Physics through Computational Thinking

Linearization

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Outline

In this module we discuss

1. Linearization. Examples of when it works, and when it fails.

General Theory of 2-d Linear Systems

The general problem is

$$\begin{aligned}\dot{x} &= a\,x + b\,y \\ \dot{y} &= c\,x + d\,y\end{aligned} \tag{1}$$

and

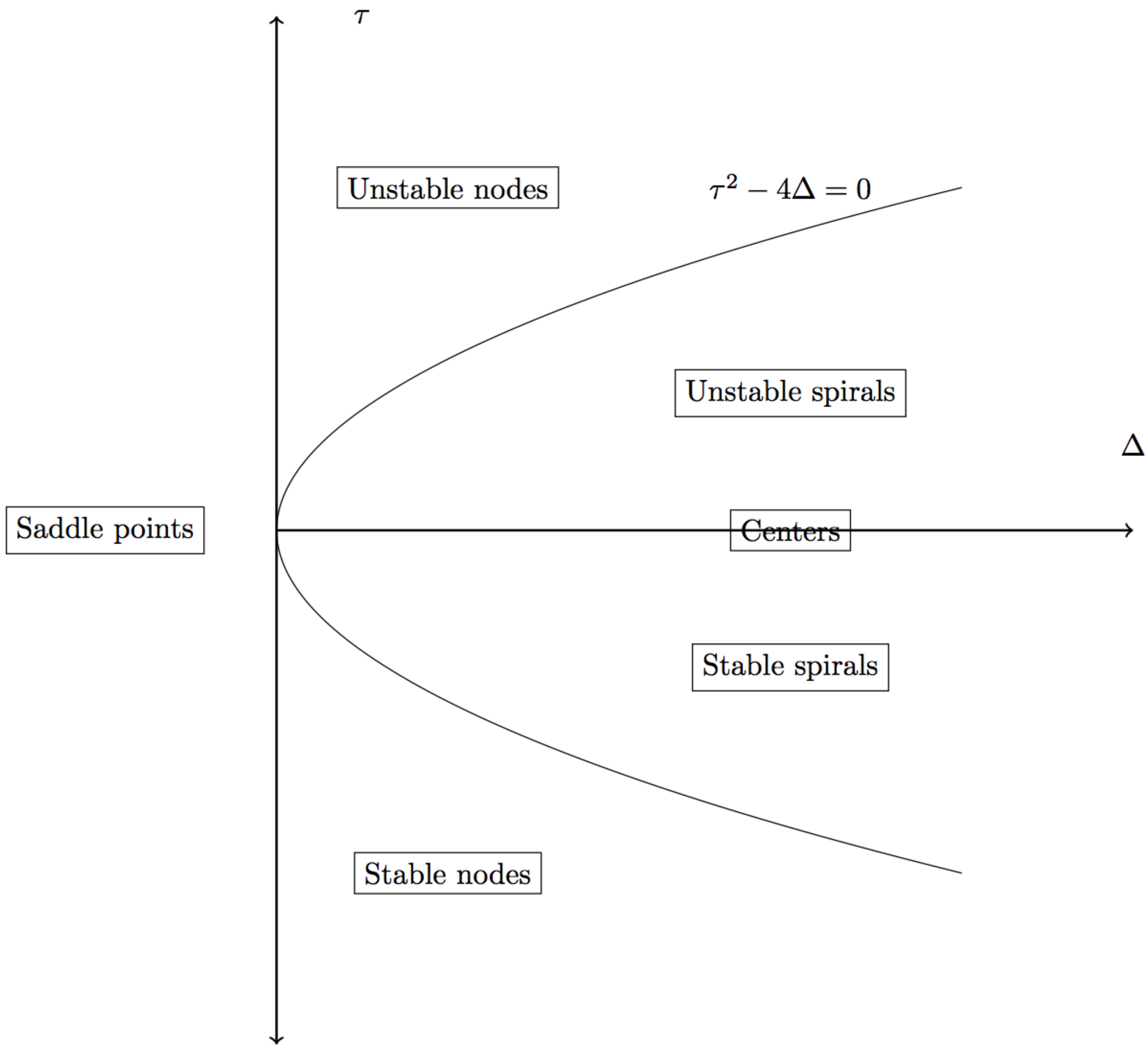
$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\,\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\,\Delta}}{2}. \tag{2}$$

The inverse relation is

$$\tau = \lambda_1 + \lambda_2 \qquad \Delta = \lambda_1\,\lambda_2\,. \tag{3}$$

We make the following observations:

- If $\Delta < 0$, then both the eigenvalues *have to* be real, and with opposite signs. Hence the fixed point is guranteed to be a saddle point.
- If $\Delta > 0$, then we have a range of different possibilities, depending on the value of τ. If $\tau^2 > 4\,\Delta$, then the eigenvalues are real, and therefore the fixed point is a *node*. On the other hand, if $\tau^2 < 4\,\Delta$, then the eigenvalues are complex (conjugates of each other), and the fixed point then becomes either a center or a spiral.



