CENG 216 – Numerical Computation

Interpolation - Part I

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Slide Credits

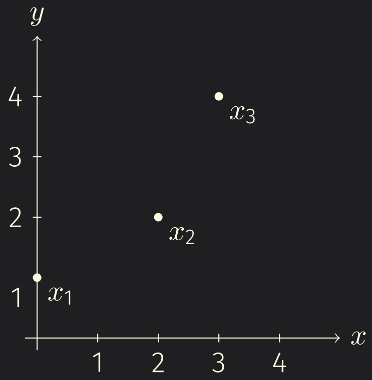
Slides are based on material from the main textbook:

“Numerical Analysis”, The new international edition, 2ed, by Timothy Sauer

Data and Interpolating Functions

A function is said to interpolate a set of data points if it passes through those points.

Def: The function $y = P(x)$ interpolates the data points $(x_1, y_1), \ldots, (x_n, y_n)$ if $P(x_i) = y_i \; \forall i = 1, \ldots, n$.



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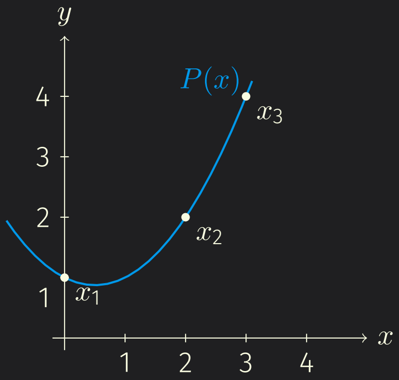

Lagrange Interpolation

To interpolate the n points $(x_1, y_1), \ldots, (x_n, y_n)$, we define the $n-1$ degree polynomials $L_k(x)\ \forall k, k = 1, \ldots, n$:

$$L_k(x) = \frac{(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}.$$

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Note that

$$L_k(x_k) = \quad \text{and } L_k(x_j) = \quad \text{for } j \neq k.$$

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Then the interpolating polynomial of degree $n-1$ for these data points is

$$P_{n-1}(x) = y_1 L_1(x) + y_2 L_2(x) + \dots + y_n L_n(x),$$

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since

$$P_{n-1}(x_k) = y_1.0 + y_2.0 + \dots + y_k.1 + \dots + y_n.0 = y_k\ \forall k = 1, \dots, n.$$

Interpolate points $(0, 1), (2, 2),$ and $(3, 4)$.

Example

Interpolate points $(0, 1), (2, 2),$ and $(3, 4)$.

$$P_2(x) = 1\frac{(x-2)(x-3)}{(0-2)(0-3)} + 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)}$$

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\begin{aligned}
P_2(x) &= 1\frac{(x-2)(x-3)}{(0-2)(0-3)} + 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)} \\
&= \frac{1}{6}(x-2)(x-3) - x(x-3) + \frac{4}{3}x(x-2)
\end{aligned}
$$

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&= \frac{1}{2}x^2 - \frac{1}{2}x + 1
\end{aligned}$$

Example

Interpolate points $(0,1),(2,2),$ and $(3,4)$.

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P_2(x) &= 1\frac{(x-2)(x-3)}{(0-2)(0-3)} + 2\frac{(x-0)(x-3)}{(2-0)(2-3)} + 4\frac{(x-0)(x-2)}{(3-0)(3-2)} \\
&= \frac{1}{6}(x-2)(x-3) - x(x-3) + \frac{4}{3}x(x-2) \\
&= \frac{1}{2}x^2 - \frac{1}{2}x + 1
\end{aligned}$$

We can check the interpolation property:

$$P_2(0) = 1,\ P_2(2) = 2,\ \text{and } P_3(3) = 4.$$

Thm: Main Theorem of Polynomial Interpolation
Let $(x_1, y_1), \ldots (x_n, y_n)$ be n points in the plane with distinct x_i. Then there exists **one and only one** polynomial of degree $n-1$ or less that satisfies $P(x_i) = y_i \; \forall i = 1, \ldots, n$.

