CENG 216 – Numerical Computation

Solving Equations

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Slide Credits

Slides are based on material from the main textbook:

"Numerical Analysis", The new international edition, 2ed, by Timothy Sauer

Introduction

Iterative Numerical Solution of $f(x) = 0$

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  - Complexity: Evaluation of $f(x_i)$ and possibly $f'(x_i), f''(x_i), \ldots$

The Bisection Method

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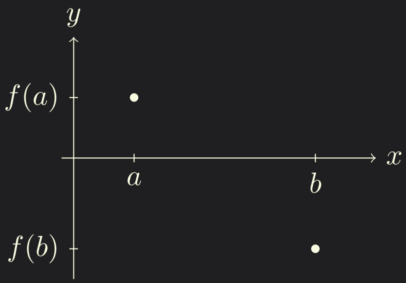

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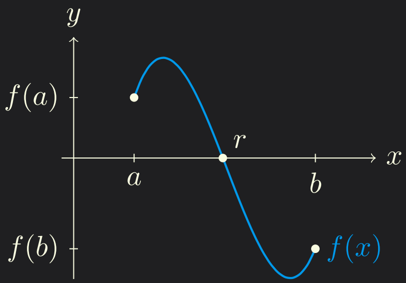

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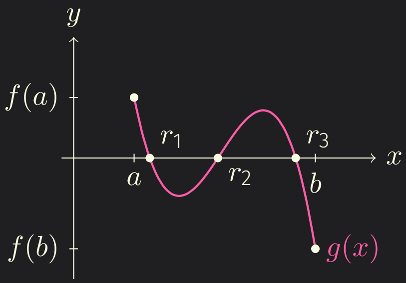

