Physics through Computational Thinking

Linear superposition of oscillations

Auditya Sharma and Ambar Jain
Dept. of Physics, IISER Bhopal

Outline

In this lecture you will

1. study the superposition of two SHMs with same frequencies.
2. learn about how beat phenomena occur when a superposition of two SHMs of slightly different frequencies are involved.
3. Lissajous figures. How to visualize vibratory motion along perpendicular directions.

The Superposition Principle

When a system is subject to *linear* forces, the superposition principle holds. An arbitrary linear superposition of different solutions is itself a solution. A familiar example of a linear ODE is the simple harmonic oscillator

$$\ddot{x} + \omega^2 x = 0. \tag{1}$$

If each of $x_1$and $x_2$are solutions of the above differential equation, then any arbitrary combintion

$$x = A\, x_1 + B\, x_2 \tag{2}$$

is also a solution of the differential equation. As we have already seen, one way of writing the most general solution is:

$$x(t) \;=\; A\cos(\omega\, t) + B\sin(\omega\, t) \tag{3}$$

where each of $\cos(\omega\, t)$ and $\sin(\omega\, t)$ are independently the solutions of the ODE. As a passing remark, we also observe that the general solution has two free constants: the same number as the order of the differential equation. This is a general feature.

Another way of writing the general solution is (notice again the presence of two constants):

$$x(t) \;=\; A\cos(\omega\, t + \phi). \tag{4}$$

Suppose we superpose two such signals, both with the same frequency:

$$\begin{aligned} x_1(t) \;&=\; A_1\cos(\omega\, t + \phi_1) \\ x_2(t) \;&=\; A_2\cos(\omega\, t + \phi_2) \end{aligned} \tag{5}$$

to create the signal:

$$x(t) \;=\; x_1(t) + x_2(t). \tag{6}$$

Let us play with a set of random parameters to see how such a superposition would pan out.

```
A₁ = 1;
ω = 1;
A₂ = 3;
ϕ₁ = RandomReal[] π;
ϕ₂ = RandomReal[] π;
f[t_] = A₁ Cos[ω t + ϕ₁];
g[t_] = +A₂ Cos[ω t + ϕ₂];
h[t_] = f[t] + g[t];
Plot[{f[t], g[t], h[t]}, {t, 0, 10 π}, PlotLegends -> "Expressions"];
```

What we observe is that the resultant also seems to be sinusoidal, and most importantly it is a signal with the *same* frequency ω! So the resulting signal must also be expressible in terms of the canonical general form, and this is indeed true.

Homework: Show that $x(t) = A\cos(\omega t + \phi)$, where $A^2 = {A_1}^2 + {A_2}^2 + 2 A_1 A_2 \cos(\phi_2 - \phi_1)$ and $\phi = \phi_1 + \beta$ where β is calculated from the expression $A\sin(\beta) = A_2 \sin(\phi_2 - \phi_1)$.

As a quick check, we can look at the simple particular case when the amplitudes of both the signals are the same. Then we have

$$\begin{aligned} x(t) &= A_1[\cos(\omega t + \phi_1) + \cos(\omega t + \phi_2)] \\ &= 2 A_1\left[\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\cos\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right)\right] \end{aligned} \tag{7}$$

implying that

$$\begin{aligned} A &= 2 A_1 \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \\ \phi(t) &= \frac{\phi_1 + \phi_2}{2}. \end{aligned} \tag{8}$$

Superposition of two signals of different frequencies

The natural continuation would be to try to superpose two signals of different frequencies:

$$\begin{aligned} x_1(t) &= A_1 \cos(\omega_1 t) \\ x_2(t) &= A_2 \cos(\omega_2 t) \end{aligned} \tag{9}$$

The resultant signal of such a superposition may not even be periodic if the frequencies $\omega_1$and ω_2 are not *commensurate*. If the resultant signal is periodic with period T, then we must simultaneously have

$$\begin{aligned} \omega_1 T = 2 n_1 \pi, \\ \omega_2 T = 2 n_2 \pi. \end{aligned} \tag{10}$$

where n_1, n_2 are integers. This is possible if and only if

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} \tag{11}$$

is a rational fraction.

