

# 1 Description

Crowd dynamics simulation in 2-Dimensional continuous space.

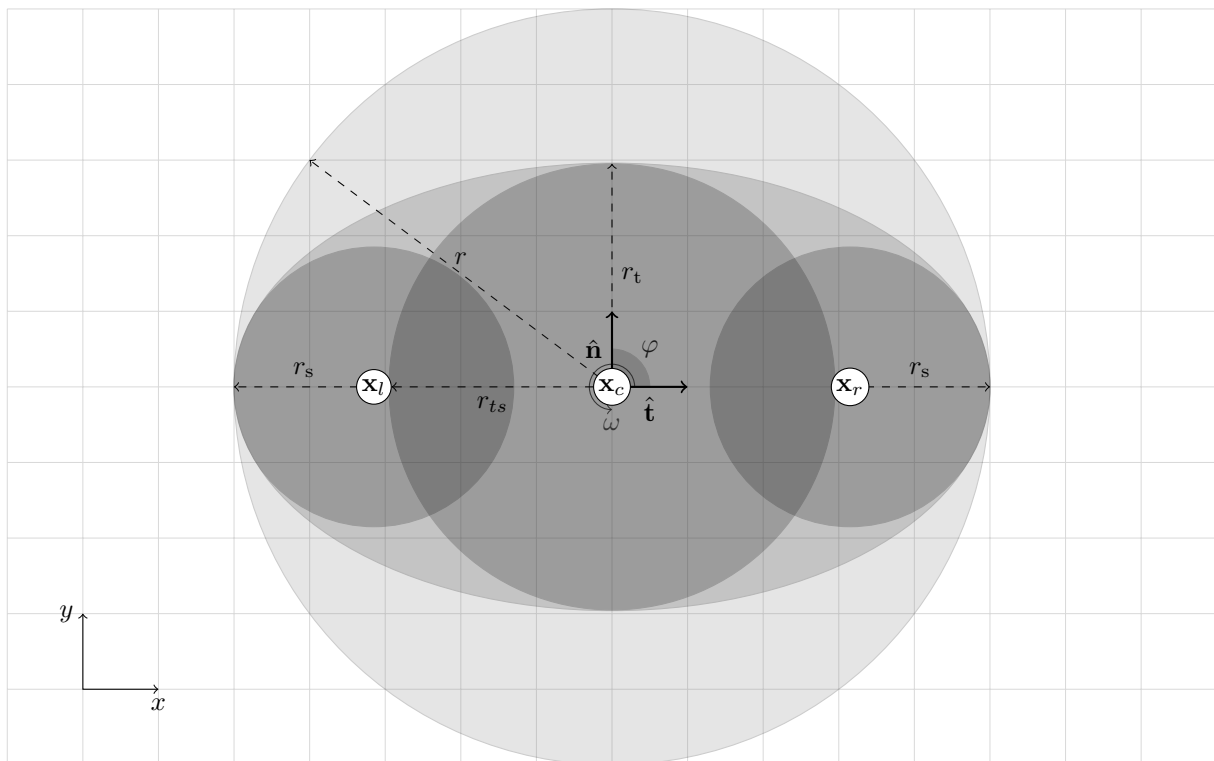
$$\mathbf{x} = (x, y) \in \mathbb{R}^2$$

Some operations for 2D geometry

Notation	Name	Return
$\ \mathbf{x}\ $	<code>hypot</code>	$\in [0, \infty)$
$\text{angle}(\mathbf{x})$	<code>arctan2</code>	$\in [-\pi, \pi]$
$R(90^\circ) \cdot \mathbf{x}$		$(-y, x)$
$R(-90^\circ) \cdot \mathbf{x}$		$(y, -x)$

## 2 Constants

Symbol	Unit	Value	Explanation
$\Delta t$	s	0.01 – 0.001	Timestep
$\tau_{adj}$	s	0.5	Characteristic time in which agent adjusts its movement.
$k$	N	1.5	Social force scaling constant.
$\tau_0$	s	3.0	Interaction time horizon.
$\mu$	kg/s <sup>2</sup>	$1.2e + 05$	Compression counteraction constant.
$\kappa$	kg/(m s)	$2.4e + 05$	Sliding friction constant.
$A$	N	$2.0e + 03$	Scaling coefficient for social force.
$B$	m	0.08	Coefficient for social force.
$\ \mathbf{f}_{max}\ $	N		Force magnitude limit.



