EE281 - Sinusoidal Steady State Analysis

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Sinusoidal Steady-State Analysis

- Sinusoidal signals in
  - Alternating current (AC) circuits
  - Nature: Waves, motion
  - Mechanical vibrations
  - Rotating mechanical systems (Gearbox, …)
  - …

- **Sinusoidal signal** has the from of
  - cosine or sine function

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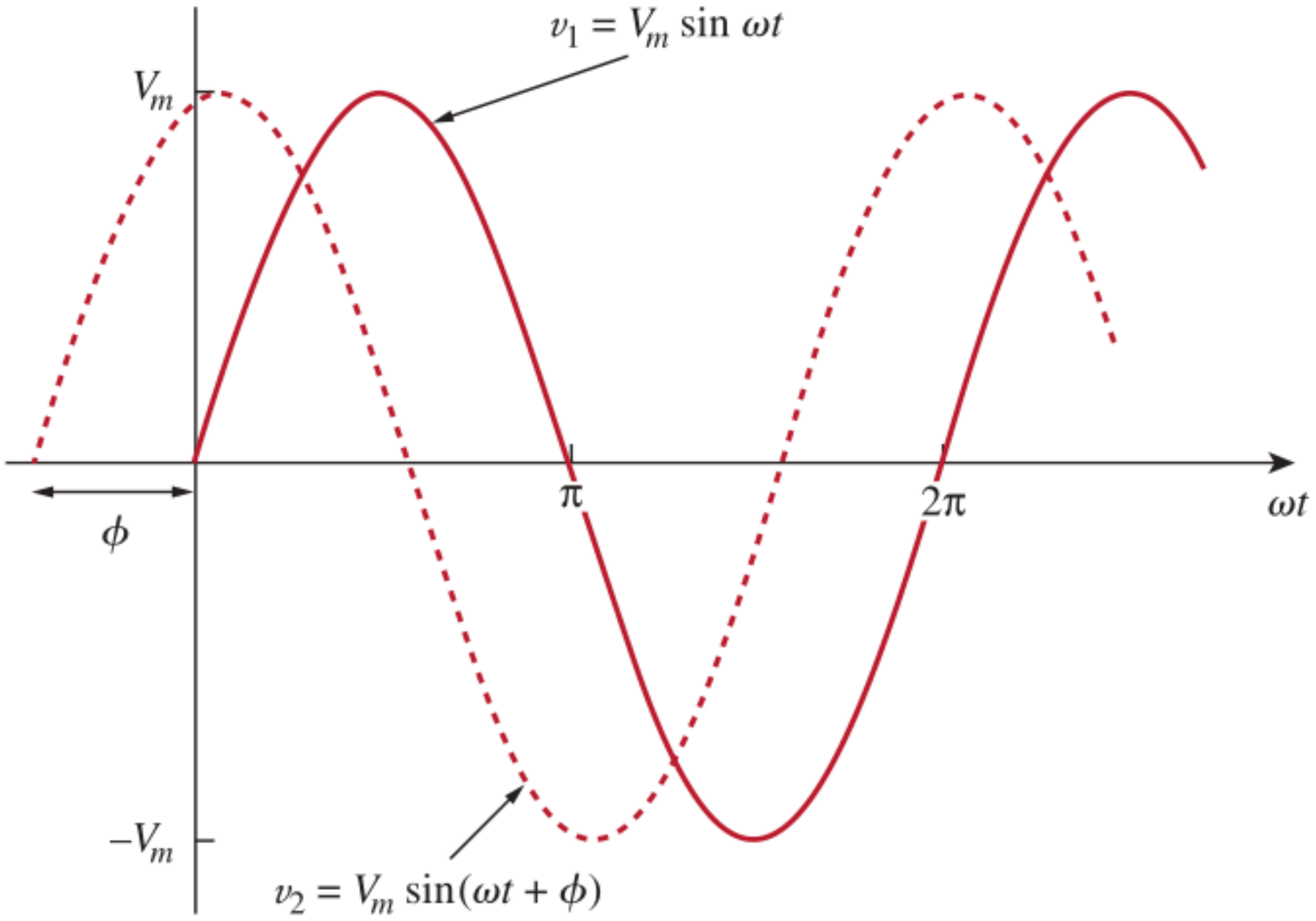
$$v(t) = M_c\cos(\omega t) + M_s\sin(\omega t) = M_p\sin(\omega t + \phi)$$

