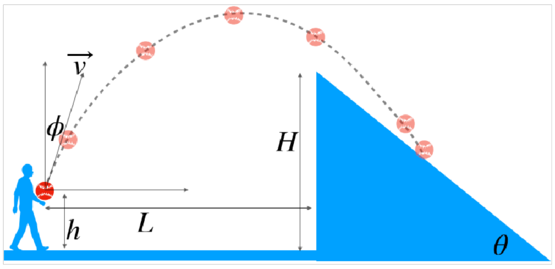
Practice Problems: Week 2 (Submission not required)

1. Consider the situation demonstrated in the image below. A man standing a distance L away from the cliff throws a ball with speed v at an angle ϕ with the vertical. Height of the cliff is H and its slope makes an angle θ with the horizontal.



After laying down a suitable coordinate system

(a) Find the equation of the trajectory of the ball.

(b) Find the equation of the slope of the hill.

(c) Non-dimensionalize both the equations. How many free parameters are left in the problem?

(d) Construct a Manipulate command by varying all the free parameters in a suitable range. Play with parameters to find out in what cases you have ball land somewhere on the slope. What is the condition for which ball will hit the edge of the cliff (top of the cliff).

(e) Validate your observations in part **(d)** by doing algebraic calculations.

2. Consider the following ways of writing the solution of the simple harmonic oscillator.

$$\begin{aligned} x(t) &= A\cos(\omega t) + B\sin(\omega t) \\ x(t) &= C\,e^{i\omega t} + D\,e^{-i\omega t} \\ x(t) &= F\cos(\omega t + \phi) \\ x(t) &= G\sin(\omega t + \psi) \end{aligned} \tag{1}$$

Find $C,\ D,\ F,\ G,\ \phi$ and ψ in terms of A and B.

3. (a) Show that the solution of the exact pendulum equation

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{2}$$

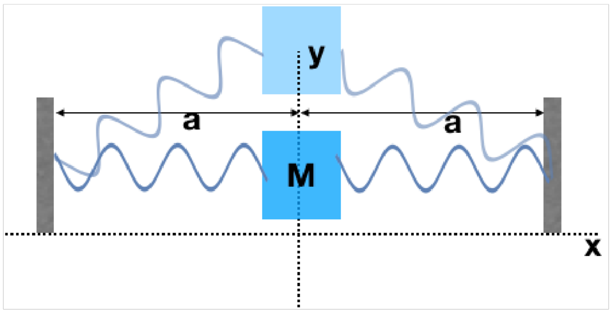
can be written in the form of the following indefinite integral :

$$t = \int d\theta\, \frac{1}{\sqrt{A - \omega^2\cos\theta}} \tag{3}$$

where A is an integration constant. The other integration constant will emerge from performing the indefinite integral, thus it is hidden in the integral. Interpret A, that is, what is physical meaning of the constant A in this expression.

(b) Derive the solution of the form of eqn. (2) by making small angle approximation in eqn. (4).

4. Transverse Oscillations in Spring System: For the spring mass system shown below, where the motion of the mass is constraint along y-axis, show that for the small oscillations the system is a simple harmonic oscillator. Take the spring constant of each spring to be k and natural length of each spring to be a_0.



(a) Show that the potential for this system is given by

$$V(y) = k\, a_0^2 \left(\sqrt{\frac{a^2}{a_0^2} + \frac{y^2}{a_0^2}} - 1 \right)^2$$

(b) Rewrite this potential after non-dimensionalization: $\frac{V}{k\, a_0^2} = \mathcal{V}$ and $\frac{y}{a_0} = Y$, to obtain

$$\mathcal{V}(Y) = \left(\sqrt{r^2 + Y^2} - 1 \right)^2$$

where $r = a / a_0$, a dimensionless number. Now make a plot $\mathcal{V}(y)$ for $\frac{1}{2} \leq r \leq 2$. You may Manipulate r to see the effect of changing r on the shape of the potential. For what value of r you have stable minima at $Y = 0$? What happens at $r = 1$? Is the potential quadratic for $r = 1$? What about $r > 1$ and $r < 1$?

(c) Obtain the equation of motion for this system.

(d) Find the frequency ω_y and the time period T_y for small oscillation in y-direction assuming $r > 1$.

(e) When the motion of the mass is constrained along the x-axis, we obtained $\omega_x = \sqrt{\frac{2\,k}{m}}$ (as calculated in one of the videos this week).

Show that $\frac{\omega_y}{\omega_x} = \sqrt{1 - \frac{1}{r}}$ for $r > 1$.

5. Fibonacci Sequence: Let f_n represent the n^{th} Fibonacci number, where $f_0 = 0$ and $f_1 = 1$.

(a) Write a For loop to calculate sequence $\frac{f_{n+1}}{f_n}$ for $n = 1$ to $n = 100$. Does this sequence converge? If yes to what value?

(b) Write a For loop to produce the sequence f_n for for $n = -100$ to $n = 100$. Note that f_n are well-defined negative n, they simply satisfy the equation

$$f_{n+1} = f_n + f_{n-1}$$
$$\Rightarrow\ f_{n-1} = f_{n+1} - f_n$$

where you can use the first equation to generate positive n sequence and the second equation to generate negative n sequence, starting from f_0 and f_1.

(c) What is the ratio $\frac{f_{-(n+1)}}{f_{-n}}$ in the large n limit (large positive n). How does it compare with $\frac{f_{n+1}}{f_n}$ in the large n limit (large positive n).

(d) Mathematica has a built-in function to calculate Fibonacci numbers. Use **Fibonacci[n]** in Mathematica to get the n^{th} Fibonacci number. Compare your results using this function. You can repeat parts **(a)** to **(c)** with Table and Fibonacci functions to see if you get the same result.

(e) Interestingly, **Fibonacci[n]** is continuous function. Make a Plot of this from $n = -10$ to $+10$. Using **ListPlot** and **Show**, overlay the discrete Fibonacci sequence on this continuous plot. What do you find?