**Figure 1:** Circle, ellipse and three circle representations of an agent.

### 3 Agents

$$h = d - \tilde{r}$$

#### 3.1 Properties

**Table 1:** Shoulder, torso and total radii.

	Total		Torso		Shoulder
	$r$	$\pm$	$k_t = \frac{r_t}{r}$	$k_s = \frac{r_s}{r}$	$k_{ts} = \frac{r_{ts}}{r}$
adult	0.255	0.035	0.5882	0.3725	0.6275
child	0.210	0.015	0.5714	0.3333	0.6667
eldery	0.250	0.020	0.6000	0.3600	0.6400
female	0.240	0.020	0.5833	0.3750	0.6250
male	0.270	0.020	0.5926	0.3704	0.6296

**Table 2:** Properties

$r$	m		Total radius
$r_t$	m		Torso radius
$r_s$	m		Shoulder radius
$r_{ts}$	m		Distance from torso to shoulder
$m$	kg	80	Mass
$I$	kg · m <sup>2</sup>	4.0	Rotational moment
$\mathbf{x}$	m		Position
$\mathbf{v}$	m/s		Velocity
$v_0$	m/s		Goal velocity
$\hat{\mathbf{e}}_0$			Goal direction
$\hat{\mathbf{e}}$			Target direction
$\varphi$	rad	$[0, 2\pi]$	Body angle
$\omega$	rad/s		Angular velocity
$\varphi_0$	rad		Target angle
$\omega_0$	rad/s	$4\pi$	Max angular velocity
$p$		$0 - 1$	Herding tendency

#### 3.2 Models

##### 3.2.1 Circular

**Table 3:** Relative

$\tilde{\mathbf{x}} = \mathbf{x}_i - \mathbf{x}_j$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_i - \mathbf{v}_j$	Relative velocity
$d = \ \tilde{\mathbf{x}}\ $	Distance
$\hat{\mathbf{n}} = \tilde{\mathbf{x}}/d$	Normal vector
$\hat{\mathbf{t}} = R(-90^\circ) \cdot \hat{\mathbf{n}}$	Tangent vector

Total radius and relative distance

$$\tilde{r} = r_i + r_j$$

##### 3.2.2 Three circles

$$\mathbf{x}_r = \mathbf{x}_c + \hat{\mathbf{t}}r_{ts}$$

$$\mathbf{x}_l = \mathbf{x}_c - \hat{\mathbf{t}}r_{ts}$$

$$\hat{\mathbf{t}} = [-\sin(\varphi) \quad \cos(\varphi)]$$

$$\mathbf{r}_{tot} = \begin{bmatrix} r_t & r_s & r_s \end{bmatrix}_i + \begin{bmatrix} r_t \\ r_s \\ r_s \end{bmatrix}_j$$

