ECE/CS 559 - Neural Networks Lecture Notes #2
Mathematical models for the neuron, Neural network architectures

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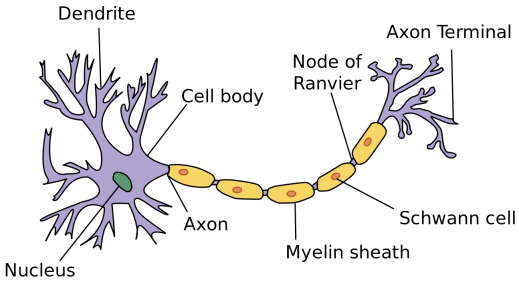
1 Mathematical models for the neuron

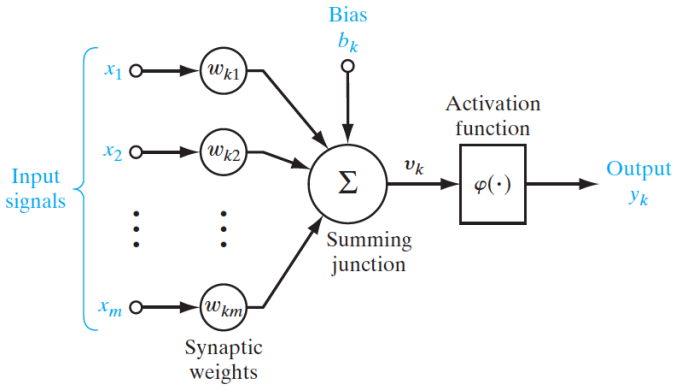
1.1 The biology of the neuron



(Image taken from Wikipedia)

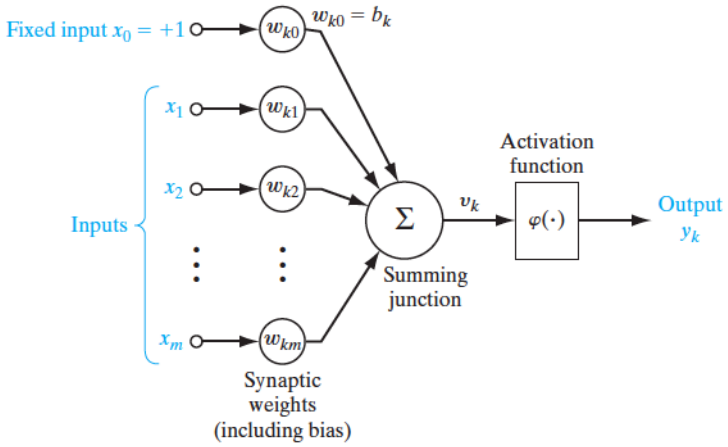
- Neuron: A highly-specialized cell that can transmit and receive information through electrical/chemical signaling. The fundamental building block of the brain.
- Synapse: The structure that connects two given neurons, i.e. the functional unit that mediates the interactions between neurons.
- A chemical synapse: Electrical signal through axon of one neuron $\rightarrow$ chemical signal / neurotransmitter $\rightarrow$ postsynaptic electrical signal at the dendrite of another neuron.
- Information flows through dendrites to axons: Dendrite (of neuron A) $\rightarrow$ Cell body (Soma) $\rightarrow$ Axon $\rightarrow$ Synapses $\rightarrow$ Dendrite (we are now at neuron B) $\rightarrow$ Cell body (Soma) $\rightarrow$ Axon $\rightarrow$ Synapses $\rightarrow \cdots$.
- A neuron can receive thousands of synaptic contacts and it can project onto thousands of target neurons.
- A synapse can impose excitation or inhibition (but not both) on the target neuron. The "strength" of excitation/inhibition may vary from one synapse to another.
- There are thousands of different types of neurons. There are, however, 3 main categories.
  - Sensory neurons: Conveys sensory (light, sound, touch) information.
  - Interneurons: Convey information between different types of neurons.
  - Motor neurons: Transmit muscle/gland control information from the brain/spinal cord.
- The soma is a few to a hundred micrometers in diameter. Axons are generally much longer, and can can in fact be up to a meter long! (e.g. neurons of the sciatic nerve).
- Speed of neural impulses: A few to a hundred meters per second.
- Around 10^{11} neurons in the brain, 10^{12} neurons in the entire nervous system. Also, an average a neuron makes thousands of connections to other neurons, resulting in around 10^{14} synapses in the brain.

