Kinematic and Dynamic Analyses of Planar 3 Degree of Freedom Hopping Robot

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1 Introduction

1.1 Motivation

We want to design a closed-chain robot to study legged locomotion. A closed-chain legged robot has more mass concentrated at its top, compared to an open-chain robot. This is advantageous for legged locomotion for two reasons:

- Less mass near the foot increases agility and reduces actuation requirements at the hip.
- The robot behaves like an inverted pendulum during stance, so concentrating the mass at the top of the pendulum increases stability.

The remainder of this paper derives a mathematical model of a particular planar 3-degree-of-freedom (DOF) closed-chain robot. Having 3 DOF is desirable because a planar legged robot with 3 DOF can fully control its position and orientation during stance and flight. [1]

These analyses closely follow [1], especially Chapters 6 ("Inverse Kinematics") and 7 ("Kinematics of Closed Chains").

1.2 Proof: mobility

This proof is simply a verification that the robot shown in Figure 1 has 3 degrees of freedom, using Grübler's formula:

[1] The interface between the foot and ground can be viewed as an unactuated joint, inviting questions about underactuation and controllability for this particular robot.

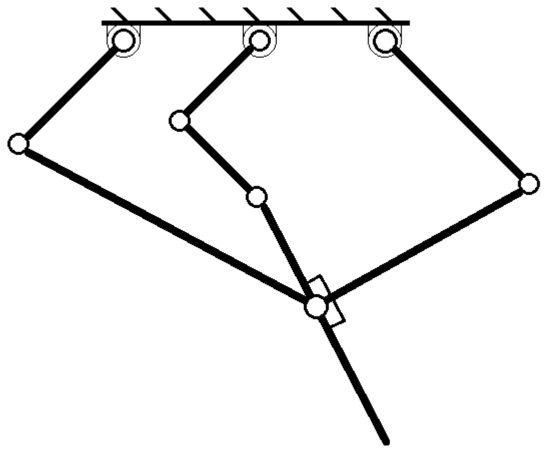

Figure 1: Schematic diagram of the planar robot.

$$\text{DOF} = m(N - 1 - J) + \sum_{i=1}^{J} f_i, \tag{1}$$

where $m = 3$ for planar mechanisms, N is the number of links (including the base), J is the number of joints, and f_i is the number of freedoms at the i^{th} joint. From Figure 1, $N = 8$, $J = 9$, and $f_i = 1$ for all i. Note that the lowest joint in the figure is really two joints; one joint connects the left coupler to the middle link, and the other connects the right coupler to the middle link. Thus, according to Grübler's formula, the robot has 3 degrees of freedom.

2 Forward kinematics

Closed-chain linkages can be decomposed into groups of open-chain linkages (hereafter referred to as "open sub-chains") whose motions are constrained by loop-closure equations. Our robot comprises three open sub-chains, shown in Figure 2: the θ-chain is on the left, the ϕ-chain is in the middle, and the ψ-chain is on the right. We will first examine the forward kinematics for each open sub-chain and then develop loop-closure equations.

