PROFESSOR: Last time, we began to address the issue of building continuous time signals out of a linear combination of complex exponentials. And for the class of periodic signals specifically, what this led to was the Fourier series representation for periodic signals. Let me just summarize the results that we developed last time. For periodic signals, we had the continuous-time Fourier series, where we built the periodic signal out of a linear combination of harmonically related complex exponentials. And what that led to was what we referred to as the synthesis equation.

And we briefly addressed the issue of when this in fact builds, when this in fact is a complete representation of the periodic signal, and in essence, what we presented was conditions either for x of t being square integrable or x of t being absolutely integrable. Then, the other side of the Fourier series is what I referred to as the analysis equation. And the analysis equation was the equation that told us how we get the Fourier series coefficients from x of t. And so this equation together with the synthesis equation represent the Fourier series description for periodic signals.

Now what we'd like to do is extend this idea to provide a mechanism for building non-periodic signals also out of a linear combination of complex exponentials. And the basic idea behind doing this is very simple and also very clever as I indicated last time. Essentially, the thought is the following, if we have a non-periodic signal or aperiodic signal, we can think of constructing a periodic signal by simply periodically replicating that aperiodic signal.

So for example, if I have an aperiodic signal as I've indicated here, I can consider building a periodic signal, where I simply take this original signal and repeat it at multiples of some period t 0. Now, two things to recognize about this. One is that the periodic signal is equal to the aperiodic signal over one period. And the second is

that as the period goes to infinity then, in fact, the periodic signal goes to the aperiodic signal. So the basic idea then is to use the Fourier series to represent the periodic signal, and then examine the Fourier series expression as we let the period go to infinity.

Well, let's quickly see how this develops in terms of the associated equations. Here, again, we have the periodic signal. And what we want to inquire into is what happens to the Fourier series expression for this as we let the period go to infinity. As that happens, whatever Fourier series representation we end up with will correspond also to a representation for this aperiodic signal.

Well, let's see. The Fourier series synthesis expression for the periodic signal expresses x tilde of t, the periodic signal as a linear combination of harmonically related complex exponentials with the fundamental frequency omega 0 equaled to 2 pi divided by the period. And the analysis equation tells us what the relationship is for the coefficients in terms of the periodic signal.

Now, I indicated that the periodic signal and the aperiodic signal are equal over one period. We recognize that this integration, in fact, only occurs over one period. And so we can re-express this in terms of our original aperiodic signal. So this tells us the Fourier series coefficients in terms of x of t.

Now, if we look at this expression, which is the expression for the Fourier coefficients of the aperiodic signal, one of the things to recognize is that in effect what this represents are samples of an integral, where we can think of the variable omega taking on values that are integer multiples of omega 0. Said another way, let's define a function, as I've indicated here, which is this integral, where we may think of omega as being a continuous variable and then the Fourier series coefficients correspond to substituting for omega k omega 0.

Now, one reason for doing that, as we'll see, is that in fact, this will turn out to provide us with a mechanism for a Fourier representation of x of t. And this, in fact, then, is an envelope of the Fourier series coefficients. In other words, t 0 the period times the coefficients is equal to this integral add integer multiples of omega 0.

So this, in effect, tells us how to get the Fourier series coefficients of the periodic signal in terms of samples of an envelope. And that will become a very important notion shortly. And that, in effect, will correspond to an analysis equation to represent the aperiodic signal.

Now, let's look at the synthesis equation. Recall that in the synthesis our strategy is to build a periodic signal and let the period go to infinity. Well, here is the expression for the synthesis of the periodic signal now expressed in terms of samples of this envelope function, and where I've simply used the fact or the substitution that t 0 is 2 pi over omega 0, and so I have an omega 0 here and a 1 over 2 pi. And the reason for doing that, as we'll see in a minute, is that this then turns into an integral.

Specifically, then, the synthesis equation that we have is what I've indicated here. We would now want to examine this as the period goes to infinity, which means that omega 0 becomes infinitesimally small. And without dwelling on the details, and with my suggesting that you give this a fair amount of reflection, in fact, what happens as the period goes to infinity is that this summation approaches an integral over omega, where omega 0 becomes the differential in omega, and the periodic signal, of course, approach is the aperiodic signal.

