IE 407

- Introduction to OR
- Introduction to Linear Programming

Middle East Technical University
Department of Industrial Engineering

IE 407 - FUNDAMENTALS OF OPERATIONAL RESEARCH

Fall 2021

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Lectures: Tuesday, 13:40 – 16:30, IE 102

Course Web Page: Go to https://odtuclass.metu.edu.tr/ and login. If you are on the roster, IE 407 will be listed under “Courses”; click on it to have access to everything that’s been posted.

Textbook: W.L. Winston, *Operations Research* (4th ed.), Thomson Brooks/Cole, 2004.

Reference Books:

- F.S. Hillier, G.J. Lieberman, *Introduction to Operations Research* (5th ed.), McGraw-Hill, 1990.
- H.A. Taha, *Operations Research* (7th ed.), Prentice Hall, 2003.
- H.M. Wagner, *Principles of Operations Research with Application to Managerial Decisions* (2nd ed.), Prentice Hall, 1975.

Course Objectives: At the end of the course, the students will:

- have an understanding of the general principles of linear programming.
- acquire the skills to formulate and build linear programming models, understand how to solve linear programming problems (graphical solution and computer solution) and how to make sensitivity analysis.
- become familiar with the special types of linear programming problems such as transportation, transshipment, assignment and network problems.
- acquire the skills to formulate integer programming models.
- acquire the skills to formulate and solve problems that involve sequential interrelated decisions.

Tentative Course Outline:

- Introduction and history of OR.
- Methodology and applications.
- Definition of linear programming.
- Examples of linear programming.
- Graphical method and sensitivity analysis.
- Mathematical programming software solutions and interpretation of results.
- Dual problem and economic interpretation.
- Examples of and solution methods for transportation, transshipment, and assignment models.
- Examples of network models (shortest path problems, maximum flow problems, minimum cost problems, minimum spanning tree) and solution methods.
- Examples of integer programming models (set-covering and knapsack problems, either-or and if-then constraints, piecewise linear functions) and solution methods (branch-and-bound method).
- Examples of and solution methods for deterministic and stochastic dynamic programming models.

Grading: Course grade will be based on:

- Assignments + Quizzes (30%)
- Midterm exam (30%)
- Final exam (40%)

This syllabus is subject to change based on the needs of the class and at the discretion of the instructor. Students will be notified and held responsible for any changes.

Introduction to OR

What is Operations Research (OR)?

- Operations Research (OR) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.

- System is an organization of interdependent components that work together to accomplish the goal of the system.
  - Ex: Ford Motor Company is a system whose goal consists of maximizing the profit that can be earned by producing quality vehicles.

What is Operations Research (OR)?

- History of OR
  - Origin: During WWII.
    - The application of mathematics and the scientific method to military operations was called operations research.
    - Ex: Deployment of radar, management of staff, etc.
  - After WWII, many new applications emerged.
    - Ex: Factory scheduling, transportation logistics, product design, etc.
  - Today, it has a broader meaning.
    - A scientific approach to decision making.
    - A structured approach to framing, formulation, and solving of complex problems.
    - OR is concerned with optimal decision making in and modeling of deterministic and probabilistic systems that originate from real life.

OR Characteristics

- 1. Scientific method

- 2. Systems approach
  - A system is organized assembly of components.
  - A system does something.
  - Each component contributes towards the behavior of the system and is not independent.
  - Groups of components lead to subsystem.
  - A system has outside – i.e., functions in an environment.

- 3. Multi-disciplinary

OR Methodology / Phases of an OR Project / Phases of Model-Building Process

- 1. Problem identification and formulation
  - Specify the objectives/limitations and the parts of the organization
- 2. Observing the system and data collection
  - Determine parameters affecting the problem
  - Collect data to estimate problem parameters
- 3. Building a mathematical model
- 4. Model verification
  - Check whether the model is an accurate representation of reality
- 5. Solving the model and performing sensitivity analysis
- 6. Presenting the results to the decision maker
- 7. Implementation and control

Mathematical Models

- Our focus will be on optimization models.
- An **optimization model** is a **prescriptive model** that determines optimal **decisions.**
  - An **objective function** is maximized or minimized.
  - Decisions must be feasible with respect to **constraints**.
- Basic components of optimization models:
  - Objective function
  - Decision variables
  - Constraints

Project Examples

- Designing a waste management system
- Locating a new ice cream plant
- Redesigning the sales territories of a pharmaceutical company
- Optimizing refinery operations

Optimizing Refinery Operations

Step 1 Klingman et al. wanted to minimize the cost of operating CITGO's refineries.

Step 2 The Lake Charles, Louisiana, refinery was closely observed in an attempt to estimate key relationships such as:

1 How the cost of producing each of CITGO's products (motor fuel, no. 2 fuel oil, turbine fuel, naptha, and several blended motor fuels) depends on the inputs used to produce each product.