1.2 A mathematical model for the neuron



(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

- Input signals are from dendrites of other neurons.
- The synaptic weights correspond to the synaptic strengths: positivity/negativity $\rightarrow$ excitation/inhibition.
- The summing unit models the operation of the cell body (soma).
- The nonlinearity $\varphi(\cdot)$ (activation function) models the axon operation.
- The output may be connected to dendrite of another neuron through another synapse.
- $v_k = \sum_{j=1}^{n} w_{kj} x_j + b_k$ is called the induced local field of neuron k.
- $y_k = \varphi(v_k) = \varphi\left(\sum_{j=1}^{n} w_{kj} x_j + b_k\right)$.
- Alternatively, we may consider a fixed input $x_0 = 1$ with weight $w_{k0} = b_k$:



(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

- $y_k = \varphi\left(\sum_{j=0}^{n} w_{kj} x_j\right)$. Note that now the summation starts from index 0.

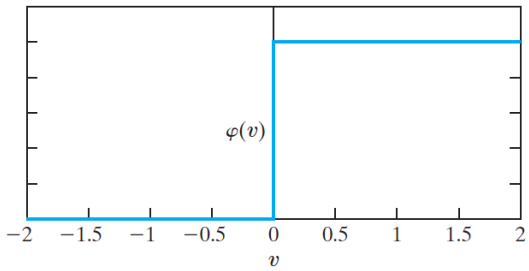
1.3 Types of activation function

Typically, $\varphi(\cdot)$ has bounded image (e.g. $[0, 1]$ or $[-1, 1]$), and thus is also called a squashing function. It limits the amplitude range of the neuron output.

1.3.1 Step function

- Threshold function (or the Heaviside/step function):

$$\varphi(v) = \begin{cases} 1, & v \geq 0 \\ 0, & v < 0 \end{cases}.$$



(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

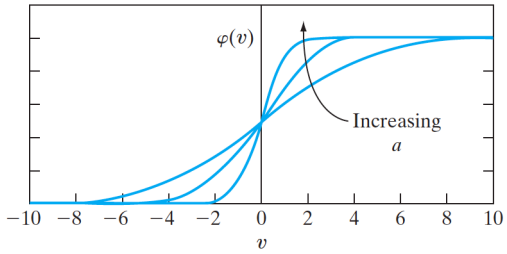
Such a neuron is referred to as the McCulloch-Pitts model (1943).

1.3.2 Sigmoid function

- The sigmoid function is defined as

$$\varphi(v) = \frac{1}{1 + \exp(-av)},$$

where a is called the slope parameter.



(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

- As $a \to \infty$ the sigmoid function approaches the step function.
- Unlike the step function, the sigmoid function is continuous and differentiable. Differentiability turns out to be a desirable property of an activation function, as we shall see later.

1.3.3 Signum function

$$\varphi(v) = \operatorname{sgn}(v) = \begin{cases} 1, & v \geq 0 \\ 0, & v = 0 \\ -1, & v < 0 \end{cases}.$$

1.3.4 Hyperbolic tangent

$$\varphi(v) = \tanh(av) = \frac{e^{av} - e^{-av}}{e^{av} + e^{-av}}.$$

for some parameter $a > 0$. Approaches the signum function as $a \to \infty$.

2 Neural network architectures

Having introduced our basic model of a (mathematical) neuron, we now introduce the different neural network architectures that we will keep revisiting throughout the course.

