CENG 216 – Numerical Computation

Systems of Equations - Part II

Mustafa Özuysal
mustafaozuysal@iyte.edu.tr

March 13, 2022

İzmir Institute of Technology

Slide Credits

Slides are based on material from the main textbook:

"Numerical Analysis", The new international edition, 2ed, by Timothy Sauer

Introduction

Sources of Error

There are two primary sources of error when solving a system of equations using finite precision real numbers:

- Ill-conditioning: The system is not exactly singular, but it is nearly so.

Sources of Error

There are two primary sources of error when solving a system of equations using finite precision real numbers:

- Ill-conditioning: The system is not exactly singular, but it is nearly so. $\rightarrow$ There is not much to do, we should avoid such systems if we can.

There are two primary sources of error when solving a system of equations using finite precision real numbers:

- **Ill-conditioning**: The system is not exactly singular, but it is nearly so. $\rightarrow$ There is not much to do, we should avoid such systems if we can.
- **Swamping**: The system is well-conditioned but the order of equations causes numerical problems.

There are two primary sources of error when solving a system of equations using finite precision real numbers:

- Ill-conditioning: The system is not exactly singular, but it is nearly so. $\rightarrow$ There is not much to do, we should avoid such systems if we can.
- Swamping: The system is well-conditioned but the order of equations causes numerical problems.$\rightarrow$ Solve by pivoting.

Sources of Error: Ill-conditioning

- For root finding, we defined forward and backward errors as the **absolute value** of the difference between two scalars (for example the estimated and the true values of the root). For a system of equations, we need the equivalent of **absolute value** for vectors.

- For root finding, we defined forward and backward errors as the absolute value of the difference between two scalars (for example the estimated and the true values of the root). For a system of equations, we need the equivalent of absolute value for vectors.
- **Def:** Infinity (maximum) norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$ is $\|\mathbf{x}\|_\infty = \max |x_i|, i = 1, \ldots, n$.

- For root finding, we defined forward and backward errors as the absolute value of the difference between two scalars (for example the estimated and the true values of the root). For a system of equations, we need the equivalent of absolute value for vectors.
- Def: Infinity (maximum) norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top$ is $\|\mathbf{x}\|_\infty = \max |x_i|, i = 1, \ldots, n$.
- Def: Let $\mathbf{x}_a$ be an approximate solution to $\mathtt{A}\mathbf{x} = \mathbf{b}$. Then the residual is $\mathbf{r} = \mathbf{b} - \mathtt{A}\mathbf{x}_a$, the backward error is $\|\mathbf{r}\|_\infty = \|\mathbf{b} - \mathtt{A}\mathbf{x}_a\|_\infty$, and the forward error is $\|\mathbf{x} - \mathbf{x}_a\|_\infty$.

Example

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [1, 1]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Example

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [1, 1]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ with true solution } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [1, 1]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ with true solution } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Backward Error } &= \|\mathbf{b} - \mathbf{A}\mathbf{x}_a\|_\infty = \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\|_\infty = 3 \end{aligned}$$

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [1, 1]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ with true solution } \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Backward Error } &= \|\mathbf{b} - \mathbf{A}\mathbf{x}_a\|_\infty = \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\|_\infty = 3 \\ \text{Forward Error } &= \|\mathbf{x}_a - \mathbf{x}\|_\infty = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\|_\infty = 1 \end{aligned}$$

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [-1, 3.0001]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$$

Example 2

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [-1, 3.0001]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$$

$$\begin{aligned} \text{Backward Error } &= \|\mathbf{r}\|_\infty = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} \right\|_\infty = 0.0001 \end{aligned}$$

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [-1, 3.0001]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -0.0001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -0.0001 \end{bmatrix}$$

$$\rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 1 \end{matrix}$$

$$\text{Backward Error } = \|\mathbf{r}\|_\infty = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} \right\|_\infty = 0.0001$$

Example 2

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [-1, 3.0001]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -0.0001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -0.0001 \end{bmatrix}$$

$$\to \begin{matrix} x_1 = 1 \\ x_2 = 1 \end{matrix}$$

$$\text{Backward Error } = \|\mathbf{r}\|_\infty = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty$$