We plot some functions to explore this.

```
Clear["Global`*"]

A₁ = 1;
ω₁ = 1;
ω₂ = Sqrt[2];
A₂ = 3;
f[t_] = A₁ Cos[ω₁ t];
g[t_] = A₂ Cos[ω₂ t];
h[t_] = f[t] + g[t];
Plot[{h[t]}, {t, 0, 14 π}, PlotLegends -> "Expressions"];
```

```
A1 = 1;
ω1 = 1;
ω2 = 2;
A2 = 3;
ϕ1 = 0;
ϕ2 = RandomReal[] π;
f[t_] = A1 Cos[ω1 t + ϕ1];
g1[t_] = A2 Cos[ω2 t];
g2[t_] = A2 Cos[ω2 t + ϕ2];
h1[t_] = f[t] + g1[t];
h2[t_] = f[t] + g2[t];
Plot[{h1[t], h2[t]}, {t, 0, 10 π}, PlotLegends -> "Expressions"];
```

Beats

If the two frequencies that are superposed are close to each other, we can get beat phenomena. For simplicity let us assume that the amplitudes of the two signals are identical. Then we are superposing

$$\begin{aligned} x(t) &= A\,[\cos(\omega_1\, t) + \cos(\omega_2\, t)] \\ &= 2\,A\left[\cos\left(\frac{(\omega_1 + \omega_2)\, t}{2}\right)\cos\left(\frac{(\omega_1 - \omega_2)\, t}{2}\right)\right] \end{aligned} \tag{12}$$

```
Clear["Global`*"]

A1 = 1;
ω1 = 2 π / 600;
ω2 = 2 π / 700;
A2 = 1;
f[t_] = A1 Cos[ω1 t];
g[t_] = A2 Cos[ω2 t];
h[t_] = f[t] + g[t];
Plot[{f[t], g[t]}, {t, 0, 4000}, PlotLegends -> "Expressions"];
envfunc[t_] = 2 A1 Cos[(ω1 - ω2) t/2];
Plot[{envfunc[t], h[t]}, {t, 0, 10 000}, PlotLegends -> "Expressions"];
```

Note 1: Although superposition of such signals would be possible for any pair of frequencies, beats would be heard only if the two frequencies are relatively close to each other. Beats would be heard for example when two tuning forks with frequencies very close to each other are simultaneous struck. The ear is able to perceive beats in a scenario where the very large frequency modulation of the amplitude $\frac{\omega_1 + \omega_2}{2}$ can be skipped.

Note 2: Naively one might think that the beat frequency should be $\frac{\omega_1 - \omega_2}{2}$. However a closer inspection of the superposed plot reveals that in fact there are two envelopes each of frequency $\frac{\omega_1 - \omega_2}{2}$, with a perfect phase shift as to halve the time period, or double the perceived frequency. It will be seen that the peak-to-peak change happens with a frequency of $\omega_1 - \omega_2$, which is the beat frequency that is perceived by the ear.

Superposition of two signals along perpendicular directions

Suppose we superpose two harmonic signals operating in perpendicular directions. We would now have to analyze the motion in two dimensions. The general scenario would be:

$$\begin{aligned} x(t) &= A_1 \cos(\omega_1 t + \phi_1) \\ y(t) &= A_2 \cos(\omega_2 t + \phi_2) \end{aligned} \tag{13}$$

