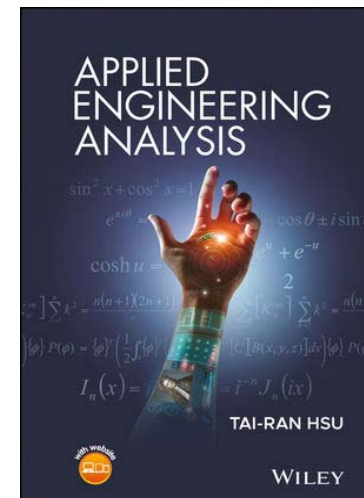**Applied Engineering Analysis**
- slides for class teaching*

Chapter 7

Application of First-order Differential Equations in Engineering Analysis

* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, published by John Wiley & Sons, 2018

Chapter Learning Objectives

- Learn to solve typical first order ordinary differential equations of both homogeneous and non-homogeneous types with or without specified conditions.

- Learn the definitions of essential physical quantities in fluid mechanics analyses.

- Learn the Bernoulli's equation relating the driving pressure and the velocities of fluids in motion.

- Learn to use the Bernoulli's equation to derive differential equations describing the flow of non-compressible fluids in large tanks and funnels of given geometry.

- Learn to find time required to drain liquids from containers of given geometry and dimensions.

- Learn the Fourier law of heat conduction in solids and Newton's cooling law for convective heat transfer in fluids.

- Learn how to derive differential equations to predict required times to heat or cool small solids by surrounding fluids.

- Learn to derive differential equations describing the motion of rigid bodies under the influence of gravitation.

Introduction on Differential Equations

<u>Types of Differential equations</u>:

We have learned in Chapter 2 that differential equations are the equations that involve "derivatives."

They are used extensively in mathematical modeling of engineering and physical problems.

There are generally two types of differential equations used in engineering analysis. These are:

1. Ordinary differential equations (ODE): Equations with functions that involve only one variable and with different orders of "ordinary" derivatives , and
2. Partial differential equations (PDE): Equations with functions that involve more than one variable and with different orders of "partial" derivatives.

<u>How differential equations are derived</u>?

They are derived from the three fundamental laws of physics of which most engineering analyses involve. These laws are:

(1) The law of conservation of mass,
(2) The law of conservation of energy, and
(3) The law of conservation of momentum.

Differential equations for mechanical engineering:

For mechanical engineering analyses, frequently used laws of physics include the following:

- The Newton's laws for statics, dynamics and kinematics of solids.
- The Fourier's law for heat conduction in solids.
- The Newton cooling law for convective heat transfer in fluids.
- The Bernoulli's principle for fluids in motion.
- Fick's law for diffusion of substances with different densities
- Hooke's law for deformable solids

7.2 **Review of Solution Methods for First Order Differential Equations**

In "real-world," there are many physical quantities that can be represented by functions involving only one of the four independent variables e.g., (x, y, z, t), in which variables (x,y,z) For space and variable t for time.

First order differential equations are the equations that involve highest order derivatives of order **one** . They are often called "the 1st order differential equations

Examples of first order differential equations:

Function σ(x)= the stress in a uni-axial stretched metal rod with tapered cross section (Fig. a), or Function v(x)=the velocity of fluid flowing in a straight channel with varying cross-section (Fig. b):

Figure a

σ(x)

x

Figure b

v(x)

x
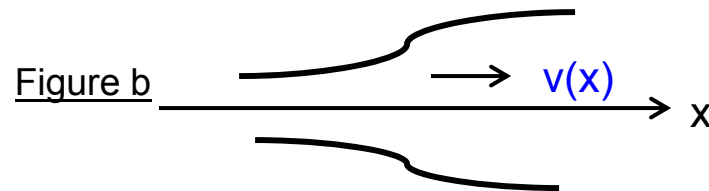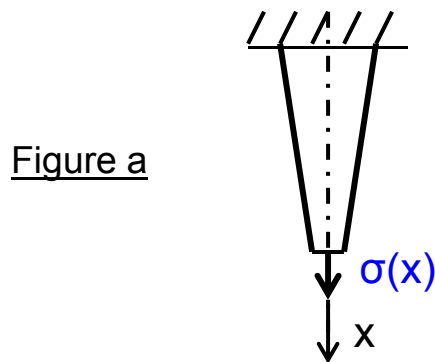
Mathematical modeling using differential equations involving these functions are classified as First Order Differential Equations

Typical form of the first order differential equations:

$$h(u)\frac{du(x)}{dx} = g(x)$$

(7.1)

in which h(u) and g(x) are given functions.

By re-arranging the terms in Equation (7.1) the following form with the left-hand-side (LHS) involves function of u (or constants) only, and the right-hand-side (RHS) consists of function of the variable g(x) (or constants) only :

h(u) du = g(x) dx

Solution u(x) of Equation (7.1) may be obtained by integrating both sides of the above equality and resulting in:

$$\int h(u)du = \int g(x)dx + c$$

(7.2)

where c is the integration constant to be determined by given specified condition for the problem.

Example 7.1 Solve the following first order ordinary differential equation: $x\frac{du(x)}{dx} + u^2 = 4$

Solution:

By re-arranging the terms in the DE into the separated form with the LHS involving the function u(x) and the RHS with the variable x:

$$\frac{du}{4 - u(x)^2} = \frac{dx}{x}$$ followed by integrating both sides to get: $\frac{1}{4}\ell n\frac{2+u}{2-u} = \ell n\,x + c$ with c=integration constant

The solution u in the above expression may be re-written in the following form:

$$u(x) = \frac{2(c'x^4 - 1)}{c'x^4 + 1}$$ with c′ being another arbitrary constant to be determined by specified condition

7.2.2 **Solution of linear, homogeneous equations**

Typical form of the equation:

$$\frac{du(x)}{dx} + p(x)u(x) = 0 \qquad (7.3)$$

The **solution** u(x) in the above equation is:

$$u(x) = \frac{K}{F(x)} \qquad (7.4)$$

where K = constant to be determined by given condition, and the function F(x) has the form:

$$F(x) = e^{\int p(x)dx} \qquad (7.5)$$

in which the function p(x) is given in the differential equation in Equation (7.3)

Example 7.2

Solve the following first order ordinary differential equation:

$$\frac{du(x)}{dx} = u(x)\sin x \qquad\qquad \text{(a)}$$

Solution:

We will first re-arrange the terms in Equation (a) in the following way:

$$\frac{du(x)}{dx} - (\sin x)u(x) = 0$$

By comparison, we have: p(x) = -sin x, which leads to the integral: $\int p(x)dx = \cos x$

and hence have: $F(x) = e^{\int p(x)dx} = e^{\cos x}$ from Equation (7.5)

The solution of the differential equation in Equation (a) is available in Equation (7.4), or in the form:

$$|u(x)| = \frac{e^c}{F(x)} = Ke^{-\cos x}$$

where K is an arbitrary constant to be determined by an appropriate prescribed condition.

7.2.3 **Solution of linear Non-homogeneous equations**:

Typical differential equation:

$$\frac{du(x)}{dx} + p(x)u(x) = g(x) \qquad (7.6)$$

The appearance of function g(x) in Equation (7.6) makes the DE non-homogeneous

The solution of ODE in Equation (7.6) is similar to the solution of homogeneous equation in a little more complex form than that for the homogeneous equation in (7.3):

$$u(x) = \frac{1}{F(x)} \int F(x)\, g(x)\, dx + \frac{K}{F(x)} \qquad (7.7)$$

where function F(x) can be obtained from Equation (7.5) as:

$$F(x) = e^{\int p(x)\,dx} \qquad (7.5)$$

where the integral in Equation (7.5) $F(x) = e^{\int p(x)\,dx}$ is called the "integration factor,"

and K is the integration constant.

Example 7.3

Solve the following first order non-homogeneous differential equation:

$$x^2 \frac{du(x)}{dx} + 2u(x) = 5x$$

Solution:

By re-arranging the terms, we get:

$$\frac{du(x)}{dx} + \frac{2}{x^2} u(x) = \frac{5}{x} \qquad\qquad \text{(a)}$$

By comparison of Equations (a) and (7.6), we get:

$$p(x) = \frac{2}{x^2} \qquad and \qquad g(x) = \frac{5}{x}$$

The integration factor in Equation (7.5) is $\quad F(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x^2} dx} = e^{-\frac{2}{x}}$

The solution of Equation (a), u(x) is obtained by substituting the above integration factor into Equation (7.7) and resulted in:

$$u(x) = \frac{1}{F(x)} \int F(x)\, g(x)\, dx + \frac{K}{F(x)} = \frac{1}{e^{-\frac{2}{x}}} \int e^{-\frac{2}{x}} \left(\frac{5}{x} \right) dx + \frac{K}{e^{-\frac{2}{x}}} = \frac{5}{e^{-\frac{2}{x}}} \int x^{-1} e^{-\frac{2}{x}} dx + Ke^{\frac{2}{x}}$$

The integration constant K in the above solution of this ODE can be determined by specified conditions, as will be demonstrated in the next example - Example 7.4.

Example 7.4

Solve the following differential equation:

$$\frac{du(x)}{dx} + 2u(x) = 2 \qquad\qquad\qquad (a)$$

with a given condition: u(0) = 2   (b)

Solution:

By comparison of Equation (a) with the typical form in Equation (7.6), we will have: p(x) = 2 and g(x) = 2.

Thus, from Equation (7.5), we get: $F(x) = e^{\int p(x)dx} = e^{\int 2dx} = e^{2x}$

Following Equation (7.7), we have the solution of Equation (a) as

$$u(x) = \frac{1}{F(x)}\int F(x)\,g(x)\,dx + \frac{K}{F(x)} = \frac{1}{e^{2x}}\int\left(e^{2x}\right)(2)dx + \frac{K}{e^{2x}} = 1 + \frac{K}{e^{2x}} \qquad (c)$$

The integration constant K may be determined by using the specified condition in Equation (b) with u(x) = 2 at x = 0. Thus by substituting this condition into Equation (c), we will get the relationship: $1 + \dfrac{K}{e^{2x}}\bigg|_{x=0} = 2$ from which we may obtain K = 1

Consequently, we have the complete solution of u(x) in Equation (a) to be:

$$u(x) = 1 + \frac{1}{e^{2x}} = 1 + e^{-2x}$$

**Application of First Order Differential Equation to Fluid Mechanics Analysis**

Fundamental Principles of Fluid Mechanics Analysis:

Fluids - A substance with **mass** but **no** shape

Compressible
(Gases)

Non-compressible
(Liquids)
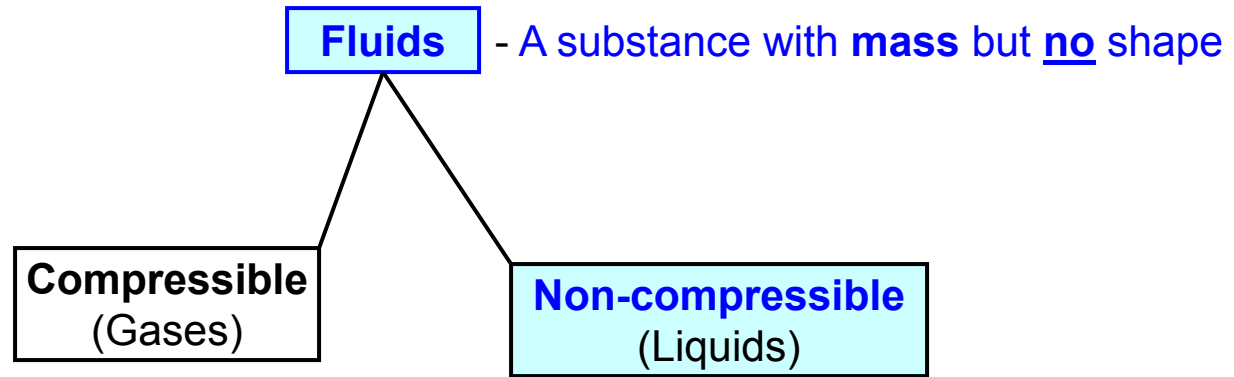
Moving of a fluid requires:

- A conduit, e.g., tubes, pipes, channels, etc.
- Driving **pressure**, or by gravitation, i.e., difference in "head"
- Fluid flows with a **velocity** v from higher pressure (or elevation) to lower pressure (or elevation)
- Fluid flows from higher elevation to low elevation

Higher pressure (or elevation)

Fluid velocity, v

Cross-sectional
Area, A

Lower pressure (or elevation)
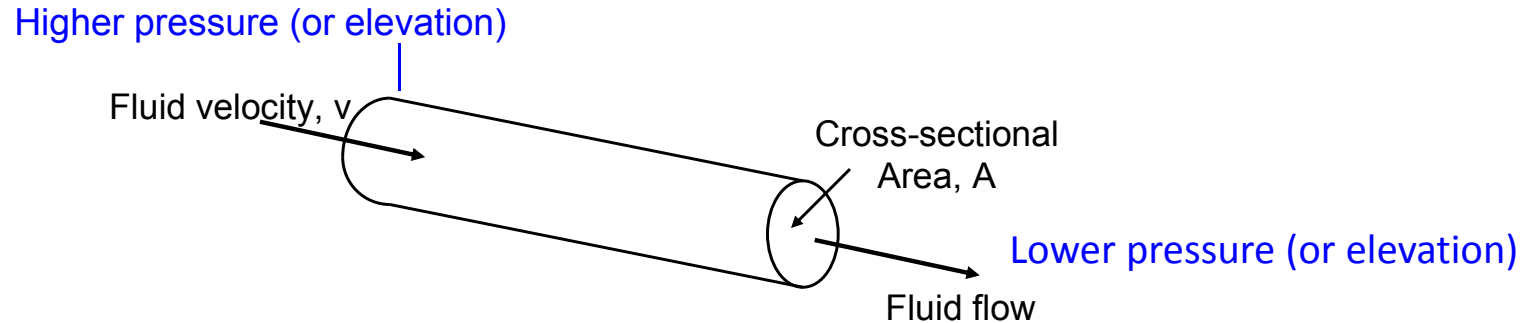
Fluid flow

The total mass flow, $\quad Q = \rho\,A\,v\,\Delta t \qquad$ kg $\hspace{5cm}$ (7.8a)

Total mass flow rate, $\dot{Q} = \dfrac{Q}{\Delta t} = \rho\,A\,v \qquad (kg\,/\sec)$ $\hspace{3cm}$ (7.8b)

Total volumetric flow rate, $\dot{V} = \dfrac{\dot{Q}}{\rho} = Av \qquad (m^3\,/\sec)$ $\hspace{2cm}$ (7.8c)

Total volumetric flow, $\;V = \dot{V}\,\Delta t = A\,v\,\Delta t \qquad (m^3)$ $\hspace{2.5cm}$ (7.8d)

in which ρ = mass density of fluid (g/m³),
$\qquad$ A = cross-sectional area (m²),
$\qquad$ v = velocity (m/s), and Δt = duration of fluid flow (s).
The units associated with the above quantities are (kg) for kilograms, (m) for meters and (sec) for seconds.