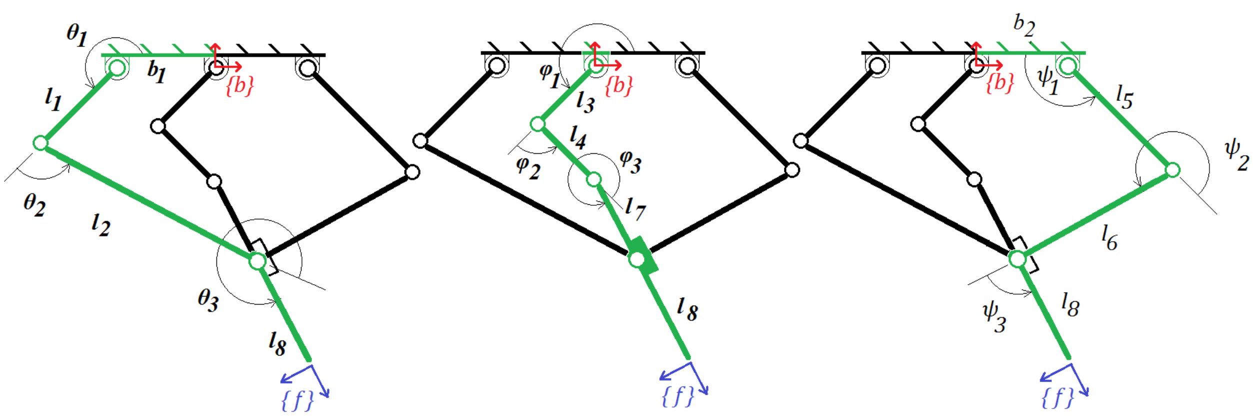

Figure 2: The planar robot comprises three open sub-chains, referred to as the θ-, ϕ-, and ψ-chains, from left to right.

2.1 Open sub-chain forward kinematics

The forward kinematics of each open sub-chain relate the foot frame $\{f\}$ to the base frame $\{b\}$. The forward kinematics of the θ-chain are

$$\begin{bmatrix} x_f \\ y_f \\ \measuredangle_f \end{bmatrix} = \begin{bmatrix} -b_1 + l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_8 \cos(\theta_1+\theta_2+\theta_3) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_8 \sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{bmatrix}. \quad (2)$$

The forward kinematics of the ϕ-chain are

$$\begin{bmatrix} x_f \\ y_f \\ \measuredangle_f \end{bmatrix} = \begin{bmatrix} l_3 \cos\phi_1 + l_4 \cos(\phi_1+\phi_2) + (l_7+l_8) \cos(\phi_1+\phi_2+\phi_3) \\ l_3 \sin\phi_1 + l_4 \sin(\phi_1+\phi_2) + (l_7+l_8) \sin(\phi_1+\phi_2+\phi_3) \\ \phi_1+\phi_2+\phi_3 \end{bmatrix}. \quad (3)$$

Lastly, the forward kinematics of the ψ-chain are

$$\begin{bmatrix} x_f \\ y_f \\ \measuredangle_f \end{bmatrix} = \begin{bmatrix} b_2 + l_5 \cos\psi_1 + l_6 \cos(\psi_1+\psi_2) + l_8 \cos(\psi_1+\psi_2+\psi_3) \\ l_5 \sin\psi_1 + l_6 \sin(\psi_1+\psi_2) + l_8 \sin(\psi_1+\psi_2+\psi_3) \\ \psi_1+\psi_2+\psi_3 \end{bmatrix}. \quad (4)$$

Thus, if we know the joint positions of any of the three open sub-chains, we can calculate the end-effector pose. Although this formulation is straightforward, it is not particularly useful on its own, because both the actuated and unactuated joint positions in that chain must be known in order to calculate the end-effector pose. It would be more useful to have a mapping from the positions of only the actuated joints to the end-effector pose.

One way to develop such a mapping is to first solve for the unactuated joint positions as functions of the actuated joint positions, using loop closure equations. Then, once all the joint positions are known, we can choose any of the three open sub-chains and use the corresponding forward kinematics equations to calculate the end-effector pose.

2.2 Numerical forward kinematics for a closed-chain robot

The forward kinematics method described above requires solving a system of nonlinear equations (with potentially many solutions), so a numerical root-finding algorithm may be more appropriate than seeking an analytical solution. One such algorithm is the Newton-Raphson method.

2.2.1 Newton-Raphson root-finding method

Consider a nonlinear equation $g(q) = \dots$, for which we seek the roots, i.e., we wish to solve $g(q) = 0$. Linearize the equation by writing a Taylor series expansion of $g(q)$ about the point q_0:

$$g(q) = g(q_0) + \left.\frac{\partial g}{\partial q}\right|_{q_0} (q - q_0) + \text{h.o.t.} \quad (5)$$

Discarding the higher-order terms, the roots are approximately

$$q = q_0 - \left(\left.\frac{\partial g}{\partial q}\right|_{q_0} \right)^{-1} g(q_0) \quad . \quad (6)$$

Taking the new q as q_0, this method can be iterated until some convergence threshold criterion is satisfied.

