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Assignment - 4.

1. If $u = x(1-y)$ and $v = xy$ find $\frac{\partial(x,y)}{\partial(u,v)}$.

Soln.

Let, $f_1 = x - xy - u$

$f_2 = xy - v$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(u,v)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}} = \cancel{No}$$

$$= \frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}}$$

$$= \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$\begin{vmatrix} (1-y) & -x \\ y & x \end{vmatrix}$$

$$= \frac{1}{(1-y)x + xy} = \frac{1}{1-\cancel{y}(x)\cancel{+x(y)}}$$

$= 1.$

$$= \frac{1}{u+v} = \frac{1}{x}$$

2. If $x = v^2+w^2$, $y = w^2+u^2$, $z = u^2+v^2$ prove that $J \cdot J' = 1$.

Soln.

Given:- $x = v^2+w^2$

$y = w^2+u^2$

$z = u^2+v^2$.

$$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} \end{array}$$

$$= -2v(-4uw) + 2w(4uv)$$

$$= 8uvw + 8uvw = 16uvw.$$

For $J' = \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Let $f_1 = v^2 + w^2 - x = 0$,
$f_2 = u^2 + w^2 - y = 0$.
$f_3 = v^2 + u^2 - z = 0$.

$$\cancel{\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \end{vmatrix}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

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$$= (1) \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}}$$

$$= (-1) \frac{\begin{array}{c} + \quad\quad - \quad\quad + \\ \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \end{array}}{\begin{array}{c} + \quad\quad - \quad\quad + \\ \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} \end{array}}$$

$$\overset{(-1)}{=} \frac{(-1)\,(1)}{-2v(-4uw) + 2w(4uv)}$$

$$= \frac{(-1)\,-1}{8uvw + 8uvw} = \frac{+1}{16uvw}$$

$$J' = \cancel{16}\,\frac{1}{16uvw}.$$

$$\therefore J \cdot J' = 16uvw \times \frac{1}{16uvw} = 1 \text{ is proved.}$$

3). If $u+v^2 = x$, $v+w^2 = y$, $w+u^2 = z$ find $\frac{\partial u}{\partial x}$.

Soln:-

$$f_1 \equiv u - x + v^2$$
$$f_2 \equiv v + w^2 - y$$
$$f_3 \equiv w + u^2 - z.$$

$$\therefore \frac{\partial u}{\partial x} = - \frac{\left(\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}\right)}{\left(\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}\right)}$$

$$\therefore \frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)} = \begin{vmatrix} f_{1x} & f_{1v} & f_{1w} \\ f_{2x} & f_{2v} & f_{2w} \\ f_{3x} & f_{3v} & f_{3w} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 0 & +1 & 2w \\ 0 & 0 & +1 \end{vmatrix}$$

$$= -1(+1-0) \qquad = -1$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} f_{1u} & f_{1v} & f_{1w} \\ f_{2u} & f_{2v} & f_{2w} \\ f_{3u} & f_{3v} & f_{3w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & +2v & 0 \\ 0 & 1 & 2w \\ 2u & 0 & 1 \end{vmatrix}$$

$$\cancel{= 1(1-0) + 2v(4uw)}$$

$$= 1(1-0) - 2v(-4uw)$$

$$= 1 + 8uvw$$

$$\therefore \left(\frac{\partial u}{\partial x}\right) = (-1)\left[\frac{-1}{1+8uvw}\right] = \frac{1}{1+8uvw}$$

4. If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Soln

$$f_1 = v^2 + w^2 - x$$
$$f_2 = w^2 + u^2 - y$$
$$f_3 = u^2 + v^2 - z.$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \left[\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \right]$$

$$\therefore \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$\begin{array}{cccc} & + & - & + \\ = & \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \end{array}$$

$$= -1(1-0) = -1$$

$$\therefore \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$= \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$$= -2v(0-4uw) + 2w(4uv)$$
$$= 8uvw + 8uvw$$
$$= 16uvw.$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{16uvw}.$$

5. Verify whether given functions are functionally dependent. If so find the relation between them

$$u = \frac{y-x}{1+xy}, \quad v = \tan^{-1}y - \tan^{-1}x.$$

$\rightarrow$ $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

① $\dfrac{\partial u}{\partial x} = \dfrac{(1+xy)(-1) - (y-x)y}{(1+xy)^2}$

$= \dfrac{-1-xy-y^2+xy}{(1+xy)^2}$

$= \dfrac{-1-\cancel{xy}-y^2+\cancel{xy}}{(1+xy)^2} = -\dfrac{(1+y^2)}{(1+xy)^2}$

②. $\dfrac{\partial u}{\partial y} = \dfrac{(1+xy)(1) - (y-x)x}{(1+xy)^2}$

$= 1+xy - xy + x^2 = \dfrac{1+x^2}{(1+xy)^2}$.

