Notes on PRB.67.092408

1 spin field effect transistor

A type of quantum field effect transistor that operates purely on the flow of spin current in the absence of charge current. The rotating field induces a time-independent dc spin current, and at the same time generates no charge cur- rent. The physical principle of our SFET is due to a spin flip mecha- nism provided by the field.

2 Hamiltonian

A rotating magnetic field is

$$B_x = B_0 \mathrm{sin}\theta \; \cos(\omega \mathrm{t}) \tag{1}$$

$$B_y = B_0 \mathrm{sin}\theta \; \sin(\omega \mathrm{t}) \tag{2}$$

$$B_z = B_0 \mathrm{cos}\theta. \tag{3}$$

The Hamiltonian of system is

$$\begin{aligned} H = & \sum_{k,\sigma,\alpha=L,R} \epsilon_k C^+_{k\alpha\sigma} C_{k\alpha\sigma} + \sum_{n\sigma} \left[\epsilon_n + \sigma B_0 \cos\theta\right] d^+_{n\sigma} d_{n\sigma} \\ & + H'(t) + \sum_{k,n,\sigma,\alpha=L,R} \left[T_{k\alpha n} C^+_{k\alpha\sigma} d_{n\sigma} + \text{c.c.}\right] \end{aligned} \tag{4}$$

We assume that there are multiple orbits in the scattering region, which is different from the original paper, in which only one orbit is considered. The level energy is

$$\epsilon_n(t) = \epsilon^0_n - qV_g(t), \tag{5}$$

which can be controlled by the gate voltage V_g. Energy level of lead $\epsilon_{Lk} = \epsilon_{Rk} = \epsilon_k$.

A counterclock-wise rotating field allows a spin-down electron to absorb a photon and flip to spin-up, and it does not allow a spin-up electron to absorb a photon and flip to spin-down.

$$H'(t) = \sum_n \gamma \left[\exp(-i\omega t) d^+_{n\uparrow} d_{n\downarrow} + \exp(i\omega t) d^+_{n\downarrow} d_{n\uparrow}\right] \tag{6}$$

$$\gamma = B_0 \sin\theta \tag{7}$$

We solve the transport properties (charge and spin currents) of the model in both adiabatic and nonadiabatic regimes using the standard Keldysh nonequilibrium Green's function technique.

3 Operator evolution

EoM of $d_\sigma^\dagger$ is

$$\partial_{t'} d_{n\sigma}^\dagger(t') = i[H, d_{n\sigma}^\dagger] \tag{8}$$

central dot part:

$$[\sum_{n'\sigma'} [\epsilon_{n'} + \sigma B_0 \cos\theta]\, d_{n'\sigma'}^\dagger d_{n'\sigma'}, d_{n\sigma}^\dagger] = (\epsilon_n + \sigma B_0 \cos\theta) d_{n\sigma}^\dagger \tag{9}$$

rotating magnetic field part:

$$\begin{aligned}[d_{n'\uparrow}^\dagger d_{n'\downarrow}, d_{n\sigma}^\dagger] &= d_{n'\uparrow}^\dagger \{d_{n'\downarrow}, d_{n\sigma}^\dagger\} - \{d_{n'\uparrow}^\dagger, d_{n\sigma}^\dagger\} d_{n'\downarrow} \\ &= d_\uparrow^\dagger \delta_{nn'} \delta_{\sigma\downarrow},\end{aligned} \tag{10}$$

$$\begin{aligned}[d_{n'\downarrow}^\dagger d_{n'\uparrow}, d_{n\sigma}^\dagger] &= d_{n'\downarrow}^\dagger \{d_{n'\uparrow}, d_{n\sigma}^\dagger\} - \{d_{n'\downarrow}^\dagger, d_{n\sigma}^\dagger\} d_{n'\uparrow} \\ &= d_{n'\uparrow}^\dagger \delta_{nn'} \delta_{\sigma\uparrow}.\end{aligned} \tag{11}$$