$$\begin{aligned} \mathbf{d} &= \left\| \begin{bmatrix} \mathbf{x}_c & \mathbf{x}_r & \mathbf{x}_l \end{bmatrix}_i - \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_r \\ \mathbf{x}_l \end{bmatrix}_j \right\| \\ &= \left\| \begin{bmatrix} 0 & \hat{\mathbf{t}}r_{ts} & -\hat{\mathbf{t}}r_{ts} \end{bmatrix}_i - \begin{bmatrix} 0 \\ \hat{\mathbf{t}}r_{ts} \\ -\hat{\mathbf{t}}r_{ts} \end{bmatrix}_j + (\mathbf{x}_i - \mathbf{x}_j) \right\| \\ &= \left\| \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} (\hat{\mathbf{t}}r_{ts})_i - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} (\hat{\mathbf{t}}r_{ts})_j + \tilde{\mathbf{x}} \right\| \\ &= \left\| \mathbf{k} (\hat{\mathbf{t}}r_{ts})_i - \mathbf{k}^T (\hat{\mathbf{t}}r_{ts})_j + \tilde{\mathbf{x}} \right\| \\ &= \left\| \mathbf{c}_i - \mathbf{c}_j^T + \tilde{\mathbf{x}} \right\| \end{aligned}$$

$$\mathbf{h} = \mathbf{d} - \mathbf{r}_{tot}$$

1. Find

$$h = \min(\mathbf{h})$$

and track minimizing values

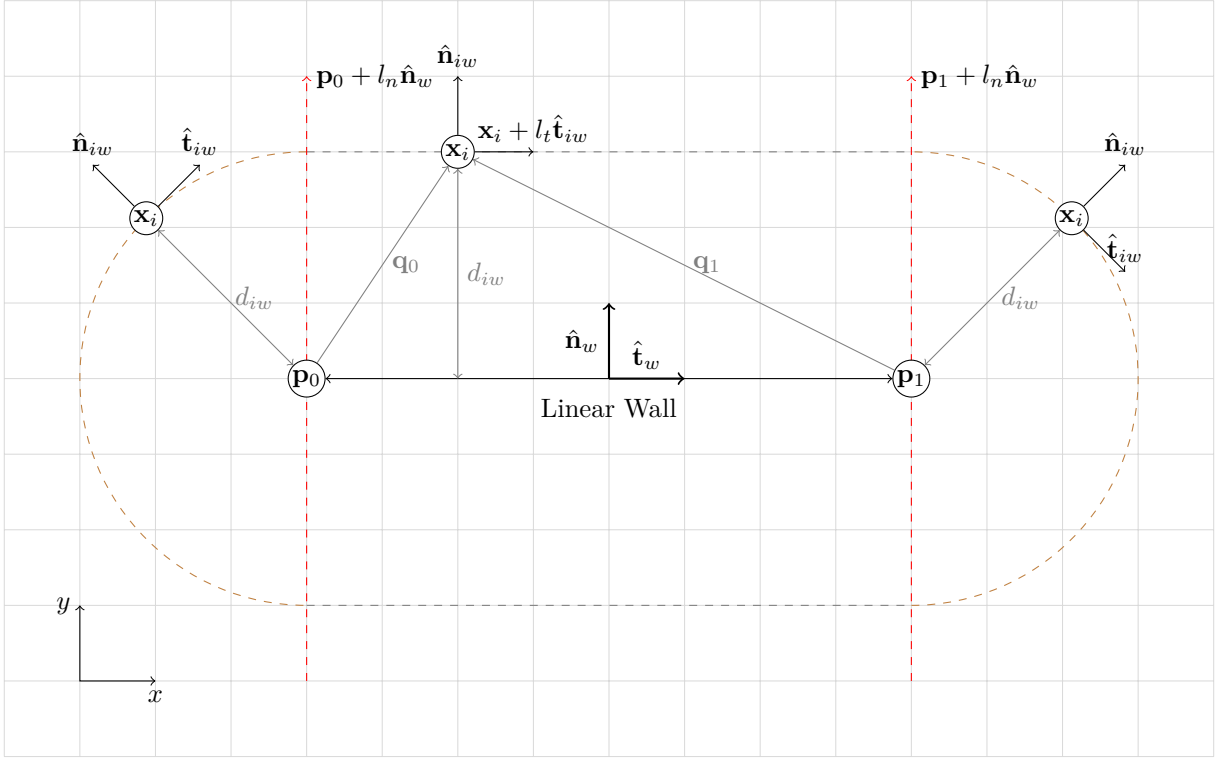
$$\hat{\mathbf{e}}_{ij}, k_i, k_j, r_i, r_j$$

2.

