Modelling Physical Phenomena, Planetary Motion and Conservation Laws

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Outline

What is a Differential Equation?

- A differential equation is an equation that related one or more unknown functions and their derivatives.
- In applications, the functions generally represent physical quantities, the derivatives represent their rate of changes, and the differential equation defines a relationship between the two.
- Two major division of differential equations: ODE and PDE.

ODE and PDE

- An ordinary differential equation (ODE) is an equation which has a dependent variable, which is an unknown function of an independent variable, its derivative, and some given functions of the independent variable.
- Examples for ODE: $y = x + \frac{dy}{dx}$, $(\frac{dy}{dx})^3 + \frac{d^2x}{dy^2} - x\frac{dy}{dx} = y$, etc.
- A partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives.
- Examples for PDE: $\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} = 6u(\frac{\partial u}{\partial x})^2 - \frac{\partial^3 u}{\partial x^3}$, etc.

In this presentation, we only focus on modeling using ordinary differential equations.

Physical Quantities

- Base or Fundamental Quantities:
  The quantities that are distinct in nature, which cannot be defined in terms of other quantities.
  1. Length
  2. Time
  3. Mass
  4. Temperature
  5. Amount of substance
  6. Electric current
  7. Luminous Intensity
- Other physical quantities can be defined using these base quantities (some function of base quantites).
- For instance, a derived physical quantity can be the rate of change of one quantity w.r.t another quantity (i.e., the derivative of one quanity w.r.t another), or the integral of one quantity w.r.t another quantity.

General Method to Model Physical Phenomena using Differential Equations

- Assign variables to physical quantities.
- Some physical quantities will be functions (derivatives or integrals) of other physical quanitities related to the phenomena.
- Form equations using physical laws. These equations will be the differential equation model of the physical phenomena.
- The key to modeling is to choose the variables in such a way that the differential equations formed using the physical laws are simple to analyze.
- Now, the physical phenomena can be analyzed by analyzing the differential equations.

In Kinematics

- Displacement is usually denoted by the variable s.
- Velocity is the rate of change of displacement. i.e., $v = \frac{ds}{dt}$.
- Acceleration is the rate of change of velocity. i.e., $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
- Acceleration can also be written as $a = \frac{dv}{dt} = \frac{dv}{dt}\frac{ds}{ds} = v\frac{dv}{ds}$.
- These definitions are the most basic equations of kinematics. Every other relation is derived from them.
- Assuming a is constant, solving the differential equations $a = \frac{dv}{dt}$, $v = \frac{ds}{dt}$ and $a = v\frac{dv}{ds}$, we get the equations of one dimensional motion with constant acceleration: $v = u + at$, $s = ut + \frac{1}{2}at^2$ and $v^2 - u^2 = 2as$, where u denotes the initial velocity.

Newtonian Mechanics

- Momentum is defined as $p = mv$.
- Force is defined as the rate of change of momentum. i.e., $F = \frac{dp}{dt} = m\frac{dv}{dt}$ (m is constant).
- Newton's Second Law: Net force acting on a body of mass m is $F = m\frac{dv}{dt} = ma$.
- Work done is denoted by W and is defined as $W = Fs$. (more precisely, $dW = Fds$)
- NOTE: The quantities velocity, acceleration, momentum, force, etc are all vectors, althought here it is shown as a scalar (here, it is assumed for one dimensional physical systems). Work done is technically the dot product of F and s.
- The work done on a body is stored as energy in it.
- Force can also be written as the anti-gradient of potential energy.

$$F = -\frac{dU}{dx} \quad \text{(considering 1-D motion)}$$

Example of a Mechanical System

Let us say a ball of mass m is thrown vertically up in the air with an initial velocity of u upward, where the viscous force coefficient is b and the acceleration due to gravity is g downward. Our goal is to find how the velocity and the position of the ball changes with time.

- Let the variable x denote the position, v denote the velocity and a denote the acceleration of the ball at any instant t. Assume that the initial position of the ball as $x = 0$, and the positive axis directed upwards and the negative axis directed downwards.
- The force due to air friction (viscous force) at any instant is proportional to the magnitude of velocity (bv in this case), directed opposite to the direction of the velocity.
- The other force acting on the ball is the gravitational force, which is mg downward.

