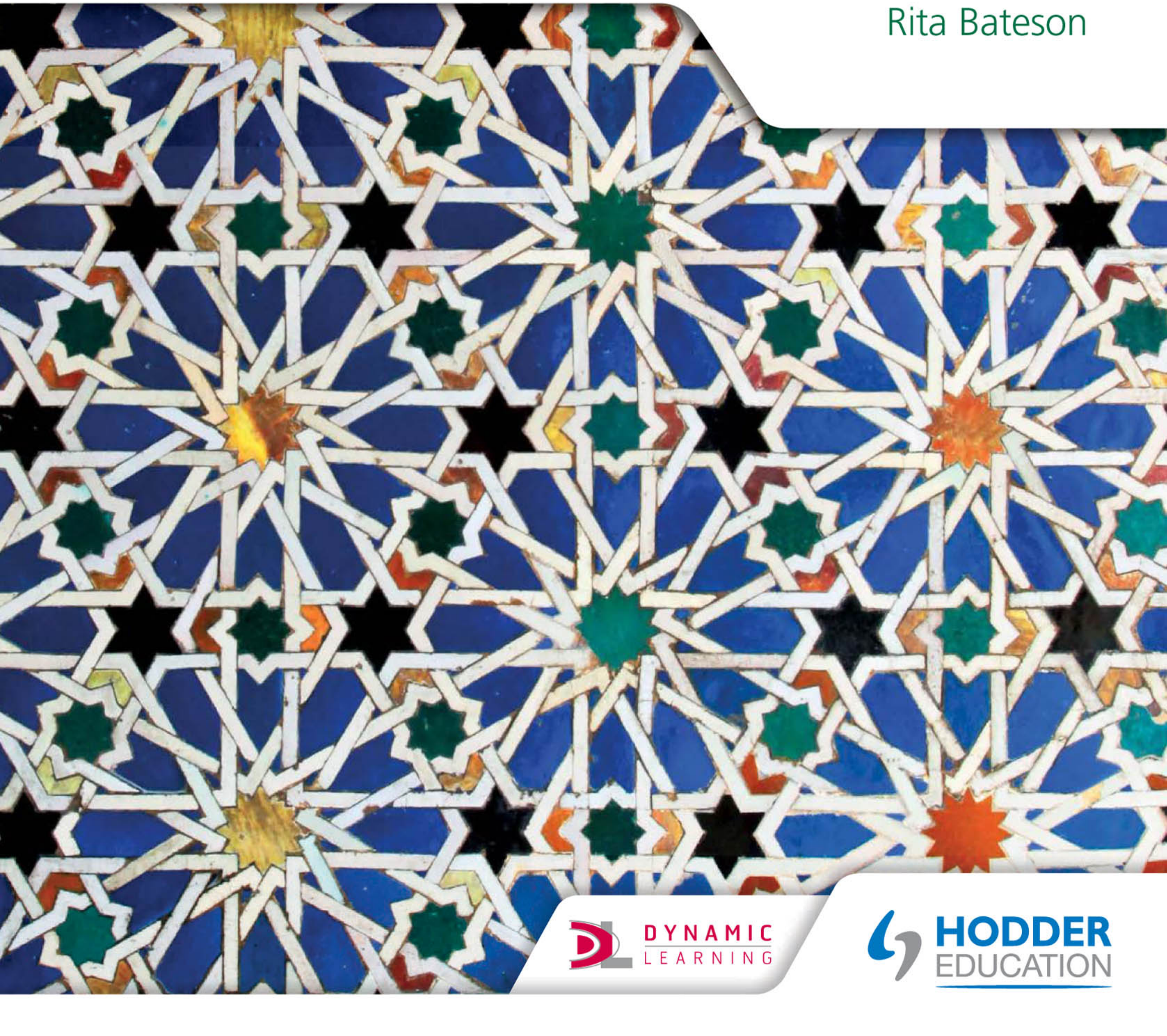
MYP by Concept

4&5

Mathematics

Rita Bateson



DYNAMIC LEARNING

HODDER EDUCATION

MYP by Concept 4&5

Mathematics

Rita Bateson
Series editor: Paul Morris

DYNAMIC LEARNING

HODDER EDUCATION
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Author's acknowledgements

With thanks to my first and most enthusiastic maths teacher, my Dad, Sean, and to my ever supportive mother, Eileen as well as my wonderful siblings and friends who made me laugh (and feature) throughout. Special thanks to my proof-reading brother Sean and to one of the toughest mathematics teachers I ever had, Sarah Gibbons.

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To my "hero" mathematics teacher, Mrs. Solan and all the wonderful educators and students I have worked with in schools, at the IB and at the International School of Hamburg (especially Irina and Andy).

Most of all, this book is dedicated to my incredible daughter Ellie and amazing, patient and kind husband Andrew, without whom nothing would ever get done.

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Contents

How to use this book

Each chapter is framed with a *Key concept* and a *Related concept* and is set in a *Global context*.


Form Patterns Globalization and sustainability

1 In how many different ways can we express the same thing?

Numbers in different **forms** give us a variety of ways to predict **patterns** and think about problems of **global significance**.

CONSIDER THESE QUESTIONS:

Factual: How are numbers sets defined? How and why do we group numbers? What is meant by approximate and exact?

Conceptual: How do number systems expand our understanding? What patterns can we see in different number forms and operations?

Debatable: Were numbers invented or discovered? Is there a best form for a number? Can the form of a number mislead or affect our decisions? Can rounding help or hinder decision-making?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how to express numbers in a variety of forms and why we do this.
- **Explore** situations where different levels of accuracy or detail of numbers is required.
- **Take action** by engaging and educating the school community in the role of numbers in our interconnected global community.

2 Mathematics for the IB MYP 4&5: *by Concept*

Welcome to Hodder Education's *MYP by Concept* series! Each chapter is designed to lead you through an *inquiry* into the concepts of mathematics and how they interact in real-life global contexts.

The *Statement of Inquiry* provides the framework for this inquiry, and the *Inquiry questions* then lead us through the exploration as they are developed through each chapter.

KEY WORDS

Key words are included to give you access to vocabulary for the topic. **Glossary** terms are highlighted and, where applicable, **search terms** are given to encourage independent learning and research skills.

As you explore, activities suggest ways to learn through *action*.

ATL

- Activities are designed to develop your *Approaches to Learning* (AtL) skills.

Detailed information or explanation of certain points are given whenever necessary. Key *Approaches to Learning* skills for MYP Mathematics are highlighted whenever we encounter them.

Assessment opportunities in this chapter:

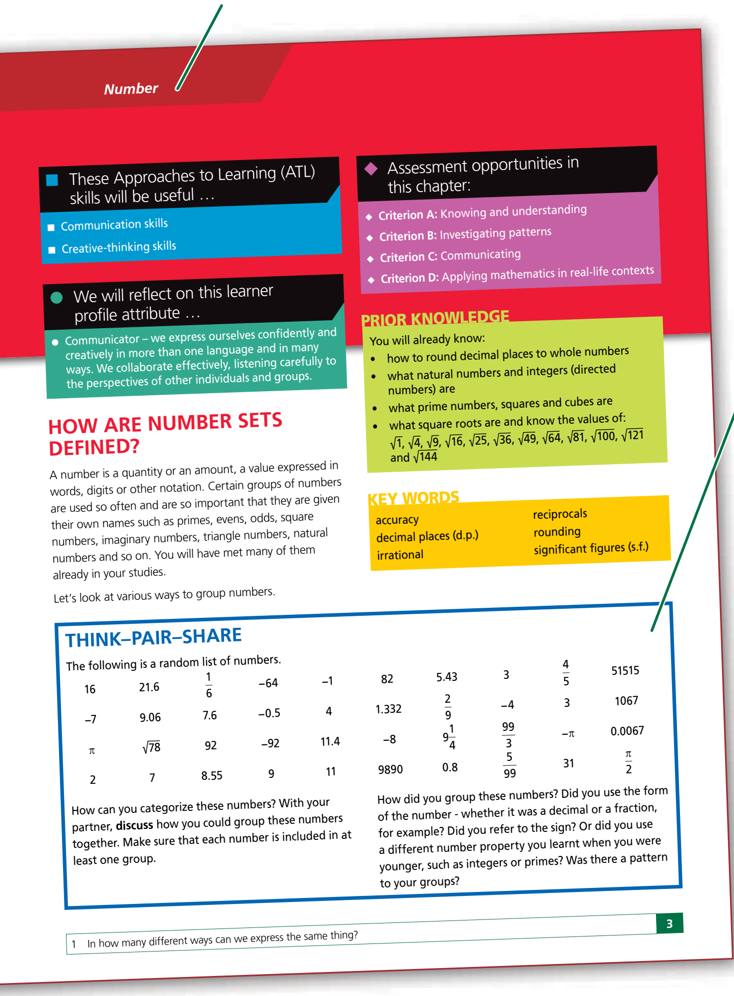
- Certain parts of the activities are *formative* as they allow you to practise certain of the MYP Mathematics Assessment Criteria. Other activities can be used by you or your teachers to assess your achievement against all parts of an assessment criteria.

Worked examples and practice questions are given in colour-coded boxes to show the level of difficulty:

Problem **Complex**

Challenging

Each chapter covers one of the four branches of mathematics identified in the MYP Mathematics skills framework.


Number

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Creative-thinking skills

We will reflect on this learner profile attribute …

- Communicator – we express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

Assessment opportunities in this chapter:

- Criterion A: Knowing and understanding
- Criterion B: Investigating patterns
- Criterion C: Communicating
- Criterion D: Applying mathematics in real-life contexts

PRIOR KNOWLEDGE

You will already know:

- how to round decimal places to whole numbers
- what natural numbers and integers (directed numbers) are
- what prime numbers, squares and cubes are
- what square roots are and know the values of: $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, $\sqrt{25}$, $\sqrt{36}$, $\sqrt{49}$, $\sqrt{64}$, $\sqrt{81}$, $\sqrt{100}$, $\sqrt{121}$ and $\sqrt{144}$

HOW ARE NUMBER SETS DEFINED?

A number is a quantity or an amount, a value expressed in words, digits or other notation. Certain groups of numbers are used so often and are so important that they are given their own names such as primes, evens, odds, square numbers, imaginary numbers, triangle numbers, natural numbers and so on. You will have met many of them already in your studies.

Let's look at various ways to group numbers.

KEY WORDS

accuracy
decimal places (d.p.)
irrational
reciprocals
rounding
significant figures (s.f.)

THINK–PAIR–SHARE

The following is a random list of numbers.

16	21.6	$\frac{1}{6}$	−64	−1	82	5.43	3	$\frac{4}{5}$	51515
−7	9.06	7.6	−0.5	4	1.332	$\frac{2}{9}$	−4	3	1067
π	$\sqrt{78}$	92	−92	11.4	−8	$9\frac{1}{4}$	$\frac{99}{3}$	$-\pi$	0.0067
2	7	8.55	9	11	9890	0.8	$\frac{5}{99}$	31	$\frac{\pi}{2}$

How can you categorize these numbers? With your partner, **discuss** how you could group these numbers together. Make sure that each number is included in at least one group.

How did you group these numbers? Did you use the form of the number - whether it was a decimal or a fraction, for example? Did you refer to the sign? Or did you use a different number property you learnt when you were younger, such as integers or primes? Was there a pattern to your groups?

1 In how many different ways can we express the same thing? 3

We have incorporated Visible Thinking – ideas, framework, protocol and thinking routines – from Project Zero at the Harvard Graduate School of Education into many of our activities.

Both standard and extended are included in this book. Extended is signposted.

You are prompted to consider your conceptual understanding in a variety of activities throughout each chapter.

Finally, at the end of each chapter, you are asked to reflect back on what you have learnt with our *Reflection table*, maybe to think of new questions brought to light by your learning.

Use this table to reflect on your own learning in this chapter.						
Questions we asked	Answers we found	Any further questions now?				
Factual						
Conceptual						
Debatable						
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?				
		Novice	Learner	Practitioner	Expert	

Hint

In some of the activities, we provide Hints to help you work on the assignment. This also introduces you to the new Hint feature in the on-screen assessment. These Hints will give additional guidance or shortcuts to improve your proficiency.

! Take action

! While the book provides many opportunities for action and plenty of content to enrich the conceptual relationships, you must be an active part of this process. Guidance is given to help you with your own research, including how to carry out research, how to make change in the world informed by Mathematics, and how to link and develop your study of Mathematics to the global issues in our twenty-first century world.

Links to:

Like any other subject, Mathematics is just one part of our bigger picture of the world. Links to other subjects are discussed.

We will reflect on this learner profile attribute …

- Each chapter has a *IB Learner Profile* attribute as its theme, and you are encouraged to reflect on these too.

1 In how many different ways can we express the same thing?

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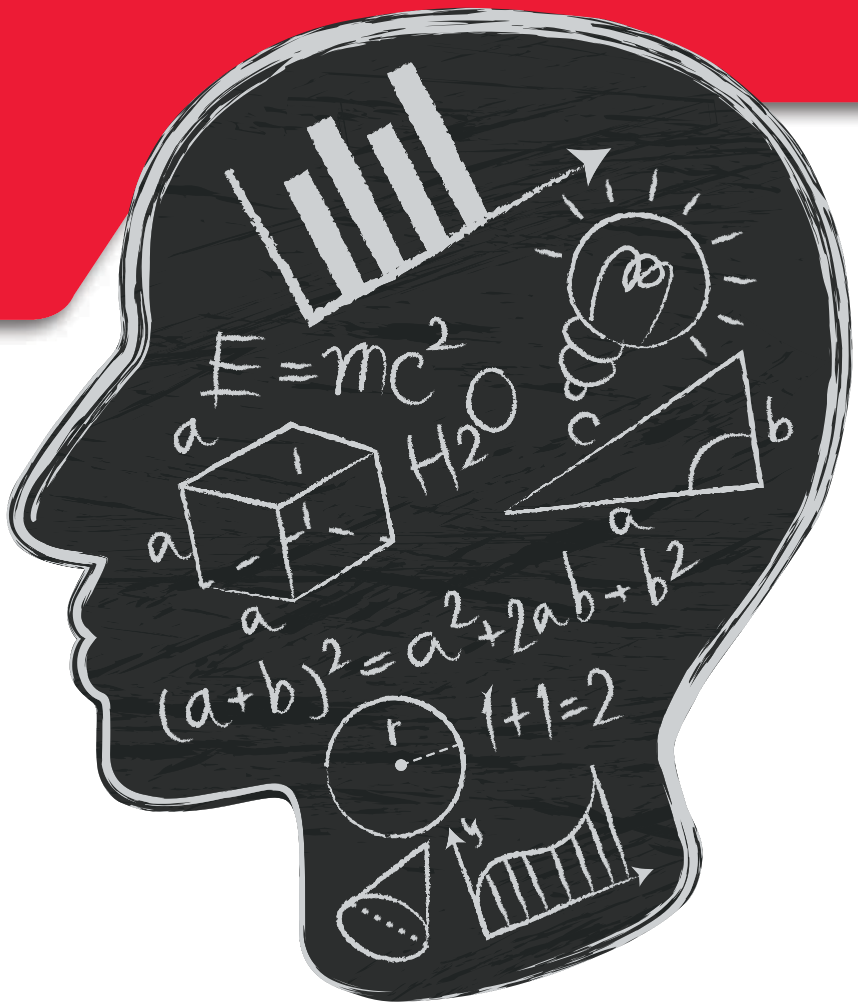

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How and why do we group numbers?

Links to: Language acquisition

Why are the letters representing integers (**Z**) and rational numbers (**Q**) not what you would expect them to be? What language do they come from? According to the definition below, if a rational number is one that can be written as a fraction or that can be made by dividing one integer by another, what must an **irrational number** be?

Hint

If you are using a search engine, look for websites about the ***history of mathematical notation*** *or* ***number sets****.*

What does this tell us about the possible language spoken by the mathematicians who invented the names? Can you find who named them and when?

WHAT IS A NUMBER SET?

Some sets of numbers are used so frequently and are so important in mathematics that they have been given a name and a symbol.

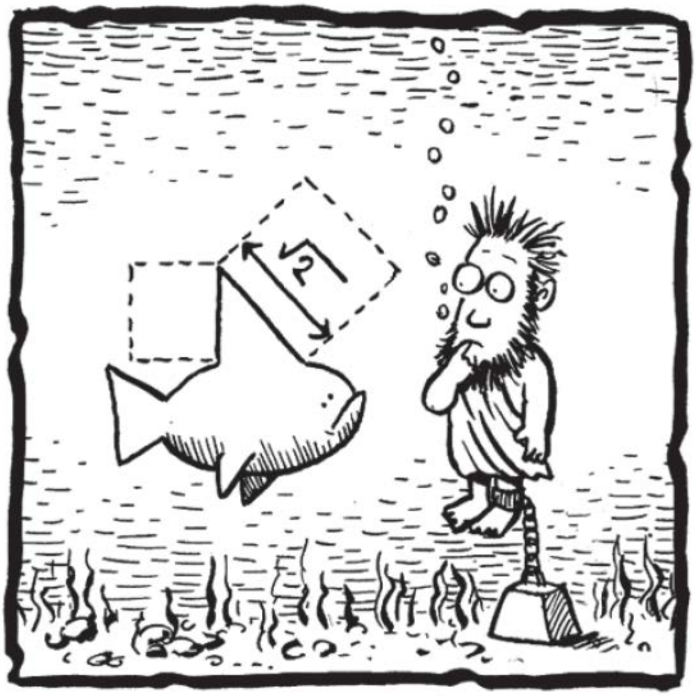
Table 1.1

Number set	Symbol	Definition	What it includes	Examples
Natural numbers	**N**	All the whole, positive numbers	All the whole positive numbers, including zero, up to infinity. It does not include decimals or fractions	0, 1, 2, … 55, 56, … 0 1 2 3 4 5 6 7 8 9 10
Integers	**Z**	All the whole, positive and negative numbers	Every whole number no matter what sign is in front of it. It does not include decimals or fractions.	0, -1, -2, -3, … and 1, 2, 3, … −5 −4 −3 −2 −1 0 1 2 3 4 5
Rational numbers	**Q**	All numbers which you can make by dividing one integer by another	All fractions, decimals and whole numbers, of any sign. This does not include numbers divided by zero.	0.2, -0.2, $\frac{22}{7}$, $4\frac{1}{8}$, −3 −2 −1 0 1 2 3 4 $-3\frac{1}{8}$ $-\frac{1}{2}$ $\frac{2}{5}$ $2=\frac{2}{1}$ $3\frac{1}{2}$ This also includes whole numbers as those can be divided by 1 or written as a fraction over 1 e.g. $4 = \frac{4}{1}$
Real numbers	**R**	All rational and irrational numbers	Any number anywhere on the number line. This includes non-terminating and non-repeating decimals such as π or $\sqrt{45}$	The real numbers = **N** and **Z** and **Q** and (any other real numbers which can exist but don't belong to the other sets)

The discovery of irrational numbers is attributed (credited) to the Pythagoreans in 5th century BCE. According to their philosophy, 'all is number', number meant positive integers, with 1 being the 'unit' by which all other numbers were measured.

Ironically, a member of the Pythagorean society, Hippasus, using their famous theorem discovered that the side of the square and its diagonal were incommensurable (there wasn't a common unit that would divide them both).

The tale tells that the other Pythagoreans threw him off the boat to keep this discovery of irrational numbers a secret. Nevertheless, the irrational numbers found their way out of the dark. The notation $\sqrt{a}$ that we use today appeared much later, around the 16th century.



Hippasus

DISCUSS

Knowing this story, whether it is anecdotal (a story people tell without proof) or factual, challenges how we think about what numbers like irrationals mean.

If Hippasus actually invented irrational numbers and was silenced, could they ever have then been invented later? Were they waiting to be discovered? What do you think?

I USED TO THINK … BUT NOW, I THINK …

Now that we have discussed numbers a lot, **identify** one thing that has changed the way you thought about something. Complete the following sentences:

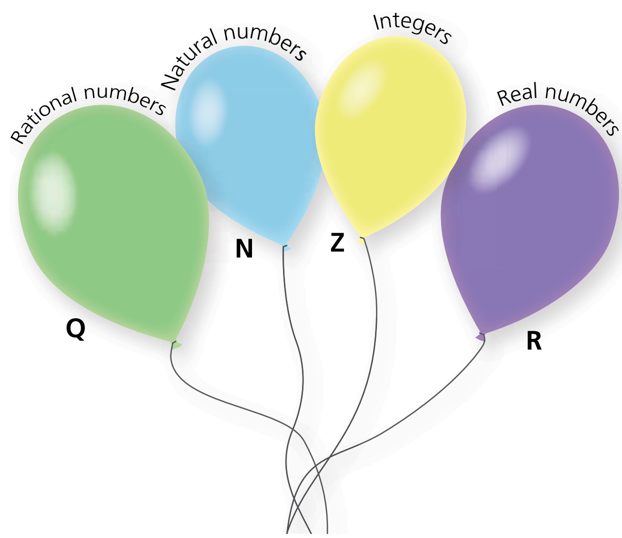
- **I used to think …**
- **But now, I think …**

ACTIVITY: Number sets

ATL

- Communication skills: Use and interpret a range of discipline-specific terms and symbols

Using the definitions in Table 1.1, place these numbers in the correct balloon. Include each number in as many possible sets as you can.



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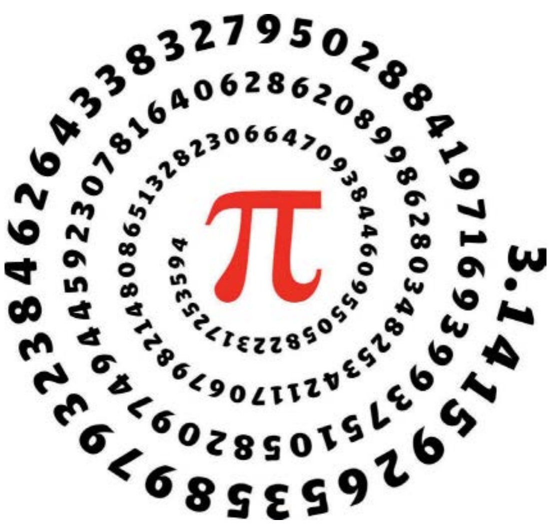
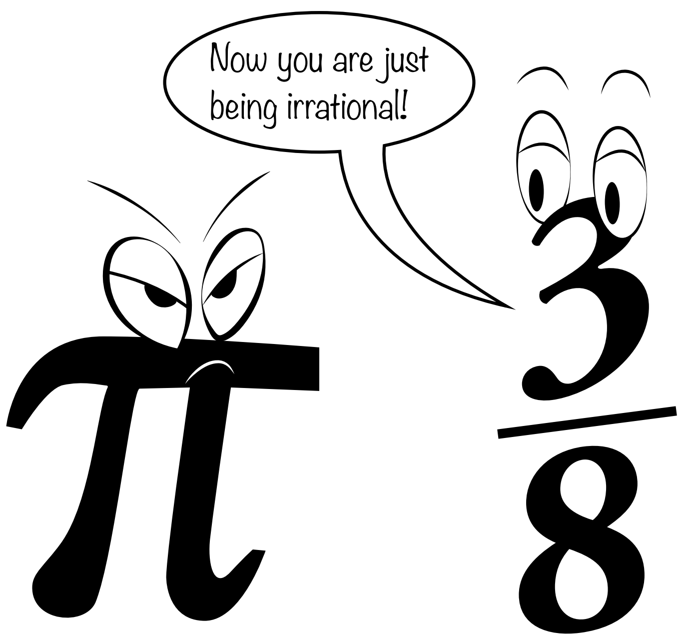
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding.

How do number systems expand our understanding?

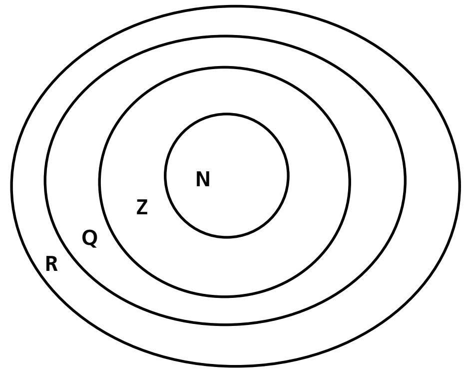


■ Pi is the loneliest number, it goes on and on and on....



■ What do we mean when we say a number is irrational?

THINK–PAIR–SHARE



From the definitions given, the set of natural numbers **N** are all the whole positive numbers. Where does this end?

Likewise, the elements in the integer set **Z** are all the whole positive and negative numbers in existence. How many is that? Is that more or less or the same as the set of natural numbers?

And the rational numbers? And the real numbers? Where do these sets 'end'? How many elements do they have? Which is the largest set?

Discuss with your partner. You could refer to the diagram above to support or **explain** your ideas.

Can you draw any other parallels? What does it mean when we call someone irrational?

We can see whole numbers appearing in the real natural world: eight cows, four people, two hundred monkeys. Why are negative numbers not considered natural too?

WHAT DO WE MEAN WHEN SOMETHING IS AN ELEMENT OF A SET?

Each set has a definition and when a number meets these conditions we say it is an **element** of the set. This number **belongs to** this set.

$\in$ is the symbol for 'is an element or is a member of'

So, the mathematical notation $3 \in \mathbf{N}$ means that 3 is an element of the natural numbers, 3 *is* a natural number.

$4.3 \in \mathbf{Q}$ means that 4.3 is a rational number.

$0.5 \notin \mathbf{N}$ means that 0.5 is **not** a natural number or does not belong to this set.

Ramanujan $\in$ {greatest mathematicians of all time!}

MEET A MATHEMATICIAN: SRINIVASA RAMANUJAN (1887–1920)

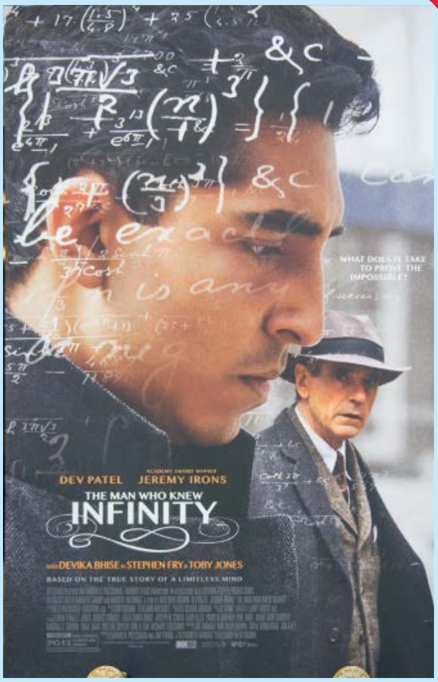
Learner Profile: Communicator

Srinivasa Ramanujan was born in 1887 in Madras, South India. He was a bright and confident young man who dreamed of changing the world of mathematics. By the age of 13 he was discovering sophisticated mathematical theorems on his own. His love of mathematics was so overwhelming that it bordered on an obsession. This contributed to his failing out of college twice due to his total focus on his mathematical work and nothing else. In the hope of furthering his mathematical studies, he wrote to many leading universities and professors of the day, introducing himself and communicating his work.

Many disregarded or ignored him but an open-minded professor, G.H. Hardy, invited him to Cambridge, UK to study and collaborate further. In the UK he experienced many challenges, including racist and snobbish attitudes but continued to produce some mind-blowing mathematics. His work includes major contributions to number theory, mathematical analysis and infinite series. Sadly, he died in 1920 at the age of 32.