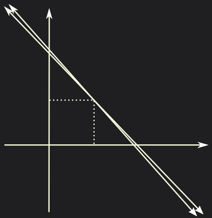
$$= \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} \right\|_\infty = 0.0001$$

$$\text{Forward Error } = \|\mathbf{x}_a - \mathbf{x}\|_\infty = \left\| \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty = 2.0001$$

Ex: Find the backward and forward errors for the approximate solution $\mathbf{x}_a = [-1, 3.0001]^\top$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} \text{ with true solution } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Backward Error } &= \|\mathbf{r}\|_\infty = 0.0001 \\ \text{Forward Error } &= \|\mathbf{x}_a - \mathbf{x}\|_\infty = 2.0001 \end{aligned}$$



Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.

Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- Def: Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.

Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- Def: Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.
- Def: Error Magnification Factor$= \frac{\frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}}{\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}}$.

Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- Def: Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.
- Def: Error Magnification Factor$= \frac{\frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}}{\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}}$.

Values for our example:

$$\text{Relative Backward Error} = \frac{0.0001}{2.0001} \approx 0.00005 = 0.005\%$$

Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- Def: Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.
- Def: Error Magnification Factor$= \frac{\frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}}{\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}}$.

Values for our example:

$$\text{Relative Backward Error} = \frac{0.0001}{2.0001} \approx 0.00005 = 0.005\%$$

$$\text{Relative Forward Error} = \frac{2.0001}{1} = 2.0001 \approx 200\%$$

Error Magnification Factor

- Def: Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- Def: Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.
- Def: Error Magnification Factor$= \frac{\frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}}{\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}}$.

Values for our example:

$$\text{Relative Backward Error} = \frac{0.0001}{2.0001} \approx 0.00005 = 0.005\%$$

$$\text{Relative Forward Error} = \frac{2.0001}{1} = 2.0001 \approx 200\%$$

$$\text{Error Magnification Factor} = \frac{2.0001}{0.00005} \approx 40004.0001$$

Error Magnification Factor

- **Def:** Relative Backward Error$= \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.
- **Def:** Relative Forward Error$= \frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}$.
- **Def:** Error Magnification Factor$= \frac{\frac{\|\mathbf{x}-\mathbf{x}_a\|_\infty}{\|\mathbf{x}\|_\infty}}{\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}}$.

Values for our example:

$$\text{Relative Backward Error} = \frac{0.0001}{2.0001} \approx 0.00005 = 0.005\%$$

$$\text{Relative Forward Error} = \frac{2.0001}{1} = 2.0001 \approx 200\%$$

$$\text{Error Magnification Factor} = \frac{2.0001}{0.00005} \approx 40004.0001$$

Large error magnification factor$\rightarrow$ ill-conditioned system

Matrix Condition Number

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\mathrm{cond}\,(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.

Matrix Condition Number

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\mathrm{cond}(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.
- **Def:** The matrix infinity norm for $n \times n$ matrix $\mathtt{A}$ is

$$\|\mathtt{A}\|_\infty = \text{maximum absolute row sum}$$

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\operatorname{cond}(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.
- **Def:** The matrix infinity norm for $n \times n$ matrix $\mathtt{A}$ is

$$\|\mathtt{A}\|_\infty = \text{maximum absolute row sum}$$

- **Thm:** $\operatorname{cond}(\mathtt{A}) = \|\mathtt{A}\|_\infty \left\|\mathtt{A}^{-1}\right\|_\infty$

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\operatorname{cond}(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.
- **Def:** The matrix infinity norm for $n \times n$ matrix $\mathtt{A}$ is

$$\|\mathtt{A}\|_\infty = \text{maximum absolute row sum}$$

- **Thm:** $\operatorname{cond}(\mathtt{A}) = \|\mathtt{A}\|_\infty \left\|\mathtt{A}^{-1}\right\|_\infty$

$$\mathtt{A} = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \to \|\mathtt{A}\|_\infty = 2.0001$$

