Physics through Computational Thinking

Driven oscillations: variations

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Outline

In this module we will look at
variants of the problem of driven oscillations.

```
Clear["Global`*"]
```

Numerical Solution with the RK4 Method

- Lets recall how we can bring a higher order differential equation into the canonical form:

$$\begin{aligned} \dot{x} &= f(t, x, y, z) \\ \dot{y} &= g(t, x, y, z) \\ \dot{z} &= h(t, x, y, z) \end{aligned} \tag{1}$$

- Next we define the column vectors X and F as

$$X = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ f \\ g \\ h \end{pmatrix} \tag{2}$$

- Then the coupled ODEs can be written as

$$\dot{X} = F \tag{3}$$

- The RK4 method is given by

$$\begin{aligned} R_1 &= F(X_n) \\ R_2 &= F\left(X_n + \frac{h}{2} R_1\right) \\ R_3 &= F\left(X_n + \frac{h}{2} R_2\right) \\ R_4 &= F(X_n + h R_3) \end{aligned} \tag{4}$$

$$X_{n+1} = X_n + h \frac{R_1 + 2 R_2 + 2 R_3 + R_4}{6} \tag{5}$$

- Here we have copied its implementation.

```
rk4[F_, X0_, tf_, nMax_] := Module[{h, datalist, prev, rate1, rate2, rate3, rate4, next},
  h = (tf - X0[[1]]) / nMax // N;
  For[datalist = {X0},
   Length[datalist] ≤ nMax,
   AppendTo[datalist, next],
   prev = Last[datalist];
   rate1 = F@prev;
   rate2 = F@(prev + h/2 rate1);
   rate3 = F@(prev + h/2 rate2);
   rate4 = F@(prev + h rate3);
   next = prev + h/6 (rate1 + 2 rate2 + 2 rate3 + rate4);
  ];
  Return[datalist];
 ]
```

The General Driven Oscillator.

We have looked at the problem of a harmonic oscillator that is subjected to a periodic driving external force. A more general external force can be handled theoretically, and it is of interest to study various special cases.

This would correspond to a differential equation of the type:

$$m \frac{d^2 x}{d t^2} + k\, x = F(t), \tag{6}$$

where $F(t)$ is a generic external force. For simplicity, we assume that the particle is at rest at the origin at time $t = 0$. Let us consider the following cases:

Exercise

(b) $F(t) = F_0$.

Solution

$$\begin{aligned} &\omega \text{ scale}: \quad \omega_0 = \sqrt{\frac{k}{m}} \\ &t \text{ scale}: \quad \frac{1}{\omega_0} = \sqrt{\frac{m}{k}} \\ &\text{acceleration scale}: \quad \frac{F_0}{m} \\ &x \text{ scale}: \quad \frac{F_0}{k} \end{aligned} \tag{7}$$

Making the transformation:

$$\begin{aligned} x &\longrightarrow \frac{F_0}{k}\, x \\ t &\longrightarrow \frac{1}{\omega_0}\, t \end{aligned} \tag{8}$$

we get

$$m \frac{F_0}{k} {\omega_0}^2 \frac{d^2 x}{d t^2} = -k \frac{F_0}{k} x + F_0$$
$$\Rightarrow \frac{d^2 x}{d t^2} = -x + 1 \tag{9}$$

After non-dimensionalization, there is *no* free parameter left in the problem.

