Computer Graded Assignment: Week 2

1. Let f_n represent Fibonacci numbers with $f_0 = 0$ and $f_1 = 1$. The ratio $\frac{f_{-n}}{f_n}$ for large n approaches

□ a negative constant

□ a positive constant

□ an oscillatory function of n with period π

☑ an oscillatory function of n with period 2

Solution: You can use the built-in Fibonacci[n] as explained in the practice problem. The plot shows an oscillatory function. Plotting $\sin\left(\pi n - \frac{\pi}{2}\right)$ alongside confirms that $\frac{f_{-n}}{f_n}$ converges $\sin\left(\pi n - \frac{\pi}{2}\right)$ in large n limit, which has the period 2.

In[∘]:=

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Plot[{Fibonacci[-n]/Fibonacci[n], Sin[π n - π/2]}, {n, 1, 20}]
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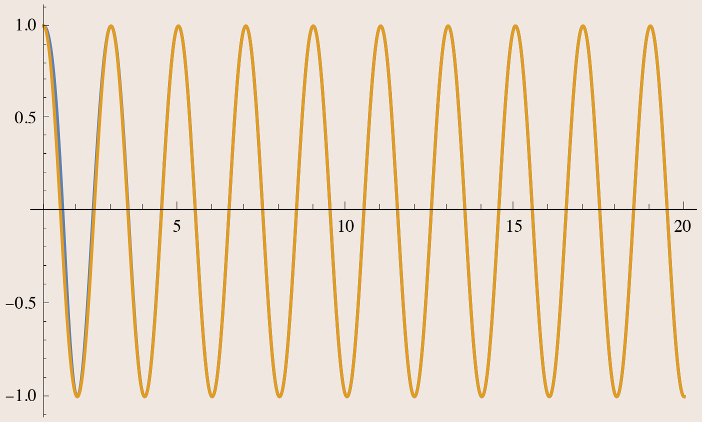
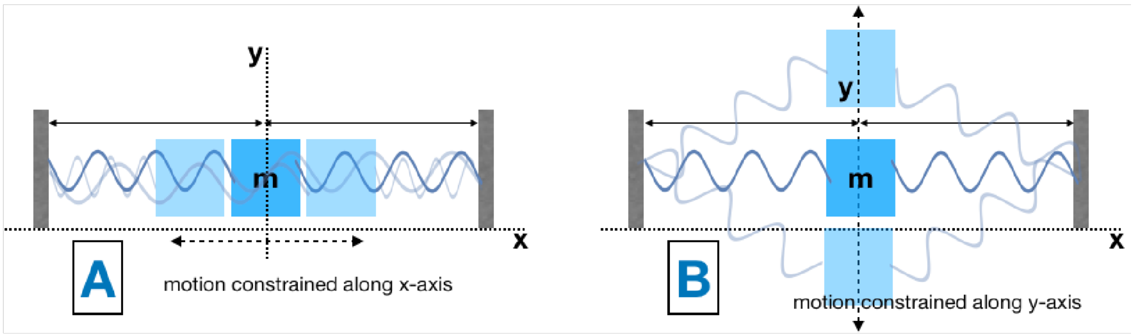
Out[∘]=



2. For the spring mass systems shown below, both the systems have mean position of the block of mass m in the center and is connected by ideal springs of spring constant k and ideal length $a_0$on each side stretched to length $2\,a_0$ at mean position. System A is constrained to oscillate horizontally while system B is constrained to oscillate vertically.



Ignoring effect of gravity, if frequency of small oscillation for system A is ω_A and in system B is ω_B, then the ratio ω_A/ω_B is ________ .

☑ **Answer Range: 1.40 to 1.42 .**

Solution: For small oscillations, equation of motion for system A is

$$\ddot{x} = -\frac{2\,k}{m}\,x$$

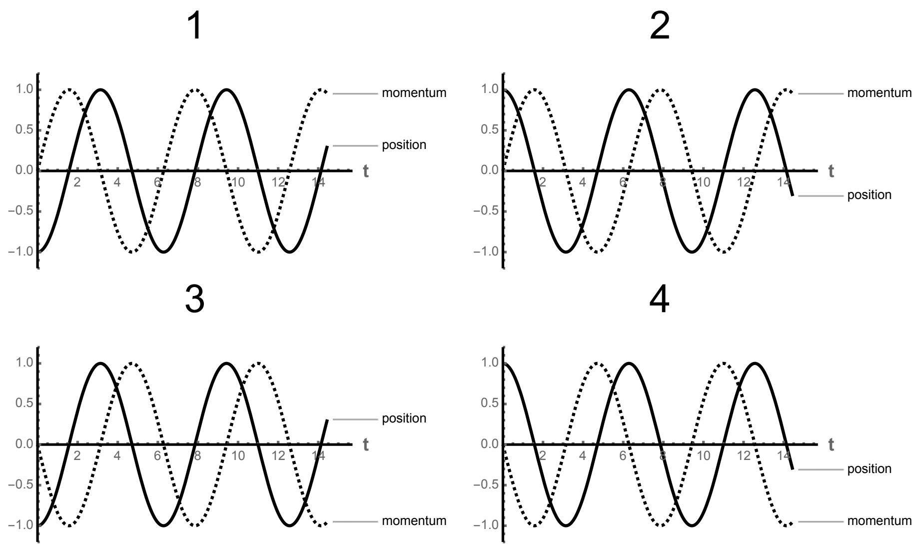
For small oscillations, equation of motion for system B is