- In case the velocity varies with time, i.e., v = v(t):
  Then the change of volumetric flow becomes: ΔV = A v(t) Δt, $\qquad$ (7.9)
  with Δt = time duration for the fluid flow

(It is the mathematical expression of the law of physics that relates the driving pressure and velocity in a moving non-compressible fluid)
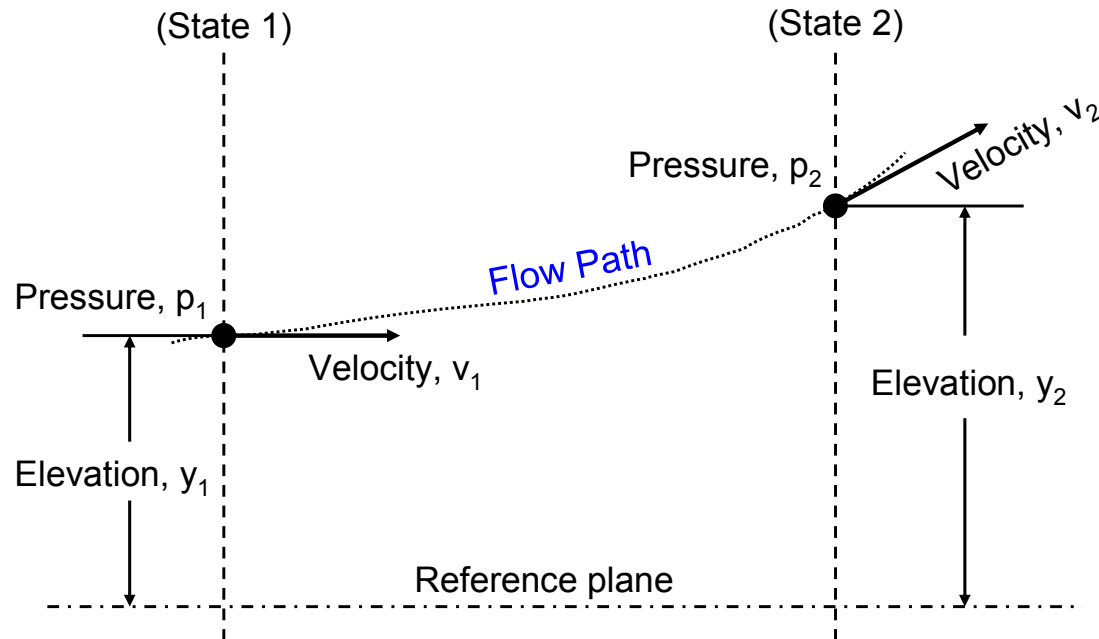

Using the **Law of conservation of energy**, or the First Law of Thermodynamics, for the energies of the fluid at State 1 and State 2, we can derive the following expression relating driving pressure (p) and the resultant velocity of the flow (v):

The Bernoulli Equation:

$$\frac{v_1^2}{2g} + \frac{p_1}{\rho g} + y_1 = \frac{v_2^2}{2g} + \frac{p_2}{\rho g} + y_2$$

Application of Bernoulli equation in liquid (water) flow in a LARGE reservoir:

We assume the friction between the moving fluids and their containing wall is negligible in all cases.
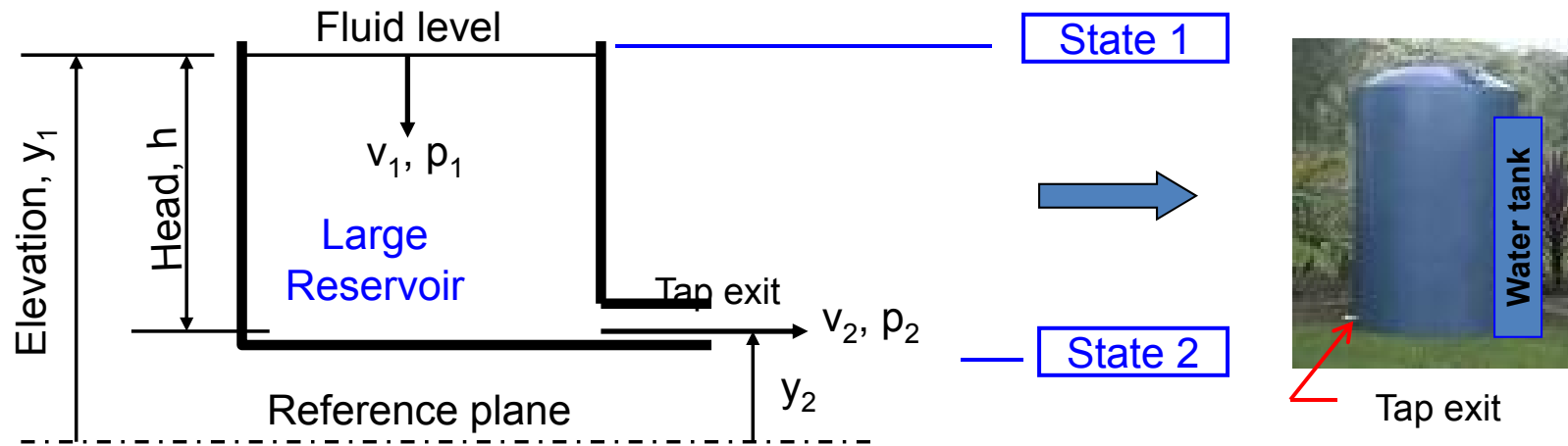
From the Bernoullis's equation, we have:

$$\frac{v_1^2 - v_2^2}{2g} + \frac{p_1 - p_2}{\rho g} + (y_1 - y_2) = 0 \qquad (7.10)$$

If the difference of elevations between State 1 and 2 is not too large, we can have: $p_1 \approx p_2$

Also, because it is a LARGE reservoir (or tank), we realize that $v_1 << v_2$, or $v_1 \approx 0$

Equation (7.10) can thus be reduced to the form: $-\dfrac{v_2^2}{2g} + 0 + h = 0$ $\quad$ *with* $h = y_1 - y_2$

from which, we may express the **exit velocity** of the liquid at the tap, i.e. v_2 to be:

$$v_2 = \sqrt{2gh} \qquad (7.11)$$

7.3.3 **The Continuity Equation**

This equation is derived from the law of conservation of mass. By referring to the situation depicted in figure below, the non-compressible fluids (for most liquids) flow from a section of a pipeline with larger cross-sectional area A_1 to a section with smaller cross-sectional area A_2.
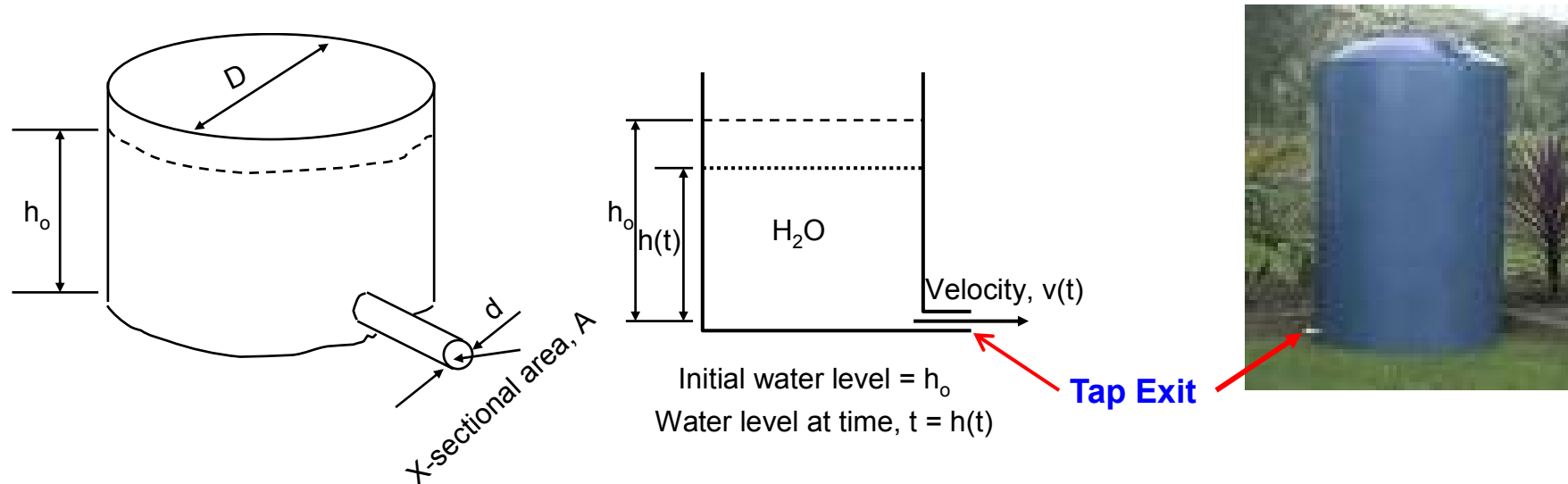
The rate of volumetric flow follows the rule:

$$q = A_1v_1 = A_2v_2 \quad m^3/s \qquad\qquad (7.12)$$
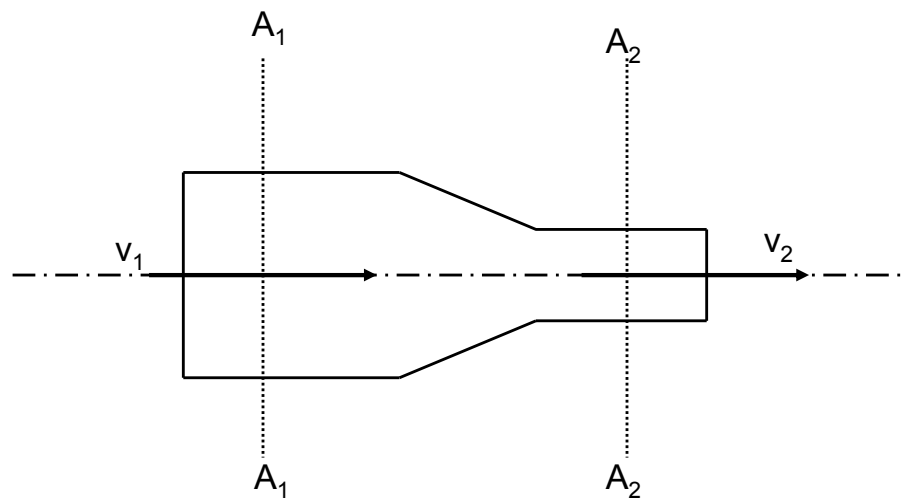
Liquid Flow in Reservoirs, Tanks and Funnels

7.4.1 **Derivation of differential equations and drainage of a large tank or reservoir:**

h_o

D

d

X-sectional area, A

h_o

$h(t)$

H_2O

Velocity, v(t)

Initial water level = h_o

Water level at time, t = h(t)

Tap Exit

The physical condition for math modeling: The **law of conservation of mass requires**:

The total volume of water leaving the tank at the exit during Δt: (ΔV_{exit}) =
The total volume of water supplied by the tank during Δt: (ΔV_{tank})

We have from Equation (7.8d): $\Delta V = A\,v(t)\,\Delta t$, in which v(t) is the velocity of moving fluid

Thus, the volume of water leaving the tap exit is:

$$\Delta V_{exit} = A\,v(t)\,\Delta t = \left(\frac{\pi d^2}{4}\right)\sqrt{2g\,h(t)}\,\Delta t \qquad \text{(a)}$$

where $v(t) = \sqrt{2g\,h(t)}$ is the instantaneous velocity of the water at the exit following the expression in Equation (7.11), and h(t) is the instantaneous water level in the reservoir.

Next, we need to formulate the water supplied by the tank, ΔV_{tank}:
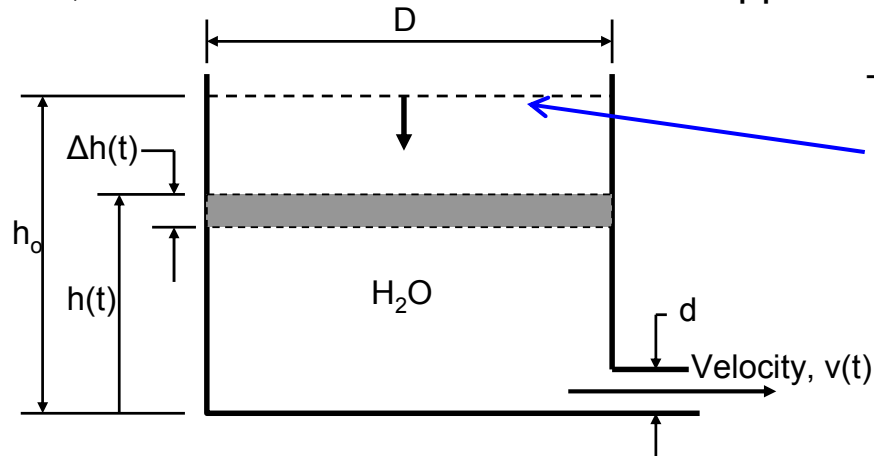
The given initial water level in the tank is h_o

The water level keeps dropping after the tap exit is opened, and the reduction of water level is CONTINUOUS with time t

Let the water level at time t be h(t)

We let $\Delta h(t)$ = amount of drop of water level during time increment Δt

Then, the volume of water LOSS in the circular tank is: $\Delta V_{tank} = -\dfrac{\pi D^2}{4} \Delta h(t)$ (b)

(**Caution:** a "-" sign is given to ΔV_{tank} , b/c of the LOSS of water volume with **increasing time** Δt

The total volume of water leaving the tank during Δt (ΔV_{exit}) in Equation (a) =
    The total volume of water supplied by the tank during Δt (ΔV_{tank}) in Equation (b):

Thus, by equating Equation (a) = Equation (b), we get:

$$\frac{\pi d^2}{4} \sqrt{2g\,h(t)}\, \Delta t = -\frac{\pi D^2}{4} \Delta h(t) \qquad (c)$$

or by re-arranging the above: $\dfrac{\Delta h(t)}{\Delta t} = -\left[h(t)\right]^{1/2} \left(\dfrac{d^2}{D^2}\right)\sqrt{2g}$ (d)

If the process of draining is indeed **CONTINUOUS**, i.e., $\Delta t \rightarrow 0$, we will have Equation (d) expressed in the "differential" rather than "difference" form as follows:

$$\frac{dh(t)}{dt} = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)\sqrt{h(t)} \tag{7.13}$$

with an initial condition of h(0) = h$_o$ $\hspace{2cm}$ (f)

Equation (7.13) is the 1st order differential equation for the CONTINUOUS draining of a water tank.

The solution of Equation (7.13) can be obtained by separating the solution h(t) and the variable t by re-arranging the terms in the following way:

$$\frac{dh(t)}{\sqrt{h(t)}} = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)dt$$

Upon integrating on both sides: $\int h^{-1/2}\,dh = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)\int dt + c$ where c = integration constant

from which, we obtain the solution of Equation (7.13) to be: $2h^{1/2} = -\sqrt{2g}\left(\frac{d^2}{D^2}\right)t + c$

The constant $c = 2\sqrt{h_o}$ is determined from the initial condition in Equation (f).

The complete solution of Equation (7.13) with the initial condition in Equation (f) is thus:

$$h(t) = \left[-\sqrt{\frac{g}{2}}\left(\frac{d^2}{D^2}\right)t + \sqrt{h_o}\right]^2 \tag{7.14}$$

The solution in Equation (g) will allow us to determine the water level in the tank at any given instant, t.

One may imagine that PHYSICALLY, the time required to drain the tank is the time **t$_e$.**

Mathematically, it can be expressed as **h(t$_e$) = 0:** ⟶ $0 = \left[-\sqrt{\frac{g}{2}} \left(\frac{d^2}{D^2} \right) t_e + \sqrt{h_o} \right]^2$

We may solve the above algebraic expression for t$_e$ to be:

$$t_e = \frac{D^2}{d^2} \sqrt{\frac{2h_o}{g}} \qquad s \qquad\qquad (7.15)$$

Numerical example:

Tank diameter, D = 12" = 1 ft.
Drain pipe diameter, d = 1" = 1/12 ft.
Initial water level in the tank, h$_o$ = 12" = 1 ft.
Gravitational acceleration, g = 32.2 ft/sec.
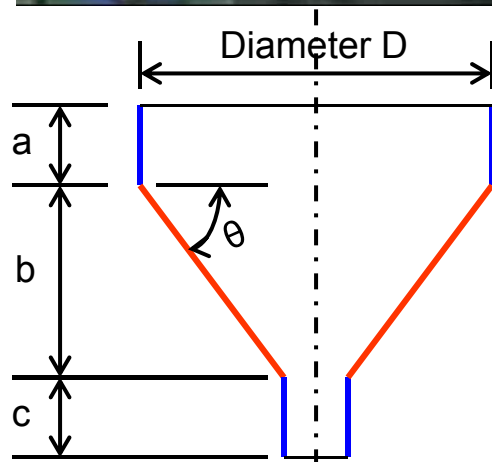The time required to empty the tank is: $\quad t_e = \left(\frac{1}{\frac{1}{12}} \right)^2 \sqrt{\frac{2x1}{32.2}} = 35.89 \qquad sec\,onds$

So, now you have learned how to determine the time required to drain a large "fish tank" or a "process tank" or a "swimming pool," Make sure you do.