Let us assume that the initial conditions for this problem in dimensionless units is given by $x(0)=0$ and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved exactly analytically for all times. It turns out that the solution is:

$$x(t) = 1 - \cos(t). \tag{10}$$

- The differential equation in canonical form is:

$$\begin{gathered} \frac{d x}{d t} = v \\ \frac{d v}{d t} = -x + 1 \\ x(0) = 0 \\ v(0) = 0 \end{gathered} \tag{11}$$

- So in vector form we have:

$$X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ -x + 1 \end{pmatrix}$$
$$\dot{X} = F \tag{12}$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, -x + 1};
initial = {0, 0, 0};
solx[t_] = -Cos[t] + 1;
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 40, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 20}, PlotRange -> Full, PlotStyle -> Red]];
```

Therefore, the constant external force simply shifts the origin about which oscillations happen!

A linearly increasing external force

We are studying differential equations of the type:

$$m \frac{d^2 x}{dt^2} + k\,x = F(t), \tag{13}$$

where $F(t)$ is a generic external force. Let us now consider an external force that is linear in time.

Exercise

(b) $F(t) = a\,t$.

Solution

$$\begin{aligned}
&\omega \text{ scale}: \quad \omega_0 = \sqrt{\frac{k}{m}} \\
&t \text{ scale}: \quad \frac{1}{\omega_0} = \sqrt{\frac{m}{k}} \\
&\text{acceleration scale}: \quad \frac{a\,t}{m} \\
&x \text{ scale}: \quad \frac{a\,t}{k} = \frac{a}{k}\sqrt{\frac{m}{k}} = \frac{a}{m\,\omega_0{}^3}
\end{aligned} \tag{14}$$

Making the transformation:

$$\begin{aligned}
x &\longrightarrow \frac{a}{m\,\omega_0{}^3}\,x \\
t &\longrightarrow \frac{1}{\omega_0}\,t
\end{aligned} \tag{15}$$

we get

$$m \frac{a}{m\,\omega_0^3}\,\omega_0^2 \frac{d^2 x}{d t^2} = -k \frac{a}{m\,\omega_0^3}\,x + a \frac{1}{\omega_0}\,t$$
$$\Rightarrow \frac{d^2 x}{d t^2} = -x + t \tag{16}$$

After non-dimensionalization, there is *no* free parameter left in the problem.

Let us assume that the initial conditions for this problem in dimensionless units is given by $x(0)$=0 and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved exactly analytically for all times. The solution turns out to be:

$$x(t) = t - \sin(t). \tag{17}$$

- The differential equation in canonical form is:

$$\frac{d x}{d t} = v$$
$$\frac{d v}{d t} = -x + t$$
$$x(0) = 0$$
$$v(0) = 0 \tag{18}$$

- So in vector form we have:

$$X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ -x + t \end{pmatrix}$$
$$\dot{X} = F \tag{19}$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, -x + t};
initial = {0, 0, 0};
solx[t_] = -Sin[t] + t;
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 40, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 20}, PlotRange -> Full, PlotStyle -> Red]];
```

A quadratically increasing force

We are solving the differential equation:

$$m \frac{d^2 x}{d t^2} + k x = F(t), \tag{20}$$

where $F(t)$ is a generic external force.

Exercise

(c) $F(t) = a t^2$.

Solution

$$\begin{aligned}
&\omega \text{ scale}: && \omega_0 = \sqrt{\frac{k}{m}} \\
&t \text{ scale}: && \frac{1}{\omega_0} = \sqrt{\frac{m}{k}} \\
&\text{acceleration scale}: && \frac{a t^2}{m} \\
&x \text{ scale}: && \frac{a t^2}{k} = \frac{a}{k} \frac{1}{\omega_0^2} = \frac{a}{m \omega_0^4}
\end{aligned} \tag{21}$$

Making the transformation:

$$\begin{aligned}
x &\longrightarrow \frac{a}{m \omega_0^4} x \\
t &\longrightarrow \frac{1}{\omega_0} t
\end{aligned} \tag{22}$$

we get

$$m \frac{a}{m\,\omega_0^4}\,\omega_0^2 \frac{d^2 x}{d t^2} = -k \frac{a}{m\,\omega_0^4}\, x + a \frac{1}{\omega_0^2}\, t^2$$
$$\Rightarrow \frac{d^2 x}{d t^2} = -x + t^2$$

After non-dimensionalization, there is *no* free parameter left in the problem.

