Physics through Computational Thinking

Solutions to Computer Graded Assignment: Week 6

1. Consider the following differential equations:

- a) $\frac{d^2 x}{d t^2} + \frac{d x}{d t} + x = t$
- b) $\frac{d^2 x}{d t^2} + \frac{d x}{d t} + x = \cos(t)$
- c) $\frac{d^2 x}{d t^2} + \frac{d x}{d t} + \cos(x) = 0$
- d) $\frac{d x}{d t} + x^2 = 0$

Which of the above is nonlinear?

☐ a alone.

☐ a and b.

☐ a and c.

☑ c and d.

■ Solution

The equations c) and d) have the nonlinear terms involving x, namely: $\cos(x)$ and x^2. So they are nonlinear.

2. We saw that the steady solution of the differential equation:

$$\frac{d^2 x}{d t^2} = -x + \cos(\omega t)$$

tends to mimic the forcing term. What happens to the steady state solution if the drive is of the form $\cos(\omega t + \phi)$. Assume a steady-state solution of the form: $x_{ss} = C \cos(\omega t + \phi_{ss})$. Which of the following is true?

☑ $C(\omega) = \frac{1}{1-\omega^2}$, $\phi_{ss} = \phi$.

☐ C is a function of both ω and ϕ, and $\phi_{ss} = \phi$.

☐ $C(\omega) = \frac{1}{1-\omega^2}$, and $\phi_{ss} = 0$.

☐ $C(\omega, \phi)$ can be found out only numerically, and $\phi_{ss} = \phi$.

■ Solution

The given differential equation is

$$\frac{d^2 x}{dt^2} = -x + \mathrm{Cos}(\omega t)$$

The solution of the above equation has the form

$$x_{ss} = C\, \mathrm{Cos}(\omega t + \phi_{ss})$$

Substituting this solution to the differential equation results in

$C(\omega) = \frac{1}{1-\omega^2}$, $\phi_{ss} = \phi$

3. Suppose we are solving the differential equation with a sinusoidal drive given by:

$$\frac{d^2 x}{d t^2} = -x + \sin(\omega t).$$

Find the general solution of this differential equations. Now, plug in the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$, and determine the free constants. Using a proper limiting procedure, find the solution at resonance, by taking the limit $\omega \to 1$. The resonant solution is:

☐ $x(t) = \sin(t)$

☐ $x(t) = \frac{1}{2}\sin(t) + \frac{1}{2} t\cos(t)$

☐ $x(t) = \frac{1}{2}\sin(t) + \frac{1}{2} t\cos(t) + t\sin(t)$

☑ $x(t) = \frac{3}{2}\sin(t) - \frac{1}{2} t\cos(t)$

■ Solution

The solution for the differential equation is

$x = \frac{(1-\omega-\omega^2)\sin(t)+\sin(\omega t)}{1-\omega^2}$.

Taking the limit as $\omega \to 1$ results in

In[1]:=
```
Limit[((1 - ω - ω^2) Sin[t] + Sin[ω t])/(1 - ω^2), {ω → 1}]
```

Out[1]=
```
{1/2 (-t Cos[t] + 3 Sin[t])}
```

4. Suppose we are solving the differential equation with a sinusoidal drive given by:

$$\frac{d^2 x}{d t^2} = -x + \sin(\omega t) + \cos(\omega t),$$

with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$. Suppose we run the system at $\omega = 0.8$. Tweak the eulerGen code appropriately to evaluate the position of the particle at times $t = 10$, and $t = 40$. Take nMax = 20000.

