MATH 487 Deterministic Operations Research

GoldenOrbWeaver

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1 Linear Programming

Definition 1.1. **Linear programming**: The optimization of a linear function subject to linear constraints.

Example. Suppose a starving artist is trying to plan a food budget. He is health conscious and wants a healthy diet that includes the following: at least 70 g of protein per day, at least 100 g of carbohydrates per day, exactly 15 mg of vitamin D per day, but no more than 75 g of fat per day.

Five foods to choose from (fix formatting later):

Food	Protein	Carbohydrates	Vitamin D	Fat	Cost
Hamburger	10g/oz	2g/oz	.5mg/oz	8g/oz	\$0.20/oz
Milk	2g/oz	3 g/oz	4mg/oz	2g/oz	\$0.02/oz
Cereal	3g/oz	23g/oz	2mg/oz	1g/oz	\$0.10/oz
Ch. N S	2g/oz	2g/oz	0 mg/oz	0.5g/oz	\$0.03/oz
Eggs	6g/egg	4g/egg	1mg/egg	5g/egg	\$0.10/egg

Question: How can he meet dietary goals while minimizing cost?

Answer. Set up **decision variables**:
H, M, C, CNS, and E are oz (or number) per day

Constraints:
Protein: $p = 10H + 2M + 3C + 2CNS + 6E \geq 70$
Carbs: $c = 2H + 3M + 23C + 2CNS + 4E \geq 100$
Vitamin D: $0.5H + 4M + 2C + E = 15$
Fat: $f = 8H + 2M + 1C + 0.5CNS + 5E \leq 75$
Nonnegativity: $H, M, C, CNS, E \geq 0$

Subject to these requirements, we wish to minimize cost:

$$cost = 20H + 2M + 10C + 3CNS + 10E$$

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Definition 1.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, then f is called linear $\iff$ f is of the form

$$f(x_1, x_2, \dots x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_0$$

for some constraints

$$a_1, a_2, \dots a_n \text{ and } b_0$$

Definition 1.3. A **linear equation** is an equation of the form $f(x_1, \dots, x_n) = a$ constant.

Definition 1.4. A **linear inequality** is an inequality of the form $f(x_1, \dots, x_n) \leq$ a constant, or $f(x_1, \dots, x_n) \geq$ a constant.

Definition 1.5. A **linear constraint** is either a linear equation or a linear inequality.

Definition 1.6. A **linear program** is the optimization of a linear function subject to linear constraints.

Example. The Furniture Problem

Suppose you are in charge of a furniture factory. Your plant makes tables and chairs out of iron, wood, and labor.

Product	Iron (lbs)	Wood (ft)	Labor (hrs)	Profit ($)
Table	1	20	16	80
Chair	2	15	5	40

Suppose that your plant has access to 100 lbs of iron/day, 1000 lbs of wood/day, and it has 80 employees and thus 640 labor hours/day. What should their production plan be?

Answer. First, we need to decide on the decision variables. These should have two properties:

1. The direction manager must have control over them
2. Designation of optimal values solves the problem

We select two, T and C, the number of tables and chairs produced per day respectively.

Next, we need to select our objective function. Since we wish to maximize profit, our objective function is:

$$profit = \Pi = 80T + 40C$$

We also need to figure out constraints:

$$Iron : T + 2C \leq 100 Wood : 20T + 15C \leq 1000 Labor : 16T + 5C \leq 640 Nonnegativity : T, C \geq 0$$

We have a linear program:

$$max_{T,C} 80T + 40C s.t.$$

⊛

Remark. When there are two decision variables, we can graphically solve a linear program.

ADD DRAWING later

Definition 1.7. The **feasible region** of a linear program is the set of all points that satisfy all constraints.

Geometrically, we wish to find the highest isoprofit that intersects the feasible region, grazing the side of it. This will occur at a vertex (unless a constraint line is parallel to the isoprofit line). We can check all the vertices or we can analyze the slopes of the constraints and find the vertex of constraints with slopes above and below the slope of the isoprofits.

