Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6th Edition)**
by Kenneth H. Rosen

Slides were prepared by Dr. Michael P. Frank
for COT 3100 course in University of Florida

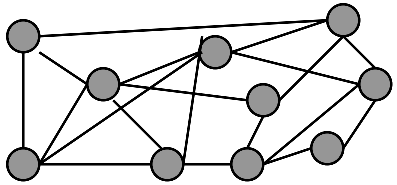
Graphs

Rosen 6th ed., Chapter 9

~32 slides, ~1 lecture

Graphs

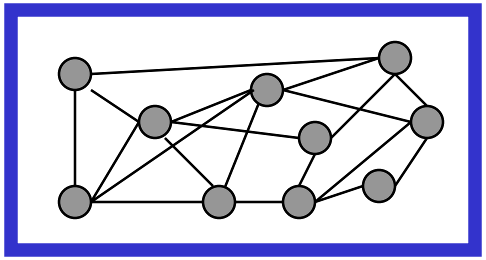
- *A particular class of discrete structures that is useful for representing relations and has a convenient webby-looking graphical representation.*



- The graphs can be used to examine finite structures and to analyze relationships and applications in many different settings.

Simple Graphs

- Correspond to symmetric binary relations R.
- A *simple graph* $G=(V,E)$ consists of:
  - a set V of *vertices* or *nodes* (V corresponds to the universe of the relation R),
  - a set E of *edges* / *arcs* / *links*: unordered pairs of elements $u,v \in V$, such that uRv.

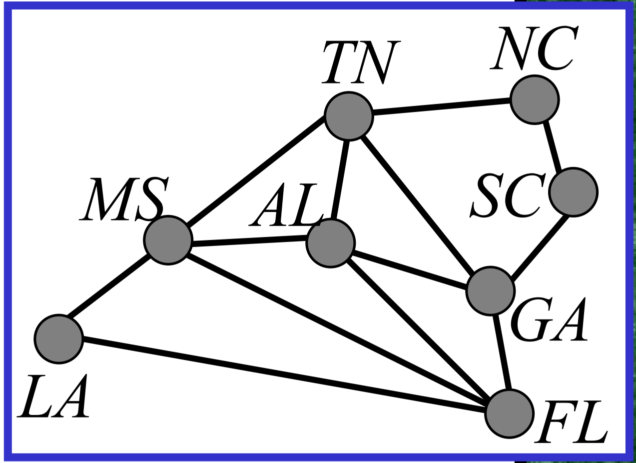


Visual Representation of a Simple Graph

Example of a *Simple Graph*

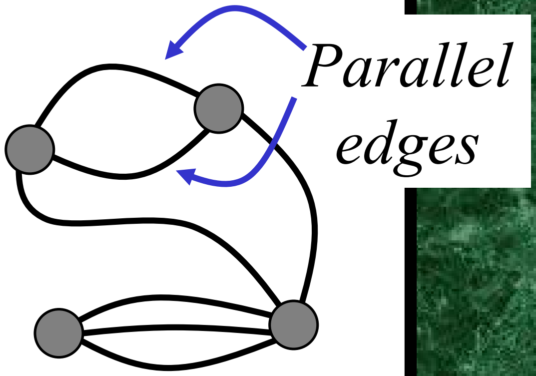
- Let V be some states in the U.S.:

  V={FL, GA, AL, MS, LA, SC, TN, NC}
- Let E={{FL,GA},{FL,AL},{FL,MS},{FL,LA}, {GA,AL},{AL,MS},{MS,LA}, {GA,SC},{GA,TN},{SC,NC}, {NC,TN},{MS,TN},{MS,AL}}
- In a simple graph, no two edges connect the same pair of vertices.



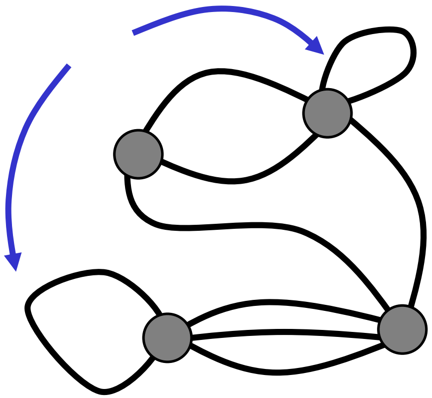
Multigraphs

- Like simple graphs, but there may be *more than one* edge connecting two given nodes.
- A *multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges (as primitive objects), and a function $f: E \rightarrow \{\{u,v\} | u,v \in V \wedge u \neq v\}$.
- E.g., nodes are cities, edges are segments of major highways.



