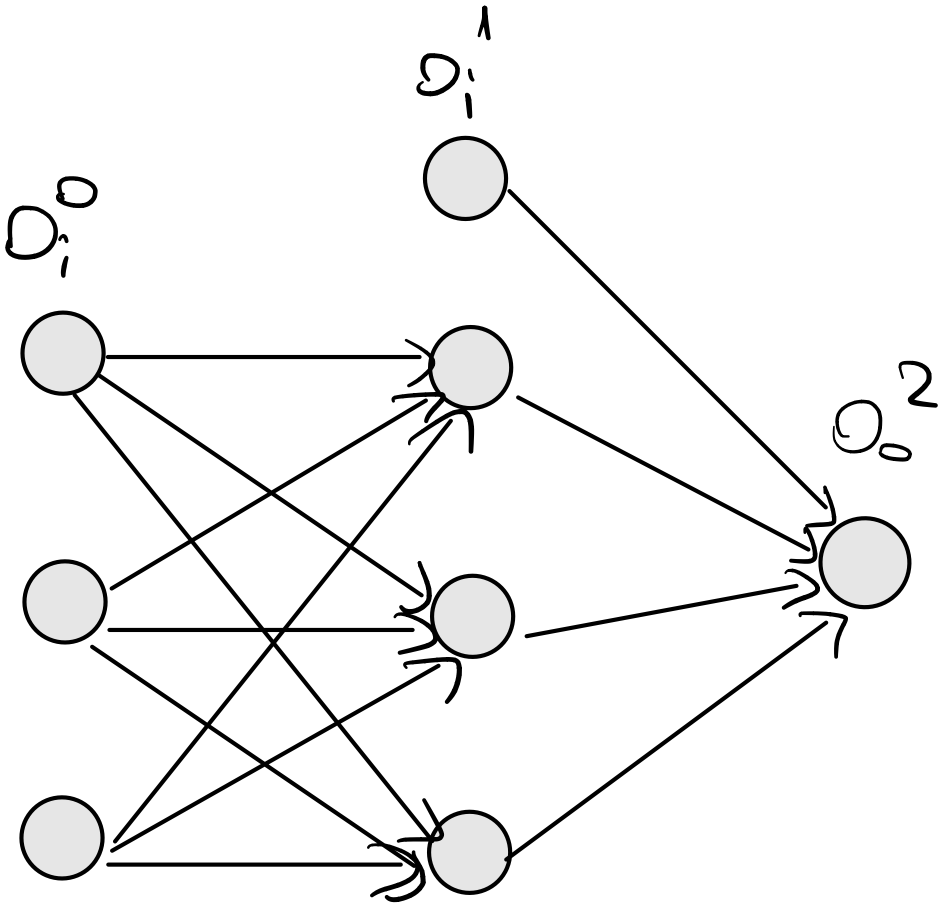
Assignment 1



Regression Part

$$O_o^{(2)} = \sum_{k=0}^{3} O_k^{(1)} \cdot a_{ko}^{(1)}$$

$$O_k^{(1)} = \sigma\left(\sum_{i=0}^{2} O_i^{(0)} \cdot a_{ik}^{(0)}\right) \text{ hence}$$

$$O_0^{(2)} = \sum_{k=0}^{3} \left(\frac{1}{1 + e^{-\left(\sum_{i=0}^{2} O_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) a_{k0}^{(1)}$$

$$SE = Loss = \left(y - O_0^{(2)}\right)^2$$ where y is label

Two Update Rules:

First Layer $\rightarrow$ $a_{ik}^{(0)} = a_{ik}^{(0)} - \alpha \dfrac{\partial SE(y, O_0^{(2)})}{\partial a_{ik}^{(1)}}$

Second Layer $\rightarrow$ $a_{k0}^{(1)} = a_{k0}^{(1)} - \alpha \dfrac{\partial SE(y, O_0^{(2)})}{\partial a_{k0}^{(1)}}$

I will find the derivatives for both of the layers respectively.

