Spin current in NM-QD-MIL system

For system consists of quantum dot(QD) sandwiched by a left normal metal(NM) lead and a right magnetic insulating lead(MIL), the Hamiltonian is

$$H = H_{\mathrm{L}} + H_{\mathrm{QD}} + H_{\mathrm{R}} + H_{\mathrm{T}}. \tag{1}$$

$$H_{\mathrm{L}} = \sum_{k\sigma} \varepsilon_{k\sigma} c^{\dagger}_{k\sigma} c_{k\sigma}, \tag{2}$$

$$H_{\mathrm{QD}} = \sum_{\sigma} \varepsilon_{\sigma} d^{\dagger}_{\sigma} d_{\sigma}, \tag{3}$$

$$H_{\mathrm{R}} \approx \sum_{q} \hbar w_q a^{\dagger}_q a_q, \tag{4}$$

$$H_{\mathrm{T}} = V_{\mathrm{L}} + V_{\mathrm{R}} \tag{5}$$

V$_L$ is the coupling between left lead and QD, while V$_R$ is the coupling between right lead and QD.

$$V_{\mathrm{L}} = \sum_{k\sigma} (t_{k\sigma} c^{\dagger}_{k\sigma} d_{\sigma} + t^{*}_{k\sigma} d^{\dagger}_{\sigma} c_{k\sigma}) \tag{6}$$

$$V_{\mathrm{R}} = -\sum_{q} J_q \left[S^{-}_q d^{\dagger}_{\uparrow} d_{\downarrow} + S^{+}_q d^{\dagger}_{\downarrow} d_{\uparrow} \right]. \tag{7}$$

where $S^{-}_q \approx \sqrt{2S_0} a^{\dagger}_q, S^{+}_q \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling between the QD and MIL.

1 Spin-dependent current in left lead

The spin-dependent current flow out of left lead is $I_{L\sigma}$:

$$I_{L\sigma} = \frac{d}{dt} \langle N_{L\sigma} \rangle \tag{8}$$

in which, $N_{L\sigma} = \sum_k c^{\dagger}_{k\sigma} c_{k\sigma}$ Heisenberg equation:

$$\frac{d}{dt} \langle N_{L\sigma} \rangle = \frac{i}{\hbar} \langle [H, N_{L\sigma}] \rangle \tag{9}$$

$$[H, N_{L\sigma}] = [H_T, N_{L\sigma}] = \sum_k \left(t^*_{k\sigma} d^\dagger_\sigma c_{k\sigma} - t_{k\sigma} c^\dagger_{k\sigma} d_\sigma \right) \tag{10}$$

so, the spin-dependent current

$$I_{L\sigma} = \frac{i}{\hbar} \sum_k \left(t^*_{k\sigma} \langle d^\dagger_\sigma c_{k\sigma} \rangle - t_{k\sigma} \langle c^\dagger_{k\sigma} d_\sigma \rangle \right). \tag{11}$$

Namely, the spin-up current is

$$I_{L\uparrow} = \frac{i}{\hbar} \sum_k \left(t^*_{k\uparrow} \langle d^\dagger_\uparrow c_{k\uparrow} \rangle - t_{k\uparrow} \langle c^\dagger_{k\uparrow} d_\uparrow \rangle \right), \tag{12}$$

the spin-down current is

$$I_{L\downarrow} = \frac{i}{\hbar} \sum_k \left(t^*_{k\downarrow} \langle d^\dagger_\downarrow c_{k\downarrow} \rangle - t_{k\downarrow} \langle c^\dagger_{k\downarrow} d_\downarrow \rangle \right), \tag{13}$$

The charge current in left lead is defined as

$$I_e = e(I_{L\uparrow} + I_{L\downarrow}). \tag{14}$$

The spin current in left lead is defined as

$$I_{LS} = \frac{1}{2}(I_{L\uparrow} - I_{L\downarrow}) \tag{15}$$

Substitute the spin-dependent current in, we get the spin current in left lead

$$I_{LS} = \frac{i}{2\hbar} \sum_k \left(t^*_{k\uparrow} \langle d^\dagger_\uparrow c_{k\uparrow} \rangle - t_{k\uparrow} \langle c^\dagger_{k\uparrow} d_\uparrow \rangle - t^*_{k\downarrow} \langle d^\dagger_\downarrow c_{k\downarrow} \rangle + t_{k\downarrow} \langle c^\dagger_{k\downarrow} d_\downarrow \rangle \right) \tag{16}$$

2 magnonic current in right lead

The magnonic current in right lead is

$$I_{RS} = \frac{d\langle N_R \rangle}{dt} = \frac{d}{dt} \langle \sum_q a^\dagger_q a_q \rangle, \tag{17}$$

From the Heisenberg equation, we have

$$\frac{d}{dt} \langle \sum_q a^\dagger_q a_q \rangle = \frac{i}{\hbar} \langle [H, \sum_q a^\dagger_q a_q] \rangle. \tag{18}$$

We have

$$
\begin{aligned}
&[H,\sum_q a_q^\dagger a_q] \\
=&[V_R,\sum_q a_q^\dagger a_q] \\
=&-\sum_q J_q\Big(\big[S_q^- d_\uparrow^\dagger d_\downarrow,\sum_{q'} a_{q'}^\dagger a_{q'}\big]+\big[S_q^+ d_\downarrow^\dagger d_\uparrow,\sum_{q'} a_{q'}^\dagger a_{q'}\big]\Big) \\
=&\sum_q J_q\big(S_q^- d_\uparrow^\dagger d_\downarrow - S_q^+ d_\downarrow^\dagger d_\uparrow\big)
\end{aligned}
\tag{19}
$$