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while (b−a)/2 > TOL do
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        stop
    end
end
```

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- The final interval $[a, b]$ contains the root, which is approximately at $\frac{a+b}{2}$.

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$r_c = 0.682617, \quad \text{max. error} = \pm 0.000977$

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- At $i = n$: Absolute Error $< \frac{b-a}{2^{n+1}}$
- Number of required function evaluations is $n + 2$.
- **Def:** A solution is correct to p decimal places if the error is less than $0.5 \cdot 10^{-p}$.

Find the number of iterations of the bisection method required to compute a root of the function $f(x) = \cos x - x$ in $[0, 1]$ to six correct decimal places.

Example

Find the number of iterations of the bisection method required to compute a root of the function $f(x) = \cos x - x$ in $[0, 1]$ to six correct decimal places.

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$$\frac{1}{2^{n+1}} < 0.5 \cdot 10^{-6} \to n > \frac{6}{\log_{10} 2} \approx \frac{6}{0.301} \approx 19.9$$

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$$\frac{1}{2^{n+1}} < 0.5 \cdot 10^{-6} \to n > \frac{6}{\log_{10} 2} \approx \frac{6}{0.301} \approx 19.9$$

Take $n = 20$.

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$$x_2 = \cos(1.0) = 0.5403023058681398$$

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$$\begin{aligned} x_1 &= \cos(0.0) = 1.0 \\ x_2 &= \cos(1.0) = 0.5403023058681398 \\ &\vdots \\ x_k &= \cos(0.739085133215) = 0.739085133215 \end{aligned}$$

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$$y^3 = 1.0 - y$$
$$y = \sqrt[3]{1.0 - y} = q(y)$$

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$$
\begin{aligned}
0 &= z^3 + z - 1.0 \\
z &= 1 - z^3 \\
3z^3 + z &= 1 + 2z^3 \\
(3z^2 + 1)z &= 1 + 2z^3 \\
