

Physics through Computational Thinking

Visual Thinking and Non-dimensionalization

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Outline

In this lecture you will

1. learn to translate physics problems to represent visually after suitably non-dimensionalizing the equations

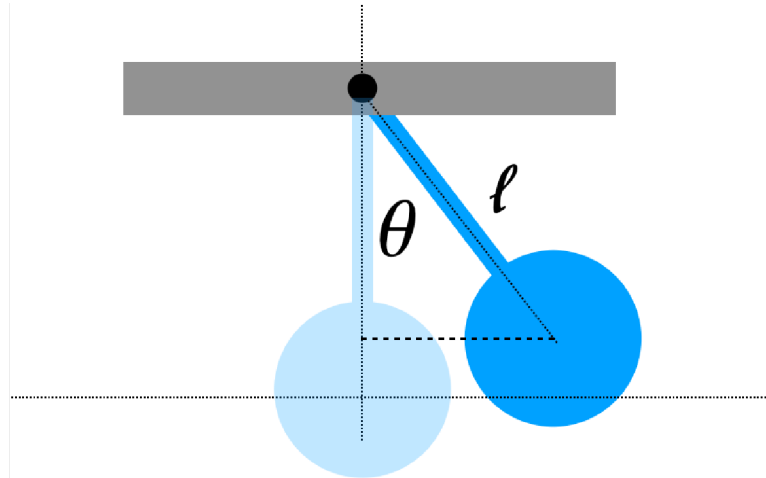
2. apply skills of visual thinking to solve a physics problems

3. apply skills of visual thinking to interpret results from graphs

(non-)Dimensional Analysis and Visualization - The Simple Pendulum

“The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction -- Sidney Coleman”

Example 1: Consider the pendulum of mass m shown in the figure. Find the potential and plot it as a function of θ .



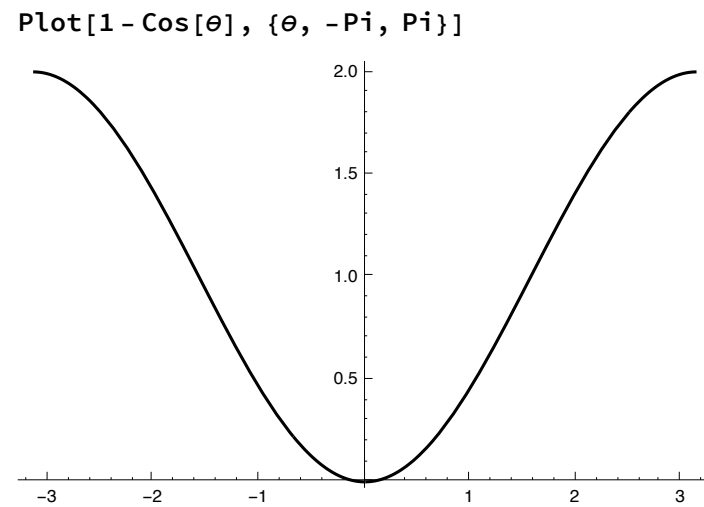
Solution: [Define] The potential is given by

$$V(\theta) = m g \ell (1 - \cos \theta) \quad (1)$$

In order to plot this as a function of θ , we need to convert it into **abstract mathematical form** by removing physical dimensions from the problem. Its easy! We measure the potential in units of the natural potential scale present in the problem that is $m g \ell$. Thus we can re-write the potential as [Translate]