So the resulting equation that we get out of the original Fourier series synthesis equation is the equation that I indicate down here, x of t synthesized in terms of this integral, which is what the Fourier series approaches as omega 0 goes to 0. And we had previously that x of omega was in fact an envelope function. And we have then the corresponding Fourier transform analysis equation, which tells us how we arrive at that envelope in terms of x of t.

So we now have an analysis equation and a synthesis equation, which in effect expresses for us how to build x of t in terms of infinitesimally finely spaced complex exponentials. The strategy to review it, and which I'd like to illustrate with a succession of overlays, was to begin with our aperiodic signal, as I indicate here, and then we constructed from that a periodic signal. And this periodic signal has a Fourier series, and we express the Fourier series coefficients of this as samples of

an envelope function. The envelope function is what I indicate on the curve below.

So this is the envelope of the Fourier series coefficients. For example, if the period t 0 was four times t1, then the Fourier series coefficients that we would end up with is this set of samples of the envelope. If instead we doubled that period, then the Fourier series coefficients that we end up with are more finely spaced. And as t 0 continues to increase, we get more and more finely spaced samples of this envelope function, and as t 0 goes to infinity in fact, what we get is every single point on the envelope, and that provides us with the representation for the aperiodic signal.

Let me, just to really emphasize the point, show this example once again. But now, let's look at it dynamically on the computer display. So here we have the square wave, and below it, the Fourier series coefficients. And we now want to look at the Fourier series coefficients as the period of the square wave starts to increase.

And what we see is that these look like samples of an envelope. And in fact, the envelope of the Fourier series coefficients is shown in the bottom [? trace, ?] and to emphasize in fact that it is the envelope let's superimpose it on top of the Fourier series coefficients that we've generated so far. OK. Now, let's increase the period even further, and we'll see the Fourier series coefficients fill in under that envelope function even more. And in fact, as the period gets large enough, what we begin to get a sense of is that we're sampling more and more finely this envelope. And in fact, in the limit, as the period goes off to infinity, the samples basically will represent every single point on the envelope.

Well, this is about as far as we want to go. Let's once again, plot the envelope function, and again, to emphasize that we've generated samples of that, let's superimpose that on the Fourier series coefficients. So what we have then is now our Fourier transform representation, the continuous time Fourier transform with the synthesis equation expressed as an integral, as I've indicated here, and this integral is what the Fourier series sum went to as we let the period go to infinity or the frequency go to zero. The corresponding analysis equation, which we have here,

the analysis equation being the expression for the envelope of the Fourier series coefficients for the periodically replicated signal.

And in shorthand notation, we would think of x of t and [? its ?] Fourier transform as a pair, as I've indicated here. And the Fourier transform, as we'll emphasize in several examples, and certainly as is consistent with the Fourier Series, is a complex valued function even when x of t is real. So with x of t real, we end up with a Fourier transform, which is a complex function. Just as the Fourier series coefficients were complex for a real value time function. So we could alternatively, as with the Fourier series, express the Fourier transform in terms of it's real part and imaginary part, or alternatively, in terms of its magnitude and its angle.

All right, now let's look at an example of a time function in its Fourier transform. And so let's consider an example, which in fact is an example worked out in the text. It's example 4.7 in the text. And this is our old familiar friend the exponential. It's Fourier transform is the integral from minus infinity to plus infinity, x of t, e to the minus j omega t dt. And so, if we substitute in x of t and combine these two exponentials together, these two exponentials combined are e to the minus t times a plus j omega.

And if we carry out the integration of this, we end up with the expression indicated here and provided now, and this is important, provided that a is greater than 0, then at the upper limit, this exponential becomes 0. At the lower limit, of course, it's one. And so what we have finally is for the Fourier transform expression 1 over a plus j omega.

Now, this Fourier transform as I indicated is a complex valued function. Let's just take a look at what it looks like graphically. We have the expression for the Fourier transform pair, e to the minus a t times [? a ?] [? step ?]. And its Fourier transform is 1 over a plus j omega. And I indicated that that's true for a greater than 0.

Now, in the expression that we just worked out, if a is less than 0, in fact, the expression doesn't converge e to the minus a t for a negative as t goes to infinity, blows up, and so in fact the Fourier transform doesn't converge except for the case

where a is greater than 0 And in fact, there is a more detailed discussion of convergence issues in the text. The convergence issues are very much the same for the Fourier transform as they are for the Fourier series. And in fact, that's not surprising, because we developed the Fourier transform out of a consideration of the Fourier series. So the convergence conditions as you'll see as you refer in detail to the text relate to whether the time function is absolutely integrable under one set of conditions and square integrable under another set of conditions.

