Izmir Institute of Technology

CENG 115

Discrete Structures

Slides are based on the Text
***Discrete Mathematics & Its Applications* (6th Edition)**
by Kenneth H. Rosen

Module #3:
Sets

Rosen 6th ed., § § 2.1-2.2
~33 slides, ~1.5 lectures

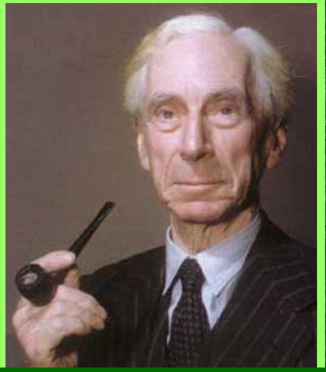
Introduction to Set Theory

- A *set* is an *unordered* collection of *objects*.
- The objects in a *set* are called the *elements*. A set is said to *contain* its elements.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous (frequently seen) in computer science and systems.

Naive Set Theory

- Basic definition: Any collection of objects that we can describe constitutes a set.
- This intuitive definition may lead to *paradoxes* or *logical inconsistencies*, shown by B. Russell.
- These "sets" mathematically *cannot* exist.

  E.g. let $S = \{ X \mid X \notin X \}$. Is $S \in S$?
- More sophisticated set theories fix this problem.
- For this course, we do not use sophisticated theories, but use the intuitive one, also known as the *naive set theory*.



Bertrand Russell
1872-1970

Basic notations for sets

- We can denote a set S in writing by listing all of its elements in curly braces:
  - $\{1, 2, 3, \ldots, 99\}$ is the set of positive integers less than 100.
- *Set builder notation*: For any predicate and domain, $\{x|P(x)\}$ is *the set of all x such that* $P(x)$. E.g. $O=\{x \in \mathbf{Z}^+ |\ x$ is odd and $x<10\}$
  is the set of odd positive integers less than 10.

Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- It does not matter *how the set is defined.*
- For example: The set $\{1, 2, 3, 4\}$ =
  $\{x \mid x$ is an integer where $x>0$ and $x<5\}$ =
  $\{x \mid x$ is a positive integer whose square is greater than 0 and less than $25\}$
- $\{a, b, c\} = \{a, c, b\} = \{b, a, c\}$
  since the sets are inherently *unordered.*

Cardinality and Finiteness

- $|S|$ (read "the *cardinality* of S") is a measure of how many different elements S has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,
  $|\{\{1,2,3\},\{4,5\}\}| = \underline{\quad 2 \quad}$
- If $|S|\in\mathbf{N}$, then we say S is *finite*.
  Otherwise, we say S is *infinite*.
- Can you give an example of an infinite set?

Some Infinite Sets

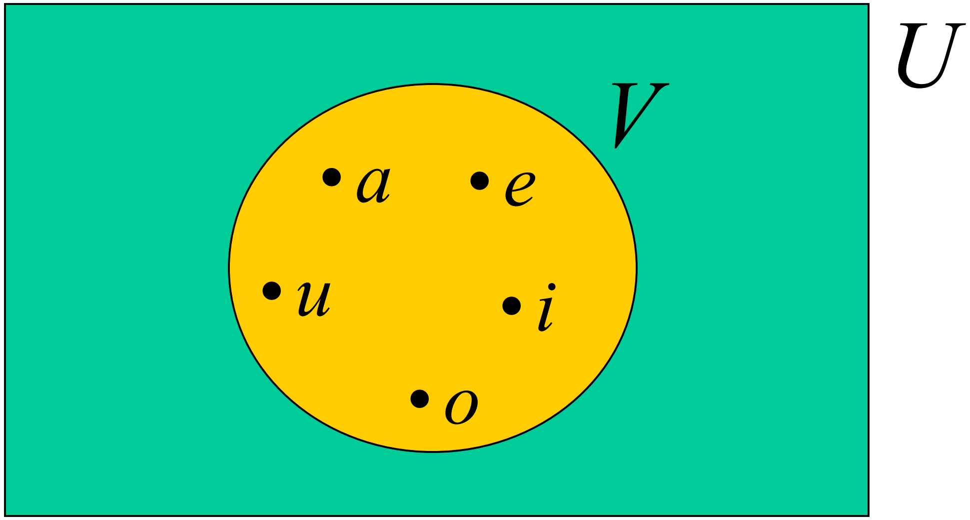
- Symbols for some special infinite sets:

  N = {0, 1, 2, …} The natural numbers.