Then

$$[H'(t), d_{n\sigma}^\dagger] = \gamma(e^{-i\omega t} d_{n\uparrow}^\dagger \delta_{\sigma\downarrow} + e^{i\omega t} d_{n\downarrow}^\dagger \delta_{\sigma\uparrow}) \tag{12}$$

Coupling part

$$\sum_{k,n',\sigma',\alpha=L,R} \left[T_{n'k\alpha} C_{k\alpha\sigma'}^\dagger d_{n'\sigma'} + \text{c.c.}\ , d_{n\sigma}^\dagger\right] = \sum_{k,\alpha=L,R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger \tag{13}$$

Substitute these three parts into Eq. (8), we get

$$\partial_{t'} d_{n\sigma}^\dagger(t') = i[(\epsilon_n^0 + \sigma B_0 \cos\theta) d_{n\sigma}^\dagger + \gamma(e^{-i\omega t} d_{n\uparrow}^\dagger \delta_{\sigma\downarrow} + e^{i\omega t} d_{n\downarrow}^\dagger \delta_{\sigma\uparrow}) + \sum_{k,\alpha=L,R} T_{k\alpha n} C_{k\alpha\sigma}^\dagger] \tag{14}$$

Equation of motion of particle operator $\hat{N}_{\alpha k\sigma}$ in the lead α is

$$\begin{aligned}\frac{d}{dt}\hat{N}_{\alpha k\sigma} = \frac{i}{\hbar}[H, C_{\alpha k\sigma}^\dagger C_{\alpha k\sigma}] &= \left[\sum_{k',\sigma',\alpha'=L,R} \left[T_{k'\alpha'} C_{k'\alpha'\sigma'}^+ d_{\sigma'} + \text{c.c.}\right], C_{\alpha k\sigma}^\dagger C_{\alpha k\sigma}\right] \\ &= \frac{i}{\hbar} \sum_{k',\sigma',\alpha'=L,R} \left[-T_{k'\alpha'} C_{k'\alpha'\sigma'}^+ d_{\sigma'} \delta_{\alpha\alpha'} \delta_{kk'} \delta_{\sigma\sigma'} + \text{c.c.}\right] \\ &= \frac{i}{\hbar}[-T_{k\alpha} C_{k\alpha\sigma}^\dagger d_\sigma + T_{k\alpha}^* d_\sigma^\dagger C_{k\alpha\sigma}]\end{aligned} \tag{15}$$

4 Charge current

So, the charge current due to L(R) lead with spin σ is given by

$$\begin{aligned}I_{L\sigma}(t) &= e\langle \frac{d}{dt}\hat{N}_\sigma(t)\rangle \\ &= \frac{ie}{\hbar} \sum_{kn\alpha\in L} (\langle -T_{k\alpha n} C_{k\alpha\sigma}^\dagger(t) d_{n\sigma}(t)\rangle + \langle T_{k\alpha n}^* d_{n\sigma}^\dagger(t) C_{k\alpha\sigma}(t)\rangle)\end{aligned} \tag{16}$$

Define the lesser Green's function

$$G_{n\sigma',k\alpha\sigma}^<(\tau,\tau') = i\langle C_{k\alpha\sigma}^\dagger(\tau') d_{n\sigma'}(\tau)\rangle, \tag{17}$$

the current becomes

$$I_{L\sigma}(t) = \frac{-e}{\hbar} \sum_{kn\alpha \in L} (T_{k\alpha n} G^{<}_{n,k\alpha\sigma}(t,t) - T^{*}_{k\alpha n} G^{<}_{k\alpha,n}(t,t))) \tag{18}$$

More generally, we define the contour Green's function

$$G_{n\sigma',k\alpha\sigma}(\tau,\tau') = -i\langle d_{n\sigma'}(\tau) C^{\dagger}_{k\alpha\sigma}(\tau')\rangle. \tag{19}$$

EoM of operator $C^{\dagger}_{k\alpha\sigma}$ is

$$\partial_{t'} C^{\dagger}_{k\alpha\sigma}(t') = i[H, C^{\dagger}_{k\alpha\sigma}] = i(\varepsilon_k C^{\dagger}_{k\alpha\sigma} + \sum_n T_{k\alpha n} d^{\dagger}_{n\sigma}) \tag{20}$$