Definition 1.8. An **integer program** is a linear program where all of the decision variables must have integer values.

Example. Blending Model A scrap metal operator reviews an order for 24 lbs of tin, 15 lbs of copper, and 20 lbs of aluminum. She can buy two types of scrap metal which she can melt down:

Type	Tin	Copper	Aluminum	Cost ($0.01/lb)
Metal 1	40%	50%	10%	20
Metal 2	40%	10%	50%	10

Only 50 lbs of Metal 1 are available.

How can she meet the order most effectively?

Answer. Decision variables:
M_1 = Amount of metal 1 to buy (lbs)
M_2 = Amount of metal 2 to buy (lbs)
Linear program:
min $20M_1 + 10M_2 = cost$

s.t. $0.4M_1 + 0.4M_2 \geq 24$ (tin)
$0.5M_1 + 0.1M_2 \geq 15$ (copper)
$0.1M_1 + 0.5M_2 \geq 20$ (aluminum)
$M_1 \leq 50$ (availability)
Since there are only 2 decision variables, we can solve this graphically: ⊛

Example. Transportation Problems

Goods are located at sources and needed to be shipped to destinations. There is a per unit cost to ship from any particular source to an particular destination. The objective is to minimize the cost...

Suppose that the Frank Perdue Chicken Co. has 2000 tons of chickens on inventory, 500 of which are on a farm near San Francisco, 500 on a farm near Houston, and 1000 on a farm near Detroit. They wish to ship the chicken to four superstores located in New York, Los Angeles, Kansas City, and Miami. Demand is NYC 300 tons, LA 900 tons, KC 600 tons, and Mia 200 tons. The shipping costs per ton are:

From/To	NY	LA	KC	Mia
SF	80	10	65	80
Hou	30	50	20	20
Det	30	100	50	50

Define $x_{i,j}$ = the tonnage of chicken shipped from i to j
Linear program:

$$min_x 80x_{11} + 10x_{12} + 65x_{13} + 80x_{14} + 30x_{21} + 50x_{12} + 20x_{23} + 20x_{24} + 30x_{31} + 100x_{32} + 50x_{33} + 50x_{34}$$

s.t. $x_{11} + x_{12} + x_{13} + x_{14} \leq 500$
$x_{21} + x_{22} + x_{23} + x_{24} \leq 500$
$x_{31} + x_{32} + x_{33} + x_{34} \leq 1000$
$x_{11} + x_{21} + x_{31} \geq 300$
$x_{12+x_{22}+x_{32}\geq 900}$
$x_{13} + x_{23} + x_{33} \geq 600$
$x_{14} + x_{24} + x_{34} \geq 200$
$x_{i,j} \geq 0$

Remark. Note the specific special structure of the constraint matrix. This allows for specialized algorithms to solve transportation problems.

Remark. In this particular problem, the sum of the supplies at sources equals the sum of the demands at destinations. This implies that for any feasible solution, all the constraints hold with equality. In general transportation problems, the total supply at sources is greater than or equal to total demand at sinks.

Remark. In general, the transportation problem has the form:
$max_x \sum_{i=1}^{I} \sum_{j=1}^{J} C_{ij} X_{ij}$
s.t. $\sum_{i=1}^{I} X_{ij} \leq S_i$ for $i = 1, \dots, I$
$\sum_{j=1}^{J} X_{ij} \geq D_j$ for $j = 1, \dots, J$
$X_{ij} \geq 0$ for all i, j
where C_{ij} = cost per unit shipped from source i to destination j
I = number of sources
J = number of destinations
S_i = on hand at source i
d_j = demand at destination j

Example. Daisy Drugs manufactures two drugs, Drug 1 and Drug 2. The drugs are produced by blending together two chemicals: Chemical ! and Chemical 2. By weight, Drug 1 must contain at least 65% Chemical 1, and Drug 2 must contain at least 55 % Chemical 1. Both drugs, by weight, are completely composed of Chemical 1 and Chemical 2. Drug 1 sells for $6/oz and Drug 2 sells for $4/oz. Chemicals 1 and 2 are produced by one of two production processes:

Process	Raw Mat./hr	Labor/hr	Chemical 1	Chemical 2
I	3	2	3	3
II	2	3	3	1

A total of 120 hours of skilled labor, 100 oz of raw material are available. Formulate a LP which can be used to maximize Daisy's sale revenue.