Example of a Mechanical System (cont.)

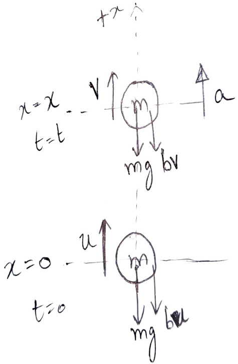


Figure: Free-body diagram representing all the forces on the ball at any time t.

- From Newton's second law, the net force on the ball at any instant must sum up to ma.

$$-mg - bv = ma$$

$$\implies -mg - bv = m\frac{dv}{dt}$$

- This is the differential equation that helps us find everything related to this system. Solving it by separating the variables and using the initial values, we get

$$v = ue^{\frac{-bt}{m}} - \frac{mg}{b}(1 - e^{\frac{-bt}{m}})$$

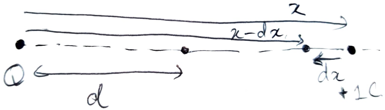
- Once we have v as a function of t, we can get a and x by differentiating and integrating v w.r.t t respectively.

Electrostatics

- Electric current (i) is defined as the rate of flow of charge (q). i.e., $i = \frac{dq}{dt}$.
- Defining electric charge in terms of the base quantity (electric current), $q = \int idt$.
- Coulomb's Law: Force on an electric charge q due to the charge Q is given by $F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{d^2}$, where d is the distance between Q and q.
- Electric field due to a point charge Q at a point d distance from it is defined as $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{d^2}$.
- So, the electric force on a charge q can also be written as $F = Eq$, where E is the electric field at that point due to all the other charges.

Electrostatics

- Electric potential due to a point charge Q at a point d distance from it is the work done in bringing an unit positive charge from infinity to the point at a distance d.



- Let dW be the work done by the external force in moving the charge $+1C$ from x to $x + dx$.

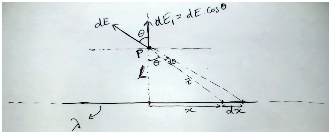
$$dW = -Fdx \implies dW = \frac{-1}{4\pi\epsilon_0}\frac{Q}{x^2}dx$$

- Integrating this from $x = \infty$ to $x = d$, we get the total work done, which is the required electric potential (V).

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{d^2}$$

Modelling example in Electrostatics

Consider a line of charge of infinite length, with charge density λ coulombs per unit length. Our goal is to find the magnitude of electric field due to the line of charge at a point L distance from it.



- Take a infinitesimal length dx of the infinite line charge at a distance (or at an angle θ w.r.t the perpendicular about the point P).
- Let the electric field due to this small length be dE (shown in the figure). The total electric field at P is the summation of the vertical components ($dEcos\theta$) of all such dE.
- The infinitesimal charge present in the length dx is λdx. The electric field due to this charge is $dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{z^2}$.

Modelling example in Electrostatics(cont.)

- From the figure $z = \frac{L}{cos\theta}$ and $x = L(tan\theta) \implies dx = L(sec^2\theta)d\theta$.

$$\implies dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda L(sec^2\theta)d\theta}{\frac{L^2}{cos^2\theta}}$$

$$\implies dE = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{L}d\theta$$

- The total electric field at P is $\int dE_1 = \int dE(cos\theta)$.

$$E_1 = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4\pi\epsilon_0}\frac{\lambda}{L}cos\theta d\theta$$

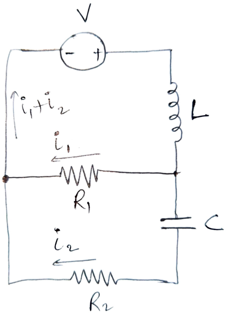
$$\boxed{E_1 = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{L}}$$

Electricity

- Ohm's law: The voltage drop (V) across a resistor R is directly proportional to the current (i) flowing through the resistor. i.e., $V = iR$
- Kirchoff's junction law: Net current flowing through a junction is zero.
- Kirchoff'f loop or voltage law: The algebraic sum of all the potential differences along a closed loop is zero.
- Using these three basic laws and the definition of the physical quantities, any electrical circuit can be modelled using differential equations.
- Circuit elements and their V-I relationships:
    - Resistor: $V = iR$
    - Capacitor: $i = C\frac{dV}{dt} \implies V = \frac{1}{C}\int_0^t idt + V(0)$, where $V(0)$ is the voltage at $t = 0$.
    - Inductor: $V = L\frac{di}{dt} \implies i = \frac{1}{L}\int_0^t Vdt + i(0)$, where $i(0)$ is the current at $t = 0$.