Since the extreme values that x can take are $+A_1$ and $-A_1$ and the extreme values of y are $+A_2$ and $-A_2$, we can be sure that the motion here would be confined within a rectangle of width $2\,A_1$ and height $2\,A_2$ centred at the origin. A fascinating wide variety of motions is possible by playing with the parameters.

```
Clear["Global`*"]

A₁ = 1;
ω₁ = (1 + RandomReal[]) π;
ω₂ = (1 + RandomReal[]) π;
A₂ = 1;
ϕ₁ = (1 + RandomReal[]) π;
ϕ₂ = (1 + RandomReal[]) π;
x[t_] = A₁ Cos[ω₁ t + ϕ₁];
y[t_] = A₂ Cos[ω₂ t + ϕ₂];



ParametricPlot[{ x[t], y[t]}, {t, 0, 10}, AxesLabel → {"x", "y"}];
```

Since we have made the frequencies and the phases all random, in general the motion along x and the y directions are incommensurate. Therefore, there is no chance for periodic motion. The trajectory would be some very complicated one, but if the evolution is carried for sufficiently long, it is going to fill up the entire rectangle for all practical purposes. This is related to a deep concept in statistical mechanics called *ergodicity*. In statistical mechanics, one is often interested in the kind of dynamics that can explore the whole of phase space. Here y would have the status of velocity, and the trajectories in the X-Y plane would then be called phase portraits. Ergodicity means that the fundamental postulates of equilibrium statistical mechanics holds, and therefore the laws of statistical mechanics hold. In the absence of ergodicity, the laws of statistical mechanics are violated. These questions are still part of an active field of research.

Equal Frequencies.

Let us play with a different set of parameters. Suppose we set the two frequencies to be the same. What do we see?

```
Clear["Global`*"]

A1 = 1;
ω1 = π;
ω2 = π;
A2 = 1;
ϕ1 = (1 + RandomReal[]) π;
ϕ2 = (1 + RandomReal[]) π;
x[t_] = A1 Cos[ω1 t + ϕ1];
y[t_] = A2 Cos[ω2 t + ϕ2];


ParametricPlot[{ x[t], y[t]}, {t, 0, 1000}, AxesLabel → {"x", "y"}];
```

Analysis

If we put the two frequencies to be the same ($\omega_1 = \omega_2 = \omega$), the motion is guaranteed to be periodic. This is true because when we add two signals with identical frequency, the resultant signal must also have exactly the same frequency. Without loss of generality we can put the phase $\phi_1 = 0$. So the phase of the second signal is the same as the difference in the phases $\delta = \phi_2 - \phi_1 = \phi_2$. The two signals we have are

$$\begin{aligned} x(t) &= A_1 \cos(\omega\, t) \\ y(t) &= A_2 \cos(\omega\, t + \delta) \end{aligned} \tag{14}$$

There are some special cases which will yield immediate analytical results. If we take $\delta = \frac{\pi}{2}$, then it is straightforward to eliminate time and write down the equation of the trajectory:

$$\frac{x^2(t)}{A_1{}^2} + \frac{y^2(t)}{A_2{}^2} = 1, \tag{15}$$

the equation of the ellipspe! So it is not really a surprise that we see so many ellipses. As we change δ, all this does is to change the orientation of the major and minor axes of the ellipse. The circle is a special ellipse when the major and minor axes are both of the same size.

Lissajous Figures.

If we make the frequencies commensurate, periodic motion would result even if the phases are kept completely random.

```
Clear["Global`*"]

A₁ = 1;
ω₁ = π;
ω₂ = π ;
A₂ = 1;
ϕ₁ = (1 + RandomReal[]) π;
ϕ₂ = (1 + RandomReal[]) π;
x[t_] = A₁ Cos[ω₁ t + ϕ₁];
y[t_] = A₂ Cos[ω₂ t + ϕ₂];


ParametricPlot[{ x[t], y[t]}, {t, 0, 1000}, AxesLabel → {"x", "y"}];
```

Homework: Play with the parameters to generate Lissajous figures that are:
a) Straight line.
b) Circle.
c) Ellipse of a desired orientation.
d) More complicated shapes, but still periodic.
e) Aperiodic motion.