Studio Worksheets for Math 1554

School of Mathematics, Georgia Institute of Technology

The Studio Worksheets are meant to be used by all sections of MATH 1554. The exact pacing through the questions in the worksheets may vary from class to class.

"I never teach my pupils. I only attempt to provide the conditions in which they can learn."
- Albert Einstein

"A problem isn't finished just because you've found the right answer."
- Yōko Ogawa

Worksheet 1.1, Systems of Linear Equations

Worksheet Solutions

A primary goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. To help us meet this goal, **solutions are not are provided for worksheets**. This is intentional: most upper level courses do not have recitations and solutions for everything. Students in this class are encouraged to ask questions they may have on Piazza, office hours, and with their peers. Calculators and software are also great ways to check your work. All of these methods develop skills that are transferable to higher level courses, and beyond.

Worksheet Exercises

Recitations are meant to be active: students are encouraged to work with other students in recitation. As students are working through exercises, the TA should circulate around the room, helping students. Students may be asked to present their work using a document camera or write a solution on a whiteboard.

1. **Written Explanation Exercise** Answer each the following in one or two sentences.
   (a) What does it mean for a linear system to be consistent?
   (b) How can we determine whether a linear system is consistent?

2. Indicate whether the statements are true or false.
   (a) If a linear system has more equations than unknowns, then the system cannot have a unique solution.
   (b) If a linear system has more unknowns than equations, then the system cannot have a unique solution.

3. For what values of A and B, if any, does the system have (a) an infinite number of solutions? (b) no solutions? (c) exactly one solution?

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ Ax_1 + Bx_2 &= 2 \end{aligned}$$

   For the case where there are an infinite number of solutions, sketch the set of solutions.

4. Sketch the set of solutions to the linear system.

$$\begin{aligned} x - 4y &= 0 \\ -2x + 8y &= 0 \end{aligned}$$

Worksheet 1.2, Row Reduction and Echelon Forms

Worksheet Exercises

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1. **Written Explanation Exercise** Answer each the following in one or two sentences.

   (a) What are some of the differences between echelon form and row reduced echelon form (RREF)? List at least three.

   (b) How can we use row reduction to determine whether an augmented matrix corresponds to a consistent system?

2. Which matrices are in RREF? In echelon form?

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 5 & 5 \end{pmatrix}$$

3. List all 3×2 matrices in RREF. Use $*$ for entries that can be arbitrary.

4. Indicate whether the statements are true or false.

   a) A linear system, whose 3×5 coefficient matrix has three pivotal columns, must be consistent.

   b) The echelon form of a coefficient matrix is unique.

5. For any three distinct points in the plane, no two on a vertical line, there is a second degree polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ that passes through those three points. Construct the polynomial that passes through $(1, 12)$, $(2, 15)$, and $(3, 16)$. That is, solve

$$\begin{aligned} p(1) &= 12 = a_0 + a_1 + a_2 \\ p(2) &= 15 = a_0 + 2a_1 + 4a_2 \\ p(3) &= 16 = a_0 + 3a_1 + 9a_2 \end{aligned}$$

   (The coefficent matrix is an instance of the *Vandermonde matrix*.)

Worksheet 1.3 and 1.4, Vector Equations and The Matrix Equation

Worksheet Exercises

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1. **Written Explanation Exercise**
   (a) What does the span of a set of vectors represent?
   (b) How do we determine whether a vector is in the span of a set of vectors?

2. Indicate whether the statements are true or false.
   a) If the equation $A\vec{x} = \vec{b}$ is inconsistent, then $\vec{b}$ is not in the set spanned by the columns of A.
   b) If the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ has a pivot position in every row, then the equation $A\vec{x} = \vec{b}$ must be consistent.
   c) There are exactly three vectors in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

3. Span$\{\vec{v}_1, \vec{v}_2\}$ is equal to which of the expressions below?
   i) Span$\{\vec{v}_1, \vec{v}_2, 3\vec{v}_1\}$
   ii) Span$\{\vec{v}_1, 3\vec{v}_1\}$
   iii) Span$\{\vec{v}_1, \vec{v}_2, 3\vec{v}_1 + 2\vec{v}_2\}$

4. For what values of h is $\vec{b}$ in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$?

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} -6 \\ -17 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ 2 \\ h \end{pmatrix}$$

5. Sketch the span of the columns of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \end{pmatrix}$.

Worksheet 1.5, Solution Sets of Linear Systems

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The worksheet for this week is shorter; we are assuming that there will be a quiz this week.

Worksheet Exercises

1. **Written Explanation Exercise**

   (a) When a homogeneous system has a nontrivial solution, what properties does that system have? List at least two.