Tapered funnels are common pieces of equipment used in many process plants, such as, in wine bottling and food processing plants, as illustrated below:

Design of tapered funnels involves the determination of configurations, i.e. the tapered angle Ө, and the diameters D and lengths of sections of the funnel (a, b and c) for the intended liquid contents.

It is also required the determination on the time required to empty the contained liquid.

Formulations on Water or Liquid Level in Tapered Funnels
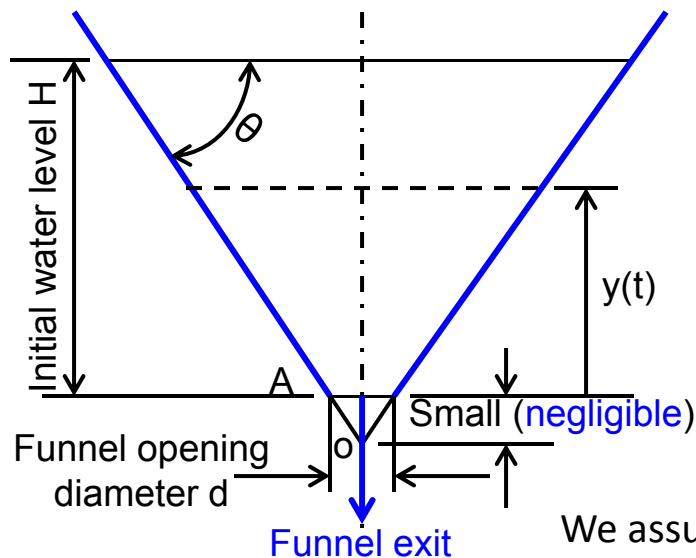
The physical solution we are seeking is the water level at ANY given time t, $y(t)$ after the water is let to flow from the exit of the funnel at $t = 0^+$.

The "real" funnel has an outline of frustum cone with smaller circular opening in the bottom end "A" allowing the contained liquid to leave the funnel. In the subsequent analysis, we assume the funnel has an outline shape of "right cone" with its tip at "O" Instead.

We assume the initial water level in the funnel = H

Once the funnel exit is open, and the liquid begins to flow. The liquid level in the funnel at given time t is represented by the function $y(t)$.

We will use the same principle in Section 7.4 .1 to formulate the expression for $y(t)$ for the straight tank:

The total volume of liquid leaving the funnel during Δt (ΔV_{exit})   =
The total volume of liquid supplied by the funnel during Δt (ΔV_{funnel})

Determine the Instantaneous Water Level in a Tapered Funnel, y(t):
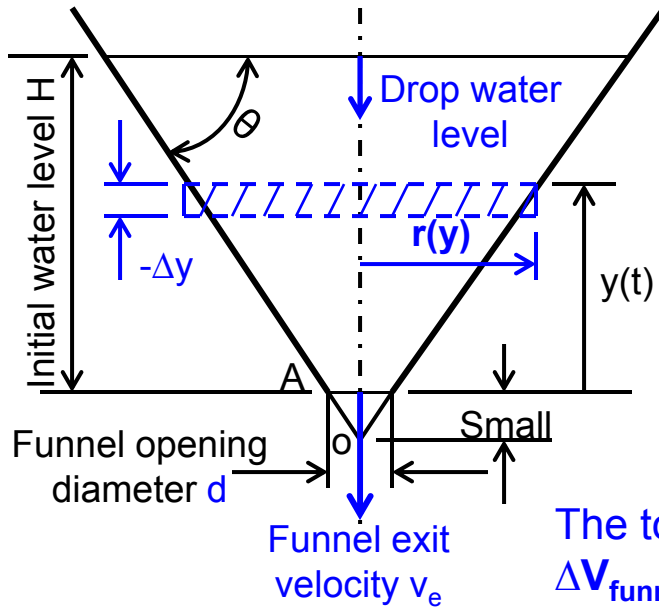
The total volume of water leaving the tank during Δt:
$$(\Delta V_{exit}) = (A_{exit})(v_e)(\Delta t)$$

But from Equation (7.11), we have the exit velocity v_e to be:
$$v_e = \sqrt{2\,g\,y(t)}$$

which leads to the exit volume of water to be:

$$\Delta V_{exit} = \frac{\pi d^2}{4}\sqrt{2\,g\,y(t)}\;\Delta t \qquad (7.16)$$

The total volume of water supplied by the tank during Δt:
ΔV_{funnel} = volume of the cross-hatched "water disk" in the diagram

$$\Delta V_{funnel} = - [\pi\,(r(y)^2)]\,(\Delta y) \qquad (7.17)$$

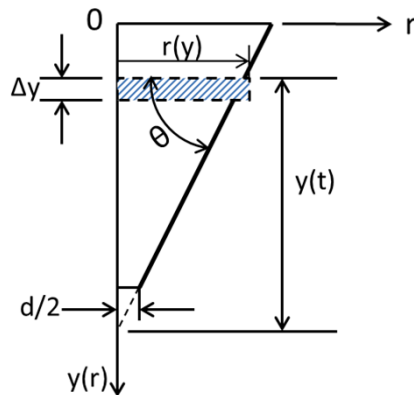A "-ve" sign to indicate decreasing ΔV_{funnel} with increasing $y(t)$

From the diagram in the left, we have: $r(y) = \dfrac{y(t)}{\tan\theta}$ \qquad (7.18)

Hence by equating (7.16) and (7.17) with r(y) given in (7.18), we have:

$$\frac{\pi d^2}{4}\sqrt{2\,g\,y(t)}\;\Delta t = -\pi\,\frac{[y(t)]^2}{\tan^2\theta}\,\Delta y \qquad (7.19)$$
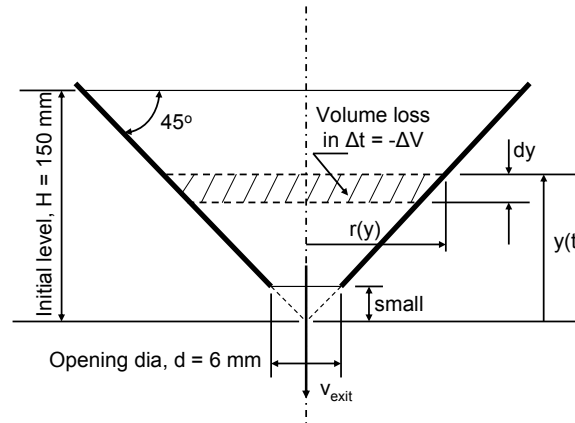
For a CONTINUOUS variation of y(t), we have $\Delta t \to 0$, we will have the differential equation for y(t) as:

$$\frac{[y(t)]^{\frac{3}{2}}}{\tan^2\theta}\,\frac{dy(t)}{dt} = -\frac{d^2}{4}\sqrt{2g} \qquad (7.20)$$

Example 7.6

Determine the time required to drain a tapered funnel with a taper angle 45° as shown in figure below. The funnel contains water with an initial level H = 150 mm.

Solution:

According to Equation (7.12), the rate of volumetric flow of water leaving the funnel, q_e (or $\dot{V}$ in ft³/s) is $q_e = A_e \, v_{exit}$, in which the cross-sectional area of the funnel at the exit: $A_e = \dfrac{\pi d^2}{4}$

But as we have, from the Bernoulli equation and Equation (7.11), the exit velocity at the bottom of the funnel to be:
$$v_{exit} = \sqrt{2g \, y(t)}$$

We thus have the total volumetric flow of water continuously leaving the funnel during time interval Δt (or dt in a continuous flow process) to be:

$$\Delta V_e = q_e \, \Delta t = \frac{\pi d^2}{4}\sqrt{2g \, y(t)} \, \Delta t \tag{a}$$

The water supply in this case can be related to the drop of water level at the "upstream" in the funnel as expressed by the following equation:

Example 7.6 – Cont'd

The "loss" of volume of water: $\Delta V_{lost} = -\left(\pi r^2\right)dy = -\pi y^2\, dy$ (b)

By referring to geometry of the funnel with a tapered angle $\theta = 45°$, we can relate the instantaneous radius of the funnel, r(y) to the instantaneous water level y(t) to be:

$$r = \frac{y}{\tan 45^o} = y \qquad \text{as in Equation (b)}$$

By the law of conservation of mass, we should have ΔV_e in Equation (a) = ΔV_{lost} in Equation (b), which leads to the following equality: $\frac{\pi d^2}{4}\sqrt{2g\, y(t)}\, dt = -\pi y^2\, dy$

from which we may derive the differential equation for determining the instantaneous water level in the funnel y(t) by the following differential equation:

$$\frac{y^2}{\sqrt{y}}\frac{dy}{dt} + \frac{d^2}{4}\sqrt{2g} = 0 \qquad\qquad (c)$$

We may solve this separable differential equation in (c) by re-arranging its terms to be:

$$\frac{y^2}{\sqrt{y}}dy = -\frac{d^2}{4}\sqrt{2g}\, dt$$

and solve the equation to get: $\frac{2}{5}y^{5/2} = -\frac{d^2}{4}\sqrt{2g}\, t + c$ with an integration constant c, which can be determined by the given initial condition of y(0) = H=150 mm, resulting in: $c = \frac{2}{5}H^{5/2}$

We thus have the complete solution of the differential equation to be:

$$\frac{2}{5}\left(y^{5/2} - H^{5/2}\right) = -\frac{d^2}{4}\sqrt{2g}\, t \qquad \text{or} \qquad [y(t)]^{\frac{5}{2}} = -\frac{5d^2}{8}\sqrt{2g}\, t + H^{\frac{5}{2}} \qquad (d)$$

The time required to drain the funnel is obtained by letting y(t$_e$)=0 in Equation (d) and solve for t$_e$

$$t_e = \frac{8H^{5/2}}{5d^2\sqrt{2g}} \qquad\qquad (e)$$

Example 7.7

Consider a circular funnel made up by a straight section on its top and a lower tapered section as illustrated in Figure 7.10 (a), with the dimensions indicated in Figure 7.10 (b).
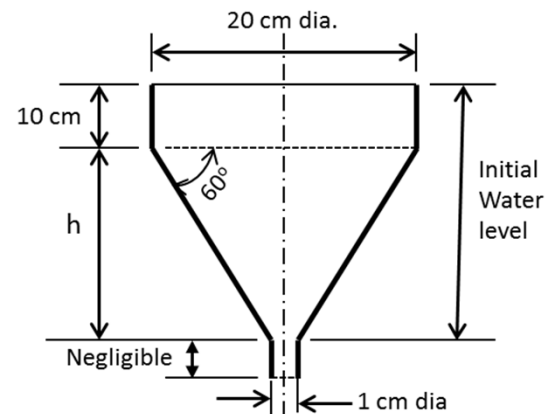

Figure 7.10(a)

Figure 7.10(b)

Use the integration method to determine: (a) the volume of water it contains, and (b) the time required to drain this funnel system from its initial level as indicated in Figure 7.10 (b).

Solution:

We will first establish the initial water level in the funnel system by computing the length of the tapered section of the funnel in Figure 7.10(b). This is done by relating the tapered angle $\theta = 60°$ and the radius of the circular straight portion of the funnel of 10 cm, and with the relation of $h = 10 \tan(60°) = 17.32$ cm. We thus have the initial water level in the funnel to be:

$$y(t)\big|_{t=0} = y(0) = H_0 = 10 + 17.32 = 27.35 \; cm$$ in which y(t) is the water level in the funnel with

the origin of the coordinate $y = 0$ located at the lower end of the tapered section of the funnel.

Example 7.7 – Cont'd

20 cm dia.

10 cm

h

60°

Negligible

1 cm dia

Initial Water level
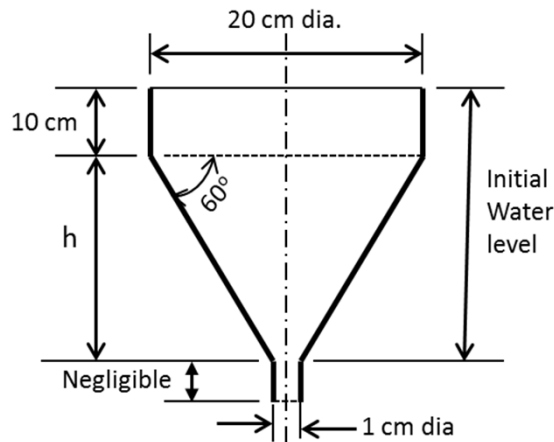
We realize that the funnel system consists of two portions: (1) The straight cylindrical portion [Portion (I)] on the top, and (2) The tapered portion [Portion (II)] at the lower portion of the funnel system.

Both sections share the same exit at the bottom of the system.

(a) Compute the volume of the compound funnel:

The volume of the straight section V_1 can be obtained by using Equation (2.17) as follows:

$$V_1 = \int_0^{10} \pi \left(\frac{20}{2}\right)^2 dy = 100\pi \left. y \right|_0^{10} = 1000\pi = 3140 \ cm^3$$

The volume of the tapered portion of the funnel system (V_2) can be computed using Equation (2.16) with the profile of the funnel defined by the x-y coordinates shown in following Figure 7.11.
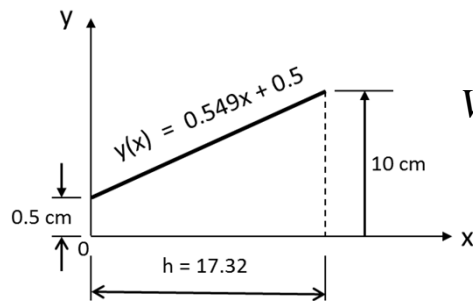

$y(x) = 0.549x + 0.5$

10 cm

0.5 cm

h = 17.32

Figure 7.11
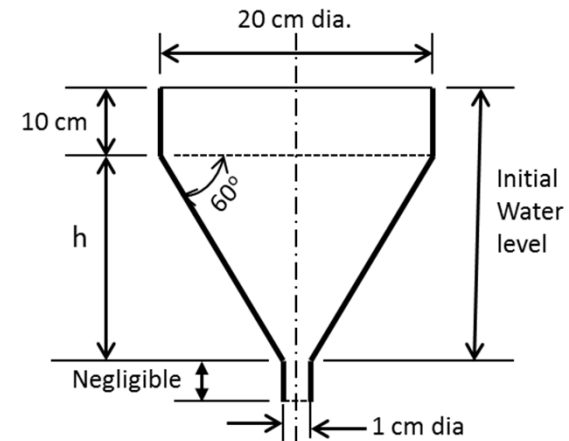
$$V_2 = \pi \int_0^{17.32} y(x)dx = \pi \int_0^{17.32} (0.549x + 0.5)^2 dx = 606.3\pi = 1903.78 \ cm^3$$

The total volume of the compound funnel is $V = V_1 + V_2 = 5043.78$ cm^3

Example 7.7 – Cont'd

(b) Time required to drain the funnel:

We will compute the time required to drain each section, from which we may obtain the total time required to drain the entire funnel.

20 cm dia.

10 cm

60°

h

Initial Water level

Negligible

1 cm dia

We may use the formula derived for draining a straight cylindrical tank presented in Equation (7.15) for t_{e1} by setting the diameter of the exit hose to be 1 cm as shown in the above Figure 7.10(b). We thus calculate t_{e1} with D = 20 cm, d = 1 cm, h_0 = 10 cm and g = 9.81 cm/s^2, and result in:

$$t_{e1} = \frac{D^2}{d^2}\sqrt{\frac{2h_0}{g}} = \left(\frac{20}{1}\right)^2\sqrt{\frac{2x10}{981}} = 57\ s$$

The time required to drain the lower tapered section of the funnel system requires the following derivation of the differential equation for the solution of instantaneous water level first, as presented in Figure 7.12.