Matrix Condition Number

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\operatorname{cond}(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.
- **Def:** The matrix infinity norm for $n \times n$ matrix $\mathtt{A}$ is

$$\|\mathtt{A}\|_\infty = \text{maximum absolute row sum}$$

- **Thm:** $\operatorname{cond}(\mathtt{A}) = \|\mathtt{A}\|_\infty \left\|\mathtt{A}^{-1}\right\|_\infty$

$$\mathtt{A} = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \to \|\mathtt{A}\|_\infty = 2.0001$$

$$\mathtt{A}^{-1} = \begin{bmatrix} -10000 & 10000 \\ 10001 & -10000 \end{bmatrix} \to \left\|\mathtt{A}^{-1}\right\|_\infty = 20001$$

- **Def:** The condition number of a square matrix $\mathtt{A}$, $\operatorname{cond}(\mathtt{A})$, is the maximum possible error magnification factor for solving $\mathtt{A}\mathbf{x} = \mathbf{b}$ for all right hand sides $\mathbf{b}$.
- **Def:** The matrix infinity norm for $n \times n$ matrix $\mathtt{A}$ is

$$\|\mathtt{A}\|_\infty = \text{maximum absolute row sum}$$

- **Thm:** $\operatorname{cond}(\mathtt{A}) = \|\mathtt{A}\|_\infty \|\mathtt{A}^{-1}\|_\infty$

$$\mathtt{A} = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \rightarrow \|\mathtt{A}\|_\infty = 2.0001$$

$$\mathtt{A}^{-1} = \begin{bmatrix} -10000 & 10000 \\ 10001 & -10000 \end{bmatrix} \rightarrow \|\mathtt{A}^{-1}\|_\infty = 20001$$

$$\operatorname{cond}(\mathtt{A}) = (2.0001)(20001) = 40004.1$$

Sources of Error: Swamping

Swamping Example

Consider the system

$$\begin{aligned} 10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

Consider the system

$$10^{-20}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 4$$

Exact Arithmetic:

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{subtract } 10^{20}\times \text{ row 1}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

Swamping Example

Consider the system

$$10^{-20}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 4$$

Exact Arithmetic:

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{subtract } 10^{20}\times \text{ row 1}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

$$x_2 = \frac{4-10^{20}}{2-10^{20}}, \text{ and } x_1 = 10^{20}\left(1 - \frac{4-10^{20}}{2-10^{20}}\right) = \frac{-2\times 10^{20}}{2-10^{20}} \approx 2$$

Consider the system

$$10^{-20}x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 4$$

Exact Arithmetic:

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow[\text{from row 2}]{\text{subtract } 10^{20}\times \text{ row 1}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2-10^{20} & 4-10^{20} \end{array}\right]$$

$$x_2 = \frac{4-10^{20}}{2-10^{20}}, \text{ and } x_1 = 10^{20}\left(1 - \frac{4-10^{20}}{2-10^{20}}\right) = \frac{-2\times 10^{20}}{2-10^{20}} \approx 2$$

$$[x_1, x_2]^\top \approx [2, 1]^\top$$

Swamping Example

Consider the system

$$\begin{aligned} 10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

Exact Arithmetic: $[x_1, x_2]^\top \approx [2, 1]^\top$

Consider the system

$$\begin{aligned} 10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

Exact Arithmetic: $[x_1, x_2]^\top \approx [2, 1]^\top$

IEEE double-precision:

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & \underbrace{2 - 10^{20}}_{\text{rounds to } -10^{20}} & \underbrace{4 - 10^{20}}_{\text{rounds to } -10^{20}} \end{array}\right] \to \begin{array}{l} x_2 = 1 \\ x_1 = 0 \end{array}$$

Swamping Example

Consider the system

$$
\begin{aligned}
10^{-20}x_1 + x_2 &= 1 \\
x_1 + 2x_2 &= 4
\end{aligned}
$$

Exact Arithmetic: $[x_1, x_2]^\top \approx [2, 1]^\top$
IEEE double-precision: $[x_1, x_2]^\top \approx [0, 1]^\top$

