Notes on quantum transport in mesoscopic systems

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(Dated: 2020/12/1)

CONTENTS

I. BASICS

A. Magnon

A magnon is a quasiparticle, a collective excitation of the electrons' spin structure in a crystal lattice. In the equivalent wave picture of quantum mechanics, a magnon can be viewed as a quantized spin wave. Magnons carry a fixed amount of energy and lattice momentum, and are spin-1, indicating they obey boson behavior.

B. Hall effect

1. *Conventional Hall effect*

2. *Quantum Hall effect*

3. *Spin Hall effect*

Occurs in paramagnetic systems as a result of spin-orbit interaction, refers to generation of pure spin current transverse to an applied electric field, even in the absence of magnetic field.

Similar to charge accumulation at sample edges in conventional Hall effect, spin accumulation is expected in spin Hall effect.

- extrinsic spin Hall effect: originates from asymmetric scattering for spin-up and spin-down.

- intrinsic spin Hall effect: originates from band structures without scattering.[1]

4. *Fractional Hall effect*

5. *Anomalous Hall effect*

II. SPIN CURRENT IN NM/NM/FI SYSTEM

A. Model Hamiltonion

For system consists of non-magnetic metal (NM) region sandwiched by a left NM lead and a right ferromagnetic insulating lead (FI), the Hamiltonian is

$$H = H_{\rm L} + H_{\rm C} + H_{\rm R} + H_{\rm T} + H_{\rm sd}. \tag{1}$$

$$H_{\rm L} = \sum_{k\sigma} (\varepsilon_{k\sigma} - \mu_\sigma) c^\dagger_{k\sigma} c_{k\sigma}, \tag{2}$$

$$H_{\rm C} = \sum_{n\sigma} \varepsilon_\sigma d^\dagger_{n\sigma} d_{n\sigma}, \tag{3}$$

$$H_{\rm R} \approx \sum_{q} \hbar w_q a^\dagger_q a_q, \tag{4}$$

are Hamiltonians of left lead, central region and right lead respectively. $H_{\rm T}$ is the hopping between the left lead and the central region, while $H_{\rm sd}$ is the exchange coupling between right lead and central region,

$$H_{\rm T} = \sum_{nk\sigma} (t_{nk\sigma} c^\dagger_{k\sigma} d_{n\sigma} + t^*_{nk\sigma} d^\dagger_{n\sigma} c_{k\sigma}), \tag{5}$$

$$H_{\rm sd} = -\sum_{qnn'} J_{qnn'} a^\dagger_q d^\dagger_{n\uparrow} d_{n'\downarrow} + \text{ h.c.} \tag{6}$$

Here $S^-_q \approx \sqrt{2S_0} a^\dagger_q, S^+_q \approx \sqrt{2S_0} a_q$ are in the momentum space and $J_{qnn'}$ denotes the effective exchange coupling between the NM central region and FI lead. We can further partition H into H_0, Hamiltonian of the isolated leads and central region, and coupling terms H'. So the unperturbed Hamiltonian is given by

$$H_0 = H_L + H_R + H_d \tag{7}$$

which is quadratic. This is very important since Wick theorem can be only used if H_0 is quadratic. The interacting term is in H'

$$H' = H_T + H_{sd}. \tag{8}$$

B. Verifying (spin) current continuity condition in two-terminal NM/NM/FI system

1. *Spin-dependent current in left lead*

The spin-dependent number operator of the left NM lead is $\hat{N}_{L\sigma} = \sum_k c^\dagger_{k\sigma} c_{k\sigma}$. The spin-dependent current flowing out of left lead is $I_{L\sigma}$:

$$I_{L\sigma} = \frac{d}{dt} \langle \hat{N}_{L\sigma} \rangle. \tag{9}$$

Heisenberg equation:

$$\frac{d}{dt} \langle \hat{N}_{L\sigma} \rangle = \frac{i}{\hbar} \langle [H, \hat{N}_{L\sigma}] \rangle, \tag{10}$$

$$[H, N_{L\sigma}] = [H_T, N_{L\sigma}] = \sum_{nk} \left(t^*_{nk\sigma} d^\dagger_{n\sigma} c_{k\sigma} - t_{nk\sigma} c^\dagger_{k\sigma} d_{n\sigma} \right), \tag{11}$$

so, the spin-dependent current

$$I_{L\sigma} = \frac{i}{\hbar} \sum_{nk} \left(t^*_{nk\sigma} \langle d^\dagger_{n\sigma} c_{k\sigma} \rangle - t_{nk\sigma} \langle c^\dagger_{k\sigma} d_{n\sigma} \rangle \right). \tag{12}$$

Namely, the spin-up current is

$$I_{L\uparrow} = \frac{i}{\hbar} \sum_{nk} \left(t^*_{nk\uparrow} \langle d^\dagger_{n\uparrow} c_{k\uparrow} \rangle - t_{nk\uparrow} \langle c^\dagger_{k\uparrow} d_{n\uparrow} \rangle \right), \tag{13}$$

the spin-down current is

$$I_{L\downarrow} = \frac{i}{\hbar} \sum_{nk} \left(t^*_{nk\downarrow} \langle d^\dagger_{n\downarrow} c_{k\downarrow} \rangle - t_{nk\downarrow} \langle c^\dagger_{k\downarrow} d_{n\downarrow} \rangle \right), \tag{14}$$

The charge current in the left lead is defined as

$$I^e_L = e(I_{L\uparrow} + I_{L\downarrow}). \tag{15}$$

The spin current in the left lead is defined as

$$I^s_L = \frac{1}{2}(I_{L\uparrow} - I_{L\downarrow}). \tag{16}$$

Substitute the spin-dependent current in, we get the spin current in left lead

$$I^s_L = \frac{i}{2\hbar} \sum_{nk} \left(t^*_{nk\uparrow} \langle d^\dagger_{n\uparrow} c_{k\uparrow} \rangle - t_{nk\uparrow} \langle c^\dagger_{k\uparrow} d_{n\uparrow} \rangle - t^*_{nk\downarrow} \langle d^\dagger_{n\downarrow} c_{k\downarrow} \rangle + t_{nk\downarrow} \langle c^\dagger_{k\downarrow} d_{n\downarrow} \rangle \right). \tag{17}$$

2. Magnonic current in right lead

The magnonic current in right lead is

$$I^s_R = \frac{d\langle \hat{N}_R \rangle}{dt} = \frac{d}{dt} \langle \sum_q a^\dagger_q a_q \rangle, \tag{18}$$

From the Heisenberg equation, we have

$$\frac{d}{dt} \langle \sum_q a^\dagger_q a_q \rangle = \frac{i}{\hbar} \langle [H, \sum_q a^\dagger_q a_q] \rangle. \tag{19}$$

We have

$$\begin{aligned} & [H, \sum_q a^\dagger_q a_q] \\ =& [H_{\mathrm{sd}}, \sum_q a^\dagger_q a_q] \\ =& -\sum_{qnn'} J_{qnn'} \Big(\big[a^\dagger_q d^\dagger_{n\uparrow} d_{n'\downarrow}, \sum_{q'} a^\dagger_{q'} a_{q'} \big] + \big[a_q d^\dagger_{n'\downarrow} d_{n\uparrow}, \sum_{q'} a^\dagger_{q'} a_{q'} \big] \Big) \\ =& \sum_{qnn'} J_{qnn'} \big(a^\dagger_q d^\dagger_{n\uparrow} d_{n'\downarrow} - a_q d^\dagger_{n'\downarrow} d_{n\uparrow} \big). \end{aligned} \tag{20}$$

So, the magnon current reads

$$I^s_R = \frac{i}{\hbar} \langle \sum_{qnn'} J_{qnn'} \big(a^\dagger_q d^\dagger_{n\uparrow} d_{n'\downarrow} - a_q d^\dagger_{n'\downarrow} d_{n\uparrow} \big) \rangle. \tag{21}$$

3. Spin-dependent current in the central region

Similarly, the spin-dependent current in the central region is defined as

$$I_{C\sigma} = \langle \frac{d\hat{N}_{C\sigma}}{dt} \rangle = \sum_n \langle \frac{d}{dt} d^\dagger_{n\sigma} d_{n\sigma} \rangle, \tag{22}$$

Heisenberg equation:

$$\frac{d}{dt} d^\dagger_{n\sigma} d_{n\sigma} = \frac{i}{\hbar}[H, d^\dagger_{n\sigma} d_{n\sigma}]. \tag{23}$$

Specifically, we have

$$[H, d^\dagger_{n\sigma} d_{n\sigma}] = [H_{\mathrm{T}}, d^\dagger_{n\sigma} d_{n\sigma}] + [H_{\mathrm{sd}}, d^\dagger_{n\sigma} d_{n\sigma}] \tag{24}$$

in which,

$$\begin{aligned} [H_{\mathrm{T}}, d^\dagger_{n\sigma} d_{n\sigma}] &= \sum_{k'\sigma'} \big(t_{nk'\sigma'} \big[c^\dagger_{k'\sigma'} d_{n\sigma'}, d^\dagger_{n\sigma} d_{n\sigma} \big] + t^*_{nk'\sigma'} \big[d^\dagger_{n\sigma'} c_{k'\sigma'}, d^\dagger_{n\sigma} d_{n\sigma} \big] \big) \\ &= \sum_{k'\sigma'} \big(t_{nk'\sigma'} c^\dagger_{k'\sigma'} d_{n\sigma'} \delta_{\sigma\sigma'} - t^*_{nk'\sigma'} d^\dagger_{n\sigma'} c_{k'\sigma'} \delta_{\sigma\sigma'} \big) \\ &= \sum_k \big(t_{nk\sigma} c^\dagger_{k\sigma} d_{n\sigma} - t^*_{nk\sigma} d^\dagger_{n\sigma} c_{k\sigma} \big) \end{aligned} \tag{25}$$

We change the summation index from k' to k in the last line , which doesn't change the result.

$$[H_{\mathrm{sd}}, d^\dagger_{n\sigma} d_{n\sigma}] = - \sum_{qn'n''} J_{qn'n''} \big(a^\dagger_q \big[d^\dagger_{n'\uparrow} d_{n''\downarrow}, d^\dagger_{n\sigma} d_{n\sigma} \big] + a_q \big[d^\dagger_{n''\downarrow} d_{n'\uparrow}, d^\dagger_{n\sigma} d_{n\sigma} \big] \big) \tag{26}$$

in which,

$$\begin{aligned} \big[d^\dagger_{n'\uparrow} d_{n''\downarrow}, d^\dagger_{n\sigma} d_{n\sigma} \big] &= d^\dagger_{n'\uparrow} \big[d_{n''\downarrow}, d^\dagger_{n\sigma} d_{n\sigma} \big] + \big[d^\dagger_{n'\uparrow}, d^\dagger_{n\sigma} d_{n\sigma} \big] d_{n''\downarrow} \\ &= d^\dagger_{n'\uparrow} d_{n\sigma} \delta_{nn''} \delta_{\sigma\downarrow} - d^\dagger_{n\sigma} d_{n''\downarrow} \delta_{nn'} \delta_{\sigma\uparrow} \end{aligned} \tag{27}$$

For the simplicity, we consider the central region as a single level and suppress index n. We have

$$[H_{\mathrm{sd}}, d^\dagger_\sigma d_\sigma] = - \sum_q J_q \big[S^-_q \big(d^\dagger_\uparrow d_\downarrow \delta_{\sigma\downarrow} - d^\dagger_\uparrow d_\downarrow \delta_{\sigma\uparrow} \big) + S^+_q \big(d^\dagger_\downarrow d_\uparrow \delta_{\sigma\uparrow} - d^\dagger_\downarrow d_\uparrow \delta_{\sigma\downarrow} \big) \big] \tag{28}$$

The spin-dependent current is

$$I_{C\uparrow} = \frac{i}{\hbar} \big[H, d^\dagger_\uparrow d_\uparrow \big] \tag{29}$$

$$I_{C\downarrow} = \frac{i}{\hbar} \big[H, d^\dagger_\downarrow d_\downarrow \big] \tag{30}$$

which gives

$$I_{C\uparrow} = \frac{i}{\hbar} \big[\sum_k \big(t_{k\uparrow} c^\dagger_{k\uparrow} d_\uparrow - t^*_{k\uparrow} d^\dagger_\uparrow c_{k\uparrow} \big) + \sum_q J_q \big(S^-_q d^\dagger_\uparrow d_\downarrow - S^+_q d^\dagger_\downarrow d_\uparrow \big) \big] \tag{31}$$

and

$$I_{C\downarrow} = \frac{i}{\hbar} \big[\sum_k \big(t_{k\downarrow} c^\dagger_{k\downarrow} d_\downarrow - t^*_{k\downarrow} d^\dagger_\downarrow c_{k\downarrow} \big) + \sum_q J_q \big(S^+_q d^\dagger_\downarrow d_\uparrow - S^-_q d^\dagger_\uparrow d_\downarrow \big) \big]. \tag{32}$$

The spin current in central dot is

$$I^s_C = \frac{1}{2} (I_{C\uparrow} - I_{C\downarrow}) \tag{33}$$

4. Verifying continuity condition

In this subsection we consider only one single level in the central region. Since the right lead is a insulating lead, there is no charge current flow through it, so the charge current is

$$I_R^e = 0 \tag{34}$$

where the subscript R denotes the right lead, while the subscript e denotes the charge current. The charge current flows in left lead and the central region is summed up to zero,

$$I_e = e(\sum_\sigma I_{L\sigma} + \sum_\sigma I_{C\sigma}) = 0 \tag{35}$$

The spin current in the left lead and the central region:

$$I_L^s + I_C^s = \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big). \tag{36}$$

