CENG 216 – Numerical Computation

Interpolation - Part II

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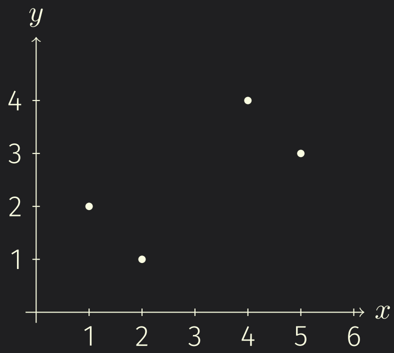
Slide Credits

Slides are based on material from the main textbook:

"Numerical Analysis", The new international edition, 2ed, by Timothy Sauer

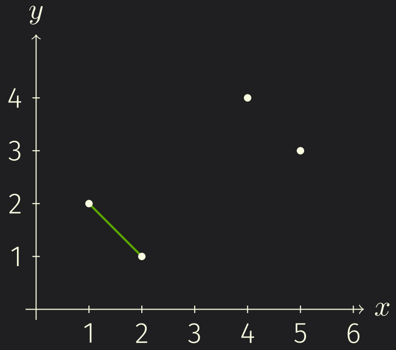
Cubic Splines

Spline Idea: Instead of fitting a single high degree polynomial to all data points, fit multiple low degree polynomials to subsets of data points.



Spline Idea: Instead of fitting a single high degree polynomial to all data points, fit multiple low degree polynomials to subsets of data points.

The simplest is to use **line segments** for interpolating between two consecutive data points $\to$ **linear spline**



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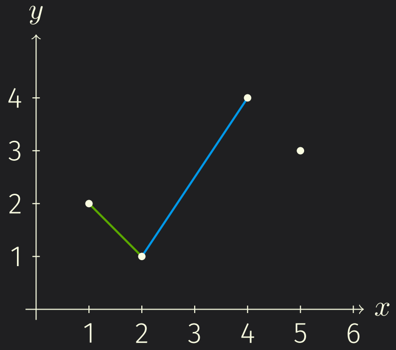

$$S_1(x) = 2 - (x - 1) \qquad \text{on } [1, 2]$$

$$S_2(x) = 1 + \frac{3}{2}(x - 2) \qquad \text{on } [2, 4]$$

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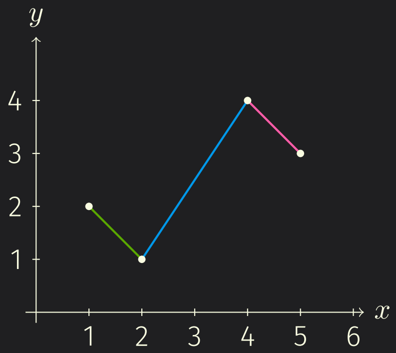
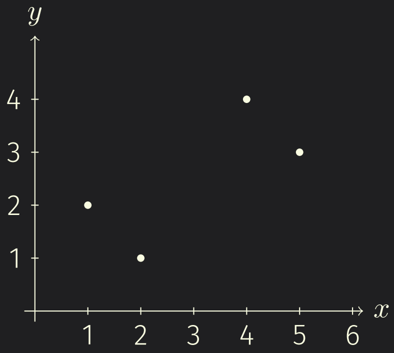


$$\begin{aligned} S_1(x) &= 2 - (x - 1) && \text{on } [1, 2] \\ S_2(x) &= 1 + \frac{3}{2}(x - 2) && \text{on } [2, 4] \\ S_3(x) &= 4 - (x - 4) && \text{on } [4, 5] \end{aligned}$$

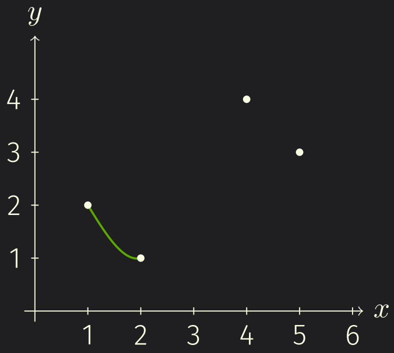
To ensure smoothness of the curve, we can use a higher order curve for each interval.



Cubic Spline

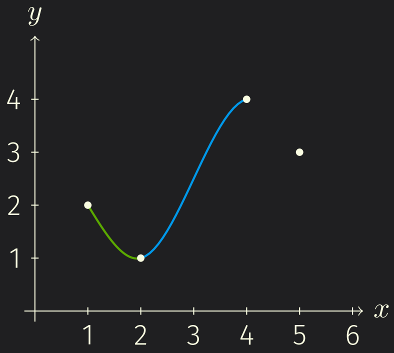
To ensure smoothness of the curve, we can use a higher order curve for each interval.

