Notes on PRB.67.092408

1 spin field effect transistor

A type of quantum field effect transistor that operates purely on the flow of spin current in the absence of charge current. The rotating field induces a time-independent dc spin current, and at the same time generates no charge cur- rent. The physical principle of our SFET is due to a spin flip mecha- nism provided by the field.

2 Hamiltonian

A rotating magnetic field is

$$B_x = B_0 \mathrm{sin}\theta\ \cos(\omega \mathrm{t}) \tag{1}$$

$$B_y = B_0 \mathrm{sin}\theta\ \sin(\omega \mathrm{t}) \tag{2}$$

$$B_z = B_0 \mathrm{cos}\theta. \tag{3}$$

The Hamiltonian of system is

$$\begin{aligned} H = &\sum_{k,\sigma,\alpha=L,R} \epsilon_k C^+_{k\alpha\sigma} C_{k\alpha\sigma} + \sum_{n\sigma} \left[\epsilon_n + \sigma B_0 \cos\theta\right] d^+_{n\sigma} d_{n\sigma} \\ &+ H'(t) + \sum_{k,n,\sigma,\alpha=L,R} \left[T_{k\alpha n} C^+_{k\alpha\sigma} d_{n\sigma} + \text{c.c.}\right] \end{aligned} \tag{4}$$

We assume that there are multiple orbits in the scattering region, which is different from the original paper, in which only one orbit is considered. The level energy is

$$\epsilon_n(t) = \epsilon_n^0 - qV_g(t), \tag{5}$$

which can be controlled by the gate voltage V_g. Energy level of lead $\epsilon_{Lk} = \epsilon_{Rk} = \epsilon_k$.

A counterclock-wise rotating field allows a spin-down electron to absorb a photon and flip to spin-up, and it does not allow a spin-up electron to absorb a photon and flip to spin-down.

$$H'(t) = \sum_n \gamma \left[\exp(-i\omega t) d^+_{n\uparrow} d_{n\downarrow} + \exp(i\omega t) d^+_{n\downarrow} d_{n\uparrow}\right] \tag{6}$$

$$\gamma = B_0 \sin\theta \tag{7}$$

We solve the transport properties (charge and spin currents) of the model in both adiabatic and nonadiabatic regimes using the standard Keldysh nonequilibrium Green's function technique.

3 Operator evolution

EoM of $d^\dagger_\sigma$ is

$$\partial_{t'} d^\dagger_{n\sigma}(t') = i[H, d^\dagger_{n\sigma}] \tag{8}$$

central dot part:

$$[\sum_{n'\sigma'} \left[\epsilon_{n'} + \sigma B_0 \cos\theta\right] d^\dagger_{n'\sigma'} d_{n'\sigma'}, d^\dagger_{n\sigma}] = (\epsilon_n + \sigma B_0 \cos\theta) d^\dagger_{n\sigma} \tag{9}$$

rotating magnetic field part:

$$\begin{aligned} [d^\dagger_{n'\uparrow} d_{n'\downarrow}, d^\dagger_{n\sigma}] &= d^\dagger_{n'\uparrow}\{d_{n'\downarrow}, d^\dagger_{n\sigma}\} - \{d^\dagger_{n'\uparrow}, d^\dagger_{n\sigma}\} d_{n'\downarrow} \\ &= d^\dagger_\uparrow \delta_{nn'} \delta_{\sigma\downarrow}, \end{aligned} \tag{10}$$

$$
\begin{aligned}
[d^\dagger_{n'\downarrow} d_{n'\uparrow}, d^\dagger_{n\sigma}] &= d^\dagger_{n'\downarrow}\{d_{n'\uparrow}, d^\dagger_{n\sigma}\} - \{d^\dagger_{n'\downarrow}, d^\dagger_{n\sigma}\} d_{n'\uparrow} \\
&= d^\dagger_{n'\uparrow}\delta_{nn'}\delta_{\sigma\uparrow}.
\end{aligned} \tag{11}
$$

Then

$$
[H'(t), d^\dagger_{n\sigma}] = \gamma(e^{-i\omega t} d^\dagger_{n\uparrow}\delta_{\sigma\downarrow} + e^{i\omega t} d^\dagger_{n\downarrow}\delta_{\sigma\uparrow}) \tag{12}
$$

Coupling part

$$
\sum_{k,n',\sigma',\alpha=L,R} \left[T_{n'k\alpha} C^\dagger_{k\alpha\sigma'} d_{n'\sigma'} + \text{c.c.}, d^\dagger_{n\sigma}\right] = \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma} \tag{13}
$$