The magnon current is

$$I_R^s = \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big), \tag{37}$$

in which, I_L^s, I_C^s, I_R^s is defined earlier. Thus, we have

$$\begin{aligned} I_L^s + I_C^s &= I_R^s \\ &= \frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big). \end{aligned} \tag{38}$$

C. Spin current in two-terminal NM/NM/FI system

1. Spin current from the right lead

Rewrite current using Green's functions – The spin current can by obtained from either the left NM lead or the right FI lead. From the right lead, the spin current is Eq. (21),

$$I_R^s = \frac{i}{\hbar}\langle\sum_{qnn'} J_{qnn'}\big(d_{n\uparrow}^\dagger d_{n'\downarrow} a_q^\dagger - a_q d_{n'\downarrow}^\dagger d_{n\uparrow}\big)\rangle. \tag{39}$$

Now we define the Green's function on the Keldysh contour

$$G_{nn',q}(\tau,\tau') = -i\langle T_c S d_{n\uparrow}^\dagger(\tau) d_{n'\downarrow}(\tau) a_q^\dagger(\tau')\rangle, \tag{40}$$

$$G_{nn',q}(\tau,\tau') = -i\langle T_c S a_q(\tau) d_{n'\downarrow}^\dagger(\tau') d_{n\uparrow}(\tau')\rangle, \tag{41}$$

Here we define $J_v = J_{qnn'}$ for sake of convenience and further assume that the spin-flip scattering occurs at same interfacial sites, namely $n = n'$. So that

$$\begin{aligned} I_R^s(t) &= -\frac{1}{\hbar}\sum_{nq} J_{nq} G_{n,q}^<(t,t) + \frac{1}{\hbar}\sum_v J_{nq} G_{n,q}^<(t,t) \\ &= i\sum_{nq} J_{nq}\langle d_{n\uparrow}^\dagger(t) d_{n\downarrow}(t) a_q^\dagger(t)\rangle + h.c. \end{aligned} \tag{42}$$

where $G_{n,q}(\tau,\tau')$ is the Green's function connecting the right lead and the central scattering region. Here T_c is the time ordering operator and S is the S-matrix defined as

$$S = \exp(-i\int_c d\tau H'(\tau))$$

where H' is the interacting coupling term defined above. $\langle O\rangle$ is the expectation value of O over ground state of H_0 which is non-interacting. This is very important since Wick theorem can be used if H_0 is non-interacting.

Decouple Green's functions – The next step is decouple the d operator and a_q in Green's function $G_{d,R}$. We can do this by defining the d-operator Green's function and a_q-operator Green's function,

$$G_{nn'}(\tau,\tau') = -i\langle T_c S d^{\dagger}_{n\uparrow}(\tau) d_{n\downarrow}(\tau) d^{\dagger}_{n'\downarrow}(\tau') d_{n'\uparrow}(\tau')\rangle. \tag{43}$$

We can decouple it as follows (missing $\sum_{n'}$?)

$$-i\partial_{\tau'} G_{n,q}(\tau,\tau') = \omega_q G_{nn',q}(\tau,\tau') - J^*_{qn'} G_{nn'}, \tag{44}$$

or

$$G_{n,q} g^{-1}_{Rq} = -J^*_{qn'} G_{nn'}, \tag{45}$$

or

$$G_{n,q}(\tau,\tau') = -J^*_{qn'} \int G_{nn'}(\tau,\tau_1)\, g_{Rq}(\tau_1,\tau')\, d\tau_1. \tag{46}$$

Here $g_{Rq} = (-i\partial_\tau - \omega_q)^{-1}$ is the Green's function of the right lead. The minus befor J originates from the minus in H_{sd}. There are two derivations for Eq.(46). The rigorous one follows from p188 of Jahou's book. Here we derivated it from equation of motion (EoM) method. The rules of analytic continuation gives ($\sum_{n'}$?)

$$G^<_{n,q}(t,t') = -\sum_{n'} J^*_{qn'} \int_{-\infty}^{\infty} dt_1 [G^r_{nn'}(t,t_1)\, g^<_{Rq}(t_1,t') + G^<_{nn'}(t,t_1)\, g^a_{Rq}(t_1,t')], \tag{47}$$

$$G^<_{n,q}(t,t') = -\sum_{n'} J^*_{qn'} \int_{-\infty}^{\infty} dt_1 [g^r_{Rq}(t,t_1)\, G^<_{nn'}(t_1,t') + g^<_{Rq}(t,t_1)\, G^a_{nn'}(t_1,t')]. \tag{48}$$

Substituting Eq. (47) into Eq. (42), we get

$$\begin{aligned} I^s_R(t) &= \frac{1}{\hbar}\sum_{qnn'} J_{nq} J^*_{qn'} \int dt' \mathrm{Tr}[G^r_{nn'}(t,t') g^<_{Rq}(t',t) + G^<_{nn'}(t,t') g^a_{Rq}(t',t)] + h.c. &(49)\\ &= \frac{1}{\hbar}\sum_{nn'} \int dt' \mathrm{Tr}[G^r_{nn'}(t,t')\Sigma^<_R(t',t) + G^<_{nn'}(t,t')\Sigma^a_R(t',t)] + h.c. &(50)\end{aligned}$$

The matrix element of the right lead self-energies are defined as $\Sigma^r_{R,nn'} = \sum_{qnn'} J_{qn} g^r_{Rq} J^*_{qn'}$ and $\Sigma^<_R = i f_R \Gamma_R$. Γ_R is assumed Ohmic in calculation. For DC driving force, after Fourier transformation, the above equation can take another form using $G^r - G^a = G^> - G^<$

$$I^s_R = \sum_{nn'} \int dE \mathrm{Tr}[(G^>_{nn'} - G^<_{nn'})\Sigma^<_R + G^<_{nn'}(\Sigma^a_R - \Sigma^r_R)]\ . \tag{51}$$

Evaluate central region GF – In the following we calculate G_d using S-matrix expansion. G_d is the matrix form of GF defined in Eq. (43). First of all, it is easy to see that in the absence of the right lead, we have

$$G_d(\tau,\tau') = -iG_{L\uparrow}(\tau',\tau) G_{L\downarrow}(\tau,\tau')$$

where

$$G_{L\sigma} = g_{d\sigma} + g_{d\sigma}\Sigma_{L\sigma} g_{d\sigma} + ... = 1/[g^{-1}_{d\sigma} - \Sigma_{L\sigma}]$$

where $g_{d,nm\uparrow} = -i\langle d_{n\uparrow} d^{\dagger}_{m\uparrow}\rangle$, $g_{d,nm\downarrow} = -i\langle d_{n^+\downarrow} d^{\dagger}_{m^+\downarrow}\rangle$, and

$$\Sigma_{L,nm\uparrow} = \sum_k t^*_{k\uparrow n} t_{k\uparrow m} g_{Lk\uparrow}, \quad \Sigma_{L,nm\downarrow} = \sum_k t^*_{k\downarrow n^+} t_{k\downarrow m^+} g_{Lk\downarrow} \tag{52}$$

Here $g_L = (-i\partial_\tau - \epsilon_k)^{-1}$ is the Green's function of left lead.See derivation in Ligy's notebook P.224.

We now treat the right lead without presence of the left lead. We get

$$\begin{aligned} G_d(\tau,\tau') &= -ig_{d\uparrow}(\tau',\tau)g_{d\downarrow}(\tau,\tau') \\ &+[-ig_{d\uparrow}(\tau_1,\tau)g_{d\downarrow}(\tau,\tau_1)][\Sigma_{Rq}(\tau_1,\tau_2)][-ig_{d\uparrow}(\tau',\tau_2)g_{d\downarrow}(\tau_2,\tau')]+... \\ &= -ig_{d\uparrow}(\tau',\tau)g_{d\downarrow}(\tau,\tau') - ig_{d\uparrow}(\tau_1,\tau)g_{d\downarrow}(\tau,\tau_1)\Sigma_{Rq}(\tau_1,\tau_2)G_d(\tau_2,\tau') \end{aligned} \tag{53}$$

where we have summed all the RPA diagrams. See derivation in Ligy's notebook P.216. When we take left lead back, we replace $g_{d\sigma}$ by $G_{L\sigma}$ (reason we can do this because the central region and left lead are the same ?).

$$G_d(\tau,\tau') = -iG_{L\uparrow}(\tau',\tau)G_{L\downarrow}(\tau,\tau') - iG_{L\uparrow}(\tau_1,\tau)G_{L\downarrow}(\tau,\tau_1)\Sigma_{Rq}(\tau_1,\tau_2)G_d(\tau_2,\tau') \tag{54}$$

Analytic continuation – Then we use analytic continuation to get all GFs in real-time current expansions. The analytic continuation theorem on Eq.(54) yields,

$$G_d^< = -iG_{L\uparrow}^> G_{L\downarrow}^<(1+\Sigma_{Rq}^a G_d^a) - i(G_{L\uparrow}^a G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^r + G_{L\uparrow}^a G_{L\downarrow}^r)(\Sigma_{Rq}^< G_d^a + \Sigma_{Rq}^r G_d^<) \tag{55}$$

and

$$G_d^r = -i(G_{L\uparrow}^a G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^r + G_{L\uparrow}^a G_{L\downarrow}^r)(1+\Sigma_{Rq}^r G_d^r)$$

$$G_d^a = -i(G_{L\uparrow}^r G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^a + G_{L\uparrow}^r G_{L\downarrow}^a)(1+\Sigma_{Rq}^a G_d^a)$$

Denoting $(G_{L\uparrow}G_{L\downarrow})^r = G_{L\uparrow}^a G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^r + G_{L\uparrow}^a G_{L\downarrow}^r$, we find

$$G_d^r = -i/[1+i(G_{L\uparrow}G_{L\downarrow})^r\Sigma_{Rq}^r](G_{L\uparrow}G_{L\downarrow})^r \tag{56}$$

Solving $G_d^<$ in Eq.(55) and using Eq.(56), we obtain

$$G_d^< = -i(G_d^r\Sigma_{Rq}^r+1)G_{L\uparrow}^> G_{L\downarrow}^<(1+\Sigma_{Rq}^a G_d^a) + G_d^r\Sigma_{Rq}^< G_d^a \tag{57}$$

which resembles the Keldysh equation. Similarly we have

$$G_d^> = -i(G_d^r\Sigma_{Rq}^r+1)G_{L\uparrow}^< G_{L\downarrow}^>(1+\Sigma_{Rq}^a G_d^a) + G_d^r\Sigma_{Rq}^> G_d^a \tag{58}$$

Final expression for DC – Substituting these GFs in Eq. (21), for dc spin bias or temperature gradient, the Green's function depends only on the time difference, i.e., $G(t-t')$. Hence after Fourier transform, we have

$$[(G_d^> - G_d^<)\Sigma_R^<](t,t) = \int dE\ (G_d^>(E) - G_d^<(E))\Sigma_R^<(E)$$

where in energy domain,

$$G_d^>(E) - G_d^<(E) \quad = i(G_d^r\Sigma_{Rq}^r+1)(G_{L\uparrow}^< G_{L\downarrow}^> - G_{L\uparrow}^> G_{L\downarrow}^<)(1+\Sigma_{Rq}^a G_d^a) + G_d^r(\Sigma_{Rq}^> - \Sigma_{Rq}^<)G_d^a, \tag{59}$$

in which, $G_{L\sigma}^< = G_{L\sigma}^r\Sigma_{L\sigma}^< G_{L\sigma}^a = iG_{L\sigma}^r\Gamma_{L\sigma}f_{L\sigma}G_{L\sigma}^a \equiv iD_{L\sigma}f_{L\sigma}$ and

$$\begin{aligned} G_{L\sigma}^>(E) &= -(G_{L\sigma}^<(E))^\dagger \\ &= G_{L\sigma}^r(E)\Sigma_{L\sigma}^>(E)G_{L\sigma}^a(E) \\ &= iD_{L\sigma}(f_{L\uparrow}(E)-1) \end{aligned} \tag{60}$$

where, $\Sigma_{L\sigma}^> = i(f_{L\sigma}-1)\Gamma_{L\sigma}$. Hence,

$$G_{L\uparrow}^< G_{L\downarrow}^> - G_{L\uparrow}^> G_{L\downarrow}^< = D_{L\uparrow}D_{L\downarrow}[(f_{L\uparrow}-1)f_{L\downarrow} - (f_{L\downarrow}-1)f_{L\uparrow}] = D_{L\uparrow}D_{L\downarrow}(f_{L\uparrow}-f_{L\downarrow}), \tag{61}$$

$$G_d^> - G_d^< = i(G_d^r\Sigma_{Rq}^r+1)D_{L\uparrow}D_{L\downarrow}(f_{L\uparrow}-f_{L\downarrow})(1+\Sigma_{Rq}^a G_d^a) + G_d^r(-i\Gamma_R)G_d^a, \tag{62}$$

$$G_d^< = -i(G_d^r\Sigma_{Rq}^r+1)(-D_{L\uparrow}D_{L\downarrow}(f_{L\uparrow}-1)f_{L\downarrow})(1+\Sigma_{Rq}^a G_d^a) + G_d^r\Sigma_{Rq}^< G_d^a \tag{63}$$