The equation-of-motion for the time-ordered Green function

$$\begin{aligned} -i\frac{\partial}{\partial t'} G^{t}_{n\sigma',k\alpha\sigma}\left(t,t'\right) &= \delta(t-t')\langle\{d_{n\sigma'}, C^{\dagger}_{k\alpha\sigma}\}\rangle - \langle T_c d_{n\sigma'} \partial_{t'} C^{\dagger}_{k\alpha\sigma}\rangle \\ &= \varepsilon_k G^{t}_{n\sigma',k\alpha\sigma}\left(t,t'\right) + \sum_m T^{*}_{k\alpha m} G^{t}_{n\sigma',m\sigma}(t,t') \end{aligned} \tag{21}$$

So, we have

$$(-i\frac{\partial}{\partial t'} - \varepsilon_k) G^{t}_{n\sigma',k\alpha\sigma}\left(t,t'\right) = \sum_m T^{*}_{k\alpha m} G^{t}_{n\sigma',m\sigma}(t,t') \tag{22}$$

in which

$$G^{t}_{n\sigma',m\sigma}(t,t') = -i\langle T_c d_{n\sigma'}(t) d^{\dagger}_{m\sigma}(t')\rangle. \tag{23}$$

Similarly, we evaluate the EoM for free Green's function $g^{t}_{k\alpha\sigma}(t,t')$ in lead α (note that $H = \sum_{k\sigma\alpha} \epsilon_k C^{\dagger}_{k\alpha\sigma} C_{k\alpha\sigma}$).

$$\begin{aligned} -i\frac{\partial}{\partial t'} g^{t}_{k\alpha\sigma}\left(t,t'\right) &= \delta(t-t')\langle\{C_{k\alpha\sigma}, C^{\dagger}_{k\alpha\sigma}\}\rangle - \langle T_c C_{k\alpha\sigma} \partial_{t'} C^{\dagger}_{k\alpha\sigma}\rangle \\ &= \delta(t-t') + \varepsilon_k g^{t}_{k\alpha\sigma}\left(t,t'\right), \end{aligned} \tag{24}$$

we have

$$(-i\frac{\partial}{\partial t'} - \varepsilon_k) g^{t}_{k\alpha\sigma}\left(t,t'\right) = \delta(t-t'). \tag{25}$$

Substitute Eq. (25) into Eq. (22) and integrate on both sides, we get an equation analogous to Jauho's notation [2],

$$G_{n,k\alpha}(\tau,\tau') = \sum_m \int d\tau_1 G_{nm}\left(\tau,\tau_1\right) t^{*}_{k\alpha m} g_{k\alpha}\left(\tau_1,\tau'\right),$$

we have

$$G^{t}_{n\sigma',k\alpha\sigma}(t,t') = \sum_m \int dt_1 G_{n\sigma',m\sigma}\left(t,t_1\right) T^{*}_{k\alpha m} g^{t}_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{26}$$

When there is only one orbit presents, this equation reduces to

$$G^{t}_{\sigma',k\alpha\sigma}(t,t') = \int dt_1 G_{\sigma',\sigma}\left(t,t_1\right) T^{*}_{k\alpha} g^{t}_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{27}$$

Since the contour Green's function has the same structure as real-time Green's function, the we have relation

$$G_{n\sigma',k\alpha\sigma}(\tau,\tau') = \sum_m \int d\tau_1 G_{n\sigma',m\sigma}\left(\tau,\tau_1\right) T^{*}_{k\alpha m} g_{k\alpha\sigma}\left(\tau_1,\tau'\right) \tag{28}$$

where $G_{n\sigma',k\alpha\sigma}(\tau,\tau')$ is contour Green's function defined in Eq. (17), and similarly the contour Green's function for non-interacting lead is defined as

$$g_{k\alpha\sigma}(\tau,\tau') = -i\langle T_c C_{k\alpha\sigma}(\tau) C^{\dagger}_{k\alpha\sigma}(\tau')\rangle \tag{29}$$

