Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
***Discrete Mathematics & Its Applications* (6th Edition)**
by Kenneth H. Rosen

Module #8:

The Growth of Functions

Rosen 6^{th} ed., § 3.2

The Growth of Functions (§ 3.2)

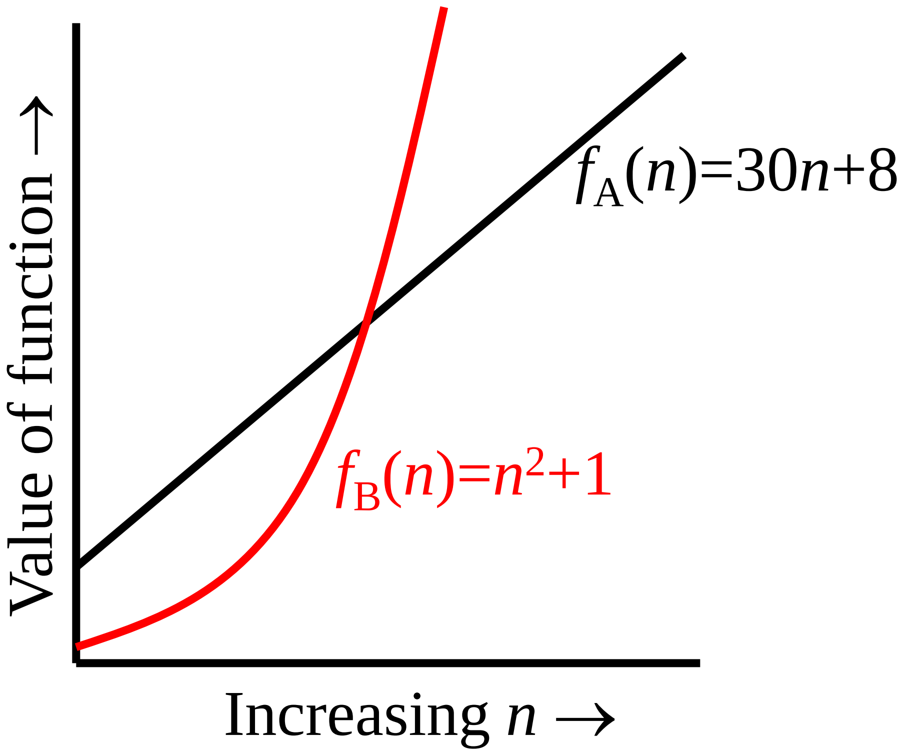
- For functions over numbers, we often need to know a rough measure of *how fast a function grows*.
- If $f(x)$ is *faster growing* than $g(x)$, then $f(x)$ always eventually becomes larger than $g(x)$ *in the limit* (for large enough values of x).
- Useful in engineering for showing that one design *scales* better or worse than another.

Motivation

- Suppose you are designing a web site to process user data (*e.g.*, financial records).
- Suppose database program A takes $f_A(n)=30n+8$ microseconds to process any n records, while program B takes $f_B(n)=n^2+1$ microseconds to process n records.
- Which program do you choose, knowing you'll want to support many users? A

Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



Concept of order of growth

- We say $f_A(n)=30n+8$ is *order n*, or $O(n)$. It is, at most, roughly *proportional* to n.
- $f_B(n)=n^2+1$ is *order* n^2, or $O(n^2)$. It is roughly proportional to n^2.
- Any $O(n^2)$ function is faster-growing than any $O(n)$ function.
- For large numbers of user records, the $O(n^2)$ function will always take more time.