2. Indicate whether the statements are true or false.

   (a) A non-trivial solution $\vec{x}$ to $A\vec{x} = \vec{0}$ has all non-zero entries.

   (b) If $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$, then $A(\vec{x} - \vec{y}) = \vec{0}$.

   (c) Any 3×2 matrix A with two pivotal positions has a non-trivial solution to $A\vec{x} = 0$.

3. **Example Construction**

   (a) Give an example of a non-zero 2×3 matrix A such that $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a solution of $A\vec{x} = \vec{0}$.

   (b) Give an example of a non-trivial solution to $A\vec{x} = \vec{0}$, where $A = \begin{pmatrix} 2 & 5 \\ 0 & 0 \\ 4 & 10 \end{pmatrix}$.

4. Express the solution to $A\vec{x} = \vec{0}$ in parametric vector form, where $A = \begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

Worksheet 1.7, Linear Independence

Worksheet Exercises

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1. **Written Explanation Exercise**

   (a) How are span and linear dependence related to each other?

   (b) Suppose T is a linear map.
   (a) If $v_1, \ldots, v_k$ are dependent, why are $T(v_1), \ldots, T(v_k)$ dependent?
   (b) If $v_1, \ldots, v_k$ are independent, need $T(v_1), \ldots, T(v_k)$ be independent?

2. In the problems below, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$. Which of the collections of vectors below are linearly independent?

   a) $\{\vec{v}_1, \vec{v}_2, \vec{0}\}$

   b) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_2\}$

   c) $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$

3. For what values of h are the columns of A linearly dependent?

$$A = \begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 4 & 7 & h \end{bmatrix}$$

4. A 5×3 matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ has all non-zero columns, and $\vec{a}_3 = 5\vec{a}_1 + 7\vec{a}_2$. Identify a non-trivial solution to $A\vec{x} = \vec{0}$.

5. Fill in the blanks.

   (a) The columns of a 7×5 matrix are linearly independent. How many pivots does the matrix have?

   (b) If the columns of a 3×7 matrix span $\mathbb{R}^3$, how many pivots does the matrix have?

Worksheet 1.8, An Introduction to Linear Transforms

Worksheet Exercises

1. **Written Explanation Exercise**
   Suppose $T(x) = Ax$ for all x where A is a matrix and T is onto.
   (a) What can we say about pivotal rows of A?
   (b) What can we say about the existence of solutions to $Ax = b$?

2. Let A be an 3×4 matrix. What must c and d be if we define the linear transformation $T : \mathbb{R}^c \to \mathbb{R}^d$ by $T(\vec{x}) = A\vec{x}$?

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that
   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$
   Construct a matrix A so that $T(\vec{x}) = A\vec{x}$ for all vectors $\vec{x}$.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that
   $$T\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \neq \vec{0}.$$
   Identify a non-trivial solution $\vec{x}$ to $T\vec{x} = \vec{0}$.

5. Let T_A be the linear transformation with the matrix below. Match each choice of A on the left with the geometric description of the action of T_A on the right.

$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$	rotation by 90^o
$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	A shear
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	Projection onto y axis
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	dilation by 1/2

Worksheet 1.9, Linear Transforms

1. Indicate whether the statements are true or false.

   (a) If A is a 3×2 matrix then the map $x \mapsto Ax$ cannot be one-to-one.

   (b) If A is a 2×3 matrix then the map $x \mapsto Ax$ cannot be onto.

   (c) $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $A\vec{x} = \vec{0}$ only has the trivial solution.

2. Construct the standard matrix of the linear transformation T.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^4$, where $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\\4\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\6\\1\\8\end{bmatrix}$

   (b) T is a vertical shear given by $T(\vec{e}_2) = 2\vec{e}_2$ and $T(\vec{e}_1) = \vec{e}_1 - 2\vec{e}_2$.

   (c) A matrix $A \in \mathbb{R}^{2\times 2}$ such that $T(\vec{x}) = A\vec{x}$. T is a linear transformation that first reflects vectors across the line $x_1 = x_2$, then rotates them counterclockwise by π radians about the origin, then reflects them across the line $x_2 = 0$.

Worksheet 2.1, Matrix Operations

1. **Written Explanation Exercise**
   For square matrices A, B, is it always true that $(A+B)^2 = A^2+2AB+B^2$? Explain why/why not.

2. Consider:
$$A = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \quad B = \begin{pmatrix}1 & h\\k & 1\end{pmatrix}$$
   For what values (if any) of $k \in \mathbb{R}$ and $h \in \mathbb{R}$:

   (a) do matrices A and B commute?

   (b) is the product AB equal to I_2?

   (c) is the product AB equal to the 2×2 zero matrix $0_{2\times 2}$?