2.1 Single-layer feedforward networks

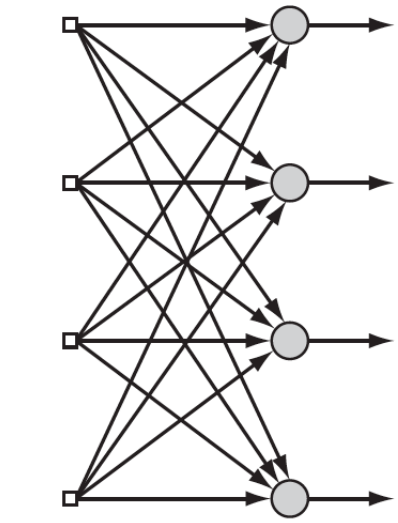


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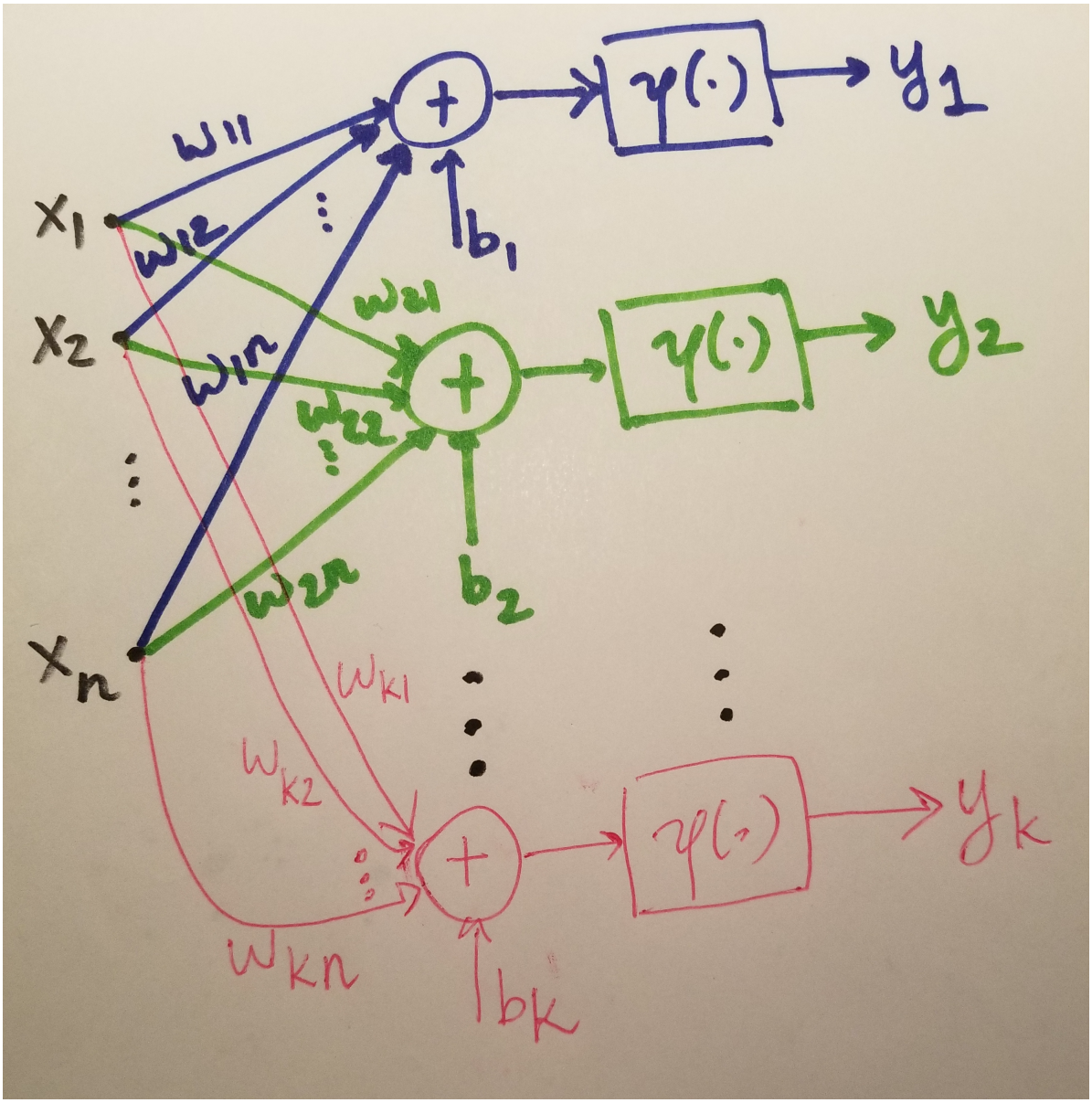
Figure 1: Feedforward network with a single layer of neurons