Modelling example in Electric Circuits

Consider the circuit given below. Our goal here is to find the currents i_1 and i_2.



- KVL on upper loop:

$$\boxed{V - L\frac{d(i_1+i_2)}{dt} - i_1R_1 = 0}$$

- KVL on lower loop:

$$i_1R_1 - \frac{q_2}{C} - i_2R_2 = 0$$

$$\boxed{i_1R_1 - \frac{1}{C}\int_0^t i_2 dt - i_2R_2 = 0}$$

(Assuming the initial voltage across the capacitor to be zero)

- Solving these two differential equations, we can find i_1 and i_2.

Other physical phenomena

Other than the phenomena mentioned so far, all the other fields of physics use differential equations to model the physical system or phenomena. Some of the examples are:

- In Magnetism: finding magnetic field due to a varying current, effects of varying magnetic field, etc.
- In Fluid Mechanics: finding pressure when the density is varying, time taken for a tank to empty through a pipe, etc.
- In Heat Conduction and Thermodynamics: finding the time taken for a block to reach a given temperature when heated at one end, finding the heat current when the conductivity is varying with length, etc.
- In Planetary Motion (discussed later)

Modelling conservation laws and planetary motion are discussed in the upcoming slides.

Laws of Conservation

Definition and Types

A conservation law can be stated as follows:

- A particular measurable property of an isolated physical system does not change as the system evolves over time.
- So, the basic idea is that any particle interaction must not change the total energy, total mass, and total charge of the particles, given any condition they are in. This is also known as "the particle physics conservation laws" or "the conservation laws in nuclear physics".

We study more or less the following kinds of laws of conservation:

- Mass-energy Conservation: The total mass and energy of particles before and after the exchange must be the same.
- Momentum Conservation: Momentum, defined as $p = mv$, where $m =$ mass of particle, $v =$ velocity of particle. The total momentum of the system's particles before and after the impact are the same.
- Charge Conservation: The total charge of the particles before and after the exchange is the same.

Laws of Conservation

Motive

When we formulate a mathematical model for a ‘continuum’ physical system, the following are the basic steps:

1. To identify appropriate conservation laws (eg., mass, energy, momentum, etc).
2. Identify their corresponding densities and fluxes.
3. Writing corresponding equations using conservation.
4. Closing the system of equations by proposing appropriate relationships between fluxes and the densities.

The first, second and fourth have physical requirements, the third one is the mathematical one and this is the one which we would be focusing on.

Laws of Conservation

Motive

Once a model is formulated, another step gets involved: -

- Analyzing and validating the model
- Comparing its predictions with observations
- Correcting them whenever needed

In any 'continuum' modeling, there are several scales:

- Visible scales: Densities and fluxes (Mathematical variables in the model vary)
- Invisible scales: The micro-scales that have been averaged in obtaining the model

For example, considering the first scale, when we talk about 'river flow':

- We can say that volume density (volume per unit length), A, as "density", u as the average flow velocity, then $Q = uA$.
- Here, Q is the volume flux of water down the river (volume per unit time).
- This follows conservation of volume, since water is incompressible.

Conservation forms and Differential forms

- Now coming to the mathematical part, the conservation laws and the differential laws are inter changeable.

$$u_t + (f(u))_x = 0 \quad \underleftrightarrow{(\text{if } u \in C^1)} \quad u_t + f'(u)u_x = 0$$

  where, f = flux function; $f'(u) = c(u)$

- The equation towards left: Conservation Form
- The equation towards right: Differential Form

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t))$$

- This is the integral form.