Consider the system

$$\begin{aligned} 10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

Exact Arithmetic: $[x_1, x_2]^\top \approx [2, 1]^\top$
IEEE double-precision: $[x_1, x_2]^\top \approx [0, 1]^\top$
IEEE double-precision with Row Exchange:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & \underbrace{1 - 2 \times 10^{-20}}_{\text{rounds to 1}} & \underbrace{1 - 4 \times 10^{-20}}_{\text{rounds to 1}} \end{array}\right] \rightarrow \begin{array}{l} x_2 = 1 \\ x_1 = 2 \end{array}$$

Swamping Example

Consider the system

$$
\begin{aligned}
10^{-20}x_1 + x_2 &= 1\\
x_1 + 2x_2 &= 4
\end{aligned}
$$

Exact Arithmetic: $[x_1, x_2]^\top \approx [2, 1]^\top$
IEEE double-precision: $[x_1, x_2]^\top \approx [0, 1]^\top$
IEEE double-precision with Row Exchange: $[x_1, x_2]^\top \approx [2, 1]^\top$ ✓

$$
\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & \underbrace{1 - 2 \times 10^{-20}}_{\text{rounds to 1}} & \underbrace{1 - 4 \times 10^{-20}}_{\text{rounds to 1}} \end{array}\right] \to \begin{aligned} x_2 &= 1 \\ x_1 &= 2 \end{aligned}
$$

The PA=LU Factorization

Look for the biggest pivot in the column even when the pivot is not zero:

$$|a_{pj}| \geq |a_{ij}|\,,\text{ for } j \leq i \leq n \to \text{ exchange row } p \text{ and row } i$$

Partial Pivoting

Look for the biggest pivot in the column even when the pivot is not zero:

$$|a_{pj}| \geq |a_{ij}|, \text{ for } j \leq i \leq n \rightarrow \text{ exchange row } p \text{ and row } i$$

Partial pivoting solves both the problem with zero pivots and also the swamping problem.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right]$$

Gaussian Elimination with Partial Pivoting

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

Gaussian Elimination with Partial Pivoting

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 2}]{\text{Subtract } -\frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 2}]{\text{Subtract } -\frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } \frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 2}]{\text{Subtract } -\frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } \frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \xrightarrow[\text{rows 2 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Gaussian Elimination with Partial Pivoting

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 2}]{\text{Subtract } -\frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } \frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \xrightarrow[\text{rows 2 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } -\frac{1}{2}\times \text{ row 2}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Gaussian Elimination with Partial Pivoting

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{rows 1 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 2}]{\text{Subtract } -\frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } \frac{1}{2}\times \text{ row 1}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \xrightarrow[\text{rows 2 and 3}]{\text{Exchange}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{from row 3}]{\text{Subtract } -\frac{1}{2}\times \text{ row 2}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \rightarrow \begin{array}{l} x_3 = -1 \\ x_2 = 1 \\ x_1 = 1 \end{array}$$

- We need to find a matrix form that exchanges two rows.

Permutation Matrices

- We need to find a matrix form that exchanges two rows.
- **Def:** A **permutation matrix** is an $n \times n$ matrix with all zeros except a single one in every row and column.

Permutation Matrices

- We need to find a matrix form that exchanges two rows.
- **Def:** A **permutation matrix** is an $n \times n$ matrix with all zeros except a single one in every row and column.
- All possible 2×2 permutation matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- We need to find a matrix form that exchanges two rows.
- **Def:** A **permutation matrix** is an $n \times n$ matrix with all zeros except a single one in every row and column.
- All possible 2×2 permutation matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- All possible 3×3 permutation matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thm: If $\mathtt{P}$ is an $n \times n$ permutation matrix that corresponds to a particular row exchange applied to the identity matrix, then $\mathtt{PA}$ is the matrix which is the result of applying the same row exchanges to $\mathtt{A}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}.$$

To calculate the PA=LU factorization (with pivoting), we carry out elimination with the following differences:

- We start with an identity permutation matrix $\mathtt{P} = \mathtt{I}$.

