential singularity at $z = 0$.

Q3. Suppose $f \in \mathcal{S}$ satisfies $f(\mathrm{i}z) = \mathrm{i}f(z)$. Show that $f(\Delta)$ contains $B(0, 1/\sqrt{2})$.

Proof. It suffices to show that there is a $g \in \mathcal{S}$ such that $f(z)^4 = g(z^4)$; then it would follow from the Koebe 1/4 Theorem that if w is not in the image of f, then w^4 is not in the image of g, so that $|w^4| \geq 1/4 \Rightarrow |w| \geq 1/\sqrt{2}$.

It remains to produce such a g. There are many ways to do this: one can use covering space theory as shown on a previous solution set, or argue directly using the Taylor coefficients a_n of f as follows. Writing $f(z) = z + a_2 z^2 + \cdots$ as usual, the condition $f(\mathrm{i}z) = \mathrm{i}f(z)$ tells us that if $n \not\equiv 1 \pmod 4$, then $a_n = 0$, so

$$f(z) = z(1 + a_5 z^4 + a_9 z^8 + \cdots).$$

Since f converges on Δ, we have by the Cauchy-Hadamard Formula that $\overline{\lim}\, |a_n|^{1/n} \leq 1$, and this same bound tells us that the function

$$h(z) := \sum_{n=0}^{\infty} a_{4n+1} z^n = 1 + a_5 z + a_9 z^2 + \cdots$$

converges on Δ as well. It remains to define

$$g(z) := z h(z)^4.$$

The relation $f(z)^4 = g(z^4)$ follows by construction; and $g(0) = 0$ and $g'(0) = 1$ are clear. To show that g is univalent, suppose $g(z) = g(w)$ for $z, w \in \Delta$. Pick any $z_0, w_0 \in \Delta$ such that $z = z_0^4$ and $w = w_0^4$. Then

$$f(z_0)^4 = g(z_0^4) = g(z) = g(w) = g(w_0^4) = f(w_0)^4,$$

so that for some $k \in \{0, 1, 2, 3\}$ we have

$$f(z_0) = \mathrm{i}^k f(w_0) = f(\mathrm{i}^k w_0),$$

where in the last step we have used $f(\mathrm{i}z) = \mathrm{i}f(z)$. Since $f \in \mathcal{S}$, it follows that $z_0 = \mathrm{i}^k w_0$, so that $z = w$ as needed. ∎

Remark 3. The bound $1/\sqrt{2}$ is optimal: the image of $f(z) = z \cdot (1 - z^4)^{-1/2}$ doesn't contain $(1 + \mathrm{i})/2$, or any other fourth root of $-1/4$ (why?).

Remark 4. The same technique shows for any $n \geq 1$ that if $f \in \mathcal{S}$ satisfies $f(\zeta_n z) = \zeta_n f(z)$, then $f(\Delta)$ contains $B(0, 2^{-2/n})$, and again this bound is optimal, as shown by $f(z) = z \cdot (1 - z^n)^{-2/n}$.

Q4. Let $f(z)$ be an entire function such that $f(z)$ is never zero and $f^{-1}(1)$ is finite. Prove that f is constant.

Solution. Since $f(z)$ is never zero, we may write $f(z) = \exp(g(z))$ for an entire function $g(z)$. Since $f^{-1}(1) = g^{-1}(2\pi \mathrm{i}\mathbb{Z})$ is finite, g misses infinitely many values in $2\pi \mathrm{i}\mathbb{Z}$, and hence in particular is constant, by Picard's Little Theorem. Therefore, f is constant too. ■

Q5. Where are the 9 flexes of the cubic curve $V \subset \mathbb{CP}^2$ defined by $x^3 + y^3 = 1$? How many of these are real?

