1 Linear wall

1.1 Properties

$\mathbf{p}_0$	Start point
$\mathbf{p}_1$	End point
$h_{iw} = d_{iw} - r_i$	
$l_w = \|\mathbf{p}_1 - \mathbf{p}_0\|$	Length
$\hat{\mathbf{t}}_w = (\mathbf{p}_1 - \mathbf{p}_0)/l_w$	
$\hat{\mathbf{n}}_w = R(90°) \cdot \hat{\mathbf{t}}_w$	

1.2 Absolute distance

Solving linear system of equations determining the position of the agent $\mathbf{x}_i$ in relation to wall

$$\begin{cases} \mathbf{p}_0 + l_{n_0}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_0}\hat{\mathbf{t}}_w \\ \mathbf{p}_1 + l_{n_1}\hat{\mathbf{n}}_w = \mathbf{x}_i + l_{t_1}\hat{\mathbf{t}}_w \end{cases}$$

$$\begin{cases} l_{n_0}\hat{\mathbf{n}}_w - l_{t_0}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_0 = \mathbf{q}_0 \\ l_{n_1}\hat{\mathbf{n}}_w - l_{t_1}\hat{\mathbf{t}}_w = \mathbf{x}_i - \mathbf{p}_1 = \mathbf{q}_1 \end{cases}$$

In matrix form

$$\begin{bmatrix} l_{n_0} & l_{n_1} \\ l_{t_0} & l_{t_1} \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} \mathbf{q}_0 & \mathbf{q}_1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{n}}_w & -\hat{\mathbf{t}}_w \end{bmatrix} = \begin{bmatrix} -t_1 & -t_0 \\ t_0 & -t_1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{t_0^2 + t_1^2}\begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix} = \begin{bmatrix} -t_1 & t_0 \\ -t_0 & -t_1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\mathbf{n}}_w \\ -\hat{\mathbf{t}}_w \end{bmatrix} = \mathbf{A}^T$$

Conditions

$$l_n = l_{n_0} \vee l_{n_1} = \hat{\mathbf{n}}_w \cdot \mathbf{q}_0 \vee \hat{\mathbf{n}}_w \cdot \mathbf{q}_1$$

$$l_t = l_{t_1} + l_{t_0} = -\hat{\mathbf{t}}_w \cdot \mathbf{q}_1 - \hat{\mathbf{t}}_w \cdot \mathbf{q}_0$$

Distance between agent and linear wall

$$d_{iw} = \begin{cases} \|\mathbf{q}_0\| & l_t > l_w \\ |l_n| & \text{otherwise} \\ \|\mathbf{q}_1\| & l_t < -l_w \end{cases}$$

Normal vector away from the wall

$$\hat{\mathbf{n}}_{iw} = \begin{cases} \hat{\mathbf{q}}_0 & l_t > l_w \\ \text{sign}(l_n)\hat{\mathbf{n}}_w & \text{otherwise} \\ \hat{\mathbf{q}}_1 & l_t < -l_w \end{cases}$$

1.3 Velocity relative distance

$\tilde{\mathbf{x}} = \mathbf{x}_{iw}$	Relative position
$\tilde{\mathbf{v}} = \mathbf{v}_{iw} = \mathbf{v}_i$	Relative velocity
$\tilde{r} = r_{iw}$	Total radius
$d = \|\tilde{\mathbf{x}}\|$	Distance
$h = d - \tilde{r}$	Relative distance

$$\mathbf{q}_0 = \mathbf{p}_0 - \mathbf{x}$$
$$\mathbf{q}_1 = \mathbf{p}_1 - \mathbf{x}$$
$$\hat{\mathbf{n}}_{iw} = -\text{sign}(\hat{\mathbf{n}}_w \cdot \mathbf{q}_0)\hat{\mathbf{n}}_w$$

$$\boldsymbol{\alpha} = [\text{angle}(\mathbf{q}_0), \text{angle}(\mathbf{q}_1), \text{angle}(\hat{\mathbf{n}}_{iw})]$$
$$\varphi = \text{angle}(\mathbf{v})$$
$$\boldsymbol{\alpha}_2 = \boldsymbol{\alpha} - \varphi \mod 2\pi$$

$$i = (\arg\min(\boldsymbol{\alpha}_2), \arg\max(\boldsymbol{\alpha}_2))$$

Intersection

$$\mathbf{x} + a\mathbf{v} = \mathbf{p}_0 + b(\mathbf{p}_1 - \mathbf{p}_0), \quad a \in \mathbb{R}^+, \quad b \in [0, 1]$$
$$[\mathbf{v}, -\mathbf{p}] \cdot [a, b] = \mathbf{q}_0, \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0$$

2 Motion

2.1 Differential systems

$$m\frac{d^2}{dt^2}\mathbf{x}(t) = \mathbf{f}(t)$$

$$I\frac{d^2}{dt^2}\varphi(t) = M(t)$$

2.2 Social force model

$$\mathbf{f}_i(t) = \mathbf{f}_i^{adj} + \sum_{j\neq i}\left(\mathbf{f}_{ij}^{soc} + \mathbf{f}_{ij}^c\right) + \sum_w\left(\mathbf{f}_{iw}^{soc} + \mathbf{f}_{iw}^c\right) + \boldsymbol{\xi}_i$$

2.2.1 Adjusting force

$$\mathbf{f}^{adj} = \frac{m}{\tau^{adj}}(v_0 \cdot \hat{\mathbf{e}} - \mathbf{v})$$

