Physics through Computational Thinking

Linear systems: Insights from the Phase Space picture

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Outline

In this module we will cover

1. Coupled ODEs, and the Phase space picture.
2. Linear systems, and some examples.

Coupled ODEs, Phase Space and Fixed Point

- For equations of the type, which have no explicit time dependence,

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{1}$$

- where RHS does not depend on time explicitly, we can get a lot of insight about the problem looking for trajectory in the ***phase space***, that is x–y space, since dynamics is completely dependent on the x and y.
- To obtain the phase space plot we don't need to know the solution of the equations. We just need to know in what direction is the flow at each point (x, y) in the phase space. Knowing this flow traces out a trajectory for us. Knowing the trajectory can guide us to anticipate and understand the solution before we even solve the equations.
- We can accomplish this plot using **VectorPlot** or **StreamPlot** of the vector field $(\dot{x}, \dot{y}) = (f, g)$.
- If there exist a point (x_0, y_0) such that $f(x_0, y_0) = g(x_0, y_0) = 0$, then it is called a fixed point in the phase space. If a system arrives at this point it will stay their forever. A fixed point can be stable, unstable or a saddle point.

Example-1: Center

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned} \tag{2}$$

- Solution of this equation is of course:

$$\begin{aligned} x(t) &= A\cos(t + t_0) \\ y(t) &= -A\sin(t + t_0) \\ x^2 + y^2 &= A^2 \end{aligned} \tag{3}$$

- We can see the circular flow by simply making a **StreamPlot**

```
StreamPlot[{y, -x}, {x, -1, 1}, {y, -1, 1}];
```

Example-2: Unstable Fixed Point: Source

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \end{aligned} \tag{4}$$

- Solution of this equation is of course:

$$
\begin{aligned}
x(t) &= A\, e^{t} \\
y(t) &= B\, e^{t} \\
y &= \frac{B}{A}\, x
\end{aligned}
\tag{5}
$$

- We can see the expected straightline trajectory by simply making a **StreamPlot**

```
StreamPlot[{x, y}, {x, -1, 1}, {y, -1, 1}];
```

Example-3: Saddle Point

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= x
\end{aligned}
\tag{6}
$$

- Solution of this equation can be obtained by

$$
\begin{aligned}
&\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x}{y} \\
&\Rightarrow y\, dy = x\, dx \\
&\Rightarrow\ y^2 - x^2 = \text{const.}
\end{aligned}
\tag{7}
$$

- The asymptotes of phase space trajectory are lines $y = \pm x$.
- We can see the flow by simply making a **StreamPlot**

```
StreamPlot[{y, x}, {x, -1, 1}, {y, -1, 1}];
```

Linear Systems

A *two-dimensional linear system* has the specific form:

$$\begin{aligned} \dot{x} &= a\,x + b\,y \\ \dot{y} &= c\,x + d\,y \end{aligned} \tag{8}$$

where a, b, c, d are paramters of the system. A compact vector notation for this system of coupled differential equations is:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x}, \tag{9}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{10}$$

Question:

Which of the following systems is linear?

(a) $\dot{x} = -x,\ \ \dot{y} = -y$

(b) $\dot{x} = -y,\ \ \dot{y} = -x$

(c) $\dot{x} = y,\ \ \dot{y} = -\sin(x)$

(d) $\dot{x} = y,\ \ \dot{y} = -a\,x - y$

(e) $m\,\ddot{x} + k\,x = 0.$

General Solution

For the very special *two-dimensional linear system* given by

$$\begin{aligned} \dot{x} &= a\,x \\ \dot{y} &= d\,y \end{aligned} \tag{11}$$

the solution is obviously

$$x = e^{a\,t}$$
$$y = e^{d\,t}. \tag{12}$$

So we ask whether the general solution can be of the form

$$\boldsymbol{x} = \boldsymbol{e}^{\lambda\,t}\,\boldsymbol{v} \tag{13}$$

where $\boldsymbol{v}$ is some time-independent special vector to be determined, and λ is a special scalar to be determined. Plugging this solution in, we have

$$\boldsymbol{A}\,\boldsymbol{v} = \lambda\,\boldsymbol{v}\,, \tag{14}$$

which is an eigenvalue equation. The eigenvalues of the two-dimensional matrix are conveniently expressed in terms of the trace and the determinant of the matrix as:

