Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6^{th} Edition)**
by Kenneth H. Rosen

Module #5:
Sequences and Summations

Rosen 6^{th} ed., § 2.4

Sequences

- A *sequence* or *series* is a discrete structure to represent an ordered list (like 1 4 9 16 25 ..)
  - Each element in a sequence has an associated *index* number.
  - A sequence may be infinite.
- Sequences are used for many purposes, especially to solve counting problems (Ch.7)

x_i : i^2

$x_1 = 1 \longrightarrow 1$

$x_2 = 2 \longrightarrow 4$

$x_3 = 3 \longrightarrow 9$

Sequences

- Formally: A *sequence* is a *generating function* from a subset of integers to a set S.
- We use the notation a_n to denote the image of the integer n. We call n the *index* and a_n a *term* of the sequence.

  W about the general term?
- If we choose the subset as natural numbers, then generating function is $f:\mathbf{N}\rightarrow S$.

  Our terms follow: a_0, a_1, a_2 ,..

Sequences

- If we choose the subset as positive integers, then generating function is f:$\mathbf{Z}^+ \rightarrow S$.

Then the terms are a_1, a_2, a_3 , ...

- We use the notation $\{a_n\}$ to describe a sequence (series).

General term $a_n = n^2 + 2$

$a_0 = 2$ $a_2 = 6$

$a_1 = 3$ $a_3 = 11$ ....

Sequence Examples

- Example of an infinite series:
  - $\{a_n\} = a_1, a_2, \ldots$, where $a_n = f(n) = 1/n$.
  - Then $\{a_n\} = 1, 1/2, 1/3, \ldots$ → give values to n!
- An arithmetic progression is another example:
  - The sequence is $c, c+d, c+2d, c+3d \ldots$
  - What is $f(n)$? $f(n) = c+d\cdot n$ where $n=0,1,2,\ldots$ $c+dn$ → $n = 0, 1, 2, 3, 4 \ldots$

Example with Repetitions

- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note 0 is an index) where $b_n = (-1)^n$.

  $\{b_n\} = 1, -1, 1, -1, \ldots$
- Note: $\{b_n\}$ denotes an infinite sequence of 1's and –1's, *not* the 2-element set $\{1, -1\}$.

Recognizing Sequences

- Sometimes, you' re given the first few terms and you are asked to find the sequence' s generating function?
- Examples: What' s the next number?
  - 1,3,5,7,9,… 11 (the 6th smallest odd number >0)
  - 2,3,5,7,11,... 13 (the 6th smallest prime number)
  - 5,11,17,23,29,... arithmetic progression $5+6n$
  - 1,7,25,79,241,... 3^n-2

Summations

- A *summation* is a notation for the sum of all terms in a (possibly infinite) sequence.

- Given $\{a_n\}$, we use the notation $\sum_{i=j}^{k} a_i$

upper bound

term

index

lower bound

$\sum$

to represent $a_j + a_{j+1} + \ldots + a_k$

$\prod_{i=j}^{k} a_i = a_j \times a_{j+1} \times \ldots a_k$

increasing the index up to k from j.

Summation Notation

$\{a_n\} = x^2 + 2$

$\sum_{i=1}^{5} a_i = 1+2+3+4+5$

$3+6+11+18+27$

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

- Here, *j* is an integer to denote *lower bound* (or *limit*),
- $k \geq j$ is an integer to denote *upper bound*.
- *i* is called the *index of summation*.

Simple Summation Example

(i^2+1) for 2 for 3 for 4

$$\sum_{i=2}^{4}(i^2+1) = (2^2+1)+(3^2+1)+(4^2+1)$$

$$= (4+1)+(9+1)+(16+1)$$

$$= 5+10+17$$

$$= 32$$

Generalized Summations

- For an infinite series, we may write:

$$\sum_{i=j}^{\infty} a_i :\equiv a_j + a_{j+1} + \ldots$$

- An infinite series with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \ldots = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots = 2$$

- What about $\sum_{i=0}^{n} 2^i$ $\xrightarrow[\text{Conv.}]{?}$ $2^{n+1} - 1$

$n = 2, 3$

$2^0 + 2^1 + 2^2 = 7$

$n = 3 : \Sigma = 15$

Generalized Summations

- To sum a function over all members of a set $X=\{x_1, x_2, \ldots\}$: $$\sum_{x\in X} f(x) :\equiv f(x_1) + f(x_2) + \ldots$$
- Or, if $X=\{x|P(x)\}$, we may just write:

$$\sum_{P(x)} f(x) :\equiv f(x_1) + f(x_2) + \ldots$$

E.g.: $$\sum_{(x\text{ is prime})\wedge x<10} x^2 = 2^2 + 3^2 + 5^2 + 7^2$$

Summation Manipulations

- Sometimes, it is useful to apply some arithmetic laws to manipulate summations.

