Physics through Computational Thinking
Lecture-3

Visual Thinking-3

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Outline

In this lecture you will

1. learn to plot various functions and identify their salient properties

2. learn to plot 2-D and 3-D plots such as vector plots, streamline plots and contour plots.

10-Jan-2019

Properties of Functions: Zeros, divergences, extrema and asymptotes

Zero

Zero: A function $f(x)$ has zeros at points x^* where $f(x^*) = 0$. Identifying these points should be the first step in sketching a function.

$$\begin{aligned} f(x) &= x^2 - 3\,x + 2 \\ &= (x-2)\,(x-1). \end{aligned} \tag{1}$$

Factorization (when possible) helps immediately identify the zeros.

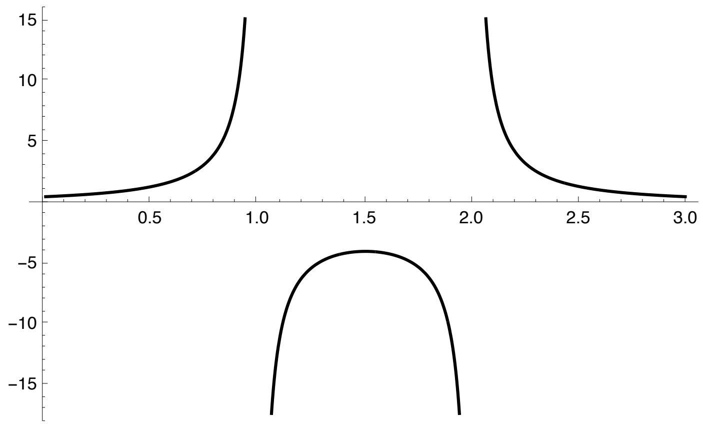
```
Plot[x^2 - 3 x + 2, {x, 0, 3}];
```

Divergence

Divergence: A function $f(x)$ has divergences (or singularities) at points x^* where $\frac{1}{f(x^*)}$has a zero. Identifying these points (and sometimes the form of the divergence nearby) helps us figure out where and how the function blows up.

$$f(x) = \frac{1}{(x-2)\,(x-1)}. \tag{2}$$

```
Plot[1/((x - 1) (x - 2)), {x, 0, 3}]
```



Extrema

Extrema: A function $f(x)$ has extrema at points x^* where $f'(x^*) = 0$. Further work would be essential to clarify if the point is a minimum or a maximum or an inflection point. Identifying these points help in getting the broad shape of the function. Let us take the example of the same function $f(x) = x^2 - 3x + 2$. Here it turns out that rather than factorization, it is more useful to `complete the squares'.

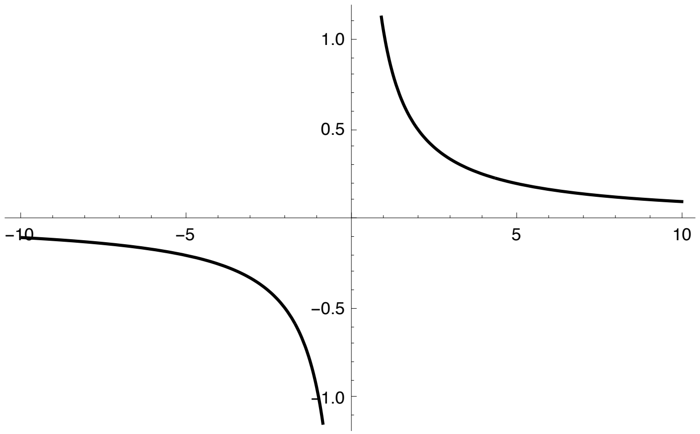
$$\begin{aligned} f(x) &= x^2 - 3x + 2 \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}. \end{aligned} \tag{3}$$

```
Plot[x^2 - 3 x + 2, {x, 0.5, 2.5}, PlotRange → {-0.3, 0.3}];
```

Asymptote

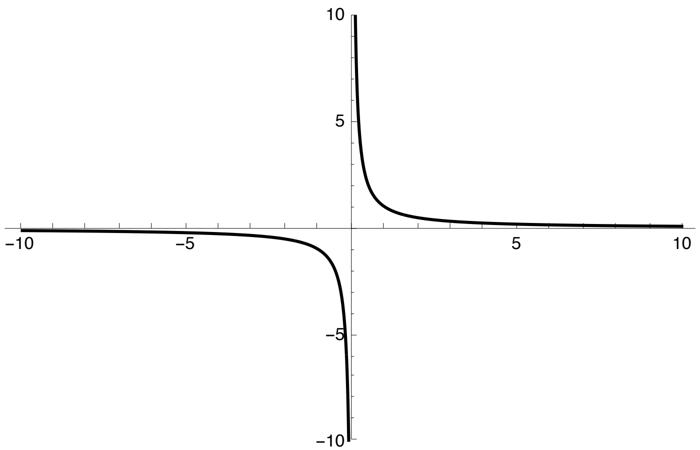
Asymptote: An asymptote is a curve that a function $f(x)$ approaches arbitrarily closely in some limit. A familiar example is the curve $f(x) = \frac{1}{x}$, which asymptotically approaches the X-axis as $x \to \infty$, and asymptotically approaches the Y-axis as $x \to 0$.

```
Plot[1/x, {x, -10, 10}]
```