> **Remark.** There is a more complicated problem than the furniture problem because we now have to model a multi-step production process. We have 1 (raw materials and labor) -¿ 2 (chemicals) -¿ 3 (drugs).

Decision variables:
D_1 = amount of Drug 1 produced (oz)
D_2 = amount of Drug 2 produced (oz)
C_1 = amount of Chemical 1 produced (oz)
C_2 = amount of Chemical 2 produced (oz)
P_1 = number of hours used with Process I
P_2 = number of hours used with Process II
C_1D_1 = amount of Chemical 1 used in Drug 1 (oz)
C_2D_1 = amount of Chemical 2 used in Drug 1 (oz)
C_1D_2 = amount of Chemical 1 used in Drug 2 (oz)
C_2D_2 = amount of Chemical 2 used in Drug 2 (oz)

Linear program:
max $6D_1 + 4D_2$
s.t. $\frac{C_1D_1}{C_1D_1+C_2D_1} \geq 0.65$
=> $C_1D_1 \geq 0.65(C_1D_1 + C_2D_1)$
=> $0.35C_1D_1 - 0.65C_2D_1 \geq 0$
$0.45C_1D_2 - 0.55C_2D_2 \geq 0$ (Since we need constraints to be linear)
$C_1 - C_1D_1 - C_1D_2 \geq 0$
$C_2 - C_2D_1 - C_2D_2 \geq 0$
$D_1 = C_1D_1 + C_2D_2$
$D_2 = C_1D_2 + C_2D_2$

The constraints defining C_1, C_1, D_1, and D_2 are sometimes called linking constraints. We still need constraints linking 1 to 2:
$C_1 - 3P_1 - 3P_2 = 0$
$C_2 - 3P_1 - P_2 = 0$

We also have resource constraints:
$3P_1 + 2P_2 \leq 100$
$2P_1 + 3P_2 \leq 120$

And of course nonnegativity:
$D_1, D_2, C_1, C_2, C_1D_1, C_1D_2, C_2D_1, C_2D_2, P_1, P_2 \geq 0$

Definition 1.9. **The Simplex Method** is a way of solving an LP with more than 2 decision variables...

Preliminaries:

1. 3 Classes of LPs
   (a) LPs with finite optima
   (b) Infeasible LPs, where $F.R. = \emptyset$
   (c) Unbounded LPs, where either:
      i. min problems where we can send $z^* \to -\infty$
      ii. max problems where we can send $z^* \to \infty$

   We desire that an algorithm tells us which of these classes our LP is in and if an optimum exists, what it is.

Types of linear programs:

1. LPs in general form
   (a) max or min problem
   (b) constraints are $\leq$, $\geq$, or $=$ (never ¡ or ¿)
   (c) RHS of constraints can be negative, positive or O
   (d) The variables can be nonnegative, nonpositive, or unrestricted
2. LPs in standard form. Similar to LPs in general form except:
   (a) Constraints are all $=$
   (b) RHS of constraints are all positive or 0
   (c) Variables are all nonnegative

Proposition 1.1. Any LP in general form can be rewritten as an equivalent LP in standard form.

Proof. We demonstrate a procedure on an example, which can be used on any example. The steps can be used in any order.

