Standard Distribution:

Discrete Distribution:

→ Binomial Distribution
→ Poisson Distribution
→ Geometric Distribution

Continuous Distribution:

→ Uniform Distribution
→ Exponential Distribution
→ Normal Distribution

Binomial Distribution:

A random variable X is said to follow a binomial distribution with the parameters n, p & it assumed only non-negative values & its PDF is given by

$$P(X=x) = \begin{cases} nC_x \, p^x q^{n-x} & , \; x = 0,1,2,\ldots\ldots \quad (q = 1-p) \\ 0 & , \; \text{otherwise} \end{cases}$$

Here n = number of trails.
p = Probability of success
q = Probability of failure

If X follows a binomial distribution then it can be denoted as

$X \sim B(n, p)$

$p + q = 1$

General Binomial Formula:

$$(x+y)^n = x^n + nC_1 x^{n-1} y + nC_2 x^{n-2} y^2 + \ldots\ldots + y^n.$$

1) Find the moment generating function of binomial distribution, hence find mean + variance.

Soln:

We know that the probability density function of binomial distribution is

$$P(X=x) = nC_x p^x q^{n-x} \quad , \quad x = 0,1,2,\dots n$$

$$MGF \Rightarrow M_X(t) = \sum e^{tx} P(X=x)$$

$$= \sum_{x=0}^{n} e^{tx} \cdot nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x e^{tx} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (e^t p)^x q^{n-x}$$

$$= nC_0 (e^t p)^0 q^{n-0} + nC_1 (e^t p)^1 q^{n-1} +$$

$\cdot\ nC_0 = 1$

$nC_n = 1$

$$nC_2 (e^t p)^2 q^{n-2} + \dots + nC_n (e^t p)^n q^{n-n}$$

$$= (1)(1) q^n + nC_1 (e^t p)^1 q^{n-1} + nC_2 (e^t p)^2 q^{n-2} + (1)(e^t p)^n (1)$$

$$= q^n + nC_1 (e^t p)^1 q^{n-1} + nC_2 (e^t p^2) q^{n-2} + (e^t p)^n$$

$$= (q + e^t p)^n$$

$$= (x + y)^n \Rightarrow \text{when } x = q, \; y = e^t p$$

is the general binomial formula.

$$M_X(t) = (q + e^t p)^n$$

$$\text{Mean} = E(X) = \frac{d}{dt}\left[M_X(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[(q+e^t p)^n\right]_{t=0}$$

$$= \left[n(q+e^t p)^{n-1} \cdot pe^t\right]_{t=0} \longrightarrow (1)$$

$$= \left[n(q+e^0 p)^{n-1} \cdot pe^0\right]$$

$$= \left[n(q+p)^{n-1} \cdot p\right]$$

$$= \left[n(1)^{n-1} \cdot p\right] \qquad [\because p+q=1]$$

$$\text{Mean} = E(x) = np$$

$$E(x^2) = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[n(q+e^t p)^{n-1} \cdot pe^t\right]_{t=0} \qquad [\text{From } (1)]$$

$$= \frac{d}{dt}\left[np\underbrace{(q+e^t p)^{n-1}}_{u}\underbrace{e^t}_{v}\right]_{t=0} \qquad uv = uv' + vu'$$

$$= np\left[(q+e^t p)^{n-1} e^t + e^t (n-1)(q+e^t p)^{n-2}(0+pe^t)\right]_{t=0}$$

$$= np\left[(q+e^0 p)^{n-1} e^0 + e^0 (n-1)(q+e^0 p)^{n-2}(0+pe^0)\right]$$

$$= np\left[(q+p)^{n-1} + (n-1)(q+p)^{n-2}(p)\right]$$

$$= np\left[1 + (n-1)(1)(p)\right] \qquad \because p+q=1$$

$$= np\left[1+pn-p\right]$$

$$= np\left[np+q\right] \qquad \because 1-p=q$$

$$E(x^2) = n^2p^2 + npq$$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= n^2p^2 + npq - (np)^2$$
$$= \cancel{n^2p^2} + npq - \cancel{n^2p^2}$$
$$= npq$$

$$\boxed{\begin{aligned} M_X(t) &= (q + e^t p)^n \\ \text{Mean} &= np \\ \text{Variance} &= npq \\ S.D &= \sqrt{npq} \end{aligned}}$$

1) The mean & variance of the binomial distribution are 4 & 3 find $P(X \geq 1)$

<u>Soln:</u>

$$\text{Mean} = np = 4 \rightarrow (1)$$
$$\text{Variance} = npq = 3 \rightarrow (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{3}{4}$$

$$\boxed{q = \frac{3}{4}}$$

We know that $p + q = 1$

$$p + \frac{3}{4} = 1$$
$$p = 1 - \frac{3}{4}$$
$$\boxed{p = \frac{1}{4}}$$

$$(1) \Rightarrow np = 4$$
$$n\left(\frac{1}{4}\right) = 4$$
$$\boxed{n = 16}$$

$\therefore$ The Probability density function of binomial distribution is

$$P(X=x) = nC_x\, p^x q^{n-x}$$

$$P(X=x) = 16C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x}$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - [P(X=0)]$$

$$= 1 - \left[16C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16-0}\right]$$

$$= 1 - \left[(1)(1)\left(\frac{3}{4}\right)^{16}\right]$$

$$= 1 - [(0.75)^{16}]$$

$$= 1 - [0.0100226]$$

$$= 0.98998$$

2) Find the probability of tossing a fair coin 5 times that will appear

i) 3 heads

ii) 3 tails

iii) Atleast one head

iv) not more than one head.

