

Spin field effect transistor is a type of quantum field effect transistor that operates purely on the flow of spin current in the absence of charge current. The rotating field induces a time-independent dc spin current, and at the same time generates no charge current. The physical principle of our SFET is due to a spin flip mechanism provided by the field.

## 1 Hamiltonian

A rotating magnetic field is

$$B_x = B_0 \sin \theta \cos(\omega t) \quad (1)$$

$$B_y = B_0 \sin \theta \sin(\omega t) \quad (2)$$

$$B_z = B_0 \cos \theta. \quad (3)$$

The Hamiltonian of system is

$$H = \sum_{k, \sigma, \alpha=L,R} \epsilon_k C_{k\alpha\sigma}^+ C_{k\alpha\sigma} + \sum_{n\sigma} [\epsilon_n + \sigma B_0 \cos \theta] d_{n\sigma}^+ d_{n\sigma} + H'(t) + \sum_{k, n, \sigma, \alpha=L,R} [T_{k\alpha n} C_{k\alpha\sigma}^+ d_{n\sigma} + \text{c.c.}] \quad (4)$$

We assume that there are multiple orbits in the scattering region, which is different from the original paper, in which only one orbit is considered. The level energy is

$$\epsilon_n(t) = \epsilon_n^0 - qV_g(t), \quad (5)$$

which can be controlled by the gate voltage  $V_g$ . Energy level of lead  $\epsilon_{Lk} = \epsilon_{Rk} = \epsilon_k$ .

A counterclock-wise rotating field allows a spin-down electron to absorb a photon and flip to spin-up, and it does not allow a spin-up electron to absorb a photon and flip to spin-down.

$$H'(t) = \sum_n \gamma \left[ \exp(-i\omega t) d_{n\uparrow}^+ d_{n\downarrow} + \exp(i\omega t) d_{n\downarrow}^+ d_{n\uparrow} \right] \quad (6)$$

$$\gamma = B_0 \sin \theta \quad (7)$$

We solve the transport properties (charge and spin currents) of the model in both adiabatic and nonadiabatic regimes using the standard Keldysh nonequilibrium Green's function technique.

## 2 Operator evolution

EoM of  $d_\sigma^\dagger$  is

$$\partial_{t'} d_{n\sigma}^\dagger(t') = i[H, d_{n\sigma}^\dagger] \quad (8)$$

central dot part:

$$\left[ \sum_{n'\sigma'} [\epsilon_{n'} + \sigma B_0 \cos \theta] d_{n'\sigma'}^\dagger d_{n'\sigma'}, d_{n\sigma}^\dagger \right] = (\epsilon_n + \sigma B_0 \cos \theta) d_{n\sigma}^\dagger \quad (9)$$

rotating magnetic field part:

$$\begin{aligned} [d_{n'\uparrow}^\dagger d_{n'\downarrow}, d_{n\sigma}^\dagger] &= d_{n'\uparrow}^\dagger \{d_{n'\downarrow}, d_{n\sigma}^\dagger\} - \{d_{n'\uparrow}^\dagger, d_{n\sigma}^\dagger\} d_{n'\downarrow} \\ &= d_{n\uparrow}^\dagger \delta_{nn'} \delta_{\sigma\downarrow}, \end{aligned} \quad (10)$$

$$\begin{aligned}
[d_{n'\downarrow}^\dagger d_{n'\uparrow}, d_{n\sigma}^\dagger] &= d_{n'\downarrow}^\dagger \{d_{n'\uparrow}, d_{n\sigma}^\dagger\} - \{d_{n'\downarrow}^\dagger, d_{n\sigma}^\dagger\} d_{n'\uparrow} \\
&= d_{n'\uparrow}^\dagger \delta_{nn'} \delta_{\sigma\uparrow}.
\end{aligned} \tag{11}$$

Then

$$[H'(t), d_{n\sigma}^\dagger] = \gamma(e^{-i\omega t} d_{n\uparrow}^\dagger \delta_{\sigma\downarrow} + e^{i\omega t} d_{n\downarrow}^\dagger \delta_{\sigma\uparrow}) \tag{12}$$

Coupling part

$$\sum_{k,n',\sigma',\alpha=L,R} \left[ T_{n'k\alpha} C_{k\alpha\sigma'}^\dagger d_{n'\sigma'} + \text{c.c.}, d_{n\sigma}^\dagger \right] = \sum_{k,\alpha=L,R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger \tag{13}$$

Substitute these three parts into Eq. (8), we get

$$\partial_{t'} d_{n\sigma}^\dagger(t') = i[(\epsilon_n^0 + \sigma B_0 \cos \theta) d_{n\sigma}^\dagger + \gamma(e^{-i\omega t} d_{n\uparrow}^\dagger \delta_{\sigma\downarrow} + e^{i\omega t} d_{n\downarrow}^\dagger \delta_{\sigma\uparrow}) + \sum_{k,\alpha=L,R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger] \tag{14}$$

Equation of motion of particle operator  $\hat{N}_{\alpha k\sigma}$  in the lead  $\alpha$  is

$$\begin{aligned}
\frac{d}{dt} \hat{N}_{\alpha k\sigma} &= \frac{i}{\hbar} [H, C_{\alpha k\sigma}^\dagger C_{\alpha k\sigma}] = \left[ \sum_{k',\sigma',\alpha'=L,R} [T_{k'\alpha'} C_{k'\alpha'\sigma'}^\dagger d_{\sigma'} + \text{c.c.}], C_{\alpha k\sigma}^\dagger C_{\alpha k\sigma} \right] \\
&= \frac{i}{\hbar} \sum_{k',\sigma',\alpha'=L,R} [-T_{k'\alpha'} C_{k'\alpha'\sigma'}^\dagger d_{\sigma'} \delta_{\alpha\alpha'} \delta_{kk'} \delta_{\sigma\sigma'} + \text{c.c.}] \\
&= \frac{i}{\hbar} [-T_{k\alpha} C_{k\alpha\sigma}^\dagger d_{\sigma} + T_{k\alpha}^* d_{\sigma}^\dagger C_{k\alpha\sigma}]
\end{aligned} \tag{15}$$

