1 Linear wall

1.1 Properties

$\mathbf{p}_0$	Start point
$\mathbf{p}_1$	End point
$h_{iw} = d_{iw} - r_i$	
$l_w = \lVert \mathbf{p}_1 - \mathbf{p}_0 \rVert$	Length
$\hat{\mathbf{t}}_w = (\mathbf{p}_1 - \mathbf{p}_0)/l_w$	
$\hat{\mathbf{n}}_w = R(90°) \cdot \hat{\mathbf{t}}_w$	

1.2 Absolute distance

Solving linear system of equations determining the position of the agent $\mathbf{x}_i$ in relation to wall

$$\begin{cases} \mathbf{p}_0 + l_{n_0}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_0}\hat{\mathbf{t}}_w \\ \mathbf{p}_1 + l_{n_1}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_1}\hat{\mathbf{t}}_w \end{cases}$$

$$\begin{cases} l_{n_0}\hat{\mathbf{n}}_w - l_{t_0}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_0 = \mathbf{q}_0 \\ l_{n_1}\hat{\mathbf{n}}_w - l_{t_1}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_1 = \mathbf{q}_1 \end{cases}$$

In matrix form

$$\begin{bmatrix} l_{n_0} & l_{n_1} \\ l_{t_0} & l_{t_1} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \hat{\mathbf{n}}_w & -\hat{\mathbf{t}}_w \end{bmatrix} = \begin{bmatrix} -t_1 & -t_0 \\ t_0 & -t_1 \end{bmatrix} \\ \mathbf{A}^{-1} &= \frac{1}{t_0^2 + t_1^2} \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} = \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\mathbf{n}}_w \\ -\hat{\mathbf{t}}_w \end{bmatrix} = \mathbf{A}^T \end{aligned}$$

Conditions

$$\begin{aligned} l_n &= l_{n_0} \vee l_{n_1} = \hat{\mathbf{n}}_w \cdot \mathbf{q}_0 \vee \hat{\mathbf{n}}_w \cdot \mathbf{q}_1 \\ l_t &= l_{t_1} + l_{t_0} = -\hat{\mathbf{t}}_w \cdot \mathbf{q}_1 - \hat{\mathbf{t}}_w \cdot \mathbf{q}_0 \end{aligned}$$

Distance between agent and linear wall

$$d_{iw} = \begin{cases} \lVert \mathbf{q}_0 \rVert & l_t > l_w \\ |l_n| & \text{otherwise} \\ \lVert \mathbf{q}_1 \rVert & l_t < -l_w \end{cases}$$

Normal vector away from the wall

$$\hat{\mathbf{n}}_{iw} = \begin{cases} \hat{\mathbf{q}}_0 & l_t > l_w \\ \operatorname{sign}(l_n)\hat{\mathbf{n}}_w & \text{otherwise} \\ \hat{\mathbf{q}}_1 & l_t < -l_w \end{cases}$$

1.3 Velocity relative distance

$\tilde{\mathbf{x}} = \mathbf{x}_{iw}$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_{iw} = \mathbf{v}_i$	Relative velocity
$\tilde{r} = r_{iw}$	Total radius
$d = \lVert \tilde{\mathbf{x}} \rVert$	Distance
$h = d - \tilde{r}$	Relative distance

$$\begin{aligned} \mathbf{q}_0 &= \mathbf{p}_0 - \mathbf{x} \\ \mathbf{q}_1 &= \mathbf{p}_1 - \mathbf{x} \\ \hat{\mathbf{n}}_{iw} &= -\operatorname{sign}(\hat{\mathbf{n}}_w \cdot \mathbf{q}_0)\hat{\mathbf{n}}_w \end{aligned}$$

$$\begin{aligned} \boldsymbol{\alpha} &= [\operatorname{angle}(\mathbf{q}_0), \operatorname{angle}(\mathbf{q}_1), \operatorname{angle}(\hat{\mathbf{n}}_{iw})] \\ \varphi &= \operatorname{angle}(\mathbf{v}) \\ \boldsymbol{\alpha}_2 &= \boldsymbol{\alpha} - \varphi \mod 2\pi \end{aligned}$$

$$i = (\arg\min(\boldsymbol{\alpha}_2), \arg\max(\boldsymbol{\alpha}_2))$$

Intersection

$$\begin{aligned} \mathbf{x} + a\mathbf{v} &= \mathbf{p}_0 + b(\mathbf{p}_1 - \mathbf{p}_0), \quad a \in \mathbb{R}^+, \quad b \in [0, 1] \\ [\mathbf{v}, -\mathbf{p}] \cdot [a, b] &= \mathbf{q}_0, \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0 \end{aligned}$$