☐ x(10)=1.23454, x(40)=3.49856

☐ x(10)=4.56781, x(40)=8.98382

☐ x(10)=2.34987, x(40)=1.24567

☑ $x(10) = 5.34553$, $x(40) = 4.84318$

■ Solution

```
w4 = 0.8;
id[t_, distance_, velocity_] = 1;
distanceDot[t_, distance_, velocity_] = velocity;
velocityDot4[t_, distance_, velocity_] = -distance + Sin[w4 * t] + Cos[w4 * t];
initial4 = {0, 0, 1};
data4 = eulerGen[{id, distanceDot, velocityDot4},
   initial4, 10, 20 000];
(*Change t=10, t=40 here for different times*)
ListPlot[data4⟦ ;; , 1 ;; 2⟧, Joined → True, PlotMarkers → None, PlotRange → Full];
Last[data4⟦ ;; , 1 ;; 2⟧];
```

5. Suppose we are solving the differential equation with a sinusoidal drive given by:

$\frac{d^2 x}{d t^2} = -x + \sin(\omega t) + \cos(\omega t)$,

with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$. Suppose we run the system at $\omega = 0.8$. Tweak the eulerImp code appropriately to evaluate the position of the particle at times $t = 10$, and $t = 40$. Take nMax = 20 000.

□ x(10)=1.23454, x(40)=3.49856

□ x(10)=4.56781, x(40)=8.98382

□ x(10)=2.34987, x(40)=1.24567

☑ $x(10) = 5.33972$, $x(40) = 4.79101$

■ Solution

```
w5 = 0.8;
id[t_, distance_, velocity_] = 1;
distanceDot[t_, distance_, velocity_] = velocity;
velocityDot5[t_, distance_, velocity_] = -distance + Sin[w5 * t] + Cos[w5 * t];
initial5 = {0, 0, 1};
data5 = eulerImp[{id, distanceDot, velocityDot5},
   initial5, 10, 20 000];
(*Change t=10, t=40 here for different times*)
ListPlot[data5⟦ ;; , 1 ;; 2⟧, Joined → True, PlotMarkers → None, PlotRange → Full];
Last[data5⟦ ;; , 1 ;; 2⟧];
```

6. Suppose we look at a falling body that is simultaneously subject to a drag force that is proportional to the speed, and that is quadratic in the speed. We would have a differential equation of the form:

$$m \frac{d^2 y}{d t^2} = m g - k_1 \frac{d y}{d t} - k_2 \left(\frac{d y}{d t}\right)^2.$$

Nondimensionalize this differential equation. How many free parameters are left after non-dimensionalization?

□ 0

☑ 1

□ 2

□ 3

■ Solution

The given differential equation is

$$\mathrm{m}\,\frac{d^2 y}{\mathrm{dt}^2} = \mathrm{mg} - k_1\,\frac{\mathrm{dy}}{\mathrm{dt}} - k_2\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^2$$

The non dimensionalizing scales are $t \to \frac{m}{k_1}\,t$, $\;y \to \frac{m^2\,g}{k_1{}^2}\,y$.

There is only one free parameter in the system i.e. $\gamma = \frac{k_2\,m^2\,g^2}{k_1{}^2}$.

7. Consider a falling body which is subjected to a drag force that is proportional to the fourth power of the speed. We have the differential equation:

$$m\,\frac{d^2\,y}{d\,t^2} = m\,g - k\left(\frac{d\,y}{d\,t}\right)^4.$$

Nondimensionalize this differential equation using the time scale: $\left(\frac{m}{k\,g^3}\right)^{1/4}$ and using the distance scale: $\sqrt{\frac{m}{k\,g}}$. If the $y(t = 0) = 0$, and the body is at rest at time $t = 0$, use the rk4 routine with nMax = 300, to find the distance covered by the particle at time $t = 4$.

☐ 2.34

☑ 3.43

☐ 5.78

☐ 6.32

■ Solution

The given differential equation of harmonic oscillator is

$$m\,\frac{d^2\,y}{\mathrm{dt}^2} = \mathrm{mg} - k\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^4$$

The given differential equation after non − dimenionalising it with time scale $\left(\frac{m}{\mathrm{kg}^3}\right)^{\frac{1}{4}}$ and length scale $\sqrt{\frac{m}{\mathrm{kg}}}$ is

$$\frac{d^2\,y}{\mathrm{dt}^2} = 1 - \left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^4.$$