2.2.2 Loop-closure equations

Refer to Figures 1 and 2 and observe that the open sub-chains intersect at a common joint, measured by θ_3 and ψ_3. Consequently, there are three loops: the θ-ϕ loop, the ϕ-ψ loop, and the ψ-θ loop. We can describe these three loops with two sets of equations:

$$g(\theta, \phi, \psi) := \begin{bmatrix} T_{bf}(\theta) - T_{bf}(\phi) \\ -T_{bf}(\phi) + T_{bf}(\psi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

where $T_{bf}(\cdot)$ represents the twist from the base frame $\{b\}$ to the end-effector frame $\{f\}$ along the corresponding open sub-chain; this is just a compact representation of the earlier open sub-chain forward kinematic equations, so the left and right sides of the equation are 6×1 column vectors.

For a given set of actuated joint angles $q_a = [\theta_1, \phi_1, \psi_1]^{\mathrm{T}}$, we seek unactuated joint angles ($q_u = [\theta_2, \theta_3, \phi_2, \phi_3, \psi_2, \psi_3]^{\mathrm{T}}$), i.e., the roots of the loop-closure equations, so the iterative Newton-Raphson root-finding equation is

$$\left[\theta_2^{k+1}, \theta_3^{k+1}, \phi_2^{k+1}, \phi_3^{k+1}, \psi_2^{k+1}, \psi_3^{k+1}\right]^{\mathrm{T}} = q_u^{k+1} = q_u^k - J_c^{-1} g\left(q_a, q_u\right) \ , \tag{8}$$

where J_c, the *constraint Jacobian*, is the matrix of partial derivatives of the loop-closure expression with respect to the elements of q_u. Since the loop-closure equation was expressed as a 6×1 column vector, the constraint Jacobian is a square 6×6 matrix.

2.2.3 Constraint Jacobian

As mentioned earlier, the constraint Jacobian J_c is the matrix of partial derivatives of the loop-closure expression with respect to the unactuated joint positions q_u. Differentiating the loop-closure equation above, we get the following 6×6 matrix:

$$J_c = \begin{bmatrix} \frac{\partial T_{bf}(\theta)}{\partial \theta_2} & \frac{\partial T_{bf}(\theta)}{\partial \theta_3} & -\frac{\partial T_{bf}(\phi)}{\partial \phi_2} & -\frac{\partial T_{bf}(\phi)}{\partial \phi_3} & 0 & 0 \\ 0 & 0 & -\frac{\partial T_{bf}(\phi)}{\partial \phi_2} & -\frac{\partial T_{bf}(\phi)}{\partial \phi_3} & \frac{\partial T_{bf}(\psi)}{\partial \psi_2} & \frac{\partial T_{bf}(\psi)}{\partial \psi_3} \end{bmatrix} . \tag{9}$$

2.2.4 Numerical forward kinematics algorithm

Now that we have the constraint Jacobian, we can develop an iterative numerical forward kinematics algorithm.

Algorithm 1 Newton-Raphson algorithm for closed-chain forward kinematics

function NRFK(q_a, q_u^0, ε) ▷ compute end-effector pose T_{bf} from actuated joint positions
  Initialize $r > \varepsilon$ ▷ measure of convergence
  while $r > \varepsilon$ **do** ▷ convergence threshold
    $q_u^{new} = q_u^0 - J_c^{-1} g\left(q_a, q_u^0\right)$
    $r = \frac{\left|g(q_a, q_u^0) - g(q_a, q_u^{new})\right|}{\left|g(q_a, q_u^0)\right|}$ ▷ update measure of convergence
    $q_u^0 = q_u^{new}$
  end while ▷ we now have approximate values for q_u
  $\theta = \left[q_a(1) \quad q_u^{new}(1) \quad q_u^{new}(2)\right]^{\mathrm{T}}$
  return $\left[x_f \quad y_f \quad \measuredangle_f\right]^{\mathrm{T}} = T_{bf}(\theta)$ ▷ $T_{bf}(\theta)$ is the end-effector pose computed using the θ-chain
end function

In summary, this algorithm finds approximate unactuated joint positions from the actuated joint positions and uses the positions along one open sub-chain to compute the end-effector pose. Note that this algorithm uses the θ-chain to compute the end-effector pose, but the other open sub-chains are equally viable.