2 Motion

2.1 Differential systems

$$m\frac{d^2}{dt^2}\mathbf{x}(t) = \mathbf{f}(t)$$

$$I\frac{d^2}{dt^2}\varphi(t) = M(t)$$

2.2 Social force model

$$\mathbf{f}_i(t) = \mathbf{f}_i^{adj} + \sum_{j\neq i}\left(\mathbf{f}_{ij}^{soc} + \mathbf{f}_{ij}^{c}\right) + \sum_{w}\left(\mathbf{f}_{iw}^{soc} + \mathbf{f}_{iw}^{c}\right) + \boldsymbol{\xi}_i$$

2.2.1 Adjusting force

$$\mathbf{f}^{adj} = \frac{m}{\tau^{adj}}(v_0 \cdot \hat{\mathbf{e}} - \mathbf{v})$$

2.2.2 Social force

$$\mathbf{f}^{soc} = A\exp\left(-\frac{h}{B}\right)\hat{\mathbf{n}}$$

$$\begin{aligned}\mathbf{f}^{soc} &= -\nabla_{\tilde{\mathbf{x}}}E(\tau)\\ &= -\nabla_{\tilde{\mathbf{x}}}\left(\frac{k}{\tau^2}\exp\left(-\frac{\tau}{\tau_0}\right)\right)\\ &= -\left(\frac{k}{a\tau^2}\right)\left(\frac{2}{\tau}+\frac{1}{\tau_0}\right)\exp\left(-\frac{\tau}{\tau_0}\right)\left(\tilde{\mathbf{v}} - \frac{a\tilde{\mathbf{x}} + b\tilde{\mathbf{v}}}{d}\right),\end{aligned}$$

where

$$\begin{aligned}a &= \tilde{\mathbf{v}}\cdot\tilde{\mathbf{v}}\\ b &= -\tilde{\mathbf{x}}\cdot\tilde{\mathbf{v}}\\ c &= \tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}} - \tilde{r}^2\\ d &= \sqrt{b^2 - ac}, \quad b^2 - ac > 0\\ \tau &= \frac{b-d}{a}.\end{aligned}$$

$$\|R(\varphi)[r_t(\tilde{\mathbf{x}} + \tau\tilde{\mathbf{v}})\cdot\hat{\mathbf{e}}_n, r\|(\tilde{\mathbf{x}} + \tau\tilde{\mathbf{v}})\times\hat{\mathbf{e}}_n\|]\|$$

2.2.3 Contact force

$$\mathbf{f}^c = -h\cdot\left(\mu\cdot\hat{\mathbf{n}} - \kappa\cdot(\mathbf{v}\cdot\hat{\mathbf{t}})\hat{\mathbf{t}}\right) + c_n\cdot(\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}, \quad h < 0$$

2.2.4 Random Fluctuation

Uniformly distributed random fluctuation force

$$\boldsymbol{\xi} = f\cdot\begin{bmatrix}\cos(\varphi) & \sin(\varphi)\end{bmatrix},$$

where

$$f \in [0, f_{max}], \quad \varphi \in [0, 2\pi]$$

2.3 Rotational motion

$$M_i(t) = M_i^{adj} + \sum_{j\neq i}\left(M_{ij}^{soc} + M_{ij}^{c}\right) + \sum_{w}\left(M_{iw}^{soc} + M_{iw}^{c}\right) + \eta_i(t)$$

2.3.1 Adjusting torque

$$M^{adj} = \frac{I}{\tau}\left((\varphi(t) - \varphi^0)\omega^0 - \omega(t)\right)$$

2.3.2 Social torque

$$\mathbf{M}^{soc} = \mathbf{r}^{soc}\times\mathbf{f}^{soc}$$

2.3.3 Contact torque

$$\mathbf{M}^{c} = \mathbf{r}^{c}\times\mathbf{f}^{c}$$

2.3.4 Related equations

$$\mathbf{M} = \mathbf{r}\times\mathbf{f} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ R_1 & R_2 & R_3\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$M = \begin{vmatrix}R_1 & R_2\\ f_1 & f_2\end{vmatrix} = R_1\cdot f_2 - R_2\cdot f_1$$