  - or can be even in complex form

$$v(t) = cos(\omega t) + j\sin(\omega t) = e^{j\omega t}$$

- **Phase:** $v(t) = V_m \sin(\omega t + \boxed{\phi})$



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$$M_c cos(\omega t) + M_s \sin(\omega t) = M \cos(\phi)\cos(\omega t) - M \sin(\phi)\sin(\omega t)$$

$$M_c = M \cos(\phi)$$

$$M_s = -M \sin(\phi)$$

$$\Rightarrow$$

$$M = \sqrt{M_c^2 + M_s^2}$$

$$\phi = \mathrm{atan2}\left(-M_s, M_m\right)$$

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$$\text{atan2}(y, x) = \begin{cases} \text{atan}\left(\frac{y}{x}\right) & \text{if} x > 0 \\ \text{atan}\left(\frac{y}{x}\right) + \pi & \text{if} x < 0 \ \& \ y \geq 0 \\ \text{atan}\left(\frac{y}{x}\right) - \pi & \text{if} x < 0 \ \& \ y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \ \& \ y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \ \& \ y < 0 \\ \text{undefined} & \text{if } x = 0 \ \& \ y = 0 \end{cases}$$

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$$\rightarrow$$

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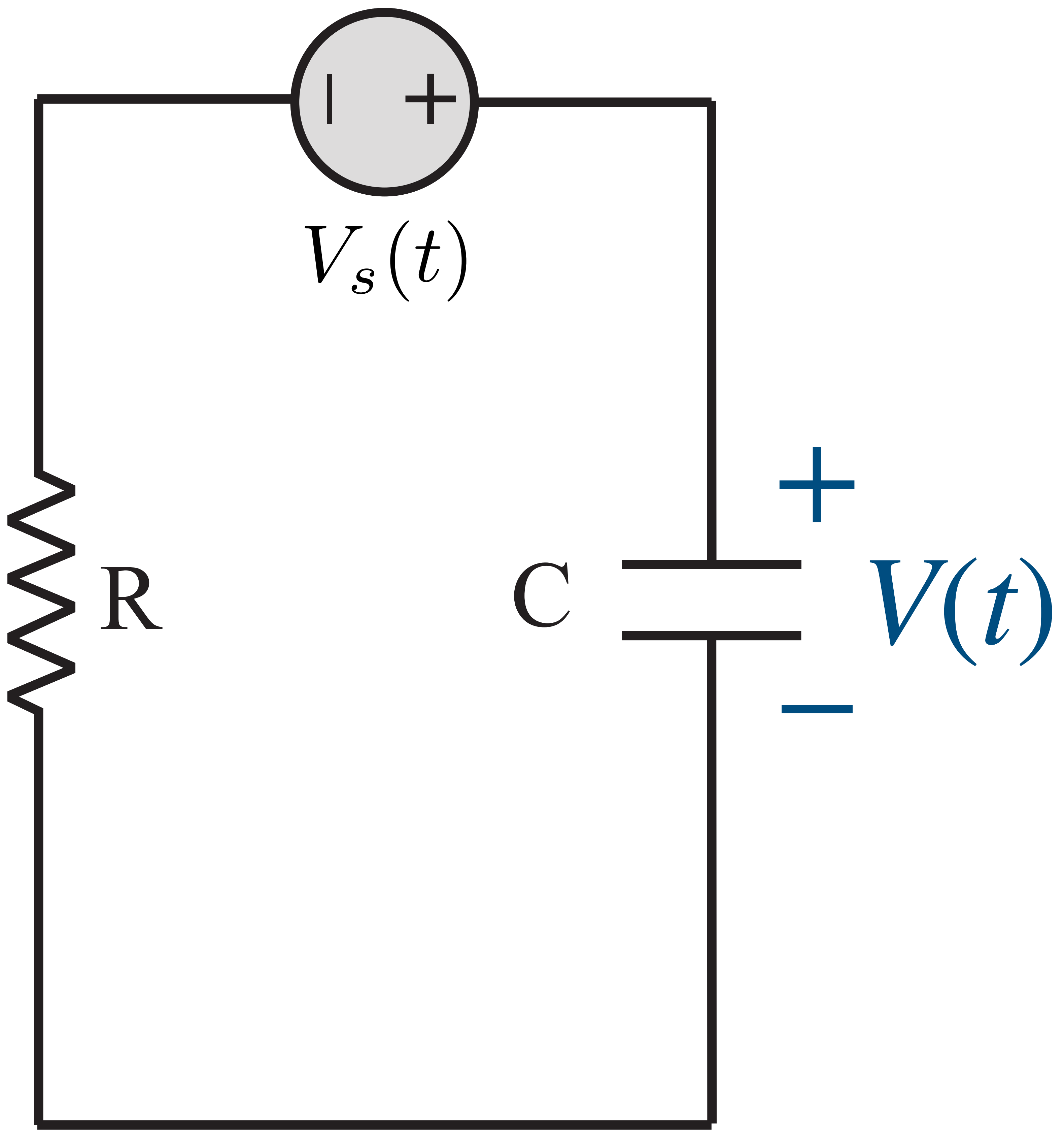
$$V + RC\,\dot{V} = V_s(t)$$

