Predicate Logic (§ 1.3)

- Propositional logic treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate logic* distinguishes the *subject* of a sentence from its *predicate*.

  (Predicate -> Yüklem, Subject -> Özne)
- For instance, when the followings are given:

  "Arf2 is a computer at IYTE." "Arf2 is under attack."

  Predicate logic allows us to conclude:

  "There is a computer at IYTE which is under attack."

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object/entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true of the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

More About Predicates

- Convention:
  Lowercase variables *x*, *y*, *z*... denote objects/entities, uppercase variables *P*, *Q*, *R*… denote predicates.
- Keep in mind that the *result of applying* a predicate *P* to an object *x* is the *proposition* *P*(*x*). But the predicate *P* **itself** (*e.g*. *P*="is sleeping") is **not** a proposition (not a complete sentence).

  E.g. if *P*(*x*) = "*x* is a prime number",
  then *P*(3) is the *proposition* "3 is a prime number."

Propositional Functions

- Predicate logic *generalizes* the notion of a predicate to include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.

  E.g. let $P(x,y,z)$ = "x gave y the grade z",
  then if x="Mike", y="Mary", z="A",
  then $P(x,y,z)$ = "Mike gave Mary the grade A."

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \land (\mathbf{1}+1>\mathbf{1}) \land (\mathbf{2}+1>\mathbf{2}) \land \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse* (also called '*domain*').

Quantifier Expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of discourse satisfy a given predicate.
- "∀" is the FOR∀LL or *universal* quantifier. $\forall x\, P(x)$ means *for all* x in the u.d., P holds.
- "∃" is the ∃XISTS or *existential* quantifier. $\exists x\, P(x)$ means *there exists* an x in the u.d. (that is, 1 or more) such that $P(x)$ is true.

The Universal Quantifier ∀

- Example:
  Let the u.d. of x be parking spaces in Alsancak.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$,
  $\forall x\, P(x)$, is the *proposition*:
  - "All parking spaces in Alsancak are full."
  - *i.e.*,"Every parking space in Alsancak is full."
  - *i.e.*,"For each parking space in Alsancak,that space is full."

The Existential Quantifier ∃

- Example:
  Let the u.d. of x be parking spaces in Alsancak.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space in Alsancak is full."
  - "There is a parking space in Alsancak that is full."
  - "At least one parking space in Alsancak is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning, the bounds of x are undefined).
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression and *binds* one or more of the free variables, to produce an expression having one or more *bound variables*.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable, and one bound variable. [Which is which?] [y is free, x is bound]
- "$P(x)$, where x=3" is another way to bind x.

More Conventions

- Sometimes the universe of discourse is restricted within the quantification:
  - $\forall x{>}0\ P(x) \equiv \forall x\ (x{>}0 \rightarrow P(x))$
    "For all *x* that are greater than zero, $P(x)$."
  - $\exists x{>}0\ P(x) \equiv \exists x\ (x{>}0 \wedge P(x))$
    "There is an *x* greater than zero such that $P(x)$."
- E.g. $\forall y{\neq}0\ (y^3{\neq}0)$ says that
  "The cube of every nonzero number is nonzero."

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$
  $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this?

DeMorgan's

Quantifier Equivalence Laws

- $\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$

Prove $\forall x\, P(x) \Leftrightarrow \neg\exists x\, \neg P(x)$

Prove $\exists x\, P(x) \Leftrightarrow \neg\forall x\, \neg P(x)$

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$
- Exercise:
  See if you can prove the last two by yourself.
  – What propositional equivalence laws would you use?

Quantifier Exercise

Negate this sentence: "*All dogs have fleas.*"

a) "No dogs have fleas." ?

b) "There exists a dog that does not have fleas." ?

Solution:

Let $D(x)$ ="*being a dog*" and $F(x)$="*having fleas*"

"*All dogs have fleas.*" $=\forall x(D(x)\rightarrow F(x))$

$\neg(\forall x(D(x)\rightarrow F(x))) \Leftrightarrow \exists x\, \neg(D(x)\rightarrow F(x)) \Leftrightarrow$

$\exists x\, \neg\, (\neg D(x) \vee F(x)) \Leftrightarrow \exists x\, (D(x) \wedge \neg F(x))$ which is

"There exists a dog that does not have fleas."

Quantifier Exercise

Solution 2:

If the context lets you to define the U.D. of x, then let x represent the dogs and $F(x)$ = "*having fleas*".

"*All dogs have fleas.*" = $\forall x\ F(x)$

$\neg(\forall x\ F(x)) \Leftrightarrow \exists x\ \neg F(x)$ which is

"There exists a dog that does not have fleas."

Examples from Lewis Carroll

From Carroll's book 'Symbolic Logic':

$P(x)$=being lion $Q(x)$=being fierce $R(x)$=drink coffee

- "All lions are fierce."

  $\forall x\ (\ P(x) \rightarrow Q(x)\ )$

- "Some lions do not drink coffee."

