Izmir Institute of Technology

CENG 115

Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6th Edition)**
by Kenneth H. Rosen

Module #4:
Functions

Rosen 6th ed., § 2.3

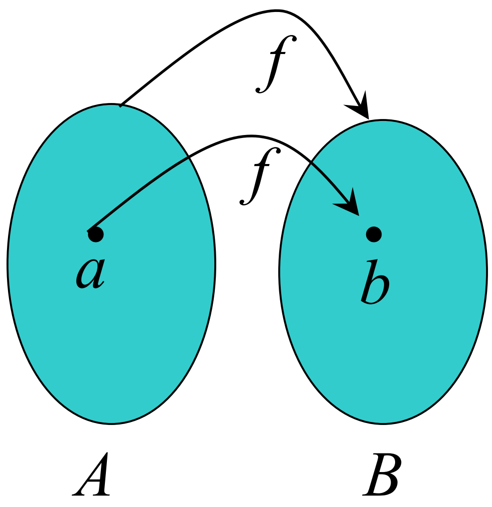
~20 slides, ~1 lecture

Function: Formal Definition

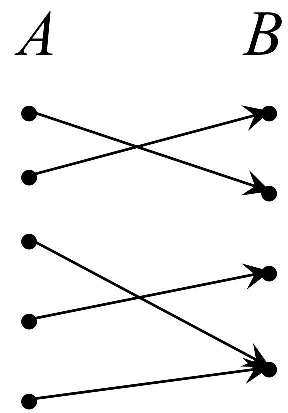
- For any nonempty sets A, B, we say that a *function* f from A to B ($f{:}A \rightarrow B$) is an assignment of exactly one element of B to each element of A.
- Functions are different from *relations* (ordered n-tuples)
- Functions are sometimes also called *mappings* or *transformations*.

Graphical Representations

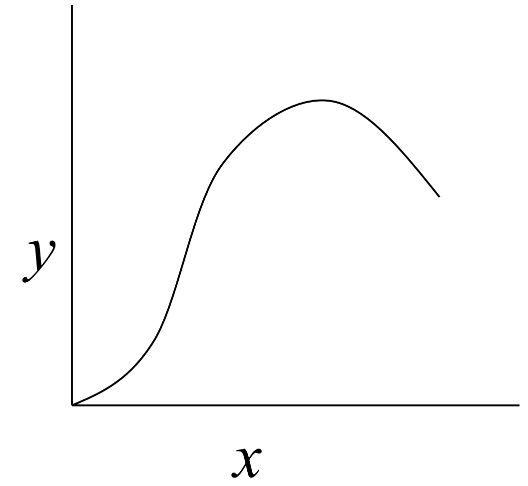
- Functions can be represented graphically in several ways:



Venn-like diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

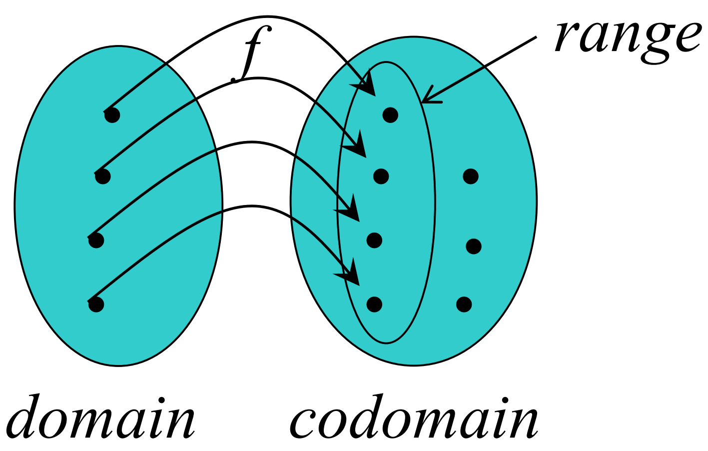
- A *predicate* can be viewed as a function from *objects* to *truth values*:
  P :≡ "is a nice creature",
  P(Garfield) = "Garfield is a nice creature"
  $P(\text{Garfield}) \in \{\mathbf{T},\mathbf{F}\}$
- A *set operator* such as $\cap$,$\cup$ can be viewed as a function from pairs of sets to sets.
  - Example: $\cap((\{1,3\},\{3,4\})) = \{3\}$

Some Function Terminology

- If $f:A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
  - The *range* of f is the set of all images of elements in A.
  - The *range* $R\subseteq B$ of f is $\{b \mid \exists a\, f(a)=b\}$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The range is the *particular* set of values in the codomain that the function *actually* maps to.