z &= \frac{1 + 2z^3}{3z^2 + 1} = r(z)
\end{aligned}
$$

Root Finding Example

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0	0.50000000	0.50000000	0.50000000

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1	0.87500000	0.79370053	0.71428571

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1	0.87500000	0.79370053	0.71428571
2	0.33007813	0.59088011	0.68317972

Root Finding Example

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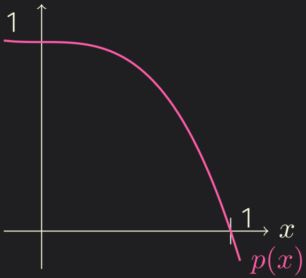
10	0.00000000	0.67625892	0.68232780
11	1.00000000	0.68664554	0.68232780
12	0.00000000	0.67922234	0.68232780
$\vdots$	$\vdots$	$\vdots$	$\vdots$
24	0.00000000	0.68227157	0.68232780
25	1.00000000	0.68236807	0.68232780

Cobweb Diagrams



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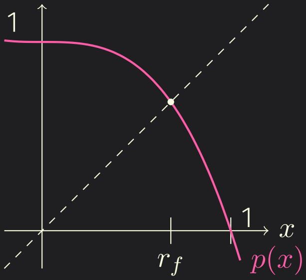

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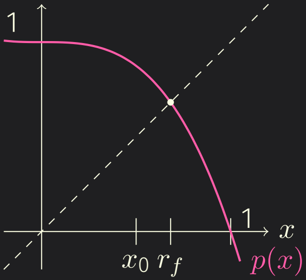

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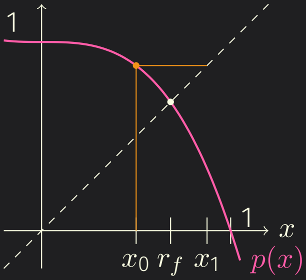

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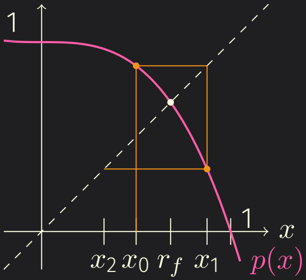

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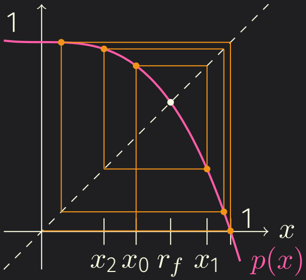

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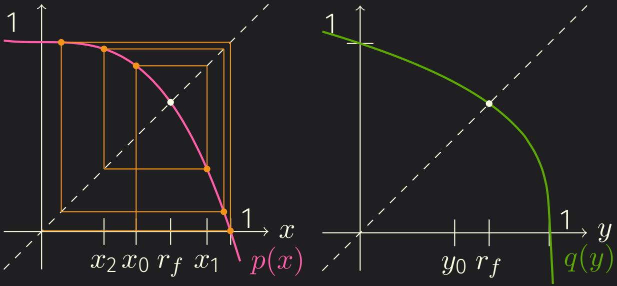

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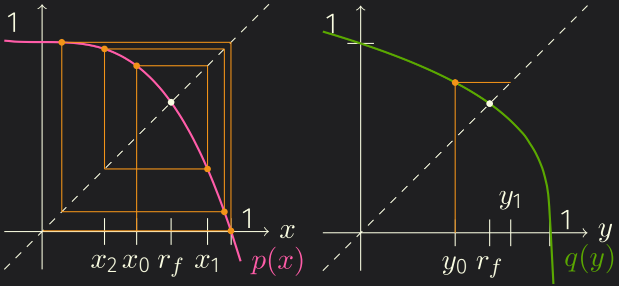

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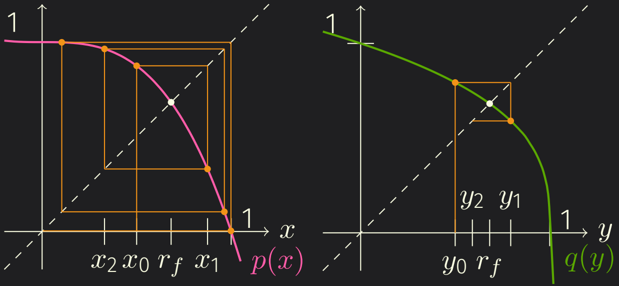

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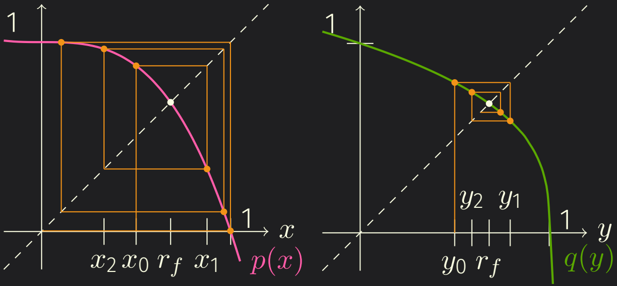

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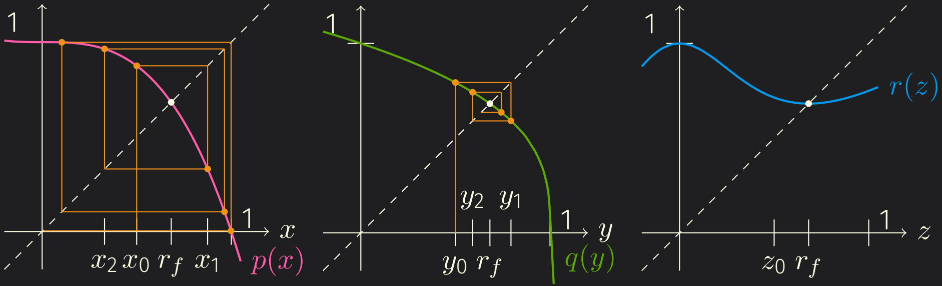

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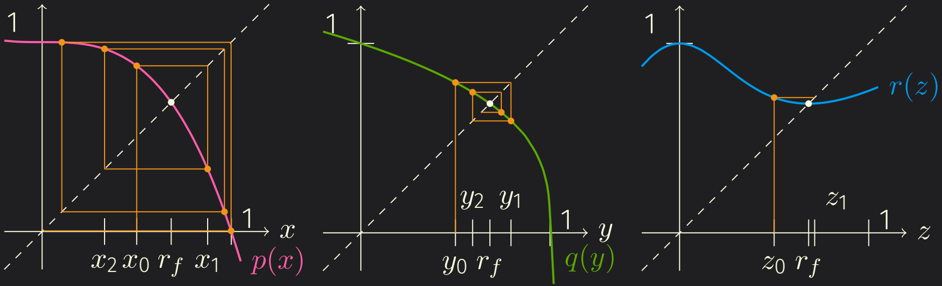

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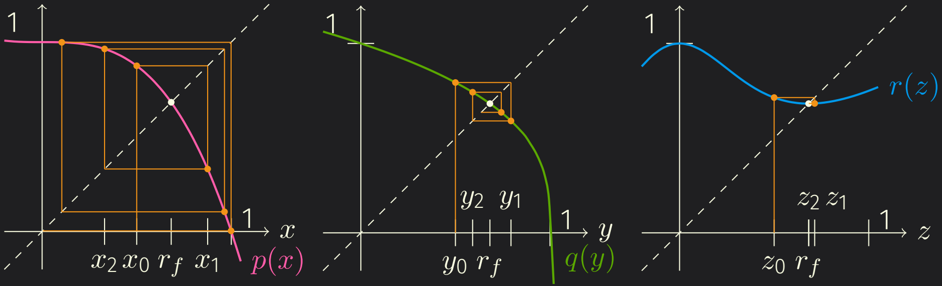