Now we evaluate the spin current. Note that $\Sigma_R^a - \Sigma_R^r = \Sigma_R^< - \Sigma_R^> = i\Gamma_R$. From Eqs.(62) and (63), we find

$$\begin{aligned} (G_d^> - G_d^<)\Sigma_R^< + G_d^<(\Sigma_R^a - \Sigma_R^r) =& -(G_d^r\Sigma_R^r+1)D_{L\uparrow}D_{L\downarrow}(f_{L\uparrow}-f_{L\downarrow})(1+\Sigma_R^a G_d^a)f_R\Gamma_R + G_d^r f_R\Gamma_R G_d^a\Gamma_R \\ & -(G_d^r\Sigma_R^r+1)D_{L\uparrow}D_{L\downarrow}(f_{L\uparrow}-1)f_{L\downarrow}(1+\Sigma_R^a G_d^a)\Gamma_R - G_d^r f_R\Gamma_R G_d^a\Gamma_R \\ =& -[(f_{L\uparrow}-f_{L\downarrow})f_R + (f_{L\uparrow}-1)f_{L\downarrow}](G_d^r\Sigma_R^r+1)D_{L\uparrow}D_{L\downarrow}(1+\Sigma_R^a G_d^a)\Gamma_R \end{aligned} \tag{64}$$

Given $f_{L\sigma}(\epsilon) = 1/[\exp(\beta_L(\epsilon - \mu_\sigma)) + 1]$ and considering the fact that spin down spectrum of the left lead is shifted up, we have the identity $(f_{L\uparrow} - 1)f_{L\downarrow} = -(f_{L\uparrow} - f_{L\downarrow})f_L^B$ where $f_L^B(\omega) = 1/[\exp(\beta_L(\omega + \Delta\mu_s)) - 1]$ is the effective Bose distribution of the left lead and $\mu_s = \mu_\uparrow - \mu_\downarrow$, from which we arrive at the final result

$$I_{sR} = \int d\omega \rho_R(\omega)(f_R^B(\omega) - f_L^B(\omega)) \int dE (f_{L\uparrow}(E) - f_{L\downarrow}(E + \omega)) \mathrm{Tr}[A(E, \omega)] \tag{65}$$

where

$$A(E, \omega) = [G_d^r(E)\Sigma_{Rq}^r(\omega) + 1]D_{L\uparrow}(E)D_{L\downarrow}(E + \omega)[1 + \Sigma_{Rq}^a(\omega)G_d^a(E)] \tag{66}$$

When $\Delta\mu_s = T_L - T_R = 0$, there is no spin current.

2. *Spin current from the left lead*

Now we calculate the spin current from the left lead. The first several procedules are the same with NM system. The spin current in the left lead is given by

$$I_L^s = (1/2)\partial_t N_s = (1/2)(I_\uparrow - I_\downarrow)$$

where

$$I_\sigma = \mathrm{Tr}[(G_{d\sigma}^r - G_{d\sigma}^a)\Sigma_{L\sigma}^< + G_{d\sigma}^<(\Sigma_{L\sigma}^a - \Sigma_{L\sigma}^r)].$$

The left self-energy is defined previously. However, the d-operator is different from the one defined earlier, which is

$$[G_{d\sigma}]_{nm} = -i\langle T_c S d_{n\sigma} d_{m\sigma}^\dagger \rangle$$

and the factor of 1/2 comes from spin of electron while spin of magnon is 1. We first calculate $G_{d\uparrow}$ using S-matrix expansion. In the absence of the left lead, the second order term in J_q is

$$\begin{aligned} &\langle T_c d_{n\uparrow}(\tau) d_{m\uparrow}^\dagger(\tau') \sum_{n_1 m_1 q_1} [J_{q_1} a_{q_1}(\tau_1) d_{m_1\downarrow}^\dagger(\tau_1) d_{n_1\uparrow}(\tau_1)\delta + h.c.] \\ &\sum_{n_2 m_2 q_2} [J_{q_2} a_{q_2}(\tau_2) d_{m_2\downarrow}^\dagger(\tau_2) d_{n_2\uparrow}(\tau_2)\delta + h.c.]\rangle \end{aligned}$$

After taking care of the sign, we find

$$\begin{aligned} G_{d\uparrow}(\tau, \tau') &= g_{d\uparrow}(\tau, \tau') + \sum_q g_{d\uparrow}(\tau, \tau_1)[i g_{d\downarrow}(\tau_1, \tau_2)\Sigma_{Rq}(\tau_2, \tau_1)] g_{d\uparrow}(\tau_2, \tau') + ... \\ &= g_{d\uparrow}(\tau, \tau') + \sum_q g_{d\uparrow}(\tau, \tau_1)[i g_{d\downarrow}(\tau_1, \tau_2)\Sigma_{Rq}(\tau_2, \tau_1)] G_{d\uparrow}(\tau_2, \tau') \end{aligned}$$

When both leads are present, we have

$$G_{d\uparrow}(\tau, \tau') = G_{L\uparrow}(\tau, \tau') + G_{L\uparrow}(\tau, \tau_1) i \sum_q G_{L\downarrow}(\tau_1, \tau_2)\Sigma_{Rq}(\tau_2, \tau_1) G_{L\uparrow}(\tau_2, \tau') \tag{67}$$

where $G_{L\sigma}$ is defined previously. In particular, in $G_{L\downarrow}$ the spin down energy spectra of QD and the left lead have been shifted up by ω_q as before. For $I_\downarrow$, we expand the S-matrix up to the 2nd order,

$$\begin{aligned} X = \;& \langle T_c d_{n\downarrow}(\tau) d_{m\downarrow}^\dagger(\tau') \sum_{n_1 m_1 q_1} [J_{q_1} a_{q_1}(\tau_1) d_{m_1\downarrow}^\dagger(\tau_1) d_{n_1\uparrow}(\tau_1)\delta + h.c.] \\ &\sum_{n_2 m_2 q_2} [J_{q_2} a_{q_2}(\tau_2) d_{m_2\downarrow}^\dagger(\tau_2) d_{n_2\uparrow}(\tau_2)\delta + h.c.]\rangle \\ &= \langle T_c \sum_{q n_1 m_1 n_2 m_2} J_q^2 d_{n\downarrow}(\tau) d_{m_1\downarrow}^\dagger(\tau_1) d_{n_1\uparrow}(\tau_1) d_{n_2\uparrow}^\dagger(\tau_2) a_q(\tau_1) a_q^\dagger(\tau_2) d_{m_2\downarrow}(\tau_2) d_{m\downarrow}^\dagger(\tau')\delta\rangle \end{aligned}$$

We use m_2^- to denote the constraint $\epsilon_{n_2} - \omega_q = \epsilon_{m_2}$. We obtain

$$X = \langle T_c \sum_{qm_1m_2} J_q^2 d_{n\downarrow}(\tau) d^\dagger_{m_1\downarrow}(\tau_1) d_{m_1^-\uparrow}(\tau_1) d^\dagger_{m_2^-\uparrow}(\tau_2) a_q(\tau_1) a^\dagger_q(\tau_2) d_{m_2\downarrow}(\tau_2) d^\dagger_{m\downarrow}(\tau')\delta\rangle$$

and this in turn gives

$$G_{d\downarrow}(\tau,\tau') = g_{d\downarrow}(\tau,\tau') + \sum_q g_{d\downarrow}(\tau,\tau_1)[ig_{d\uparrow}(\tau_1,\tau_2)\Sigma_{Rq}(\tau_2,\tau_1)]G_{d\downarrow}(\tau_2,\tau')$$

Now we see that the up spin spectrum has been shifted *down* by ω_q. When both leads are considered, we find

$$G_{d\downarrow}(\tau,\tau') = G_{L\downarrow}(\tau,\tau') + i\sum_q G_{L\downarrow}(\tau,\tau_1)G_{L\uparrow}(\tau_1,\tau_2)\Sigma_{Rq}(\tau_2,\tau_1)G_{L\downarrow}(\tau_2,\tau')$$

We emphasize that the up spin energy spectra of QD and left leads have been shifted down in contrast to the case of $I_\uparrow$ and there is no shift in down spin spectrum. Notice that Eqs.(54) and (67) are structurally very similar, hence we can skip some of intermediate steps in derivation. After analytic continuation, we have

$$G^<_{d\uparrow} = G^<_{L\uparrow} + G^<_{L\uparrow}\bar{\Sigma}^a G^a_{d\uparrow} + G^r_{L\uparrow} i\sum_q G^<_{L\downarrow}\Sigma^>_{Rq}G^a_{L\uparrow} + G^r_{L\uparrow}\bar{\Sigma}^r G^<_{d\uparrow}, \tag{68}$$

$$G^>_{d\uparrow} = G^>_{L\uparrow} + G^>_{L\uparrow}\bar{\Sigma}^a G^a_{d\uparrow} + G^r_{L\uparrow} i\sum_q G^>_{L\downarrow}\Sigma^>_{Rq}G^a_{L\uparrow} + G^r_{L\uparrow}\bar{\Sigma}^r G^>_{d\uparrow}. \tag{69}$$

For $G^r_{d\uparrow}$, we have

$$G^r_{d\uparrow} = G^r_{L\uparrow} + G^r_{L\uparrow}\bar{\Sigma}^r G^r_{d\uparrow} = 1/[(G^r_{L\uparrow})^{-1} - \bar{\Sigma}^r] \tag{70}$$

where $\bar{\Sigma}(\tau_1,\tau_2) = i\sum_q G_{L\downarrow}(\tau_1,\tau_2)\Sigma_{Rq}(\tau_2,\tau_1)$. We contracted terms into Dyson equation form, so that the term in the middle has the meaning of self-energy. Combining Eqs.(68) and Eqs.(70), we obtain

$$G^<_{d\uparrow} = (G^r_{d\uparrow}\bar{\Sigma}^r + 1)G^<_{L\uparrow}(1 + \bar{\Sigma}^a G^a_{d\uparrow}) + iG^r_{d\uparrow}\sum_q G^<_{L\downarrow}\Sigma^>_{Rq}G^a_{d\uparrow}$$

As before, the down spin spectra of H_L and QD are shifted in the expression of $G^<$ and $G^>$. Similar expression can be written for $G^<_{d\downarrow}$.

DC spin current from the left lead – Since there is no charge current in the left lead, we should have $I_\uparrow = I_\downarrow$. In energy domain,

$$I_\uparrow = \int dE \mathrm{Tr}[(G^>_{d\uparrow}(E) - G^<_{d\uparrow}(E))\Sigma^<_{L\uparrow}(E) + G^<_{d\uparrow}(E)(\Sigma^a_{L\uparrow}(E) - \Sigma^r_{L\uparrow}(E))] \tag{71}$$

note that

$$\begin{aligned}(G^>_{d\uparrow} - G^<_{d\uparrow})\Sigma^<_{L\uparrow} + G^<_{d\uparrow}(\Sigma^a_{L\uparrow} - \Sigma^r_{L\uparrow}) \quad &= \sum_q [(f_{L\uparrow} - f_{L\downarrow})f_R + (f_{L\uparrow} - 1)f_{L\downarrow}] \\ &\times G^r_{d\uparrow}(D_{L\downarrow}\Gamma_{Rq})G^a_{d\uparrow}\Gamma_{L\uparrow}\end{aligned}$$

Given $f_{L\sigma}(\epsilon) = 1/[\exp(\beta_L(\epsilon - \mu_\sigma)) + 1]$, we have the identity $(f_{L\uparrow} - 1)f_{L\downarrow} = -(f_{L\uparrow} - f_{L\downarrow})f^B_L$, we still have the same identity $(f_{L\uparrow} - 1)f_{L\downarrow} = -(f_{L\uparrow} - f_{L\downarrow})f^B_L$. Hence

$$I_\uparrow = -\int d\omega \rho_R(\omega)(f^B_R(\omega) - f^B_L(\omega))\int dE (f_{L\uparrow}(E) - f_{L\downarrow}(E+\omega))\mathrm{Tr}[A(E,\omega)] \tag{72}$$

where

$$A(E,\omega) = G^r_{d\uparrow}(D_{L\downarrow}\Gamma_{Rq})G^a_{d\uparrow}\Gamma_{L\uparrow}$$

For $I_\downarrow$, we have

$$\begin{aligned}\int dE[(G^>_{d\downarrow} - G^<_{d\downarrow})\Sigma^<_{L\downarrow} + G^<_{d\downarrow}(\Sigma^a_{L\downarrow} - \Sigma^r_{L\downarrow})] \quad &= -\int dE \sum_q [(f_{L\downarrow} - f_{L\uparrow})f_R + (f_{L\downarrow} - 1)f_{L\uparrow}] \\ &\times G^r_{d\downarrow}(D_{L\uparrow}\Gamma_{Rq})G^a_{d\downarrow}\Gamma_{L\downarrow}\end{aligned}$$

Note that the up spin spectra of QD and left lead are shifted down so that after transformation from E to $E+\omega$, $G_\uparrow$, $\Sigma_\uparrow$, and $f_\uparrow$ will be transformed to $G_\downarrow$, $\Sigma_\downarrow$, and $f_\downarrow$, respectively. Hence $I_\uparrow = I_\downarrow$ and $I_{sL} = I_\uparrow$. It is easy to see that up to the 2nd order in J_q, the spin current is conserved, i.e., we have $I_{sL} + I_{sR} = 0$.

