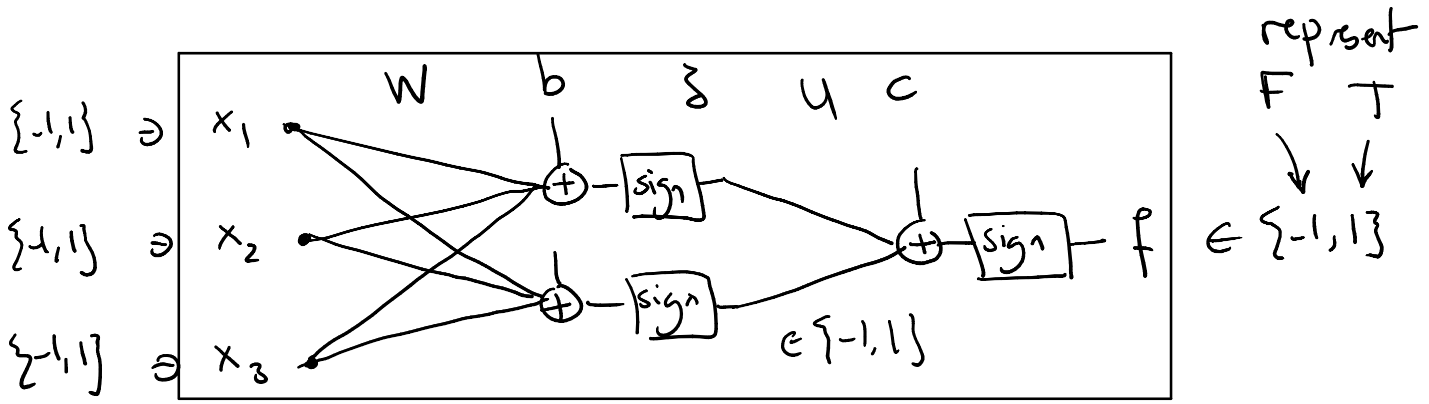
Homework 2 Solution

## Question 1

a) To make a network implement $(x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge x_3)$,

we need 2 layers, one for the ands and one for the or.

Since there are 2 and clauses, we can use a $\boxed{3-2-1}$ NN:




- You can use the formulas we saw in lecture, but let's redo!
- To implement $x_1 \wedge x_2 \wedge x_3$ we can threshold $x_1 + x_2 + x_3$. This sum can be $-3 \quad -1 \quad 1 \quad 3$

all $-1$ — one 1 — two 1s — all three 1s
and is F here — threshold — and is T here

We want thus to threshold between 1 and 3, say 2.

So, $x_1 + x_2 + x_3 > 2$ represents $x_1 \wedge x_2 \wedge x_3$.

- Since $x_3$ is negated for us, we just flip its sign:

$$x_1 + x_2 - x_3 > 2 \iff 1 \cdot x_1 + 1 \cdot x_2 - 1 \cdot x_3 - 2 > 0$$

$$x_1 + x_2 - x_3 > 2 \iff \underbrace{1}_{w_{11}}\cdot x_1 + \underbrace{1}_{w_{12}}\cdot x_2 \underbrace{-1}_{w_{13}}\cdot x_3 \underbrace{-2}_{b_1} > 0$$

This way, $z_1$ will be the $\pm 1$ T/F representation of the clause $(x_1 \wedge x_2 \wedge \neg x_3)$.

- For the $(\neg x_2 \wedge x_3)$ clause, we eliminate $x_1$ by setting its weight to 0. To represent an <u>and</u> clause with two terms, $x_2 \wedge x_3$, look at the sum $x_2 + x_3$, can be $-2, 0, 2$

$$\underbrace{-2}_{\substack{\text{both } -1\\ \text{and F}}},\ \underbrace{0}_{\pm 1},\ \underbrace{2}_{\substack{\text{both } +1\\ \text{and T}}}$$

threshold

We need to threshold between 0 & 2, say 1.

So, $x_2 + x_3 > 1$ represents $x_2 \wedge x_3$.

