Izmir Institute of Technology

CENG 115

Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6th Edition)**
by Kenneth H. Rosen

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for COT 3100 course in University of Florida

Module #2 – Part 1:
Rules of Inference

Rosen 6th ed., § 1.5

Proofs & Inference Rules (§ 1.5)

- In mathematics, *proofs* are *correct* (logically valid) and *complete* (clear, detailed) arguments that establish the truth of mathematical statements.
- Proof methods can be formalized in terms of *rules of logical inference*.
- We will review both correct reasonings and incorrect reasonings (fallacies) and also we will see several proof methods.

Inference Rules - Terminology

- An *argument* in propositional logic is a sequence of propositions.
- All but the final proposition in the argument are called *premises/hypotheses*.
- The final proposition is called *conclusion*.
- An argument is *valid* if the truth of all its premises implies that the conclusion is true.

Inference Rules - General Form

- *Inference Rule* –
  - Pattern establishing that if we know that a set of *hypotheses/premises* are all true, then the *conclusion* is true.
- *premise* 1
  <u>*premise* 2 ...</u>
  ∴ *conclusion*

"∴" means "therefore"

Examples of *Argument*

If it snows today, then we will go skiing.

<u>It is snowing now</u>.

∴ We will go skiing.

If $\sqrt{2} > 3/2$, then $(\sqrt{2})^2 > (3/2)^2$

$\underline{\sqrt{2} > 3/2}$

∴ $(\sqrt{2})^2 > (3/2)^2$

A valid but false argument

Rules of Inference

- $$\left.\begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}\right\}$$ Rule of *modus ponens*

  "the mode of affirming"

- $$\left.\begin{array}{c} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}\right\}$$ Rule of *modus tollens*

  "the mode of denying"

Rules of Inference

- $$\frac{p}{\therefore\ p \vee q}$$ Rule of Addition
- $$\frac{p \wedge q}{\therefore\ p}$$ Rule of Simplification
- $$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$ Rule of Conjunction

Examples

It is below freezing now.

∴ It is below freezing or raining now.

(Rule of addition)

If you have a password, you can log on to network.

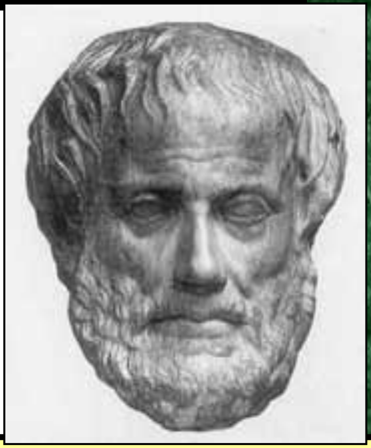
You have a password.

∴ You can log on to network.

(Modus ponens)

More Inference Rules

- $$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$ Rule of hypothetical syllogism
- $$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$ Rule of disjunctive syllogism
- $$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$ Resolution



Aristotle
(ca. 384-322 B.C.)

Inference Rules & Implications

- Each logical inference rule corresponds to an implication that is a tautology.
- Inference rule: Corresponding tautology:

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \land q \end{array} \quad \longrightarrow \quad [(p) \land (q)] \rightarrow (p \land q)$$

$$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore\ q \end{array} \quad \longrightarrow \quad [(p) \land (p \rightarrow q)] \rightarrow q$$

Building Arguments

- Given a number of premises $p_1, p_2, \ldots, p_n$ and applying some inference rules to premises yields a new valid statement (the conclusion).
- This demonstrates *if* the premises are true, *then* the conclusion is true.

Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will canoe."
  "If we canoe, then we will be home early."
- Given these premises, reach the conclusion **"We will be home early"** using inference rules.

Example *cont.*

- Let us adopt the following abbreviations:
  - *sunny* = **"It is sunny"**; *cold* = **"It is cold"**; *swim* = **"We will swim"**; *canoe* = **"We will canoe"**; *early* = **"We will be home early"**.
- Then, the premises can be written as:

(1) $\neg sunny \wedge cold$ (2) $swim \rightarrow sunny$
(3) $\neg swim \rightarrow canoe$ (4) $canoe \rightarrow early$

(1) $\neg sunny \land cold$

(2) $swim \rightarrow sunny$

(3) $\neg swim \rightarrow canoe$

(4) $canoe \rightarrow early$

Example *cont.*

Step	Proved by
1. $\neg sunny \wedge cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2,3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4,5.
7. $canoe \rightarrow early$	Premise #4.
8. $early$	Modus ponens on 6,7.

Inference Rules-Another Question

- Test the validity of:

  "For my husband's birthday, I bring him gifts."

  "Today is my husband's birthday or I work late in office."

  "I did not bring my husband gifts today."

  "Therefore, today I worked late."

Common Fallacies

- A *fallacy* is an inference rule or other proof method that lead to invalid argument.
  - May yield a false conclusion!
- Fallacy of *affirming the conclusion*:
  - "$p\rightarrow q$ is true, and q is true, so p must be true." (No, because $\mathbf{F}\rightarrow\mathbf{T}$ is true.)
- Fallacy of *denying the hypothesis*:
  - "$p\rightarrow q$ is true, and p is false, so q must be false." (No, again because $\mathbf{F}\rightarrow\mathbf{T}$ is true.)