Substitute these three parts into Eq. (8), we get

$$
\partial_{t'} d^\dagger_{n\sigma}(t') = i[(\epsilon^0_n + \sigma B_0 \cos\theta) d^\dagger_{n\sigma} + \gamma(e^{-i\omega t} d^\dagger_{n\uparrow}\delta_{\sigma\downarrow} + e^{i\omega t} d^\dagger_{n\downarrow}\delta_{\sigma\uparrow}) + \sum_{k,\alpha=L,R} T_{k\alpha n} C^\dagger_{k\alpha\sigma}] \tag{14}
$$

Equation of motion of particle operator $\hat{N}_{\alpha k\sigma}$ in the lead α is

$$
\begin{aligned}
\frac{d}{dt}\hat{N}_{\alpha k\sigma} = \frac{i}{\hbar}[H, C^\dagger_{\alpha k\sigma} C_{\alpha k\sigma}] &= \left[\sum_{k',\sigma',\alpha'=L,R} \left[T_{k'\alpha'} C^+_{k'\alpha'\sigma'} d_{\sigma'} + \text{c.c.}\right], C^\dagger_{\alpha k\sigma} C_{\alpha k\sigma}\right] \\
&= \frac{i}{\hbar} \sum_{k',\sigma',\alpha'=L,R} \left[-T_{k'\alpha'} C^+_{k'\alpha'\sigma'} d_{\sigma'} \delta_{\alpha\alpha'}\delta_{kk'}\delta_{\sigma\sigma'} + \text{c.c.}\right] \\
&= \frac{i}{\hbar}[-T_{k\alpha} C^\dagger_{k\alpha\sigma} d_\sigma + T^*_{k\alpha} d^\dagger_\sigma C_{k\alpha\sigma}]
\end{aligned} \tag{15}
$$

4 Charge current

So, the charge current due to L(R) lead with spin σ is given by

$$
\begin{aligned}
I_{L\sigma}(t) &= e\langle \frac{d}{dt}\hat{N}_\sigma(t)\rangle \\
&= \frac{ie}{\hbar} \sum_{kn\alpha\in L} (\langle -T_{k\alpha n} C^\dagger_{k\alpha\sigma}(t) d_{n\sigma}(t)\rangle + \langle T^*_{k\alpha n} d^\dagger_{n\sigma}(t) C_{k\alpha\sigma}(t)\rangle)
\end{aligned} \tag{16}
$$

Define the lesser Green's function

$$
G^<_{n\sigma',k\alpha\sigma}(\tau,\tau') = i\langle C^\dagger_{k\alpha\sigma}(\tau') d_{n\sigma'}(\tau)\rangle, \tag{17}
$$

the current becomes

$$
I_{L\sigma}(t) = \frac{-e}{\hbar} \sum_{kn\alpha\in L} (T_{k\alpha n} G^<_{n,k\alpha\sigma}(t,t) - T^*_{k\alpha n} G^<_{k\alpha,n}(t,t))) \tag{18}
$$

More generally, we define the contour Green's function

$$
G_{n\sigma',k\alpha\sigma}(\tau,\tau') = -i\langle d_{n\sigma'}(\tau) C^\dagger_{k\alpha\sigma}(\tau')\rangle. \tag{19}
$$

EoM of operator $C^\dagger_{k\alpha\sigma}$ is

$$
\partial_{t'} C^\dagger_{k\alpha\sigma}(t') = i[H, C^\dagger_{k\alpha\sigma}] = i(\varepsilon_k C^\dagger_{k\alpha\sigma} + \sum_n T_{k\alpha n} d^\dagger_{n\sigma}) \tag{20}
$$

The equation-of-motion for the time-ordered Green function

$$
\begin{aligned}
-i\frac{\partial}{\partial t'} G^t_{n\sigma',k\alpha\sigma}(t,t') &= \delta(t-t')\langle\{d_{n\sigma'}, C^\dagger_{k\alpha\sigma}\}\rangle - \langle T_c d_{n\sigma'} \partial_{t'} C^\dagger_{k\alpha\sigma}\rangle \\
&= \varepsilon_k G^t_{n\sigma',k\alpha\sigma}(t,t') + \sum_m T^*_{k\alpha m} G^t_{n\sigma',m\sigma}(t,t')
\end{aligned} \tag{21}
$$

So, we have

$$
(-i\frac{\partial}{\partial t'} - \varepsilon_k) G^t_{n\sigma',k\alpha\sigma}(t,t') = \sum_m T^*_{k\alpha m} G^t_{n\sigma',m\sigma}(t,t') \tag{22}
$$

in which

$$G^t_{n\sigma',m\sigma}(t,t') = -i\langle T_c d_{n\sigma'}(t) d^\dagger_{m\sigma}(t')\rangle. \tag{23}$$