2.2.2 Social force

$$\mathbf{f}^{soc} = A\exp\left(-\frac{h}{B}\right)\hat{\mathbf{n}}$$

$$\mathbf{f}^{soc} = -\nabla_{\tilde{\mathbf{x}}}E(\tau)$$
$$= -\nabla_{\tilde{\mathbf{x}}}\left(\frac{k}{\tau^2}\exp\left(-\frac{\tau}{\tau_0}\right)\right)$$

$$= -\left(\frac{k}{a\tau^2}\right)\left(\frac{2}{\tau} + \frac{1}{\tau_0}\right)\exp\left(-\frac{\tau}{\tau_0}\right)\left(\tilde{\mathbf{v}} - \frac{a\tilde{\mathbf{x}} + b\tilde{\mathbf{v}}}{d}\right),$$

where

$$a = \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}$$
$$b = -\tilde{\mathbf{x}} \cdot \tilde{\mathbf{v}}$$
$$c = \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} - \tilde{r}^2$$
$$d = \sqrt{b^2 - ac}, \quad b^2 - ac > 0$$
$$\tau = \frac{b - d}{a}.$$

2.2.3 Contact force

$$\mathbf{f}^c = -h \cdot \left(\mu \cdot \hat{\mathbf{n}} - \kappa \cdot (\mathbf{v} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}\right) + c_n \cdot (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}, \quad h < 0$$

2.2.4 Random Fluctuation

Uniformly distributed random fluctuation force

$$\boldsymbol{\xi} = f \cdot \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \end{bmatrix},$$

where

$$f \in [0, f_{max}], \quad \varphi \in [0, 2\pi]$$

2.3 Rotational motion

$$M_i(t) = M_i^{adj} + \sum_{j\neq i}\left(M_{ij}^{soc} + M_{ij}^c\right) + \sum_w\left(M_{iw}^{soc} + M_{iw}^c\right) + \eta_i(t)$$

2.3.1 Adjusting torque

$$M^{adj} = \frac{I}{\tau}\left((\varphi(t) - \varphi^0)\omega^0 - \omega(t)\right)$$

2.3.2 Social torque

$$\mathbf{M}^{soc} = \mathbf{r}^{soc} \times \mathbf{f}^{soc}$$

2.3.3 Contact torque

$$\mathbf{M}^c = \mathbf{r}^c \times \mathbf{f}^c$$

2.3.4 Related equations

$$\mathbf{M} = \mathbf{r} \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R_1 & R_2 & R_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$M = \begin{vmatrix} R_1 & R_2 \\ f_1 & f_2 \end{vmatrix} = R_1 \cdot f_2 - R_2 \cdot f_1$$

3 Spatial game

Spatial game for egress congestion.

3.1 Game matrix

$T_i = \lambda_i/\beta$	Estimated evacuation time
λ_i	Number of other agents closer to the exit
β	Capacity of the exit
$T_{ij} = (T_i + T_j)/2$	Average evacuation time
T_{ASET}	Available safe egress time
$T_0(= T_{ASET})$	Time difference between T_{ASET} and T_i before agents start playing the game
$C > 0$	Cost of conflict

Number of other agents closer to the exit can be solved

$$\lambda = \arg\text{sort} \, \|\mathbf{p}_0 - \mathbf{x}\|$$

where $\mathbf{p}_0$ is the center of the exit

Cost function

$$u(T_i, T_{ASET}), \quad u'(T_i) \geq 0, \quad u''(T_i) \geq 0$$

Increase/decrease in cost

$$\Delta u(T_{ij}) = u(T_{ij}) - u(T_{ij} - \Delta T) \approx u'(T_{ij})\Delta T$$

	Impatient	Patient
Impatient	C, C	$-\Delta u(T_{ij}), \Delta u(T_{ij})$
Patient	$\Delta u(T_{ij}), -\Delta u(T_{ij})$	$0, 0$

	Impatient	Patient
Impatient	$\frac{C}{\Delta u(T_{ij})}, \frac{C}{\Delta u(T_{ij})}$	$-1, 1$
Patient	$1, -1$	$0, 0$

i) Prisoner's dilemma (PD)

$$0 < \frac{C}{\Delta u(T_{ij})} \leq 1$$

ii) Hawk-dove (HD)

$$\frac{C}{\Delta u(T_{ij})} > 1$$

Assumptions

1. Game is not played
$$T_{ij} \leq T_{ASET} - T_0$$

2. Cost function starts to increase quadratically
$$T_{ij} > T_{ASET} - T_0$$

3. Game turns into prisoner's dilemma
$$u'(T_{ASET}) = C, \quad T_{ij} \geq T_{ASET}$$

Cost function that meets the assumptions

$$u(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{2T_0}(T_{ij} - T_{ASET} + T_0)^2 & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Derivative

$$u'(T_{ij}) = \begin{cases} 0 & T_{ij} \leq T_{ASET} - T_0 \\ \frac{C}{T_0}(T_{ij} - T_{ASET} + T_0) & T_{ij} > T_{ASET} - T_0 \end{cases}$$

Loss/gain of overtaking

$$\Delta u(T_{ij}) \cong u'(T_{ij})\Delta T = \frac{C}{T_0}(T_i - T_{ASET} + T_0)\Delta T$$

Value parameter of the game matrix

$$\frac{C}{\Delta u(T_{ij})} \cong \frac{T_0}{T_{ij} - T_{ASET} + T_0}$$

3.2 Settings and best-response dynamics

	Unit	Value	
		4	von Neumann neighborhood
		8	Moore neighborhood
r_n	m	0.40	Distance to agent that is considered as neighbor
v_i			Loss defined by game matrix
S			Set of strategies {Patient, Impatient}
s			Strategy $\in$ {Patient, Impatient}

The best-response strategy

$$s_i^{(t)} = \arg\min_{s_i' \in S} \sum_{j \in N_i} v_i\left(s_i', s_j^{(t-1)}; T_{ij}\right)$$

$s_j^{(t-1)}$ strategy neighbor played on period $t-1$

Updating strategy using poisson process.