Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6th Edition)**
by Kenneth H. Rosen

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for COT 3100 course in University of Florida

Module #12: **Inductive Proofs**

Rosen 6th ed., § 4.1-4.2

§ 4.1: Mathematical Induction

- A powerful, rigorous technique for proving that a predicate $P(n)$ is true for *every* natural number n, no matter how large.
- Based on a predicate-logic inference rule:

$$\begin{array}{l} P(0) \\ \forall k{\geq}0\ (P(k) \rightarrow P(k+1)) \\ \hline \therefore \forall n{\geq}0\ P(n) \end{array}$$

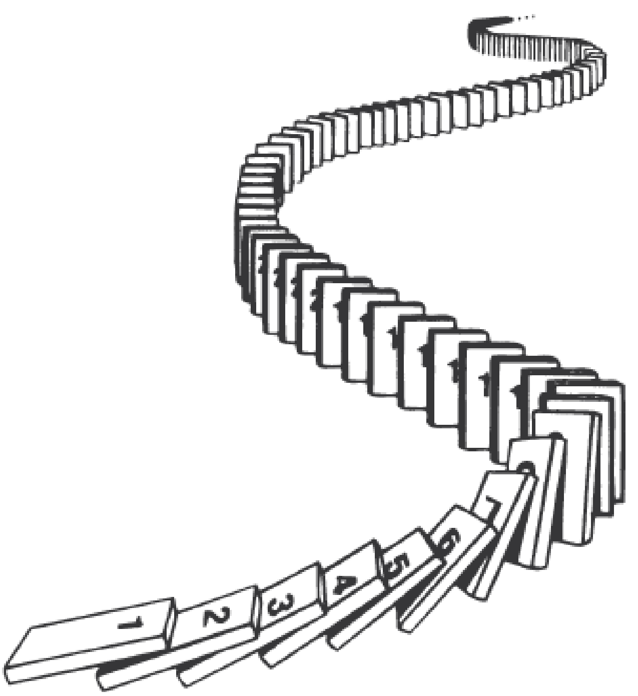
"The First Principle of Mathematical Induction"

Validity of Induction

- Given $\forall k \geq 0\ (P(k) \rightarrow P(k+1))$ (premise #2) trivially implies $\forall k \geq 0\ (k<n) \rightarrow (P(k) \rightarrow P(k+1))$, or $(P(0) \rightarrow P(1)) \wedge (P(1) \rightarrow P(2)) \wedge \ldots \wedge (P(n-1) \rightarrow P(n))$.
- Repeatedly applying the hypothetical syllogism rule to adjacent implications n-1 times gives $P(0) \rightarrow P(n)$; which with $P(0)$ (premise #1) and *modus ponens* gives $P(n)$. Thus $\forall n \geq 0\ P(n)$.

Analogy with Domino Effect

- Let $P(n)$: n^{th} domino is knocked over
- If you show:
  1) $P(1)$ is true.
  2) $\forall k \geq 1\ (P(k) \rightarrow P(k+1))$ is true.
- This implies $\forall n\ P(n)$, meaning all dominoes are knocked over.



Outline of an Inductive Proof

- Want to prove $\forall n\, P(n)$…
- *Base case* (or *basis step*): Prove $P(0)$.
- *Inductive step*: Prove $\forall k\, P(k) \rightarrow P(k+1)$.
  - *E.g.* use a direct proof:
  - Let $k \in \mathbf{N}$, assume $P(k)$. (*inductive hypothesis*)
  - Under this assumption, prove $P(k+1)$.
- Inductive inference rule then gives $\forall n\, P(n)$.

Generalizing Induction

- *Base case* does not have to be $P(0)$.
- Can be used to prove $\forall n \geq c\ P(n)$ for a given constant $c \in \mathbf{Z}$, where $c \neq 0$.
  – In this circumstance, the base case is to prove $P(c)$ rather than $P(0)$, and the inductive step is to prove $\forall k \geq c\ (P(k) \rightarrow P(k+1))$.

