Akhmediev Breather and Peregrine Soliton

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1 Introduction

The existence of rogue waves (RW) has been predicted and observed in a variety of physical settings like oceanography, hydrodynamics, plasma physics and nonlinear optics. They capture a lot of interest in oceanographic field because they are responsible for many maritime disasters. RWs are unpredictable with a peak amplitude pretty higher than the average level of the background, they appear seemingly out of nowhere from a 'calm' initial condition and they disappear in a while without a trace.

A mathematical description of RW can be provided through Peregrine soliton (PS) that is a localized wave in space and time, it was firstly derived as a rational solution of the NLSE in the context of plane wave modulation instability by Howell Peregrine in 1983. PS is the limit case of others more general classes of solutions, one of these is the Akhmediev breather (AB) a one parameter family solutions localized in space and periodic in time. The AB solution was discovered by Nail Akhmediev in 1980s. AB and PS were experimentally observed in optics just in 2011.

PS and AB are solutions of the focusing non linear Schrödinger equation 1+1D (NLSE), thus the case of negative group velocity dispersion coefficient $\beta_2 < 0$. The NLS is exactly solvable by the Inverse Scattering Technique (IST). Solitons in general, AB and PS included, arise from non linear and dispersive interaction and they are central objects of nonlinear science.

In this experience i want to investigate the following basic aspects: space-time and space-frequency behaviour of the AB and PS; observations of space and time width characteristics of AB; error between the simulation and the analytical expression; PS as limit of AB for $a \to \frac{1}{2}$; The code and all the results are available on my Google Drive.

2 Non linear Schrödinger equation

The universal model to describe propagation of the slowly varying envelope $F(z,T)$ in a medium with non-linear cubic response and dispersion effects is the NLSE 1+1D.

$$i\frac{\partial F}{\partial z} - \frac{\beta_2}{2}\frac{\partial^2 F}{\partial T^2} + \gamma |F|^2 F = 0 \tag{1}$$

Eq1 is governed by two parameters: β_2 $\mathrm{s^2\,m^{-1}}$ the group velocity dispersion coefficient and γ $\mathrm{W\,m^{-1}}$ related to $\hat{\chi}^3$ $\mathrm{m^2\,V^{-2}}$ that accounts for non linear cubic response; z is the space coordinate and T is a moving time reference. Associated to these two quantities there are two lengths: the dispersion length $L_D = \frac{T_0^2}{|\beta_2|}$ m is the length up to which the effects or group velocity dispersion are negligible and T_0 is a time duration, while the nonlinear length $L_{NL} = \frac{1}{\gamma P_0}$ m represents the length up to which the effects of non-linearity are negligible, P_0 W represents the peak power of the pulse.

By setting $\xi = \frac{z}{L_D}$, $\tau = \frac{T}{T0}$, $N^2 = \frac{L_D}{L_{NL}}$ and assuming that $\beta_2 = -1$, $\gamma = 1$ Eq1 can be rewritten in a form that is a starting point in the study of AB and PS.

$$i\frac{\partial F}{\partial \xi} + \frac{1}{2}\frac{\partial^2 F}{\partial \tau^2} + |F|^2 F = 0 \tag{2}$$

3 Akhmediev breather

AB solutions are a one parameter family of localized in space and periodic in time solutions of (2).

$$F(\xi, \tau) = \left[\frac{(1-4a)\cosh(b\xi) + \sqrt{(2a)}\cos(\Omega\tau) + ib\sinh(b\xi)}{\sqrt{(2a)}\cos(\Omega\tau) - \cosh(b\xi)} \right] e^{i\xi} \tag{3}$$

Ω is the dimensionless modulation frequency, linked to time evolution, it determines the time period of the breather.
$a = \frac{1}{2}\left(1 - \frac{\Omega^2}{4}\right)$, $a\left[0, \frac{1}{2}\right)$ determines the frequency that experience gain,
$b = \sqrt{8a(1-2a)}$ determines the instability growth, linked to space evolution.