2 The amount of energy needed to produce each product. This required the installation of a new metering system.

3 The yield associated with each input–output combination. For example, if 1 gallon of crude oil would yield .52 gallons of motor fuel, then the yield would equal 52%.

4 To reduce maintenance costs, data were collected on parts inventories and equipment breakdowns. Obtaining accurate data required the installation of a new database-management system and integrated maintenance-information system. A process control system was also installed to accurately monitor the inputs and resources used to manufacture each product.

Step 3 Using linear programming (LP), a model was developed to optimize refinery operations. The model determines the cost-minimizing method for mixing or blending together inputs to produce desired outputs. The model contains **constraints** that ensure that inputs are blended so that each output is of the desired quality. Blending constraints are discussed in Section 3.8. The model ensures that plant capacities are not exceeded and allows for the fact that each refinery may carry an inventory of each end product. Sections 3.10 and 4.12 discuss inventory constraints.

Step 4 To validate the model, inputs and outputs from the Lake Charles refinery were collected for one month. Given the actual inputs used at the refinery during that month, the actual outputs were compared to those predicted by the model. After extensive changes, the model's predicted outputs were close to the actual outputs.

Step 5 Running the LP yielded a daily strategy for running the refinery. For instance, the model might, say, produce 400,000 gallons of turbine fuel using 300,000 gallons of crude 1 and 200,000 gallons of crude 2.

Steps 6 and 7 Once the database and process control were in place, the model was used to guide day-to-day refinery operations. CITGO estimated that the overall benefits of the refinery system exceeded $50 million annually.

Introduction to Linear Programming

Example: Giapetto's Woodcarving

Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains.

- A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14.
- A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10.

The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing.

- A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor.
- A train requires 1 hour of finishing and 1 hour of carpentry labor.

Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours.

Demand for trains is unlimited, but at most 40 soldiers are bought each week.

Giapetto wants to maximize weekly profit (revenues-costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

	Soldiers	Trains
Selling price	\$27	\$21
Costs		
Raw material	\$10	\$9
Labor + OH	\$14	\$10
Labor		
Carpentry (80 hrs)	1hr	1hr
Finishing (100 hrs)	2hrs	1hr
Demand	≤ 40	unlimited

We specify the objective function, decision variables, and the restrictions.
(constraints)

Then, we write the mathematical model.

Obj. fnc: Maximize $\underbrace{\text{profit}}$

weekly revenue − raw mat. cost − labor & OH cost

Decision variables: How many trains & soldiers to produce each week?

x_1: # of soldiers produced each week
x_2: " " trains " " "

$z = ?x_1 + ?x_2 \Rightarrow z = 3x_1 + 2x_2$

\$27 − (\$10+\$14) = \$3

\$21 − (\$9 + \$10) = \$2

Constraints:

- Labor constraints
  - Carpentry: $x_1 + x_2 \leq 80$
  - Finishing: $2x_1 + x_2 \leq 100$

- Demand constraints: $x_1 \leq 40$

- Sign constraints (set constraints): $x_1 \geq 0$, $x_2 \geq 0$

Then, our model is:

Linear Programming Model

Max $z = 3x_1 + 2x_2$

subject to

(s.to, s.t.)

$x_1 + x_2 \leq 80$ (carpentry)

$2x_1 + x_2 \leq 100$ (finishing)

$x_1 \leq 40$ (demand)

$$\left.\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \end{array}\right\} \text{(sign const.)}$$

where x_1: # of soldiers produced each week

x_2: # of trains " " "

Linear Function & Linear Inequalities

- A function $f(x_1, x_2, \dots, x_n)$ of $x_1, x_2, \dots, x_n$ is a **linear function** if and only if for some set of constants $c_1, c_2, \dots, c_n$, $f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$.

- For any linear function $f(x_1, x_2, \dots, x_n)$ and any number b, the inequalities $f(x_1, x_2, \dots, x_n) \leq b$ and $f(x_1, x_2, \dots, x_n) \geq b$ are **linear inequalities**.

Linear Programming Problem

- A **linear programming problem** (linear program, LP) is an optimization problem for which we do the following:

1. We attempt to maximize (or minimize) a <u>linear</u> function of the decision variables. The function that is to be maximized or minimized is called the <u>objective function</u>.
2. The values of the decision variables must satisfy a set of <u>constraints</u>. Each constraint must be a <u>linear equation</u> or <u>linear inequality</u>.
3. A <u>sign restriction</u> is associated with each variable. For any variable x_i, the sign restriction specifies that x_i must be either nonnegative ($x_i \geq 0$) or unrestricted in sign (urs).