### 3 Charge current

So, the charge current due to L(R) lead with spin  $\sigma$  is given by

$$\begin{aligned}
I_{L\sigma}(t) &= e \left\langle \frac{d}{dt} \hat{N}_\sigma(t) \right\rangle \\
&= \frac{ie}{\hbar} \sum_{kn\alpha \in L} (\langle -T_{k\alpha n} C_{k\alpha\sigma}^\dagger(t) d_{n\sigma}(t) \rangle + \langle T_{k\alpha n}^* d_{n\sigma}^\dagger(t) C_{k\alpha\sigma}(t) \rangle)
\end{aligned} \tag{16}$$

Define the lesser Green's function

$$G_{n\sigma',k\alpha\sigma}^<(\tau, \tau') = i \langle C_{k\alpha\sigma}^\dagger(\tau') d_{n\sigma'}(\tau) \rangle, \tag{17}$$

the current becomes

$$I_{L\sigma}(t) = \frac{-e}{\hbar} \sum_{kn\alpha \in L} (T_{k\alpha n} G_{n,k\alpha\sigma}^<(t, t) - T_{k\alpha n}^* G_{k\alpha,n}^<(t, t)) \tag{18}$$

More generally, we define the contour Green's function

$$G_{n\sigma',k\alpha\sigma}(\tau, \tau') = -i \langle d_{n\sigma'}(\tau) C_{k\alpha\sigma}^\dagger(\tau') \rangle. \tag{19}$$

EoM of operator  $C_{k\alpha\sigma}^\dagger$  is

$$\partial_{t'} C_{k\alpha\sigma}^\dagger(t') = i[H, C_{k\alpha\sigma}^\dagger] = i(\varepsilon_k C_{k\alpha\sigma}^\dagger + \sum_n T_{k\alpha n} d_{n\sigma}^\dagger) \tag{20}$$

The equation-of-motion for the time-ordered Green function

$$\begin{aligned}
-i \frac{\partial}{\partial t'} G_{n\sigma',k\alpha\sigma}^t(t, t') &= \delta(t - t') \langle \{d_{n\sigma'}, C_{k\alpha\sigma}^\dagger\} \rangle - \langle T_c d_{n\sigma'} \partial_{t'} C_{k\alpha\sigma}^\dagger \rangle \\
&= \varepsilon_k G_{n\sigma',k\alpha\sigma}^t(t, t') + \sum_m T_{k\alpha m}^* G_{n\sigma',m\sigma}^t(t, t')
\end{aligned} \tag{21}$$

So, we have

$$(-i \frac{\partial}{\partial t'} - \varepsilon_k) G_{n\sigma',k\alpha\sigma}^t(t, t') = \sum_m T_{k\alpha m}^* G_{n\sigma',m\sigma}^t(t, t') \tag{22}$$

in which

$$G_{n\sigma',m\sigma}^t(t,t') = -i\langle T_c d_{n\sigma'}(t) d_{m\sigma}^\dagger(t') \rangle. \quad (23)$$

Similarly, we evaluate the EoM for free Green's function  $g_{k\alpha\sigma}^t(t,t')$  in lead  $\alpha$  (note that  $H = \sum_{k\sigma\alpha} \epsilon_k C_{k\alpha\sigma}^\dagger C_{k\alpha\sigma}$ ).

$$\begin{aligned} -i \frac{\partial}{\partial t'} g_{k\alpha\sigma}^t(t,t') &= \delta(t-t') \langle \{C_{k\alpha\sigma}, C_{k\alpha\sigma}^\dagger\} \rangle - \langle T_c C_{k\alpha\sigma} \partial_{t'} C_{k\alpha\sigma}^\dagger \rangle \\ &= \delta(t-t') + \varepsilon_k g_{k\alpha\sigma}^t(t,t'), \end{aligned} \quad (24)$$

we have

$$(-i \frac{\partial}{\partial t'} - \varepsilon_k) g_{k\alpha\sigma}^t(t,t') = \delta(t-t'). \quad (25)$$

Substitute Eq. (25) into Eq. (22) and integrate on both sides, we get an equation analogous to Jauho's notation [2],

$$G_{n,k\alpha}(\tau, \tau') = \sum_m \int d\tau_1 G_{nm}(\tau, \tau_1) t_{k\alpha m}^* g_{k\alpha}(\tau_1, \tau'),$$

we have

$$G_{n\sigma',k\alpha\sigma}^t(t,t') = \sum_m \int dt_1 G_{n\sigma',m\sigma}(t,t_1) T_{k\alpha m}^* g_{k\alpha\sigma}^t(\tau_1, \tau'). \quad (26)$$

When there is only one orbit presents, this equation reduces to

$$G_{\sigma',k\alpha\sigma}^t(t,t') = \int dt_1 G_{\sigma',\sigma}(t,t_1) T_{k\alpha}^* g_{k\alpha\sigma}^t(\tau_1, \tau'). \quad (27)$$

