Izmir Institute of Technology

# CENG 115
# Discrete Structures

**Slides are based on the Text**
***Discrete Mathematics & Its Applications*** **(6th Edition)**
**by Kenneth H. Rosen**

# Module #6
# **Cardinality & Infinite Sets**

Rosen 6th ed., § 2.4

# Cardinality

- In § 2.1, we defined *cardinality* of a finite set to be the number of elements in the set.
- Using what we learned about *functions* in § 2.3, it's possible to formally define cardinality for infinite sets !

# Equal Cardinalities

- For any two (possibly infinite) sets $A$ and $B$, we say that $A$ and $B$ *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (one-to-one correspondence) from $A$ to $B$.
- We will now split infinite sets into two groups:
  - Those with the same cardinality with the set of positive integers.
  - Those with different cardinality.

# Countable versus Uncountable

- If a set $S$ has same cardinality with the set of positive integers, i.e. $|S|=|\mathbf{Z}^+|$, we say $S$ is a *countable* set. Else, $S$ is *uncountable*.
- Intuition behind "**countable**": we can *enumerate* the elements of $S$ with a sequence where *any* individual element of $S$ will eventually be *counted*. Examples: **N**, **Z**.
- **Uncountable**: No sequence can cover all the elements of $S$. Its cardinality is different from $\mathbf{Z}^+$ Examples: **R**, $\mathbf{R}^2$, P(**N**)

# Countable Sets: Example

- **Theorem:** The set of odd positive integers is a countable set.
- To prove, you need to exhibit a function from $\mathbf{Z}^+$ to the set of odd positive integers and show that it is a bijective function.
  - The generating function $f(n)=2n-1$, generates the set of odd positive integers from $\mathbf{Z}^+$ $(1,2,3,4,5,..) \rightarrow (1,3,5,7,9,..)$
  - $f$ is one-to-one and onto. So it is bijective.

# Countable Sets: Example 2

- **Theorem:** The set **Z** is countable.

  **Proof:** Consider $f : \mathbf{Z}^+ \rightarrow \mathbf{Z}$ where

  $f(n) = n/2$ when $n$ is even

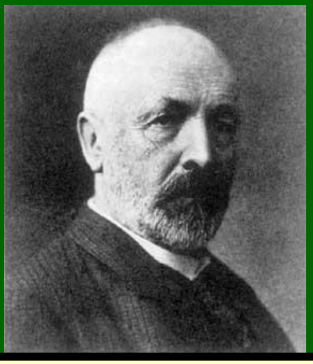
  $f(n) = -(n-1)/2$ when $n$ is odd

  It generates $(1,2,3,4,5,..) \rightarrow (0,1,-1,2,-2,..)$

  Is this function bijective?

  Yes.

# Uncountable Sets: Example



Georg Cantor
1845-1918

**Theorem:** The open interval
$[0,1) :\equiv \{r \in \mathbf{R} | \; 0 \leq r < 1\}$ is uncountable.

**Proof** by *diagonalization*: (Cantor, 1891)
(based on proof by contradiction)

- Assume there is a series $\{r_i\} = r_1, r_2, \ldots$ containing *all* elements $r \in [0,1)$.
- Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index. *(continued on next slide)*

# Uncountability of Real Numbers

A decimal representation of real numbers:

$$r_1 = 0.d_{1,1}\, d_{1,2}\, d_{1,3}\, d_{1,4}\, d_{1,5}\, d_{1,6}\, d_{1,7}\, d_{1,8} \ldots$$
$$r_2 = 0.d_{2,1}\, d_{2,2}\, d_{2,3}\, d_{2,4}\, d_{2,5}\, d_{2,6}\, d_{2,7}\, d_{2,8} \ldots$$
$$r_3 = 0.d_{3,1}\, d_{3,2}\, d_{3,3}\, d_{3,4}\, d_{3,5}\, d_{3,6}\, d_{3,7}\, d_{3,8} \ldots$$
$$r_4 = 0.d_{4,1}\, d_{4,2}\, d_{4,3}\, d_{4,4}\, d_{4,5}\, d_{4,6}\, d_{4,7}\, d_{4,8} \ldots$$

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· E.g. $r_1 = 0.237688...$

Now, consider a real number generated by taking all digits $d_{i,i}$ and replacing them with *different* digits. That real number does not exist in the list!

# Uncountability of Real Numbers

- Assumption is that a series generates the list of reals:
  $r_1 = 0.301948571\ldots$
  $r_2 = 0.103918481\ldots$
  $r_3 = 0.039194193\ldots$
  $r_4 = 0.918237461\ldots$
- Now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103… can't be on the list anywhere! It is a contradiction! Even such a series exists there still be a real number that is not in the list.

# Transfinite Numbers

- We saw that both $\mathbf{Z}^+$ and $\mathbf{R}$ are infinite, but $\mathbf{R}$ is an uncountable set and $|\mathbf{R}|>|\mathbf{Z}^+|$.
- Cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of positive integers, $\aleph_0 :\equiv |\mathbf{Z}^+|$, is the *first transfinite cardinal* number.
- $|\mathbf{R}|=\aleph_1$, the *second transfinite cardinal*.

Source: http://en.wikipedia.org/wiki/Transfinite_number

# Countable vs. Uncountable

- You should:
  - Know how to define "same cardinality" in the case of infinite sets.
  - Know the definitions of *countable* and *uncountable*.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.