Notes on quantum transport in mesoscopic systems

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Contents

0.1 Basics

0.1.1 Magnon

A magnon is a quasiparticle, a collective excitation of the electrons' spin structure in a crystal lattice. In the equivalent wave picture of quantum mechanics, a magnon can be viewed as a quantized spin wave. Magnons carry a fixed amount of energy and lattice momentum, and are spin-1, indicating they obey boson behavior.

0.1.2 Hall effect

Conventional Hall effect

Quantum Hall effect

Spin Hall effect

Occurs in paramagnetic systems as a result of spin-orbit interaction, refers to generation of pure spin current transverse to an applied electric field, even in the absence of magnetic field.

Similar to charge accumulation at sample edges in conventional Hall effect, spin accumulation is expected in spin Hall effect.

- extrinsic spin Hall effect: originates from asymmetric scattering for spin-up and spin-down.
- intrinsic spin Hall effect: originates from band structures without scattering.[1]

Fractional Hall effect

Anomalous Hall effect

0.2 Formulas in PRB. 88, 220406(R) (2013)

0.2.1 Formula 1

System Hamiltonion:

$$H = H_L + H_R + H_{sd}. \tag{1}$$

Left lead is metallic

$$H_L = \sum_{k\sigma} (\varepsilon_{k\sigma} - \mu_\sigma) c^\dagger_{k\sigma} c_{k\sigma}, \tag{2}$$

right lead is insulating magnetic

$$H_R \approx \sum_q \hbar w_q a^\dagger_q a_q + \text{ constant }. \tag{3}$$

The interfacial electron-magnon interaction is described by

$$H_{sd} = -\sum_{k,q} J_q \left[S^-_q c^\dagger_{k\uparrow} c_{k+q\downarrow} + S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow} \right] \tag{4}$$

where $S^-_q \approx \sqrt{2S_0} a^\dagger_q, S^+_q \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling at the interface. The magnonic spin current operator can be obtained by

$$\hat{I}_S = \frac{d\hat{N}_R}{dt} = \frac{d}{dt} \sum_q a^\dagger_q a_q, \tag{5}$$

the magnonic spin current is obtained by taking average over the nonequilibrium ground state $|\psi_0\rangle$ of the interacting system H:

$$I_S = \frac{dN_R}{dt} = \frac{d}{dt} \langle \sum_q a^\dagger_q a_q \rangle. \tag{6}$$

Using the Heisenberg equation, we get

$$I_S = \frac{i}{\hbar} \langle [H_{sd}, \sum_q a^\dagger_q a_q] \rangle. \tag{7}$$

$$[H_{sd}, \sum_q a^\dagger_q a_q] = [-\sum_{k,q} J_q \left(S^-_q c^\dagger_{k\uparrow} c_{k+q\downarrow} + S^+_q c^\dagger_{k+q\downarrow} c_{k\uparrow} \right), \sum_q a^\dagger_q a_q], \tag{8}$$

in which,

$$[a^\dagger_q, \sum_{q'} a^\dagger_{q'}, a_{q'}] = \delta_{qq'} [a^\dagger_q, a^\dagger_{q'} a_{q'}] = [a^\dagger_q, a^\dagger_q a_q] = a^\dagger_q [a^\dagger_q, a_q] = -a^\dagger_q. \tag{9}$$

Similarly,

$$[a_q, \sum_{q'} a^+_{q'} a_{q'}] = [a_q, a^\dagger_q a_q] = a_q. \tag{10}$$

So,

$$
\begin{aligned}
I_S &= \frac{i}{\hbar}\langle -\sum_{kq} J_q \left(-S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow} + S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow} \right)\rangle \\
&= \frac{i}{\hbar}\sum_{kq} J_q \left(\langle S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\rangle - \langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\rangle \right).
\end{aligned}
\tag{11}
$$

0.2.2 Formula 2

$$
\frac{d}{dt}\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle = \frac{i}{\hbar}\left\langle \left[H_L + H_{sd} + H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right]\right\rangle. \tag{12}
$$

The rhs. of eq. (12) is decomposed into 3 terms. The first term reads

$$
\begin{aligned}
\left\langle \left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right]\right\rangle &= [\sum_{k'\sigma} \left(\varepsilon_{k'\sigma} - \mu_\sigma\right) c_{k'\sigma}^\dagger c_{k'\sigma}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}] \\
&= S_q^+ [\sum_{k'\sigma} \left(\varepsilon_{k'\sigma} - \mu_\sigma\right) c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] c_{k\uparrow} \\
&\quad + S_q^+ c_{k+q\downarrow}^\dagger [\sum_{k'\sigma} \left(\varepsilon_{k'\sigma} - \mu_\sigma\right) c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}]
\end{aligned}
\tag{13}
$$