We now verify that Eq. (72) is equivelant to Eq. (10) in PRB 106, 205303 (2022). First note that Eq. (16) in the PRB is the above Eq. (72). Eq. (72) can be transformed from Eq. (12) in the PRB (see Ligy's note for derivation). Moreover, Eq. (12) can be transformed from Eq. (11).

$$
\begin{aligned}
I_{sR} &= \int \frac{dE}{2\pi} Tr\left[G_\uparrow^r(E)\bar{\Sigma}_{R\uparrow}^<(E) + G_\uparrow^<(E)\bar{\Sigma}_{R\uparrow}^a(E) + h.c.\right] \\
&= \int \frac{dE}{2\pi} Tr\left[\left(G_\uparrow^r(E) - G_\uparrow^a(E)\right)\bar{\Sigma}_{R\uparrow}^<(E) + G_\uparrow^<(E)\left(\bar{\Sigma}_{R\uparrow}^a(E) - \bar{\Sigma}_{R\uparrow}^r(E)\right)\right] \\
&= \int \frac{dE}{2\pi} Tr\left[\underbrace{-G_\uparrow^a(E)\left(\Sigma_{\alpha\uparrow}^a(E) - \Sigma_{\alpha\uparrow}^r(E)\right)G_\uparrow^r(E)}_{\left(G_\uparrow^r(E)-G_\uparrow^a(E)\right)}\bar{\Sigma}_{R\uparrow}^<(E) + \underbrace{G_\uparrow^r(E)\Sigma_{\alpha\uparrow}^< G_\uparrow^a(E)}_{G_\uparrow^<(E)}\left(\bar{\Sigma}_{R\uparrow}^a(E) - \bar{\Sigma}_{R\uparrow}^r(E)\right)\right] \\
&= \int \frac{dE}{2\pi} Tr\left[\underbrace{-G_\uparrow^a(E)\left(\Sigma_{\alpha\uparrow}^a(E) - \Sigma_{\alpha\uparrow}^r(E)\right)G_\uparrow^r(E)}_{R\not\subset\alpha}\bar{\Sigma}_{R\uparrow}^<(E) + \underbrace{G_\uparrow^r(E)\Sigma_{\alpha\uparrow}^< G_\uparrow^a(E)}_{R\not\subset\alpha}\left(\bar{\Sigma}_{R\uparrow}^a(E) - \bar{\Sigma}_{R\uparrow}^r(E)\right)\right] \\
&= -\int \frac{dE}{2\pi} Tr\left[\underbrace{G_\uparrow^a(E)\, i\Gamma_\alpha(E)\, G_\uparrow^r(E)}_{R\not\subset\alpha}\bar{\Sigma}_{R\uparrow}^<(E) - \underbrace{G_\uparrow^r(E)\, i f_\alpha \Gamma_\alpha(E)\, G_\uparrow^a(E)}_{R\not\subset\alpha}\left(\bar{\Sigma}_{R\uparrow}^a(E) - \bar{\Sigma}_{R\uparrow}^r(E)\right)\right] \\
&= -\int \frac{dE}{2\pi} Tr\left[\underbrace{G_\uparrow^r(E)\, i\Gamma_\alpha(E)\, G_\uparrow^a(E)}_{R\not\subset\alpha}\bar{\Sigma}_{R\uparrow}^<(E) + \underbrace{G_\uparrow^r(E)\, f_\alpha \Gamma_\alpha(E)\, G_\uparrow^a(E)}_{R\not\subset\alpha} 2\,\mathrm{Im}\,\bar{\Sigma}_{R\uparrow}^a(E)\right] \\
&= -\int \frac{dE}{2\pi} Tr\left[\underbrace{G_\uparrow^r(E)\,\Gamma_\alpha(E)\, G_\uparrow^a(E)}_{R\not\subset\alpha}\left(i\bar{\Sigma}_{R\uparrow}^<(E) + f_\alpha 2\,\mathrm{Im}\,\bar{\Sigma}_{R\uparrow}^a(E)\right)\right] \qquad Eq.(10)
\end{aligned}
$$

D. Linear response regime

$$
\begin{aligned}
f_R(\omega) - f_L^B(\omega) &= \frac{1}{e^{\beta_R \omega} - 1} - \frac{1}{e^{\beta_L(\omega+\Delta\mu_s)} - 1} \\
&= \frac{e^{\beta_L(\omega+\Delta\mu_s)} - e^{\beta_R\omega}}{[e^{\beta_R\omega} - 1][e^{\beta_L(\omega+\Delta\mu_s)} - 1]} \\
&= \frac{e^{\beta_L\omega}[e^{\beta_L\Delta\mu_s} - e^{(\beta_R-\beta_L)\omega}]}{e^{\beta_L\omega}[e^{(\beta_R-\beta_L)\omega} - e^{-\beta_L\omega}][e^{\beta_L(\omega+\Delta\mu_s)} - 1]} \\
&= \frac{e^{\beta_L\Delta\mu_s} - e^{(\beta_R-\beta_L)\omega}}{[e^{(\beta_R-\beta_L)\omega} - e^{-\beta_L\omega}][e^{\beta_L(\omega+\Delta\mu_s)} - 1]}.
\end{aligned} \tag{73}
$$

$$
f_{L\uparrow}(E) - f_{L\downarrow}(E+\omega) = \frac{1}{e^{\beta_L(E-\mu_{L\uparrow})} + 1} - \frac{1}{e^{\beta_L(E+\omega-\mu_{L\downarrow})} + 1}. \tag{74}
$$

Here $\Delta\mu_s = \mu_{L\uparrow} - \mu_{L\downarrow}$ as before, and $\beta_R - \beta_L = \frac{\Delta T}{k_B T_L T_R}$.

1. $\Delta\mu_s = 0$ and $\Delta T \to 0$ limit

If $\Delta\mu_s = 0$, we have

$$\begin{aligned} f_R(\omega) - f_L^B(\omega) &= \frac{e^{\beta_L \Delta\mu_s} - e^{(\beta_R-\beta_L)\omega}}{[e^{(\beta_R-\beta_L)\omega} - e^{-\beta_L\omega}][e^{\beta_L(\omega+\Delta\mu_s)} - 1]} \\ &\overset{\Delta\mu_s=0}{\to} \frac{1 - e^{(\beta_R-\beta_L)\omega}}{[e^{(\beta_R-\beta_L)\omega} - e^{-\beta_L\omega}][e^{\beta_L\omega} - 1]} \end{aligned} \tag{75}$$

$$f_{L\uparrow}(E) - f_{L\downarrow}(E+\omega) = \frac{1}{e^{\beta_L(E-\mu_L)} + 1} - \frac{1}{e^{\beta_L(E+\omega-\mu_L)} + 1}. \tag{76}$$

In $\Delta T \to 0$ limit, $\beta_R - \beta_L \to 0$,

$$e^{(\beta_R-\beta_L)\omega} = 1 + \omega\beta_L^2 k_B \Delta T + O(\Delta T^2), \tag{77}$$

then,

$$\begin{aligned} f_R(\omega) - f_L^B(\omega) &\overset{\Delta T\to 0}{\to} \frac{\omega k_B \beta_L^2}{[1 - e^{-\beta_L\omega}][1 - e^{\beta_L\omega}]}\Delta T \\ &\overset{\Delta T\to 0}{\to} \frac{-\omega k_B \beta_L^2}{4\sinh^2(\beta_{\rm L}\omega/2)}\Delta T = \frac{-\omega k_B \beta_L^2}{2[\cosh(\beta_{\rm L}\omega) - 1]}\Delta T. \end{aligned} \tag{78}$$

Note that

$$\sinh^2(\frac{\rm x}{2}) = \frac{1}{2}[\cosh({\rm x}) - 1].$$

$$f_{L\uparrow}(E) - f_{L\downarrow}(E+\omega) \overset{\Delta T\to 0}{\to} 0. \tag{79}$$

When $\omega = 0$, the Eq. (79) is 0, thus the total coefficient is not singular.

2. $\Delta T = 0$ limit, and $\Delta\mu_s \to 0$

When $\Delta T = 0$, $\beta_R - \beta_L = 0$. The Eq. (73) reduces to

$$\begin{aligned} f_R(\omega) - f_L^B(\omega) &= \frac{e^{\beta_L \Delta\mu_s} - e^{(\beta_R-\beta_L)\omega}}{[e^{(\beta_R-\beta_L)\omega} - e^{-\beta_L\omega}][e^{\beta_L(\omega+\Delta\mu_s)} - 1]} \\ &\overset{\Delta T=0}{\to} \frac{e^{\beta_L\Delta\mu_s} - 1}{[1 - e^{-\beta_L\omega}][e^{\beta_L(\omega+\Delta\mu_s)} - 1]} \\ &\overset{\Delta\mu_s\to 0}{\to} \frac{e^{\beta_L\Delta\mu_s} - 1}{[1 - e^{-\beta_L\omega}][e^{\beta_L\omega} - 1]} \\ &\overset{\Delta\mu_s\to 0}{\to} \frac{-\beta_L\Delta\mu_s}{[1 - e^{-\beta_L\omega}][1 - e^{\beta_L\omega}]} \\ &\overset{\Delta\mu_s\to 0}{\to} \frac{\beta_L}{4\sinh^2(\beta_{\rm L}\omega/2)}\Delta\mu_s = \frac{\beta_L}{2[\cosh(\beta_{\rm L}\omega) - 1]}\Delta\mu_s. \end{aligned} \tag{80}$$

$$\begin{aligned} f_{L\uparrow}(E) - f_{L\downarrow}(E+\omega) &= \frac{1}{e^{\beta_L(E-\mu_{L\uparrow})} + 1} - \frac{1}{e^{\beta_L(E+\omega-\mu_{L\downarrow})} + 1} \\ &= \frac{e^{\beta_L(E+\omega-\mu_{L\downarrow})} - e^{\beta_L(E-\mu_{L\uparrow})}}{[e^{\beta_L(E-\mu_{L\uparrow})} + 1][e^{\beta_L(E+\omega-\mu_{L\downarrow})} + 1]} \\ &= \frac{e^{\beta_L(E-\mu_{L\uparrow})}[e^{\beta_L(\omega+\Delta\mu_s)} - 1]}{[e^{\beta_L(E-\mu_{L\uparrow})} + 1]e^{\beta_L(E-\mu_{L\uparrow})}[e^{\beta_L(\omega+\Delta\mu_s)} + e^{-\beta_L(E-\mu_{L\uparrow})}]} \end{aligned} \tag{81}$$

3. linear response regime

In summary, we have

$$f_R(\omega) - f_L^B(\omega) = \frac{-\omega k_B \beta^2 \Delta T + \beta \Delta\mu_s}{2[\cosh(\beta_\mathrm{L}\omega) - 1]} \tag{82}$$

Additionally, when $\omega \to 0$,

$$\lim_{\omega\to 0} \omega f_R(\omega) = \frac{\omega}{e^{\beta_R \omega} - 1} = \frac{1}{\beta_R}.$$

(1) For $\Delta\mu_s = 0$, when $\omega \to 0$,

$$\lim_{\omega\to 0} \omega f_L^B(\omega) = \frac{\omega}{e^{\beta_L(\omega+\Delta\mu_s)} - 1} = \frac{1}{\beta_L}.$$

Then for $\Delta\mu_s = 0$ and limits $\Delta T \to 0$, $\omega \to 0$,

$$\begin{aligned} \omega[f_R(\omega) - f_L^B(\omega)] &= \frac{1}{\beta_R} - \frac{1}{\beta_L} \\ &= -k_B \Delta T. \end{aligned} \tag{83}$$

(2) For $\Delta T = 0$, when $\omega \to 0$ and $\Delta\mu_s \to 0$,

$$\lim_{\omega\to 0} \omega f_L^B(\omega) = \frac{\omega}{e^{\beta_L(\omega+\Delta\mu_s)} - 1} = \frac{1}{\beta_L e^{\beta_L \Delta\mu_s}}.$$

Then for $\Delta T = 0$ and limits $\Delta\mu_s \to 0$, $\omega \to 0$,

$$\omega[f_R(\omega) - f_L^B(\omega)] = \frac{1}{\beta_R} - \frac{1}{\beta_L e^{\beta_L \Delta\mu_s}} \tag{84}$$

E. Ways to reduce time-consuming

1. Low temperature

In transport problem, the Landauer type formula has term of the difference of two Fermionic distribution. Generally, this constrains the range of integrating variable to $[-\frac{T}{2}, \frac{T}{2}]$ or $[\mu_1, \mu_2]$.