Solution. The flexes of a smooth projective curve $V \subset \mathbb{CP}^2$ defined by the vanishing of a homogenous polynomial $F \in \mathbb{C}[X, Y, Z]$ are located at the points of intersection of V with the variety defined by the vanishing of the Hessian determinant

$$\mathrm{Hess}(F) := \det \begin{bmatrix} \partial^2 F/\partial X^2 & \partial^2 F/\partial X \partial Y & \partial^2 F/\partial X \partial Z \\ \partial^2 F/\partial X \partial Y & \partial^2 F/\partial Y^2 & \partial^2 F/\partial Y \partial Z \\ \partial^2 F/\partial X \partial Z & \partial^2 F/\partial Y \partial Z & \partial^2 F/\partial Z^2 \end{bmatrix}.$$

Taking $F = X^3 + Y^3 - Z^3$ yields

$$\mathrm{Hess}(F) = \det \begin{bmatrix} 6X & 0 & 0 \\ 0 & 6Y & 0 \\ 0 & 0 & -6Z \end{bmatrix} = -216XYZ.$$

Therefore, the flexes of V are given by its intersection with the "coordinate axes" $X = 0, Y = 0$ and $Z = 0$. These are the six finite points $(\omega^k, 0)$ and $(0, \omega^k)$ for $k = 0, 1, 2$, where $\omega = \zeta_3 = e^{2\pi i/3}$, and the three points at infinity given by in homogenous coordinates by $[1 : -\omega^k : 0]$ for $k = 0, 1, 2$. Of these, three are real, namely $(1, 0), (0, 1)$ and the point at infinity $[1 : -1 : 0]$, as is geometrically clear as well: see Figure 1. ■

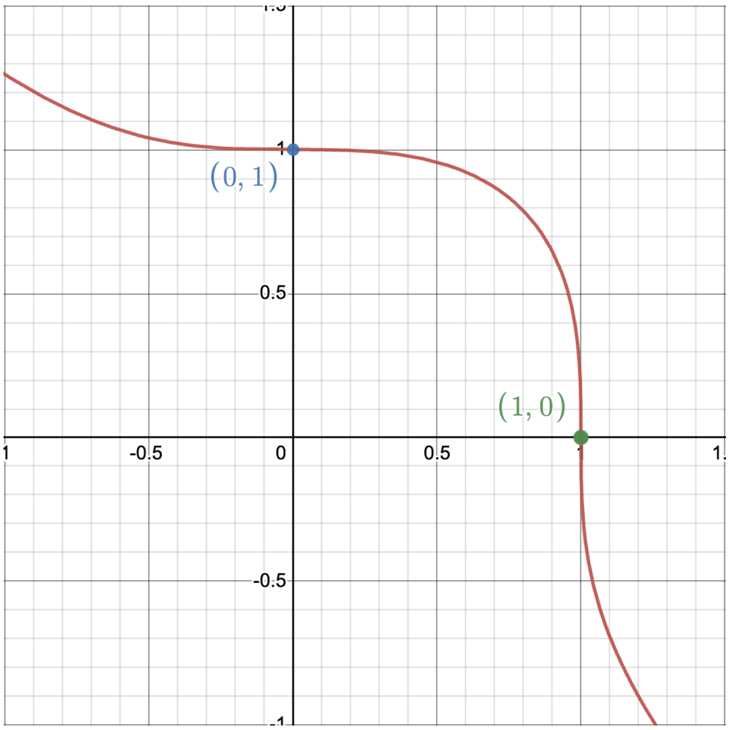

Figure 1: A Desmos graph of the (real points of the) curve $x^3 + y^3 = 1$.

Remark 5. As observed on a previous problem set, not all the 9 flexes of a smooth cubic curve $V \subset \mathbb{CP}^2$ can be real, as a consequence of the Sylvester-Gallai Theorem. In fact, for a smooth cubic V defined over $\mathbb{R}$, there are always exactly three real flexes; see [2, §2.1-2.2].

Q6. Let L be the length in the hyperbolic metric of the closed geodesic γ on $X = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ that makes a figure 8 around 0 and 1. Show that $L = \log\bigl(17 + 2\sqrt{2}\bigr)$. (Hint: show that γ corresponds to a matrix of trace 6 in $\pi_1(X) = \Gamma(2)$.).