Definition: O(g), *at most order g*

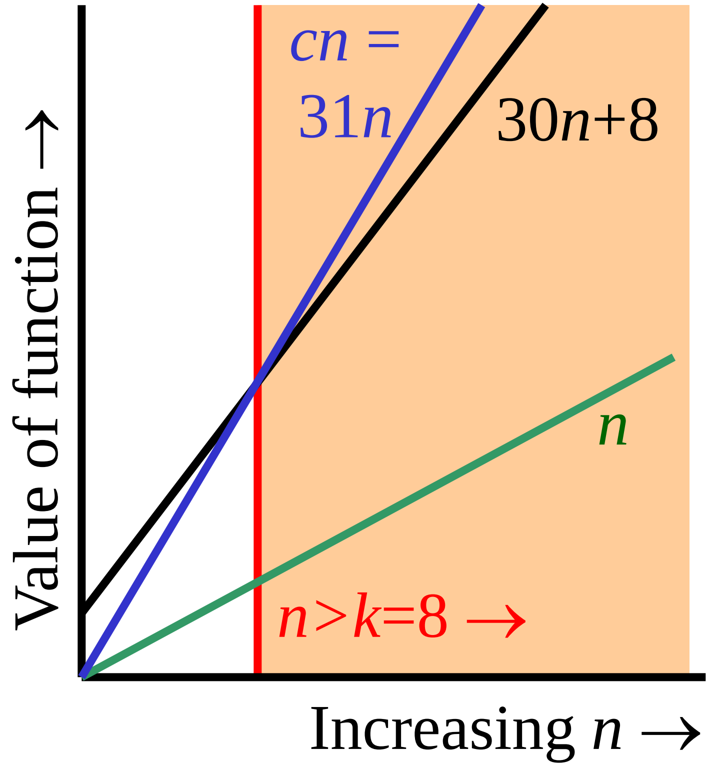
- Let g be any function $\mathbf{R}\rightarrow\mathbf{R}$.
- We say that $f(x)$ is $O(g(x))$ if $\exists c,k$: $\forall x>k$: $|f(x)| \le c|g(x)|$
  - When x is greater than k, function f is at most a constant c times g (*i.e.*, proportional to g).
- "f is *at most order g*", or "f is big-O of g"
- Sometimes f=O(g) or $f\in$O(g) are also used.

"Big-O" Proof Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c,k: \forall n>k: 30n+8 \leq cn$.
    - Let $c=31$, $k=8$. So, when we consider $n>k=8$, $cn = 31n = 30n + n > 30n+8$, that is $30n+8 < cn$.
- Show that n^2+1 is $O(n^2)$.
  - Show $\exists c,k: \forall n>k: n^2+1 \leq cn^2$.
    - Let $c=2$, $k=1$. For $n>k=1$, $cn^2 = 2n^2 = n^2+n^2 > n^2+1$, or $n^2+1< cn^2$.

Big-O example, graphically

- $30n+8$ is *nowhere* less than n.
- $30n+8$ is not less than $31n$ *everywhere*.
- But it *is* less than $31n$ <u>everywhere to the right of $n=8$</u>.



$30n+8$ is $O(n)$

- This (c,k) pair are called witnesses.

Points about Big-O proof

- Note that f is $O(g)$ as long as a pair of c and k exists that satisfy the definition.
- Important: The particular c, k, values that make the statement true are *not* unique: **Any larger value of *c* and/or *k* will also work.**
- You are **not** required to find the smallest c and k values that work. But you should **prove** that the values you choose do work.
- **Class exercise**: Show that $7x^2$ is $O(x^3)$.

Useful Facts about Big-O

- Big-O, as a relation, is transitive:
  $f \in O(g) \wedge g \in O(h) \rightarrow f \in O(h)$
- When a polynomial function is used, the leading term (n) dominates its growth:
  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$,
  then $f \in O(x^n)$.
- Sums of functions:
  If $g \in O(f)$ and $h \in O(f)$, then $g+h \in O(f)$.

More Big-O facts

- Operations with a constant:
  $\forall a>0$, $O(af) = O(f+a) = O(f-a) = O(f)$
- Combination of functions:
  $f_1 \in O(g_1) \wedge f_2 \in O(g_2) \rightarrow$
  1) $f_1 f_2 \in O(g_1 g_2)$
  2) $f_1+f_2 \in O(g_1+g_2)$
     $= O(\max(g_1,g_2))$
     $= O(g_1)$ if $g_2 \in O(g_1)$

