Machine Learning for Data Mining

Linear Algebra Review

Andres Mendez-Vazquez

May 14, 2015

Outline

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What is a Vector?

A ordered tuple of numbers

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Expressing a magnitude and a direction

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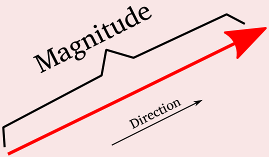

Outline

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Vector Spaces

Definition

A vector is an element of a vector space

Vector Space V

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1. If $\boldsymbol{x}, \boldsymbol{y} \in V$ then $\boldsymbol{x} + \boldsymbol{y} \in V$.
2. If $\boldsymbol{x} \in V$ then $\alpha \boldsymbol{x} \in V$ for any scalar α.
3. There exists $0 \in V$ then $\boldsymbol{x} + 0 = \boldsymbol{x}$ for any $\boldsymbol{x} \in V$.

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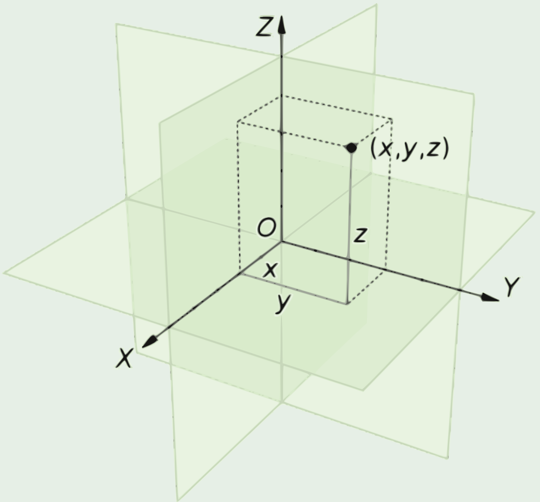
Classic Example

Euclidean Space $\mathbb{R}^3$



Span

Definition

The span of any set of vectors $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ is defined as:

$$\text{span}\left(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\right) = \alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + ... + \alpha_n\boldsymbol{x}_n$$

What Examples can you Imagine?

Give it a shot!!!

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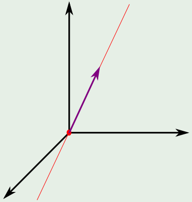
Subspaces of $\mathbb{R}^n$

A line through the origin in $\mathbb{R}^n$



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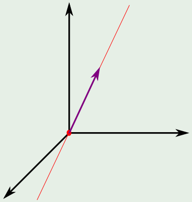

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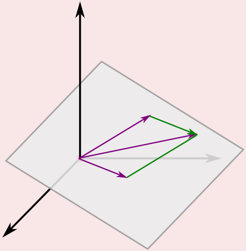

Outline

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Linear Independence and Basis of Vector Spaces

Fact 1

A vector $\boldsymbol{x}$ is a linearly independent of a set of vectors $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ if it does not lie in their span.

Fact 2

A set of vectors is linearly independent if every vector is linearly independent of the rest.

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The Rest

1. A basis of a vector space V is a linearly independent set of vectors whose span is equal to V
2. If the basis has d vectors then the vector space V has dimensionality d

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Norm of a Vector

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Properties

1. Homogeneity: $\|\alpha \boldsymbol{x}\| = \alpha \|x\|$.
2. Triangle inequality: $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.
3. Point Separation $\|x\| = 0$ if and only if $x = 0$.

4. Euclidean or ℓ_2-norm : $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

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1. Manhattan or ℓ_1-norm : $\|x\|_1 = \sum_{i=1}^{d} |x_i|$.
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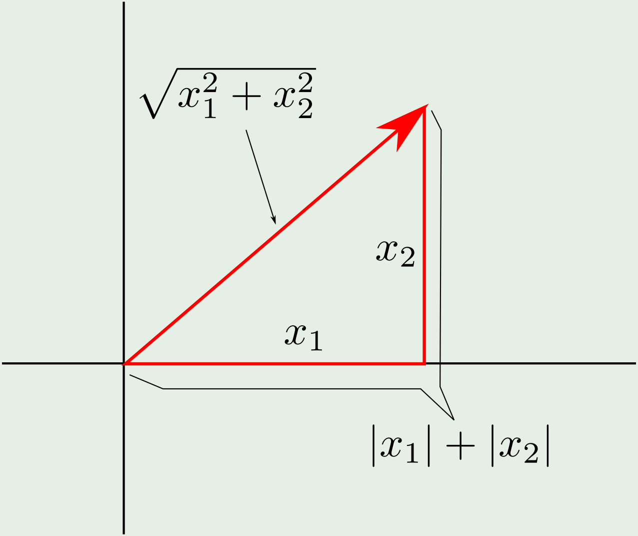
Examples