$$\ddot{y} = -\frac{k}{m} y$$

Thus we have

$$\frac{\omega_A^2}{\omega_B^2} = \frac{\frac{2k}{m}}{\frac{k}{m}} = 2 \qquad \Rightarrow \quad \frac{\omega_A}{\omega_B} = \sqrt{2} \simeq 1.414$$

3. For a simple harmonic oscillator, which of the following set of curves represent the position and momentum as a function of time when at $t = 0$ the oscillator was at extremum position.



☐ 2 and 3

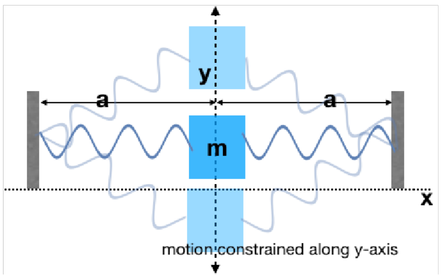
☑ 1 and 4

☐ 1 only

☐ 2 only

Solution: Initial condition tells us that position should be at a maxima or a minima at $t = 0$, therefore momentum should vanish at $t = 0$. Furthermore, momentum should be derivative of position. This leaves 1 and 4 as the only viable solutions.

4. Consider the oscillator shown below made up of block of mass m and ideal springs of natural length a_0 and spring constant k. Oscillator is constrained to move along the y–axis as shown in the figure.



Small y expansion of the potential for $r = \frac{a}{a_0} > 1$ has the form:

$$V(y) = V_0 + \alpha\, y^2 + \beta\, y^4 + \ldots$$

where α and β are constants. The ratio α / β is

□ $2\, a_0^2 (r-1)^2\, r$

☑ $4\, a_0^2 (r-1)\, r^2$

□ $4\, a_0^2\, \frac{(r-1)}{r}$

□ $2\, a_0^2\, \frac{r^2}{r-1}$

Solution: The potential for the system is given by

$$V(y) = k\left(\sqrt{a^2 + y^2} - a_0\right)^2 = k\left(\sqrt{a^2 + y^2} - a_0\right)^2$$

You have to make Taylor expansion and then take the ratio of coefficients. You can do this manually or use Mathematica like shown below (something new for you to learn):

In[∘]:=
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series = Series[k (Sqrt[a^2 + y^2] - a0)^2, {y, 0, 4}]
```

Out[∘]=

$$\left(\sqrt{a^2} - a0\right)^2 k + \frac{\left(\sqrt{a^2} - a0\right) k\, y^2}{\sqrt{a^2}} + \frac{\sqrt{a^2}\, a0\, k\, y^4}{4\, a^4} + O[y]^5$$

Reading of Series Coefficients we have

In[∘]:=
```
SeriesCoefficient[series, 2]
```

Out[∘]=

$$\frac{\left(\sqrt{a^2} - a0\right) k}{\sqrt{a^2}}$$

In[∘]:=
```
SeriesCoefficient[series, 4]
```

Out[∘]=

$$\frac{\sqrt{a^2}\, a0\, k}{4\, a^4}$$

Taking the ratio

In[∘]:=
```
Simplify[SeriesCoefficient[series, 2] / SeriesCoefficient[series, 4] /. a → r a0, Assumptions → {r > 1, a0 > 0}]
```

Out[∘]=

$$4\, a0^2\, (-1 + r)\, r^2$$

5. The sequence x_n that obeys the equation $x_{n+1} = \frac{1}{2} x_n + 2 \log(x_n)$ converges to (given $x_0 > 2$)

□ 4.53640

☑ 8.61317

□ 16.9989

□ 26.0935

Solution: Lets write a For loop to do this. Note we use decimal points in the body so that we get floating point numbers and not symbolic expressions. Try various initial values for $x_0 > 2$. You will get the same result.

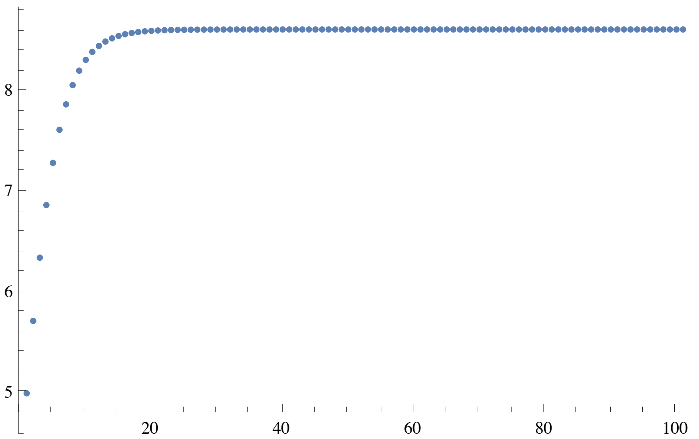
In[∘]:=