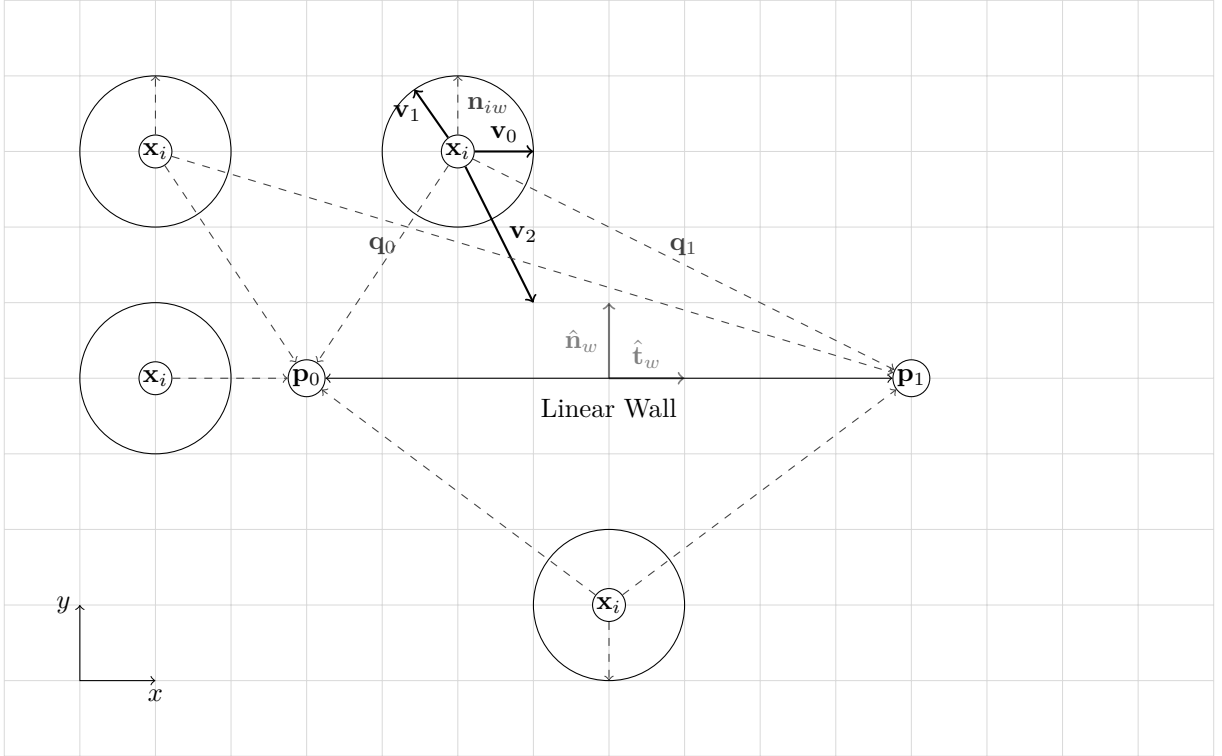
$$\mathbf{r}_i^{moment} = \mathbf{x}_i^c + k_i \cdot \hat{\mathbf{t}}_i r_i^{ts} + r_i \hat{\mathbf{e}}_{ij}$$

$$\mathbf{r}_j^{moment} = \mathbf{x}_j^c + k_j \cdot \hat{\mathbf{t}}_j r_j^{ts} - r_j \hat{\mathbf{e}}_{ij}$$