Note that,

$$
[\sum_{k'\sigma} c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k+q\downarrow}^\dagger] = \sum_{k'\sigma} c_{k'\sigma}^\dagger \delta_{k',k+q}\delta_{\sigma\downarrow} = c_{k+q\downarrow}^\dagger, \tag{14}
$$

$$
[\sum_{k'\sigma} c_{k'\sigma}^\dagger c_{k'\sigma}, c_{k\uparrow}] = -\sum_{k'\sigma} \{c_{k'\sigma}^\dagger, c_{k\uparrow}\} c_{k'\sigma} = -c_{k\uparrow}. \tag{15}
$$

Eq. (14) (15) are derived using equity

$$
\begin{aligned}
[AB, C] &= A[B,C] + [A,C]B \\
&= A\{B,C\} - \{A,C\}B.
\end{aligned}
\tag{16}
$$

So,

$$
\left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} + \mu_\uparrow - \mu_\downarrow\right) S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{17}
$$

If $\mu_\uparrow = \mu_\downarrow$, then eq. (17) reduces to

$$
\left[H_L, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] = \left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow}\right) S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{18}
$$

The second term $H_R = \sum_q \hbar w_q a_q^\dagger a_q$, then using eq. (10), we get

$$
\left[H_R, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] = -\hbar\omega_q S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}. \tag{19}
$$

The third term in eq. (12) reads

$$
\begin{aligned}
\left[H_{sd}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] &= -J_q \left[S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}, S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right] \\
&= J_q \left[S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}, S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\right]
\end{aligned}
\tag{20}
$$

Combine these three terms, we get

$$
\begin{aligned}
\frac{d}{dt}\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle &= \frac{i}{\hbar}\left(\varepsilon_{k+q\downarrow} - \varepsilon_{k\uparrow} - \hbar\omega_q\right)\left\langle S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}\right\rangle \\
&\quad + \frac{i}{\hbar} J_q \left\langle \left[S_q^+ c_{k+q\downarrow}^\dagger c_{k\uparrow}, S_q^- c_{k\uparrow}^\dagger c_{k+q\downarrow}\right]\right\rangle,
\end{aligned}
\tag{21}
$$

which is also eq. (2) in PRB. 88, 220406(R) (2013).

0.3 Spin current in NM-QD-MIL system

For system consists of quantum dot(QD) sandwiched by a left normal metal(NM) lead and a right magnetic insulating lead(MIL), the Hamiltonian is

$$H = H_{\rm L} + H_{\rm QD} + H_{\rm R} + H_{\rm T}. \tag{22}$$

$$H_{\rm L} = \sum_{k\sigma} \varepsilon_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma}, \tag{23}$$

$$H_{\rm QD} = \sum_{\sigma} \varepsilon_{\sigma} d^\dagger_{\sigma} d_{\sigma}, \tag{24}$$

$$H_{\rm R} \approx \sum_{q} \hbar w_q a^\dagger_q a_q, \tag{25}$$

$$H_{\rm T} = V_{\rm L} + V_{\rm R} \tag{26}$$

V$_L$ is the coupling between left lead and QD, while V$_R$ is the coupling between right lead and QD.

$$V_{\rm L} = \sum_{k\sigma} (t_{k\sigma} c^\dagger_{k\sigma} d_\sigma + t^*_{k\sigma} d^\dagger_\sigma c_{k\sigma}) \tag{27}$$

$$V_{\rm R} = -\sum_q J_q \left[S^-_q d^\dagger_\uparrow d_\downarrow + S^+_q d^\dagger_\downarrow d_\uparrow \right]. \tag{28}$$

where $S^-_q \approx \sqrt{2S_0} a^\dagger_q, S^+_q \approx \sqrt{2S_0} a_q$ are in the momentum space and J_q denotes the effective exchange coupling between the QD and MIL.

0.3.1 Spin-dependent current in left lead

The spin-dependent current flow out of left lead is $I_{L\sigma}$:

$$I_{L\sigma} = \frac{d}{dt} \langle N_{L\sigma} \rangle \tag{29}$$

in which, $N_{L\sigma} = \sum_k c^\dagger_{k\sigma} c_{k\sigma}$ Heisenberg equation:

$$\frac{d}{dt} \langle N_{L\sigma} \rangle = \frac{i}{\hbar} \langle [H, N_{L\sigma}] \rangle \tag{30}$$

$$[H, N_{L\sigma}] = [H_T, N_{L\sigma}] = \sum_k \left(t^*_{k\sigma} d^\dagger_\sigma c_{k\sigma} - t_{k\sigma} c^\dagger_{k\sigma} d_\sigma \right) \tag{31}$$

so, the spin-dependent current

$$I_{L\sigma} = \frac{i}{\hbar} \sum_k \left(t^*_{k\sigma} \langle d^\dagger_\sigma c_{k\sigma} \rangle - t_{k\sigma} \langle c^\dagger_{k\sigma} d_\sigma \rangle \right). \tag{32}$$