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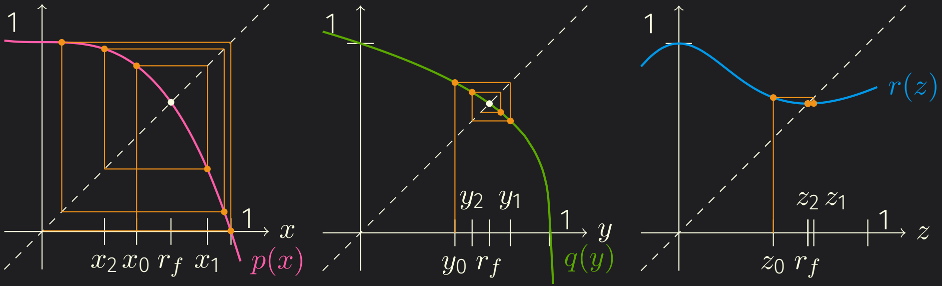

- **Def:** Let $e_i = |x_i - r|$ be the error at step i. If

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = s < 1$$

then the method is linearly convergent with rate s.

Linear Convergence of Fixed Point Iterations

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- $f(x) = \cos x$, $f'(x) = -\sin x$, $f'(0.739085133215) \approx -0.67$.

Example

Find the fixed points of $2.8x - x^2 = f(x)$.

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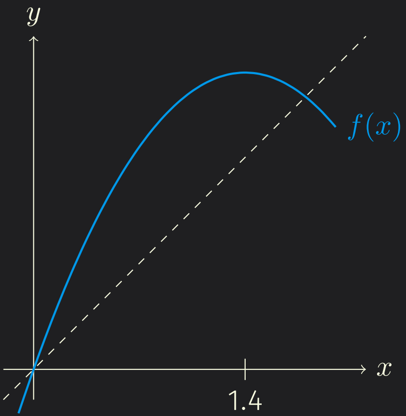

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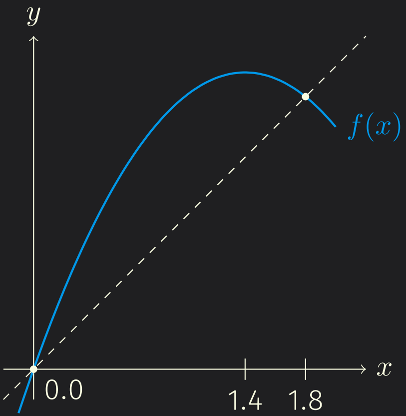

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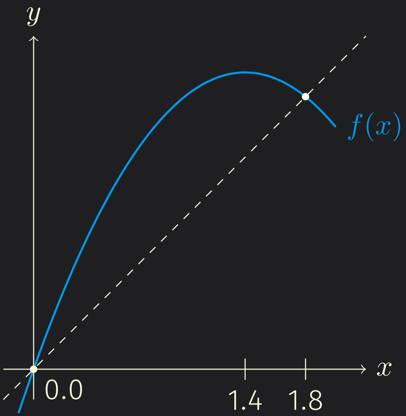

$x_0 = 0.1000$

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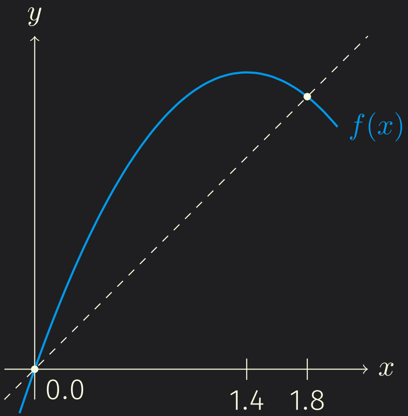

$$x_0 = 0.1000$$
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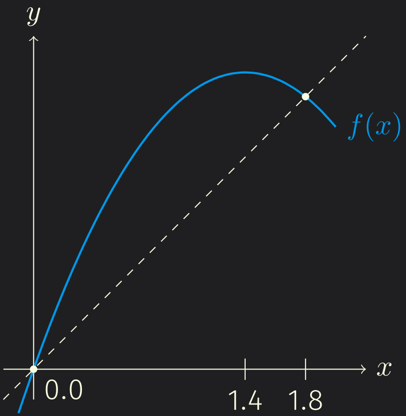

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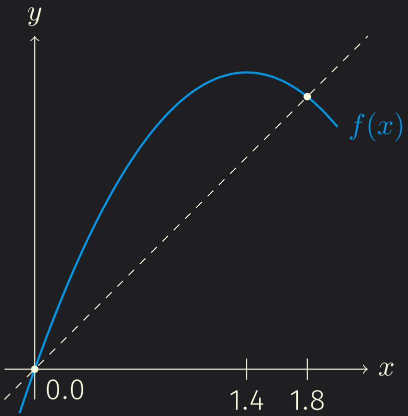

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$$f'(x) = 2.8 - 2x$$

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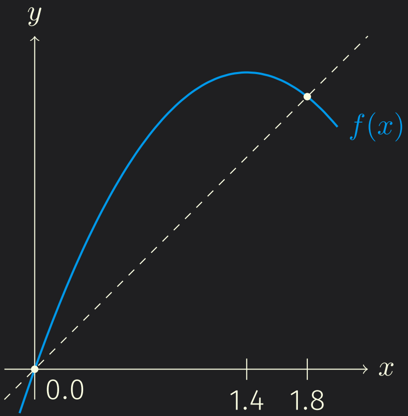

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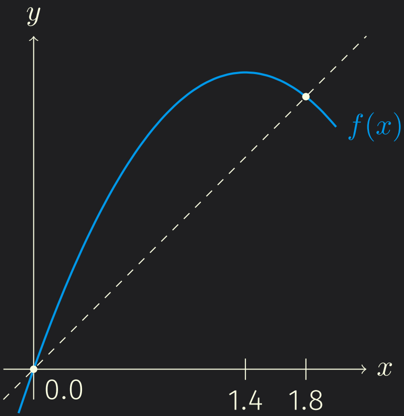

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f'(x) &= 2.8 - 2x\\
f'(1.8) &= -0.8\\
f'(0.0) &= 2.8
\end{aligned}
$$

$$|x_{i+1} - x_i| < \mathit{TOL}$$

Stopping Criteria

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$$\frac{|x_{i+1} - x_i|}{|x_{i+1}|} < \mathit{TOL}$$

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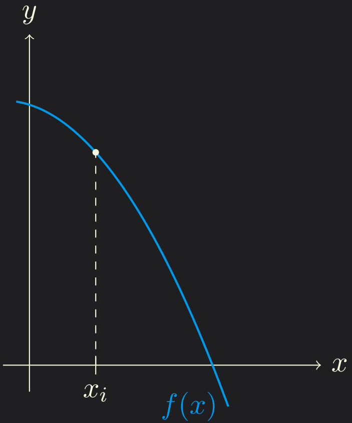
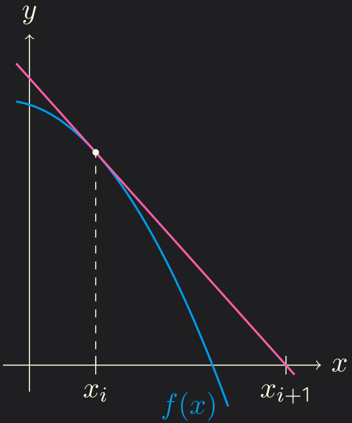
$$\frac{|x_{i+1} - x_i|}{\max\left(|x_{i+1}|, \theta\right)} < \mathit{TOL}, \text{ with } \theta > 0.0$$

Newton's Method

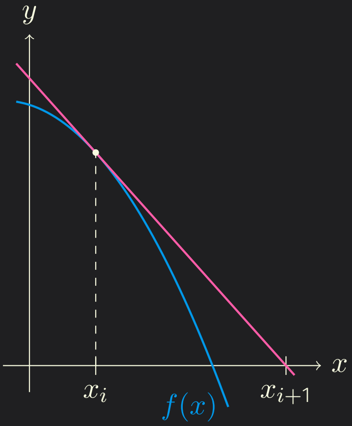