Let's increase the *y*-range of the plot and examine it whether the curve $\frac{1}{x}$ approaches *y*-axis. We will do this by invoking an option for the **Plot** function called **PlotRange**

```
Plot[1/x, {x, -10, 10}, PlotRange → {-10, 10}]
```



Properties of Functions

Example-1

For the function $f(x)$ shown below find the extremum points if any and find the nature of the extremum:

$$f(x) = x\,|x| \tag{4}$$

```
Plot[{x Abs[x]}, {x, -4, 4}]
```

Example-2

For the function $f(x)$ shown below find the behaviour of the function in various regions and identify domains of continuity and differentiability.

$$f(x) = |x|^{1/3} \tag{5}$$

```
Plot[{Abs[x]^(1/3)}, {x, -1, 1}]
```

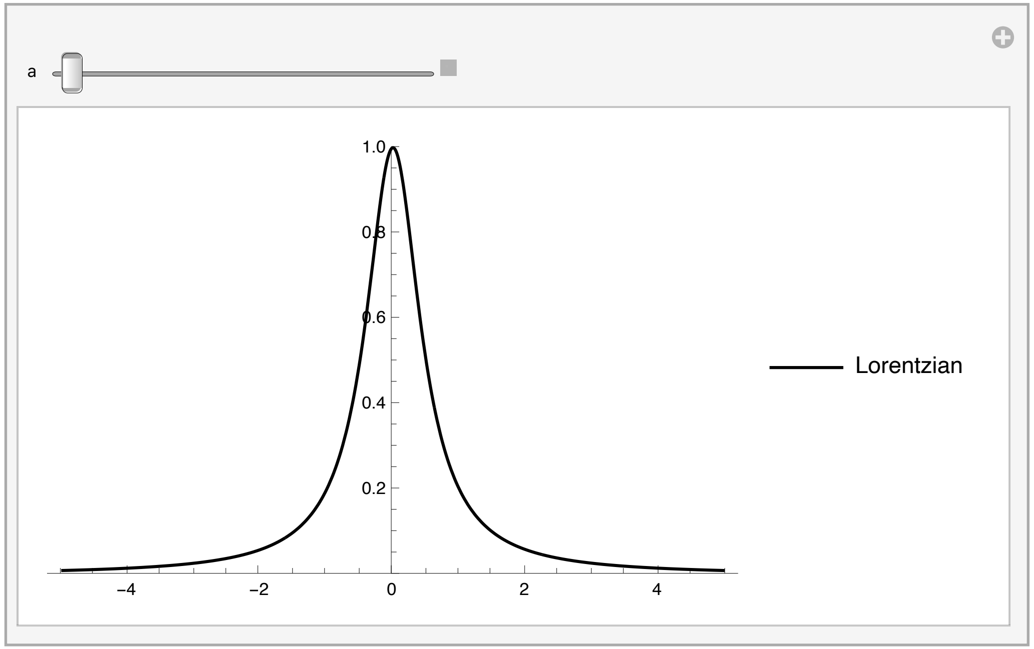
Example-3

For the famous function known as ***Lorentzian*** shown below, plot the function and identify what happens by changing the parameter a. What is the asymptotic behaviour of the function?

$$f_{\text{Lorentzian}}(x) = \frac{a^2}{x^2 + a^2} \tag{6}$$

In[•]:=
```
Manipulate[Plot[{a^2/(x^2 + a^2)}, {x, -5, 5}, PlotRange → {0, 1}, PlotLegends → {"Lorentzian", "a²/x²"}], {a, 0.5, 2, 0.1}]
```

Out[•]=



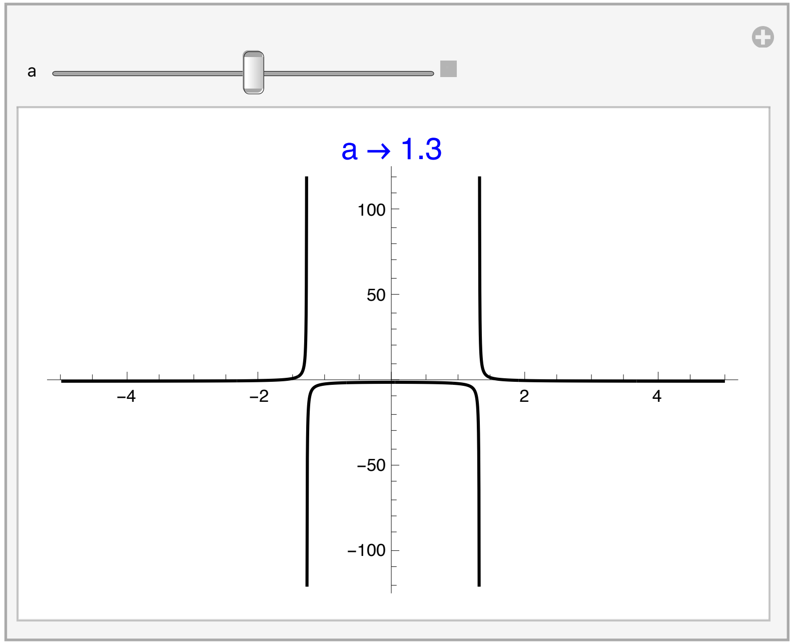
Example-4

Let’s change the sign of a^2 in the Lorentzian to get the not-so-famous function shown below. Can you plot this and identify the divergences?