Properties of LP

- Proportionality: Contribution of each variable is proportional to the value of the variable: $c_1 x_1$
- Additivity: Total contribution is additive: $c_1 x_1 + c_2 x_2$
- Divisibility: Non-integer numbers are permissible
- Certainty: Each parameter is known in advance (i.e., deterministic)

Feasible Region and Optimal Solution

- The **feasible region** for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions.
  - feasible point vs. infeasible point

- For a maximization problem, an **optimal solution** to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.
  - If an LP has a feasible solution → unique optimal solution, alternative optimal solutions, or unbounded
  - If an LP has no optimal solution → infeasible or unbounded

Example: Feed Mix Problem

- Consider the following feed mix problem where the decision maker (DM) is a farm owner.
- Weekly feed consumption: At least 20,000 lbs
- Ingredients for the mix: Limestone, corn, soybean
- Unit cost ($/lb): 0.04 (limestone), 0.15 (corn), 0.4 (soybean)
- Nutrition requirements:
  - Calcium: 0.8%-1.2%
  - Protein: ≥ 22%
  - Fiber: ≤ 5%

Nutrition percentage:

	Limestone	Corn	Soybean
Calcium	0.38	0.001	0.002
Protein	-	0.09	0.5
Fiber	-	0.02	0.08

- Farmer wants to determine how much ingredient should be used in the mix to satisfy the mix requirements at the minimum cost of mix.

Decision variables

x_1: lb of limestone to be used in the mix

x_2: " " corn " " " " " "

x_3: " " soybean " " " " " "

Objective

Minimize cost

$\hookrightarrow z = 0.04x_1 + 0.15x_2 + 0.4x_3$

Constraints

- Weekly consumption: $x_1 + x_2 + x_3 \geq 20{,}000$
- Calcium requirement: $0.38x_1 + 0.001x_2 + 0.002x_3 \geq 0.008(x_1 + x_2 + x_3)$

$0.38x_1 + 0.001x_2 + 0.002x_3 \leq$ **0.012** $(x_1 + x_2 + x_3)$

Simplified: $0.372x_1 - 0.007x_2 - 0.006x_3 \geq 0$

$0.368x_1 - 0.011x_2 - 0.010x_3 \leq 0$

- Protein requirement: $0.09x_2 + 0.5x_3 \geq 0.22(x_1 + x_2 + x_3)$

  $\hookrightarrow -0.22x_1 - 0.13x_2 + 0.28x_3 \geq 0$

- Fiber requirement: $0.02x_2 + 0.08x_3 \leq 0.05(x_1 + x_2 + x_3)$

  $\hookrightarrow -0.05x_1 - 0.03x_2 + 0.03x_3 \leq 0$

- Sign const: $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

The model:

$$\text{Min } z = 0.04x_1 + 0.15x_2 + 0.4x_3$$

s.to

$$x_1 + x_2 + x_3 \geq 20{,}000 \quad \text{(weekly requirement)}$$

$$\left.\begin{array}{l} 0.372x_1 - 0.007x_2 - 0.006x_3 \geq 0 \\ 0.368x_1 - 0.011x_2 - 0.01x_3 \leq 0 \end{array}\right\} \text{(calcium)}$$

$$0.22x_1 + 0.13x_2 - 0.28x_3 \leq 0 \quad \text{(protein)}$$

$$0.05x_1 + 0.03x_2 - 0.03x_3 \geq 0 \quad \text{(fiber)}$$

$$x_i \geq 0 \quad i = 1, 2, 3 \quad \text{(sign const.)}$$

Example: Work Scheduling Problem

- A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in the table.

Day	Number of Full-time Employees Required
1 = Monday	17
2 = Tuesday	13
3 = Wednesday	15
4 = Thursday	19
5 = Friday	14
6 = Saturday	16
7 = Sunday	11

- Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday.
- The post office wants to meet its daily requirements using only fulltime employees. Formulate an LP that the post office can use to minimize the number of full-time employees who must be hired.

Obj: Minimize the # of employees to be hired

Decision variables:

x_i: # of employees starting to work on day i

i = 1 (Monday), 2 (Tuesday), ..., 7 (Sunday)

The model is:

$$\text{Min } z = x_1 + x_2 + \dots + x_7$$

s.to

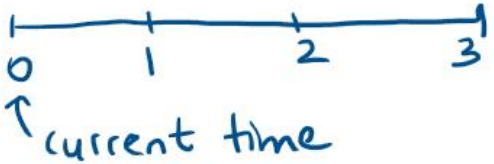
$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 \quad \text{(Monday)}$$
$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \quad \text{(Tuesday)}$$
$$x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \quad \text{(Wednesday)}$$
$$x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 \quad \text{(Thursday)}$$
$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 \quad \text{(Friday)}$$
$$x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 \quad \text{(Saturday)}$$
$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \quad \text{(Sunday)}$$
$$x_i \geq 0 \quad i = 1, 2, \dots, 7 \quad \text{(Sign const.)}$$

Example: Investment Planning

- Finco Investment Corporation must determine investment strategy for the firm during the next three years. Currently (time 0), $100,000 is available for investment.
- Investments A, B, C, D, and E are available. The cash flow associated with investing $1 in each investment is given in the table. For example, $1 invested in investment B requires a $1 cash outflow at time 1 and returns 50¢ at time 2 and $1 at time 3.