After analytic continuation, the current is formulated as

$$\begin{aligned} I_{\alpha\sigma}(t) = -\frac{e}{\hbar}\int dt_1\,\mathrm{Tr}\left[G^r(t,t_1)\,\Sigma^<_\alpha(t_1,t)\right. \\ \left.+G^<(t,t_1)\,\Sigma^a_\alpha(t_1,t)\right] + h.c. \end{aligned} \tag{30}$$

Following Eq. (224) in WangJ's note, its Fourier transformation is

$$\begin{aligned} I_{\alpha\sigma}(\omega) = -\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi}\mathrm{Tr}\left[G^r\left(E+\omega,E'\right)\Sigma^<_\alpha\left(E',E\right)\right. \\ \left.+\,G^<(E+\omega,E')\Sigma^a_\alpha(E',E)\right] + c.c. \end{aligned} \tag{31}$$

Here, $G^{r,<} \equiv G^{r,<}_{n\sigma',m\sigma}$, notation Tr means sum over QD level index n, and matrix element

$$\Sigma^\gamma_{\alpha,mn}(t_1,t_2) = \sum_k T^*_{k\alpha m}(t_1) g^\gamma_{k\alpha}(t_1,t_2) T_{k\alpha n}(t_2), \tag{32}$$

and

$$\Sigma^\gamma_{\alpha,mn}(t_1,t_2) = \sum_k T^*_{k\alpha m}(t_1)\, g^\gamma_{k\alpha}(t_1,t_2)\, T_{k\alpha n}(t_2) \tag{33}$$

in which, $g_{k\sigma}$ is the free propagator of lead, and γ is $\{>, <, \mathrm{r}, \mathrm{a}\}$. From Keldysh equation, we have (different from Eq. (33) in Chap. II ?)

$$G^< = G^r \Sigma^< G^a \tag{34}$$

whose Fourier transformation gives

$$G^<(E_1,E_2) = \int \frac{dE}{2\pi}\frac{dE'}{2\pi} G^r(E_1,E)\Sigma^<(E,E')G^a(E',E_2) \tag{35}$$

5 Calculate G^r

For G^r, the Dyson equation gives (refer to [3])

$$G^r(t_1,t_2) = G^{0r}(t_1-t_2) + \int dt\; G^{0r}(t_1-t)H'(t)G^r(t,t_2) \tag{36}$$

in which, G^{0r} is not the free propagator of central dot, but the equilibrium Green's function when the pumping potential H is set to zero, i.e. the hamiltonian is $H - H'$, not $\varepsilon_n d^\dagger_{n\sigma} d_{n\sigma}$. Multiply $e^{iE_1 t_1 - iE_2 t_2}$ and integrate on both sides, we have

$$\begin{aligned} G^r(E_1,E_2) \equiv& \int dt_1 dt_2 e^{iE_1t_1 - iE_2t_2} G^r(t_1,t_2) \\ =& 2\pi G^{0r}(E_1)\delta(E_1-E_2) \\ &+\int dt_1 G^{0r}(t_1-t)e^{iE_1(t_1-t)}H'(t)e^{i(E_1-E)t}\iint dt dt_2 G^r(t,t_2)e^{iEt-E_2t_2} \\ =& 2\pi G^{0r}(E_1)\delta(E_1-E_2) + \int\frac{dE}{2\pi}G^{0r}(E_1)H'(E_1-E)G^r(E,E_2), \end{aligned} \tag{37}$$

in which, we inserted the inverse Fourier transformation

$$H'(t) = \int \frac{dE}{2\pi} e^{-i(E_1-E)t} H'(E_1-E). \tag{38}$$

5.1 Calcualte G^{0r}

Using Eq. (9) and Eq. (13), we have

$$\partial_t d^\dagger_{n\sigma} = i(\epsilon_n + \sigma B_0 \cos\theta) d^\dagger_{n\sigma} + i \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{39}$$

$$\partial_t d_{n\sigma} = -i(\epsilon_n + \sigma B_0 \cos\theta) d_{n\sigma} - i \sum_{k,\alpha=L,R} T^*_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{40}$$

then

$$d^\dagger_{n\sigma}(t) = d^\dagger_{n\sigma}(0) e^{i(\epsilon_n + \sigma B_0 \cos\theta)t} + it \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{41}$$

$$d_{n\sigma}(t) = d_{n\sigma}(0) e^{-i(\epsilon_n + \sigma B_0 \cos\theta)} - it \sum_{k,\alpha=L,R} T^*_{k\alpha n} C_{k\alpha\sigma} \tag{42}$$