$$f(x) = \frac{1}{x^2 - a^2} \tag{7}$$

Do this on paper and pen before we test it out on the computer. Discuss in groups of three.

```
Manipulate[Plot[1/(x^2 - a^2), {x, -5, 5}, PlotRange → Full, PlotLabel → Style["a" → a, 16, Blue]], {a, 0.5, 2, 0.1}]
```



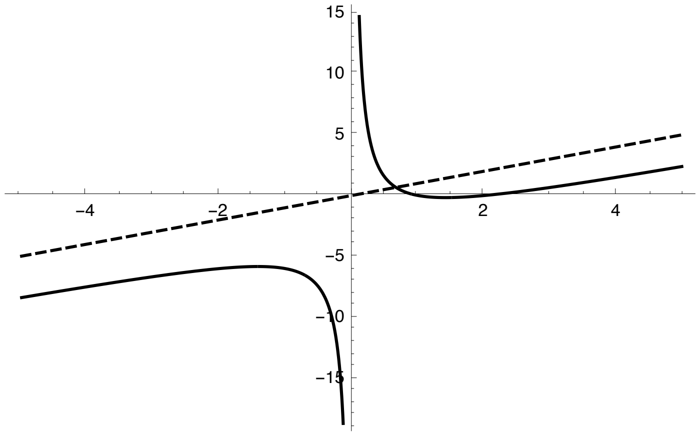
Example-5

For the function below, find the asymptotes.

$$f(x) = \frac{(x-1)(x-2)}{x} \tag{8}$$

Do this on paper and pen before we test it out on the computer. Discuss in groups of three.

```
Plot[{(x - 1) (x - 2) / x, x}, {x, -5, 5}]
```



Example-6

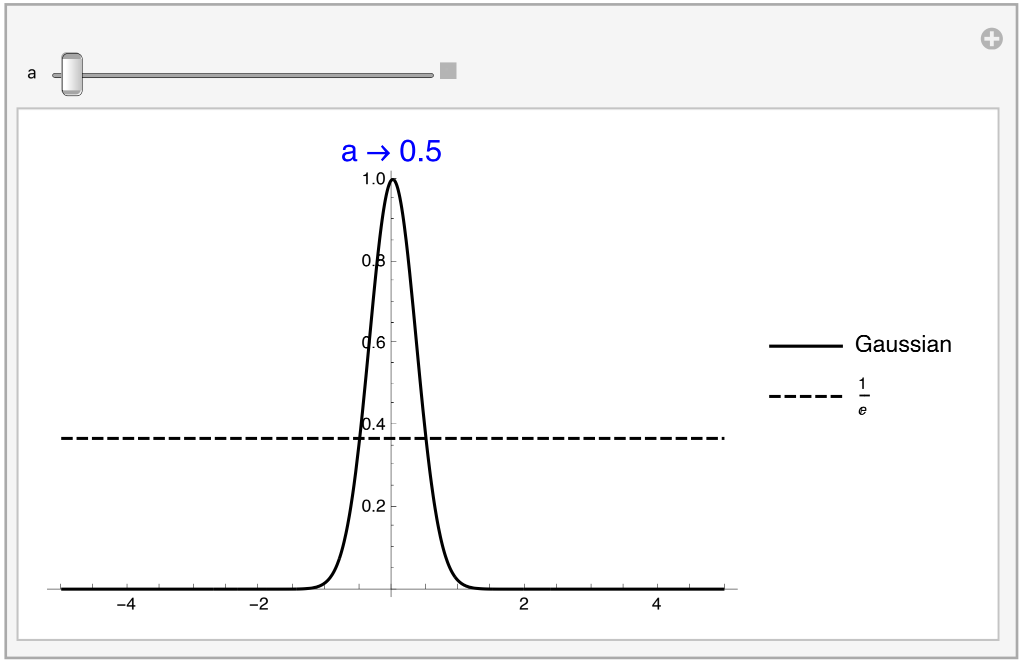
Here is another famous function: ***Gaussian***

$$f_{\text{Gaussian}}(x) = e^{-x^2/a^2} \tag{9}$$

Study the properties of this function and explore the role of parameter a.

In[∘]:=
```
Manipulate[Plot[{Exp[-x^2/a^2], Exp[-1]}, {x, -5, 5}, PlotRange → Full, PlotLabel → Style["a" → a, 16, Blue],
  PlotLegends → {"Gaussian", e^-1, "Lorentzian"}], {a, 0.5, 2, 0.1}]
```

Out[∘]=



Hyperbolic Trigonometric Functions

Definitions of Hyperbolic Trigonometric functions

function	definition	WL usage	Notes
$\sinh(x)$	$\frac{1}{2}(e^x - e^{-x})$	Sinh[x]	
$\cosh(x)$	$\frac{1}{2}(e^{-x} + e^x)$	Cosh[x]	
$\tanh(x)$	$\frac{e^x - e^{-x}}{e^{-x}+e^x}$	Tanh[x]	
$\operatorname{cosech}(x)$	$\frac{2}{e^x - e^{-x}}$	Csch[x]	$\frac{1}{\sinh(x)}$
$\operatorname{sech}(x)$	$\frac{2}{e^{-x}+e^x}$	Sech[x]	$\frac{1}{\cosh(x)}$
$\coth(x)$	$\frac{e^{-x}+e^x}{e^x - e^{-x}}$	Coth[x]	$\frac{1}{\tanh(x)}$

Problem

(a) For $\sinh(x)$, $\operatorname{sech}(x)$, $\tanh(x)$ and $\coth(x)$, make the plots in suitable ranges for the function. Identify the salient features, such as extrema, zeros, asymptotes, discontinuities, derivative discontinuities.

