NAME: KEY

Math 213

1st Midterm Exam, March 26th, 2022, 11:00-12:00

IMPORTANT

1. Write down your name and surname on top of each page of your work. **2.** Show all your work. Correct answers without justification will not get credit. **3.** Unless otherwise stated, you may use any method from classwork to solve the problems. **4.** Calculators are not allowed. **5.** All electronic devices are to be kept shut and out of sight. **6.** All cellphones are to be left on the instructor's desk before the exam begins.

Q1	Q2	Q3	Q4	TOT
25 pts	25 pts	25 pts	25 pts	100 pts

Q1.

15pts a) Find the general solution of the differential equation

$y' + 2xy = 2xe^{-x^2}$.

Linear eq

Integrating factor $IF = e^{\int 2x\,dx} = e^{x^2}$

$$\Rightarrow \underbrace{y' e^{x^2} + 2xye^{x^2}}_{\frac{d}{dx}(ye^{x^2})} = 2x$$

$$\Rightarrow \quad ye^{x^2} = \int 2x\,dx = x^2 + C$$

$$\boxed{y = x^2 e^{-x^2} + Ce^{-x^2}}$$

10pts b) Find a constant-coefficient homogeneous linear differential equation for which

$y(x) = 6 + 3xe^x - 5cos3x$

is a solution.

$$\boxed{D(D-1)^2(D^2+9)\,y = 0}$$

NAME:

Q2. Find the general solution of the differential equation

$(D^3 + 7D^2 + 19D + 13)y = 0$

where D is the differential operator $\frac{d}{dx}$.

5 pts

Auxiliary eq $m^3 + 7m^2 + 19m + 13 = 0$

$m = -1 \Rightarrow -1 + 7 - 19 + 13 = 0$

$\therefore$ $m+1$ is a factor

10 pts

$$\begin{array}{l|l} m^3 + 7m^2 + 19m + 13 & m+1 \\ \hline -\cancel{m^3} + m^2 & m^2 + 6m + 13 \\ \quad 6m^2 + 19m & \\ \quad -6m^2 + 6m & \\ \quad\quad 13m + 13 & \end{array}$$

$\therefore$ The auxiliary eq is

$(m+1)(m^2 + 6m + 13) = 0$

5 pts

roots $m = \frac{-6 \pm \sqrt{36 - 4 \cdot 13}}{2} = \frac{-6 \pm \sqrt{36-52}}{2}$

$= \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$

5 pts

$\therefore$ The general soln is

$$\boxed{y = C_1 e^{-x} + C_2 e^{-3x} \cos 2x + C_3 e^{-3x} \sin 2x}$$

NAME:

Q3. Find a particular solution of the differential equation
$y'' + 4y = sin2x$.

Annihilator method:

10 pts

$(D^2+4)y = \sin 2x \Rightarrow (D^2+4)^2 y = 0$

$y = \cancel{C_1 \sin 2x} + \cancel{C_2 \cos 2x} + C_3 x\sin 2x + C_4 x\cos 2x$

because these functions are in the soln space of $(D^2+4)y=0$

$\therefore \; y_p = A x\sin 2x + B x\cos 2x$

10 pts

$y_p' = A\sin 2x + 2Ax\cos 2x + B\cos 2x - 2Bx\sin 2x$

$y_p'' = 4A\cos 2x - 4Ax\sin 2x - 4B\sin 2x - 4Bx\cos 2x$

$y_p'' + 4y_p = 4A\cos 2x - \cancel{4Ax\sin 2x} - 4B\sin 2x - \cancel{4Bx\cos 2x}$
$+ \cancel{4Ax\sin 2x} + \cancel{4Bx\cos 2x}$

$= 4A\cos 2x - 4B\sin 2x = \sin 2x$

5 pts

Matching the coefficients,

$4A = 0 \Rightarrow A = 0$
$-4B = 1 \quad\; B = -1/4$

$$\boxed{y_p = -\frac{1}{4} x\cos 2x}$$

NAME:

Q4. Given the initial value problem

$$y' - 2xy = 0\,,\quad y(0) = 1$$

(a) Find the recursion relation for a power series expansion about $x = 0$ for the solution of this problem.
(b) Solve the recursion relation and find the power series solution to the problem.

(a) 15 pts

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots \Rightarrow y(0) = \boxed{a_0 = 1}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - 2xy = \sum_{1}^{\infty} n a_n x^{n-1} - 2\sum_{0}^{\infty} a_n x^{n+1} = 0$$

$n-1 \rightarrow n$ $\quad$ $n+1 \rightarrow n$ $\quad$ Shift the index

$n \rightarrow n+1$ $\quad$ $n \rightarrow n-1$

$$\sum_{0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{1}^{\infty} a_{n-1} x^n = 0$$

$$a_1 + \sum_{1}^{\infty}\left[(n+1) a_{n+1} - 2a_{n-1}\right] x^n = 0$$

$\therefore$ Matching the coefficients, $\boxed{a_1 = 0}$ and

$(n+1) a_{n+1} - 2a_{n-1} = 0$ $\quad \Rightarrow \boxed{a_3 = a_5 = \cdots = 0}$

$\therefore$ The recursion relation is

$$a_{n+1} = \frac{2}{n+1} a_{n-1} \qquad n = 1, 2, \ldots$$

(b) 10 pts

$n = 1 \quad a_2 = \frac{2}{2} a_0 = 1$

$n = 3 \quad a_4 = \frac{2}{4} a_2 = \frac{1}{2!}$

$n = 5 \quad a_6 = \frac{2}{6} a_4 = \frac{2}{6 \cdot 2} = \frac{1}{3!}$

$n = 7 \quad a_8 = \frac{2}{8} a_6 = \frac{2}{8 \cdot 3!} = \frac{1}{4!}$

$\vdots$

$$a_{2m} = \frac{1}{m!}$$

$$\Rightarrow \quad y = 1 + \sum_{1}^{\infty} \frac{x^{2m}}{m!}.$$

or $$\boxed{y = \sum_{0}^{\infty} \frac{x^{2m}}{m!}.}$$