OK, now, if we plot the Fourier transform, let's first consider the shape of the time function. And as I indicated, we're restricting the time function so that the exponential factor a is positive. In other words, e to the minus a t decays as t goes to infinity. The magnitude of the Fourier transform is as I indicate here and the phase below it. And there are a number of things we can see about the magnitude and phase of the Fourier transform for this example, which in fact we'll see in the next lecture are properties that apply more generally. For example, the fact that the Fourier transform magnitude is an even function of frequency, and the phase is an odd function of frequency.

Now, let me also draw your attention to the fact that on this curve we have both positive frequencies and negative frequencies. In other words, in our expression for the Fourier transform, it requires both omega positive and omega negative. This, of course, was exactly the same in the case of the Fourier series. And the reason you should recall and keep in mind is related to the fact that we're building our signals out of complex exponentials, which require both positive values of omega and negative values of omega.

Alternatively, if we chosen other representation, which turns out notationally to be much more difficult, namely sines and cosines, then we would in fact only consider positive frequencies. So it's important to keep in mind that, in our case, both with the Fourier series and the Fourier transform, we deal and require both positive and negative frequencies in order to build our signals.

Now, in the graphical representation that I've shown here, I've chosen a linear

amplitude scale and a linear frequency scale. And that's one graphical representation for the Fourier transform that we'll typically use. There's another one that very commonly arises, which I'll just briefly indicate for this example. And that is what's referred to as a bode plot in which the magnitude is displayed on a log amplitude and log frequency scale. And the phase is displayed on a log frequency scale. Let me show you what I mean.

Here is the general expression for the bode plot. The bode plot expresses for us the amplitude in terms of the logarithm to the base 10 of the magnitude. And it also expresses the angle in both cases expressed as a function of a logarithmic frequency axis. So here is the amplitude as I've displayed it. And this is a log magnitude scale, a logarithmic frequency scale as indicated by the fact that as we move in equal increments along this axis, we change frequency by a factor of 10.

And similarly, what we have is a display for the phase again on a log frequency scale. And I indicated that there is a symmetry to the Fourier transform, and so in fact, we can infer from this particular picture what it looks like for the negative frequencies as well as for the positive frequencies. Now, what we've done so far is to develop the Fourier transform on the basis, the Fourier transform of an aperiodic signal on the basis of periodically repeating it and recognizing that the Fourier series coefficients are samples of an envelope and that these become more finely spaced as frequency increases.

And in fact, we can go back to our original equation in which we developed an envelope function, and what we had indicated is that the Fourier series coefficients were samples of this envelope. We then defined this envelope as the Fourier transform of this aperiodic signal. So that provided us with a way-- and it was a mechanism-- for getting a representation for an aperiodic signal.

Now, suppose that we have instead a periodic signal, are there, in fact, some statements that we can make about how the Fourier series coefficients of that are related to the Fourier transform of something. Well, in fact, this statement tells us exactly how to do that. What this statement says is that, in fact, the Fourier series

coefficients are samples of the Fourier transform of one period?

So if we now consider a periodic signal, we can in fact get the Fourier series coefficients of that periodic signal by considering the Fourier transform of one period. Said another way, the Fourier series coefficients are proportional to samples of the Fourier transform of one period. So if we consider this a periodic signal, computed as Fourier transform, and selected these samples that I indicate here, namely samples equally spaced in omega by integer multiples of omega 0, then in fact, those would be the Fourier series coefficients.

So we can go back to our example previously that involved the square wave. And now, in this case, we could argue that if in fact it was the periodic signal that we started with, we could get the Fourier series coefficients of that by thinking about the Fourier transform of one period, which I indicate here. And then the Fourier series coefficients of the periodic signal, in fact, are the appropriate set of samples of this envelope.

All right, now, we have a way of getting the Fourier series coefficients from the Fourier transform of one period. We originally derived the Fourier transform of one period from the Fourier series. What would, in fact, be nice is if we could incorporate the Fourier series and the Fourier transform within a common framework. And in fact, it turns out that there is a very convenient way of doing that almost by definition.

Essentially, if we consider what the equation for the synthesis looks like in both cases, we can in effect define a Fourier transform for the periodic signal, which we know is represented by its Fourier series coefficients. We can define a Fourier transform, and the definition of the Fourier transform is as an impulse train, where the coefficients in the impulse train are proportional, with a proportionality factor of 2 pi for a more or less a bookkeeping reason, proportional to the Fourier series coefficients.