$$V_s(t) = \cos(\omega t)V$$



$$V + RC\,\dot{V} = V_s(t)$$

$$V(t) = V_h(t) + V_p(t)$$

$$V_{ss}(t) = \lim_{t\to\infty}\left(\cancelto{0}{V_h(t)} + V_p(t)\right) = V_p(t)$$

$$V_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\left[A + B\omega\tau - 1\right]\cos(\omega t) + \left[B - A\omega\tau\right]\sin(\omega t) = 0$$

$$A + \gamma B - 1 = 0$$

$$B - \gamma A = 0$$

$$\rightarrow \quad A = \frac{1}{1+\gamma^2} \qquad B = \frac{\gamma}{1+\gamma^2}$$

$$V_{ss}(t) = A\cos(\omega t) + B\sin(\omega t) = M\cos(\omega t + \phi)$$

$$A = \frac{1}{1+\gamma^2}$$

$$M = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{1+\gamma^2}}$$

$$\phi = -\operatorname{atan}(\gamma)$$

$$B = \frac{\gamma}{1+\gamma^2}$$

$$V_{ss}(t) = \frac{1}{\sqrt{1+\gamma^2}}\cos(\omega t - \operatorname{atan}(\gamma))$$

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$$V_p(t) = Me^{j\omega t+\phi}$$

$$Me^{j\omega t+\phi} + j\,\tau\omega Me^{j\omega t+\phi} = e^{j\omega t}$$

$$\left[Me^{j\phi} + j\ \gamma Me^{j\phi}\right] e^{j\omega t} = e^{j\omega t}$$

$$M\left[1 + j\ \gamma\right] e^{j\phi} = 1$$

$$Me^{j\phi} = \frac{1}{1 + j\ \gamma}$$

$$M = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{1 + \gamma^2}}$$

$$\phi = -\operatorname{atan}(\gamma)$$

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Let's assume that $V_s(t) = e^{j\omega t}$

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$$Me^{j\omega t + \phi} + j\tau\omega Me^{j\omega t + \phi} = e^{j\omega t}$$

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$$M\left[1 + j\,\gamma\right] e^{j\phi} = 1$$

$$Me^{j\phi} = \frac{1}{1 + j\,\gamma}$$

$$M = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{1 + \gamma^2}}$$

$$\phi = -\operatorname{atan}(\gamma)$$

Same magnitude scale and same phase diff.