3. A is an $n \times n$ matrix that has elements a_{ij} where
$$a_{ij} = \begin{cases}0, & \text{when } i+j \text{ is odd}\\ 1, & \text{when } i+j \text{ is even}\end{cases}$$
   For $n \geq 2$, how many pivot columns does A have?

Worksheet 2.2 and 2.3, Invertible Matrices

Worksheet Exercises

1. Consider the sequence of row operations that reduce matrix A to the identity.

$$A = \underbrace{\begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 8 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 8 & 1 \end{pmatrix}}_{E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2 E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3 E_2 E_1 A} = I_3$$

   Construct the elementary matrices E_1, E_2, E_3.

2. Indicate whether the statements are true or false. A is an $n \times n$ matrix.

   (a) If $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y} \in \mathbb{R}^n$, then A cannot be invertible.
   (b) If for some $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has more than one solution, then A is invertible.
   (c) Every elementary matrix is invertible.

3. Compute the inverse of the matrix, where $c \in \mathbb{R}$. For what values of c does the matrix have an inverse?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & -1 & c \end{bmatrix}$$

4. Let A be an $n \times n$ matrix. Which statements guarantee that A is invertible?

   (a) Every vector in $\mathbb{R}^n$ is in the span of the columns of A.
   (b) $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.
   (c) Matrix A can be row reduced to the identity matrix.
   (d) The range of the linear transform $\vec{x} \to A\vec{x}$ is $\mathbb{R}^n$.

5. Two reasons that a matrix is not invertible are:

   (a) One column is a multiple of another column.
   (b) One column is the sum of other columns.

   By inspection, identify which of the reasons above apply to these matrices.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -7 & 0 & 5 \\ 3 & 0 & -2 \\ 10 & 0 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 14 & 21 \\ 5 & 10 & 15 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

Worksheet 2.4 and 2.5
Partitioned Matrices and Matrix Factorizations

Worksheet Exercises

1. A and B are $n \times n$ invertible matrices, I_n is the $n \times n$ identity matrix, and 0 is the $n \times n$ zero matrix. Construct an expression for X in terms of A and B.

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 0 & X & B \\ A & B & 0 \end{pmatrix} \begin{pmatrix} X \\ I_n \\ BA \end{pmatrix} = B^2 + BAX$$

2. Compute the LU factorization for

$$A = \begin{bmatrix} -1 & 5 & 3 & 1 \\ 1 & -10 & -3 & 1 \\ 0 & -5 & 0 & 2 \end{bmatrix}$$

3. Complete the LU factorization of A and use it to solve $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$$

4. **Written Explanation Exercise** What is the LU decomposition good for? Your reasoning should involve computational efficiency.

Worksheet 2.6, 2.7, 2.8: The Leontif Input-Output Model, Computer Graphics, Subspaces of $\mathbb{R}^n$

Worksheet Exercises

1. An economy contains three sectors: X, Y, Z. For each unit of output,

   - X requires .2 units from X, .1 units from Y, and .1 units from Z
   - Y requires 0 units from X, .2 units from Y, and .1 units from Z
   - Z requires 0 units from X, 0 units from Y, and .2 units from Z

   Construct the consumption matrix for this economy. What production level is required to satisfy a final demand of 80 units of X, 60 units of Y, and 160 units of Z?

2. Rectangle S is determined by the data points, $(1,1),(3,1),(3,2),(1,2)$. Transform T reflects points through the line $y = 2 - x$.

   (a) Represent the data with a matrix, D. Use homogeneous coordinates.

   (b) Use matrix multiplication to determine the image of S under T.

   (c) Sketch S and its image under T.

3. Transform $T_A = A\vec{x}$ rotates points in $\mathbb{R}^2$ about the point $(1,2)$. Construct a standard matrix for the transform using homogeneous coordinates. Leave your answer as a product of three matrices.

4. Construct the matrix for the transformation that performs a rotation in $\mathbb{R}^3$ about the x-axis by π radians.

5. A has the reduced echelon form below. Construct a basis for Col$A$ and for Nul$A$.

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 & \vec{a}_6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 10 & 0 & 13 \\ 0 & 0 & 1 & -3 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Worksheet 2.8, 2.9, Dimension and Rank

The worksheet for this week is shorter; we are assuming that there will be a quiz this week.

Worksheet Exercises

1. Construct a 3×3 matrix A with two pivotal columns, so that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the null space of A.

   (a) All 2×4 matrices have a non-trivial null space.

   (b) A 4×2 matrix with two pivot columns can have a non-trivial null space.

   (c) If the columns of a 6×6 matrix A are a basis for $\mathbb{R}^6$, the null space of A is the zero vector.

2. A is an $n \times n$ matrix that has elements a_{ij} where

$$a_{ij} = \begin{cases} 0, & \text{when } i + j \text{ is odd} \\ 1, & \text{when } i + j \text{ is even} \end{cases}$$

   Suppose $n \geq 2$.

