1.8 06/26/24 - Smoothness, Multiplicity, Tangent Lines

Today, we will talk about smoothness of algebraic curves. What should smoothness mean–i.e. what should it mean to say that a curve $C \subset \mathbb{A}^2_k$ is smooth at a point $P \in C$? One definition is that at each point, we have a well-defined tangent direction, i.e. that the curve is well-approximated by a linear polynomial. Certainly, whatever this notion is, it should be invariant under affine changes of coordinates, so we may focus on the case when $P = (0,0)$, and then considering a few examples naturally leads us to the following definition.

Definition 1.8.1.

(a) A polynomial $f(x,y) \in k[x,y]$ is said to be homogeneous of degree $d \geq 0$ if in the ring $k[x,y,t]$, we have the polynomial identity

$$f(tx,ty) = t^d f(x,y).$$

This is equivalent to saying that in an expression of the form $f(x,y) = \sum_{i,j\geq 0} a_{i,j}x^iy^j$ with $a_{i,j} \in k$, we have $a_{i,j} = 0$ unless $i + j = d$. For each $d \geq 0$, the set of all polynomials in $k[x,y]$ of degree d will be denoted by $k[x,y]_d$.

(b) Any $f(x,y) \in k[x,y]$ can be written uniquely as

$$f = f_0 + f_1 + \cdots + f_d,$$

where $d = \deg f \geq 0$, and for each i with $0 \leq i \leq d$, the polynomial $f_i \in k[x,y]$ is homogeneous of degree i. If $0 \neq f$, then there is a unique smallest index i_0 such that $f_{i_0} \neq 0$; in this case, we define the multiplicity of f at the origin $O = (0,0)$, written $m_O(f)$, and the initial part of f, written $\text{in}(f)$, to be, respectively,

$$m_O(f) = i_0 \text{ and } \text{in}(f) := f_{i_0}.$$

Example 1.8.2. If $f(x,y) = y^2 - x^3$, then $m_O(f) = 2$ with $\text{in}(f) = y^2$.

We say that a function $F : \mathbb{A}^2_k \to k$ is homogeneous of degree $d \geq 0$ if for all $(p,q) \in \mathbb{A}^2_k$ and $t \in k$, we have $F(tp,tq) = t^dF(p,q)$. If a polynomial $f \in k[x,y]$ is homogeneous of degree $d \geq 0$, then so is the associated function F_f, and the converse holds if k is infinite. Note that the zero polynomial $0 \in k[x,y]$ is homogenous of degree d for every $d \geq 0$, and for each $d \geq 0$, the subset $k[x,y]_d \subset k[x,y]$ is a vector subspace of dimension $d+1$ with basis $x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d$, with $k[x,y] = \bigoplus_{d\geq 0} k[x,y]_d$. Finally, if $f \in k[x,y]_d$ and $g \in k[x,y]_e$, then $fg \in k[x,y]_{d+e}$. This structure on $k[x,y]$ is called the structure of a graded k-algebra.

Lemma 1.8.3. If $k = \overline{k}$, then for any $d \geq 0$ and $f \in k[x,y]_d$, there are homogeneous linear polynomials $\ell_1, \ldots, \ell_d \in k[x,y]_1$ such that $f = \ell_1\ell_2\cdots\ell_d$. If f is nonzero, then these factors are uniquely determined up to reordering and nonzero scalars.

Proof. Write $f = \sum_{i=0}^d a_ix^{d-i}y^i$. If $f \neq 0$, let i_0 be the least index such that $a_{i_0} \neq 0$. Since $k = \overline{k}$, we can factor the polynomial $f(t,1) = \sum_{i=i_0}^d a_it^{d-i}$ of degree $d - i_0$ as

$$f(t,1) = \sum_{i=i_0}^{d} a_it^{d-i} = a_{i_0}\prod_{j=1}^{d-i_0}(t - \alpha_j)$$

for some $\alpha_j \in k$, and then taking $a_{i_0}^{-1}\ell_1 = \ell_2 = \cdots = \ell_{i_0} = y$ and $\ell_{i_0+j} = x - \alpha_jy$ for $j = 1, \ldots, d - i_0$ suffices. Uniqueness is clear because $k[x,y]$ is a UFD, and each ℓ_j is prime. ■

Definition 1.8.4.