③. $\dfrac{\partial v}{\partial x} = \dfrac{-1}{1+x^2}$

④. $\dfrac{\partial v}{\partial y} = \dfrac{1}{1+y^2}$.

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-(1+y^2)}{(1+xy)^2} & \frac{1+x^2}{(1+xy)^2} \\ -\frac{1}{(1+x^2)} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= -\frac{(1+y^2)}{(1+xy)^2}\frac{1}{(1+y^2)} + \frac{(1+x^2)}{(1+xy^2)(1+x^2)}$$

$$= 0.$$

$\therefore$ The given function is functionally dependent.

$$u = \frac{y-x}{\cancel{x+y}\ 1+xy} \qquad v = \tan^{-1}y - \tan^{-1}x$$

$$\therefore v = \tan^{-1}\left(\frac{y-x}{1+xy}\right)$$

$$\therefore v = \tan^{-1}(u)$$

$\therefore u = \tan v$. is required relation

6. Find extreme values of $f(x,y) = x^3 + y^3 - 3axy$, $a > 0$.

→

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0 \quad \text{--- (2)}$$

Now, $3x^2 - 3ay = 0$.

$$\cancel{3}x^2 = \cancel{3}ay$$

$$y = \frac{x^2}{a} \quad \text{--- (3)}$$

$$3y^2 = 3ax.$$

$$y^2 = ax.$$

$$\left(\frac{x^2}{a}\right)^2 = ax.$$

$$x^4 = a^3 x$$

$$x^4 - a^3 x = 0.$$

$$x(x^3 - a^3) = 0.$$

$x = 0$ or $(x-a)(x^2 + ax + a^2) = 0$

$x = 0$, $y = 0$ $\qquad$ $x = a$, $y = a$.

i) At $(0,0)$.

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \cdot \partial y} = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y.$$

$$rt - s^2 = 0 - (-3a)^2 = -9a^2 < 0.$$

$\therefore$ Hence at the pt. $(0,0)$ $f(x,y)$ is neither maximum nor minimum.

ii) At (a,a)

$$rt - s^2 = (6a)(6a) - (-3a)^2$$
$$= 27a > 0$$

and $r = 6a > 0$.

$\therefore$ at pt (a,a) function has ~~maxima~~ minima.

$\therefore$ ~~Maximum~~ Minimum value $= f(a,a)$

$= a^3 + a^3 - 3a \cdot a \cdot a$

$= 2a^3 - 3a^3$

$\therefore$ ~~Maximum~~ Minimum value, $= -a^3$

7. Examine for maxima and minima of the function and find their extreme values,

$f = x^2 + y^2 + 6x + 12 = 0.$

$\rightarrow$ $\frac{\partial f}{\partial x} = 2x + 6 = 0.$

$x = -3$

$\frac{\partial f}{\partial y} = 2y = 0$

$\therefore y = 0.$

$r = \frac{\partial^2 f}{\partial x^2} = 2$

$s = \frac{\partial^2 f}{\partial x \cdot \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

At pt. $(-3, 0)$

$\therefore rt - s^2 = 2 \times 2 - 0^2 = 4$

and $r = 2 > 0.$

$\therefore$ At pt $(-3, 0)$ we get minima.

$\therefore$ Minimum value $= f(-3, 0)$

$= \cancel{(-3)^2 + (0)^3 - 3\times 0}$

$= (-3)^2 + 0^2 + 6\times(-3) + 12$

$= 9 - 18 + 12$

$= -9 + 12 = 3$

$\therefore$ Minimum value, $= 3$

8) Using Lagrange's method divides 24 into three parts such that the continued product of the first, square of the second and cube of third may be maximum.

→

First part $= x$.

Second part $= y$.

Third part $= z$.

