1.13 07/08/24 - Parametric Projective Curves, Pascal's Theorem, and More on Conics

Today, I want to prove Bézout's theorem for conics, and derive some delicious applications. For this, I will need to talk about parametric projective curves.

1.13.1 Parametric Projective Curves and Bézout's Theorem for a Conic

In the affine case, we defined a parametric curve to be an image of $\mathbb{A}^1_k$ under two rational functions. In the projective case, we can always clear denominators and work with $\mathbb{P}^1_k$ instead. This leads us to

Definition 1.13.1. A parametric projective algebraic curve is the image of a map $\Psi : \mathbb{P}^1_k \to \mathbb{P}^2_k$ of the form

$$\Psi[U : V] = [F_1(U,V) : F_2(U,V) : F_3(U,V)],$$

where $F_i(U,V) \in k[U,V]$ for $i = 1,2,3$ are homogeneous polynomials of the same degree, not all zero.

This definition corresponds to the affine one by considering $\mathbb{A}^1_k \subset \mathbb{P}^2_k$ as the set where $V \neq 0$ with coordinate $t = U/V$, in which case the affine part of this parametric projective curve is given by

$$t \mapsto \left(\frac{F_1(t,1)}{F_3(t,1)}, \frac{F_2(t,1)}{F_3(t,1)}\right),$$

which is a parametric affine curve. One can show, either using techniques similar to those in §1.3 or by reducing to the affine case, that a parametric projective algebraic curve is, in fact, a projective algebraic curve (at least when not all F_i are proportional, in which case the image is a single point). I will not do this here, but I encourage you to carry this out yourselves.

Remark 1.13.2. Note that I did not ask for the $F_j(U,V)$ to not have a common root on $\mathbb{P}^1_k$, because if they did, then I would very easily be able to just cancel this common factor from each $F_j(U,V)$. This is a manifestation of the completeness of projective curves–projective curves have no holes, and rational maps out a smooth projective curve always eextends to a regular morphism out of it. As usual, if this doesn't make sense, please ignore it.

Example 1.13.3. The smooth conic defined by $YZ - X^2 \in k[X,Y,Z]_2$ can be parametrized via the map Ψ given as

$$\Psi[U : V] = [UV : U^2 : V^2].$$

This is the projective version of affine parametrization $t \mapsto (t, t^2)$ of the parabola defined by $y - x^2 = 0$. By Corollary 1.12.14, this gives us a parametrization of every smooth conic curve. In particular, every smooth conic curve admits a parametrization.

From this parametrization, we can now prove Bézout's theorem for a conic.

Theorem 1.13.4. If k is an algebraically closed field, $C \subset \mathbb{P}^2_k$ is a conic (i.e. a curve with $\deg C = 2$), and $D \subset \mathbb{P}^2_k$ a curve of degree $d \geq 1$ such that C and D do not share a component, then

$$\sum_{P \in C \cap L} i_P(C,D) = 2d.$$

Of course, $\deg C = 2$ and so $2d = (\deg C)(\deg D)$.

Proof. By Corollary 1.12.14, we can choose coordinates such that C is either the union of the two lines C_X and C_Y, or that $C = C_{YZ-X^2}$; make a change of coordinates so that we are working with this coordinate system. In the first case, neither of the two lines C_X or C_Y can be contained in D, and we are done by additivity of intersection multiplicity and Theorem 1.12.12 (make sure you believe this!). In the second case, C is irreducible and Example 1.13.3 tells us that we can parametrize C as the image of the map

$$[U : V] \mapsto [UV : U^2 : V^2].$$

If F is a minimal polynomial for D, then F is a homogeneous polynomial of degree d, and the intersection points of C and D correspond exactly to $[U_0 : V_0] \in \mathbb{P}^1_k$ such that

$$F(U_0V_0, U_0^2, V_0^2) = 0.$$

Now $F(UV, U^2, V^2) \in k[U, V]_{2d}$ is a homogeneous polynomial of degree $2d$. If it is identically zero, then we conclude that $C \subset D$, contrary to assumption that C and D do not share any components; therefore, this polynomial is not identically zero, and so has exactly $2d$ roots counted with multiplicity, again by Lemma 1.8.3. I will again leave it to the reader to check, perhaps using techniques similar to those from Example 1.9.12, that the intersection multiplicity of C and D at a point $[U_0V_0 : U_0^2 : V_0^2]$ agrees with the multiplicity of $[U_0 : V_0]$ as a root of $F(U_0V_0, U_0^2, V_0^2) = 0$.[33] ■

We are now ready for some delicious applications!