As he was initially mostly self-taught, his dedication and output is even more impressive. Ramanujan's dedication to thinking about, and communicating, the true nature of numbers has given us some of the most exciting mathematics of the 20th century. It really shows us that mathematical ability and success can sometimes lie outside of the traditional academic routes.

To learn more about him, his work and G.H. Hardy, read *The man who knew infinity* by Robert Kanigel or watch the movie adaptation.



The Man Who Knew Infinity

! Take action

- ! Why not watch the movie or read the book as a group, in your class or after school? **Discuss** your thoughts and impressions of Ramanujan afterwards.

WHO'S GOT THE POWER?

Powers, exponents and indices are all different names given to the following:

x^n this is the power or exponent or index; this term is read as 'x to the power of n'.

For consistency, we will call it 'power' or 'exponent' from now on but any of the names would be acceptable and you should try to feel comfortable with any and all of them. You will already be familiar with squares and cubes as powers, as well as **higher orders of powers**.

Remember that: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$

and that: $q^4 = q \times q \times q \times q$

These powers can be extremely important to help us solve real-world problems. For example, if many people are accessing or streaming a huge amount of data from a website at the same, we can calculate the total usage and the server size necessary.

ACTIVITY: The element of

ATL

- Communication skills: Understand and use mathematical notation

Caroline has been practicing the 'element of' symbol but has made some mistakes in the following statements. Can you correct her mistakes by changing the $\in$ to $\notin$ where appropriate?

$6 \in \mathbf{R}$	$\pi \in \mathbf{Z}$	$\sqrt{3} \in \mathbf{R}$
$5.2 \in \mathbf{N}$	$-2 \in \mathbf{N}$	$-\frac{38}{50} \in \mathbf{Q}$

Give these powers as a simple natural number:

a 2^6 **b** 5^3 **c** 12^2

Hint

When you are trying to input a power into a calculator, app or in a program such as Excel, you will often use the symbol ^, for example 2^6 will be input as 2^6

This is a programming notation, not a strictly mathematical one, and as such is not accepted in a final answer.

Solve:

a $2^5 \times 3^2 \div 1^{10}$ **b** $9^{10} \div 5^{10}$ **c** $52^1 \div 7^2$

Hint

You will see here that any number to the power of 1 remains the same.

Any number to the power of 0 is equal to 1.

Rules: $x^1 = x$ and $x^0 = 1$

This is true for all numbers to the power of zero, which is undefined.

Solve:

a $(2^{10} - 10^2) \times (32 \times 0.5)$

b $(2843^0 \times 7819^0)$

c $(80^1 \times 0.05^1 \div 22^0)^2$

How does it work with letters or algebraic terms?

$x^3 = x \times x \times x$ and $x^4 = x \times x \times x \times x$

If the power indicates how many times you multiply them, it must follow that if we multiply these …

$x^3 \times x^4 = (x \times x \times x)(x \times x \times x \times x)$ count up the number of x terms

So $x^3 \times x^4 = x^7$

and $x^a \times x^b = x^{a+b}$

Solve:

a $a^7 \times a^3$ **b** $b^{10} \times b^2$ **c** $c^4 \times c^3$

Multiplying different algebraic terms with different letters is also easy. See if you follow the patterns from these examples.

$ab \times ab = a \times a \times b \times b = a^2b^2$

$x^2y^4 \times x^4y^2 = (x \times x \times y \times y \times y \times y) \times (x \times x \times x \times x \times y \times y) = x^6y^6$

$rs \times st = r \times s \times s \times t = rs^2t$

Hint

Best communication is always to list the coefficient or number part first, followed by any letters, in alphabetical order. This shows you can use appropriate forms of mathematical representation to present information (Criterion Cii).

Show that

We have seen that when we multiply like terms with powers, that we simply *add* the powers, for example

$h^3 \times h^{10} = h^{13}$

Show that $6^3 \div 6^1 = 6^2$

'Show that' means you should begin with the left-hand side (LHS) and show all working out to get you to the answer on the right-hand side (RHS).

Show that $d^6 \div d^4 = d^2$

From your working (hence), or otherwise **describe** how to divide like terms with powers. Remember to **verify**, or **prove**, the rule by checking with numbers.

Some exponent multiplication (and division) can be made more challenging by including a coefficient and/or a negative or minus sign:

Example

$(14x^2 yz) \times (2y^2z^2)$

Solution

$(14x^2 yz) \times (2y^2 z^2) = (14 \times 2)(x^2)(y \times y^2)(z \times z^2)$ group the like terms together

$= (28)(x^2)(y^3)(z^3)$

$= 28\, x^2\, y^3\, z^3$

$-ijk \times jk \times -ik$

Solution

$-ijk \times jk \times -ik = (- \times -)\,(i \times i)\,(j \times j)\,(k \times k)$ as minus × minus = plus, or $- \times - = +$

$= i^2\, j^2\, k^2$

$t^4 \times t^6 \times -2t^{10} \times rst \times r^{10}$

Solution

$t^4 \times t^6 \times -2t^{10} \times rst \times r^{10} = (t^4 \times t^6 \times t^{10} \times t)(-2)(r \times r^{10})(s)$

$= (t^{21})(-2)(r^{10})(s)$ now rearrange to put the coefficient, including the sign, first

$= -2(t^{21})(r^{10})(s)$ now rearrange to alphabetize

$= -2r^{10}st^{21} \longleftarrow$ this is the best form of communication

PRACTICE QUESTIONS

a $-11mno \times 3m^2$

b $4pq \times 7qr \times 2pr$

c $32s^2 \times 18r^2 \times t^2$

d $2uv \times 100v^3 \times u^0\, v^0$

e $25w^5 \times 2w^6r^7 \times 10r^3\, w^4$

f $-3wfx \times -2fxw \times -4xfw$

Powers and brackets

$(x^4)^2 \neq x^6$

but is actually

$(x^4)(x^4) = x^8$

AND

$(a^m)(a^n) = a^{m+n}$

BUT

$(a^m)^n = a^{mn}$

Hint

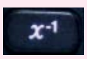
For best marks in Criterion Biii, always **prove**, or **verify** and **justify**, the rules you have discovered with numbers you haven't already been given. This shows the rule works not only for these particular numbers but also as a general rule.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

HOW CAN POWERS BE NEGATIVE?

Now we are familiar with positive powers and how to operate with them (non-addition, multiplication, division, powers of powers), what does a negative power mean? What does its form tell us?

On your calculator, app or GDC, find the button which looks like 

On some models, it may be a second function or require a shift button to get to it. Some of the more popular calculator buttons look like 

Reciprocals

This operation is called a reciprocal and appears in algebra notation as follows:

$$a^{-1} = \frac{1}{a^1} \text{ or } \frac{1}{a}$$

Write the following reciprocals as fractions:

w^{-1}

9^{-1}

$(ab)^{-1}$

$2a^{-1}$

You will learn more about the reciprocal function in Chapter 9.

It follows from the rule above that a^{-2} must be $\frac{1}{a^2}$. Notice how the minus (negative) is gone but the power is not.

Let's see this in action with numbers:

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$$

Verify (check this) by inputting 2^{-2} into your calculator to see if you get 0.25.

All of these are **equivalent** forms of the **same** number.

ACTIVITY: Negative power

ATL

- Communication skills: Use and interpret terms and symbols; Understand and use mathematical notation

Using your calculator, find the answer to these two powers:

$2^1 =$

and $2^{-1} =$

What do you notice? What happens to the 2 when it has a negative power?

Now **find** $3^1 =$

$3^{-1} =$

Does this agree with what you noticed before? If so, how does it support what you noticed before? If not, how is it different? Has it changed your observation?

Next **find** $4^1 =$

$4^{-1} =$

What pattern(s) can you **identify** now?

Let's try one more …

$10^1 =$

$10^{-1} =$

Can you write a general rule for any number to the power of –1?

(Remember to **verify** it with a number not given above.)

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns.

ACTIVITY: Reciprocals practice

ATL

- Communication skills: Reading, writing and using language to gather and communicate information; Use and interpret a range of discipline-specific terms and symbols; Understand and use mathematical notation

Complete the table below:

Term with negative power	Fraction with power	Fraction	Decimal
10^{-2}	$\frac{1}{10^2}$	$\frac{1}{100}$	0.01
2^{-4}			
3^{-3}			
6^{-2}			
8^{-3}			
7^{-1}			
4^{-4}			
5^{-3}			
9^{-3}			
10^{-5}			
a^{-2}			
b^{-4}			
c^{-5}			
d^{-1}			
$(e^{-3})^2$			

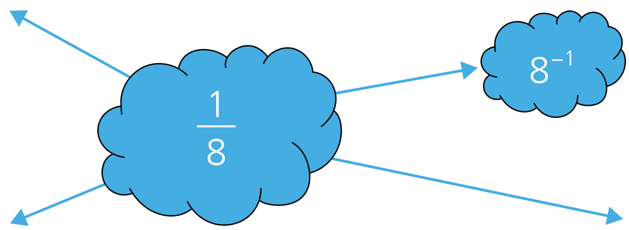
ACTIVITY: Mind mapping

ATL

- Creative-thinking skills: Generating novel ideas and considering new perspectives; Use brainstorming and visual diagrams to generate new ideas and inquiries

This is a self-assessment activity to see how well you have understood the different forms of powers and reciprocals. The more forms you can master the higher your level of achievement.

Give as many different forms for $\frac{1}{8}$ as you can think of. Be creative and use positive and negative powers or other forms of number.



How many could you think of? Here's a mark scheme:

- **I got between 2 and 4** ☺ **Level 1–2**
- **I got either 5 or 6** ☺☺ **Level 3–4**
- **I came up with between 7 and 10** ☺☺☺☺ **Level 5–6**
- **I got more than 10 forms for $\frac{1}{8}$** 😀 **Level 7–8**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

EXTENDED

You have seen above that

$$a^{-m} = \frac{1}{a^m}$$

The following must also be true.

$$a^m = \frac{1}{a^{-m}}$$

Can you show that this is true, using your knowledge of *dividing fractions*?

Simplify these terms:

$b^{22}b^{-7}$ $\quad$ $b^{-2}b^{-4}$ $\quad$ $(b^{-3})(b)(b^{10})(b^{-8})$

Can rounding help or hinder decision-making?

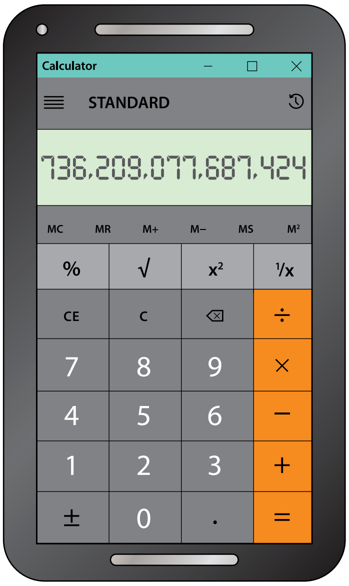
WHAT HAPPENS IF A NUMBER IS TOO DETAILED?

Accuracy can be an important concept. It is important that we check the facts, figures and answers that we give and receive are correct and true, as far as possible. The same is true for our mathematical assumptions and answers. But *correct* or *true* and *accurate* or *precise* are often confused in daily language.

The mathematical definition for accuracy is the nearness of a calculation to the true value while precision refers to the number of digits or decimals given. For much of this chapter we will be talking about the precision of numbers and answers.

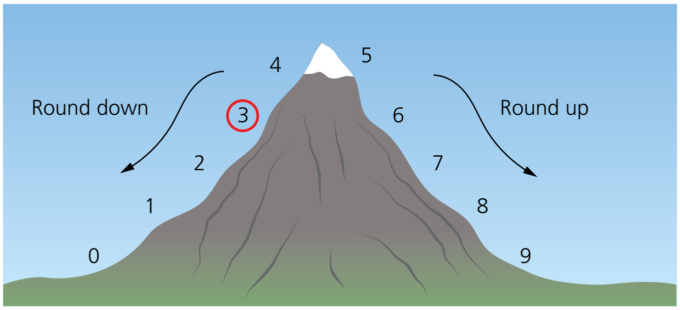
But why wouldn't it be near to its true value? Often a calculated value will give a long list of digits or decimal places. This can be time-consuming to write out, re-enter in the calculator over and over or simply too much information to be useful.

When people are making a persuasive argument or comparing different ideas, numbers which are very specific and detailed can be unhelpful. Journalists, politicians, lawyers and other jobs which require communication, have to be able to give numbers in context, to the most **appropriate** significant figure.



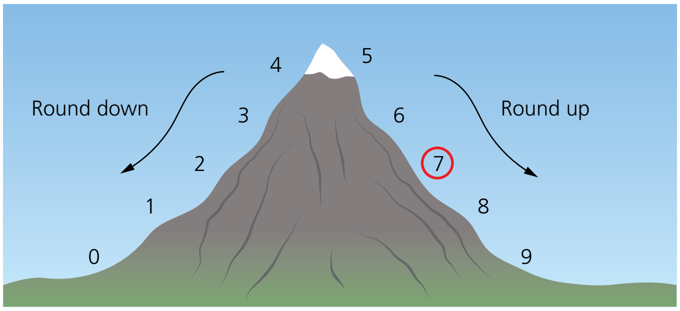
Let's look at some examples:

Example 1 Round 27.3 to a whole number



So, rounding 27.3 to a whole number, round down to 27.

Example 2 Rounding 2675 to the nearest 100



Identify what digits you have to round:

$\underbrace{2675}$

to the nearest 1 0 0

So, the 6 in 675 will either be 600 or 700

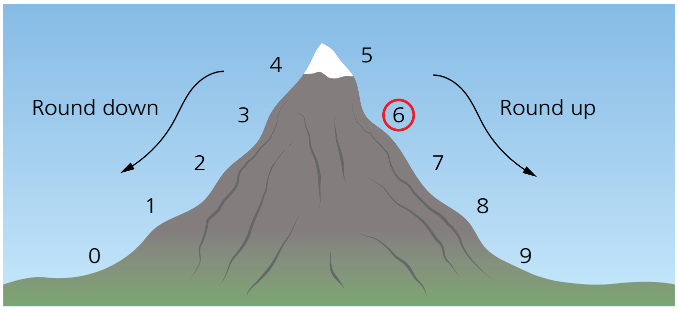
We look to the next digit:

2675

7 is on the 'round up' side of the mountain.

So, 2675 is 2700 to the nearest 100.

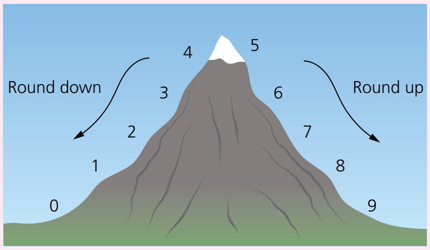
Example 3 Round 0.345679 to 3 d.p.



She'll be coming *round* the mountain ♫ …

You will already be familiar with the rules for rounding. The image of a rounding mountain is a visual way to represent what is happening when we want to round to a whole number, the nearest thousand, to two decimal places, etc.

So, if you want to round a number, if the next digit is less than 5, round down, if the next digit is greater or equal to 5, round upwards.



■ The ROUNDING mountain

0.345⁝679 the dotted line shows 3 d.p. separated from the numbers to round

∴ 0.345⁝679

6 indicates that we must round up

So, 0.345679 will be 0.346 to 3 d.p.

Be careful!

This might seem like a trivial (non-important) thing but it is important to self-check … which is larger, 0.276 or 0.31?

Some students carry forward the misconception that the numbers after the decimal act like whole numbers. They would see the 0.276 as larger because there are more figures in the number or as if they are comparing two hundred and seventy-six (276) with thirty-one (31). Make sure you don't have this wrong idea!

WHAT IS THE DIFFERENCE BETWEEN SF AND DP?

The number of decimal places simply means the decimal digits given after the decimal point. Significant figures indicate how many digits between the first and the last non-zero figure.

So, 1.987 has three decimal places but it has four significant figures, counting from this first digit, 1, to the last decimal digit, 7.

There are also three significant figures in 213 000 as we count 2, 1 and 3 as significant figures. Why do you think we exclude the zeros when counting?

There is only one significant figure 0.003, as the zeros are not included (unless they are in the middle of the number and not affecting the beginning or end.)

PRACTICE EXERCISES

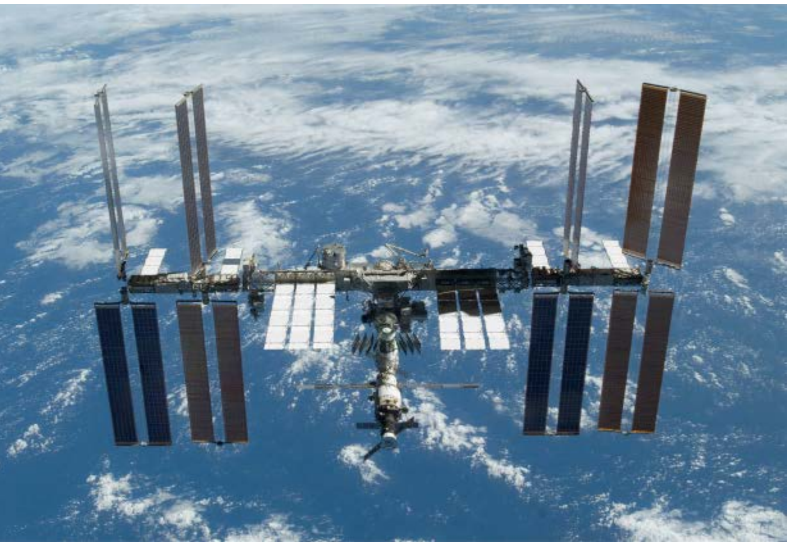
Round the following numbers to

		a 2 d.p.	b 1 d.p	c 3 s.f.	d 2 s.f.
		Solution			
1	1234.56	a 1234.56	b 1234.6	c 1230	d 1200
2	7039.288	a 7039.29	b 7039.3	c 7040	d 7000
		Solution			
3	1 384 509.319	a 1 384 509.32	b 1384509.3	c 1 380 000	d 1 400 000
4	0.088 2622	a 0.09	b 0.1	c 0.0883	d 0.088
		Solution			
5	0.172 309 77	a 0.17	b 0.2	c 0.172	d 0.17
6	10.9732	a 10.97	b 11.0	c 11.0	d 11

What patterns can we see in different number forms and operations?

WHEN DO WE HAVE TO WORK WITH EXTREMELY LARGE NUMBERS?

A real-life example of extremely large numbers would be the International Space Station (ISS). It costs approximately €10 000 (euro) for every kilogramme to get resources into a low earth orbit. The ISS has to be refuelled and restocked every six months. Cargo supply missions are flown to the space station, where computer equipment, food and other important supplies are needed. It returns with waste, dirty laundry, broken equipment and scientific data dumps.



The International Space Station

How would we **calculate** the cost for sending each of the following into orbit?

Item	Total weight (kg)
Fresh water	1500
Dried food	200
Exercise equipment	1150
Photographs and letters	3.5
Repair kits	400
Medical supplies	50
Clean clothing	95
Scientific equipment	100

Table 1.2 Cargo supplies to the ISS

Solution

Item	Total weight (kg) × Cost per kilogram	Cost (€)
Fresh water	1500 × 10 000	15 000 000
Dried food	200 × 10 000	2 000 000
Exercise equipment	1150 × 10 000	11 500 000
Photographs and letters	3.5 × 10 000	35 000
Repair kits	400 × 10 000	4 000 000
Medical supplies	50 × 10 000	500 000
Clean clothing	95 × 10 000	950 000
Scientific equipment	100 × 10 000	1 000 000

Table 1.3 Cargo supplies to the ISS and how much it would all cost

How much is the total cost if they decide to send all the above on a single mission? And what is the total cost over 5 years? A staggering €349 850 000!

If you were working with these numbers and calculating with them, you would get very bored of entering and writing 0s all the time. Mathematicians have developed a method of scientific notation to represent these numbers in a more efficient form.

We saw in the ISS example that numbers can very quickly become extremely large in real-world contexts. In science and economics, they can also become extremely small. With globalization, the advances of the internet, rapid expansion of data and population growth, we need to be able to measure and calculate bigger numbers than ever before.

Some other examples of extremely large numbers include:

- The budgets of nations and countries
- International finance and aid packages
- Space travel, measurement and discovery
- Distance calculations on GPS, satellites and underwater exploration
- Force calculations in Physics
- Molecular number calculations in Chemistry such as Avogadro's number
- Decoding information in DNA and genome sequencing
- Social media users and information shared per second
- Other internet data and 'lifelogging'
- Carbon emissions.

Some examples of extremely small numbers include:

- Nanotechnologies and particles
- Revenue from clicks on advertisements online
- Molecular, DNA and genome sizes
- Blood testing for doping in sport (parts per million)
- Pollution measurements – particles in air
- Stock market and currency fluctuations (irregular changes)
- Computer chip sizes.

2011 Total emissions country rank	Country	2011 Total carbon dioxide emissions from the consumption of energy (million metric tons)	2011 Per capita CO_2 emissions from the consumption of energy (metric tons of CO_2 per person)
1	China	8715.31	6.52
2	United States	5490.63	17.62
3	Russia	1788.14	12.55
4	India	1725.76	1.45
5	Japan	1180.62	9.26
6	Germany	748.49	9.19
7	Iran	624.86	8.02
8	South Korea	610.95	12.53
9	Canada	552.56	16.24
10	Saudi Arabia	513.53	19.65

Source: ***www.ucsusa.org/global_warming/science_and_impacts/science/each-countrys-share-of-co2.html#.V2fYnLsrJD9***

Table 1.4 CO_2 emissions

THINK–PAIR–SHARE

Think about the different MYP subjects you are studying. What examples of extremely large and small numbers can you think of?

! Take action

- ! Your mission is to educate people about the incredible scale of numbers these days – how we need to be aware of the incredibly large and small as our digital world gets ever bigger and our computing storage and technology gets smaller.
- ! **Research** instances of extremely large and small numbers and create a display to communicate how these numbers affect us all globally. Use your creativity and engage your audience by making these extreme numbers interesting and relevant.

HOW DO MATHEMATICIANS AND SCIENTISTS HANDLE THESE HUGE OR TINY NUMBERS?

While the goal of journalists, politicians and online news is to make facts accessible and easy to understand (so they often round or simplify), the goal for mathematicians and scientists is to be as accurate, precise and correct as possible. So how can we make these extreme numbers easier to work with, mathematically?

First let's consider the idea that the zeros at the start or end of the number can be represented as a power of 10 because:

$10^6 = 1\,000\,000$ and $10^{-6} = 0.000\,001$

and so on for lots and lots of zeros.

So the **standard form** for handling such large or small numbers is to convert them into a number between 1 and 10 and remove all the zeros by having orders of magnitude/powers.

Hint

When you see these types of questions: 'Give your answer in the form of $a \times 10^n$ where $n \in \mathbb{Z}$ and $1 \le a < 10$, you will know they are asking for standard form.

Let's take 26 800 000 as an example. The only way to make this a number between 1 and 10 is to move the decimal place until we get a number that lies in this range.

For example,

26 800 000.0 put in the decimal point

Now move it until it shows a number between 1 and 10.

3 2 1

2.6 8 0 0 0 0 0. stop at 2.6 as this is > 1 but < 10

How many times did we move the decimal to the left?

7 6 5 4 3 2 1

2.6 8 0 0 0 0 0

We moved it 7 times so the correct standard form is

$2.68\cancel{00000} \times 10^7$

2.68×10^7

Let's take another real-world example:

In 2006 Warren Buffett, one of the US's richest people, donated $4 350 000 000 to charity.

This number is both unwieldy (hard to work with) and so large that it is difficult to imagine. Outside of mathematics and sciences, this number would most often be referred to 4.35 billion dollars. But in standard form, or scientific notation, it would be:

4 3 5 0 0 0 0 0 0 0.

↓

(4.35×10^9) … 7 zeros

+ 2 hops to get 4.35

9

So 10^9 is the order of magnitude.

If Mr Buffett decided to double his contribution to global charitable causes, it is an easy calculation to complete in this form, i.e.

$(4.35 \times 10^9) \times 2 = 8.7 \times 10^9$ dollars

But what about an extremely small number?

0.000 000 004 4

4.4 would be the number between 1 and 10 but how do we get there?

0.0 0 0 0 0 0 0 0 4.4

We 'jumped' 9 times to get to 4.4, but we jumped to the *right* this time so the standard form would be

4.4×10^{-9}

Here's another example. A popular website hosts an emergency and natural disaster service so people can check in to say they are safe and let their loved ones around the world know they are ok. A powerful earthquake hit a city of almost 4 million people. Following the earthquake, the website's servers showed 2.4×10^2 people 'checked in' every minute on average for the first six hours after the disaster. How many people used the service?

$(2.4 \times 10^2) \times 60 \times 6 = 8.64 \times 10^4$ people

$= 86\,400$ people were confirmed safe.

Let's take another real-world example but this time it will be 'out of this world'. A comet is travelling extremely fast at 204 000 metres per hour. How far will it travel in 4500 hours?

If we use the formula from Physics:

distance = speed × time

distance = 204 000 × 4500

distance = 918 000 000

Let's get rid of the zeros

distance = 918×10^6

But 918 is not between 1 and 10 and does not satisfy standard form, so let's move the decimal

∴ distance = 9.18×10^8

The power has increased by 2 because we moved the decimal twice.

So the final answer is that the comet travelled 9.18×10^8 metres in that time.

Hint

It helps when you are working with all these zeros to cover up numbers with your finger while counting or when you have counted them. It stops you double counting or getting confused.

PRACTICE QUESTIONS

Convert these numbers into standard form

a	70 000 000	d	0.000 000 9
b	50 000 000 000	e	0.0008
c	13 000 000	f	0.03
g	0.000 352	j	125 000 000
h	0.000 000 000 777	k	52 340
i	0.000 202 2	l	93 000 000
m	5316.72	p	0.007 100
n	1094.73	q	0.002 22
o	2.0059	r	32.0007

Don't forget! A zero within a number is not going to disappear as it would change the value of that number.

1 002 500 000 000 000 becomes 1.0025×10^{15}

In what way would this number change if you accidentally removed the 00? How would it make the answer wrong?

Convert these numbers out of standard form into very large or small numbers.

a	7×10^3	d	9×10^{-2}
b	5×10^6	e	3×10^{-3}
c	8.2×10^4	f	2×10^{-9}
g	8.5×10^{-2}	j	6.25×10^5
h	9.73×10^5	k	5.323×10^{-3}
i	2.52×10^{-2}	l	6.324×10^{11}
m	6.022×10^{23}	p	3.14159×10^2
n	5.005×10^5	q	0.222×10^{22}
o	5.005×10^{-5}	r	3.0007×10^4

Working with radicals and surds

Just as with powers and algebraic terms, like radical terms can be added or subtracted while unlike radicals cannot.