Similarly, we evaluate the EoM for free Green's function $g^t_{k\alpha\sigma}(t,t')$ in lead α (note that $H = \sum_{k\sigma\alpha} \epsilon_k C^\dagger_{k\alpha\sigma} C_{k\alpha\sigma}$).

$$\begin{aligned} -i\frac{\partial}{\partial t'} g^t_{k\alpha\sigma}\left(t,t'\right) &= \delta(t-t')\langle\{C_{k\alpha\sigma}, C^\dagger_{k\alpha\sigma}\}\rangle - \langle T_c C_{k\alpha\sigma}\partial_{t'} C^\dagger_{k\alpha\sigma}\rangle \\ &= \delta(t-t') + \varepsilon_k g^t_{k\alpha\sigma}\left(t,t'\right), \end{aligned} \tag{24}$$

we have

$$(-i\frac{\partial}{\partial t'} - \varepsilon_k) g^t_{k\alpha\sigma}\left(t,t'\right) = \delta(t-t'). \tag{25}$$

Substitute Eq. (25) into Eq. (22) and integrate on both sides, we get an equation analogous to Jauho's notation [2],

$$G_{n,k\alpha}(\tau,\tau') = \sum_m \int d\tau_1 G_{nm}\left(\tau,\tau_1\right) t^*_{k\alpha m} g_{k\alpha}\left(\tau_1,\tau'\right),$$

we have

$$G^t_{n\sigma',k\alpha\sigma}(t,t') = \sum_m \int dt_1 G_{n\sigma',m\sigma}\left(t,t_1\right) T^*_{k\alpha m} g^t_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{26}$$

When there is only one orbit presents, this equation reduces to

$$G^t_{\sigma',k\alpha\sigma}(t,t') = \int dt_1 G_{\sigma',\sigma}\left(t,t_1\right) T^*_{k\alpha} g^t_{k\alpha\sigma}\left(\tau_1,\tau'\right). \tag{27}$$

Since the contour Green's function has the same structure as real-time Green's function, the we have relation

$$G_{n\sigma',k\alpha\sigma}(\tau,\tau') = \sum_m \int d\tau_1 G_{n\sigma',m\sigma}\left(\tau,\tau_1\right) T^*_{k\alpha m} g_{k\alpha\sigma}\left(\tau_1,\tau'\right) \tag{28}$$

where $G_{n\sigma',k\alpha\sigma}(\tau,\tau')$ is contour Green's function defined in Eq. (17), and similarly the contour Green's function for non-interacting lead is defined as

$$g_{k\alpha\sigma}(\tau,\tau') = -i\langle T_c C_{k\alpha\sigma}(\tau) C^\dagger_{k\alpha\sigma}(\tau')\rangle \tag{29}$$

After analytic continuation, the current is formulated as

$$\begin{aligned} I_{\alpha\sigma}(t) &= -\frac{e}{\hbar}\int dt_1 \,\mathrm{Tr}\left[G^r\left(t,t_1\right)\Sigma^<_\alpha\left(t_1,t\right) + G^<\left(t,t_1\right)\Sigma^a_\alpha\left(t_1,t\right) + c.c.\right] \\ &= -\frac{e}{\hbar}\int dt_1 \,\mathrm{Tr}\left[G^r\left(t,t_1\right)\Sigma^<_\alpha\left(t_1,t\right) + G^<\left(t,t_1\right)\Sigma^a_\alpha\left(t_1,t\right)\right. \\ &\qquad\qquad \left. + G^a\left(t_1,t\right)(-1)\Sigma^<_\alpha\left(t,t_1\right) + (-1)G^<\left(t_1,t\right)\Sigma^r_\alpha\left(t,t_1\right)\right] \\ &= -\frac{e}{\hbar}\int dt_1 \,\mathrm{Tr}\left[G^r\left(t,t_1\right)\Sigma^<_\alpha\left(t_1,t\right) + G^<\left(t,t_1\right)\Sigma^a_\alpha\left(t_1,t\right)\right. \\ &\qquad\qquad \left. - \Sigma^<_\alpha\left(t,t_1\right)G^a\left(t_1,t\right) - \Sigma^r_\alpha\left(t,t_1\right)G^<\left(t_1,t\right)\right] \end{aligned} \tag{30}$$