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Thm: Main Theorem of Polynomial Interpolation
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Uniqueness Proof: Let $P(x)$ and $Q(x)$ be distinct $n-1$ degree polynomials that interpolate the n points. $H(x) = P(x) - Q(x)$ is of degree $n-1$ and $H(x_i) = 0 \; \forall i = 1, \ldots, n \to H(x)$ has n distinct zeros. Fundamental Theorem of Algebra states that polynomial of degree d has at most d zeros or its is the zero polynomial$\to H(x) = 0 = P(x) - Q(x) \to P(x) = Q(x)$ ※

Example

Find polynomial of degree three or less interpolating $(0,2),(1,1),(2,0)$ and $(3,-1)$.

$$
\begin{aligned}
P(x) = & 2\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + 1\frac{x(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\
& + 0\frac{x(x-1)(x-3)}{(2-0)(2-1)(2-3)} - 1\frac{x(x-1)(x-2)}{(3-0)(3-1)(3-2)}
\end{aligned}
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Example

Find polynomial of degree three or less interpolating $(0, 2), (1, 1), (2, 0)$ and $(3, -1)$.

$$\begin{aligned} P(x) =& 2\frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} + 1\frac{x(x-2)(x-3)}{(1-0)(1-2)(1-3)} \\ & + 0\frac{x(x-1)(x-3)}{(2-0)(2-1)(2-3)} - 1\frac{x(x-1)(x-2)}{(3-0)(3-1)(3-2)} \\ & = -x + 2, \end{aligned}$$

which is a line and the points are colinear. There is no parabola (degree=2) and no cubic (degree=3) passing through the same points.

Simpler way of writing the interpolating polynomial.

Def: Denote by $f[x_1 \ldots x_n]$, the coefficient of the x^{n-1} term in the unique polynomial that interpolates $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

$$
\begin{aligned}
P(x) = & f[x_1] + f[x_1\, x_2](x - x_1) + f[x_1\, x_2\, x_3](x - x_1)(x - x_2) + \ldots \\
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$$f[x_k] = f(x_k)$$

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f[x_k] = f(x_k)
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$$
f[x_k\, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}
$$

Newton's Divided Differences

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$$f[x_k] = f(x_k)$$

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$$f[x_k\, x_{k+1}\, x_{k+2}] = \frac{f[x_{k+1}\, x_{k+2}] - f[x_k\, x_{k+1}]}{x_{k+2} - x_k}$$

Simpler way of writing the interpolating polynomial.

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$$\begin{aligned} f[x_k] &= f(x_k) \\ f[x_k\, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\ f[x_k\, x_{k+1}\, x_{k+2}] &= \frac{f[x_{k+1}\, x_{k+2}] - f[x_k\, x_{k+1}]}{x_{k+2} - x_k} \\ f[x_k\, x_{k+1}\, x_{k+2}\, x_{k+3}] &= \frac{f[x_{k+1}\, x_{k+2}\, x_{k+3}] - f[x_k\, x_{k+1}\, x_{k+2}]}{x_{k+3} - x_k} \\ \cdots &= \cdots \end{aligned}$$

Newton's Divided Differences: The Notation

Interpolate $(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3))$.

$$\begin{array}{c|l}
x_1 & f[x_1] = f(x_1) \\
 & \\
x_2 & f[x_2] = f(x_2) \\
 & \\
x_3 & f[x_3] = f(x_3)
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x_2 & f[x_2] = f(x_2) & & \\
 & & f[x_2\, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} & \\
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& \qquad f[x_1\, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} \\
x_2 & f[x_2] = f(x_2) \qquad\qquad f[x_1\, x_2\, x_3] = \frac{f[x_2\, x_3]-f[x_1\, x_2]}{x_3-x_1} \\
& \qquad f[x_2\, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} \\
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x_1 & f[x_1] = f(x_1) & & \\
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x_2 & f[x_2] = f(x_2) & & f[x_1\, x_2\, x_3] = \frac{f[x_2\, x_3] - f[x_1\, x_2]}{x_3 - x_1} \\
 & & f[x_2\, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} & \\
x_3 & f[x_3] = f(x_3) & &
\end{array}
$$

