Contents

1 Agents

1.1 Properties

Table 1: Shoulder, torso and total radii.

	Total		Torso	Shoulder	
	r	$\pm$	$k_t = \frac{r_t}{r}$	$k_s = \frac{r_s}{r}$	$k_{ts} = \frac{r_{ts}}{r}$
adult	0.255	0.035	0.5882	0.3725	0.6275
child	0.210	0.015	0.5714	0.3333	0.6667
eldery	0.250	0.020	0.6000	0.3600	0.6400
female	0.240	0.020	0.5833	0.3750	0.6250
male	0.270	0.020	0.5926	0.3704	0.6296

Table 2: Properties

r	m		Total radius
r_t	m		Torso radius
r_s	m		Shoulder radius
r_{ts}	m		Distance from torso to shoulder
m	kg	80	Mass
I	$\mathrm{kg} \cdot \mathrm{m}^2$	4.0	Rotational moment
$\mathbf{x}$	m		Position
$\mathbf{v}$	m/s		Velocity
v_0	m/s		Goal velocity
$\hat{\mathbf{e}}_0$			Goal direction
$\hat{\mathbf{e}}$			Target direction
φ	rad	$[-\pi, \pi]$	Body angle
ω	rad/s		Angular velocity
φ_0	rad	$[-\pi, \pi]$	Target angle
ω_0	rad/s	0.4π	Max angular velocity
p		$0-1$	Herding tendency

1.2 Models

1.2.1 Circular

Table 3: Relative

$\tilde{\mathbf{x}} = \mathbf{x}_i - \mathbf{x}_j$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_i - \mathbf{v}_j$	Relative velocity

$d = \lVert\tilde{\mathbf{x}}\rVert$	Distance
$\hat{\mathbf{n}} = \tilde{\mathbf{x}}/d$	Normal vector
$\hat{\mathbf{t}} = R(-90°) \cdot \hat{\mathbf{n}}$	Tangent vector

Total radius and relative distance

$$\tilde{r} = r_i + r_j$$

$$h = d - \tilde{r}$$

1.2.2 Three circles

$$\begin{aligned}
\mathbf{x}_r &= \mathbf{x}_c + \hat{\mathbf{t}} r_{ts} \\
\mathbf{x}_l &= \mathbf{x}_c - \hat{\mathbf{t}} r_{ts} \\
\hat{\mathbf{t}} &= \begin{bmatrix} -\sin(\varphi) & \cos(\varphi) \end{bmatrix}
\end{aligned}$$

$$\mathbf{r}_{tot} = \begin{bmatrix} r_t & r_s & r_s \end{bmatrix}_i + \begin{bmatrix} r_t \\ r_s \\ r_s \end{bmatrix}_j$$

$$\begin{aligned}
\mathbf{d} &= \left\lVert \begin{bmatrix} \mathbf{x}_c & \mathbf{x}_r & \mathbf{x}_l \end{bmatrix}_i - \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_r \\ \mathbf{x}_l \end{bmatrix}_j \right\rVert \\
&= \left\lVert \begin{bmatrix} 0 & \hat{\mathbf{t}} r_{ts} & -\hat{\mathbf{t}} r_{ts} \end{bmatrix}_i - \begin{bmatrix} 0 \\ \hat{\mathbf{t}} r_{ts} \\ -\hat{\mathbf{t}} r_{ts} \end{bmatrix}_j + (\mathbf{x}_i - \mathbf{x}_j) \right\rVert \\
&= \left\lVert \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \left(\hat{\mathbf{t}} r_{ts}\right)_i - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \left(\hat{\mathbf{t}} r_{ts}\right)_j + \tilde{\mathbf{x}} \right\rVert \\
&= \left\lVert \mathbf{k} \left(\hat{\mathbf{t}} r_{ts}\right)_i - \mathbf{k}^T \left(\hat{\mathbf{t}} r_{ts}\right)_j + \tilde{\mathbf{x}} \right\rVert \\
&= \left\lVert \mathbf{c}_i - \mathbf{c}_j^T + \tilde{\mathbf{x}} \right\rVert
\end{aligned}$$

$$\mathbf{h} = \mathbf{d} - \mathbf{r}_{tot}$$

1. Find

$$h = \min(\mathbf{h})$$

and track minimizing values

$$\hat{\mathbf{e}}_{ij}, k_i, k_j, r_i, r_j$$

2.