(a) Given a curve $C \subset \mathbb{A}^2_k$, we define the **multiplicity of C at the origin** $O = (0,0)$ to be

$$m_O(C) := m_O(f_C),$$

where $f_C \in k[x,y]$ is any minimal polynomial for C. If $\operatorname{in}(f_C) = \ell_1 \cdots \ell_m$ is the factorization of $\operatorname{in}(f_C)$ into linear factors as in Lemma 1.8.3, where $m := m_O(C)$, then we define the **tangent lines to C at O** to be the lines $L_j := C_{\ell_j}$ for $j = 1, \ldots, m$. (These need not all be distinct, and are independent of the choice of f_C.) Finally, the **tangent cone to C at O** is define to be

$$\mathrm{TC}_O\, C := C_{\operatorname{in}(f)} = L_1 \cup L_2 \cup \cdots \cup L_m.$$

(b) Given a curve $C \subset \mathbb{A}^2_k$ and an arbitrary point $P \in \mathbb{A}^2_k$, we define the **multiplicity of C at P** to be

$$m_P(C) := m_O(\phi^{-1}C),$$

where $\phi : \mathbb{A}^2_k \to \mathbb{A}^2_k$ is any affine change of coordinates such that $\phi(O) = P$. We define the tangent lines to C at P to be the lines $\phi(L_j)$ for $j = 1, \ldots, m$ where $m = m_P(C)$, and similarly the tangent cone to C at P to be

$$\mathrm{TC}_P C = \phi(\mathrm{TC}_O(\phi^{-1}C)).$$

(c) Given a curve $C \subset \mathbb{A}^2_k$ and point $P \in \mathbb{A}^2_k$, we have $m_P(C) \geq 1$ iff $P \in C$, in which case we say that P is a **smooth point** of C iff $m_p(C) = 1$. The curve C is said to be **smooth** if every $P \in C$ is a smooth point. A point $P \in C$ that is not a smooth point is called a **singular point** or **multiple point** of C.[a]

[a] Outside of mathematics, the terms "singular" and "multiple" are usually antonyms; in this case, they are not, because "singular" here means "exceptional" or "extraordinary" (see Lemma **??**), while "multiple" means "of higher (i.e. > 1) multiplicity".

Note that a smooth point on a curve has a unique tangent line, which we will denote by $\mathrm{T}_P C$. The coordinate-invariance of smoothness and multiplicity is baked into the definition–if we can show that it is well-defined. To do this, we need that if $\phi : \mathbb{A}^2_k \to \mathbb{A}^2_k$ is an affine change of coordinates such that $\phi(O) = O$, then for any polynomial $f \in k[x,y]$ we have $m_O(f) = m_O(\phi^*(f))$. By considering the homogeneous parts separately, this reduces to showing

Lemma 1.8.5. If $\phi : \mathbb{A}^2_k \to \mathbb{A}^2_k$ is an affine change of coordinates such that $\phi(O) = O$, and if $0 \neq f \in k[x,y]$ is homogeneous of degree $n \geq 0$, then so is $\phi^*(f)$.

