ECE/CS 559 - Fall 2019 - Midterm #1.

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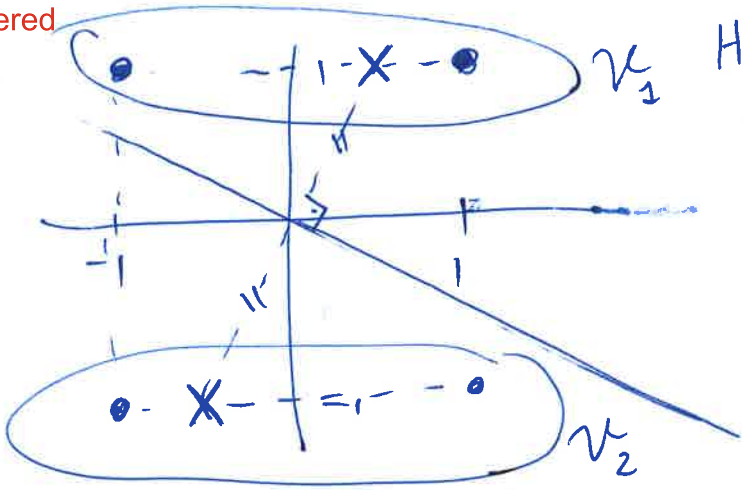
Q1 (10 pts). Consider applying the k-means algorithm to the set of vectors $\mathcal{C} = \{[\begin{smallmatrix}-1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}], [\begin{smallmatrix}-1\\-1\end{smallmatrix}]\}$ with initial centers $[\begin{smallmatrix}0.5\\1\end{smallmatrix}], [\begin{smallmatrix}-0.5\\-1\end{smallmatrix}]$. What are the resulting centers after the algorithm converges? Recall that, given S is the input (training set), and c_i are the centers, the k-means algorithm relies on the update $c_i \leftarrow \frac{1}{|V_i|}\sum_{x\in V_i} x$, where $V_i = \{x \in S : \|x - c_i\| \leq \|x - c_j\|, \forall j\}$, and $|V_i|$ is the number of elements in V_i.

Not Covered



Hence, at iteration 1, the centers will be updated to $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-1\end{pmatrix}$.

These centers will remain the same at subsequent iterations as V_1 & V_2 does not change.

Q2 (30 pts). Let u be the step activation function with $u(x) = 1$ if $x \geq 0$, and $u(x) = 0$, otherwise. Consider the perceptron $y = u(w_0 + w_1x_1 + w_2x_2)$, where w_1 and w_2 are the weights for inputs x_1 and x_2, respectively, w_0 is the perceptron bias, and y is the perceptron output. Let $\mathcal{C}_0 = \{[0 \;\; 2], [2 \;\; 0]\}$, and $\mathcal{C}_1 = \{[1 \;\; 1]\}$. The desired output for class $\mathcal{C}_0$ is 0, and the desired output for class $\mathcal{C}_1$ is 1. Correspondingly, let $d(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathcal{C}_0$, and otherwise, let $d(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{C}_1$.

(a) **(10 pts)** If possible, find w_0, w_1, w_2 that can separate $\mathcal{C}_0$ and $\mathcal{C}_1$ (i.e., provide the desired output for all 4 possible input vectors). Otherwise, prove that no choice of weights can separate the two classes.

We need

$w_0 + 2w_2 < 0 \quad (i)$

$w_0 + 2w_1 < 0 \quad (ii)$

$w_0 + w_1 + w_2 \geq 0 \quad (iii)$

$-\dfrac{(i) - (ii)}{2} + (iii)$ leads to $0 > 0$

a contradiction.

So no choice of weights can separate the classes.

(b) **(10 pts)** Recall that the perceptron training algorithm relies on the update $\mathbf{w} \leftarrow \mathbf{w} + \eta(d(\mathbf{x}) - y)\begin{bmatrix}1 & \mathbf{x}\end{bmatrix}$, where $\mathbf{w} = \begin{bmatrix}w_0 & w_1 & w_2\end{bmatrix}$ is the weight vector. Let $\eta = 1$ and the initial weight vector be given by $\mathbf{w} = \begin{bmatrix}-1 & 0 & 0\end{bmatrix}$. Calculate the updated weights after one epoch of training.

With extended notation (biases)

$$C_0 = \left\{ \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix} \right\}, \quad C_1 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

Epoch 1.

Feed $\begin{pmatrix}1\\0\\2\end{pmatrix}$. $y = u\left(\begin{pmatrix}-1\\0\\0\end{pmatrix}^T \begin{pmatrix}1\\0\\2\end{pmatrix}\right) = 0 =$ desired output

So, no update.

Feed $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $y = u\left(\begin{pmatrix}-1\\0\\0\end{pmatrix}^T \begin{pmatrix}1\\2\\0\end{pmatrix}\right) = 0 =$ desired output

So, again, no update.

Feed $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $y = u\left(\begin{pmatrix}-1\\0\\0\end{pmatrix}^T \begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 0 \neq$ desired output

So update $w \leftarrow \begin{pmatrix}-1\\0\\0\end{pmatrix} + 1\cdot(1-0)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$.

$\therefore$ Find weights after one epoch of training

$$\begin{pmatrix}0\\1\\1\end{pmatrix}.$$

(c) **(10 pts)** Will the weights provided by the algorithm (as setup in (b)) eventually converge after a sufficiently larger number of epochs? Justify your answer.

No. ~~Because~~ If converges $\Rightarrow \exists w$ to separate the two classes. However, the classes are not linearly separable.