Namely, the spin-up current is

$$I_{L\uparrow} = \frac{i}{\hbar} \sum_k \left(t^*_{k\uparrow} \langle d^\dagger_\uparrow c_{k\uparrow} \rangle - t_{k\uparrow} \langle c^\dagger_{k\uparrow} d_\uparrow \rangle \right), \tag{33}$$

the spin-down current is

$$I_{L\downarrow} = \frac{i}{\hbar}\sum_k \left(t^*_{k\downarrow}\langle d^\dagger_\downarrow c_{k\downarrow}\rangle - t_{k\downarrow}\langle c^\dagger_{k\downarrow} d_\downarrow\rangle \right), \tag{34}$$

The charge current in left lead is defined as

$$I_e = e(I_{L\uparrow} + I_{L\downarrow}). \tag{35}$$

The spin current in left lead is defined as

$$I_{LS} = \frac{1}{2}(I_{L\uparrow} - I_{L\downarrow}) \tag{36}$$

Substitute the spin-dependent current in, we get the spin current in left lead

$$I_{LS} = \frac{i}{2\hbar}\sum_k \left(t^*_{k\uparrow}\langle d^\dagger_\uparrow c_{k\uparrow}\rangle - t_{k\uparrow}\langle c^\dagger_{k\uparrow} d_\uparrow\rangle - t^*_{k\downarrow}\langle d^\dagger_\downarrow c_{k\downarrow}\rangle + t_{k\downarrow}\langle c^\dagger_{k\downarrow} d_\downarrow\rangle \right) \tag{37}$$

0.3.2 magnonic current in right lead

The magnonic current in right lead is

$$I_{RS} = \frac{d\langle N_R\rangle}{dt} = \frac{d}{dt}\langle \sum_q a^\dagger_q a_q\rangle, \tag{38}$$

From the Heisenberg equation, we have

$$\frac{d}{dt}\langle \sum_q a^\dagger_q a_q\rangle = \frac{i}{\hbar}\langle [H, \sum_q a^\dagger_q a_q]\rangle. \tag{39}$$

We have

$$\begin{aligned} &[H, \sum_q a^\dagger_q a_q] \\ =&[V_R, \sum_q a^\dagger_q a_q] \\ =&-\sum_q J_q\Big(\big[S^-_q d^\dagger_\uparrow d_\downarrow, \sum_{q'} a^\dagger_{q'} a_{q'}\big] + \big[S^+_q d^\dagger_\downarrow d_\uparrow, \sum_{q'} a^\dagger_{q'} a_{q'}\big]\Big) \\ =&\sum_q J_q\big(S^-_q d^\dagger_\uparrow d_\downarrow - S^+_q d^\dagger_\downarrow d_\uparrow\big) \end{aligned} \tag{40}$$

So, the magnon current reads

$$I_{RS} = \frac{i}{\hbar}\sum_q J_q\big(\langle S^-_q d^\dagger_\uparrow d_\downarrow\rangle - \langle S^+_q d^\dagger_\downarrow d_\uparrow\rangle\big). \tag{41}$$

0.3.3 spin-dependent current in QD

Similarly, the spin-dependent current in the central QD is defined as

$$I_{C\sigma} = \langle \frac{dN_{C\sigma}}{dt}\rangle = \langle \frac{d}{dt} d^\dagger_\sigma d_\sigma\rangle, \tag{42}$$

Heisenberg equation:

$$\frac{d}{dt} d^\dagger_\sigma d_\sigma = \frac{i}{\hbar}[H, d^\dagger_\sigma d_\sigma]. \tag{43}$$

Specifically, we have

$$[H, d_\sigma^\dagger d_\sigma] = [V_L, d_\sigma^\dagger d_\sigma] + [V_R, d_\sigma^\dagger d_\sigma] \tag{44}$$

in which,

$$\begin{aligned}
[V_L, d_\sigma^\dagger d_\sigma] &= \sum_{k'\sigma'} \left(t_{k'\sigma'} \left[c_{k'\sigma'}^\dagger d_{\sigma'}, d_\sigma^\dagger d_\sigma \right] + t_{k'\sigma'}^* \left[d_{\sigma'}^\dagger c_{k'\sigma'}, d_\sigma^\dagger d_\sigma \right] \right) \\
&= \sum_{k'\sigma'} \left(t_{k'\sigma'} c_{k'\sigma'}^\dagger d_{\sigma'} \delta_{\sigma\sigma'} - t_{k'\sigma'}^* d_{\sigma'}^\dagger c_{k'\sigma'} \delta_{\sigma\sigma'} \right) \\
&= \sum_{k} \left(t_{k\sigma} c_{k\sigma}^\dagger d_{\sigma} - t_{k\sigma}^* d_{\sigma}^\dagger c_{k\sigma} \right)
\end{aligned} \tag{45}$$

We change the summation index from k' to k in the last line , which doesn't change the result.

$$[V_R, d_\sigma^\dagger d_\sigma] = -\sum_q J_q \left(S_q^- \left[d_\uparrow^\dagger d_\downarrow, d_\sigma^\dagger d_\sigma \right] + S_q^+ \left[d_\downarrow^\dagger d_\uparrow, d_\sigma^\dagger d_\sigma \right] \right) \tag{46}$$

in which,

$$\begin{aligned}
\left[d_\uparrow^\dagger d_\downarrow, d_\sigma^\dagger d_\sigma \right] &= d_\uparrow^\dagger \left[d_\downarrow, d_\sigma^\dagger d_\sigma \right] + \left[d_\uparrow^\dagger, d_\sigma^\dagger d_\sigma \right] d_\downarrow \\
&= d_\uparrow^\dagger d_\downarrow \delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow \delta_{\sigma\uparrow}
\end{aligned} \tag{47}$$

