Notes on PRB.67.092408

1 spin field effect transistor

A type of quantum field effect transistor that operates purely on the flow of spin current in the absence of charge current. The rotating field induces a time-independent dc spin current, and at the same time generates no charge cur- rent. The physical principle of our SFET is due to a spin flip mecha- nism provided by the field.

2 Hamiltonian

A rotating magnetic field is

$$B_x = B_0 \sin\theta \, \cos(\omega \mathrm{t}) \tag{1}$$

$$B_y = B_0 \sin\theta \, \sin(\omega \mathrm{t}) \tag{2}$$

$$B_z = B_0 \cos\theta. \tag{3}$$

The Hamiltonian of system is

$$\begin{aligned} H = &\sum_{k,\sigma,\alpha=L,R} \epsilon_k C^+_{k\alpha\sigma} C_{k\alpha\sigma} + \sum_{n\sigma} \left[\epsilon_n + \sigma B_0 \cos\theta\right] d^+_{n\sigma} d_{n\sigma} \\ &+ H'(t) + \sum_{k,n,\sigma,\alpha=L,R} \left[T_{k\alpha n} C^+_{k\alpha\sigma} d_{n\sigma} + \mathrm{c.c.}\right] \end{aligned} \tag{4}$$

We assume that there are multiple orbits in the scattering region, which is different from the original paper, in which only one orbit is considered. The level energy is

$$\epsilon_n(t) = \epsilon^0_n - qV_g(t), \tag{5}$$

which can be controlled by the gate voltage V_g. Energy level of lead $\epsilon_{Lk} = \epsilon_{Rk} = \epsilon_k$.

A counterclock-wise rotating field allows a spin-down electron to absorb a photon and flip to spin-up, and it does not allow a spin-up electron to absorb a photon and flip to spin-down.

$$H'(t) = \sum_n \gamma \left[\exp(-i\omega t) d^+_{n\uparrow} d_{n\downarrow} + \exp(i\omega t) d^+_{n\downarrow} d_{n\uparrow}\right] \tag{6}$$

$$\gamma = B_0 \sin\theta \tag{7}$$

We solve the transport properties (charge and spin currents) of the model in both adiabatic and nonadiabatic regimes using the standard Keldysh nonequilibrium Green's function technique.

3 Operator evolution

EoM of $d^\dagger_\sigma$ is

$$\partial_{t'} d^\dagger_{n\sigma}(t') = i[H, d^\dagger_{n\sigma}] \tag{8}$$

central dot part:

$$[\sum_{n'\sigma'} \left[\epsilon_{n'} + \sigma B_0 \cos\theta\right] d^\dagger_{n'\sigma'} d_{n'\sigma'}, d^\dagger_{n\sigma}] = (\epsilon_n + \sigma B_0 \cos\theta) d^\dagger_{n\sigma} \tag{9}$$

rotating magnetic field part:

$$\begin{aligned} [d^\dagger_{n'\uparrow} d_{n'\downarrow}, d^\dagger_{n\sigma}] &= d^\dagger_{n'\uparrow}\{d_{n'\downarrow}, d^\dagger_{n\sigma}\} - \{d^\dagger_{n'\uparrow}, d^\dagger_{n\sigma}\} d_{n'\downarrow} \\ &= d^\dagger_{\uparrow} \delta_{nn'} \delta_{\sigma\downarrow}, \end{aligned} \tag{10}$$

$$
\begin{aligned}
[d^\dagger_{n'\downarrow} d_{n'\uparrow}, d^\dagger_{n\sigma}] &= d^\dagger_{n'\downarrow}\{d_{n'\uparrow}, d^\dagger_{n\sigma}\} - \{d^\dagger_{n'\downarrow}, d^\dagger_{n\sigma}\} d_{n'\uparrow} \\
&= d^\dagger_{n'\uparrow}\delta_{nn'}\delta_{\sigma\uparrow}.
\end{aligned} \tag{11}
$$

Then

$$
[H'(t), d^\dagger_{n\sigma}] = \gamma(e^{-i\omega t} d^\dagger_{n\uparrow}\delta_{\sigma\downarrow} + e^{i\omega t} d^\dagger_{n\downarrow}\delta_{\sigma\uparrow}) \tag{12}
$$

Coupling part

$$
\sum_{k,n',\sigma',\alpha=L,R} \left[T_{n'k\alpha} C^\dagger_{k\alpha\sigma'} d_{n'\sigma'} + \text{c.c.}\ , d^\dagger_{n\sigma}\right] = \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{13}
$$

