Physics through Computational Thinking

Linear systems: Insights from the Phase Space picture

Auditya Sharma and Ambar Jain
Dept. of Physics, IISER Bhopal

Outline

In this module we cover

1. General theory of linear systems.
2. Study the example of love affairs using the general theory.

General Theory

The general problem is

$$\begin{aligned}\dot{x} &= a\,x + b\,y \\ \dot{y} &= c\,x + d\,y\end{aligned} \tag{1}$$

and

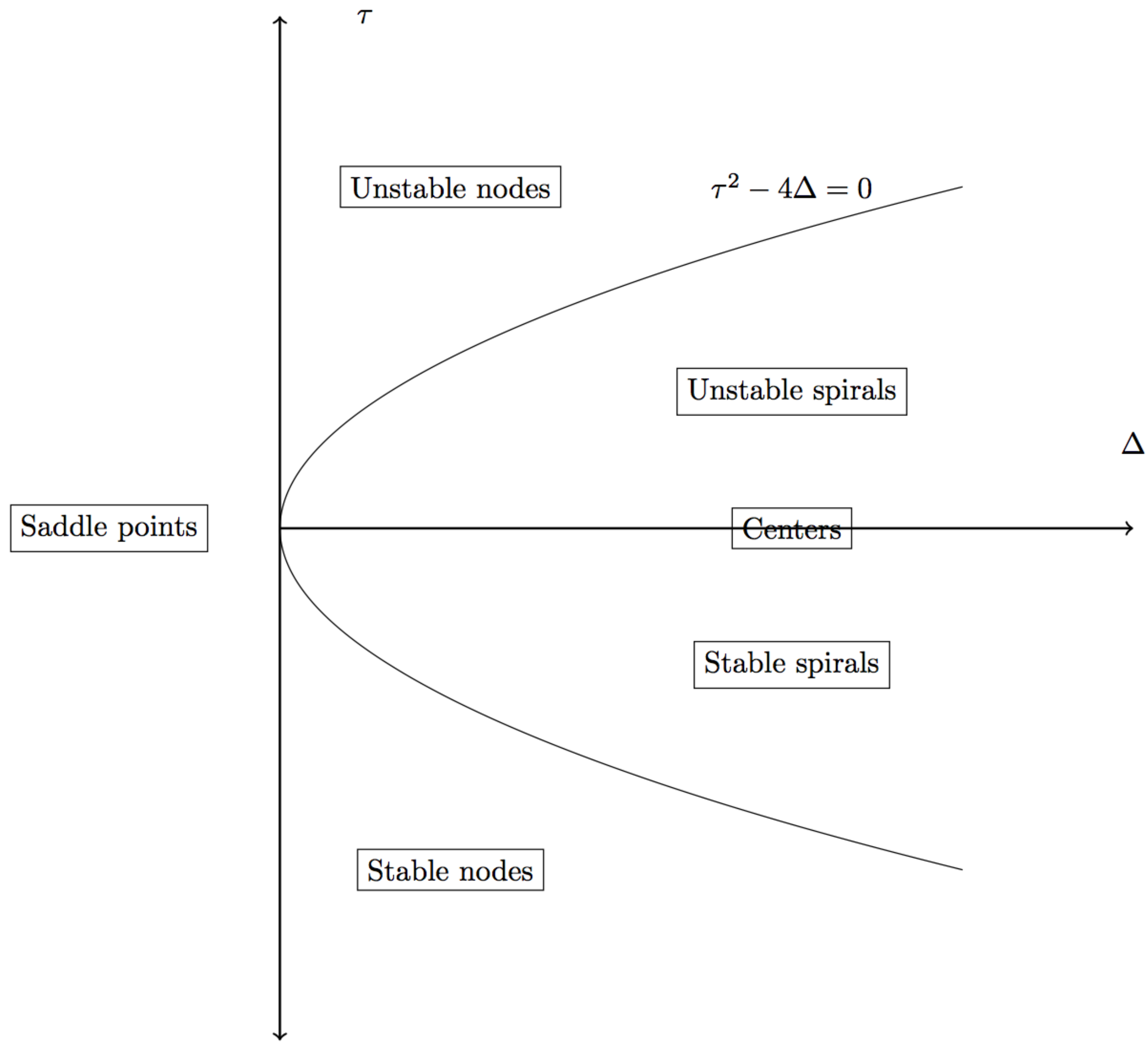
$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\,\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\,\Delta}}{2}. \tag{2}$$

The inverse relation is

$$\tau = \lambda_1 + \lambda_2 \qquad \Delta = \lambda_1\,\lambda_2\,. \tag{3}$$

We make the following observations:

- If $\Delta < 0$, then both the eigenvalues *have to* be real, and with opposite signs. Hence the fixed point is guranteed to be a saddle point.
- If $\Delta > 0$, then we have a range of different possibilities, depending on the value of τ. If $\tau^2 > 4\,\Delta$, then the eigenvalues are real, and therefore the fixed point is a *node*. On the other hand, if $\tau^2 < 4\,\Delta$, then the eigenvalues are complex (conjugates of each other), and the fixed point then becomes either a center or a spiral.