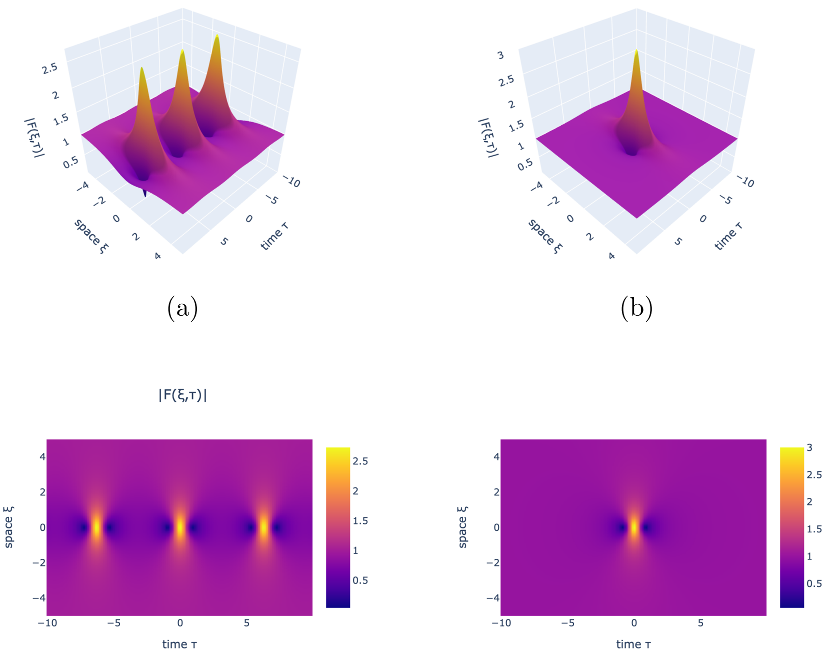
4 Peregrine soliton

PS is a rational, a part from a simple exponential factor, solution of focusing NLSE, it is localized in time and in space and it is the limit case of AB when $a \to \frac{1}{2}$ thus it is an AB when the period tends to infinity.

$$F(\xi, \tau) = \left[1 - \frac{4(2i\xi)}{1 + 4\tau^2 + 4\xi^2} \right] e^{i\xi} \tag{4}$$

5 Space-time

In Fig1 are shown the space-time behaviour of $F(\xi, \tau)$ of AB Fig1a, 1c and PS, Fig1b, Fig1d. It is clear the space localization property of both, they appears around $\xi = 0$. AB shows periodicity in time with period related to its parameter in this way $T_{AB} = \frac{2\pi}{\Omega}$, while PS is localized also in time around $\tau = 0$. The main characteristic is that starting from a pretty uniform background $F(\xi = \xi_{start}, \tau)$ one or more peaks arise, they reach higher value than the background and then they disappear coming back to the initial 'calm' situation.



(a) (b)

(c) (d)

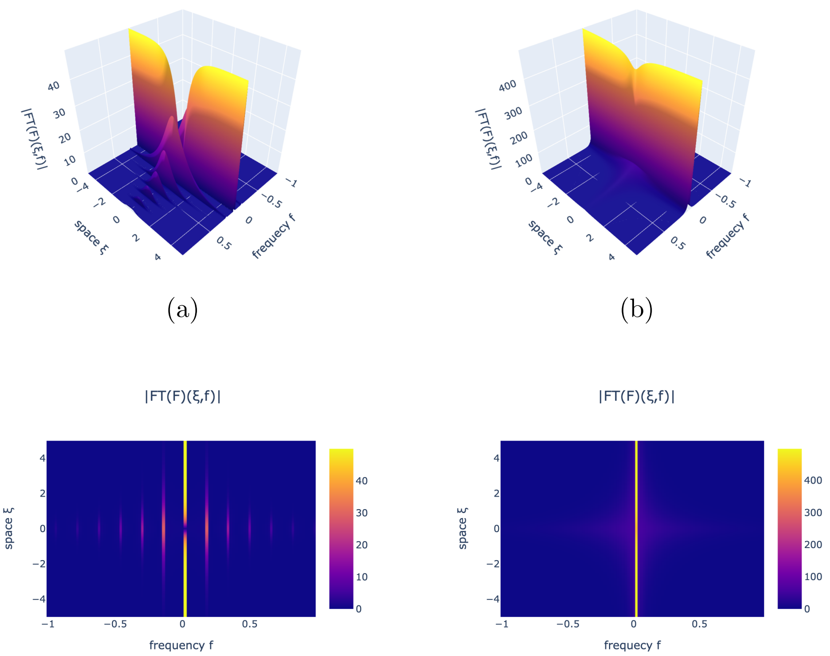
Figure 1: Space-time figures. 3D plots of AB (a) and PS (b). Color-map plots of AB (c) and PS (d). AB is obtained for $\Omega = 1$.