Let us assume that the initial conditions for this problem in dimensionless units is given by $x(0)=0$ and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved exactly analytically for all times. The solution is:

$$x(t) = t^2 - 2 + 2\cos(t). \tag{24}$$

- The differential equation in canonical form is:

$$\begin{gathered} \frac{d x}{d t} = v \\ \frac{d v}{d t} = -x + t^2 \\ x(0) = 0 \\ v(0) = 0 \end{gathered} \tag{25}$$

- So in vector form we have:

$$\begin{gathered} X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ -x + t^2 \end{pmatrix} \\ \dot{X} = F \end{gathered} \tag{26}$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, -x + t^2};
initial = {0, 0, 0};
solx[t_] = 2 Cos[t] + t^2 - 2;
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 10, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 8}, PlotRange -> Full, PlotStyle -> Red]];
```

An exponentially decaying force

We are solving the differential equation:

$$m \frac{d^2 x}{d t^2} + k\,x = F(t), \tag{27}$$

where $F(t)$ is a generic external force.

Exercise

(d) $F(t) = F_0\, e^{-\alpha t}$.

Solution

$$\begin{aligned}
&\omega \text{ scale}: \quad \omega_0 = \sqrt{\tfrac{k}{m}} \\
&t \text{ scale}: \quad \tfrac{1}{\omega_0} = \sqrt{\tfrac{m}{k}} \\
&\text{acceleration scale}: \quad \tfrac{F_0}{m} \\
&x \text{ scale}: \quad \frac{F_0}{k}
\end{aligned} \tag{28}$$

Making the transformation:

$$\begin{aligned}
x &\longrightarrow \tfrac{F_0}{k}\, x \\
t &\longrightarrow \tfrac{1}{\omega_0}\, t
\end{aligned} \tag{29}$$

we get

$$m \frac{F_0}{k} \omega_0{}^2 \frac{d^2 x}{d t^2} = -k \frac{F_0}{k} x + F_0 e^{\frac{-\alpha}{\omega_0} t}$$
$$\Rightarrow \frac{d^2 x}{d t^2} = -x + e^{-\lambda t}$$

where $\lambda = \frac{\alpha}{\omega_0}$ is a new dimensionless parameter in the system.

Therefore, after non-dimensionalization, there is *one* free parameter left in the problem.

Let us assume that the initial conditions for this problem in dimensionless units is given by $x(0)=0$ and $\dot{x}(0) = 0$.

This is a second order differential equation which can be solved exactly analytically for all times. The solution is:

$$x(t) = \frac{1}{1+\lambda^2} \left(e^{-\lambda t} - \cos(t) + \lambda \sin(t)\right). \tag{31}$$

Let us solve numerically the special case $\lambda=1$.

- The differential equation in canonical form is:

$$\begin{gathered} \frac{d x}{d t} = v \\ \frac{d v}{d t} = -x + e^{-t} \\ x(0) = 0 \\ v(0) = 0 \end{gathered} \tag{32}$$

- So in vector form we have:

$$\begin{gathered} X = \begin{pmatrix} t \\ x \\ v \end{pmatrix} \qquad F = \begin{pmatrix} 1 \\ v \\ -x + e^{-t} \end{pmatrix} \\ \dot{X} = F \end{gathered} \tag{33}$$

- So we proceed to define the functions and the initial vector:

```
rateFunc[{t_, x_, v_}] = {1, v, -x + Exp[-t]};
initial = {0, 0, 0};
solx[t_] = (Exp[-t] - Cos[t] + Sin[t]) / 2.0;
```

- Now we are ready to invoke the rk4 function:

```
data = rk4[rateFunc, initial, 100, 300];

ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full];

Show[ ListPlot[data[[ ;; , 1 ;; 2]], Joined -> True, PlotMarkers -> None, PlotRange -> Full],
  Plot[solx[t], {t, 0, 80}, PlotRange -> Full, PlotStyle -> Red]];
```