   (a) What is the rank of A?

   (b) Give a basis for the column space of A.

3. Which of the following, if any, are subspaces of $\mathbb{R}^3$? For those that are subspaces, what is the dimension of the subspace?

   (a) $\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 + x_2 = 4\}$

   (b) $\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 + x_2 + x_3 = 0,\ x_1 + 2x_2 = 0\}$

   (c) $\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,|\, x_1 \leq x_2 \leq x_3\}$

   (d) The nullspace of $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}$.

Worksheet 3.1 to 3.3, Determinants

Worksheet Exercises

1. **Written Explanation Exercise**
   Discuss the computational efficiency of computing $\det(A)$ by cofactor expansion and by row operations. Which method is computationally better if A is $n \times n$ and n is large? (Compare how many arithmetic operations it takes).

2. Use a determinant to identify all values of t and k such that the are the matrices are singular. Assume that h and k must be real numbers.

   (a) $A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} - t\, I_2$

   (b) $B = \begin{pmatrix} 0 & 1 & t \\ -3 & 10 & 0 \\ 0 & 5 & k \end{pmatrix}$

3. Let $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix}$ be a 4×4 matrix whose determinant is equal to 2. Compute the value of the determinant $\begin{vmatrix} \vec{d} & \vec{b} & 3\vec{c} & \vec{a} \end{vmatrix}$.

4. R is the parallelogram determined by $\vec{p}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $\vec{p}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, what is the area of the image of R under the map $\vec{x} \mapsto A\vec{x}$?

5. $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transformation that first rotates vectors in $\mathbb{R}^2$ counterclockwise by θ radians about the origin, then reflects them through the line $x_1 = x_2$. By inspection, what is the value of the determinant of A? You should compute its value to check your answer.

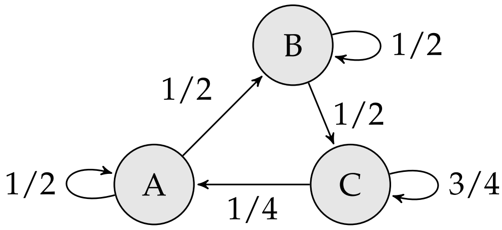
Worksheet 4.9, Applications to Markov Chains

Worksheet Exercises

1. Determine whether P and Q are a regular stochastic matrices.

$$P = \begin{pmatrix} .8 & 0 \\ .2 & 1 \end{pmatrix} \quad Q = \frac{1}{4}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$$

2. Consider the Markov chain below.



What is the transition matrix? Calculate the steady-state vector.

3. Suppose there are two cities, X and Y. Every year,

   - 70% of the people from X stay in X, the remaining 30% move to Y.
   - 40% of the people from Y stay in Y, the remaining 60% move to X.

   The initial populations of X and Y are 100 and 200, respectively.

   (a) What is the stochastic matrix that represents this situation?

   (b) After a long period of time, what is the population in city X?

4. **Written Explanation Exercise** Let P be a stochastic $n \times n$ matrix with positive entries. Give two methods of finding the steady state solution.

5. A mouse lives in a maze that has at least three rooms. Each room is connected to at least one other room (in other words, every room is connected). At every hour the mouse moves from the room where it is in, to one of the rooms it is connected to, with equal probability.

   (a) Design any mouse maze and its corresponding transition matrix P.

   (b) Is your P regular stochastic?

   (c) In the long run, is there a room that the mouse is more likely to be in at a given time? If so, which room?

Note: this problem is related to the PageRank problem that we explore later in this class.

Worksheet 5.1 and 5.2: Eigenvectors and Eigenvalues, The Characteristic Equation

Worksheet Exercises

1. If possible, give an example of:

   (a) a 2×2 matrix, $A \in \mathbb{R}^{2\times 2}$, whose eigenvalues have non-zero imaginary components.

   (b) a non-zero 2×2 matrix, $A \in \mathbb{R}^{2\times 2}$, that is not triangular, but has a zero eigenvalue.

2. Determine whether $\vec{u}$ and $\vec{v}$ are eigenvectors of A. If so, what are their eigenvalues? Do not find the characteristic polynomial.

$$A = \begin{pmatrix} -3 & -3 & 2 \\ 6 & 4 & 0 \\ 5 & 3 & 0 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

3. T is a linear transformation in $\mathbb{R}^2$. Without constructing A, identify one eigenvalue of A.

   (a) T reflects points across the line $x_1 = -x_2$.

   (b) T projects points onto one of the coordinate axes.