Now, $x + y + z = 24$ [first condⁿ]

$f(x, y, z) = x y^2 z^3$ [second condⁿ]

Now,

$F = u + \lambda \phi$

$= x y^2 z^3 + \lambda(x + y + z - 24)$

$$\frac{\partial f}{\partial x} = y^2 z^3 + \lambda = 0$$
$$\lambda = -y^2 z^3 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 2xyz^3 + \lambda = 0$$
$$\lambda = -2xyz^3 \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 + \lambda = 0$$
$$\lambda = -3xy^2z^2 \quad \text{--- (3)}$$

from (1), (2) and (3)

$$-y^2z^3 = -2xyz^3 = -3xy^2z^2$$

Now, divide by xy^2z^3.

$$\frac{1}{x} = \frac{2}{y} = \frac{3}{z} = k$$

$$\therefore x = \frac{1}{k}, \quad y = \frac{2}{k}, \quad z = \frac{3}{k}.$$

$$\therefore \quad x + y + z - 24 = 0.$$

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} = 24$$

$$\frac{6}{k} = 24$$

$$\therefore k = \frac{1}{4}$$

$\therefore$

$x = 4, \quad y = 8, \quad z = 12$

$\therefore$ Maximum value $= x y^2 z^3$
$= 4 \times 64 \times 12 \times 12 \times 12$
$= 4,42,368$

9) Use Lagrange's method to find the minimum distance from origin to the plane $3x + 2y + z = 12$.

→ $3x + 2y + z = 12$.

Let $P(x, y, z)$ be any point on given plane.

$\therefore$ distance from origin $O(0, 0)$ is given by.

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$\therefore$ $OP^2 = x^2 + y^2 + z^2$.

Let $u = x^2 + y^2 + z^2$.

and $\phi = 3x + 2y + z - 12 = 0$.

consider,

$$F = u + \lambda\phi.$$
$$= (x^2+y^2+z^2) + \lambda(3x+2y+z-12)$$

$$\therefore \frac{\partial F}{\partial x} = 2x + 3\lambda = 0.$$
$$\therefore \lambda = \frac{-2x}{3}$$

$$\therefore \frac{\partial F}{\partial y} = 2y + \lambda 2 = 0.$$
$$\lambda = -\frac{2y}{2}$$

$$\therefore \frac{\partial F}{\partial z} = 2z + \lambda = 0.$$
$$\lambda = -2z.$$

equating all values of λ.

$$-\frac{2x}{3} = -y = -2z = k$$

$$\therefore x = \frac{-3k}{2}, \quad y = -k, \quad z = \frac{-k}{2}$$

$$\therefore P(x,y,z) = P\left(\frac{-3k}{2}, -k, \frac{-k}{2}\right)$$

$\therefore$ These point P, must satisfy eq^n of given plane.

$$\therefore 3x + 2y + z = 12.$$

$$3\left(\frac{-3k}{2}\right) + 2(-k) + \left(\frac{-k}{2}\right) = 0.$$

$$\therefore k = \frac{-12}{7}.$$

$$\therefore P\left(\frac{-3k}{2}, -k, \frac{-k}{2}\right)$$

$$\therefore P\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}.\right)$$

$\therefore$ Min^m distance from origin is given by.

$$OP = \sqrt{x^2+y^2+z^2}.$$

$$= \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2}.$$

$$OP. = 3.207 \text{ units.}$$

10. The area of a triangle ABC is calculated from the formula $\Delta = \frac{1}{2} bc\sin A$. Errors of 1%, 2%, 3% respectively are made in measuring b, c and A. If the correct value of A is 30°, find the error in the calculated value of Δ.

Sol^n

$$\Delta = \frac{1}{2} bc \sin A.$$

$$\log \Delta = \log\frac{1}{2} + \log b + \log c + \log(\sin A).$$

$$\frac{d\Delta}{\Delta} = \frac{db}{b} + \frac{dc}{c} + \frac{\cos A}{\sin A} \cdot dA.$$

$$100\frac{d\Delta}{A} = 100\frac{db}{b} + 100\frac{dc}{c} + 100 \frac{\cot A}{\cancel{dA}} \cdot \frac{dA}{A} \cdot A$$

$$\% A = 1\% + 2\% + \cot A \cdot 3 \times \frac{\pi}{6}$$

$$= 3 + 3 \times \cot\frac{\pi}{6} \times \frac{\pi}{6}$$

$$= 3.71 \%$$