$$\begin{aligned}
\mathbf{r}_i^{moment} &= \mathbf{x}_i^c + k_i \cdot \hat{\mathbf{t}}_i r_i^{ts} + r_i \hat{\mathbf{e}}_{ij} \\
\mathbf{r}_j^{moment} &= \mathbf{x}_j^c + k_j \cdot \hat{\mathbf{t}}_j r_j^{ts} - r_j \hat{\mathbf{e}}_{ij}
\end{aligned}$$

3. Return $(\tilde{\mathbf{x}}, r_{tot}, h, \mathbf{r}_i^{moment}, \mathbf{r}_j^{moment})$

2 Linear wall

2.1 Properties

$\mathbf{p}_0$	Start point
$\mathbf{p}_1$	End point
$h_{iw} = d_{iw} - r_i$	
$l_w = \|\mathbf{p}_1 - \mathbf{p}_0\|$	Length
$\hat{\mathbf{t}}_w = (\mathbf{p}_1 - \mathbf{p}_0) / l_w$	
$\hat{\mathbf{n}}_w = R(90°) \cdot \hat{\mathbf{t}}_w$	

2.2 Absolute distance

Solving linear system of equations determining the position of the agent $\mathbf{x}_i$ in relation to wall

$$\begin{cases} \mathbf{p}_0 + l_{n_0}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_0}\hat{\mathbf{t}}_w \\ \mathbf{p}_1 + l_{n_1}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_1}\hat{\mathbf{t}}_w \end{cases}$$

$$\begin{cases} l_{n_0}\hat{\mathbf{n}}_w - l_{t_0}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_0 = \mathbf{q}_0 \\ l_{n_1}\hat{\mathbf{n}}_w - l_{t_1}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_1 = \mathbf{q}_1 \end{cases}$$

In matrix form

$$\begin{bmatrix} l_{n_0} & l_{n_1} \\ l_{t_0} & l_{t_1} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \hat{\mathbf{n}}_w & -\hat{\mathbf{t}}_w \end{bmatrix} = \begin{bmatrix} -t_1 & -t_0 \\ t_0 & -t_1 \end{bmatrix} \\ \mathbf{A}^{-1} &= \frac{1}{t_0^2 + t_1^2} \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} = \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mathbf{n}}_w \\ -\hat{\mathbf{t}}_w \end{bmatrix} = \mathbf{A}^T \end{aligned}$$

Conditions

$$\begin{aligned} l_n &= l_{n_0} \vee l_{n_1} = \hat{\mathbf{n}}_w \cdot \mathbf{q}_0 \vee \hat{\mathbf{n}}_w \cdot \mathbf{q}_1 \\ l_t &= l_{t_1} + l_{t_0} = -\hat{\mathbf{t}}_w \cdot \mathbf{q}_1 - \hat{\mathbf{t}}_w \cdot \mathbf{q}_0 \end{aligned}$$

Distance between agent and linear wall

$$d_{iw} = \begin{cases} \|\mathbf{q}_0\| & l_t > l_w \\ |l_n| & \text{otherwise} \\ \|\mathbf{q}_1\| & l_t < -l_w \end{cases}$$

Normal vector away from the wall

$$\hat{\mathbf{n}}_{iw} = \begin{cases} \hat{\mathbf{q}}_0 & l_t > l_w \\ \operatorname{sign}(l_n)\hat{\mathbf{n}}_w & \text{otherwise} \\ \hat{\mathbf{q}}_1 & l_t < -l_w \end{cases}$$