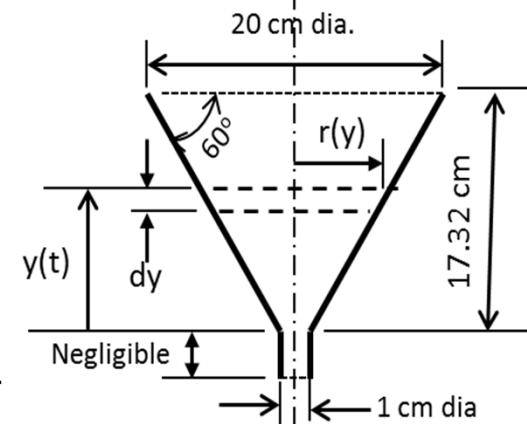We may use Equation (7.20) derived for tapered funnel with taper angle θ in the following form:

20 cm dia.

60°   r(y)

y(t)   dy

17.32 cm

Negligible

1 cm dia

Figure 7.12

$$\frac{[y(t)]^{\frac{3}{2}}}{\tan^2\theta}\frac{dy(t)}{dt} = -\frac{d^2}{4}\sqrt{2g}$$

(7.20)

Example 7.7 – Cont'd

20 cm dia.

60°

r(y)

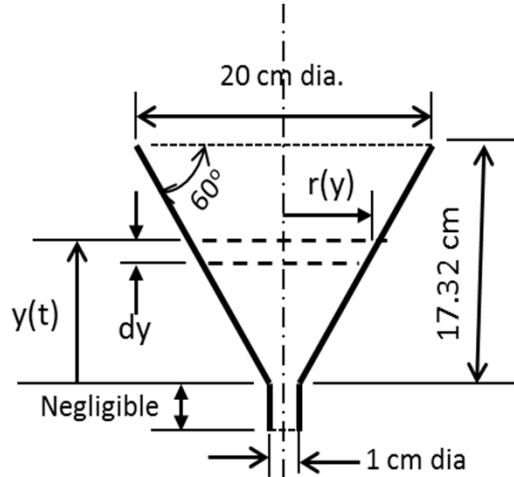17.32 cm

y(t)

dy

Negligible

1 cm dia

Figure 7.12

By solving the differential equation $\dfrac{[y(t)]^{\frac{3}{2}}}{\tan^2\theta}\dfrac{dy(t)}{dt} = -\dfrac{d^2}{4}\sqrt{2g}$

that we derived to solve for the instantaneous water level in the tapered section of the funnel system with $\theta = 60°$, $d = 1$ cm and $g = 981$ cm/s^2 , we get the differential equation that fits the current situation as:

$$[y(t)]^{\frac{3}{2}}\frac{dy(t)}{dt} = -33.22$$

with a solution: $[y(t)]^{\frac{5}{2}} = -83.047t + c$

The integration constant c in the above expression may e obtained by using the condition y(0)=17.32 cm, leading to c = 1248.45, we thus obtain the instantaneous water level in the tapered section of the funnel to be:

$$[y(t)]^{\frac{5}{2}} = -83.047t + 1248.45 \qquad\qquad (c)$$

The physical situation for an empty tapered funnel corresponding to a math model of having $y(t_{e2}) = 0$ in Equation (c), from which we may solve for t_{e2} from the following equation: $[y(t_{e2})]^{\frac{5}{2}} = 0 = -83.047t_{e2} + 1248.45$ from which we obtain:

$$t_{e2} = \frac{1248.45}{83.047} = 15.03\ s$$

We may thus solve for the total time required to drain the compound funnel to be:

$$t_e = t_{e1} + t_{e2} = 57 + 15.03 = 72.03\ s$$

Analysis on drainage of a funnel in winery: To design a funnel that will fill a wine bottle with estimated time to fill the bottle

Design objective: To provide SHORTEST time in draining the funnel for fastest bottling process

Given

Max space limit

?

?

?

Given diameter

Bottle with given Geometry and dimensions
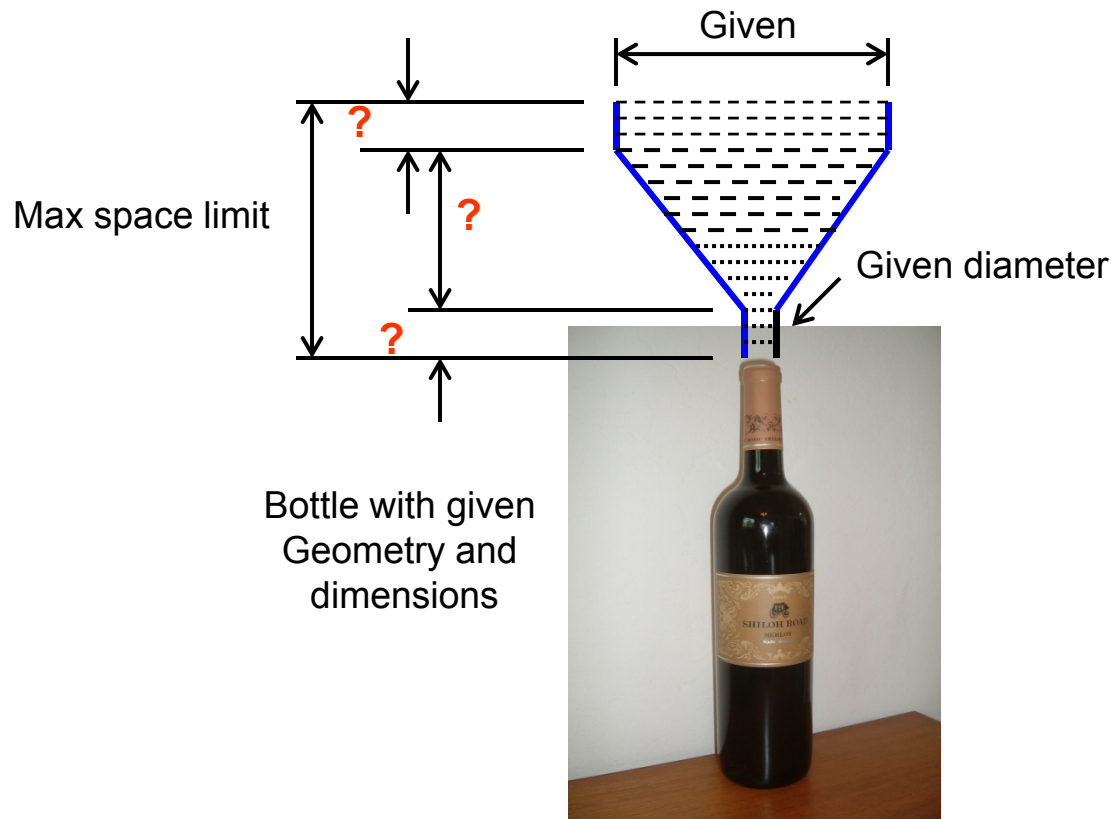
Example 7.8

Design a circular funnel with a taper angle 60 degrees to fill the wine bottle described in Example 2.10 in Chapter 2 (P. 45).

The designer will configure and set the dimensions of the radius and the length H of the funnel with the diameter of the exit to be 1.5 cm, as illustrated in Figure 7.13 (a). Also determine the time required to fill the bottle with dimensions of the bottle as stipulated in Figure 2.33b.

Figure 7.13 Filling a wine bottle by a funnel
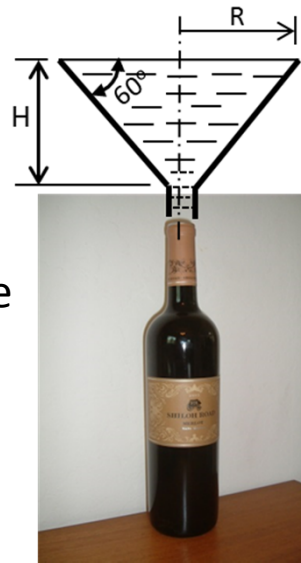
(a) The wine bottle and funnel
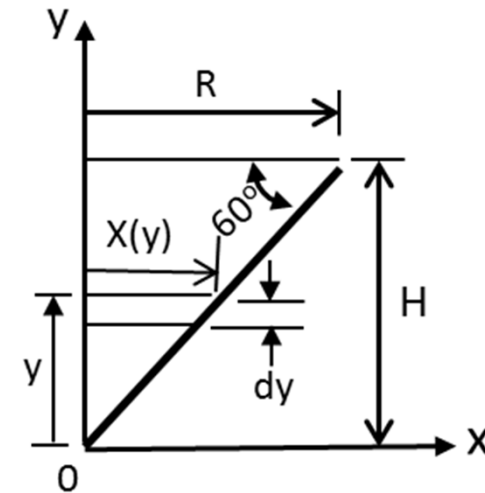
(b) Volume of a tapered funnel

Solution:

(a) Determine the dimensions of the funnel: We will first set the volume of wine content in the bottle to be 841.52 cm³ as determined in Example 2.10.

The purpose of this analysis is to design a funnel having the same volume as the bottle.

Example 7.8- Cont'd

To achieve this purpose, we will need to determine the maximum radius R of the funnel relating to the length H using the given taper angle of the funnel of θ = 60°. This can be done by the geometric relation of: tan60° = H/R, from which we get the following relationship of R and H as:

$$R = H/\tan60° = 0.577H \qquad\qquad (a)$$

We will then determine both R and H for the funnel with a containment volume of 841.52 cm³.

By referring to Figure 7.13(b), we would have the volume of the funnel using Equation (2.17) with R(y) = 0.577y to be:

$$V_f = \pi \int_0^H [R(y)]^2 \, dy = \pi \int_0^H (0.577y)^2 \, dy = (0.577)^2 \pi \frac{y^3}{3}\bigg|_0^H = 0.3485H^3$$

But the **volume of the funnel = volume of the bottle**, $V_f = 0.3464 H^3 = 841.52$ cm³ which leads to the diameter of the funnel to be: D = 2R = 2x(0.577x13.416) = 15.4821 cm.

(b) **Time required to drain the funnel (**or time required to fill the bottle**):**

Now that we have determined the geometry, the dimensions and the volume of the tapered funnel, we may proceed to find the time required to drain this funnel as follows:

We may use Equation (7.20) to determine the required time to drain the tapered funnel. Thus, by substituting θ = 60° and d = 1.5 cm into Equation (7.20), we will have the differential equation for the instantaneous wine lever in the funnel y(t) during the drainage process to be:

$$\frac{[y(x)]^{\frac{3}{2}}}{(\tan 60°)^2} \frac{dy(t)}{dt} = -\frac{(1.5)^2}{4}\sqrt{2x981} \quad \text{or} \quad [y(t)]^{\frac{3}{2}} \frac{dy(t)}{dt} = -74.7513 \qquad\qquad (b)$$

Example 7.8- Cont'd

The solution y(t) of Equation (b) may be obtained by integrating both sides of Equation (b) resulting in:

$$[y(t)]^{2.5} = -186.8783t + C \qquad \text{(c)}$$

The integration constant c in Equation (c) may be determined with the initial condition of $y(0) = H = 13.416$ cm in Part (a) in the solution. We may thus obtain: c = 634.9654.

The solution of the differential equation in (b) thus has the form of:

$$[y(t)]^{2.5} = -186.8783\, t + 634.9654 \qquad \text{(d)}$$

If we let the time required to fill the wine bottle = t_e. This time t_e is the same as the time required to empty the funnel. Consequently, we may obtain the value of te to be the required time to drain the funnel and fill the wine bottle in Figure 7.13(a). We may thus solve for t_e from the following Equation (d) to give:

$$y(t_e) = -186.8783(t_e) + 634.9654 = 0 \qquad \text{(e)}$$

By solving t_e from Equation (e), we have the time required to fill the bottle to be: $t_e = 3.42$ s

7.5 **Applications of First Order Differential Equations in Heat Transfer Analysis**

Physics of heat transmission in substances in three(3) distinct modes:

- **Heat conduction in solids**
- **Heat convection in fluids**
- **Radiation of heat in space**

Heat transfer in engineering analysis determines the temperature fields [or temperature distributions (or temperature variations)] in solid structures or fluids in engineering systems. Temperature fields in an engineering system can induce significant stresses, known as "thermal stresses" in which the temperature fields exist. Ignorance of these induced thermal stresses may cause overall failure of the engineering systems due to over-stresses in the engineering systems.

Other significant consequences to structures or einginnering systems with excessive temperature effects may include: (1) significant deterioration of material properties (e.g., Young's modulus, yield strength, ultimate tensile strength, etc.), at elevated temperatures, and (2) creep deformation of the materials –leading to creep failure of structure. Neither of these effects can be ignored in the engineering analyses.

7.5.1 **Fourier's Law of Heat Conduction in Solids**

● Heat flows in SOLIDS by conduction
● Heat flows from the part of solid at higher temperature toward the part with low temperature
   - a situation similar to water flow from higher elevation to low elevation
● Thus, there is definite relationship between the amount of heat flow (Q) and the temperature difference (ΔT) in the solid
● Relating the Q and ΔT is what the Fourier's law of heat conduction governs

Derivation of Fourier's Law of Heat Conduction:

Let us consider a solid slab: With its left surface maintained at temperature T_a and its right surface at T_b
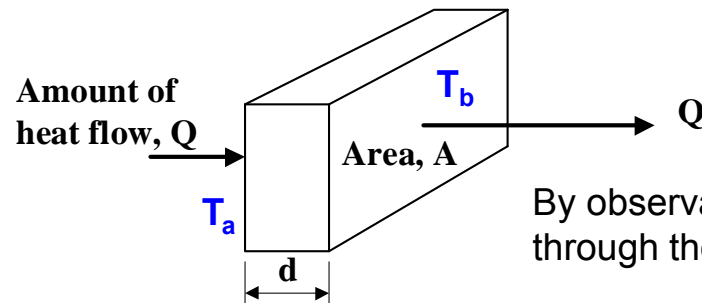
Heat will flow from the left to the right surface if

$$T_a > T_b$$

By observations, we can formulate the total amount of heat flow (**Q**) through the thickness of the slab relating to the following parameters:

$$Q \propto \frac{A(T_a - T_b)t}{d}$$

where A = the area to which heat flows; t = time allowing heat flow; and d = the distance of heat flow.

Replacing the $\propto$ sign in the above expression by an = sign and a constant k, leads to:

$$Q = k \frac{A(T_a - T_b)t}{d} \qquad (7.21)$$

The constant k in Equation (7.21) is called "thermal conductivity" – treated as a property of the solid Material with a unit: Btu/in-s-^{o}F in traditional system, or W/m-^{o}C in the SI or metric system.

The amount of total heat flow in a solid (Q) as expressed in Equation (7.21) is useful, but make less engineering sense without specifying the area A and time t in the heat transfer process.

Consequently, the "**Heat flux**" (**q**) – a sense of the intensity of heat conduction is used more frequently in engineering analyses. From Equation (7.21), we may define the heat flux q as:

$$q = \frac{Q}{At} = k\frac{(T_a - T_b)}{d} \qquad (7.22)$$

Note: Heat flux q is a vector quantity and it has a unit of: Btu/in²-s, or W/m² or J/m², or N/m-s in SI system

We realize Equation (7.22) is derived from a situation of heat flowing through a thickness of a slab with distinct temperatures at both surfaces maintained with temperatures T_a and T_b..

