Computer Graded Assignment: Week 4

1. The solution of the initial value problem

$$\ddot{x}(t) = -x - \dot{x} \qquad x(0) = 1 \qquad \dot{x}(0) = 0$$

is given by

- ☐ $\frac{1}{\sqrt{3}} e^{-t/2} \cos\left(\frac{\pi}{6} + \frac{2}{\sqrt{3}} t\right)$
- ☐ $\frac{1}{\sqrt{3}} e^{-t/2} \cos\left(\frac{\pi}{6} - \frac{2}{\sqrt{3}} t\right)$
- ☑ $\frac{2}{\sqrt{3}} e^{-t/2} \cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2} t\right)$
- ☐ $\frac{2}{\sqrt{3}} e^{-t/2} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} t\right)$

Solution: You can solve the equation using DSolve or use the general solution of the form: $A\, e^{-\alpha\, t}\cos(\beta\ t+\phi)$ and differential equation and boundary condition to fix the constants.

In[∘]:=
```
sol = DSolve[{x''[t] == -x[t] - x'[t], x[0] == 1, x'[0] == 0}, x[t], t]
```

Out[∘]=

$$\left\{\left\{x[t] \to \frac{1}{3} e^{-t/2} \left(3 \, \text{Cos}\left[\frac{\sqrt{3}\, t}{2}\right] + \sqrt{3}\, \text{Sin}\left[\frac{\sqrt{3}\, t}{2}\right]\right)\right\}\right\}$$

This matches with one of the solutions given above.

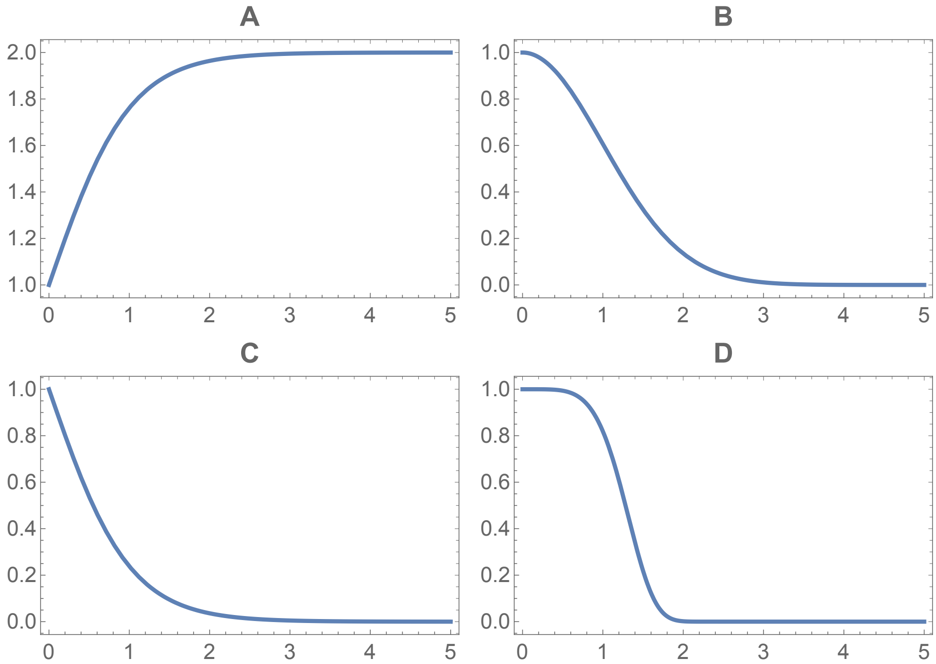
In[∘]:=

$$\frac{2}{\sqrt{3}} \text{Cos}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{2} t\right] e^{-t/2} - x[t] \text{ /. sol}[[1]] \text{ // Simplify}$$

Out[∘]=
```
0
```

2. Which of the following plots below represent the solution of the following initial value problem?

$$\dot{x}\,(t) = -x\, t; \qquad x(0) = 1 \qquad t \in [0,\ 5]$$



☐ A

☑ B

☐ C

☐ D

Solution: Solution of this is IVP is a Gaussian.

