Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
***Discrete Mathematics & Its Applications* (6th Edition)**
by Kenneth H. Rosen

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for COT 3100 course in University of Florida

Module #15:
Recurrence Relations

Rosen 6th ed., § 7.1-7.2

§ 7.1: Recurrence Relations

- A *recurrence relation* (or just *recurrence*) for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements $a_0, \ldots, a_{n-1}$ of the sequence, for all $n \geq n_0$.
  - A recursive definition, without the base cases.
- A non-recursively described (closed-form) sequence is said to *solve* a given recurrence relation if it's terms satisfy the recurrence relation.
  - A given recurrence relation may have many solutions.

Recurrence Relation Example

- Consider the recurrence relation

  $$a_n = 2a_{n-1} - a_{n-2} \ (n \geq 2).$$

- Which of the following are solutions?

  $a_n = 3n$ $\quad 3n \stackrel{?}{=} 2 \cdot 3(n\text{-}1) - 3(n\text{-}2)$ $\quad$ Yes

  $a_n = 2^n$ $\quad$ No

  $a_n = 5$ $\quad$ Yes

Example Applications

- Recurrence relation for growth of a bank account with *P%* interest per given period:

$$M_n = M_{n-1} + (P/100)M_{n-1}$$

- Growth of a population in which each organism yields 1 new organism at every period starting 2 periods after its birth.

$$P_n = P_{n-1} + P_{n-2} \quad \text{(Fibonacci)}$$

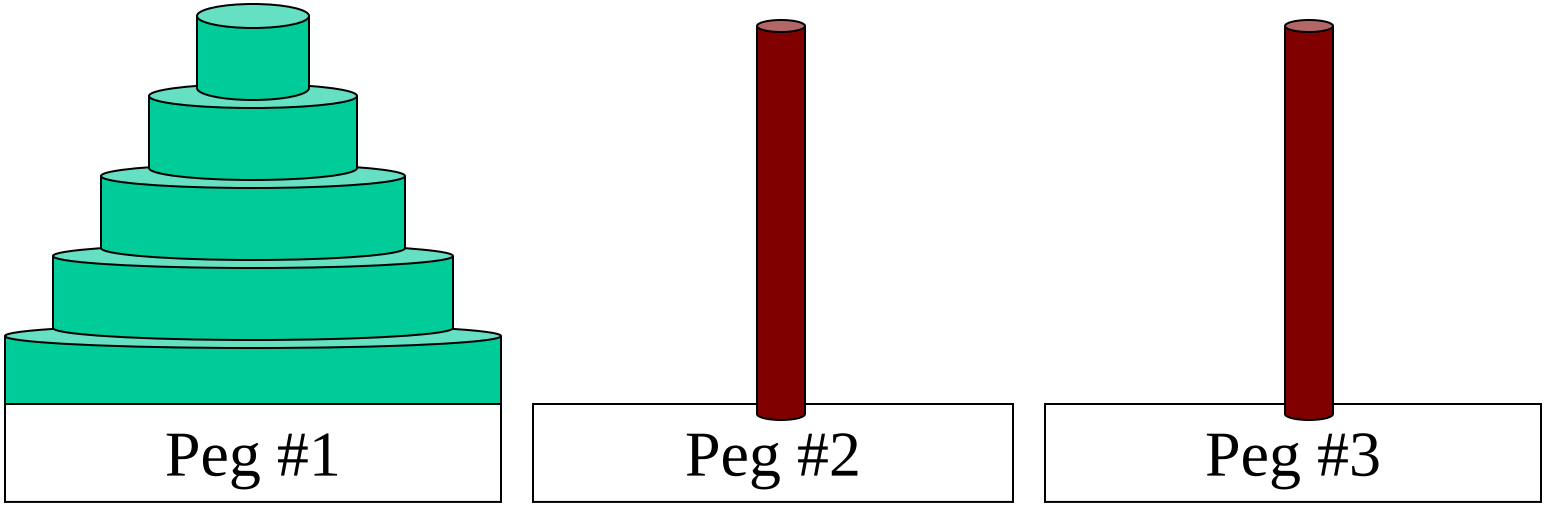
Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Solving Compound Interest

- A *solution* to a *recurrence* is a closed-form equation.
- $M_n = M_{n-1} + (P/100)M_{n-1}$
  $= (1 + P/100)\, M_{n-1}$
  $= r\, M_{n-1}$ (let $r = 1 + P/100$)
  $= r\,(r\, M_{n-2})$
  $= r \cdot r \cdot (r\, M_{n-3})$ …and so on to…
  $= r^n\, M_0$