Example ℓ_1-norm and ℓ_2-norm



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Inner Product

Definition

The inner product between u and v

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i.$$

It is the projection of one vector onto the other one

Remark: It is related to the Euclidean norm: $\langle u, u \rangle = \|u\|_2^2$.

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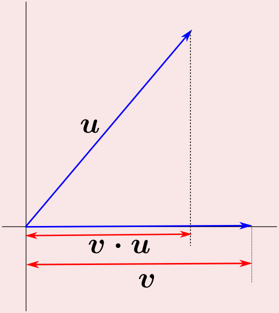

Properties

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if $u \cdot v > 0$, u and v are aligned

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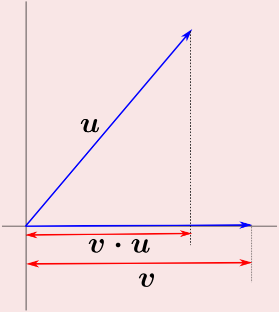

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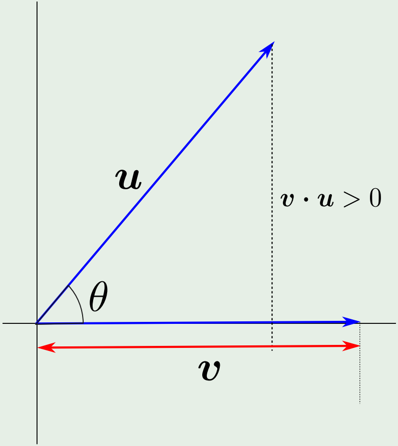

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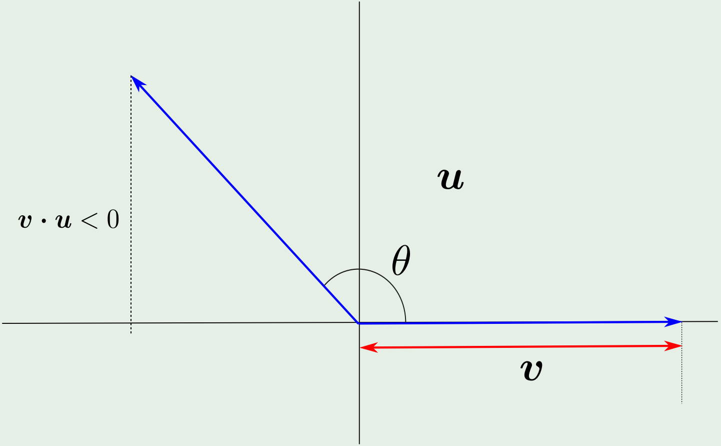

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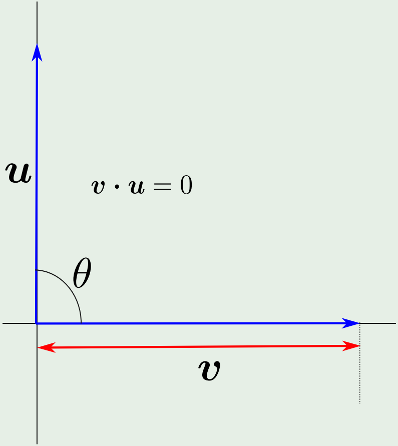

Definitions involving the norm

Orthonormal

The vectors in orthonormal basis have unit Euclidean norm and are orthonorgonal.