$$\frac{\partial SE\left(y, o_0^{(2)}\right)}{\partial a_{ik}^{(0)}} =$$

$$\frac{\partial \left(y - \sum_{k=0}^{3} \left(\frac{1}{1 + e^{-\left(\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}\right)}} \right) a_{k0}^{(1)} \right)^2}{\partial a_{ik}^{(0)}}$$

$$= 2\left(y - \sum_{k=0}^{3} \left(1 + e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right)^{-1} a_{k0}^{(1)} \right)$$

$$\cdot - \left(\underbrace{-a_{k0}^{(1)} \left(1 + e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right)^{-2}}_{\text{derivative}} \underbrace{\left(e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right)}_{\text{chain1}} \underbrace{\left(\sum_{i=0}^{2} o_i^{(0)} \right)}_{\text{chain2}} \right)$$

Rule:

$$a_{ik} = a_{ik} - \alpha \left(2\left(y - \sum_{k=0}^{3} \left(1 + e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right)^{-1} a_{k0}^{(1)} \right) \cdot \left(-a_{k0}^{(1)} \left(1 + e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right)^{-2} \left(-e^{\sum_{i=0}^{2} o_i^{(0)} \cdot a_{ik}^{(0)}} \right) \left(\sum_{i=0}^{2} o_i^{(0)} \right) \right) \right)$$

for a_{k0};

$$a_{k0} = a_{k0} - \alpha \frac{\partial SE(y, O_0^{(2)})}{\partial a_{k0}}$$

$$\frac{\partial SE(y, O_0^{2})}{\partial a_{k0}} = \frac{\partial \left(y - \sum_{k=0}^{3} \left(\frac{1}{1+e^{-\left(\sum_{i=0}^{2} a_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) a_{k0}^{(1)} \right)^2}{\partial a_{k0}}$$

$$= \underbrace{2\left(y - \sum_{k=0}^{3} \left(\frac{1}{1+e^{-\left(\sum_{i=0}^{2} a_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) a_{k0}^{(1)} \right)}_{\text{derivative}} \cdot \underbrace{\left(-\sum_{k=0}^{3} \left(\frac{1}{1+e^{-\left(\sum_{i=0}^{2} a_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) \right)}_{\text{chain}}$$

Hence;

$$a_{k0} = a_{k0} - \alpha \left(2\left(y - \sum_{k=0}^{3} \left(\frac{1}{1+e^{-\left(\sum_{i=0}^{2} a_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) a_{k0}^{(1)} \right) \cdot \left(-\sum_{k=0}^{3} \left(\frac{1}{1+e^{-\left(\sum_{i=0}^{2} a_i^{0} \cdot a_{ik}^{(0)}\right)}} \right) \right) \right)$$

Classification Part

Update Rule: $w = w - \alpha \dfrac{\partial \mathcal{E}(x)}{\partial w}$

$$\mathcal{E}(x) = CE(\ell, \ell') = -\sum_i l_i \cdot \log(l_i')$$

where $\ell = [l_0, l_1, l_2]$

$\ell' = [o_0^2, o_1^2, o_2^2)$

$$o_n^2 = \text{softmax}(x_n^{(2)}, x^{(2)})$$

$$o_n^2 = \frac{e^{x_n^{(2)}}}{\sum_{s=0} e^{x_s^{(2)}}}$$

$$X_n^{(2)} = \sum_{k=0} O_k^{(1)} \cdot a_{kn}^{(1)}$$

$$O_k^{(1)} = \sigma\left(\sum_{i=0} O_i^{(0)} \cdot a_{ik}^{(0)}\right)$$

$$O_k^{(1)} = \frac{1}{1 + e^{-\left(\sum_{i=0} O_i^{(0)} \cdot a_{ik}^{(0)}\right)}}$$

$$X_n^{(2)} = \sum_{k=0} \left(\frac{1}{1 + e^{-\left(\sum_{i=0} O_i^{(0)} \cdot a_{ik}^{(0)}\right)}} \right) a_{kn}^{(1)}$$

So now, time for O_n^2

$$\frac{e^{\left(\sum_{k=0}\left(\frac{1}{1+e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right)}}\right) a_{kn}^{(1)}\right)}}{\sum_{s=0}\left(e^{\left(\sum_{k=0}\left(\frac{1}{1+e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right)}}\right) a_{ks}^{(1)}\right)}\right)}$$

difference

$$= o_n^2$$

For the sake of simplicity I will call A and B for the nominator and denominator respectively.