(b) Using the **Manipulate** command explore the effects of parameter a in functions $\sinh(a\,x)$, $\operatorname{sech}(a\,x)$, $\tanh(a\,x)$ and $\coth(a\,x)$

(c) Compare the functions $\frac{1}{x^2+1}$, e^{-x^2} and $\operatorname{sech}(x)$ on the same plot. What is the difference between them for large x?

Vector Fields

- A vector field in 3-dimensions is a function represented by a 3-tuple of functions

$$\vec{v}(x,\ y,\ z) = \left(v_x(x,\ y,\ z),\ \ v_y(x,\ y,\ z),\ \ v_z(x,\ y,\ x)\right) \tag{10}$$

- It can also be represented as (where $\vec{r}$ is the position coordinate)

$$\vec{v}(\vec{r}) = \left(v_x(\vec{r}),\ \ v_y(\vec{r}),\ \ v_z(\vec{r})\right) \tag{11}$$

- or in the unit vector notation:

$$\vec{v}(\vec{r}) = v_x(\vec{r})\,\hat{i} + \ v_y(\vec{r})\,\hat{j} + v_z(\vec{r})\,\hat{k} \tag{12}$$

- In two dimensions $\left(\vec{r} = x\,\hat{i} + y\,\hat{j}\right)$

$$\vec{v}(\vec{r}) = v_x(\vec{r})\,\hat{i} + \ v_y(\vec{r})\,\hat{j} \tag{13}$$

- It is straightforward to generalize it to n-dimensions but it will be difficult to think about plots in n-dimensions. So for now let's focus on 2 and 3-dimensions.

Example-1

(a) For the vector fields given below find their **divergence** and **curl**:

$$\begin{gathered}\vec{v} = x\,\hat{i} + y\,\hat{j}\\ \vec{u} = y\,\hat{i} + x\,\hat{j}\\ \vec{w} = -y\,\hat{i} + x\,\hat{j}\end{gathered} \tag{14}$$

(b) By creating a vector plot verify the results you found for divergence and curl of these functions.

(c) If these vector fields represented flow of a fluid, make a streamline plot to demonstrate it:

Sol (a):

$$\begin{gathered}\vec{\nabla}\cdot\vec{v} = 2\\ \vec{\nabla}\times\vec{v} = 0\\ \vec{\nabla}\cdot\vec{u} = 0\end{gathered}$$

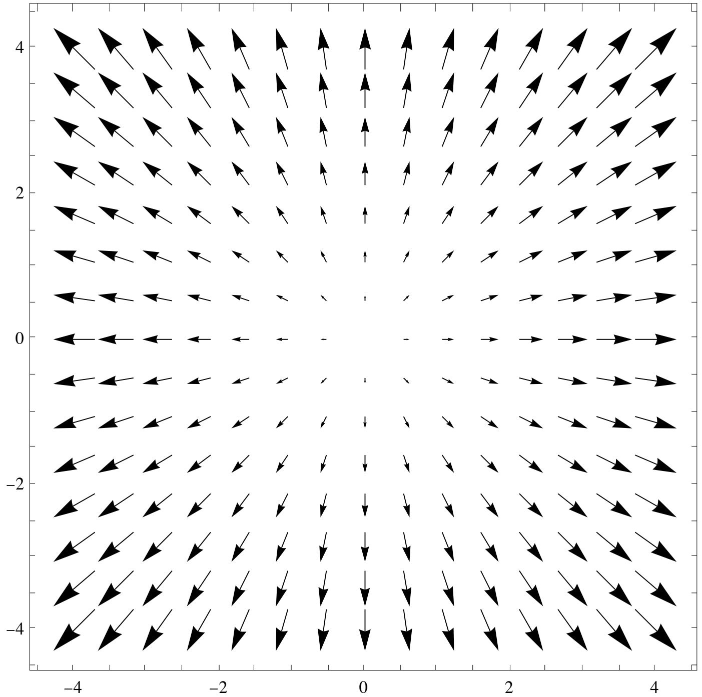
(15)

$$\vec{\nabla}\times\vec{u}=0$$
$$\vec{\nabla}\cdot\vec{w}=0$$
$$\vec{\nabla}\times\vec{w}=2\,\hat{k}$$

Sol (b): In Mathematica, we will do this by invoking **VectorPlot,** where a vector field is represented by a 2-tuple or 3-tuple as given below:

