Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
***Discrete Mathematics & Its Applications* (6th Edition)**
by Kenneth H. Rosen

Slides were prepared by Dr. Michael P. Frank
for COT 3100 course in University of Florida

Relations

Rosen 6th ed., Chapter 8

Binary Relations

- Let A, B be any sets. A binary relation R from A to B, is a subset of $A \times B$.
  - *E.g.* $<$ can be seen as $\{(n,m) \mid n < m\}$
- $(a,b) \in R$ means that a is related to b (by R)
- Also written as $a \, R \, b$
  - *E.g.* $a < b$ mean $(a,b) \in <$

Relations on a Set

- A (binary) relation from a set A to itself is called a *relation on the set A*.
- *E.g.*, the **“<”** relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- Let $A=\{1,2,3,4\}$. Which ordered pairs are in relation $R=\{(a,b) \mid a \text{ divides } b\}$?
- Relations can be represented as sets of pairs. E.g. $A=\{0,1,2\}$ $B=\{x,y\}$ $R=\{(0,x),(0,y),(2,x)\}$ Then, $0\ R\ x$ and $0\ R\ y$ and $2\ R\ x$

Reflexivity

- A relation R on A is *reflexive* if $\forall a\ (a,a)\in R$.
  - *E.g.* the relation $\geq\ :\equiv \{(a,b) \mid a\geq b\}$ is reflexive.
  - Other examples of reflexive relations:

    =, 'have same cardinality', $\leq$, $\geq$, $\subseteq$

Example

E.g. Consider the following relations on {1, 2, 3, 4}

R1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R2 = {(1,1), (1,2), (2,1)}

R3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}

R4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

R5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

R6 = {(3,4)}

Which of these relations are reflexive? R3, R5

Symmetry

- A binary relation R on A is *symmetric* if $\forall a,b((a,b)\in R \leftrightarrow (b,a)\in R)$.
  - *Eg*. = (equality) is symmetric. < is not. ≤ is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *asymmetric* if $\forall a,b((a,b)\in R \rightarrow (b,a)\notin R)$.
  - *Eg*. < is asymmetric, "likes" is not.
  - ≤ is not asymmetric, since it is true for (a,a)

Example

- Example: Which of the following relations are symmetric?

R1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R2 = {(1,1), (1,2), (2,1)}

R3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}

R4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

R2 and R3

Example

- Consider the relation $x \leq y$
- Is it symmetrical? No.
- Is it asymmetrical? No.
- Is it reflexive? Yes.

Antisymmetry

- Consider the relation $x \leq y$
  - It is not symmetric. (For instance, $5 \leq 6$ but not $6 \leq 5$)
  - It is not asymmetric. (For instance, $5 \leq 5$)
  - (You *might* say it's *nearly* asymmetric, since the only symmetries occur when $x=y$)
- This is called antisymmetry: the only symmetrical pairs $(x,y),(y,x)$ in the relation are ones where $x=y$.
- A binary relation R on A is *antisymmetric* iff $\forall a,b((a,b) \in R \wedge (b,a) \in R) \rightarrow a=b)$. *E.g.* $\leq$, $\geq$, $\subseteq$

Transitivity

- A relation R is *transitive* if (for all a,b,c)

$$((a,b)\in R \wedge (b,c)\in R) \rightarrow (a,c)\in R.$$

- Which of the relations in example are transitive?

R1 = {(1,1), (1,2), (2,1)}

R2 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}

R3 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

R4 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

R5 = {(3,4)}   R3,R4 and R5

§ 8.3: Representing Relations

- So far, we have seen some ways to represent n-ary relations such as ordered pairs (pairs etc.)
- Two other methods to represent binary relations:
  - Zero-one matrices.
  - Directed graphs.
- An important reason to choose among these is some calculations are easier with one of these representations.