So,

$$[V_R, d_\sigma^\dagger d_\sigma] = -\sum_q J_q \left[S_q^- \left(d_\uparrow^\dagger d_\downarrow \delta_{\sigma\downarrow} - d_\uparrow^\dagger d_\downarrow \delta_{\sigma\uparrow} \right) + S_q^+ \left(d_\downarrow^\dagger d_\uparrow \delta_{\sigma\uparrow} - d_\downarrow^\dagger d_\uparrow \delta_{\sigma\downarrow} \right) \right] \tag{48}$$

The spin-dependent current is

$$I_{C\uparrow} = \frac{i}{\hbar} \left[H, d_\uparrow^\dagger d_\uparrow \right] \tag{49}$$

$$I_{C\downarrow} = \frac{i}{\hbar} \left[H, d_\downarrow^\dagger d_\downarrow \right] \tag{50}$$

which gives

$$I_{C\uparrow} = \frac{i}{\hbar} \Big[\sum_k \left(t_{k\uparrow} c_{k\uparrow}^\dagger d_\uparrow - t_{k\uparrow}^* d_\uparrow^\dagger c_{k\uparrow} \right) + \sum_q J_q \left(S_q^- d_\uparrow^\dagger d_\downarrow - S_q^+ d_\downarrow^\dagger d_\uparrow \right) \Big] \tag{51}$$

and

$$I_{C\downarrow} = \frac{i}{\hbar} \Big[\sum_k \left(t_{k\downarrow} c_{k\downarrow}^\dagger d_\downarrow - t_{k\downarrow}^* d_\downarrow^\dagger c_{k\downarrow} \right) + \sum_q J_q \left(S_q^+ d_\downarrow^\dagger d_\uparrow - S_q^- d_\uparrow^\dagger d_\downarrow \right) \Big]. \tag{52}$$

The spin current in central dot is

$$I_{CS} = \frac{1}{2} (I_{C\uparrow} - I_{C\downarrow}) \tag{53}$$

0.3.4 Verifying continuity condition

Charge current

Since the right lead is a insulating lead, there is no charge current flow through it, so the charge current is

$$I_{Re} = 0 \tag{54}$$

where the subscript R denotes the right lead, while the subscript e denotes the charge current.

Meanwhile, the charge current flows in left lead and QD is

$$I_e = e\left(\sum_\sigma I_{L\sigma} + \sum_\sigma I_{C\sigma}\right) = 0 \tag{55}$$

Spin current

Spin current in the left lead and QD:

$$I_{LS} + I_{CS} = \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big). \tag{56}$$

The magnon current is

$$I_{RS} = \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big), \tag{57}$$

in which, I_{LS}, I_{CS}, I_{RS} is defined earlier. Thus, we have

$$\begin{aligned} I_{LS} + I_{CS} &= I_{RS} \\ &= \frac{i}{\hbar} \sum_q J_q \big(\langle S_q^- d_\uparrow^\dagger d_\downarrow \rangle - \langle S_q^+ d_\downarrow^\dagger d_\uparrow \rangle \big). \end{aligned} \tag{58}$$

0.3.5 Perturbation expansion of $G_d(\tau, \tau)$

When neglecting left lead, the hamiltonian is

$$H = H_{QD} + H_R + H_{sd} \tag{59}$$

Expand S-matrix up to the second order of J, we have

$$\begin{aligned} &G_d(\tau, \tau) = -i \langle T_C S s_q^+(\tau) s_q^-(\tau') \rangle \\ =& -i \sum_k g_{k\uparrow}(\tau', \tau) g_{k+q\downarrow}(\tau, \tau') \\ &+ \int_c d\tau_1 \int_c d\tau_2 \sum_{kq_1} J_{q_1}^2 g_{Rq_1}(\tau_2, \tau_1) g_{k\uparrow}(\tau', \tau_2) g_{k+q_1\downarrow}(\tau_2, \tau_1) g_{k\uparrow}(\tau_1, \tau) g_{k+q\downarrow}(\tau, \tau') \\ &+ \int_c d\tau_1 \int_c d\tau_2 \sum_{kq_1} J_{q_1}^2 g_{Rq_1}(\tau_2, \tau_1) g_{k\uparrow}(\tau', \tau) g_{k+q\downarrow}(\tau, \tau_1) g_{k+q-q_1\uparrow}(\tau_1, \tau_2) g_{k+q\downarrow}(\tau_2, \tau') \\ &- \int_c d\tau_1 \int_c d\tau_2 \sum_{kk'} J_q^2 g_{Rq}(\tau_2, \tau_1) g_{k\uparrow}(\tau_1, \tau) g_{k+q\downarrow}(\tau, \tau_1) g_{k'\uparrow}(\tau', \tau_2) g_{k'+q\downarrow}(\tau_2, \tau') \end{aligned} \tag{60}$$