2.3 Velocity relative distance

$\tilde{\mathbf{x}} = \mathbf{x}_{iw}$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_{iw} = \mathbf{v}_i$	Relative velocity
$\tilde{r} = r_{iw}$	Total radius
$d = \|\tilde{\mathbf{x}}\|$	Distance
$h = d - \tilde{r}$	Relative distance

$$\begin{aligned} \mathbf{q}_0 &= \mathbf{p}_0 - \mathbf{x} \\ \mathbf{q}_1 &= \mathbf{p}_1 - \mathbf{x} \\ \hat{\mathbf{n}}_{iw} &= -\operatorname{sign}(\hat{\mathbf{n}}_w \cdot \mathbf{q}_0)\hat{\mathbf{n}}_w \end{aligned}$$

$$\begin{aligned} \boldsymbol{\alpha} &= [\operatorname{angle}(\mathbf{q}_0), \operatorname{angle}(\mathbf{q}_1), \operatorname{angle}(\hat{\mathbf{n}}_{iw})] \\ \varphi &= \operatorname{angle}(\mathbf{v}) \\ \boldsymbol{\alpha}_2 &= \boldsymbol{\alpha} - \varphi \mod 2\pi \end{aligned}$$

$$i = (\arg\min(\boldsymbol{\alpha}_2), \arg\max(\boldsymbol{\alpha}_2))$$

Intersection

$$\mathbf{x} + a\mathbf{v} = \mathbf{p}_0 + b(\mathbf{p}_1 - \mathbf{p}_0), \quad a \in \mathbb{R}^+, \quad b \in [0, 1]$$

$$[\mathbf{v}, -\mathbf{p}] \cdot [a, b] = \mathbf{q}_0, \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0$$

3 Motion

3.1 Social force

Total force exerted on the agent is the sum of movement adjusting, social and contact forces between other agents and wall.

$$\mathbf{f}_i(t) = \mathbf{f}_i^{adj} + \sum_{j \neq i} \left(\mathbf{f}_{ij}^{soc} + \mathbf{f}_{ij}^{c} \right) + \sum_{w} \left(\mathbf{f}_{iw}^{soc} + \mathbf{f}_{iw}^{c} \right) + \boldsymbol{\xi}_i$$

3.1.1 Adjusting force

Force adjusting agent's movement towards desired in some characteristic time

$$\mathbf{f}^{adj} = \frac{m}{\tau^{adj}} (v_0 \cdot \hat{\mathbf{e}} - \mathbf{v})$$

3.1.2 Social force

Psychological force for collision avoidance. Naive velocity independent equation

$$\mathbf{f}^{soc} = A \exp\left(-\frac{h}{B}\right) \hat{\mathbf{n}}$$

Improved velocity dependent algorithm

$$\begin{aligned}
\mathbf{f}^{soc} &= -\nabla_{\tilde{\mathbf{x}}} E(\tau) \\
&= -\nabla_{\tilde{\mathbf{x}}} \left(\frac{k}{\tau^2} \exp\left(-\frac{\tau}{\tau_0}\right) \right) \\
&= -\left(\frac{k}{a\tau^2}\right) \left(\frac{2}{\tau} + \frac{1}{\tau_0}\right) \exp\left(-\frac{\tau}{\tau_0}\right) \left(\tilde{\mathbf{v}} - \frac{a\tilde{\mathbf{x}} + b\tilde{\mathbf{v}}}{d}\right),
\end{aligned}$$

where

$$\begin{aligned}
a &= \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}} \\
b &= -\tilde{\mathbf{x}} \cdot \tilde{\mathbf{v}} \\
c &= \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} - \tilde{r}^2 \\
d &= \sqrt{b^2 - ac}, \quad b^2 - ac > 0 \\
\tau &= \frac{b - d}{a}.
\end{aligned}$$