Since the contour Green's function has the same structure as real-time Green's function, the we have relation

$$G_{n\sigma',k\alpha\sigma}(\tau, \tau') = \sum_m \int d\tau_1 G_{n\sigma',m\sigma}(\tau, \tau_1) T_{k\alpha m}^* g_{k\alpha\sigma}(\tau_1, \tau') \quad (28)$$

where  $G_{n\sigma',k\alpha\sigma}(\tau, \tau')$  is contour Green's function defined in Eq. (17), and similarly the contour Green's function for non-interacting lead is defined as

$$g_{k\alpha\sigma}(\tau, \tau') = -i\langle T_c C_{k\alpha\sigma}(\tau) C_{k\alpha\sigma}^\dagger(\tau') \rangle \quad (29)$$

After analytic continuation, the current is formulated as

$$\begin{aligned} I_{\alpha\sigma}(t) &= -\frac{e}{\hbar} \int dt_1 \text{Tr} [G^r(t, t_1) \Sigma_\alpha^<(t_1, t) + G^<(t, t_1) \Sigma_\alpha^a(t_1, t) + c.c.] \\ &= -\frac{e}{\hbar} \int dt_1 \text{Tr} [G^r(t, t_1) \Sigma_\alpha^<(t_1, t) + G^<(t, t_1) \Sigma_\alpha^a(t_1, t) \\ &\quad + G^a(t_1, t) (-1) \Sigma_\alpha^<(t, t_1) + (-1) G^<(t_1, t) \Sigma_\alpha^r(t, t_1)] \\ &= -\frac{e}{\hbar} \int dt_1 \text{Tr} [G^r(t, t_1) \Sigma_\alpha^<(t_1, t) + G^<(t, t_1) \Sigma_\alpha^a(t_1, t) \\ &\quad - \Sigma_\alpha^<(t, t_1) G^a(t_1, t) - \Sigma_\alpha^r(t, t_1) G^<(t_1, t)] \end{aligned} \quad (30)$$

Following Eq. (224) in WangJ's note, its Fourier transformation is

$$\begin{aligned} I_{\alpha\sigma}(\omega_1) &= -\frac{e}{\hbar} \int \frac{dE}{2\pi} \frac{dE'}{2\pi} \text{Tr} [G^r(E + \omega_1, E') \Sigma_\alpha^<(E', E) + G^<(E + \omega_1, E') \Sigma_\alpha^a(E', E) + c.c.] \\ &= -\frac{e}{\hbar} \int \frac{dE}{2\pi} \frac{dE'}{2\pi} \text{Tr} [G^r(E + \omega_1, E') \Sigma_\alpha^<(E', E) + G^<(E + \omega_1, E') \Sigma_\alpha^a(E', E) \\ &\quad - \Sigma_\alpha^<(E + \omega_1, E') G^a(E', E) - \Sigma_\alpha^r(E + \omega_1, E') G^<(E', E)] \end{aligned} \quad (31)$$

Here,  $G^{r,<} \equiv G_{n\sigma',m\sigma}^{r,<}$ , notation Tr means sum over QD level index  $n$ , and matrix element

$$\Sigma_{\alpha,mn}^\gamma(t_1, t_2) = \sum_k T_{k\alpha m}^*(t_1) g_{k\alpha}^\gamma(t_1, t_2) T_{k\alpha n}(t_2), \quad (32)$$

and

$$\Sigma_{\alpha,mn}^{\gamma}(t_1, t_2) = \sum_k T_{k\alpha m}^* (t_1) g_{k\alpha}^{\gamma}(t_1, t_2) T_{k\alpha n}(t_2) \quad (33)$$

in which,  $g_{k\sigma}$  is the free propagator of lead, and  $\gamma$  is  $\{>, <, r, a\}$ . From Keldysh equation, we have (different from Eq. (33) in Chap. II ?)

$$G^< = G^r \Sigma^< G^a \quad (34)$$

whose Fourier transformation gives

$$G^<(E_1, E_2) = \int \frac{dE}{2\pi} \frac{dE'}{2\pi} G^r(E_1, E) \Sigma^<(E, E') G^a(E', E_2) \quad (35)$$

## 4 Calculate $G^r$

For  $G^r$ , the Dyson equation gives (refer to [3])

$$G^r(t_1, t_2) = G^{0r}(t_1 - t_2) + \int dt G^{0r}(t_1 - t) H'(t) G^r(t, t_2) \quad (36)$$

in which,  $G^{0r}$  is not the free propagator of central dot, but the equilibrium Green's function when the pumping potential  $H$  is set to zero, i.e. the hamiltonian is  $H - H'$ , not  $\varepsilon_n d_{n\sigma}^\dagger d_{n\sigma}$ . Multiply  $e^{iE_1 t_1 - iE_2 t_2}$  and integrate on both sides, we have

$$\begin{aligned} G^r(E_1, E_2) &\equiv \int dt_1 dt_2 e^{iE_1 t_1 - iE_2 t_2} G^r(t_1, t_2) \\ &= 2\pi G^{0r}(E_1) \delta(E_1 - E_2) \\ &\quad + \int dt_1 G^{0r}(t_1 - t) e^{iE_1(t_1 - t)} H'(t) e^{i(E_1 - E)t} \iint dt dt_2 G^r(t, t_2) e^{iEt - E_2 t_2} \\ &= 2\pi G^{0r}(E_1) \delta(E_1 - E_2) + \int \frac{dE}{2\pi} G^{0r}(E_1) H'(E_1 - E) G^r(E, E_2), \end{aligned} \quad (37)$$

in which, we inserted the inverse Fourier transformation

$$H'(t) = \int \frac{dE}{2\pi} e^{-i(E_1 - E)t} H'(E_1 - E). \quad (38)$$

### 4.1 Calcualte $G^{0r}$

Using Eq. (9) and Eq. (13), we have

$$\partial_t d_{n\sigma}^\dagger = i(\epsilon_n + \sigma B_0 \cos \theta) d_{n\sigma}^\dagger + i \sum_{k, \alpha=L, R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger \quad (39)$$