4. For what values of k (if any) does A have one real eigenvalue of algebraic multiplicity 2?

$$A = \begin{pmatrix} -4 & k \\ 2 & -2 \end{pmatrix}$$

5. tr(A) is the sum of the elements on the main diagonal of A. If tr$(A) = 2$, det$(A) = 1$, and $A \in \mathbb{R}^{2\times 2}$, compute the eigenvalues of A. *Hint: let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6. **Written Explanation Exercise** If $Av = \lambda v$ with $v \neq 0$ and A is invertible, can you find an eigenvalue/eigenvector of A^{-1}? Can A has a zero eigenvalue?

Worksheet 5.3, Diagonalization

Worksheet Exercises

1. Recall from lecture: matrix A is diagonalizable if it can be written as $A = PDP^{-1}$.

   (i) P is a matrix whose columns are ________________________ .

   (ii) D is a ______________ matrix.

   (iii) The elements on the main diagonal of D are ______________ .

   (iv) A diagonal matrix is a matrix that ___________________ .

   (v) The geometric multiplicity of an eigenvalue is:

   (vi) A matrix can be diagonalized when the geometric multiplicities of all the eigenvalues:

2. If possible, construct P and D so that $A = PDP^{-1}$. Eigenvalues of A are given.

   (a) $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \lambda = 2, 2, 5$

   (b) $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad \lambda = -2, 5$

   (c) $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \quad \lambda = 3, 3$

3. If possible, give an example of:

   (a) A singular 2×2 matrix in echelon form that can be diagonalized.

   (b) A singular 2×2 matrix in echelon form that cannot be diagonalized.

   (c) An invertible 2×2 matrix in echelon form that can be diagonalized.

   (d) An invertible 2×2 matrix in echelon form that cannot be diagonalized.

4. Indicate whether the statements are true or false.

   (a) if A is diagonalizable, then so is A^2.

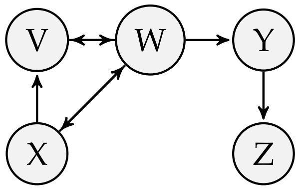
   (b) if A^2 is diagonalizable, then so is A.

5. **Written Explanation Exercise** Give an example of an upper triangular 4×4 matrix A such that 0 is its only eigenvalue and such that its eigenspace is 3-dimensional. Explain why the eigenspace has dimension 3.

Worksheet 10.2, The Steady-State Vector and Page Rank

Worksheet Exercises

1. A set of web pages link to each other according to this diagram.



   (a) Create the transition matrix, P, for this web.
   (b) Construct the Google Matrix for this web, G. Use damping factor $p = 0.85$.
   (c) During an exam, to determine the page ranks of each page in the web a you would be given the steady-state vector of G. Because you are not taking an exam right now, compute the steady-state vector and page ranks of each page on the web. You can use software.

   Hint: For a web with only two pages that are linked to each other, we can compute the steady state using MATLAB or Octave using these commands:

   ```
   Pstar=1/2*[0 1;1 0]
   K = 1/2*ones(2)
   p = 0.85
   G = p*Pstar + (1-p)*K
   G ^ (100)
   ```

   There are many free online Octave compilers.

2. Suppose p and q are real numbers on the open interval $(0, 1)$, and

$$A = \begin{pmatrix} p & 1-q \\ 1-p & q \end{pmatrix}$$

   (a) Is A stochastic? Is A regular?
   (b) By inspection, what is one eigenvalue of A?
   (c) Compute the steady-state vector of A.
   (d) Compute the limit $\lim_{n\to\infty} A^n$

3. Consider the dynamical system $\vec{x}_k = A\vec{x}_{k-1},\ k = 1, 2, 3, \ldots$, where

$$A = \begin{pmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

   The eigenvalues of A are 1 and $\frac{1}{4}$. Analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

Worksheet 5.5, Complex Eigenvalues

Worksheet Exercises

1. Indicate whether the statements are true or false.
   (a) There exists a real 2×2 matrix with the eigenvalues i and $2i$.
   (b) Every real 3×3 matrix must have a real eigenvalue.

2. A is a composition of a rotation and a scaling. Give the angle of rotation, ϕ, and the scale factor, r.
$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

3. Let $A = \begin{pmatrix} 4 & -1 \\ 2 & 6 \end{pmatrix}$. Find an invertible matrix P and a rotation-dilation matrix C such that $A = PCP^{-1}$.

4. Matrix A is a 2×2 matrix that satisfies the equality
$$A^2 + 2A = -6I_2$$
I_2 is the 2×2 identity matrix. Compute the eigenvalues of A.

5. **Written Explanation Exercise** Can a 7×7 matrix have 2 real eigenvalues and 5 non-real eigenvalues? If A is an $n \times n$ matrix and n is odd, why does A have a real eigenvalue?