2.3 Actuation requirements

We would now like to solve the inverse dynamics problem: what motor speeds/torques are required to generate a desired end-effector speed/force? This problem can be solved using the *actuator Jacobian*.

2.3.1 Actuator Jacobians

The actuator Jacobian maps generalized actuator velocities and forces to end-effector velocities and forces. Returning to the loop-closure equation (equation 7) and differentiating with respect to all joint positions (q_a and q_u), we get

$$\begin{bmatrix} J_\theta & -J_\phi & 0 \\ 0 & -J_\phi & J_\psi \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = 0_{6\times1} \ , \tag{10}$$

where each $J \in \mathbb{R}^{3\times 3}$ is the Jacobian of the corresponding open sub-chain. Expanding this equation is straightforward and will help with the next step. When fully expanded, equation 10 becomes

$$\begin{bmatrix} J_{\theta_{1,1}} & J_{\theta_{1,2}} & J_{\theta_{1,3}} & -J_{\phi_{1,1}} & -J_{\phi_{1,2}} & -J_{\phi_{1,3}} & 0 & 0 & 0 \\ J_{\theta_{2,1}} & J_{\theta_{2,2}} & J_{\theta_{2,3}} & -J_{\phi_{2,1}} & -J_{\phi_{2,2}} & -J_{\phi_{2,3}} & 0 & 0 & 0 \\ J_{\theta_{3,1}} & J_{\theta_{3,2}} & J_{\theta_{3,3}} & -J_{\phi_{3,1}} & -J_{\phi_{3,2}} & -J_{\phi_{3,3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_{\phi_{1,1}} & -J_{\phi_{1,2}} & -J_{\phi_{1,3}} & J_{\psi_{1,1}} & J_{\psi_{1,2}} & J_{\psi_{1,3}} \\ 0 & 0 & 0 & -J_{\phi_{2,1}} & -J_{\phi_{2,2}} & -J_{\phi_{2,3}} & J_{\psi_{2,1}} & J_{\psi_{2,2}} & J_{\psi_{2,3}} \\ 0 & 0 & 0 & -J_{\phi_{3,1}} & -J_{\phi_{3,2}} & -J_{\phi_{3,3}} & J_{\psi_{3,1}} & J_{\psi_{3,2}} & J_{\psi_{3,3}} \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \\ \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \\ \dot\psi_1 \\ \dot\psi_2 \\ \dot\psi_3 \end{bmatrix} = 0_{6\times 1} \ , \tag{11}$$

which can be rearranged into an actuated part, $H_a \in \mathbb{R}^{6\times 3}$, and an unactuated part, $H_u \in \mathbb{R}^{6\times 6}$, such that

$$\begin{bmatrix} H_a(q_a, q_u) & H_u(q_a, q_u) \end{bmatrix} \begin{bmatrix} \dot q_a \\ \dot q_u \end{bmatrix} = 0_{6\times 1} \ , \tag{12}$$

or equivalently,

$$\dot q_u = -H_u^{-1} H_a \dot q_a \ . \tag{13}$$

From equation 11, it is apparent that

$$H_a(q_a, q_u) = \begin{bmatrix} J_{\theta_{1,1}} & -J_{\phi_{1,1}} & 0 \\ J_{\theta_{2,1}} & -J_{\phi_{2,1}} & 0 \\ J_{\theta_{3,1}} & -J_{\phi_{3,1}} & 0 \\ 0 & -J_{\phi_{1,1}} & J_{\psi_{1,1}} \\ 0 & -J_{\phi_{2,1}} & J_{\psi_{2,1}} \\ 0 & -J_{\phi_{3,1}} & J_{\psi_{3,1}} \end{bmatrix} \text{, and} \tag{14}$$