By looking one period of AB for all spaces and PS evolution, they seem very similar. Starting from a uniform background level $F(\xi = \xi_{start}, \tau)$ there is an initial growth with the formation of a pulse. In a first stage the pulse undergoes a compression and increases its peak value till a maximum at $\xi = 0$ then the pulse stars to broaden and decrease the peak till the shape disappear again in the background with no trace. In the case of PS the peak reach 3 times the level of the background in amplitude, 9 times in intensity, it is higher than all possible values taken by AB as will be shown later.

6 Space-frequency

Spectra of $F(\xi,\tau)$ is $\hat{F}(\xi,f)=\int_{-\infty}^{+\infty}F(\xi,\tau)e^{-i2\pi f\tau}d\tau$ it is obtained by applying Fourier transform operator with respect to time.

In Fig2 are shown the space-frequency behaviour of AB Fig2a, 2c and PS, Fig2b, Fig2d.



(a) (b)

(c) (d)

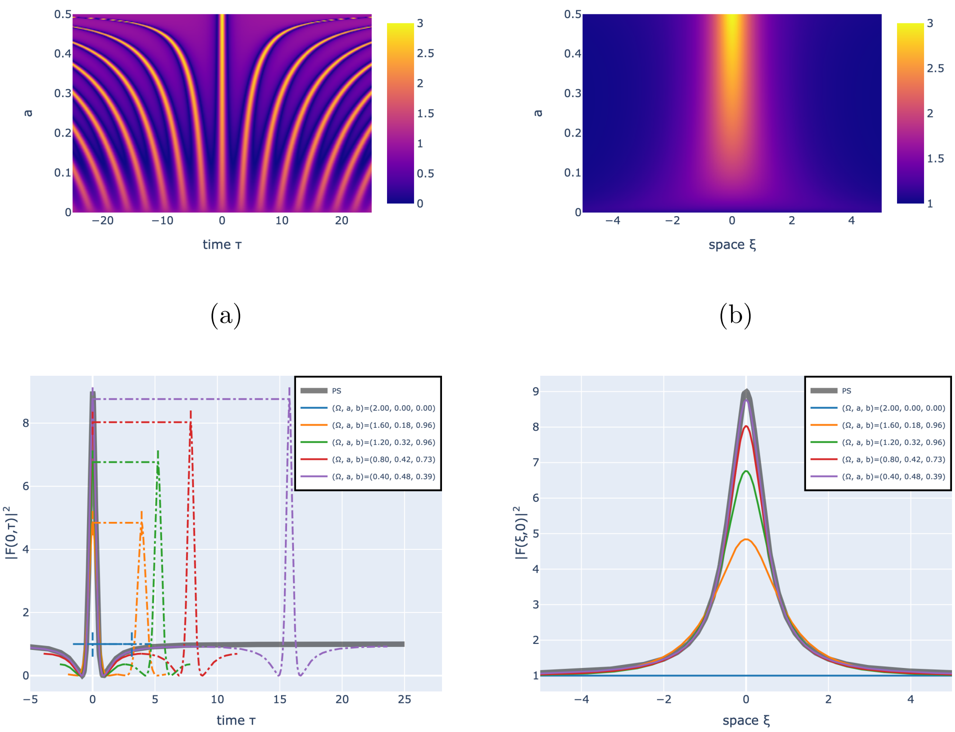
Figure 2: Space-frequency figures. 3D plots of AB (a) and PS (b). Color-map plots of AB (c) and PS (d). AB is obtained for $\Omega=1$.