0.4 Checking formulas in notes

0.4.1 Hamiltonian

$$H = H_L + H_R + H_d + H_T + H_{sd} \tag{61}$$
$$H_L = \sum_{k\sigma} \epsilon_{k\sigma,L} c^\dagger_{k\sigma} c_{k\sigma} \tag{62}$$
$$H_R = \sum_{q} \omega_q a^\dagger_q a_q \tag{63}$$
$$H_d = \sum_{n\sigma} \epsilon_{n\sigma} d^\dagger_{n\sigma} d_{n\sigma} \tag{64}$$
$$H_T = \sum_{k\sigma n} \left(t_{k\sigma n} c^\dagger_{k\sigma} d_{n\sigma} + t^*_{k\sigma n} d^\dagger_{n\sigma} c_{k\sigma} \right) \tag{65}$$
$$H_{sd} = -\sum_{qnm} J_q \left(d^\dagger_{n\uparrow} d_{m\downarrow} a^\dagger_q + a_q d^\dagger_{m\downarrow} d_{n\uparrow} \right) \delta\left(\epsilon_{n\uparrow} - \epsilon_{m\downarrow} - \omega_q\right) \tag{66}$$
$$s^+_q = \sum_{nm} d^\dagger_{n\uparrow} d_{m\downarrow} \delta_{\uparrow\downarrow} \tag{67}$$
$$s^-_q = \sum_{nm} d^\dagger_{m\downarrow} d_{n\uparrow} \delta_{\uparrow\downarrow} \tag{68}$$

check operators

$$i\dot{a}_q = \omega_q a_q - J_q s^+_q \tag{69}$$
$$i\dot{c}_{k\sigma} = \epsilon_{k\sigma,L} c_{k\sigma} + \sum_{k'} t_{k\sigma n} d_{n\sigma} \tag{70}$$
$$i\dot{d}_{n\uparrow} = \epsilon_{n\uparrow} d_{n\uparrow} + \sum_k t^*_{k\uparrow n} c_{k\uparrow} - \sum_{q,m} J_q a^\dagger_q d_{m\downarrow} \delta(\epsilon_{n\uparrow} - \epsilon_{m\downarrow} - \omega_q) \tag{71}$$
$$i\dot{d}_{n\downarrow} = \epsilon_{n\downarrow} d_{n\downarrow} + \sum_k t^*_{k\downarrow n} c_{k\downarrow} - \sum_{q,m} J_q a_q d_{m\uparrow} \delta(\epsilon_{m\uparrow} - \epsilon_{n\downarrow} - \omega_q) \tag{72}$$

0.4.2 spin current ???

$$-i\partial_{\tau'} G_{d,R}\left(\tau,\tau'\right) = \omega_q G_{d,R}\left(\tau,\tau'\right) - J_q G_d \tag{73}$$

or

$$G_{d,R} g^{-1}_{Rq} = -J_q G_d \tag{74}$$

or

$$G_{d,R}\left(\tau,\tau'\right) = -J_q \int G_d\left(\tau,\tau_1\right) g_{Rq}\left(\tau_1,\tau'\right) d\tau_1 \tag{75}$$

the minus before J_q originates from the minus in H_{sd}.

$$\begin{aligned} I_s &= i\sum_q J_q \left(\left\langle s^+_q a^\dagger_q \right\rangle - \left\langle a_q s^-_q \right\rangle \right) \\ &= \sum_q J_q (G^<_{d,R}(t,t') - G^<_{R,d}(t,t')) \\ &= -2\mathrm{Re} \sum_{\mathrm{q}} \int \mathrm{Tr}\left[\mathrm{G^r_d}\left(\mathrm{t},\mathrm{t_1}\right) \Sigma^<_{\mathrm{R}}\left(\mathrm{t_1},\mathrm{t'}\right) + \mathrm{G^<_d}\left(\mathrm{t},\mathrm{t_1}\right) \Sigma^{\mathrm{a}}_{\mathrm{R}}\left(\mathrm{t_1},\mathrm{t'}\right) \right] \mathrm{dt_1} \\ &= -2\mathrm{Re} \sum_{\mathrm{q}} \int \mathrm{Tr}\left[\left(\mathrm{G^>_d} - \mathrm{G^<_d}\right) \Sigma^<_{\mathrm{Rq}} + \mathrm{G^<_d}\left(\Sigma^{\mathrm{a}}_{\mathrm{Rq}} - \Sigma^{\mathrm{r}}_{\mathrm{Rq}}\right) \right] \mathrm{dt_1} \end{aligned} \tag{76}$$

$$\Sigma_{Rq}(\tau,\tau') = J^2_q g_{Rq}(\tau,\tau') \tag{77}$$

0.4.3 Calculation of G_d

Definition:

$$\begin{aligned} G_d\left(\tau,\tau'\right) &= -i\left\langle T_c S s_q^{+}(\tau) s_q^{-}\left(\tau'\right)\right\rangle \\ &= -i\sum_{mnm'n'}\langle T_c S d_{n\uparrow}^{\dagger} d_{m\downarrow} d_{m'\downarrow}^{\dagger} d_{n'\uparrow}\rangle \delta\left(\epsilon_{n\uparrow}-\epsilon_{m\downarrow}-\omega_q\right)\delta\left(\epsilon_{n'\uparrow}-\epsilon_{m'\downarrow}-\omega_q\right) \end{aligned} \tag{78}$$