Following Eq. (224) in WangJ's note, its Fourier transformation is

$$\begin{aligned} I_{\alpha\sigma}(\omega_1) &= -\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi}\mathrm{Tr}\left[\mathrm{G^r}\left(\mathrm{E}+\omega_1,\mathrm{E'}\right)\Sigma^<_\alpha\left(\mathrm{E'},\mathrm{E}\right) + \mathrm{G^<}(\mathrm{E}+\omega_1,\mathrm{E'})\Sigma^{\mathrm{a}}_\alpha(\mathrm{E'},\mathrm{E}) + \mathrm{c.c.}\right] \\ &= -\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi}\mathrm{Tr}[\mathrm{G^r}\left(\mathrm{E}+\omega_1,\mathrm{E'}\right)\Sigma^<_\alpha\left(\mathrm{E'},\mathrm{E}\right) + \mathrm{G^<}(\mathrm{E}+\omega_1,\mathrm{E'})\Sigma^{\mathrm{a}}_\alpha(\mathrm{E'},\mathrm{E}) \\ &\qquad\qquad - \Sigma^<_\alpha\left(E+\omega_1,E'\right)G^a\left(E',E\right) - \Sigma^r_\alpha\left(E+\omega_1,E'\right)G^<\left(E',E\right)]. \end{aligned} \tag{31}$$

Here, $G^{r,<} \equiv G^{r,<}_{n\sigma',m\sigma}$, notation Tr means sum over QD level index n, and matrix element

$$\Sigma^\gamma_{\alpha,mn}(t_1,t_2) = \sum_k T^*_{k\alpha m}(t_1) g^\gamma_{k\alpha}(t_1,t_2) T_{k\alpha n}(t_2), \tag{32}$$

and

$$\Sigma_{\alpha,mn}^{\gamma}(t_1,t_2)=\sum_k T_{k\alpha m}^*(t_1)\,g_{k\alpha}^{\gamma}(t_1,t_2)\,T_{k\alpha n}(t_2) \tag{33}$$

in which, $g_{k\sigma}$ is the free propagator of lead, and γ is {¿, ¡, r, a}. From Keldysh equation, we have (different from Eq. (33) in Chap. II ?)

$$G^{<}=G^{r}\Sigma^{<}G^{a} \tag{34}$$

whose Fourier transformation gives

$$G^{<}(E_1,E_2)=\int\frac{dE}{2\pi}\frac{dE'}{2\pi}G^{r}(E_1,E)\Sigma^{<}(E,E')G^{a}(E',E_2) \tag{35}$$

5 Calculate G^r

For G^r, the Dyson equation gives (refer to [3])

$$G^{r}(t_1,t_2)=G^{0r}(t_1-t_2)+\int dt\;G^{0r}(t_1-t)H'(t)G^{r}(t,t_2) \tag{36}$$

in which, G^{0r} is not the free propagator of central dot, but the equilibrium Green's function when the pumping potential H$'$ is set to zero, i.e. the hamiltonian is $H-H'$, not $\varepsilon_n d_{n\sigma}^{\dagger}d_{n\sigma}$. Multiply $e^{iE_1t_1-iE_2t_2}$ and integrate on both sides, we have

$$\begin{aligned}G^{r}(E_1,E_2)\equiv&\int dt_1dt_2e^{iE_1t_1-iE_2t_2}G^{r}(t_1,t_2)\\=&2\pi G^{0r}(E_1)\delta(E_1-E_2)\\&+\int dt_1G^{0r}(t_1-t)e^{iE_1(t_1-t)}H'(t)e^{i(E_1-E)t}\iint dtdt_2G^{r}(t,t_2)e^{iEt-E_2t_2}\\=&2\pi G^{0r}(E_1)\delta(E_1-E_2)+\int\frac{dE}{2\pi}G^{0r}(E_1)H'(E_1-E)G^{r}(E,E_2),\end{aligned} \tag{37}$$

in which, we inserted the inverse Fourier transformation

$$H'(t)=\int\frac{dE}{2\pi}e^{-i(E_1-E)t}H'(E_1-E). \tag{38}$$

5.1 Calcualte G^{0r}

Using Eq. (9) and Eq. (13), we have

$$\partial_t d_{n\sigma}^{\dagger}=i(\epsilon_n+\sigma B_0\cos\theta)d_{n\sigma}^{\dagger}+i\sum_{k,\alpha=L,R}T_{k\alpha n}C_{k\alpha\sigma}^{\dagger} \tag{39}$$

$$\partial_t d_{n\sigma}=-i(\epsilon_n+\sigma B_0\cos\theta)d_{n\sigma}-i\sum_{k,\alpha=L,R}T_{k\alpha n}^*C_{k\alpha\sigma}^{\dagger} \tag{40}$$

then

$$d_{n\sigma}^{\dagger}(t)=d_{n\sigma}^{\dagger}(0)e^{i(\epsilon_n+\sigma B_0\cos\theta)t}+it\sum_{k,\alpha=L,R}T_{k\alpha n}C_{k\alpha\sigma}^{\dagger} \tag{41}$$

$$d_{n\sigma}(t)=d_{n\sigma}(0)e^{-i(\epsilon_n+\sigma B_0\cos\theta)}-it\sum_{k,\alpha=L,R}T_{k\alpha n}^*C_{k\alpha\sigma} \tag{42}$$