2. Physical consideration

Usually, only several subbands or transverse modes are investigated, which suggests the integrating range of $[-\frac{T}{2}, E_{i+1}]$ to include i subbands, with E_i the ith subband energy threshold. When integrating a very high energy,

3. Repalcing repeating calculations by interpolation

If a complex manipulation, like matrix inverse, is contained in a loop, we can take the matrix inverse out of loop, and replace it with an interpolation of inversed matrice calculated earlier. Interpolating by

$$f(c) = \frac{f(a) - f(b)}{a - b} \times (c - a) + f(a). \tag{85}$$

4. $Tr[\Gamma'_{Rq} D_{L\uparrow}(E) D_{L\downarrow}(E+\omega)\Gamma'_{Rq}]$

Γ'_{Rq} is block matrix of dimension $n_{wid}n_{len} \times n_{wid}n_{len}$, with only $I_{n_{wid}\times n_{wid}}$ block. For n_wid=3, n_len=5, we have

$$\Gamma'_L = \begin{bmatrix} \Gamma^L_{3\times3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{86}$$

$$\Gamma'_{Rq} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{3\times3} \end{bmatrix}. \tag{87}$$

$$D_{L\uparrow} = G^r_{L\uparrow}\Gamma_{L\uparrow}G^a_{L\uparrow} = \begin{bmatrix} G^r_{11}\Gamma^L_{3\times3} & 0 & 0 & 0 \\ G^r_{21}\Gamma^L_{3\times3} & 0 & 0 & 0 \\ G^r_{31}\Gamma^L_{3\times3} & 0 & 0 & 0 \\ G^r_{41}\Gamma^L_{3\times3} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G^a_{11} & G^a_{12} & G^a_{13} & G^a_{14} \\ G^a_{21} & G^a_{22} & G^a_{23} & G^a_{24} \\ G^a_{31} & G^a_{32} & G^a_{33} & G^a_{34} \\ G^a_{41} & G^a_{42} & G^a_{43} & G^a_{44} \end{bmatrix}. \tag{88}$$

Then,

$$\Gamma'_{Rq} D_{L\uparrow} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G^r_{41}\Gamma^L_{3\times3}G^a_{11} & G^r_{41}\Gamma^L_{3\times3}G^a_{12} & G^r_{41}\Gamma^L_{3\times3}G^a_{13} & G^r_{41}\Gamma^L_{3\times3}G^a_{14} \end{bmatrix} \tag{89}$$

and

$$D_{L\downarrow}\Gamma'_{Rq} = \begin{bmatrix} 0 & 0 & 0 & \mathrm{x} \\ 0 & 0 & 0 & \mathrm{x} \\ 0 & 0 & 0 & \mathrm{x} \\ 0 & 0 & 0 & \mathrm{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G^r_{11}\Gamma^L_{3\times3}G^a_{14} \\ 0 & 0 & 0 & G^r_{21}\Gamma^L_{3\times3}G^a_{14} \\ 0 & 0 & 0 & G^r_{31}\Gamma^L_{3\times3}G^a_{14} \\ 0 & 0 & 0 & G^r_{41}\Gamma^L_{3\times3}G^a_{14} \end{bmatrix} \tag{90}$$

So

$$\Gamma'_{Rq} D_{L\uparrow} D_{L\downarrow}\Gamma'_{Rq} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A(E,\omega) \end{bmatrix}. \tag{91}$$

$$[A(E,\omega)]_{ij} = G^r_{41}\Gamma^L_{3\times3}G^a_{1i}G^r_{j1}\Gamma^L_{3\times3}G^a_{14}, \tag{92}$$

the advanced Green's function is related to the retarded Green's function by

$$G^a = [G^r]^\dagger, \tag{93}$$

which gives

$$[A(E,\omega)]_{ij} = G^r_{41}\Gamma^L_{3\times3}[G^r]^\dagger_{1i}G^r_{j1}\Gamma^L_{3\times3}[G^r]^\dagger_{14}. \tag{94}$$

Here G^r_{ij} is a 3×3 matrix in full matrix G^r of dimension 12×12, and $\{i,j\}\in[1,2,3,4]$.

$$Tr[A(E,\omega)] = \sum_i G^r_{41}\Gamma^L_{3\times3}[G^r]^\dagger_{1i}G^r_{i1}\Gamma^L_{3\times3}[G^r]^\dagger_{14}. \tag{95}$$

5. *interpolate on $Tr[A(E,\omega)]$*

The interpolation to trA reduces computation from 10000 points to 1000 points.

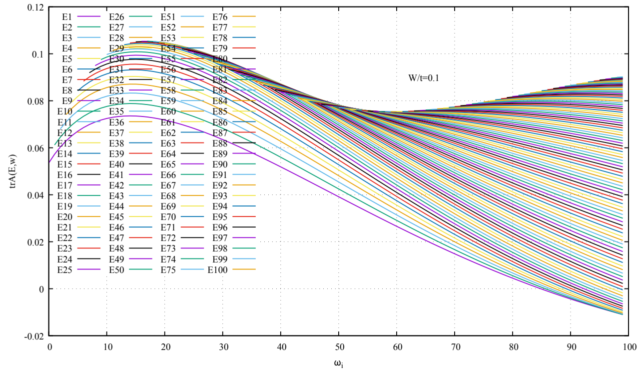

FIG. 1: TrA for the clean NM-NM-FI system.

III. SPIN CURRENT IN ALTERMAGNETIC METAL/ALTERMAGNETIC METAL/FERROMAGNETIC INSULATOR SYSTEM

A. Hamiltonian of a two-terminal AM/AM/FI system

For system consists of an altermagnetic metal (AM) region sandwiched by a left AM lead and a right ferromagnetic insulating (FI) lead, the Hamiltonian is

$$H = H_{\mathrm{L}} + H_{\mathrm{C}} + H_{\mathrm{R}} + H_{\mathrm{T}} + H_{\mathrm{sd}}, \tag{96}$$

in which, the left lead and the central region is described by

$$H_e(\mathbf{k}) = tk^2 + 2t_J k_x k_y \sigma_z. \tag{97}$$

Then we have

$$H_{\mathrm{L}} = \sum_{k\sigma} \epsilon_\sigma(k) c^\dagger_{k\sigma} c_{k\sigma}, \tag{98}$$

$$H_{\mathrm{C}} = \sum_{n\sigma} \varepsilon_{n\sigma} d^\dagger_{n\sigma} d_{n\sigma}, \tag{99}$$

$$H_{\mathrm{R}} \approx \sum_{q} \hbar w_q a^\dagger_q a_q, \tag{100}$$

are Hamiltonians of left lead, central region and right lead respectively. The electron dispersion $\epsilon_\sigma(k) = tk^2 + 2\sigma t_J k_x k_y$, with $\sigma = \pm 1$ for spin-$\uparrow$ and spin-$\downarrow$. H_{T} is the hopping between the left lead and the central region, while H_{sd} is the exchange coupling between right lead and central region,

$$H_{\mathrm{T}} = \sum_{nk\sigma} (t_{nk\sigma} c^\dagger_{k\sigma} d_{n\sigma} + t^*_{nk\sigma} d^\dagger_{n\sigma} c_{k\sigma}), \tag{101}$$

$$H_{\mathrm{sd}} = -\sum_{qnn'} J_{qnn'} a^\dagger_q d^\dagger_{n\uparrow} d_{n'\downarrow} + \text{ h.c.} \tag{102}$$

B. DC spin current of a two-terminal AM/AM/FI system

Since the system Hamiltonian H remains unchanged, the spin current formulas in this system is the same as those in NM/NM/FI system. Left lead electron currents in a two-terminal NM/FI system were

$$I_\uparrow = \int \frac{dE}{2\pi} \mathrm{Tr}[(G^r_\uparrow(E) - G^a_\uparrow(E))\Sigma^<_{L\uparrow}(E) + G^<_\uparrow(E)(\Sigma^a_{L\uparrow}(E) - \Sigma^r_{L\uparrow}(E))], \tag{103}$$

$$I_\downarrow = \int \frac{dE}{2\pi} \mathrm{Tr}[(G^r_\downarrow(E) - G^a_\downarrow(E))\Sigma^<_{L\downarrow}(E) + G^<_\downarrow(E)(\Sigma^a_{L\downarrow}(E) - \Sigma^r_{L\downarrow}(E))]. \tag{104}$$

In above equations,

$$G^r_\sigma = g^r_{d\sigma} + g^r_{d\sigma}\Sigma^r_\sigma G^r_\sigma = 1/[(g^r_{d\sigma})^{-1} - \Sigma^r_\sigma], \tag{105}$$

with $\Sigma^r_\sigma = \Sigma^r_{L\sigma} + \bar{\Sigma}^r_{R\sigma}$ and

$$\begin{aligned} \bar{\Sigma}^<_{R\sigma}(E) &= \int d\omega\ G^r_{L\bar{\sigma}}(\bar{E})[i\Gamma_{L\bar{\sigma}}(\bar{E})f_{L\bar{\sigma}}(\bar{E})]G^a_{L\bar{\sigma}}(\bar{E})(1+f^B_R(\omega))\Gamma_R(\omega), & (106)\\ \bar{\Sigma}^r_{R\sigma}(E) &= \int d\omega G^r_{L\bar{\sigma}}(\bar{E})[f^B_R(\omega) - \Gamma_{L\bar{\sigma}}(\bar{E})f_{L\bar{\sigma}}(\bar{E})G^a_{\bar{\sigma}}(\bar{E})(i/2)]\Gamma_R(\omega). & (107)\end{aligned}$$

Spin current from the right FI lead

$$I^s_R = \int \frac{dE}{R\pi}\mathrm{Tr}[(G^r_\uparrow(E) - G^a_\uparrow(E))\bar{\Sigma}^<_{R\uparrow}(E) + G^<_\uparrow(E)(\bar{\Sigma}^a_{R\uparrow}(E) - \bar{\Sigma}^r_{R\uparrow}(E))]. \tag{108}$$

C. DC spin current expression in four-terminal AM/FI system

Similarly, the electron current flowing out of metallic lead-α ($\alpha = 1, 3, 4$) is given by

$$\begin{aligned} I_{\alpha\uparrow} &= \int \frac{dE}{2\pi}\mathrm{Tr}[(G^r_\uparrow(E) - G^a_\uparrow(E))\Sigma^<_{\alpha\uparrow}(E) + G^<_\uparrow(E)(\Sigma^a_{\alpha\uparrow}(E) - \Sigma^r_{\alpha\uparrow}(E))], & (109)\\ I_{\alpha\downarrow} &= \int \frac{dE}{2\pi}\mathrm{Tr}[(G^r_\downarrow(E) - G^a_\downarrow(E))\Sigma^<_{\alpha\downarrow}(E) + G^<_\downarrow(E)(\Sigma^a_{\alpha\downarrow}(E) - \Sigma^r_{\alpha\downarrow}(E))]. & (110)\end{aligned}$$

In above equations,

$$G^r_\sigma = g^r_{d\sigma} + g^r_{d\sigma}\Sigma^r_\sigma G^r_\sigma = 1/[(g^r_{d\sigma})^{-1} - \Sigma^r_\sigma], \tag{111}$$

with $\Sigma^r_\sigma = \Sigma^r_{1\sigma} + \Sigma^r_{3\sigma} + \Sigma^r_{4\sigma} + \bar{\Sigma}^r_{2\sigma} = \Sigma^r_{\mathrm{M}\sigma} + \bar{\Sigma}^r_{2\sigma}$. In BA, we have

$$\begin{aligned} \bar{\Sigma}^<_{2\sigma}(E) &= \int d\omega\ G^r_{\mathrm{M}\bar{\sigma}}(\bar{E})[\textstyle\sum_{\alpha\neq 2} i\Gamma_{\alpha\bar{\sigma}}(\bar{E})f_{\alpha\bar{\sigma}}(\bar{E})]G^a_{\mathrm{M}\bar{\sigma}}(\bar{E})(1+f^B_2(\omega))\Gamma_2(\omega), & (112)\\ \bar{\Sigma}^r_{2\sigma}(E) &= \int d\omega G^r_{\mathrm{M}\bar{\sigma}}(\bar{E})[f^B_2(\omega) - \textstyle\sum_{\alpha\neq 2}\Gamma_{\alpha\bar{\sigma}}(\bar{E})f_{\alpha\bar{\sigma}}(\bar{E})]G^a_{\mathrm{M}\bar{\sigma}}(\bar{E})(i/2)\Gamma_2(\omega). & (113)\end{aligned}$$

Here $\bar{E} = E - \sigma\omega$. $G^r_{M\sigma} = 1/[(g^r_{d\sigma})^{-1} - \Sigma^r_{\mathrm{M}\sigma}]$ is the Green's function of the central region with the three metallic leads only. For SCBA, the corresponding extrpolation is $G^r_{\mathrm{M}\sigma} \to G^r_\sigma$ and adding $\Sigma^{r,<}_2$ in summation $\sum_\alpha$.