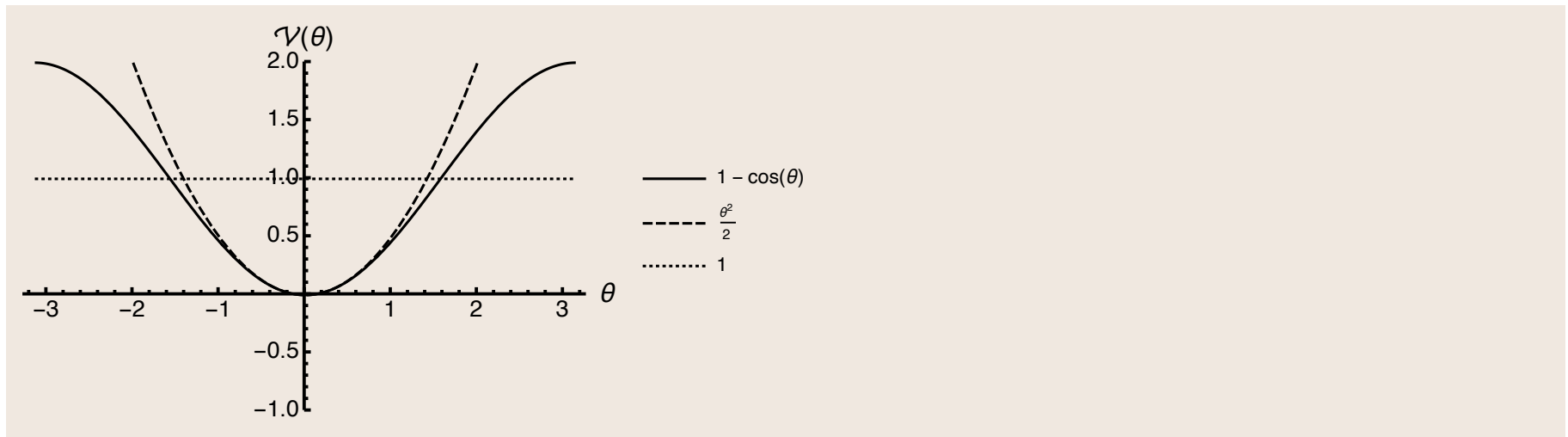
$$\mathcal{V}(\theta) \equiv \frac{V(\theta)}{m g \ell} = 1 - \cos \theta \quad (2)$$

Now the resulting right hand side is dimensionless and depends only on dimensionless variable θ . Thus we plot our dimensionless potential $\mathcal{V}(\theta)$ with respect to θ [Compute]



- Let me make my plot prettier by labelling the axes and put some styling. I will leave it as an **Homework** exercise for you to figure out how this is done. Just some tinkering with the code below will help you understand that. **[Compute]**

```
Plot[{{1 - Cos[θ],  $\frac{\theta^2}{2}$ , 1}, {θ, -Pi, Pi}, PlotRange → {-1, 2}, AxesLabel → {Style[θ, 16], Style[V[θ], 16]},  
AxesStyle → Thick, TicksStyle → Directive["Label", 14], PlotLegends → "Expressions"]
```



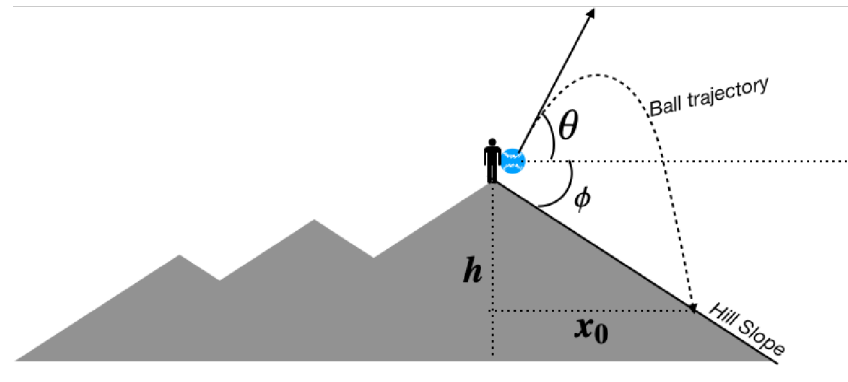
- We note that the potential near $\theta \approx 0$ behaves like a quadratic function with a stable minima, thus we expect the pendulum to behave like Simple Harmonic Oscillator for small oscillations.