Put the LP below into an equivalent LP in standard form:

$max\ 6x_1 + x_2 - x_3 - x_4 = z$
s.t. $x_1 - x_2 \leq 5$
$x_1 + 2x_2 + x_3 - x_4 \geq 2$
$2x_2 + x_3 = -4$
$x_1, x_4 \geq 0$
$x_2 \leq 0$
x_3 unrestricted

Step 1: Change "max" problem to "min" problem, if necessary, by multiplying the objective function by -1:

$min - 6x_1 - x_2 + x_3 + x_4 = -z$ s.t. $x_1 - x_2 \leq 5$
$x_1 + 2x_2 + x_3 - x_4 \geq 2$
$2x_2 + x_3 = -4$
$x_1, x_4 \geq 0$
$x_2 \leq 0$
x_3 unrestricted

Step 2: Multiply any constraints with negative RHS by -1. (This changes $\leq$ constraints to $\geq$ and vice versa):

$min - 6x_1 - x_2 + x_3 + x_4 = -z$ s.t. $x_1 - x_2 \leq 5$
$x_1 + 2x_2 + 2x_3 - x_4 \geq 2$
$-2x_2 - x_3 = 4$
$x_1, x_4 \geq 0$

$x_2 \leq 0$
x_3 unrestricted

Step 3: Add slack variables so that all constraints are =:
$min - 6x_1 - x_2 + x_3 + x_4 = -z$ s.t. $x_1 - x_2 + S_1 = 5$
$x_1 + 2x_2 + 2x_3 - x_4 - S_2 = 2$
$-2x_2 - x_3 = 4$
$x_1, x_4 \geq 0$
$x_2 \leq 0$
x_3 unrestricted
$S_1, S_2 \geq 0$

Step 4: Replace any nonpositive variable x_i with a new variable x_i^*, where $x_i' = -x_i$:
$min - 6x_1 + x_2' + x_3 + x_4 = -z$ s.t. $x_1 + x_2' + S_1 = 5$
$x_1 - 2x_2' + 2x_3 - x_4 - S_2 = 2$
$2x_2' - x_3 = 4$
$x_1, x_2', x_4, S_1, S_2 \geq 0$
x_3 unrestricted

Step 5: Replace any unrestricted variables x_i with the difference of two nonnegative variables $x_i' - x_i''$:
$min - 6x_1 + x_2' + x_3' - x_3'' + x_4 = -z$ s.t. $x_1 + x_2' + S_1 = 5$
$x_1 - 2x_2' + 2x_3' - 2x_3'' - x_4 - S_2 = 2$
$2x_2' - x_3' + x_3'' = 4$
$x_1, x_2', x_3', x_3'' x_4, S_1, S_2 \geq 0$

This is now in standard form

□

Note. Suppose we solved the linear program in standard form that we just found using the simplex method and get an optimal answer of:
$x_1 = 0,\ x_2' = 5,\ x_3' = 6,\ x_3'' = 0,\ x_4 = 0,\ S_1 = 0,\ S_2 = 0,\ -z = 11$
Then we can translate back and find that the solution of the linear program in general form is:
$x_1 = 0,\ x_2 = -5,\ x_3 = 6,\ x_4 = 0,\ z = -11$

Definition 1.10. An $m \times n$ linear program is in **canonical form** if it is in standard form and there is a distinguished set of m variables called **basic variables** for which:

1. Each basic variable has coefficient 1 in one constraint and 0 in the others.
2. Each basic variable has coefficient 0 in the objective function.
3. Each constraint has exactly 1 basic variable with coefficient 1.

Example. Recall the furniture problem:
max $80T + 40C = z$
s.t. $T + 2C \leq 100$
$20T + 15C \leq 1000$
$16T + 5C \leq 640$
$T, C \geq 0$

We start by converting this to a linear program in standard form:
min $-80T - 40C = -z$
s.t $T + 2C + S_1 = 100$
$20T + 15C + S_2 = 1000$

$16T + 5C + S_3 = 640$
$T, C, S_1, S_2 \geq 0$

This happens to also be in canonical form with the set of basic variables or basis=$\{S_1, S_2, S_3\}$

Remark. We were lucky that our standard form turned out to be a canonical form. In most examples this won't be the case.

Remark. If we have a canonical form LP, it is easy to see what is called an associated **basic feasible solution**. In this case, $T = C = 0$ (nonbasic variables are 0), $S_1 = 100$, $S_2 = 1000$, and $S_3 = 640$ (basic variables are RHS).