  Z = {…, -2, -1, 0, 1, 2, …} The integers.

  R = The "real" numbers.

Venn Diagrams



John Venn
1834-1923

U, **the universal set**, contains all the objects under consideration.

Basic Set Relations: Element of

- $x \in S$ ("x is in S") is the proposition that object x is an $\in$*lement* or *member* of set S.
  - *e.g.* $3 \in \mathbf{N}$, "a" $\in \{x \mid x \text{ is a letter of the alphabet}\}$
  - $S = T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$
    "Two sets are equal iff they have all the same members."
- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"

The Empty Set

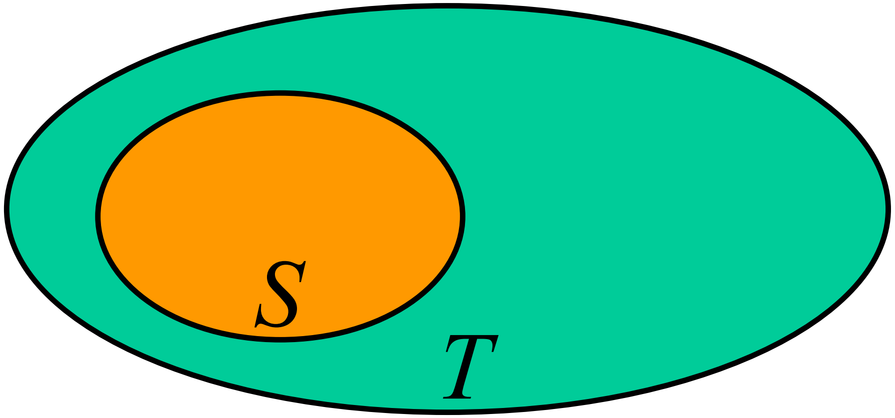
- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\}$
- No matter the domain of discourse, we have the axiom $\neg\exists x\colon x\in\varnothing$.

Subsets

- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
- $S \subseteq T \Leftrightarrow \forall x\ (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S$, $S \subseteq S$.
- $S \not\subseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x (x \in S \wedge x \notin T)$

Proper (Strict) Subsets

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ and $T \not\subseteq S$.
- $\forall x\,(x \in S \rightarrow x \in T) \wedge \exists x(x \in T \wedge x \notin S)$



Venn Diagram equivalent of $S \subset T$

Example:
$\{1,2\} \subset \{1,2,3\}$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S=\{x \mid x \subseteq \{1,2,3\}\}$
  then $S=\{\varnothing,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}\}$
- Note that $\{1,3\} \neq \{\{1\},\{3\}\}$

The *Power Set* Operation

- The *power set* P(S) of a set S is the set of all subsets of S. P(S) = $\{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Note that for finite S, $|P(S)| = 2^{|S|}$.
- It turns out that $|P(\mathbf{N})| > |\mathbf{N}|$.
  There are different sizes of infinite sets!

Review: Set Notations So Far

- Variable objects x, y, z; sets S, T, U.
- Set builder notation, $\{x|P(x)\}$.
- $\in$ relational operator, and the empty set $\varnothing$.
- Set relations $=$, $\subseteq$, $\subset$, $\not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets **N**, **Z**, **R**.
- Power sets P(S).

Ordered *n*-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in \mathbf{N}$, an *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that the *first* element is a_1, the *second* element is a_2, *etc.*
- Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.
- Empty sequence, singlets, pairs, triples, quadruples, quin<u>tuples</u>, …, n-tuples.