Example

$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$\sqrt{11} + \sqrt{11} + \sqrt{11} = 3\sqrt{11}$

$5\sqrt{7} - 3\sqrt{7} = 2\sqrt{7}$

While radicals of different numbers cannot be added or subtracted in this form for example:

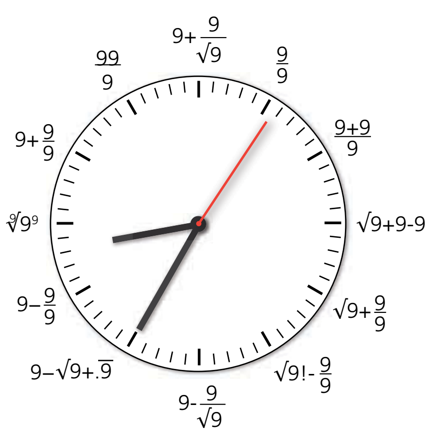
$\sqrt{2} + \sqrt{5} = \sqrt{2} + \sqrt{5}$ this cannot be simplified further

$\sqrt{6} + \sqrt{4} + \sqrt{4} + \sqrt{6} = 2\sqrt{4} + 2\sqrt{6}$ … collecting like terms

$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} = \pm 1 + \sqrt{2} + \sqrt{3} \pm 2$
none of the terms were alike but we know that $\sqrt{4}$ has the solutions 2 and –2 and $\sqrt{1}$ has the solutions 1 and –1

WHY IS IT ALL SO IRRATIONAL?

Radicals and surds



You will have learnt the symbol for radicals or roots as $\sqrt{x}$, for example $\sqrt{16}$ or $\sqrt{100}$. This $\sqrt{\ }$ symbol denotes a square root, while $\sqrt[3]{\ }$ indicates a cube root, $\sqrt[4]{\ }$ a fourth root and so on. A cubed root is a number which when cubed gives the original number. For example, cube root of 64 ($\sqrt[3]{64}$) is 4, because 4 × 4 × 4 is 64.

Some square roots have integer solutions such as $\sqrt{4} = 2$ and –2. Many other radicals, or roots, do not have whole number solutions, for example $\sqrt{5} \approx 2.236\,067\,977\,499\,7$.

As we learnt earlier, these cannot be written as fractions or ratios because they are **non-terminating decimals**. Many radicals are irrational numbers. Sometimes it may be easier to carry out calculations or give answers in **surd form**. Surd form means you can leave the roots symbol in your answer.

This is said to be its most **exact form** because with any decimal form of the radical there has been some rounding of the decimal places.

How do we use radicals, roots and surds with algebraic terms (letters)?

The procedure is exactly the same as shown earlier:

$\sqrt{x} + \sqrt{x} = 2\sqrt{x}$

$3\sqrt{y} - 2\sqrt{y} = 1\sqrt{y}$

but $\sqrt{x} + \sqrt{y} = \sqrt{x} + \sqrt{y}$

How do we multiply radicals and surds?

Remember multiplying letter terms in algebra? You will have learnt that

four times x, or four lots of $x = 4x$

or x multiplied by $z = xz$

Similarly, this works with numbers and letters in surd form:

$\sqrt{3} \times \sqrt{5} = \sqrt{15}$

$\sqrt{10} \times \sqrt{2} = \sqrt{20}$

$\sqrt{b} \times \sqrt{d} = \sqrt{bd}$

$\sqrt{e} \times -\sqrt{f} = -\sqrt{ef}$

Same for multiplying a radical by an integer:

$3 \times \sqrt{14} = 3\sqrt{14}$

$6 \times \sqrt{7} = 6\sqrt{7}$

$g \times \sqrt{h} = g\sqrt{h}$

Examples

$\sqrt{12} \times \sqrt{2} = \sqrt{24}$ spoiler: this can be simplified further

$\sqrt{12} \times \sqrt{10} = \sqrt{120}$

$\sqrt{x} \times \sqrt{a} = \sqrt{ax}$ list the terms in alphabetical order

$\sqrt{7} \times \sqrt{6} \times \sqrt{2} = \sqrt{84}$

$\sqrt{l} \times \sqrt{m} \times \sqrt{n} = \sqrt{lmn}$

$4\sqrt{o} \times 2\sqrt{p} = 8\sqrt{op}$

$-\sqrt{6} \times \sqrt{5} = -\sqrt{30}$

$-11\sqrt{5} \times 3\sqrt{u} = -33\sqrt{5u}$

$\sqrt[4]{f} \times \sqrt[4]{g} \times \sqrt[4]{h} = \sqrt[4]{fgh}$

How can we simplify surds?

If $\sqrt{50} \div \sqrt{5} = \sqrt{10}$ then $\sqrt{b} \div \sqrt{c} = \sqrt{\dfrac{b}{c}}$

PRACTICE QUESTIONS

1 Try these questions:

a $\sqrt{8} + \sqrt{8} =$

b $3\sqrt{3} + 3\sqrt{3} =$

c $11\sqrt{2} + 2\sqrt{2} - 10\sqrt{2} =$

d $\sqrt{7} + \sqrt{8} + \sqrt{8} =$

e $\sqrt{a} + \sqrt{a} + \sqrt{a} + \sqrt{a} =$

f $8\sqrt{6} - 6\sqrt{6} =$

g $\sqrt{5} + 2\sqrt{5} + 3\sqrt{5} + 4\sqrt{5} + 5\sqrt{5} =$

h $\sqrt{12} + 2\sqrt{12} + 6\sqrt{12} + 12\sqrt{12} =$

i $\sqrt{47} + \sqrt{33} + 3\sqrt{33} + 47\sqrt{33} - 11\sqrt{33} =$

j $2\sqrt{c} + 2\sqrt{d} - \sqrt{c} =$

k $18\sqrt{7} - 11\sqrt{7} - 4\sqrt{3} - 6\sqrt{3} - 2\sqrt{7} =$

l $4\sqrt{10} - 6\sqrt{10} =$

m $\sqrt{6} + \sqrt{6} + \sqrt{36} - 6\sqrt{6} =$

n $\frac{1}{2}\sqrt{21} + \frac{3}{2}\sqrt{21} + \sqrt{12} + 2\sqrt{12} =$

o $\sqrt{b} - \sqrt{b} + \sqrt{a} + 2\sqrt{a} =$

p $\sqrt[3]{g} + \sqrt[3]{g} =$

2 **Show** how to multiply these radicals and surds:

a $\sqrt{28} \times \sqrt{2} =$

b $\sqrt{2} \times \sqrt{17} =$

c $\sqrt{f} \times \sqrt{g} =$

d $2\sqrt{h} \times 4\sqrt{i} =$

e $\sqrt{6} \times \sqrt{6v} =$

f $6\sqrt{7} \times 3\sqrt{2} =$

g $-\sqrt{5x} \times \sqrt{2y} =$

h $1\sqrt{3} \times -\sqrt{20} =$

i $\sqrt[3]{m} \times \sqrt[3]{o} \times \sqrt[3]{n} =$

3 **Explain** in your own words how to divide radicals/surds.

Hint

If a question asks you to leave your answer in surd form or as $a\sqrt{b}$ form, you should leave the root symbols in your answer. By turning it into a decimal number, you have made it less exact and not followed instructions fully. This will mean you cannot reach the highest communication levels.

Speaking of communication …

In an earlier example:

$\sqrt{12} \times \sqrt{2} = \sqrt{24}$ spoiler: this can be simplified further

What was meant by this?

So, $\sqrt{24}$ is a surd but it is not in its **simplest** form. If it is possible to remove a known square and convert it into a whole number (or take it out from under the square root) is considered a more elegant answer.

And as we know that one of the factors of 24 is an easily recognizable square, i.e. 4 then we can simplify in the following way:

$\sqrt{24} = \sqrt{(4)(6)}$

which is also

$= \sqrt{4}\sqrt{6}$

so $= 2\sqrt{6}$ is the simplest form of this surd.

Example

Simplify $\sqrt{32}$.

Solution

$\sqrt{32} = \sqrt{(16)(2)}$

$= \sqrt{16}\sqrt{2}$

$= 4\sqrt{2}$

Simplify $\sqrt{640}$.

Solution

$\sqrt{640} = \sqrt{(64)(10)}$

$= \sqrt{64}\sqrt{10}$

$= 8\sqrt{10}$

Write $\sqrt{500}$ in simplest terms.

Solution

$\sqrt{500} = \sqrt{(25)(20)}$

$= \sqrt{25}\sqrt{20}$

$= 5\sqrt{20}$

But is this as simple as possible? There is another whole (square) number which we can remove: 4, which is a factor of 20

So,

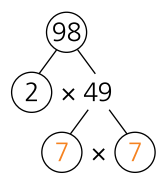
$5\sqrt{20} = 5\sqrt{(4)(5)}$

$= 5\sqrt{4}\sqrt{5}$

$= 5(2)\sqrt{5}$

$= 10\sqrt{5}$

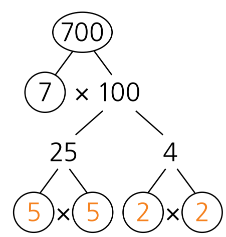
Remember prime factor trees from primary school?



So $\sqrt{98}$

$= \sqrt{2 \times 49}$

$= 7\sqrt{2}$



$\sqrt{700}$

$= \sqrt{7 \times 100}$

$= 10\sqrt{7}$

Also

$\sqrt{7 \times 5 \times 5 \times 2 \times 2}$

$= (5)(2)\sqrt{7}$

$= 10\sqrt{7}$

If it helps to you to **identify** the largest square numbers possible, you can use these trees in this way.

Of course, it also really helps if you know the square numbers up to at least 144 for easy and quick recognition.

See the summative problems at the end of this chapter for more practice questions.

Play this interactive game: Is this a prime number?
http://isthisprime.com/game/

EXTENDED MATHEMATICS: FRACTIONAL EXPONENTS

The following pages are designed for students undertaking the Extended Mathematics course in MYP but can also be studied as enrichment or extension for Standard Mathematics students. The more forms you know, the better! Each one is a form of communication and helps deepen your appreciation of number patterns and expression.

Radicals (or roots) can also be expressed in a different form

$a^{\frac{1}{2}} = \sqrt{a}$

Where the power of $\frac{1}{2}$ is another way of expressing a square root.

Examples

$64^{\frac{1}{2}} = \sqrt{64} = 8$

$21^{\frac{1}{2}} = \sqrt{21}$

$\sqrt{21} \approx 4.58$ correct to 3 significant figures

$$196^{-\frac{1}{2}} = \frac{1}{196^{\frac{1}{2}}} = \frac{1}{\sqrt{196}}$$

$\frac{1}{14}$ or 0.071

So now we know that the power of a half is the same as the square root of something. What about the power of a third? Or a quarter?

$a^{\frac{1}{3}} = \sqrt[3]{a}$ and …

$a^{\frac{1}{4}} = \sqrt[4]{a}$ and …

$a^{\frac{1}{5}} = \sqrt[5]{a}$

And the general form is

$a^{\frac{1}{n}} = ?$

Most calculators and apps have a $\sqrt{\square}$ button for square roots, a $\sqrt[3]{\square}$ button for cube roots and a $\sqrt[x]{\square}$ button to find all other roots.

You could also use the fractional power by using the y^x button or x^y button and inserting the fractional power. Both of these methods should give you the same answer, if inputted correctly. Ask your teacher for help if you cannot use any of the buttons above.

For the purposes of the following questions, all of the answers will be an $\in$ **N**. Try to do as many without the calculator as possible to practice your number and memory skills.

Other important rules

Combining what we know about negative powers as well as powers with fractions, means that:

$$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$$

PRACTICE QUESTIONS: Simplify

1	**a** $25^{\frac{1}{2}}$	**b** $8^{\frac{1}{3}}$	**c** $225^{\frac{1}{2}}$
2	**a** $121^{\frac{1}{2}}$	**b** $27^{\frac{1}{3}}$	**c** $64^{\frac{1}{3}}$
3	**a** $64^{\frac{1}{2}}$	**b** $32^{\frac{1}{5}}$	**c** $625^{\frac{1}{4}}$
4	**a** $1^{\frac{1}{2}}$	**b** $256^{\frac{1}{2}}$	**c** $512^{\frac{1}{3}}$
5	**a** $49^{\frac{1}{2}}$	**b** $100\,000^{\frac{1}{5}}$	**c** $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$

ACTIVITY: Extended Mathematics: Logs and bases

ATL

- Media literacy skills: Locate, organize, analyse, evaluate, synthesize and ethically use information from a variety of media (including digital and social media and online networks)

Calculator practice: Push the button!

Find the 'log' button. Use this button to find the answers to the questions below:

- **Find log 10**
- **Find log 100**
- **Find log 10 000 000**
- **Find log 1 000 000 000 000**

What do you notice?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns.

IS IT REALLY *ALL* ABOUT THAT BASE?

Let's start by looking at a simple equation:

$$2^3 = 8$$

What does the equation in this form tell us?

power

$$2^3 = 8$$

base   number

We read this aloud as '2 to the power of 3 is equal to 8'. We know this to be true because $2 \times 2 \times 2 = 8$

But what if 3 was unknown?

$$2^x = 8$$

How could we find x in this case? The question becomes

2 to the power of what gives an answer of 8?

We can represent this using **logarithms**. It is written in the following form:

$\log_2 8 = x$ log 8 to the base 2 is equal to what?

Is there a visual way to imagine what we are looking for?

Imagine there is an empty box between the base and the number. We need to fill that box with the correct power to get to the bigger number.

Alternatively, you could imagine them as stairs – how do you get from the bottom step (the base) to the final number? What power would get you there?

$\log_2 \blacksquare\, 8$ 2 to the power of what = 8?

$\log_2 3 = 8$ the missing number is 3

so $\log_2 8 = 3$

Example 1

$\log_2 16 = ?$

Let's zoom in on the numbers 2 and 16. What power would 2 need to equal 16? How many times must you multiply it by itself to get 16?

$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 16$

$\therefore \log_2 16 = 5$

Example 2

$\log_5 125 = ?$

5 to the power of what, would give an answer of 125?

$5 \times 5 \times 5 = 125$

$\therefore \log_5 125 = 3$

Example 3

Find

$\log_4 1024$

$4 \times 4 = 16$ $16 \times 4 = 64$ $64 \times 4 = 256$ $256 \times 4 = 1024$

$\therefore \log_4 1024 = 5$

PRACTICE QUESTIONS

Find:

1	$\log_3 9$	**2**	$\log_4 16$	**3**	$\log_{10} 10000$
4	$\log_5 625$	**5**	$\log_{13} 169$	**6**	$\log_{11} 11$
7	$\log_5 1$	**8**	$\log_9 9$	**9**	$\log_8 32768$

Hint

Remember:

$\log_a b = c$ means that $a^c = b$

$\log_a a = 1$ because the power of 1 always remains the same $\rightarrow a^1 = a$

$\log_b 1 = 0$ because anything to the power of zero $= 1 \rightarrow b^0 = 1$

Other laws of logs

Law 1: Addition

$$\log x + \log y = \log xy$$

for example $\log 2 + \log 5 = \log 10$

Law 2: Subtraction

$$\log x - \log y = \log \frac{x}{y}$$

for example $\log 10 - \log 5 = \log 2$

Law 3: Multiplication

$$n \log m = \log m^n$$

for example $3 \log 2 = \log 2^3 = \log 8$

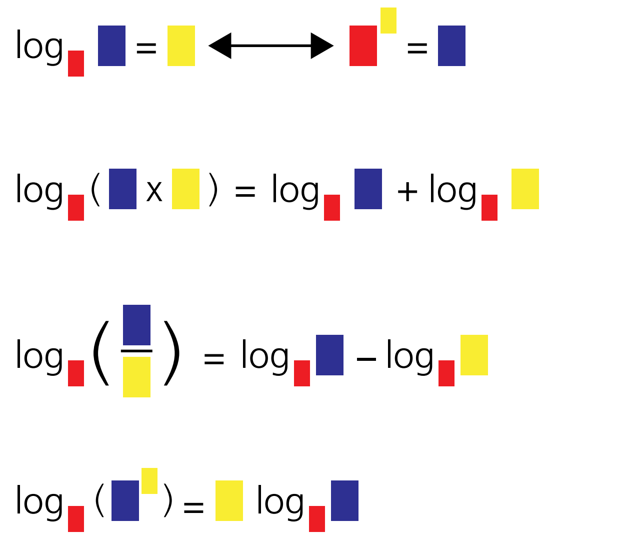
Practice without bases:

Log 7 + log 10 =

Log 52 – log 4 =

Log 4 + log 9 + log 10 – log 30 =

Here's another way to visualize the laws of logs using colours instead of numbers:



Source: Visual log laws *by Brittany Bordewyk*

If logs are the inverses of powers, **explain** how the first two laws relate to the following power laws:

$x^a \times x^b = x^{a+b}$ when you multiply, add the powers

$\frac{y^a}{y^b} = y^{a-b}$ when you divide, subtract the powers

COLOUR–IMAGE–SYMBOL

Think about what you have learnt about logarithms, what they do and the laws that govern them. **Summarize** what you have studied by choosing a colour, an image and a symbol to represent logs.

(Be honest, did you pick brown as the colour for logs?)

To learn more about logarithmic functions and what they look like, see Chapter 9 on Functions.

SOME SUMMATIVE PROBLEMS TO TRY

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D

1 The music streaming service Spotify pays a royalty to a musician or a band each time someone streams their song. They pay $0.0084 per streaming of a song. At the end of 2015, the five most played songs were:

1. Major Lazer - 'Lean On (feat. MØ & DJ Snake)' - 586 million streams
2. Ed Sheeran - 'Thinking Out Loud' - 556m streams
3. OMI - 'Cheerleader - Felix Jaehn Remix Radio Edit' - 507m streams
4. Mark Ronson - 'Uptown Funk' - 498m streams
5. Hozier - 'Take Me To Church' - 476m streams

a Do you think these numbers have been rounded off? Why do you think that?

b Would any simplification of the numbers affect the amount of money received by the artists? Why or why not?

c How much was paid to each of these artists? **Show** your calculations. Choose an appropriate level of accuracy for your answer.

d Give your answer in standard form also.

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION C

2 **Redraw** Table 1.4 on page 15 by adding additional columns for each rounding of Columns 3 and 4.

Level 1–2 Round all the values in Column 3 and 4 to 1 decimal place.

Level 3–4 Round all the values in Column 3 and 4 to 1 d.p. and whole numbers.

Level 5–6 Round all the values in Column 3 and 4 to 1 d.p., whole numbers and to the nearest 100.

Level 7–8 Round all the values as above and to an appropriate degree of accuracy for a news report.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

3 **Explain** what these symbols mean?

a $\mathbb{N}$

b $\mathbb{Z}$

c $\in$

4 **Match** the numbers to the correct reciprocal.

a 5 **b** 11 **c** 10

i $\frac{1}{10}$ **ii** $\frac{1}{11}$ **iii** $\frac{1}{5}$

5 **Convert** these numbers into scientific notation

a 5 673 000 000

b 0.000 0087

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

6 **Explain** what these symbols mean:

a $\mathbb{Q}$ **b** $\mathbb{R}$

7 **Complete** the following with a correct set.

$5 \in$ ______

8 **Convert** these numbers into scientific notation.

a 5 000 073 000 000

b 0.000 000 0807

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

9 **Complete** the following with a correct set.

$5.6 \in$ ______

10 **Convert** these numbers into scientific notation.

a 5 673 000 000 × 304 700 000 000

b 0.000 0087 × 0.000 000 0099

EXTENDED

11 **a** $\log_{27} 729$

b $\log_{100} 1$

c $\log_{122} 122$

12 **Simplify**

a $\log 8 + \log 3$

b $\log 24 - \log 8$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

13 Complete the following with the correct set.

$\sqrt{6} \in$ ______

14 Convert these numbers into scientific notation.

a $(5\,673\,000\,000)^2$

b $(0.000\,008\,7)^{-1}$

EXTENDED

15 Find

a $\log_2 \frac{1}{2}$

b $\log_3 \frac{1}{27}$

c $\log_{100} 10\,000$

16 Simplify $\log 6 - \log 2 + 3 \log 2$

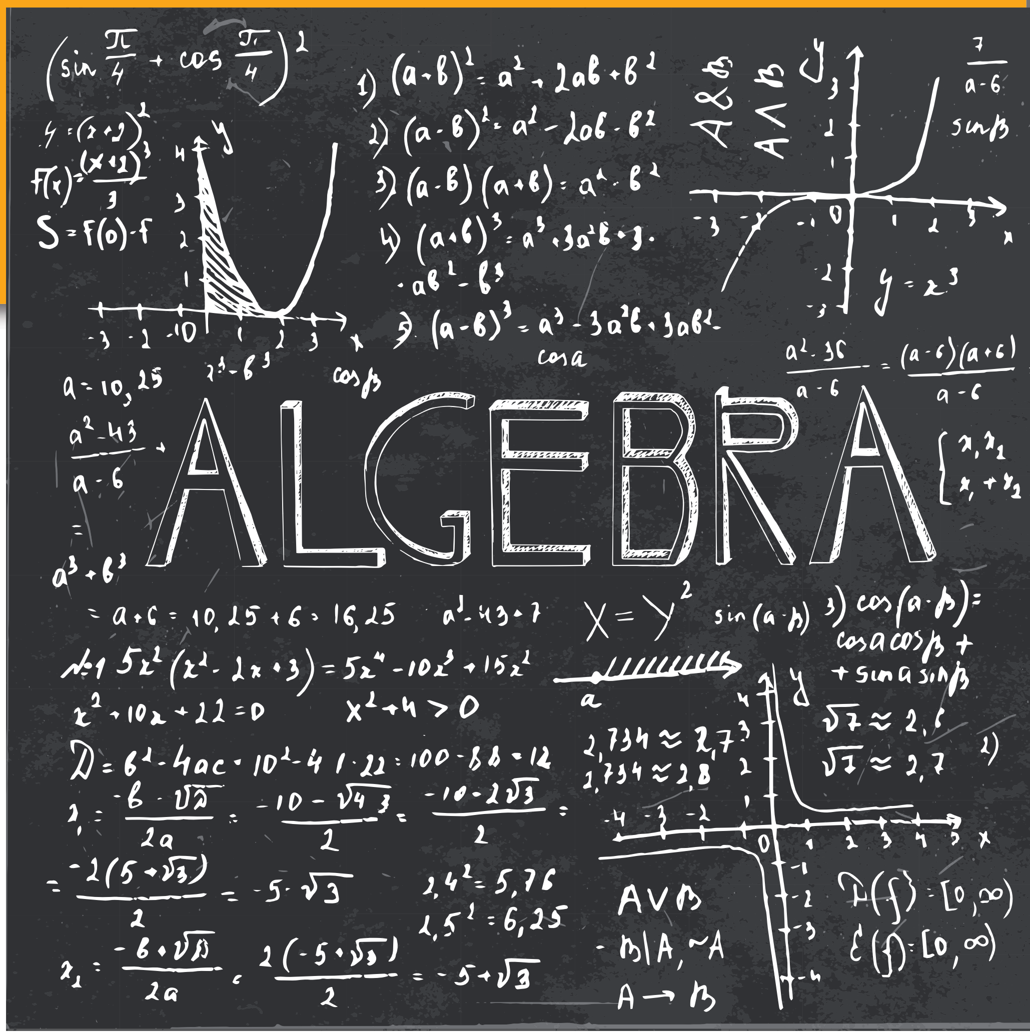
17 Solve $\log x + \log 7 = \log 42$

Reflection

<table>
<tr><th colspan="6">Use this table to reflect on your own learning in this chapter.</th></tr>
<tr><td>Questions we asked</td><td>Answers we found</td><td colspan="4">Any further questions now?</td></tr>
<tr><td>Factual: How are numbers sets defined? How and why do we group numbers? What is meant by approximate and exact?</td><td></td><td colspan="4"></td></tr>
<tr><td>Conceptual: How do number systems expand our understanding? What patterns can we see in different number forms and operations?</td><td></td><td colspan="4"></td></tr>
<tr><td>Debatable: Were numbers invented or discovered? Is there a best form for a number? Can the form of a number mislead or affect our decisions? Can rounding help or hinder decision-making?</td><td></td><td colspan="4"></td></tr>
<tr><td rowspan="2">Approaches to Learning you used in this chapter:</td><td>Description – what new skills did you learn?</td><td colspan="4">How well did you master the skills?</td></tr>
<tr><td></td><td>Novice</td><td>Learner</td><td>Practitioner</td><td>Expert</td></tr>
<tr><td>Communication skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Media literacy skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Creative-thinking skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Learner profile attribute</td><td colspan="5">How did you demonstrate your skills as a communicator in this chapter?</td></tr>
<tr><td>Communicator</td><td colspan="5"></td></tr>
</table>

2 Why does algebra look so clever?

Finding and expressing things **in common** helps us to **simplify** and improve **relationships**.



CONSIDER THESE QUESTIONS:

Factual: How do we factorize expressions? How can equations be solved? What is meant by an unknown?

Conceptual: How does simplification allow us to find an unknown? How can relationships be expressed algebraically?

Debatable: What does simplification cost in terms of accuracy? Can we say one method is better than another?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how to simplify, factorize and solve equations.
- **Explore** what it means to be a mathematical communicator in the 'language' of algebra**.**
- **Take action** by creating new ways to use algebra in everyday situations and special occasions.

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Organization skills
- Affective skills
- Creative-thinking skills
- Critical-thinking skills

We will reflect on this learner profile attribute …

- Thinker – we use critical-thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions.

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating

Read this article (**www.newscientist.com/blogs/culturelab/2012/01/the-mathematics-behind-sherlock-holmes.html**) and work out the mathematics behind Sherlock Holmes!

PRIOR KNOWLEDGE

You will already know:

- how to simplify expressions by collecting like terms
- how to multiply algebraic terms
- what a radical or surd form is
- how to substitute terms.

KEY WORDS

coefficient	factorize	transpose
expand	inverse	

STEPPING INTO THE UNKNOWN

Algebra is a language of simplification and communication and yet somehow has come to be seen by people outside of mathematics as something really hard that only intelligent people can do or understand. Look at movies and TV shows showing geniuses, geeks or scientists. In the background you will always see boards covered in letters, numbers and equal signs. Movies often use algebra in the background as a way to communicate that someone is smart or that a location is a place of learning or that genius is somehow involved. Books, illustrations and images online offer the same message. Algebra = hard + clever.

Why would this be? Algebra has been developed by mathematicians to simplify, not to confuse. It is a tool for exploring patterns. The purpose of this communication is to show and solve relationships - not to exclude people or convince them of your genius. Sometimes the boards in the background are just a jumble of nonsense symbols and sometimes they are very subtle communications, showing a famous theorem or mathematical joke (yes, there is such a thing). **https://youtu.be/ReOQ300AcSU**

In this chapter we will focus on decoding the language of algebra to help you communicate in, and understand, this language – to develop your own mathematical identity and to decipher what is real mathematics on TV and what isn't.

In *Mathematics for the IB MYP 1* and *MYP 3* you will have studied what an algebraic expression is and how it is used to show relationships, express unknown quantities and to model real-world problems.

What is meant by an unknown?

ASSIGNING LETTERS OR SYMBOLS TO UNKNOWNS

An unknown is assigned a letter as we do not yet know the value or values of that term. The most common letter to use is x and this algebraic use has entered the English language in a variety of ways – the X-factor is a phrase used to explain someone with a quality that is hard to explain or define. Generation X is often used as a name for the generation born between the 1960s and 1980s. The 'X' refers to an unknown variable or to a desire not to be defined, to have the freedom to define their own identities for themselves.

Why is x so regularly used though? Why would x be chosen as the unknown?

Watch this TED Talk: **www.ted.com/talks/terry_moore_why_is_x_the_unknown?language=en**

But we don't use x or y only. We can technically use any letter of the alphabet or we could even make up our own symbols. Sometimes Greek letters are used to represent the unknowns or variables in algebra. Some of them have been given specific purposes too, for example θ is often used as an angle notation. You will learn about δ, σ and Σ in Diploma Mathematics.

