Matrix Decompositions

Tuesday, 18 February, 2020 2:50 PM

Source: https://people.duke.edu/~ccc14/sta-663/LinearAlgebraMatrixDecompWithSolutions.html

Matrix Decompositions are an important step in solving linear systems in a computationally efficient manner.

LU Decompositions and Gaussian Elimination

LU = 'Lower Upper'.

1) LU decompositions of matrix A: A = LU , where L is lower triangular and U is upper triangular.
2) LU decompositions basically gaussian elimination, but only work with matrix A rather than augmented matrix.
3) Goal: factorization of a square matrix into two triangular matrices; L and U, such that product of L and U = A.

Source: https://www.geeksforgeeks.org/l-u-decomposition-system-linear-equations/

Steps for L U Decomposition:

Given a set of linear equations, convert into matrix form A X = C where A = coefficient, X = variable matrix, C= matrix of numbers on the right-hand side of the equations.

Reduce coefficient matrix A to row echelon form using Gaussian elimination. Matrix obtained is U.

To find L, two methods:

- Assume the remaining elements as some artificial variables, make equations using A = L U and solve them to find those artificial variables
- Remaining elements are the multiplier coefficients because of which the respective positions became zero in the U matrix

Additional info:

Now have A (the nXn coefficient matrix), L (the nXn lower triangular matrix), X (the nX1 matrix of variables) and C (the nX1 matrix of numbers on the right-hand side of the equations).

The given system of equations is A X = C. We substitute A = L U. Thus, we have L U X = C.

We put Z = U X, where Z is a matrix or artificial variables and solve for L Z = C first and then solve for U X = Z to find X or the values of the variables, which was required.

Recommendation system links:

https://heartbeat.fritz.ai/recommender-systems-with-python-part-iii-collaborative-filtering-singular-value-decomposition-5b5dcb3f242b

https://beckernick.github.io/matrix-factorization-recommender/

https://www.kaggle.com/gohngeejuay/recommender-systems-in-python-101/edit

Math: https://www.geeksforgeeks.org/l-u-decomposition-system-linear-equations/

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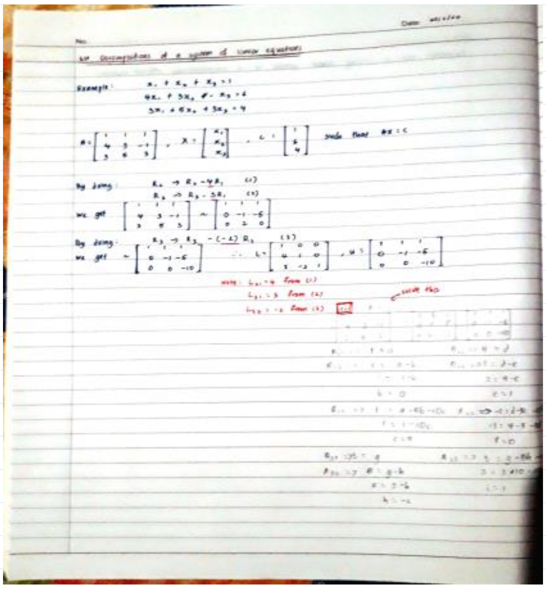


LU Decompositions of a system of linear equations

Example: $x_1 + x_2 + x_3 = 1$, $4x_1 + 3x_2 - x_3 = 6$, $3x_1 + 5x_2 + 3x_3 = 4$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$ such that $AX = C$

By doing: $R_2 \to R_2 - 4R_1$ (1); $R_3 \to R_3 - 3R_1$ (2)

we get $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$

By doing: $R_3 \to R_3 - (-2)R_2$ (3)

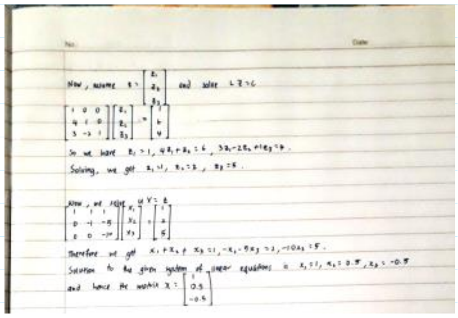
we get $\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$ $\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

note: $L_{21} = 4$ from (1)

$L_{31} = 3$ from (2)

$L_{32} = -2$ from (3)

Additional info: Solving systems of linear equations after using L U Decompositions



Now, assume $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ and solve $LZ = C$

$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

So we have $z_1 = 1$, $4z_1 + z_2 = 6$, $3z_1 - 2z_2 + z_3 = 4$.