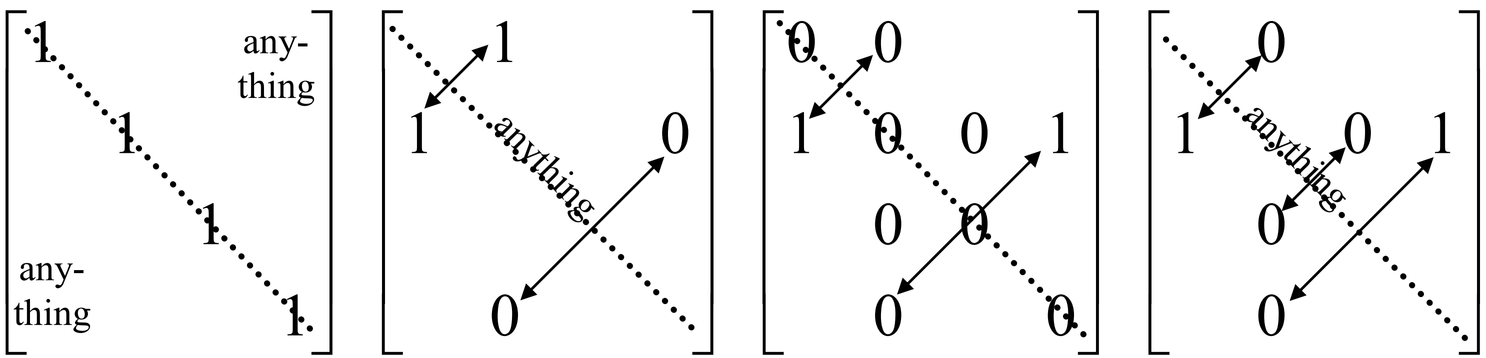
Using Zero-One Matrices

- To represent a binary relation $R{:}A\times B$ by an $|A|\times|B|$ 0-1 matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i,b_j)\in R$.
- *E.g.*, Let A={Joe,Fred,Mark} and B={Susan,Mary,Sally}. Suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- Then the 0-1 matrix representation of the relation Likes:$A\times B$ is:

$$\begin{array}{r c} & \begin{array}{ccc}\text{Susan} & \text{Mary} & \text{Sally}\end{array} \\ \begin{array}{r}\text{Joe}\\ \text{Fred}\\ \text{Mark}\end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{array}$$

Zero-One Reflexive, Symmetric

- Recall: *Reflexive*, *symmetric*, *asymmetric,* and *anti-symmetric* relations.



Reflexive: only 1's on diagonal	*Symmetric*: all identical across diagonal	*Asymmetric*: 0's on diagonal + 1's are 0's across diagonal	*Anti-symmetric*: all 1's are 0's across diagonal

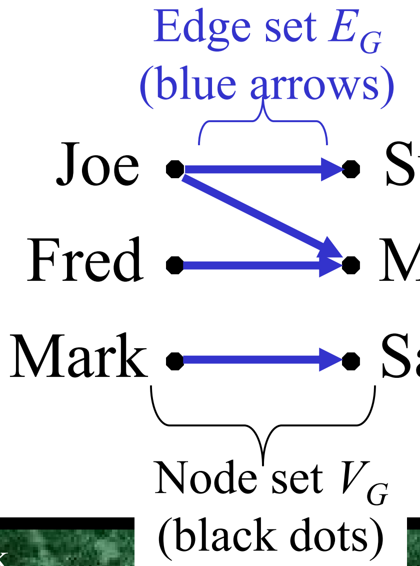
Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set E_G of *edges (arcs)*. Represented using dots for nodes, and arrows for edges. $R:A\times B$ can be represented as a graph $G_R=(V_G=A\cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

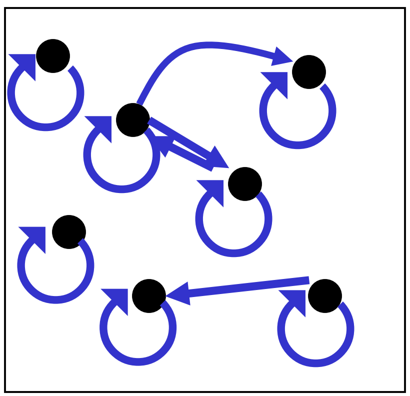
$$\begin{array}{c c} & \begin{array}{ccc} \text{Susan} & \text{Mary} & \text{Sally} \end{array} \\ \begin{array}{c} \text{Joe} \\ \text{Fred} \\ \text{Mark} \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Graph rep. G_R:

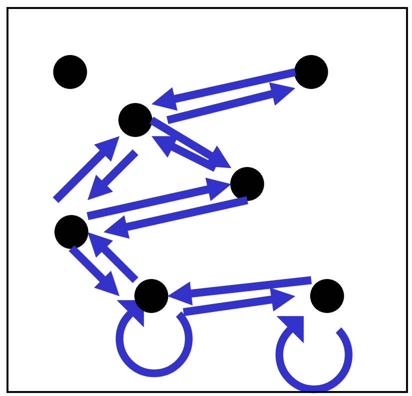


Digraph Reflexive, Symmetric

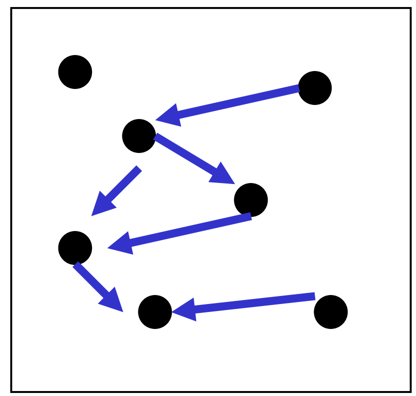
- Many properties of a relation are easily determined by inspection of its graph.



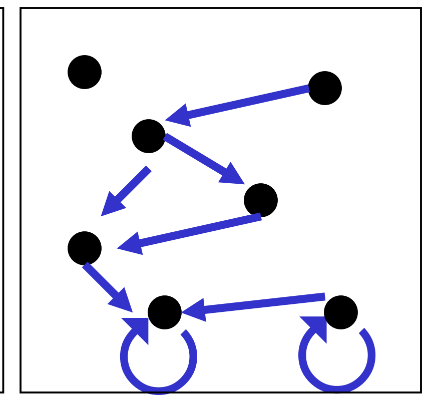
Reflexive: Every node has a self-loop



Symmetric: Every link is bidirectional



Asymmetric: No link is bidirectional, no self-loop



Anti-symmetric: No link is bidirectional

§ 8.4: Closures of Relations

- For any property X, the "X closure" of a set A is defined as the "smallest superset" of A that has property X. More specifically,
  - The *reflexive closure* of a relation R on A is the smallest superset of R that is reflexive.
  - The *symmetric closure* of R is the smallest superset of R that is symmetric.
  - The *transitive closure* of R is the smallest superset of R that is transitive.

Reflexive Closure

- Let R be a relation on the set A. The reflexive closure of R is $S = R \cup \{(a, a), \forall a \in \mathrm{A}\}$.
- Example:

  Let $R = \{(a,b)\ (a,c)\ (b,d)\ (d,e)\ (a,a)\}$,

  then the reflexive closure of R is

$\{(a,b)\ (a,c)\ (b,d)\ (d,e)\ (a,a)\ (b,b)\ (c,c)\ (d,d)\ (e,e)\}$

Symmetric Closure

- Let R be a relation on the set A. The symmetric closure of R is $S = R \cup R^{-1}$
- Example:

  Let $R = \{(a,b)\ (a,c)\ (b,d)\ (d,e)\}$,

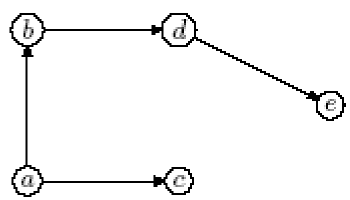
  then the symmetric closure of R is

$\{(a,b)\ (a,c)\ (b,d)\ (d,e)\ (b,a)\ (c,a)\ (d,b)\ (e,d)\}$

Transitive Closure

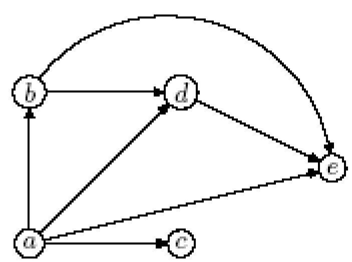
- The *transitive closure* of R (notation: R^*) is obtained by "*repeatedly*" adding (a,c) to R for each (a,b),(b,c) in R.
- *E.g:* $R=\{(a,b)\ (a,c)\ (b,d)\ (d,e)\}$

	a	b	c	d	e
a	0	1	1	0	0
b	0	0	0	1	0
c	0	0	0	0	0
d	0	0	0	0	1
e	0	0	0	0	0



- $R^*=\{(a,b)\ (a,c)\ (b,d)\ (d,e)\ (a,d)\ (b,e)\ (a,e)\ \}$

	a	b	c	d	e
a	0	1	1	1	1
b	0	0	0	1	1
c	0	0	0	0	0
d	0	0	0	0	1
e	0	0	0	0	0



Warshall's Algorithm

- Fast algorithms are available for calculating R^* , especially *Warshall's algorithm*. This algorithm uses a matrix representation.

