CENG 216 – Numerical Computation

Systems of Equations - Part I

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Slide Credits

Slides are based on material from the main textbook:

“Numerical Analysis”, The new international edition, 2ed,
by Timothy Sauer

Systems of Equations

- Previous weeks: Find the root of a single equation

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- Next two weeks: Find a common solution to sets of equations

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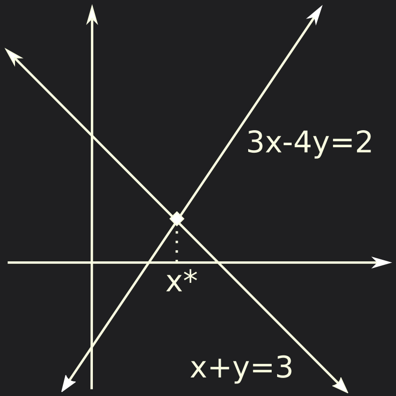
- This week (Part I): Sections 2.1 and 2.2 from the main textbook

A System of Equations

Given the system

$$x + y = 3$$
$$3x - 4y = 2,$$

our aim is to find the common solution x^* to both equations.



Gaussian Elimination

Naive Gaussian Elimination

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In matrix notation:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{Subtract } 3\times\text{row 1}} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right] \rightarrow y = 1, x = 2$$

Example

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$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y - 2z &= 3 \\ -3x + y + z &= -6 \end{aligned} \quad \rightarrow \quad \begin{aligned} x + \ 2y \ - \ \ z \ &= \ 3 \\ -3y \qquad &= -3 \\ -2z &= -4 \end{aligned} \xrightarrow{\text{backsubstitution}} \begin{aligned} z &= 2 \\ y &= 1 \\ x &= 3 \end{aligned}$$

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$$\begin{aligned}
\frac{2}{3}500^3 t_{op} &= 1\,\mathrm{sec.} \\
500^2 t_{op} &= \frac{\frac{3}{2}}{500}\,\mathrm{sec.} \\
&\approx 0.003\,\mathrm{sec.}
\end{aligned}$$

The LU Factorization

Matrix Form of Gaussian Elimination

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Example

Find the LU factorization of $\mathtt{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$.

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$$\mathtt{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \mathtt{A}.$$

Back substitution with LU

$$\mathtt{A}\mathbf{x} = \mathbf{b} \rightarrow \mathtt{LU}\mathbf{x} = \mathbf{b}$$

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$$\mathtt{A}\mathbf{x} = \mathbf{b} \rightarrow \mathtt{L}\mathtt{U}\mathbf{x} = \mathbf{b}$$

1. Define $\mathbf{c} = \mathtt{U}\mathbf{x}$ and solve $\mathtt{L}\mathbf{c} = \mathbf{b}$ for $\mathbf{c}$.

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2. Solve $\mathtt{U}\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$.

Example

$$\overbrace{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}}^{\mathtt{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}}^{\mathbf{b}}$$

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$$\begin{aligned} c_1 &= 3 \\ \rightarrow c_2 &= -3 \\ c_3 &= -6 + 9 - 7 = -4 \end{aligned}$$

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$$\to c_2 = -3$$

$$c_3 = -6 + 9 - 7 = -4$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{\mathbf{U}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}}_{\mathbf{c}} \quad \begin{aligned} x_3 &= 2 \\ \to x_2 &= 1 \\ x_1 &= 3 + 2 - 2 = 3. \end{aligned}$$

Solving multiple equations with the same $\mathtt{A}$ but different $\mathbf{b}$.

$$
\begin{aligned}
\mathtt{A}\mathbf{x} &= \mathbf{b}_1 \\
\mathtt{A}\mathbf{x} &= \mathbf{b}_2 \\
&\vdots \\
\mathtt{A}\mathbf{x} &= \mathbf{b}_k
\end{aligned}
$$

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$$\left.\begin{aligned} \mathtt{A}\mathbf{x} &= \mathbf{b}_1 \\ \mathtt{A}\mathbf{x} &= \mathbf{b}_2 \\ &\vdots \\ \mathtt{A}\mathbf{x} &= \mathbf{b}_k \end{aligned}\right\} \begin{array}{r} \text{cost of naive} \\ \text{Gaussian elimination} \end{array} \to \frac{2}{3}kn^3$$

Complexity of LU Decomposition

Solving multiple equations with the same $\mathtt{A}$ but different $\mathbf{b}$.

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Solving k equations with LU factorization means that we need to run the costly elimination step ($\frac{2}{3}n^3$ operations) only once. For some applications this can be done in an offline preprocessing step and the solutions can be performed online as data becomes available.

Ex: Prove that $\mathtt{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU factorization although it is invertible.

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Next week, we will generalize LU factorization to handle these kinds of matrices.