Answer the following questions by analyzing the plot **[Interpret]**

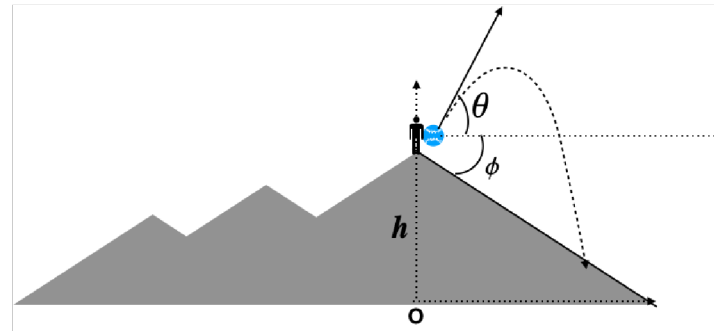
- **Question:** What does $V(\theta) = 1$ represent? What physical value of the potential does it correspond to? What is the angle of maximum deflection θ_{\max} for $V = 1$.
- **Question:** By eyeballing the picture above estimate the maximum energy of the pendulum for which it may still qualify for a simple harmonic oscillator.
- **Question:** By eyeballing the picture above estimate the maximum deflection angle θ_{\max} for which the pendulum may still qualify for a simple harmonic oscillator.

(non-)Dimensional Analysis and Visualization - 2

Example 2: Consider the following problem from your course on Mechanics. A man throws a ball from the top of the hill of height h and angle of inclination ϕ , as shown in the figure below. Lets assume that the ball is thrown at an angle θ with speed v . Ignore the height of the man compared to the hill. Find the distance x_0 where the ball hits the hill slope for the first time. For what values of θ , ϕ , v and h there is a solution? Under what condition there is no solution? Explore the problem visually using graphics.



Define: First we need to lay out a coordinate system.



1. We identify the trajectory of the ball as a quadratic function from the information that projectiles follow parabolic trajectories or more appropriately that most general trajectory of a particle under constant acceleration is parabolic. Thus, we can write:

$$y_{\text{ball}}(x) = a x^2 + b x + c$$

(3)

2. Equation of the hill is represented by a straight line

$$y_{\text{hill}}(x) = \alpha x + \beta \quad (4)$$

3. We need to determine coefficients, a , b , c , α and β by using some known data points. After that intersection of these two lines give me a solution.

Translate: Now we will extract appropriate information out of the problem and so that we can compute these coefficients and find second point of intersection. We will express this information in mathematical form:

(i) Position: $y_{\text{ball}}(0) = h$

(ii) Velocity: $\left. \frac{dy_{\text{ball}}}{dt} \right|_{x=0} = v \sin \theta$

(iii) Acceleration: $\left. \frac{d^2 y_{\text{ball}}}{dt^2} \right|_{x=0} = -g$

(iv) Hill Top: $y_{\text{hill}}(0) = h$

(v) Hill Angle: $\frac{dy_{\text{hill}}(0)}{dx} = -\tan \phi$

Compute: Using the five pieces of information we can determine the coefficients a , b , c , α and β .

$$\begin{aligned} \text{(i)} & \Rightarrow c = h \\ \text{(ii)} & \Rightarrow b = \tan \theta \\ \text{(iii)} & \Rightarrow a = \frac{-g}{2 v^2 \cos^2 \theta} \\ \text{(iv)} & \Rightarrow \beta = h \\ \text{(v)} & \Rightarrow \alpha = -\tan \phi \end{aligned} \quad (5)$$

The equations now become

$$y_{\text{ball}}(x) = \frac{-g}{2 v^2 \cos^2 \theta} x^2 + \tan \theta x + h \quad (6)$$

$$y_{\text{hill}}(x) = -\tan \phi x + h \quad (7)$$

At this point you can solve for intersection of two curves and find where they intersect next. We will explore this problem by visualization and and cross-check with analytical result.

h is the only length scale in the problem, so we will non-dimensionalize y and x in units of h so that we can make a plot. This gives

$$\begin{aligned}\frac{y_{\text{ball}}(x)}{h} &= \frac{-g h}{2 v^2 \cos^2 \theta} \left(\frac{x}{h}\right)^2 + \tan \theta \frac{x}{h} + 1 \\ \Rightarrow Y_{\text{ball}}(X) &= \frac{-\gamma}{\cos^2 \theta} X^2 + \tan \theta X + 1\end{aligned}\quad (8)$$

where Y and X are dimensionless coordinates measured in units of h , and $\gamma = \frac{g h}{2 v^2}$. For the hill, we get

$$\begin{aligned}\frac{y_{\text{hill}}(x)}{h} &= -\tan \phi \frac{x}{h} + 1 \\ \Rightarrow Y_{\text{hill}}(X) &= -\tan \phi X + 1\end{aligned}\quad (9)$$