- **Procedure** *Warshall* ($\mathbf{M}_R$: n x n 0-1 matrix)
  $\mathbf{W} := \mathbf{M}_R$
  for $k := 1$ **to** n
      for $i := 1$ **to** n
          for $j := 1$ **to** n
              $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
  return W {this represents R^*}

note: $w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$

§ 8.6: Partial Orderings

- A relation R on A is called a *partial ordering* or *partial order* iff it is reflexive, antisymmetric, and transitive.
  - We often use a symbol looking something like $\preccurlyeq$ (or analogous shapes) for such relations.
- A set A together with a partial order $\preccurlyeq$ on A is called a *partially ordered set* or *poset* and is denoted $(A, \preccurlyeq)$.

Partial Orderings

- Examples:
  - The relation $\geq$

    is a partial ordering on the set of integers
  - The divisibility relation $|$

    is a partial ordering on the set of positive integers
  - The subset relation $\subseteq$

    is a partial ordering on the power set of a set S.

Total Orderings

- The elements a and b of a poset $(S, \preccurlyeq)$ are called comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are called incomparable.
- Example:

  In the poset $(Z^+, |)$

  - 3 and 9 are comparable? Yes, because $3 \mid 9$.
  - 5 and 7 are comparable? No, because $5 \nmid 7$ and $7 \nmid 5$.

Total Orderings

- If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable
  - S is called a *totally ordered set*
  - $\preccurlyeq$ is called a total order
- Example
  - $(Z, \leq)$ is totally ordered
  - $(Z^+, |)$ is not totally ordered

Hasse Diagram

To construct a Hasse Diagram of a partial ordering:

1) Construct a digraph representation of the poset (A, R) so that all arcs point up (except the loops).

2) Eliminate all loops since we know that the poset is reflexive.

3) Eliminate all arcs that are redundant because of transitivity. We know that the poset is transitive.

4) Eliminate the arrows at the ends of arcs. Since every edge points upward, we do not show directions.

Hasse Diagrams (1)

- We are able to simplify the diagram because we know that the relation is a partial order, so the ‘missing’ information can be inferred.

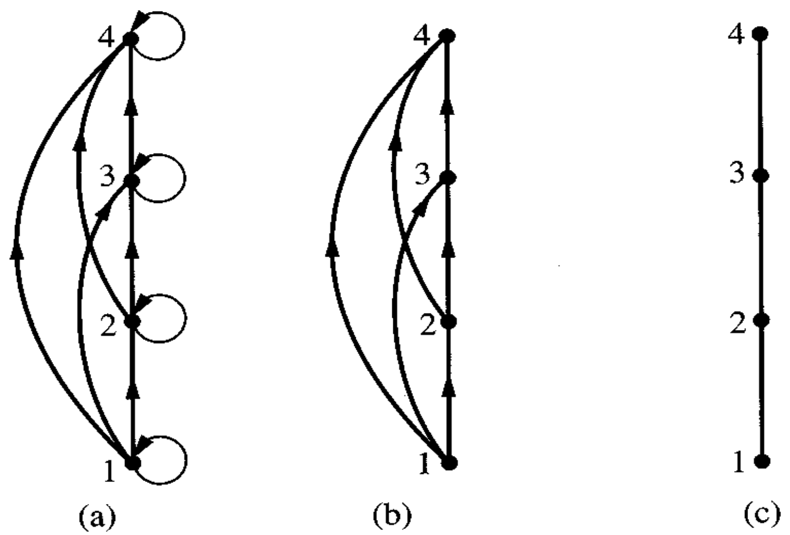


FIGURE 2 Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤).

Hasse Diagrams (2)