3.1.3 Contact force

Physical contact force

$$\mathbf{f}^c = -h \cdot \left(\mu \cdot \hat{\mathbf{n}} - \kappa \cdot (\mathbf{v} \cdot \hat{\mathbf{t}}) \hat{\mathbf{t}} \right), \quad h < 0$$

with damping

$$\mathbf{f}^c = -h \cdot \left(\mu \cdot \hat{\mathbf{n}} - \kappa \cdot (\mathbf{v} \cdot \hat{\mathbf{t}}) \hat{\mathbf{t}} \right) + c_n \cdot (\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}, \quad h < 0$$

3.1.4 Random Fluctuation

Uniformly distributed random fluctuation force

$$\boldsymbol{\xi} = f \cdot \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \end{bmatrix},$$

where

$$f \in [0, f_{max}], \quad \varphi \in [0, 2\pi]$$

3.2 Rotational

Total torque exerted on agent, is the sum of adjusting contact and social torques

$$M_i(t) = M_i^{adj} + \sum_{j \neq i} \left(M_{ij}^{soc} + M_{ij}^{c} \right) + \sum_{w} \left(M_{iw}^{soc} + M_{iw}^{c} \right) + \eta_i(t)$$

3.2.1 Adjusting torque

Torque adjusting agent's rotational motion towards desired

$$M^{adj} = \frac{I}{\tau} \left((\varphi(t) - \varphi^0) \omega^0 - \omega(t) \right)$$

3.2.2 Social torque

Torque from social forces acting with other agent or wall

$$\mathbf{M}^{soc} = \mathbf{r}^{soc} \times \mathbf{f}^{soc}$$

3.2.3 Contact torque

Torque from contact forces acting with other agent or wall

$$\mathbf{M}^{c} = \mathbf{r}^{c} \times \mathbf{f}^{c}$$

3.2.4 Related equations

Torque calculated using cross product

$$\mathbf{M} = \mathbf{r} \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R_1 & R_2 & R_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

which in two dimensions is

$$M = \begin{vmatrix} R_1 & R_2 \\ f_1 & f_2 \end{vmatrix} = R_1 \cdot f_2 - R_2 \cdot f_1$$

4 Integrators

4.1 Differential systems

Position and velocity

$$m\frac{d^2}{dt^2}\mathbf{x}(t) = \mathbf{f}(t)$$

Rotational motion

$$I\frac{d^2}{dt^2}\varphi(t) = M(t)$$

4.2 Excplicit Euler Method

Updating using discrete time step Δt

$$\begin{aligned} t_0 &= 0 \\ t_1 &= t_0 + \Delta t \\ &\vdots \\ t_k &= t_{k-1} + \Delta t \end{aligned}$$

Acceleration on an agent

$$\begin{aligned} a_k &= \mathbf{f}_k / m \\ \mathbf{v}_{k+1} &= \mathbf{v}_k + a_k \Delta t \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_{k+1} \Delta t \end{aligned}$$

Angular acceleration

$$\begin{aligned} \alpha_k &= M_k / I \\ \omega_{k+1} &= \omega_k + \alpha_k \Delta t \\ \varphi_{k+1} &= \varphi_k + \omega_{k+1} \Delta t \end{aligned}$$

4.3 Velocity verlet

Velocity verlet algorithm

$$\begin{aligned} \mathbf{v}_{k+\frac{1}{2}} &= \mathbf{v}_k + \frac{1}{2} a_k \Delta t \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_{k+\frac{1}{2}} \Delta t \\ \mathbf{v}_{k+1} &= \mathbf{v}_{k+\frac{1}{2}} + \frac{1}{2} a_{k+1} \Delta t \end{aligned}$$

or more simply

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_k \Delta t + \frac{1}{2} a_k \Delta t^2 \\ \mathbf{v}_{k+1} &= \mathbf{v}_k + \frac{1}{2}\left(a_k + a_{k+1}\right) \Delta t \end{aligned}$$

5 Navigation

5.1 Theory

Navigation algorithm is a function that takes at least coordinate $\mathbf{x}$ as an argument and returns an unit vector $\hat{\mathbf{e}}$ that is used as target direction for the agent

$$f(\mathbf{x}, \ldots) \to \hat{\mathbf{e}}$$

5.2 Manual construction

5.3 Fluid flow

One way to find suitable function is to solve how *incompressible*, *irrotational* and *inviscid* fluid (ideal fluid) would flow out of the constructed space.