	Cash Flow ($) at Time*			
	0	1	2	3
A	−1	+0.50	+1	0
B	0	−1	+0.50	+1
C	−1	+1.2	0	0
D	−1	0	0	+1.9
E	0	0	−1	+1.5

- To ensure that the company's portfolio is diversified, Finco requires that at most $75,000 be placed in any single investment.
- In addition to investments A–E, Finco can earn interest at 8% per year by keeping uninvested cash in money market funds. Returns from investments may be immediately reinvested. For example, the positive cash flow received from investment C at time 1 may immediately be reinvested in investment B. Finco cannot borrow funds, so the cash available for investment at any time is limited to cash on hand. Formulate an LP that will maximize cash on hand at time 3.



Decision Variables

x_i: amount of money invested in alternative i, $i = A, B, C, D, E$

s_t: " " " " at bank at time t, $t = 0, 1, 2, 3$

$$\text{Max } z = s_3$$

s.to

$x_A + x_C + x_D + s_0 = 100{,}000$ (current time)

$0.5x_A + 1.2x_C + 1.08s_0 - x_B - s_1 = 0$ (time 1)

$x_A + 0.5x_B + 1.08s_1 - x_E - s_2 = 0$ (time 2)

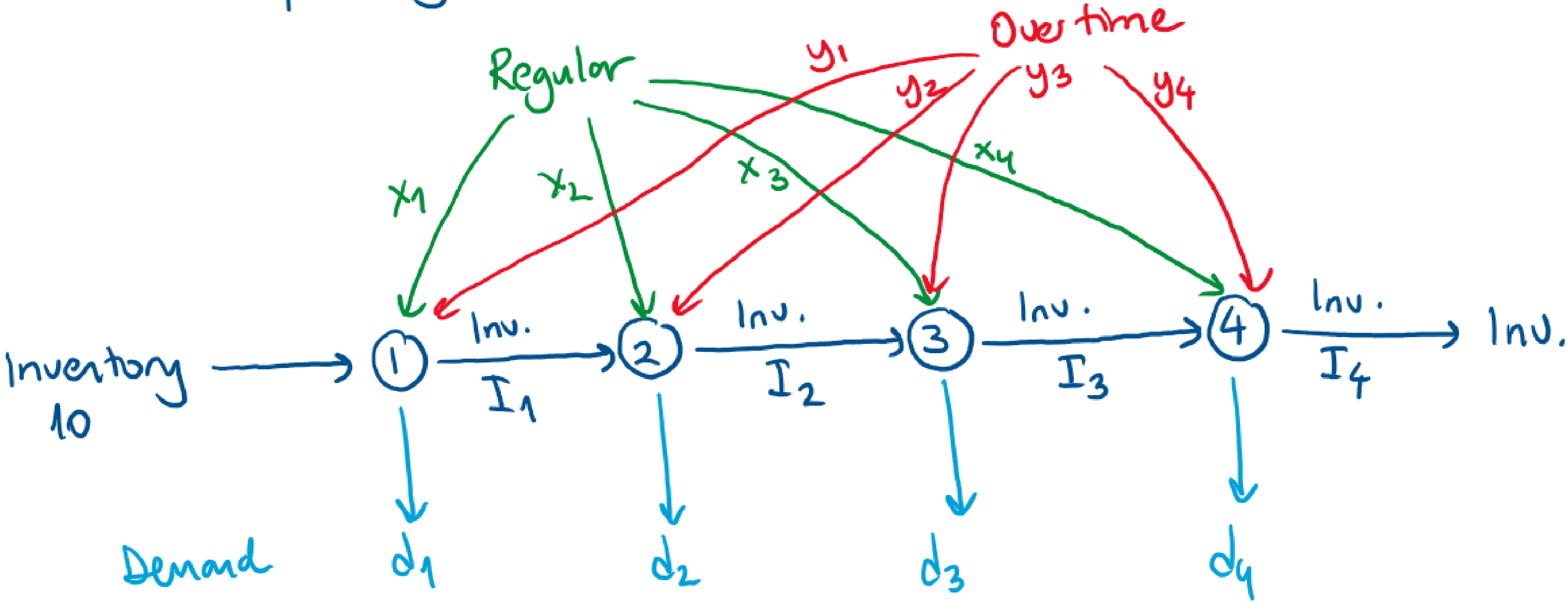
$x_B + 1.9x_D + 1.5x_E + 1.08s_2 - s_3 = 0$ (time 3)

$x_i \leq 75{,}000 \quad i = A, B, C, D, E$ (max. limit on an investment)

$$\left.\begin{array}{ll} x_i \geq 0 & i = A, B, C, D, E \\ s_t \geq 0 & t = 0, 1, 2, 3 \end{array}\right\} \text{(sign constraints)}$$