3. Return  $(\tilde{\mathbf{x}}, r_{tot}, h, \mathbf{r}_i^{moment}, \mathbf{r}_j^{moment})$



**Figure 2:** Absolute distance from a linear wall.



**Figure 3:** Velocity dependent distance from a linear wall.

## 4 Linear wall

### 4.1 Properties

$\mathbf{p}_0$	Start point
$\mathbf{p}_1$	End point
<hr/>	
$h_{iw} = d_{iw} - r_i$	
$l_w = \ \mathbf{p}_1 - \mathbf{p}_0\ $	Length
$\hat{\mathbf{t}}_w = (\mathbf{p}_1 - \mathbf{p}_0) / l_w$	
$\hat{\mathbf{n}}_w = R(90^\circ) \cdot \hat{\mathbf{t}}_w$	

### 4.2 Absolute distance

Solving linear system of equations determining the position of the agent  $\mathbf{x}_i$  in relation to wall

$$\begin{cases} \mathbf{p}_0 + l_{n_0} \hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_0} \hat{\mathbf{t}}_w \\ \mathbf{p}_1 + l_{n_1} \hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_1} \hat{\mathbf{t}}_w \end{cases}$$

$$\begin{cases} l_{n_0} \hat{\mathbf{n}}_w - l_{t_0} \hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_0 = \mathbf{q}_0 \\ l_{n_1} \hat{\mathbf{n}}_w - l_{t_1} \hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_1 = \mathbf{q}_1 \end{cases}$$

In matrix form

$$\begin{bmatrix} l_{n_0} & l_{n_1} \\ l_{t_0} & l_{t_1} \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{n}}_w & -\hat{\mathbf{t}}_w \end{bmatrix} = \begin{bmatrix} -t_1 & -t_0 \\ t_0 & -t_1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{t_0^2 + t_1^2} \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} = \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\mathbf{n}}_w \\ -\hat{\mathbf{t}}_w \end{bmatrix} = \mathbf{A}^T$$

Conditions

$$l_n = l_{n_0} \vee l_{n_1} = \hat{\mathbf{n}}_w \cdot \mathbf{q}_0 \vee \hat{\mathbf{n}}_w \cdot \mathbf{q}_1$$

$$l_t = l_{t_1} + l_{t_0} = -\hat{\mathbf{t}}_w \cdot \mathbf{q}_1 - \hat{\mathbf{t}}_w \cdot \mathbf{q}_0$$

Distance between agent and linear wall

$$d_{iw} = \begin{cases} \|\mathbf{q}_0\| & l_t > l_w \\ |l_n| & \text{otherwise} \\ \|\mathbf{q}_1\| & l_t < -l_w \end{cases}$$

Normal vector away from the wall

$$\hat{\mathbf{n}}_{iw} = \begin{cases} \hat{\mathbf{q}}_0 & l_t > l_w \\ \text{sign}(l_n) \hat{\mathbf{n}}_w & \text{otherwise} \\ \hat{\mathbf{q}}_1 & l_t < -l_w \end{cases}$$

### 4.3 Velocity relative distance

$\tilde{\mathbf{x}} = \mathbf{x}_{iw}$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_{iw} = \mathbf{v}_i$	Relative velocity
$\tilde{r} = r_{iw}$	Total radius
$d = \ \tilde{\mathbf{x}}\ $	Distance
$h = d - \tilde{r}$	Relative distance

$$\mathbf{q}_0 = \mathbf{p}_0 - \mathbf{x}$$

$$\mathbf{q}_1 = \mathbf{p}_1 - \mathbf{x}$$

$$\hat{\mathbf{n}}_{iw} = -\text{sign}(\hat{\mathbf{n}}_w \cdot \mathbf{q}_0) \hat{\mathbf{n}}_w$$

$$\boldsymbol{\alpha} = [\text{angle}(\mathbf{q}_0), \text{angle}(\mathbf{q}_1), \text{angle}(\hat{\mathbf{n}}_{iw})]$$

$$\varphi = \text{angle}(\mathbf{v})$$

$$\boldsymbol{\alpha}_2 = \boldsymbol{\alpha} - \varphi \mod 2\pi$$

$$i = (\arg \min(\boldsymbol{\alpha}_2), \arg \max(\boldsymbol{\alpha}_2))$$

Intersection

$$\mathbf{x} + a\mathbf{v} = \mathbf{p}_0 + b(\mathbf{p}_1 - \mathbf{p}_0), \quad a \in \mathbb{R}^+, \quad b \in [0, 1]$$

$$[\mathbf{v}, -\mathbf{p}] \cdot [a, b] = \mathbf{q}_0, \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0$$