In a situation in which the temperature variation **within** the solid is **CONTINUOUS**, by a function **T(x),** as illustrated in the figure below:
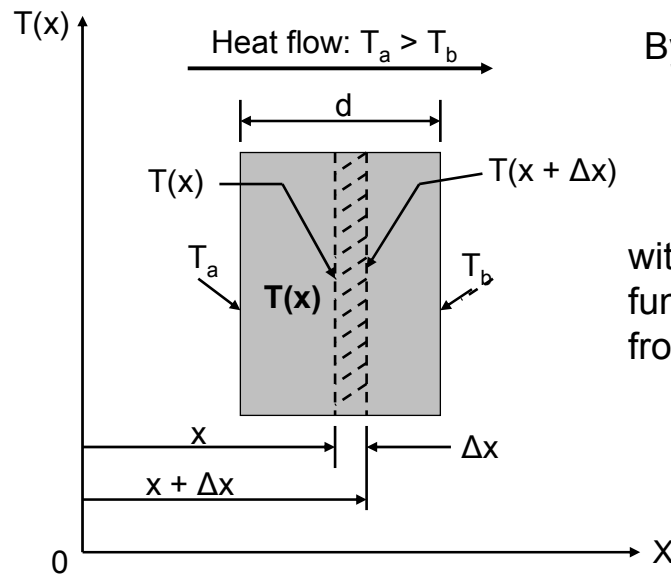
By following the expression in Equation (7.22), we will have:

$$q = k\frac{T(x) - T(x + \Delta x)}{\Delta x} = -k\frac{T(x + \Delta x) - T(x)}{\Delta x} \qquad (7.23)$$

within the solid slab, if function T(x) is a CONTINUOUS varying function w.r.t variable x, meaning $\Delta x \rightarrow 0$, We will have the following from Equation (7.23):

$$q(x) = \lim_{\Delta x \to 0}\left[-k\frac{T(x + \Delta x) - T(x)}{\Delta x}\right] = -k\frac{dT(x)}{dx} \qquad (7.24)$$

Equation (7.24) is the mathematical expression of Fourier's Law of Heat Conduction in the x-direction

Example 7.9:

A one meter long metal rod is thermally insulated in its circumference. The terminal temperatures of the rod are measured to be 100°C and 20°C as illustrated in the figure below. Determine the heat fluxes in the rod. The rod is made of two different materials of copper and aluminum.

Solution:

A materials handbook indicates the following material properties:

Thermal conductivities (k) for copper and aluminum to be: k_{cu} = **3.95 w/cm-°C** and
k_{al} = **2.36 w/cm-°C**   respectively.

Thermally insulated

Temperature, T(x)   X

Heat flow

X = 0
T = 100°C

X = 100 cm
T = 20°C
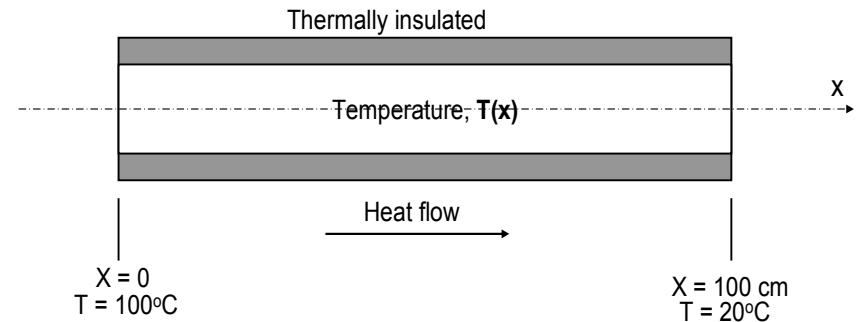
The differential equation for the problem can be expressed in a slightly different form from a first order differential equation in (7.24) to be: $\dfrac{dT(x)}{dx} = -\dfrac{q}{k_{cu}} = -\dfrac{q}{3.95}$ for the copper wire.

The solution of the above differential equation is: $T(x) = -\dfrac{q}{3.95}x + c$ where c=integration constant

The boundary condition: T(0) = 100°C at the left end of the rod is used to determine c = 100, which expresses the heat flux in the rod to be: $T(x) = -\dfrac{q}{3.95}x + 100$

The other condition T(100) = 20°C at the other end of the rod is used to determine the heat flux in the rod to be q = 3.16 w/cm² from the above expression.

The same procedure is followed to find the corresponding heat flux in aluminum rod to be 1.89 w/cm².

Example 7.10

Heat is transferred at the rate of 10 kW at the left end of a metal rod as shown in Figure 7.17. Determine the temperature distribution in the rod if the right end of the rod at x=2m is held at 50°C. The cross sectional area of the rod is 1200 mm² and the thermal conductivity k = 100 kW/m-°C
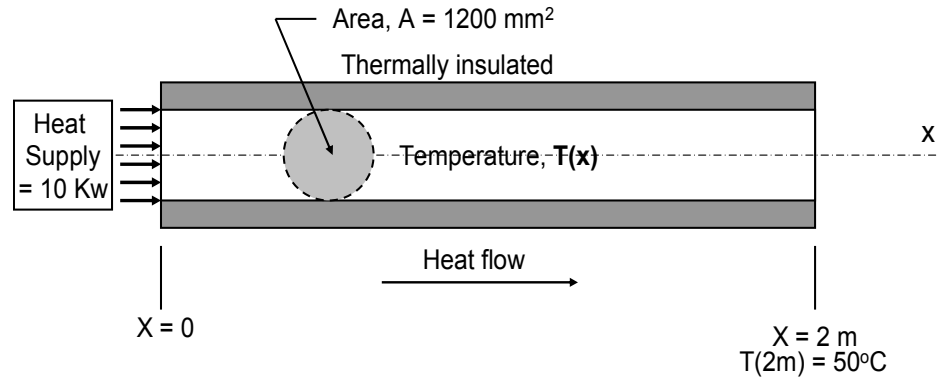

Figure 7.17 Temperature Variation in a Metal Rod with Heat Flow

Solution:

We have the total heat flow in the rod per unit time to be: $Q = qA = -kA\dfrac{dT(x)}{dx}$ (a)

with the end condition, T(x) = 50°C at x = 2 m   (b)

The temperature distribution T(x) in the rod may be obtained from the differential equation in Equation (a) in the following form:

$$\frac{dT(x)}{dx} = -\frac{Q}{kA} = -\frac{10}{100(1200 x 10^{-6})} = -83.33 \ ^{o}C/m \qquad \text{(c)}$$

Solving Equation (c) leads to the following relation: T(x) = -83.33x + c

in which c is the integration constant to be determined by the end condition specified in Equation (b), resulting in a value of c = 216.67. We will thus have the temperature variation in the rod to be:

$$T(x) = 216.67 - 83.33x$$

Heat Flux in Space: | **Expressions in 3-dimensional form**

- Heat flows in the direction of decreasing temperature in a solid. So, it is a vector quantity.
- In solids with temperature variations in all direction, heat will flow in ALL directions
- So, in general, there can be 3-dimensional heat flow in solids
- This leads to 3-dimensional formulation of heat flux
- Heat flux **q**(**r**,t) is a **vector** quantity, with **r** = position vector, representing (x, y, z) in Chapter 3

Position vector:
r: (x,y,z)
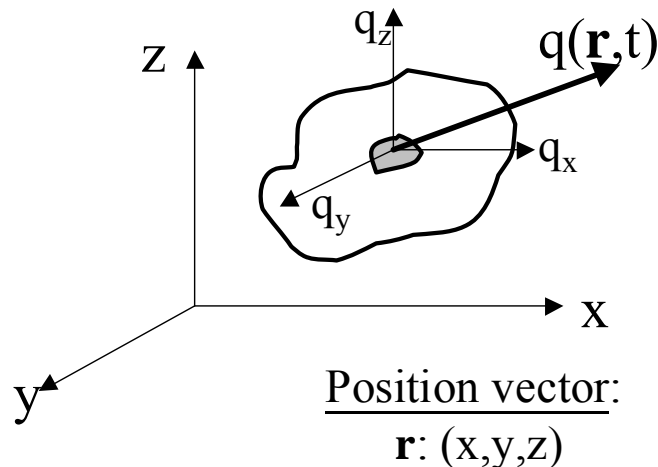
In general, the heat flux vector in the Fourier Law of heat conduction can be expressed as:

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t) \qquad (7.25)$$

The magnitude of vector **q**(**r**,t) is:

$$q(x, y, z, t) = \sqrt{q_x^2 + q_y^2 + q_z^2} \qquad (7.26)$$

with the components along respective x-, y- and z-coordinates:

$$q_x = -k_x \frac{\partial T(x, y, z, t)}{\partial x} \qquad (7.27a)$$

$$q_y = -k_y \frac{\partial T(x, y, z, t)}{\partial y} \qquad (7.27b)$$

$$q_z = -k_z \frac{\partial T(x, y, z, t)}{\partial z} \qquad (7.27c)$$
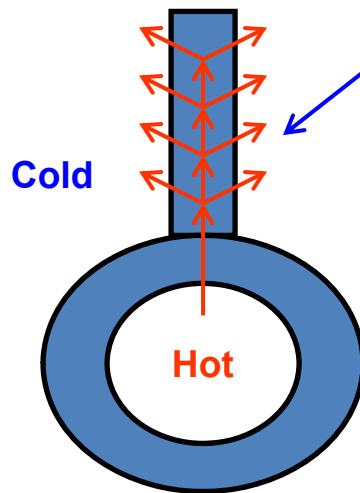
Examples of Heat Flux in a 2-D Plane

We have learned from Equation (7.21) that amount of heat flow in solids is proportional to the cross-sectional area for heat transmission (A), which means that more heat can flow with more area for the heat flow. Following are two such examples for facilitating more heat flow in machines by adding more surface areas to facilitate more heat flow.

(1) Tubes with longitudinal fins are common in many tubular heat exchangers and boilers for effective heat exchange between the hot fluids inside the tube and cooler fluids outside:
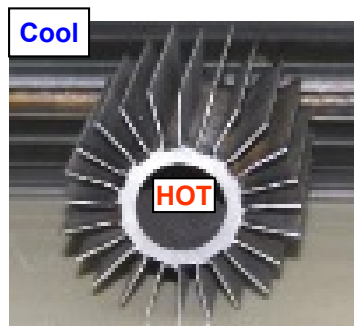
The heat inside the tube (figure in the left) flows along the longitudinal plate-fins to facilitate more effective cooling of the contacting fluid outside the tube.

It is desirable to analyze how effective heat can flow in the cross-section of these longitudinal fins.

(2) Additional heat transfer areas (fins) attached to the exterior of a motorcycle engine are also used to conduct excessive heat generated inside of the engine for better cooling effect of the a motorcycle engine by the outside cooling air:

Cross-section of a tube with longitudinal fins with only one fin shown

Cooling fins

Examples of "Heat Spreaders" in Microelectronics Cooling

The same principle of adding more areas to facilitate heat dissipation of the heat-generating integrated circuits "chips" to the outside cooling air is illustrated below:

Heat Spreader
(rectangular fins)

IC Chips
(heat generators)
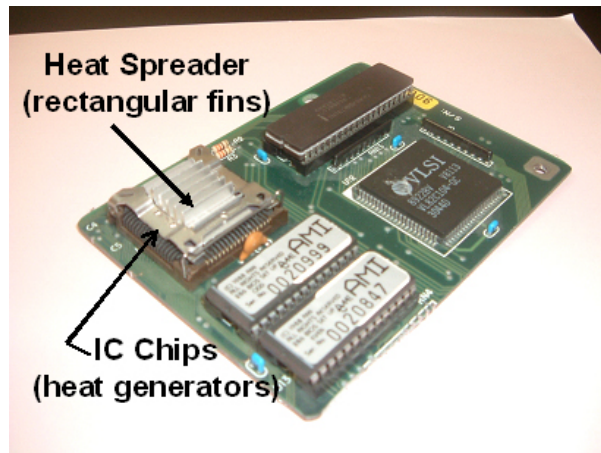
(a) A typical printed circuit board

IC Chip

Cooling Fan
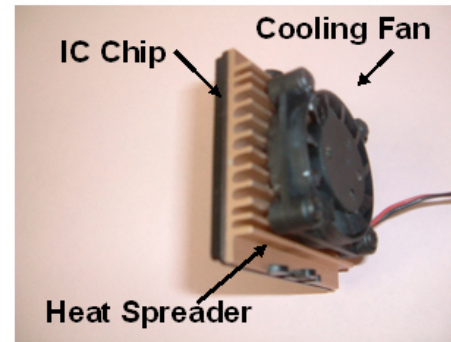
Heat Spreader

(b) Dissipation of heat in an IC chip

Heat spreader of common cross-sections:

q   q   q

Heat Source e.g., IC chip
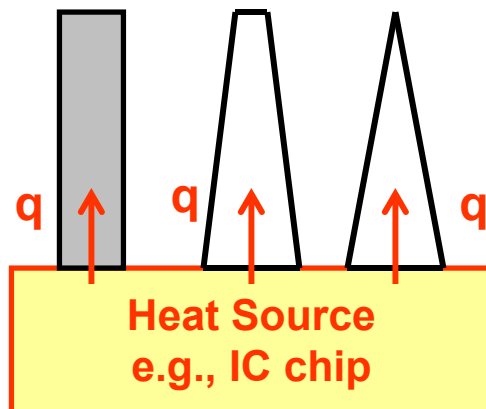
Heat flow in a 2-dimensional plane, by Fourier's Law in x-y plane

For **one-dimensional** heat flow, Equation (7.24):

$$q(x) = -k\frac{dT(x)}{dx}$$

$$q(x) = +k\frac{dT(x)}{dx}$$

0

T(x)

0

T(x)
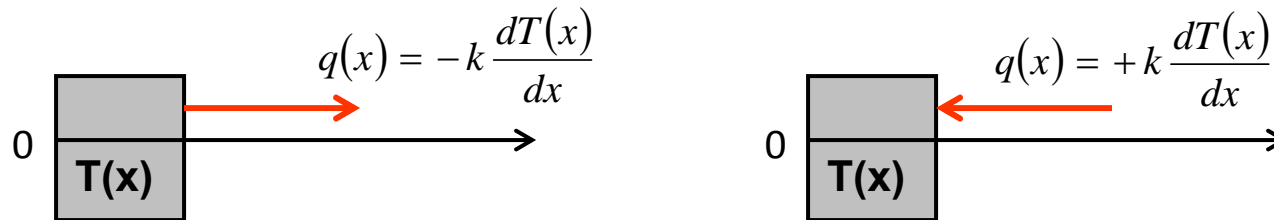
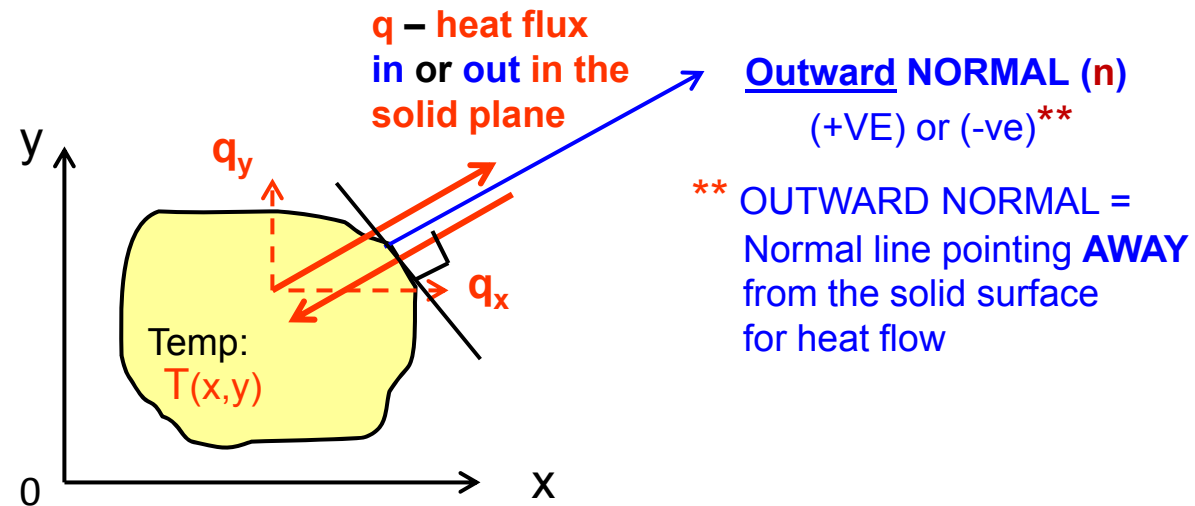**NOTE: The sign attached to q(x) changes with change of direction of heat flow!!**