Range vs. Codomain - Example

- Suppose that: "f is a function from students in the class to the set of grades {A,B,C,D,F}"
- At this point, you know f's codomain is: {A,B,C,D,F}, and its range is unknown.
- Suppose at the end of the term, I announce that all the grades are A or B.
- Then the range of f is {A,B}, and its codomain is still {A,B,C,D,F}!.

Operators (general definition)

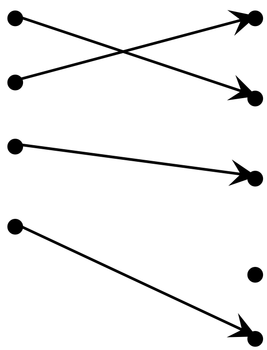
- An n-ary *operator* over the set S is any function from the set of ordered n-tuples of elements of S, to S itself.
- *E.g.*, if $S=\{\mathbf{T},\mathbf{F}\}$, $\neg$ can be seen as a unary operator, and $\wedge,\vee$ are binary operators on S.
- Another example: $\cup$ and $\cap$ are binary operators on the set of all sets.
- Another example: $+, \times$ ("plus","times") are binary operators over **R.**

One-to-One Functions

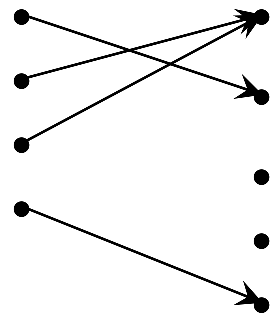
- A function is *one-to-one* (1-1), or *injective*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f{:}A\rightarrow B$,
    "f is injective" $:\equiv (\neg\exists x,y{:}\ x\neq y \wedge f(x)=f(y))$.
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain? **May be larger!**

One-to-One Illustration

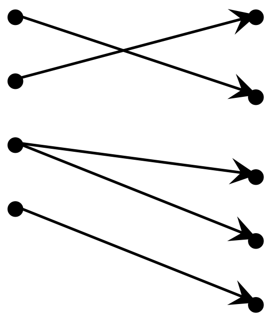
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!
(but a relation)

Sufficient Conditions for 1-1ness

- For functions f over numbers,
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
  - *Inverse is not necessarily true. E.g.* $1/x$

Onto (Surjective) Functions

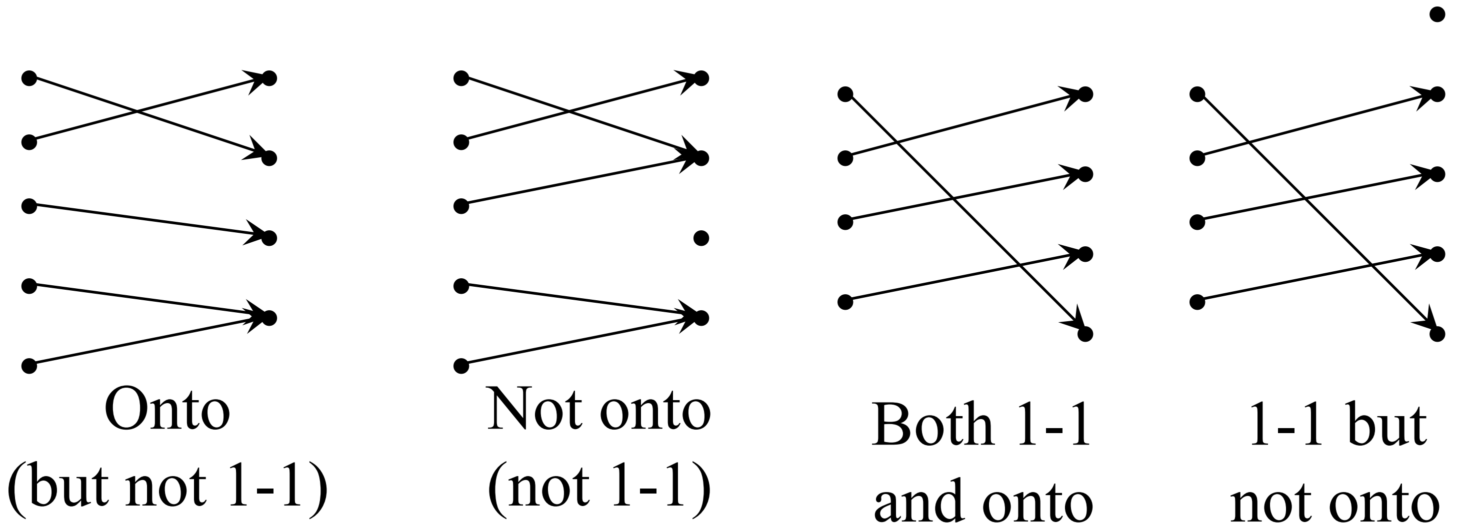
- A function $f{:}A \rightarrow B$ is *onto* or *surjective* iff its range is equal to its codomain:

  $$\forall b \in B,\ \exists a \in A{:}\ f(a)=b$$

- An *onto* function maps set A onto (over, covering) the *entirety* of set B, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