For AB till a certain space position the spectra shows only the zero frequency component with non zero value and this indicates the presence of a uniform background. From that position it starts to have equally spaced peaks that becomes more and more visible in high frequency when ξ approaches 0, the values where these peaks reach the maximum. The presence of equally spaced peaks in the spectra means that something periodic is happening in time and the extension of the peaks to high frequency indicates that the phenomena show some 'fast' variations.
About PS from the spectra the non periodicity can be observed, in this case around $\xi=0$ there is a continuous shape centered around $f=0$.

7 Space and time width properties

Now I want to investigate some characteristics of AB by varying a parameter, in particular I am interested in time periodicity, width at half power in both time and space, the minimum of $F(\xi = 0, \tau)$, its time distance from the maximum and the values of the minimum and the maximum.

In Fig3a and Fig3b are reported two color-map that show how $F(\xi = 0, \tau)$ and $F(\xi, \tau = 0)$ behaves as functions of a. $F(\xi = 0, \tau)$ is almost uniform for $a = 0$, as a starts to increase the function becomes periodic with peaks with higher distance one to each ad higher value as $a \to \frac{1}{2}$ in the limit the period tends to ∞ and the peak intensity to 9. The same for $F(\xi, \tau = 0)$ but it is not periodic it just shows one peak for each values of a.

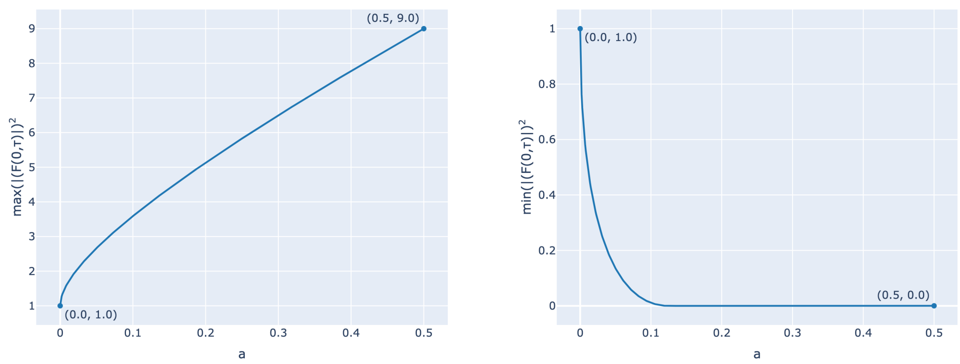


(a) (b) (c) (d)

Figure 3: (a) $F(0, \tau)$ and (a) $F(\xi, 0)$ of AB by varying a from 0 to $\frac{1}{2}$. Some traces of $F(0, \tau)$ (c) and $F(\xi, 0)$ (d) for some values of a.

Fig3c shows the first period, zero centered, in solid line and second period in dashed line of $F(\xi = 0, \tau)$ for some values of a reported in legend. It is highlighted that the time between two consecutive peaks for each function corresponds to $\frac{2\pi}{\Omega}$, the expected period. In Fig3c are reported functions $F(\xi, \tau = 0)$ for for the same values of the parameter a. As a increases the pulses $F(\xi = 0, \tau)$ and $F(\xi, \tau = 0)$ become shorter, for the first can also be observed the presence of min values lower than the background level, they are not present in the second. In these two figures there is a first intuition that the limit for $a \to \frac{1}{2}$ of $F(\xi = 0, \tau)$ and $F(\xi, \tau = 0)$ of AB is the PS, this will be made more clear later.