Note Hamiltonian (97) is not in pseudo-spin space, σ_z labels the true spin-space, therefore the spin degree of freedom is spit. $G_\uparrow$, $G_\downarrow$ and $\Sigma_{\alpha\uparrow}$, $\Sigma_{\alpha\downarrow}$ even $g_{d\uparrow}$ and $g_{d\downarrow}$ are no longer degenerate. For a spin-diagonal Hamiltonian (97), we can calculate the Green's function of spin-σ Hamiltonian seperately. The GF of central region with $H_{C\sigma}$ is

$$g^r_{d\sigma} = 1/[E - H_{C\sigma}], \tag{114}$$

where $H_{C\sigma}$ is the submatrix of H_C is spin-space representation. Considering the continuity equity of electron current and spin current, we should have

$$\sum_{\alpha\sigma} I_{\alpha\sigma} = 0, \tag{115}$$

$$\sum_\alpha I^s_\alpha = 0. \tag{116}$$

IV. FORMULAS IN PRB. 88, 220406(R) (2013)

A. Formula 1

System Hamiltonion:

$$H = H_L + H_R + H_{sd}. \tag{117}$$

Left lead is metallic

$$H_L = \sum_{k\sigma}(\varepsilon_{k\sigma} - \mu_\sigma)\, c^\dagger_{k\sigma}c_{k\sigma}, \tag{118}$$

right lead is insulating magnetic

$$H_R \approx \sum_q \hbar w_q a_q^\dagger a_q + \text{ constant } . \tag{119}$$

The interfacial electron-magnon interaction is described by

$$H_{sd} = -\sum_{k,q} J_q \left[S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \tag{120}$$

where $S_q^- \approx \sqrt{2S_0} a_q^\dagger, S_q^+ \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling at the interface. The magnonic spin current operator can be obtained by

$$\hat{I}_S = \frac{d\hat{N}_R}{dt} = \frac{d}{dt} \sum_q a_q^\dagger a_q, \tag{121}$$

the magnonic spin current is obtained by taking average over the nonequilibrium ground state $|\psi_0\rangle$ of the interacting system H:

$$I_S = \frac{dN_R}{dt} = \frac{d}{dt} \langle \sum_q a_q^\dagger a_q \rangle. \tag{122}$$

Using the Heisenberg equation, we get

$$I_S = \frac{i}{\hbar} \langle [H_{sd}, \sum_q a_q^\dagger a_q] \rangle. \tag{123}$$

$$[H_{sd}, \sum_q a_q^\dagger a_q] = [-\sum_{k,q} J_q \left(S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right), \sum_q a_q^\dagger a_q], \tag{124}$$

in which,

$$[a_q^\dagger, \sum_{q'} a_{q'}^\dagger a_{q'}] = \delta_{qq'} [a_q^\dagger, a_{q'}^\dagger a_{q'}] = [a_q^\dagger, a_q^\dagger a_q] = a_q^\dagger [a_q^\dagger, a_q] = -a_q^\dagger. \tag{125}$$

Similarly,

$$[a_q, \sum_{q'} a_{q'}^+ a_{q'}] = [a_q, a_q^\dagger a_q] = a_q. \tag{126}$$

So,

$$\begin{aligned} I_S &= \frac{i}{\hbar} \langle -\sum_{kq} J_q \left(-S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right) \rangle \\ &= \frac{i}{\hbar} \sum_{kq} J_q \left(\langle S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} \rangle - \langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \rangle \right). \end{aligned} \tag{127}$$

B. Formula 2

$$\frac{d}{dt} \left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right\rangle = \frac{i}{\hbar} \left\langle \left[H_L + H_{sd} + H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \right\rangle. \tag{128}$$

The rhs. of eq. (128) is decomposed into 3 terms. The first term reads

$$\begin{aligned} \left\langle \left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \right\rangle &= [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma) c_{k'\sigma}^\dagger c_{k'\sigma}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}] \\ &= S_q^+ [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma) c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] c_{k\uparrow} \\ &\quad + S_q^+ c_{k+q\downarrow}^\dagger [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma) c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}] \end{aligned} \tag{129}$$

Note that,

$$[\sum_{k'\sigma} c^{\dagger}_{k'\sigma}c_{k'\sigma}, c^{\dagger}_{k+q\downarrow}] = \sum_{k'\sigma} c^{\dagger}_{k'\sigma}\delta_{k',k+q}\delta_{\sigma\downarrow} = c^{\dagger}_{k+q\downarrow}, \tag{130}$$

$$[\sum_{k'\sigma} c^{\dagger}_{k'\sigma}c_{k'\sigma}, c_{k\uparrow}] = -\sum_{k'\sigma}\{c^{\dagger}_{k'\sigma}, c_{k\uparrow}\}c_{k'\sigma} = -c_{k\uparrow}. \tag{131}$$

Eq. (130) (131) are derived using equity

$$\begin{aligned}[AB, C] &= A[B, C] + [A, C]B \\ &= A\{B, C\} - \{A, C\}B.\end{aligned} \tag{132}$$

So,

$$\left[H_L, S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} + \mu_{\uparrow} - \mu_{\downarrow}\right) S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}. \tag{133}$$

If $\mu_{\uparrow} = \mu_{\downarrow}$, then eq. (133) reduces to

$$\left[H_L, S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow}\right) S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}. \tag{134}$$

The second term in eq. (128) reads

$$\begin{aligned}\left[H_{sd}, S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right] &= -J_q\left[S^{-}_{q}c^{\dagger}_{k\uparrow}c_{k+q\downarrow}, S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right] \\ &= J_q\left[S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}, S^{-}_{q}c^{\dagger}_{k\uparrow}c_{k+q\downarrow}\right]\end{aligned} \tag{135}$$

The third term $H_R = \sum_q \hbar w_q a^{\dagger}_q a_q$, then using eq. (126), we get

$$\left[H_R, S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right] = -\hbar\omega_q S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}. \tag{136}$$

Combine these three terms, we get

$$\begin{aligned}\frac{d}{dt}\left\langle S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right\rangle = &\frac{i}{\hbar}\left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} - \hbar\omega_q\right)\left\langle S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}\right\rangle \\ &+ \frac{i}{\hbar}J_q\left\langle\left[S^{+}_{q}c^{\dagger}_{k+q\downarrow}c_{k\uparrow}, S^{-}_{q}c^{\dagger}_{k\uparrow}c_{k+q\downarrow}\right]\right\rangle,\end{aligned} \tag{137}$$

which is also eq. (2) in supplementary material of PRB. 88, 220406(R) (2013).

V. NONEQUILIBRIUM GREEN'S FUNCTION TECHNIQUE

A. Demonstrative Hamiltonian

$$\hat{H} = H_{lead} + H_{dot} + H_T \tag{138}$$

$$H_{lead} = \sum_{k\alpha}\epsilon_{k\alpha}\hat{C}^{\dagger}_{k\alpha}\hat{C}_{k\alpha} \tag{139}$$

$$\epsilon_{k\alpha} = \epsilon^{(0)}_{k\alpha} + qv_{\alpha} \tag{140}$$

$$H_{dot} = \sum_{n}\left(\epsilon_n + qU_n\right)d^{\dagger}_n d_n \tag{141}$$

$$U_n = \sum_m V_{nm} < d^{\dagger}_m d_m > \tag{142}$$

$$H_T = \sum_{k\alpha n}\left[t_{k\alpha n}\hat{C}^{\dagger}_{k\alpha}\hat{d}_n + t^{*}_{k\alpha n}\hat{d}^{\dagger}_n\hat{C}_{k\alpha}\right] \tag{143}$$

B. Current definition

We use the Hamiltonian in WangJian's notes. Equation of motion of particle operator $\hat{N}_{\alpha k\sigma}$ in the lead α is

$$\begin{aligned}\frac{d}{dt}\hat{N}_\alpha &= \frac{i}{\hbar}[H,\sum_k c^\dagger_{\alpha k}c_{\alpha k}] = \left[\sum_{k'n,\alpha'=L,R}\left[t_{k'\alpha'}c^\dagger_{k'\alpha'}d_n + \text{c.c.}\right],\sum_k c^\dagger_{\alpha k}c_{\alpha k}\right] \\ &= \frac{i}{\hbar}\sum_{kk',n,\alpha'=L,R}\left[-t_{k'\alpha'}c^\dagger_{k'\alpha'}d_n\delta_{\alpha\alpha'}\delta_{kk'} + \text{c.c.}\right] \\ &= \frac{i}{\hbar}\sum_{kn}[-t_{k\alpha}c^\dagger_{k\alpha}d_n + t^*_{k\alpha}d^\dagger_n c_{k\alpha}]\end{aligned} \tag{144}$$

So, the charge current is given by

$$\begin{aligned}I_\alpha(t) &= e\langle\frac{d}{dt}\hat{N}_\alpha(t)\rangle \\ &= \frac{ie}{\hbar}\sum_{kn}(\langle -t_{k\alpha}c^\dagger_{k\alpha}(t)d_n(t)\rangle + \langle t^*_{k\alpha}d^\dagger_n(t)c_{k\alpha}(t)\rangle)\end{aligned} \tag{145}$$

Define the lesser Green's function

$$G^<_{\sigma',k\alpha\sigma}(t,t') = i\langle c^\dagger_{k\alpha\sigma}(t')d_{\sigma'}(t)\rangle \tag{146}$$

the charge current is written as

$$I_L(t) = \frac{-e}{\hbar}\sum_{kn\alpha\in L}(t_{k\alpha n}G^<_{n,k\alpha\sigma}(t,t) - t^*_{k\alpha n}G_{k\alpha,n}(t,t)\rangle) \tag{147}$$

More generally, we define the contour Green's function

$$G_{n,k\alpha}(\tau,\tau') = -i\langle d_n(\tau)c^\dagger_{k\alpha}(\tau')\rangle. \tag{148}$$

Following Jauho's notation [2], when the electron in the lead is non-interacting, $\mathrm{G}_{n,k\alpha\sigma}(\tau,\tau')$ is related to G_{nm} and $g_{k\alpha}$ by the following contour integral

$$G_{n,k\alpha}(\tau,\tau') = \sum_m\int d\tau_1 G_{nm}(\tau,\tau_1)\,t^*_{k\alpha m}g_{k\alpha}(\tau_1,\tau') \tag{149}$$

where

$$G_{nm}(\tau_1,\tau_2) \equiv -i\langle T_c\left[d_n(\tau_1)\,d^\dagger_m(\tau_2)\right]\rangle \tag{150}$$

$$g_{k\alpha}(\tau_1,\tau_2) \equiv -i\langle T_c\left[c_{k\alpha}(\tau_1)\,c^\dagger_{k\alpha}(\tau_2)\right]\rangle_0. \tag{151}$$

Using the theorem of analytic continuation, we have

$$\begin{aligned}G^<_{n,k\alpha}(t,t') = \sum_m\int dt_1 &\left[G^r_{nm}(t,t_1)\,t^*_{k\alpha m}g^<_{k\alpha}(t_1,t')\right. \\ &\left.+G^<_{nm}(t,t_1)\,t^*_{k\alpha m}g^a_{k\alpha}(t_1,t')\right].\end{aligned} \tag{152}$$

This gives the term in current

$$\begin{aligned}&\textstyle\sum_{kn}t_{k\alpha n}G^<_{n,k\alpha}(t,t') = \sum_{kmn}\int dt_1 t_{k\alpha n}t^*_{k\alpha m} \\ &\times\left[G^r_{nm}(t,t_1)\,g^<_{k\alpha}(t_1,t') + G^<_{nm}(t,t_1)\,g^a_{k\alpha}(t_1,t')\right] \\ &= \textstyle\sum_n\int dt_1\left[G^r(t,t_1)\,\Sigma^<_\alpha(t_1,t') + G^<(t,t_1)\,\Sigma^a_\alpha(t_1,t')\right]_{nn}\end{aligned} \tag{153}$$

matrix element of the self-energy Σ_α due to lead α is

$$\Sigma^{\gamma}_{\alpha,mn}(t_1,t_2) = \sum_k t^*_{k\alpha m}(t_1) g^{\gamma}_{k\alpha}(t_1,t_2) t_{k\alpha n}(t_2). \tag{154}$$

Here, the matrix index are m, n, which is index for energy level of central scattering area. Substitute **??** in charge current, we have

$$\begin{aligned} I_\alpha(t) = -\frac{e}{\hbar}\int dt_1 \operatorname{Tr}\big[G^r(t,t_1)\,\Sigma^<_\alpha(t_1,t) \\ +G^<(t,t_1)\,\Sigma^a_\alpha(t_1,t)\big] + h.c. \end{aligned} \tag{155}$$

where the summation over index n is abbreviated in to matrix summation notation Tr, and summation index k goes into self-energy matrix Σ_α.