For **two-dimensional** heat flow, Equation (7.25):

y

q_y

q_x

Temp:
T(x,y)

x

q – heat flux in or out in the solid plane

Change of sign in the General form of Fourier's Law of Heat Conduction:

$$\mathbf{q}(\mathbf{r}, t) = \pm k\nabla T(\mathbf{r}, t)$$
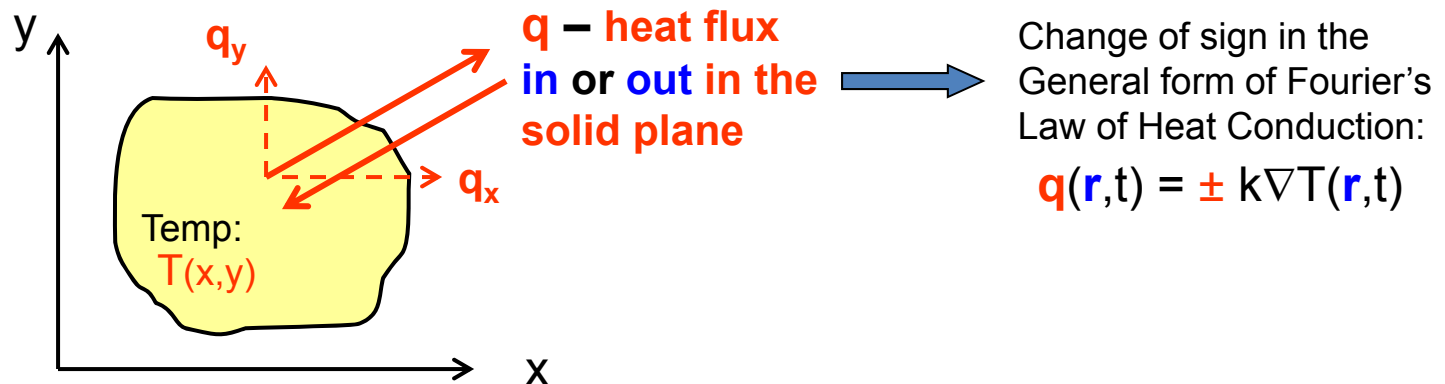
Question: How to add a CORRECT sign in heat flux equations??

Guideline for Assigning Right Signs in Heat Flux Formulations in 2-D Planes:

q – heat flux in or out in the solid plane

Outward NORMAL (n)
(+VE) or (-ve)**

q_y

y

q_x

Temp:
T(x,y)

0

x

** OUTWARD NORMAL =
Normal line pointing **AWAY**
from the solid surface
for heat flow

Sign of Outward Normal (n)?	q along n?	Sign of q in Fourier Law
+	Yes	-
+	No	**+**
-	Yes	**+**
-	No	-

Example 7.12: Show the heat fluxes across the four edges of a rectangular block with +ve or −ve sign in terms of the available temperature distribution of $T(x,y)$.
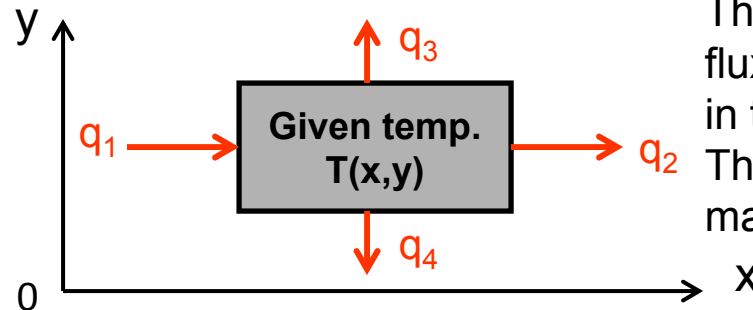
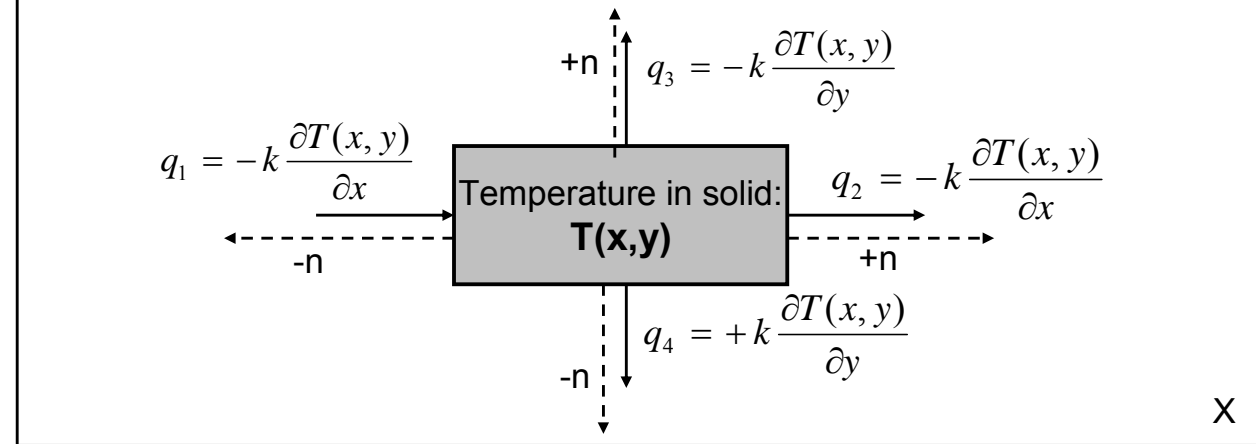The specified directions of heat fluxes are prescribed as shown in the left figure.
Thermal conductivity of the material k is given.

Solution:

We may use the inserted table to express the heat fluxes q_1, q_2, q_3 and q_4 as shown in the following figure.

Sign of outward normal, n	q along n?	Sign of q in Fourier law
Case 1: +	yes	-
Case 2: +	no	+
Case 3: -	yes	+
Case 4: -	no	-

$$q_3 = -k\frac{\partial T(x,y)}{\partial y}$$

$$q_1 = -k\frac{\partial T(x,y)}{\partial x}$$

Temperature in solid: $T(x,y)$

$$q_2 = -k\frac{\partial T(x,y)}{\partial x}$$

$$q_4 = +k\frac{\partial T(x,y)}{\partial y}$$

Numerical analyses of heat fluxes leaving a heat spreader of a half-triangular cross-section (P. 224):.
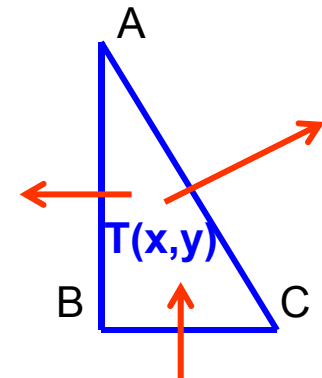
Given: $T(x,y) = 100 + 5xy^2 - 3x^2y$ °C

Thermal conductivity $k = 0.021$ W/cm-°C
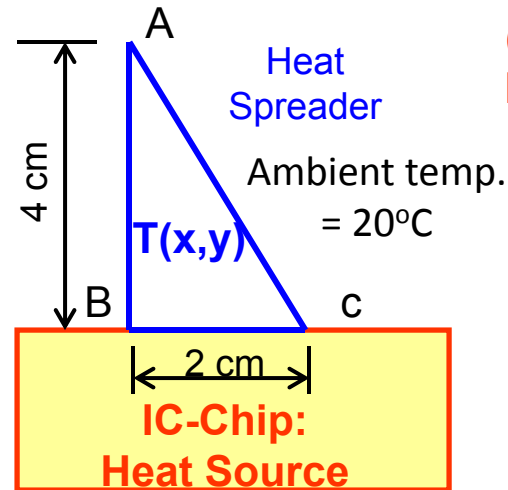
Heat flow in the Spreader: (normal to the areas of heat flows:

Ambient temp. = 20°C

Heat Spreader

Ambient temp. = 20°C

4 cm

T(x,y)

A

B

C

2 cm

IC-Chip: Heat Source

A

T(x,y)

B

C

Solution:

Set coordinate system and Identify outward normals as shown in the following sketch:
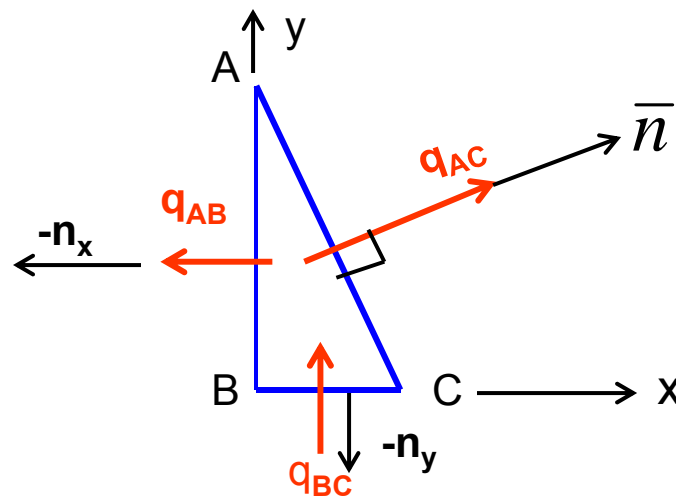
A. Heat flux across surface BC:

The direction of heat flow is known (from heat source to the spreader)

-ve n $\longrightarrow$ Case 3 or 4; q_{BC} is not along n $\longrightarrow$ Case 4 with –ve sign

$$q_{bc} = -k\left.\frac{\partial T(x,y)}{\partial y}\right|_{y=0} = -0.021\left.\frac{\partial(100+5xy^2-3x^2y)}{\partial y}\right|_{y=0} = -0.021(10xy-3x^2)\Big|_{y=0} = 0.063x^2 \ w/cm^2$$

B. Heat flux across surface AB:

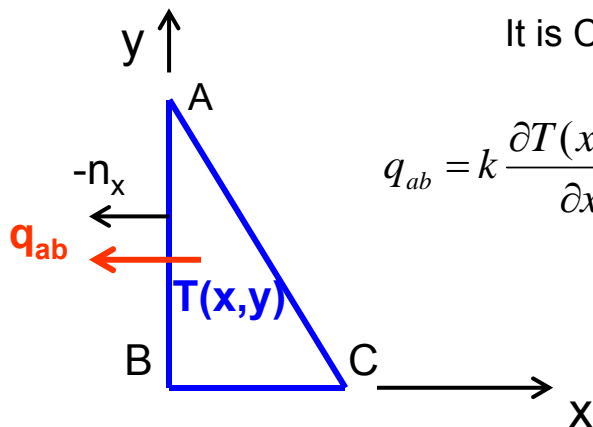We need to verify the direction of heat flow across this surface first::
By checking the temperature at the two terminal points of the Edge AB:

At Point B (x = 0 and y = 0) the corresponding temperature is:

$$(100+5xy^2-3x^2y)\Big|_{\substack{x=0\\y=0}} = 100^oC > 20^oC, \ the \ ambient \ temperature$$

The same temperature at terminal A
So, heat flows from the spreader to the surrounding.

It is Case 3 in Fourier law, we thus have:

$$q_{ab} = k\left.\frac{\partial T(x,y)}{\partial x}\right|_{x=0} = 0.021\left.\frac{\partial(100+5xy^2-3x^2y)}{\partial x}\right|_{x=0} = 0.105y^2 \ w/cm^2$$

C. Heat flux across surface AC:

Surface AC is an inclined surface, so we need to handle this situation by the following procedure:

Use the same technique as in Case B, we may find the temperature at both terminal A and C to be 100°C > 20°C, the ambient temperature. So heat leaves the surface AC to the ambient.
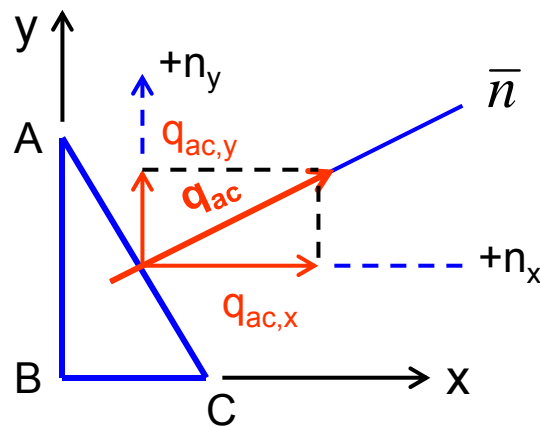
Based on the direction of the components of the heat flow and outward normal, we recognize Case 1 for both $q_{ac,x}$ and $q_{ac,y}$. Thus we have:

$$q_{ac,x} = -k \frac{\partial T(x,y)}{\partial x}\Big|_{\substack{x=1 \\ y=2}} = -0.021(5y^2 - 6xy)\Big|_{\substack{x=1 \\ y=2}} = -0.021(20-12) = -0.168 \ w/cm^2$$

and

$$q_{ac,y} = -k \frac{\partial T(x,y)}{\partial y}\Big|_{\substack{x=1 \\ y=2}} = -0.021(10xy - 3x^2)\Big|_{\substack{x=1 \\ y=2}} = -0.021(20-3) = -0.357 \ w/cm^2$$

The heat flux across the mid-point of surface AC at x = 1 cm and y = 2 cm is:

$$\vec{q}_{ac} = \vec{q}_{ac,x} + \vec{q}_{ac,y} = \sqrt{(-0.168)^2 + (-0.357)^2} = 0.3945 \ w/cm^2$$

Review of Newton's Cooling Law for Heat Convection in Fluids

- Heat flows (transmission) in fluid by CONVECTION
- Heat flows from higher temperature end to low temperature end in fluids
- Motion of fluids causes heat convection
- As a rule-of-thumb, the amount of heat transmission by convection
  is proportional to the velocity of the moving fluid

7.5.4 Mathematical expression of heat convection – The Newton's Cooling Law

A fluid of non-uniform temperature in a container causes convective heat transfer:
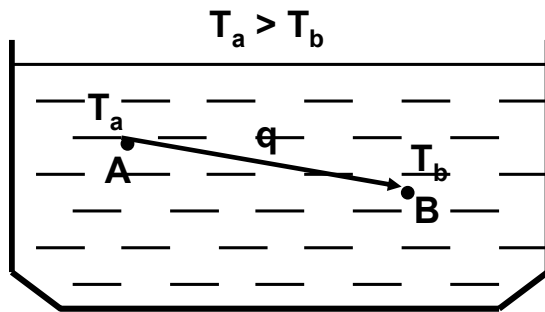
Heat flows from T_a to T_b with $T_a > T_b$. The heat flux between A and B can be expressed by:

$$q \propto (T_a - T_b) = h(T_a - T_b) \qquad (7.28)$$

where **h = heat transfer coefficient** (W/m²-ºC)

The heat transfer coefficient h in Equation (7.28) is normally determined by empirical expression, with its values relating to the Reynolds number (Re) of the moving fluid. The Reynolds number is expressed as:

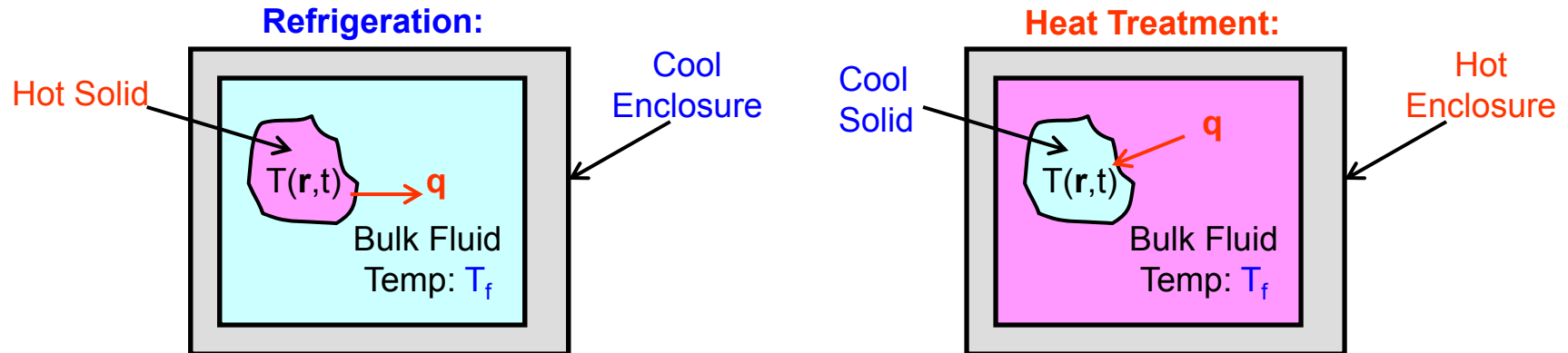$$Re = \frac{\rho L v}{\mu}$$

with ρ = mass density of the fluid; L = characteristic length of the fluid flow, e.g., the diameter of a circular pipe, or the length of a flat plate; v = velocity of the moving fluid; μ = dynamic viscosity of the fluid

Heat Transfer in Solids Submerged in Fluids

● There are numerous cases in reality, in which solids are in contact with fluids at different temperatures:

Refrigeration:

Hot Solid

$T(r,t)$ **q**

Cool Enclosure

Bulk Fluid
Temp: T_f

Heat Treatment:

Cool Solid

$T(r,t)$ **q**

Hot Enclosure

Bulk Fluid
Temp: T_f

● In such cases, there is heat flow between the contacting solid and the surrounding fluid.
● But the physical laws governing heat flow in solids is the Fourier's Law and that in fluids by the Newton's Cooling Law

So, mathematical modeling for the contacting surface in this situation requires the use of **both** Fourier's Law and Newton's Cooling Law:

Mathematical Modeling of Small Solids in Refrigeration and Heating

Bulk environmental temperature = T_f

Surface area, A

Solid $T(t)$

Initial solid temperature, T_o
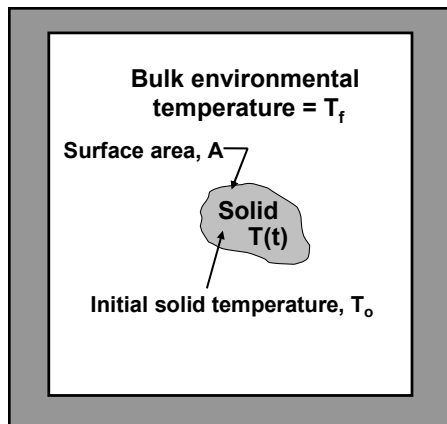
We will formulate a simplified case involving heat exchange between the solids and the surrounding fluids at different temperature with the **Following assumptions::**
● the solid is initially at temperature T_o
● the solid is so small that it has uniform temperature throughout its volume, so its temperature varies with time t only , i.e., $T = T(t)$
● the time t begins at the instant at which the solid is submerged in the fluid maintained at a different constant temperature T_f
● variation of temp. in the solid is attributed by the heat supplied, or removed by the surrounding fluid

Derivation of Math Model for Heat Transfer in Solids Submerged in Fluids

Bulk environmental temperature = T_f

Surface area, A

Solid
T(t)

Initial solid temperature, T_o

Solid Temp
T(t)

Bulk Fluid

q

Bulk Fluid
Temp T_f

$T_s(t)$

$\overline{n}$
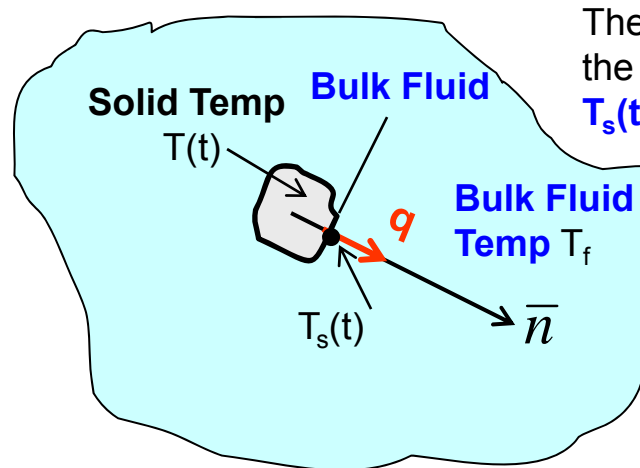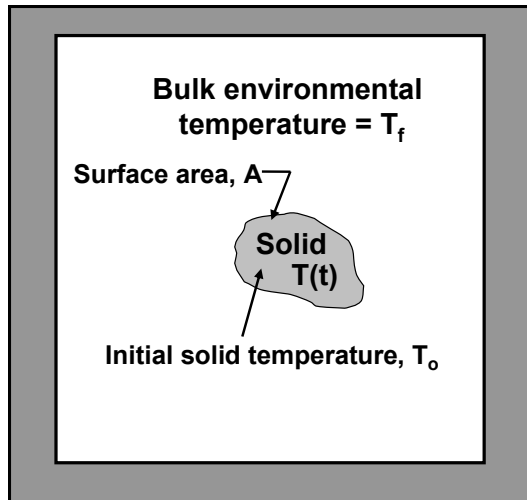
The solid is small, so the surface temperature $T_s(t) = T(t)$, the solid temperature

Heat flows in the fluid, when $T(t) \neq T_f$, T_f = the bulk fluid temp.

Heat flows in the fluid follows the Newton Cooling Law expressed in Equation (7.28), or::

(7.29)
$$q = h [T_s(t) - T_f] = h [T(t) - T_f]$$

where h = heat transfer coefficient between the solid and the bulk fluid

From the First Law of Thermodynamics, the heat required to produce temperature change in a solid $\Delta T(t)$ during time period Δt can be obtained by the following physical relationship:

Change in internal energy of the small solid during Δt	=	Net heat flow from the small solid to the surrounding fluid during Δt

$$- \rho c V \Delta T(t) \quad = \quad Q = q A_s \Delta t = h A_s [T(t) - T_f] \Delta t \qquad \text{(a)}$$

where ρ = mass density of the solid;   c = specific heat of solid
        V = volume of the solid; A_s = contacting surface between solid and bulk fluid

From Equation (a), we express the rate of temperature change in the solid to be:

$$\frac{\Delta T(t)}{\Delta t} = -\frac{h}{\rho c V} A_s \left[T(t) - T_f \right] \qquad \text{(b)}$$

Bulk environmental
temperature = T_f

Surface area, A

Solid
T(t)

Initial solid temperature, T_o
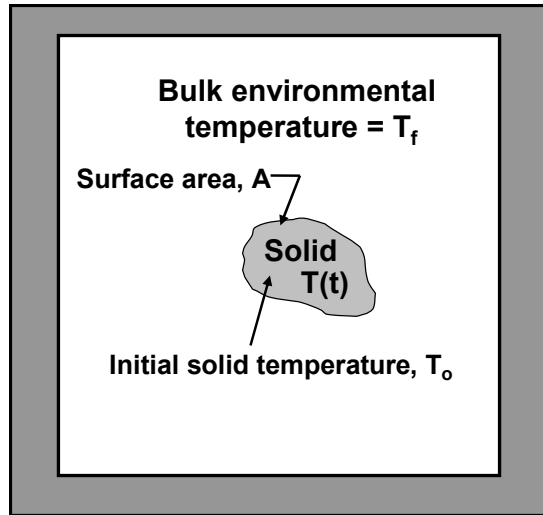
Since h , ρ, c and V on the right-hand-side of Equation (b) are constants, we may lump these constants to let:

$$\alpha = \frac{h}{\rho c V} \qquad \text{(7.30)}$$

with a unit (/m²-s)

Equation (b) is thus expressed as:
$$\frac{\Delta T(t)}{\Delta t} = -\alpha A_s \left[T(t) - T_f \right] \qquad \text{(7.27)}$$

Since the change of the temperature of the submerged solid T(t) is CONTINOUS with respect to time t, i.e., $\Delta t \rightarrow o$, and if we replace the contact surface area A_s to a generic symbol A, we can express Equation (3.27) in the form of a 1ˢᵗ order differential equation as follows::

$$\frac{dT(t)}{dt} = -\alpha A \left[T(t) - T_f \right] \qquad \text{(7.31)}$$

with a given initial condition:
$$T(t)\big|_{t=0} = T(o) = T_0$$

Example 7.13:

A small solid at temperature of 80°C is to be cooled in a cooling chamber maintained at 5°C. Determine the time required for the temperature of the solid to reach 8°C if the proportionality constant in Equation (7.31) $\alpha = 0.002/m^2$-s with a contacting surface area A = 0.2 m²

Bulk environmental temperature = T_f

Surface area, A

Solid T(t)

Initial solid temperature, T_o
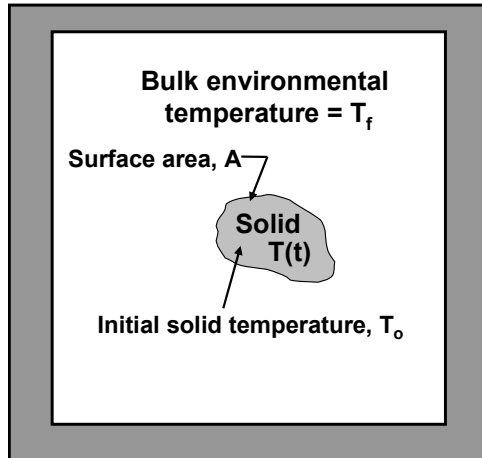
Solution:

We have $T_o = 80°C$, $T_f = 5°C$, $\alpha = 0.002/m^2$-s and A = 0.2 m²

Substituting the above into Equation (7.31) will lead to the following 1st order differential equation:

$$\frac{dT(t)}{dt} = -(0.002)(0.2)[T(t) - 5] = -0.0004[T(t) - 5] \qquad (a)$$

with an initial condition: T(0) = 80°C   (b)

Equation (a) can be re-written as: $\dfrac{dT(t)}{T(t) - 5} = -0.0004\, dt$ (c)

Integrating both sides of Equation (c): $\displaystyle\int \frac{dT(t)}{T(t) - 5} = -0.0004 \int dt + c_1$ (c)

leads to the solution: $T(t) - 5 = e^{-0.0004t + c_1} = c e^{-0.0004t}$ (d)

The integration constant c in Equation (d) can be obtained by the condition T(0) = 80°C in Equation (b) Results in c = 75. Consequently, the solution T(t) in Equation (d) is:

$$T(t) = 5 + 75 e^{-0.0004t} \qquad (e)$$

If we let t_e = required time for the solid to drop its temperature from 80°C to 8°C, we should have:

$$T(t_e) = 8 = 5 + 75 e^{-0.0004 t_e} \qquad (f)$$

Solve Equation (f) for t_e = 8047 s or 2.24 h

What would you do if the required time to cool down the solid is too long?

Example 7.14

A Silicon Valley company produces microcomputer chips and is required to perform thermal cycling tests of the chips it has produced as a part of its reliability testing. In this particular series of tests, each cycle involves heating and cooling of the chips between terminal temperatures and the durations as illustrated in the figure below:
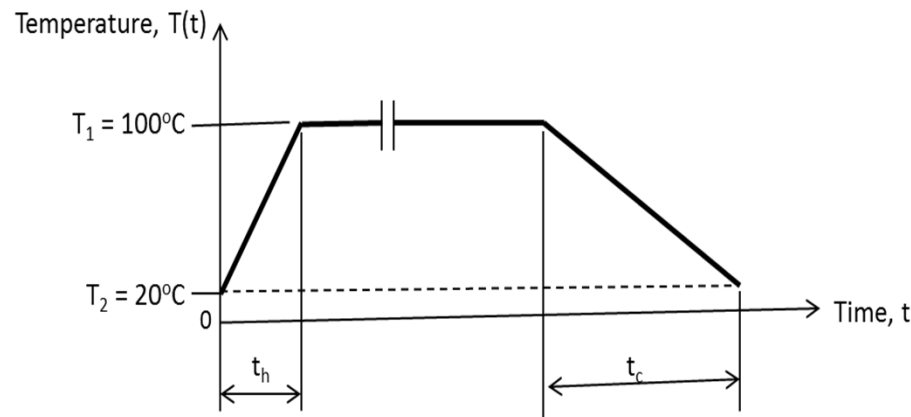
In the present example, each chip has an overall surface area of 8×10^{-4} m^2 with both the heating and cooling chambers providing the heating and cooling conditions as specified in the following Table:

Chambers	Proportional coefficients (α), /m²-s	Chamber ambient temperature (T_f), °C
Heating	0.8	$T_h = 150$
Cooling	0.2	$T_c = -20$

The procedure for this particular thermal cycling testing is to first heat the chip from an initial temperature of 20°C to 100°C in the heating chamber for a period of t_h, followed by a cooling process to have the chip cooled down to 20°C in a cooling chamber for a period of t_c. The subsequent cycles will have the chip's temperature cycling between $T_2 = 20$°C and $T_1 = 100$°C as illustrated in the above figure. Determine the required times t_h and t_c for the heating and cooling of the chips in the first cycle of the planned thermal cycling testing

Example 7.14-Cont'd

Solution:

We will use the first order differential equation in Equation (7.31) to solve the instantaneous temperature in the chip T(t) as follows:

$$\frac{dT(t)}{dt} = -6.4x10^{-4}[T(t)-150] \qquad \text{(a)}$$

(A) For the heating portion of the thermal cycling test:

We have from conditions Table the values of $\alpha = 0.8/m^2$-s and $T_f = 150°C$ together with $A = 8x10^{-4}$ m^2 and the initial temperature of the chip $T_0 = 20°C$. Substituting these given conditions into Equation (a) will result in the following differential equation for the solution T(t):

The time required to heat the chips to 100°C may be determined by the relation that $T(t_h) = T_1 = 100°C$, which will lead to the following expression:

$$\frac{dT(t)}{dt} = -6.4x10^{-4}[T(t)-150] \qquad \text{(b)}$$

with condition: $\quad T(t)\big|_{t=0} = T(0) = T_0 = 20 \qquad \text{(c)}$

By integrating both sides of the Equation (b), we get:

$$\int \frac{dT(t)}{T(t)-150} = -6.4x10^{-4}\int dt + c_1 \qquad \text{(d)}$$

The integration constant c_1 is determined by the initial condition in Equation (c). We thus obtain the following expression for function T(t) as:

$$\text{(e)}$$

$$T(t) = 130e^{-6.4x10^{-4}t} + 150$$

The time required to heat the chips to 100°C may be determined by the relation that $T(t_h) = T_1 = 100°C$, which will lead to the following expression: $100 = -130e^{-6.4x10^{-6.4x10^{-4}}t_h} + 150$

from which we solve for $\quad t_h = \dfrac{\ell n\left(\dfrac{150-100}{130}\right)}{-6.4x10^{-4}} = 1493s \ or \ 24.88\,min$

Example 7.14-Cont'd

(b) For the cooling portion of the thermal cycling:

We have the proportional coefficient α = 0.2 /m²-s and T_f = -20°C in the cooling process

We will use the same Equation (a) for the solution T(t) with the following expression:

$$\frac{dT(t)}{dt} = -0.2x(8x10^{-4})[T(t)-(-20)] = -1.6x10^{-4}[T(t)+20] \tag{f}$$

The solution of T(t) in Equation (f) has the following form:

$$\ell n[T(t)+20] = -1.6x10^{-4}\,t + c_2$$

In which c_2 is the integration constant to be determined by the initial condition of $T(t)\Big|_{t=0} = T(0) = T_1 = 100°C$, from which we have c_2 = 4.7875. We thus have the solution:

$$T(t) = 120e^{-1.6x10^{-4}t} - 20 \tag{g}$$

from which we may express the temperature of the chip at time t_c to be:

$$T(t_c) = 20 = 120e^{-1.6x10^{-4}t_c} - 20$$

Solve for t_c from the above equation and obtain t_c = 6813 s or 113.55 minutes or 1.89 hour.