Substitute these three parts into Eq. (8), we get

$$
\partial_{t'} d^\dagger_{n\sigma}(t') = i[(\epsilon^0_n + \sigma B_0 \cos\theta) d^\dagger_{n\sigma} + \gamma(e^{-i\omega t} d^\dagger_{n\uparrow}\delta_{\sigma\downarrow} + e^{i\omega t} d^\dagger_{n\downarrow}\delta_{\sigma\uparrow}) + \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma}] \tag{14}
$$

Equation of motion of particle operator $\hat{N}_{\alpha k\sigma}$ in the lead α is

$$
\begin{aligned}
\frac{d}{dt}\hat{N}_{\alpha k\sigma} = \frac{i}{\hbar}[H, C^\dagger_{\alpha k\sigma} C_{\alpha k\sigma}] &= \left[\sum_{k',\sigma',\alpha'=L,R} \left[T_{k'\alpha'} C^+_{k'\alpha'\sigma'} d_{\sigma'} + \text{c.c.}\right], C^\dagger_{\alpha k\sigma} C_{\alpha k\sigma}\right] \\
= \frac{i}{\hbar} \sum_{k',\sigma',\alpha'=L,R} &\left[-T_{k'\alpha'} C^+_{k'\alpha'\sigma'} d_{\sigma'} \delta_{\alpha\alpha'}\delta_{kk'}\delta_{\sigma\sigma'} + \text{c.c.}\right] \\
= \frac{i}{\hbar}[-T_{k\alpha} C^\dagger_{k\alpha\sigma} d_\sigma &+ T^*_{k\alpha} d^\dagger_\sigma C_{k\alpha\sigma}]
\end{aligned} \tag{15}
$$

4 Charge current

So, the charge current due to L(R) lead with spin σ is given by

$$
\begin{aligned}
I_{L\sigma}(t) &= e\langle \frac{d}{dt}\hat{N}_\sigma(t)\rangle \\
&= \frac{ie}{\hbar}\sum_{kn\alpha\in L} (\langle -T_{k\alpha n} C^\dagger_{k\alpha\sigma}(t) d_{n\sigma}(t)\rangle + \langle T^*_{k\alpha n} d^\dagger_{n\sigma}(t) C_{k\alpha\sigma}(t)\rangle)
\end{aligned} \tag{16}
$$

Define the lesser Green's function

$$
G^<_{n\sigma',k\alpha\sigma}(\tau,\tau') = i\langle C^\dagger_{k\alpha\sigma}(\tau') d_{n\sigma'}(\tau)\rangle, \tag{17}
$$

the current becomes

$$
I_{L\sigma}(t) = \frac{-e}{\hbar}\sum_{kn\alpha\in L} (T_{k\alpha n} G^<_{n,k\alpha\sigma}(t,t) - T^*_{k\alpha n} G^<_{k\alpha,n}(t,t))) \tag{18}
$$

More generally, we define the contour Green's function

$$
G_{n\sigma',k\alpha\sigma}(\tau,\tau') = -i\langle d_{n\sigma'}(\tau) C^\dagger_{k\alpha\sigma}(\tau')\rangle. \tag{19}
$$

EoM of operator $C^\dagger_{k\alpha\sigma}$ is

$$
\partial_{t'} C^\dagger_{k\alpha\sigma}(t') = i[H, C^\dagger_{k\alpha\sigma}] = i(\varepsilon_k C^\dagger_{k\alpha\sigma} + \sum_n T_{k\alpha n} d^\dagger_{n\sigma}) \tag{20}
$$

The equation-of-motion for the time-ordered Green function

$$
\begin{aligned}
-i\frac{\partial}{\partial t'} G^t_{n\sigma',k\alpha\sigma}(t,t') &= \delta(t-t')\langle\{d_{n\sigma'}, C^\dagger_{k\alpha\sigma}\}\rangle - \langle T_c d_{n\sigma'} \partial_{t'} C^\dagger_{k\alpha\sigma}\rangle \\
&= \varepsilon_k G^t_{n\sigma',k\alpha\sigma}(t,t') + \sum_m T^*_{k\alpha m} G^t_{n\sigma',m\sigma}(t,t')
\end{aligned} \tag{21}
$$

So, we have

$$
(-i\frac{\partial}{\partial t'} - \varepsilon_k) G^t_{n\sigma',k\alpha\sigma}(t,t') = \sum_m T^*_{k\alpha m} G^t_{n\sigma',m\sigma}(t,t') \tag{22}
$$

in which

$$G^t_{n\sigma',m\sigma}(t,t') = -i\langle T_c d_{n\sigma'}(t) d^\dagger_{m\sigma}(t')\rangle. \tag{23}$$