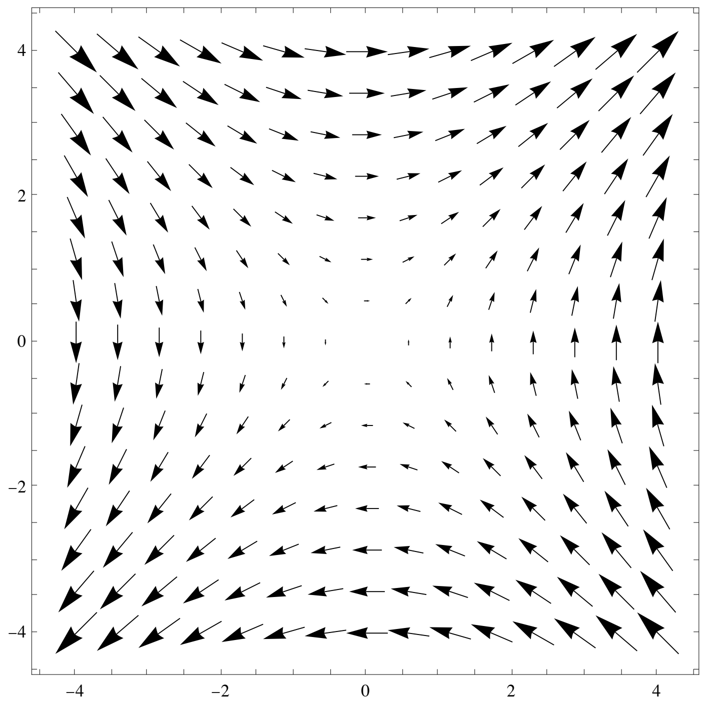
$$\begin{aligned}\vec{v}&=\{x,\,y\}\\ \vec{u}&=\{y,\,x\}\\ \vec{w}&=\{-y,\,x\}\end{aligned} \tag{16}$$

```
VectorPlot[{x, y}, {x, -4, 4}, {y, -4, 4}]
```



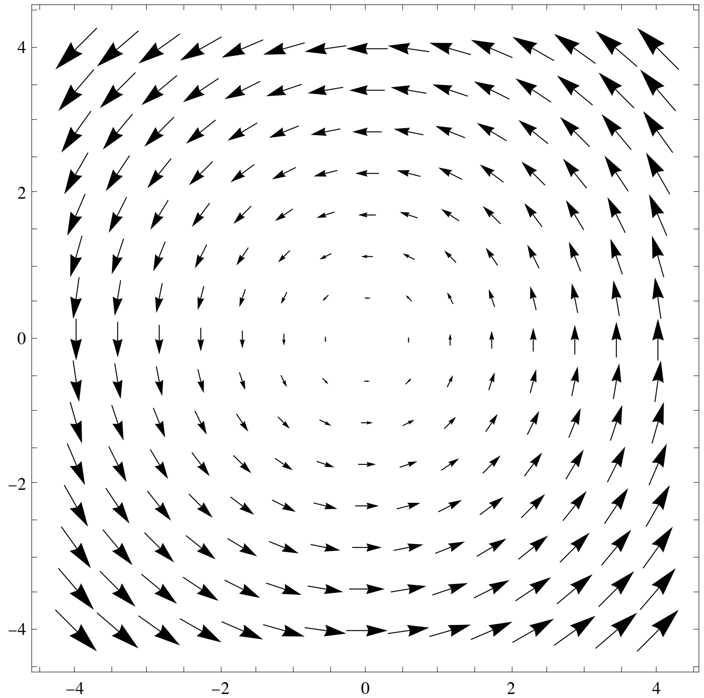
Divergent

```
VectorPlot[{y, x}, {x, -4, 4}, {y, -4, 4}]
```



Neither convergent nor divergent.

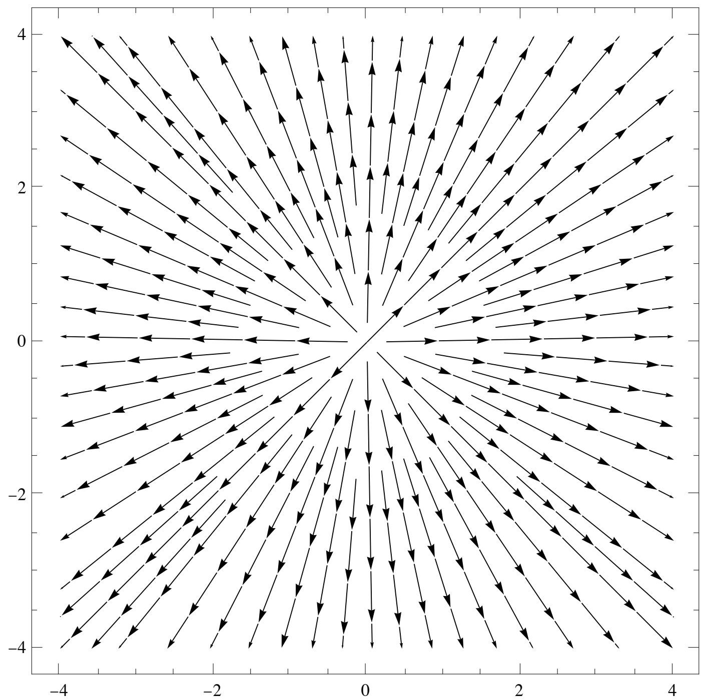
```
VectorPlot[{-y, x}, {x, -4, 4}, {y, -4, 4}]
```



