CENG 115 Fall 2023 Homework 2

Note: You are expected to bring your handwritten answers to the class (your own section) on 28th December. Assignments submitted afterwards will not be evaluated.

1. **(25 points)**
Use mathematical induction to prove that
$$1-2+2^2-2^3+\ldots+(-1)^n 2^n = \frac{2^{n+1}(-1)^n+1}{3}$$
for all positive integers n.
Hint: Do not forget to define and prove basis step and inductive step separately.

Answer:
Basis step:
$$P(1)\text{: } 1-2=\frac{2^2(-1)+1}{3}$$
Inductive step:
$$1-2+2^2+\ldots+(-1)^{k+1}2^{k+1}=\frac{2^{k+1}(-1)^k+1}{3}+(-1)^{k+1}2^{k+1}=\frac{2^{k+1}(-1)^k+1+3(-1)^{k+1}2^{k+1}}{3}$$
$$=\frac{2^{k+1}(-1)^k(1+3(-1))+1}{3}=\frac{2^{k+1}(-1)^k(-2)+1}{3}=\frac{2^{k+2}(-1)^{k+1}+1}{3}.$$

2. **(20 points)**
Each machine part made in a factory is stamped with a code of the form “letter-digit-digit-digit”, where the digits can be repeated. For instance, W065. At least how many parts should be produced to make sure that at least four of them have the same code stamped on them?

Answer:
There are 26·10·10·10=26000 different starting pairs. So, given 78000 parts none of the codes may be stamped on four parts. But with 78001 parts, by the generalized pigeonhole principle, at least $\left\lceil \frac{78001}{26000} \right\rceil = 4$ parts should have the same code stamped on them.

3. **(15 points)** How many different strings can be made by reordering the letters of CORRECT?

Answer:
There would be 7! strings if all letters were different. Since two C’s can be interchanged, half of those strings are same as the other half. Also two R’s can be interchanged, half of the remaining strings are also same. Therefore, there are $7!/(2!\,2!) = 1260$ different such strings.

4. **(15 points)** How many different solutions are there to the equation:

$$x_1+x_2+x_3+x_4=17 \qquad \text{where } x_i \geq 1$$

Answer: 4 of the 17 units reserved for all $x_i \geq 1$. Then C(13+4-1,4-1)=C(16,3) = 560

5. **(25 points)** Solve the following recurrence relation:

$$a_n = 10a_{n-1} - 25a_{n-2} \text{ where } a_0=3,\quad a_1=4.$$

Note: Here, solving means finding a closed-form (non-recursive) equation for a_n .

Answer:
$r^2 - 10r + 25 = 0$
The left side factors as $(r - 5)(r - 5)$, yielding the root 5 twice. The general solution to the given recurrence relation is
$a_n = c5^n + dn5^n$
Using the initial conditions $a_0 = 3$ and $a_1 = 4$ yields the system of equations
$c = 3$
$5c+5d = 4$ -> $d = -11/5$
So, $c = 3$, $d = -11/5$.
The answer is $a_n = 3 \cdot 5^n - (11/5) \cdot n \cdot 5^n$