3 Integrators

3.1 Velocity verlet

Velocity verlet algorithm

$$\begin{aligned}
\mathbf{v}_{k+\frac{1}{2}} &= \mathbf{v}_k + \frac{1}{2} a_k \Delta t \\
\mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_{k+\frac{1}{2}} \Delta t \\
\mathbf{v}_{k+1} &= \mathbf{v}_{k+\frac{1}{2}} + \frac{1}{2} a_{k+1} \Delta t
\end{aligned}$$

or more simply

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_k \Delta t + \frac{1}{2} a_k \Delta t^2 \\
\mathbf{v}_{k+1} &= \mathbf{v}_k + \frac{1}{2} \left(a_k + a_{k+1} \right) \Delta t
\end{aligned}$$

versus

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_k \Delta t + \frac{1}{2} a_k \Delta t^2 \\
\mathbf{v}_{k+1} &= \mathbf{v}_k + a_k \Delta t
\end{aligned}$$

4 Spatial game

Spatial game for egress congestion.

4.1 Game matrix

$T_i = \lambda_i / \beta$	Estimated evacuation time
λ_i	Number of other agents closer to the exit
β	Capacity of the exit
$T_{ij} = (T_i + T_j)/2$	Average evacuation time
T_{ASET}	Available safe egress time
$T_0 (= T_{ASET})$	Time difference between T_{ASET} and T_i before agents start playing the game
$C > 0$	Cost of conflict

Number of other agents closer to the exit can be solved

$$\lambda = \operatorname{argsort} \|\mathbf{p}_0 - \mathbf{x}\|$$

where $\mathbf{p}_0$ is the center of the exit

Cost function

$$u(T_i, T_{ASET}), \quad u'(T_i) \geq 0, \quad u''(T_i) \geq 0$$

Increase/decrease in cost

$$\Delta u(T_{ij}) = u(T_{ij}) - u(T_{ij} - \Delta T) \approxeq u'(T_{ij}) \Delta T$$

	Impatient	Patient
Impatient	C, C	$-\Delta u(T_{ij}), \Delta u(T_{ij})$
Patient	$\Delta u(T_{ij}), -\Delta u(T_{ij})$	$0, 0$

	Impatient	Patient
Impatient	$\frac{C}{\Delta u(T_{ij})}, \frac{C}{\Delta u(T_{ij})}$	$-1, 1$
Patient	$1, -1$	$0, 0$

i) Prisoner's dilemma (PD)

$$0 < \frac{C}{\Delta u(T_{ij})} \leq 1$$

ii) Hawk-dove (HD)

$$\frac{C}{\Delta u(T_{ij})} > 1$$

Assumptions

1. Game is not played
$$T_{ij} \leq T_{ASET} - T_0$$
2. Cost function starts to increase quadratically
$$T_{ij} > T_{ASET} - T_0$$
3. Game turns into prisoner's dilemma
$$u'(T_{ASET}) = C, \quad T_{ij} \geq T_{ASET}$$

Cost function that meets the assumptions

$$u(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{2T_0} \left(T_{ij} - T_{ASET} + T_0\right)^2 & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Derivative

$$u'(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{T_0} \left(T_{ij} - T_{ASET} + T_0\right) & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Loss/gain of overtaking

$$\Delta u(T_{ij}) \approxeq u'(T_{ij}) \Delta T = \frac{C}{T_0} \left(T_i - T_{ASET} + T_0\right) \Delta T$$

Value parameter of the game matrix

$$\frac{C}{\Delta u(T_{ij})} \approxeq \frac{T_0}{T_{ij} - T_{ASET} + T_0}$$

4.2 Settings and best-response dynamics

	Unit	Value	
		4	von Neumann neighborhood
		8	Moore neighborhood
r_n	m	0.40	Distance to agent that is considered as neighbor
v_i			Loss defined by game matrix
S			Set of strategies $\{\text{Patient}, \text{Impatient}\}$
s			Strategy $\in \{\text{Patient}, \text{Impatient}\}$

The best-response strategy

$$s_i^{(t)} = \arg\min_{s_i' \in S} \sum_{j \in N_i} v_i \left(s_i', s_j^{(t-1)}; T_{ij}\right)$$

$s_j^{(t-1)}$ strategy neighbor played on period $t-1$

Updating strategy using poisson process.