When we are trying to learn how to read and write the algebraic communication, the analogy of language is often used to understand the structures.

- *Letters* stand for numbers or variables and they combine to give terms. They stand alone with a positive or negative sign associated with it. These terms are like words, standing alone or in small groups.
- *Expressions* show the relationships between those numbers and variables. Expressions are like phrases or sentences; in fact, we often use the word expression when talking about English literature.

Greek symbols

Why would we use Greek symbols especially? Do you recognize any of these symbols already? Where from?

α	Alpha	ν	Nu
β	Beta	ξ	Xi
γ	Gamma	o	Omicron
δ	Delta	π	Pi
ε	Epsilon	ρ	Rho
ζ	Zeta	σ	Sigma
η	Eta	τ	Tau
θ	Theta	υ	Upsilon
ι	Iota	ϕ	Phi
κ	Kappa	χ	Chi
λ	Lambda	ψ	Psi
μ	Mu	ω	Omega

Links to: Sciences (Physics)

Some of these variables have been given specific meanings such as ρ for density, α for radioactive alpha particles and γ for gamma radiation.

- *Equations* show two expressions which are equal, either in every situation or in certain situations (depending on the values). These are like sentences that use multiple characters or use a simile or metaphor.
- *Formulae* are well known equations that have been proven to be a true relationship and can be used to identify quantities. They are like a catchphrase or motto that everyone recognizes and that means something to you.
- An algebraic *calculation, solution* or *derivation* is like a paragraph or short story that ties all of these features together.

THINK–PUZZLE–EXPLORE

- **What do you think you know about equations?**
- **What questions or puzzles do you have about equations?**
- **How can you explore this topic further?**

Let's see this in action:

Variable	x	(the cost of) a downloaded song
Term	$4x$	(the cost of) four downloaded songs
Expression	$4x + 7.64$	(the cost of) four downloaded songs and a remaining balance of \$7.64
Equation	$4x + 7.64 = 10$	you downloaded 4 songs and have a balance of \$7.64 from a \$10 card

Solution

$4x + 7.64 = 10$

$4x = 10 - 7.64$

$4x = 2.36$

$x = \dfrac{2.36}{4}$

$x = 0.59$

So each song cost \$0.59 or 59 cents.

An equation:

- MUST have an equals sign
- MUST state two expressions
- MUST have an unknown quantity, denoted by a letter value
- MAY include a constant term (a number without an unknown term)
- MAY have more than one unknown (see Chapters 3 and 8 for more on these)
- WILL have at least one solution.

Equations are statements of quantitative equality between expressions. The equality depends on certain values. These values could have a permanent definition or be equivalent involving several variables where we can use one expression to find the other, for example:

area = ½ bh or $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let's first revise the simple operations in algebra: addition and subtraction, multiplication and division.

ACTIVITY: Algebra revision

ATL

- Communication skills: Use and interpret a range of mathematical rules, terms and symbols

You will already know many or all of these rules. Fill in the blanks:

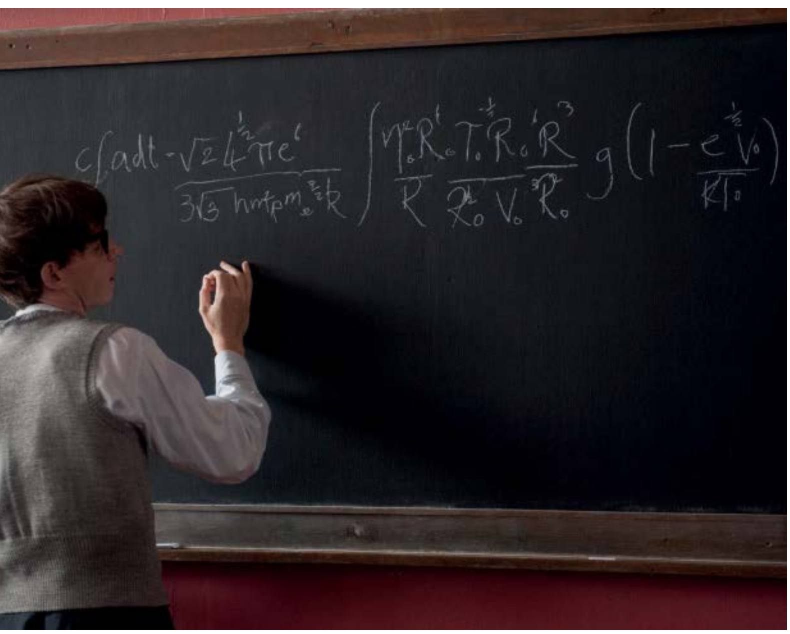
$a + a + a = 3a$	This is known as collecting __________ terms
$a + b + c = a + b + c$	This cannot be simplified further because __________
$4d + d = 5d$	If a term doesn't seem to have a coefficient, we treat it like which number? __________
$6e - 2e - e - 5e = -2e$	Subtraction works just like __________
$f \times g \times h = fgh$	Multiplying terms is also known as the p__________t
$2f \times -3g \times h = -6fgh$	You also need to multiply the co__________
$j \times j \times j \times j = j^4$	A term with a power of 4 means you multiplied the letter by __________ 4 times
$k \times k \times k = k^3$	Why isn't this 3k? __________
$\dfrac{3got}{6go} = \dfrac{3\cancel{go}t}{6\cancel{go}} = \dfrac{3t}{6} = \dfrac{1t}{2} = \dfrac{t}{2}$	When dividing terms you can __________ like terms
$\dfrac{2xy}{z}$	One quantity divided by another is known as a q__________

And from Chapter 1, we also know how to handle algebraic terms with powers and roots, see pages 8 and 16. All of these operations allow us to simplify expressions and handle equations so we can find solutions to abstract and real-world problems.

SUBSTITUTION

The act of replacing one thing with another is called substitution. If we substitute a player in sports, the new player takes the place and role of the previous one. In algebra if we input a value for a variable, or a known number into an unknown, this is known as substitution.

Substitution gives you a value for an expression in a certain situation, given certain conditions.



■ Substitution is when an actor takes the role of a character; in this case given the specific condition of the movie "The Theory of Everything", Eddie Redmayne is 'substituting' for, or taking the place of, Stephen Hawking.

Examples

Find

$3y + 4$ if $y = 1$

Solution

$= 3(1) + 4$	replace y with 1 everywhere it appears in the equation
$= 3 + 4$	carry out every operation
$= 7$	

Evaluate $a = -10b + 5$ when $b = 3$.

Solution

$a = -10(3) + 5$

$a = -30 + 5$

$a = -25$

Find $y = 3t^2 + t - 1$ when

i $t = 7$ **ii** $t = 3.5$ **iii** $t = -5$

Solution

i $= 3(7)^2 + 7 - 1$

$= 3(49) + 6$ order of operations

$= 147 + 6$

$y = 153$

ii $y = 3t^2 + t - 1$

$y = 3(3.5)^2 + 3.5 - 1$

$y = 3(12.25) + 2.5$

$y = 36.75 + 2.5$

$y = 39.25$

iii $y = 3t^2 + t - 1$

$y = 3(-5)^2 + (-5) - 1$

$y = 3(25) - 6$

$y = 69$

PRACTICE QUESTIONS

1 Evaluate

a $2e + 4f$ where $e = 3$, $f = 1$

b $2e + 4f$ where $e = 0$, $f = 7$

c $12e - 4f$ where $e = 3$, $f = 15$

d $12e - 4f$ where $e = -2$, $f = 1$

2 Evaluate

a $12e + 4f - e + 6f$ where $e = 3$, $f = 1$

b $12e + 4f - e + 6f$ where $e = -1$, $f = 4.5$

c $12e \times 4f$ where $e = 3$, $f = 1$

d $\frac{4ef}{f}$ where $e = 1$, $f = 600$

3 Evaluate

a $2e^2 + 4f^2 - e + 6f$ where $e = 3$, $f = 1$

b $\frac{3e^4f^3}{ef}$ where $e = -5$, $f = -2$

c $\frac{3e^4f^3}{ef}$ where $e = -6.7$, $f = -2.4$

d $\left(\frac{3e^4f^3}{ef}\right)^2$ where $e = -5$, $f = -2$

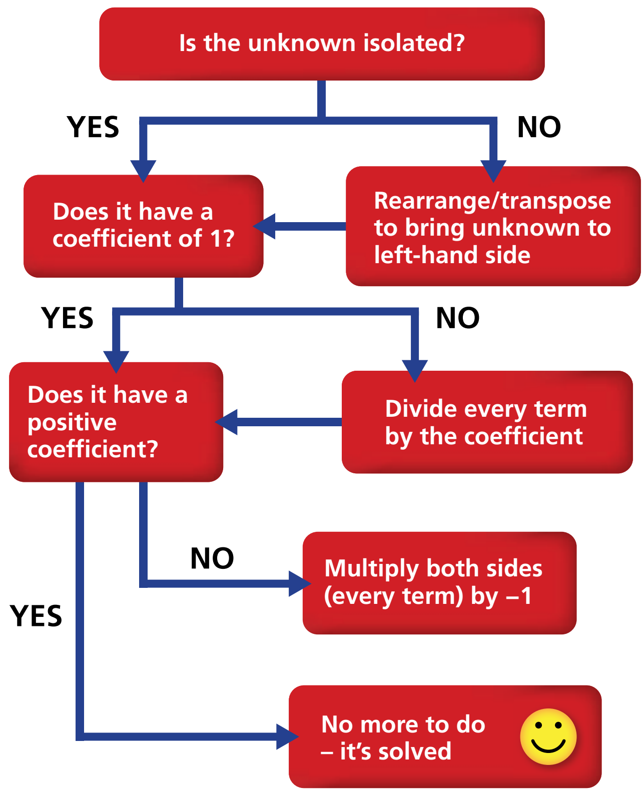
REFLECTION

Were there skills you already know in algebra that you could use to simplify in the questions above? How did you, or could you, make these expressions simpler before you substituted? Does this affect the *relationship* between the terms?

How can I be sure I have solved correctly?

Before you finish a question, particularly in a test or exam if you have enough time, it is a good idea to self-check. How can you find out if it is the right answer?

Substitute it into the original equation to see if it works/satisfies the original.



How does simplification allow us to find an unknown?

HOW CAN WE VISUALIZE LINEAR EQUATIONS?

Equalities make two expressions equal and allow us to find the unknowns. But what is happening in real terms when we are moving numbers across equals and inversing operations?

You will have solved simple equations before, perhaps by making use of the concepts of balance or maintaining relationships by inverse. There are many ways to visualize the steps to rearrange and isolate the variable to allow you to solve it. Two particular ideas of balance or logical sequence are commonly used; you will probably have seen at least one of these methods in your earlier algebra studies.

What does that mean exactly? To solve, we want to isolate a single variable, with a coefficient of +1. We rearrange or '**transpose**' all the other terms so we can find the value of that letter.

Method 1

In this method you see the expression in an equation as a sequence of operations which have to be *undone* or *reversed*. The reversing operation is called inversing. Sometimes the equals sign is viewed as a portal or bridge – when you cross it you 'do the opposite' and examples sometimes show you arrows travelling across the equals.

Example 1

$x + 5 = 10$

$x + 5 = 10$

-5 inverse of $+5$

$\therefore x = 10 - 5$

$x = 5$

Example 2

$3x = 12$

$3x = 12 \div 3$

multiply × 3 becomes ÷ 3 on the other side

$x = 4$

Example 3

$5x - 2 = 23$

Isolate the x-term first by moving -2

$5x - 2 = 23 + 2$

$5x = 25$

Now find $1x$ by dividing the RHS by 5

$5x = 25$

$\div 5$

$\therefore x = 5$

Example 4

$\frac{x}{4} + 2 = 5$

Isolate the x-term by moving +2 over

$\frac{x}{4} + 2 = 5 - 2$

$\frac{x}{4} = 3$

Now move the ÷ 4 to the other side

$\frac{x}{4} = 3$

$\times 4$

$\therefore x = 12$

Example 5

$\frac{18x + 3}{6} = 3.5$

As the entire LHS is a fraction, move the ÷6 first

$\frac{18x + 3}{6} = 3.5$

$\times 6$

$18x + 3 = 21$

-3

$18x = 18$

$\therefore x = 1$

Method 2

The other common method is to view the equation as a balance which has to be maintained at all times – what you 'do' to one side, you must 'do' to the other. You wish to simplify the relationship, not to change it fundamentally and put it out of balance.

Example 1

$x + 5 = 10$

$-5 \qquad -5$

$x = 5$

Example 2

$\frac{3x}{3} = \frac{12}{3}$

$x = 4$

In this method, each step towards isolating the singular variable is a balanced operation on each side, students often show the operation they are doing under both sides to explain their method. This becomes less necessary over time as your reasoning becomes clear by the steps themselves and not the operations needed.

You will grow more confident and your lines of reasoning will become more concise.

Example 3

$5x - \cancel{2} = 23$

$\cancel{+2} \qquad +2$

$\frac{5x}{5} = \frac{25}{5}$

$x = 5$

Example 4

$\frac{x}{4} + \cancel{2} = 5$

$\cancel{-2} \qquad -2$

$\frac{x}{\cancel{4}} = 3$

$\cancel{\times 4} \qquad \times 4$

$x = 12$

Example 5

$\frac{18x + 3}{\cancel{6}} = 3.5 \qquad \times 6$

$\cancel{\times 6}$

$18x + \cancel{3} = 21$

$\cancel{-3} \qquad -3$

$\frac{18x}{18} = \frac{18}{18}$

$x = 1$

Hint

Some people like to use arrows to show what they have done or write the operation on the side to help with communication. As you become more confident this may not be necessary but at first it might be a helpful support.

Example 6

$2x^2 - 2 = 240$

$2x^2 - 2 = 240 \qquad + 2$

$2x^2 = 242 \qquad \div 2$

$x^2 = 121 \qquad \sqrt{}$

$x = \sqrt{121}$

$x = 11$

$2x^2 - 2 = 240$

$2x^2 = 242 \qquad | + 2$

$x^2 = 121 \qquad | \div 2$

$x = \sqrt{121} \qquad | \sqrt{}$

$x = 11$

Example 7

$6 - x^2 = 2$	where $x \in \mathbf{N}$
$-x^2 = 2 - 6$	isolate x term
$-x^2 = -4$	
$x^2 = 4$	multiply $\times -1$
$x = \sqrt{4}$	inverse the square by using square root
$\underline{x = 2}$	

We can check by substituting this value into the original equation

$6 - x^2 = 2 \qquad x = 2$?

$6 - (2)^2 = 2$

$6 - 4 = 2$

$2 = 2 \qquad ✓$

Now you know the answer is correct.

What happens if my substitution shows I've got the wrong answer?

Don't worry – the most common cause of mistakes in algebra calculations is in rearranging or transposing the equation. Did you use the correct inverse to move an operation? The second most common mistake is an error in positive or negative signs. Go back and check each line (this is one of the reasons why laying out your lines of reasoning is important).

Remember everyone makes mistakes in calculations, including your teacher and professional mathematicians! If you are thorough and patient, you will find the error(s) and can correct them. This will help with your ATL: Organization skills – Managing tasks effectively and managing a positive state of mind.

How can equations be solved?

WHAT IF THE EQUATION IS MORE COMPLEX?

What do you do when an equation has many terms and some more complex operations to inverse?

This will not be a problem if you approach the equation logically. Treat each term and operation individually and in the correct order. Solving equations with multiple steps is a good opportunity to practice your self-management (affective) skills for both managing time and tasks effectively, as well as managing a calm and positive state of mind.

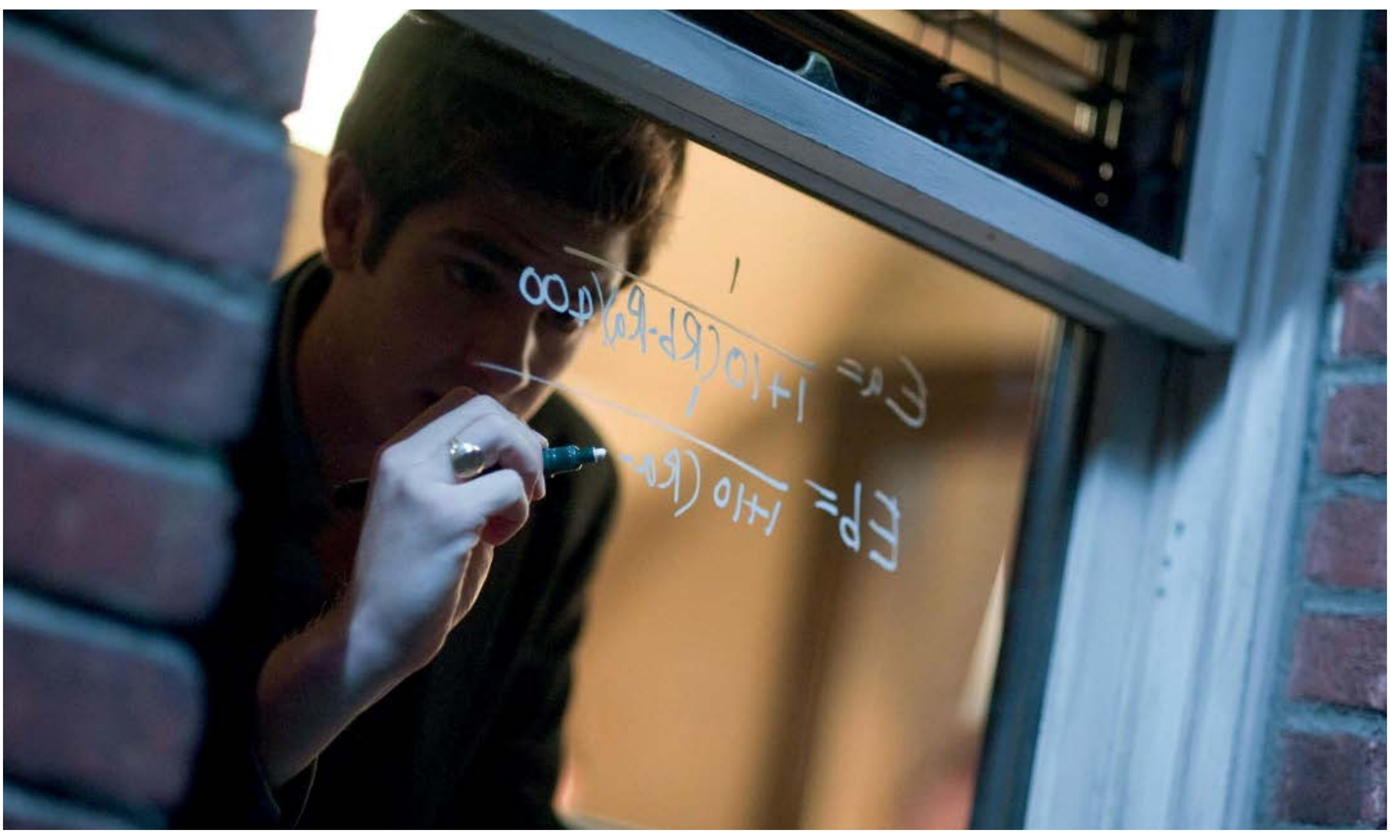
IS THERE A SPECIFIC ORDER TO UNDO OPERATIONS?

We know that there is an order to carry out operations – this is known variously as BEDMAS, BIMDAS, PEDMAS, etc. This tells you to remove brackets before adding, etc. See Chapter 1 from Mathematics for the IB MYP 1: by Concept for more details. We will use BEDMAS for consistency going forwards.

DISCUSS

Watch this video on Grandi's series:
www.youtube.com/watch?v=PCu_BNNI5x4

In pairs, **discuss** what you've learned.



Don't mathematicians ever use paper?

ACTIVITY: BEDMAS

ATL

- Affective skills: Practise analysing and attributing causes for failure

Alfonso says that he thinks that if he uses BEDMAS that it will help him to solve equations correctly. Mary thinks that he is partially right but she thinks it needs to be in reverse (backwards). Look at their calculations and **determine** whose method is correct.

$x^2 + 4 = 29$ $\qquad x \in \mathbf{N}$

Alfonso	Mary
BEDMAS FORWARDS	BEDMAS BACKWARDS
no brackets	none
exponents	✓ + 4
	$x^2 + 4 = 29$
$x^2 + 4 = 29$	$x^2 = 29 - 4$
$\sqrt{\ }$ $\quad x + 2 = \sqrt{29}$	$x^2 = 25$
A	$x = \sqrt{25}$
$x = \sqrt{29} - 2$	$x = 5$
$x = 3.39$	

Who is correct?
How could you check?

Testing both solutions:

$x^2 + 4 = 29$

Alfonso says	Mary says
$x = 3.39$	$x = 5$
$(3.39)^2 + 4 = 29$	$(5)^2 + 4 = 29$
$(13.221) + 4 = 29$	$25 + 4 = 29$
$17.221 \neq 29$	$29 = 29$
✗	✓

Clearly Mary's method was correct.

Alfonso made some serious errors in his assumptions and calculations. Can you **identify** what they were?

Assessment opportunities

- This activity can be assessed with Criterion C: Communicating.

PRACTICE QUESTIONS

1. $x^2 + 4 = 29$
2. $\frac{x}{3} = 4$
3. $x^3 - 1 = 2$
4. $2y^3 = 128$
5. $6z^2 + 3 = 27$
6. $\frac{2y - 1}{4} = 10$
7. $-\frac{2}{x + 4} = 8$
8. $a^3 + 1 = 28$
9. $2^x + 3 = 19$

HOW DO YOU EXPAND BRACKETS OR PARENTHESES?

An important algebraic operation is the ability to add or remove brackets. Removing brackets from an expression by multiplying is called **expansion**. Brackets can also be called parentheses.

In some questions, you may need to **expand** before you can **solve**. For example:

$6(4d + 11) = 78$

OR

$-3(x^2 + 2) = -306$

HOW DO WE EXPAND?

One of the aims for MYP Mathematics is for students 'to develop confidence, perseverance, and independence in mathematical thinking and problem-solving'. To support this aim, it is important for you to see that there is often more than one way to solve a problem. This means you can choose your preferred way, while being familiar with the others. There is no one official way to do it and all you have to do is to make sure you communicate it clearly and well. People learn, and approach problems, in different ways. Look at the following three ways to expand and see which one seems to you to be the most logical way of looking at expansion.

How to expand

Question: Expand 3(b + 2)

Method 1: Using arrows

$3(b + 2)$ use the arrows to make sure you multiply *every* term

$3(b) + 3(2)$

Answer $3b + 6$

Method 2: visualising/modelling the terms as length

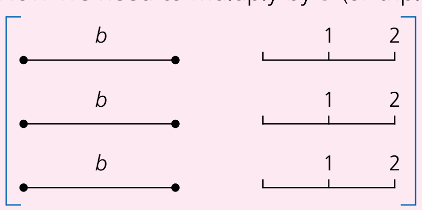
This method is sometimes a good option for visual learners although it has some **limitations** (more on that later). For this method you must visualize that the terms inside the brackets are lengths. You don't have to draw them to scale – it's just a visual tool. You don't even have to draw them (it can take a while unless you just do a rough sketch), you can imagine them if it helps.

$3(b + 2)$

This means multiplying these lengths three times.

Let's picture b + 2 as a length.

Now we need to multiply by 3 (or triple it).



add these together

$3b \quad + \quad 6$

Method 3: the grid

For this method, we will borrow a trick from long multiplication.

One way to multiply is to lay out a grid to multiply terms that are easily done first and add them at the end.

×	**20**	**7**
10	200	70
4	80	28

So, to **find** 27 × 14, we add

$200 + 70 + 80 + 28 = 378$

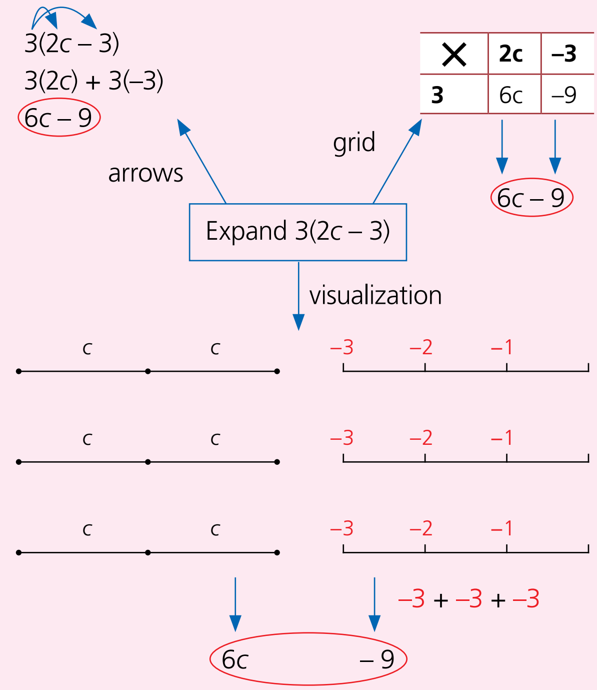
Let's try the same trick with the algebra terms for 3(b + 2).

×	***b***	**2**
3	$3b$	6

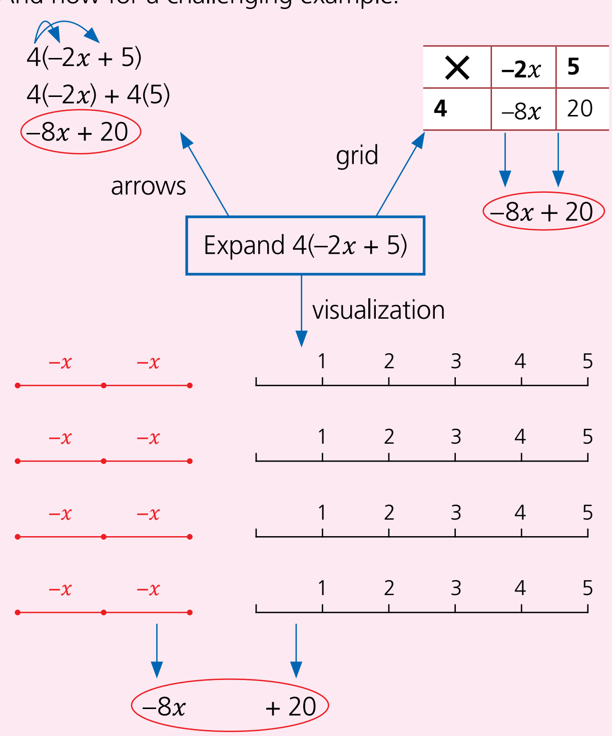
This method gives us the same answer as the previous two: $3b + 6$

Let's try a more complex problem with all three methods.

Expand 3(2c – 3)



And now for a challenging example:



PRACTICE QUESTIONS

Expand:

1. $3(b + 2)$
2. $3(b - 2)$
3. $3(-b + 2)$
4. $3(-b - 2)$
5. $-3(b + 2)$
6. $-3(-b - 2)$
7. $b(b + 2)$
8. $-b(b + 2)$
9. $3b(b + 2)$
10. $3b(b - 2)$

What makes you say that?

ATL

- Organization skills: Use appropriate strategies for organizing complex information

Post-method analysis: Give one thing you like or dislike about each method.

Nominate your favourite method. What makes you say that?

Assessment opportunities

- This activity can be assessed with Criterion A: Knowing and understanding.

WHAT HAPPENS IF WE HAVE TWO BRACKETS TO EXPAND?