Solution. We present two solutions:

(a) Given any local isometry $Y \to X$ of Riemannian manifolds, geodesics in X lift locally to geodesics in Y.[1] Therefore, to calculate the length of this geodesic, it suffices to find a lift to $\mathbb{H}$ under the universal covering map $\lambda : \mathbb{H} \to X$ and compute its length. Here we can choose the basepoint $\mathrm{i} \in \mathbb{H}$ and $1/2 \in X$, and then the lift of this geodesic is the part of the circle $|z - 1| = \sqrt{2}$ from $z = \mathrm{i}$ to $z = (12 + \mathrm{i})/5$; see Figure 2. On a previous problem set, we showed that the length of the geodeisc $c + r\mathrm{e}^{\mathrm{i}\theta}$ from θ_1 to θ_2 for $c \in \mathbb{R}, r \in \mathbb{R}_+$ and $0 < \theta_1 < \theta_2 < \pi$ is given by
$$L = \log(\csc\theta - \cot\theta)\big|_{\theta_1}^{\theta_2}.$$
Taking $c = 1, r = \sqrt{2}$, $\theta_1 = \tan^{-1}(1/7)$ and $\theta_2 = 3\pi/4$, we get $L = \log\bigl(17 + 2\sqrt{2}\bigr)$ as needed.[2]

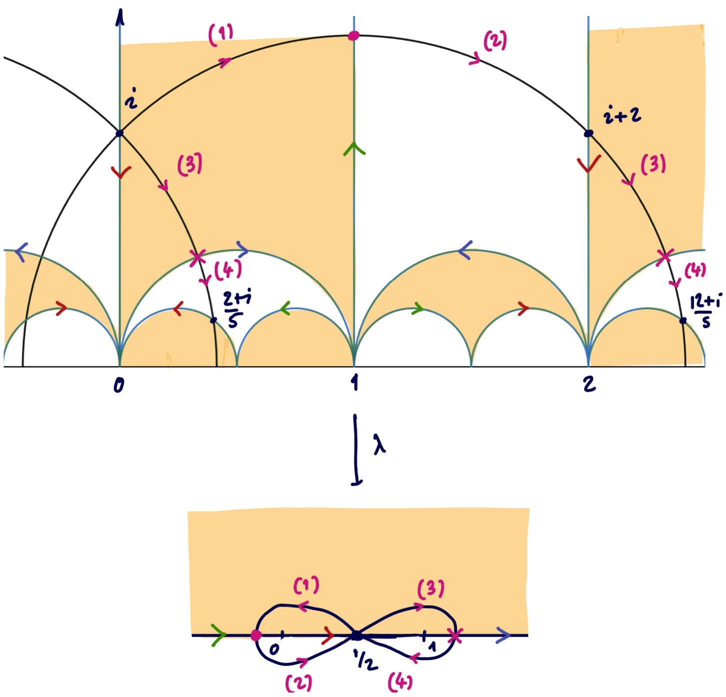

Figure 2: A lift of the geodeisc to $\mathbb{H}$. The region $\lambda^{-1}(\mathbb{H})$ is shaded in yellow, and the red, blue, and green arrows denote the preimages of the corresponding real segements $(0,1)$, $(1,\infty)$, and $(-\infty, 0)$ respectively. The points marked in solid black dots form the $\Gamma(2)$ orbit of i, those in solid pink dots the orbit of $1 + \sqrt{2}\mathrm{i}$, and those in pink crosses the orbit of $(1 + \sqrt{2}\mathrm{i})/3$.

[1] This is because the geodesic equations are differential equations–they are local in nature. A local isometry of Riemmannian manifolds without boundary $Y \to X$ is a covering map whenever Y is complete and X is connected (so certainly in our case where $Y = \mathbb{H}$ and $X = \mathbb{C} \setminus \{0, 1\}$), although we don't need this result here.

[2] One can also employ the symmetry of the situation to their advantage to take $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$ to get the half length
$$\frac{1}{2}L = \log\frac{\sqrt{2}+1}{\sqrt{2}-1},$$
which gives the same answer.