$$\partial_t d_{n\sigma} = -i(\epsilon_n + \sigma B_0 \cos \theta) d_{n\sigma} - i \sum_{k, \alpha=L, R} T_{k\alpha n}^* C_{k\alpha\sigma}^\dagger \quad (40)$$

then

$$d_{n\sigma}^\dagger(t) = d_{n\sigma}^\dagger(0) e^{i(\epsilon_n + \sigma B_0 \cos \theta)t} + it \sum_{k, \alpha=L, R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger \quad (41)$$

$$d_{n\sigma}(t) = d_{n\sigma}(0) e^{-i(\epsilon_n + \sigma B_0 \cos \theta)t} - it \sum_{k, \alpha=L, R} T_{k\alpha n}^* C_{k\alpha\sigma} \quad (42)$$

Note  $\epsilon_n$  is time-dependent,  $G^{0r}(t, t')$  depends only on time difference. The hamiltonian  $H - H'$  cannot flip spin, so in the spin space  $\{d_{1,\uparrow}^\dagger, d_{1,\downarrow}^\dagger, d_{2,\uparrow}^\dagger, d_{2,\downarrow}^\dagger, \dots; d_{1,\uparrow}, d_{1,\downarrow}, d_{2,\uparrow}, d_{2,\downarrow}, \dots\}$ ,  $G^{0r}$  is diagonal

$$G^{0r}(E) = \bigotimes_n G_n^{0r}(E) \quad (43)$$

$$G_n^{0r}(E) = \begin{pmatrix} G_{n\uparrow}^{0r}(E) & 0 \\ 0 & G_{n\downarrow}^{0r}(E) \end{pmatrix}. \quad (44)$$

## 4.2 Calcualte $G^r$

In spin space

$$G^r \equiv \begin{pmatrix} G_{n\uparrow,m\uparrow}^r & G_{n\uparrow,m\downarrow}^r \\ G_{n\downarrow,m\uparrow}^r & G_{n\downarrow,m\downarrow}^r \end{pmatrix} \quad (45)$$

is a matrix of  $2N$  dimension, where  $N$  is the total number of levels in central area.  $H'$  in Eq. (36) is a matrix of same dimension, given by

$$H' = \bigotimes_n H'_n, \quad (46)$$

$$H'_n = \begin{pmatrix} 0 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & 0 \end{pmatrix}. \quad (47)$$

For simplicity, we consider only one energy level and neglect level index. we have

$$\begin{aligned} H'(E_1 - E) &= \int dt e^{i(E_1 - E)t} H'(t) \\ &= \begin{pmatrix} 0 & \gamma \int dt e^{i(E_1 - E - \omega)t} \\ \gamma \int dt e^{i(E_1 - E + \omega)t} & 0 \end{pmatrix} \\ &= 2\pi\gamma \begin{pmatrix} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{pmatrix} \end{aligned} \quad (48)$$

Substitute these equations into Eq. (37), we get

$$\begin{aligned} \begin{pmatrix} G_{\uparrow,\uparrow}^r(E_1, E_2) & G_{\uparrow,\downarrow}^r(E_1, E_2) \\ G_{\downarrow,\uparrow}^r(E_1, E_2) & G_{\downarrow,\downarrow}^r(E_1, E_2) \end{pmatrix} &= \begin{pmatrix} G_{\uparrow}^{0r}(E_1) & 0 \\ 0 & G_{\downarrow}^{0r}(E_1) \end{pmatrix} 2\pi\delta(E_1 - E_2) \\ &\quad + \gamma \int dE \begin{pmatrix} G_{\uparrow}^{0r}(E_1) & 0 \\ 0 & G_{\downarrow}^{0r}(E_1) \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{pmatrix} \begin{pmatrix} G_{\uparrow,\uparrow}^r & G_{\uparrow,\downarrow}^r \\ G_{\downarrow,\uparrow}^r & G_{\downarrow,\downarrow}^r \end{pmatrix} \\ &= \begin{pmatrix} G_{\uparrow}^{0r}(E_1) & 0 \\ 0 & G_{\downarrow}^{0r}(E_1) \end{pmatrix} 2\pi\delta(E_1 - E_2) \\ &\quad + \begin{pmatrix} G_1(E_1, E_2) & G_2(E_1, E_2) \\ G_3(E_1, E_2) & G_4(E_1, E_2) \end{pmatrix}. \end{aligned} \quad (49)$$

where we have omitted the independent variables  $(E, E_2)$  of  $G_{n\sigma,m\sigma'}^r$  for sake of convenience. In the above equations  $G_1, G_2, G_3, G_4$  are

$$G_1(E_1, E_2) = \gamma \int dE G_{\uparrow}^{0r}(E_1) \delta(E_1 - E - \omega) G_{\downarrow,\uparrow}^r(E, E_2) \quad (50)$$

$$G_2(E_1, E_2) = \gamma \int dE G_{\uparrow}^{0r}(E_1) \delta(E_1 - E - \omega) G_{\downarrow,\downarrow}^r(E, E_2) \quad (51)$$

$$G_3(E_1, E_2) = \gamma \int dE G_{\downarrow}^{0r}(E_1) \delta(E_1 - E + \omega) G_{\uparrow,\uparrow}^r(E, E_2) \quad (52)$$

$$G_4(E_1, E_2) = \gamma \int dE G_{\downarrow}^{0r}(E_1) \delta(E_1 - E + \omega) G_{\uparrow,\downarrow}^r(E, E_2) \quad (53)$$