In[∘]:=
```
sol = DSolve[{x'[t] == -x[t] t, x[0] == 1}, x[t], t]
```

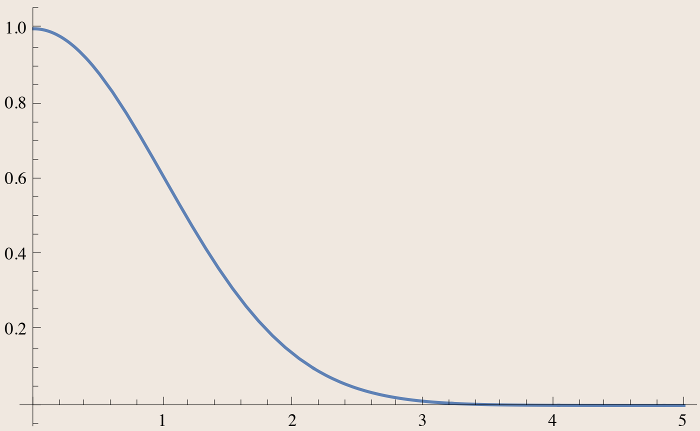
Out[∘]=

$$\left\{\left\{x[t] \to e^{-\frac{t^2}{2}}\right\}\right\}$$

Its plot is given by

In[∘]:=
```
Plot[x[t] /. sol[[1]], {t, 0, 5}]
```

Out[∘]=



3. The relevant time scales governing the dynamics in the LCR circuit are

☑ $\frac{L}{R}$

☑ $R\,C$

☑ $\sqrt{L\,C}$

☐ $\sqrt{\frac{1}{L\,C} - \frac{R^2}{4\,L^2}}$

☐ $\frac{L}{R^2\,C}$

Solution: L/R is damping time scale, $\sqrt{\mathrm{LC}}$ is oscillation scale, $R\,C$ is time scale associated with capacitor charge and discharge. It is also obtained by $\frac{\left(\sqrt{L\,C}\right)^2}{L/R}$.

4. The rate at which average energy decreases in an underdamped LCR circuit of small resistance, with instantaneous charge Q in the capacitor, is of the order of

☑ $\frac{R\,Q^2}{L\,C}$

☐ $\frac{Q^2}{C\,\sqrt{L\,C}}$

☐ $\frac{Q^2}{C}\sqrt{\frac{1}{L\,C} - \frac{R^2}{4\,L^2}}$

☐ $\frac{Q^2}{R\,C^2}$

Solution: You can obtain this by dimensional analysis (as well as physics of damping) as explained in the video.

5. Identify the time scales in the system whose equation of motion are given by

$$\ddot{\theta}(t) = -\alpha \sin\theta - \beta\,\dot{\theta} - \gamma\,|\dot{\theta}|\,\dot{\theta}$$

☐ $\frac{1}{\sqrt{\alpha}}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$

☐ $\frac{1}{\sqrt{\alpha}}$, β and $\frac{1}{\gamma}$

☐ $\sqrt{\alpha}$ and $\frac{1}{\beta}$

☑ $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\beta}$

Solution: α has dimensions of $1/T^2$, β has dimensions of $1/T$ and γ is dimensionless.

6. For a spring mass system moving on a surface with non-zero but small friction, the amplitude decreases

☑ linearly with time

☐ as t^2

☐ as $\sqrt{t}$

☐ as $\log(t)$

Solution: This was explained in the video tutorial where we predicted this behavior from physics of damping and then verified computationally.

7. For Euler's method the local error is of the order of (where h is the step size in time)

☐ $\sqrt{h}$

☐ h

☑ h^2

☐ h^3

Solution: We explained this is the video tutorial.

8. An LCR circuit is critically damped when $\frac{L}{R^2 C}$ is

☑ $\frac{1}{4}$

☐ $\frac{1}{2}$

☐ 2

☐ 4

Solution: When $\frac{L}{R^2 C} = \frac{1}{4}$, the oscillation frequency of underdamped oscillator vanishes.

9. Consider an oscillator, whose equation of motion in terms of coordinate $\theta(t)$ is given by

$$\ddot{\theta}(t) = -\theta - 3\,\dot{\theta}\,.$$

Its general solution is a linear combination of

☐ $\exp\left(\frac{-t}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\,t\right)$ and $\exp\left(\frac{-t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\,t\right)$

☐ $\exp(-t)$ and $t\exp(-t)$

☐ $\exp\left(-\left(2+\sqrt{3}\right)t\right)$ and $\exp\left(-\left(2-\sqrt{3}\right)t\right)$

☑ $\exp\left(-\frac{3+\sqrt{5}}{2}\,t\right)$ and $\exp\left(-\frac{3-\sqrt{5}}{2}\,t\right)$

Solution: This is example of an underdamped oscillator. We can easily check using DSolve.

In[∘]:=
```
sol = DSolve[{x''[t] == -x[t] - 3 x'[t]}, x[t], t]
```

Out[∘]=

$$\left\{\left\{x[t] \to e^{\left(-\frac{3}{2}-\frac{\sqrt{5}}{2}\right)t}\,c_1 + e^{\left(-\frac{3}{2}+\frac{\sqrt{5}}{2}\right)t}\,c_2\right\}\right\}$$

10. Consider the initial value problem given by $\dot{x} = 4\,t + 3\,x$ and $x(0) = 0$. If we numerically solve this problem using Euler's method with $x_0 = x(0) = 0$ and $h = 0.3$, then the value of x_3 is ___________.

Range: 1.402 to 1.406

S o l u t i o n :
$x_0 = 0$
$x_1 = x_0 + h\,f(t_0, x_0) = 0$
$x_2 = x_1 + h\,f(t_1, x_1) = 0 + 0.3\,f(0.3, 0) = 0.3 \times 1.2 = 0.36$
$x_3 = x_2 + h\,f(t_2, x_2) = 0.36 + 0.3\,f(0.6, 0.36) = 0.36 + 0.3\,(2.4 + 1.08) = 1.404$