We now have a b.f.s. for the LP, but this isn't optimal. We can raise C while reducing S_1, S_2, S_3 to maintain constraints, but we can't do this indefinitely, only up to $\frac{100}{2}$ for the iron constraint, $\frac{1000}{15}$ for the wood constraint, and $\frac{640}{5}$ for the labor constraint. Therefore, we should make C basic and S_1 nonbasic.

Theorem 1.1. Minimum-Ratio Test
In general, if we've decided that x_{j^*} should enter the basis, then the current basic variable of the i^*th row should leave the basis where i^* minimizes $\frac{b_i}{a_{ij^*}}$ over all i s.t. $a_{ij^*} > 0$.

To make C basic and S_1 nonbasic while retaining canonical form:

1. Multiply equation (1) by 1/2, obtaining (1').
2. Subtract $15 * (1')$ from (2), obtaining (2')
   Subtract $15 * (1')$ from (3), obtaining (3')
   Subtract $-40 * (1')$ from (obj), obtaining (obj')

Doing the first step:

$$\frac{1}{2}T + C + \frac{1}{2}S_1 = 50$$

Doing the second step:

$$20T + 15C + S_2 = 1000 \tag{1}$$
$$-15(\frac{1}{2}T + C + \frac{1}{2}S_1 = 50 \tag{2}$$
$$\frac{25}{2}T - \frac{15}{2}S_1 + S_2 = 250 \tag{3}$$

$$16T + 5C + S_3 = 640 \tag{4}$$
$$-5(\frac{1}{2}T + C + \frac{1}{2}S_1 = 50) \tag{5}$$
$$\frac{27}{2}T - \frac{5}{2}S_1 + S_3 = 390 \tag{6}$$

$$-80T - 40C = -Z \tag{7}$$
$$+40(\frac{1}{2}T + C + \frac{1}{2}S_1 = 50) \tag{8}$$
$$-60T + 20S_1 = -Z + 2000 \tag{9}$$

Our new LP is:
max $-60T + 20S_1 = -Z + 2000$
s.t. $\frac{1}{2}T + C + \frac{1}{2}S_1 = 50$
$\frac{25}{2}T - \frac{15}{2}S_1 + S_2 = 250$
$\frac{27}{2}T - \frac{5}{2}S_1 + S_3 = 390$

$T, C, S_1, S_2, S_3 \geq 0$

We have a new canonical form with basis C, S_2, S_3. The associated b.f.s is $(T, C, S_1, S_2, S_3) = (0, 50, 0, 250, 390)$.

Note that we can read the associated Z value from the canonical form (here $Z = 2000$.)

Remark. In general, if we wish to enter x_{j*} into the basis and remove the basic variable from the i^*th row from the basis, then the canonical form tableau changes as follows:

Note. No prime means before pivot and prime means after pivot.

1. $a'_{i^*j^*} = 1$
2. $a'_{i^*j} = \frac{a_{i^*j}}{a_{i^*j^*}}$ for $j = j^*$
3. $a'_{ij^*} = 0$ for $i \neq i^*$
4. $a'_{ij} = a_{ij} - \frac{a_{ij^*} * a_{i^*j}}{a_{i^*j^*}}$ for $i \neq i^*$ and $j \not{j}^*$
5. $b'_{i^*} = \frac{b_{i^*}}{a_{i^*j^*}}$
6. $b'_i = b_i - \frac{b_{i^*}a_{ij^*}}{a_{i^*j^*}}$ for $i \neq i^*$
7. $C'_{j^*} = 0$
8. $C'_j = C_j - \frac{C_{j^*}a_{i^*j}}{a_{i^*j^*}}$ for $j \neq j^*$
9. $(\pm Z) + Z'_0 = (\pm Z) + Z_0 - \frac{b_{i^*}C_{j^*}}{a_{i^*j^*}}$

The operation which generates a new canonical form from an old canonical form is called a pivot on $a_{i^*j^*}$

An easy way to do this is to use the **box method**. First, write the original LP in **tableau form**:
ADD A DIAGRAM FOR THIS