Big-O example

- Give smallest big-O estimate for

$$(n\log n + n^2)(n^3 + 2)$$

- Solution: $(n\log n + n^2)(n^3 + 2) =$

$$(O(n\log n) + O(n^2)) \cdot O(n^3) =$$

$$O(n^2) \cdot O(n^3) =$$

$$O(n^5)$$

Ordering of Functions

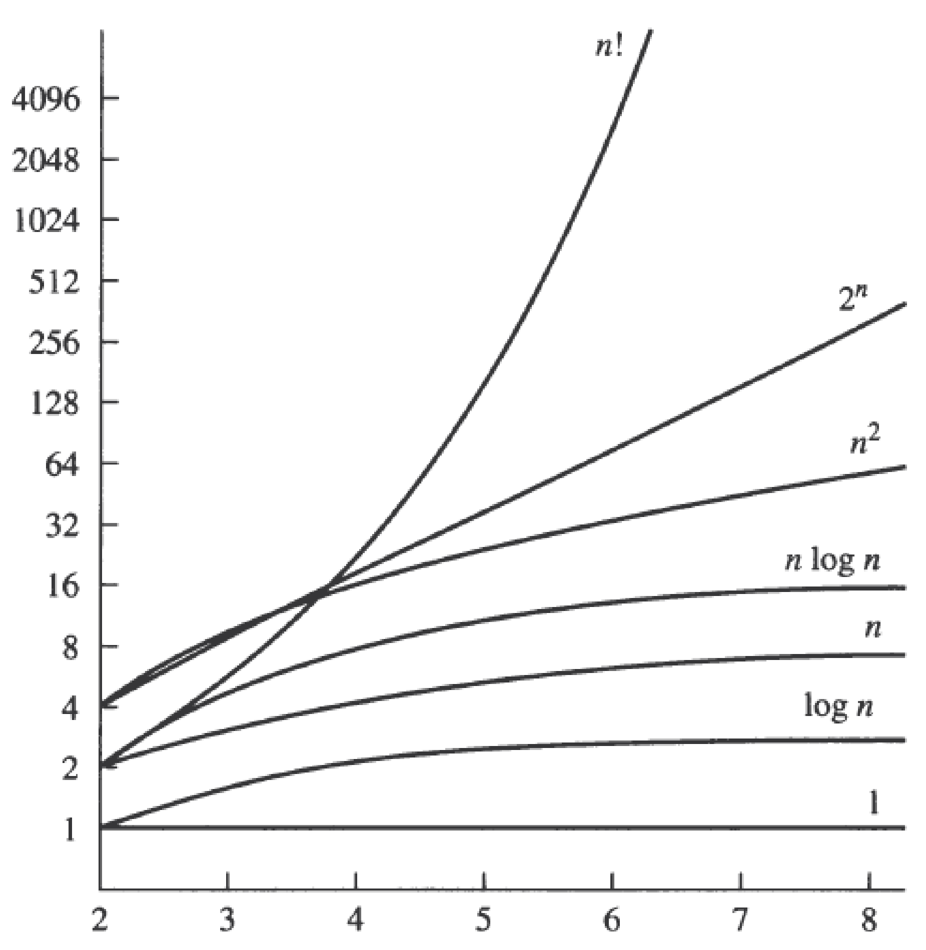
- Let's write $f<g$ for

  g is higher order than f

  and let $k>1$ then:

  $1 < \log n < n < n \log n$

  $< n^k < k^n < n! < n^n$



Definition: $\Omega(g)$, *at least order g*

- Let g be any function $\mathbf{R}\rightarrow\mathbf{R}$.
- We say that $f(x)$ is $\Omega(g(x))$ if $\exists c,k\colon \forall x>k\colon |f(x)| \geq c|g(x)|$
  - When x is greater than k, function f is at least a constant c times g (*i.e.*, proportional to g).
- This serves as a lower bound for $f(x)$.
- "*f* is *at least order g*", or "*f* is big-Omega of *g*"

