EE 5111: Estimation Theory

Jan - May 2021

Mini Project 1

February 14, 2021

1 Problem

Consider the following OFDM-like system model:

$$\mathbf{y} = \mathbf{XFh} + \mathbf{n}, \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^{512}$ are the set of observations, $\mathbf{X}$ is a 512 dimensional diagonal matrix with known symbols, $\mathbf{h}$ is the L tap time domain channel vector, $\mathbf{F}$ is the $512 \times L$ matrix performing IDFT[1] and $\mathbf{n}$ is complex Gaussian noise with variance σ^2.
For the following set of experiments, generate a set of random bits and modulate them as QPSK symbols to generate[2] $\mathbf{X}$. $\mathbf{h}$ is a multipath Rayleigh fading channel vector with an exponentially decaying power-delay profile $\mathbf{p}$ where $p[k] = e^{-\lambda(k-1)}, k = 1, 2...L$. That is, each component of $\mathbf{h}$ will be $h[k] = \dfrac{1}{\|\mathbf{p}\|_2}(a[k] + ib[k])p[k]$, where $a[k], b[k] \sim \mathcal{N}(0, \frac{1}{2});\; k = 1, 2...L$. Here, λ is the decay factor (and choose $\lambda = 0.2$ for your simulations). Now, perform the following experiments on the described problem set up.

1. Estimate $\mathbf{h}$ using least squares method of estimation with $L = 32$.[3]
2. Now, suppose that $\mathbf{h}$ is sparse with just 6 non zero taps. Assuming that you know the non zero locations, estimate $\mathbf{h}$ using Least squares with the sparsity information.
3. Next, introduce guard band of 180 symbols on either side[4], i.e. now we have reduced number of observations. For this case:
   - a Repeat (1),(2) for the above set up.
   - b Apply regularization and redo least squares. Use various values of α for regularization with $\alpha\mathbf{I}$ and compare the estimation results.

[1] $\mathbf{F}(i, j) = e^{\frac{j2\pi(i-1)(j-1)}{512}}; i = 1, \cdots, 512, j = 1, \cdots, L$

[2] $\mathbf{X}_{i,i} \in \{1 + 1j, -1 + 1j, 1 - 1j, -1 - 1j\}$

[3] Note that you are dealing with complex data now and hence the least squares estimate for the model $\mathbf{y} = \mathbf{Xb}$ shall now be $\hat{\mathbf{b}} = (\mathbf{X}^H\mathbf{X})^{-1}\mathbf{X}^H\mathbf{y}$

[4] Suppress to zero the first and last 180 symbols in $\mathbf{X}$

4. Perform least squares estimation on $\mathbf{h}$ with the following linear constraints :

$$\begin{aligned} h[1] &= h[2] \\ h[3] &= h[4] \\ h[5] &= h[6] \end{aligned}$$

For each of the above experiments, you have to compare $\mathbb{E}[\hat{\mathbf{h}}]$ and $\mathbf{h}$, theoretical and simulated MSE of estimation, all averaged over 10000 random trials. (Generate different instances of $\mathbf{X}$ and $\mathbf{n}$ for each trial.) Repeat the experiments for $\sigma^2 = \{0.1, 0.01\}$ for each case. Plot $\hat{\mathbf{h}}$ and $\mathbf{h}$ for one trial in each of the above cases.

5. Next, for the scenarios in question 2 and 3, compare the results with the estimates you get from the following steps :

   – Step 1 :

   Algorithm 1: To find the non-zero locations of the sparse vector $\mathbf{h}$ (support estimate).

   Input: Observation $\mathbf{y}$, matrix $\mathbf{A} = \mathbf{XF}$, sparsity $k_o = 6$
   Initialize $\mathcal{S}^0_{omp} = \phi$, $k = 1$, $\mathbf{r}^0 = \mathbf{y}$
   for $k \leftarrow 1$ *to* k_0 **do**
       Identify the next column as $t_k = \underset{j}{\text{argmax}}|\mathbf{A}_j^H \mathbf{r}^{k-1}|$
       Expand the current support as $\mathcal{S}^k_{omp} = \mathcal{S}^{k-1}_{omp} \cup t_k$
       Update residual: $\mathbf{r}^k = [\mathbf{I}_{512} - \mathbf{P}_k]\,\mathbf{y}$ where $\mathbf{P}_k = \mathbf{A}_{\mathcal{S}^k_{omp}} \mathbf{A}^\dagger_{\mathcal{S}^k_{omp}}$.
       Increment $k \rightarrow k + 1$
   end
   Output: Support estimate $\hat{\mathcal{S}} = \mathcal{S}^k_{omp}$

   – Step 2 : Now that you know the non-zero locations of $\mathbf{h}$, estimate $\mathbf{h}$ using least squares.

   **In the algorithm* $\mathbf{A}_j$ *is the* j^{th} *column of matrix* $\mathbf{A}$, $\mathbf{A}_\mathcal{S}$ *denotes the sub-matrix of* $\mathbf{A}$ *formed using the columns indexed by* $\mathcal{S}$ *and* $\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$ *is the Moore-Penrose pseudo inverse of* $\mathbf{A}$. *Also,* $\mathbf{I}_N$ *is the* N *dimensional identity matrix.*

2 Submission guidelines

You need to submit the solutions for this problem no later than March 1, 2021 (11:59 pm). Upload a compressed file containing the program/programs your team has written for the mini project in Moodle. The viva for each team will be conducted jointly on a date and time convenient for all the members.