

## 2.5 Exercise Sheet 5

### 2.5.1 Standard Exercises/Numerical and Exploration

**Exercise 2.5.1.** Given a nonempty finite set  $S \subset \mathbb{P}_k^2$  of points in  $\mathbb{P}_k^2$ , let  $d(S)$  be the smallest degree of a curve  $C \subset \mathbb{P}_k^2$  through  $S$ , i.e. such that  $C \supset S$ . Let's investigate the relationship between  $S$ , its size  $n := \#S$ , and the integer  $d(S)$ .

- (a) Show that if  $n \in \{1, 2\}$ , then  $d(S) = 1$ .
- (b) Show that if  $n \in \{3, 4\}$ , then  $d(S) \in \{1, 2\}$ . When does each case hold?
- (c) Show that if  $n = 5$ , then  $d(S) \in \{1, 2\}$ , or equivalently that given any five distinct points  $P_1, \dots, P_5 \in \mathbb{P}_k^2$ , there is at least one (possibly reducible) conic  $C \subset \mathbb{P}_k^2$  passing through each  $P_i$ .
- (d) Show that, in general, we have

$$1 \leq d(S) \leq \left\lceil \frac{\sqrt{9+8n}-3}{2} \right\rceil,$$

where  $\lceil \cdot \rceil$  denotes the ceiling function. (Hint: When does a system of  $N$  linear equations in  $M$  variables always have a solution that is not identically zero?)

- (e) Show that the bound in (d) is sharp in general: for each  $n \geq 1$ , come up with a collection  $S$  of  $n$  points such that  $d(S)$  equals the upper bound from (d). Can you characterize the sets  $S$  for which this equality holds?

**Exercise 2.5.2.** Let  $k$  be a field.

- (a) Suppose  $\text{ch } k \neq 2$ , and consider the collection of 9 points  $S := \{(i, j) \in \mathbb{A}_k^2 : 0 \leq i, j \leq 2\}$ . How many distinct cubic curves  $C \subset \mathbb{A}_k^2$  pass through  $S$ ? (Hint: by Exercise 2.5.1(d), there is at least one such  $C$ . Does your answer change if the question is about projective cubics instead? Does the choice of base field matter? Can you come up with an analog if  $\text{ch } k = 2$ ?)
- (b) Can you formulate an analog of (a) for a configuration of  $n^2$  points

$$S := \{(i, j) \in \mathbb{A}_k^2 : 0 \leq i, j \leq n-1\},$$

where  $n \geq 2$  is any integer (say when  $\text{ch } k = 0$  for convenience)?

**Exercise 2.5.3 (More on Pascal).** (Adapted from [3] Exercise 5.31[.]) If in Pascal's Theorem, we let some adjacent vertices coincide (the side being tangent), then we get many new theorems.

- (a) State and sketch what happens if  $P_1 = P_2$ ,  $P_3 = P_4$  and  $P_5 = P_6$ .
- (b) Let  $P_1 = P_2$  and the other four points be distinct. Deduce a rule for constructing a tangent to a given conic at a given points, using only a straight-edge.

**Exercise 2.5.4.** Let  $C \subset \mathbb{P}_k^2$  be a curve of degree  $d$  over an algebraically closed field  $k$ .

- (a) Make sense of the following statement: a "general" line  $L \subset \mathbb{P}_k^2$  intersects  $C$  in exactly  $d$  distinct points.
- (b) Suppose  $\text{ch } k = 0$ . Given a "general" point  $P \in \mathbb{P}_k^2$ , how many tangents to  $C$  pass through  $P$  in terms of  $d$ ?

(Hint: How is this exercise is related to Exercises 2.5.8, 2.5.9 and 2.5.10?)

**Exercise 2.5.5.** Let  $k$  be an algebraically closed field, and let  $C \subset \mathbb{P}_k^2$  be a smooth cubic curve.

- (a) Show that  $C$  has exactly 9 inflection points. This set is usually denoted as  $C[3]$ . (Hint: Exercise 2.4.5. You may assume  $\text{ch } k \neq 2, 3$  for convenience, but the result is true in general—can you come up with a characteristic-independent proof?)