Big-Omega Example

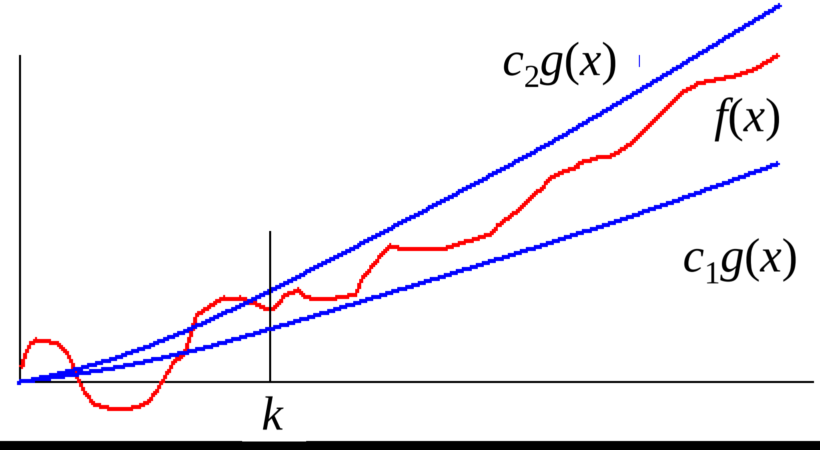
- $f(x)=8x^3+5x^2+7$. $g(x)=x^3$.
- $f(x)$ is $\Omega(g(x))$.
- Can you give a (c,k) pair to satisfy the definition of Ω?
- Also note that $g(x)$ is $O(f(x))$.

Definition: $\Theta(g)$, *exactly order g*

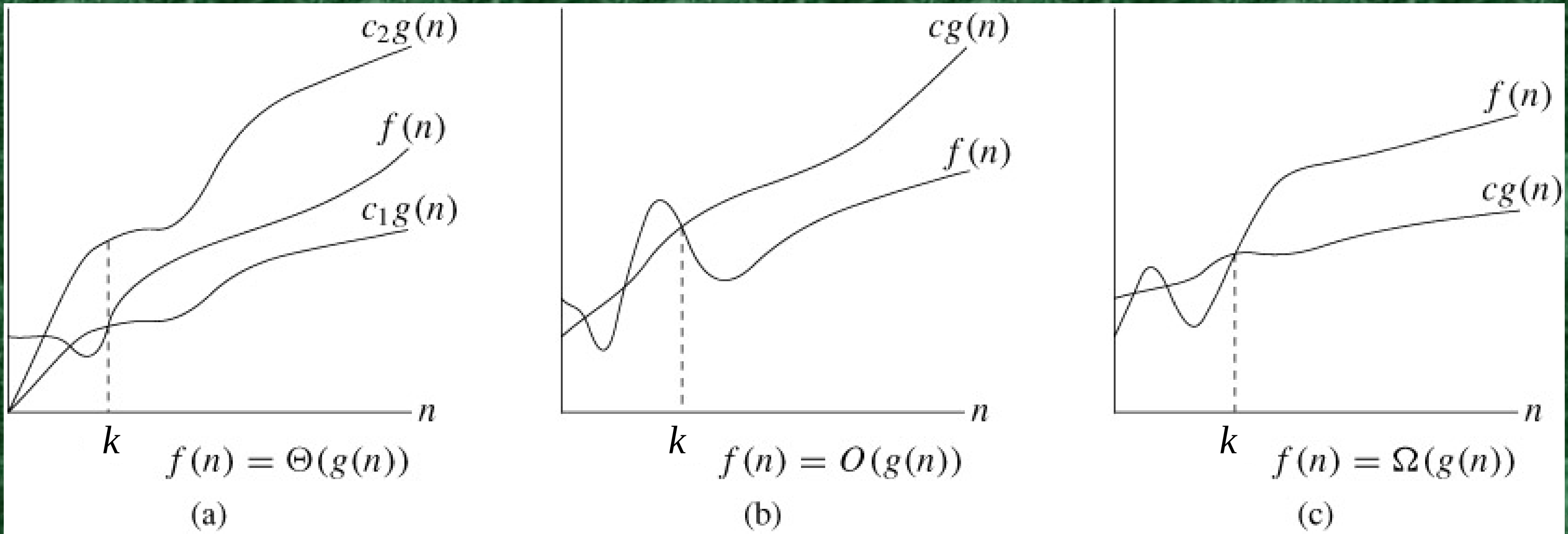
- If $f \in O(g)$ and $g \in O(f)$ then g and f are of the same order, we say "f is *order of* g" or "f is big-Theta of g" and write f is $\Theta(g)$.
- Another equivalent definition:
  $f \in \Theta(g)$ if
  $\exists c_1, c_2, k \;\; \forall x > k: \;\; |c_1 g(x)| \le |f(x)| \le |c_2 g(x)|$
- E.g. $3x^2 + 8x \log x$ is $\Theta(x^2)$
  Can you give an example of (c_1, c_2, k)?