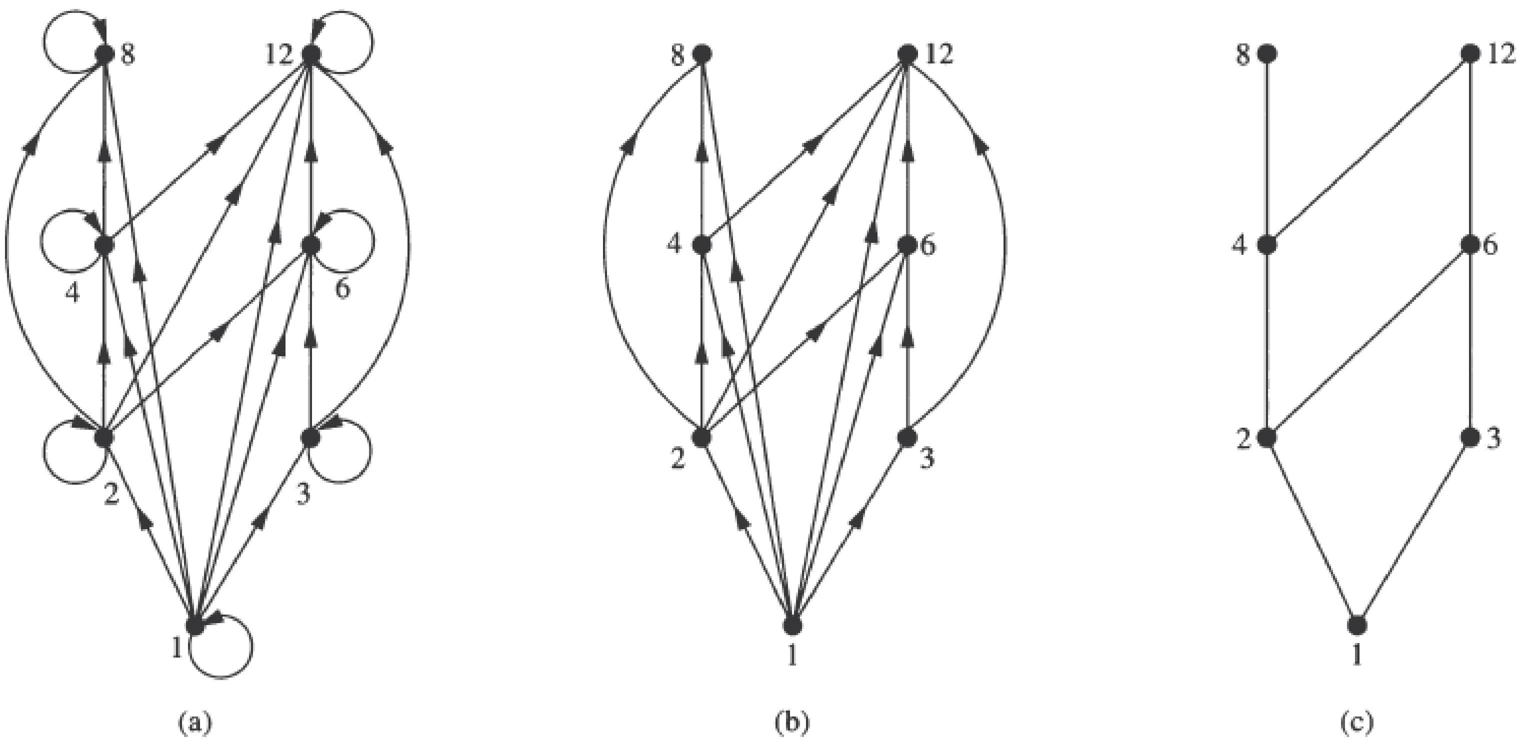


FIGURE 3 **Constructing the Hasse Diagram of ($\{1, 2, 3, 4, 6, 8, 12\}$, $|$).**

Hasse Diagrams (3)

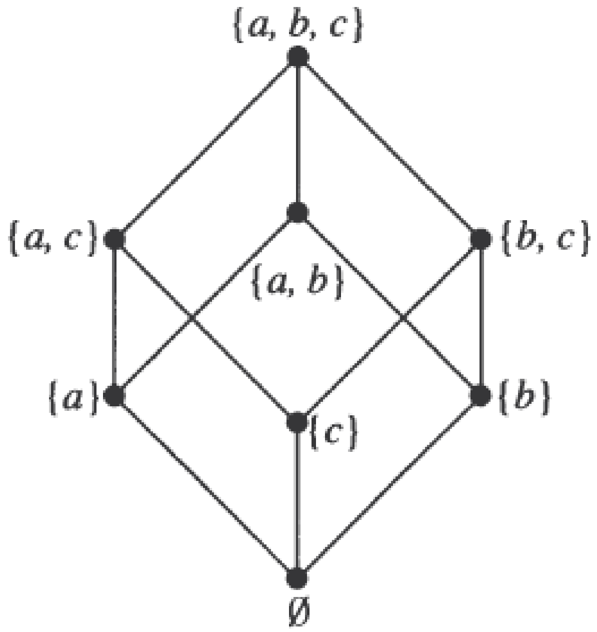


FIGURE 4 The Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$.