Similarly, we evaluate the EoM for free Green's function $g^t_{k\alpha\sigma}(t,t')$ in lead α (note that $H = \sum_{k\sigma\alpha} \epsilon_k C^\dagger_{k\alpha\sigma} C_{k\alpha\sigma}$).

$$\begin{aligned} -i\frac{\partial}{\partial t'} g^t_{k\alpha\sigma}\left(t,t'\right) &= \delta(t-t')\langle\{C_{k\alpha\sigma}, C^\dagger_{k\alpha\sigma}\}\rangle - \langle T_c C_{k\alpha\sigma}\partial_{t'} C^\dagger_{k\alpha\sigma}\rangle \\ &= \delta(t-t') + \varepsilon_k g^t_{k\alpha\sigma}\left(t,t'\right), \end{aligned} \tag{24}$$

we have

$$(-i\frac{\partial}{\partial t'} - \varepsilon_k) g^t_{k\alpha\sigma}\left(t,t'\right) = \delta(t-t'). \tag{25}$$

Substitute Eq. (25) into Eq. (22) and integrate on both sides, we get an equation analogous to Jauho's notation [2],

$$G_{n,k\alpha}(\tau,\tau') = \sum_m \int d\tau_1 G_{nm}\left(\tau,\tau_1\right) t^*_{k\alpha m} g_{k\alpha}\left(\tau_1,\tau'\right),$$

we have

$$G^t_{n\sigma',k\alpha\sigma}(t,t') = \sum_m \int dt_1 G_{n\sigma',m\sigma}\left(t,t_1\right) T^*_{k\alpha m} g^t_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{26}$$

When there is only one orbit presents, this equation reduces to

$$G^t_{\sigma',k\alpha\sigma}(t,t') = \int dt_1 G_{\sigma',\sigma}\left(t,t_1\right) T^*_{k\alpha} g^t_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{27}$$

Since the contour Green's function has the same structure as real-time Green's function, the we have relation

$$G_{n\sigma',k\alpha\sigma}(\tau,\tau') = \sum_m \int d\tau_1 G_{n\sigma',m\sigma}\left(\tau,\tau_1\right) T^*_{k\alpha m} g_{k\alpha\sigma}\left(\tau_1,\tau'\right) \tag{28}$$

where $G_{n\sigma',k\alpha\sigma}(\tau,\tau')$ is contour Green's function defined in Eq. (17), and similarly the contour Green's function for non-interacting lead is defined as

$$g_{k\alpha\sigma}(\tau,\tau') = -i\langle T_c C_{k\alpha\sigma}(\tau) C^\dagger_{k\alpha\sigma}(\tau')\rangle \tag{29}$$

After analytic continuation, the current is formulated as

$$\begin{aligned} I_{\alpha\sigma}(t) = -\frac{e}{\hbar}\int dt_1 \operatorname{Tr}\big[&G^r\left(t,t_1\right)\Sigma^<_\alpha\left(t_1,t\right) \\ &+G^<\left(t,t_1\right)\Sigma^a_\alpha\left(t_1,t\right)\big] + h.c. \end{aligned} \tag{30}$$

Following Eq. (224) in WangJ's note, its Fourier transformation is

$$\begin{aligned} I_{\alpha\sigma}(\omega_1) = -\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi}\operatorname{Tr}\big[&G^r\left(E+\omega_1,E'\right)\Sigma^<_\alpha\left(E',E\right) \\ &+ G^<(E+\omega_1,E')\Sigma^a_\alpha(E',E)\big] + c.c. \end{aligned} \tag{31}$$

Here, $G^{r,<} \equiv G^{r,<}_{n\sigma',m\sigma}$, notation Tr means sum over QD level index n, and matrix element

$$\Sigma^\gamma_{\alpha,mn}(t_1,t_2) = \sum_k T^*_{k\alpha m}(t_1) g^\gamma_{k\alpha}(t_1,t_2) T_{k\alpha n}(t_2), \tag{32}$$

and

$$\Sigma^\gamma_{\alpha,mn}\left(t_1,t_2\right) = \sum_k T^*_{k\alpha m}\left(t_1\right) g^\gamma_{k\alpha}\left(t_1,t_2\right) T_{k\alpha n}\left(t_2\right) \tag{33}$$

in which, $g_{k\sigma}$ is the free propagator of lead, and γ is {>, <, r, a}. From Keldysh equation, we have (different from Eq. (33) in Chap. II ?)