Definition: $\Theta(g)$, *exactly order g*

- A function $f(x)$ is $\Theta(g(x))$ if there exist positive constants c_1 and c_2 such that $f(x)$ can be *sandwiched* between $c_1 g(x)$ and $c_2 g(x)$, for points beyond k.



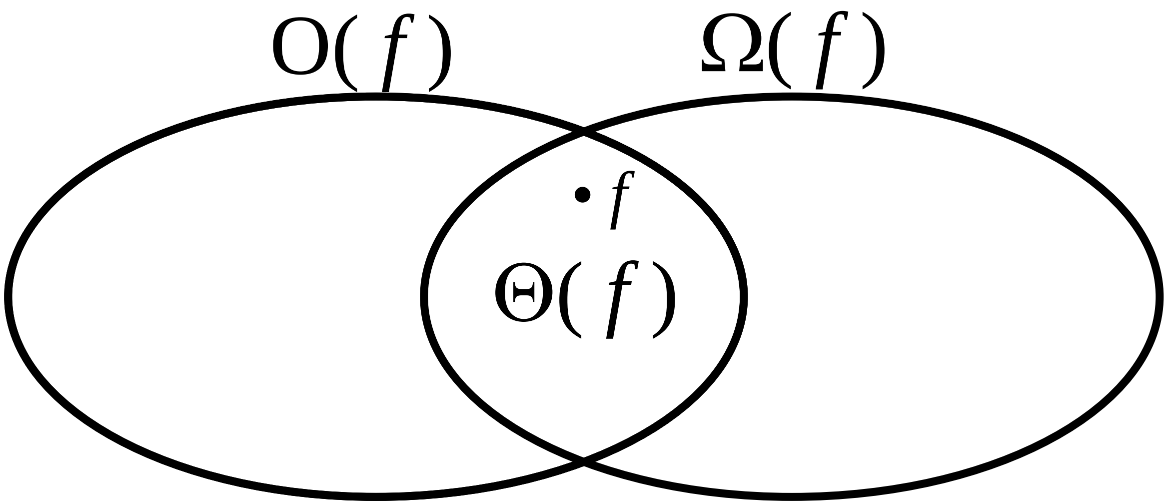
$\Theta(g)$, $O(g)$, and $\Omega(g)$



$$f(n) \text{ is } \Theta(g(n)) \Leftrightarrow (f(n) \text{ is } O(g(n))) \wedge (f(n) \text{ is } \Omega(g(n)))$$

Relations Between Θ, O, and Ω

- Subset relations between order-of-growth sets.



- Big-Omega and big-Theta notations are introduced by Donald Knuth (1938-...)

Θ example

- Determine whether: $\left(\sum_{i=1}^{n} i\right) \overset{?}{\in} \Theta(n^2)$
- Quick solution:

$$\left(\sum_{i=1}^{n} i\right) = n(n+1)/2$$

$$= n\,\Theta(n)/2$$

$$= n\,\Theta(n)$$

$$= \Theta(n^2)$$

Θ example

- Home exercise: Which one is correct?

a) $\left(\sum_{i=1}^{n}(2i-1)^2\right) \in \Theta(n^2)$

b) $\left(\sum_{i=1}^{n}(2i-1)^2\right) \in \Theta(n^3)$

c) $\left(\sum_{i=1}^{n}(2i-1)^2\right) \in \Theta(n^4)$