(b) Fixing basepoints $x \in X$ and $\tilde{x} \in \mathbb{H}$ related by $\pi(\tilde{x}) = x$, the covering map π gives us an isomorphism

$$\rho : \pi_1(X, x) \to \Gamma(2)$$

given by the following recipe: given a class $[\gamma] \in \pi_1(X, x)$ represented by a loop $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$, lift γ to a path $\tilde{\gamma} : [0,1] \to \mathbb{H}$ with $\tilde{\gamma}(0) = \tilde{x}$; then ρ takes $[\gamma]$ to the unique $T \in \Gamma(2)$ such that $\tilde{\gamma}(1) = T\tilde{x}$. Given a closed geodesic γ containing x, we can parametrize γ to give us a class $[\gamma] \in \pi_1(X, x)$; then the element $\rho[\gamma] \in \Gamma(2)$ is hyperbolic.[3] A different choice of parametrization yields a conjugate element in $\Gamma(2)$, and this gives us a well-defined map

$$\rho : \{\text{closed geodesics in } X \text{ through } x\} \to \{\text{hyperbolic conjugacy classes in } \Gamma(2)\}.$$

The key here is:

Theorem 0.0.1. This map ρ is a bijection, and the length L of a geodesic γ is related to the trace of $\rho[\gamma]$ by

$$\operatorname{Tr} \rho[\gamma] = 2\cosh(L/2). \tag{1}$$

Proof. An inverse map can be constructed as follows: given a hyperbolic conjugacy class in $\Gamma(2)$, pick a representative T. Now T has two fixed points in $\Gamma(2)$; by conjugating this representative if needed, we may assume that the geodesic connecting these two points passes through $\tilde{x}$. Then the image of this geodesic in $\mathbb{H}$ under π is a closed geodesic in X through x; it is then easy to see that these maps give inverse bijections. Finally, to see the last claim, note that the length L of γ is the distance in $\mathbb{H}$ between z and $\rho[\gamma]z$ for any $z \in \mathbb{H}$. Since $\rho[\gamma]$ is hyperbolic, we may conjugate it so it looks like $z \mapsto \lambda z$ for some $\lambda \in \mathbb{R}_+$ i.e. has a matrix representative of the form $\begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{bmatrix}$. Then (1) follows from the equations $\operatorname{Tr} \rho[\gamma] = \lambda^{1/2} + \lambda^{-1/2}$ and $L = \log \lambda$. The verification of all the omitted details is left to the reader. ∎

Let's carry this program out explicitly in our case. Take $x = 1/2$ and $\tilde{x} = \mathrm{i}$ as before. Then you can check that under the homomorphism

$$\rho : \pi_1(X, 1/2) \to \Gamma(2)$$

described above, the loop around 0 (made of arcs labelled 1 and 2 in Figure 2) lifts to the geodesic $|z - 1| = \sqrt{2}$ between i and $\mathrm{i} + 2$ and hence maps to the matrix

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

whereas the loop around 1 (made of arcs labelled 3 and 4 in Figure 2) lifts to the geodesic $|z + 1| = \sqrt{2}$ between i and $(2 + \mathrm{i})/5$ and hence maps to the matrix

$$S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$[4]

Then the geodesic making a figure 8 corresponds to the concatenation of these, and corresponds to the matrix

$$\rho[\gamma] = ST = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix},$$

[3] This follows from the classification of isometries of $\mathbb{H}$; see the section in the notes dates 11/23/23 preceding Theorem 2.25.

[4] Already, this gives the fascinating result that $\Gamma(2)$ is a free group on two generators generated by T and S.

which has trace 6. Therefore, the length L of this geodesic is given by

$$6 = 2\cosh(L/2),$$

which yields

$$L = 2\log\left(3 + 2\sqrt{2}\right) = \log\left(17 + 2\sqrt{2}\right)$$

as needed. ■

Q7. Prove that $\lambda(\mathrm{i}/2) = 12\sqrt{2} - 16$. (Hint: Letting $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, [the value] $\lambda(\tau)$ is the cross-ratio of the critical values (suitably ordered) of any degree two map $f_\tau : X_\tau \to \hat{\mathbb{C}}$. For the square torus $\tau = \mathrm{i}$, choose f_i so its critical values are the roots of $z^4 + 1 = 0$. Then one can choose $f_{\mathrm{i}/2}(z) = (f_{\mathrm{i}}(z) + f_{\mathrm{i}}(z)^{-1})/2$, and find that is critical values are $\{-1, -1/\sqrt{2}, 1/\sqrt{2}, 1\}$.)