$$G^< = G^r\Sigma^< G^a \tag{34}$$

whose Fourier transformation gives

$$G^<(E_1,E_2) = \int \frac{dE}{2\pi}\frac{dE'}{2\pi} G^r(E_1,E)\Sigma^<(E,E')G^a(E',E_2) \tag{35}$$

5 Calculate G^r

For G^r, the Dyson equation gives (refer to [3])

$$G^r(t_1, t_2) = G^{0r}(t_1 - t_2) + \int dt\ G^{0r}(t_1 - t)H'(t)G^r(t, t_2) \tag{36}$$

in which, G^{0r} is not the free propagator of central dot, but the equilibrium Green's function when the pumping potential H is set to zero, i.e. the hamiltonian is $H - H'$, not $\varepsilon_n d^\dagger_{n\sigma} d_{n\sigma}$. Multiply $e^{iE_1t_1 - iE_2t_2}$ and integrate on both sides, we have

$$\begin{aligned} G^r(E_1, E_2) \equiv & \int dt_1 dt_2 e^{iE_1t_1 - iE_2t_2} G^r(t_1, t_2) \\ =& 2\pi G^{0r}(E_1)\delta(E_1 - E_2) \\ &+ \int dt_1 G^{0r}(t_1 - t)e^{iE_1(t_1-t)} H'(t)e^{i(E_1-E)t} \iint dt dt_2 G^r(t, t_2)e^{iEt - E_2t_2} \\ =& 2\pi G^{0r}(E_1)\delta(E_1 - E_2) + \int \frac{dE}{2\pi} G^{0r}(E_1)H'(E_1 - E)G^r(E, E_2), \end{aligned} \tag{37}$$

in which, we inserted the inverse Fourier transformation

$$H'(t) = \int \frac{dE}{2\pi} e^{-i(E_1-E)t} H'(E_1 - E). \tag{38}$$

5.1 Calcualte G^{0r}

Using Eq. (9) and Eq. (13), we have

$$\partial_t d^\dagger_{n\sigma} = i(\epsilon_n + \sigma B_0 \cos\theta)d^\dagger_{n\sigma} + i \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{39}$$

$$\partial_t d_{n\sigma} = -i(\epsilon_n + \sigma B_0 \cos\theta)d_{n\sigma} - i \sum_{k,\alpha=L,R} T^*_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{40}$$

then

$$d^\dagger_{n\sigma}(t) = d^\dagger_{n\sigma}(0)e^{i(\epsilon_n + \sigma B_0 \cos\theta)t} + it \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{41}$$

$$d_{n\sigma}(t) = d_{n\sigma}(0)e^{-i(\epsilon_n + \sigma B_0 \cos\theta)} - it \sum_{k,\alpha=L,R} T^*_{k\alpha n} C_{k\alpha\sigma} \tag{42}$$

Note ϵ_n is time-dependent, $G^{0r}(t, t')$ depends only on time difference.

$$\begin{aligned} G^{0r}_{n\sigma}(t, t') =& - i\theta(t - t')\langle\{d_{n\sigma}(t), d^\dagger_{n\sigma}(t')\}\rangle \\ =& ? \end{aligned} \tag{43}$$

The hamiltonian $H - H'$ cannot flip spin, so in the spin space $\{d^\dagger_{1,\uparrow}, d^\dagger_{1,\downarrow}, d^\dagger_{2\uparrow}, d^\dagger_{2\downarrow}, \cdots; d_{1\uparrow}, d_{1\downarrow}, d_{2\uparrow}, d_{2\downarrow}, \cdots\}$, G^{0r} is diagonal

$$G^{0r}(E) = \bigotimes_n G^{0r}_n(E) \tag{44}$$

$$G^{0r}_n(E) = \begin{pmatrix} G^{0r}_{n\uparrow}(E) & 0 \\ 0 & G^{0r}_{n\downarrow}(E) \end{pmatrix}. \tag{45}$$

5.2 Calcualte G^r

In spin space

$$G^r \equiv \begin{pmatrix} G^r_{n\uparrow,m\uparrow} & G^r_{n\uparrow,m\downarrow} \\ G^r_{n\downarrow,m\uparrow} & G^r_{n\downarrow,m\downarrow} \end{pmatrix} \tag{46}$$

is a matrix of $2N$ dimension, where N is the total number of levels in central area. H' in Eq. (36)is a matrix of same dimension, given by

$$H' = \bigotimes_n H'_n, \tag{47}$$

$$H'_n = \left(\begin{array}{cc} 0 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & 0 \end{array} \right). \tag{48}$$

