CENG 216 – Numerical Computation

Introduction

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February 16, 2022

İzmir Institute of Technology

Course Information

People

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Time	Monday, 09:45–12:30
Teams Code	`e65du56`

Grading

Assignment	Grade Percentage
Homeworks	40%
Midterm Exam	30%
Final Exam	30%

Main Textbook:
Numerical Analysis, The new international edition, 2ed,
Timothy Sauer

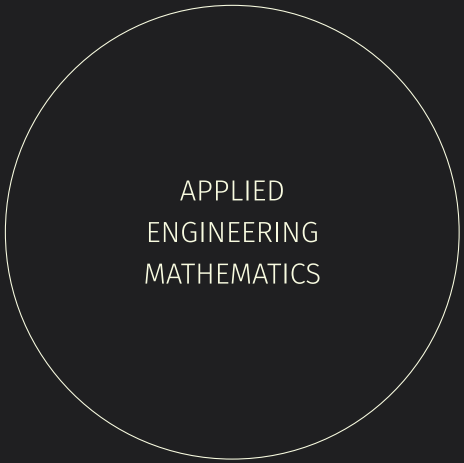
Secondary Textbooks:
Numerical Algorithms: Methods for Computer Vision, Machine Learning, and Graphics, J. Solomon
https://people.csail.mit.edu/jsolomon/share/book/numerical_book.pdf
Matrix Computations (4th Ed.), G. H. Golub and C. F. Van Loan

Course Contents

This course covers the fundamentals of numerical computation starting with finite representation of real numbers and investigation of errors resulting from the discrete and approximate nature of the computation using such a representation.

The course includes topics from numerical linear algebra, interpolation, numerical differentiation and integration, numerical solution of ordinary differential equations, basics of numerical optimization, and generation of random numbers with their application to numerical problems.

Numerical Computation



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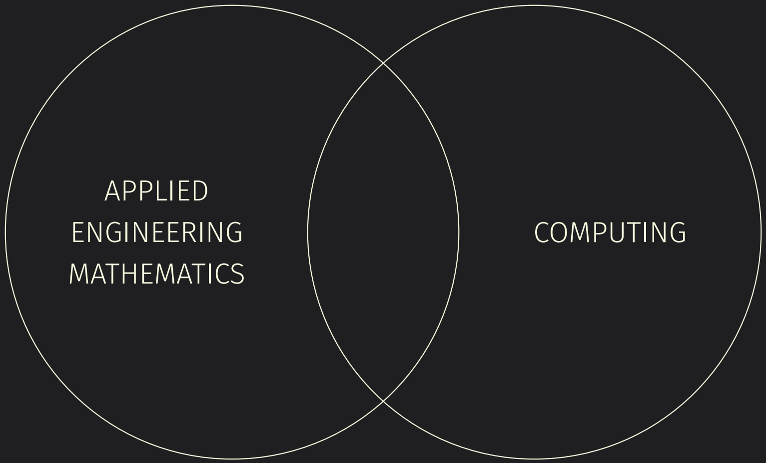

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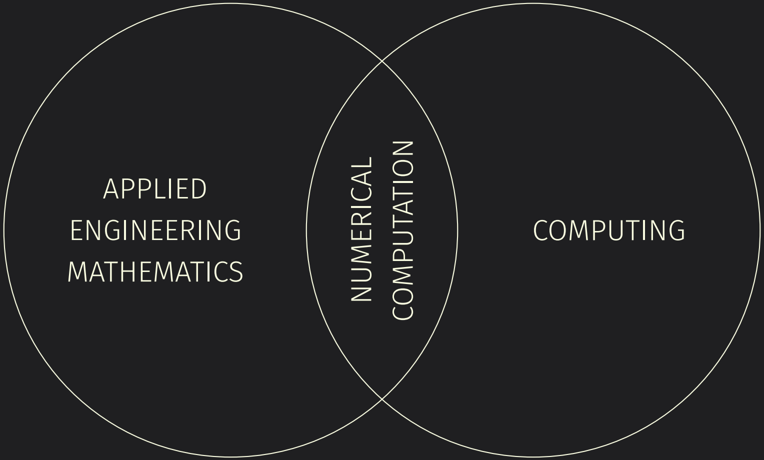

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- Numerical Differentiation and Integration (Chapter 5)
- Ordinary Differential Equations (Chapter 6)

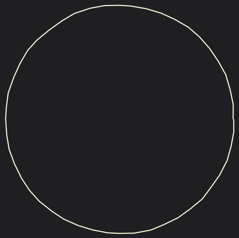
- To be able to explain the effects of the finite representation of real numbers on the implementation of a given algorithm.
- To be able to derive the numerical error in computations and compare the numerical error of different algorithms for the same problem.
- To be able to solve numerical problems requiring differentiation, integration, interpolation and/or optimization.

Learning Outcomes II

- To be able to apply iterative solutions to numerical problems.
- To be able to derive linear/nonlinear systems for a given problem description.
- To be able to select and apply a numerical algorithm to a given linear/nonlinear system.