So, the  $G_1$  term

$$\begin{aligned} G_{\uparrow,\uparrow}^r(E_1, E_2) &= G_{\uparrow}^{0r}(E_1) 2\pi\delta(E_1 - E_2) + G_1(E_1, E_2) \\ &= 2\pi G_{\uparrow}^{0r}(E_1) \delta(E_1 - E_2) + \gamma G_{\uparrow}^{0r}(E_1) G_{\downarrow,\uparrow}^r(E_1 - \omega, E_2) \end{aligned} \quad (54)$$

The  $G_2$  term

$$\begin{aligned} G_{\uparrow,\downarrow}^r(E_1, E_2) &= G_2(E_1, E_2) \\ &= \gamma G_{\uparrow}^{0r}(E_1) G_{\downarrow,\downarrow}^r(E_1 - \omega, E_2) \end{aligned} \quad (55)$$

The  $G_3$  term

$$\begin{aligned} G_{\downarrow,\uparrow}^r(E_1, E_2) &= G_3(E_1, E_2) \\ &= \gamma G_{\downarrow}^{0r}(E_1) G_{\uparrow,\uparrow}^r(E_1 + \omega, E_2) \end{aligned} \quad (56)$$

and the  $G_4$  term

$$\begin{aligned} G_{\downarrow,\downarrow}^r(E_1, E_2) &= g_{\downarrow}^r(E_1) 2\pi \delta(E_1 - E_2) + G_4(E_1, E_2) \\ &= 2\pi G_{\downarrow}^{0r}(E_1) \delta(E_1 - E_2) + \gamma G_{\downarrow}^{0r}(E_1) G_{\uparrow,\downarrow}^r(E_1 + \omega, E_2) \end{aligned} \quad (57)$$

After collecting terms, we get

$$\begin{aligned} G_{\uparrow,\uparrow}^r(E_1, E_2) &= 2\pi G_{\uparrow}^{0r}(E_1) \delta(E_1 - E_2) + \gamma G_{\uparrow}^{0r}(E_1) \gamma G_{\downarrow}^{0r}(E_1 - \omega) G_{\uparrow,\uparrow}^r(E_1, E_2) \\ &= \frac{2\pi G_{\uparrow}^{0r}(E_1) \delta(E_1 - E_2)}{1 - \gamma^2 G_{\uparrow}^{0r}(E_1) G_{\downarrow}^{0r}(E_1 - \omega)} \end{aligned} \quad (58)$$

and

$$G_{\downarrow,\downarrow}^r(E_1, E_2) = \frac{2\pi G_{\downarrow}^{0r}(E_1) \delta(E_1 - E_2)}{1 - \gamma^2 G_{\downarrow}^{0r}(E_1) G_{\uparrow}^{0r}(E_1 + \omega)} \quad (59)$$

and

$$\begin{aligned} G_{\uparrow,\downarrow}^r(E_1, E_2) &= \gamma G_{\uparrow}^{0r}(E_1) G_{\downarrow,\downarrow}^r(E_1 - \omega, E_2) \\ &= \frac{2\pi \gamma G_{\uparrow}^{0r}(E_1) G_{\downarrow}^{0r}(E_1 - \omega) \delta(E_1 - \omega - E_2)}{1 - \gamma^2 G_{\downarrow}^{0r}(E_1 - \omega) G_{\uparrow}^{0r}(E_1)} \end{aligned} \quad (60)$$

$$\begin{aligned} G_{\downarrow,\uparrow}^r(E_1, E_2) &= \gamma G_{\downarrow}^{0r}(E_1) G_{\uparrow,\uparrow}^r(E_1 + \omega, E_2) \\ &= \frac{2\pi \gamma G_{\downarrow}^{0r}(E_1 + \omega) G_{\uparrow}^{0r}(E_1) \delta(E_1 + \omega - E_2)}{1 - \gamma^2 G_{\uparrow}^{0r}(E_1 + \omega) G_{\downarrow}^{0r}(E_1)} \end{aligned} \quad (61)$$

Define

$$g_{\sigma}^{r,a}(E) \equiv \frac{G_{\sigma}^{0r,a}(E)}{1 - \gamma^2 G_{\sigma}^{0r,a}(E) G_{\bar{\sigma}}^{0r,a}(E + \bar{\sigma}\omega)}, \quad (62)$$

and denote

$$G^r(E_1, E_2) = \begin{pmatrix} G_{11}^r(E_1, E_2) & G_{12}^r(E_1, E_2) \\ G_{21}^r(E_1, E_2) & G_{22}^r(E_1, E_2) \end{pmatrix} \quad (63)$$

then we can write

$$G_{11}^r(E_1, E_2) = 2\pi g_{\uparrow}^r(E_1) \delta(E_1 - E_2). \quad (64)$$

$$G_{22}^r(E_1, E_2) = 2\pi g_{\downarrow}^r(E_1) \delta(E_1 - E_2). \quad (65)$$

$$G_{12}^r(E_1, E_2) = 2\pi \gamma g_{\uparrow}^r(E_1) G_{\downarrow}^{0r}(E_1 - \omega) \delta(E_1 - \omega - E_2). \quad (66)$$

$$G_{21}^r(E_1, E_2) = 2\pi \gamma g_{\downarrow}^r(E_1) G_{\uparrow}^{0r}(E_1 + \omega) \delta(E_1 + \omega - E_2). \quad (67)$$

Thus we get  $G^r$  in Eq. (45).