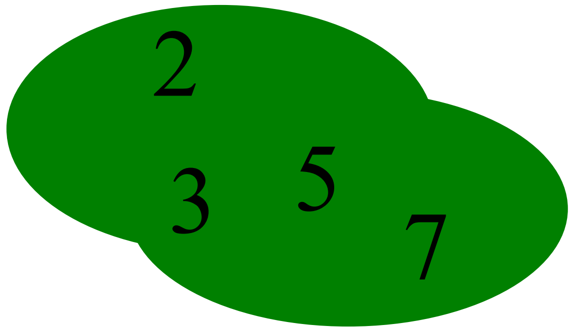
Cartesian Products of Sets

- For sets A, B, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Note that for finite A, B, $|A \times B| = |A| \cdot |B|$.
- A subset R of $A \times B$ is called a **relation** from set A to set B.
- Extends to $A_1 \times A_2 \times \ldots \times A_n \ldots$

René Descartes
(1596-1650)

Start § 2.2: The Union Operator

- For sets A, B, their $\cup$*nion* $A\cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
- Formally, $\forall A,B$: $A\cup B = \{x \mid x\in A \vee x\in B\}$.
- E.g. $\{2,3,5\}\cup\{3,5,7\} = \{2,3,5,7\}$



The Intersection Operator

- For sets A, B, their *intersection* $A\cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.
- Formally, $\forall A,B$: $A\cap B\equiv\{x \mid x\in A \wedge x\in B\}$.
- $A\cap B$ is a subset of A **and** it is a subset of B: $\forall A, B$: $(A\cap B \subseteq A) \wedge (A\cap B \subseteq B)$
- $\{a,b,c\}\cap\{2,3\}$ = $\underline{\ \varnothing\ }$
- $\{2,4,6\}\cap\{3,4,5\}$ = $\underline{\ \{4\}\ }$

Disjoint sets

- Two sets A, B are called *disjoint* (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \emptyset$)
- Example: the set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

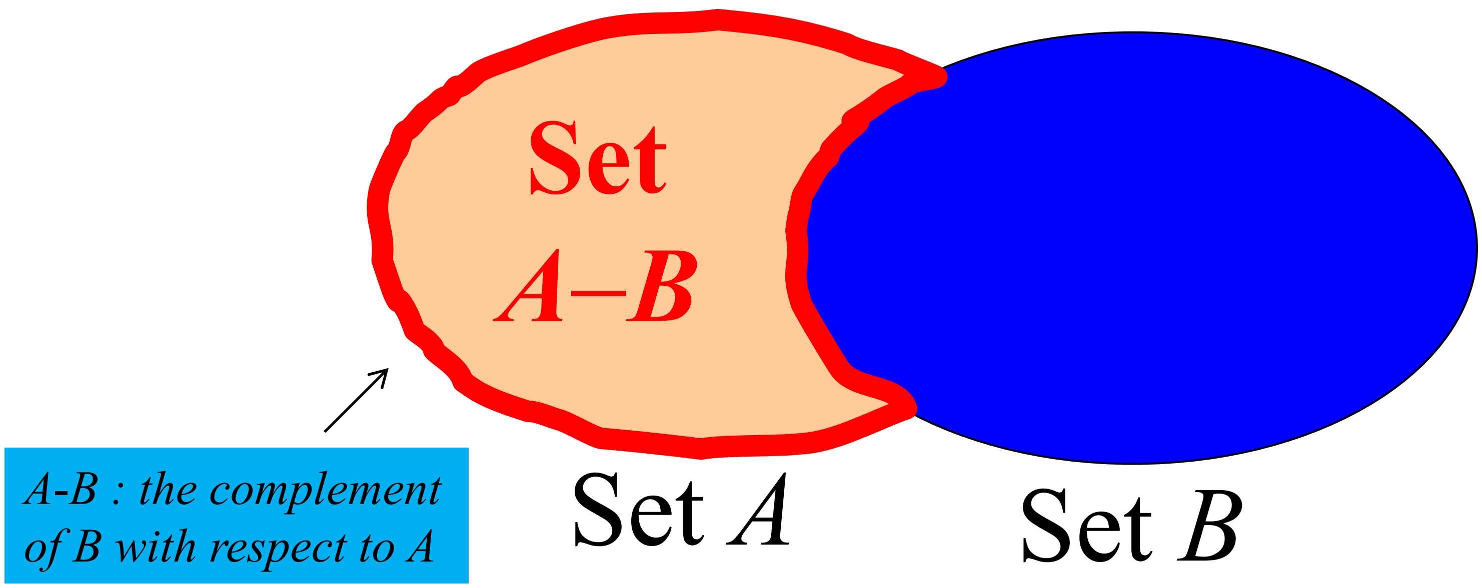
- How many elements are in $A \cup B$?
  $|A \cup B| = |A| + |B| - |A \cap B|$
- Example: How many students speak English or German? Consider set $L = E \cup G$,
  $E = \{s \mid s \text{ speaks English}\}$
  $G = \{s \mid s \text{ speaks German}\}$
- Some students speak both!
  $|L| = |E \cup G| = |E| + |G| - |E \cap G|$