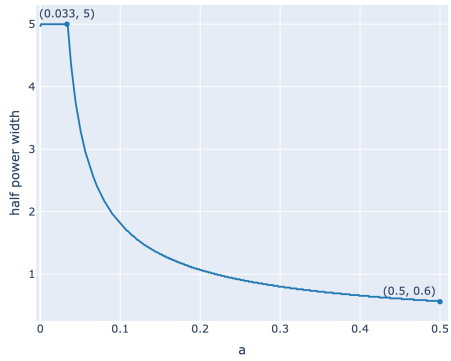
The traces of minimum and maximum values of $|F(\xi = 0, \tau)|^2$ as function of a that goes from 0 to $\frac{1}{2}$ are reported Fig4b and Fig4a. Both start from value 1, the max value increases to 9 and the min value decreases to 0 more fast than the increasing of the maximum.



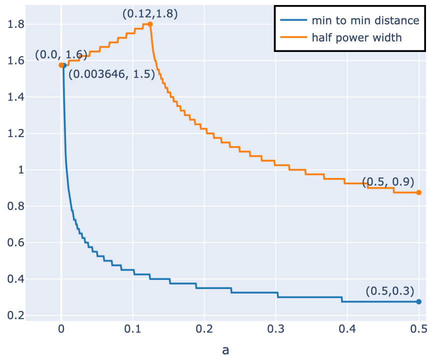
(a) (b)

Figure 4: Plots of $\max |F(0, \tau)|^2$ (a) and $\min |F(0, \tau)|^2$ (b) as functions of parameter a for AB.

I define half power width in space ξ_0 and half power width in time τ_0 as the positive space or time which the intensity is half of the maximum when this condition can be satisfied. I also define the time between the minimum and the maximum τ_{min}. In Fig5a is shown ξ_0. For a less than 0.03 the value coincide with 5 that is the extent of ξ in the code, in this case ξ_0 is undefined because $F(\xi, 0)$ doesn't go below half of its max value; after that the value ξ_0 decreases to 0.6. In Fig5b are reported two traces, the blue corresponds to τ_0 and the red to τ_{min}. The first shows an initial constant part, where τ_0 is not defined, as commented for ξ_0, for a less than 0.003 and then it decreases from value 1 to 0.3. The τ_{min} reported in Fig5b show an initial linear increase till $a = 0.12$ followed by a very fast decrease that becomes slower and it reach the values of 0.9. The 'step effect' in Fig5 is a quantization effect due to the adopted time and space steps, i verified that the average value of the jumps for ξ_0 is around 0.0153 very near to ξ step that is 0.0133 and for τ_0 is 0.022 and for τ_{min} is 0.031 and the time step is 0.0249.



(a)



(b)

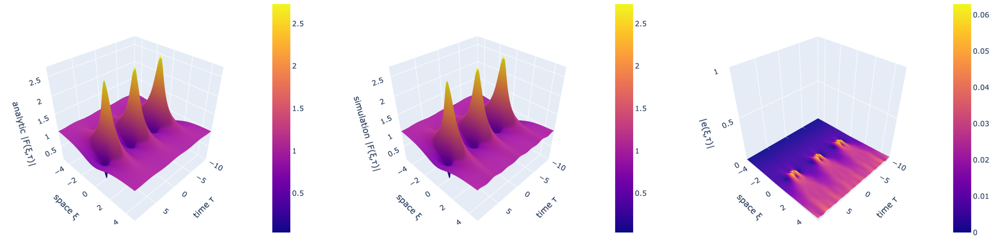
Figure 5: Width of AB by varying a: (a) half power width in space; (a) width in time, two different width one is half power width, blue line, the other width between two consecutive minimum, red line.