We can use all of the previous techniques, if we modify them slightly for the two brackets. Each term in the first bracket must be multiplied by each term in the second bracket. Watch out for positive and negative signs.

Expand $(x + 2)(x + 3)$

Method 1: using arrows

As long as you remember to multiply each term, the order is unimportant. Here are **two** possibilities, each with simple memory devices to assist you:

1a

$(x + 2)(x + 3)$

FIRST OUTSIDE INSIDE LAST

multiply the first terms of each

→ the outside terms of each

→ then the inside terms of each

→ and then the last terms

First Outside Inside Last : FOIL

F$(x)(x)$ + O$(x)(3)$ + I$(2)(x)$ + L$(2)(3)$

$x^2 + 3x + 2x + 6$

like terms

Answer $x^2 + 5x + 6$

1b

$(x + 2)(x + 3)$

Another combination of arrows looks this. Make a few tiny additions and you get a …

$(x + 2)(x + 3)$

smiley face!

Again $x^2 + 6 + 2x + 3x$

like terms

Answer $x^2 + 5x + 6$

Method 2: visualizing/modelling the terms as areas or lengths

When we had the single lengths on the previous page, we could imagine 'three lots of …' or 'four times …'. This would be hard to visualize 'x times', or 'x lots of …'

2

$(x + 2)(x + 3)$

$\rightarrow (x + 2)$

	x	1	2
x	x^2	x	x
1	x	1	1
2	x	1	1
3	x	1	1

$x^2 + 5x + 6$

$\downarrow$

$(x + 3)$

Answer $x^2 + 5x + 6$

Method 3: the grid

The grid method remains the same, with an additional row for the additional term in the second bracket. Watch out for the minus signs!

$(x + 2)(x + 3)$

×	x	**2**
x	x^2	$2x$
3	$3x$	6

As with the other methods, you get four terms which need to be simplified. The orange shaded boxes are always the like terms.

So the final answer is

$x^2 + 5x + 6$

As we know from before, each of these methods will get you the right answer and is equally valid.

WHAT MAKES YOU SAY THAT?

Do you still have a preferred method or has it changed? Why?

DOES THIS EXPANSION GET MORE COMPLICATED?

The simple answer is yes! The terms can get larger, positive and negative signs can be more complicated and there may even be a term outside the bracket which will have to multiplied.

Let's look at a more complex example:

$\boxed{(2x + 3)(3x + 1)}$

Face

$(2x + 3)(3x + 1)$

$6x^2 + 3 + 9x + 2x$

$6x^2 + 11x + 3$

FOIL

$(2x + 3)(3x + 1)$

$6x^2 + 2x + 9x + 3$

$6x^2 + 11x + 3$

Arrows

grid

×	**3x**	**1**
2x	$2x^2$	$2x$
3	$9x$	3

$6x^2 + 9x + 2x + 3$

$6x^2 + 11x + 3$

area

	x	x	1	2	3	→ $(2x + 3)$
x	x^2	x^2	x	x	x	
x	x^2	x^2	x	x	x	
x	x^2	x^2	x	x	x	
1	x	x	1	1	1	

↓

$(3x + 1)$ $\quad 6x^2 + 11x + 3$

$6x^2 + 11x + 3$

And now a challenging one, with negative (minus) values

$\boxed{(2x + 3)(x - 1)}$

Face

$(2x + 3)(x - 1)$

$2x^2 + 2x + 3x - 3$

$2x^2 + x - 3$

FOIL

$(2x + 3)(x - 1)$

$2x^2 - 2x + 3x - 3$

$2x^2 + x - 3$

Arrows

grid

×	$2x$	3
x	$2x^2$	$+3x$
-1	$-2x$	-3

$2x^2 + 3x - 2x - 3$

$2x^2 + x - 3$

area

	x	x	1	2	3
x	x	x^2	x	x	x
-1	$-x$	$-x$	-1	-1	-1

indicate negative with dotted line

BUT 2 of the x's will fill the negative x's

x^2	x^2	x	x	x
$-x$	$-x$	-1	-1	-1

cancelling

So, that leaves

x^2	x^2	x		
		-1	-1	-1

$2x^2 + x - 3$

Links to: Art

What is the concept of negative space in Art? How could it relate to the concept of spatial relationships as shown by this diagram?

PRACTICE QUESTIONS

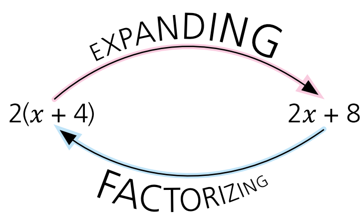
Expand:

1. $(x + 3)(x + 5)$
2. $(x + 2)(x + 2)$
3. $(x + 8)(x + 10)$
4. $(x + 7)(x + 9)$
5. $(2x + 1)(x + 1)$
6. $(x + 4)(3x + 1)$

How do we factorize expressions?

WHAT HAVE WE GOT IN COMMON?

Factorizing is writing an expression as a **product**. It can be considered as the reverse process of expanding.



It is important to find common factors in each term to 'take it out' and move the factor to make a product.

Example 1

Factorize $6x + 42$

First **identify** the **highest common factor** (remember that phrase from primary mathematics?)

the highest common factor is 6 (2 and 3 are also common factors)

$$6x + 42$$

$$\cancel{6}(x) \qquad \cancel{6} \times 7$$

So, bring the six 'outside' a bracket

$6(x + 7)$

Example 2

Factorize $15x + 500$

$$15x + 500$$

$$5 \times 3 \times x \qquad 5 \times 100$$

$$5 \times 20$$

$$2 \times 10$$

$$5 \times 2$$

Bring 5 outside, all remaining factors go (inside).

So $\qquad 5(3x + 100)$

Example 3

$$11x^2 \quad + \quad 3xy$$

$$11 \times x \times x \qquad 3 \times x \times y$$

Factorization: $\qquad x(11x + 3y)$

If you expand the factorization, you should be able to get the *original* expression.

Let's check

$x(11x + 3y)$

$x(11x) + x(3y)$

$= 11x^2 + 3xy$ ✓

Example 4

Factorize

$$4a^2 - 16ab$$

$$4 \times a \times a \qquad 4 \times 4 \times a \times b$$

$$4a(a - 4b)$$

Hint

Remember to self-check!

Example 5

$$6e^2 \quad + \quad 3e$$

$$6 \times e \times e \qquad 3 \times e$$

$$3 \times 2$$

The highest common factor is $3e$

This leaves only a 1 for the second term.

$3e\,(\,2e + 1\,)$

Example 6

$$-3x^2y^3z \quad - \quad 6y^2z^2$$

$$3 \times x \times x \times y \times y \times y \times z \qquad 3 \times 2 \times y \times y \times z \times z$$

The common factors are $3 \times y \times y \times z$

So, $\quad 3y^2z\,(\,-x^2y \; - \; 2z\,)$

but there is still one more thing they have in common

$3y^2z\,(\,-\,x^2y\,-\,2z\,)$

So, the final answer is

$-3y^2z\,(\,x^2y \; + \; 2z\,)$

FACTOR TREES

The factor trees have been used here to illustrate the process, you don't need to do them if you find them unnecessary or take too long. Knowing your times tables and factors well will help you speed up with factorization.

PRACTICE QUESTIONS

Factorize:

1. $9x + 810$
2. $4y + 400$
3. $20z + 8$
4. $15a + 75$
5. $12b + 60$
6. $7c - 49$
7. $2d^2 + 3d$
8. $5ef + 7de$
9. $13fgh - 6ghi$
10. $10ijk - 50jk$
11. $lmn - 5m^2n$
12. $320p + 100p^2$
13. $q^2r^2 + 2qr^3$
14. $-15xyz - x^2yz^3$
15. $19s^5t^6 + 133s^6$
16. $-3u^{10}v^{10} - 21u^8v^5$
17. $11x^3 - 220x^2$
18. $14y^5z + 126yz^5$

Links to: Language acquisition

Quad usually indicates four – quadruped, quad bike, quadrilateral. Why do you think these expressions are called *quad*ratic when they only have three terms?

QUADRATIC EXPRESSIONS

Now that you can factorize a single bracket, it is time to turn your attention to the more advanced double bracket.

Let's refresh:

Expand

$(x + 12)(x + 2)$

Whichever method you choose will give you the answer:

$x^2 + 14x + 24$

Factorizing these **quadratic expressions** involves several steps, which we will practice quite a lot. For more on quadratic equations, see Chapter 8. Spoiler: you will see lots of projectiles like snowboarders jumping because visually these model the quadratic equations very closely.

Example 1

Factorize $x^2 + 4x + 4$.

- Step 1: $(\qquad)(\qquad)$
- Step 2: $x^2 + 4x + 4$ The factors for x^2 must be $x \times x$
  $(x \qquad)(x \qquad)$
- Step 3: $x^2 + 4x + 4$ The factors for 4 are either 4×1 or 2×2

  Let's try 4×1
  $(x + 4)(x + 1)$ Could these expand to $x^2 + 4x + 4$?
- Step 4: $x^2 + 4x + 4$ Now we check the x terms
  $(x + 4)(x + 1)$ we multiply to see if we get $+4x$

$(x + 4)(x + 1)$ Remember the smiley face?

$4x$

$1x$

So $4x + 1x = 5x$ ✗ doesn't match

So this was the wrong choice of factors ☹ Let's try again!

- Step 3: $x^2 + 4x + 4$
  $(x + 2)(x + 2)$ try again with the only other pair of factors
- Step 4: $x^2 + 4x + 4$ test to see if we get $+4x$

$(x + 2)(x + 2)$

$2x$

$2x$

So $2x + 2x = 4x$ ✓ This works.

Procedure pro

So we know this form of expression $ax^2 + bx + c$ is the result of expanding two brackets, both of which have an x-term and a constant. We will have to do some guessing when it comes to choosing the factors to try – you may pick the wrong ones and you will find that out when you test it. No need to worry. Simply cross it through and try again with a different pair of factors. It won't take long and you won't be penalized for making a mistake.

Where do we begin?

Step 1: write two sets of brackets with lots of space to write the terms in
$(\qquad\qquad)(\qquad\qquad)$

Step 2: look at the first term (the one with x^2 and choose a pair of factors

Step 3: look at the last term (the constant number) and choose a pair of factors

Step 4: look at the middle term (the one with the x) to test the factors chosen.

Example 2

Factorize $x^2 + 9x + 20$

- Step 1: $(\quad)(\quad)$
- Step 2: $x^2 + 9x + 20$ the factors for x^2 must be $x \times x$
  $(x \quad)(x \quad)$
- Step 3: $x^2 + 9x + 20$ the factors for 20 are 1 × 10, 2 × 10, 4 × 5

  Let's try 4 × 5

  $(x + 4)(x + 5)$ Could these expand to $x^2 + 9x + 20$?
- Step 4: $x^2 + 9x + 20$ now we check the x terms

  $(x + 4)(x + 5)$ we multiply to see if we get $+9x$

$(x + 4)(x + 5)$

$4x$

$5x$

$4x + 5x = 9x$ ✓

We got it first time!

Example 3

Factorize $x^2 + 2x - 3$

- Step 1: $(\quad)(\quad)$
- Step 2: $x^2 + 2x - 3$ the factors for x^2 must be $x \times x$
  $(x \quad)(x \quad)$
- Step 3: $x^2 + 2x - 3$ the factors for –3 are (–3 × 1) or (–1 × 3)

  Remember (– × +) = –

  Let's try –3 × 1

  $(x - 3)(x + 1)$ Could these expand to $x^2 + 2x - 3$?
- Step 4: $x^2 + 2x - 3$ now we check the x terms

  $(x - 3)(x + 1)$ we multiply to see if we get $+2x$

$(x - 3)(x + 1)$

$-2x$

$1x$

$-3x + 1x = -2x$

we need to find $+2x$

So ✗ not correct.

This was the wrong choice of factors. No worries - try again!

- Step 3: $x^2 + 2x - 3$
  $(x + 3)(x - 1)$ try again with the only other pair of factors
- Step 4: $x^2 + 2x - 3$ test to see if we get +2

$(x + 3)(x - 1)$

$3x$

$-1x$

$3x - 1x = 2x$ ✓ These are the correct factors.

PRACTICE QUESTIONS

Factorize these quadratic expressions:

1	$x^2 + 4x + 3$	**6**	$x^2 + 8x + 12$
2	$x^2 + 9x + 8$	**7**	$x^2 + 2x + 1$
3	$x^2 + 10x + 25$	**8**	$x^2 + 7x + 10$
4	$x^2 + 14x + 49$	**9**	$x^2 + 14x + 13$
5	$x^2 + 7x + 12$		

SEE–THINK–WONDER

Did you identify any **patterns** to help you choose the correct factors? What is the purpose of patterns?

1	$x^2 + 18x + 32$	**6**	$x^2 + 29x + 100$
2	$x^2 + 10x + 9$	**7**	$x^2 + 19x + 78$
3	$x^2 + 22x + 121$	**8**	$x^2 + 19x + 18$
4	$x^2 + 20x + 100$	**9**	$x^2 + 11x + 18$
5	$x^2 + 25x + 100$		

1	$x^2 + 3x - 4$	**6**	$x^2 + 5x - 66$
2	$x^2 + 3x - 10$	**7**	$x^2 - 5x - 66$
3	$x^2 + x - 6$	**8**	$x^2 + 25x + 66$
4	$x^2 - x - 6$	**9**	$x^2 + 30x - 1800$
5	$x^2 + 6x - 40$		

HOW CAN SQUARES BE PERFECT OR DIFFERENT?

On the previous page, patterns were mentioned as a method to help you factorize quadratic expressions. What do the following terms have in common?

4, 16, 25, 100, 400, x^2, y^2, b^2…?

These terms are all examples of square numbers. There is a special case for factorizing when the expression contains the difference of two square numbers.

Example 1

square number

$x^2 - 9$

square number difference (subtraction)

But how to factorize?

$(x \quad)(x \quad)$

as x^2 can be factorized

and

$(x \quad 3)(x \quad 3) \longrightarrow$ we use the square roots of 9

And we recall that + × − = −

positive × negative = negative

So $(x + 3)(x - 3)$

or $(x - 3)(x + 3)$

Remember the order is unimportant as it will be the same once expanded.

Example 2

$e^2 - 144$

$(e \quad)(e \quad)$ … e^2

$(e \quad 12)(e \quad 12)$ … 144

$(e - 12)(e + 12)$ in any order

Remember that quadratics aren't limited to x^2, other letters are also possible. It's the x^2 which makes it a quadratic – the same is true for c^2 or t^2, and so on.

Example 3

$4x^2 - 169$

$(2x \quad)(2x \quad)$ … $4x^2$

$(2x \quad 13)(2x \quad 13)$ both squares

$(2x + 13)(2x - 13)$

We can express this as a general rule, using algebraic notation.

$a^2 - b^2 = (a + b)(a - b)$

Can you **explain** the pattern in words?

PRACTICE QUESTIONS

Factorize these expressions:

1 $x^2 - 4$	**2** $x^2 - 25$	**3** $x^2 - 100$
4 $a^2 - 36$	**5** $b^2 - 121$	**6** $c^2 - 1$
7 $9x^2 - 144$	**8** $x^2 - 81$	**9** $100d^2 - 16$

The next set includes an unfamiliar element. See if you can work out what you need to do, based on the information given so far.

1 $64b^2 - c^2$	**2** $36x^2 - 49y^2$	**3** $x^2y^2 - 4$

ACTIVITY: Perfect square trinomials

ATL

- Affective skills: Mindfulness – Practise focus and concentration

First, **expand** $(x + 1)^2$

The squared symbol means we multiply it by itself

So $(x + 1)^2 = (x + 1)(x + 1)$

by any method $= x^2 + 1x + 1x + 1$

$= x^2 + 2x + 1$

For $(x + 1)^2 = x^2 + \underline{2}x + \underline{1}$

Now **expand** $(x + 2)^2 =$

and $(x + 3)^2 =$

and $(x + 4)^2 =$

Look carefully at the coefficients and the constant terms. What do you notice?

Predict what the expansion of $(x + 5)^2$ will be:

x^2 + _____ x + _____

What makes you say that?

Now check your prediction by expanding $(x + 5)^2$. Were you correct?

Predict the expansion of $(x + 6)^2$ and test your prediction by expanding

Can you **describe** this pattern as a general rule?

Verify it for a number greater than 6.

What problems did you encounter in this activity? Was the pattern obvious or did it take a while to emerge? Did you seek support from your teacher or others? What have you learned about patience in finding patterns?

Assessment opportunities

- This activity can be assessed with Criterion B: Investigating patterns.

DISCUSS

When communication is too good!

Several years ago, staff at Plymouth University in the UK hung up these posters in exam halls to discourage students from cheating in examinations.

Search online using the terms **university**, **cheating** and **poster**. Click on 'images' to find the poster.

The problem was that the mathematics was valid and correct and some students noticed this. Some principled students notified the staff, while others didn't. Eventually the posters were removed. This was a good lesson for the University about the importance of understanding what's going on in the background.

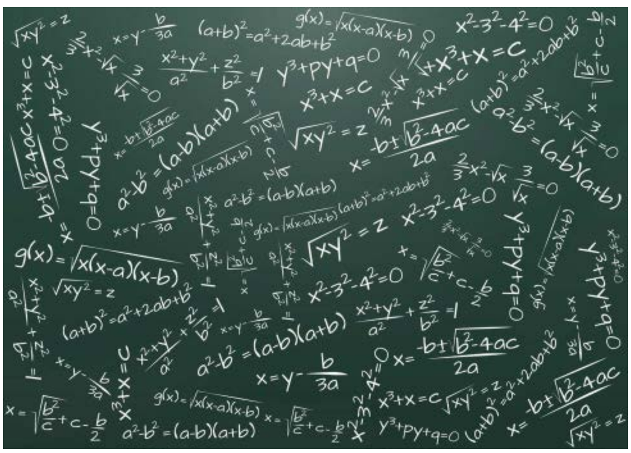
How could critical-thinking and a better knowledge of mathematics have helped the situation? What advice would you give to Plymouth University for the future?

ACTIVITY: Backdrop

ATL

- Critical-thinking skills: Interpret data, and Creative-thinking skills: Apply existing knowledge to generate new ideas, products or processes

You are working on a video shoot in a school and one scene is taking place in a classroom, showing a teacher and her class. The background needs to be convincing and your team has found the image below on the internet. They would like to put it on the whiteboard to make it realistic.



Analyse the image and **determine** whether it would be a good display for the board. Give your team clear and detailed feedback on their suggestion.

Design your own calculation(s) that could be displayed on the board which would communicate more effectively and authentically than this one. Personalize it and see if you can 'hide' a message or some numbers that mean something to you in there. https://youtu.be/98unLjZRc_8

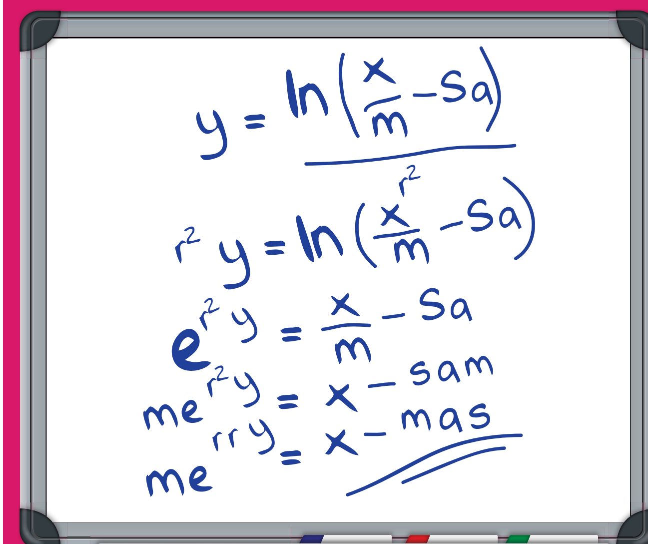
Read this interesting article: www.wsj.com/articles/the-big-bang-theory-has-hidden-jokes-down-to-a-science-1443808102 to see how the TV programme The Big Bang Theory hides mathematics jokes in its episodes.

Assessment opportunities

- This activity can be assessed with Criterion C: Communicating.

Take action

- You are likely to be coming up to a break from school now. Perhaps it is nearly winter break or a religious festival such as Christmas or Hanukah. Why not devise a seasonal message using algebra like these below? You could create a card, a flipbook or post as a photo montage.



- X-mas algebra

- Christmas algebra message

MEET A MATHEMATICIAN: YOU!

Learner Profile: Thinker; Communicator

Throughout this chapter you have been working hard on acquiring and developing the skills to become fluent in algebra. You are well on your way to becoming an excellent communicator of mathematical ideas. As communicators, we express ourselves confidently and creatively in more than one way in algebra and can often be understood by those who may only speak a different language but can 'read' mathematics.

Mathematics is often considered a universal language and this work will provide you with the knowledge, skills and attitudes necessary to pursue further studies in mathematics as well as to be a life-long critical and logical thinker.

SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1-2

1 **Find** $3a + 3$ when $a = 5$

2 **Solve**

a $b + 7 = 19$

b $2c + 1 = 5$

c $x^2 = 100$

3 **Explain** the term 'expand'.

4 **Expand**

a $4(c - 2)$ b $5(-d + 2)$

5 **Expand**

a $(x + 8)(x + 9)$ b $(2x + 3)(x + 1)$

6 Complete the following equation:

$x^2 + 4x + 3 = (x + 1)(x + \_\_\_)$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3-4

7 **Find** $3a + b$ when $a = 5$, $b = 7$

8 **Solve**

a $2d - 1 = 11$ b $e + 3 = 2.5$

9 **Expand**

a $3a(-b - 2)$ b $-3(-u - 2)$

10 **Expand**

a $(3x + 5)(5x + 3)$ b $(x + 11)(2x - 1)$

11 **Explain** the term 'factorize'.

12 **Factorize** $x^2 + 8x + 15$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5-6

13 **Find** $3a - 3b + 1$ when $a = 5$, $b = -5$

14 **Solve**

a $\dfrac{4f + 2}{3} = 9$ c $3j^2 = 27$

b $\dfrac{y}{3} + 1 = 3$ d $\sqrt{h} + 2 = 5$

15 **Expand**

a $b(v + 2u)$ b $-e(s + 2g)$

16 **Expand**

a $(9x - 8)(3x - 7)$ b $(8x - 4)^2$

17 **Explain** what is meant by factorizing a quadratic.

18 **Factorize** $2x^2 + 5x - 12$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7-8

19 **Find** $(3a + 3)^3$ when $a = -2$

20 **Solve**

a $4x + 7 = 2x - 1$

b $5k + 8 = \frac{2k - 4}{4}$

c $2^{x+1} = 32$

d $\frac{10}{x} = 4$

e $\frac{2}{x} = \frac{4}{x + 3}$

21 **Expand**

a $-3b(b + c + cd + 2)$

b $3bc^4(b + c + d - 2)$

22 **Expand**

a $(2x + 10)^2$

b $3(x - 7)^2$

c $-2(4 - x)^2$

23 Is factorizing a quadratic expression **different** to factorizing the difference of two squares? **Explain** your answer.

24 **Simplify** and **factorize** $2x^2 + 20x - 42 - 10(x - 3)$

Extended challenging question

25 **Factorize** these expressions and simplify.

a $\frac{7x}{14x - 21}$

b $\frac{x + 3}{x^2 + 7x + 12}$

To find out more about hidden maths jokes, have a look at these links:

http://onenerdlaughing.com/3

www.telegraph.co.uk/culture/books/11118343/The-Simpsons-One-big-numbers-game.html

Check out these entrepreneurs:

https://j2kun.svbtle.com/authenticity-of-background-math

and

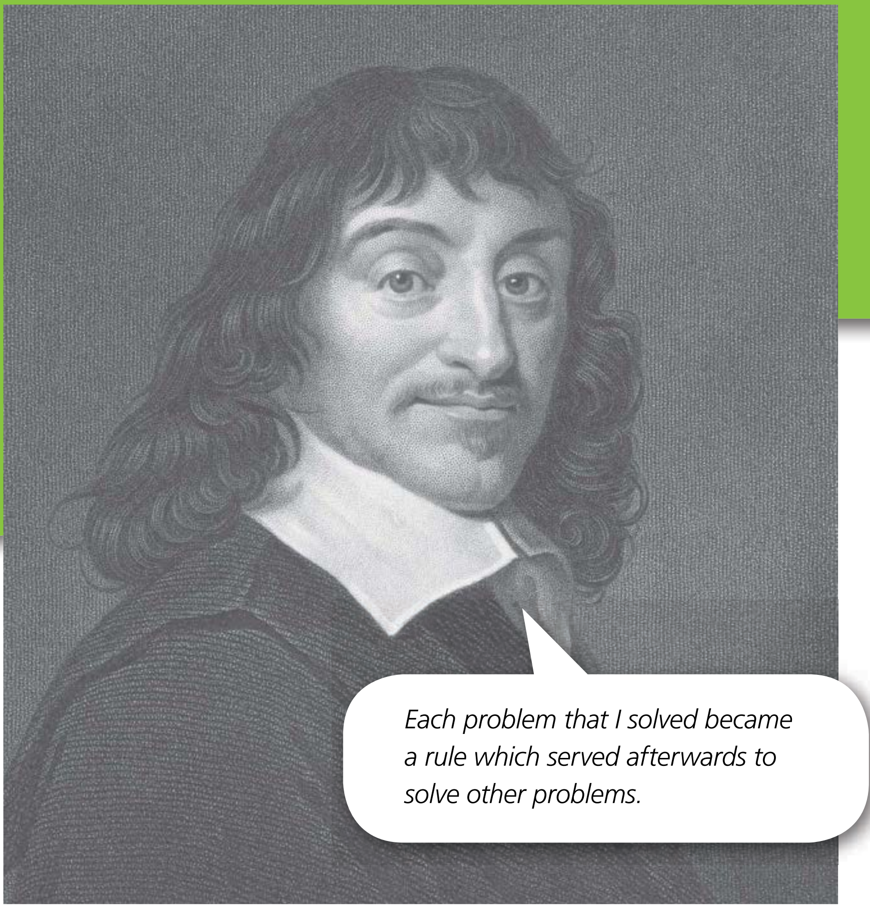
www.updownandnatural.com/2012/04/babys-bum-is-a-genius-wordless-wednesday.html

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: How do we factorize expressions? How can equations be solved? What is meant by an unknown?					
Conceptual: How does simplification allow us to find an unknown? How can relationships be expressed algebraically?					
Debatable: What does simplification cost in terms of accuracy? Can we say one method is better than another?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Communication skills					
Organization skills					
Affective skills					
Creative-thinking skills					
Critical-thinking skills					
Learner profile attribute	How did you demonstrate your skills as a thinker in this chapter?				
Thinker					

3 Can you walk the line?

Mathematical knowledge is built through **logical structures**, developed over **time** and transferred to **equivalent** situations.



René Descartes

CONSIDER THESE QUESTIONS:

Factual: What do the terms gradient and intercept mean? What if either are equal to zero? How do we plot lines? How do we solve simultaneous equations?

Conceptual: How can equivalent forms help us in different ways? What information can equivalent forms of lines give us? Can logic help us apply algebra to real-life? Where are these lines in time and space? Where is the invisible algebra around us? Are there other paths to find the same information? What else can we learn from linear equations? How can technology help us find the same information quicker?