Note ϵ_n is time-dependent, $G^{0r}(t,t')$ depends only on time difference.

$$\begin{aligned} G^{0r}_{n\sigma}(t,t') =& - i\theta(t-t')\langle\{d_{n\sigma}(t), d^\dagger_{n\sigma}(t')\}\rangle \\ =& ? \end{aligned} \tag{43}$$

The hamiltonian $H - H'$ cannot flip spin, so in the spin space $\{d^\dagger_{1,\uparrow}, d^\dagger_{1,\downarrow}, d^\dagger_{2\uparrow}, d^\dagger_{2\downarrow}, \cdots; d_{1\uparrow}, d_{1\downarrow}, d_{2\uparrow}, d_{2\downarrow}, \cdots\}$, G^{0r} is diagonal

$$G^{0r}(E) = \bigotimes_n G^{0r}_n(E) \tag{44}$$

$$G^{0r}_n(E) = \begin{pmatrix} G^{0r}_{n\uparrow}(E) & 0 \\ 0 & G^{0r}_{n\downarrow}(E) \end{pmatrix}. \tag{45}$$

5.2 Calcualte G^r

In spin space

$$G^r \equiv \begin{pmatrix} G^r_{n\uparrow,m\uparrow} & G^r_{n\uparrow,m\downarrow} \\ G^r_{n\downarrow,m\uparrow} & G^r_{n\downarrow,m\downarrow} \end{pmatrix} \tag{46}$$

is a matrix of $2N$ dimension, where N is the total number of levels in central area. H' in Eq. (36)is a matrix of same dimension, given by

$$H' = \bigotimes_n H'_n, \tag{47}$$

$$H'_n = \begin{pmatrix} 0 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & 0 \end{pmatrix}. \tag{48}$$

For simplicity, we consider only one energy level and neglect level index. we have

$$\begin{aligned} H'(E_1 - E) =& \int dt e^{i(E_1-E)t} H'(t) \\ =& \begin{pmatrix} 0 & \gamma \int dt e^{i(E_1-E-\omega)t} \\ \gamma \int dt e^{i(E_1-E+\omega)t} & 0 \end{pmatrix} \\ =& 2\pi\gamma \begin{pmatrix} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{pmatrix} \end{aligned} \tag{49}$$

Substitute these equations into Eq. (37), we get

$$
\begin{aligned}
\begin{pmatrix} G^r_{\uparrow,\uparrow}(E_1,E_2) & G^r_{\uparrow,\downarrow}(E_1,E_2) \\ G^r_{\downarrow,\uparrow}(E_1,E_2) & G^r_{\downarrow,\downarrow}(E_1,E_2) \end{pmatrix} &= \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} 2\pi\delta(E_1-E_2) \\
&\quad + \gamma \int dE \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & \delta(E_1-E-\omega) \\ \delta(E_1-E+\omega) & 0 \end{pmatrix} \begin{pmatrix} G^r_{\uparrow,\uparrow} & G^r_{\uparrow,\downarrow} \\ G^r_{\downarrow,\uparrow} & G^r_{\downarrow,\downarrow} \end{pmatrix} \\
&= \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} 2\pi\delta(E_1-E_2) \\
&\quad + \begin{pmatrix} G_1(E_1,E_2) & G_2(E_1,E_2) \\ G_3(E_1,E_2) & G_4(E_1,E_2) \end{pmatrix}.
\end{aligned}
\tag{50}
$$

where we have omitted the independent variables (E, E_2) of $G^r_{n\sigma,m\sigma'}$ for sake of convenience. In the above equations G_1, G_2, G_3, G_4 are