`https://en.wikipedia.org/wiki/Conservative_vector_field#Irrotational_flows`

`https://en.wikipedia.org/wiki/Inviscid_flow`

`https://en.wikipedia.org/wiki/Euler_equations_(fluid_dynamics)`

5.4 Combination

6 Spatial game

Spatial game for egress congestion.

6.1 Game matrix

$T_i = \lambda_i / \beta$	Estimated evacuation time
λ_i	Number of other agents closer to the exit
β	Capacity of the exit
$T_{ij} = (T_i + T_j)/2$	Average evacuation time
T_{ASET}	Available safe egress time
$T_0 (= T_{ASET})$	Time difference between T_{ASET} and T_i before agents start playing the game
$C > 0$	Cost of conflict

Number of other agents closer to the exit can be solved

$$\lambda = \operatorname{argsort} \|\mathbf{p}_0 - \mathbf{x}\|$$

where $\mathbf{p}_0$ is the center of the exit

Cost function

$$u(T_i, T_{ASET}), \quad u'(T_i) \geq 0, \quad u''(T_i) \geq 0$$

Increase/decrease in cost

$$\Delta u(T_{ij}) = u(T_{ij}) - u(T_{ij} - \Delta T) \approxeq u'(T_{ij}) \Delta T$$

	Impatient	Patient
Impatient	C, C	$-\Delta u(T_{ij}), \Delta u(T_{ij})$
Patient	$\Delta u(T_{ij}), -\Delta u(T_{ij})$	$0, 0$

	Impatient	Patient
Impatient	$\frac{C}{\Delta u(T_{ij})}, \frac{C}{\Delta u(T_{ij})}$	$-1, 1$
Patient	$1, -1$	$0, 0$

i) Prisoner's dilemma (PD)

$$0 < \frac{C}{\Delta u(T_{ij})} \leq 1$$

ii) Hawk-dove (HD)

$$\frac{C}{\Delta u(T_{ij})} > 1$$

Assumptions

1. Game is not played
$$T_{ij} \leq T_{ASET} - T_0$$
2. Cost function starts to increase quadratically
$$T_{ij} > T_{ASET} - T_0$$
3. Game turns into prisoner's dilemma
$$u'(T_{ASET}) = C, \quad T_{ij} \geq T_{ASET}$$

Cost function that meets the assumptions

$$u(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{2T_0} \left(T_{ij} - T_{ASET} + T_0\right)^2 & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Derivative

$$u'(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{T_0} \left(T_{ij} - T_{ASET} + T_0\right) & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Loss/gain of overtaking

$$\Delta u(T_{ij}) \approxeq u'(T_{ij}) \Delta T = \frac{C}{T_0} \left(T_i - T_{ASET} + T_0\right) \Delta T$$

Value parameter of the game matrix

$$\frac{C}{\Delta u(T_{ij})} \approxeq \frac{T_0}{T_{ij} - T_{ASET} + T_0}$$

6.2 Settings and best-response dynamics

	Unit	Value	
		4	von Neumann neighborhood
		8	Moore neighborhood
r_n	m	0.40	Distance to agent that is considered as neighbor
v_i			Loss defined by game matrix
S			Set of strategies $\{\text{Patient}, \text{Impatient}\}$
s			Strategy $\in \{\text{Patient}, \text{Impatient}\}$

The best-response strategy

$$s_i^{(t)} = \arg\min_{s_i' \in S} \sum_{j \in N_i} v_i \left(s_i', s_j^{(t-1)}; T_{ij}\right)$$

$s_j^{(t-1)}$ strategy neighbor played on period $t-1$

Updating strategy using poisson process.