Debatable: Does the best form change depending on when and where you are at the time? How do lines change over time?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

René Descartes was one of the most important and influential mathematicians of all times. Born in 1596, he was a philosopher and a soldier, as well as a key contributor in the Scientific Revolution. Descartes created the standard coordinate system, called the Cartesian system or plane in his honour. Cartesian comes from (Des)Cartes. This allows algebraic equations to modelled as geometric shapes in space. He is often referred to as the father of analytical geometry which bridges algebra and geometry.

IN THIS CHAPTER, WE WILL …

- **Find out** how algebra can be expressed visually, in geometry, and vice versa.
- **Explore** everyday problems which can be represented as a linear equation.
- **Take action** by raising appreciation of, and celebrating the life and work of René Descartes.

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Transfer skills
- Critical-thinking skills

We will reflect on this learner profile attribute …

- Reflective – we thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

THINK–PAIR–SHARE

- **Read the Descartes quotation. In pairs, discuss what you think it means.**
- **Now search Descartes and find out about his life and works.**

KEY WORDS

gradient
intercept
intersection
quadrant
simultaneous

PRIOR KNOWLEDGE

You will already know:

- how to simplify, substitute into and solve equations
- how to plot points on an x–y grid (Cartesian plane)
- the difference between constants, coefficients and variables.

WHO SAID ANYTHING ABOUT TWO UNKNOWNS?

In Chapter 2, we simplified and solved algebraic equations. These equations mostly had one possible answer, for example $3 + x = 7$ clearly has one correct answer, $x = 4$.

Let's compare that to an equation with two unknowns:

$$x + y = 5$$

Can you think of any numbers that would satisfy (work for) for this situation?

If	$x = 2$, $y = 3$ then $x + y = 5$	this works
or	$x = 5$, $y = 0$ then $x + y = 5$	this works too
or	$x = 2.5$, $y = 2.5$ then $x + y = 5$	and this one …

There seems to be multiple solutions for this equation. Here are some more:

$x = 0$, $y = 5$ $\quad$ $x = 1$, $y = 4$ $\quad$ $x = 2$, $y = 3$

$x = 3$, $y = 2$ $\quad$ $x = 4$, $y = 1$ $\quad$ $x = 6$, $y = -1$

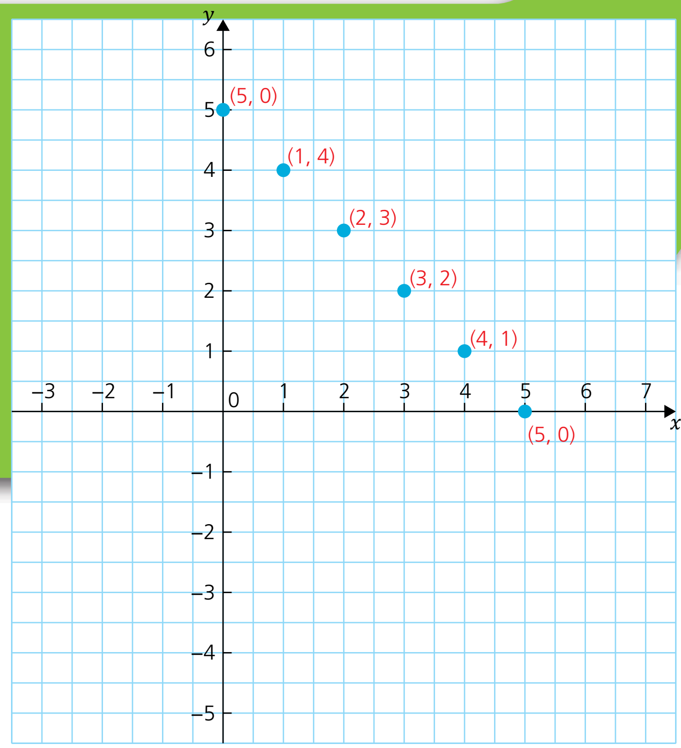
You can see from the last pair that negatives are also possible, as well as decimals (before). We know from studying geometry that we can plot these points on a coordinate plane because:

$x = 2$, $y = 3$ can also be written as (2, 3)

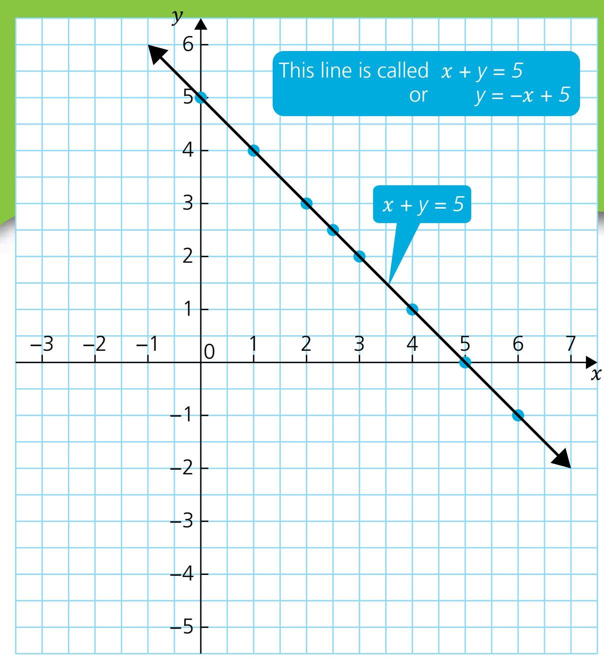
$x = 3$, $y = 2$ can be written and plotted as (3, 2), etc.

So let's plot the following pairs of values that we found already:

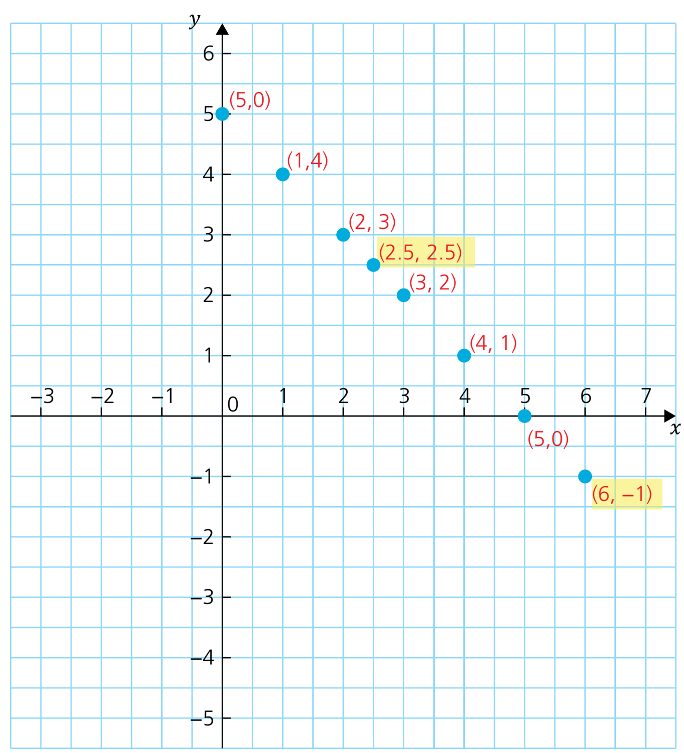
(0, 5) (1, 4) (2, 3) (3, 2) (4, 1) (5, 0) – we will look at the six points with natural numbers first.



■ Plotting the values



What do you notice about the points? What do you think will happen when we plot the other points (6, –1) and (2.5, 2.5)?



■ Plot further values

As you can see, the points form a line. All possible pairs of solutions for the equation $x + y = 5$ will be on the line too. We call equations like this **linear equations** for this reason.

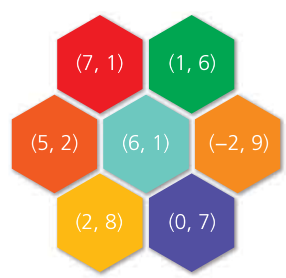
Equations have important properties and we can tell a lot from the shapes they form when plotted as graphs. They have certain important points that indicate a change or something significant. With linear equations, it can be useful to know if they are increasing or decreasing as well as how quickly and where they cross the axes.

ACTIVITY: Equations into lines

■ ATL

- Transfer skills: Combine knowledge, understanding and skills to create solutions

1 **For the equation $x + y = 7$, which points are *not* on the line?**



2 **Plot and label the following lines on graph paper. Remember to find and mark at least five points to draw the line accurately.**

a $x + y = 3$

b $x + y = 4$

c $x + y = 10$

d $x - y = 2$

e $x + y = -2$

◆ Assessment opportunities

◆ This activity can be assessed with Criterion A: Knowing and understanding, and Criterion C: Communicating.

What else can we learn from the linear equation?

Equations like $x - 2y = 6$ or $x + y = -7$ are written in the form:

$ax + by = c$ where a, b are called coefficients

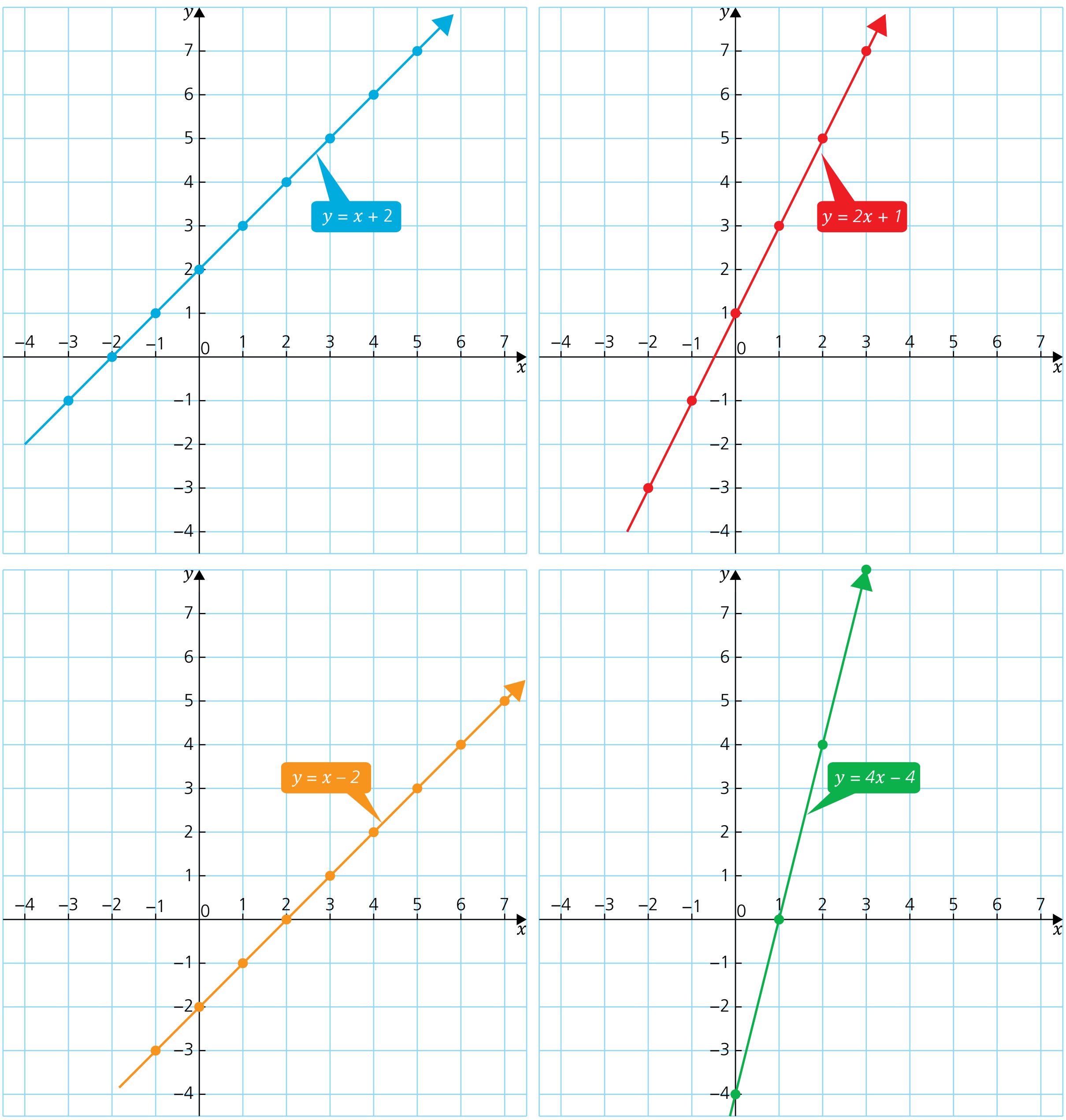
and c is a constant term, or number.

While this is elegant and correct, there is an equivalent form of the same equation which tells a lot more about the line and much more quickly.

This form is known as

$y = mx + c$ where m is a coefficient and c is a constant term, or number.

But what does this tell us? Let's look at some examples:



■ Linear equations

What did you notice about the *c* or constant value in relation to the graph? What did you notice about the value of *m*? This form of the equation tells you the following:

$$y = \underbrace{m}_{\text{gradient}}x + \underbrace{c}_{\substack{\text{intercept}\\ \text{(cuts the } y\text{-axis)}}}$$

and remember

$$y = mx + c$$

these are still the variables

So this form will instantly tell us two things:

1. how quickly the line is increasing (or decreasing if the m-value is a negative or minus number). This is called the **gradient**;
2. and where the line crosses the y-axis. This is called the **y-intercept**.

Remember that x and y are called variables and we represent these varying values on the axes of the same name (x-axis and y-axis). We also refer to them as the independent variable and the dependent variable, especially in experiments and sampling.

www.active-maths.co.uk/algebra/graphs/index.html

Note: many national systems use the notation $y = mx + b$ which is equivalent and also perfectly correct. However, to stay true to IB notation and to prepare you for Diploma Mathematics, we will use $y = mx + c$ only. In the same way, you will often see the word **slope** instead of gradient. We will use both terms at first but will gradually move to using gradient only.

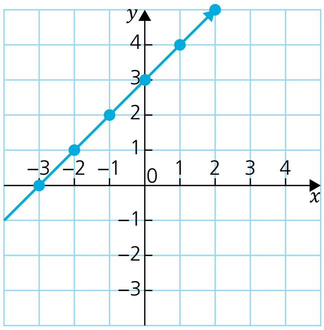
What does a gradient of 0 mean? What do we usually call this?

ACTIVITY: Linear equations

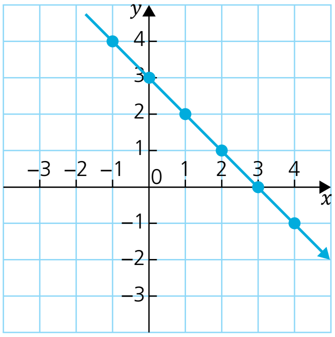
Match the equation to the correct image.

1 $y = 3x$
2 $y = x + 3$
3 $y = -x + 3$

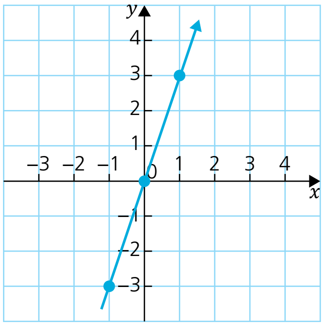
a



b



c



WHY DO WE REARRANGE? AND HOW?

To do this we will rely on rearranging skills from algebra. You will have to isolate the y-term (or variable) and make sure that the coefficient is 1 (not –1).

Example 1

$x + y = 5$

Isolate the y-term

$x + y = 5$

$y = 5 - x$

rearrange to put x-term first in expression

$y = -x + 5$

check there is no sign/coefficient in front of y ✓

Answer $y = -x + 5$

So this tells us that the line has a gradient of –1 and the line crosses the y-axis at (0, 5).

Example 2

$2x + y = 11$

$2x + y = 11$

$y = 11 - 2x$

$y = -2x + 11$

$m = -2$ the line is decreasing

$c = 11$ The y-intercept is at (0,11)

Example 3

$4x + 2y = -6$

$2y = -6 - 4x$

$2y = -4x - 6$

but this is not yet $y =$

$2y = -4x - 6 \div 2$

$$\frac{2y}{2} = \frac{-4x}{2} - \frac{6}{2}$$

$y = -2x - 3$

$m = -2$ $c = -3$

The line has a negative gradient, with a value of –2. The y-intercept is at (0, –3).

Example 4

$3x - y = 9$

$-y = -3x + 9$

but $-y \neq y$

$+y = +3x - 9$ Multiply or divide each term by –1

$y = 3x - 9$

$m = 3$ $c = -9$

ACTIVITY: Gradients

Copy and complete the following table:

$ax + by = c$	$y = mx + c$	Gradient	y-intercept		x-intercept	
$x + y = 9$	$y = -x + 9$	$m = -1$	$y = 9$	(0,9)	$x = 9$	(9,0)
$x + y = 3$						
$x + y = 8$						
$2x + y = 4$						
$3x + y = -2$						
$x - y = 6$						
$3x - 6y = 9$						

Example 5

$-x - 2y = 4$

$-2y = x + 4$

$\div -2$

$$\frac{-2y}{-2} = \frac{x}{-2} + \frac{4}{-2}$$

$$y = -\frac{x}{2} - 2$$

$m = -\frac{1}{2}$ $c = -2$

This form is also particularly useful for graphing the equation on a GDC or App. More on this on the next page where we will explore the idea of gradient further.

$y = mx + c$ form does not tell us the x-intercept, i.e. where the line crosses the x-axis but we can find this easily. If it crosses the x-axis, then the y-value must be equal to 0.

To find the x-intercept, substitute $y = 0$ and solve.

So, if $2x + 3y = 12$, then if $y = 0$

$2x + 3(0) = 12$

$2x = 12$

$x = 6$ the line crosses the axis at this point or the x-intercept is (6, 0)

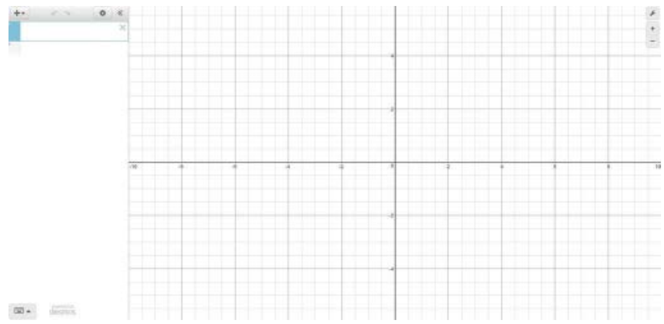
How can technology enable us to find the same info quicker?

Before technology made function plotting almost instantaneous, students spent a lot of time using the equation to generate specific values for y, finding points to plot and then connecting the points. This process involved choosing values for x, substituting them into the equation and gathering y-values as we practiced earlier. The x-values chosen are often the integers between –2 and 2 or –3 to 3. These values are around the **origin** (0, 0) and will show the y-intercept also.

These skills are still important but it is even more important to know what the equation tells you. The function isn't just a line connecting these certain **arbitrary values**. It is *every single possible* point on the line *and* in the original equation.

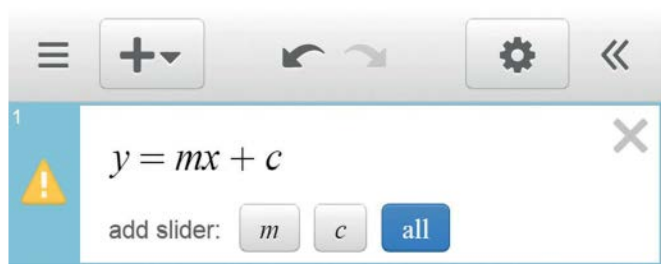
ACTIVITY: Desmos graphing (modelling)

Desmos is an online graphing tool. There are many similar online apps which can show you a function or equation. In this activity we will change the **parameters** of the equation and see what happens to the location of the line in Cartesian space.



■ Screenshot from Desmos

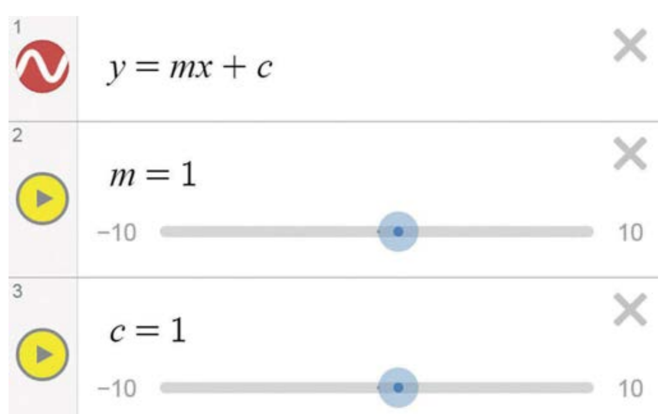
Type in the general form of a linear equation $y = mx + c$



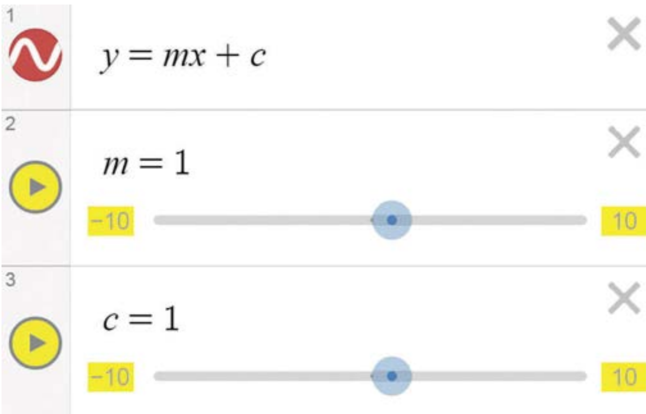
■ Type $y = mx + c$

Click all

By clicking this button, sliders will appear which allow you to change the values of both m and c of the line.



First press play on the gradient (*m*), then play the second slider, changing intercept (*c*). Then play them *simultaneously*. What happens to the line?



Change the values for the range - choose your own values. Recommendation: try values much larger than ±10 to maximize the effect.

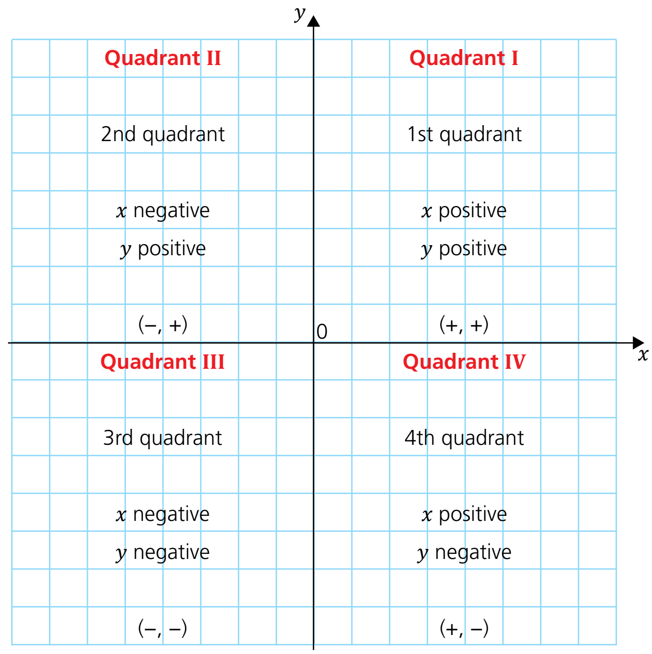
THINK–PUZZLE–EXPLORE

What is happening when you change these parameters? What happens to the lines? Use as many mathematical terms as you can to **explain** what is happening to the orientation of the line at different times of the animations.

Assessment opportunities

- This activity can be assessed with Criterion B: Investigating patterns.

Here are some more important things to note when you are working with graphs on the Cartesian plane:



- Note that equivalent names, including roman numerals, are used in many textbooks and online sources.

Where are these lines in time and space?

ACTIVITY: Cartesian plane

Draw a Cartesian plane and use it to answer the following questions:

Can you **find** or **draw** a line which passes through:

- **the origin (0, 0)?**
- **two quadrants only?**
- **three quadrants only?**
- **Quadrant I, III and IV but not II?**
- **all four quadrants?**

Reflection

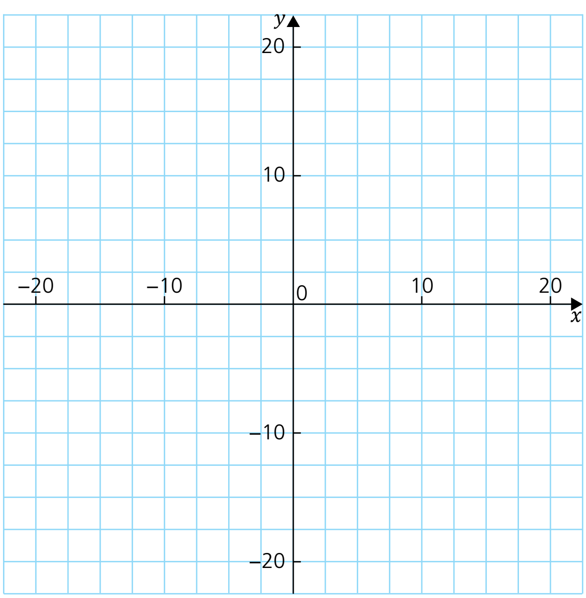
Can you draw any conclusions about lines and quadrants from this activity?

Assessment opportunities

- This activity can be assessed with Criterion A: Knowing and understanding, and Criterion C: Communicating.

ACTIVITY: Inquiry

You can use DESMOS, your GDC or you could draw the lines yourself on graphing paper to complete this activity. For each line described, find the line itself and draw it on the plane. Make sure to include the equation of the line by labelling it.



■ The Cartesian plane

Find a line which:

- **goes through the origin (0, 0)**
- **which has an intercept at (0, 2)**
- **which goes through (3, 3)**
- **has a gradient of 2**
- **has a gradient of –3**
- **passes through (0, 4) and has a gradient of +1**
- **crosses through 3 quadrants**
- **has a gradient of -0.5**
- **has a y- and x-intercept at the same point (has the same value).**

Where do the lines end? How far do they **extend** and in which direction? **Discuss** with your partner and try to come to a final decision. Then share with the class to **compare** your conclusions. Make sure to use logic and mathematical words in your discussion.

ATL

- Media-literacy skills: Using technology.

Assessment opportunities

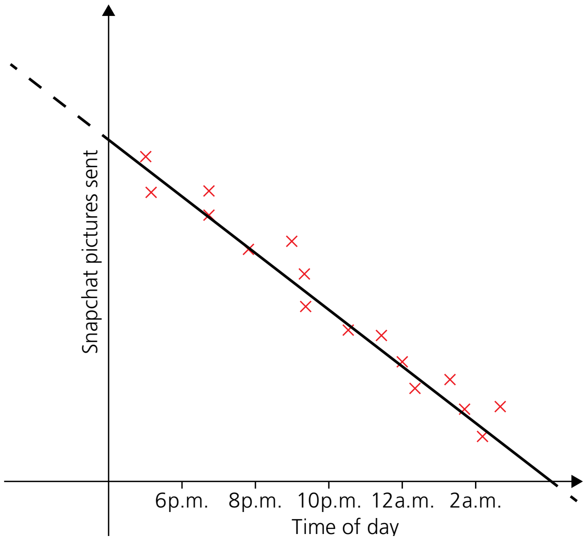
- This activity can be assessed with Criterion C: Communicating.

ACTIVITY: Plotted data or graphed equation?

ATL

- Critical-thinking skills: Use mathematical communications to explore systems and issues

Is this plotted data or a graphed equation?



■ What does this image show you?

