Physics through Computational Thinking

Random walks

Auditya Sharma and Ambar Jain
Dept. of Physics, IISER Bhopal

Outline

In this module we

1. introduce the one dimensional random walk problem.

Random Walk

Consider a drunkard whose motion is confined to the X axis. For simplicity, let us assume that after every unit of time, he moves one step either to the right or to the left with probabilities p and q respectively. If he starts at the origin, the question is how far is his typical distance after N units of time have elapased? We can address the more general question. What is the probability that after N steps, the drunkard is at the coordinate m? Let us call this probability $P_N(m)$. If we define $n_1$to be the number of steps taken to the right, and n_2 to be the number of steps taken to the left, then we have the relations:

$$\begin{gathered} N = n_1 + n_2 \\ m = n_1 - n_2 \end{gathered} \tag{1}$$

Suppose we assume that the drunkard has zero memory and that every step is completely independent of the previous step and is only characterized by the probabilities p and q, then we can go ahead and solve this problem analytically.

The number of ways in which N steps can be composed of n_1 right steps and n_2 left steps is given by

$$\binom{N}{n_1} \tag{2}$$

For each of these possibilities, the probability is simply given by

$$p^{n_1} q^{n_2}. \tag{3}$$

Therefore the overall probability of finding the random walker at position m after N steps is given by

$$P_N(m) = \binom{N}{n_1} p^{n_1} q^{n_2} \tag{4}$$

Since

$$\begin{gathered} n_1 = \frac{N+m}{2} \\ n_2 = \frac{N-m}{2} \end{gathered} \tag{5}$$

we can rewrite the final solution as

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\, p^{\left(\frac{N+m}{2}\right)}\, q^{\left(\frac{N-m}{2}\right)} \tag{6}$$

Homework

Is the probability $P_N(m)$ normalized? How can we check this?

Mean and variance

We have the condition

$$p + q = 1. \tag{7}$$

Invoking the binomial theorem, we have

$$Z = (p+q)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1}\, q^{N-n_1} = \sum_{n_1=0}^{N} P_N(n_1) \tag{8}$$

The mean number of steps taken to the right is given by

$$\begin{aligned} < n_1 > &= \sum_{n_1=0}^{N} n_1\, P_N(n_1) \\ &= p\, \frac{\partial Z}{\partial p} \end{aligned} \tag{9}$$

Therefore the mean number of steps taken to the right is given by

$$< n_1 > = N\, p \tag{10}$$

Taking the second partial derivative with respect to p we have

$$\frac{\partial^2 Z}{\partial p^2} = \sum_{n_1=0}^{N} n_1 (n_1 - 1)\, \frac{N!}{n_1!\,(N-n_1)!}\, p^{n_1-2}\, q^{N-n_1} \tag{11}$$

Therefore

$$\begin{aligned} p^2\, \frac{\partial^2 Z}{\partial p^2} &= \sum_{n_1=0}^{N} {n_1}^2\, P_N(n_1) - \sum_{n_1=0}^{N} n_1\, P_N(n_1) \\ &= < {n_1}^2 > - < n_1 > \end{aligned} \tag{12}$$

This in turn yields (after putting $p+q=1$)

$$N(N-1)\,p^2 = \ <n_1{}^2> - N\,p \tag{13}$$

Therefore

$$<n_1{}^2> \ = N(N-1)\,p^2 + N\,p \tag{14}$$

The variance is a useful quantity to study:

$$\sigma^2 = \ <n_1{}^2> - <n_1>^2 = N(N-1)\,p^2 + N\,p - N^2\,p^2 \tag{15}$$

which after some manipulation can be shown to be:

$$\sigma^2 = \ N\,p\,q. \tag{16}$$

Mean and variance of net displacement

We have the condition

$$N = n_1 + n_2$$
$$m = n_1 - n_2 \tag{17}$$

from which we have

$$m = 2\,n_1 - N \tag{18}$$

Therefore

$$\begin{aligned} < m > \; &= \; 2 < n_1 > - N \\ &= \; 2\,N\left(p - \frac{1}{2}\right) \end{aligned} \tag{19}$$

Also the variance is now given by

$$\begin{aligned} < m^2 > - < m >^2 &= \; < (2\,n_1 - N)^2 > - < 2\,n_1 - N >^2 \\ &= \left(4 < n_1{}^2 > -4\,N \; < n_1 > +N^2\right) - \left(4 < n_1 >^2 -4\,N < n_1 > +N^2\right) \\ &= 4\left(< n_1{}^2 > - < n_1 >^2\right) \end{aligned} \tag{20}$$

Therefore

$$< m^2 > - < m >^2 = \; 4\,N\,p\,q \tag{21}$$

Special Case: The unbiased random walk

The unbiased random walk when $p = \; q \; = \; \frac{1}{2}$, and where the drunkard is equally likely to move to the right or to the left deserves special attention.

The mean and variance in displacement after N steps is now

$$\begin{aligned} < m > \; &= 2\,N\left(p - \frac{1}{2}\right) \; = \; 0 \\ < m^2 > - < m >^2 &= \; 4\,N\,p\,q \; = \; N. \end{aligned} \tag{22}$$

Equivalently

$$< m^2 > \, = N, \tag{23}$$

which is an important result. Physically what it means is that although the random walker takes N steps the *typical* displacement is only of O($\sqrt{N}$). This fact finds application in a variety of fields ranging from error-analysis to the stock market to polymer physics to Brownian motion.

The probability distribution for the unbiased walk is

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\left(\frac{1}{2}\right)^N \tag{24}$$

In the limit of large N, n_1 and n_2, it is reasonable to assume that m is much smaller than N, and with the help of a powerful tool called Stirling's approximation, the limiting procedure can be carried out to yield

$$P_N(m) \approx \sqrt{\frac{1}{2\pi N}} \exp\left(\frac{-m^2}{2N}\right). \tag{25}$$

Homework

Use the Stirling formula:

$$\ln(n!) = \left(n + \frac{1}{2}\right)\ln(n) - n + \frac{1}{2}\ln(2\pi) + O\left(n^{-1}\right) \tag{26}$$

to derive the above result.