y
x
x_i
$f(x)$

Newton-Raphson Approach



$$y - f(x_i) = f'(x_i)(x - x_i)$$

Newton-Raphson Approach



$$y - f(x_i) = f'(x_i)(x - x_i)$$

$$y_1 = 0 \implies x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

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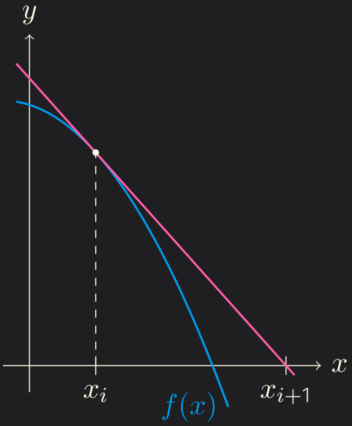

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Alternatively,

$$0 = f\,\overbrace{(x_i + \Delta x_i)}^{x_{i+1}}$$

$$0 = f(x_i) + \Delta x_i f'(x_i) + H.O.T$$

$$\Delta x_i \approx -\frac{f(x_i)}{f'(x_i)}$$

Example

Find a root of the equation $x^3 + x - 1$.

$$f'(x) = 3x^2 + 1$$

Example

Find a root of the equation $x^3 + x - 1$.

$$\begin{aligned} f'(x) &= 3x^2 + 1 \\ x_{i+1} &= x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1} \\ &= \frac{2x_i^3 + 1}{3x_i^2 + 1} \end{aligned}$$

Example

Find a root of the equation $x^3 + x - 1$.

$$f'(x) = 3x^2 + 1$$

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}$$

$$= \frac{2x_i^3 + 1}{3x_i^2 + 1}$$

$$x_0 = -0.7$$

$$x_1 = 0.12712551$$

$$\vdots$$

$$x_6 = 0.68232780$$

$$x_7 = 0.68232780$$

Quadratic Concergence of Newton's Method

- **Def:** Iterations are quadratically convergent if

$$M = \lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} < \infty$$

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- If $f'(r) = 0$, which happens in case of double roots, convergence is linear. See Modified Newton's Method in the textbook.

Apply Newton's Method to $f(x) = 4x^4 - 6x^2 - \frac{11}{4}$ starting at $x_0 = \frac{1}{2}$

Example of Non-convergence

Apply Newton's Method to $f(x) = 4x^4 - 6x^2 - \frac{11}{4}$ starting at $x_0 = \frac{1}{2}$

$$x_{i+1} = x_i - \frac{4x_i^4 - 6x_i^2 - \frac{11}{4}}{16x_i^3 - 12x_i}$$

Apply Newton's Method to $f(x) = 4x^4 - 6x^2 - \frac{11}{4}$ starting at $x_0 = \frac{1}{2}$

$$x_{i+1} = x_i - \frac{4x_i^4 - 6x_i^2 - \frac{11}{4}}{16x_i^3 - 12x_i}$$