- (b) Show that the set  $C[3]$  is not contained in a line, but any line passing through any two points in  $C[3]$  passes through a third point in  $C[3]$ . Why does this not violate the Sylvester-Gallai Theorem?
- (c) Suppose that  $\text{ch } k \neq 3$ . Show that by a projective change of coordinates, we can bring  $C[3]$  to be the nine points

$$\begin{aligned} &[0 : 1 : -1], [-1 : 0 : 1], [1 : -1 : 0], \\ &[0 : 1 : \zeta], [\zeta : 0 : 1], [1 : \zeta : 0], \\ &[0 : 1 : \bar{\zeta}], [\bar{\zeta} : 0 : 1], [1 : \bar{\zeta} : 0], \end{aligned}$$

where  $\zeta$  and  $\bar{\zeta}$  are the roots (in  $k$ ) of  $t^2 - t + 1 = 0$ .

- (d) Keeping the hypothesis that  $\text{ch } k \neq 3$ , every cubic curve passing through the 9 points from (c) has the equation

$$F_\Lambda = \lambda(X^3 + Y^3 + Z^3 + 3) + 3\mu XYZ \in k[X, Y, Z]$$

for some  $\Lambda := [\lambda : \mu] \in \mathbb{P}_k^1$ . This curve is singular iff  $\Lambda \in \{[0 : 1], [1 : -1], [1 : \zeta], [1 : \bar{\zeta}]\}$ , and in each of these cases the curve  $C_\Lambda := C_{F_\Lambda}$  degenerates into a product of three lines. If  $C_\Lambda$  is irreducible, then the flexes of  $C_\Lambda$  are exactly the 9 points above.

- (e) Conclude, using either (b) or both (c) and (d), that if  $k = \mathbb{C}$ , then

$$\#(C[3] \cap C(\mathbb{R})) \leq 3,$$

i.e. at most three of the flexes of a complex smooth cubic curve can be real. Come up with a curve  $C$  for which this bound is achieved. Can this intersection have fewer than 3 points? Can it have exactly 2?

**Exercise 2.5.6.** If  $f, g \in k[x, y]$  are nonconstant polynomials and  $P \in \mathbb{A}_k^2$ , then

$$i_P(f, g) \geq m_P(f) \cdot m_P(g).$$

When does equality hold? (This is a very hard exercise, and you may not be able to do it with the tools we have developed so far; nonetheless, it is very valuable to work out special cases. Try doing the case when  $f$  or  $g$  is linear. Next, try the case when  $m_P(f) = 1$  or  $m_P(g) = 1$ . Finally, see how far you can extend your techniques to the next (or general) case; once you've done that, see [3] §3.3, Theorem 3].)

## 2.5.2 PODASIPs

Prove or disprove and salvage if possible the following statements.

**Exercise 2.5.7 (Converse of Pascal's Theorem).** If the intersection points of opposite sides of a hexagon lie on a straight line, then the vertices of the hexagon lie on a conic.

**Exercise 2.5.8.** (Adapted from [3] Exercise 5.26.) If  $C \subset \mathbb{P}_k^2$  is a curve of degree  $n \geq 1$ , and  $P \in \mathbb{P}_k^2$  a point of multiplicity  $m := m_P(C) \geq 0$ , then for all lines  $L$  through  $P$ , except possibly finitely many,  $L$  intersects  $C$  in  $n - m$  distinct points other than  $P$ .

**Exercise 2.5.9.** Given a curve  $C \subset \mathbb{P}_k^2$  and a point  $P \in \mathbb{P}_k^2$ , there is at least one tangent line  $L$  to  $C$  that does not pass through  $P$ .

**Exercise 2.5.10 (Dual Curve).** Let  $C \subset \mathbb{P}_k^2$  be a curve. Let

$$C^* := \{L \in \mathbb{P}_k^{2*} : L \text{ is tangent to } C \text{ at some point } P \in C\} \subset \mathbb{P}_k^{2*}.$$

Then  $C^* \subset \mathbb{P}_k^{2*}$  is a curve, and  $C^{**} = C$ . (Hint: Can you work out a few examples in low degrees? What is the relationship between the degrees of  $C$  and  $C^*$ ?)