For simplicity, we consider only one energy level and neglect level index. we have

$$\begin{aligned} H'(E_1 - E) &= \int dt e^{i(E_1-E)t} H'(t) \\ &= \left(\begin{array}{cc} 0 & \gamma \int dt e^{i(E_1-E-\omega)t} \\ \gamma \int dt e^{i(E_1-E+\omega)t} & 0 \end{array} \right) \\ &= 2\pi\gamma \left(\begin{array}{cc} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{array} \right) \end{aligned} \tag{49}$$

Substitute these equations into Eq. (37), we get

$$\begin{aligned} \left(\begin{array}{cc} G^r_{\uparrow,\uparrow}(E_1, E_2) & G^r_{\uparrow,\downarrow}(E_1, E_2) \\ G^r_{\downarrow,\uparrow}(E_1, E_2) & G^r_{\downarrow,\downarrow}(E_1, E_2) \end{array} \right) &= \left(\begin{array}{cc} G^{0r}_\uparrow(E_1) & 0 \\ 0 & G^{0r}_\downarrow(E_1) \end{array} \right) 2\pi\delta(E_1 - E_2) \\ &\quad + \gamma \int dE \left(\begin{array}{cc} G^{0r}_\uparrow(E_1) & 0 \\ 0 & G^{0r}_\downarrow(E_1) \end{array} \right) \\ &\quad \times \left(\begin{array}{cc} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{array} \right) \left(\begin{array}{cc} G^r_{\uparrow,\uparrow} & G^r_{\uparrow,\downarrow} \\ G^r_{\downarrow,\uparrow} & G^r_{\downarrow,\downarrow} \end{array} \right) \\ &= \left(\begin{array}{cc} G^{0r}_\uparrow(E_1) & 0 \\ 0 & G^{0r}_\downarrow(E_1) \end{array} \right) 2\pi\delta(E_1 - E_2) \\ &\quad + \left(\begin{array}{cc} G_1(E_1, E_2) & G_2(E_1, E_2) \\ G_3(E_1, E_2) & G_4(E_1, E_2) \end{array} \right). \end{aligned} \tag{50}$$

where we have omitted the independent variables (E, E_2) of $G^r_{n\sigma,m\sigma'}$ for sake of convenience. In the above equations G_1, G_2, G_3, G_4 are

$$G_1(E_1, E_2) = \gamma \int dE G^{0r}_\uparrow (E_1)\, \delta(E_1 - E - \omega) G^r_{\downarrow,\uparrow}(E, E_2) \tag{51}$$

$$G_2(E_1, E_2) = \gamma \int dE G^{0r}_\uparrow (E_1)\, \delta(E_1 - E - \omega) G^r_{\downarrow,\downarrow}(E, E_2) \tag{52}$$

$$G_3(E_1, E_2) = \gamma \int dE G^{0r}_\downarrow (E_1)\, \delta(E_1 - E + \omega) G^r_{\uparrow,\uparrow}(E, E_2) \tag{53}$$

$$G_4(E_1, E_2) = \gamma \int dE G^{0r}_\downarrow (E_1)\, \delta(E_1 - E + \omega) G^r_{\uparrow,\downarrow}(E, E_2) \tag{54}$$

So, the G_1 term

$$\begin{aligned} G^r_{\uparrow,\uparrow}(E_1, E_2) &= G^{0r}_\uparrow (E_1)\, 2\pi\delta\,(E_1 - E_2) + G_1\,(E_1, E_2) \\ &= 2\pi G^{0r}_\uparrow (E_1)\, \delta\,(E_1 - E_2) + \gamma G^{0r}_\uparrow (E_1)\, G^r_{\downarrow,\uparrow}(E_1 - \omega, E_2) \end{aligned} \tag{55}$$

The G_2 term

$$\begin{aligned} G^r_{\uparrow,\downarrow}(E_1, E_2) &= G_2\,(E_1, E_2) \\ &= \gamma G^{0r}_\uparrow (E_1)\, G^r_{\downarrow,\downarrow}(E_1 - \omega, E_2) \end{aligned} \tag{56}$$