To summarize, the two abstract equations that we are dealing with now are

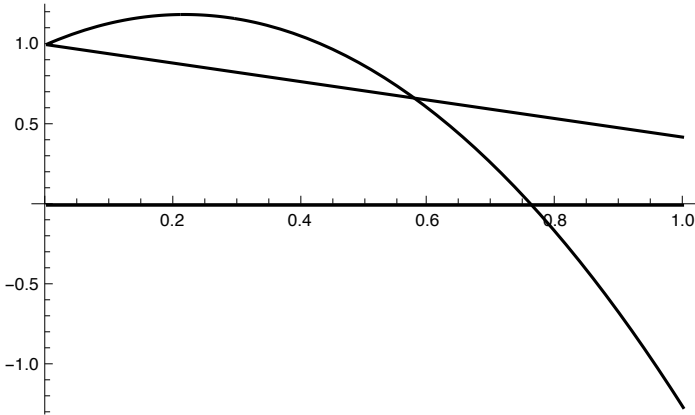
$$\begin{aligned}Y_{\text{ball}}(X) &= \frac{-\gamma}{\cos^2 \theta} X^2 + \tan \theta X + 1 \\ Y_{\text{hill}}(X) &= -\tan \phi X + 1\end{aligned}\quad (10)$$

which have got three independent dimensionless parameters: γ , θ and ϕ . Few things to note here:

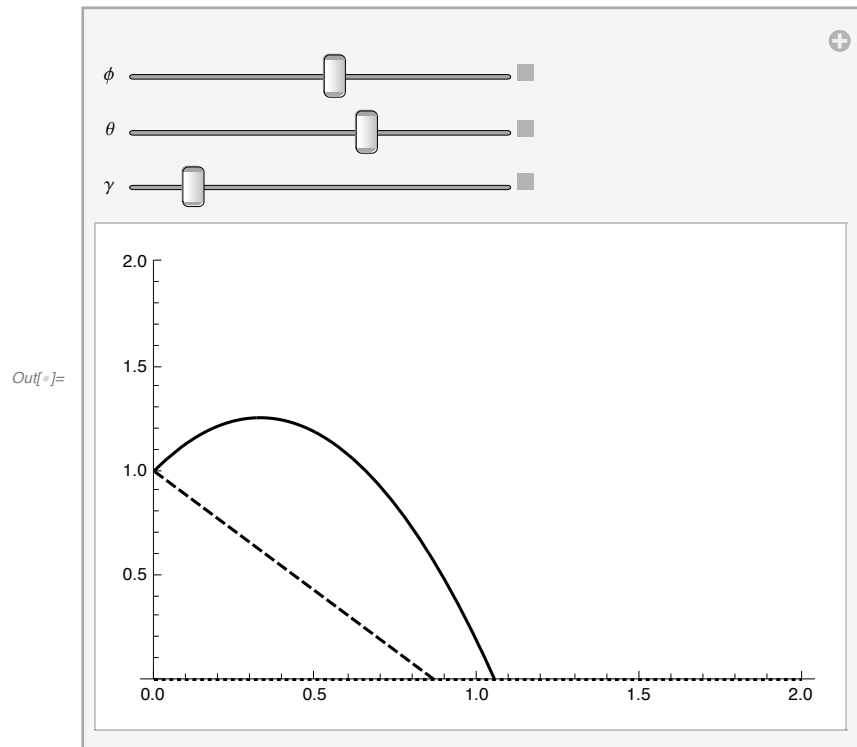
- γ is a measure of inverse of speed in units of $\sqrt{g h}$.
- All the dimensionless parameters that define the problem for physically relevant values are typically order 1 numbers. Its always easy to analyze a problem in terms of order 1 numbers.
- The effects created by changing v and h are not independent. γ can be changed either by changing v or h .

