Physics through Computational Thinking

Falling Bodies

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Outline

In this module we will look at
falling bodies and differential equations that arise from there.

Free Fall

```
Clear["Global`*"]
```

Let us begin with the very simple problem of free fall of a body starting at rest. The differential equation involved is

$$\frac{d^2 y}{dt^2} = g. \tag{1}$$

Integrating, we have

$$v = \frac{dy}{dt} = g\,t + c_1 \tag{2}$$

Since the body started at rest, clearly the speed at time $t = 0$ is zero. Therefore, $c_1 = 0$. We have

$$v = \frac{dy}{dt} = g\,t \tag{3}$$

If we set the starting position at $y = 0$, another round of integration, gives us the familiar result

$$y(t) = \frac{1}{2} g\,t^2. \tag{4}$$

Fall with air resistance

Next, let us consider a situation where air exerts a resisting force that is proportional to the velocity of the falling body. The differential equation now becomes

$$m \frac{d^2 y}{d t^2} = m g - k \frac{d y}{d t}. \tag{5}$$

Exercise

(a) Non-dimensionalize the equation by choosing suitable scales expressing the equation in dimensionless quantities.

(b) How many free parameters are left in the equation after non-dimensionalization?

Solution

$$\begin{aligned} a \text{ scale}: &\quad g \\ v \text{ scale}: &\quad \frac{m g}{k} \\ t \text{ scale}: &\quad \frac{m g}{k} \frac{1}{g} = \frac{m}{k} \\ y \text{ scale}: &\quad \frac{m g}{k} \frac{m}{k} = \frac{m^2 g}{k^2}. \end{aligned} \tag{6}$$

Making the transformation:

$$\begin{aligned} y &\longrightarrow \frac{m^2 g}{k^2} y \\ t &\longrightarrow \frac{m}{k} t \end{aligned} \tag{7}$$

we get

$$m \frac{m^2 g}{k^2 \left(\frac{m}{k}\right)^2} \frac{d^2 y}{d t^2} = m g - k \frac{m^2 g}{k^2 \frac{m}{k}} \frac{d y}{d t}. \tag{8}$$

$$\Rightarrow \frac{d^2 y}{dt^2} = 1 - \frac{dy}{dt}$$

After non-dimensionalization, there is *no* free parameter left in the problem!

Let us assume that the initial conditions for this problem in dimensionless units is given by $y(0) = 0$ and $\dot{y}(0) = 0$.

This is a second order differential equation which can be solved exactly. The method involves realizing that it is really a first-order differential equation in the velocity. Defining $v = \frac{dy}{dt}$, we have

$$\frac{dv}{dt} = 1 - v. \tag{9}$$

This equation can be solved by the method of separation of variables:

$$\frac{dv}{1 - v} = dt. \tag{10}$$

Integrating we have

$$-\log(1 - v) = t + c. \tag{11}$$

Since $v(0) = 0$, we get $c = 0$. Therefore, we have

$$v = 1 - e^{-t}. \tag{12}$$

A plot of this function is very instructive.

```
vfunc[t_] = 1 - e^-t;
Plot[vfunc[t], {t, 0, 4}, PlotRange → Automatic, AxesLabel → {t, v}];
```

Another integration (followed by the use of the initial condition $y(0) = 0$) allows us to write down the distance covered as a function of time as:

$$y = t + e^{-t} - 1. \tag{13}$$

Plotting this function we have

```
yfunc[t_] = t + ⅇ^-t - 1;
Plot[yfunc[t], {t, 0, 4}, PlotRange → Automatic, AxesLabel → {t, y}];
```

Numerical Solution with the RK4 Method

- Lets recall how we can bring a higher order differential equation into the canonical form:

$$\begin{aligned}\dot{x} &= f(t, x, y, z)\\ \dot{y} &= g(t, x, y, z)\\ \dot{z} &= h(t, x, y, z)\end{aligned} \tag{14}$$

- Next we define the column vectors X and F as

$$X = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ f \\ g \\ h \end{pmatrix} \tag{15}$$

- Then the coupled ODEs can be written as

$$\dot{X} = F \tag{16}$$

- The RK4 method is given by

$$\begin{aligned}R_1 &= F(X_n)\\ R_2 &= F\left(X_n + \frac{h}{2} R_1\right)\\ R_3 &= F\left(X_n + \frac{h}{2} R_2\right)\\ R_4 &= F(X_n + h\, R_3)\end{aligned} \tag{17}$$

$$X_{n+1} = X_n + h\,\frac{R_1 + 2\,R_2 + 2\,R_3 + R_4}{6} \tag{18}$$

- Here we have copied its implementation.

```
rk4[F_, X0_, tf_, nMax_] := Module[{h, datalist, prev, rate1, rate2, rate3, rate4, next},
  h = (tf - X0[[1]]) / nMax // N;
  For[datalist = {X0},
   Length[datalist] ≤ nMax,
   AppendTo[datalist, next],
   prev = Last[datalist];
   rate1 = F@prev;
   rate2 = F@(prev + h/2 rate1);
   rate3 = F@(prev + h/2 rate2);
   rate4 = F@(prev + h rate3);
   next = prev + h/6 (rate1 + 2 rate2 + 2 rate3 + rate4);
  ];
  Return[datalist];
 ]
```