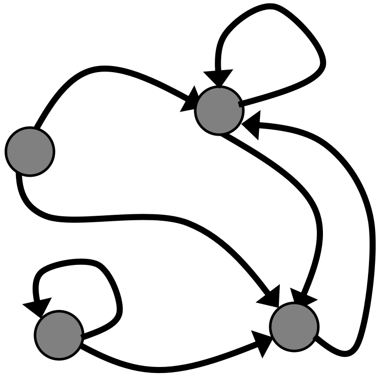
Pseudographs

- Like a multigraph, but edges connecting a node to itself are allowed.
- A *pseudograph* $G=(V, E, f)$ where $f: E \rightarrow \{\{u,v\} | u,v \in V\}$. Edge $e \in E$ is a *loop* if $f(e)=\{u,u\}=\{u\}$.
- *E.g.*, nodes are campsites in a state park, edges are hiking paths in the forest.



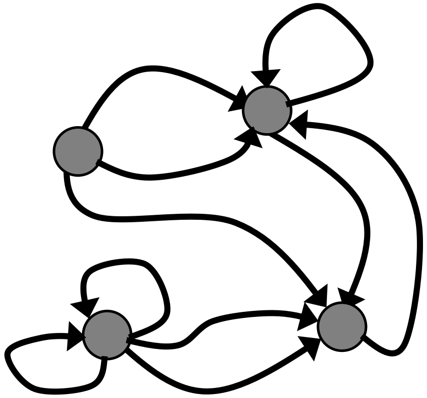
Directed Graphs

- Correspond to arbitrary binary relations R, which need not be symmetric.
- A *directed graph* (V,E) consists of a set of vertices V and a binary relation E on V.
- *E.g.*: V = people, $E=\{(x,y) \mid x \text{ loves } y\}$



Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- A *directed multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges, and a function $f:E\rightarrow V\times V$.
- E.g., V=web pages, E=hyperlinks. *The WWW is a directed multigraph.*



§9.2: Graph Terminology

- *Adjacent, connects, degree, initial, terminal, in-degree, out-degree, complete graphs, cycles, bipartite graphs.*

Adjacency

Let G be an undirected graph with edge set E. Let $e \in E$ be (or map to) the pair $\{u,v\}$. Then we say:

- u, v are *adjacent* / *neighbors* / *connected*.
- Edge e *connects* u and v.

Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The *degree* of v, $\deg(v)$, is its number of connecting edges.
- A vertex with degree 0 is *isolated*.

Handshaking Theorem

- Let G be an undirected graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- Summation of degrees is an even number.

Directed Adjacency

- Let G be a directed graph, and let e an edge of G that is (or maps to) (u,v). Then we say:
  - u is *adjacent to* v, v is *adjacent from* u
  - e *connects* u *to* v, e *goes from* u *to* v
  - the *initial vertex* of e is u
  - the *terminal vertex* of e is v

Directed Degree

- Let G be a directed graph, v a vertex of G.
  - The *in-degree* of v, $\deg^-(v)$, is the number of edges going to v.
  - The *out-degree* of v, $\deg^+(v)$, is the number of edges coming from v.
  - The *degree* of v, $\deg(v) \equiv \deg^-(v) + \deg^+(v)$, is the sum of v's in-degree and out-degree.

Directed Handshaking Theorem

- Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v\in V}\deg^-(v)=\sum_{v\in V}\deg^+(v)=\frac{1}{2}\sum_{v\in V}\deg(v)=|E|$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.

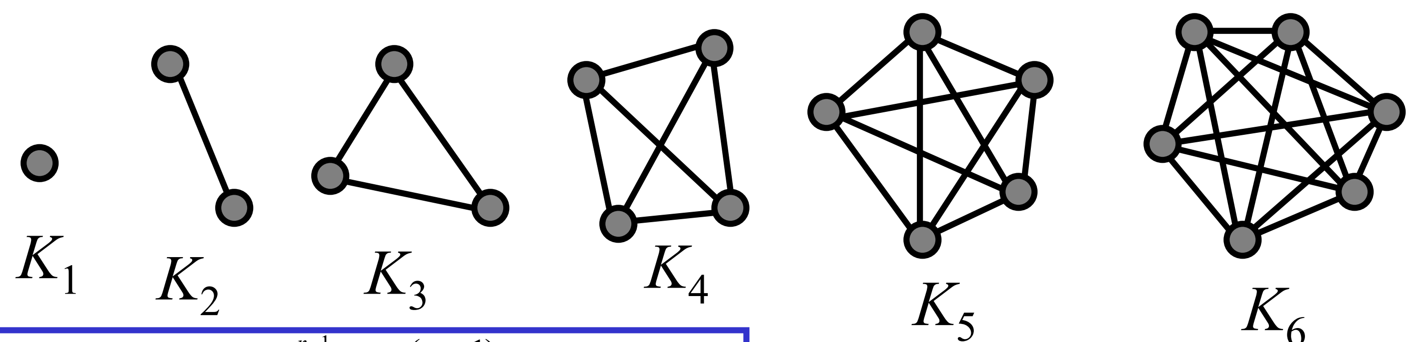
Special Graph Structures

Special cases of undirected graph structures:

- Complete graphs K_n
- Cycles C_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$

Complete Graphs

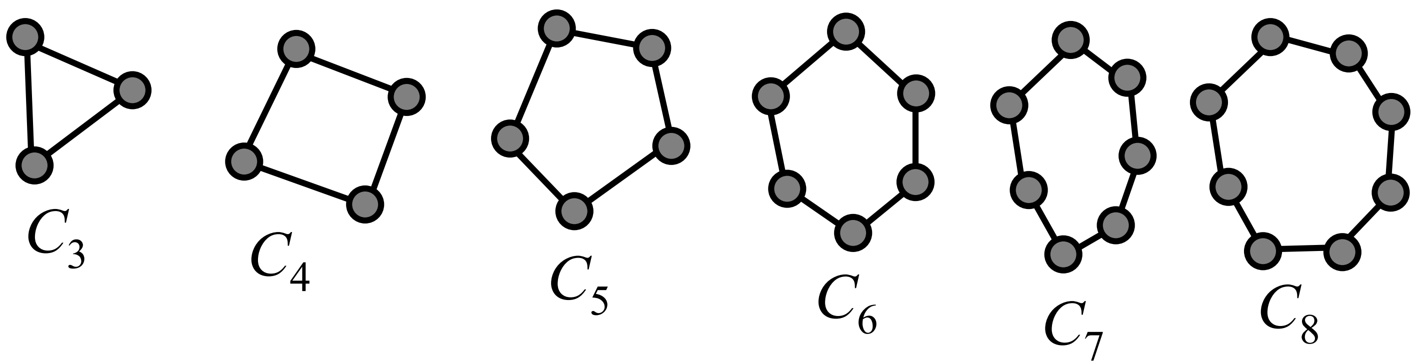
- For any $n \in \mathbf{N}$, a *complete graph* on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node: $\forall u,v \in V$: $u \neq v \leftrightarrow \{u,v\} \in E$.



Note that K_n *has* $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ *edges.*

Cycles

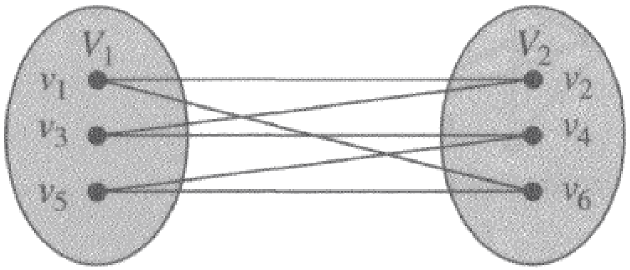
- For any $n \geq 3$, a *cycle* on n vertices, C_n, is a simple graph where $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.



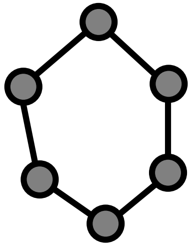
How many edges are there in C_n?

Bipartite Graphs

- A simple graph G is called **bipartite** if its V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2
- An example:



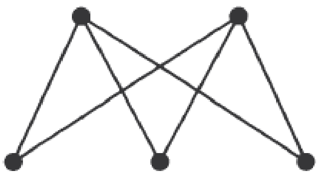
Did you notice that this is C_6 ?



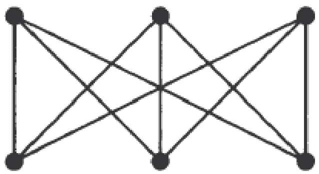
C_6

Complete Bipartite Graphs

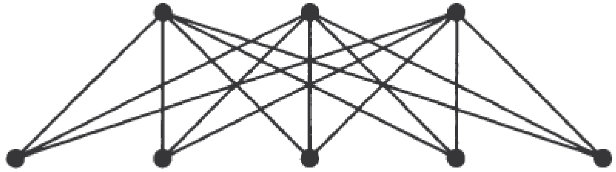
- When every vertices in V_1 is connected to all vertices in $V_{2.}$