$$G_1(E_1,E_2) = \gamma \int dE G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E-\omega)G^r_{\downarrow,\uparrow}(E,E_2) \tag{51}$$

$$G_2(E_1,E_2) = \gamma \int dE G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E-\omega)G^r_{\downarrow,\downarrow}(E,E_2) \tag{52}$$

$$G_3(E_1,E_2) = \gamma \int dE G^{0r}_{\downarrow}(E_1)\,\delta(E_1-E+\omega)G^r_{\uparrow,\uparrow}(E,E_2) \tag{53}$$

$$G_4(E_1,E_2) = \gamma \int dE G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E+\omega)G^r_{\uparrow,\downarrow}(E,E_2) \tag{54}$$

So, the G_1 term

$$
\begin{aligned}
G^r_{\uparrow,\uparrow}(E_1,E_2) &= G^{0r}_{\uparrow}(E_1)\,2\pi\delta(E_1-E_2) + G_1(E_1,E_2) \\
&= 2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E_2) + \gamma G^{0r}_{\uparrow}(E_1)\,G^r_{\downarrow,\uparrow}(E_1-\omega,E_2)
\end{aligned}
\tag{55}
$$

The G_2 term

$$
\begin{aligned}
G^r_{\uparrow,\downarrow}(E_1,E_2) =& G_2(E_1,E_2) \\
=& \gamma G^{0r}_{\uparrow}(E_1)\,G^r_{\downarrow,\downarrow}(E_1-\omega,E_2)
\end{aligned}
\tag{56}
$$

The G_3 term

$$
\begin{aligned}
G^r_{\downarrow,\uparrow}(E_1,E_2) &= G_3(E_1,E_2) \\
&= \gamma G^{0r}_{\downarrow}(E_1)\,G^r_{\uparrow,\uparrow}(E_1+\omega,E_2)
\end{aligned}
\tag{57}
$$

and the G_4 term

$$
\begin{aligned}
G^r_{\downarrow,\downarrow}(E_1,E_2) &= g^r_{\downarrow}(E_1)\,2\pi\delta(E_1-E_2) + G_4(E_1,E_2) \\
&= 2\pi G^{0r}_{\downarrow}(E_1)\,\delta(E_1-E_2) + \gamma G^{0r}_{\uparrow}(E_1) G^r_{\uparrow,\downarrow}(E_1+\omega,E_2)
\end{aligned}
\tag{58}
$$

After collecting terms, we get

$$
\begin{aligned}
G^r_{\uparrow,\uparrow}(E_1,E_2) &= 2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E_2) + \gamma G^{0r}_{\uparrow}(E_1)\,\gamma G^{0r}_{\downarrow}(E_1-\omega) G^r_{\uparrow,\uparrow}(E_1,E_2) \\
&= \frac{2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1)\,G^{0r}_{\downarrow}(E_1-\omega)}
\end{aligned}
\tag{59}
$$

and

$$G_{\downarrow,\downarrow}^{r}(E_1,E_2)=\frac{2\pi G_{\downarrow}^{0r}\left(E_1\right)\delta\left(E_1-E_2\right)}{1-\gamma^2 G_{\uparrow}^{0r}\left(E_1\right)G_{\uparrow}^{0r}\left(E_1+\omega\right)} \tag{60}$$

and

$$\begin{aligned}G_{\uparrow,\downarrow}^{r}\left(E_1,E_2\right)&=\gamma G_{\uparrow}^{0r}\left(E_1\right)G_{\downarrow,\downarrow}^{r}(E_1-\omega,E_2)\\&=\frac{2\pi\gamma G_{\uparrow}^{0r}\left(E_1\right)G_{\downarrow}^{0r}\left(E_1-\omega\right)\delta\left(E_1-\omega-E_2\right)}{1-\gamma^2 G_{\uparrow}^{0r}\left(E_1-\omega\right)G_{\uparrow}^{0r}\left(E_1\right)}\end{aligned} \tag{61}$$

$$\begin{aligned}G_{\downarrow,\uparrow}^{r}\left(E_1,E_2\right)&=\gamma G_{\downarrow}^{0r}\left(E_1\right)G_{\uparrow,\uparrow}^{r}(E_1+\omega,E_2)\\&=\frac{2\pi\gamma G_{\uparrow}^{0r}\left(E_1+\omega\right)G_{\downarrow}^{0r}\left(E_1\right)\delta(E_1+\omega-E_2)}{1-\gamma^2 G_{\uparrow}^{0r}\left(E_1+\omega\right)G_{\downarrow}^{0r}\left(E_1\right)}\end{aligned} \tag{62}$$

Thus we get G^r in Eq. (46).