Worksheet 6.1, Inner Product, Length, and Orthogonality

The worksheet for this week is shorter; we are assuming that there will be a quiz this week.

Worksheet Exercises

1. Fill in the blanks.

   (a) The distance between the vector $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the line spanned by $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\boxed{}$.

   (b) If W is the plane spanned by the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, a basis of $W^\perp$ is given by $\vec{w} = \begin{pmatrix} \\ \\ \end{pmatrix}$.

   (c) If $V = \{\vec{x} \in \mathbb{R}^3 \,|\, x_1 + x_2 = x_3\}$, then $\dim V = \boxed{}$, and $\dim V^\perp = \boxed{}$.

2. W is the set of all vectors of the form $\begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$. Which of the vectors are in $W^\perp$?

$$\vec{u} = \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

3. True or False.

   (a) If $\vec{x} \in \text{Null}(A)$, then $\vec{x}$ is orthogonal to the rows of matrix A.

   (b) If $\vec{u}$ and $\vec{v}$ are non-zero orthogonal vectors, then they are linearly independent.

Worksheet 6.2, Orthogonal Sets

Worksheet Exercises

1. Indicate whether the statements are true or false.

   (a) If the columns of an $n \times n$ matrix A are orthonormal, then the linear mapping $\vec{x} \mapsto A\vec{x}$ preserves lengths.

   (b) If P is a stochastic matrix, then the columns of P have unit length.

2. Write $\vec{y}$ as the sum of a vector parallel to $\vec{u}$ and a vector perpendicular to $\vec{u}$.

$$\vec{y} = \begin{pmatrix} -1 \\ -5 \\ 10 \end{pmatrix}, \qquad \vec{u} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

3. Find the coordinates for $\vec{v}$ in the subspace spanned by the orthogonal vectors $\vec{u}_1$ and $\vec{u}_2$.

$$\vec{v} = \begin{pmatrix} 0 \\ -5 \\ -3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 6 \\ -7 \\ -10 \end{pmatrix}$$

4. Give examples of the following.

   (a) A matrix, A, in RREF, such that $\dim\left((\mathrm{Row}(A))^\perp\right) = 1$ and $\dim\left((\mathrm{Col}(A))^\perp\right) = 2$.

$$A = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

   (b) Two linearly independent vectors in $\mathbb{R}^3$, $\vec{u}$ and $\vec{v}$, such that $\vec{u} \cdot \vec{x} = \vec{v} \cdot \vec{x} = 0$, where $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. $\qquad \vec{u} = \begin{pmatrix} \quad \\ \\ \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} \quad \\ \\ \end{pmatrix}$

   (c) A 3×3 matrix in RREF, A, such that $(\mathrm{Null}\, A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

$$A = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

   (d) A non-zero vector, $\vec{w}$, whose projection onto $\mathrm{Col}(A)$ is $\vec{w}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\vec{w} = \begin{pmatrix} \quad \\ \\ \end{pmatrix}$$

Worksheet 6.3 and 6.4: Orthogonal Projections, The Gram-Schmidt Process

1. $\vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

   (a) Determine whether $\vec{u}_1$ and $\vec{u}_2$

      i. are linearly independent

      ii. are mutually orthogonal

      iii. are orthonormal

      iv. span $\mathbb{R}^3$

   (b) Is $\vec{y}$ in W = Span$\{\vec{u}_1, \vec{u}_2\}$?

   (c) Compute the vector, $\widehat{y} \in W$, that most closely approximates $\vec{y}$.

   (d) Construct a vector, $\vec{z}$, that is in $W^\perp$.

2. Compute the QR decomposition of $A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix}$.

3. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for subspace V. Classify each set as a basis for V, an orthogonal basis for V, or not a basis for V.

   (a) $\{3\vec{v}_3, 2\vec{v}_1, \vec{v}_2\}$

   (b) $\{(\vec{v}_1 + \vec{v}_2), (\vec{v}_1 - \vec{v}_2), \vec{v}_3\}$

   (c) $\{(\vec{v}_1 + \vec{v}_2), (\vec{v}_1 - \vec{v}_2), (\vec{v}_3 - \vec{v}_1)\}$

4. Indicate whether the statements are true or false.

   (a) If $\vec{y}$ is in subspace W, the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$.

   (b) If $\vec{x}$ is orthogonal to $\vec{v}$ and $\vec{w}$, then $\vec{x}$ is also orthogonal to $\vec{v} - \vec{w}$.

5. If possible, give an example of:

   (a) two linearly independent vectors that are orthogonal to $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

   (b) a subspace of $\mathbb{R}^3$, S, such that $\dim(S^\perp) = 2$.