  $\exists x\ (\ P(x) \wedge \neg R(x)\ )$

C. Lutwidge Dodgson a.k.a. 'Lewis Carroll' 1832-1898. Author of 'Alice in Wonderland'

Note: We can't write $\exists x\ (\ P(x) \rightarrow \neg R(x)\ )$ for the second statement, because $P(x) \rightarrow \neg R(x)$ would be true even for one creature that is not a lion, even if every lion drinks coffee. That is not correct.

Defining New Quantifiers

As their name implies, quantifiers can be used to express that a predicate is true of any given *quantity* (number) of objects.

Define $\exists! x\, P(x)$ to mean "$P(x)$ is true for *exactly one* x in the universe of discourse."

$\exists! x\, P(x) \Leftrightarrow \exists x\, (P(x) \wedge \neg\exists y\, (P(y) \wedge y \neq x))$

"There is an x such that $P(x)$, where there is no y such that P(y) and y is other than x."

Some Number Theory Examples

- Let U.D. = the *natural numbers* 0, 1, 2, …
- "A number x is *even*, $E(x)$, if and only if it is equal to 2 times some other number."
  $\forall x\, (E(x) \leftrightarrow (\exists y\;\; x=2y))$
- "A number is *prime*, $P(x)$, iff it's greater than 1 and it isn't the product of two non-unity numbers."
  $\forall x\, (P(x) \leftrightarrow (x>1 \wedge \neg\exists yz\, (x=yz \wedge y\neq 1 \wedge z\neq 1)))$

Goldbach's Conjecture (unproven)

Using $E(x)$ and $P(x)$ from previous slide,

$$\forall x\ (x>2 \land E(x)) \rightarrow$$
$$(\exists p\ \exists q\ P(p) \land P(q) \land p+q = x).$$

means that

"Every even number greater than 2
is the sum of two primes."

Exercise

- Rewrite the following statement in the language of mathematical logic:

  - "All people have two parents."

  given Parent(x, y): x is the parent of y.

$$x, y, z \in People$$
$$\forall x \ \exists y, z : \left(Parent(y, x) \wedge Parent(z, x) \wedge (y \neq z)\right)$$

Exercise

- Rewrite the following statement in the language of mathematical logic:

  - "A grandparent of a person is a parent of a parent of that person."

  given Parent(x, y): x is the parent of y.

  Note: To make a definition, use IFF ($\leftrightarrow$) operator.

  $$x, y, z \in People$$

  $$\forall x, y : \left(Grandparent(x, y) \leftrightarrow \left(\exists z : \left(Parent(x, z) \wedge Parent(z, y)\right)\right)\right)$$

Exercise

- Rewrite the following statement in the language of mathematical logic:
  - "An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents."

Note: You can inductively use the Ancestor(·, ·) predicate in the definition condition itself.

$$x, y, z \in People$$
$$\forall x, y : \left(Ancestor(x, y) \leftrightarrow \left(Parent(x, y) \vee \left(\exists z : \left(Ancestor(x, z) \wedge Parent(z, y)\right)\right)\right)\right)$$

Nesting of Quantifiers (§ 1.4)

Order of quantifiers is important unless all quantifiers are the same type.

E.g.: Let x & y be people and $L(x,y)$=“x likes y”

Then $\exists y\, L(x,y)$ =
“There is someone whom x likes.”

Then $\forall x \exists y\, L(x,y)$ =
“Everyone has someone whom they like.”

Hint: Think $\forall x \exists y\, L(x,y)$ as $\forall x(\exists y\, L(x,y))$

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x \exists y\, R(x,y)$= Everyone has *someone* to rely on.

$\exists y \forall x\, R(x,y)$= There exists a person whom *everyone* relies upon (including himself)!

$\exists x \forall y\, R(x,y)$= There's someone who relies upon *everybody* (including himself).

$\forall y \exists x\, R(x,y)$= Everyone has *someone* who relies upon them.

$\forall x \forall y\, R(x,y)$= *Everyone* relies upon *everybody*, (including themselves)!

Quantifier Exercise

Let x represents people and y represents stores. Using predicates:

$S(x,y)$: "x shops in y"

$T(x)$: "x is a student"

Then, $\exists y \forall x\ (T(x) \rightarrow \neg S(x,y))$ means that

"There exists a store in which no student shops".

Home exercise: $\forall x \exists y\ (T(x) \rightarrow \neg S(x,y))$ means?

End of § 1.3-1.4, Predicate Logic

- From these sections you should have learned:
  - Predicates P, Q, R, … are functions mapping objects x to propositions $P(x)$.
  - Multi-argument predicates $P(x, y)$.
  - Quantifiers: $[\forall x\ P(x)]$:≡"For all x's, $P(x)$."
    $[\exists x\ P(x)]$:≡ "There is an x such that $P(x)$."
  - Universes of discourse, bound & free variables
  - Simple reasoning with quantifiers
  - Nested quantifiers