- We just count the number of layers consisting of neurons (not the layer of source nodes as no computation is performed there). Thus, the network in Fig. 1 is called a single-layer network.
- Also, note that the information flow is only over one direction, i.e. the input layer of source nodes project directly onto the output layer of neurons (according to the non-linear transformations as specified by the neurons.). There is no **feedback** of the network's output to network's input. We thus say that the network in Fig. 1 is of feedforward type.
- Let us try to formulate the input-output relationship of the network. Putting in the symbols, we have the following diagram:



We have

$$
\begin{aligned}
y_1 &= \phi\left(b_1 + \sum_{i=1}^{n} w_{1i}x_i\right) \\
y_2 &= \phi\left(b_2 + \sum_{i=1}^{n} w_{2i}x_i\right) \\
&\vdots \\
y_k &= \phi\left(b_k + \sum_{i=1}^{n} w_{ki}x_i\right)
\end{aligned}
$$

We can rewrite all k equations via a single equation:

$$y_j = \phi\left(b_j + \sum_{i=1}^{n} w_{ji}x_i\right), \; j = 1, \ldots, k$$

Here w_{ji} is the weight from input i to Neuron j. We can further rewrite everything in a simple matrix form. Define

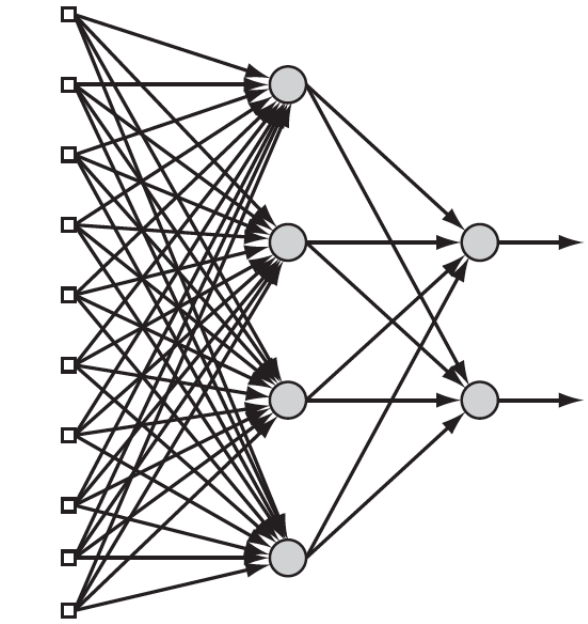
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}, \mathbf{W}' = \begin{bmatrix} b_1 & w_{11} & \cdots & w_{1n} \\ b_2 & w_{21} & \cdots & w_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_k & w_{k1} & \cdots & w_{kn} \end{bmatrix}, \mathbf{x}' = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, the above input output relationship can just be written as $\mathbf{y} = \phi(\mathbf{W}'\mathbf{x}')$ in the sense that ϕ is applied component-wise. Sometimes biases are treated separately. Defining

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}, \mathbf{W} = \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ w_{21} & \cdots & w_{2n} \\ \vdots & \vdots & \vdots \\ w_{k1} & \cdots & w_{kn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

we can write $\mathbf{y} = \phi(\mathbf{W}'\mathbf{x}') = \phi(\mathbf{W}\mathbf{x} + \mathbf{b})$.