C. Free propagators

Here we assume a time-dependent external voltage v_α. The free Green's functions of lead electrons are (XXX)

$$g^<_{k\sigma}(t,t') \equiv i\left\langle c^\dagger_{k\sigma}(t')\, c_{k\sigma}(t)\right\rangle = i f(\varepsilon^{(0)}_k) e^{-i\int_{t'}^{t} dt_1 \varepsilon_{k\sigma}(t_1)} \tag{156}$$

$$g^>_{k\sigma}(t,t') \equiv -i\left\langle c_{k\sigma}(t) c^\dagger_{k\sigma}(t')\right\rangle = i\left[f(\varepsilon_k) - 1\right] e^{-i\varepsilon_{k\sigma}(t-t')} \tag{157}$$

$$g^r_{k\sigma}(t) \equiv -i\theta(t)\left\langle \left[c_{k\sigma}(t), c^\dagger_{k\sigma}(t')\right]_+\right\rangle = -i\theta(t) e^{-i\varepsilon_{k\sigma}(t-t')} \tag{158}$$

$$g^a_{k\sigma}(t) \equiv i\theta(-t)\left\langle \left[c_{k\sigma}(t), c^\dagger_{k\sigma}(t')\right]_+\right\rangle = i\theta(-t) e^{-i\varepsilon_{k\sigma}(t-t')} \tag{159}$$

Using the relation

$$\int dt e^{i\omega t} = 2\pi\delta(\omega), \tag{160}$$

Fourier transformation gives

$$g^<_{k\sigma}(\omega) = 2\pi i f(\varepsilon_{k\sigma})\,\delta(\omega - \varepsilon_{k\sigma}) = i f(\varepsilon_{k\sigma})\, A_0(k,\omega) \tag{161}$$

$$g^>_{k\sigma}(\omega) = 2\pi i\left[f(\varepsilon_{k\sigma}) - 1\right]\delta(\omega - \varepsilon_{k\sigma}) \tag{162}$$

$$g^r_{k\sigma}(\omega) = -i\int_{-\infty}^{\infty} dt e^{i\omega t}\theta(t) e^{-i\epsilon_{k\sigma} t} = -i\int_0^{\infty} dt e^{i(\omega-\epsilon_{k\sigma})t} = \frac{-i}{i(\omega-\varepsilon_{k\sigma})} e^{i(\omega-\varepsilon)}\big|_0^{+\infty} \tag{163}$$

To make the integral converge at the upper limit, we let $\omega \to \omega + i0^+$, where 0^+ is a positive infinitesimal, which yields

$$g^r_{k\sigma}(\omega) = \frac{1}{\omega - \varepsilon_{k\sigma} + i0^+}. \tag{164}$$

Similarly,

$$g^a_{k\sigma}(\omega) = \frac{1}{\omega - \varepsilon_{k\sigma} - i0^+}. \tag{165}$$

Then we have

$$g_{k\sigma}^{r}(\omega)-g_{k\sigma}^{a}(\omega)=-2\pi i\delta(\omega-\varepsilon_{k\sigma}) \tag{166}$$

The fermion spectral function is defined as

$$\begin{aligned}A_0(k\sigma,\omega) &= i\left[g_{k\sigma}^{r}(\omega)-g_{k\sigma}^{a}(\omega)\right]\\ &= -2\Im\mathrm{m}\left[g_{k\sigma}^{r}(\omega)\right]\\ &= 2\pi\delta\left(\omega-\varepsilon_{k\sigma}\right)\end{aligned} \tag{167}$$

where the following relation are used

$$\frac{1}{x\pm i\eta}=P\frac{1}{x}\mp i\pi\delta(x),\quad \eta=0^+, \tag{168}$$

$$\Im\mathrm{m}\left[g_{k\sigma}^{r}(\omega)\right]=-\pi\delta(\omega-\varepsilon_k). \tag{169}$$

D. DC case

$$G^{\gamma}\left(t,t_1\right)=G^{\gamma}\left(t-t_1\right) \tag{170}$$

and

$$\Sigma^{\gamma}\left(t,t_1\right)=\Sigma^{\gamma}\left(t-t_1\right) \tag{171}$$

where

$$\gamma=<,>,r,a. \tag{172}$$

Recall that

$$\left[G^{<}\right]^{\dagger}(E)=-G^{<}(E) \tag{173}$$

$$\left[G^{r}\right]^{\dagger}=G^{a} \tag{174}$$

and using equation (221) in WangJian's note, we have charge current for DC bias

$$\begin{aligned}I_\alpha=-\frac{e}{\hbar}\int\frac{dE}{2\pi}\,\mathrm{Tr}&\left[\left(G^r(E)-G^a(E)\right)\Sigma_\alpha^{<}(E)\right.\\ &\left.+G^{<}(E)\left(\Sigma_\alpha^a(E)-\Sigma_\alpha^r(E)\right)\right]\end{aligned} \tag{175}$$

Substitute free propagators in, we have

$$\Sigma_{\alpha,mn}^{<}(t-t_1)=\sum_k t_{k\alpha m}^{*}(t_1)g_{k\alpha}^{<}(t_1-t_2)t_{k\alpha n}(t_2)=i\sum_k t_{k\alpha m}^{*}(t_1)f(\epsilon_{k\alpha})e^{-i\varepsilon_{k\alpha(t-t_1)}}t_{k\alpha n}(t_2) \tag{176}$$

Fourier transformation gives (dependent variable $\epsilon_{k\alpha}$ not ω?, check Eq.(71) in WangJ's note Chap2?)

$$\Sigma_{\alpha,mn}^{<}(E)=2\pi i\sum_k t_{k\alpha m}^{*}f(\varepsilon_{k\alpha})t_{k\alpha n}\delta(E-\varepsilon_{k\alpha}) \tag{177}$$

$$\Sigma_\alpha^a(E)-\Sigma_\alpha^r(E)=\sum_k t_{k\alpha m}^{*}(g_{k\alpha}^{a}(E)-g_{k\alpha}^{r}(E))t_{k\alpha n} \tag{178}$$

which according to Eq. (166), we have

$$\Sigma_\alpha^a(E)-\Sigma_\alpha^r(E)=2\pi i\sum_k t_{k\alpha m}^{*}\delta(E-\epsilon_{k\alpha})t_{k\alpha n}. \tag{179}$$

Define a level-width function:

$$\Gamma_{\alpha,mn}(E) = \sum_k 2\pi t^*_{k\alpha m} t_{k\alpha n} \delta\left(E - \varepsilon_{k\alpha}\right) \tag{180}$$

So it gives equations(the fermion distribution is factorized out of summation k?)

$$\Sigma^<_{\alpha,mn}(E) = i f(\varepsilon_{k\alpha}) \Gamma_{\alpha,mn}(E) \tag{181}$$

and

$$\Sigma^a_\alpha(E) - \Sigma^r_\alpha(E) = i\Gamma_{\alpha,mn}(E) \tag{182}$$

Then Eq. (175) can be written as

$$\begin{aligned} I_\alpha =& -\frac{e}{\hbar}\int \frac{dE}{2\pi} \operatorname{Tr}\left[\left(G^r(E) - G^a(E)\right)\left(i f(\varepsilon_{k\alpha} \Gamma_{\alpha,mn}(E))\right)\right. \\ & + G^<(E)(i\Gamma_{\alpha,mn}(E)) \\ =& -\frac{ie}{\hbar}\int \frac{dE}{2\pi} \operatorname{Tr}\left[\Gamma_{\alpha,mn}(E)([G^r(E) - G^a(E)]\, f(\varepsilon_{k\alpha}) + G^<(E))\right] \end{aligned} \tag{183}$$

In steady state, $I = I_L = -I_R$, or $I = I_L + I_R = (I_L - I_R)/2$, this leads to the general expression for the dc-current

$$\begin{aligned} I = & -\frac{\mathrm{i}e}{2\hbar}\int \frac{\mathrm{d}\varepsilon}{2\pi} \operatorname{Tr}\left\{\left[\mathbf{\Gamma}^L(\varepsilon) - \mathbf{\Gamma}^R(\varepsilon)\right]\mathbf{G}^<(\varepsilon)\right. \\ & \left. + \left[f_L(\varepsilon)\mathbf{\Gamma}^L(\varepsilon) - f_R(\varepsilon)\mathbf{\Gamma}^R(\varepsilon)\right]\left[\mathbf{G}^{\mathrm{r}}(\varepsilon) - \mathbf{G}^{\mathrm{a}}(\varepsilon)\right]\right\} \end{aligned} \tag{184}$$

if the left and right line-width functions are proportional to each other,

$$\mathbf{\Gamma}^L(\varepsilon) = \lambda \mathbf{\Gamma}^R(\varepsilon) \tag{185}$$

and fix the arbitrary parameter x, i.e. $x = 1/(1+\lambda)$, gives

$$\begin{aligned} J &= \frac{1e}{\hbar}\int \frac{\mathrm{d}\varepsilon}{2\pi}\left[f_L(\varepsilon) - f_R(\varepsilon)\right]\mathcal{T}(\varepsilon) \\ \mathcal{T}(\varepsilon) &= \operatorname{Tr}\left\{\frac{\Gamma^L(\varepsilon)\Gamma^R(\varepsilon)}{\Gamma^L(\varepsilon) + \Gamma^R(\varepsilon)}\left[\mathbf{G}^{\mathrm{r}}(\varepsilon) - \mathbf{G}^{\mathrm{a}}(\varepsilon)\right]\right\} \end{aligned} \tag{186}$$

Despite the apparent similarity of (12.27) to the Landauer formula, it is important to bear in mind that, in general, there is no immediate connection between the quantity $\mathcal{T}(\varepsilon)$ and the transmission coefficient $T(\varepsilon)$.

E. Another way to get $G_{n,k\alpha}(\tau,\tau')$(Dyson equation + Keldysh equation)

Denote G_0 the Green's function of the isolated quantum dot and leads corresponding to the Hamiltonian H_0, and G the Green's function of the open system corresponding to H, one has the Dyson equation

$$G = G_0 + G_0\Sigma G \tag{187}$$

Use the theorem of analytic continuation on Dyson equation, we get the Keldysh equation (in matrix representation)

$$G^{<,>} = G_0^{<,>} + G^r\Sigma^r G_0^{<,>} + G^{<,>}\Sigma^a G_0^a + G^r\Sigma^{<,>}G_0^a \tag{188}$$

or

$$G^{<,>} = G_0^{<,>} + G_0^r\Sigma^r G^{<,>} + G_0^{<,>}\Sigma^a G^a + G_0^r\Sigma^{<,>}G^a \tag{189}$$

or

$$G^< = G^r\left(G_0^r\right)^{-1} G_0^< \left(G_0^a\right)^{-1} G^a + G^r\Sigma^< G^a \tag{190}$$

See Eq. (77) in WangJian's notes.

F. With spin index

The demonstrative current of lead β with spin σ is [**?**]

$$I_{\beta\sigma} = \frac{e}{h}\sum_{k,i,j}\int d\omega V_{ki\beta\sigma}V^*_{kj\beta\sigma}\Big\{\left[G^r_{i\sigma,j\sigma}(\omega) - G^a_{i\sigma,j\sigma}(\omega)\right]g^<_{k\beta\sigma}(\omega) - \left[g^r_{k\beta\sigma}(\omega) - g^a_{k\beta\sigma}(\omega)\right]G^<_{i\sigma,j\sigma}(\omega)\Big\}. \tag{191}$$

Substitute free propagators into current formula, we have

$$I_{\beta\sigma} = \frac{ie}{h}\sum_{i,j}\int d\omega\Gamma_{ij\beta\sigma}(\omega)\left\{\left[G^r_{i\sigma,j\sigma}(\omega) - G^a_{i\sigma,j\sigma}(\omega)\right]f_\beta(\omega) + G^<_{i\sigma,j\sigma}(\omega)\right\} \tag{192}$$

self-energy of lead α is

$$\Sigma^<_\alpha(\omega) = i\Gamma_\alpha\left(\omega - qv_\alpha\right)f_\alpha(\omega) \tag{193}$$

VI. NOTES ON PROGRAMS

A. NM/10x10SC/NM

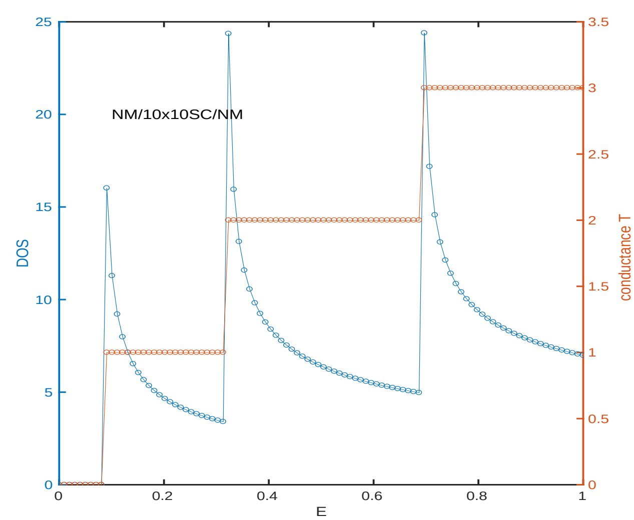

FIG. 2: NM/10x10/NM system, 2D simple cubic lattice connected to two normal metal leads.

B. NM/10x10SC/MI

C. Perturbation expansion of $G_d(\tau,\tau)$

When neglecting left lead, the hamiltonian is

$$H = H_{QD} + H_R + H_{sd} \tag{194}$$

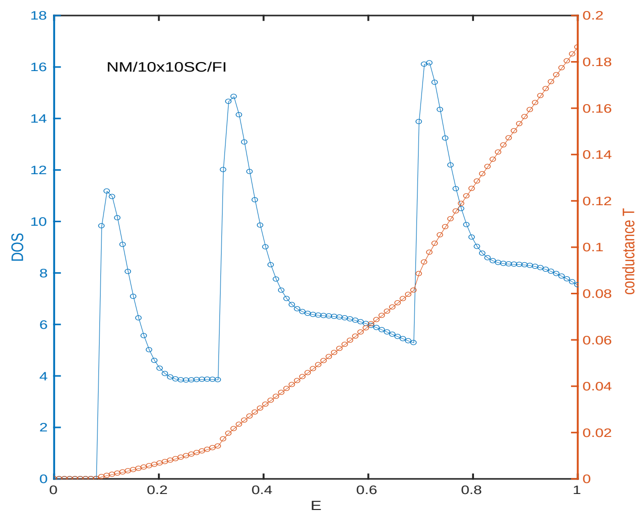

FIG. 3: NM/10x10/MI system, 2D simple cubic lattice connected to a normal metal lead and a magnetic insulator lead with Ohmic spectra.

