Notes on quantum transport in mesoscopic systems

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1 Basics

1.1 Magnon

A magnon is a quasiparticle, a collective excitation of the electrons' spin structure in a crystal lattice. In the equivalent wave picture of quantum mechanics, a magnon can be viewed as a quantized spin wave. Magnons carry a fixed amount of energy and lattice momentum, and are spin-1, indicating they obey boson behavior.

2 Formulas in PRB. 88, 220406(R) (2013)

2.1 Formula 1

System Hamiltonion:

$$H = H_L + H_R + H_{sd}. \tag{1}$$

Left lead is metallic

$$H_L = \sum_{k\sigma} \left(\varepsilon_{k\sigma} - \mu_\sigma\right) c_{k\sigma}^\dagger c_{k\sigma}, \tag{2}$$

right lead is insulating magnetic

$$H_R \approx \sum_q \hbar w_q a_q^\dagger a_q + \text{ constant } . \tag{3}$$

The interfacial electron-magnon interaction is described by

$$H_{sd} = -\sum_{k,q} J_q \left[S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \tag{4}$$

where $S_q^- \approx \sqrt{2S_0} a_q^\dagger, S_q^+ \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling at the interface. The magnonic spin current operator can be obtained by

$$\hat{I}_S = \frac{d\hat{N}_R}{dt} = \frac{d}{dt} \sum_q a_q^\dagger a_q, \tag{5}$$

the magnonic spin current is obtained by taking average over the nonequilibrium ground state $|\psi_0\rangle$ of the interacting system H:

$$I_S = \frac{dN_R}{dt} = \frac{d}{dt}\langle\sum_q a_q^\dagger a_q\rangle. \tag{6}$$

Using the Heisenberg equation, we get

$$I_S = \frac{i}{\hbar}\langle[H_{sd}, \sum_q a_q^\dagger a_q]\rangle. \tag{7}$$

$$[H_{sd}, \sum_q a_q^\dagger a_q] = [-\sum_{k,q} J_q \left(S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right), \sum_q a_q^\dagger a_q], \tag{8}$$

in which,

$$[a_q^\dagger, \sum_{q'} a_{q'}^\dagger, a_{q'}] = \delta_{qq'}[a_q^\dagger, a_{q'}^\dagger a_{q'}] = [a_q^\dagger, a_q^\dagger a_q] = a_q^\dagger[a_q^\dagger, a_q] = -a_q^\dagger. \tag{9}$$

Similarly,

$$[a_q, \sum_{q'} a_{q'}^+ a_{q'}] = [a_q, a_q^\dagger a_q] = a_q. \tag{10}$$

So,

$$\begin{aligned} I_S &= \frac{i}{\hbar}\langle -\sum_{kq} J_q \left(-S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right)\rangle \\ &= \frac{i}{\hbar}\sum_{kq} J_q \left(\langle S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\rangle - \langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\rangle\right). \end{aligned} \tag{11}$$

2.2 Formula 2

$$\frac{d}{dt}\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle = \frac{i}{\hbar}\left\langle\left[H_L + H_{sd} + H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right]\right\rangle. \tag{12}$$

The rhs. of eq. (12) is decomposed into 3 terms. The first term reads

$$\begin{aligned} \left\langle\left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right]\right\rangle &= [\sum_{k'\sigma}(\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}] \\ &= S_q^+[\sum_{k'\sigma}(\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] c_{k\uparrow} \\ &\quad + S_q^+ c_{k+q\downarrow}^\dagger[\sum_{k'\sigma}(\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}] \end{aligned} \tag{13}$$

Note that,

$$[\sum_{k'\sigma} c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] = \sum_{k'\sigma} c_{k'\sigma}^\dagger \delta_{k',k+q}\delta_{\sigma\downarrow} = c_{k+q\downarrow}^\dagger, \tag{14}$$

$$[\sum_{k'\sigma} c^\dagger_{k'\sigma} c_{k'\sigma}, c_{k\uparrow}] = -\sum_{k'\sigma}\{c^\dagger_{k'\sigma}, c_{k\uparrow}\} c_{k'\sigma} = -c_{k\uparrow}. \tag{15}$$