- The differential equation corresponding to the falling body subject to air resistance can be recast into canonical form as:

$$\begin{gathered} \frac{dx}{dt} = v \\ \frac{dv}{dt} = 1 - v \\ x(0) = 0 \\ v(0) = 0 \end{gathered} \tag{19}$$

- So in vector form we have:

$$\begin{gathered} X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ 1 - v \end{pmatrix} \\ \dot{X} = F \end{gathered} \tag{20}$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, 1 - v};
initial = {0, 0, 0};
solx[t_] = t + e^-t - 1;
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 4, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 4}, PlotRange -> Full, PlotStyle -> Red]];
```

Fall with air resistance quadratic in velocity

Next, we could consider variants of the resisting force that is more complicated functions of the velocity of the falling body. One natural extension that can be considered is that of a resistance that is quadratic in velocity. The differential equation would now be nolinear:

$$m \frac{d^2 y}{d t^2} = m g - k \left(\frac{d y}{d t} \right)^2. \tag{21}$$

Exercise

(a) Non-dimensionalize the equation by choosing suitable scales expressing the equation in dimensionless quantities.

(b) How many free parameters are left in the equation after non-dimensionalization?

Solution

$$\begin{aligned} &a \text{ scale}: \quad g \\ &v \text{ scale}: \quad \sqrt{\frac{m g}{k}} \\ &t \text{ scale}: \quad \sqrt{\frac{m g}{k}} \frac{1}{g} = \sqrt{\frac{m}{k g}} \\ &y \text{ scale}: \quad \sqrt{\frac{m g}{k}} \sqrt{\frac{m}{k g}} = \frac{m}{k}. \end{aligned} \tag{22}$$

Making the transformation:

$$\begin{aligned} y &\longrightarrow \frac{m}{k} y \\ t &\longrightarrow \sqrt{\frac{m}{k g}}\, t \end{aligned} \tag{23}$$

we get

$$m \frac{m}{k\left(\frac{m}{k g}\right)} \frac{d^2 y}{d t^2} = m g - k \frac{m^2}{k^2 \frac{m}{k g}} \left(\frac{d y}{d t}\right)^2.$$

$$\Rightarrow \frac{d^2 y}{d t^2} = 1 - \left(\frac{d y}{d t}\right)^2$$

Once again, after non-dimensionalization, there is *no* free parameter left in the problem.

Let us assume that the initial conditions for this problem in dimensionless units is given by $y(0) = 0$ and $\dot{y}(0) = 0$.

This is a second order differential equation which can be solved exactly like before. The method again involves converting it into a first-order differential equation in the velocity. Defining $v = \frac{dy}{dt}$, we have

$$\frac{d v}{d t} = 1 - v^2. \tag{25}$$

This equation can be solved by the method of separation of variables:

$$\frac{d v}{1 - v^2} = d t . \tag{26}$$

So

$$\left(\frac{1}{1 - v} + \frac{1}{1 + v}\right) d v = 2 d t . \tag{27}$$

Integrating we have

$$-\log(1 - v) + \log(1 + v) = 2 t + c. \tag{28}$$

Since $v(0) = 0$, we get $c = 0$. Therefore, we have

$$\frac{1 + v}{1 - v} = e^{2 t}. \tag{29}$$

Thus the solution is

$$v = \frac{e^{2t}-1}{e^{2t}+1} = \frac{\frac{e^t-e^{-t}}{2}}{\frac{e^t+e^{-t}}{2}} = \tanh(t) \tag{30}$$

A plot of this function is very instructive.

```
vfunc[t_] = Tanh[t];
Plot[vfunc[t], {t, 0, 4}, PlotRange → Automatic, AxesLabel → {t, v}];
```

Another integration (followed by the use of the initial condition $y(0) = 0$) allows us to write down the distance covered as a function of time as:

$$y = \log(\cosh(t)) \tag{31}$$

Plotting this function we have

```
yfunc[t_] = Log[Cosh[t]];
Plot[yfunc[t], {t, 0, 4}, PlotRange → Automatic, AxesLabel → {t, y}];
```

Numerical Solution with the RK4 Method

```
rk4[F_, X0_, tf_, nMax_] := Module[{h, datalist, prev, rate1, rate2, rate3, rate4, next},
  h = (tf - X0[[1]]) / nMax // N;
  For[datalist = {X0},
   Length[datalist] ≤ nMax,
   AppendTo[datalist, next],
   prev = Last[datalist];
   rate1 = F@prev;
   rate2 = F@(prev + h/2 rate1);
   rate3 = F@(prev + h/2 rate2);
   rate4 = F@(prev + h rate3);
   next = prev + h/6 (rate1 + 2 rate2 + 2 rate3 + rate4);
  ];
  Return[datalist];
 ]
```

- The differential equation corresponding to the falling body subject to air resistance can be recast into canonical form as:

$$\begin{gathered} \frac{dx}{dt} = v \\ \frac{dv}{dt} = 1 - v^2 \\ x(0) = 0 \\ v(0) = 0 \end{gathered} \tag{32}$$

- So in vector form we have:

$$X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ 1 - v^2 \end{pmatrix}$$

$$\dot{X} = F$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, 1 - v^2};
initial = {0, 0, 0};
solx[t_] = Log[Cosh[t]];
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 4, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 4}, PlotRange -> Full, PlotStyle -> Red]];
```