τ
Unstable nodes
$\tau^2 - 4\Delta = 0$
Unstable spirals
Δ
Saddle points
Centers
Stable spirals
Stable nodes

A Nonlinear system: Linearization

Consider the system

$$\begin{aligned}\dot{x} &= x + e^{-y}\\ \dot{y} &= -y\end{aligned} \tag{4}$$

To find the fixed points of this system, we must simultaneously put

$$\begin{aligned}x + e^{-y} &= 0\\ -y &= 0\end{aligned} \tag{5}$$

which gives the unique solution $(-1, 0)$. For this problem, qualitative arguments already provide a lot of information. We see that the second equation is decoupled, and its solution is $y(t) = e^{-t}$. Therefore as $t \to \infty$, $y(t) \to 0$. So in the limit of large times the first differential equation becomes $\dot{x} = x + 1$, which would give diverging $x$so it must be unstable atleast along one direction. Linearization confirms our intuition. We can write down the Jacobian at (-1,0) as

$$J = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \tag{6}$$

which has a Δ = -1, and thus predicts a saddle. This is indeed confirmed by a direct study of the phase portrait:

```
StreamPlot[{x + Exp[-y], -y}, {x, -4, 4}, {y, -4, 4}];
```

A Nonlinear system where Linearization fails

Consider the system

$$\dot{x} = -y + a\,x\left(x^2 + y^2\right)$$
$$\dot{y} = x \;+\; a\,y\left(x^2 + y^2\right) \tag{7}$$

To find the fixed points of this system, we must simultaneously put

$$-y + a\,x\left(x^2 + y^2\right) = \;0$$
$$x \;+\; a\,y\left(x^2 + y^2\right) \;=\; 0 \tag{8}$$

which gives the origin (0, 0) as the fixed point. The Jacobian at the origin is found to be:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

which has $\tau = 0$, and $\Delta = 1$, so the origin is always a center independent of the value of a, according to linearization. Let us check if this is reasonable by plotting the phase space diagram

```
a = -1;
StreamPlot[{-y + a x (x^2 + y^2), x + a y (x^2 + y^2)}, {x, -4, 4}, {y, -4, 4}];

a = 1;
StreamPlot[{-y + a x (x^2 + y^2), x + a y (x^2 + y^2)}, {x, -4, 4}, {y, -4, 4}];

a = 0;
StreamPlot[{-y + a x (x^2 + y^2), x + a y (x^2 + y^2)}, {x, -4, 4}, {y, -4, 4}];
```

In fact, by choosing a to be negative, positive or zero, we can get either a stable spiral or an unstable spiral or a center respectively! In this case, linearization has failed us. For this specific problem, it turns out that an exact solution is possible if we move to polar coordinates.

$$x = \;r\cos(\theta)$$
$$y \;=\; r\sin(\theta) \tag{10}$$

Since

$$x^2 + y^2 = \;r^2 \tag{11}$$

$$x\,\dot{x} + y\,\dot{y} = r\,\dot{r} \tag{12}$$

So

$$\begin{aligned} r\,\dot{r} &= x\left(-y + a\,x\left(x^2 + y^2\right)\right) + y\left(x + a\,y\left(x^2 + y^2\right)\right) \\ &= a\left(x^2 + y^2\right)^2 = a\,r^4 \end{aligned} \tag{13}$$

It can also be shown that

$$\dot{\theta} = \frac{x\,\dot{y} - y\,\dot{x}}{r^2} \tag{14}$$

which in the end yields the relations

$$\begin{aligned} \dot{r} &= a\,r^3 \\ \dot{\theta} &= 1 \end{aligned} \tag{15}$$

Now it is obvious that depending on whether a is negative, positive or zero, the dynamics is going to be a stable spiral, unstable spiral or neutral center dynamics.

Moral

If linearization predicts one of the borderline cases, this is called marginal, and may not actually be applicable. However if linearization predicts a non-boderline case like a source, sink or saddle, then this is robust even for the nonlinear model!