Rotation field.
Non-zero curl
Zero divergence

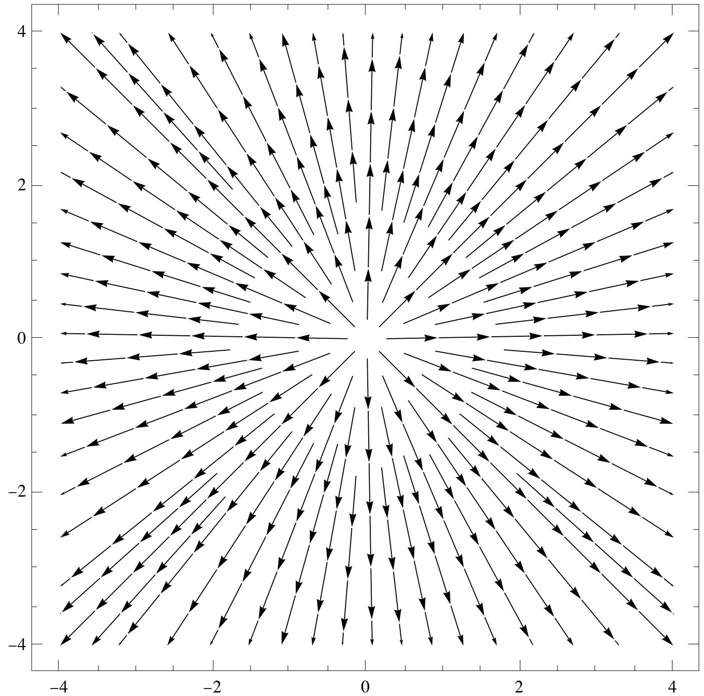
Sol (c): Streamline Plots are very similar to vector plots but they represent flow of a fluid or ***field lines***, as in electric field lines and magnetic field lines. It only shows the direction of the flow at each point. They can be made by **StreamPlot** function:

```
StreamPlot[{x, y}, {x, -4, 4}, {y, -4, 4}]
```



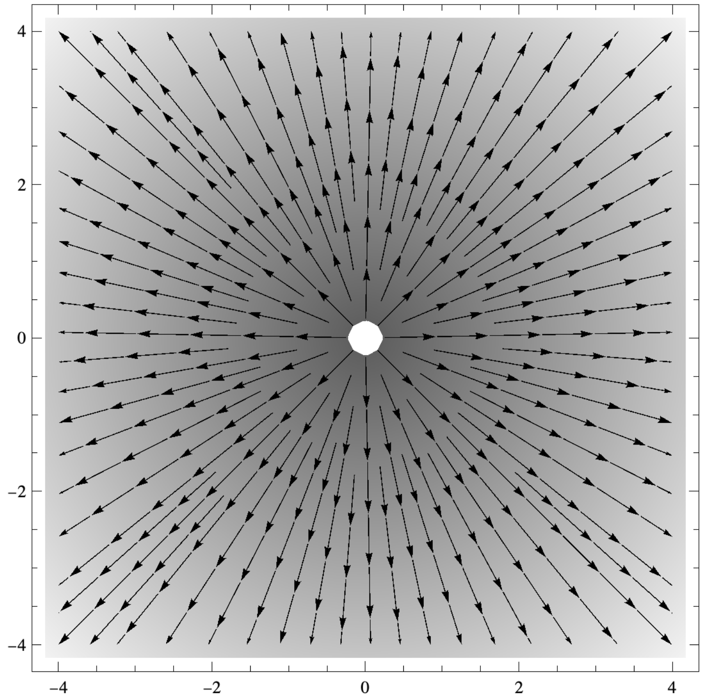
To avoid the funny business at origin where the flow direction is “confusing”, we can avoid the region by using the option known as **RegionFunction,** which puts a suitable constraint on the coordinates x and y, vector field components v_x, v_y and norm of the vector field $n = \left\| \vec{v} \right\|$. In this case we subject the plot to constraint that $x^2 + y^2 > 0.05$:

```
StreamPlot[{x, y}, {x, -4, 4}, {y, -4, 4}, RegionFunction → Function[{x, y, vx, vy, n}, x^2 + y^2 > 0.05]]
```



Using **StreamDensityPlot**, you can also get the idea about the norm of the vector field as well. Lighter the background implies bigger is the vector field. StreamDensityPlot is an alternate to **VectorPlot**

```
StreamDensityPlot[{x, y}, {x, -4, 4}, {y, -4, 4}, RegionFunction → Function[{x, y, vx, vy, n}, x^2 + y^2 > 0.05]]
```

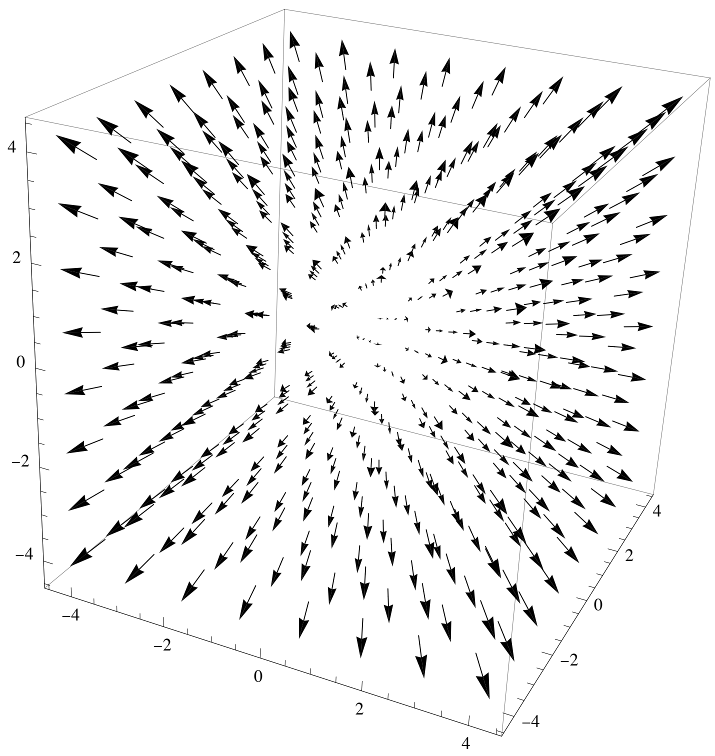


Example-2

Lets take a look at a 3-dimensional example:

$$\begin{gathered}\vec{v} = x\,\hat{i} + y\,\hat{j} + z\,\hat{j} \\ \vec{w} = -y\,\hat{i} + x\,\hat{j} + z\,\hat{k}\end{gathered} \tag{17}$$

```
VectorPlot3D[{x, y, z}, {x, -4, 4}, {y, -4, 4}, {z, -4, 4}, VectorScale → 0.06]
```



Field Lines and Equipotential Surfaces

- We will now explore some advanced plotting methods through a few basic electrodynamics problems.
- Electric field for a point charge q located at a point P whose position is represented by $\vec{r}_q$ is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_q}{\left|\vec{r}-\vec{r}_q\right|^3} \tag{18}$$

- The potential due to this point charge at any point is given by

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\left|\vec{r}-\vec{r}_q\right|} \tag{19}$$

Example: Dipole

Consider an electric dipole made by two charges $+q$ and $-q$ placed at origin and $(a, 0)$. The electric field for this set-up is given by

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} - \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-a\,\hat{i}}{\left|\vec{r}-a\,\hat{i}\right|^3} \tag{20}$$

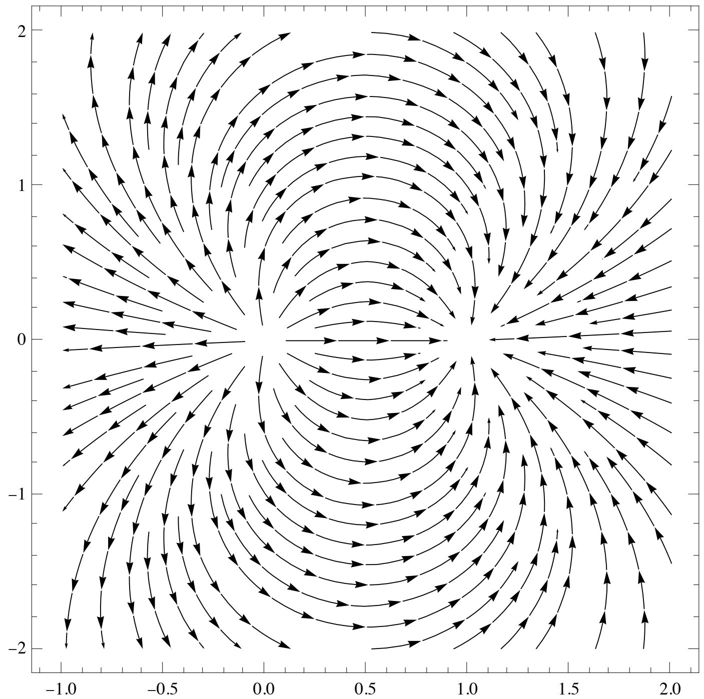
Non-dimensionalizing the electric field, we get

$$\begin{gathered}\frac{\vec{E}}{\frac{q}{4\pi\epsilon_0 a^2}} = \frac{(\vec{r}/a)}{(r/a)^3} - \frac{\frac{\vec{r}}{a}-\hat{i}}{\left|\frac{\vec{r}}{a}-\hat{i}\right|^3} \\ \vec{\mathcal{E}} = \left\{\frac{X}{\sqrt{X^2+Y^2}} - \frac{X-1}{\sqrt{(X-1)^2+Y^2}}, \frac{Y}{\sqrt{X^2+Y^2}} - \frac{Y}{\sqrt{(X-1)^2+Y^2}}\right\}\end{gathered} \tag{21}$$

where in the second line we have written the result in terms of dimensionless $X = x/a$ and $Y = y/a$.

In[∘]:=
```
fieldLines = StreamPlot[{x/r^3 - (x - 1)/r1^3, y/r^3 - y/r1^3} /. r → Sqrt[x^2 + y^2] /. r1 → Sqrt[(x - 1)^2 + y^2],
  {x, -1, 2}, {y, -2, 2},
  RegionFunction → Function[{x, y, vx, vy, n}, x^2 + y^2 > 0.01 && (x - 1)^2 + y^2 > 0.01]]
```

Out[∘]=



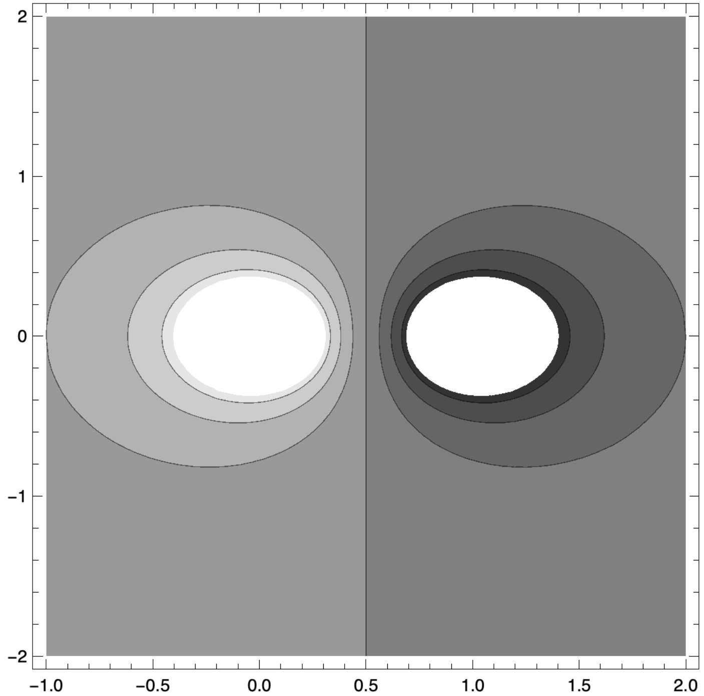
For potential also we will non-dimensionalize and use the ContourPlot to create contour plots