Example: Production Planning

- Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters (one quarter = three months).
- The demand during each of the next four quarters is as follows: first quarter, 40 sailboats; second quarter, 60 sailboats; third quarter, 75 sailboats; fourth quarter, 25 sailboats. Sailco must meet demands on time.
- At the beginning of the first quarter, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during that quarter. For simplicity, we assume that sailboats manufactured during a quarter can be used to meet demand for that quarter.
- During each quarter, Sailco can produce up to 40 sailboats with regular-time labor at a total cost of $400 per sailboat. By having employees work overtime during a quarter, Sailco can produce additional sailboats with overtime labor at a total cost of $450 per sailboat.
- At the end of each quarter (after production has occurred and the current quarter’s demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred. Use linear programming to determine a production schedule to minimize the sum of production and inventory costs during the next four quarters.



x_t: # of sailboats produced by regular time labor in period t, $t=1,2,3,4$

y_t: # of sailboats produced by overtime labor in period t, $t=1,2,3,4$

I_t: # of sailboats kept in inventory in period t (at the end of period t), $t=1,2,3,4$

$I_{t-1} + x_t + y_t = I_t + d_t$ (parameter)

$\underbrace{I_{t-1} + x_t + y_t}_{\text{input}} - \underbrace{I_t}_{\text{inv.}} = \underbrace{d_t}_{\text{demand}}$

$$\text{Min } z = \sum_{t=1}^{4} (400 x_t + 450 y_t + 20 I_t)$$

s.to

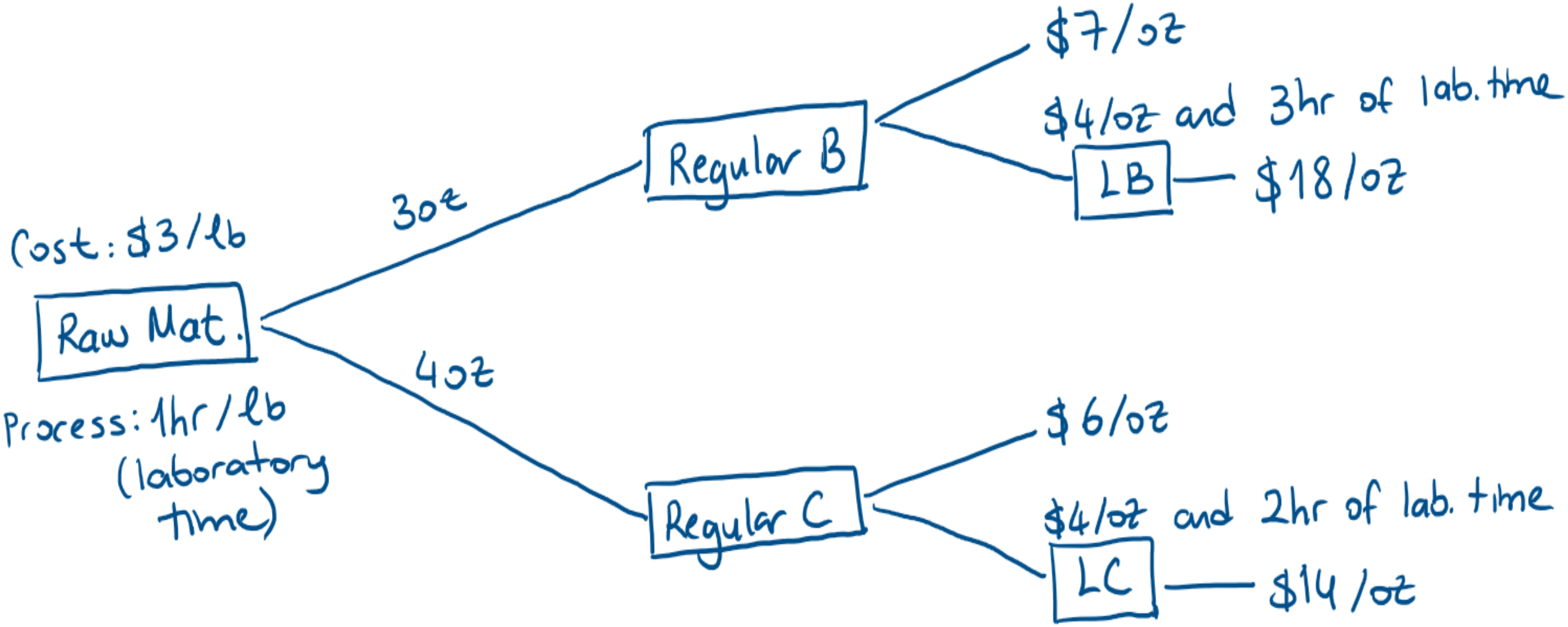
$$\left.\begin{aligned} 10 + x_1 + y_1 - I_1 &= 40 \\ I_1 + x_2 + y_2 - I_2 &= 60 \\ I_2 + x_3 + y_3 - I_3 &= 75 \\ I_3 + x_4 + y_4 - I_4 &= 25 \end{aligned}\right\} \text{(Inventory balance equations)}$$