To express a vector x in an orthonormal basis

For example, given $x = \alpha_1 b_1 + \alpha_2 b_2$

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\langle \boldsymbol{x}, \boldsymbol{b}_1 \rangle &= \langle \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2, \boldsymbol{b}_1 \rangle \\
&= \alpha_1 \langle \boldsymbol{b}_1, \boldsymbol{b}_1 \rangle + \alpha_2 \langle \boldsymbol{b}_2, \boldsymbol{b}_1 \rangle \\
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Linear Operator

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A linear operator $\mathcal{L} : U \to V$ is a map from a vector space U to another vector space V satisfies:

- $\mathcal{L}(\boldsymbol{u}_1 + \boldsymbol{u}_2) = \mathcal{L}(\boldsymbol{u}_1) + \mathcal{L}(\boldsymbol{u}_2)$

Something Notable

If the dimension n of U and m of V are finite, $\mathcal{L}$ can be represented by $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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Thus, product of

The product of two linear operator can be seen as the multiplication of two matrices

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$$= \begin{pmatrix} \sum_{i=1}^{n} a_{1i} b_{i1} & \sum_{i=1}^{n} a_{1i} b_{i2} & \cdots & \sum_{i=1}^{n} a_{1i} b_{ip} \\ \sum_{i=1}^{n} a_{2i} b_{i1} & \sum_{i=1}^{n} a_{2i} b_{i2} & \cdots & \sum_{i=1}^{n} a_{2i} b_{ip} \\ & & \cdots & \\ \sum_{i=1}^{n} a_{mi} b_{i1} & \sum_{i=1}^{n} a_{mi} b_{i2} & \cdots & \sum_{i=1}^{n} a_{mi} b_{ip} \end{pmatrix}$$

Note: if A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

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Transpose of a Matrix

The transpose of a matrix is obtained by flipping the rows and columns

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As always, we have the identity operator

The identity operator in matrix multiplication is defined as

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

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Column Space, Row Space and Rank

Let A be an $m \times n$ matrix

We have the following spaces...

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Important facts

Something Notable

The column and row space of any matrix have the same dimension.

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The dimension is the rank of the matrix.

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Range and Null Space

Range

- Set of vectors equal to $A\boldsymbol{u}$ for some $\boldsymbol{u} \in \mathbb{R}^n$.

$$\text{Range}\,(A) = \{\boldsymbol{x} | \boldsymbol{x} = A\boldsymbol{u} \text{ for some } \boldsymbol{u} \in \mathbb{R}^n\}$$

- It is a linear subspace of $\mathbb{R}^m$ and also called the column space of A.

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Null Space

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$$\text{Null Space}\,(A) = \{\boldsymbol{u} | A\boldsymbol{u} = \boldsymbol{0}\}$$

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We have another interpretation of the matrix-vector product

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\begin{aligned}
A\boldsymbol{u} = &\left(\boldsymbol{A}_1 \;\; \boldsymbol{A}_2 \;\; \cdots \;\; \boldsymbol{A}_n\right)\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\
= &u_1\boldsymbol{A}_1 + u_2\boldsymbol{A}_2 + \cdots + u_n\boldsymbol{A}_n
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&= u_1 \boldsymbol{A}_1 + u_2 \boldsymbol{A}_2 + \cdots + u_n \boldsymbol{A}_n
\end{aligned}
$$

Thus

- The result is a linear combination of the columns of A.
- Actually, the **range** is the **column space**.

Range and Column Space

We have another interpretation of the matrix-vector product

$$
\begin{aligned}
A\boldsymbol{u} = &\left(\boldsymbol{A}_1 \;\; \boldsymbol{A}_2 \;\; \cdots \;\; \boldsymbol{A}_n\right)\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\
= &u_1\boldsymbol{A}_1 + u_2\boldsymbol{A}_2 + \cdots + u_n\boldsymbol{A}_n
\end{aligned}
$$

Thus

- The result is a linear combination of the columns of A.
- Actually, the **range** is the **column space**.

Matrix Inverse

Something Notable

- For an $n \times n$ matrix A: rank + dim(null space) $= n$.
- if dim(null space)$= 0$ then A is full rank.
- In this case, the action of the matrix is invertible.
- The inversion is also linear and consequently can be represented by another matrix A^{-1}.
- A^{-1} is the only matrix such that $A^{-1}A = AA^{-1} = I$.

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Orthogonal Matrices

Definition

An orthogonal matrix U satisfies $U^T U = I$.

Properties

U has orthonormal columns.

In addition

Applying an orthogonal matrix to two vectors does not change their inner product:

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\langle U\boldsymbol{u}, U\boldsymbol{v}\rangle &= (U\boldsymbol{u})^T U\boldsymbol{v} \\
&= \boldsymbol{u}^T U^T U\boldsymbol{v} \\
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Example

A classic one

Matrices representing rotations are orthogonal.

Outline

Trace and Determinant

Definition (Trace)

The trace is the sum of the diagonal elements of a square matrix.