Eq. (14) (15) are derived using equity

$$\begin{aligned} [AB, C] &= A[B, C] + [A, C]B \\ &= A\{B, C\} - \{A, C\}B. \end{aligned} \tag{16}$$

So,

$$\left[H_L, S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} + \mu_\uparrow - \mu_\downarrow\right) S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}. \tag{17}$$

If $\mu_\uparrow = \mu_\downarrow$, then eq. (17) reduces to

$$\left[H_L, S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow}\right) S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}. \tag{18}$$

The second term $H_R = \sum_q \hbar w_q a^\dagger_q a_q$, then using eq. (10), we get

$$\left[H_R, S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right] = -\hbar\omega_q S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}. \tag{19}$$

The third term in eq. (12) reads

$$\begin{aligned} \left[H_{sd}, S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right] &= -J_q \left[S^-_q c^\dagger_{k\uparrow} c_{k+q\downarrow}, S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right] \\ &= J_q \left[S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}, S^-_q c^\dagger_{k\uparrow} c_{k+q\downarrow}\right] \end{aligned} \tag{20}$$

Combine these three terms, we get

$$\begin{aligned} \frac{d}{dt}\left\langle S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right\rangle &= \frac{i}{\hbar}\left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} - \hbar\omega_q\right)\left\langle S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}\right\rangle \\ &+ \frac{i}{\hbar} J_q \left\langle\left[S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow}, S^-_q c^\dagger_{k\uparrow} c_{k+q\downarrow}\right]\right\rangle, \end{aligned} \tag{21}$$

which is also eq. (2) in PRB. 88, 220406(R) (2013).

3 Spin current in NM-QD-MIL system

For system consists of quantum dot(QD) sandwiched by a left normal metal(NM) lead and a right magnetic insulating lead(MIL), the Hamiltonian is

$$H = H_\mathrm{L} + H_\mathrm{QD} + H_\mathrm{R} + H_\mathrm{T}. \tag{22}$$

$$H_\mathrm{L} = \sum_{k\sigma} \varepsilon_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma}, \tag{23}$$

$$H_\mathrm{QD} = \sum_\sigma \varepsilon_\sigma d^\dagger_\sigma d_\sigma, \tag{24}$$

$$H_{\rm R} \approx \sum_q \hbar w_q a_q^\dagger a_q, \tag{25}$$

$$H_{\rm T} = V_{\rm L} + V_{\rm R} \tag{26}$$

V_L is the coupling between left lead and QD, while V_R is the coupling between right lead and QD.

$$V_{\rm L} = \sum_{k\sigma} (t_{k\sigma} c_{k\sigma}^\dagger d_\sigma + t_{k\sigma}^* d_\sigma^\dagger c_{k\sigma}) \tag{27}$$

$$V_{\rm R} = -\sum_q J_q \left[S_q^- d_\uparrow^\dagger d_\downarrow + S_q^+ d_\downarrow^\dagger d_\uparrow \right]. \tag{28}$$

where $S_q^- \approx \sqrt{2S_0} a_q^\dagger, S_q^+ \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling between the QD and MIL.

3.1 Spin-dependent current in left lead

The spin-dependent current flow out of left lead is $I_{L\sigma}$:

$$I_{L\sigma} = \frac{d}{dt} \langle N_{L\sigma} \rangle \tag{29}$$

in which, $N_{L\sigma} = \sum_k c_{k\sigma}^\dagger c_{k\sigma}$ Heisenberg equation:

$$\frac{d}{dt} \langle N_{L\sigma} \rangle = \frac{i}{\hbar} \langle [H, N_{L\sigma}] \rangle \tag{30}$$

$$[H, N_{L\sigma}] = [H_T, N_{L\sigma}] = \sum_k \left(t_{k\sigma}^* d_\sigma^\dagger c_{k\sigma} - t_{k\sigma} c_{k\sigma}^\dagger d_\sigma \right) \tag{31}$$

so, the spin-dependent current

$$I_{L\sigma} = \frac{i}{\hbar} \sum_k \left(t_{k\sigma}^* \langle d_\sigma^\dagger c_{k\sigma} \rangle - t_{k\sigma} \langle c_{k\sigma}^\dagger d_\sigma \rangle \right). \tag{32}$$

