ST5218_Tut5

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Question 3:

$$E(I_1I_2) = P(I_1 = 1, I_2 = 1) = P(Y_1 \leq F^{-1}(u_1), Y_2 \leq F^{-1}(u_2)) = C(u_1, u_2)$$

Quetion 4:

$$C(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\{-\frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{2(1-\rho^2)}\} ds_1 s_2$$

where $\Phi^{-1}(\cdot)$ is the quantile or inverse function of standard normal.

$$\Phi_1(x) = \Phi(\frac{x-\mu_1}{\sigma_1}), \quad \Phi_2(x) = \Phi(\frac{x-\mu_2}{\sigma_2})$$

The combined joint distribution is

$$\begin{aligned}
C((\Phi_1(x_1), \Phi_2(x_2), \rho) &= \int_{-\infty}^{\Phi^{-1}(x_1)} \int_{-\infty}^{\Phi^{-1}(x_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\{-\frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{2(1-\rho^2)}\} ds_1 s_2 \\
&= \int_{-\infty}^{\frac{x_1-\mu_1}{\sigma_1}} \int_{-\infty}^{\frac{x_2-\mu_2}{\sigma_2}} \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\{-\frac{s_1^2 - 2\rho s_1 s_2 + s_2^2}{2(1-\rho^2)}\} ds_1 s_2 \\
&= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{1}{2\sigma_1\sigma_2\pi\sqrt{(1-\rho^2)}} exp\{-\frac{\frac{(t_1-\mu_1)^2}{\sigma_1^2} + \frac{(t_2-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(s_1-\mu_1)(s_2-\mu_2)}{\sigma-1\sigma_2}}{2(1-\rho^2)}\} dt_1 t_2
\end{aligned}$$

where $t_1 = s_1\sigma_1 + \mu_1 \quad t_2 = s_2\sigma_2 + \mu_2$

Question 5:

Because

$$P(X' < x') = P(X < x' - 1)$$

Then let it be u

$$F_X^{-1}(u) = F_{X'}^{-1}(u) - 1$$

Similarly

$$P(Y' < y') = P(-Y < y' - 1) = P(Y > -y' + 1) = 1 - P(Y \leq -y' + 1)$$

Then let it be v

$$F_Y^{-1}(1 - v) = 1 - F_{Y'}^{-1}(v)$$

Finally

$$\begin{aligned}
C_{X',Y'}(u,v) &= P(X' < F_{X'}^{-1}(u), Y' < F_{Y'}^{-1}(v)) \\
&= P(X + 1 < F_{X'}^{-1}(u), -Y + 1 < F_{Y'}^{-1}(v)) \\
&= P(X + 1 < F_X^{-1}(u) + 1, Y > 1 - F_{Y'}^{-1}(v)) \\
&= P(X < F_X^{-1}(u), Y > F_Y^{-1}(1 - v)) \\
&= P(X < F_X^{-1}(u)) - P(X < F_X^{-1}(u), Y < F_Y^{-1}(1 - v)) \\
&= u - P(X < F_X^{-1}(u), Y < F_Y^{-1}(1 - v)) \\
&= u - C_{X,Y}(u, 1 - v)
\end{aligned}$$

Question 6:

The CDF of (X_1, X_2) is

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

Then

$$F(y_1, y_2) = C(F_1(\frac{y_1 - b}{a}), F_2(\frac{y_2 - d}{c}))$$

Thus the density is

$$\begin{aligned}
f(y_1, y_2) &= c(F_1(\frac{y_1 - b}{a}), F_2(\frac{y_2 - d}{c}))\frac{d}{dy_1}F_1(\frac{y_1 - b}{a})\frac{d}{dy_2}F_2(\frac{y_2 - d}{c}) \\
&= c(F_1(\frac{y_1 - b}{a}), F_2(\frac{y_2 - d}{c}))\frac{1}{a}f_1(\frac{y_1 - b}{a})\frac{1}{c}f_2(\frac{y_2 - d}{c})
\end{aligned}$$

Question 7:

a)

$$f_{new}(y_1, y_2) \geq 0$$

$$\begin{aligned}
\int\int f_{new}(y_1, y_2) dy_1 dy_2 &= \int\int c_{new}(F_1 y_1, F_2(y_2)) f_1(y_1) f_2(y_2) dy_1 dy_2 \\
&= \int\int c_{new}(F_1 y_1, F_2(y_2)) dF_1(y_1) dF_2(y_2) d \\
&= \int_0^1 \int_0^1 c_{new}(u_1, u_2) du_1 du_2 \\
&= 1
\end{aligned}$$

where $u_1 = F_1(y_1), \quad u_2 = F_2(y_2)$

b)

Because $c_{new}(F_1(y_1)) = 1$

Thus

$$\begin{aligned}
f_{new,1}(y_1) &= \int f_{new}(y_1, y_2) dy_2 \\
&= \int c_{new}(F_1(y_1), F_2(y_2)) f_1(y_1) f_2(y_2) dy_2 \\
&= \int c_{new}(F_1(y_1), F_2(y_2)) f_1(y_1) dF_2(y_2) \\
&= \int_0^1 \int_0^1 c_{new}(F_1(y_1), u_2) f_1(y_1) du_2 \\
&= c_{new}(F_1(y_1)) f_1(y_1) \\
&= f_1(y_1)
\end{aligned}$$

Question 8:

a)

$L_\lambda = \lim_{q \to 0} \frac{C(q,q)}{q} = \lim_{q \to 0} q = 0$

b),c)

1, $\frac{1}{2}$

d)