Phasor Trans.

$$V + \tau \dot{V} = V_s(t) \longrightarrow M\left[1 + j\omega\tau\right] e^{j\phi} = 1$$

Phasor Notation & Transformation

$$\cos(\omega t) \implies e^{j\omega t} \implies 1$$

$$M\cos(\omega t + \phi) \implies Me^{j\phi}e^{j\omega t} \implies Me^{j\phi} \Leftrightarrow M\angle\phi \Leftrightarrow x + yj$$

$$x = M\cos(\phi) \qquad y = M\cos(\phi)$$

$$M = \sqrt{x^2 + y^2} \qquad \phi = \mathrm{atan2}(y, x)$$

Derivative & Integration in Phasor Notation

$$V(t) = M\cos(\omega t + \phi) \implies Me^{j\phi}e^{j\omega t} \implies V(j\omega) = Me^{j\phi}$$

$$V_d(t) = \frac{d}{dt}V(t) \implies V_d(j\omega) = ??? \qquad V_i(t) = \int V(t)dt \implies V_i(j\omega) = ???$$

$$\frac{d}{dt}\left(Me^{j\phi}e^{j\omega t}\right) = j\omega\left(Me^{j\phi}e^{j\omega t}\right) \implies V_d(j\omega) = j\omega\ V(j\omega)$$

$$\int\left(Me^{j\phi}e^{j\omega t}\right)dt = \frac{1}{j\omega}\left(Me^{j\phi}e^{j\omega t}\right) \implies V_i(j\omega) = \frac{1}{j\omega}\ V(j\omega)$$

Phasor Notation & Transformation

$$\cos(\omega t) \implies 1$$

$$M\cos(\omega t + \phi) \implies Me^{j\phi} \Leftrightarrow M\angle\phi \Leftrightarrow x + yj$$

$$\frac{d}{dt}V(t) \implies V_d(j\omega) = j\omega\ V(j\omega)$$

$$\int V(t)dt \implies V_i(j\omega) = \frac{1}{j\omega}\ V(j\omega)$$

Complex Algebra

$$z_1 = x_1 + y_1 j = M_1 \angle \phi_1 \ , \ z_2 = x_2 + y_2 j = M_2 \angle \phi_2$$

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- Summation (and subtraction is similar)

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + (y_1 + y_2) j \\ &= (M_1 \cos(\phi_1) + M_2 \cos(\phi_2)) + (M_1 \sin(\phi_1) + M_2 \sin(\phi_2)) i \end{aligned}$$

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- Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) j \\ &= M_1 M_2 e^{j(\phi_1 + \phi_2)} \end{aligned}$$