```
For[xlist = {5}, Length[xlist] ≤ 100,
 AppendTo[xlist, next], next = 1/2.0 Last[xlist] + 2.0 Log[Last[xlist]]]
Last[xlist]
ListPlot[xlist, PlotRange → Full]
```

Out[∘]=

```
8.61317
```

Out[∘]=



6. The anharmonicity in the potential of a simple pendulum is of the order of

□ θ

□ θ^2

□ θ^3

☑ θ^4

Solution: Potential of the simple pendulum is $m\,g\,\ell\,(1 - \cos(\theta)) = m\,g\,\ell\left(\frac{\theta^2}{2} + \frac{\theta^4}{24} + \ldots\right)$. Therefore anharmonicity is order θ^4.

7. For the Lenard-Jones type potential between two particles separated by a distance r, given by

$$U(r) = \frac{\alpha}{r^6} - \frac{\beta}{r^3}$$

where α and β are positive constants, a suitable length scale and energy scale in the problem is represented by

☑ $E = \frac{\beta^2}{\alpha}$ and $\ell = \left(\frac{\alpha}{\beta}\right)^{1/3}$

□ $E = \frac{\beta^4}{\alpha^2}$ and $\ell = \left(\frac{\beta}{\alpha}\right)^{1/3}$

□ $E = \left(\frac{\beta^2}{\alpha}\right)^{1/3}$ and $\ell = \left(\frac{\beta}{\alpha}\right)^{1/3}$

□ $E = \frac{\beta^4}{\alpha^2}$ and $\ell = \frac{\alpha^2}{\beta}$

Solution: Dimension of α is same as energy$\times$length^6 and dimension of β is energy$\times$length^3. Using this we conclude $\frac{\alpha}{\beta} \sim$ length^3 and $\frac{\beta^2}{\alpha} \sim$ energy.

8. For a particle of mass μ and angular momentum ℓ in a central force potential, the potential is given by

$$U(r) = \alpha\, r^6$$

where α is a positive constant. If the particle is in a circular orbit, the time period of the oscillation is of the order of

- ☐ $\frac{\mu^{1/2}\,\alpha^{1/4}}{\ell^{1/2}}$
- ☐ $\frac{\mu^{1/2}}{\ell^{1/2}\,\alpha^{1/2}}$
- ☑ $\frac{\mu^{3/4}}{\ell^{1/2}\,\alpha^{1/4}}$
- ☐ $\frac{\mu^{3/4}\,\alpha^{1/4}}{\ell^{1/2}}$

Solution: Use dimensional analysis. $\mu \sim M$. $\ell \sim M\,L^2\,T^{-1}$ and $\alpha \sim \frac{M\,L^2\,T^{-2}}{L^6} = M\,L^{-4}\,T^{-2}$. Therefore T has dimensions of $\frac{\mu^{3/4}}{\ell^{1/2}\,\alpha^{1/4}}$.

9. Find the sum of the series give below to 6 significant digits

$$s = \sum_{n=1}^{\infty} \frac{1}{n^{2\log n}}$$

s = _____________ .

☑ **Answer Range: 1.50138 to 1.50158**

Solution: Sum the series to a finite number of terms and look for convergence. This series actually converges pretty fast. By the time we reach nmax = 20 first 6 digits are fixed.

In[∘]:=

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Manipulate[Total[Table[1/n^(2 Log[n]) // N, {n, 1, nmax}]], {nmax, 2, 40, 2}]
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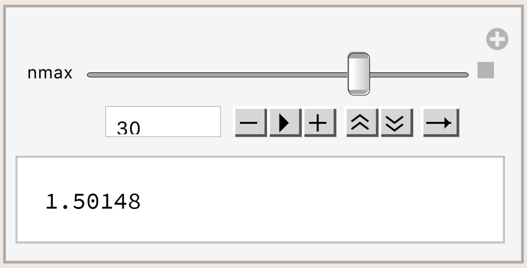
Out[∘]=



10. For Fibonacci numbers f_n, the ratio

$$r = \frac{f_n}{f_{n+1}}$$

up to six significant digits converges to

☑ **Answer Range: 0.61802 to 0.61804**

Solution:

In[∘]:=

```
Manipulate[Fibonacci[n]/Fibonacci[n + 1] // N, {{n, 20}, 1, 200}]
```

Out[∘]=