Note ϵ_n is time-dependent, $G^{0r}(t,t')$ depends only on time difference.

$$\begin{aligned}G_{n\sigma}^{0r}(t,t')=&-i\theta(t-t')\langle\{d_{n\sigma}(t),d_{n\sigma}^{\dagger}(t')\}\rangle\\=&?\end{aligned} \tag{43}$$

The hamiltonian $H-H'$ cannot flip spin, so in the spin space $\{d_{1,\uparrow}^{\dagger},\,d_{1,\downarrow}^{\dagger},d_{2\uparrow}^{\dagger},d_{2\downarrow}^{\dagger},\cdots;\,d_{1\uparrow},\,d_{1\downarrow},d_{2\uparrow},d_{2\downarrow},\cdots\}$, G^{0r} is diagonal

$$G^{0r}(E)=\bigotimes_n G_n^{0r}(E) \tag{44}$$

$$G_n^{0r}(E)=\begin{pmatrix}G_{n\uparrow}^{0r}(E)&0\\0&G_{n\downarrow}^{0r}(E)\end{pmatrix}. \tag{45}$$

5.2 Calcualte G^r

In spin space

$$G^r \equiv \begin{pmatrix} G^r_{n\uparrow,m\uparrow} & G^r_{n\uparrow,m\downarrow} \\ G^r_{n\downarrow,m\uparrow} & G^r_{n\downarrow,m\downarrow} \end{pmatrix} \tag{46}$$

is a matrix of $2N$ dimension, where N is the total number of levels in central area. H' in Eq. (36)is a matrix of same dimension, given by

$$H' = \bigotimes_n H'_n, \tag{47}$$

$$H'_n = \begin{pmatrix} 0 & \gamma e^{-i\omega t} \\ \gamma e^{i\omega t} & 0 \end{pmatrix}. \tag{48}$$

For simplicity, we consider only one energy level and neglect level index. we have

$$\begin{aligned} H'(E_1 - E) &= \int dt e^{i(E_1-E)t} H'(t) \\ &= \begin{pmatrix} 0 & \gamma \int dt e^{i(E_1-E-\omega)t} \\ \gamma \int dt e^{i(E_1-E+\omega)t} & 0 \end{pmatrix} \\ &= 2\pi\gamma \begin{pmatrix} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{pmatrix} \end{aligned} \tag{49}$$

Substitute these equations into Eq. (37), we get

$$\begin{aligned} \begin{pmatrix} G^r_{\uparrow,\uparrow}(E_1, E_2) & G^r_{\uparrow,\downarrow}(E_1, E_2) \\ G^r_{\downarrow,\uparrow}(E_1, E_2) & G^r_{\downarrow,\downarrow}(E_1, E_2) \end{pmatrix} &= \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} 2\pi\delta(E_1 - E_2) \\ &+ \gamma \int dE \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} \\ &\times \begin{pmatrix} 0 & \delta(E_1 - E - \omega) \\ \delta(E_1 - E + \omega) & 0 \end{pmatrix} \begin{pmatrix} G^r_{\uparrow,\uparrow} & G^r_{\uparrow,\downarrow} \\ G^r_{\downarrow,\uparrow} & G^r_{\downarrow,\downarrow} \end{pmatrix} \\ &= \begin{pmatrix} G^{0r}_{\uparrow}(E_1) & 0 \\ 0 & G^{0r}_{\downarrow}(E_1) \end{pmatrix} 2\pi\delta(E_1 - E_2) \\ &+ \begin{pmatrix} G_1(E_1, E_2) & G_2(E_1, E_2) \\ G_3(E_1, E_2) & G_4(E_1, E_2) \end{pmatrix}. \end{aligned} \tag{50}$$

where we have omitted the independent variables (E, E_2) of $G^r_{n\sigma,m\sigma'}$ for sake of convenience. In the above equations G_1, G_2, G_3, G_4 are

$$G_1(E_1, E_2) = \gamma \int dE G^{0r}_{\uparrow}(E_1)\, \delta(E_1 - E - \omega) G^r_{\downarrow,\uparrow}(E, E_2) \tag{51}$$

$$G_2(E_1, E_2) = \gamma \int dE G^{0r}_{\uparrow}(E_1)\, \delta(E_1 - E - \omega) G^r_{\downarrow,\downarrow}(E, E_2) \tag{52}$$

$$G_3(E_1, E_2) = \gamma \int dE G^{0r}_{\downarrow}(E_1)\, \delta(E_1 - E + \omega) G^r_{\uparrow,\uparrow}(E, E_2) \tag{53}$$

$$G_4(E_1, E_2) = \gamma \int dE G^{0r}_{\downarrow}(E_1)\, \delta(E_1 - E + \omega) G^r_{\uparrow,\downarrow}(E, E_2) \tag{54}$$