Complex Algebra

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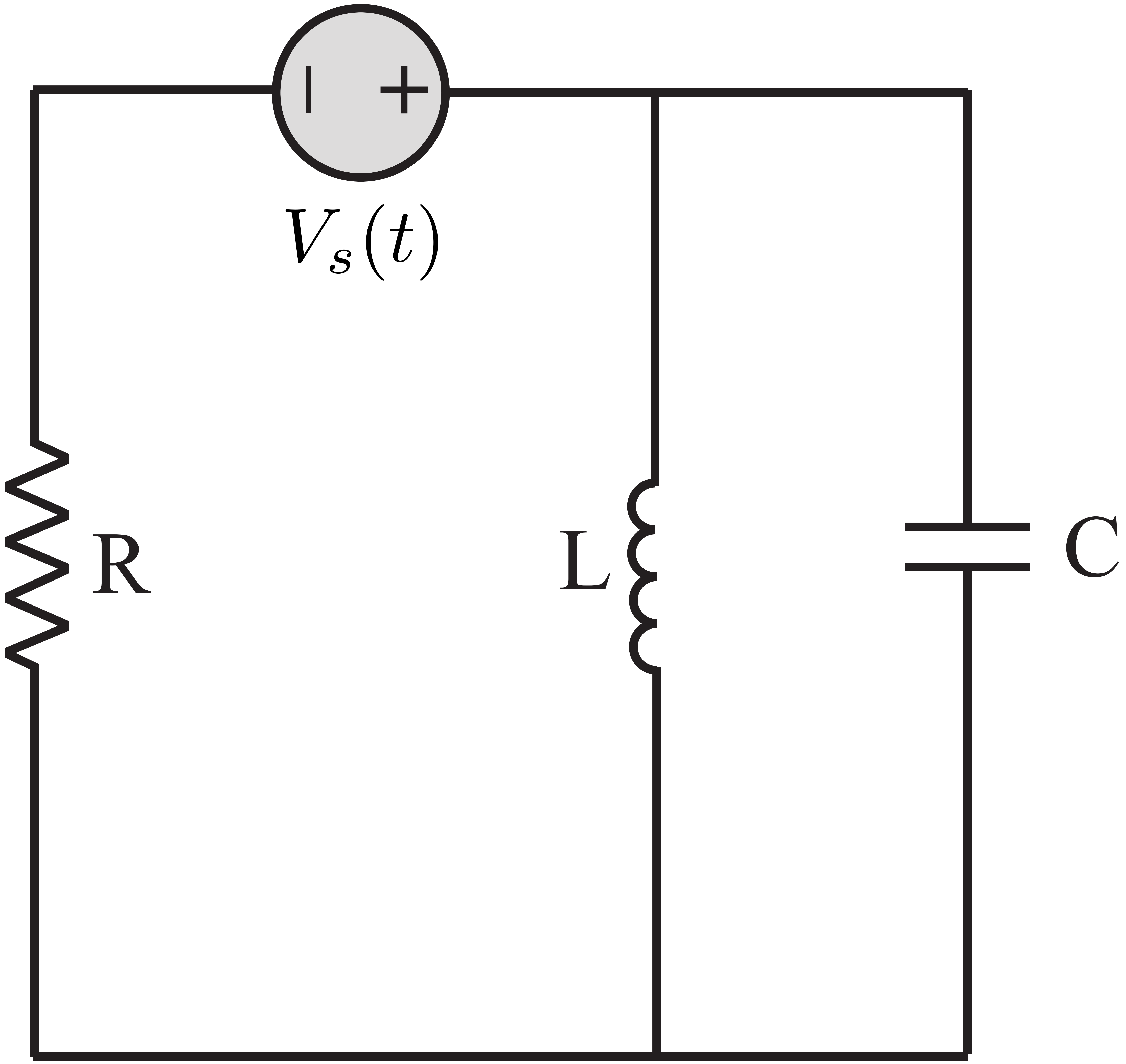
- Division

$$\begin{aligned} z_1 / z_2 &= (x_1 + y_1 j) \frac{x_2 - y_2 j}{x_2^2 + y_2^2} = ... \\ &= \frac{M_1}{M_2} e^{j(\phi_1 - \phi_2)} = \frac{M_1}{M_2} \angle (\phi_1 - \phi_2) \end{aligned}$$

$L = 0.4H$, $C = 0.1F$, $R = 1\Omega$

$$\ddot{I} + \frac{1}{RC}\dot{I} + \frac{1}{LC}I = \frac{V_s}{RLC}$$

$V_s = \cos(\omega t)V$



$L = 0.4H$, $C = 0.1F$, $R = 1\Omega$

$V_s = \cos(\omega t)V$

$$\ddot{I} + 10\,\dot{I} + 25\,I = 25V_s(t)$$

$$(-\omega^2 + 10\omega\, j + 25)I(j\omega) = 25$$

$$I(j\omega) = \frac{25}{(25 - \omega^2) + 10\omega\, j}$$

$$|I(j\omega)| = \frac{25}{(25-\omega^2)^2 + 100\omega^2}$$

$$I(j\omega) = \frac{25}{(25-\omega^2) + 10\omega\, j}$$

$$|I(j\omega)| = \frac{25}{\omega^2 + 25}$$

$$\angle(I(j\omega) = -\angle\big((25-\omega^2) + 10\omega\, j\big)$$

$$\angle(I(j\omega) = -\operatorname{atan2}\left(10\omega\ , (25-\omega^2)\right)$$

Let $\omega = \{0, 5, \infty\}$ rad/s

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$$|I(j0)| = 1\,,\ |I(j5)| = \frac{1}{2}\,,\ \&\ |I(j\infty)| = 0$$

$$\angle(I(j0) = 0\,,\ \angle(I(j5) = -\frac{\pi}{2}\,,\ \&\ \angle(I(j\infty) = -\pi$$