  E.g. $$\sum_{i=1}^{n}(ax_i + by_i) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$$

  uses

$$\sum_x cf(x) = c\sum_x f(x)$$ (Distributive law.)

$$\sum_x \big(f(x) + g(x)\big) = \sum_x f(x) + \sum_x g(x)$$ (Commutativity.)

Summation Manipulations

- Index shifting: $$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$

  $f(i)$

  $i-n=j$

  $i=i-n$

  $i=j+n$

  E.g. $$\sum_{i=1}^{5} i^2 = \sum_{i=0}^{4} (i+1)^2$$

  $1 \to i+1=1$

- Order reversal: $$\sum_{i=j}^{k} f(i) = \sum_{i=0}^{k-j} f(k-i)$$
- Grouping: $$\sum_{i=1}^{2k} f(i) = \sum_{i=1}^{k} \big(f(2i) + f(2i-1)\big)$$

  look for → ?

Example: Geometric Progression

- A *geometric progression* is a series of the form $ar^0, ar^1, ar^2, ar^3, \ldots, ar^n$, where $a,r \in \mathbf{R}$.
- The sum of such a series is given by:

$$S = \sum_{i=0}^{n} ar^i$$

power series

- We can reduce this to *closed form* by clever manipulation of summations. Let's start...

Geometric Sum Derivation

$$\boxed{S=\sum_{i=0}^{n} ar^i}$$

$i=0:n$

$f \rightarrow ar^i$

$a\sum_{i=0}^{n} r^i = S$

$$rS = r\underbrace{\sum_{i=0}^{n} ar^i}_{=S} = \sum_{i=0}^{n} rar^i$$

(multiplying sum with r, then distributive property)

$1+i=i+1$

$$= \sum_{i=0}^{n} arr^i = \sum_{i=0}^{n} ar^1 r^i = \sum_{i=0}^{n} ar^{1+i}$$

(some organization)

$$= \sum_{i=1}^{n+1} ar^{1+(i-1)} = \boxed{\sum_{i=1}^{n+1} ar^i}$$

(index shifting, increase index by 1)

Derivation continued..

$$rS = \boxed{\sum_{i=1}^{n+1} ar^i} = \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1}$$

(remove term $i=n+1$ from summation)

It seems we trying to find the original boundries

$$= ar^0 + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - ar^0$$

(add and subtract $i=0$ term)

$$= \left(\sum_{i=0}^{0} ar^i\right) + \left(\sum_{i=1}^{n} ar^i\right) + ar^{n+1} - a$$

(some organization)

$a(\_\ \_\ \_)$

$$= \left(\sum_{i=0}^{n} ar^i\right) + a(r^{n+1}-1) = \boxed{S + a(r^{n+1}-1)} = r.S$$

Concluding derivation…

$$\boxed{rS = S + a(r^{n+1} - 1)}$$

$$rS - S = a(r^{n+1} - 1)$$

$$S(r-1) = a(r^{n+1} - 1)$$

$$\boxed{S = a\left(\frac{r^{n+1} - 1}{r - 1}\right)} \qquad \text{when } r \neq 1$$

When $r = 1$, $S = \sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} a1^i = \sum_{i=0}^{n} a \cdot 1 = (n+1)a$

Result in function notation

$$\sum_{i=0}^{n} ar^{i} = \begin{cases} a\left(\dfrac{r^{n+1}-1}{r-1}\right) & \text{if } r \neq 1 \\[2ex] (n+1)a & \text{if } r = 1 \end{cases}$$

!

Nested Summations

- These are multi-layered summations.
- Example for a double summation:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}\left(\sum_{j=1}^{3} ij\right) = \sum_{i=1}^{4} i\left(\sum_{j=1}^{3} j\right) = \sum_{i=1}^{4} i(1+2+3)$$

$$= \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6(1+2+3+4)$$

$$= 6 \cdot 10 = 60$$

Some Shortcut Expressions

$\sum_{k=0}^{n} ar^k = a(r^{n+1}-1)/(r-1), r \neq 1$	Geometric series.
$\sum_{k=1}^{n} k = n(n+1)/2$	Gauss' s method.
$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$	Quadratic series.
$\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$	Cubic series.

Using the Shortcuts

- Example: Evaluate $\sum_{k=50}^{100} k^2$.
  - Use series splitting.
  - Solve for desired summation.
  - Apply quadratic series rule.
  - Evaluate.

$$\sum_{k=1}^{100} k^2 = \left(\sum_{k=1}^{49} k^2\right) + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \left(\sum_{k=1}^{100} k^2\right) - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6}$$

$$= 338350 - 40425.$$

Summations: Conclusion

- You need to know:
  - How to read, write & evaluate summation expressions like:

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulation laws we covered.
  - Shortcut closed-form formulas, & how to use them.