6 Calculate $G^<$

6.1 Calcualte Σ_α

In Eq. (32),

$$\Sigma_{\alpha}^{\gamma}\left(t_1,t_2\right)=\sum_k T_{k\alpha}^{*}\left(t_1\right)g_{k\alpha}^{\gamma}\left(t_1,t_2\right)T_{k\alpha}\left(t_2\right). \tag{63}$$

Here we consider parametric pumping, thus no bias presents in the leads as demonstrated in the system Hamiltonian, i.e. $\Sigma_a(t_1,t_2)\to\Sigma_a(t_1-t_2)$. Using free propagators $g_{k\alpha\sigma}^{\gamma}$, Fourier-transform to

$$\Sigma_{\alpha\sigma}^{<}(E_1,E_2)=2\pi\Sigma_{\alpha\sigma}^{<}(E_1)\delta(E_1-E_2), \tag{64}$$

in which

$$\Sigma_{\alpha\sigma}^{<}(E)=if(E)\Gamma_{\alpha}\left(E\right). \tag{65}$$

The linewidth function Γ is defined as

$$\Gamma_{\alpha}(E)\equiv 2\pi\sum_k T_{k\alpha}^{*}T_{k\alpha}\delta\left(E-\epsilon_k\right), \tag{66}$$

thus we have Σ_σ, which is a number in spin space, not a matrix, since $\Sigma^<$ is independent of spin σ.

6.2 Calculate G^a

We have relation

$$G_{\sigma,\sigma'}^{a}\left(E_1,E_2\right)=(G_{\sigma',\sigma}^{r}\left(E_2,E_1\right))^{*}, \tag{67}$$

$$G^r=\begin{pmatrix}G_{\uparrow,\uparrow}^{r} & G_{\downarrow,\uparrow}^{r}\\ G_{\uparrow,\downarrow}^{r} & G_{\downarrow,\downarrow}^{r}\end{pmatrix} \tag{68}$$

so in spin space,

$$G^a=\begin{pmatrix}(G_{\uparrow,\uparrow}^{r})^{*} & (G_{\uparrow,\downarrow}^{r})^{*}\\ (G_{\downarrow,\uparrow}^{r})^{*} & (G_{\downarrow,\downarrow}^{r})^{*}\end{pmatrix} \tag{69}$$

6.3 Calculate $G^<$

Substitute $G^r, \Sigma^<, G^a$ into Eq. (34), we get $G^<$,

$$\begin{aligned} G^<(E_1,E_2) &= \int \frac{dE}{2\pi}\frac{dE'}{2\pi} G^r(E_1,E)\Sigma^<(E,E')G^a(E',E_2) \\ &= \int \frac{dE}{2\pi} G^r\left(E_1,E\right)\Sigma^<\left(E\right)G^a\left(E,E_2\right) \\ &= \int \frac{dE}{2\pi} if(E)\Gamma(E)G^r\left(E_1,E\right)G^a\left(E,E_2\right) \end{aligned} \tag{70}$$

$$G^r\left(E_1,E\right)G^a\left(E,E_2\right) = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{71}$$