8 Analytic solution and numeric simulation

AB and PS analytical solutions of equation Eq2 are given in Eq3, Eq4. These two can be verified by simulations for example by applying split-step Fourier method (SSFM) to solve Eq2. I want to verify first if the simulation confirm the analytical expression, and check the agreement between the two results; second that the PS il the limit for $a \to \frac{1}{2}$ of AB.

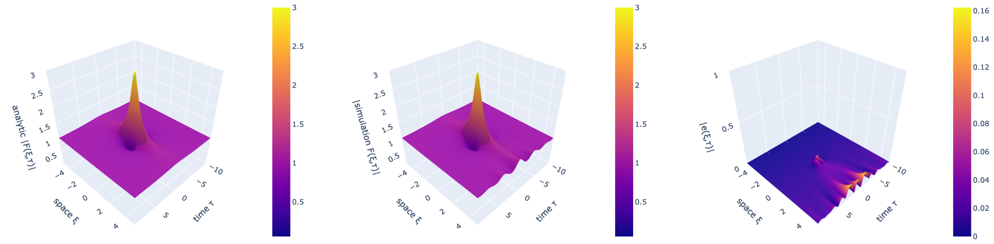
The initial setup for the numeric calculations has space ξ that goes from -5 to 5 with 501 points and time τ that goes from -25 to 25 with 501 points. To avoid unwanted and unrealistic effects that appear near the time borders, the result is cropped for $-10 < \tau < 10$.

To investigate difference between AB analytic solution F_A and simulated F_S results i generated both and plotted in Figa and Figb, from the figures the first impression is that the two surfaces are very similar, by looking at ξ near to 5 in the simulation an oscillation can be seen, seems that as ξ increases also the error increases. This is confirmed by Figc that shows the error $e = |F_{analytic} - F_{simulation}|^2$, for ξ near to zero there is no error and as it increases also the error increases, in any case from the plot can be seen the value of the error is small. Fig7 leads to the same consideration about PS.



(a) (b) (c)

Figure 6: Comparison between space-time results (a) obtained by applying analytic formula and simulation (b) with split-step Fourier method on AB. In (c) the error.



(a) (b) (c)

Figure 7: Comparison between space-time results (a) obtained by applying analytic formula and simulation (b) with split-step Fourier method on PS. In (c) the error.

The described procedure for AB is repeated by varying a from 0 to $\frac{1}{2}$ ten times and collecting the mean value of e. This is also done by varying the number of space and time points in different combinations. In Tab1 are reported the ten different values of a (Ω and b) used to perform the tests and Tab2 reports the obtained mean values of e. The mean value of e is always less than 10^{-3} except for few cases in the row no.1 and row no.8. In Tab3 are reported the results obtained from PS, they coincide with the row no.0 of Tab2 this should mean that for $a \to \frac{1}{2}$ AB tends to PS. A more exact proof of this limiting behaviour can be obtained by evaluating the error between analytic solution of AB for $a \to \frac{1}{2}$ and PS, I reported the results of this test in Tab4, the error is low and seems that it does not depend on the choice of the steps, this is what I expect because I used the analytical expression. Thus simulations seems to converges to analytical expressions and analytical expression of AB goes to PS so i can conclude that at the limit they coincide.

The space step has a strong impact on the error, in SSFM simulations the validity of the result increases as the space step decreases, in fact can be seen that decreasing the space step leads to a noticeable decreasing in the mean value of the error. Also lowering the time step leads to the same kind of behavior but the error doesn't change so much. Both steps should be chosen 'small' enough to have the desired performances.

