Izmir Institute of Technology

CENG 115
Discrete Structures

Slides are based on the Text
Discrete Mathematics & Its Applications **(6th Edition)**
by Kenneth H. Rosen

Slides were prepared by Dr. Michael P. Frank
for COT 3100 course in University of Florida

Module #14: **Combinatorics**

Rosen 6th ed., § 5.1-5.3, 5.5

~25 slides, ~1 lecture

Combinatorics

- The study of the number of ways to put things together into various combinations.
- *E.g.* In a contest entered by 100 people, how many different top-10 outcomes could occur?
- *E.g.* If a password is 6-8 letters and/or digits, how many passwords can there be?

Sum and Product Rules

- Let m be the number of ways to do task 1 and n be the number of ways to do task 2.
- Assume that no way to do task 1 simultaneously also accomplishes task 2.
- The *sum rule*: The task "do either task 1 or task 2, but not both" can be done in $m+n$ ways.
- The *product rule*: The task "do both task 1 and task 2" can be done in $m \cdot n$ ways.

Example for Sum and Product Rules

- For a computer engineering course, there are three lists containing 23, 25 and 19 projects respectively.
- If a student has to choose one project topic using the lists, there are 23+25+19=57 different ways.
- If a student has to choose a project from each list, there are 23x25x19=10925 different ways.

Set Theoretic Version

- If A is the set of ways to do task 1, and B is the set of ways to do task 2, and if A and B are disjoint, then:
- The ways to do either task 1 or 2 are $A \cup B$, and $|A \cup B| = |A| + |B|$
- The ways to do both task 1 and 2 can be represented as $A \times B$, and $|A \times B| = |A| \cdot |B|$

Inclusion-Exclusion Principle

- Suppose that $k \leq m$ of the ways of doing task 1 also simultaneously accomplish task 2. (i.e. k ways do task 1 and task 2 together)
- Then the number of ways to accomplish "Do either task 1 or task 2" is $m+n-k$.
- Set theory: If A and B are not disjoint, then $|A\cup B|=|A|+|B|-|A\cap B|$.

Inclusion-Exclusion Example

- Let the following are the rules for passwords:
  - Passwords must be 2 characters long.
  - Each password character must be a letter a-z, a digit 0-9, or one of the 10 punctuation characters !@#$%^&*().
  - Each password must contain at least 1 digit or punctuation character.
- How many valid passwords are there?

Inclusion-Exclusion Example

- A legal password has a digit or punctuation character in position 1 **or** position 2.
- Some cases overlap, apply Inc.-Exc. Principle.
- (# of passwords with Dig.$\vee$Punc. symbol in position #1) = $(10+10)\cdot(10+10+26) = 920$
- (# w. D$\vee$P sym. in pos. #2): also $20\cdot 46 = 920$
- (# w. D$\vee$P sym. both places): $20\cdot 20 = 400$
- Answer: $920+920-400 = 1440$

Another Example

- How many positive integers less than 100 are divisible by 7 or 11?

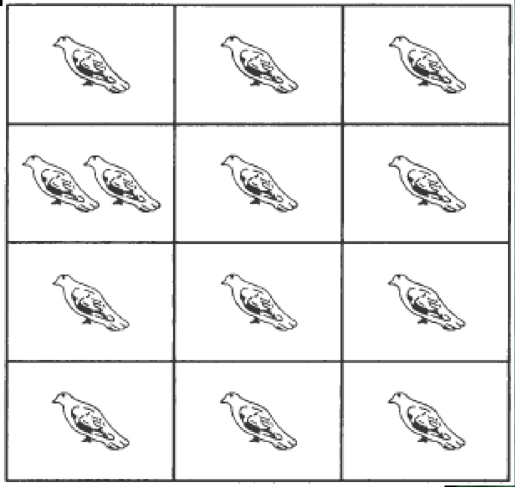
Answer:

11,22,33..99   9 integers divisible by 11

7,14,21,..,77,84,91,98   14 integers divisible by 7

But.. 77 is divisible by both (it is in both lists), so answer is 9+14-1=22

§ 5.2 Pigeonhole Principle



- A.k.a. Dirichlet drawer principle
- If $\geq k+1$ objects are placed into k boxes, then at least 1 box has ≥ 2 objects.
- With function definition:

  If f:$A \rightarrow B$ and $|A| \geq |B|+1$, then some element of B has ≥ 2 pre-images under f.

  I.e., f is not one-to-one.

Example of Pigeonhole Principle

- There are 101 possible numeric grades (0%-100%) rounded to the nearest integer.
- There are >101 students registered to CENG115.
- Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - I.e., the function from students to rounded grades is *not* a one-to-one function.

Generalized Pigeonhole Principle

- If N objects are placed into k boxes, then at least one box contains at least $\lceil N/k \rceil$ objects.
- *E.g.*, there are N=70 students in this class. There are k=52 weeks in the year.

Result by P.P.: There must be at least 1 week during which at least **$\lceil 70/52 \rceil = \lceil 1.35 \rceil = 2$** students in the class have a birthday.