- How many different functions are there when |A|=m, |B|=n?

Function Composition Operator

- For functions g:$A \rightarrow B$ and f:$B \rightarrow C$, there is a special operator called *compose* ("$\circ$").
  - It <u>composes</u> (creates) a new function out of f,g by applying f to the result of g.
  - $(f \circ g)$:$A \rightarrow C$, where $(f \circ g)(a) = f(g(a))$.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

Bijections and Inverse Function

- A function f is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f{:}A\rightarrow B$, there exists an *inverse of f*, written $f^{-1}{:}B\rightarrow A$, which is the unique function s.t. $f^{-1}(b)=a$ when $f(a)=b$.
- $f^{-1} \circ f = I$, the Identity Function.

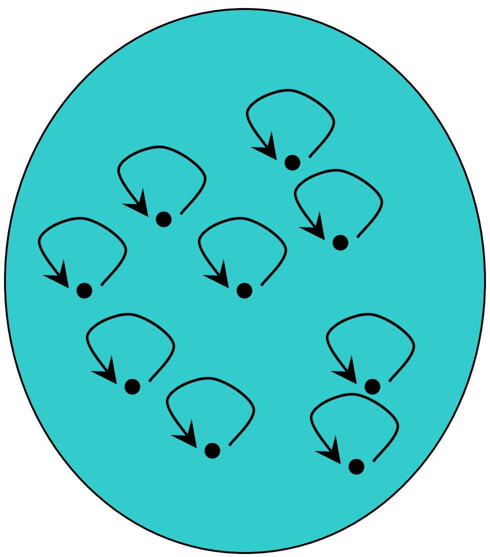
The Identity Function

- For any domain A, the *identity function* $I{:}A\rightarrow A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a\in A$: $I(a)=a$.
- Some identity functions you've seen:

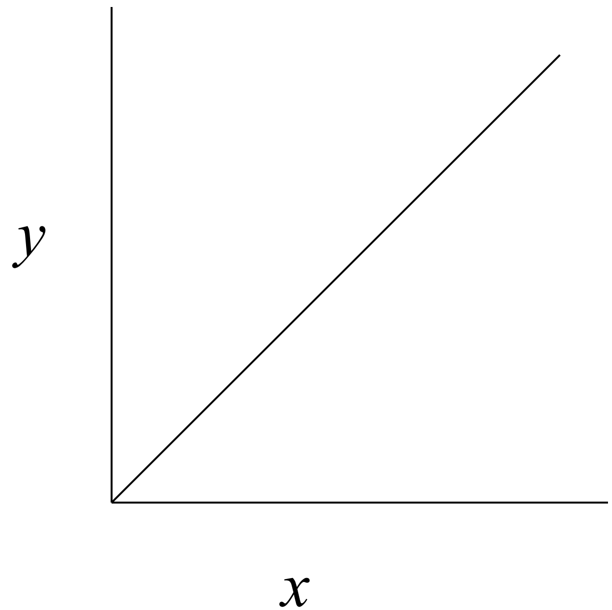
  +ing 0, ·ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\emptyset$, $\cap$ing with U.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:

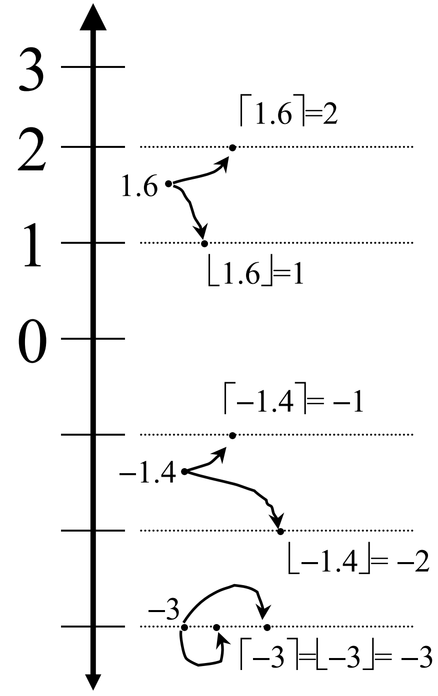


Domain and range



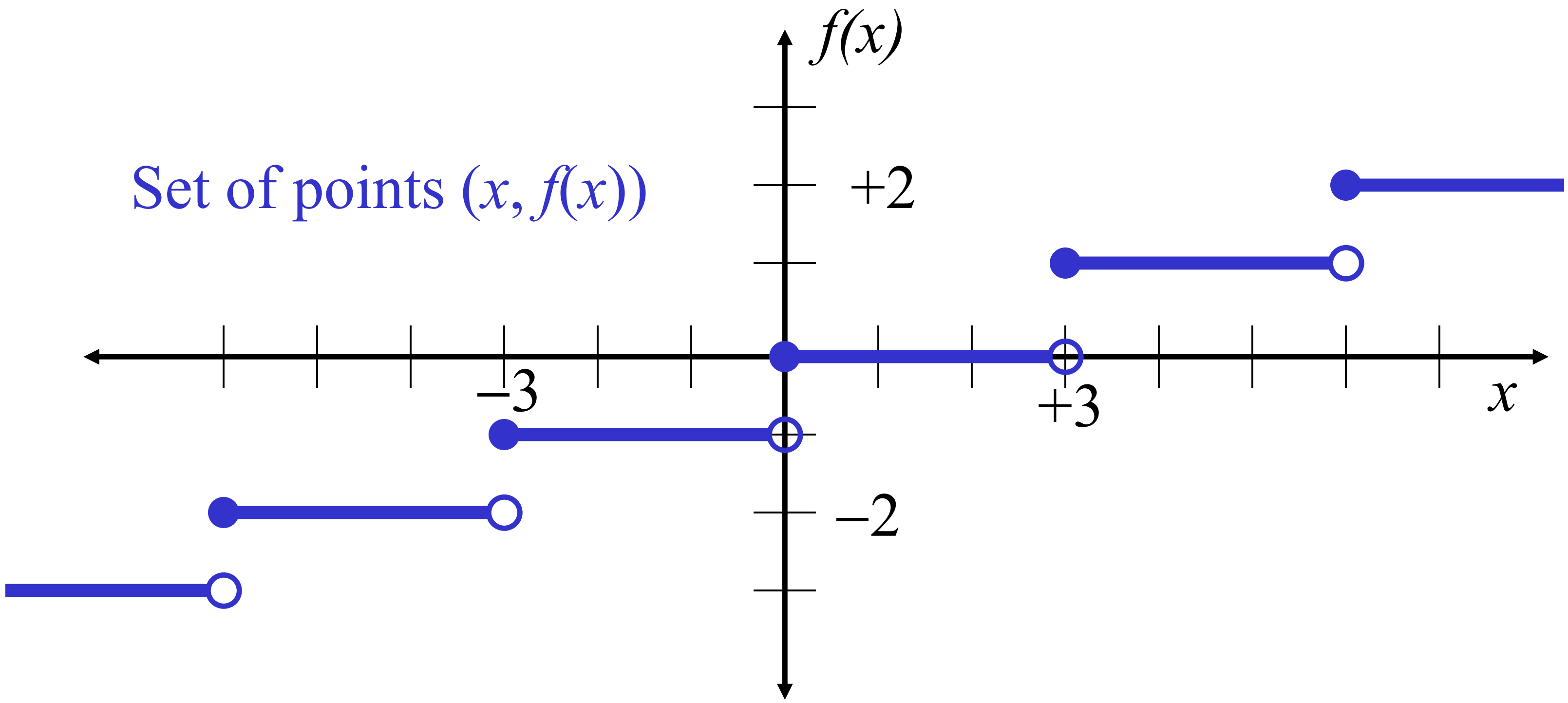
Some Important Functions

- In discrete math, we frequently use the following functions:
  - $\lfloor x \rfloor$ ("floor of x") is the largest (most positive) integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of x") is the smallest (most negative) integer $\geq x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of § 2.3 (Functions)

- Function variables f, g, h, …
- Notations: $f{:}A\rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1}, binary operators $+$, $-$, *etc.*, and $\circ$.
- The $\mathbf{R}\rightarrow\mathbf{Z}$ functions $\lfloor x\rfloor$ and $\lceil x\rceil$.