Expand S-matrix up to the second order of J, we have

$$\begin{aligned}
&G_d(\tau,\tau) = -i\langle T_C S s_q^+(\tau) s_q^-(\tau')\rangle \\
=&-i\sum_k g_{k\uparrow}(\tau',\tau) g_{k+q\downarrow}(\tau,\tau') \\
+&\int_c d\tau_1 \int_c d\tau_2 \sum_{kq_1} J_{q_1}^2 g_{Rq_1}(\tau_2,\tau_1) g_{k\uparrow}(\tau',\tau_2) g_{k+q_1\downarrow}(\tau_2,\tau_1) g_{k\uparrow}(\tau_1,\tau) g_{k+q\downarrow}(\tau,\tau') \\
+&\int_c d\tau_1 \int_c d\tau_2 \sum_{kq_1} J_{q_1}^2 g_{Rq_1}(\tau_2,\tau_1) g_{k\uparrow}(\tau',\tau) g_{k+q\downarrow}(\tau,\tau_1) g_{k+q-q_1\uparrow}(\tau_1,\tau_2) g_{k+q\downarrow}(\tau_2,\tau') \\
-&\int_c d\tau_1 \int_c d\tau_2 \sum_{kk'} J_q^2 g_{Rq}(\tau_2,\tau_1) g_{k\uparrow}(\tau_1,\tau) g_{k+q\downarrow}(\tau,\tau_1) g_{k'\uparrow}(\tau',\tau_2) g_{k'+q\downarrow}(\tau_2,\tau')
\end{aligned} \tag{195}$$

VII. TIGHT-BINDING METHOD

Tight-binding coupling t is

$$t = \frac{\hbar^2}{2ma^2},$$

in which m is the effective mass of election in the lattice, assuming $m = 0.08m_e$, which is mass of electron. a is lattice distance. $\hbar = 1.0545e-34J\cdot s, m_e = 9.10938370e-31$kg, so t is in unit of J.

Boltzman constant $k_B = 1.3806504^{-23}J/K, e = 1.602176634^{-19}C, \hbar = 1.0545e-34J\cdot s, m_e = 9.10938370e-31$kg

A. Transportation in a electron wave guide

An electron wave guide is a device analogous to light wave guide, in which only small number of electron wave modes can propagate. Reference to exercise 1.3 and 1.4 in S. Datta's book. For case one, in y direction, the wave guide is constrained in a hard-well potential. $U(y<-W/2) = U(y>W/2) = \infty, U(-W/2<y<W/2) = 0$, leads to the quantization of electron states.

$$k_y = \frac{i\pi}{W}, \qquad \text{for i is integers.} \tag{196}$$

W is the width of wave guide and central area, a is lattice constant of central lattice, n_{wid} is number of lattice points in y direction.

$$W = n_{\text{wid}} \times a \tag{197}$$

To get a propagate wave instead of a decaying wave, the k_x must be a real number, or $k_x^2 > 0$. The total injection energy of an election is

$$E = \frac{\hbar^2(k_x^2 + k_y^2)}{2m} = \frac{\hbar^2 k_x^2}{2m} + \frac{i^2\hbar^2\pi^2}{2mW^2}. \tag{198}$$

So the threshold for ith subband or transverse mode is

$$E_i = \frac{i^2\hbar^2\pi^2}{2mW^2}, \tag{199}$$

which is 0.537 meV for first subband, effective mass $m = 0.07m_e$, and width $W = 100nm$.

APPENDIX A: ANALYTIC CONTINUATION

We list here all the analytic continuations used in this work. For $C = AB$ (matrix multiplication), we have[?]

$$C^< = A^r B^< + A^< B^a, \quad \text{and} \quad C^> = A^r B^> + A^> B^a \tag{200}$$

and

$$C^r = A^r B^r, \quad \text{and} \quad C^a = A^a B^a \tag{201}$$

For $C(\tau, \tau') = A(\tau, \tau')B(\tau, \tau')$ or $C = A.B$ (the Hadamard matrix product), we have[?]

$$C^< = A^<.B^<, \quad \text{and} \quad C^> = A^>.B^> \tag{202}$$

and

$$\begin{aligned} C^r &= A^r.B^< + A^<.B^r + A^r.B^r, \\ C^a &= A^a.B^< + A^<.B^a + A^a.B^a \end{aligned} \tag{203}$$

For $C(\tau, \tau') = A(\tau, \tau')B(\tau', \tau)$ or $C = A.\tilde{B}$ where $\tilde{B}(t_1, t_2) \equiv B(t_2, t_1)$, we have[?]

$$C^< = A^<.\tilde{B}^>, \quad \text{and} \quad C^> = A^>.\tilde{B}^< \tag{204}$$

$$\begin{aligned} C^r &= A^<.\tilde{B}^a + A^r.\tilde{B}^<, \\ C^a &= A^<.\tilde{B}^r + A^a.\tilde{B}^< \end{aligned} \tag{205}$$

From Eqs. (204) and (205), one can easily check the relation $C^> - C^< = C^r - C^a$ which must be satisfied.

APPENDIX B: DERIVATION OF DECOUPLED GF

To show Eq.(??), we follow exactly the derivation of Ref.? p188. We examine jth order term in the S-matrix expansion and use Wick theorem,

$$\begin{aligned} &\langle T_c s_q^+(\tau) a_q^\dagger(\tau') [J_q a_q(\tau_1) s_q^-(\tau_1) + h.c.] ... [J_q a_q(\tau_j) s_q^-(\tau_j) + h.c.] \rangle \\ &= \langle T_c a_q(\tau_1) a_q^\dagger(\tau') \rangle J_q \langle T_c s_q^+(\tau) s_q^-(\tau_1) [J_q a_q(\tau_2) s_q^-(\tau_2) + h.c.] ... [J_q a_q(\tau_j) s_q^-(\tau_j) + h.c.] \rangle \\ &+ \langle T_c a_q(\tau_2) a_q^\dagger(\tau') \rangle J_q \langle T_c s_q^+(\tau) s_q^-(\tau_2) [J_q a_q(\tau_1) s_q^-(\tau_1) + h.c.] ... [J_q a_q(\tau_j) s_q^-(\tau_j) + h.c.] \rangle \\ &+ \text{remaining j} - 2 \text{ terms} \end{aligned} \tag{206}$$

Note that there are j terms that are all the same. The extra factor of j combined with $(-i)^j/j!$ giving rise to $-i(-i)^{j-1}/(j-1)!$ and we can reconstruct a new S-matrix. Remember that there is a factor of $(-i)^j$ in the jth expansion of S-matrix, hence we have an extra $-i$ in S-matrix reconstruction. Notice that we did not do anything for $j = 0$ term in S-matrix expansion, we then have

$$G_{d,R}(\tau, \tau') = g_{d,R}(\tau, \tau') + J_q \int G_d(\tau, \tau_1) g_{Rq}(\tau_1, \tau') d\tau_1 \tag{207}$$

where $g_{d,R}(\tau, \tau') = -i\langle T_c s_q^+(\tau) a_q^\dagger(\tau') \rangle$ is simply zero.

APPENDIX C: DERIVATION OF $\Sigma_{2\sigma}^{r,<}$

$\bar{\Sigma}_{R\sigma}(\tau_1, \tau_2) = i \sum_q G_{L\bar{\sigma}}(\tau_1, \tau_2) \Sigma_{Rq}(\tau_2, \tau_1)$. Analytic continuation gives

$$\bar{\Sigma}_{R\sigma}^r(t_1, t_2) = i \int dt G_{L\bar{\sigma}}^r(t_1, t') \Sigma_{Rq}^r(t', t_2). \tag{208}$$

After Fourier transformation, we have

APPENDIX D: CONTINUATION ON EQ. (55)

Eq. (55) is

$$G_d(\tau,\tau') = -iG_{L\uparrow}(\tau',\tau)G_{L\downarrow}(\tau,\tau') - iG_{L\uparrow}(\tau_1,\tau)G_{L\downarrow}(\tau,\tau_1)\Sigma_{Rq}(\tau_1,\tau_2)G_d(\tau_2,\tau'). \tag{209}$$

We define

$$A(\tau_1,\tau') \equiv \int d\tau_2 \Sigma_R\left(\tau_1,\tau_2\right) G_d\left(\tau_2,\tau'\right) \tag{210}$$

$$B(\tau,\tau_1) \equiv G_{L\uparrow}\left(\tau_1,\tau\right) G_{L\downarrow}\left(\tau,\tau_1\right) \tag{211}$$

$$C(\tau,\tau') \equiv -iG_{L\uparrow}\left(\tau_1,\tau\right) G_{L\downarrow}\left(\tau,\tau_1\right) A(\tau_1,\tau') \rightarrow \tag{212}$$

$$C(\tau,\tau') = -i\int d\tau_1 B(\tau,\tau_1)A(\tau_1,\tau') \tag{213}$$

$$D(\tau,\tau') \equiv -iG_{L\uparrow}\left(\tau',\tau\right) G_{L\downarrow}\left(\tau,\tau'\right) \tag{214}$$

So, we have

$$G_d(\tau,\tau') = D + C \tag{215}$$

Using the analytic continuation theorem, we have

$$D^{<} = -iG^{>}_{L\uparrow}G^{<}_{L\downarrow} \tag{216}$$

$$C^{<} = -i(B^r A^{<} + B^{<} A^a) \tag{217}$$

where

$$B^r = G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow} \tag{218}$$

$$A^{<} = \Sigma^r_R G^{<}_d + \Sigma^{<}_R G^a_d \tag{219}$$

$$B^{<} = G^{>}_{L\uparrow}G^{<}_{L\downarrow} \tag{220}$$

$$A^a = \Sigma^a_R G^a_d \tag{221}$$

Then, the analytic continuation theorem on Eq.(40) yields

$$G^{<}_d = -iG^{>}_{L\uparrow}G^{<}_{L\downarrow} - i\left[(G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^{<}_d + \Sigma^{<}_R G^a_d) + (G^{>}_{L\uparrow}G^{<}_{L\downarrow})(\Sigma^a_R G^a_d)\right] \tag{222}$$

Similarly,

$$\begin{aligned} C^r &= -iB^r A^r \\ &= -i(G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^r_d) \end{aligned} \tag{223}$$

$$D^r = -i(G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow}), \tag{224}$$

we have

$$\begin{aligned} G^r_d &= -i(G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow}) - i(G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^r_d) \\ &= \frac{-i\left(G_{L\uparrow}G_{L\downarrow}\right)^r}{1 + i\left(G_{L\uparrow}G_{L\downarrow}\right)^r \Sigma^r_R} \end{aligned} \tag{225}$$

$$\left(G_{L\uparrow}G_{L\downarrow}\right)^r \equiv G^a_{L\uparrow}G^{<}_{L\downarrow} + G^{>}_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow} \tag{226}$$

Now we calculate G_d^a.

$$C^a = -iB^a A^a \tag{227}$$

$$B^a = G_{L\uparrow}^r G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^a + G_{L\uparrow}^r G_{L\downarrow}^a \tag{228}$$

$$D^a = -i(G_{L\uparrow}^r G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^a + G_{L\uparrow}^r G_{L\downarrow}^a) \tag{229}$$

So we have

$$G_d^a = -i(G_{L\uparrow}^r G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^a + G_{L\uparrow}^r G_{L\downarrow}^a) - i(G_{L\uparrow}^r G_{L\downarrow}^< + G_{L\uparrow}^> G_{L\downarrow}^a + G_{L\uparrow}^r G_{L\downarrow}^a)(\Sigma_R^a G_d^a) \tag{230}$$

From Eq.(222) we have

$$\begin{aligned}
G_d^< &= -iG_{L\uparrow}^> G_{L\downarrow}^< \left(1 + \Sigma_R^a G_d^a\right) - i\left(G_{L\uparrow} G_{L\downarrow}\right)^r \left(\Sigma_q^< G_d^a + \Sigma_R^r G_d^<\right) \\
&= \frac{-iG_{L\uparrow}^> G_{L\downarrow}^< \left(1 + \Sigma_R^a G_d^a\right) - i\left(G_{L\uparrow} G_{L\downarrow}\right)^r \Sigma_R^< G_d^a}{1 + i\left(G_{L\uparrow} G_{L\downarrow}\right)^r \Sigma_R^r} \\
&= \frac{-iG_{L\uparrow}^> G_{L\downarrow}^< \left(1 + \Sigma_R^a G_d^a\right)}{1 + i\left(G_{L\uparrow} G_{L\downarrow}\right)^r \Sigma_R^r} + G_d^r \Sigma_R^< G_d^a \\
&= -i(G_d^r \Sigma_R^r + 1) G_{L\uparrow}^> G_{L\downarrow}^< \left(1 + \Sigma_R^a G_d^a\right) + G_d^r \Sigma_R^< G_d^a
\end{aligned} \tag{231}$$

Similarly,

$$G_d^> = -i\left(G_d^r \Sigma_R^r + 1\right) G_{L\uparrow}^< G_{L\downarrow}^> \left(1 + \Sigma_R^a G_d^a\right) + G_d^r \Sigma_R^> G_d^a \tag{232}$$

[1] Y, K, Kato. Observation of the Spin Hall Effect in Semiconductors[J]. Science, 2004.
[2] Antti-Pekka Jauho, Quantum Kinetics in Transport and Optics of Semiconductors, P188.