$x_t \leq 40 \quad \forall t$ (regular prod. capacity)

$x_t, y_t, I_t \geq 0 \quad \forall t$ (sign constraints)

Example: Production Process Modeling

- Rylon Corporation manufactures Brute and Chanelle perfumes.
- The raw material needed to manufacture each type of perfume can be purchased for $3 per pound. Processing 1 lb of raw material requires 1 hour of laboratory time. Each pound of processed raw material yields 3 oz of Regular Brute Perfume and 4 oz of Regular Chanelle Perfume. Regular Brute can be sold for $7/oz and Regular Chanelle for $6/oz.
- Rylon also has the option of further processing Regular Brute and Regular Chanelle to produce Luxury Brute, sold at $18/oz, and Luxury Chanelle, sold at $14/oz. Each ounce of Regular Brute processed further requires an additional 3 hours of laboratory time and $4 processing cost and yields 1 oz of Luxury Brute. Each ounce of Regular Chanelle processed further requires an additional 2 hours of laboratory time and $4 processing cost and yields 1 oz of Luxury Chanelle.
- Each year, Rylon has 6,000 hours of laboratory time available and can purchase up to 4,000 lb of raw material.
- Formulate an LP that can be used to determine how Rylon can maximize profits. Assume that the cost of the laboratory hours is a fixed cost.



Each year: 6000 hours of lab time
$\leq$ 4000 lb of raw material

Nothing is mentioned about demand $\Rightarrow$ Assume: We sell what we produce!

Decision variables:

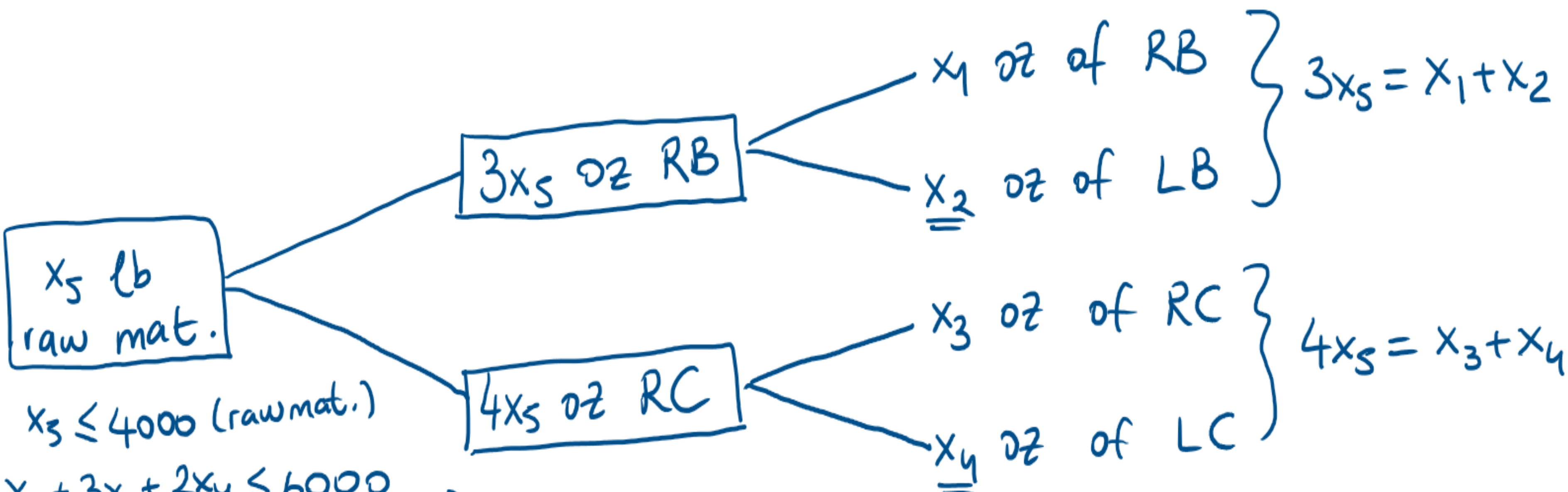
x_1: # of oz of RB sold annually

x_2: " " " " LB " "

x_3: " " " " RC " "

x_4: " " " " LC " "

x_5: " " " " raw material purchased annually



$x_3 \leq 4000$ (raw mat.)