Namely, the spin-up current is

$$I_{L\uparrow} = \frac{i}{\hbar} \sum_k \left(t_{k\uparrow}^* \langle d_\uparrow^\dagger c_{k\uparrow} \rangle - t_{k\uparrow} \langle c_{k\uparrow}^\dagger d_\uparrow \rangle \right), \tag{33}$$

the spin-down current is

$$I_{L\downarrow} = \frac{i}{\hbar} \sum_k \left(t_{k\downarrow}^* \langle d_\downarrow^\dagger c_{k\downarrow} \rangle - t_{k\downarrow} \langle c_{k\downarrow}^\dagger d_\downarrow \rangle \right), \tag{34}$$

The charge current in left lead is defined as

$$I_e = e(I_{L\uparrow} + I_{L\downarrow}). \tag{35}$$

The spin current in left lead is defined as

$$I_{LS} = \frac{1}{2}(I_{L\uparrow} - I_{L\downarrow}) \tag{36}$$

Substitute the spin-dependent current in, we get the spin current in left lead

$$I_{LS} = \frac{i}{2\hbar}\sum_k \Big(t^*_{k\uparrow}\langle d^\dagger_\uparrow c_{k\uparrow}\rangle - t_{k\uparrow}\langle c^\dagger_{k\uparrow} d_\uparrow\rangle - t^*_{k\downarrow}\langle d^\dagger_\downarrow c_{k\downarrow}\rangle + t_{k\downarrow}\langle c^\dagger_{k\downarrow} d_\downarrow\rangle\Big) \tag{37}$$

3.2 magnonic current in right lead

The magnonic current in right lead is

$$I_{RS} = \frac{d\langle N_R\rangle}{dt} = \frac{d}{dt}\langle\sum_q a^\dagger_q a_q\rangle, \tag{38}$$

From the Heisenberg equation, we have

$$\frac{d}{dt}\langle\sum_q a^\dagger_q a_q\rangle = \frac{i}{\hbar}\langle[H, \sum_q a^\dagger_q a_q]\rangle. \tag{39}$$

We have

$$\begin{aligned}
&[H, \sum_q a^\dagger_q a_q] \\
=&[V_R, \sum_q a^\dagger_q a_q] \\
=&-\sum_q J_q\Big(\big[S^-_q d^\dagger_\uparrow d_\downarrow, \sum_{q'} a^\dagger_{q'} a_{q'}\big] + \big[S^+_q d^\dagger_\downarrow d_\uparrow, \sum_{q'} a^\dagger_{q'} a_{q'}\big]\Big) \\
=&\sum_q J_q\big(S^-_q d^\dagger_\uparrow d_\downarrow - S^+_q d^\dagger_\downarrow d_\uparrow\big)
\end{aligned} \tag{40}$$

So, the magnon current reads

$$I_{RS} = \frac{i}{\hbar}\sum_q J_q\big(\big\langle S^-_q d^\dagger_\uparrow d_\downarrow\big\rangle - \big\langle S^+_q d^\dagger_\downarrow d_\uparrow\big\rangle\big). \tag{41}$$

3.3 spin-dependent current in QD

Similarly, the spin-dependent current in the central QD is defined as

$$I_{C\sigma} = \langle\frac{dN_{C\sigma}}{dt}\rangle = \langle\frac{d}{dt}d^\dagger_\sigma d_\sigma\rangle, \tag{42}$$

Heisenberg equation:

$$\frac{d}{dt}d_\sigma^\dagger d_\sigma = \frac{i}{\hbar}[H, d_\sigma^\dagger d_\sigma]. \tag{43}$$

Specifically, we have

$$[H, d_\sigma^\dagger d_\sigma] = [V_L, d_\sigma^\dagger d_\sigma] + [V_R, d_\sigma^\dagger d_\sigma] \tag{44}$$

in which,

$$\begin{aligned}
[V_L, d_\sigma^\dagger d_\sigma] &= \sum_{k'\sigma'} \big(t_{k'\sigma'}\big[c_{k'\sigma'}^\dagger d_{\sigma'}, d_\sigma^\dagger d_\sigma\big] + t_{k'\sigma'}^*\big[d_{\sigma'}^\dagger c_{k'\sigma'}, d_\sigma^\dagger d_\sigma\big]\big) \\
&= \sum_{k'\sigma'} \big(t_{k'\sigma'} c_{k'\sigma'}^\dagger d_{\sigma'}\delta_{\sigma\sigma'} - t_{k'\sigma'}^* d_{\sigma'}^\dagger c_{k'\sigma'}\delta_{\sigma\sigma'}\big) \\
&= \sum_{k} \big(t_{k\sigma} c_{k\sigma}^\dagger d_{\sigma} - t_{k\sigma}^* d_{\sigma}^\dagger c_{k\sigma}\big)
\end{aligned} \tag{45}$$