Solving, we get $z_1 = 1$, $z_2 = 2$, $z_3 = 5$.

Now, we solve $UX = Z$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

Therefore we get $x_1 + x_2 + x_3 = 1$, $-x_2 - 5x_3 = 2$, $-10x_3 = 5$.

Solution to the given system of linear equations is $x_1 = 1$, $x_2 = 0.5$, $x_3 = -0.5$

and hence the matrix $X = \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$

Reasons for matrix decompositions in movie recommendation:

Matrix factorization can be used to discover features underlying the interactions between two different kinds of entities

From <https://heartbeat.fritz.ai/recommender-systems-with-python-part-iii-collaborative-filtering-singular-value-decomposition-5b5dcb3f242b>

- One advantage of employing matrix factorization for recommender systems is the fact that it can incorporate implicit feedback—information that's not directly given but can be derived by analysing user behaviour—such as items frequently bought or viewed
- If we can discover these kinds of latent features (like genre or actors and directors), we should be able to predict a rating with respect to a certain user and a certain item, because the features associated with the user should match with the features associated with the item.

Notes to make: Matrix Decompositions

- Matrix Decompositions
- Cosine similarity

QR Decomposition

Source: http://mathworld.wolfram.com/QRDecomposition.html

Other references: https://genomicsclass.github.io/book/pages/qr_and_regression.html

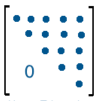
Given a matrix A, its QR decomposition is a matrix decomposition of the form:

A = Q R

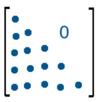
Where R is an upper triangular matrix and Q is an orthogonal matrix, ie one satisfying

$Q^T Q = I$ where Q^T is the transpose of Q and I is the identitiy matrix.

This matrix can be used to solve linear systems of equations.



Upper Triangular Matrix



Lower Triangular Matrix

Cholesky Decomposition

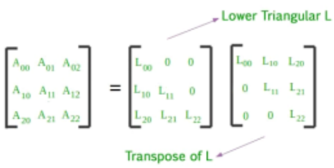
Source: https://www.geeksforgeeks.org/cholesky-decomposition-matrix-decomposition/

Given a symmetric positive defined matrix A, the Cholesky decomposition is an upper triangular matrix U with strictly positive diagonal entries such that:

$A = U^T U$

The Cholesky decomposition is roughly twice as efficient as the LU decomposition for solving systems of linear equations.

From <https://www.geeksforgeeks.org/cholesky-decomposition-matrix-decomposition/>



Singular Value Decomposition

Source: https://blog.statsbot.co/singular-value-decomposition-tutorial-52c695315254

https://www.quora.com/What-is-an-intuitive-explanation-of-singular-value-decomposition-SVD

To find a SVD of A, we must find V, \Sigma and U such that:

A=UΣV^T

1. V must diagonalize A^TA

1.1. V_i are eigenvectors of A^TA

2. Σ where Σ_ii are singular values of A.

3. U must diagonalize AA^T

3.1 u_i are eigenvectors of AA^T.

If A has rank r then:

1. vi⋯vr forms an orthonormal basis for the range of A^T

2. ui⋯ur form an orthonormal basis for the range of A

3. Rank of A is equal to the number of nonzero entries of S. From the form of this factorization

We see that we can express A another way, it can be shown that A can be written as a sum of Rank = 1 matrixes.

A=∑ri=1σiuiVTi

From <https://www.quora.com/What-is-an-intuitive-explanation-of-singular-value-decomposition-SVD>

We know that by construction σi monotonic decreasing, the significance/weight of the nth term decreases. This means that the summation to k<r is an approximation A^ of rank k for the matrix A.