6. **Written Explanation Exercise** Let $u_1, \ldots, u_k$ be an orthonormal family of vectors in $\mathbb{R}^n$. Explain why applying the Gram-Schmidt process to the pivotal columns of the $n \times (n+k)$ matrix $A = [u_1 \ldots u_k e_1 \ldots e_n]$ gives an orthonormal basis of $\mathbb{R}^n$ that contains $u_1, \ldots, u_k$.

Worksheet 6.5 and 6.6: Least-Squares Problems, Applications to Linear Models

Worksheet Exercises

1. Fill in the blanks. These questions concern the least squares solution $\hat{x}$ to $A\vec{x} = \vec{b}$.

   (a) If $A = QR$, then $A^T A =$ _______.

   (b) If the columns of A are linearly independent, then $\widehat{x} =$ _______$A^T\vec{b}$.

   (c) If $\vec{b}$ is in the column space of A, then $A\widehat{x} =$ _______.

   (d) If $A = QR$ and R is invertible, then $\widehat{x} =$ _______$Q^T\vec{b}$.

2. These questions concern the least squares solution $\hat{x}$ to $A\vec{x} = \vec{b}$. Indicate whether the statements are true or false.

   (a) The solution $\hat{x}$ is chosen so that $A\hat{x}$ is close as possible to $\vec{b}$.

   (b) If $\vec{y} \neq \widehat{x}$ then $\|A\widehat{x} - \vec{b}\| < \|A\vec{y} - \vec{b}\|$.

   (c) If the columns of A are linearly independent, then the least squares solution is unique.

3. Use the QR decomposition to calculate the least squares solution to $A\vec{x} = \vec{b}$.

$$A = QR = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

4. **Written Explanation Exercise** Explain step by step how to find the best fit line for a collection of n data points $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \dots, \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ in $\mathbb{R}^2$. Why is the best fit line unique?

5. Four points in $\mathbb{R}^3$ with coordinates (x, y, z) are given in the table below.

x	-1	0	2	1
y	2	1	0	2
z	9	1	0	-1

   Determine the coefficients c_1 and c_2 for the plane $z = c_1x + c_2y$ that best fits the data. *Hint: normal equations.*

Worksheet 7.1 Diagonalization of Symmetric Matrices

Worksheet Exercises

1. Construct a spectral decomposition for $A = PDP^T$.

$$A = \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$$

2. If possible, give an example of:

   (a) a matrix $A \in \mathbb{R}^{2\times 2}$ that is diagonalizable but not orthogonally diagonalizable

   (b) a matrix $A \in \mathbb{R}^{2\times 2}$ that is orthogonally diagonalizable but not invertible.

   (c) any matrix that is diagonalizable but not invertible.

3. Indicate whether the statements are true or false.

   (a) If A is orthogonally diagonalizable, then so is A^2.

   (b) For any matrix $A \in \mathbb{R}^{m\times n}$, AA^T and A^TA are symmetric matrices.

4. The only eigenvalues of A are 7 and -2.

$$A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

   Construct matrices P and D for the orthogonal diagonalization $A = PDP^T$. *Hints: you may want to use Gram-Scmhidt to construct P. And P is an orthogonal matrix, so what do the lengths of each column of P need to be?*

Worksheet 7.2 Quadratic Forms

Worksheet Exercises

1. Indicate whether the statements are true or false.

   (a) $3x^2 - 8xy + 3y^2 \geq 0$ for all real values of x and y.

   (b) The quadratic form $Q = -x^2 - 2xy - y^2$ is negative semi definite.

2. Assume $\vec{x} \in \mathbb{R}^3$. Construct the matrix of the quadratic form, and classify the quadratic form.
$$Q = 4x_3^2 - 2x_1x_2$$

3. Make a change of variable that transforms the quadratic form into another quadratic form that has no cross-product term.
$$Q = x_1^2 - 12x_1x_2 - 4x_2^2$$

4. **Written Explanation Exercise** Let $Q(x) = Ax \cdot x$ where A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Explain how to find a change of variables so that Q becomes $P(y) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$.

Worksheet 7.3 Constrained Optimization

Worksheet Exercises

1. Indicate whether the statements are true or false.

   (a) The largest value of a positive definite quadratic form $\vec{x}^T A\vec{x}$ is the largest eigenvalue of A.

   (b) The largest value of a positive definite quadratic form $\vec{x}^T A\vec{x}$ subject to $||\vec{x}|| = 1$ is the largest value on the diagonal of A.