Example

Traffic Flow

- Let us define: $\rho(x,t)$ = vehicle density, $\rho = 0$, empty and $\rho = 1$, packed
  $m(t) = \int_a^b \rho(x,t)dx$ = number of vehicles in $[a,b]$
  $\frac{d}{dt}m(t) = \text{Influx} - \text{Ouflux}$
  $$\frac{d}{dt}m(t) = f(\rho(a,t)) - f(\rho(b,t))$$
- Equation: $\rho_t + (f(\rho))_x = 0$, where $f(\rho) = v.\rho$, where v = velocity.
- Velocity Function: $v = v(\rho) = 1 - \rho$
- Flux Function: $f(\rho) = \rho(1-\rho)$
- Velocity of Information: $c(\rho) = f'(\rho) = 1 - 2\rho$

Conservation of Energy

- As the First Law of Thermodynamics can be stated as:
  The total energy change in system equals the difference between the heat transferred to the system and the work done by the system on its surroundings.

  $$\frac{dE}{dt} = Q' - W'$$

  where, $Q =$ heat transfer, $W =$ work done
- The above equation is used when the work is assumed to be done by the system.
- When the work is done on the system by its surroundings, then the equation basically is:

  $$\frac{dE}{dt} = Q' + W'$$

Conservation of Energy

- The *energy per unit mass* contained in a system is of three parts: internal, kinetic and potential. Thus, we can write the total energy equation as follows:

$$e_t = e(Total) = e(Internal) + e(Kinetic) + e(Potential)$$

  - $e(Internal) = e(T)$
  - $e(Kinetic) = \frac{1}{2}v^2$, where $v =$ velocity of the particle
  - $e(Potential) = gz$, where $z =$ height, $g =$ Gravitational Acceleration

- Therefore, the total energy can be written as: -

$$E = \int\int_{\Omega}\int \rho e_t d\Omega = \int\int_{\Omega}\int \rho[e(T) + \frac{1}{2}V^2 + gz]d\Omega$$

Conservation of Momentum

- Newton's First Law of Motion or the Inertial Law states that:
  The momentum of a system is constant if no external forces are acting on the system.

$$p_1 = p_2$$

- Let M_1 and M_2 be the masses of the two particles moving in the same line of motion. Let u_1 and u_2 be the initial velocities of two particles moving in the same line of motion and let v_1 and v_2 be the final velocities of the particles after the collision of the particles, respectively.
- Now, as momentum, $p = Mv$, Conservation of Momentum states that

$$M_1 u_1 + M_2 u_2 = M_1 v_1 + M_2 v_2$$

Conservation of Momentum

- Newton's Second Law of Motion states that:
  The acceleration of an object depends upon the net force acting on the object and the mass of the object.
- Now, for $x =$ distance, $u =$ velocity, $a =$ acceleration, $t =$ time, $m =$ mass, $\rho =$ density, $A =$ area, $F =$ force, $p =$ pressure

$$F = ma$$

- The above can be written as:

$$F = ma = m\frac{du}{dt} \implies -[(pA_2) - pA_1] = m(\frac{u_2 - u_1}{dt})$$

$$\implies -[(p + (\frac{dp}{dx})dx)A - pA = m(\frac{u + (\frac{du}{dx})dx - u}{dt})$$

$[m = \rho dxA] \implies -(\frac{dp}{dx})dxA = m(\frac{du}{dx})\frac{dx}{dt}$, where $\frac{dx}{dt} = u$
Therefore, the differential form can be written as:

$$-\frac{dp}{dx} = \rho u\frac{du}{dx}$$

Conservation of Charge

- The conservation of charge states that the electrical charges cannot be created or destroyed. Therefore, the total charges in the system always remains the same irrespective of any given condition.
- Integral Form:
  The integral formulation of conservation of charge is: -

$$\int_A j.da = -\frac{d}{dt}\int_V \rho dv = -\frac{dQ}{dt}$$

  where,
  j : current density, ρ : volumetric charge density, Q : total charge inside the volume, A : surface area of the volume and V : volume
- Differential Form
  The differential formulation of conservation of charge is:

$$\Delta j = -\frac{d\rho}{dt}$$

Planetary Motion

Introduction

- Ancient Greek astronomers knew the existence of five "stars", Mercury, Venus, Mars, Jupiter, and Saturn. To them, the apparent motion of these objects was following an irregular path, and was not normal compared to the observable regular pathways of the other stars. The view that the Earth was in the centre of the Universe was also deeply rooted in their religious beliefs.
- Copernicus, however, concluded that these planets revolve around the sun in circular orbits. Later, based on Tycho Brahe's observations, Kepler concluded that the speculated circular orbits were not in agreement with those observations, and that elliptical orbits were the answer.

Kepler's Laws

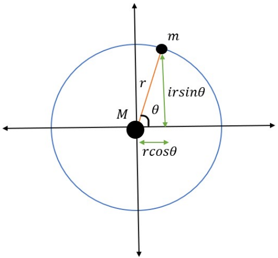
Based on the motion of the planets about the sun, Kepler devised a set of three classical laws, called Kepler's laws of planetary motion, that describe the orbits of all bodies satisfying these two conditions:

- The orbit of each planet around the sun is an ellipse with the sun at one focus.
- Each planet moves so that an imaginary line drawn from the sun to the planet sweeps out equal areas in equal times.
- The ratio of the squares of the periods of any two planets about the sun is equal to the ratio of the cubes of their average distances from the sun.

Derivation of the Orbit

Newton's Universal Law of Gravitation states that the gravitational force between the Sun M and a planet m is proportional to the product of the two masses and inversely proportional to the distance between them. Gravity is an attractive force. Therefore, M pulls m towards itself.

$$F = -G\frac{mM}{r^2}$$



From Newton's Second Law of Motion,

$$a = -G\frac{M}{r^2}e^{i\theta} \tag{1}$$

We also have,

$$z(t) = re^{i\theta} \implies v(t) = \frac{dz}{dt} = ire^{i\theta}\frac{d\theta}{dt} + e^{i\theta}\frac{dr}{dt}$$

$$a(t) = e^{i\theta}(-r(\frac{d\theta}{dt})^2 + \frac{d^2r}{dt^2} + i(\frac{d}{dt}(r^2\frac{d\theta}{dt}))) \tag{2}$$

Derivation of the Orbit (cont.)

Equating real and imaginary parts of (1) and (2):

$$G\frac{M}{r^2} = -r(\frac{d\theta}{dt})^2 + \frac{d^2r}{dt^2} \tag{3}$$

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 0 \tag{4}$$

From equations (3) and (4), we get the 2^{nd} order ODE of the orbit ($r = \frac{1}{s}$ and c is a constant)

$$\frac{d^2s}{d\theta^2} = -s + \frac{GmM}{c^2}$$

The solution for this equation will be of the form

$$s = Asin\theta + Bcos\theta + \frac{GmM}{c^2} \tag{5}$$

Derivation of the Orbit (cont.)

To simplify the equation, we assume that M and m are the closest to each other when m is on the positive real axis. That means, s has a local maximum at $\theta = 0 \implies \frac{ds}{d\theta} = 0$ and $\frac{d^2s}{d\theta^2} \leq 0$
This gives us our final equation:

$$s = Bcos\theta + \frac{GmM}{c^2} \text{ where } B \leq 0$$

$$\frac{1}{r} = Bcos\theta + \frac{GmM}{c^2} \implies r = \frac{c^2/GmM}{1+(c^2B/GmM)cos\theta}$$

Let $a = c^2/GmM$ and $e = aB$. Then,

$$r = \frac{a}{1+ecos\theta}$$

where a is the semi-major axis and e is the eccentricity of the ellipse.

Conclusion

- We see that almost every system or phenomena shown in this presentation has differential equations to represent the behaviour and other parameters. By studying differential equations and the ways to analyze or solve them, we are learning how to analyze real world phenomena, and ways to solve real world problems.
- From the above slides, we can conclude that differential equations have significant amount of role in deriving and stating various physical phenonmenon, laws of conservation, planetary motion and various other phenonmenon.

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Thank You!