Example of a Fallacy

- Premises
  - If you do every problem in this book, then you will learn discrete mathematics.
  - You learned discrete mathematics.
- Conclusion
  - Therefore, you did every problem in this book.

An Example from Lewis Carroll

Premises:

- All lions are fierce. $\forall x\ (\ P(x) \rightarrow Q(x)\ )$
- Some lions do not drink coffee. $\exists x\ (P(x) \wedge \neg R(x))$

Conclusion: Some fierce creatures do not drink coffee.

Solution:

$$\begin{array}{l} \forall x\ (\ P(x) \rightarrow Q(x)\ ) \\ \exists x\ (P(x) \wedge \neg R(x)) \\ \hline \therefore\ \exists x\ (Q(x) \wedge \neg R(x)) \end{array}$$

Inference Rules for Quantifiers

- $\dfrac{\forall x\, P(x)}{\therefore P(o)}$ Universal instantiation (any object *o* in U.D.)
- $\dfrac{P(o)}{\therefore \forall x\, P(x)}$ (for an arbitrary object *o* in U.D.) Universal generalization
- $\dfrac{\exists x\, P(x)}{\therefore P(o)}$ Existential instantiation (for some object *o*, not arbitrary)
- $\dfrac{P(o)}{\therefore \exists x\, P(x)}$ (for some object *o*) Existential generalization

Combining Rules of Inference and Quantified Expressions

- Modus ponens with universal instantiation

  $\forall x \, (P(x) \rightarrow Q(x))$

  $P(a)$, a is a particular object in the domain

  $\therefore Q(a)$

- Modus tollens with universal instantiation

  $\forall x \, (P(x) \rightarrow Q(x))$

  $\neg Q(a)$, a is a particular object in the domain

  $\therefore \neg P(a)$

Combining Rules of Inference and Quantified Expressions

- Premise: Everyone in this room has registered to CENG115 course.

  Premise: Ahmet is in this room.

  Conclusion: Ahmet has registered to CENG115.

- Solution: Let $R(x)$: x is in this room
  and $C(x)$: x has registered to CENG115.

  Premises: (1) $\forall x\ (\ R(x)\rightarrow C(x)\ )$ (2) $R(\text{Ahmet})$

  (3) Instantiation from (1): $R(\text{Ahmet})\rightarrow C(\text{Ahmet})$

  (4) Modus ponens from (2) and (3): $C(\text{Ahmet})$

Example

- Definitions:

  S :≡ Socrates (ancient Greek philosopher);
  $H(x)$:≡ "x is human";
  $M(x)$:≡ "x is mortal".

- Premises: $H(\mathrm{S})$ *Socrates is human.*

  $\forall x\, H(x) \rightarrow M(x)$ *All humans are mortal.*

- Some conclusions:

  $H(\mathrm{S}) \rightarrow M(\mathrm{S})$ *If Socrates is human then he is mortal.*
  $\neg H(\mathrm{S}) \vee M(\mathrm{S})$ *Socrates is inhuman or mortal.*

Another Example

- Definitions: $H(x) :\equiv$ "x is human"; $M(x) :\equiv$ "x is mortal"; $G(x) :\equiv$ "x is a god"
- Premises:
  - $\forall x\, H(x) \rightarrow M(x)$ ("Humans are mortal") and
  - $\forall x\, G(x) \rightarrow \neg M(x)$ ("Gods are immortal").
- Show that $\neg\exists x\, (H(x) \wedge G(x))$
  ("No human is a god.")

The Derivation

- $\forall x\ H(x) \rightarrow M(x)$ and $\forall x\ G(x) \rightarrow \neg M(x)$.
- $\forall x\ \neg M(x) \rightarrow \neg H(x)$ **[Contrapositive.]**
- $\forall x\ [G(x) \rightarrow \neg M(x)] \wedge [\neg M(x) \rightarrow \neg H(x)]$
- $\forall x\ G(x) \rightarrow \neg H(x)$ **[Transitivity of $\rightarrow$.]**
- $\forall x\ \neg G(x) \vee \neg H(x)$ **[Definition of $\rightarrow$.]**
- $\forall x\ \neg(G(x) \wedge H(x))$ **[DeMorgan's law.]**
- $\neg \exists x\ G(x) \wedge H(x)$ **[An equivalence law.]**

Home Exercise

- Premises:
  - If someone does not have eight legs, then he/she is not an insect.
  - George is an insect.
- Can you reach conclusion:

  George has eight legs.
- Please use predicate logic and universal instantiation to solve.

Solution

- Premises:

  (1) $\neg EL(x) \rightarrow \neg I(x)$

  If someone does not have eight legs, then he/she is not an insect.

  (2) $I(G)$ Universal instantiation for ‘George is an insect’.

- Solution:

  (3) $I(x) \rightarrow EL(x)$ Contrapositive of (1).

  (4) $I(G) \rightarrow EL(G)$ Instantiation of (3).

  (5) $EL(G)$ Modus ponens from (4) and (2): George has eight legs.

End of § 1.5

- We have seen:
  - Inference Rules
  - What is a Formal Proof
  - Combining Inference Rules and Quantified Expressions