2. Calculate the maximum and minimum values of the quadratic form Q subject to the constraint $||\vec{x}|| = 1$. Identify where this maximum is obtained.

$$Q(\vec{x}) = 4x_1^2 + x_2^2 + 4x_1x_2 + 3x_3^2, \quad \vec{x} \in \mathbb{R}^3$$

3. Calculate the maximum and minimum values of the quadratic form Q subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u} = 0$.

$$Q(\vec{x}) = 4x_1^2 + x_2^2 + 4x_1x_2 + 3x_3^2, \quad \vec{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

4. If possible, give examples of the following.

   (a) A quadratic form $Q : \mathbb{R}^3 \mapsto \mathbb{R}$, that has maximum value 12, subject to the constraint that $||\vec{x}|| = 1$.

   (b) A quadratic form $Q : \mathbb{R}^3 \mapsto \mathbb{R}$, that has a maximum value 4 at two distinct locations, subject to the constraint that $||\vec{x}|| = 1$.

Worksheet 7.4, The Singular Value Decomposition

Worksheet Exercises

1. Indicate whether the statements are true or false.

   (a) Every matrix has a singular value decomposition.

   (b) If A is symmetric, then its factorization $A = UDU^T$ is also its SVD.

   (c) The maximum value of $||A\vec{x}||$ subject to $||\vec{x}|| = 1$ is σ_1.

2. Construct the SVD of
$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$$

3. Find a unit vector $\vec{x}$ for which $A\vec{x}$ has maximum length
$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$
   Also compute the condition number of A.

4. By inspection, construct an SVD of the diagonal matrix
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$
   The SVD of a matrix is not unique: how many different SVDs can you create from the matrix above?

5. **Written Explanation Exercise** Let A be an $m \times n$ matrix of rank r. If r is much smaller than m and n, explain how the following version of the singular value decomposition
$$A = \sigma_1 u_1 v_1^\top + \cdots + \sigma_r u_r v_r^\top$$
   gives an efficient way to store A (this is called data compression).

Worksheet: Final Exam Review

Worksheet Exercises

1. Indicate whether the statements are possible (P) or impossible (I). For the statements that are possible, give an example.

	P	I	example
a) T_A is a linear transformation $T_A = A\vec{x}$, where $A \in \mathbb{R}^{N\times N}$, $N \geq 2$, and T_A is one-to-one but not onto.	◯	◯	
b) $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a negative semi-definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A.	◯	◯	
c) A is a square 4×4 matrix, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim(\text{Row}(A - \lambda I)^{\perp}) = 0$.	◯	◯	
d) Matrix A is $n \times n$, $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, and $\dim(\text{Null}A) = 0$.	◯	◯	
e) The Gram Schmidt algorithm applied to exactly N vectors in $\mathbb{R}^N$ produces an orthogonal basis with dimension $N - 1$.	◯	◯	
g) Matrix A is 3×2, has singular values σ_1 and σ_2, and $\sigma_1 = \sigma_2$.	◯	◯	

2. Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent and span subspace S.

   (a) Fill in the circles next to sets of vectors that span S. Leave the other circles empty.

   ◯ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, 2\vec{v}_3, \vec{0}\}$

   ◯ $\{\vec{v}_1, \vec{v}_1 - \vec{v}_2, \vec{v}_3\}$

   (b) Fill in the circles next to sets of vectors that form a basis for S. Leave the other circles empty.

   ◯ $\{\vec{v}_1, \vec{v}_2, 3\vec{v}_3, \vec{0}\}$

   ◯ $\{\vec{v}_1, \vec{v}_1 - \vec{v}_2, \vec{v}_3\}$

3. An example of an $N \times N$ matrix whose nullspace is equal to its column space is:

4. Fill in the blanks.

   (a) If $M > N$ and for all $\vec{b} \in \mathbb{R}^N$ there is at most one $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = \vec{b}$, and $A \in \mathbb{R}^{M\times N}$, how many pivot columns does A have? $\boxed{}$

   (b) Consider the row operations that reduce A to I_3.

$$A = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}}_{E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2 E_1 A} = I_3$$

   By inspection, $E_1 = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$ and $E_2 = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$.

   (c) For what values of $k \in \mathbb{R}$ (if any) is $A = \begin{pmatrix} 3 & 0 \\ k & 3 \end{pmatrix}$ diagonalizable? $\boxed{}$

   (d) An eigenvector of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is $\vec{v}_1 = \begin{pmatrix} 0 \\ -2 \\ 2 \\ 0 \end{pmatrix}$. What is the eigenvalue associated with eigenvector $\vec{v}_1$? $\boxed{}$

5. If possible, give an example of the following, or write *not possible*.

   (a) A matrix A in RREF such that $T_A : \mathbb{R}^4 \to \mathbb{R}^3$. The vectors $u = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ are a basis for the range of T. $A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$.

   (b) A non-zero matrix, A, that is in RREF, and satisfies $\dim\left((\mathrm{Row}(A))^\perp\right) = 1$ and $\dim\left((\mathrm{Col}(A))^\perp\right) = 1$. $A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$.