Interpret: We will now plot and interpret our results:

$$\text{Plot}\left[\left\{\frac{-\gamma}{\cos[\theta]^2}x^2 + \tan[\theta]x + 1, -\tan[\phi]x + 1, 0\right\} /. \left\{\phi \rightarrow \frac{\text{Pi}}{6}, \theta \rightarrow \frac{\text{Pi}}{3}, \gamma \rightarrow 1\right\}, \{x, 0, 1\}\right]$$



`In[]:= Manipulate[Plot[{ $\frac{-\gamma}{\cos[\theta]^2} x^2 + \tan[\theta] x + 1$, $-\tan[\phi] x + 1$, 0}, {x, 0, 2}, PlotRange -> {0, 2}], {{ ϕ , $\frac{\text{Pi}}{6}$ }, 0, $\frac{\text{Pi}}{2}$ }, {{ θ , $\frac{\text{Pi}}{3}$ }, 0, $\frac{\text{Pi}}{2}$ }, {{ γ , 1}, $\frac{1}{2}$, 2}]`



Homework:

- Find X_0 and x_0 for a given θ , ϕ and γ .
- For a fixed θ and ϕ , Find the critical value of γ below which there is no solution. Verify that the plot above agrees with your result.

Parametric plot for trajectory

Example (Problem adapted from Kleppner & Kolenkow): A bead moves outward with constant speed u along the spoke of a wheel. It starts from the center at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is constant. Find the trajectory of the particle and plot it.

Solution: Velocity and Acceleration in the polar coordinates is given by

$$\begin{aligned}\vec{v} &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \\ \vec{a} &= (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\theta}\end{aligned}\quad (11)$$

From the problem we identify

$$\begin{aligned}\dot{r} &= u \\ \dot{\theta} &= \omega \\ r &= u t \\ \theta &= \omega t\end{aligned}\quad (12)$$

For trajectory, we have

$$\begin{aligned}x &= r \cos(\theta) = u t \cos(\omega t) \\ y &= r \sin(\theta) = u t \sin(\omega t)\end{aligned}\quad (13)$$

We non-dimensionalize time and distance by noting that $\frac{1}{\omega}$ is constant with dimensions of time while u/ω is a constant with dimensions of distance. Thus we can define,

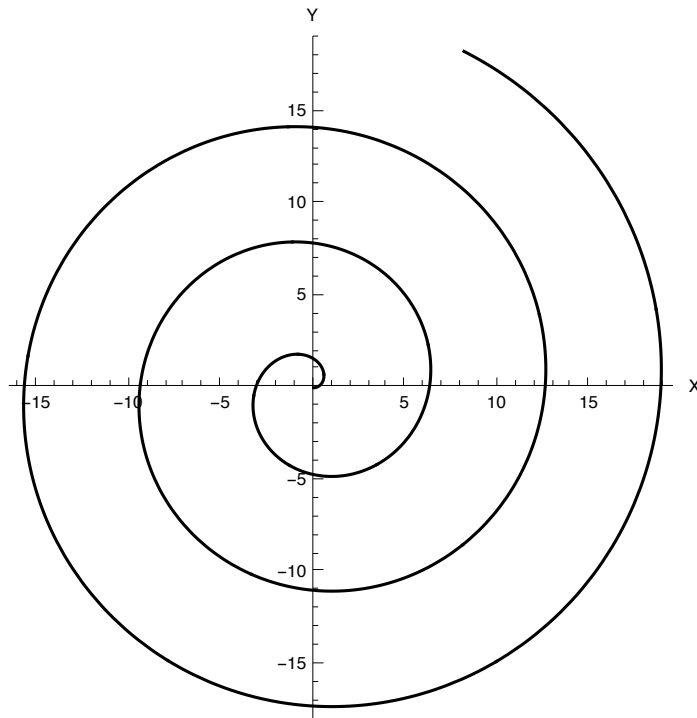
$$\begin{aligned}X &= \frac{\omega}{u} x \\ Y &= \frac{\omega}{u} y \\ T &= \omega t\end{aligned}\quad (14)$$

Therefore, in terms of dimensionless variables

$$\begin{aligned}X &= T \cos(T) \\ Y &= T \sin(T)\end{aligned}\quad (15)$$

We plot the trajectory using **ParametricPlot** function:

```
ParametricPlot[{T Cos[T], T Sin[T]}, {T, 0, 20}, AxesLabel -> {"X", "Y"}]
```



After non-dimensionalization, the equation for trajectory became scaleless and we got a unique solution.

Question: Interpret what is the effect of changing ω and u on this trajectory? This one solution in the plot contains all the solutions corresponding to various values of u and ω . Can you explain how?