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\,\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\,\Delta}}{2} \tag{15}$$

where

$$\tau = \mathrm{Tr}(A) = a + d$$
$$\Delta = \det(A) = a\,d - b\,c. \tag{16}$$

Suppose $\boldsymbol{v_1}, \boldsymbol{v_2}$are respectively the relevant eigenvectors. In general $\lambda_1 \neq \lambda_2$and this means that the two eigenvectors are linearly independent, and therefore *span* the space. This means that any initial condition $\boldsymbol{x_0}$can be written in the form

$$\boldsymbol{x_0} = c_1\,\boldsymbol{v_1} + c_2\,\boldsymbol{v_2}\ . \tag{17}$$

Once this is done, the general solution is immediately written down as

$$\boldsymbol{x(t)} = c_1\,\boldsymbol{e}^{\lambda_1\,t}\,\boldsymbol{v_1} + c_2\,\boldsymbol{e}^{\lambda_2\,t}\,\boldsymbol{v_2} \tag{18}$$

is the general solution. Since it is a linear system, an arbitrary linear combination of the two linearly independent solutions is guaranteed to be a solution. And there is a uniqueness theorem which says that if you can find *one* solution, then that *is* the solution. So, we have actually found the full general solution!

Example:

Consider the following linear system:

$$\dot{x} = x + y, \quad \dot{y} = 4\,x - 2\,y$$

subject to the initial condition $(x_{0,}\, y_0) = (2, -3)$.

Let us look at the phase portrait of this system:

```
StreamPlot[{x + y, 4 x - 2 y}, {x, -4, 4}, {y, -4, 4}];
```

The matrix here is

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}. \tag{19}$$

We have

$$\begin{aligned} \tau &= \mathrm{Tr}(A) = -1 \\ \Delta &= \det(A) = -6. \end{aligned} \tag{20}$$

Therefore the eigenvalues are

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\,\Delta}}{2} = 2, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\,\Delta}}{2} = -3. \tag{21}$$

The corresponding eigenvectors are

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\; \boldsymbol{v_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{22}$$

The general solution is:

$$\boldsymbol{x(t)} = c_1\, \boldsymbol{e}^{2\,t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, \boldsymbol{e}^{-3\,t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}. \tag{23}$$

Imposing the intial conditions, we can find the constants

$$c_1 = 1, \qquad c_2 = 1. \tag{24}$$

So the solution is

$$\begin{aligned} x(t) &= e^{2t} + e^{-3t} \\ y(t) &= e^{2t} - 4e^{-3t} \end{aligned} \tag{25}$$

Love Affairs!

Now let us study a model that is due to Strogatz (1998) which is analyzed in some detail in his book “Nonlinear Dynamics and Chaos”.

Romeo and Juliet are in a relationship. Let $R(t)$ be a measure of the love that Romeo feels for Juliet at time t and let $J(t)$ be a measure of the love Juliet feels towards Juliet. The greater the magnitude, the greater is the intensity of the feeling. However, the sign of the quantity determines whether it is love or hate. If it is positive then, it is love, and if it is negative it is hate. We model the dynamics of their relationship as a coupled linear system with fixed rate constants that determine the evolution of their feelings towards each other depending on the amount of love/hate each of them has for the other at that instant of time. One instance of this story has the following sytem:

$$\dot{R} = J, \quad \dot{J} = -2\,R.$$

Romeo is a simple man. He loves Juliet and if she shows any kind of reciprocation, his love for her increases. On the other hand if Juliet stays away, Romeo also prefers to keep his distance. But Juliet on the other hand, has a more complicated tendency. Whenever Romeo expresses his love for her, she has a tendency to run away and hide. But on the other hand, if he withdraws and keeps his distance, Juliet’s feelings for him are suddenly awakened. What can we say about their love affair, given our knowledge of linear systems?! Let us start by making a phase portrait:

```
StreamPlot[{y, -2 x}, {x, -2, 2}, {y, -2, 2}];
```

In this case, their love affair is akin to a harmonic oscillator! A never-ending cycle of love and hate. There is one quadrant where both R and J are positive, so atleast one-fourth of their cyclical dynamics they manage to both love each other.

The most general linear model of such a love affair is given by

$$\dot{R} = a\,R + b\,J, \quad \dot{J} = c\,R + d\,J,$$

where the magnitudes and (very importantly) the signs of the parameters $a_{,} b$, c, d determine the nature of the affair. We will return to this general problem after we have described the general theory of linear systems.