2.2 Multilayer feedforward networks



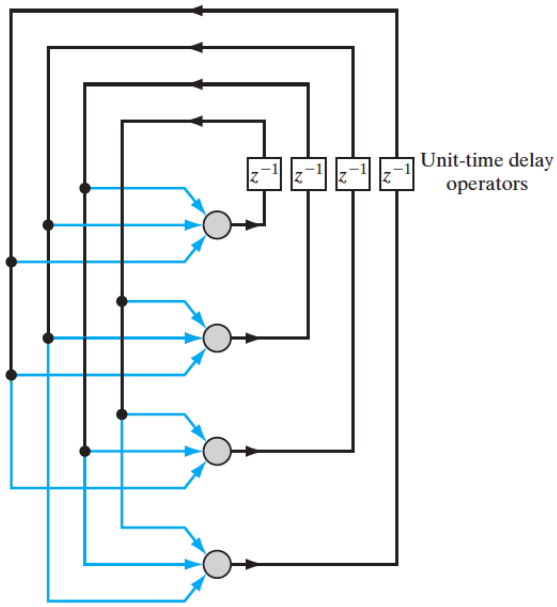
(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

Figure 2: Fully-connected 2-layer feedforward network with one hidden layer and one output layer.

- We can add more layers to the feedforward network in Fig. 1.
- The end-result is a multilayer network with one or more hidden layers.
- Hidden layers refer to those layers that are not seen directly from either the input or the output of the network.

- We call a network with m source nodes in its input layer, h_1 hidden nodes in its first hidden layer, h_2 nodes in its second hidden layer, ..., h_K nodes in its Kth hidden layer, and finally q nodes in its output layer a m-h_1-h_2-$\cdots$-h_K-q network. For example, the network in Fig. 2 is called a 10-4-2 network as it has 10 source nodes in its input layer, 4 nodes in the hidden layer, and 2 nodes in its output layer.
- Fully-connected network: Every node in each layer of the network is connected to every other node in the adjacent layer. Example: The network is Fig. 2 is fully connected. Otherwise, the network is partially connected.
- Deep network: Many (usually assumed to be > 1) hidden layers. Shallow network: The opposite.
- As will be made more precise later on, theoretically, only one hidden node is sufficient for almost any application provided that one can afford a large number of neurons. On the other hand, a deep network can perform the same tasks as a shallow network with the extra advantage of possibly a fewer number of neurons. Hence, deep networks, provided that they can be properly designed, may be better suited for complex practical applications.
- The input-output relationships may be formulated in a similar manner as the single-layer network discussed previously. For example, consider a two-layer network with n inputs, L_1 neurons in the first layer, and L_2 neurons in the second (output) layer. Let $\mathbf{x} \in \mathbb{R}^{n\times 1}$ be the vector of inputs, $\mathbf{W}_1 \in \mathbb{R}^{L_1\times n}$ be the matrix of weights connecting the input layer to the first layer of neurons (where the ith row jth column of $\mathbf{W}_1$ corresponds to the weight between input node j and neuron i of the first layer), $\mathbf{b}_1 \in \mathbb{R}^{L_1\times 1}$ be the vector of biases for the first layer of neurons, $\mathbf{W}_2 \in \mathbb{R}^{L_2\times L_1}$ be the matrix of weights connecting the first layer of neurons to the second layer of neurons (where the ith row jth column of $\mathbf{W}_2$ corresponds to the weight between neuron j of the first layer and neuron i of the second layer), and $\mathbf{b}_2 \in \mathbb{R}^{L_2\times 1}$ be the vector of biases for the second layer of neurons, and $\mathbf{y} \in \mathbb{R}^{L_2\times 1}$ be the vector of outputs. Then, we have $\mathbf{y} = \phi(\mathbf{W}_2\phi(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2)$.

2.3 Recurrent networks



(Image taken from our coursebook: S. Haykin, "Neural Networks and Learning Machines," 3rd ed.)

Figure 3: Recurrent network with no self-feedback loops and no hidden neurons.

- What distinguishes recurrent networks from feedforward networks is that they incorporate **feedback**.
- In Fig. 3, the boxes with z^{-1} represent the unit discrete time delays.
- No self-feedback: The output of a given neuron is not fed back to its own input.
- One may have variations of the structure in Fig. 3. We shall discuss these variations and the details of recurrent networks later on.