And the interpolating polynomial is

$$P_2(x) = f[x_1] + f[x_1\, x_2](x - x_1) + f[x_1\, x_2\, x_3](x - x_1)(x - x_2)$$

Example

Use divided differences to find the interpolating polynomial passing through $(0,1), (2,2),$ and $(3,4)$.

$$\begin{array}{c|c} 0 & 1 \\ \\ 2 & 2 \\ \\ 3 & 4 \end{array}$$

Use divided differences to find the interpolating polynomial passing through $(0,1), (2,2),$ and $(3,4)$.

$$
\begin{array}{c|cc}
0 & 1 & \\
 & & \frac{2-1}{2-0} = \frac{1}{2} \\
2 & 2 & \\
 & & \\
3 & 4 &
\end{array}
$$

Example

Use divided differences to find the interpolating polynomial passing through $(0,1),(2,2),$ and $(3,4)$.

$$
\begin{array}{c|lll}
0 & 1 & & \\
 & & \frac{2-1}{2-0} = \frac{1}{2} & \\
2 & 2 & & \\
 & & \frac{4-2}{3-2} = \frac{2}{1} = 2 & \\
3 & 4 & &
\end{array}
$$

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\begin{array}{c|cccc}
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2 & 2 & & & \frac{2-\frac{1}{2}}{3-0} = \frac{1}{2} \\
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\end{array}
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And the interpolating polynomial is

$$P_2(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2)$$

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3 & 4 & &
\end{array}
$$

And the interpolating polynomial is

$$
\begin{aligned}
P_2(x) &= 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2) \\
&= 1 + \frac{1}{2}x^2 - \frac{2}{2}x \\
&= \frac{1}{2}x^2 - \frac{1}{2}x + 1
\end{aligned}
$$

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$$\begin{array}{c|c}
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 & \\
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 & \\
3 & 4 \\
 & \\
1 & 0
\end{array}$$

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\begin{array}{c|cccc}
0 & 1 & & & \\
 & & \frac{1}{2} & & \\
2 & 2 & & \frac{1}{2} & \\
 & & 2 & & \\
3 & 4 & & & \\
 & & & & \\
1 & 0 & & &
\end{array}
$$

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0 & 1 & & & \\
 & & \frac{1}{2} & & \\
2 & 2 & & \frac{1}{2} & \\
 & & 2 & & \\
3 & 4 & & & \\
 & & \frac{0-4}{1-3} = 2 & & \\
1 & 0 & & &
\end{array}
$$

Use divided differences to find the interpolating polynomial passing through $(0,1),(2,2),(3,4),$ and $(1,0)$.

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\begin{array}{c|cccc}
0 & 1 & & & \\
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 & & \frac{0-4}{1-3} = 2 & & \\
1 & 0 & & &
\end{array}
$$

And the interpolating polynomial is

$$P_2(x) = \underbrace{1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2)}_{\text{same as before}} - \frac{1}{2}(x-0)(x-2)(x-3)$$

Find an approximating polynomial to $f(x) = \sin(x)$ by interpolating at four equally spaced points on $[0, \frac{\pi}{2}]$.