Definition (Determinant)

The determinant of a square matrix A, denoted by $|A|$, is defined as

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} is determinant of matrix A without the row i and column j.

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Special Case

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$|A| = ad - bc$$

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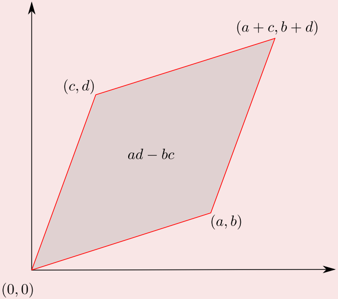

Properties of the Determinant

Basic Properties

- $|A| = \left|A^T\right|$
- $|AB| = |A|\,|B|$
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- If A is invertible, then $\left|A^{-1}\right| = \frac{1}{|A|}$.

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Eigenvalues and Eigenvectors

Eigenvalues

An eigenvalue λ of a square matrix A satisfies:

$$A\boldsymbol{u} = \lambda \boldsymbol{u}$$

for some vector , which we call an eigenvector.

Properties

Geometrically the operator A expands when $(\lambda > 1)$ or contracts $(\lambda < 1)$ eigenvectors, but does not rotate them.

Null Space relation

If u is an eigenvector of A, it is in the null space of $A - \lambda I$, which is consequently not invertible.

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Given the previous relation

The eigenvalues of A are the roots of the equation $|A - \lambda I| = 0$

Remark: We do not calculate the eigenvalues this way

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Eigenvalues and eigenvectors can be complex valued, even if all the entries of A are real.

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Eigendecomposition of a Matrix

Given

Let A be an $n \times n$ square matrix with n linearly independent eigenvectors $\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_n$ and eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$

We define the matrices

$$\boldsymbol{P} = \left(\boldsymbol{p}_1 \ \boldsymbol{p}_2 \cdots \boldsymbol{p}_n\right)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

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We have that A satisfies

$$A\boldsymbol{P} = \boldsymbol{P}\Lambda$$

In addition

$\boldsymbol{P}$ is full rank.

Thus, inverting it yields the eigendecomposition

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Properties of the Eigendecomposition

We have that

- Not all matrices are diagonalizable/eigendecomposition. Example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $Trace\left(A\right) = Trace\left(\Lambda\right) = \sum_{i=1}^{n} \lambda_i$
- $|A| = |\Lambda| = \Pi_{i=1}^{n} \lambda_i$
- The rank of A is equal to the number of nonzero eigenvalues.
- If λ is anonzero eigenvalue of A, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvector.
- The eigendecompositon allows to compute matrix powers efficiently:

$$A^m = \left(P\Lambda P^{-1}\right)^m = P\Lambda P^{-1} P\Lambda P^{-1} P\Lambda P^{-1} \ldots P\Lambda P^{-1} = P\Lambda^m P^{-1}$$

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Eigendecomposition of a Symmetric Matrix

When A symmetric, we have

- If $A = A^T$ then A is symmetric.
- The eigenvalues of symmetric matrices are real.
- The eigenvectors of symmetric matrices are orthonormal.
- Consequently, the eigendecomposition becomes $A = U \Lambda U^T$ for Λ real and U orthogonal.
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We can see the action of a symmetric matrix on a vector $\boldsymbol{u}$ as...

We can decompose the action $A\boldsymbol{u} = U\Lambda U^T$u as

- Projection of $\boldsymbol{u}$ onto the column space of A (Multiplication by U^T).
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Final equation

$$A\boldsymbol{u} = \sum_{i=1}^{n} \lambda_i \langle \boldsymbol{U}_i, \boldsymbol{u} \rangle \boldsymbol{U}_i$$

It would be great to generalize this to all matrices!!!

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Outline

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Singular Value Decomposition

Every Matrix has a singular value decomposition

$$A = U\Sigma V^T$$

Where

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Properties of the Singular Value Decomposition

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$$A^T A = V \Sigma U^T U^T \Sigma V^T = V \Sigma^2 V^T$$

Second

The rank of a matrix is equal to the number of nonzero singular values.

Third

The largest singular value σ_1 is the solution to the optimization problem:

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It can be verified that the largest singular value satisfies the properties of a norm, it is called the spectral norm of the matrix.

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In statistics analyzing data with the singular value decomposition is called **Principal Component Analysis**.

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