Formulas in PRB. 88, 220406(R) (2013)

1 Formula (1)

System Hamiltonion:

$$H = H_L + H_R + H_{sd}. \tag{1}$$

Left lead is metallic

$$H_L = \sum_{k\sigma} (\varepsilon_{k\sigma} - \mu_\sigma) c_{k\sigma}^\dagger c_{k\sigma}, \tag{2}$$

right lead is insulating magnetic

$$H_R \approx \sum_q \hbar w_q a_q^\dagger a_q + \text{ constant }. \tag{3}$$

The interfacial electron-magnon interaction is described by

$$H_{sd} = -\sum_{k,q} J_q \left[S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \tag{4}$$

where $S_q^- \approx \sqrt{2S_0} a_q^\dagger, S_q^+ \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling at the interface. The magnonic spin current operator can be obtained by

$$\hat{I}_S = \frac{d\hat{N}_R}{dt} = \frac{d}{dt} \sum_q a_q^\dagger a_q, \tag{5}$$

the magnonic spin current is obtained by taking average over the nonequilibrium ground state $|\psi_0\rangle$ of the interacting system H:

$$I_S = \frac{dN_R}{dt} = \frac{d}{dt} \langle \sum_q a_q^\dagger a_q \rangle. \tag{6}$$

Using the Heisenberg equation, we get

$$I_S = \frac{i}{\hbar} \langle [H_{sd}, \sum_q a_q^\dagger a_q] \rangle. \tag{7}$$

$$[H_{sd}, \sum_q a_q^\dagger a_q] = [-\sum_{k,q} J_q \left(S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right), \sum_q a_q^\dagger a_q], \tag{8}$$

in which,

$$[a_q^\dagger, \sum_{q'} a_{q'}^\dagger, a_{q'}] = \delta_{qq'} [a_q^\dagger, a_{q'}^\dagger a_{q'}] = [a_q^\dagger, a_q^\dagger a_q] = a_q^\dagger [a_q^\dagger, a_q] = -a_q^\dagger. \tag{9}$$

Similarly,

$$[a_q, \sum_{q'} a_{q'}^+ a_{q'}] = [a_q, a_q^\dagger a_q] = a_q. \tag{10}$$

So,

$$\begin{aligned} I_S &= \frac{i}{\hbar} \langle -\sum_{kq} J_q \left(-S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right) \rangle \\ &= \frac{i}{\hbar} \sum_{kq} J_q \left(\langle S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} \rangle - \langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \rangle \right). \end{aligned} \tag{11}$$

2 Formula (2)

$$\frac{d}{dt} \left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right\rangle = \frac{i}{\hbar} \left\langle \left[H_L + H_{sd} + H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \right\rangle. \tag{12}$$

The rhs. of eq. (12) is decomposed into 3 terms. The first term reads

$$\begin{aligned} \left\langle \left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] \right\rangle &= [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}] \\ &= S_q^+ [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] c_{k\uparrow} \\ &\quad + S_q^+ c_{k+q\downarrow}^\dagger [\sum_{k'\sigma} (\varepsilon_{k'\sigma} - \mu_\sigma)\, c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}] \end{aligned} \tag{13}$$

Note that,

$$[\sum_{k'\sigma} c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] = \sum_{k'\sigma} c_{k'\sigma}^\dagger \delta_{k',k+q} \delta_{\sigma\downarrow} = c_{k+q\downarrow}^\dagger, \tag{14}$$

$$[\sum_{k'\sigma} c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}] = -\sum_{k'\sigma} \{c_{k'\sigma}^\dagger, c_{k\uparrow}\} c_{k'\sigma} = -c_{k\uparrow}. \tag{15}$$

Eq. (14) (15) are derived using equity

$$\begin{aligned} [AB, C] &= A[B, C] + [A, C]B \\ &= A\{B, C\} - \{A, C\}B. \end{aligned} \tag{16}$$

So,

$$\left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right] = (\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} + \mu_\uparrow - \mu_\downarrow)\, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{17}$$

If $\mu_\uparrow = \mu_\downarrow$, then eq. (17) reduces to

$$\left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow}\right) S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{18}$$

The second term $H_R = \sum_q \hbar w_q a_q^\dagger a_q$, then using eq. (10), we get

$$\left[H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] = -\hbar\omega_q S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{19}$$

The third term in eq. (12) reads

$$\begin{aligned}\left[H_{sd}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] &= -J_q \left[S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] \\ &= J_q \left[S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}, S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\right]\end{aligned} \tag{20}$$

Combine these three terms, we get

$$\begin{aligned}\frac{d}{dt}\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle = &\frac{i}{\hbar}\left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} - \hbar\omega_q\right)\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle \\ &+ \frac{i}{\hbar} J_q \left\langle \left[S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}, S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\right]\right\rangle,\end{aligned} \tag{21}$$

which is also eq. (2) in PRB. 88, 220406(R) (2013).