And the validity of this is, more or less, can be seen essentially by substitution. Specifically, here is then the synthesis equation for the Fourier transform if we

substitute this definition for the Fourier transform of the periodic signal into this expression then when we do the appropriate bookkeeping and interchange the order of summation and integration the impulse integrates out to the exponential factor that we want.

So we have the exponential factor. We have the Fourier series coefficients. The 2 pis take care of each other, and what we're left with is the synthesis equation for aperiodic signal in terms of the Fourier transform, or in terms of its Fourier series coefficients.

Now, we can just see this in terms of a simple example. If we consider the example of a symmetric square wave, then in effect what we're saying is that for this symmetric square wave, this has a set of Fourier series coefficients, which we worked out previously and which I indicate on this figure with a bar graph. And really all that we're saying is that, whereas these Fourier series coefficients are indexed on an integer variable k, and [? they're ?] bars not impulses. If we simply redefine or define the Fourier transform of the periodic signal as an impulse train, where the weights of the impulses are 2 pi times the corresponding Fourier series coefficients, then this, in fact, is what we would use as the Fourier transform of the periodic signal.

Now, we've kind of gone back and forth, and maybe even it might seem like we've gone around in circles. So let me just try to summarize the various relationships and steps that we've gone through, keeping in mind that one of our objectives was first to develop a representation for aperiodic signals and then attempt to incorporate within one framework both periodic and aperiodic signals.

We began with an aperiodic signal. And the strategy was to develop a Fourier representation by constructing a periodic signal for which that was one period. And then we let the period go to infinity, as I indicate here.

So we have an aperiodic signal. We construct a periodic signal, x tilde of t for which one period is the aperiodic signal. X tilde of t, the periodic signal, has a Fourier series, and as its period increases that approaches the aperiodic signal, and the

Fourier series of that approaches the Fourier transform of the original aperiodic signal. So that was the first step we took.

Now, the second thing that we recognize is that once we have the concept of the Fourier transform, we can, in fact, relate the Fourier series coefficients to the Fourier transform of one period. So the second statement that we made was that if in fact we're trying to represent a periodic signal, we can get the Fourier series coefficients of that by computing the Fourier transform of one period and then samples of that Fourier transform are, in fact, the Fourier series coefficients for the periodic signal.

Then, the third step that we took was to inquire as to whether there is a Fourier transform that can appropriately be defined for the periodic signal, and the mechanism for doing that was to recognize that if we simply defined the Fourier transform of the periodic signal as an impulse train, where the impulse heights or areas were proportional to the Fourier series coefficients, then, in fact, the Fourier transform synthesis equation reduced to the Fourier series synthesis equation.

So the third step, then, was with a periodic signal. The Fourier transform of that periodic signal, defined as an impulse train, where the heights or areas of the impulses are proportional to the Fourier series coefficients, provides us with a mechanism for combining it together the concepts or notation of the Fourier series and Fourier transform.

So if we just took a very simple example, here is an example in which we have an aperiodic signal, which is just an impulse, and its Fourier transform is just a constant. We can think of a periodic signal associated with this, which is this signal periodically replicated with a spacing t 0. The Fourier transform of this is a constant. And this, of course, has a Fourier series representation.

So the Fourier transform of the original impulse is just a constant. The Fourier transform of the periodic signal is an impulse train, where the heights of the impulses are proportional to the Fourier series coefficients. And, of course, we could previously have computed the Fourier series coefficients for that impulse train, and

those Fourier series coefficients are as I've shown here.

So in both of these cases, these in effect represent just a change in notation, where here we have a bar graph, and here we have an impulse train. And both of these simply represent samples of what we have above, which is the Fourier transform of the original aperiodic signal.

Once again, I suspect that kind of moving back and forth and trying to straighten out when we're talking about periodic and aperiodic signals may require a little mental gymnastics initially. Basically, what we've tried to do is incorporate within one framework a representation for both aperiodic and periodic signals, and the Fourier transform provides us with a mechanism to do that.

In the next lecture, I'll continue with the discussion of the continuous-time Fourier transform in particular focusing on a number of its properties, some of which we've already seen, namely the symmetry properties. We'll see lots of other properties that relate, of course, both to the Fourier transform and to the Fourier series. Thank you.