τ
Unstable nodes
$\tau^2 - 4\Delta = 0$
Unstable spirals
Δ
Saddle points
Centers
Stable spirals
Stable nodes

Examples

Now let us investigate a bunch of examples with the aid of our method of classification.

Example 1

$$\dot{x} = y, \quad \dot{y} = -2\,x - 3\,y.$$

$$\begin{aligned} \tau &= \mathrm{Tr}(A) = -3 \\ \Delta &= \det(A) = 2. \end{aligned} \tag{4}$$

The expectation is that it will be a stable node. Let us check with:

```
StreamPlot[{y, -2 x - 3 y}, {x, -2, 2}, {y, -2, 2}];
```

Example 2

$$\dot{x} = 5\,x + 10\,y, \quad \dot{y} = -\,x - y.$$

$$\begin{aligned} \tau &= \mathrm{Tr}(A) = 4 \\ \Delta &= \det(A) = 5. \end{aligned} \tag{5}$$

The expectation is that it will be a unstable spiral. Let us check with:

```
StreamPlot[{5 x + 10 y, -x - y}, {x, -100, 100}, {y, -100, 100}];
```

Example 3: The damped harmonic oscillator

The differential equation is

$$m\,\ddot{x} + b\,\dot{x} + k\,x = 0,$$

where $b > 0$ is the damping coefficient. The differential equation can be recast into the canonical form as:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{-k}{m}\,x - \frac{b}{m}\,v \end{aligned} \tag{6}$$

Therefore we have

$$\tau = \mathrm{Tr}(A) = \frac{-b}{m}$$
$$\Delta = \det(A) = \frac{k}{m} \tag{7}$$

We immediately see that a center is possible only if $b = 0$, i.e it is a frictionless system. It is only in such a scenario that there is no loss of energy, and the eigenvalue of the matrix A is purely imaginary, and we recover the simple harmless harmonic oscillator! No saddle point is ever possible in such a system because we are always guaranteed to have a positive Δ given that the spring constant k and the mass m are necessarily positive. Also τ is necessarily a negative quantity, so we have only three possibilities:

- Stable spiral: When $b^2 < 4\,k\,m$, the eigenvalues are imaginary and we thus have oscillatory motion, but there is also loss of energy in every cycle. This corresponds to the familiar case of underdamped motion.
- Stable node: When $b^2 > 4\,k\,m$, the eigenvalues are real and there is a mixture of growing and decaying motion but with no oscillatory motion, with a continuous loss of energy. This corresponds to the familiar case of overdamped motion. The system eventually and slowly looses all its energy and settles down to equilibrium.
- Borderline case of Degenerate node: When $b^2 = 4\,k\,m$, we have the borderline case which yields critically damped motion. As we know the system decays to equilibrium without any oscillations, and the decay is fastest in this case.

```
StreamPlot[{y, -x - 2 y}, {x, -100, 100}, {y, -100, 100}];
```

Cautious Lovers

As we have seen, the most general linear model of the Romeo-Juliet love affair is given by

$$\dot{R} = a\,R + b\,J,\ \ \dot{J} = c\,R + d\,J,$$

where the magnitudes and (very importantly) the signs of the parameters a, b, c, d determine the nature of the affair. Suppose we consider a symmetric case where $c = b$, and $d = a$. Furthermore, we take a to be negative and b to be positive indicating that the two lovers are cautious. If they see reciprocation, they have a tendency to also respond positively, however they are reluctant to go too far if they are already showing a lot of love (negative a is a measure of the cautiousness). So we can ask how the relative strength of the cautiousness a and responsiveness $b$will pan out for the relationship. The matrix in question is

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{8}$$

Therefore,

$$\begin{aligned} \tau &= \mathrm{Tr}(A) = 2\,a < 0 \\ \Delta &= \det(A) = a^2 - b^2. \end{aligned} \tag{9}$$

So,

$$\tau^2 - 4\,\Delta = 4\,b^2, \tag{10}$$

which is always positive. This means that the fixed point can be either be a saddle point if Δ is negative or a stable node if Δ is positive. Let us consider the two cases and make streamplots:

```
StreamPlot[{-x + 2 y, 2 x - y}, {x, -100, 100}, {y, -100, 100}];

StreamPlot[{-2 x + y, x - 2 y}, {x, -100, 100}, {y, -100, 100}];
```