Proof. Note that ϕ is of the form $\phi(x', y') = (ax' + by', cx' + dy')$ for some $a, b, c, d \in k$ with $ad - bc \neq 0$. The claim is clear when $n = 0$, since then f is a nonzero constant and $\phi^*(f) = f$. When $n = 1$, we have $f = \lambda x + \mu y$ for some $\lambda, \mu \in k$, not both zero, and then

$$\phi^*(f) = \lambda(ax' + by') + \mu(cx' + dy') = (a\lambda + c\mu)x' + (b\lambda + d\mu)y'.$$

Now, since one of λ and μ is not zero, and since $ad - bc \neq 0$, it follows easily that at least one of $a\lambda + c\mu$ and $b\lambda + d\mu$ is nonzero (this is basic linear algebra, but can also be shown directly–how?). Therefore, we are done in this case. If $n \geq 2$, then by Lemma 1.8.3, we can write $f = \ell_1 \cdots \ell_n$ for some ℓ_j homogeneous of degree 1; then we are done by the case $n = 1$ and the observation $\phi^*(f) = \phi^*(\ell_1)\phi^*(\ell_2) \cdots \phi^*(\ell_n)$. This finishes the proof when $k = \overline{k}$ (which is the only case we care about), but in general, we can use Theorem 1.4.5 to reduce to this case. ■

Example 1.8.6. The parabola C defined by $f(x,y) = y - x^2 \in k[x,y]$ has is smooth at the point $(1,1) \in \mathbb{A}_k^2$ with tangent line L defined by the vanishing of $y - 2x + 1 = 0$.

Example 1.8.7. A curve C is said to have a simple node at P iff $m_P(C) = 2$ and C has two distinct tangent lines at P. For instance, the curve C defined by $f(x,y) = y^2 - x^2(x+1) \in k[x,y]$ over a field k with $\operatorname{ch} k \neq 2$ has a simple node at the origin, with tangent lines L_1, L_2 defined by the vanishing of $y \pm x$, and tangent cone $T_O(C) = L_1 \cup L_2$. (What happens when $\operatorname{ch} k = 2$?)

Of course, this definition is not very convenient when we want to locate all singular points of a given curve C. For this, we need a more convenient criterion. This is provided by

Theorem 1.8.8 (Jacobi Criterion)**.** Suppose we are given a curve $C \subset \mathbb{A}_k^2$ and a point $P = (p,q) \in \mathbb{A}_k^2$. Let $f \in k[x,y]$ be a minimal polynomial for C. Then

(a) $P \in C$ iff $f|_P := f(p,q) = 0$, and in this case

(b) P is a singular point of C iff

$$\left.\frac{\partial f}{\partial x}\right|_P = \left.\frac{\partial f}{\partial y}\right|_P = 0.$$

(c) If $P \in C$ is a smooth point, then the tangent line $\mathrm{T}_P C$ is defined by the vanishing of

$$\left.\frac{\partial f}{\partial x}\right|_P (x-p) + \left.\frac{\partial f}{\partial y}\right|_P (y-q) \in k[x,y].$$

Wait, what? What are these partial derivative symbols? Why can we do this over any field k? We'll discuss this more next time, but for now let's work out an example to see how conveniently Theorem 1.8.8 allows us to locate singular points of a curve C.

Example 1.8.9. If $f(x,y) = y - x^2$, then $\partial f/\partial y \equiv 1$ tells us that f is smooth everywhere. At the point $P = (t, t^2)$, the tangent line to C is given by the vanishing of

$$-2t(x-t) + 1(y-t) = y - 2tx + t^2 \in k[x,y].$$

Note that when $\operatorname{ch} k = 2$, this tangent line is always horizontal–which is incredibly weird. In general, weird stuff happens to curves of degree p in characteristic p–watch out for this over the next few weeks!

Example 1.8.10. If $f(x,y) = y^2 - x^3$, then the system of equations we need to solve for the singular points of C is

$$\begin{aligned} y^2 - x^3 &= 0, \\ -3x^2 &= 0, \\ 2y &= 0, \end{aligned}$$

which in any characteristic has the unique solution $(x,y) = (0,0)$ (check!). Therefore, the unique singular point of C is the origion O, where C has the unique tangent line $y = 0$, i.e. the x-axis.