When right lead is absent,

$$G_d\left(\tau,\tau'\right) = -i\sum_{mnm'n'} G_{L,n'n\uparrow}(\tau',\tau) G_{L,mm'\downarrow}(\tau,\tau')\delta\left(\epsilon_{n\uparrow}-\epsilon_{m\downarrow}-\omega_q\right)\delta\left(\epsilon_{n'\uparrow}-\epsilon_{m'\downarrow}-\omega_q\right) \tag{79}$$

where

$$\begin{aligned} G_{L,mn\sigma}(\tau,\tau') &= -i\langle T_c d_{m\sigma}(\tau) d_{n\sigma}^{\dagger}(\tau')\rangle \\ &= g_{n\sigma}(\tau,\tau')\delta_{mn} \\ &\quad + \iint d\tau_1 d\tau_2 g_{m\sigma}(\tau,\tau_2)\sum_k t_{k\sigma n} t_{k\sigma m}^{*} g_{k\sigma}(\tau_2,\tau_1) g_{n\sigma}(\tau_1,\tau') \\ &\quad + \cdots \\ &= g_{n\sigma}(\tau,\tau')\delta_{mn} + \iint d\tau_1 d\tau_2 g_{m\sigma}(\tau,\tau_2)\Sigma_{L,mn\sigma}(\tau_2,\tau_1) g_{n\sigma}(\tau_1,\tau') \\ &\quad + \cdots \\ &= \quad ? \end{aligned} \tag{80}$$

$$g_{n\sigma}(\tau,\tau') = -i\langle T_c d_n(\tau) d_{n\sigma}^{\dagger}(\tau')\rangle_0 \tag{81}$$

Self-energy of left lead

$$\Sigma_{L,mn\sigma}(\tau_2,\tau_1) = \sum_k t_{k\sigma n} t_{k\sigma m}^{*} g_{k\sigma}(\tau_2,\tau_1) \tag{82}$$

where

$$g_{k\sigma}(\tau_2,\tau_1) = -i\langle T_c c_{k\sigma}(\tau_2) c_{k\sigma}^{\dagger}(\tau_1)\rangle_0 \tag{83}$$

When left lead is absent,

$$\begin{aligned} G_d\left(\tau,\tau'\right) = &-i\sum_{mn} g_{n\uparrow}\left(\tau',\tau\right) g_{m\downarrow}\left(\tau,\tau'\right)\delta(\varepsilon_{n\uparrow}-\varepsilon_{m\downarrow}-\omega_q) \\ &-\int d\tau_1\int d\tau_2 \sum_{mnm'n'} g_{n\uparrow}\left(\tau_1,\tau\right) g_{m\downarrow}\left(\tau,\tau_1\right)\Sigma_{R,mnm'n'}\left(\tau_1,\tau_2\right) g_{n'\uparrow}\left(\tau',\tau_2\right) g_{m'\downarrow}(\tau_2,\tau') \\ &\quad\times\delta(\varepsilon_{n\uparrow}-\varepsilon_{m\downarrow}-\omega_{q_1})\delta(\varepsilon_{n'\uparrow}-\varepsilon_{m'\downarrow}-\omega_{q_1}) \\ &+\cdots \\ =& g_d(\tau,\tau') + \iint d\tau_1 d\tau_2 g_d(\tau,\tau_1)\Sigma_R(\tau_1,\tau_2) G_d(\tau_2,\tau') \end{aligned} \tag{84}$$

in which,

$$g_d\left(\tau,\tau'\right) = -i\sum_{mn} g_{n\uparrow}\left(\tau',\tau\right) g_{m\downarrow}\left(\tau,\tau'\right)\delta(\varepsilon_{n\uparrow}-\varepsilon_{m\downarrow}-\omega_q), \tag{85}$$

the self-energy of right lead is

$$\Sigma_{R,mnm'n'}(\tau_1,\tau_2) = -i\sum_{q1} J_{q1}^2 g_{Rq_1}(\tau_1,\tau_2)\delta(\varepsilon_{n\uparrow}-\varepsilon_{m\downarrow}-\omega_{q_1})\delta(\varepsilon_{n'\uparrow}-\varepsilon_{m'\downarrow}-\omega_{q_1}) \tag{86}$$

$$g_{Rq_1}(\tau_1,\tau_2) = -i\langle T_c a_{q_1}(\tau_1) a^\dagger_{q_1}(\tau_2)\rangle_0 \tag{87}$$