The graph shows the relationship between the number of Snapchat pictures sent over a certain period of the day, i.e. from approximately 5p.m. to almost 3 a.m.

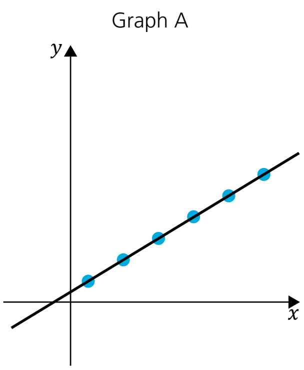
Gwyaneshwaran says this graph is a scatter graph and therefore it shows a line of best fit and not an equation. Is he correct?

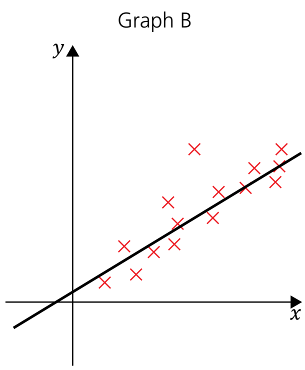
Justify your answer, using correct mathematical communication.

Assessment opportunities

- This activity can be assessed with Criterion D: Applying mathematics in real-life contexts.

What is the difference between graphs A and B?





■ Graphs A and B

The difference is between plotting graphs from data and plotting graphs of equations.

When you are plotting data, *all* you know is the measured points. These values are found by the readings during an experiment, an observation or a survey. We guess at what happens between these plots, then the very best we can get is a trend line called the **line of best fit**. We use this trend line to **extrapolate** and **interpolate** for other values.

But when we are plotting points from a formula or equation, the points are *exact*. They represent definite values on the line so all other points we interpolate and extrapolate are precise values. We know they are on the line, unlike measured data.

Links to: Physics

Have you explored and tried to measure the Earth's gravitational field strength in your Physics classes using a newton meter? In your analysis, you will find that you are drawing a 'best-fit' line when you plot the points on a graph. You can find more information in *Physics for the IB MYP 4&5: by Concept*, Chapter 2, pages 30–31.

Can logic help us apply algebra to real-life?

We can find equation calculations all around us, wherever a quantity can change and therefore give a different outcome. For example, a cell phone bill changes the more you use your phone. The price of an electronic device changes depending on the storage capacity. Your grades might change, the more you study.

In this section, we will look at how you identify what is going to change (the variable) and by how much. We also look for constants that will not change over time and whether these constants will be added, subtracted, etc. from the equation.

Example 1

Kasia-Li is organizing the class trip to see "The Man who Knew Infinity" at a local movie theatre. The tickets cost \$11.50 per student and a \$3 fee for buying online. How can she use this information to construct an equation to represent the total cost?

Booking tickets online often involves a booking fee. This is called a flat fee because it does not change how many tickets, or what kind of tickets, you buy. We already know that this is considered a mathematical **constant**.

Therefore, the variables which *can* change are:

- **a** the number of tickets we are going to buy – this we can decide or input (the independent variable)
- **b** the final price – this depends on the number of tickets (the dependent variable).

Hint

Look for the words 'per' or 'for every' in the problem. This will give you the coefficient and the independent variable almost immediately; for example, cost per hour, price per kilogram, value per square foot, percentage, and so on.

price <u>per</u> ticket

→ usually indicates a multiplier, this is the coefficient

11.50 per ticket + booking fee

↓

$11.50x + 3$

Where x = number of students

let y = final price

$\therefore\ y = 11.50x + 3$

Now this means that Kasia-Li has an equation that will tell her, by substitution, the total cost of the trip.

If she knows that 10 students are going,

$y = 11.50x + 3$ from our previous equation

If $x = 10$ $\quad y = 11.50(10) + 3$

$y = 115 + 3$

$y = 118$

The trip will cost \$118.

But if they decide to invite their tutor and class teacher, the x value changes

$y = 11.50x + 3$

Then $x = 12$ $\quad y = 11.50(12) + 3$

$y = 138 + 3$

$y = 141$

She mentions the trip to the school principal, who wonders how much it would cost to book out the whole theatre and take all 173 students in the year group (but no teachers!).

$y = 11.50x + 3$

Then $x = 173$ $\quad y = 11.50(173) + 3$

$y = 1989.50 + 3$

$y = 1992.50$

Example 2

A moving company estimates that it usually needs a dozen large boxes per room when they are packing the contents of a house. They **always** bring half a dozen extra boxes just in case they are wrong or there is an accident. How many boxes will the company bring to pack …

a an apartment with 4 rooms?
b a large house with 8 rooms?
c a sports star's mansion of 82 rooms?
d a hotel with 15 floors of 30 rooms each?

First we find the equation. Then we can substitute all the values above into the equation to get the total number of boxes needed.

a dozen boxes per room plus half a dozen extra

12 boxes per room 12r

plus 6 extras + 6

total boxes = t

$t = 12r + 6$

So $t = 12r + 6$

a if $r = 4$ in the apartment
$t = 12(4) + 6$
$t = 48 + 6$
$t = 54$ boxes

b if $r = 8$ in the large house
$t = 12(8) + 6$
$t = 96 + 6$
$t = 102$ boxes

c if $r = 82$ in the mansion
$t = 12(82) + 6$
$t = 984 + 6$
$t = 990$ boxes

d if $r = 15 \times 30$ for the hotel
$t = 12(450) + 6$
$t = 5400 + 6$
$t = 5406$ boxes

Example 3

Albie has a summer job mowing lawns. He earns ₱10 (10 pesos) for every garden lawn that he mows. Each day he works he must spend ₱14 on fuel and ₱6 to rent the mower from his parents. Let the number of garden lawns = g and the profit = p (profit is the amount of money earned minus any expenses)

Examples

a What is the equation representing the total profit earned by Albie at the end of each day?

Solution

profit = price per garden – fuel – rent

$\therefore p = 10g - 14 - 6$

simplify $p = 10g - 20$

b How much profit will Albie make from 10 gardens?

Solution

$g = 10 \quad p = 10g - 20$

$p = 10(10) - 20$

$p = 80$

So Albie earned ₱80 profit after mowing 10 gardens.

c How many gardens did he mow if his profit was ₱60?

Solution

This time we know the value for profit but not the number of gardens. We **substitute** 60 for *p* and rearrange to find *g*.

$p = 60 \quad p = 10g - 20$

$10g - 20 = 60$

$10g = 60 + 20$

$10g = 80$

$g = 8$

Albie mowed 8 gardens.

d How many gardens will Albie have to mow before he breaks even? (Breaking even is when the money earned is exactly equal to the expenses paid out.)

Solution

At break even point, profit =0

So $10g - 20 = 0$

$$10g = 20$$

$$g = \frac{20}{10}$$

$$g = 2$$

After 2 gardens, Albie has 'broken even'.

e Albie ends the day with ₱35 profit. How many gardens would this suggest he mowed that day? **Discuss** the accuracy of this answer.

Solution

p = 35 p = $10g - 20$

$$10g - 20 = 35$$

$$10g = 35 + 20$$

$$10g = 55$$

$$g = \frac{55}{10}$$

$$g = 5.5$$

This answer indicates that Albie moved 5.5 (or 5½) gardens.

Real-world interpretation on accuracy:

It's unlikely that he mowed half a lawn, or that someone paid him for half a job.

The answer makes us wonder if Albie made a mistake with his money.

Some equations do not have a constant or a multiplier. For example: Sylvie is three years older than Omar.

Let S = Sylvie's age and O = Omar's age.

Sylvie is older, so we must add the three years (a constant) to Omar's age.

The equation is $S = O + 3$ or $O = S - 3$

We do these simpler equations in our head without knowing it. When Sylvie is 11 then Omar must be 8. If Sylvie is turning 40, then we know that Omar is 37.

Likewise, we can say that if Omar is 82, then we know that Sylvie is 85. These types of worded problems involving age seem to be difficult at first because they are worded but they are in fact easy to solve. The hard part is identifying the right variables and constants to equate.

Some equations have several variables. A trip to the movies may include different adult *and* children's tickets. The moving company decides to bring large *and* small boxes. Let's look at a real-world example that includes several terms in the equation.

A famous soccer player has just signed a lucrative deal where he will earn a £15 000 bonus for every game his team wins, £20 000 bonus for each goal he scores personally, a weekly wage of £50 000 whether he plays or not and an end of season bonus of a month's wages. He must pay a £25 000 fine for every red card and £15 000 for any yellow card. What would his final annual earnings look like as an equation?

Identify any variables + constants

Give them a suitable letter

£15 000 per game ⟶ $15\,000g$

£20 000 per goal ⟶ $20\,000s$ (s for goal scored)

£50 000 per week ⟶ $50\,000w$

£25 000 per red card ⟶ $-25\,000r$

£15 000 per yellow card ⟶ $-15\,000y$

Bonus: 4 weeks wages = $4 \times 50\,000 = 200\,000$

Equation:

$$\text{Earnings} = 15\,000g + 20\,000s + 50\,000w - 25\,000r - 15\,000y + 200\,000$$

Take out common factor of 5000

$$E = 5000(3g + 4s + 10w - 5r - 3y + 40)$$

Links to: Art; Design

The photo below shows Alli Schmeltz's string art installation. How does she use mathematics in her art?

■ Alli Schmeltz's string art installation

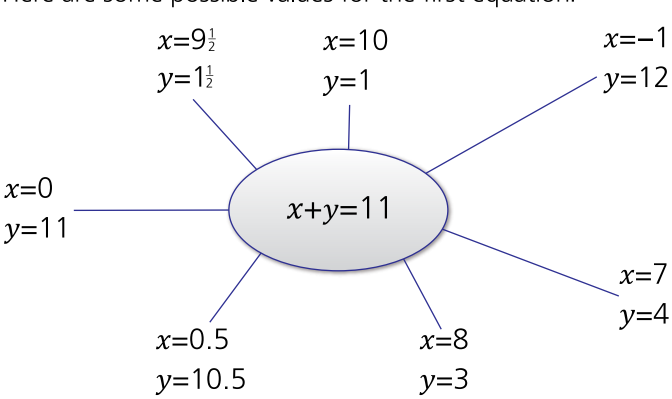
We are given two equations:

$x + y = 11$

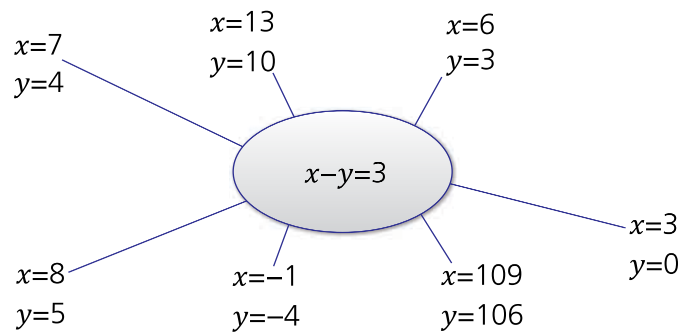
$x + y = 3$

But what does this mean?

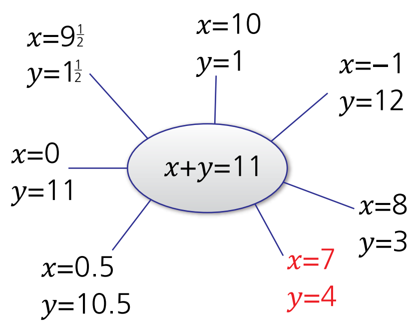
Here are some possible values for the first equation:

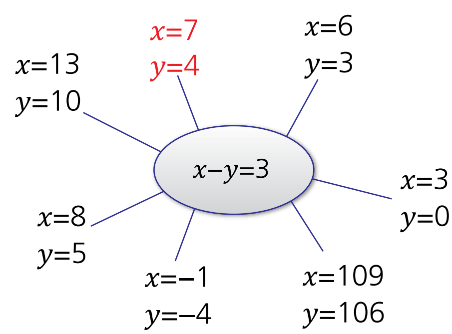


And here are some random ones for the second equation:



But look at what we notice when we compare them,





So there is one set of values ($x = 7$, $y = 4$) which works for both equations, a point which appears on both lines. Let's put these equations in a real-life situation. If you were told that Xavier and Yvonne's ages added to 11 but told nothing else, you could not logically say how old each one was.

But if you're also told that Xavier is three years older, i.e. that the difference in their ages is equal to 3, then there is only one possible solution. Xavier must be 7 and Yvonne must be 4.

When you have two equations, we can find a value for x and a value for y which works for both equations simultaneously (at the same time). These are called **simultaneous equations**.

How do we solve simultaneous equations?

If we have an equation with more than one unknown quantity, it is impossible to solve in isolation.

We have seen that if we want to know both values, we need another equation which contains both variables also. We then want to find a pair of values which satisfies both equations **simultaneously**.

By eliminating one of the variables, we can focus on the other. We need a logical method that will help us to solve for one value first and then use that value to find the second. We will use the idea of cancelling to help us, either by adding or subtracting.

Links to: Pop culture

'It's like you're always looking for a problem but there ain't enough equations for you to solve 'em' Lyrics from 'We are' – Justin Bieber, 2016

ACTIVITY: Find pairs of solutions

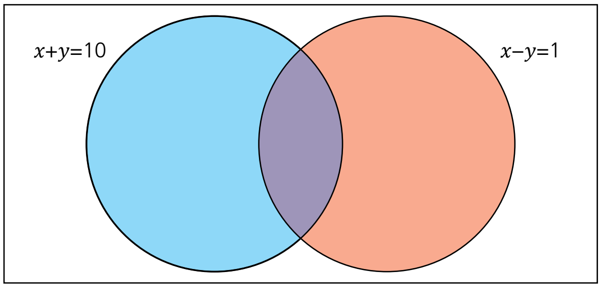
ATL

- Communication skills: Use and interpret a range of mathematical terms and forms to represent possible solutions

Copy the Venn diagram below and find possible pairs which satisfy each equation.

You can write them as $x =$, $y =$ or (x, y).

If you find values that work for both equations, put them in the intersection of the sets.



■ Venn diagram

Formative assessment opportunity: how many could you find?

I found:

At least four pairs of whole number solutions	Level 1–2
More than four pairs of whole number solutions with at least one fraction or decimal	Level 3–4
At least six pairs of whole number solutions, with at least one fraction and one decimal solution and at least one containing negative numbers	Level 5–6
More than six pairs of whole number solutions, including zero values (intercepts) several fraction and decimal solutions, several negative solutions and some more complex larger numbers	Level 7–8

Assessment opportunities

- This activity can be assessed with Criterion A: Knowing and understanding.

Procedure pro

Example 1

Solve

$x + y = 14$

$x - y = 2$

Step 1: **Identify** which terms are easier to cancel in their current form.
If we add –1 to +1, they cancel each other out.
If we add $-y$ to $+y$, they also cancel.

Step 2: Cancel them by adding (or subtracting) the equations. One set should cancel, add what remains.

$$\begin{array}{r} x + \cancel{y} = 14 \\ + \quad + \quad\quad + \\ x \cancel{-y} = 2 \\ \hline 2x = 16 \end{array}$$

Step 3: **Solve** for the remaining term

$2x = 16$

$x = \frac{16}{2}$

$x = 8$

Step 4: Now we know $x = 8$, we can substitute it into either equation to find $y =$

$x + y = 14$

$8 + y = 14$

$y = 14 - 8$

$y = 6$

So the answer is $x = 8$, $y = 6$

Want to know if your answers are correct? Why not substitute the final answers into one of the equations? What should happen? Will it happen whichever equation you choose?

Procedure pro

Hint

It doesn't matter which equation you choose to substitute into. Make your life easier (and minimize the chance for mistake) – pick the equation with the lower values or whichever looks easiest.

Example 2

Solve

$x + y = 12$

$2x - y = 9$

Step 1: We can cancel the y terms by adding

Step 2:
$$\begin{array}{r} x + \cancel{y} = 12 \\ + \quad + \quad\quad + \\ 2x \cancel{-y} = 9 \\ \hline 3x = 21 \end{array}$$

$3x = 21$

Step 3: $x = \frac{21}{3}$ **solve** ⟶ divide by 3

$x = 7$

Step 4: $x + y = 12$ simpler equation

$7 + y = 12$

$7 = 12 - y$ **solve**

$y = 5$

Answer $x = 7$, $y = 5$

PRACTICE QUESTIONS

a $x + y = 5$
$x - y = 1$

b $x + y = 6$
$-x + y = 4$

c $x + y = 11$
$-x + y = 7$

d $x + y = 10$
$x - y = 4$

e $x + y = 13$
$-x + y = 1$

f $x + y = 8$
$x - y = 0$

PRACTICE QUESTIONS

a $2x + y = 11$
$3x - y = 4$

b $x + y = 4$
$6x - y = 3$

c $-x + 4y = -5$
$x + 2y = 11$

d $2x + 4y = 30$
$10x - 5y = 25$

e $2x + 5y = 4$
$-2x + y = -4$

f $3x + 4y = 24$
$3x + y = 15$

Example 3

Solve

$2x + y = 3$

$-2x + 3y = 5$

This time it looks easiest to cancel $+2x$ and $-2x$ by adding the equations. The y-terms can't cancel as their coefficients are not the same, i.e. y and $3y$.

So
$$\begin{array}{r} \cancel{2x} + y = 3 \\ + \quad + \quad\;\; + \\ -\cancel{2x} + 3y = 5 \\ \hline 4y = 8 \end{array}$$

$$y = \frac{8}{4}$$

$$y = 2$$

$$2x + y = 3$$

$$2x + 2 = 3$$

$$2x = 3 - 2$$

$$2x = 1$$

$$x = 0.5$$

$$x = 0.5,\ y = 2$$

Don't worry if you get fractions or decimals as an answer, these are also possible and valid.

Example 4

Solve

$x + y = 19$

$2x + y = 26$

The y terms here are the easiest to cancel but as they are both positive, we can cancel them by subtracting the equations.

$$\begin{array}{r} x + y = 19 \\ - \quad - \quad\;\; - \\ 2x + y = 26 \\ \hline -x = -7 \end{array}$$

So $x = \frac{-7}{-1}$ divide by -1

$$x = 7$$

$$x + y = 19$$

$$7 + y = 19$$

$$y = 19 - 7$$

$$y = 12$$

$$x = 7,\ y = 12$$

Don't worry if you get zero as an answer. This is a valid answer and indicates that the solution just happens to be on an axis.

WHAT DO WE DO IF THE EQUATIONS GET MORE COMPLEX?

Example 5

Solve

$$3x + 7y = 23$$

$$3x + y = 29$$

$$\begin{aligned} 3x + 7y &= 23 \\ - \quad - \quad &\quad - \\ 3x + y &= 29 \end{aligned}$$

$$6y = -6$$

$$y = -1$$

$$3x + y = 29$$

$$3x + (-1) = 29$$

$$3x - 1 = 29$$

$$3x = 29 + 1$$

$$3x = 30$$

$$x = \frac{30}{3}$$

$$x = 10$$

$$x = 10 \quad y = -1$$

Example 6

Solve

$$2x + 6y = -52$$

$$5x + 6y = -40$$

$$\begin{aligned} 2x + 6y &= -52 \\ - \quad - \quad &\quad - \\ 5x + 6y &= -40 \end{aligned}$$

$$-3x = -52 - -40$$

$$-3x = -52 + 40$$

$$-3x = -12$$

$$x = \frac{-12}{-3}$$

$$x = 4$$

$$2x + 6y = -52$$

$$2(4) + 6y = -52$$

$$8 + 6y = -52$$

$$6y = -52 - 8$$

$$6y = \frac{-60}{6}$$

$$y = -10$$

$$x = 4 \quad y = -10$$

Negative numbers can be solutions too! Pay special attention to signs to make sure you cancel properly and get the right answers.

PRACTICE QUESTIONS

a $x + 2y = -1$
$2x - 2y = 10$

b $3x + y = 8$
$-3x + 10y = 47$

c $10x + 2y = 9$
$3x + 2y = 5.5$

d $2x + y = 65$
$4x - y = 85$

e $x + 4y = 45$
$2x + 4y = 48$

f $x + 5y = -11$
$4x + 5y = -14$

! Take action

! Sometimes students struggle with so many different methods and find it hard to choose one to master. Why not help someone out by tutoring or peer teaching them to find the method that best suits them until they are ready to tackle multiple methods?

COMMUNICATION CASE STUDY

A student called Padraic used to be able to identify the values for x and y mentally. He would look at the equations and try pairs of numbers quickly in his head until he found the values. He was really proud of this method and found it frustrating when he was asked to communicate this in writing.

Padraic has stumbled upon a mathematical process without knowing it which can be described as 'Guess, Test and Refine'. The problem came when the solutions were not pairs of natural numbers. If he was asked to solve for values which were negative or decimals, he could no longer 'see' them.

DISCUSS

What would you advise Padraic?

SIMULTANEOUS EQUATIONS

What if it's not possible to cancel either term? Let's say we are trying to solve this set of simultaneous equations:

$$2x + y = 14$$

$$3x - 2y = 7$$

The challenge is that none of the terms can be cancelled by adding or subtracting as they are currently because of the coefficients:

$$2x + y = 14$$

$$3x - 2y = 7$$

We can make it possible to cancel by 'scaling up' one or both equations. This means that we multiply every term in the equation by a number that would help us cancelling one set of the terms.

In this case

$2x + y = 14$ we could multiply $\times 2$

$3x - 2y = 7$ to cancel with this

$2x + y = 14$ $\quad \times 2$

$3x - 2y = 7$

$4x + 2y = 28$ … every term × 2

$3x - 2y = 7$ … unchanged

Cancel by adding, now that y-terms can

$$\begin{array}{r} 4x + \cancel{2y} = 28 \\ + \quad + \qquad + \\ 3x - \cancel{2y} = 7 \\ \hline 7x = 35 \end{array}$$

$$x = \frac{35}{7}$$

$$x = 5$$

$$2x + y = 14$$

$$2(5) + y = 14$$

$$y = 14 - 10$$

$$y = 4$$

Example 2

Solve

$$6x + y = 22$$
$$x + 3y = -2$$

We could multiply the first equation × 3 → cancel y's

or second equation × 6 → cancel x's

$$\begin{aligned} 6x + y &= 22 \\ x + 3y &= -2 \quad \times 6 \end{aligned}$$

$$\begin{aligned} 6x + y &= 22 \\ 6x + 18y &= -12 \end{aligned}$$

Cancel by subtracting

$$\begin{aligned} \cancel{6x} + y &= 22 \\ - \quad - \quad & \quad - \\ \cancel{6x} + 18y &= -12 \end{aligned}$$

$$-17y = 34$$
$$y = \frac{34}{-17}$$
$$y = -2$$

$$x + 3y = -2$$
$$x + 3(-2) = -2$$
$$x - 6 = -2$$
$$x = -2 + 6$$
$$x = 4$$

Final example

You've done a lot of questions on these pages so here's one more for good luck! What happens if we need to scale both equations to get them to cancel?

No problem ☺ – scale them both …

$$4x + 3y = 22$$
$$3x - 2y = 25$$

Should we cancel x or y?

Let's go for y's, as it looks a little easier

$$\left.\begin{aligned} 4x + 3y &= 22 \quad \times 2 \\ 3x - 2y &= 25 \quad \times 3 \end{aligned}\right\} \text{to get to } 6y$$

multiply every term

$$\begin{aligned} 8x + \cancel{6y} &= 44 \\ + \quad + \quad & \quad + \\ 9x - \cancel{6y} &= 75 \end{aligned}$$

$$17x = 119$$
$$x = \frac{119}{17}$$
$$x = 7$$

$$4x + 3y = 22$$
$$4(7) + 3y = 22$$
$$28 + 3y = 22$$
$$3y = -6$$
$$y = -2$$

PRACTICE QUESTIONS

Caution! These questions get challenging very quickly. If you want to build up your skills gradually, ask your teacher for more practice questions which slowly get harder. Take control of the pace of your own learning and don't be afraid to ask for help.

a $4x + 3y = 10$
$x - 2y = -3$

b $6x + y = 66$
$2x + 2y = 22$

c $9x + y = 26.5$
$15x + 2y = 45.5$

d $x + 5y = -7$
$5x - y = -9$

e $100x + 10y = 2310$
$20x + y = 451$

f $4x + y = 35$
$x + 10y = -391$

Are there other paths to find the same information?

In the previous method you might have noticed that the equations were in the form $ax + by = c$

What other form have we learned for the same type of equation?

If you rearrange the equations to represent both of them in $y = mx + c$, you could make them equal to one another. This is logical to do because we know the y-value must be the same for both.

Solve

$$-x + y = 4$$
$$x + y = 6$$

$-x + y = 4$	$x + y = 6$
$y = x + 4$	$y = -x + 6$
$y = mx + c$	$y = mx + c$

So make $y = y$

$$x + 4 = -x + 6$$

Solve for x $\quad x + x + 4 = 6$

$$2x + 4 = 6$$
$$2x = 6 - 4$$
$$2x = 2$$
$$x = 1$$

Substitute into either $y =$

$$y = x + 4$$
$$y = 1 + 4$$
$$y = 5$$

Example 2

$$x + y = 5$$
$$x - y = 1$$

$x + y = 5$	$x - y = 1$
$y = -x + 5$	$-y = -x + 1$
	$y = x - 1$

$$-x + 5 = x - 1$$
$$-x = x - 1 - 5$$
$$-x = x - 6$$
$$-x - x = -6$$
$$-2x = -6$$
$$x = \frac{-6}{-2}$$
$$x = 3$$
$$y = x - 1$$
$$y = 3 - 1$$
$$y = 2$$

This method can be useful if the equations are already in $y = mx + c$ form. It also works well for learners who like to make use of the fact that at that point in space, all x and y values are identical for each equation.

PRACTICE QUESTIONS

Solve by equating:

a. $-x + y = 1 \quad x + y = 13$ **b.** $x + y = 8 \quad 3x + y = 18$ **c.** $x + y = 10 \quad x - y = 4$

What do you think of this method? Do the different paths give you different information? Are they equivalent?

HOW CAN WE USE TECHNOLOGY TO FIND THE SAME INFORMATION?

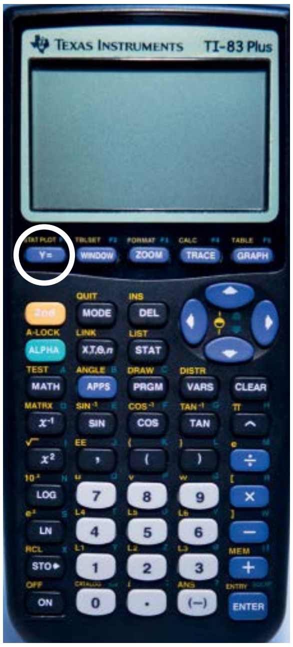
You may have started using GDCs (Graphing Display Calculators) in class and/or using online graphing apps to draw lines quickly. On pages 56–57 you carried out an activity looking at the gradient and intercepts for lines on Desmos. You will be allowed to use a GDC in Diploma Examinations so getting used to them early would be a good idea, if your school has a set. Two of the most popular GDC manufacturers are Casio and Texas Instruments (TI).

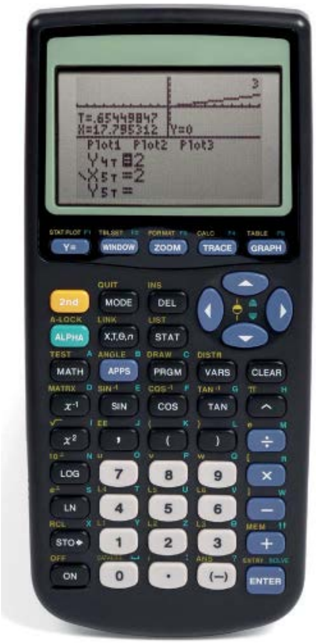
Online apps are quick, simple to use and you can cut and paste the graphs and tables into your written work easily. You will not have access to them in your exams though.