$$H_u(q_a, q_u) = \begin{bmatrix} J_{\theta_{1,2}} & -J_{\phi_{1,3}} & -J_{\phi_{1,2}} & -J_{\phi_{1,3}} & 0 & 0 \\ J_{\theta_{2,2}} & -J_{\phi_{2,3}} & -J_{\phi_{2,2}} & -J_{\phi_{2,3}} & 0 & 0 \\ J_{\theta_{3,2}} & -J_{\phi_{3,3}} & -J_{\phi_{3,2}} & -J_{\phi_{3,3}} & 0 & 0 \\ 0 & 0 & -J_{\phi_{1,2}} & -J_{\phi_{1,3}} & J_{\psi_{1,2}} & J_{\psi_{1,3}} \\ 0 & 0 & -J_{\phi_{2,2}} & -J_{\phi_{2,3}} & J_{\psi_{2,2}} & J_{\psi_{2,3}} \\ 0 & 0 & -J_{\phi_{3,2}} & -J_{\phi_{3,3}} & J_{\psi_{3,2}} & J_{\psi_{3,3}} \end{bmatrix} \ , \tag{15}$$

which is, in fact, the constraint Jacobian J_c from the previous section (see equation 9), so the velocities of the unactuated joints are

$$\dot q_u = -J_c^{-1} H_a \dot q_a \ , \tag{16}$$

assuming J_c is invertible. Observe that $-J_c^{-1} H_a$ is a 6×3 matrix that maps actuated joint velocities to unactuated joint velocities. For example,

$$\dot\theta_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(-J_c^{-1} H_a \right) \dot q_a \text{ , and} \tag{17}$$

$$\dot\theta_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \left(-J_c^{-1} H_a \right) \dot q_a \ . \tag{18}$$

We can develop a mapping from the velocities of the actuated joints to the end-effector velocity, using the θ-chain:

$$\begin{bmatrix} \dot x_f \\ \dot y_f \\ \dot\measuredangle_f \end{bmatrix} = J_\theta \left(q_a, q_u \right) \dot\theta = J_\theta \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left(-J_c^{-1} H_a \right) \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \left(-J_c^{-1} H_a \right) \end{bmatrix} \dot q_a \ , \tag{19}$$

or more simply,

$$\begin{bmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{\measuredangle}_f \end{bmatrix} = J_{a,\theta}\left(q_a, q_u\right)\dot{q}_a \ , \tag{20}$$

where $J_{a,\theta} \in \mathbb{R}^{3\times 3}$ is the θ-chain *actuator Jacobian*. Similar expressions can be derived for $J_{a,\phi} \in \mathbb{R}^{3\times 3}$ and $J_{a,\psi} \in \mathbb{R}^{3\times 3}$ by returning to equation 16 and deriving expressions for $\dot{\phi}_2$, $\dot{\phi}_3$, $\dot{\psi}_2$, and $\dot{\psi}_3$. Together, the actuator Jacobians are

$$J_{a,\theta} = J_\theta \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\left(-J_c^{-1}H_a\right) \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}\left(-J_c^{-1}H_a\right) \end{bmatrix} \tag{21}$$

$$J_{a,\phi} = J_\phi \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\left(-J_c^{-1}H_a\right) \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}\left(-J_c^{-1}H_a\right) \end{bmatrix} \tag{22}$$

$$J_{a,\psi} = J_\psi \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\left(-J_c^{-1}H_a\right) \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\left(-J_c^{-1}H_a\right) \end{bmatrix} . \tag{23}$$

2.3.2 Motor sizing

The relationship between joint speeds and end-effector speed is given above in equation 20, but in general, the relationship is

$$V = J_a\dot{\theta} \quad \text{and } \dot{\theta} = J_a^{-1}V \ . \tag{24}$$

Similarly, the relationship between joint torques and end-effector force is

$$\tau = J_a^{\mathrm{T}}F \quad \text{and } F = J_a^{-\mathrm{T}}\tau \tag{25}$$

References

[1] K. M. Lynch and F. C. Park. *Modern Robotics.* Cambridge University Press.