So, the G_1 term

$$\begin{aligned} G^r_{\uparrow,\uparrow}(E_1, E_2) &= G^{0r}_{\uparrow}(E_1)\, 2\pi\delta(E_1 - E_2) + G_1(E_1, E_2) \\ &= 2\pi G^{0r}_{\uparrow}(E_1)\, \delta(E_1 - E_2) + \gamma G^{0r}_{\uparrow}(E_1)\, G^r_{\downarrow,\uparrow}(E_1 - \omega, E_2) \end{aligned} \tag{55}$$

The G_2 term

$$\begin{aligned} G^r_{\uparrow,\downarrow}(E_1, E_2) =& G_2(E_1, E_2) \\ =& \gamma G^{0r}_{\uparrow}(E_1)\, G^r_{\downarrow,\downarrow}(E_1 - \omega, E_2) \end{aligned} \tag{56}$$

The G_3 term

$$\begin{aligned}G^r_{\downarrow,\uparrow}(E_1,E_2) &= G_3(E_1,E_2)\\ &= \gamma G^{0r}_{\downarrow}(E_1)\,G^r_{\uparrow,\uparrow}(E_1+\omega,E_2)\end{aligned} \tag{57}$$

and the G_4 term

$$\begin{aligned}G^r_{\downarrow,\downarrow}(E_1,E_2) &= g^r_{\downarrow}(E_1)\,2\pi\delta(E_1-E_2)+G_4(E_1,E_2)\\ &= 2\pi G^{0r}_{\downarrow}(E_1)\,\delta(E_1-E_2)+\gamma G^{0r}_{\downarrow}(E_1)G^r_{\uparrow,\downarrow}(E_1+\omega,E_2)\end{aligned} \tag{58}$$

After collecting terms, we get

$$\begin{aligned}G^r_{\uparrow,\uparrow}(E_1,E_2) &= 2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E_2)+\gamma G^{0r}_{\uparrow}(E_1)\,\gamma G^{0r}_{\downarrow}(E_1-\omega)G^r_{\uparrow,\uparrow}(E_1,E_2)\\ &= \frac{2\pi G^{0r}_{\uparrow}(E_1)\,\delta(E_1-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1)\,G^{0r}_{\downarrow}(E_1-\omega)}\end{aligned} \tag{59}$$

and

$$G^r_{\downarrow,\downarrow}(E_1,E_2) = \frac{2\pi G^{0r}_{\downarrow}(E_1)\,\delta(E_1-E_2)}{1-\gamma^2 G^{0r}_{\downarrow}(E_1)\,G^{0r}_{\uparrow}(E_1+\omega)} \tag{60}$$

and

$$\begin{aligned}G^r_{\uparrow,\downarrow}(E_1,E_2) &= \gamma G^{0r}_{\uparrow}(E_1)\,G^r_{\downarrow,\downarrow}(E_1-\omega,E_2)\\ &= \frac{2\pi\gamma G^{0r}_{\uparrow}(E_1)\,G^{0r}_{\downarrow}(E_1-\omega)\,\delta(E_1-\omega-E_2)}{1-\gamma^2 G^{0r}_{\downarrow}(E_1-\omega)\,G^{0r}_{\uparrow}(E_1)}\end{aligned} \tag{61}$$

$$\begin{aligned}G^r_{\downarrow,\uparrow}(E_1,E_2) &= \gamma G^{0r}_{\downarrow}(E_1)\,G^r_{\uparrow,\uparrow}(E_1+\omega,E_2)\\ &= \frac{2\pi\gamma G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)\,\delta(E_1+\omega-E_2)}{1-\gamma^2 G^{0r}_{\uparrow}(E_1+\omega)\,G^{0r}_{\downarrow}(E_1)}\end{aligned} \tag{62}$$

Define

$$g^{r,a}_{\sigma}(E) \equiv \frac{G^{0r,a}_{\sigma}(E)}{1-\gamma^2 G^{0r,a}_{\sigma}(E)G^{0r,a}_{\bar{\sigma}}(E+\bar{\sigma}\omega)}, \tag{63}$$

then we can write

$$G^r_{\uparrow,\uparrow}(E_1,E_2) = 2\pi g^r_{\uparrow}(E_1)\delta(E_1-E_2). \tag{64}$$

$$G^r_{\downarrow,\downarrow}(E_1,E_2) = 2\pi g^r_{\downarrow}(E_1)\delta(E_1-E_2). \tag{65}$$

$$G^r_{\uparrow,\downarrow}(E_1,E_2) = 2\pi\gamma G^{0r}_{\uparrow}(E_1)g^r_{\downarrow}(E_1-\omega)\delta(E_1-\omega-E_2). \tag{66}$$

$$G^r_{\downarrow,\uparrow}(E_1,E_2) = 2\pi\gamma G^{0r}_{\downarrow}(E_1)g^r_{\uparrow}(E_1+\omega)\delta(E_1+\omega-E_2). \tag{67}$$

Thus we get G^r in Eq. (46).