## 5 Calculate $G^<$

### 5.1 Calcualte $\Sigma_{\alpha}$

In Eq. (32),

$$\Sigma_{\alpha}^{\gamma}(t_1, t_2) = \sum_k T_{k\alpha}^*(t_1) g_{k\alpha}^{\gamma}(t_1, t_2) T_{k\alpha}(t_2). \quad (68)$$

Here we consider parametric pumping, thus no bias presents in the leads as demonstrated in the system Hamiltonian, i.e.  $\Sigma_a(t_1, t_2) \rightarrow \Sigma_a(t_1 - t_2)$ . Using free propagators  $g_{k\alpha\sigma}^{\gamma}$ , Fourier-transform to

$$\Sigma_{\alpha\sigma}^<(E_1, E_2) = 2\pi \Sigma_{\alpha\sigma}^<(E_1) \delta(E_1 - E_2), \quad (69)$$

in which

$$\Sigma_{\alpha\sigma}^<(E) = if(E)\Gamma_{\alpha}(E). \quad (70)$$

The linewidth function  $\Gamma$  is defined as

$$\Gamma_{\alpha}(E) \equiv 2\pi \sum_k T_{k\alpha}^* T_{k\alpha} \delta(E - \epsilon_k), \quad (71)$$

thus we have  $\Sigma_{\sigma}$ , which is a number in spin space, not a matrix, since  $\Sigma^<$  is independent of spin  $\sigma$ . Similarly,

$$\Sigma_{\alpha\sigma}^a(E_1, E_2) = 2\pi \Sigma_{\alpha\sigma}^a(E_1) \delta(E_1 - E_2), \quad (72)$$

and the retarded(advanced) self-energy is (?)

$$\Sigma_{\alpha\sigma}^{r,a}(E) = \Lambda_{\alpha}(E) \mp \frac{i}{2} \Gamma_{\alpha}(E), \quad (73)$$

since

$$\Sigma_{\alpha\sigma}^a(E) = [\Sigma_{\alpha\sigma}^r(E)]^* \quad (74)$$

and

$$\begin{aligned} \Sigma_{\alpha\sigma}^a(E) - \Sigma_{\alpha\sigma}^r(E) &= i\Gamma_{\alpha\sigma}(E) \\ &= 2i\text{Im}\{\Sigma_{\alpha\sigma}^a(E)\}. \end{aligned} \quad (75)$$

## 5.2 Calculate $G^a$

We have relation

$$G_{\sigma,\sigma'}^a(E_1, E_2) = (G_{\sigma',\sigma}^r(E_2, E_1))^*, \quad (76)$$

$$G^r(E_1, E_2) = \begin{pmatrix} G_{11}^r(E_1, E_2) & G_{12}^r(E_1, E_2) \\ G_{21}^r(E_1, E_2) & G_{22}^r(E_1, E_2) \end{pmatrix} \quad (77)$$

so in spin space,

$$G^a(E_1, E_2) = \begin{pmatrix} (G_{11}^r(E_2, E_1))^* & (G_{21}^r(E_2, E_1))^* \\ (G_{12}^r(E_2, E_1))^* & (G_{22}^r(E_2, E_1))^* \end{pmatrix} \quad (78)$$

## 5.3 Calculate $G^<$

Substitute  $G^r, \Sigma^<, G^a$  into Eq. (34), we get  $G^<$ ,

$$\begin{aligned} G^<(E_1, E_2) &= \int \frac{dE}{2\pi} \frac{dE'}{2\pi} G^r(E_1, E) \Sigma^<(E, E') G^a(E', E_2) \\ &= \int \frac{dE}{2\pi} G^r(E_1, E) \Sigma^<(E) G^a(E, E_2) \\ &= \int \frac{dE}{2\pi} if(E) \Gamma(E) G^r(E_1, E) G^a(E, E_2) \end{aligned} \quad (79)$$

$$G^r(E_1, E) G^a(E, E_2) = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \quad (80)$$

in which, the  $G_1$  term is

$$\begin{aligned} G_1 &= G_{11}^r(E_1, E) G_{11}^{r*}(E_2, E) + G_{12}^r(E_1, E) G_{12}^{r*}(E_2, E) \\ &= (2\pi)^2 g_{\uparrow}^r(E_1) g_{\uparrow}^{r*}(E_2) \delta(E_2 - E) \delta(E_1 - E) \\ &\quad + (2\pi\gamma)^2 g_{\uparrow}^r(E_1) G_{\downarrow}^{0r}(E_1 - \omega) g_{\uparrow}^{r*}(E_2) G_{\downarrow}^{0r*}(E_2 - \omega) \delta(E_1 - \omega - E) \delta(E_2 - \omega - E) \end{aligned} \quad (81)$$

the  $G_2$  term is

$$G_2 = G_{11}^r(E_1, E) G_{21}^{r*}(E_2, E) + G_{12}^r(E_1, E) G_{22}^{r*}(E_2, E) \quad (82)$$

the  $G_3$  term is

$$G_3 = G_{21}^r(E_1, E) G_{11}^r(E_2, E) + G_{22}^r(E_1, E) G_{12}^{r*}(E_2, E) \quad (83)$$

the  $G_4$  term is

$$\begin{aligned} G_4 &= G_{21}^r(E_1, E)G_{21}^{r*}(E_2, E) + G_{22}^r(E_1, E)G_{22}^{r*}(E_2, E) \\ &= (2\pi)^2 g_{\downarrow}^r(E_1) g_{\downarrow}^{r*}(E_2) \delta(E_2 - E) \delta(E_1 - E) \\ &\quad + (2\pi\gamma)^2 g_{\downarrow}^r(E_1) G_{\uparrow}^{0r}(E_1 + \omega) g_{\downarrow}^{r*}(E_2) G_{\uparrow}^{0r*}(E_2 + \omega) \delta(E_1 + \omega - E) \delta(E_2 + \omega - E) \end{aligned} \quad (84)$$