Since $x_2$ is negated, do: $-x_2 + x_3 > 1 \iff \underbrace{0}_{w_{21}}\cdot x_1 \underbrace{-1}_{w_{22}}\cdot x_2 + \underbrace{1}_{w_{23}}\cdot x_3 \underbrace{-1}_{b_2} > 0$

- Finally, in the second layer, we need to implement $z_1 \vee z_2$, the sum $z_1 + z_2$ can be $-2, 0, 2$

$$\underbrace{-2}_{\substack{\text{both } -1\\ \text{or F}}},\ \underbrace{0}_{\pm 1},\ \underbrace{2}_{\substack{\text{both } 1\\ \text{or T}}}$$

threshold

We need to threshold between $-2$ & $0$, say $-1$.

So, $z_1 \vee z_2$ is represented by $z_1 + z_2 > -1 \iff \underbrace{+1}_{u_1} z_1 + \underbrace{1}_{u_2} z_2 \underbrace{+1}_{c} > 0$

- Thus: $\boxed{W = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ -1 \end{bmatrix}}$

• Thus: $W = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} -1 \end{bmatrix}$

$u = [1 \ \ 1]$ $c = 1$

b) The analytic expression is thus:

$$f(x) = \text{sign}\left( U \cdot \text{sign}(Wx + b) + c \right)$$ or

$$f(x) = \text{sign}\left( \text{sign}(x_1 + x_2 - x_1 - 2) + \text{sign}(-x_2 + x_3) + 1 \right)$$

c) Here's pseudo code that will generate the tables

```
For x1 in {true, false}:
    For x2 in {true, false}:
        For x3 in {true, false}:
            y = (x1 and x2 and not x3) or (not x2 and x3)
            print (x1, x2, x3, y)
```

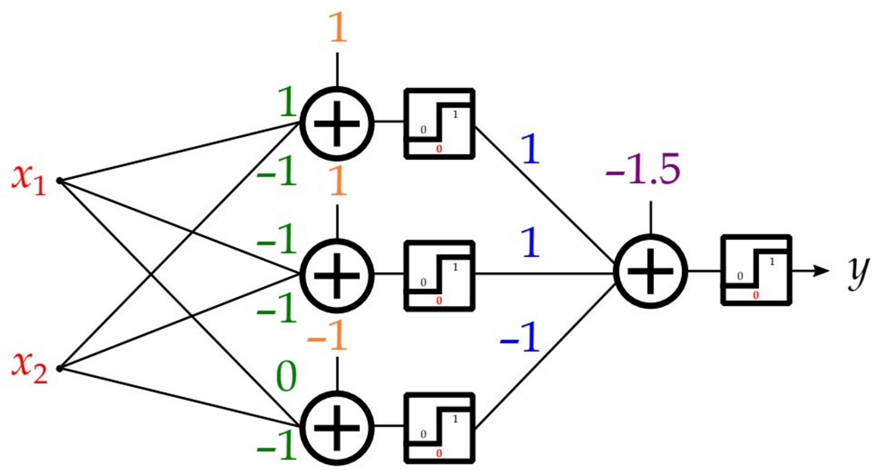
```
For i in range(8)                # use binary digits
    x1 = 2*(i/4) % 2 -1          # most significant
    x2 = 2 * (i/2) % 2 - 1
    x3 = 2 * (i/1) % 2 - 1       # least significant
    # The advantage of this is: no long nested for loops!
    y = sign( sign(x1+x2-x1-2) + sign(-x2+x3) + 1 )
    print (x1, x2, x3, y)
```

$y = \text{sign}(\text{sign}(x_1 + x_2 + x_1 - 2) + \text{sign}(-x_2 + x_3) + 1)$

print $(x_1, x_2, x_3, y)$

## Question 2

a)




$$W \in \mathbb{R}^{3\times 2} \quad b \in \mathbb{R}^3 \quad u = \mathbb{R}^{1\times 3} \quad c \in \mathbb{R}^1$$

$$W = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad u = [1 \;\; 1 \;\; -1] \quad c = -1.5$$

b) $$y = \text{step}(u\,\text{step}(Wx + b) + c)$$

expected answer (linear algebraic)