Proof. The function
$$g(w) := \int_0^w \frac{\mathrm{d}z}{\sqrt{1 + z^4}}$$
is the Schwarz-Christoffel isomorphism from Δ to the square of side length $2g(1)$ with sides aligned along the co-ordinate axes, and takes the four roots of $z^4 + 1$ to the corners of this square. Therefore, the function h defined on Δ by

$$h(w) = \frac{1}{4g(1)} g(w) + \frac{1 + \mathrm{i}}{4}$$

gives the Riemann mapfrom Δ to the square $(0, 1/2) \times (0, 1/2)\mathrm{i}$, with again the four roots of $z^4 + 1 = 0$ mapped to the corners of the square. The inverse function to $h(z)$, which we will call f_{i}, can be developed by Schwarz reflection to give a meromorphic function on the complex plane which is doubly periodic with respect to the square lattice $\mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$, and hence can be written as $A \circ \wp_{\mathrm{i}}$ for some Möbius transformation A. In particular, the critical points of f_{i} are located at the points of order 2, namely $0, 1/2, \mathrm{i}/2, (1 + \mathrm{i})/2$, and the critical values of f_{i} are the roots of $z^4 + 1$. See Figure 3.

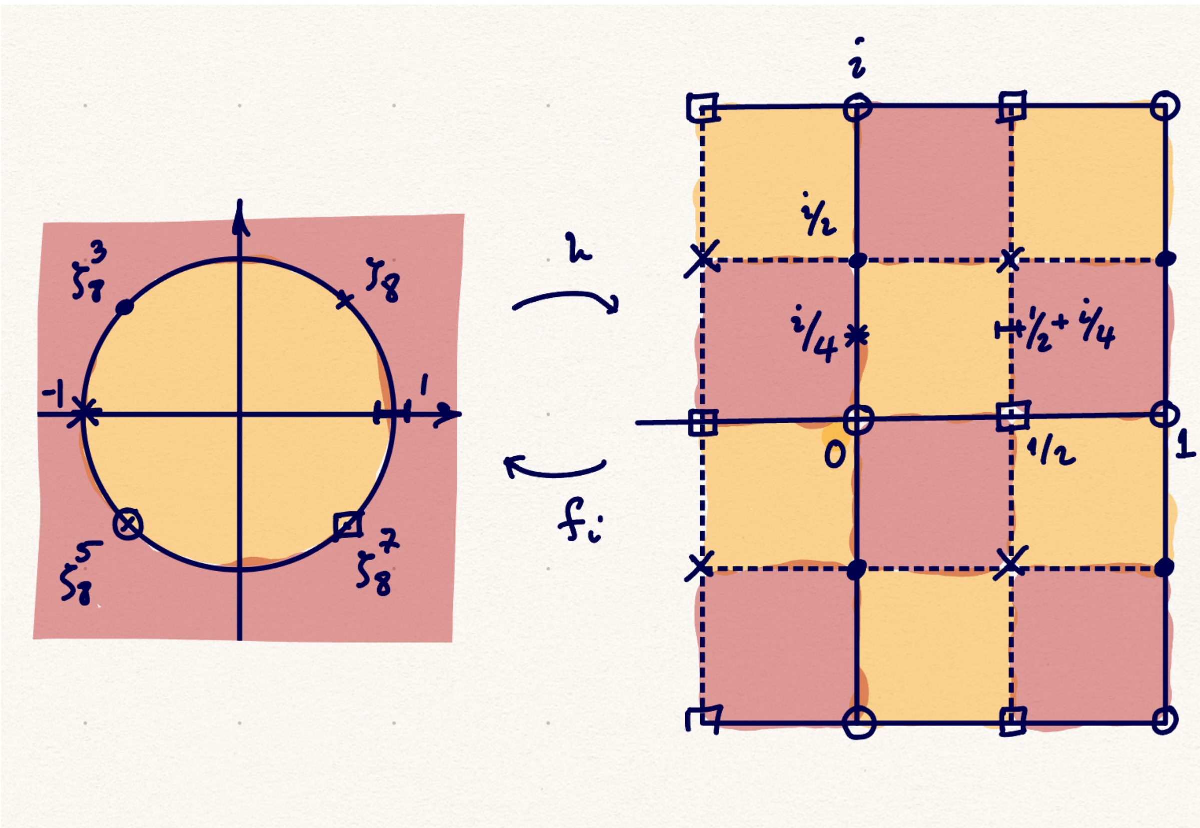

Figure 3: The inverse functions h and f_{i}, where f_{i} has been developed by Schwarz reflection as shown.