The PA=LU Factorization

To calculate the PA=LU factorization (with pivoting), we carry out elimination with the following differences:

- We start with an identity permutation matrix $\mathtt{P} = \mathtt{I}$.
- Whenever we exchange rows, we apply the same row exchange to the last $\mathtt{P}$ matrix that we have.

The PA=LU Factorization

To calculate the PA=LU factorization (with pivoting), we carry out elimination with the following differences:

- We start with an identity permutation matrix $\mathtt{P} = \mathtt{I}$.
- Whenever we exchange rows, we apply the same row exchange to the last $\mathtt{P}$ matrix that we have.
- Since the position of the row multipliers that we put in $\mathtt{L}$ will be affected by row exchanges, we keep them in the position of the zeros of the $\mathtt{A}$ matrix that appear due to elimination. We circle them to remember that these are zeros of the $\mathtt{A}$ matrix.

To calculate the PA=LU factorization (with pivoting), we carry out elimination with the following differences:

- We start with an identity permutation matrix $\mathtt{P} = \mathtt{I}$.
- Whenever we exchange rows, we apply the same row exchange to the last $\mathtt{P}$ matrix that we have.
- Since the position of the row multipliers that we put in $\mathtt{L}$ will be affected by row exchanges, we keep them in the position of the zeros of the $\mathtt{A}$ matrix that appear due to elimination. We circle them to remember that these are zeros of the $\mathtt{A}$ matrix.
- Every time we exchange rows, these multipliers will also change their position. See the example in the next slide to see how this works in practice.

Example

Find the PA=LU factorization of $\mathtt{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathtt{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathtt{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathtt{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathtt{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4} \times r_1 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4} \times r_1 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$$

Find the PA=LU factorization of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathrm{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4} \times r_1 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix} \xrightarrow[\mathrm{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathtt{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathtt{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4} \times r_1 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix} \xrightarrow[\mathtt{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

Example

Find the PA=LU factorization of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$. Set $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}]{\text{exchange } r_1 \text{ and } r_2} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow[\frac{1}{2} \times r_1 \text{ from } r_2]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{4} \times r_1 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix} \xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix}$$

$$\xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}.$$

Example

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \cdots$$

$$\xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix} \xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{\mathbf{A}} =$$

Example

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \cdots$$

$$\xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix} \xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{\mathbf{A}} =$$

Example

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \cdots$$

$$\xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix} \xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}}_{\mathbf{L}}$$

Example

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \cdots$$

$$\xrightarrow[\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}]{\text{exchange } r_2 \text{ and } r_3} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -1 & 7 \end{bmatrix} \xrightarrow[-\frac{1}{2} \times r_2 \text{ from } r_3]{\text{subtract}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{4} & 2 & 2 \\ \frac{1}{2} & -\frac{1}{2} & 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}_{\mathbf{U}}$$

To solve $\mathtt{A}\mathbf{x} = \mathbf{b}$, multiply both sides with $\mathtt{P}$ and use $\mathtt{PA} = \mathtt{LU}$:

$$\begin{aligned} \mathtt{A}\mathbf{x} = \mathbf{b} &\longrightarrow \mathtt{PA}\mathbf{x} = \mathtt{P}\mathbf{b} \\ &\longrightarrow \mathtt{LU}\mathbf{x} = \mathtt{P}\mathbf{b} \end{aligned}$$

To solve $\mathtt{A}\mathbf{x} = \mathbf{b}$, multiply both sides with $\mathtt{P}$ and use $\mathtt{PA} = \mathtt{LU}$:

$$\begin{aligned}\mathtt{A}\mathbf{x} = \mathbf{b} &\longrightarrow \mathtt{PA}\mathbf{x} = \mathtt{P}\mathbf{b}\\ &\longrightarrow \mathtt{LU}\mathbf{x} = \mathtt{P}\mathbf{b}\end{aligned}$$

- First, solve $\mathtt{L}\mathbf{c} = \mathtt{P}\mathbf{b}$ for $\mathbf{c}$.