## 5 Motion

### 5.1 Social force

Total force exerted on the agent is the sum of movement adjusting, social and contact forces between other agents and wall.

$$\mathbf{f}_i(t) = \mathbf{f}_i^{adj} + \sum_{j \neq i} (\mathbf{f}_{ij}^{soc} + \mathbf{f}_{ij}^c) + \sum_w (\mathbf{f}_{iw}^{soc} + \mathbf{f}_{iw}^c) + \boldsymbol{\xi}_i$$

#### 5.1.1 Adjusting force

Force adjusting agent's movement towards desired in some characteristic time

$$\mathbf{f}_i^{adj} = \frac{m}{\tau^{adj}} (v_0 \cdot \hat{\mathbf{e}} - \mathbf{v})$$

#### 5.1.2 Social force

Psychological force for collision avoidance. Naive velocity independent algorithm

$$\mathbf{f}^{soc} = A \exp\left(-\frac{h}{B}\right) \hat{\mathbf{n}}$$

and improved velocity dependent algorithm

$$\begin{aligned} \mathbf{f}^{soc} &= -\nabla_{\tilde{\mathbf{x}}} E(\tau) \\ &= -\nabla_{\tilde{\mathbf{x}}} \left( \frac{k}{\tau^2} \exp\left(-\frac{\tau}{\tau_0}\right) \right) \\ &= -\left( \frac{k}{a\tau^2} \right) \left( \frac{2}{\tau} + \frac{1}{\tau_0} \right) \exp\left(-\frac{\tau}{\tau_0}\right) \left( \tilde{\mathbf{v}} - \frac{a\tilde{\mathbf{x}} + b\tilde{\mathbf{v}}}{d} \right), \end{aligned}$$

where

$$\begin{aligned} a &= \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}} \\ b &= -\tilde{\mathbf{x}} \cdot \tilde{\mathbf{v}} \\ c &= \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} - \tilde{r}^2 \\ d &= \sqrt{b^2 - ac}, \quad b^2 - ac > 0 \\ \tau &= \frac{b-d}{a} > 0. \end{aligned}$$

#### 5.1.3 Contact force

Physical contact force

$$\mathbf{f}^c = -h \cdot (\mu \cdot \hat{\mathbf{n}} - \kappa \cdot (\mathbf{v} \cdot \hat{\mathbf{t}}) \hat{\mathbf{t}}), \quad h < 0$$

### 5.1.4 Random Fluctuation

Uniformly distributed random fluctuation force

$$\boldsymbol{\xi} = f \cdot [\cos(\varphi) \quad \sin(\varphi)],$$

where

$$f \in [0, f_{max}], \quad \varphi \in [0, 2\pi]$$

## 5.2 Rotational

Total torque exerted on agent  $M(t)$ , is the sum of contact, social and motivational torques

$$M_i(t) = M_i^{adj} + \sum_{j \neq i} (M_{ij}^{soc} + M_{ij}^c) + \sum_w (M_{iw}^{soc} + M_{iw}^c) + \eta_i(t)$$

### 5.2.1 Adjusting torque

$$M_i^{adj} = \frac{I_i}{\tau_i} ((\varphi_i(t) - \varphi_i^0) \omega^0 - \omega(t))$$