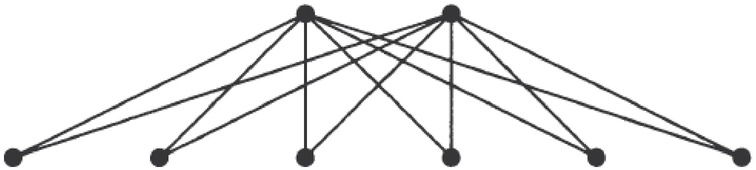
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



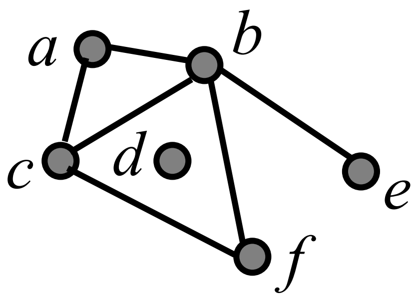
$K_{2,6}$

§9.3: Graph Representations & Isomorphism

- Graph representations:
  - Adjacency lists.
  - Adjacency matrices.
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists

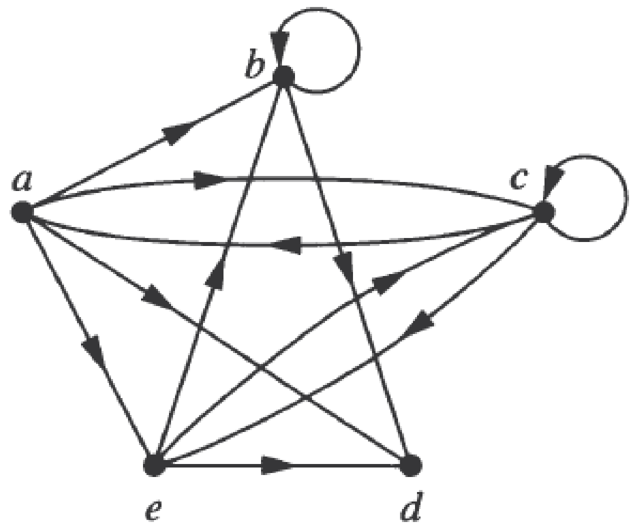
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	*Adjacent Vertices*
a	*b, c*
b	*a, c, e, f*
c	*a, b, f*
d	
e	*b*
f	*c, b*

Directed Adjacency Lists

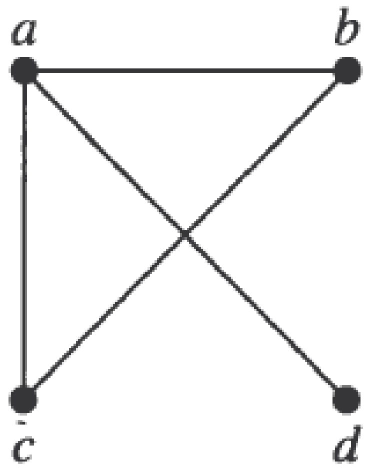
- 1 row per node, listing the terminal nodes of each edge incident from that node.



Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

- Matrix $\mathbf{A}=[a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, 0 otherwise.



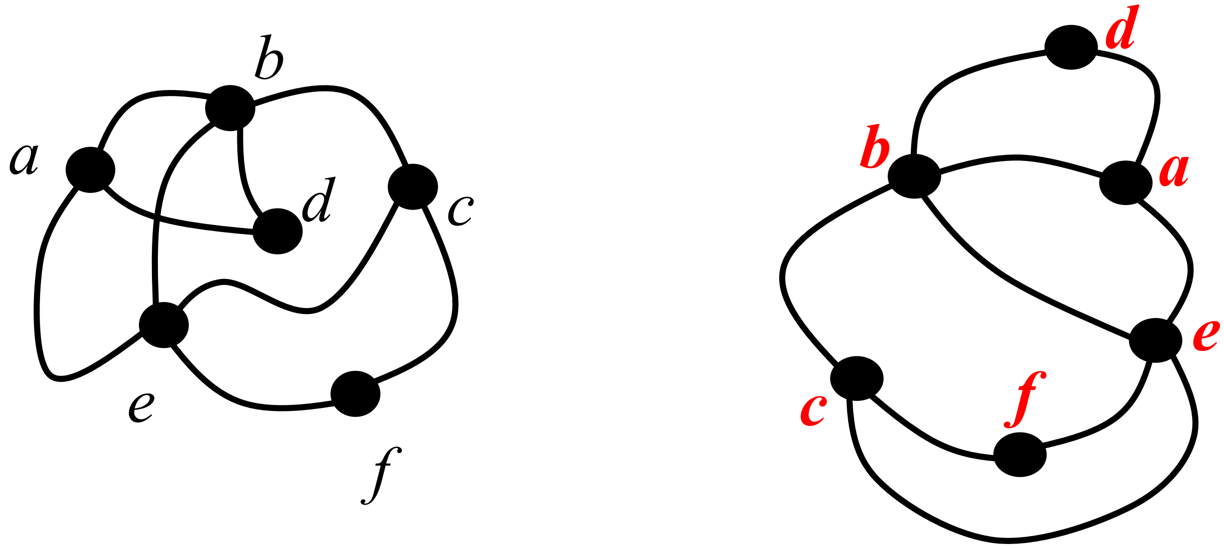
$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Graph Isomorphism