$\lambda_L = \lim_{q \to 0^+} \frac{(q^{-\theta} + q^{-\theta} - 1)^{-\theta^{-1}}}{q} = \lim_{q \to 0^+} \frac{(\frac{q^\theta}{2 - q^\theta})^{\theta^{-1}}}{q} = \lim_{q \to 0^+} \frac{1}{(2 - q^\theta)^{\theta^{-1}}} = \frac{1}{2^{\theta^{-1}}} = 2^{-\theta^{-1}}$

Quetion 9:

Because

$$F(x_1, x_2) = C(\Phi(x_1), \Phi(x_2)) = \frac{\Phi(x_1)\Phi(x_2)}{\Phi(x_1) + \Phi(x_2) - \Phi(x_1)\Phi(x_2)}$$

$$\frac{\partial}{\partial u_1} C(u_1, u_2) = \frac{u_2(u_1 + u_2 - u_1 u_2) - u_1 u_2(1 - u_2)}{(u_1 + u_2 - u_1 u_2)^2} = \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2}$$

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2) = \frac{\partial}{\partial} \frac{u_2^2}{(u_1 + u_2 - u_1 u_2)^2} = \frac{2u_1 u_2}{(u_1 + u_2 - u_1 u_2)^3}$$

Thus

$$f(x_1, x_2) = c(\Phi(x_1), \Phi(x_2))\phi(x_1)\phi(x_2) = \frac{2\Phi(x_1)\Phi(x_2)}{(\Phi(x_1) + \Phi(x_2) - \Phi(x_1)\Phi(x_2))^3} \phi(x_1)\phi(x_2)$$

Quetion 10:

a)

To prove that $C_{new}(u_1, u_2, \cdots, u_p)$ is a copula, it satisfies $(u_1, u_2, \cdots, u_p) \in [0, 1]^p$

$C_{new}(u_1, u_2, \cdots, u_p)$ is cumulative function $[0, 1]^p$, because $C_1(u_1, u_2, \cdots, u_p)$, $\quad C_2(u_1, u_2, \cdots, u_p)$ are copulas, they must satisfy the above property, and their summation also do.

Then $C_{new}(u_1, u_2, \cdots, u_p)$ has marginal CDFs that satisfy

$$\begin{aligned} C_k(u) &= C_{new}(1, \cdots, 1, u, 1, \cdots, 1) \\ &= wC_1(1, \cdots, 1, u, 1, \cdots, 1) + (1 - w)C_2(1, \cdots, 1, u, 1, \cdots, 1) \\ &= w * u + (1 - w) * u \\ &= u \end{aligned}$$

$\Rightarrow$ Marginal distributions are uniform $[0, 1]$.

Since $\{x_1, x_2, \cdots, x_n\} = \{x_(1), x_(2), \cdots, x_(n)\}$ and each rank will take some integer value from 1 to n

$$\frac{1}{n+1} \sum_{i=1}^{n} (x_i \leq x_{(j)}) = \frac{j}{n+1}$$

Thus $\{U_1, \cdots, U_n\} = \{\frac{j}{n+1}; j = 1, \cdots, n\}$

Quetion 11:

a)

```
library(tseries)
library(timeSeries)
library(fCopulae)
library(MASS)

MMM = get.hist.quote(instrument="MMM",
                     start="2008-01-01", end="2009-12-31",
                     quote=c("AdjClose"), provider="yahoo",
                     compression="d")
```

```
## time series starts 2008-01-02
## time series ends   2009-12-30
```

```
MSFT = get.hist.quote(instrument="MSFT",
                     start="2008-01-01", end="2009-12-31",
                     quote=c("AdjClose"), provider="yahoo",
                     compression="d")
```

```
## time series starts 2008-01-02
## time series ends   2009-12-30
```

```
r.MMM = as.data.frame(diff(log(MMM)))
r.MSFT = as.data.frame(diff(log(MSFT)))

FitT.MMM = fitdistr(r.MMM[,1], "t")
FitT.MSFT = fitdistr(r.MSFT[,1], "t")

FitT.MMM
```

```
##          m                s               df
##    0.0004073579   0.0149011154   3.6382468226
##   (0.0007953477) (0.0008814453) (0.6687772033)
```

```
FitT.MSFT
```

```
##          m                s               df
##    2.938186e-05   1.737255e-02   3.013202e+00
##   (9.414720e-04) (1.092891e-03) (4.949178e-01)
```

b)

```
library(copula)
n = nrow(r.MMM)
U.MMM = rank(r.MMM)/(n+1)
```

```r
U.MSFT = rank(r.MSFT)/(n+1)

data = data.frame(list(U.MMM=U.MMM, U.MSFT=U.MSFT))
data = data.matrix(data)

tCopula = tCopula(dim=2)
FIT1 = fitCopula(tCopula, data)
FIT1
```

```
## Call: fitCopula(copula, data = data)
## Fit based on "maximum pseudo-likelihood" and 503 2-dimensional observations.
## Copula: tCopula
##  rho.1     df
## 0.6086 2.4977
## The maximized loglikelihood is 141.5
## Optimization converged
```

Question 12:

```r
library(copula)
c1 = normalCopula(param=0.8)
c2 = tCopula(param=0.8)
c3 = claytonCopula(param = 0.8)
c4 = frankCopula(param= 0.8)
c5 = gumbelCopula(param= 1.6)

q = 0.000001
pCopula(c(q,q), c1)/q
```

```
## [1] 0.0996965
```

```r
pCopula(c(q,q), c2)/q
```

```
## [1] 0.4899074
```

```r
pCopula(c(q,q), c3)/q
```

```
## [1] 0.4204524
```

```r
pCopula(c(q,q), c4)/q
```

```
## [1] 1.452772e-06
```

```r
pCopula(c(q,q), c5)/q
```

```
## [1] 0.0005581296
```