### 5.2.2 Social torque

$$\mathbf{M}_i^{soc} = \mathbf{r}_i^{soc} \times \mathbf{f}_{ij}^{soc}$$

### 5.2.3 Contact torque

$$\mathbf{M}_i^c = \mathbf{r}_i^c \times \mathbf{f}_{ij}^c$$

### 5.2.4 Related equations

Torque calculated using cross product

$$\mathbf{M} = \mathbf{r} \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R_1 & R_2 & R_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

which in two dimensions is

$$M = \begin{vmatrix} R_1 & R_2 \\ f_1 & f_2 \end{vmatrix} = R_1 \cdot f_2 - R_2 \cdot f_1$$

## 6 Integrators

### 6.1 Differential systems

Position and velocity

$$m \frac{d^2}{dt^2} \mathbf{x}(t) = \mathbf{f}(t)$$

Rotational motion

$$I \frac{d^2}{dt^2} \varphi(t) = M(t)$$

### 6.2 Numerical methods

Updating using discrete time step  $\Delta t$

$$\begin{aligned} t_0 &= 0 \\ t_1 &= t_0 + \Delta t \\ &\vdots \\ t_k &= t_{k-1} + \Delta t \end{aligned}$$

### 6.3 Explicit Euler Method

Acceleration on an agent

$$\begin{aligned} a_k &= \mathbf{f}_k / m \\ \mathbf{v}_{k+1} &= \mathbf{v}_k + a_k \Delta t \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{v}_{k+1} \Delta t \end{aligned}$$

Angular acceleration

$$\begin{aligned} \alpha_k &= M_k / I \\ \omega_{k+1} &= \omega_k + \alpha_k \Delta t \\ \varphi_{k+1} &= \varphi_k + \omega_{k+1} \Delta t \end{aligned}$$

## 7 Navigation

### 7.1 Target direction

Herding behavior

$$\mathbf{e}_i = (1 - p_i) \hat{\mathbf{e}}_i^0 + p_i \langle \hat{\mathbf{e}}_j^0 \rangle_i$$

## 8 Spatial game

### 8.1 Game matrix

$T_i = \lambda_i / \beta$	Estimated evacuation time
$\lambda_i$	Number of other agents closer to the exit
$\beta$	Capacity of the exit
$T_{ij} = (T_i + T_j) / 2$	Average evacuation time
$T_{ASET}$	Available safe egress time
$T_0 = T_{ASET}$	Time difference between $T_{ASET}$ and $T_i$ before agents start playing the game
$C > 0$	Cost of conflict

Cost function

$$u(T_i, T_{ASET}), \quad u'(T_i) \geq 0, \quad u''(T_i) \leq 0$$

Increase/decrease in cost

$$\Delta u(T_{ij}) = u(T_{ij}) - u(T_{ij} - \Delta T) \approx u'(T_{ij}) \Delta T$$

	Impatient	Patient
Impatient	$C, C$	$-\Delta u(T_{ij}), \Delta u(T_{ij})$
Patient	$\Delta u(T_{ij}), -\Delta u(T_{ij})$	$0, 0$

	Impatient	Patient
Impatient	$\frac{C}{\Delta u(T_{ij})}, \frac{C}{\Delta u(T_{ij})}$	$-1, 1$
Patient	$1, -1$	$0, 0$

i) Prisoner's dilemma (PD)

$$0 < \frac{C}{\Delta u(T_{ij})} \leq 1$$

ii) Hawk-dove (HD)

$$\frac{C}{\Delta u(T_{ij})} > 1$$

Assumptions

1. Game is not played

$$T_{ij} \leq T_{ASET} - T_0$$

2. Cost function starts to increase quadratically

$$T_{ij} > T_{ASET} - T_0$$

3. Game turns into prisoner's dilemma

$$u'(T_{ASET}) = C, \quad T_{ij} \geq T_{ASET}$$

Cost function that meets the assumptions

$$u(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{2T_0} (T_{ij} - T_{ASET} + T_0)^2 & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Derivative

$$u'(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{T_0} (T_{ij} - T_{ASET} + T_0) & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Loss/gain of overtaking

$$\Delta u(T_{ij}) \cong u'(T_{ij}) \Delta T = \frac{C}{T_0} (T_i - T_{ASET} + T_0) \Delta T$$