Tower of Hanoi Example

- Problem: Get all disks from peg 1 to peg 2.
  - Only move 1 disk at a time.
  - Never put a larger disk on a smaller one.



http://towersofhanoi.info/Animate.aspx

Hanoi Recurrence Relation

- Let H_n = # moves for a stack of n disks.
- Optimal strategy:
  - Move top $n-1$ disks to spare peg. (H_{n-1} moves)
  - Move bottom disk. (1 move)
  - Move top $n-1$ to bottom disk. (H_{n-1} moves)
- Note: $H_n = 2H_{n-1} + 1$

Solving Hanoi Recurrence Relation

Question: How many moves for a stack of n disks?

$$
\begin{aligned}
H_n &= 2\,H_{n-1} + 1 \\
&= 2\,(2\,H_{n-2} + 1) + 1 && = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2\,H_{n-3} + 1) + 2 + 1 && = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\ldots \\
&= 2^{n-1} H_1 + 2^{n-2} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1 && \text{(since } H_1 = 1\text{)} \\
&= \sum_{i=0}^{n-1} 2^i \\
&= 2^n - 1
\end{aligned}
$$

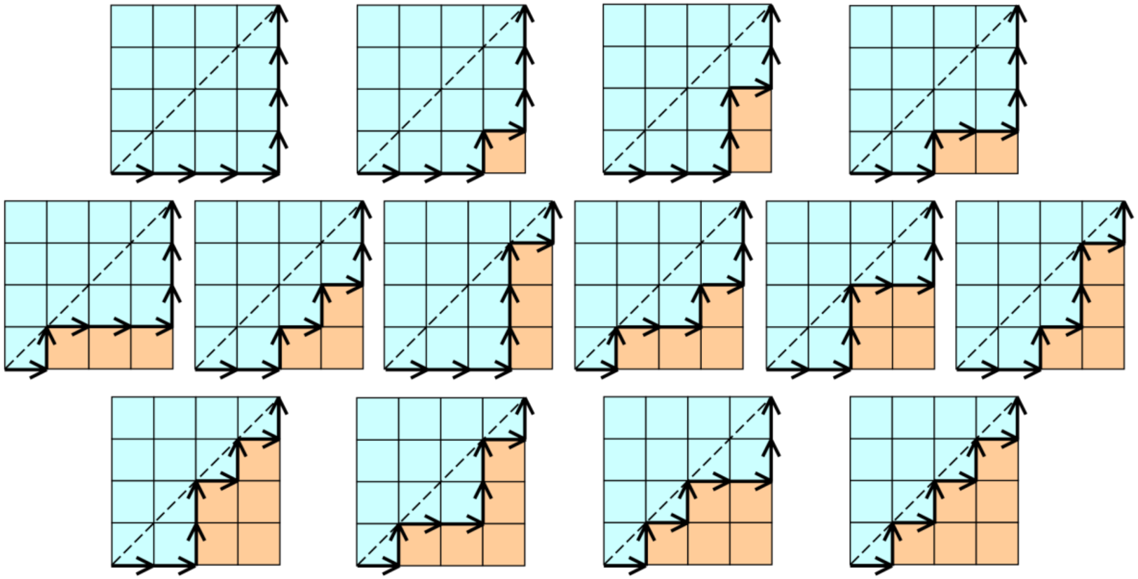
Catalan numbers

- C_n is the number of different ways $n + 1$ factors can be completely parenthesized. For example $C_3 = 5$, since we have five different parenthesizations of four factors:

  $((x_0x_1)x_2)x_3$ $\quad (x_0(x_1x_2))x_3$ $\quad (x_0x_1)(x_2x_3)$
  $x_0((x_1x_2)x_3)$ $\quad x_0(x_1(x_2x_3))$
- C_0 and C_1 are equal to 1, not surprisingly.
- $C_2 = 2$ since $x_0(x_1x_2)$ and $(x_0x_1)x_2$ are the options.
- C_n is the n^{th} Catalan number.

Catalan numbers

- Catalan numbers are ubiquitous.
  https://en.wikipedia.org/wiki/Catalan_number
- C_n is also the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal:



Catalan numbers

- Catalan numbers can be written as a recurrence:

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-2} C_1 + C_{n-1} C_0$$

with initial conditions $C_0 = 1$ and $C_1 = 1$.