1.13.2 Pascal's Theorem, Pappus's Theorem, Brocard's Theorem, etc.

Theorem 1.13.5 (Pascal). Let k be an algebraically closed field, $C \subset \mathbb{P}^2_k$ a smooth conic and $P_1, \ldots, P_6$ distinct points on C. For $i = 1, \ldots, 6$, let L_i be the line joining P_i and P_{i+1} (where $P_7 := P_1$), and for $j = 1, 2, 3$, let $Q_j := L_j \cap L_{j+3}$. Then the points $Q_1, Q_2, Q_3 \in \mathbb{P}^2_k$ are collinear, i.e. there is a line $L_0 \subset \mathbb{P}^2_k$ such that $Q_j \in L_0$ for $j = 1, 2, 3$.

Let's first make a few observations about the statement:

(a) The lines L_i are all distinct: if $L_i = L_{i'}$ for some $i \neq i'$, then this line intersects C in at least 3 distinct points, and is hence contained in C (by either Theorem 1.12.12 or 1.13.4); this would mean that C is reducible and hence (by Corollary 1.12.14 if needed) not smooth. In particular, by Proposition 1.11.3, the points Q_1, Q_2, Q_3 are uniquely determined.

(b) Each P_i lies on exactly two lines $L_{i'}$, namely L_{i-1} and L_i, and, in particular, these lines have indices that differ by 1 (modulo 6); conversely, each L_i contains exactly two points P_i and P_{i+1} of C, because again if it contained a third point of C, it would be contained in C entirely.

(c) We have $P_i \neq Q_j$ for all i, j. Indeed, let us take the indices i, j modulo 6; then $P_i = Q_j$ cannot happen because this implies that $P_i \in L_j \cap L_{j+3}$, violating the observation (b).

(d) Finally, we have $Q_1, Q_2, Q_3 \notin C$. Indeed, if some $Q_j \in C$, then $Q_j \in L_j \cap C = \{P_j, P_{j+1}\}$ implies that $Q_j = P_i$ for some i, j, violating (c).

Let's now proceed to the proof, which is rather simple given the tools we have.

[33] I'm being lazy partly because, in the proof of Pascal's Theorem (Theorem 1.13.5) below, we will only need the result that C and D intersect in at most $2d$ points unless they share a component, and this we have already proven. Also, we shall do a full proof of the general Bézout Theorem very soon.

Proof. For $i = 1, \ldots, 6$, let $\ell_i \in k[X, Y, Z]_1$ be a homogeneous linear polynomial vanishing on L_i. Consider the family D_Λ of cubic curves parametrized by $\Lambda = [\lambda : \mu] \in \mathbb{P}^1_k$, where D_Λ is defined by the vanishing of the polynomial

$$\lambda \ell_1 \ell_3 \ell_5 + \mu \ell_2 \ell_4 \ell_6.$$

Note that each curve D_Λ in this family passes through all the P_i's and Q_j's, and we have that $D_{[1:0]} = L_1 \cup L_3 \cup L_5$ and $D_{[0:1]} = L_2 \cup L_4 \cup L_6$.[34] Now pick a point $R \in C \smallsetminus \{P_1, \ldots, P_6\}$, which exists because C is infinite (Proposition 1.11.13). From observation (b) above, we conclude that $Q_0 \notin D_{[1:0]} \cup D_{[0:1]}$, from which it follows that there is a unique $\Lambda_0 \in \mathbb{P}^1_k$ such that $R \in D_{\Lambda_0}$.[35] Let $D = D_{\Lambda_0}$. Since D is a cubic curve and C and D intersect in at least 7 points, it follows from Theorem 1.13.4 that C and D share a component. Since C is irreducible (Corollary 1.12.14) and $\deg D = 3$, this can only happen if $C \subset D$ and $D = C \cup L_0$ for some line $L_0 \subset \mathbb{P}^2_k$. But now, D contains Q_1, Q_2, Q_3 (because each D_Λ does), while C does not contain Q_1, Q_2, Q_3 (this was observation (c) above), and hence $Q_1, Q_2, Q_3 \in L_0$. ■

See Figure 1.8 for a visual demonstration of the proof technique.