The key claim is:

Lemma 0.0.2. For any $z \in \mathbb{C}$, we have $f_{\mathrm{i}}(z + (\mathrm{i}/2)) = 1/f_{\mathrm{i}}(z)$.

Proof. Note that the identity $g(w) = \overline{g(\overline{w})}$ implies that $h(w) = \overline{h(\overline{w})} + (\mathrm{i}/2)$, which translates to the identity

$$f_{\mathrm{i}}\left(z + \frac{\mathrm{i}}{2}\right) = \overline{f_{\mathrm{i}}(\overline{z})}.$$

On the other hand, the construction of f_{i} by Schwarz reflection along the x-axis gives us the identity

$$f_{\mathrm{i}}(\overline{z}) = 1/\overline{f_{\mathrm{i}}(z)},$$

and combining these identities gives the result. ∎

It follows that the function $f_{\mathrm{i}/2}(z) := (f_{\mathrm{i}}(z) + f_{\mathrm{i}}(z)^{-1})/2$ is meromorphic and doubly periodic for the lattice $\mathbb{Z} \oplus \mathbb{Z}(\mathrm{i}/2)$. Next, the resulting map $f_{\mathrm{i}/2} : X_{\mathrm{i}/2} \to \hat{\mathbb{C}}$ again has degree 2; indeed, one half of the fundamental parallelogram $(0, 1/2) \times (0, 1/2)\mathrm{i}$ maps isomorphically Δ under f_{i}, and hence to isomorphically to $\hat{\mathbb{C}} \setminus [-1, 1]$ under $f_{\mathrm{i}/2}$, which tells us that any point in $\hat{\mathbb{C}} \setminus [-1, 1]$ has exactly two preimages under $f_{\mathrm{i}/2}$, one each in the squares $(0, 1/2) \times (0, 1/2)\mathrm{i}$ and $(1/2, 1) \times (0, 1/2)\mathrm{i}$. The critical values of $f_{\mathrm{i}/2}$ (in the right order) are therefore given by

$$(e_0, e_1, e_2, e_3) = \left(f_{\mathrm{i}/2}(0), f_{i/2}\left(\frac{1}{2}\right), f_{\mathrm{i}/2}\left(\frac{\mathrm{i}}{4}\right), f_{i/2}\left(\frac{1}{2} + \frac{\mathrm{i}}{4}\right)\right).$$

It remains to compute these. It is clear from Figure 3 that

$$f_{\mathrm{i}}(0) = \zeta_8^5 \Rightarrow f_{\mathrm{i}/2}(0) = -\frac{1}{\sqrt{2}} \text{ and } f_{\mathrm{i}}(1/2) = \zeta_8^7 \Rightarrow f_{\mathrm{i}/2}\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}.$$

Next, by symmetry (see Figure 3 again) it follows that

$$f_{\mathrm{i}}\left(\frac{\mathrm{i}}{4}\right) = -1$$

so that

$$f_{\mathrm{i}/2}\left(\frac{\mathrm{i}}{4}\right) = -1.$$

Similarly, we have

$$f_{\mathrm{i}}\left(\frac{1}{2} + \frac{\mathrm{i}}{4}\right) = 1 \Rightarrow f_{\mathrm{i}/2}\left(\frac{1}{2} + \frac{\mathrm{i}}{4}\right) = 1.$$

Therefore, the critical values of $f_{i/2}$ (in the right order) are given by

$$(e_0, e_1, e_2, e_3) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1, 1\right).$$

It follows that their cross-ratio is given by

$$\lambda\left(\frac{i}{2}\right) = \frac{e_1 - e_0}{e_3 - e_0} \cdot \frac{e_3 - e_2}{e_1 - e_2} = 12\sqrt{2} - 16$$

as needed. ∎

References

[1] W. Rudin, *Real and Complex Analysis.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Company, third ed., 1987.

[2] J. H. Silverman and J. H. Tate, *Rational Points on Elliptic Curves.* Undergraduate Texts in Mathematics, Springer, second ed., 2015.