Use polynomials of degree three $\rightarrow$ **cubic spline**



$$S_1(x) = 2 - \frac{13}{8}(x-1) + 0(x-1)^2 + \frac{5}{8}(x-1)^3 \qquad \text{on } [1,2]$$

To ensure smoothness of the curve, we can use a higher order curve for each interval.
Use polynomials of degree three $\rightarrow$ cubic spline



$$S_1(x) = 2 - \frac{13}{8}(x-1) + 0(x-1)^2 + \frac{5}{8}(x-1)^3 \qquad \text{on } [1,2]$$

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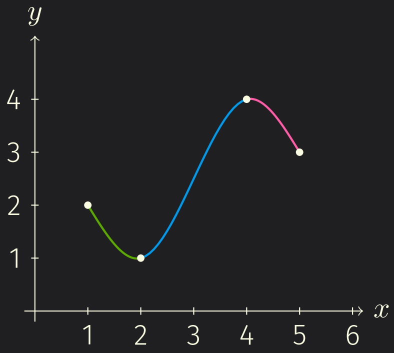
Use polynomials of degree three → **cubic spline**



$$S_1(x) = 2 - \frac{13}{8}(x-1) + 0(x-1)^2 + \frac{5}{8}(x-1)^3 \qquad \text{on } [1,2]$$

$$S_2(x) = 1 + \frac{1}{4}(x-2) + \frac{15}{8}(x-2)^2 - \frac{5}{8}(x-2)^3 \qquad \text{on } [2,4]$$

$$S_3(x) = 4 + \frac{1}{4}(x-4) - \frac{15}{8}(x-4)^2 + \frac{5}{8}(x-4)^3 \qquad \text{on } [4,5]$$

Cubic Spline: The General Form

We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

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We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

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We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

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$$S_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3]$$

We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

$$
\begin{aligned}
S_1(x) &= a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2] \\
S_2(x) &= a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3] \\
&\vdots \\
S_{n-1}(x) &= a_{n-1} + b_{n-1}(x - x_{n-1}) \\
&\quad + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \qquad \text{on } [x_{n-1}, x_n].
\end{aligned}
$$

We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

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\end{aligned}
$$

Total number of unknowns $\rightarrow$

We can interpolate between $x_1, x_2, \ldots, x_n$ with a cubic spline with components

$$\begin{aligned} S_1(x) &= a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2] \\ S_2(x) &= a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3] \\ &\vdots \\ S_{n-1}(x) &= a_{n-1} + b_{n-1}(x - x_{n-1}) \\ &\quad + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \quad \text{on } [x_{n-1}, x_n]. \end{aligned}$$

Total number of unknowns $\rightarrow 4 \times (n - 1) = 4n - 4$

Properties of Cubic Splines

Total number of unknowns $\rightarrow 4n - 4$

Total number of unknowns $\rightarrow 4n - 4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Total number of unknowns $\to 4n-4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Prop. II: $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \ldots, n-1$.

Total number of unknowns $\rightarrow 4n - 4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

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Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

- Use first part of Property I to get $a_i = y_i$ for $i = 1, \ldots, n-1$.

Total number of unknowns $\rightarrow 4n-4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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- Use first part of Property I to get $a_i = y_i$ for $i = 1, \ldots, n-1$.
- Remaining number of unknowns $\rightarrow 3n-3$.

Total number of unknowns $\rightarrow 4n - 4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Prop. II: $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \ldots, n-1$.

Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

- Use first part of Property I to get $a_i = y_i$ for $i = 1, \ldots, n-1$.
- Remaining number of unknowns $\rightarrow 3n - 3$.
- Use second part of Property I to get $n - 1$ constraints.

Properties of Cubic Splines

Total number of unknowns $\rightarrow 4n-4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

- Use first part of Property I to get $a_i = y_i$ for $i = 1, \ldots, n-1$.
- Remaining number of unknowns $\rightarrow 3n-3$.
- Use second part of Property I to get $n-1$ constraints.
- Use Property II to get $n-2$ constraints.

Total number of unknowns $\to 4n-4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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- Remaining number of unknowns $\to 3n-3$.
- Use second part of Property I to get $n-1$ constraints.
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- Use Property III to get $n-2$ constraints.

Total number of unknowns $\to 4n - 4$

Prop. I: $S_i(x_i) = y_i$ and $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Prop. II: $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \ldots, n-1$.