Hence, when both leads are present, we have

$$\begin{aligned} ?\ G_d\left(\tau,\tau'\right) = &-i\sum_{mnm'n'} G_{L,nn'\uparrow}\left(\tau',\tau\right) G_{L,mm'\downarrow}\left(\tau,\tau'\right)\delta\left(\epsilon_{n\uparrow}-\epsilon_{m\downarrow}-\omega_q\right)\delta\left(\epsilon_{n'\uparrow}-\epsilon_{m'\downarrow}-\omega_q\right) \\ &-i\sum_{mnm'n'} G_{L,nn'\uparrow}\left(\tau_1,\tau\right) G_{L,mm'\downarrow}\left(\tau,\tau_1\right)\Sigma_{R,mnm'n'}\left(\tau_1,\tau_2\right) G_d\left(\tau_2,\tau'\right)\delta\left(\epsilon_{n\uparrow}-\epsilon_{m\downarrow}-\omega_q\right) \\ &\times\delta\left(\epsilon_{n'\uparrow}-\epsilon_{m'\downarrow}-\omega_q\right) \end{aligned} \tag{88}$$

For the sack of convenience, we rewrite the above formula as follows.

$$?\ G_d\left(\tau,\tau'\right) = -iG_{L\uparrow}\left(\tau',\tau\right)G_{L\downarrow}\left(\tau,\tau'\right) - iG_{L\uparrow}\left(\tau_1,\tau\right)G_{L\downarrow}\left(\tau,\tau_1\right)\Sigma_R\left(\tau_1,\tau_2\right)G_d\left(\tau_2,\tau'\right) \tag{89}$$

0.4.4 continuation on Eq.(89)

$$A(\tau_1,\tau') \equiv \int d\tau_2 \Sigma_R\left(\tau_1,\tau_2\right)G_d\left(\tau_2,\tau'\right) \tag{90}$$

$$B(\tau,\tau_1) \equiv G_{L\uparrow}\left(\tau_1,\tau\right)G_{L\downarrow}\left(\tau,\tau_1\right) \tag{91}$$

$$C(\tau,\tau') \equiv -iG_{L\uparrow}\left(\tau_1,\tau\right)G_{L\downarrow}\left(\tau,\tau_1\right)A(\tau_1,\tau') \to \tag{92}$$

$$C(\tau,\tau') = -i\int d\tau_1 B(\tau,\tau_1)A(\tau_1,\tau') \tag{93}$$

$$D(\tau,\tau') \equiv -iG_{L\uparrow}\left(\tau',\tau\right)G_{L\downarrow}\left(\tau,\tau'\right) \tag{94}$$

So, we have

$$G_d(\tau,\tau') = D + C \tag{95}$$

Using the analytic continuation theorem, we have

$$D^< = -iG^>_{L\uparrow}G^<_{L\downarrow} \tag{96}$$

$$C^< = -i(B^r A^< + B^< A^a) \tag{97}$$

where

$$B^r = G^a_{L\uparrow}G^<_{L\downarrow} + G^>_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow} \tag{98}$$

$$A^< = \Sigma^r_R G^<_d + \Sigma^<_R G^a_d \tag{99}$$

$$B^< = G^>_{L\uparrow}G^<_{L\downarrow} \tag{100}$$

$$A^a = \Sigma^a_R G^a_d \tag{101}$$

Then, the analytic continuation theorem on Eq.(89) yields

$$G^<_d = -iG^>_{L\uparrow}G^<_{L\downarrow} - i\left[(G^a_{L\uparrow}G^<_{L\downarrow} + G^>_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^<_d + \Sigma^<_R G^a_d) + (G^>_{L\uparrow}G^<_{L\downarrow})(\Sigma^a_R G^a_d)\right] \tag{102}$$

Similarly,

$$\begin{aligned} C^r &= -iB^r A^r \\ &= -i(G^a_{L\uparrow}G^<_{L\downarrow} + G^>_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^r_d) \end{aligned} \tag{103}$$

$$D^r = -i(G^a_{L\uparrow}G^<_{L\downarrow} + G^>_{L\uparrow}G^r_{L\downarrow} + G^a_{L\uparrow}G^r_{L\downarrow}), \tag{104}$$

we have

$$
\begin{aligned}
G_d^r &= -i(G^a_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^r_{L\downarrow}+G^a_{L\uparrow}G^r_{L\downarrow})-i(G^a_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^r_{L\downarrow}+G^a_{L\uparrow}G^r_{L\downarrow})(\Sigma^r_R G^r_d)\\
&= \frac{-i\left(G_{L\uparrow}G_{L\downarrow}\right)^r}{1+i\left(G_{L\uparrow}G_{L\downarrow}\right)^r\Sigma^r_R}
\end{aligned}
\tag{105}
$$

$$
\left(G_{L\uparrow}G_{L\downarrow}\right)^r \equiv G^a_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^r_{L\downarrow}+G^a_{L\uparrow}G^r_{L\downarrow} \tag{106}
$$

Now we calculate G^a_d.