Hence;

$$O_n^2 = \frac{A_n}{B_n}$$

$$CE(\ell, \ell') = \sum_{i=0} \ell_i \cdot \log(\ell'_i)$$

For a_{kn};

$$\frac{\partial CE(\ell, \ell')}{\partial a_{kn}} = \frac{\partial \sum_{i=0} \ell_i \cdot \log(\ell'_i)}{\partial (a_{kn})}$$

$$\log(\ell'_i) = \log\left(\frac{A_i}{B}\right)$$

derivative for this is;

$$\log\left(\frac{A_i}{B}\right) \longrightarrow \text{we need to find } A_i' \text{ for chain rule}$$

no a_{kn} inside

$$= \sum_{i=0} l_i \cdot \frac{1}{\frac{A_i}{B} \cdot \ln(2)} \cdot \left(\frac{A_i'}{B_i}\right)$$

no a_{kn} inside

$$A_i' = A_i \cdot \sum_{k=0} \sigma\left(\sum_{i=0} o_i^0 \cdot a_{ik}\right)$$

$$\frac{\partial CE(l,l')}{\partial a_{kn}} = \sum_{i=0} \frac{l_i \cdot B_i \cdot \left(A_i \cdot \sum_{k=0} \sigma\left(\sum_{p=0} o_p^0 \cdot a_{pk}\right)\right)}{A_i \cdot \ln(2) \cdot B_i}$$

For a_{ik}:

$$\frac{\partial CE(\ell, \ell')}{\partial a_{ik}} = \frac{\partial \sum_{i=0} \ell_i \cdot \log(\ell'_i)}{\partial (a_{ik})}$$

$$\log(\ell'_i) = \log\left(\frac{A_i}{B_i}\right)$$

$$\frac{\partial CE(\ell, \ell')}{\partial a_{ik}} = \sum_{i=0} \ell_i \cdot \left(\frac{1}{\frac{A_i}{B_i} \cdot \ln(2)} \cdot \frac{A'_i \cdot B_i - A_i \cdot B'_i}{B_i^2} \right)$$

So, we need A'_i and B'_i with respect to a_{ik}. where

$$A_n = e^{\left(\sum_{k=0} \left(\frac{1}{1 + e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right)}} \right) a_{kn}^{(1)} \right)}$$

$$\frac{\partial A_n}{\partial a_{ik}} = \sum_{k=0} \left(\sigma^2 \left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)} \right) \cdot a_{kn}^{(1)} \right) \cdot \sum_{i=0} o_i^{(0)}$$

$$= A_n \cdot \sum_{k=0} \sigma \left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)} \right) \cdot \sum_{i=0} o_i^{(0)}$$

Time for B_n where;

$$B = \sum_{s=0} \underbrace{\left(e^{\left(\sum_{k=0} \left(\frac{1}{1 + e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right)}} \right) a_{ks}^{(1)} \right)} \right)}$$

we can see that this part is A_s

This knowledge will make our derivative much simpler

We know that $\frac{\partial A_s}{\partial a_{ik}}$ which is

$$A_s \cdot \sum_{k=0} \sigma\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right) \cdot \sum_{i=0} o_i^{(0)}$$

which makes $\frac{\partial B}{\partial a_{ik}}$ to be;

$$\sum_{s=0} A_s \cdot \sum_{k=0} \sigma\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)}\right) \cdot \sum_{i=0} o_i^{(0)}$$

Let's remind the loss function derivation.

$$\sum_{i=0} p_i \cdot \left(\frac{1}{\frac{A_i}{B} \cdot \ln(2)} \cdot \frac{A_i' \cdot B - A_i \cdot B'}{B^2}\right)$$

We know all the elements of this equation.

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$$A_n = e^{\left(\sum_{k=0} \left(\frac{1}{1 + e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)} \right)}} \right) a_{kn}^{(1)} \right)}$$

$$B = \sum_{s=0} e^{\left(\sum_{k=0} \left(\frac{1}{1 + e^{-\left(\sum_{i=0} o_i^{(0)} \cdot a_{ik}^{(0)} \right)}} \right) a_{ks}^{(1)} \right)}$$

$$\frac{\partial A_n}{\partial a_{iz}} = A_n \cdot \sum_{k=0} \sigma\left(\sum_{i=0} o_i^{(0)} \cdot a_{iz}^{(0)} \right) \cdot \sum_{i=0} o_i^{(0)}$$

$$\frac{\partial B}{\partial a_{iz}} = \sum_{s=0} A_s \cdot \sum_{k=0} \sigma\left(\sum_{i=0} o_i^{(0)} \cdot a_{iz}^{(0)} \right) \cdot \sum_{i=0} o_i^{(0)}$$