Solving Equation Systems with PA=LU

To solve $\mathtt{A}\mathbf{x} = \mathbf{b}$, multiply both sides with $\mathtt{P}$ and use $\mathtt{PA} = \mathtt{LU}$:

$$\begin{aligned} \mathtt{A}\mathbf{x} = \mathbf{b} &\longrightarrow \mathtt{PA}\mathbf{x} = \mathtt{P}\mathbf{b} \\ &\longrightarrow \mathtt{LU}\mathbf{x} = \mathtt{P}\mathbf{b} \end{aligned}$$

- First, solve $\mathtt{L}\mathbf{c} = \mathtt{P}\mathbf{b}$ for $\mathbf{c}$.
- Second, solve $\mathtt{U}\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$.

Non-linear System of Equations

So far, we have discussed only systems of linear equations in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$. What if the equations are non-linear possibly involving powers of x_i?

So far, we have discussed only systems of linear equations in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$. What if the equations are non-linear possibly involving powers of x_i?

- Intersections of arbitrary curves instead of lines $\rightarrow$ there might be multiple solution points (Think of the intersection of two circles).

Non-linear System of Equations

So far, we have discussed only systems of linear equations in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$. What if the equations are non-linear possibly involving powers of x_i?

- Intersections of arbitrary curves instead of lines $\rightarrow$ there might be multiple solution points (Think of the intersection of two circles).
- We can not have a closed form solution in general, so we will need iterative methods.

So far, we have discussed only systems of linear equations in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$. What if the equations are non-linear possibly involving powers of x_i?

- Intersections of arbitrary curves instead of lines $\rightarrow$ there might be multiple solution points (Think of the intersection of two circles).
- We can not have a closed form solution in general, so we will need iterative methods.
- We can extend iterative root-finding approaches to the simultaneous solution of multiple equations.

To derive the update rule for Newton's Method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k + \Delta_k$$

To derive the update rule for Newton's Method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k + \Delta_k$$

we linearized the function $f(x)$ around a guess for the root x_k:

$$f(\underbrace{x_k + \Delta_k}_{x_{k+1}}) \approx f(x_k) + \Delta_k f'(x_k) = 0.$$

We need to do a similar approximation to find a common root of the nonlinear system of equations,

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_N) &= 0 \\
f_2(x_1, x_2, \ldots, x_N) &= 0 \\
&\vdots \\
f_M(x_1, x_2, \ldots, x_N) &= 0
\end{aligned}
$$

We need to do a similar approximation to find a common root of the nonlinear system of equations,

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_N) &= 0 \\
f_2(x_1, x_2, \ldots, x_N) &= 0 \\
&\vdots \\
f_M(x_1, x_2, \ldots, x_N) &= 0
\end{aligned}
$$

which we can write in vector form as

$$\mathbf{f}(\mathbf{x}) = 0_M$$

where $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^M$ and 0_M is a column vector of M zeros.

The analogue of the derivative for a vector function such as $\mathbf{f}(\mathbf{x})$ is the Jacobian matrix

$$\mathrm{J}_{\mathbf{f}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_N} \end{bmatrix},$$

which is an $M \times N$ matrix storing partial derivative of each component function w.r.t. each variable dimension.