$$
C^a = -iB^aA^a \tag{107}
$$

$$
B^a = G^r_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^a_{L\downarrow}+G^r_{L\uparrow}G^a_{L\downarrow} \tag{108}
$$

$$
D^a = -i(G^r_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^a_{L\downarrow}+G^r_{L\uparrow}G^a_{L\downarrow}) \tag{109}
$$

So we have

$$
G^a_d = -i(G^r_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^a_{L\downarrow}+G^r_{L\uparrow}G^a_{L\downarrow})-i(G^r_{L\uparrow}G^<_{L\downarrow}+G^>_{L\uparrow}G^a_{L\downarrow}+G^r_{L\uparrow}G^a_{L\downarrow})(\Sigma^a_R G^a_d) \tag{110}
$$

From Eq.(102) we have

$$
\begin{aligned}
G^<_d &= -iG^>_{L\uparrow}G^<_{L\downarrow}\left(1+\Sigma^a_R G^a_d\right)-i\left(G_{L\uparrow}G_{L\downarrow}\right)^r\left(\Sigma^<_q G^a_d+\Sigma^r_R G^<_d\right)\\
&= \frac{-iG^>_{L\uparrow}G^<_{L\downarrow}\left(1+\Sigma^a_R G^a_d\right)-i\left(G_{L\uparrow}G_{L\downarrow}\right)^r\Sigma^<_R G^a_d}{1+i\left(G_{L\uparrow}G_{L\downarrow}\right)^r\Sigma^r_R}\\
&= \frac{-iG^>_{L\uparrow}G^<_{L\downarrow}\left(1+\Sigma^a_R G^a_d\right)}{1+i\left(G_{L\uparrow}G_{L\downarrow}\right)^r\Sigma^r_R}+G^r_d\Sigma^<_R G^a_d\\
&= -i(G^r_d\Sigma^r_R+1)G^>_{L\uparrow}G^<_{L\downarrow}\left(1+\Sigma^a_R G^a_d\right)+G^r_d\Sigma^<_R G^a_d
\end{aligned}
\tag{111}
$$

Similarly,

$$
G^>_d = -i\left(G^r_d\Sigma^r_R+1\right)G^<_{L\uparrow}G^>_{L\downarrow}\left(1+\Sigma^a_R G^a_d\right)+G^r_d\Sigma^>_R G^a_d \tag{112}
$$

0.4.5 DC spin current

We have

$$
G^>_d(E)-G^<_d(E) = -i\left(G^r_d\Sigma^r_R+1\right)\left(G^<_{L\uparrow}G^>_{L\downarrow}-G^>_{L\uparrow}G^<_{L\downarrow}\right)\left(1+\Sigma^a_R G^a_d\right)+G^r_d\left(\Sigma^>_R-\Sigma^<_R\right)G^a_d \tag{113}
$$

$$
\left(G^>_d-G^<_d\right)\Sigma^<_R = \left[-i\left(G^r_d\Sigma^r_R+1\right)\left(G^<_{L\uparrow}G^>_{L\downarrow}-G^>_{L\uparrow}G^<_{L\downarrow}\right)\left(1+\Sigma^a_R G^a_d\right)+G^r_d\left(\Sigma^>_R-\Sigma^<_R\right)G^a_d\right]\Sigma^<_R \tag{114}
$$

Fourier transformation

$$
G^<_d(E) = \int_{-\infty}^{+\infty} dt G^<_d(t-t')e^{iE(t-t')} \tag{115}
$$

and inverse Fourier transformation

$$
G^<_d(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega G^<_d(E)e^{-iE(t-t')} \tag{116}
$$

Then,

$$
G_{L,mn\uparrow}(E) = \tag{117}
$$

$$
G^>_{L\uparrow} = 1-f_{L\uparrow} \tag{118}
$$

$$
G^<_{L\downarrow} = f_{L\downarrow} \tag{119}
$$

$$\Sigma_R^<(t,t') = -i \sum_{q_1} J_{q_1}^2 g_{Rq_1}^<(t,t') \tag{120}$$

$$\Sigma_R^a - \Sigma_R^r = \tag{121}$$

$$G_d^< \left(\Sigma_{Rq}^a - \Sigma_{Rq}^r\right) = \tag{122}$$

? $$G_{L\sigma}^< = iG_{L\sigma}^r \Gamma_{L\sigma} f_{L\sigma} G_{L\sigma}^a \equiv iD_{L\sigma} f_{L\sigma} \tag{123}$$

The following formula exists

$$\left(f_{L\uparrow} - 1\right) f_{L\downarrow} = -\left(f_{L\uparrow} - f_{L\downarrow}\right) f_L^B \tag{124}$$

where,

$$f_{L\sigma}(\epsilon) = \frac{1}{\exp\left(\beta_L\left(\epsilon - \mu_\sigma\right)\right) + 1} \tag{125}$$

$$f_L^B(\omega) = \frac{1}{\exp\left(\beta_L\left(\omega + \Delta\mu_s\right)\right) - 1} \tag{126}$$

0.5 Nonequilibrium Green's function technique

Bibliography

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