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- Use first part of Property I to get $a_i = y_i$ for $i = 1, \ldots, n-1$.
- Remaining number of unknowns $\to 3n - 3$.
- Use second part of Property I to get $n - 1$ constraints.
- Use Property II to get $n - 2$ constraints.
- Use Property III to get $n - 2$ constraints.
- Total number of constraints $\to (3n - 5) < (3n - 3)$

Properties of Cubic Splines

Total number of unknowns $\rightarrow 4n-4$

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- Remaining number of unknowns $\rightarrow 3n-3$.
- Use second part of Property I to get $n-1$ constraints.
- Use Property II to get $n-2$ constraints.
- Use Property III to get $n-2$ constraints.
- Total number of constraints $\rightarrow (3n-5) < (3n-3)$
- We are missing 2 constraints.

Natural Splines

Natural Splines

$$S_1(x) = y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2]$$

Natural Splines

$$\begin{aligned}
S_1(x) &= y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2] \\
S_2(x) &= y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3] \\
&\vdots \\
S_{n-1}(x) &= y_{n-1} + b_{n-1}(x - x_{n-1}) \\
&\quad + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \qquad \text{on } [x_{n-1}, x_n].
\end{aligned}$$

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Prop. I: $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

$$
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S_2(x) &= y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 && \text{on } [x_2, x_3] \\
&\vdots \\
S_{n-1}(x) &= y_{n-1} + b_{n-1}(x - x_{n-1}) \\
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\end{aligned}
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Prop. I: $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Prop. II: $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \ldots, n-1$.

Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

Prop. IVa: $S''_1(x_1) = 0$ and $S''_{n-1}(x_n) = 0$ (Natural Spline)

$$
\begin{aligned}
S_1(x) &= y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2] \\
S_2(x) &= y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3] \\
&\vdots \\
S_{n-1}(x) &= y_{n-1} + b_{n-1}(x - x_{n-1}) \\
&\quad + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \qquad \text{on } [x_{n-1}, x_n].
\end{aligned}
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Prop. I: $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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\# of unknowns $\rightarrow$ $\leftarrow$ # of constraints

Natural Splines

$$S_1(x) = y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2]$$

$$S_2(x) = y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3]$$

$$\vdots$$

$$S_{n-1}(x) = y_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \quad \text{on } [x_{n-1}, x_n].$$

Prop. I: $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

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Prop. IVa: $S''_1(x_1) = 0$ and $S''_{n-1}(x_n) = 0$ (Natural Spline)

# of unknowns $\rightarrow 3n - 3$ $\leftarrow$ # of constraints

Natural Splines

$$
\begin{aligned}
S_1(x) &= y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad \text{on } [x_1, x_2] \\
S_2(x) &= y_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 \quad \text{on } [x_2, x_3] \\
&\vdots \\
S_{n-1}(x) &= y_{n-1} + b_{n-1}(x - x_{n-1}) \\
&\qquad + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 \qquad \text{on } [x_{n-1}, x_n].
\end{aligned}
$$

Prop. I: $S_i(x_{i+1}) = y_{i+1}$ for $i = 1, \ldots, n-1$.

Prop. II: $S'_{i-1}(x_i) = S'_i(x_i)$ for $i = 2, \ldots, n-1$.

Prop. III: $S''_{i-1}(x_i) = S''_i(x_i)$ for $i = 2, \ldots, n-1$.

Prop. IVa: $S''_1(x_1) = 0$ and $S''_{n-1}(x_n) = 0$ (Natural Spline)

# of unknowns $\rightarrow 3n - 3 = 3n - 5 + 2 \leftarrow$ # of constraints

Natural Spline Equation System

$$n - 1 \text{ constraints from } \textbf{Property I} \begin{cases} y_2 = S_1(x_2) \\ \quad\vdots \\ y_n = S_{n-1}(x_n) \end{cases}$$

Natural Spline Equation System

$$n-1 \text{ constraints from } \textbf{Property I} \begin{cases} y_2 = S_1(x_2) \\ \quad\vdots \\ y_n = S_{n-1}(x_n) \end{cases}$$

$$\downarrow$$

$$(\bigstar) \begin{cases} y_2 = y_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3 \\ \quad\vdots \\ y_n = y_{n-1} + b_{n-1}(x_n - x_{n-1}) \\ \qquad\quad + c_{n-1}(x_n - x_{n-1})^2 + d_{n-1}(x_n - x_{n-1})^3 \end{cases}$$

Natural Spline Equation System

$$n-1 \text{ constraints from } \textbf{Property I} \begin{cases} y_2 = S_1(x_2) \\ \quad \vdots \\ y_n = S_{n-1}(x_n) \end{cases}$$

$$\downarrow$$

$$(\bigstar) \begin{cases} y_2 = y_1 + b_1(x_2 - x_1) + c_1(x_2 - x_1)^2 + d_1(x_2 - x_1)^3 \\ \quad \vdots \\ y_n = y_{n-1} + b_{n-1}(x_n - x_{n-1}) \\ \qquad\quad + c_{n-1}(x_n - x_{n-1})^2 + d_{n-1}(x_n - x_{n-1})^3 \end{cases}$$

Note that equations in $(\bigstar)$ are linear in the $3n-3$ unknowns $b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1}$.