G.P.P. Example

- Given: There are 70 students in the class. Without knowing anybody's birthday, what is the largest value of n for which we can prove that at least n students must have been born in the same month?
- Answer: $\lceil 70/12 \rceil = \lceil 5.83 \rceil = 6$

A Proof by Pigeonhole P. (Ex.4, p.348)

- **Theorem:** $\forall n \in \mathbf{N}$, $\exists$ a multiple of n that has only 0's and 1's in its decimal expansion!
- **Proof:** Consider the n+1 decimal integers 1, 11, 111, …, $\underbrace{1\cdots1}_{\#1\text{'s is } n+1}$. There are only n possible residues mod n.

So, take the difference of two that have the same residue. The result is the answer! ☐

A Specific Example

- Let n=3. Consider 1,11,111,1111.
  - 1 mod 3 = 1
  - 11 mod 3 = 2
  - 111 mod 3 = 0 ← Lucky extra solution.
  - 1,111 mod 3 = 1

Note same residue.

- 1,111 – 1 = 1,110.
  - It has only 0's and 1's in its expansion.
  - Its residue mod 3 = 0, so it's a multiple of 3.

§ 5.3 Permutations and Combinations

- A *permutation* of a set of distinct objects S is an ordered arrangement of these objects.
- An ordered arrangement of r distinct elements of S is called an *r-permutation*.
- The number of r-permutations of a set with $n=|S|$ elements is denoted by $P(n,r)$.
  $P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$

Permutation Example

- A terrorist has planted a nuclear bomb in your city, and it is your job to disable it by cutting wires to the trigger device.
- There are 10 wires, they look the same.
  If you cut exactly the right three wires in the right order, you will disable the bomb.
- What are your chances of survival?

$P(10,3) = 10 \cdot 9 \cdot 8 = 720$,
A 1 in 720 chance of survival!

Combinations

- An r-combination of elements of a set S is an unordered selection of r elements.
- The number of r-combinations of a set with $n=|S|$ is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

- Note that $C(n,r) = C(n, n-r)$
  – Choosing the r members of S is the same thing as choosing the $n-r$ non-members.

Combination Example

- A group of 30 people have been trained as astronauts to go on a mission to Mars. How many ways to select a crew of six people?
  - The order of crew doesn't matter.

  Answer: $C(30,6) = 30! / 6! \cdot 24!$
  $= 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 / 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
  $= 593{,}775$

Permutation/Combination Examples

- How many permutations of letters ABCDEFGH contain string ABC as a block?

  Answer: Regard block ABC as one letter. 6!

- How many bit strings of length n exists?

  Answer: $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$

- How many bit strings of length n contain exactly r number of 1s?

  Answer: $C(n,r)$

§ 5.5 Generalized Permutations and Combinations

<u>*Generalized Permutations*</u>

- The number of r-permutations of a set of n objects <u>with repetition</u> allowed is n^r.

  E.g. How many length r string can be formed using English alphabet? 26^r

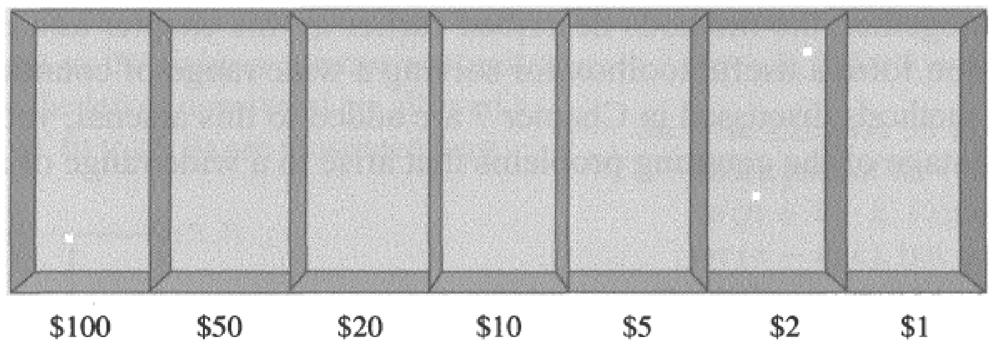
Generalized Combinations

- There are $C(r+n-1,r) = \mathrm{C}(r+n-1,n-1)$ number of r-combinations of a set of n elements when repetition of elements is allowed.
- E.g. Suppose a cookie shop has 4 different kinds of cookies. How many different ways can 6 cookies be chosen? (type of cookie matters, not individual cookies or their orders)

  Answer: $C(r+n-1, n-1)=C(6+4-1,4-1)=C(9,3)$

Generalized Combinations

- How many different ways to select 5 bills from a 7-box cash?



Answer: $C(r+n-1, n-1)=C(5+7-1,7-1)=C(11,6)$

- Note that this is analogous to place 6 bars between 5 stars

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Generalized Combinations

- How many solutions does the equation have?

  $x_1+x_2+x_3=11$

  Assume x_1 x_2 x_3 are non-negative integers.

  Answer: Note that this is analogous to place 2 bars between 11 stars. $C(r+n-1, n-1)=C(11+3-1,3-1)$ $=C(13,2)$