Soln:

Let p = probability of getting head | q = Probability of getting tail

$p = \frac{1}{2}$ | $q = \frac{1}{2}$

$$p + q = 1$$

$$n = 5$$

$$P(X=x) = nC_x\, p^x q^{n-x}$$

$$P(X=x) = 5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} \rightarrow (1)$$

i) $P(X = 3 \text{ heads})$

$$P(X=3) = 5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3}$$

$$= 5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$= \frac{5 \times 4 \times 3}{1 \times 2 \times 3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$

$= 10\,(0.5)^3 (0.5)^2$

$P(X=3) = 0.3125$

ii) $P(X = 3 \text{ tails})$ (or) $P(X = 2 \text{ heads})$

$P(X = 2 \text{ heads}) = 5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2}$

$= \frac{5 \times 4}{1 \times 2} (0.5)^2 (0.5)^3$

$= 10\,(0.5)^2 (0.5)^3$

$= 0.3125$

iii) $P(\text{atleast one head}) = P(X \geq 1)$

$= 1 - P[X < 1]$

$= 1 - P(X=0)$

$= 1 - \left[5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0}\right]$

$= 1 - \left[1(1)(0.5)^5\right]$

$= 1 - 0.03125$

$= 0.96875$

iv) $P(\text{not more than one head})$ (or) $P(X \leq 1)$

$= P(X=0) + P(X=1)$

$= 5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} + 5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1}$

$= 0.03125 + 5(0.5)(0.5)^4$

$= 0.03125 + 0.15625$

$= 0.1875$

3) When a coin is tossed 200 times find mean & standard deviation.

Soln: $n = 200, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}$

$\text{Mean} = np = 200 \times \frac{1}{2} = 100$

$\text{Variance} = npq = 200 \times \frac{1}{2} \times \frac{1}{2} = 50$

$S.D = \sqrt{\text{Var}} = \sqrt{50} = 7.07$

4) Find the probability of getting the total of 5 atleast once in 3 tosses of a pair of dice.

Soln: The sample space is given by S.

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,4), (4,5), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$n = 3$

p = Probability of getting a total 5

$= \frac{4}{36} = \frac{1}{9}$

$p + q = 1$

$\frac{1}{9} + q = 1$

$q = 1 - \frac{1}{9} = \frac{8}{9}$

$P(X=x) = nC_x \, p^x \, q^{n-x}$

$= 3C_x \left(\frac{1}{9}\right)^x \left(\frac{8}{9}\right)^{3-x}$

P (total of 5 atleast once in 3 tosses) $= P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \left[3C_0 \left(\frac{1}{9}\right)^0 \left(\frac{8}{9}\right)^3\right]$

$= 1 - \left[(1)(1)(8/9)^3\right]$

$= 1 - 0.702$

$= 0.298$

5) A dice is on '8' times to find the probability that the number of 3 will show

i) exactly 2 times

ii) atleast 2 times

iii) atmost once.

Soln: Sample space $S = \{1, 2, 3, 4, 5, 6\}$

$n = 8$

$p = 1/6$, $p = 5/6$

$$P(X=x) = nC_x \, p^x q^{n-x} = 8C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{8-x}$$

i) $$P(X=2) = 8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{8-2}$$

$$= 8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

$$= \frac{8 \times 7}{1 \times 2} \times 0.027 \times 0.334$$

$$= 0.258$$

ii) $$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[8C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{8-0} + 8C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{8-1}\right]$$

$$= 1 - \left[(1)(1)\left(\frac{5}{6}\right)^8 + 8\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^7\right]$$

$$= 1 - [0.232 + 0.370]$$

$$= 1 - 0.602$$

$$= 0.398$$

iii) $$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= 0.602$$

Poisson Distribution : [For defective items]

Poisson distribution is a limiting case of binomial distribution under the following conditions.

i) $n \rightarrow$ The number of trials is very large ($n \rightarrow \infty$)

ii) $p \rightarrow$ The probability of success in each trial is very small

iii) $np = \lambda$ is finite

(ie) $np = \lambda$ $\qquad p+q=1$

$p = \frac{\lambda}{n}$ $\qquad \frac{\lambda}{n} + q = 1$

$q = 1 - \frac{\lambda}{n}$

Definition:

If X is a discrete random variable that assume the value $X = 0, 1, 2, 3, \ldots$ such that the probability density function is given by $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ where $\lambda > 0$.

Eg:

1) The number of defective articles in a pack of finite number of articles.

2) No. of mistakes committed by a typist per page.

3) The no. of road accidents reported in a city per day.

1) Derive moment generating function of poisson distribution. Hence find mean & variance.

Soln:

$$MGF \Rightarrow M_X(t) = E(e^{tx}) = \sum e^{tx} P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda}\left[\frac{(e^t\lambda)^0}{0!} + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \cdots\right]$$