Physics through Computational Thinking

Linearization

Auditya Sharma and Ambar Jain
Dept. of Physics, IISER Bhopal

Outline

In this module we look at

1. linearization.

Linearization about the fixed point - One dimension

- Linearizing our ODE out the fixed point provides little more clarity about the stability, further we can obtain an approximate solution in the neighbourhood of the fixed point.
- The differential equation is

$$\dot{x} = f(x). \tag{1}$$

- At a fixed point x_0 $f(x_0)$=0. Invoking this fact and defining $\lambda = f'(x_0)$, the Taylor expansion of $f(x)$ near the fixed point yields

$$\begin{aligned} f(x = x_0 + \delta x) &= f(x_0) + \delta x\, f'(x_0) + O\left(\delta x^2\right) \\ &\Rightarrow \dot{x} = \lambda\, \delta x \end{aligned} \tag{2}$$

- Substituting this in the ODE, near $x = x_0$, and defining the change $\delta x(t) = x(t) - x_0$ gives a ***local linear system***

$$\dot{\delta x} = \dot{x} = \lambda\, \delta x \tag{3}$$

- which simply has a solution

$$\begin{aligned} \delta x(t) &= \delta x_0\, e^{\lambda t} \\ \Rightarrow x(t) &= x_0 + \delta x_0\, e^{\lambda t} \end{aligned} \tag{4}$$

- If $\lambda = f'(x_0)$ is positive then the solution is **unstable** (diverging away from the fixed point), while if $\lambda < 0$ then the solution is **stable** (converging into the fixed point). Indeed, this is what we saw from the plot in the precious example.

$$\begin{array}{lcc} \lambda > 0 & : & \text{unstable fixed point} \\ \lambda < 0 & : & \text{stable fixed point} \\ \lambda = 0 & : & \text{neutral fixed point (sometimes stable on one side and unstable on the other)} \end{array} \tag{5}$$

Fixed Points in two dimensions

- Many more interesting things can happen in two dimensions. Dynamics is much richer compared to one-D systems where there is simply a stable and an unstable fixed point.
- In two dimensions, we have equations of the form

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{6}$$

- which we can also write in matrix notation as

$$\dot{X} = F(X) \tag{7}$$

- where X and F are vectors of dimension 2.
- $X = X_0 = (x_0, y_0)$, is a fixed point iff

$$F(X_0) = 0 \tag{8}$$

- System sitting at the fixed point does not move.
- Linearizing near the fixed point we have

$$\begin{aligned} f(x, y) &\simeq f(x_0, y_0) + \delta x \left(\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \right) + \delta y \left(\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} \right) + \text{higher order} \\ g(x, y) &\simeq g(x_0, y_0) + \delta x \left(\left.\frac{\partial g}{\partial x}\right|_{(x_0, y_0)} \right) + \delta y \left(\left.\frac{\partial g}{\partial y}\right|_{(x_0, y_0)} \right) + \text{higher order} \end{aligned} \tag{9}$$

- Noting that $f(x_0, y_0) = g(x_0, y_0) = 0$, and defining the Jacobian matrix as

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x_0, y_0)} \tag{10}$$

- we can write the linearized system, for $X = X_0 + \delta X$, in the form of a matrix equation

$$\delta \dot{X} = J\, \delta X \tag{11}$$

- Solution of this equation is given by

$$\delta X(t) = e^{J\,t}\,\delta X_0$$
$$\Rightarrow X(t) = X(0) + e^{J\,t}\,\delta X_0 \tag{12}$$

- Since J is real 2×2 matrix, we can consider diagonalizing it by finding its eigenvalues and eigenvectors. Let's say $\lambda_{1,2}$ are its eigen values while $V_{1,2}$ are corresponding eigenvectors.
- We can write, using the linear combination property of vectors,

$$\delta X_0 = \delta\alpha_1\,V_1 + \delta\alpha_2\,V_2 \qquad \text{for some constants } \delta\alpha_{1,2} \tag{13}$$

- and

$$X_0 = \alpha_1\,V_1 + \alpha_2\,V_2 \qquad \text{for some constants } \alpha_{1,2} \tag{14}$$

- Then the solution can be written as

$$X(t) = \alpha_1\,V_1 + \alpha_2\,V_2 + \delta\alpha_1\,e^{J\,t}\,V_1 + \delta\alpha_2\,e^{J\,t}\,V_2 \tag{15}$$

- Using the eigenvalue property

$$e^{J\,t}\,V_i = e^{\lambda_i\,t}\,V_i \tag{16}$$

- we get

$$X(t) = \alpha_1\,V_1 + \alpha_2\,V_2 + \delta\alpha_1\,e^{\lambda_1\,t}\,V_1 + \delta\alpha_2\,e^{\lambda_2\,t}\,V_2$$
$$\Rightarrow X(t) \;=\; \left(\alpha_1 + \delta\alpha_1\,e^{\lambda_1\,t}\right)V_1 + \left(\alpha_2 + \delta\alpha_2\,e^{\lambda_2\,t}\right)V_2 \tag{17}$$

- Depending on the sign of λ_1 and λ_2, we can have different stability along directions V_1 and V_2. For each of the directions V_1 and V_2, we have a situation analogous to the one dimensional case.
- In general λ_1 and λ_2 are complex numbers.

Stable Fixed Point

$$\mathrm{Re}(\lambda_1) < 0 \qquad \text{and} \qquad \mathrm{Re}(\lambda_2) < 0 \tag{18}$$

Unstable Fixed Point

$$\mathrm{Re}(\lambda_1) > 0 \qquad \text{and} \qquad \mathrm{Re}(\lambda_2) > 0 \tag{19}$$

$$\begin{gathered}\textbf{Saddle Point}\\ \mathrm{Re}(\lambda_1) > 0 \qquad \text{and} \qquad \mathrm{Re}(\lambda_2) < 0\end{gathered} \tag{20}$$

$$\begin{gathered}\textbf{Center}\\ \mathrm{Re}(\lambda_1) = 0 \qquad \text{and} \qquad \mathrm{Re}(\lambda_2) = 0\end{gathered} \tag{21}$$

and so on. We have seen how a full phase diagram of all possibilities is readily obtained in terms of the trace, and determinant of the Jacobian matrix.