$x_5 + 3x_2 + 2x_4 \leq 6000$ (labor hrs.)

$$\text{Max } z = 7x_1 + \overset{18-4}{14}x_2 + 6x_3 + \overset{14-4}{10}x_4 - 3x_5$$

s.to

$$x_5 \leq 4000 \quad \text{(raw material)}$$

$$3x_2 + 2x_4 + x_5 \leq 6000 \quad \text{(lab hours)}$$

$$\left.\begin{aligned} x_1 + x_2 - 3x_5 &= 0 \\ x_3 + x_4 - 4x_5 &= 0 \end{aligned}\right\} \text{(process balance)}$$

$$x_i \geq 0 \quad \forall i \quad \text{(sign constraints)}$$

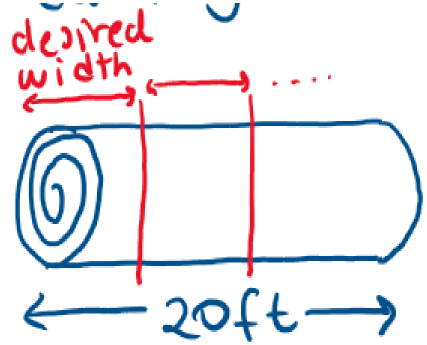
$$i = 1, 2, \ldots, 5$$

Example: Cutting Stock Problem

- A company produces paper rolls with a standard width of 20 feet (with a standard length L). Special orders with different widths are produced by cutting the standard rolls.

Order	Desired width	Desired number of rolls
1	5	150
2	7	200
3	9	300

- We want to minimize the total loss resulting from preparing the orders by cutting the rolls.



It is possible to use different knife settings.

Knife setting

Widths	1	2	3	4	5	6
5	4	2	2	1	0	0
7	0	1	0	2	1	0
9	0	0	1	0	1	2
waste (ft * L)	0	3	1	1	4	2

x_i: # of std. rolls to be cut according to setting i
$(i = 1, 2, \ldots, 6)$

\# of 5ft rolls $= 4x_1 + 2x_2 + 2x_3 + x_4$

\# of 7ft rolls $= x_2 + 2x_4 + x_5$

\# of 9ft rolls $= x_3 + x_5 + 2x_6$

y_j: # of surplus rolls of j^{th} feet rolls
($j=1$: 5 feet, $j=2$: 7 feet, $j=3$: 9 feet)

$y_1 = 4x_1 + 2x_2 + 2x_3 + x_4 - 150$

$y_2 = x_2 + 2x_4 + x_5 - 200$

$y_3 = x_3 + x_5 + 2x_6 - 300$

Total loss $= L * (3x_2 + x_3 + x_4 + 4x_5 + 2x_6 + 5y_1 + 7y_2 + 9y_3)$

↳ constant

$$\text{Min } z = 3x_2 + x_3 + x_4 + 4x_5 + 2x_6 + 5y_1 + 7y_2 + 9y_3$$

s.to

$$4x_1 + 2x_2 + 2x_3 + x_4 - y_1 = 150$$

$$x_2 + 2x_4 + x_5 - y_2 = 200$$

$$x_3 + x_5 + 2x_6 - y_3 = 300$$

$$x_i \geq 0 \quad i = 1, 2, \dots, 6$$

$$y_j \geq 0 \quad j = 1, 2, 3$$

Example: Capacity Allocation

- A company produces a final product that is assembled from three different parts: Part 1 – Part 2 – Part 3.

		Production Rate (units/hr)		
Department	Weekly Capacity (hr)	Part 1	Part 2	Part 3
1	100	8	5	10
2	80	6	12	4

- Determine the production hours to be allocated by each department to each part so that the number of final assembly units is maximized.

Final product = Part 1 + Part 2 + Part 3

<u>Decision variables:</u>

x_{ij}: # of (weekly) hours allocated by dept. i to part j
$(i=1,2\,;\, j=1,2,3)$

$$\left.\begin{array}{l} \text{Part } 1 \rightarrow 8x_{11}+6x_{21} \\ \quad " \quad 2 \rightarrow 5x_{12}+12x_{22} \\ \quad " \quad 3 \rightarrow 10x_{13}+4x_{23} \end{array}\right\} \text{Final product} \rightarrow \min\{8x_{11}+6x_{21},\, 5x_{12}+12x_{2},\, 10x_{13}+4x_{23}\}$$

$$\text{Max } z = \min\{8x_{11}+6x_{21},\, 5x_{12}+12x_{22},\, 10x_{13}+4x_{23}\}$$

Not linear!
How to linearize this?

s.to

$$\left.\begin{array}{l} x_{11}+x_{12}+x_{13} \leq 100 \\ x_{21}+x_{22}+x_{23} \leq 80 \end{array}\right\} \text{(capacity)}$$

$x_{ij} \geq 0 \quad i=1,2\,;\, j=1,2,3$ (sign const.)

Let the minimum one be y.