Introduction

Finite Numerical Representation of Shapes



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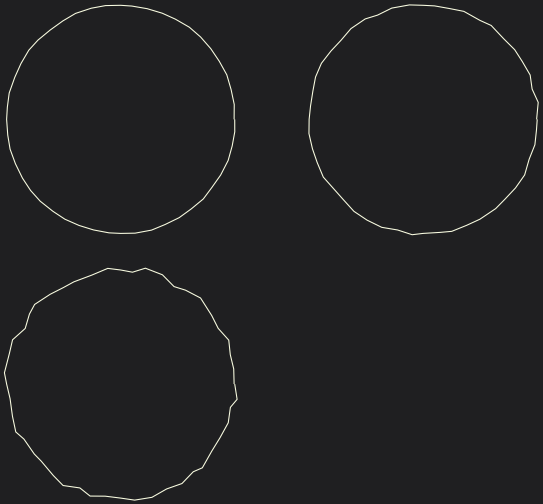

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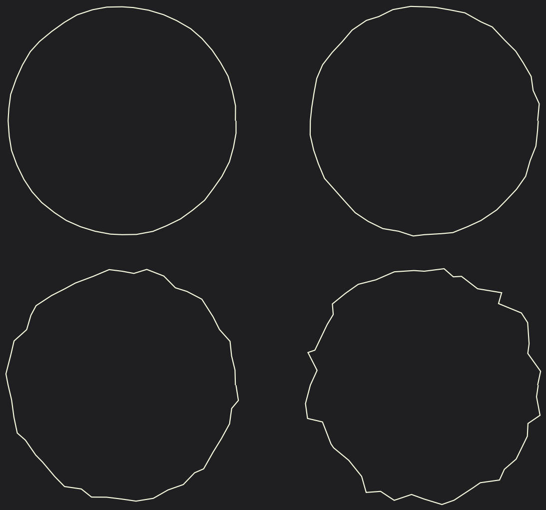

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An Old Computational Example

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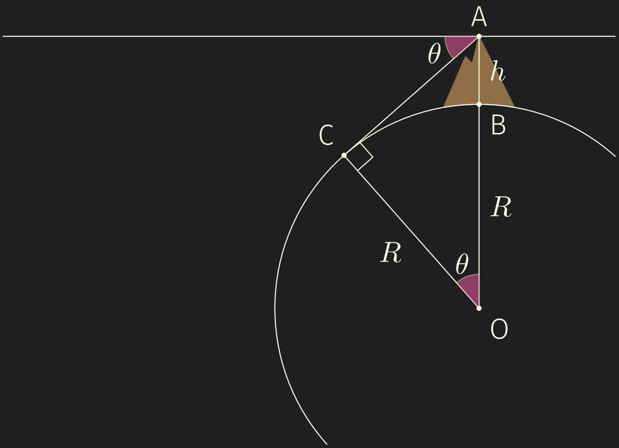

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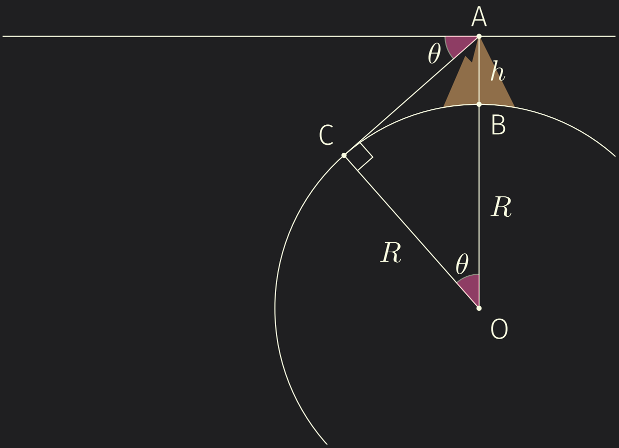

$R = \frac{h\cos\theta}{1-\cos\theta}$ →How do you compute $\cos\theta$ in the year 1000?

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Alexandrian scientist Ptolemy already tabulated some trigonometric quantities as early as the 2^{nd} century. He did his calculations based on the Pythagorean theorem and some trigonometric identities such as

$$\sin^2\theta + \sin^2(90 - \theta) = 1,$$

and

$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}}.$$

With these mathematical tools, it is possible to compute $\sin 45°, \sin 30°, \sin 75°, \sin 36°, \sin 72°$, and $\sin 3°$ accurately, which means we can compute sine for degree that is a multiple of three[1].

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However, it is not possible to compute $\sin 1°$. Ptolemy approximated it by using the following theorem

Theorem
If $\beta < \alpha$, then $\frac{\alpha}{\beta} > \frac{\sin\alpha}{\sin\beta}$.

By substituting $\alpha = 1, \beta = \frac{3}{4}$ and then $\alpha = \frac{3}{2}, \beta = 1$ we get

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$$\frac{2\sin\frac{3}{2}^{\circ}}{3} < \sin 1° < \frac{4\sin\frac{3}{4}^{\circ}}{3}$$

$$0.01745130 < \sin 1° < 0.01745279$$

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A More Accurate Solution for $\sin 1°$

Persian astronomer Al-Kāshī went further by exploiting a cubic equation[2]

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Writing $x = \sin 1° \to x = \frac{\sin 3° + 4x^3}{3}$, we can find $\sin 1°$ as the intersection point of the two simple curves $y = x$ and $y = \frac{4}{3}x^3 + \frac{\sin 3°}{3}$.

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Note We could solve the cubic equation analytically (instead of numerically as follows) but the solution was not known at the time of Al-Kāshī.

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Fixed Point Iterations for $\sin 1^\circ$

Let's start from the previous solution $x_0 = 0.01745$ and iterate with $x_{k+1} = \frac{4}{3}x_k^3 + \frac{\sin 3^\circ}{3}$ keeping 16 digits:

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We have converged to $x = 0.017452406437284$. You can verify with your favorite calculator that this is the correct value for $\sin 1^\circ$ with 16 digits.

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The course will cover all these points for numerical solutions of a variety of problems faced by computer engineers and scientists.