- $$C_3 = \underbrace{C_0 C_2}_{x_0((x_1x_2)x_3) \quad x_0(x_1(x_2x_3))} + \underbrace{C_1 C_1}_{(x_0x_1)(x_2x_3)} + \underbrace{C_2 C_0}_{((x_0x_1)x_2)x_3 \quad (x_0(x_1x_2))x_3} = 5$$

- $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$ …

- $C_n = C(2n,n) / (n+1)$ … proof is out of our scope.

§ 7.2: Solving Recurrences

- A *linear homogeneous recurrence of degree k with constant coefficients* (k-LiHoReCoCo) is a recurrence of the form
  $$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}$$
  where the c_i are all real, and $c_k \neq 0$.
- The closed-form solution is uniquely determined if k initial conditions $a_0 \ldots a_{k-1}$ are provided.

2-LiHoReCoCo Example

- Fibonacci numbers relation

  $$f_n = f_{n-1} + f_{n-2}$$

  is a linear homogeneous recurrence of degree 2. Therefore, it is a 2-LiHoReCoCo.
- The solution is uniquely determined if 2 initial conditions f_0 and f_1 are provided.

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \longleftarrow$$

See textbook for solution

Solving LiHoReCoCos

- The basic approach is looking for solutions of the form $a_n = r^n$, where r is a constant.
- This requires the *characteristic equation*:

  $$r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k},$$

  both sides are divided by r^{n-k},

  $$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

- The solutions (*characteristic roots*) can yield an explicit formula for the sequence.

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- It has the characteristic equation (C.E.):

  $$r^2 - c_1 r - c_2 = 0$$

- **Thm. 1:** If this CE has 2 roots $r_1 \neq r_2$, then

  $$a_n = A r_1^n + B r_2^n \text{ for } n \geq 0$$

  for some constants A, B.

Example

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: Use Theorem 1
  - $c_1 = 1$, $c_2 = 2$
  - Characteristic equation: $r^2 - r - 2 = 0$
  - Solutions: $r = [-(-1) \pm ((-1)^2 - 4 \cdot 1 \cdot (-2))^{1/2}] / 2 \cdot 1$
    $= (1 \pm 9^{1/2})/2 = (1 \pm 3)/2$, so $r = 2$ or $r = -1$.
  - So, $a_n = A \cdot 2^n + B \cdot (-1)^n$.

Hint: $ax^2 + bx + c = 0$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example Continued…

- To find A and B, solve the equation for the initial conditions:

$$a_0 = 2 = A\cdot 2^0 + B\,(-1)^0$$

$$a_1 = 7 = A\cdot 2^1 + B\,(-1)^1$$

Simplifying, we have the pair of equations:

$$2 = A + B$$

$$7 = 2A - B$$

which we can solve easily by substitution:

$$9 = 3A; \quad A = 3;$$

$$B = 2-A; \quad B = -1.$$

Final answer: $a_n = 3\cdot 2^n - (-1)^n$

Check: $\{a_{n\geq 0}\} = 2, 7, 11, 25, 47, 97 \ldots$

The Case of Degenerate Roots

- Now, what if the C.E. $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0?
- **Theorem 2:** Then,

  $$a_n = A r_0^n + B n r_0^n, \text{ for all } n \geq 0,$$

  for some constants A, B.

Example

- Solve the recurrence relation
  $a_n = 6a_{n-1} - 9a_{n-2}$
  given the initial conditions $a_0 = 1$, $a_1 = 6$.
- Solution:
  - $c_1 = 6$, $c_2 = -9$, characteristic equation: $r^2 - 6r + 9 = 0$.
  - Only root is r=3. Hence the solution is
    $a_n = A\,3^n + B\,n\,3^n$.
  - $a_0 = A = 1$
  - $a_1 = 3A + 3B = 6$. Then B=1.
  - Solution becomes $a_n = 3^n + n\,3^n$.

k-LiHoReCoCos

- Consider a *k*-LiHoReCoCo:
$$a_n = \sum_{i=1}^{k} c_i a_{n-i}$$
- It's C.E. is: $$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$
- **Thm.3:** If this has *k* distinct roots r_i, then the solutions to the recurrence are of the form:
$$a_n = \sum_{i=1}^{k} A_i r_i^n$$
for all $n \geq 0$, where the A_i are constants.