$$x_1 = -\frac{1}{2} \text{ and } x_2 = \frac{1}{2}$$

Limits of Accuracy

Use Bisection Method to find the root of
$f(x) = x^3 - 2x^2 + \frac{4}{3} - \frac{8}{27}$ to within six correct digits.

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Due to rounding errors $f(0.6666641) = 0$ so Bisection terminates with $r = 0.6666641$ which is close in y but not in x.

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- backward error $= |f(x_a)|$
- forward error $= |r - x_a|$

$x_a = 0.6666641 \rightarrow$ backward error $\approx \epsilon_{\text{mach}}$, forward error $\approx 10^{-5}$.

Example: Wilkinson Polynomial

$$\begin{aligned} W(x) &= (x-1)(x-2)\cdots(x-20) \\ &= x^{20} + ax^{19} + bx^{18} + \dots \end{aligned}$$

Example: Wilkinson Polynomial

$$
\begin{aligned}
W(x) &= (x-1)(x-2)\cdots(x-20) \\
&= x^{20} + ax^{19} + bx^{18} + \ldots \\
x_0 &= 16.0 \\
x_a &= 16.01468\ldots
\end{aligned}
$$

Def: A problem is called **sensitive** if small error in the input lead to large error in the solution.

Sensitivity of Root Finding

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$$\Delta r \approx -\frac{\epsilon 6^7}{5!} = -2332.6\epsilon \implies r + \Delta r = 6.0023328$$

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- We will revisit conditioning for different numerical problems throughout the course.

- A ball throwing robot arm throws a ball with initial velocity V_0 at an angle θ from the ground. Assuming that the robot arm is at $x = 0$, the ball will hit the ground at $x_f = \frac{V_0^2 \sin 2\theta}{g}$, where $g = 9.8\ \mathrm{m/s}^2$ is the gravitational constant.

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- Solve for θ in the interval $[0°, 40°]$ and calculate the required V_0.

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- After 20 iterations $\theta \approx 28.53°$, $f(\theta) \approx 6.76 \cdot 10^{-8}$, and $V_0 \approx 1.41$ m/s.