So, the matrix element of  $G^<$  is

$$G^<(E_1, E_2) = \begin{pmatrix} G_{11}^< & G_{12}^< \\ G_{21}^< & G_{22}^< \end{pmatrix} \quad (85)$$

$$\begin{aligned} G_{11}^<(E_1, E_2) &= \int \frac{dE}{2\pi} i f(E) \Gamma(E) (2\pi)^2 g_{\uparrow}^r(E_1) g_{\uparrow}^{r*}(E_2) \delta(E_2 - E) \delta(E_1 - E) \\ &\quad + \int \frac{dE}{2\pi} i f(E) \Gamma(E) (2\pi\gamma)^2 g_{\uparrow}^r(E_1) G_{\downarrow}^{0r}(E_1 - \omega) g_{\uparrow}^{r*}(E_2) G_{\downarrow}^{0r*}(E_2 - \omega) \delta(E_1 - \omega - E) \delta(E_2 - \omega - E) \\ &= 2\pi i f(E_1) \Gamma(E_1) g_{\uparrow}^r(E_1) g_{\uparrow}^{r*}(E_2) \delta(E_2 - E_1) \\ &\quad + 2\pi\gamma^2 i f(E_1 - \omega) \Gamma(E_1 - \omega) g_{\uparrow}^r(E_1) G_{\downarrow}^{0r}(E_1 - \omega) g_{\uparrow}^{r*}(E_2) G_{\downarrow}^{0r*}(E_2 - \omega) \delta(E_2 - E_1) \end{aligned} \quad (86)$$

$$\begin{aligned} G_{22}^<(E_1, E_2) &= 2\pi i f(E_1) \Gamma(E_1) g_{\downarrow}^r(E_1) g_{\downarrow}^{r*}(E_2) \delta(E_2 - E_1) \\ &\quad + 2\pi\gamma^2 i f(E_1 + \omega) \Gamma(E_1 + \omega) g_{\downarrow}^r(E_1) G_{\uparrow}^{0r}(E_1 + \omega) g_{\downarrow}^{r*}(E_2) G_{\uparrow}^{0r*}(E_2 + \omega) \delta(E_2 - E_1) \end{aligned} \quad (87)$$

## 6 Spin-up current

The spin-up current (31) is

$$\begin{aligned} I_{\alpha\uparrow, \uparrow}(\omega_1) &= -\frac{e}{\hbar} \int \frac{dE}{2\pi} \frac{dE'}{2\pi} [G_{11}^r(E + \omega_1, E') 2\pi i f(E') \Gamma_{\alpha}(E') \delta(E' - E) + G_{11}^<(E + \omega_1, E') 2\pi \Sigma_{\alpha}^a(E') \delta(E' - E) \\ &\quad - 2\pi i f(E + \omega_1) \Gamma_{\alpha}(E + \omega_1) \delta(E + \omega_1 - E') G_{11}^a(E', E) - 2\pi \Sigma_{\alpha}^r(E + \omega_1) \delta(E + \omega_1 - E') G_{11}^<(E', E)] \\ &= -\frac{e}{\hbar} \int \frac{dE}{2\pi} [G_{11}^r(E + \omega_1, E) i f(E) \Gamma_{\alpha}(E) + G_{11}^<(E + \omega_1, E) \Sigma_{\alpha}^a(E) \\ &\quad - i f(E + \omega_1) \Gamma_{\alpha}(E + \omega_1) G_{11}^a(E + \omega_1, E) - \Sigma_{\alpha}^r(E + \omega_1) G_{11}^<(E + \omega_1, E)]. \end{aligned} \quad (88)$$

Recall that

$$G_{11}^r(E_1, E_2) = 2\pi g_{\uparrow}^r(E_1) \delta(E_1 - E_2) \quad (89)$$

$$G_{11}^a(E_1, E_2) = 2\pi g_{\uparrow}^a(E_1) \delta(E_1 - E_2) \quad (90)$$

$$G_{11}^<(E_1, E_2) = 2\pi [A_{11}(E_1, E_2) + B_{11}(E_1, E_2)] \delta(E_2 - E_1), \quad (91)$$

in which

$$A_{11}(E_1, E_2) = i f(E_1) \Gamma(E_1) g_{\uparrow}^r(E_1) g_{\uparrow}^{r*}(E_2) \quad (92)$$

$$B_{11}(E_1, E_2) = \gamma^2 i f(E_1 - \omega) \Gamma(E_1 - \omega) g_{\uparrow}^r(E_1) G_{\downarrow}^{0r}(E_1 - \omega) g_{\uparrow}^{r*}(E_2) G_{\downarrow}^{0r*}(E_2 - \omega). \quad (93)$$

Substitute Eq. (89) (90) and (91) into (88), we have

$$\begin{aligned} I_{\alpha\uparrow, \uparrow}(\omega_1) &= -\frac{e}{\hbar} \int dE g_{\uparrow}^r(E + \omega_1) i f(E) \Gamma_{\alpha}(E) \delta(\omega_1) + [A_{11}(E + \omega_1, E) + B_{11}(E + \omega_1, E)] \delta(-\omega_1) \Sigma_{\alpha}^a(E) \\ &\quad - i f(E + \omega_1) \Gamma_{\alpha}(E + \omega_1) g_{\uparrow}^a(E + \omega_1) \delta(\omega_1) - \Sigma_{\alpha}^r(E + \omega_1) [A_{11}(E + \omega_1, E) + B_{11}(E + \omega_1, E)] \delta(-\omega_1) \end{aligned} \quad (94)$$