The G_3 term

$$\begin{aligned} G^r_{\downarrow,\uparrow}(E_1, E_2) &= G_3\,(E_1, E_2) \\ &= \gamma G^{0r}_\downarrow (E_1)\, G^r_{\uparrow,\uparrow}(E_1 + \omega, E_2) \end{aligned} \tag{57}$$

and the G_4 term

$$\begin{aligned} G^r_{\downarrow,\downarrow}\left(E_1,E_2\right) &= g^r_{\downarrow}\left(E_1\right)2\pi\delta\left(E_1-E_2\right)+G_4\left(E_1,E_2\right) \\ &= 2\pi G^{0r}_{\downarrow}\left(E_1\right)\delta\left(E_1-E_2\right)+\gamma G^{0r}_{\downarrow}(E_1)G^r_{\uparrow,\downarrow}(E_1+\omega,E_2) \end{aligned} \tag{58}$$

After collecting terms, we get

$$\begin{aligned} G^r_{\uparrow,\uparrow}(E_1,E_2) &= 2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E_2\right)+\gamma G^{0r}_{\uparrow}\left(E_1\right)\gamma G^{0r}_{\downarrow}(E_1-\omega)G^r_{\uparrow,\uparrow}\left(E_1,E_2\right) \\ &= \frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E_2\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \end{aligned} \tag{59}$$

and

$$G^r_{\downarrow,\downarrow}(E_1,E_2) = \frac{2\pi G^{0r}_{\downarrow}\left(E_1\right)\delta\left(E_1-E_2\right)}{1-\gamma^2 G^{0r}_{\downarrow}\left(E_1\right)G^{0r}_{\uparrow}\left(E_1+\omega\right)} \tag{60}$$

and

$$\begin{aligned} G^r_{\uparrow,\downarrow}\left(E_1,E_2\right) &= \gamma G^{0r}_{\uparrow}\left(E_1\right)G^r_{\downarrow,\downarrow}(E_1-\omega,E_2) \\ &= \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)\delta\left(E_1-\omega-E_2\right)}{1-\gamma^2 G^{0r}_{\downarrow}\left(E_1-\omega\right)G^{0r}_{\uparrow}\left(E_1\right)} \end{aligned} \tag{61}$$

$$\begin{aligned} G^r_{\downarrow,\uparrow}\left(E_1,E_2\right) &= \gamma G^{0r}_{\downarrow}\left(E_1\right)G^r_{\uparrow,\uparrow}(E_1+\omega,E_2) \\ &= \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \end{aligned} \tag{62}$$

Thus we get G^r in Eq. (46).

6 Calculate $G^<$

6.1 Calcualte Σ_α

In Eq. (32),

$$\Sigma^{\gamma}_{\alpha}\left(t_1,t_2\right) = \sum_k T^*_{k\alpha}\left(t_1\right)g^{\gamma}_{k\alpha}\left(t_1,t_2\right)T_{k\alpha}\left(t_2\right). \tag{63}$$

Here we consider parametric pumping, thus no bias presents in the leads as demonstrated in the system Hamiltonian, i.e. $\Sigma_a(t_1,t_2)\to\Sigma_a(t_1-t_2)$. Using free propagators $g^{\gamma}_{k\alpha\sigma}$, Fourier-transform to

$$\Sigma^<_{\alpha\sigma}(E_1,E_2) = 2\pi\Sigma^<_{\alpha\sigma}(E_1)\delta(E_1-E_2), \tag{64}$$

in which

$$\Sigma^<_{\alpha\sigma}(E) = if(E)\Gamma_\alpha\left(E\right). \tag{65}$$

The linewidth function Γ is defined as

$$\Gamma_\alpha(E)\equiv 2\pi\sum_k T^*_{k\alpha}T_{k\alpha}\delta\left(E-\epsilon_k\right), \tag{66}$$

thus we have Σ_σ, which is a number in spin space, not a matrix, since $\Sigma^<$ is independent of spin σ. Similarly,

$$\Sigma^a_{\alpha\sigma}(E_1,E_2) = 2\pi\Sigma^a_{\alpha\sigma}(E_1)\delta(E_1-E_2), \tag{67}$$

and the retarted(advanced) self-energy is (?)

$$\Sigma^{r,a}_{\alpha\sigma}(E) = \Lambda_\alpha(E)\mp\frac{i}{2}\Gamma_\alpha(E), \tag{68}$$

since

$$\Sigma^a_{\alpha\sigma}(E) = \left[\Sigma^r_{\alpha\sigma}(E)\right]^* \tag{69}$$

and

$$\begin{aligned} \Sigma^a_{\alpha\sigma}(E)-\Sigma^r_{\alpha\sigma}(E) &= i\Gamma_{\alpha\sigma}(E) \\ &= 2i\mathrm{Im}\{\Sigma^{\mathrm{a}}_{\alpha\sigma}(\mathrm{E})\}. \end{aligned} \tag{70}$$