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$\frac{2\pi}{6}$	0.8660
$\frac{3\pi}{6}$	1.0000

Approximating Functions by Polynomials

Find an approximating polynomial to $f(x) = \sin(x)$ by interpolating at four equally spaced points on $[0, \frac{\pi}{2}]$.

$$
\begin{array}{c|cccc}
0 & 0.0000 & & & \\
 & & 0.9549 & & \\
\frac{\pi}{6} & 0.5000 & & -0.2443 & \\
 & & 0.6990 & & -0.1139 \\
\frac{2\pi}{6} & 0.8660 & & -0.4232 & \\
 & & 0.2559 & & \\
\frac{3\pi}{6} & 1.0000 & & &
\end{array}
$$

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$$P(x) = 0.0000 + 0.9545x - 0.2443x(x - \frac{\pi}{6}) - 0.1139x(x - \frac{\pi}{6})(x - \frac{2\pi}{6})$$

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P(x) &= 0.0000 + 0.9545x - 0.2443x(x - \frac{\pi}{6}) - 0.1139x(x - \frac{\pi}{6})(x - \frac{2\pi}{6}) \\
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$[0, \frac{\pi}{2}]$ is called the fundamental domain. It is sufficient for approximating $\sin(x)$ since if $x \in [\frac{\pi}{2}, \pi] \to \sin(x) = \sin(\pi - x)$ and if $x \in [\pi, 2\pi] \to \sin(x) = -\sin(2\pi - x)$.

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x	$\sin(x)$	$P(x)$	$\lvert\sin(x) - P(x)\rvert$
1	0.8415	0.8411	0.0004
2	0.9093	0.9192	0.0009
3	0.1411	0.1428	0.0017
4	-0.7568	-0.7557	0.0011
14	0.9906	0.9928	0.0022
1000	0.8269	0.8263	0.0006

Interpolation Error

Thm: Assume that $P(x)$ is the (degree $n-1$ or less) interpolating polynomial fitting the n points $(x_1, y_1), \ldots, (x_n, y_n)$ sampled from a function $f(x)$.

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$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{n!} f^{(n)}(c),$$

where $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$ and c lies between the largest and the smallest of $x_1, x_2, \ldots, x_n$.

Example

$$
\begin{aligned}
\sin(x) - P(x) &= \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} f^{(4)}(c) \\
&= \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} \sin(c)
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|\sin(x) - P(x)| &\leq \frac{\left|(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})\right|}{24} |1|
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- At $x = 1$, the worst case error is ≈ 0.0005348 (> 0.0004 from the table of actual errors that we obtained).

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$$= \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} \sin(c)$$
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- At $x = 1$, the worst case error is ≈ 0.0005348 (> 0.0004 from the table of actual errors that we obtained).
- Error will be smaller in the middle of the interval due to smaller terms in the numerator.

Runge Phenomenon

Try to interpolate with data points

$$x \in \{-3, -2.5, -2.0, \ldots, 2.0, 2.5, 3.0\}$$

with y equal to zeros except at $x = 0$, $y = 1$.

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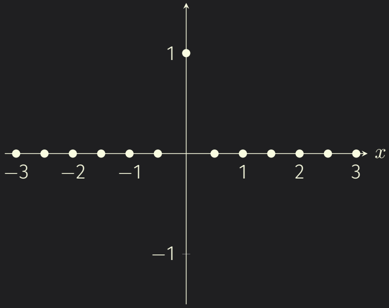

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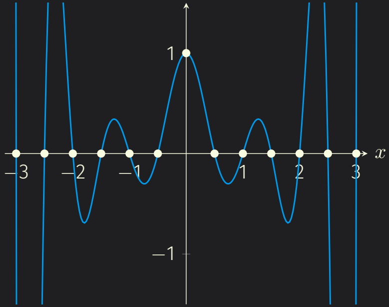

Very large errors near the ends of the interval!

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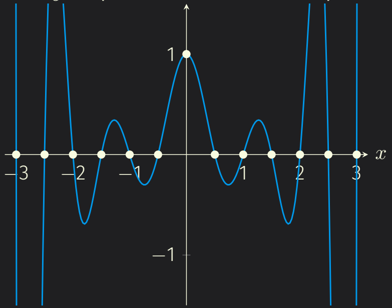

Very large errors near the ends of the interval!

$\rightarrow$ Place more points closer to the end points instead of uniformly sampling $x \rightarrow$ **Chebyshev Interpolation** (which we will skip for the course. See §3.3 for more.)