Physics through Computational Thinking

Random walks: Stirling Approximation

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Outline

In this module we

1. look at the power of Stirling’s approximation in the context of the one dimensional random walk problem.

Random Walk: Mean and variance of net displacement

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Clear["Global`*"]
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The probability distribution is

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\, p^{\left(\frac{N+m}{2}\right)} q^{\left(\frac{N-m}{2}\right)} \tag{1}$$

We have the condition

$$\begin{aligned} N &= n_1 + n_2 \\ m &= n_1 - n_2 \end{aligned} \tag{2}$$

from which we have

$$m = 2\, n_1 - N \tag{3}$$

Therefore

$$\begin{aligned} <m> &= 2 <n_1> - N \\ &= 2\, N\left(p - \frac{1}{2}\right) \end{aligned} \tag{4}$$

Also the variance is now given by

$$\begin{gathered} <m^2> - <m>^2 = <(2\, n_1 - N)^2> - <2\, n_1 - N>^2 \\ = \left(4 <n_1{}^2> -4\, N\ <n_1> +N^2\right) - \left(4 <n_1>^2 -4\, N <n_1> +N^2\right) \\ = 4\left(<n_1{}^2> - <n_1>^2\right) \end{gathered} \tag{5}$$

Therefore

$$<m^2> - <m>^2 = 4\, N\, p\, q \tag{6}$$

Special Case: The unbiased random walk

The unbiased random walk when $p = q = \frac{1}{2}$, and where the drunkard is equally likely to move to the right or to the left deserves special attention.

The mean and variance in displacement after N steps is now

$$\begin{aligned} < m > \; &= 2N\left(p - \frac{1}{2}\right) = 0 \\ < m^2 > - < m >^2 &= 4 N p q = N. \end{aligned} \tag{7}$$

Equivalently

$$< m^2 > \; = N, \tag{8}$$

which is an important result. Physically what it means is that although the random walker takes N steps the *typical* displacement is only of O($\sqrt{N}$). This fact finds application in a variety of fields ranging from error-analysis to the stock market to polymer physics to Brownian motion.

The probability distribution for the unbiased walk is

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\left(\frac{1}{2}\right)^N \tag{9}$$

In the limit of large N, n_1 and n_2, it is reasonable to assume that m is much smaller than N, and with the help of a powerful tool called Stirling's approximation, the limiting procedure can be carried out to yield

$$P_N(m) \approx \sqrt{\frac{2}{\pi N}} \exp\left(\frac{-m^2}{2N}\right). \tag{10}$$

Exercise

Use *Mathematica* to check the Stirling formula:

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln(n) - n + \frac{1}{2}\ln(2\pi) + O\left(n^{-1}\right) \tag{11}$$

- Plot on the same graph the full expression and the approximation as a function of n, and see how for larger n works out great.

- Next plot on the same graph the exact expression for the probability distribution for the unbiased walk, and the approximate one obtained after invoking the Stirling's approximation. Vary N and observe when the approximate expression becomes practically as good as the exact one.

Solution

```
plotexact = ListPlot[Table[Log[Factorial[n]], {n, 1, 10}], PlotMarkers → Style["●", 14, Red]];
plotapprox = ListPlot[Table[(n + 1/2) Log[n] - n + 1/2 Log[2 π], {n, 1, 10}], PlotMarkers → Style["◆", 10, Blue]];

Show[plotexact, plotapprox];

Manipulate[Plot[{Factorial[N]/(Factorial[(N+m)/2] Factorial[(N-m)/2]) 1/2^N, Sqrt[2/(π N)] Exp[-m^2/(2 N)]}, {m, -N, N}], {N, 10, 100}]
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