(c) Two options to shorten the times for heating and cooling: (a) For shortening time for heating: One may increase both the α-values and the bulk fluid temperature t_f, and (b) for shortening the time for cooling would involve increasing the α-values but lower the bulk fluid temperature t_f.
Both options will increase the cost for the testing because increasing α-values in the heating or cooling chambers require increasing the speed of bulk fluid moving over the solid surfaces, and increasing t_h and reducing t_c in the chambers can also be costly too.

7.6 Rigid Body Dynamics Under the Influence of Gravitation

We will demonstrate the application of 1st order differential equation in rigid body dynamics using Newton's Second Law

$$\sum F = ma$$

where F = induced dynamic force on the moving solids, M = the mass of the solids, and a = the accelerations or decelerations of the moving solids

Rigid Body Motion Under Strong Influence of Gravitation:

There are many engineering systems that involve dynamic behavior under strong influence of Gravitational acceleration of decelerations such as described in Section 3.73, or in the moving Solids in the flowing lustrations:
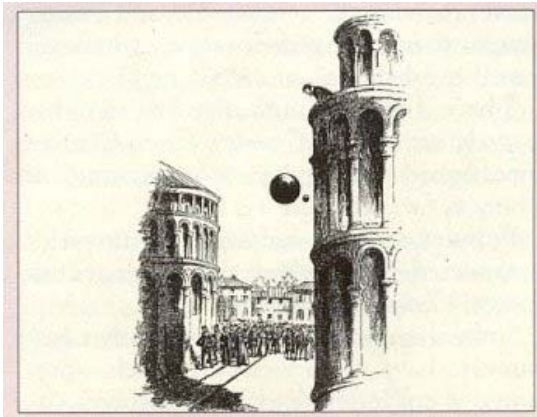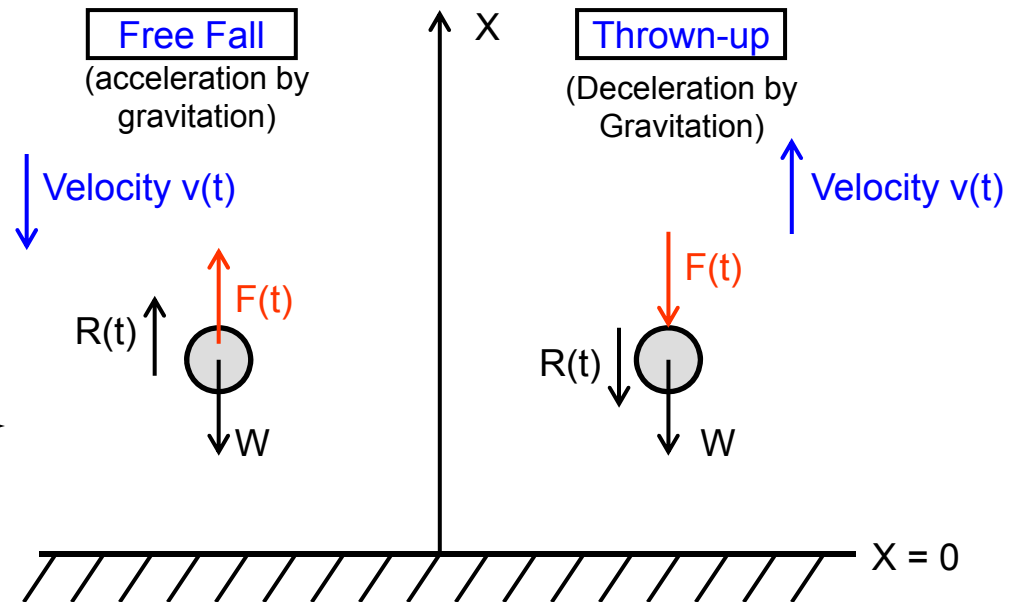

Rocket launch

The helicopter

The paratroopers

Observation on A Rigid Body in Motion Under Influence of Gravitation

Galileo Galilei

Galileo's free-fall experiment from the leaning tower in Pisa, Italy, December 1612

Free Fall
(acceleration by gravitation)

Velocity v(t)

R(t) ↑ | F(t)

↓ W

X

Thrown-up
(Deceleration by Gravitation)

Velocity v(t)

F(t)

R(t) ↓ |

↓ W

X = 0

Math Modeling →

Solution sought:
- The instantaneous position x(t) at time t
- The instantaneous velocity v(t)
- The maximum height the body can reach, and the required time with initial velocity v_o in the "thrown-up" situation

These solutions can be obtained by first deriving the mathematical expression (a differential equation in this case), and solve for the solutions

By kinematics of a moving solid:
If the instantaneous position of the solid is expressed as x(t), we will have: $v(t) = \dfrac{dx(t)}{dt}$ to be the instantaneous velocity, and $a(t) = \dfrac{dv(t)}{dt}$ to be the instantaneous acceleration (or deceleration)

Case A: Throw-up of a solid with initial velocity v_o:

X

Thrown-up

Velocity v(t)

F(t)

R(t)

W

X = 0
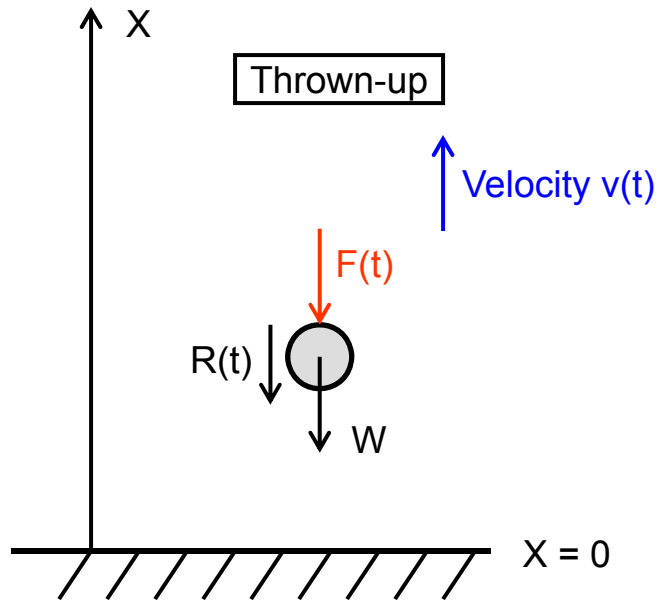
The forces acting on the up-moving solid should be in equilibrium at time t:

By using a sign convention of forces along +ve x-axis being +ve for the forces, we have:

$$\sum \mathbf{F}_x = 0$$

which leads to:

$$\sum F_x = -W - R(t) - F(t) = 0$$

with W = mg, R(t) = cv(t), and $F(t) = ma(t) = m\dfrac{dv(t)}{dt}$

A 1st order differential equation is obtained:

$$\frac{dv(t)}{dt} + \frac{c}{m}v(t) = -g \qquad (7.32)$$

with an initial condition:

$$v(t)\big|_{t=0} = v(0) = v_o \qquad (a)$$

Case A: Throw-up of a solid with initial velocity v_o – Cont'd:

The solution of Equation (7.32) is obtained by comparing it with the typical 1st order differential equation in Equation (7.6) with solution in Equation (7.7) shown below:

$$\frac{dv(t)}{dt} + p(t)u(t) = g(t) \tag{7.6}$$

with a solution:

$$v(t) = \frac{1}{F(t)} \int F(t) g(t) \, dt + \frac{K}{F(t)} \tag{7.7}$$

The present case has:

$$F(t) = e^{\int p(t)\, dt}, \quad p(t) = \frac{c}{m} \quad and \quad g(t) = -g$$

Consequently, the solution of Equation (7.32) has the form:

$$v(t) = \frac{1}{e^{\frac{ct}{m}}} \int e^{\frac{ct}{m}} \left(-g \right) dt + \frac{K}{e^{\frac{ct}{m}}} = -\frac{mg}{c} + Ke^{-\frac{c}{m}t} \tag{7.33}$$

In which the constant K is determined by the given initial condition in Equation (a), with:

$$K = v_o + \frac{mg}{c} \tag{b}$$

The complete solution of Equation (7.32) with the substitution of K in Equation (b) into Equation (3.26) for The instantaneous velocity of the rising solid:

$$v(t) = -\frac{mg}{c} + \left(v_o + \frac{mg}{c} \right) e^{-\frac{c}{m}t} \tag{7.34}$$

Case A: Throw-up of a solid with initial velocity v₀ – Cont'd:

The other solution that engineers often sought in this case is: "How long would it take for this solid reaching the maximum height and the time required to reach this stage:

We realize the fact that the solid will reduce it velocity continuously on its way up, until it reaches the maximum height, at which point the velocity will reach a zero value. Mathematically, we will have a condition that $v(t_m) = 0$ in which t_m is the required time for the upward moving solid reaching its maximum height. We thus have the following expression:

$$0 = -\frac{mg}{c} + \left(v_o + \frac{mg}{c} \right) e^{-\frac{c}{m}t_m}$$

from which, we solve for tm to be:

$$t_m = \frac{m}{c} \, ln \left(1 + \frac{v_0 \, c}{mg} \right) \tag{7.35}$$

Case B: Derivation of Math Expression for Free-Fall of a Solid

X

Free Fall

Velocity v(t)

F(t)

R(t)

W
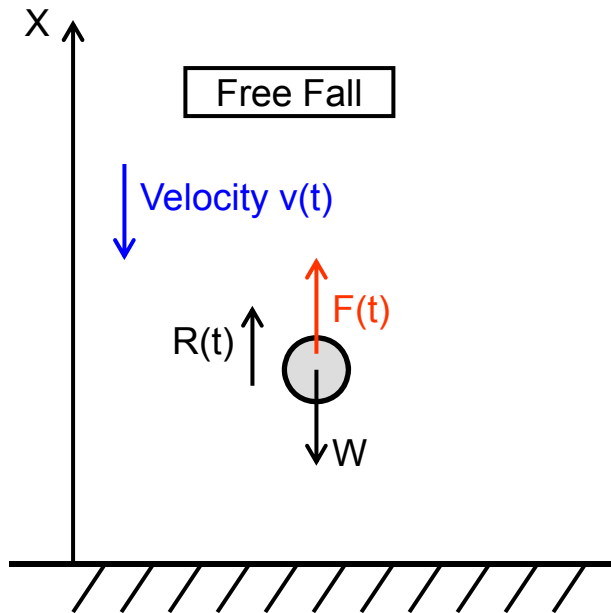
Mathematical formulation of free-fall of solids begins with instantaneous equilibrium of forces acting on the falling solid at time t:

$$\sum F_x = F(t) - w + R(t) = 0$$

We may derive a similar first order differential equation for the instantaneous velocity of the falling solid v(t) at time t following the procedure in Case A with the form:

$$\frac{dv(t)}{dt} + \frac{c}{m} v(t) = g \qquad (7.36)$$

We notice that Equation (7.36) that shows a positive g in the right-hand-side for the falling solid. Similar solution procedure in Case A may be used for the solution of Equation (7.36).

Example 7.15**: A case illustration of Free-fall of a solid:**

An armed paratrooper with ammunition weighing 322 pounds jumps from a plane with zero initial velocity, as shown in the figure in the right.

We assume that the troopers encountered negligible side wind in their descending. However, they encounter an air resistance that is 15 times the square of the descending velocity v(t), i.e., 15[v(t)]².

Determine the following:

(a) Derive the appropriate equation for the instantaneous descending velocity v(t),
(b) Solve the equation for the descending velocity.
(c) Estimate the time required for the paratrooper to reach the ground from a height of 10,000 feet.
(d) Estimate the impact velocity of the paratrooper upon touching the ground, and the corresponding momentum.

Example 7.15 – Cont'd

Solution:

(a) Derivation of the differential equation:

The total mass of the falling body m = 322/32.2 = 10 slugs,

the air resistance R(t) = cv(t) = 15[v(t)]²

The instantaneous descending velocity v(t) can be obtained by using Equation (7.36) as:

$$\frac{dv(t)}{dt} + \frac{15[v(t)]^2}{10} = 32.2 \qquad \text{(a)}$$

or in the form:

$$10\frac{dv(t)}{dt} = 322 - 15[v(t)]^2 \qquad \text{(b)}$$

with the initial condition:

$$v(t)\big|_{t=0} = v(0) = 0 \qquad \text{(c)}$$

(b) Solution for instantaneous velocity v(t):

Equation (b) may be expressed in the form after being separated for the solution:
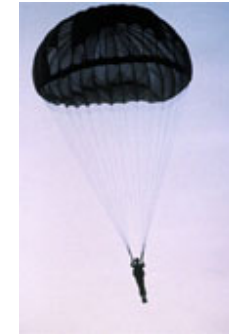
$$\frac{10}{322 - 15v^2}\, dv = dt \qquad \text{(d)}$$

where the instantaneous descending velocity of the trooper v=v(t)

Integrating both sides of Equation (d) will result in:

$$\frac{10}{139}\log\frac{4.634 + v}{4.634 - v} = t + c \qquad \text{(e)}$$

The integration constant c in Equation (e) can e determined by the initial condition in Equation (c) resulting in c=0. We thus have the solution v(t) to be:

$$v(t) = \frac{4.634(e^{13.9t} - 1)}{e^{13.9t} + 1} \qquad \text{(f)}$$

Example 7.15 – Cont'd

(c) Estimate the time required for the paratrooper to reach the
ground from a height of 10,000 feet:

Let the descending distance of the paratrooper to be X(t), in which t is the
time starting from the moment of his jumping out the carrier airplane.
The expression $v(t) = \dfrac{dX(t)}{dt}$ leads to the following expression for the

distance of descending by the paratrooper:

$$X(t) = \int_0^t v(t)dt$$

(g)

$$X(t) = \int_0^t \frac{4.634\left(e^{139t} - 1\right)}{e^{139t} + 1} dt = \left[0.667\ell n\left(1 + e^{139t}\right) - 4.634\right]_0^t$$

After we fill Equation with v(t) in Equation (f) and get the above integral for the distance that the
trooper had traveled at time t to be:

$$X(t) = 0.6667\ell n\ (1 + e^{\ 13.9t}) - 4.634t - 0.4621$$

(h)

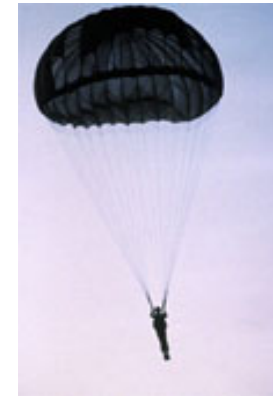Let t_g be the required time to travel a distance 10,000 feet, we will have the following relationship:

$$10000 = 0.6667\ell n\ (1 + e^{\ 13.9tg}) - 4.634t_g - 0.4621 \qquad\qquad (j)$$

Equation (j) will provide us with an approximate value of t_g = 2158.46 s 0r 35.95 minutes

(d) Estimate the impact velocity of the paratrooper upon touching the ground, and the
corresponding momentum:

We may compute the terminal velocity of the trooper to be:

$$V_f = \frac{4.634\left(e^{13.9x2158.46} - 1\right)}{e^{13.9x2158.46} + 1} \approx 4.634 \quad ft/s \text{ ,which leads to the impact momentum to be: } mv(t_g) = 46.34 \text{ ft-lb}$$