Set Difference

- For sets A, B, the *difference of A and B*, written $A–B$, is the set of all elements that are in A but not in B.
- $A - B :\equiv \{x \mid x \in A \wedge x \notin B\}$
  $= \{x \mid \neg(x \in A \rightarrow x \in B) \}$
- Also called:
  The *complement of B with respect to A*.

Set Difference

- $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ $\underline{\{1,4,6\}}$
- $\mathbf{Z} - \mathbf{N} = \{\ldots, -3, -2, -1\}$



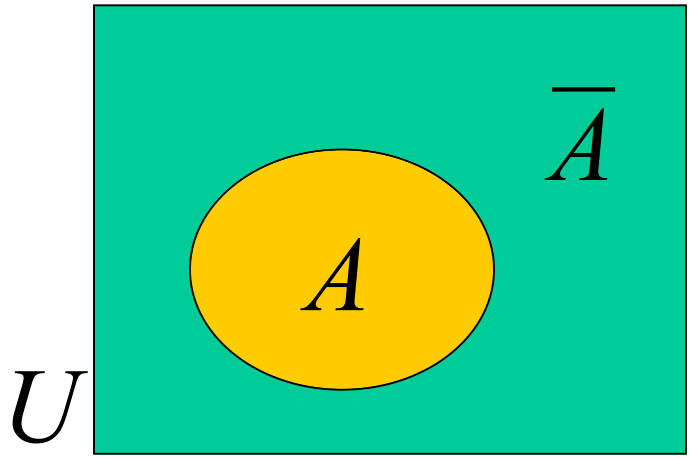
Set Complement

- We have seen the universal set U to define the *universe of discourse*.
- When the context clearly defines U, we say that for any set $A \subseteq U$, the *complement* of A, written $\overline{A}$, is the complement of A w.r.t. U, *i.e.*, it is $U-A$.
- *E.g.*, If $U=\mathbf{N}$, $\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$

Set Complement

- An equivalent definition:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities

- Identity: $A\cup\varnothing=A \quad A\cap U=A$
- Domination: $A\cup U=U \quad A\cap\varnothing=\varnothing$
- Idempotent: $A\cup A = A = A\cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A\cup B=B\cup A \quad A\cap B=B\cap A$
- Associative: $A\cup(B\cup C)=(A\cup B)\cup C$
  $A\cap(B\cap C)=(A\cap B)\cap C$
- Distributive: $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
  $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

De Morgan's Law for Sets

- Exactly analogous to (and derivable from) De Morgan's law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where *E*s are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately. (show each side is a subset of the other)
- Use set builder notation & logical equivalences.
- Use a *membership table*.

Method 1: Proving subset relation in both directions

Example: Show $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

- Show $A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
  - We know that $x\in A$, and either $x\in B$ or $x\in C$.
    - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
    - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- Show $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. …

Method 2: Builder notation and logical equivalences

Example: Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

- $\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$
- $\overline{A \cap B} = \{x \mid \neg(x \in (A \cap B)\}$
- $\overline{A \cap B} = \{x \mid \neg(x \in A \wedge x \in B)\}$
- $\overline{A \cap B} = \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$
- $\overline{A \cap B} = \{x \mid x \notin A \vee x \notin B\}$
- $\overline{A \cap B} = \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$
- $\overline{A \cap B} = \{x \mid x \in (\overline{A} \cup \overline{B})\}$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "1" to indicate membership in the derived set, "0" for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					