or

this is okay

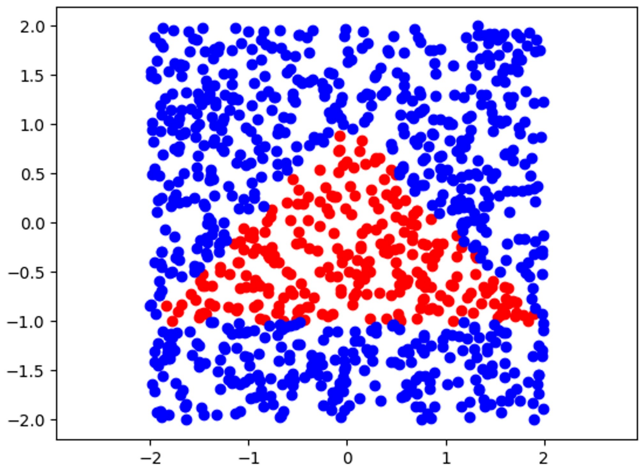
$$y = \text{step}(\text{step}(x_1 - x_2 + 1) + \text{step}(-x_1 - x_2 + 1) + \text{step}(-x_2 - 1) - 1.5)$$

c) Here's code to simulate the NN:

```
import numpy as np
import matplotlib.pyplot as plt
x1 = np.random.uniform(-2,2,1000)
x2 = np.random.uniform(-2,2,1000)
z1 = (x1-x2+1>=0)*1.
z2 = (-x1-x2+1>=0)*1.
```

```python
z3 = (-x2-1>=0)*1.
y = np.heaviside(z1+z2-z3-1.5,1)
plt.scatter(x1[np.where(y==1)],x2[np.where(y==1)],c='red')
plt.scatter(x1[np.where(y==0)],x2[np.where(y==0)],c='blue')
plt.axis('equal')
plt.show()
```

0 1 | 2 3




1 + 1 + 1

0 1 | 2 3

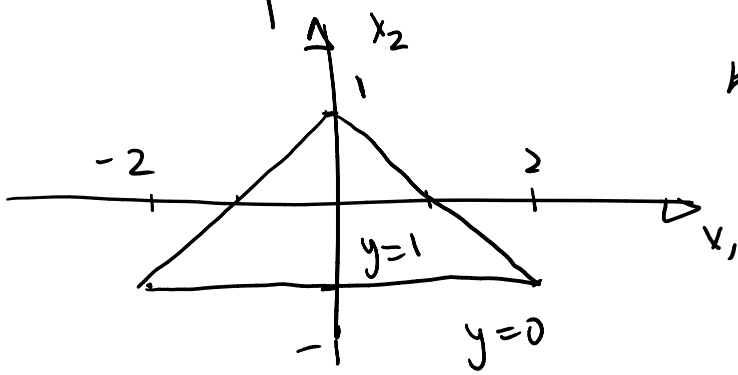
$x_2 < x_1 - 1$

$x_2 < -x_1 + 1$

and $x_2 < -1$

d) The boundary looks like a triangle. So, it's a reasonable guess to assume that's what the network represents:

acceptable answer




nerdy answer :)

- Is this all? What if we look outside of this range?

The second layer equation is $z_1 + z_2 + (1 - z_3) > 2.5$

But $z_1 + z_2 + z_3$ can be 0, 1, 2, 3

But $\delta_1 + \delta_2 + \delta_3$ can be 0, 1, 2, 3

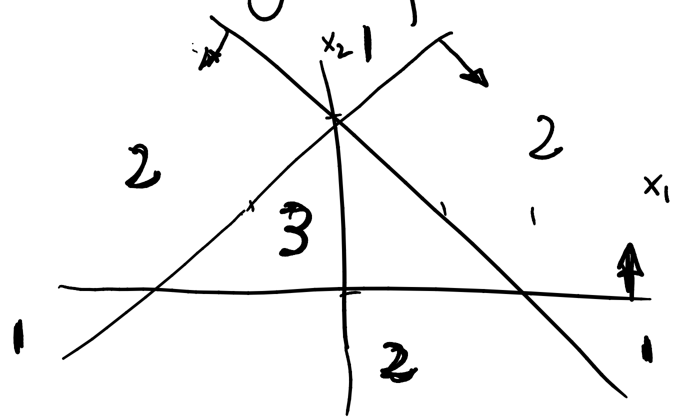
all 0, one 1, two 1s, all three 1s

threshold

Thus, the output is $y=1$ when <u>all three</u> of the half-spaces

$x_2 < x_1 - 1$, $x_2 < -x_1 + 1$, and $x_2 < -1$ (negation of $\delta_3$) are active:

These define 7 regions, with active number:




So, we indeed have only the triangle giving $y=1$