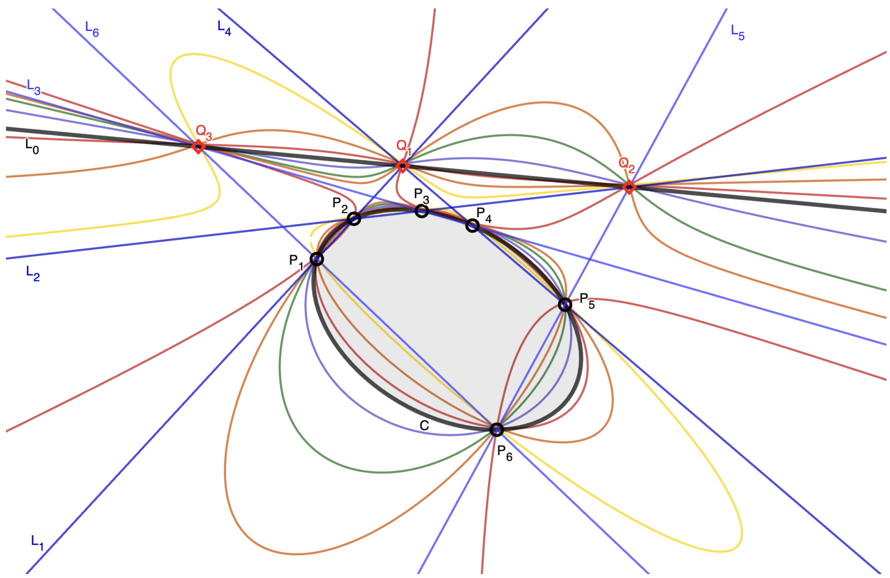

Figure 1.8: Pascal's Theorem. The conic (here ellipse) C and the line L_0 are in thick black style. The various colorful curves represent various members of the one parameter family D_Λ, one member of which is also $C \cup L_0$. Picture made with Geogebra.

Note that the actual statement of Theorem 1.13.5 does not use an ordering whatsoever on the points $P_1, \ldots, P_6$–indeed, for general fields, it does not even make sense to order points of a conic. In particular, if we start with a collection of 6 distinct unordered points on a conic C, then they can be connected into a hexagon in 60 different ways, and resulting in 60 different instances of Pascal's Theorem and 60 different "Pascal" lines; this configuration of 60 lines associated to 6 points on a hexagon is often called the Hexagrammum Mysticum. Finally, although we have proven the theorem over algebraically closed fields, it follows also immediately over all fields (e.g. over $k = \mathbb{R}$), thanks to Theorem 1.4.5 and the observation that Proposition

[34] If you were not convinced of this already, then this observation tells us that every curve D_Λ has degree 3: indeed, if it did not, then some D_Λ would be either a line or a union of two lines, neither of which can contain all the P_i's, since no three of them are collinear (why?).

[35] Indeed, if we pick a representative (X_0, Y_0, Z_0) for $R = [X_0 : Y_0 : Z_0]$, then neither of $\ell_1\ell_3\ell_5|_{(X_0,Y_0,Z_0)}$ and $\ell_2\ell_4\ell_6|_{(X_0,Y_0,Z_0)}$ is zero, and this unique Λ_0 is $\Lambda_0 = [-\ell_2\ell_4\ell_6|_{(X_0,Y_0,Z_0)} : \ell_1\ell_3\ell_5|_{(X_0,Y_0,Z_0)}] \in \mathbb{P}^1_k$.

1.11.3 does not use that the base field is algebraically closed, which implies, for instance, that if three points $Q_1, Q_2, Q_3 \in \mathbb{P}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}}$ are collinear in $\mathbb{P}^2_{\mathbb{C}}$, then they are collinear in $\mathbb{P}^2_{\mathbb{R}}$, i.e. the line joining them is real. Over the real numbers, other proofs can also be given; after all, Pascal did not actually have Bézout's Theorem. One approach involves using a variant of the classification of projetive conics over $\mathbb{R}$ (see Remark 1.12.15) to conclude that any smooth conic can be taken by a projective change of coordinates over $\mathbb{R}$ to a circle $X^2 + Y^2 - Z^2 = 0$, and then to use other techniques from Euclidean geometry (e.g.

We'll give a proof of this result at the end of this lecture.

1.13.3 More on Conics

1.13.4 Addendum: Preserving Multiplicity under Addition