Natural Spline Equation System

$$n - 2 \text{ constraints from Property II} \begin{cases} 0 = S_1'(x_2) - S_2'(x_2) \\ \quad\vdots \\ 0 = S_{n-2}'(x_{n-1}) - S_{n-1}'(x_{n-1}) \end{cases}$$

$$\downarrow$$

Natural Spline Equation System

$$n-2 \text{ constraints from } \textbf{Property II} \begin{cases} 0 = S_1'(x_2) - S_2'(x_2) \\ \quad \vdots \\ 0 = S_{n-2}'(x_{n-1}) - S_{n-1}'(x_{n-1}) \end{cases}$$

$$\downarrow$$

$$(\bigstar\bigstar) \begin{cases} 0 = b_1 + 2c_1(x_2 - x_1) + 3d_1(x_2 - x_1)^2 - b_2 \\ \quad \vdots \\ 0 = b_{n-2} + 2c_{n-2}(x_{n-1} - x_{n-2}) \\ \qquad + 3d_{n-2}(x_{n-1} - x_{n-2})^2 - b_{n-1} \end{cases}$$

Natural Spline Equation System

$$n - 2 \text{ constraints from } \textbf{Property III} \begin{cases} 0 = S_1''(x_2) - S_2''(x_2) \\ \quad \vdots \\ 0 = S_{n-2}''(x_{n-1}) - S_{n-1}''(x_{n-1}) \end{cases}$$

$$\downarrow$$

Natural Spline Equation System

$$n-2 \text{ constraints from Property } \mathbf{III} \begin{cases} 0 = S_1''(x_2) - S_2''(x_2) \\ \quad \vdots \\ 0 = S_{n-2}''(x_{n-1}) - S_{n-1}''(x_{n-1}) \end{cases}$$

$$\downarrow$$

$$(\star\star\star) \begin{cases} 0 = 2c_1 + 6d_1(x_2 - x_1) - 2c_2 \\ \quad \vdots \\ 0 = 2c_{n-2} + 6d_{n-2}(x_{n-1} - x_{n-2}) - 2c_{n-1} \end{cases}$$

Natural Spline Equation System

$$(\bigstar)\ \left\{ y_{i+1} = y_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 \right.$$

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Natural Spline Equation System

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Define $\delta_i = x_{i+1} - x_i, \Delta_i = y_{i+1} - y_i$:

$$(\bigstar\bigstar\bigstar) \left\{ \begin{array}{l} 0 = 2c_i + 6d_i\delta_i - 2c_{i+1} \\ \\ \\ \end{array} \right.$$

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$$(\star)\ \begin{cases} \Delta_i = b_i\delta_i + c_i\delta_i^2 + d_i\delta_i^3 \\ b_i = \dfrac{\Delta_i}{\delta_i} - c_i\delta_i - d_i\delta_i^2 \\ \\ \end{cases}$$

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$$(\bigstar)\ \left\{ y_{i+1} = y_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 \right.$$

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We have, for $i = 1, \ldots, n-2$:

$$(\bigstar) - (\bigstar\bigstar\bigstar) \left\{ \delta_i c_i + 2\left(\delta_i + \delta_{i+1}\right) c_{i+1} + \delta_{i+1} c_{i+2} = 3\left\{ \frac{\Delta_{i+1}}{\delta_{i+1}} - \frac{\Delta_i}{\delta_i} \right\} \right.$$

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$$\vdots$$

$$\delta_{n-2} c_{n-2} + 2\left(\delta_{n-2} + \delta_{n-1}\right) c_{n-1} + \delta_{n-1} c_n = 3\left\{ \frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}} \right\}$$

Natural Spline Equation System

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2 constraints from Property **IVa** $\left\{\begin{aligned} S''_1(x_1) &= 0 \implies c_1 = 0 \\ S''_{n-1}(x_n) &= 0 \implies c_n = 0 \end{aligned}\right.$