6.2 Calculate G^a

We have relation

$$G^a_{\sigma,\sigma'}\left(E_1,E_2\right) = \left(G^r_{\sigma',\sigma}\left(E_2,E_1\right)\right)^*, \tag{71}$$

$$G^r = \begin{pmatrix} G^r_{\uparrow,\uparrow} & G^r_{\downarrow,\uparrow} \\ G^r_{\uparrow,\downarrow} & G^r_{\downarrow,\downarrow} \end{pmatrix} \tag{72}$$

so in spin space,

$$G^a = \begin{pmatrix} (G^r_{\uparrow,\uparrow})^* & (G^r_{\uparrow,\downarrow})^* \\ (G^r_{\downarrow,\uparrow})^* & (G^r_{\downarrow,\downarrow})^* \end{pmatrix} \tag{73}$$

6.3 Calculate $G^<$

Substitute $G^r, \Sigma^<, G^a$ into Eq. (34), we get $G^<$,

$$\begin{aligned} G^<(E_1,E_2) &= \int \frac{dE}{2\pi}\frac{dE'}{2\pi} G^r(E_1,E)\Sigma^<(E,E')G^a(E',E_2) \\ &= \int \frac{dE}{2\pi} G^r\left(E_1,E\right)\Sigma^<\left(E\right)G^a\left(E,E_2\right) \\ &= \int \frac{dE}{2\pi} if(E)\Gamma(E)G^r\left(E_1,E\right)G^a\left(E,E_2\right) \end{aligned} \tag{74}$$

$$G^r\left(E_1,E\right)G^a\left(E,E_2\right) = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{75}$$

in which, the G_1 term is

$$\begin{aligned} G_1 &= G^r_{\uparrow,\uparrow}(E_1,E)G^{r,*}_{\uparrow,\uparrow}(E,E_2) + G^r_{\downarrow,\uparrow}(E_1,E)G^{r,*}_{\downarrow,\uparrow}(E,E_2) \\ &= \frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \left(\frac{2\pi G^{0r}_{\uparrow}\left(E\right)\delta\left(E-E_2\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E\right)G^{0r}_{\downarrow}\left(E-\omega\right)}\right)^* \\ &\quad + \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \left(\frac{2\pi\gamma G^{0r}_{\uparrow}\left(E+\omega\right)G^{0r}_{\downarrow}\left(E\right)\delta(E+\omega-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E+\omega\right)G^{0r}_{\downarrow}\left(E\right)}\right)^* \\ &= \frac{2\pi G^{0r}_{\uparrow}\left(E_1\right)\delta\left(E_1-E\right)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1\right)G^{0r}_{\downarrow}\left(E_1-\omega\right)} \times \frac{2\pi G^{0a}_{\uparrow}\left(E\right)\delta\left(E-E_2\right)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E\right)G^{0a}_{\downarrow}\left(E-\omega\right)} \\ &\quad + \frac{2\pi\gamma G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}\left(E_1+\omega\right)G^{0r}_{\downarrow}\left(E_1\right)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)\delta(E+\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}\left(E+\omega\right)G^{0a}_{\downarrow}\left(E\right)} \end{aligned} \tag{76}$$

the G_2 term is

$$G_2 = G^r_{\uparrow,\uparrow}(E_1,E)G^{r,*}_{\uparrow,\downarrow}(E,E_2) + G^r_{\downarrow,\uparrow}(E_1,E)G^{r,*}_{\downarrow,\downarrow}(E,E_2) \tag{77}$$

the G_3 term is

$$G_3 = G^r_{\uparrow,\downarrow}(E_1,E)G^r_{\uparrow,\uparrow}(E,E_2) + G^r_{\downarrow,\downarrow}(E_1,E)G^{r,*}_{\downarrow,\uparrow}(E,E_2) \tag{78}$$

the G_4 term is

$$G_4 = G^r_{\uparrow,\downarrow}(E_1,E)G^r_{\uparrow,\downarrow}(E,E_2) + G^r_{\downarrow,\downarrow}(E_1,E)G^{r,*}_{\downarrow,\downarrow}(E,E_2) \tag{79}$$