Q3 (40 pts): Consider a single-neuron network with input-output relationship $y = f(b + \mathbf{w}^T\mathbf{x})$, where y is the network output, f is some activation function, b is the bias term, $\mathbf{w} = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ are the synaptic weights, and $\mathbf{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is the network input.

(a) **(10 pts):** Let $E = (d - y)^2$, where d is a constant (a generic desired output). Write down the delta-learning rule (the gradient-descent update equations) for b, w_1, w_2 given learning parameter $\eta = \frac{1}{2}$.

Simple chain rule yields:

$$\begin{pmatrix} b \\ w_1 \\ w_2 \end{pmatrix} \leftarrow \begin{pmatrix} b \\ w_1 \\ w_2 \end{pmatrix} + (d-y)f'(b+w^Tx)\begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}.$$

(b) **(15 pts):** Consider the same delta-learning setup as in (a) with the activation function

$$f(v) = \begin{cases} v, & v \geq 0, \\ 0, & v < 0. \end{cases}$$

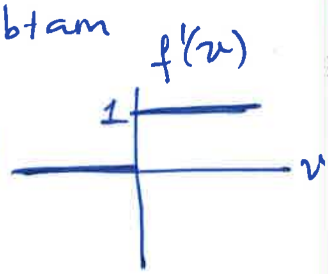
This is also known as the rectified linear unit (ReLU). Consider the training vectors $\mathbf{x}_1 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, $\mathbf{x}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, with desired outputs $d_1 = -1$, $d_2 = 2$, respectively. Find the updated bias and the updated weights after 2019 epochs of online learning given initial conditions $b = 0$, $\mathbf{w} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

Epoch 1

Show $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Local field $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -1$

$y = f(v) = 0$

According to the update rule in (a), we obtain



$$\begin{pmatrix} b \\ w_1 \\ w_2 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1-0) f'(-1) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Since $f'(-1) = 0$, we have no update.

Show $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Local field $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1$ $y = f(1) = 1$

$$\begin{pmatrix} b \\ w_1 \\ w_2 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (2-1) f'(1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Epoch 2.

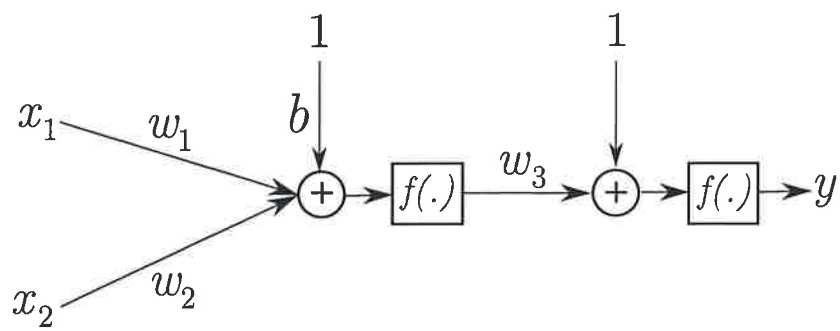
Show $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. $v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -1 \Rightarrow$ no update.

Show $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. $v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 3$, $y = f(v) = 3$

$$\begin{pmatrix} b \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + (2-3) f'(3) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

At the end of epoch 2, we end up with the same weights we began with. So after even numbered epochs, the weights are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and after odd-numbered epochs, the weights are $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. In particular, after 2019 epochs, we get $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(c) **(15 pts)** Consider now a multi-layer network as shown below.



Consider again a general f, as in (a). Let $E = (d-y)^4$. Find the gradient-descent update equations for b, w_1, w_2, w_3 given $\eta = \frac{1}{4}$.

$$y = f(1 + w_3 f(b + w_1x_1 + w_2x_2)). \quad \text{So, by chain rule.}$$

$$w_3 \longleftarrow w_3 + (d-y)^3 f'(1 + w_3 f(b + w_1x_1 + w_2x_2)) \times f(b + w_1x_1 + w_2x_2).$$

$$\begin{bmatrix} b \\ w_1 \\ w_2 \end{bmatrix} \leftarrow \begin{bmatrix} b \\ w_1 \\ w_2 \end{bmatrix} + (d-y)^3 f'(1 + w_3 f(b + w_1x_1 + w_2x_2)) \times w_3 f'(b + w_1x_1 + w_2x_2) \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}.$$

Q4 (20 pts). Let $f(x, y) = x^2 + xy + y^2$. Consider Newton's method given by the update rule $\begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} x \\ y \end{bmatrix} - H^{-1}g$, where the gradient and the Hessian are given by $g = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$ and $H = \begin{bmatrix} \frac{\partial f^2}{\partial x \partial x} & \frac{\partial f^2}{\partial x \partial y} \\ \frac{\partial f^2}{\partial y \partial x} & \frac{\partial f^2}{\partial y \partial y} \end{bmatrix}$, respectively.

(a) **(10 pts)** Calculate g and H.

$$g = \begin{bmatrix} 2x + y \\ 2y + x \end{bmatrix} \qquad H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(b) **(10 pts)** Consider the initial point $x = 2019, y = 2020$. Find the next point after one update with Newton's method.

$$H^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \quad \text{So, } H^{-1}g = \begin{bmatrix} x \\ y \end{bmatrix}$$

and any initial point $\begin{bmatrix} x \\ y \end{bmatrix}$ will end up at

$\begin{bmatrix} x \\ y \end{bmatrix} - H^{-1}g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ after one update with Newton's method. This is also the global minimum.