So, the magnon current reads

$$
I_{RS}=\frac{i}{\hbar}\sum_q J_q\big(\langle S_q^- d_\uparrow^\dagger d_\downarrow\rangle-\langle S_q^+ d_\downarrow^\dagger d_\uparrow\rangle\big). \tag{20}
$$

3 spin-dependent current in QD

Similarly, the spin-dependent current in the central QD is defined as

$$
I_{C\sigma}=\langle\frac{dN_{C\sigma}}{dt}\rangle=\langle\frac{d}{dt}d_\sigma^\dagger d_\sigma\rangle, \tag{21}
$$

Heisenberg equation:

$$
\frac{d}{dt}d_\sigma^\dagger d_\sigma=\frac{i}{\hbar}[H,d_\sigma^\dagger d_\sigma]. \tag{22}
$$

Specifically, we have

$$
[H,d_\sigma^\dagger d_\sigma]=[V_L,d_\sigma^\dagger d_\sigma]+[V_R,d_\sigma^\dagger d_\sigma] \tag{23}
$$

in which,

$$
\begin{aligned}
[V_L,d_\sigma^\dagger d_\sigma]&=\sum_{k'\sigma'}\big(t_{k'\sigma'}\big[c_{k'\sigma'}^\dagger d_{\sigma'},d_\sigma^\dagger d_\sigma\big]+t_{k'\sigma'}^*\big[d_{\sigma'}^\dagger c_{k'\sigma'},d_\sigma^\dagger d_\sigma\big]\big) \\
&=\sum_{k'\sigma'}\big(t_{k'\sigma'}c_{k'\sigma'}^\dagger d_{\sigma'}\delta_{\sigma\sigma'}-t_{k'\sigma'}^* d_{\sigma'}^\dagger c_{k'\sigma'}\delta_{\sigma\sigma'}\big) \\
&=\sum_k\big(t_{k\sigma}c_{k\sigma}^\dagger d_\sigma-t_{k\sigma}^* d_\sigma^\dagger c_{k\sigma}\big)
\end{aligned}
\tag{24}
$$

We change the summation index from k' to k in the last line , which doesn't change the result.

$$
[V_R,d_\sigma^\dagger d_\sigma]=-\sum_q J_q\big(S_q^-\big[d_\uparrow^\dagger d_\downarrow,d_\sigma^\dagger d_\sigma\big]+S_q^+\big[d_\downarrow^\dagger d_\uparrow,d_\sigma^\dagger d_\sigma\big]\big) \tag{25}
$$

in which,

$$\begin{aligned}\left[d_\uparrow^\dagger d_\downarrow, d_\sigma^\dagger d_\sigma\right] &= d_\uparrow^\dagger\left[d_\downarrow, d_\sigma^\dagger d_\sigma\right] + \left[d_\uparrow^\dagger, d_\sigma^\dagger d_\sigma\right] d_\downarrow \\ &= d_\uparrow^\dagger d_\downarrow \delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow \delta_{\sigma\uparrow}\end{aligned} \tag{26}$$

So,

$$[V_R, d_\sigma^\dagger d_\sigma] = -\sum_q J_q\left[S_q^-\left(d_\uparrow^\dagger d_\downarrow \delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow \delta_{\sigma\uparrow}\right) + S_q^+\left(d_\downarrow^\dagger d_\uparrow \delta_{\sigma\uparrow} - d_\downarrow^\dagger d_\uparrow \delta_{\sigma\downarrow}\right)\right] \tag{27}$$

The spin-dependent current is

$$I_{C\uparrow} = \frac{i}{\hbar}\left[H, d_\uparrow^\dagger d_\uparrow\right] \tag{28}$$

$$I_{C\downarrow} = \frac{i}{\hbar}\left[H, d_\downarrow^\dagger d_\downarrow\right] \tag{29}$$

which gives

$$I_{C\uparrow} = \frac{i}{\hbar}\left[\sum_k \left(t_{k\uparrow} c_{k\uparrow}^\dagger d_\uparrow - t_{k\uparrow}^* d_\uparrow^\dagger c_{k\uparrow}\right) + \sum_q J_q\left(S_q^- d_\uparrow^\dagger d_\downarrow - S_q^+ d_\downarrow^\dagger d_\uparrow\right)\right] \tag{30}$$

and

$$I_{C\downarrow} = \frac{i}{\hbar}\left[\sum_k \left(t_{k\downarrow} c_{k\downarrow}^\dagger d_\downarrow - t_{k\downarrow}^* d_\downarrow^\dagger c_{k\downarrow}\right) + \sum_q J_q\left(S_q^+ d_\downarrow^\dagger d_\uparrow - S_q^- d_\uparrow^\dagger d_\downarrow\right)\right]. \tag{31}$$

The spin current in central dot is

$$I_{CS} = \frac{1}{2}(I_{C\uparrow} - I_{C\downarrow}) \tag{32}$$

4 Verifying continuity condition

4.1 Charge current

Since the right lead is a insulating lead, there is no charge current flow through it, so the charge current is

$$I_{Re} = 0 \tag{33}$$

where the subscript R denotes the right lead, while the subscript e denotes the charge current.

Meanwhile, the charge current flows in left lead and QD is

$$I_e = e\left(\sum_\sigma I_{L\sigma} + \sum_\sigma I_{C\sigma}\right) = 0 \tag{34}$$

4.2 Spin current

Spin current in the left lead and QD:

$$I_{LS} + I_{CS} = \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big). \tag{35}$$

The magnon current is

$$I_{RS} = \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big), \tag{36}$$

in which, I_{LS}, I_{CS}, I_{RS} is defined earlier. Thus, we have

$$\begin{aligned} I_{LS} + I_{CS} &= I_{RS} \\ &= \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big). \end{aligned} \tag{37}$$