6 Calculate $G^<$

6.1 Calcualte Σ_α

In Eq. (32),

$$\Sigma^{\gamma}_{\alpha}(t_1,t_2) = \sum_k T^*_{k\alpha}(t_1)\,g^{\gamma}_{k\alpha}(t_1,t_2)\,T_{k\alpha}(t_2)\,. \tag{68}$$

Here we consider parametric pumping, thus no bias presents in the leads as demonstrated in the system Hamiltonian, i.e. $\Sigma_a(t_1,t_2)\to\Sigma_a(t_1-t_2)$. Using free propagators $g^{\gamma}_{k\alpha\sigma}$, Fourier-transform to

$$\Sigma^<_{\alpha\sigma}(E_1,E_2) = 2\pi\Sigma^<_{\alpha\sigma}(E_1)\delta(E_1-E_2), \tag{69}$$

in which

$$\Sigma^<_{\alpha\sigma}(E) = if(E)\Gamma_\alpha(E)\,. \tag{70}$$

The linewidth function Γ is defined as

$$\Gamma_\alpha(E) \equiv 2\pi \sum_k T^*_{k\alpha} T_{k\alpha} \delta\left(E - \epsilon_k\right), \tag{71}$$

thus we have Σ_σ, which is a number in spin space, not a matrix, since $\Sigma^<$ is independent of spin σ. Similarly,

$$\Sigma^a_{\alpha\sigma}(E_1, E_2) = 2\pi \Sigma^a_{\alpha\sigma}(E_1)\delta(E_1 - E_2), \tag{72}$$

and the retarted(advanced) self-energy is (?)

$$\Sigma^{r,a}_{\alpha\sigma}(E) = \Lambda_\alpha(E) \mp \frac{i}{2}\Gamma_\alpha(E), \tag{73}$$

since

$$\Sigma^a_{\alpha\sigma}(E) = [\Sigma^r_{\alpha\sigma}(E)]^* \tag{74}$$

and

$$\begin{aligned} \Sigma^a_{\alpha\sigma}(E) - \Sigma^r_{\alpha\sigma}(E) &= i\Gamma_{\alpha\sigma}(E) \\ &= 2i\mathrm{Im}\{\Sigma^{\mathrm{a}}_{\alpha\sigma}(\mathrm{E})\}. \end{aligned} \tag{75}$$

6.2 Calculate G^a

We have relation

$$G^a_{\sigma,\sigma'}(E_1, E_2) = (G^r_{\sigma',\sigma}(E_2, E_1))^*, \tag{76}$$

$$G^r(E_1, E_2) = \begin{pmatrix} G^r_{\uparrow,\uparrow}(E_1, E_2) & G^r_{\downarrow,\uparrow}(E_1, E_2) \\ G^r_{\uparrow,\downarrow}(E_1, E_2) & G^r_{\downarrow,\downarrow}(E_1, E_2) \end{pmatrix} \tag{77}$$

so in spin space,

$$G^a(E_1, E_2) = \begin{pmatrix} (G^r_{\uparrow,\uparrow}(E_2, E_1))^* & (G^r_{\uparrow,\downarrow}(E_2, E_1))^* \\ (G^r_{\downarrow,\uparrow}(E_2, E_1))^* & (G^r_{\downarrow,\downarrow}(E_2, E_1))^* \end{pmatrix} \tag{78}$$

6.3 Calculate $G^<$

Substitute $G^r, \Sigma^<, G^a$ into Eq. (34), we get $G^<$,

$$\begin{aligned} G^<(E_1, E_2) &= \int \frac{dE}{2\pi}\frac{dE'}{2\pi} G^r(E_1, E)\Sigma^<(E, E')G^a(E', E_2) \\ &= \int \frac{dE}{2\pi} G^r\left(E_1, E\right) \Sigma^<\left(E\right) G^a\left(E, E_2\right) \\ &= \int \frac{dE}{2\pi} if(E)\Gamma(E) G^r\left(E_1, E\right) G^a\left(E, E_2\right) \end{aligned} \tag{79}$$

$$G^r\left(E_1, E\right) G^a\left(E, E_2\right) = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{80}$$

