Practice Problems: Week 4 (Submission not required)

1. Solve the following differential equations using Euler's method for the given initial condition and range. Compare your numerical result with corresponding analytical result on a plot:

(a)	$\dot{x}(t) = -x\,t;$	$x(0) = 1$	$t \in [0, 5]$
(b)	$\dot{x}(t) = 1 - t;$	$x(0) = 0$	$t \in [0, 1]$
(c)	$\dot{x}(t) = \cos(t)$	$x(\pi) = 1$	$t \in [\pi, 3\pi]$
(d)	$\dot{x}(t) = \mathrm{sech}^2(t)$	$x(1) = 1$	$t \in [-1, 1]$

2. Solve the following differential equation

$$\dot{x}(t) = x^2 - t^2$$

using Euler's method for $t \in [0, 1]$ for the following initial conditions

(a) $x(0) = 1$

(b) $x(0) = 1/2$

(c) $x(0) = 0$

(d) $x(0) = -1$

This differential equation is difficult to solve analytically. Later on we will compare our solution from Euler's method with other numerical methods.

3. Find the constants A and ϕ for the underdamped harmonic oscillator solution $x(t) = A\,e^{-t/2} \cos\left(\sqrt{\frac{3}{4}}\; t + \phi\right)$, for the initial value problem

$$\ddot{x}(t) = -x - \dot{x}$$
$$x(0) = 1$$
$$\dot{x}(0) = 0$$

4. For the damped harmonic oscillator we found the following relevant time scales: $\sqrt{L\,C}$ and L/R.

(a) Show that when $2\,L/R > \sqrt{L\,C}$, the system is overdamped, that is, it does not go through oscillatory motion, that is it admits solution of the form $e^{-\alpha\,t}$.

(b) Show that when $2\,L/R < \sqrt{L\,C}$, the system is under damped, that is, it goes through several cycles of oscillation before losing significant amount of amplitude or energy.

(c) Show that when $2\,L/R = \sqrt{L\,C}$, the system is critically damped, that it is at the interface of under-damped and overdamped.

(d) Define $t_o = \sqrt{L\,C}$ and $t_d = 2\,L/R$. Using Manipulate, explore the impact of various hierarchies of these scales on the numerical solution for damped LCR circuit, which you can obtain using NDSolve or Euler's method.

5. Consider a simple pendulum made of a string of length ℓ and a solid metal bob of radius a ($a << \ell$). The bob is completely submerged in the fluid of viscosity η and the pendulum oscillates back and forth in this fluid. If the density of fluid is ρ_0 and that of the metal is ρ then

(a) Assuming slow oscillation, show that the equation of motion for the pendulum is given by (slow oscillation makes linear Stoke's law viable, that is resistive force due to viscous fluid is given by $6\,\pi\,\eta\,a\,v$, where v is the instantaneous speed of the bob.)

$$\ddot{\theta}(t) = -\left(1 - \frac{\rho_0}{\rho}\right)\frac{g}{\ell}\sin\theta - \frac{9\,\eta}{2\,\rho\,a^2}\,\dot{\theta}$$

(b) Find the oscillation and damping time scales in the problem.

(c) Assuming small and slow damping, Estimate $\frac{1}{E}\,\frac{dE}{dt}$ by dimensional analysis and physics of damping, following the procedure used in the video lectures.

(d) Non-dimensionalize the equation of motion in part (a) and show that it reduces to the form

$$\ddot{\theta}(t) = -\sin\theta - \gamma\,\dot{\theta}$$

Find the units in which we measure time and the expression for γ in terms of the constants of the problem.

(e) When the oscillation is fast, the Stoke's law is modified by a quadratic term in instantaneous speed due to vortex formed behind the fast moving bob. The resistive force modifies as

$$\vec{f}_r = -6\,\pi\,\eta\,a\,\vec{v} - \sigma\,\pi\,a^2\,\rho_0\,|\vec{v}|\,\vec{v}$$

Find the modified equation of motion and suitably non-dimensionalize it.

(f) What is the physical dimension of σ? Does adding the quadratic correction term to viscous drag add a new time scale to the problem?

(g) For the slow oscillations, show that $\dfrac{\eta}{\sigma\,\rho_0\,a\,\sqrt{g\,\ell}} >> 1$.