Omit  $\omega_1$ , we get a DC current

$$\begin{aligned} I_{\alpha\uparrow, \uparrow} &= -\frac{e}{\hbar} \int dE g_{\uparrow}^r(E) i f(E) \Gamma_{\alpha}(E) + [A_{11}(E) + B_{11}(E)] \Sigma_{\alpha}^a(E) \\ &\quad - i f(E) \Gamma_{\alpha}(E) g_{\uparrow}^a(E) - \Sigma_{\alpha}^r(E) [A_{11}(E) + B_{11}(E)] \\ &= -\frac{e}{\hbar} \int dE [g_{\uparrow}^r(E) - g_{\uparrow}^a(E)] i f(E) \Gamma_{\alpha}(E) + [A_{11}(E) + B_{11}(E)] i \Gamma_{\alpha}(E), \end{aligned} \quad (95)$$



in which,

$$A_{11}(E) = if(E)\Gamma(E)|g_{\uparrow}^r(E)|^2, \quad (96)$$

$$B_{11}(E) = \gamma^2 if(E - \omega)\Gamma(E - \omega)|g_{\uparrow}^r(E)|^2|G_{\downarrow}^{0r}(E - \omega)|^2. \quad (97)$$

Using the following relations

$$g_{\uparrow}^r(E) = \frac{G_{\uparrow}^{0r}(E)}{1 - \gamma^2 G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)}, \quad (98)$$

$$g_{\uparrow}^a(E) = \frac{G_{\uparrow}^{0a}(E)}{1 - \gamma^2 G_{\uparrow}^{0a}(E)G_{\downarrow}^{0a}(E - \omega)}, \quad (99)$$

in which,

$$G_{\uparrow}^{0r}(E) = \frac{1}{E - \varepsilon^0 - B_0 \cos \theta + i\Gamma(E)/2} \quad (100)$$

$$G_{\downarrow}^{0r}(E - \omega) = \frac{1}{E - \omega - \varepsilon^0 + B_0 \cos \theta + i\Gamma(E)/2}, \quad (101)$$

note that

$$G_{\uparrow}^{0r}(E) - G_{\uparrow}^{0a}(E) = -i\Gamma(E)|G_{\uparrow}^{0r}(E)|^2 \quad (102)$$

$$G_{\downarrow}^{0r}(E - \omega) - G_{\downarrow}^{0a}(E - \omega) = -i\Gamma(E - \omega)|G_{\downarrow}^{0r}(E - \omega)|^2, \quad (103)$$

we get

$$\begin{aligned} g_{\uparrow}^r(E) - g_{\uparrow}^a(E) &= \frac{G_{\uparrow}^{0r}(E)[1 - \gamma^2 G_{\uparrow}^{0a}(E)G_{\downarrow}^{0a}(E - \omega)] - G_{\uparrow}^{0a}(E)[1 - \gamma^2 G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)]}{|1 - \gamma^2 G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)|^2} \\ &= \frac{-i\Gamma(E)|G_{\uparrow}^{0r}(E)|^2 - i\gamma^2 \Gamma(E - \omega)|G_{\uparrow}^{0r}|^2}{|1 - \gamma^2 G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)|^2}, \end{aligned} \quad (104)$$

which cancels  $A_{11}$  term in DC current equation and yields

$$I_{\alpha\uparrow,\uparrow} = -\frac{e}{\hbar} \int dE \gamma^2 \Gamma(E - \omega) \Gamma_{\alpha}(E) |g_{\uparrow}^r(E)|^2 |G_{\downarrow}^{0r}(E - \omega)|^2 [f(E) - f(E - \omega)], \quad (105)$$

## 7 Spin-down current

From Eq. (105) we have

$$I_{\alpha\downarrow,\downarrow} = -\frac{e}{\hbar} \int dE \gamma^2 \Gamma(E + \omega) \Gamma_{\alpha}(E) |g_{\downarrow}^r(E)|^2 |G_{\uparrow}^{0r}(E + \omega)|^2 [f(E) - f(E + \omega)]. \quad (106)$$

## 8 Check charge current

In parametric pumping setup, left and right leads are identical, i.e.  $f_L(E) = f_R(E)$ , no charge current is induced. Notice that

$$\begin{aligned} g_{\uparrow}^r(E)G_{\downarrow}^{0r}(E - \omega) &= \frac{G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)}{1 - \gamma^2 G_{\uparrow}^{0r}(E)G_{\downarrow}^{0r}(E - \omega)} \\ &= G_{\uparrow}^{0r}(E)g_{\downarrow}^r(E - \omega) \end{aligned} \quad (107)$$

then the spin-up current is

$$I_{\alpha\uparrow,\uparrow} = -\frac{e}{\hbar} \int dE \gamma^2 \Gamma(E - \omega) \Gamma_{\alpha}(E) |g_{\downarrow}^r(E - \omega)|^2 |G_{\uparrow}^{0r}(E)|^2 [f(E) - f(E - \omega)]. \quad (108)$$

The integral keeps the same when the independent variable is shifted, we make  $E \rightarrow E + \omega$ ,

$$I_{\alpha\uparrow,\uparrow} = -\frac{e}{\hbar} \int dE \gamma^2 \Gamma(E) \Gamma_{\alpha}(E + \omega) |g_{\downarrow}^r(E)|^2 |G_{\uparrow}^{0r}(E + \omega)|^2 [f(E + \omega) - f(E)]. \quad (109)$$

Note that  $\Gamma(E) = \Gamma_L(E) + \Gamma_R(E) = 2\Gamma_{\alpha}(E)$ , so if  $\Gamma_{\alpha}(E + \omega) = \Gamma_{\alpha}(E - \omega)$ , we get

$$I_{\alpha e} = I_{\alpha\uparrow,\uparrow} + I_{\alpha\downarrow,\downarrow} = 0. \quad (110)$$

## References

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