The initial conditions are $y(0) = \dot{y}(0) = 0$.
The position of the body at time $t = 4$ using Rk4 code is $x(4) =$ **3.43**.

```
rateFunc7[{t_, x_, v_}] = {1, v, 1 - v^4};
initial7 = {0, 0, 0};
data7 = Rk4[rateFunc7, initial7, 4, 300];
ListPlot[data7[[ ;; , 1 ;; 2]], Joined → True, PlotMarkers → None, PlotRange → Full];
Last[data7[[ ;; , 1 ;; 2]]]
```

8. Consider the problem of a projectile that is hurled away from the surface of the Earth. After nondimensionalization, the differential equation is:

$$\frac{d^2 x}{d t^2} = \frac{-1}{x^2}.$$

The projectile starts on the surface of the Earth initially, so $x(t = 0) = 1$. It is shot out at a speed that is double the escape velocity. Use the rk4 routine with nMan = 300, to find the location of the projectile after time $t = 1$.

☐ 1.75

☐ 2.83

☑ 3.64

☐ 9.15

■ Solution

The given differential equation is $\frac{d^2 x}{dt^2} = \frac{-1}{x^2}$, along with initial conditions $x(0) = 1$, $\dot{x}(0) = 2\sqrt{2}$ (twice the escape velocity).
The position of projectile for above differential equation using Rk4 code after time t=1 with nMax=300 is x(1)=3.64.

```
rateFunc8[{t_, x_, v_}] = {1, v, -1/x^2};
initial8 = {0, 1, 2 Sqrt[2]};
data8 = Rk4[rateFunc8, initial8, 1, 300];
ListPlot[data8[[ ;; , 1 ;; 2]], Joined → True, PlotMarkers → None, PlotRange → Full];
Last[data8[[ ;; , 1 ;; 2]]];
```

9. Consider the problem of an oscillator that is subjected to an exponentially decaying external force:

$$m \frac{d^2 x}{d t^2} + k x = F_0 e^{-\alpha t}.$$

Nondimensionalize this equation using the time scale $\frac{1}{\alpha}$ and length scale $\frac{F_0}{\alpha^2 m}$. Introduce the nondimensional parameter $\gamma = \frac{k}{\alpha^2 m}$. The initial conditions are $x(t = 0) = 0$, and $\dot{x}(t = 0) = 0$. If we focus on the case $\gamma = 1$, find $x\left(t = \frac{\pi}{2}\right)$.

☐ $\sqrt{2}$

☐ $\frac{1-e^{-\frac{\pi}{2}}}{2}$

☑ $\frac{e^{-\frac{\pi}{2}}+1}{2}$

☐ $\frac{3}{2}$

■ Solution

The solution for this problem can be worked out analytically (watch video!). The answer is:

$$x(t) = \frac{1}{2}\left(\sin t - \cos t + e^{-t}\right),$$

from which the value at $t = \frac{\pi}{2}$ immediately follows.

10. A harmonic oscillator is subjected to an external force that goes as the fourth power of time. After non-dimensionalization, the differential equation is:

$$\frac{d^2 x}{d t^2} + x = t^4.$$

Use the rk4 routine with nMax = 300, to evaluate $x(t = 10)$ if the initial conditions are $x(t = 0) = 0$, and $\dot{x}(t = 0) = 0$.

☑ 8844

☐ 7645

☐ 1463

☐ 356

■ Solution

The non-dimensionalized differential equation is $\frac{d^2 x}{\mathrm{dt}^2} = -x + t^4$, along with initial conditions $x(0) = \dot{x}(0) = 0$.
The position of harmonic oscillator for above differential equation using Rk4 code at time t=10 with nMax=300 is x(1)=8844.

```
rateFunc10[{t_, x_, v_}] = {1, v, -x + t^4};
initial10 = {0, 0, 0};
data10 = Rk4[rateFunc10, initial, 10, 300];
ListPlot[data10[[ ;; , 1 ;; 2]], Joined → True, PlotMarkers → None, PlotRange → Full];
Last[data10[[ ;; , 1 ;; 2]]];
```