Homework 1 Solution

Q1

(a) With this choice we get:

$$y = \begin{cases} 1 & ; \quad \sum_i x_i \geq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Therefore y is 1 except when all x's are 0.
thus the output is True unless all inputs are False.
This is exactly the definition of logical OR:
y represents $x_1 \vee x_2 \vee \dots \vee x_g$ $\quad (\vee = \text{OR})$

(b) Say we set the first m w's to -1 and the rest to 1, and we let $b = m$, we get:

$$y = \begin{cases} 1 & ; \quad -x_1 - x_2 - \dots - x_m + x_{m+1} + \dots + x_g + m \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$y = \begin{cases} 1 & ; \quad (1-x_1) + (1-x_2) + \dots + (1-x_m) + x_{m+1} + \dots + x_m \geq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

This shows that x_1 through x_m are now inverted

$$1-x = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x=1 \end{cases} \iff \underset{\text{arithmetic inversion}}{1-x} \text{ represents } \underset{\text{logical negation}}{\neg x}$$

Then, by (a), f now represents:

$$\neg x_1 \vee \neg x_2 \vee \dots \vee \neg x_m \vee x_{m+1} \vee \dots \vee x_9$$

(c) Let case 1 = Square in top-left corner

case 2 = top right

case 3 = bottom left

case 4 = bottom right

<u>Case 1</u> is the same as $x_1 \wedge x_2 \wedge \neg x_3$

AND statement $\wedge\ x_4 \wedge x_5 \wedge \neg x_6$

$\wedge \neg x_7 \wedge x_8 \wedge \neg x_9$

1	1	0
1	1	0
0	0	0

Since we know how to represent OR statements Let's change this into an OR using De Morgan's.

$$\neg(\text{case } 1) = \neg x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4 \vee \neg x_5 \vee x_6 \vee x_7 \vee x_8 \vee x_9$$

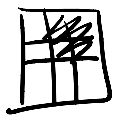
$$\underbrace{\neg(\text{case } 1)}_{z_1} = \underline{\neg x_1} \vee \underline{\neg x_2} \vee x_3 \vee \underline{\neg x_4} \vee \underline{\neg x_5} \vee x_6 \vee x_7 \vee x_8 \vee x_9$$

$$= (1-x_1) + (1-x_2) + x_3 + (1-x_4) + (1-x_5) + x_6 + x_7 + x_8 + x_9$$

$$= -x_1 - x_2 + x_3 - x_4 - x_5 + x_6 + x_7 + x_8 + x_9 + 4$$

$$w_{1.} = [-1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1 \quad 1] \qquad b_1 = 4$$

(# of negation ω ix (b))

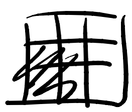
Similarly : 

$$\underline{\text{case } 2} = \neg x_1 \wedge \underline{x_2} \wedge \underline{x_3} \wedge \neg x_4 \wedge \underline{x_5} \wedge \underline{x_6} \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9$$

$$z_2 = \neg(\text{case } 2) = x_1 \vee \underline{\neg x_2} \vee \underline{\neg x_3} \vee x_4 \vee \underline{\neg x_5} \vee \underline{\neg x_6} \vee x_7 \vee x_8 \vee x_9$$

$$w_{2.} = [1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1]$$

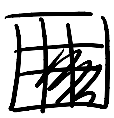
$(b_2 = 4)$

Case 3 : 

$$z_3 = \neg(\text{case } 3) = x_1 \vee x_2 \vee x_3 \vee \underline{\neg x_4} \vee \underline{\neg x_5} \vee x_6 \vee \underline{\neg x_7} \vee \underline{\neg x_8} \vee x_9$$

$$w_3 = [1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1]$$

$(b_3 = 4)$

Case 4 : 

$$z_4 = \neg(\text{case } 4) = x_1 \vee x_2 \vee x_3 \vee x_4 \vee \neg x_5 \vee \neg x_6 \vee x_7 \vee \neg x_8 \vee \neg x_9$$

$$w_4 = [1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1]$$

$(b_4 = 4)$

Putting all together

We want $y = (\text{case } 1) \vee (\text{case } 2) \vee (\text{case } 3) \vee (\text{case } 4)$

OR statement

OR statement

since we want to detect if any of these cases occurs.

$$\Rightarrow \quad y = \neg z_1 \vee \neg z_2 \vee \neg z_3 \vee \neg z_4$$

$$\Rightarrow \quad u = [-1 \quad -1 \quad -1 \quad -1] \qquad c = 4$$

Thus, we have:

<u>Layer 1</u>:

$$W = \begin{bmatrix} -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

<u>Layer 2</u>:

$$u = [-1 \quad -1 \quad -1 \quad -1] \qquad c = 4$$

<u>Note</u>: This solution is not unique. It's simply the one that follows from the steps outlined in this question.

Q2

(a) Since $\ell(x,w) = \ell(f(x,w), g(x))$, the chain rule gives:

$$\frac{\partial \ell}{\partial w} = \frac{\partial \ell}{\partial f} \cdot \frac{\partial f}{\partial w} + \frac{\partial \ell}{\partial f} \underbrace{\frac{\partial g}{\partial w}}_{0} = \frac{\partial \ell}{\partial f} \cdot \frac{\partial f}{\partial w}$$

(b) We have $\ell(f,g) = (f-g)^2 \Rightarrow \frac{\partial \ell}{\partial f} = 2(f-g)$

$$f(x,w) = wx \qquad \frac{\partial f}{\partial w} = x$$

Thus, by the chain rule in (a):

$$\frac{\partial \ell}{\partial w} = \frac{\partial \ell}{\partial f} \cdot \frac{\partial f}{\partial w} = 2(f(x,w) - g(x)) \cdot x$$

$$= 2(wx - x^3)x$$

$$= 2wx^2 - 2x^4$$

Thus: $$\frac{\partial L}{\partial w} = \int_{-1}^{1} (2wx^2 - 2x^4)\, dx$$

$$= \frac{2wx^3}{3} - \frac{2x^5}{5} \Big|_{-1}^{1} = \frac{4w}{3} - \frac{4}{5}$$

$$\frac{\partial L}{\partial w} = 0 \quad \Rightarrow \quad w = \frac{3}{5}$$

$$\frac{\partial L}{\partial w} = 0 \quad \Rightarrow \quad w = \frac{3}{5}$$

(c)