in which, the G_1 term is

$$\begin{aligned} G_1 &= G^r_{\uparrow,\uparrow}(E_1, E)G^{r,*}_{\uparrow,\uparrow}(E_2, E) + G^r_{\downarrow,\uparrow}(E_1, E)G^{r,*}_{\downarrow,\uparrow}(E_2, E) \\ &= (2\pi)^2 g^r_\uparrow\left(E_1\right) g^{r,*}_\uparrow\left(E_2\right) \delta\left(E_2 - E\right) \delta\left(E_1 - E\right) \\ &\quad + (2\pi\gamma)^2 G^{0r}_\downarrow\left(E_1\right) g^r_\uparrow\left(E_1 + \omega\right) G^{0r*}_\downarrow\left(E_2\right) g^{r*}_\uparrow\left(E_2 + \omega\right) \delta\left(E_1 + \omega - E\right) \delta\left(E_2 + \omega - E\right) \end{aligned} \tag{81}$$

the G_2 term is

$$G_2 = G^r_{\uparrow,\uparrow}(E_1, E)G^{r,*}_{\uparrow,\downarrow}(E_2, E) + G^r_{\downarrow,\uparrow}(E_1, E)G^{r,*}_{\downarrow,\downarrow}(E_2, E) \tag{82}$$

the G_3 term is

$$G_3 = G^r_{\uparrow,\downarrow}(E_1, E)G^r_{\uparrow,\uparrow}(E_2, E) + G^r_{\downarrow,\downarrow}(E_1, E)G^{r,*}_{\downarrow,\uparrow}(E_2, E) \tag{83}$$

the G_4 term is

$$G_4 = G^r_{\uparrow,\downarrow}(E_1, E)G^r_{\uparrow,\downarrow}(E_2, E) + G^r_{\downarrow,\downarrow}(E_1, E)G^{r,*}_{\downarrow,\downarrow}(E_2, E) \tag{84}$$

So, the matrix element of $G^<$ is

$$G^<(E_1,E_2) = \begin{pmatrix} G^<_{11} & G^<_{12} \\ G^<_{21} & G^<_{22} \end{pmatrix} \tag{85}$$

$$\begin{aligned} G^<_{11}(E_1,E_2) = &\int \frac{dE}{2\pi} i f(E)\Gamma(E)(2\pi)^2 g^r_\uparrow(E_1)\, g^{r,*}_\uparrow(E_2)\,\delta(E_2-E)\,\delta(E_1-E) \\ &+ \int \frac{dE}{2\pi} i f(E)\Gamma(E)(2\pi\gamma)^2 G^{0r}_\downarrow(E_1)\, g^r_\uparrow(E_1+\omega)\, G^{0r*}_\downarrow(E_2)\, g^{r*}_\uparrow(E_2+\omega)\,\delta(E_1+\omega-E)\,\delta(E_2+\omega-E) \\ = &\, 2\pi i f(E_1)\Gamma(E_1) g^r_\uparrow(E_1)\, g^{r,*}_\uparrow(E_2)\,\delta(E_2-E_1) \\ &+ 2\pi\gamma^2 i f(E_1+\omega)\Gamma(E_1+\omega) G^{0r}_\downarrow(E_1)\, g^r_\uparrow(E_1+\omega)\, G^{0r*}_\downarrow(E_2)\, g^{r*}_\uparrow(E_2+\omega)\,\delta(E_2-E_1) \end{aligned} \tag{86}$$

7 Spin-up current

The spin-up current is

$$\begin{aligned} I_{\alpha\uparrow,\uparrow}(\omega_1) = &-\frac{e}{\hbar}\int \frac{dE}{2\pi}\frac{dE'}{2\pi}[G^r_{\uparrow,\uparrow}\left(E+\omega_1,E'\right) 2\pi i f(E')\Gamma_\alpha(E')\delta(E'-E) \\ &+ G^<_{11}(E+\omega_1,E')2\pi\Sigma^a_\alpha(E')\delta(E'-E) + c.c.] \\ = &-\frac{e}{\hbar}\int \frac{dE}{2\pi}[G^r_{\uparrow,\uparrow}(E+\omega_1,E)\, i f(E)\Gamma_\alpha(E) \\ &+ G^<_{11}(E+\omega_1,E)\Sigma^a_\alpha(E) + c.c.] \end{aligned} \tag{87}$$

Assume (?)

$$\Sigma^a_\alpha(E) = \frac{i}{2}\Gamma_\alpha(E) \tag{88}$$

or

$$\Sigma^a_\alpha(E) = \sum_k T^*_{k\alpha}\frac{1}{E-\epsilon_k-i0^+}T_{k\alpha}(E) \tag{89}$$

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