The Jacobian for the system

$$\begin{aligned} x_2 - x_1^3 &= 0 \\ x_1^2 + x_2^2 - 1 &= 0 \end{aligned}$$

is

The Jacobian for the system

$$
\begin{aligned}
x_2 - x_1^3 &= 0 \\
x_1^2 + x_2^2 - 1 &= 0
\end{aligned}
$$

is

$$
\mathsf{J} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix},
$$

The Jacobian for the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

is

$$\mathsf{J} = \begin{bmatrix} -3x_1^2 & \\ & \end{bmatrix},$$

The Jacobian: The Example

The Jacobian for the system

$$
\begin{aligned}
x_2 - x_1^3 &= 0 \\
x_1^2 + x_2^2 - 1 &= 0
\end{aligned}
$$

is

$$
\mathsf{J} = \begin{bmatrix} -3x_1^2 & 1 \\ & \end{bmatrix},
$$

The Jacobian: The Example

The Jacobian for the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

is

$$\mathsf{J} = \begin{bmatrix} -3x_1^2 & 1 \\ 2x_1 & \end{bmatrix},$$

The Jacobian for the system

$$\begin{aligned} x_2 - x_1^3 &= 0 \\ x_1^2 + x_2^2 - 1 &= 0 \end{aligned}$$

is

$$\mathsf{J} = \begin{bmatrix} -3x_1^2 & 1 \\ 2x_1 & 2x_2 \end{bmatrix},$$

Now, we can expand the nonlinear system around a given point $\mathbf{x}_k$

$$\mathbf{f}(\underbrace{\mathbf{x}_k + \boldsymbol{\Delta}_k}_{\mathbf{x}_{k+1}}) \approx \mathbf{f}(\mathbf{x}_k) + \mathrm{J}_{\mathbf{f}}\boldsymbol{\Delta}_k = 0_M$$

Now, we can expand the nonlinear system around a given point $\mathbf{x}_k$

$$\mathbf{f}(\underbrace{\mathbf{x}_k + \boldsymbol{\Delta}_k}_{\mathbf{x}_{k+1}}) \approx \mathbf{f}(\mathbf{x}_k) + \mathrm{J}_{\mathbf{f}}\boldsymbol{\Delta}_k = 0_M$$

then we can compute the step size by solving the linear system

$$\mathrm{J}_{\mathbf{f}}\boldsymbol{\Delta}_k = -\mathbf{f}(\mathbf{x}_k)$$

Now, we can expand the nonlinear system around a given point $\mathbf{x}_k$

$$\mathbf{f}(\underbrace{\mathbf{x}_k + \boldsymbol{\Delta}_k}_{\mathbf{x}_{k+1}}) \approx \mathbf{f}(\mathbf{x}_k) + \mathrm{J}_{\mathbf{f}} \boldsymbol{\Delta}_k = 0_M$$

then we can compute the step size by solving the linear system

$$\mathrm{J}_{\mathbf{f}} \boldsymbol{\Delta}_k = -\mathbf{f}(\mathbf{x}_k)$$

and iterate with

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \boldsymbol{\Delta}_k.$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_0)\boldsymbol{\Delta}_0 = -\mathbf{f}(\mathbf{x}_0) \rightarrow$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_0)\boldsymbol{\Delta}_0 = -\mathbf{f}(\mathbf{x}_0) \to \begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_0)\boldsymbol{\Delta}_0 = -\mathbf{f}(\mathbf{x}_0) \rightarrow \begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_0)\boldsymbol{\Delta}_0 = -\mathbf{f}(\mathbf{x}_0) \rightarrow \begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

starting at $\mathbf{x}_0 = [1, 2]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_0)\boldsymbol{\Delta}_0 = -\mathbf{f}(\mathbf{x}_0) \rightarrow \begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_1)\mathbf{\Delta}_1 = -\mathbf{f}(\mathbf{x}_1) \rightarrow$$

Example

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_1)\boldsymbol{\Delta}_1 = -\mathbf{f}(\mathbf{x}_1) \to \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_1)\mathbf{\Delta}_1 = -\mathbf{f}(\mathbf{x}_1) \to \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_1)\boldsymbol{\Delta}_1 = -\mathbf{f}(\mathbf{x}_1) \to \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\to \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{8} \end{bmatrix}$$

Make to steps of the Newton's Method to solve the system

$$x_2 - x_1^3 = 0$$
$$x_1^2 + x_2^2 - 1 = 0$$

$\mathbf{x}_1 = [1, 1]^\top$.

$$\mathrm{J}_{\mathbf{f}}(\mathbf{x}_1)\boldsymbol{\Delta}_1 = -\mathbf{f}(\mathbf{x}_1) \rightarrow \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{8} \end{bmatrix}$$
$$\rightarrow \mathbf{x}_2 = \begin{bmatrix} \frac{7}{8} \\ \frac{5}{8} \end{bmatrix}$$