So, the matrix element of $G^<$ is

$$G^<\left(E_1,E_2\right) = \begin{pmatrix} G^<_{11} & G^<_{12} \\ G^<_{21} & G^<_{22} \end{pmatrix} \tag{80}$$

$$
\begin{aligned}
G^<_{11}(E_1,E_2) &= \int \frac{dE}{2\pi} if(E)\Gamma(E) \frac{2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1)\,G^{0r}_{\downarrow}(E_1-\omega)} \times \frac{2\pi G^{0a}_{\uparrow}(E)\,\delta(E-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}(E)\,G^{0a}_{\downarrow}(E-\omega)} \\
&\quad + \int \frac{dE}{2\pi} if(E)\Gamma(E) \frac{2\pi\gamma G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)\,\delta(E_1+\omega-E)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}(E+\omega)\,G^{0a}_{\downarrow}(E)\,\delta(E+\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}(E+\omega)\,G^{0a}_{\downarrow}(E)} \\
&= if(E_1)\Gamma(E_1) \frac{G^{0r}_{\uparrow}(E_1)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1)\,G^{0r}_{\downarrow}(E_1-\omega)} \times \frac{2\pi G^{0a}_{\uparrow}(E_1)\,\delta(E_1-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}(E_1)\,G^{0a}_{\downarrow}(E_1-\omega)} \\
&\quad + if(E_1+\omega)\Gamma(E_1+\omega) \frac{\gamma G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)} \times \frac{2\pi\gamma G^{0a}_{\uparrow}(E_1+2\omega)\,G^{0a}_{\downarrow}(E_1+\omega)\,\delta(E_1+2\omega-E_2)}{1-\gamma^2 G^{0a}_{\uparrow}(E_1+2\omega)\,G^{0a}_{\downarrow}(E_1+\omega)}
\end{aligned}
\tag{81}
$$

7 Spin-up current

Define

$$
g^{r,a}_{\sigma}(E) \equiv \frac{G^{0r,a}_{\sigma}(E)}{1-\gamma^2 G^{0r,a}_{\sigma}(E) G^{0r,a}_{\bar{\sigma}}(E+\bar{\sigma}\omega)}, \tag{82}
$$

then we can write

$$
\begin{aligned}
G^<_{11}(E_1,E_2) &= if(E_1)\Gamma(E_1) g^r_{\uparrow}(E_1) \times 2\pi\delta\,(E_1-E_2)\,g^a_{\uparrow}(E_1) \\
&\quad + if(E_1+\omega)\Gamma(E_1+\omega)\gamma g^r_{\uparrow}(E_1+\omega) G^{0r}_{\downarrow}(E_1) \times 2\pi\gamma g^a_{\uparrow}(E_1+2\omega) G^{0a}_{\downarrow}(E_1+\omega)\delta(E_1+2\omega-E_2),
\end{aligned}
\tag{83}
$$

and

$$
G^r_{\uparrow,\uparrow}(E_1,E_2) = 2\pi g^r_{\uparrow}(E_1)\delta(E_1-E_2). \tag{84}
$$

$$
G^r_{\downarrow,\downarrow}(E_1,E_2) = 2\pi g^r_{\downarrow}(E_1)\delta(E_1-E_2). \tag{85}
$$

$$
G^r_{\uparrow,\downarrow}(E_1,E_2) = 2\pi\gamma G^{0r}_{\uparrow}(E_1) g^r_{\downarrow}(E_1-\omega)\delta(E_1-\omega-E_2). \tag{86}
$$

$$
G^r_{\downarrow,\uparrow}(E_1,E_2) = 2\pi\gamma G^{0r}_{\downarrow}(E_1) g^r_{\uparrow}(E_1+\omega)\delta(E_1+\omega-E_2). \tag{87}
$$

The spin-up current is

$$
\begin{aligned}
I_{\alpha\uparrow,\uparrow}(\omega_1) &= -\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi} \left[G^r_{\uparrow,\uparrow}\left(E+\omega_1,E'\right) 2\pi i f(E')\Gamma_\alpha(E')\delta(E'-E)\right. \\
&\qquad\qquad \left. + G^<_{11}(E+\omega_1,E') 2\pi \Sigma^a_\alpha(E')\delta(E'-E)\right] + c.c. \\
&= -\frac{e}{\hbar}\int \frac{dE}{2\pi} \left[G^r_{\uparrow,\uparrow}(E+\omega_1,E)\, if(E)\Gamma_\alpha(E)\right. \\
&\qquad\qquad \left. + G^<_{11}(E+\omega_1,E)\Sigma^a_\alpha(E)\right] + c.c.
\end{aligned}
\tag{88}
$$

Assume (?)

$$
\Sigma^a_\alpha(E) = \frac{i}{2}\Gamma_\alpha(E) \tag{89}
$$

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