Student Information

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Answer 1

a.

i) It is a topology since it meets all 3 of the conditions.

ii) It is not a topology since union of the subsets {a} ∪ {b} which is {a , b} is not in the set T_2. It contradicts with the 2nd condition.

iii) It is a topology since it meets all 3 of the conditions.

iv) It is not a topology since union of the subsets $\{a, c\} \cup \{b, c\}$ which is {a, b, c} is not in the set T_4. It contradicts with the 2nd condition.

b.

i)

ii)

iii)

Answer 2

a. For f(a,b) to be injective the condition $f(a_1, b_1) = f(a_2, b_2) \longrightarrow (a_1 = a_2 \wedge b_1 = b_2\ )$ must be met.

We write that as $a_1 + b_1 = a_2 + b_2$

The left side of the equation $a_1 - a_2 = b_2 - b_1$ is a natural number and the right side of the equation must be between (-1,1) The only natural number in this interval is 0.

Thus $a_1 - a_2 = 0$ and $a_1 = a_2$ similarly $b_2 - b_1 = 0$ and $b_1 = b_2$. The function is injective.

b. For f(a,b) to be surjective the condition $\forall y \in [0, \infty), \quad \exists x \in A, \quad$ such that f(x)=y

So for y=5 there must be an x such that f(x)=y but there is no x is that is mapped to 5. Thus it is not surjective. (Proof by contradiction)

c. We are being given that g(x,y) is injective from $[0, \infty) \longrightarrow A \times (0, 1)$. As prooven in part **a.** f(x,y) is injective from $A \times (0, 1) \longrightarrow [0, \infty)$ Thus by Schroeder–Bernstein theorem their cardinality must be the same.

Answer 3

a. Set A, will be equal to the Cartesian product of two element copies of $\mathbb{Z}^+$. It is because both elements of the set will be mapped $\mathbb{Z}^+ \times \mathbb{Z}^+$ times. Then A will be equal to $\mathbb{Z}^+ \times \mathbb{Z}^+$. Cartesian product of two infinite sets that are countable is also countable.

b. In this situation the set B of all functions, will be equal to the Cartesian product of n element copies of $\mathbb{Z}^+$. From the part (a.) we now that Cartesian product of two infinitely countable sets is countable. Then taking Cartesian product of the obtained set n more times wont change the countability. $(\mathbb{Z}^+)^n$ will also be countable.

c. The set C of all functions f : $\mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ contains the set of all functions from $\mathbb{Z}^+$ to Z^+ We konow that since D is uncountable and D $\subseteq$ C, we can deduce that C is uncountable. Cantor's second diagonal argument shows that C is uncountable.

d. The set D is the set of all possible binary sequences. D is the set of all functions f : $\mathbb{Z}^+$ to $\{0,1\}$ Cantor's second diagonal argument shows that this set is uncountable.

e.

Answer 4

a.

By Stirling approximation we say n! $\approx \sqrt{2\pi n} * \frac{n^n}{e^n}$

For function n! to be $\Theta(n^n)$, condition $c_1 n^n < n! < c_2 n^n, \quad c_1, c_2 \in \mathbb{R}^+$ must be met.

We apply Stirling approximation $c_1 n^n < \sqrt{2\pi n} * \frac{n^n}{e^n} < c_2 n^n$

$\sqrt{2\pi n} * \frac{n^n}{e^n} < c_2 n^n$ is true for all n and all positive integers $c_2 \geq 3$.

$c_1 n^n < \sqrt{2\pi n} * \frac{n^n}{e^n}$ is false for any n that is sufficiently big

Therefore n! is not $\Theta(n^n)$.

b.

For function $(n + a)^b$ to be $\Theta(n^b)$, condition $c_1 n^b < (n + a)^b < c_2 n^b, \quad c_1, c_2 \in \mathbb{R}^+$ must be met.

We know that $n^b > 0$ so we can divide all sides by n^b it becomes $c_1 < \frac{(n + a)^b}{n^b} < c_2$

$c_1 < \left(1 + \frac{a}{n}\right)^b < c_2$ can be written as $c_1 < (1^+)^b < c_2$ for sufficiently big **n** .

Since b $\in \mathbb{Z}$, for any value of b, $c_1 < (1^+)^b < c_2$ can be simplified as $c_1 < 1^+ < c_2$

Therefore there exists valid c_1 and c_2 such that $(n + a)^b$ is $\Theta(n^b)$

Answer 5

a.

- **If x < y** , we can say that $(2^x - 1) \bmod (2^y - 1) = (2^x - 1)$.

Similarly $(2^{x\, mod\, y} - 1) = (2^x - 1)$.

We can conclude that $(2^x - 1) \bmod (2^y - 1) = (2^{x\, mod\, y} - 1)$ since $(2^x - 1) = (2^x - 1)$ if $x < y$

- **If** $x \geqslant y$, we can subtract $(2^y - 1)$ from the $(2^x - 1) \bmod (2^y - 1)$ without changing the result.

After subtracting we get $(2^x - 2^y) \bmod (2^y - 1) = (2^{x\, mod\, y} - 1)$. Since $x \geqslant y$ we can write the left side as $2^y(2^{x-y} - 1) \mathrm{mod} (2^y - 1)$

The 2^y's remainder will be 1 when divided by $(2^y - 1)$ so we can omit that part and write it as $(2^{x-y} - 1) \mathrm{mod} (2^y - 1) = (2^{x\, mod\, y} - 1)$

We repeat steps above k times and the equality becomes $(2^{x-ky} - 1) \mathrm{mod} (2^y - 1) = (2^{x\, mod\, y} - 1)$.

We repeat the steps until $(2^{x-ky} - 1) < (2^y - 1)$ condition is met.

As I proved before if $(2^{x-ky} - 1) < (2^y - 1)$ then $(2^{x-ky} - 1) \mathrm{mod} (2^y - 1) = (2^{x-ky} - 1)$

And finally since $(2^{x-ky} - 1) = (2^{x\, mod\, y} - 1)$ we can conclude that $(2^x - 1) \bmod (2^y - 1) = (2^{x\, mod\, y} - 1)$ for $x, y \in \mathbb{Z}^+$

b.