Value parameter of the game matrix

$$\frac{C}{\Delta u(T_{ij})} \cong \frac{T_0}{T_{ij} - T_{ASET} + T_0}$$

## 8.2 Settings and best-response dynamics

	Unit	Value	
		4	von Neumann neighborhood
		8	Moore neighborhood
$r_n$	m	0.40	Distance to agent that is considered as neighbor
$v_i$			Loss defined by game matrix
$S$			Set of strategies
			{Patient, Impatient}
$s$			Strategy $\in$ {Patient, Impatient}

The best-response strategy

$$s_i^{(t)} = \arg \min_{s'_i \in S} \sum_{j \in N_i} v_i(s'_i, s_j^{(t-1)}; T_{ij})$$

$s_j^{(t-1)}$  strategy neighbor played on period  $t - 1$

Updating strategy using poisson process.



## 9 Algorithms

### 9.1 Circular agent - agent

**Ensure:**  $i, j \in N, \quad i \neq j$

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1:  $\tilde{\mathbf{x}} = \mathbf{x}_i - \mathbf{x}_j$ 
2:  $\tilde{\mathbf{v}} = \mathbf{v}_i - \mathbf{v}_j$ 
3:  $r_{tot} = r_i + r_j$ 
4:  $d = \|\tilde{\mathbf{x}}\|$ 
5:  $h = d - r_{tot}$ 
6:
7: if  $h \leq sight$  then
8:    $\mathbf{f}_{soc}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, r_{tot}, k, \tau_0)$ 
9: end if
10:
11: if  $h < 0$  then
12:    $\hat{\mathbf{n}} = \tilde{\mathbf{x}}/d$ 
13:    $\hat{\mathbf{t}} = R(-90^\circ)\hat{\mathbf{n}}$ 
14:    $\mathbf{f}_c(\tilde{\mathbf{v}}, h, \hat{\mathbf{n}}, \hat{\mathbf{t}}, \mu, \kappa)$ 
15: end if
16:

```

### 9.2 Orientable agent - agent

**Ensure:**  $i, j \in N, \quad i \neq j$  asd

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1:  $\tilde{\mathbf{x}} = \mathbf{x}_i - \mathbf{x}_j$ 
2:  $\tilde{\mathbf{v}} = \mathbf{v}_i - \mathbf{v}_j$ 
3:  $r_{max} = r_i + r_j$ 
4:  $d = \|\tilde{\mathbf{x}}\|$ 
5:  $h_{min} = d - r_{tot}$ 
6:
7: if  $h_{min} \leq sight$  then
8:    $\tilde{\mathbf{x}}, r_{tot}, h, r_{i,j}^{moment} = \text{distance}(\text{agent}, i, j, \tilde{\mathbf{x}}, d)$ 
9:
10:   if  $h \leq sight$  then
11:      $\mathbf{f}_{soc}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, r_{tot}, k, \tau_0)$ 
12:      $M_i^{soc}(\mathbf{r}_i^{moment}, \mathbf{f}_{soc})$ 
13:      $M_j^{soc}(\mathbf{r}_j^{moment}, \mathbf{f}_{soc})$ 
14:   end if
15:
16:   if  $h < 0$  then
17:      $\hat{\mathbf{n}} = \tilde{\mathbf{x}}/d$ 
18:      $\hat{\mathbf{t}} = R(-90^\circ)\hat{\mathbf{n}}$ 
19:      $\mathbf{f}_c(\tilde{\mathbf{v}}, h, \hat{\mathbf{n}}, \hat{\mathbf{t}}, \mu, \kappa)$ 
20:      $M_i^c(\mathbf{r}_i^{moment}, \mathbf{f}_c)$ 
21:      $M_j^c(\mathbf{r}_j^{moment}, \mathbf{f}_c)$ 
22:   end if
23: end if
24:

```