- Formal definition:
  - Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f:V_1\rightarrow V_2$ such that $\forall\ a,b\in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
  - f is the "renaming" function that makes the two graphs identical.

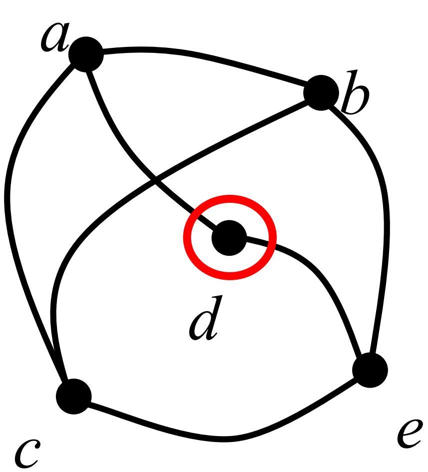
Isomorphism Example

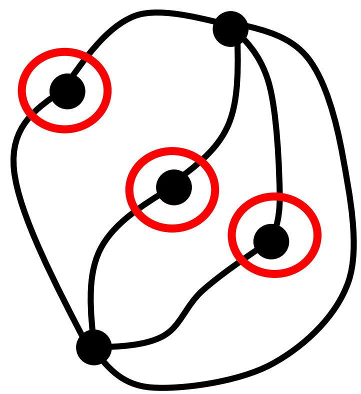
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





* *Same # of vertices*
* *Same # of edges*
* *Different # of vertices of degree 2! (1 vs 3)*

§9.5: Euler & Hamilton Paths

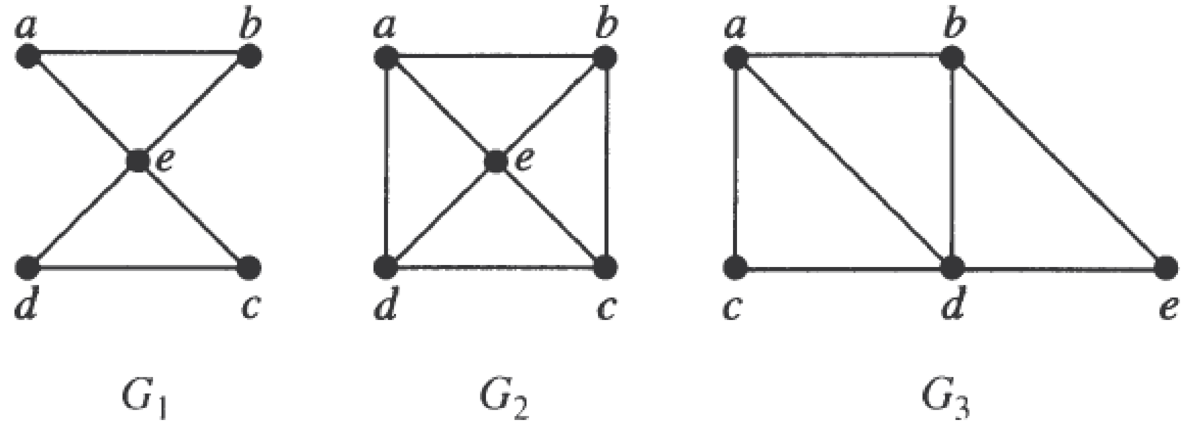
- An *Euler path* in *G* is a path that contains each edge of *G* exactly once.
- An *Euler circuit* in *G* is a circuit (path with same start and end point) that contains each edge of *G* exactly once.
- A *Hamilton path* is a path that traverses each vertex in *G* exactly once.
- A *Hamilton circuit* is a circuit that traverses each vertex in *G* exactly once.

Euler Circuits/Paths

- A connected multigraph has an Euler circuit iff each vertex has even degree.
- A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.

Euler Circuit Examples

- Which of the followings have an Euler circuit (if not, an Euler path)?



Hamilton Circuit Examples

- Which of the followings have a Hamilton circuit (if not, a Hamilton path)?