in which, the G_1 term is

$$\begin{aligned} G_1 &= G^r_{\uparrow,\uparrow}(E_1,E)G^{r,*}_{\uparrow,\uparrow}(E,E_2) + G^r_{\downarrow,\uparrow}(E_1,E)G^{r,*}_{\downarrow,\uparrow}(E,E_2) \\ &= \frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \left(\frac{2\pi G^{0r}_{\uparrow}\left(E\right)\delta\left(E-E_2\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E\right)G^{0r}_{\downarrow}\left(E-\omega\right)}\right)^* \\ &+ \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \left(\frac{2\pi\gamma G^{0r}_{\uparrow}\left(E+\omega\right)G^{0r}_{\downarrow}\left(E\right)\delta(E+\omega-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E+\omega\right)G^{0r}_{\downarrow}\left(E\right)}\right)^* \\ &= \frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \frac{2\pi G^{0a}_{\uparrow}\left(E\right)\delta\left(E-E_2\right)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E\right)G^{0a}_{\downarrow}\left(E-\omega\right)} \\ &+ \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)\delta(E+\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)} \end{aligned} \tag{72}$$

the G_2 term is

$$G_2 = G^r_{\uparrow,\uparrow}(E_1,E)G^{r,*}_{\uparrow,\downarrow}(E,E_2) + G^r_{\downarrow,\uparrow}(E_1,E)G^{r,*}_{\downarrow,\downarrow}(E,E_2) \tag{73}$$

the G_3 term is

$$G_3 = G^r_{\uparrow,\downarrow}(E_1,E)G^r_{\uparrow,\uparrow}(E,E_2) + G^r_{\downarrow,\downarrow}(E_1,E)G^{r,*}_{\downarrow,\uparrow}(E,E_2) \tag{74}$$

the G_4 term is

$$G_4 = G^r_{\uparrow,\downarrow}(E_1,E)G^r_{\uparrow,\downarrow}(E,E_2) + G^r_{\downarrow,\downarrow}(E_1,E)G^{r,*}_{\downarrow,\downarrow}(E,E_2) \tag{75}$$

So, the matrix element of $G^<$ is

$$G^<\left(E_1,E_2\right) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{76}$$

$$\begin{aligned} G_{11} &= \int \frac{dE}{2\pi} if(E)\Gamma(E)\frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \frac{2\pi G^{0a}_{\uparrow}\left(E\right)\delta\left(E-E_2\right)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E\right)G^{0a}_{\downarrow}\left(E-\omega\right)} \\ &+ \int \frac{dE}{2\pi} if(E)\Gamma(E)\frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)\delta(E+\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)} \\ &= if(E_1)\Gamma(E_1)\frac{G^{0r}_{\uparrow}\left(E_1\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \frac{2\pi G^{0a}_{\uparrow}\left(E_1\right)\delta\left(E_1-E_2\right)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E_1\right)G^{0a}_{\downarrow}\left(E_1-\omega\right)} \\ &+ if(E_1+\omega)\Gamma(E_1+\omega)\frac{\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}\left(E_1+2\omega\right)G^{0a}_{\downarrow}\left(E_1+\omega\right)\delta(E_1+2\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E_1+2\omega\right)G^{0a}_{\downarrow}\left(E_1+\omega\right)} \end{aligned} \tag{77}$$

7 Appendix

7.1 Double Fourier transformation

Note the double Fourier transformation, if

$$F(t) = \int dt_1 G_1(t, t_1) G_2(t_1, t) \tag{78}$$

then

$$F(\omega) = \int \frac{dE}{2\pi} \frac{dE'}{2\pi} G_1(E + \omega, E') G_2(E', E). \tag{79}$$

If

$$F(t_1, t_2) = \int dt G_1(t_1, t) G_2(t, t_2), \tag{80}$$

then

$$F(E_1, E_2) = \int \frac{dE}{2\pi} G_1(E_1, E) G_2(E, E_2). \tag{81}$$

If $F(t_1, t_2)$ depends only on $t_1 - t_2$,

$$F(E_1, E_2) = 2\pi F(E_1) \delta(E_1 - E_2) \tag{82}$$

Note that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1. \tag{83}$$

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x - a) dx = f(a) \tag{84}$$

8 Adiabatic regime(ω is small)

So, the charge current is given by (why? DC?)

$$dQ_{\alpha\sigma}(t)/dt = q \int \frac{dE}{2\pi} (-\partial_E f) [\Gamma_\alpha \mathbf{G}^r(t) \mathbf{\Delta G}^a(t)]_{\sigma\sigma} \tag{85}$$

References

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