Max $z = y$

s.to

$y \le 8x_{11} + 6x_{21}$
$y \le 5x_{12} + 12x_{22}$
$y \le 10x_{13} + 4x_{23}$
(requirement for obtaining the final assembled product)

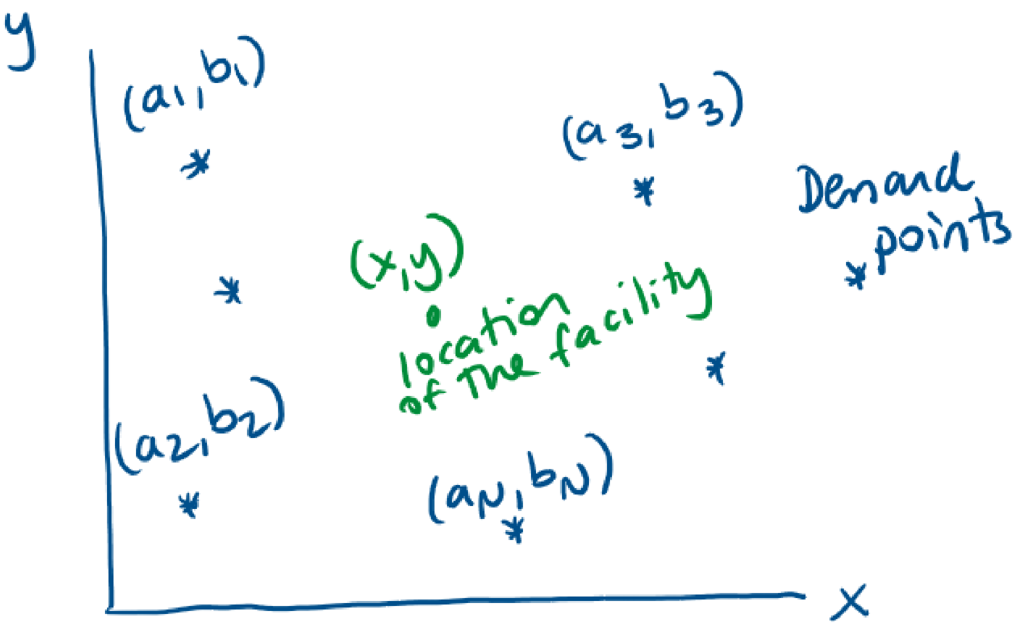
$x_{11} + x_{12} + x_{13} \le 100$
$x_{21} + x_{22} + x_{23} \le 80$
(capacity constraints)

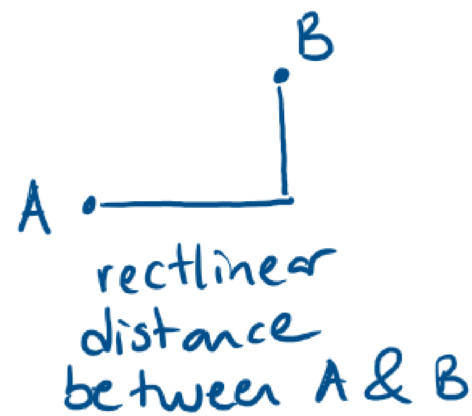
$x_{ij} \geq 0 \quad i = 1,2 \; ; \; j = 1,2,3$ (sign const.)

$y \geq 0$

Example: Facility Location Problem

- N demand points
  - a_i and b_i represent the x and y coordinates, respectively, of demand point i $(i = 1,2,\ldots,N)$
- A facility is to be located to serve these demand points.
- Formulate a linear program to find the facility location that minimizes the total rectilinear distance to the demand points.





$$\text{Min } z = \sum_{i=1}^{N} |x - a_i| + |y - b_i|$$

s.to

$$x \geq 0$$

$$y \geq 0$$

We want to make this a linear function.

How to linearize this?

Define v_i and u_i to represent $|x-a_i|$ and $|y-b_i|$, respectively.

$$\text{Min } z = \sum_{i=1}^{N} (v_i + u_i)$$

s.to

$$\left.\begin{aligned} v_i &\geq x - a_i \\ v_i &\geq a_i - x \\ u_i &\geq y - b_i \\ u_i &\geq b_i - y \end{aligned}\right\} \begin{array}{l} \forall i \\ (i = 1, 2, \ldots, N) \end{array}$$

$$\left.\begin{aligned} &v_i, u_i \geq 0 \quad \forall i \\ &x, y \geq 0 \end{aligned}\right\} \text{(sign const.)}$$

Or Define $v_i^+, v_i^-, u_i^+, u_i^-$.

$\Rightarrow$

$$\text{Min } z = \sum_{i=1}^{N} (v_i^+ + v_i^- + u_i^+ + u_i^-)$$

s.to

$$x - a_i = v_i^+ - v_i^- \quad \forall i$$

$$y - b_i = u_i^+ - u_i^- \quad \forall i$$

$$v_i^+, v_i^-, u_i^+, u_i^- \geq 0 \quad \forall i$$

$$x, y \geq 0$$

What if the objective is minimizing the maximum distance?

$$\text{Min } z$$

s.to

$$z \geq v_i^+ + v_i^- + u_i^+ + u_i^- \quad \forall i$$

$\oplus$

all the other constraints.