Natural Spline Equation System

$$
\begin{aligned}
c_1 &= 0 \\
\delta_1 c_1 + 2\left(\delta_1 + \delta_2\right) c_2 + \delta_2 c_3 &= 3\left\{\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right\} \\
&\vdots \\
\delta_{n-2} c_{n-2} + 2\left(\delta_{n-2} + \delta_{n-1}\right) c_{n-1} + \delta_{n-1} c_n &= 3\left\{\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right\} \\
c_n &= 0
\end{aligned}
$$

Natural Spline Equation System

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\delta_1 & 2(\delta_1 + \delta_2) & \delta_2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \delta_{n-2} & 2(\delta_{n-2} + \delta_{n-1}) & \delta_{n-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\
3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\
\vdots \\
3\left(\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right) \\
0
\end{bmatrix}
$$

Natural Spline Equation System

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \delta_1 & 2(\delta_1+\delta_2) & \delta_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_{n-2} & 2(\delta_{n-2}+\delta_{n-1}) & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ \vdots \\ 3\left(\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right) \\ 0 \end{bmatrix}$$

Solve the above system for $c_1, \ldots, c_n$ and use the formulas below to get b_i and d_i for $i = 1, \ldots, n-1$:

$$b_i = \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}\left(2c_i + c_{i+1}\right)$$

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i}$$

Example

Find the natural cubic spline through $(0, 3), (1, -2),$ and $(2, 1)$.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 0 \end{bmatrix}$$

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$$c_1 = c_3 = 0, c_2 = 6$$

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$$\begin{aligned} S_1(x) &= 3 - 7x + 0x^2 + 2x^3 && \text{on } [0,1] \\ S_2(x) &= -2 - 1(x-1) + 6(x-1)^2 - 2(x-1)^3 && \text{on } [1,2] \end{aligned}$$

Prop. IVa: $S_1''(x_1) = 0$ and $S_{n-1}''(x_n) = 0$ (Natural Spline)

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Prop. IVc: $S_1'(x_1) = v_n$ and $S_{n-1}'(x_n) = v_n$ (Same Tangent Line)

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Prop. IVd: $d_1 = d_{n-1} = 0$

Prop. IVe: $d_1 = d_2$ and $d_{n-2} = d_{n-1}$

Bezier Curves

End Points and Control Points

- Cubic splines lets us control the points that the curve pass through. It is difficult to control the general shape of the curve.

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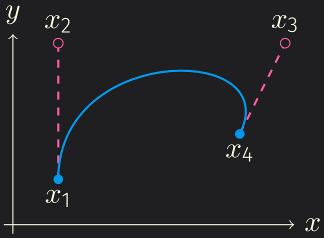
- Cubic splines lets us control the points that the curve pass through. It is difficult to control the general shape of the curve.
- A Cubic Bézier Curve passes through two given points (start and end points) and its shape is influenced by two control points, which it does not pass through.



Importance of Control in Applications

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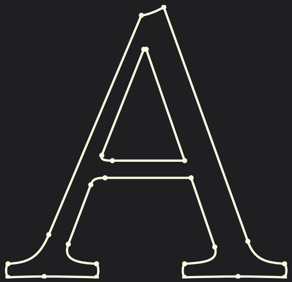
- Characters in animated movies are modelled using similar ideas and animated across frames using interpolation by parametric curves.



Given end points $(x_1, y_1), (x_4, y_4)$ and control points $(x_2, y_2), (x_3, y_3)$, the Bézier curve is defined for the parameter $0 \leq t \leq 1$ by

$$x(t) = x_1 + b_x t + c_x t^2 + d_x t^3$$
$$y(t) = y_1 + b_y t + c_y t^2 + d_y t^3,$$

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\end{aligned}
$$

where

$$
\begin{aligned}
b_x &= 3(x_2 - x_1), \quad c_x = 3(x_3 - x_2) - b_x, \quad d_x = x_4 - x_1 - b_x - c_x \\
b_y &= 3(y_2 - y_1), \quad c_y = 3(y_3 - y_2) - b_y, \quad d_y = y_4 - y_1 - b_y - c_y
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$$b_x = 3(x_2 - x_1), \quad c_x = 3(x_3 - x_2) - b_x, \quad d_x = x_4 - x_1 - b_x - c_x$$
$$b_y = 3(y_2 - y_1), \quad c_y = 3(y_3 - y_2) - b_y, \quad d_y = y_4 - y_1 - b_y - c_y$$

Note that $x(0) = x_1$, $x(1) = x_4$, and $x'(0) = 3(x_2 - x_1)$, $x'(1) = 3(x_4 - x_3)$.

Example

Find the Bézier curve $(x(t), y(t))$ through the points $(1, 1)$ and $(2, 2)$ with control points $(1, 3)$ and $(3, 3)$.

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