Table 1: (Ω, a, b) parameters used to test AB error between analytical expression and numeric simulation

	Omega	a	b
0	0.000001	0.500000	0.000001
1	0.222223	0.493827	0.220847
2	0.444445	0.475309	0.433332
3	0.666667	0.444444	0.628540
4	0.888889	0.401234	0.796273
5	1.111112	0.345679	0.923866
6	1.333334	0.277778	0.993808
7	1.555556	0.197531	0.977728
8	1.777778	0.104938	0.814440
9	2.000000	0.000000	0.000000

Table 2: Error between analytical expression and numeric simulation of AB

					$1000 \cdot \lvert e(\xi,\tau)\rvert$				
		$\Delta\xi_1$			$\Delta\xi_2$			$\Delta\xi_3$	
		0.020			0.013			0.010	
	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$
	0.020	0.013	0.010	0.020	0.013	0.010	0.020	0.013	0.010
0	0.45	0.42	0.41	0.16	0.14	0.13	0.09	0.07	0.07
1	1.19	1.16	1.14	0.97	0.95	0.94	0.93	0.91	0.91
2	1.06	1.01	0.99	0.85	0.81	0.79	0.80	0.76	0.75
3	0.81	0.74	0.71	0.47	0.42	0.39	0.37	0.33	0.31
4	0.44	0.41	0.40	0.17	0.15	0.15	0.10	0.09	0.08
5	0.36	0.33	0.32	0.18	0.16	0.15	0.12	0.10	0.09
6	0.50	0.44	0.42	0.28	0.24	0.22	0.20	0.17	0.15
7	0.67	0.56	0.51	0.41	0.33	0.29	0.32	0.25	0.22
8	1.09	0.82	0.70	0.74	0.53	0.43	0.60	0.41	0.33
9	0.13	0.13	0.13	0.06	0.06	0.06	0.03	0.03	0.03

Table 3: Error between analytical expression and numeric simulation of PS

				$1000 \cdot \lvert e(\xi,\tau)\rvert$				
		$\Delta\xi_1$			$\Delta\xi_2$		$\Delta\xi_3$	
	0.020			0.013			0.010	
$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$	$\Delta\tau_1$	$\Delta\tau_2$	$\Delta\tau_3$
0.020	0.013	0.010	0.020	0.013	0.010	0.020	0.013	0.010
0.46	0.42	0.41	0.16	0.14	0.13	0.09	0.07	0.07

Table 4: Error between PS analytical expression and analytical expression of AB for $a \to \frac{1}{2}$.

$1000 \cdot \lvert e(\xi,\tau)\rvert$								
$\Delta\xi_1$ 0.020			$\Delta\xi_2$ 0.013			$\Delta\xi_3$ 0.010		
$\Delta\tau_1$ 0.020	$\Delta\tau_2$ 0.013	$\Delta\tau_3$ 0.010	$\Delta\tau_1$ 0.020	$\Delta\tau_2$ 0.013	$\Delta\tau_3$ 0.010	$\Delta\tau_1$ 0.020	$\Delta\tau_2$ 0.013	$\Delta\tau_3$ 0.010
0.07	0.08	0.08	0.07	0.08	0.08	0.08	0.09	0.08

9 Conclusion

I investigated the basic features of Akhmediev breather and Peregrine soliton solution of focusing NLSE 1+1D. They appear from a 'calm' background and they reach an higher amplitude in absolute value than the initial condition and they disappear with no trace, this make them suitable to describe Rogue Waves events. Localization and periodicity property are shown, AB and PS are localized in space, AB is periodic in time while PS is localized also in space. These solution exhibits a peak that takes the highest intensity value of 9 in the case of PS while the background value is 1. I showed the presence of two point of minimum that goes below the background level near the peak of $F(0,\tau)$. I tried to analyze how the parameter of AB affects it shape and by playing with the parameter a from 0 to $\frac{1}{2}$ i looked at the period, $max|F(\xi,\tau)|^2$, $min|F(\xi,\tau)|^2$, τ_0, ξ_0 and i discovered that the first three increase with a while the last two are not defined for a less than a specific value, different for each one, and then they decrease while a increases.
The comparison between numerical simulation results obtained by split-step Fourier method with the analytical expressions of AK and PS shows that the two agree and the error can be made small by playing mainly on space step but also on time step parameters of the simulation. I also showed that for $a \to \frac{1}{2}$ AB tends to PS.