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r} - \frac{q}{4\pi\epsilon_0}\frac{1}{\left|\vec{r} - a\,\hat{i}\right|}$$

$$\frac{V}{\frac{q}{4\pi\epsilon_0 a}} = \frac{1}{(r/a)} - \frac{1}{\left|\frac{\vec{r}}{a} - \hat{i}\right|} = \frac{1}{\sqrt{X^2 + Y^2}} - \frac{1}{\sqrt{(X-1)^2 + Y^2}},$$

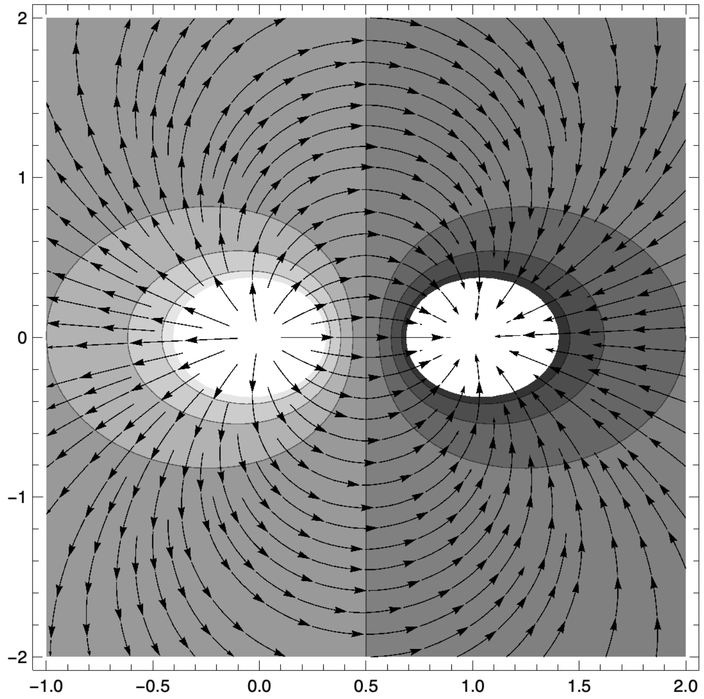
In[∘]:= equiPotential = ContourPlot$\left[\left\{\frac{1}{r} - \frac{1}{r1}\right\}$ /. r → $\sqrt{x^2 + y^2}$ /. r1 → $\sqrt{(x-1)^2 + y^2}$, {x, -1, 2}, {y, -2, 2}$\right]$

Out[∘]=



Shows regions with equal potentials.

In[•]:=
```
Show[equiPotential, fieldLines]
```

Out[•]=


Note that equi-potential contours are perpendicular to the the electric field lines.

Homework

1. Sketch the following functions, first on a piece of paper analyzing them for their zeros, divergences, extrema and asymptotes. Next cross-check your sketch by plotting the function on *Mathematica*.

Hyperbolic functions:

1. $\cosh(x)$ 2. $\sinh(x)$ 3. $\tanh(x)$ 4. $\text{cosech}(x)$ 5. $\text{sech}(x)$ 6. $\coth(x)$.

7. $\ln x$ 8. $\ln(\ln(x))$ 9. $\ln(x)/x$ 10. $\ln(e^x - 1)$ 11. $\ln\left(\frac{1-x}{1+x}\right)$ 12. $\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)$

13. $e^{-x}\cos(x)$ 14. $e^{-x}\sin(x)$ 15. $e^{-|x|}\cos(x)$ 16. $e^{-|x|}\sin(x)$ 17. $x\,e^{-x^2}$ 18. $x - 1 + e^{-x}$

19. x^x 20. $x^{1/x}$ 21. $x^{|x|}$ 22. $|x|^{|x|}$ 23. $\frac{|x|^{1/2}}{1+|x|^{1/2}}$ 24. $\frac{|x|^{\frac{1}{2}}}{e^x+1}$

25. $e^{\frac{1}{x}}$ 26. $e^{\frac{-1}{x^2}}$ 27. $x^{-12} - x^{-6}$ 28. $\cosh^{-1}(x)$ 29. $\coth^{-1}(x)$ 30. $\coth(x) - \frac{1}{x}$

2. Electric field lines of a quadrupole: Plot the electric field lines and equipotential surfaces for the quadrupolar configuration: four charges of same magnitude and alternating sign on the corners of a square of side a, that is, $+q$ at $(0, 0)$ and (a, a) while $-q$ at $(a, 0)$ and $(0, a)$. Use combination of StreamPlot and ContourPlot as shown in this lecture inside a Show function.