We change the summation index from k' to k in the last line , which doesn't change the result.

$$[V_R, d_\sigma^\dagger d_\sigma] = -\sum_q J_q\big(S_q^-\big[d_\uparrow^\dagger d_\downarrow, d_\sigma^\dagger d_\sigma\big] + S_q^+\big[d_\downarrow^\dagger d_\uparrow, d_\sigma^\dagger d_\sigma\big]\big) \tag{46}$$

in which,

$$\begin{aligned}
\big[d_\uparrow^\dagger d_\downarrow, d_\sigma^\dagger d_\sigma\big] &= d_\uparrow^\dagger\big[d_\downarrow, d_\sigma^\dagger d_\sigma\big] + \big[d_\uparrow^\dagger, d_\sigma^\dagger d_\sigma\big]d_\downarrow \\
&= d_\uparrow^\dagger d_\downarrow\delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow\delta_{\sigma\uparrow}
\end{aligned} \tag{47}$$

So,

$$[V_R, d_\sigma^\dagger d_\sigma] = -\sum_q J_q\big[S_q^-\big(d_\uparrow^\dagger d_\downarrow\delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow\delta_{\sigma\uparrow}\big) + S_q^+\big(d_\downarrow^\dagger d_\uparrow\delta_{\sigma\uparrow} - d_\downarrow^\dagger d_\uparrow\delta_{\sigma\downarrow}\big)\big] \tag{48}$$

The spin-dependent current is

$$I_{C\uparrow} = \frac{i}{\hbar}\big[H, d_\uparrow^\dagger d_\uparrow\big] \tag{49}$$

$$I_{C\downarrow} = \frac{i}{\hbar}\big[H, d_\downarrow^\dagger d_\downarrow\big] \tag{50}$$

which gives

$$I_{C\uparrow} = \frac{i}{\hbar}\big[\sum_k \big(t_{k\uparrow}c_{k\uparrow}^\dagger d_\uparrow - t_{k\uparrow}^* d_\uparrow^\dagger c_{k\uparrow}\big) + \sum_q J_q\big(S_q^- d_\uparrow^\dagger d_\downarrow - S_q^+ d_\downarrow^\dagger d_\uparrow\big)\big] \tag{51}$$

and

$$I_{C\downarrow} = \frac{i}{\hbar}\big[\sum_k \big(t_{k\downarrow}c_{k\downarrow}^\dagger d_\downarrow - t_{k\downarrow}^* d_\downarrow^\dagger c_{k\downarrow}\big) + \sum_q J_q\big(S_q^+ d_\downarrow^\dagger d_\uparrow - S_q^- d_\uparrow^\dagger d_\downarrow\big)\big]. \tag{52}$$

The spin current in central dot is

$$I_{CS} = \frac{1}{2}(I_{C\uparrow} - I_{C\downarrow}) \tag{53}$$

3.4 Verifying continuity condition

3.4.1 Charge current

Since the right lead is a insulating lead, there is no charge current flow through it, so the charge current is

$$I_{Re} = 0 \tag{54}$$

where the subscript R denotes the right lead, while the subscript e denotes the charge current.

Meanwhile, the charge current flows in left lead and QD is

$$I_e = e(\sum_\sigma I_{L\sigma} + \sum_\sigma I_{C\sigma}) = 0 \tag{55}$$

3.4.2 Spin current

Spin current in the left lead and QD:

$$I_{LS} + I_{CS} = \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big). \tag{56}$$

The magnon current is

$$I_{RS} = \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big), \tag{57}$$

in which, I_{LS}, I_{CS}, I_{RS} is defined earlier. Thus, we have

$$\begin{aligned} I_{LS} + I_{CS} &= I_{RS} \\ &= \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big). \end{aligned} \tag{58}$$

4 Nonequilibrium Green's function technique