Induction Example

- Prove that the sum of the first n odd positive integers is n^2. That is, prove:

$$\forall n \geq 1 : \underbrace{\sum_{i=1}^{n}(2i-1) = n^2}_{P(n)}$$

- Proof by induction.
  - Base case: P(1) is True. The sum of the first 1 odd positive integer is 1 which equals 1^2. (Cont…)

Example cont.

- Inductive step: Prove $\forall k \geq 1$: $P(k) \rightarrow P(k+1)$.
  - Let $k \geq 1$, assume $P(k)$, and prove $P(k+1)$.

$$\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1)$$

$$= k^2 + 2k + 1$$

By inductive hypothesis $P(k)$

$$= (k+1)^2$$

Also see Example 3 in Section 4.1

Another Induction Example

- Prove that $\forall n>0,\ n<2^n$. Let $P(n)=(n<2^n)$
  - Base case: $P(1)=(1<2^1)=(1<2)=\mathbf{T}$.
  - Inductive step: For $k>0$, prove $P(k)\rightarrow P(k+1)$.
    - Assuming $k<2^k$, prove $k+1 < 2^{k+1}$.
    - Note $k + 1 < 2^k + 1$ (by inductive hypothesis)
      $< 2^k + 2^k$ (because $1<2=2\cdot 2^0\leq 2\cdot 2^{k-1}= 2^k$)
      $= 2^{k+1}$
    - So $k + 1 < 2^{k+1}$, and we're done.

Another Induction Example

- Prove "3 divides n^3+2n when n is a positive integer".
  - Base case: $P(1)$ is true because $3 \mid 1^3+2$
  - Inductive step: For $k>0$, prove $P(k)\rightarrow P(k+1)$.
    - Assume $P(k)$ is true (ind. hyp.): $3 \mid k^3+2k$
    - Then, $(k+1)^3+2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$
    - $k^3 + 2k + 3(k^2 + k + 1)$
    - $k^3 + 2k + 3j$
    - $3m + 3j$ (by inductive hypothesis)
    - $3(m + j)$ and we're done.

§ 4.2: Strong Induction

- Also called Second Principle of Induction.
- Characterized by:

$$\frac{\begin{array}{l} P(0) \\ \forall k\geq 0\ \overbrace{(P(0)\wedge P(1)\wedge \ldots \wedge P(k))}^{\textit{Inductive hypothesis:} P(0) \text{ to } P(k) \text{ is true}} \rightarrow P(k+1) \end{array}}{\therefore \forall n\geq 0\ P(n)}$$

- Difference with 1st principle is that the inductive step uses the fact that $P(j)$ is true for all values $<k+1$, not just for k.

Example of Strong Induction

- Show that every $n>1$ can be written as a product $p_1p_2\ldots p_s$ of some series of s prime numbers. Let $P(n)$="n has that property"
- Basis step: $P(2)$ is true, let $s=1$, $p_1=2$.
- Inductive step: Let $k\geq 2$.
  Inductive hypothesis is the assumption that $\forall 2\leq j\leq k\ P(j)$ is true.
  To complete the inductive step we must show $P(k+1)$ is true under this assumption.

Example of Strong Induction

inductive step continues..

Consider k+1. There are two cases:

1) If k+1 is prime, let s=1, p_1=k+1. $P(k+1)$ is true.
2) If $k+1=ab$, where $1<a\leq k$ and $1<b\leq k$.
Since $P(j)$ is true $\forall 2\leq j\leq k$, $P(a)$ and $P(b)$ are true.
Then $a=p_1p_2\ldots p_t$ and $b=q_1q_2\ldots q_u$.
Then $n+1= p_1p_2\ldots p_t\,q_1q_2\ldots q_u$, which is a product of $t+u=s$ primes.

Another Strong Induction Example

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Basis step:
  $12=3(4)$,
  $13=2(4)+1(5)$,
  $14=1(4)+2(5)$,
  $15=3(5)$,
  so $\forall 12 \leq n \leq 15, P(n)$.

Example continues..

- Inductive step: Let $k \geq 15$ and $\forall 12 \leq j \leq k\ P(j)$ is true. Since $12 \leq k-3$, $P(k-3)$ is true.
- To form $k+1$ cents, we add a 4-cent stamp to the stamps we used to form $k-3$ cents.