You may need to re-arrange the equation to $y = mx + c$ form before you input the equation.

On a TI-calculator, look for the $y=$ button on the top left hand side.

On most Casio GDC models, you will find the graphing functions on the main menu.





We can find solutions by graphing the equations and drawing the appropriate graph on the GDC or app.

Let's try an example we had before:

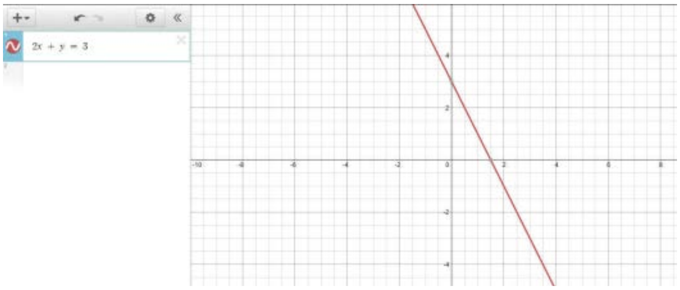
$$2x + y = 3$$

$$-2x + 3y = 5$$

If we go to Desmos (desmos.com/calculator), we can input the equation in either $ax + by = c$ or $y = mx + c$ form and it will graph the lines instantly.

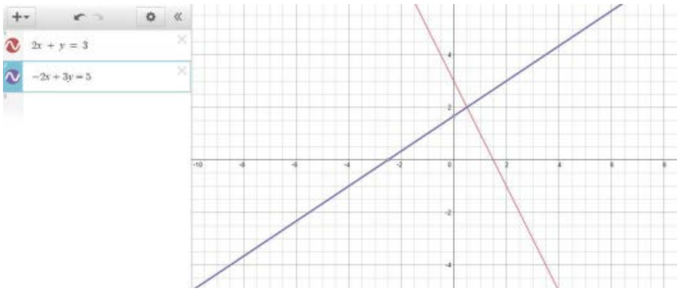
The first equation

$$2x + y = 3$$



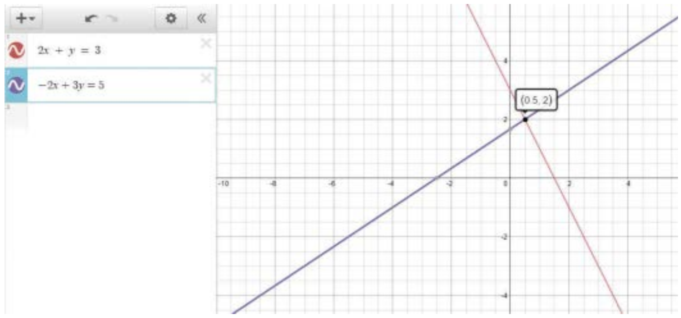
Add an item and click on expression. Enter the second equation

$$-2x + 3y = 5$$



And now we can see both equations and where they cross. At that one point in both lines the values are simultaneously identical. You can either read off the values by using the axes or click on the point of intersection to reveal the values.

Technology allows us to find these values quickly and accurately but you would be able to do this yourself easily, if a bit more slowly. To graph a line manually, substitute a range of values for x and find y. Plot these on a graph and connect the points.



■ The intersection

PRACTICE QUESTIONS

Using your most preferred or available method, **draw** a graph of the following equations and **solve** for x and y (the point of intersection):

a $2x + y = 11$
$3x - y = 4$

b $x + y = 4$
$6x - y = 3$

c $-x + 4y = -5$
$x + 2y = 11$

d $2x + 4y = 30$
$10x - 5y = 25$

e $2x + 5y = 4$
$-2x + y = -4$

f $3x + 4y = 24$
$3x + y = 15$

Could you always see the point of intersection? In which direction does it appear to be coming closer together or **converging**?

If you can't see them because the lines do not intersect in the current screen, enlarge your 'window' or zoom out.

SEE–THINK–WONDER

Is it possible that two lines might never intersect? Investigate what property of straight lines would make this case impossible.

ACTIVITY: Cheat sheets

ATL

- Communication skills: Make effective summary notes for studying; Organize and depict information logically

Cheat sheets are not intended to actually be used to cheat, they are simply an expression used for study notes. **Summarize** the four methods we have covered to solve a set of simultaneous equations:

- **Solving by cancelling terms**
- **Solving by equating expressions**
- **Graphing using technology**
- **Graphing manually (by hand).**

Include an example to represent the method.

Assessment opportunities

- This activity can be assessed with Criterion C: Communicating.

CIRCLE OF VIEWPOINTS

Consider the viewpoints of a learner who prefers one method over another and why. **Draw** a circle and split it into three equal segments. In each segment, **write down** the arguments that you think that person would say to defend their choice. When completed, shade in the segment or segments which reflect your own actual opinion.

The segments should be titled:

- **Solving algebraically by adding, subtracting and cancelling**
- **Graphing the equations as lines**
- **Rearranging the equations to make them equal.**

MEET A MATHEMATICIAN: RENÉ DESCARTES (1596–1650)

Learner profile: Reflective

In addition to his work in mathematics, although Descartes would say it was in support of his work in mathematics, he was a hugely important philosopher. He chose to question everything, including authority, previous experts and his own senses. His most famous phrase came from his philosophical work , *Discourse on Method*: 'I think, therefore I am'. He is thought of as the first thinker to prioritize reason as a means to develop knowledge of the universe.

Descartes reflected and refined his thoughts throughout his whole lifetime. He questioned, revisited and continued to seek for the universal truths that bind mathematics, sciences, philosophy and everything. His work was so important in so many fields that the town of his birth was renamed in his honour. Imagine a whole town naming itself after you?!

www.softschools.com/facts/scientists/descartes_facts/825/

! Take action

- ! Celebrate Descartes Day!
- ! Many schools around the world celebrate 'I heart Mathematics' day on February 14.

- ! Why not have **Descartes Day** as the theme for your next day?
- ! Activities could include:
  - A Descartes display board with some of Descartes most inspiring and famous quotations.

    "I think; therefore, I am"

    "It is not enough to have a good mind; the main thing is to use it well"

    and many more….
  - An Instagram account of images inspired by Descartes philosophy and mathematics, and using the hashtag #descartesday.
  - A proposal to the school as to why it would be a good and useful idea to create a 10 × 10 Cartesian grid in the playground.
  - Challenge the Individuals and societies teachers and students to a mathematics vs philosophy debate. Motions might be on whether mathematical knowledge is the highest form of knowledge or whether ALL knowledge is ultimately mathematics.

SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion C: Communicating and Criterion D: Applying mathematics in real-life contexts.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION C AND CRITERION D TO LEVEL 1–2

1 Scenario. Daniel is a pilot who flies large jumbo jets on long-haul flights from Tokyo to Dubai only. The distance per flight is approximately 8000 kilometres. To qualify for long-haul flights, he had to complete 100 000 kilometers of flight training before beginning the route.

Cliona is a small aircraft pilot who flies the workers from Iceland to oil rigs off the Faroe Islands, a journey of 400 km. These journeys can be dangerous and she completed over 400 000 kilometers of flight training before beginning this route.

- a What are the variables and which are the constants in this real-life problem?
- b How many kilometres has Daniel flown after 10 long-haul flights?
- c How many kilometres has Cliona flown after 10 short-haul flights?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION C AND CRITERION D TO LEVEL 3–4

2 **Find** an equation that describes the total kilometres flown by Daniel. Let x = number of flights and y = total kilometres flown.

3 **Find** an equation which describes the total kilometres flown by Cliona. Let x = number of flights and y = total kilometres flown.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION C AND CRITERION D TO LEVEL 5–6

4 **Draw** both equations on a graph.

5 What are the y-intercepts for each pilot? What do they tell you?

6 How accurate do you think the numbers in this situation are? Why?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION C AND CRITERION D TO LEVEL 7-8

7 At what number of flights will they have flown the same number of kilometres?

8 How correct do you think these answers are, based on what you know about flying and/or pilots? Why?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1-2

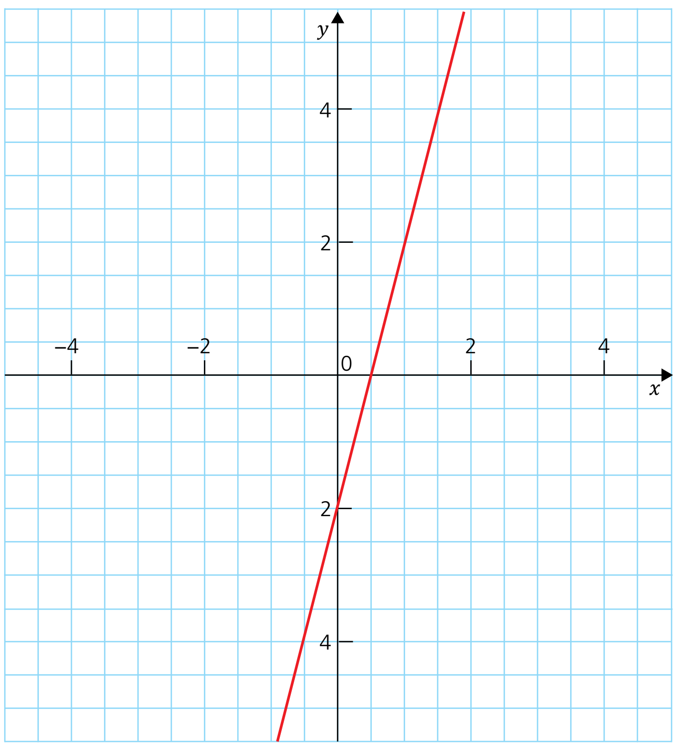
8 **Plot** the line $y = 3x + 1$.

  a What is the gradient of this line?

  b Where is the y-intercept?

  c Where is the x-intercept?

9 From graph A, **find** the y-intercept.



■ Graph A

10 If Bertha is seven years older than Cleo, what ages could they be?

11 A repairman charges \$50 an hour plus a call-out fee of \$40. Which is the constant term?

12 **Solve**

$x + y = 9$

$x - y = 2$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

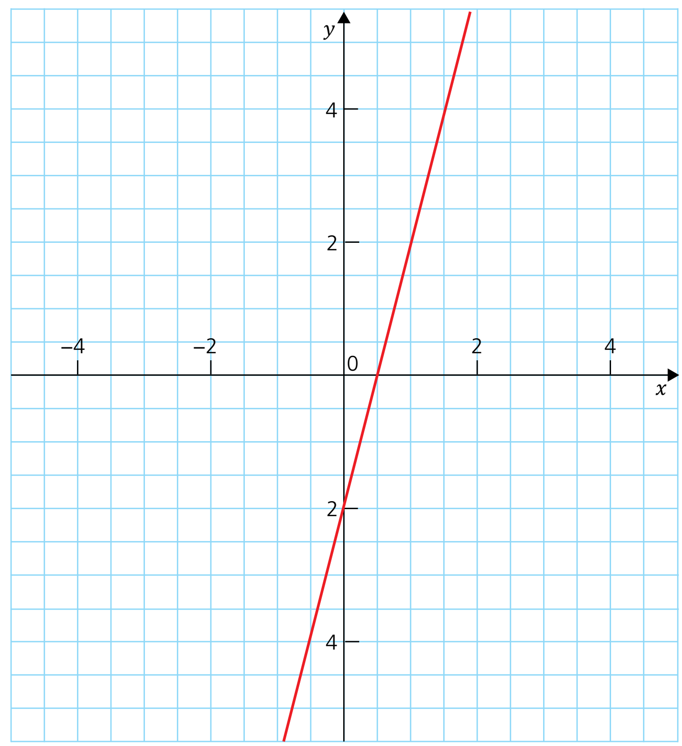
13 **Plot** the line $3x + y = 4$.

  a What is the gradient of this line?

  b Where is the y-intercept?

  c Where is the x-intercept?

14 From graph B, **find** the x-intercept.



■ Graph B

15 If Bertha is seven years older than Cleo, what letters would you assign as the variables?

16 A repairman charges \$50 an hour plus a call-out fee of \$40. Express this as an equation.

17 **Solve**

$3x + y = 6$

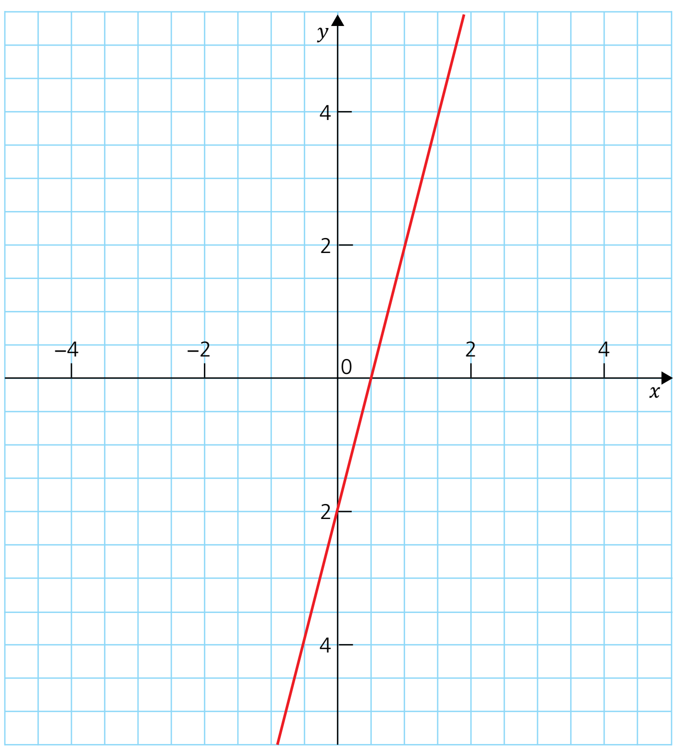
$x - y = 2$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

18 **Plot** the line $2x + y = 4$.

- **a** What is the gradient of this line?
- **b** Where is the y-intercept?
- **c** Where is the x-intercept?

19 From graph C, **find** the gradient of the line.



■ Graph C

20 If Bertha is seven years older than Cleo, what would the equation to represent this look like?

21 A repairman charges $50 an hour plus a call-out fee of $40. How much will a five-hour long job cost in total?

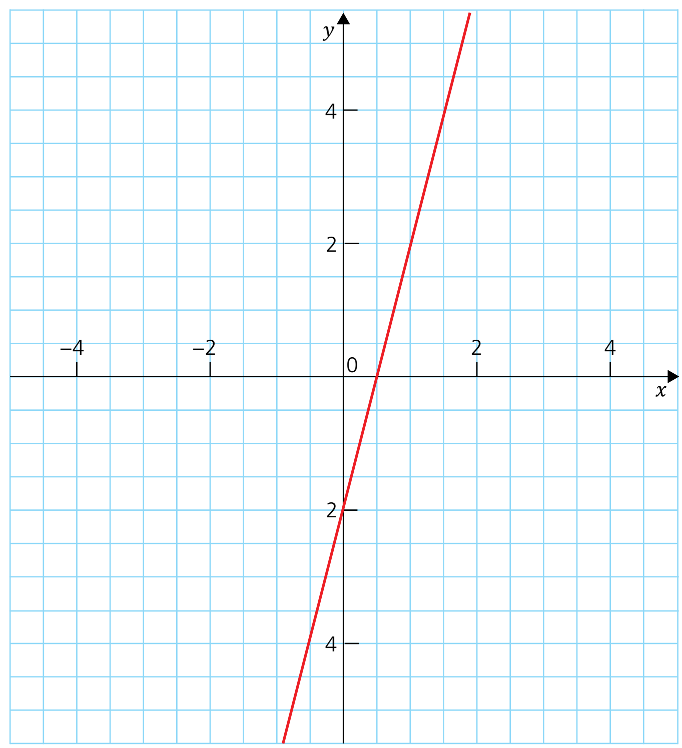
22 **Solve**

$3x + 2y = 14$

$x + 5y = 9$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

23 From graph D, **find** the equation of the line.



■ Graph D

24 Check if the point (3, 6) is on the line above.

25 If Bertha is seven years older than Cleo, use the equation of their relative ages to show that Bertha can't be 15 on Cleo's ninth birthday.

26 A repairman charges $50 an hour plus a call-out fee of $40. When he gets to your house he finds he does not have the right parts and has to come back another day. How much will your bill be for that day? **Justify** your answer by showing your working.

27 **Solve**

$3x + 4y = -1$

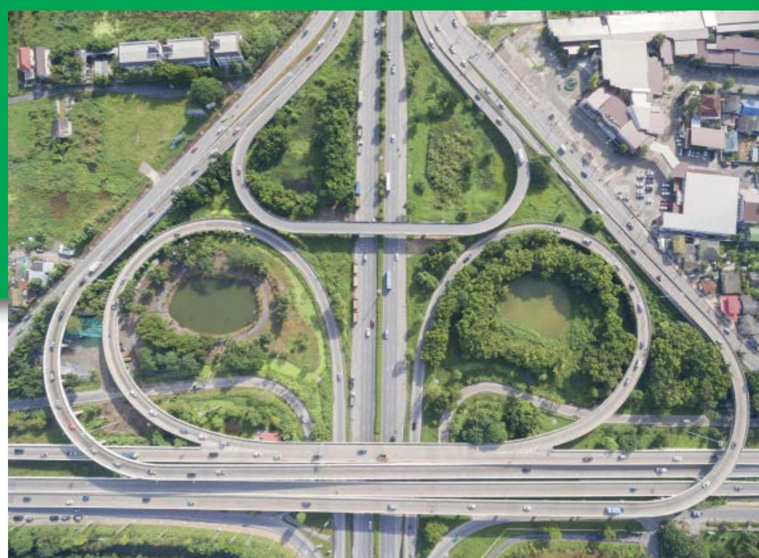
$7x + 10y = -2.$

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: What do the terms gradient and intercept mean? What if either are equal to zero? How do we plot lines? How do we solve simultaneous equations?					
Conceptual: How can equivalent forms help us in different ways? What information can equivalent forms of lines give us? Can logic help us apply algebra to real-life? Where are these lines in time and space? Where is the invisible algebra around us? Are there other paths to find the same information? What else can we learn from linear equations? How can technology help us find the same information quicker?					
Debatable: Does the best form change depending on when and where you are at the time? How do lines change over time?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Communication skills					
Transfer skills					
Critical-thinking skills					
Learner profile attribute	How did you demonstrate your reflection skills in this chapter?				
Reflective					

4 How is technical innovation changing our ideas of public and private space?

Modelling allows us to solve new spatial **relationship** problems arising from **technical innovation**.



Drone photography

CONSIDER THESE QUESTIONS:

Factual: How can we calculate unknown angles and sides? How do trigonometric relationships work?

Conceptual: Where do geometric shapes occur around us? How do their relationships give us insight into the unknown? How do I find a missing angle or side? What can we measure? What can't we measure? Can we calculate what we can't measure?

Debatable: Can anyone 'own' an angle? How much should models affect our understanding of the real world?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER WE WILL …

- **Find out** about drones and what they have to do with trigonometry.
- **Explore** Pythagoras' theorem, geometric shapes and trigonometric relationships.
- **Take action** by debating about the viewing angles and heights arising from drone use.

These Approaches to Learning (ATL) skills will be useful …

- Information literacy skills
- Critical-thinking skills
- Collaboration skills
- Creative-thinking skills
- Transfer skills
- Communication skills

We will reflect on this learner profile attribute …

- Principled – we act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying Mathematics in real-life contexts

PRIOR KNOWLEDGE

You will already know:

- that angles in a triangle sum to 180°
- that Pythagoras' theorem states that the square of the hypotenuse equals the sum of the squares of the other two sides
- how to use Pythagoras' theorem to find missing/unknown sides of right-angled triangles
- how similar triangles relate to one another
- what a Pythagorean triple is, such as 3, 4, 5 or 5, 12, 13.

KEY WORDS

adjacent	inverse
depression	opposite
elevation	ratio

THINK–PUZZLE–EXPLORE

- **What do you think you know about this topic?**
- **What questions do you have?**
- **How can you explore this topic?**

WHAT IS A DRONE?

Drones, or UAVs (unmanned aerial vehicles) are aircrafts without a human pilot on-board. These aerial vehicles are often expensive but come in a variety of sizes and are becoming more commonly used and owned by individuals.

Drones can be mounted with cameras or other devices and are often used by individuals, governments and organizations for a variety of uses and applications. They are traditionally used by the military for dangerous missions, but have become increasingly used for policing, surveillance, scientific investigation and aerial filming.

You might have seen popular videos, items in the news or viral stories as drones become more and more commonplace. The area swept out, covered by or visible to, a drone depends on its height from the earth and the viewing angle of the camera.

https://youtu.be/mxWd5OuoiRQ

DISCUSS

ATL

- Critical-thinking skills: Evaluate evidence and arguments

Now that drones can survey and record areas that previously were hard to see, what does this mean for our privacy? For our safety? For our creativity? For our discovery?

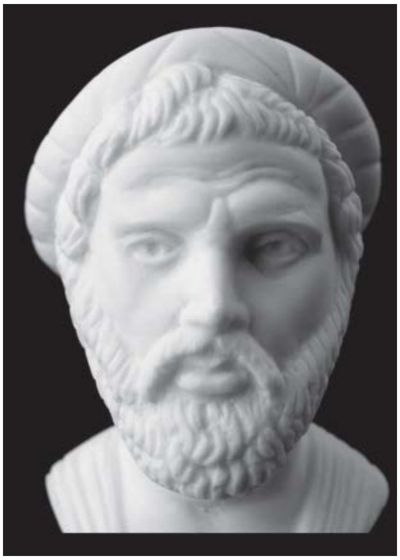
How can we calculate unknown angles and sides?

Geometric shapes are all around us. Throughout time, the relationships shown by these shapes have been used and applied to calculate quantities such as the area of a field, the distance between stars and to navigate the seas. Rules and relationships have been discovered to allow people to calculate what they can't measure.

Triangles in particular have been used throughout the ages as their relationships can help us to model distances and angles to find unknowns. As humankind has moved from the surface of the earth into the skies and the heavens, scientific innovation has brought new opportunities and challenges.

WHO AND WHAT IS PYTHAGORAS?

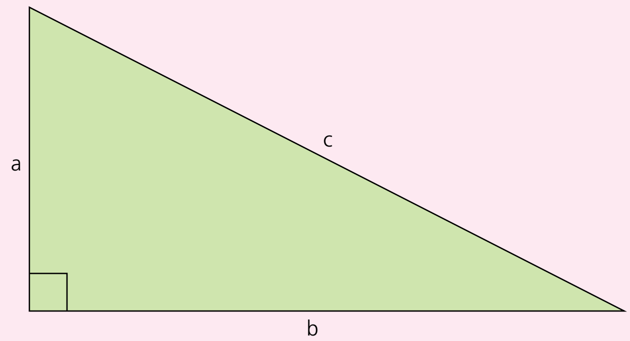
Pythagoras was an ancient Greek philosopher and mathematician who is estimated to have lived between 570 and approximately 495 BCE. The exact details of Pythagoras' life and works have been lost in the mists of time but he is mostly commonly remembered for **Pythagoras' theorem**, which is perhaps one of the oldest and most elegant relationships in geometry.



What is Pythagoras' theorem?

Remember that:

A right-angled triangle contains a right angle. The longest side opposite the right angle is called the **hypotenuse**.



We have looked at Pythagoras in Chapter 5 of *Mathematics for the IB MYP 3: by Concept*. Before we look inside a right-angled triangle to find relationships, we must revise the relationship governing the sides. You will have already learned that the square of the hypotenuse is equal to the sum of the square of the other two sides, or:

$$a^2 = b^2 + c^2 \text{ or } a^2 + b^2 = c^2 \text{ or hyp}^2 = a^2 + b^2$$

For the purposes of consistency, we will use the version of $a^2 + b^2 = c^2$ where a, b are the shorter sides and c is the hypotenuse. It is important to remember that all forms are valid and all will be accepted in assessment and examinations, if the communication is clear, consistent and correct.

Hint

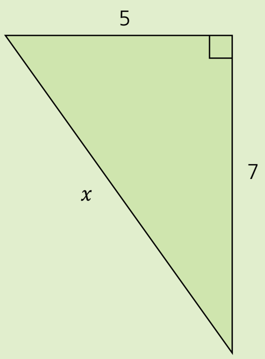
Sometimes it might be faster to write the equation as $c^2 = a^2 + b^2$ if you know you are looking for the hypotenuse.

Example

Problem 1

Calculate the value of x.

Give your answer correct to 1 d.p.



■ Triangle A

Solution

First you should **state** the theorem:

$c^2 = a^2 + b^2$

Substitute values:

$x^2 = 5^2 + 7^2$

$x^2 = 25 + 49$

$x^2 = 74$

To find x, **inverse** or undo the square:

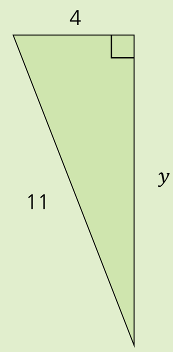
$x = \sqrt{74}$

$x = 8.602325267$ from the calculator

$x = 8.6$ to 1 decimal place

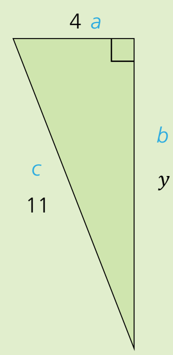
Problem 2

Calculate the length of side y.



■ Triangle B

Solution



■ Triangle B with sides labelled

Pythagoras: $c^2 = a^2 + b^2$

BUT we are looking for a shorter side

so $11^2 = 4^2 + y^2$

so $y^2 = 11^2 - 4^2$

$y^2 = 121 - 16$

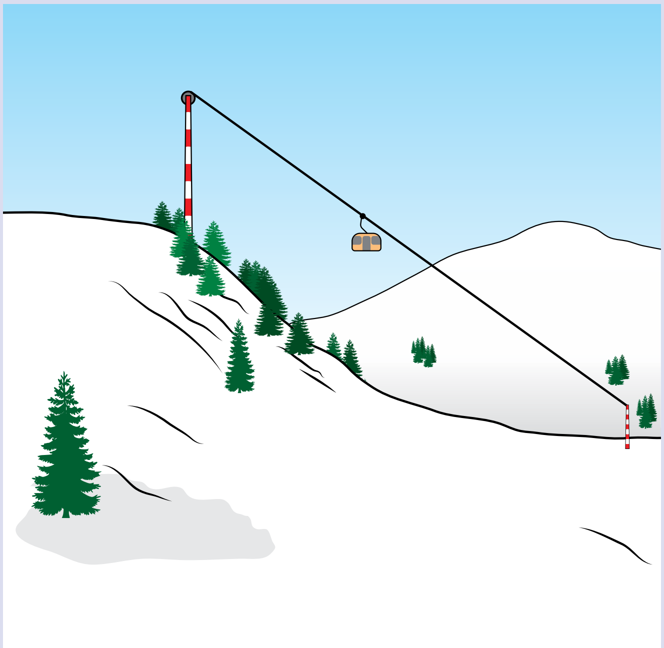
$y^2 = 105$

$y = \sqrt{105}$

$y = 10.24695 \ldots$

$y \approx 10.2$

Example

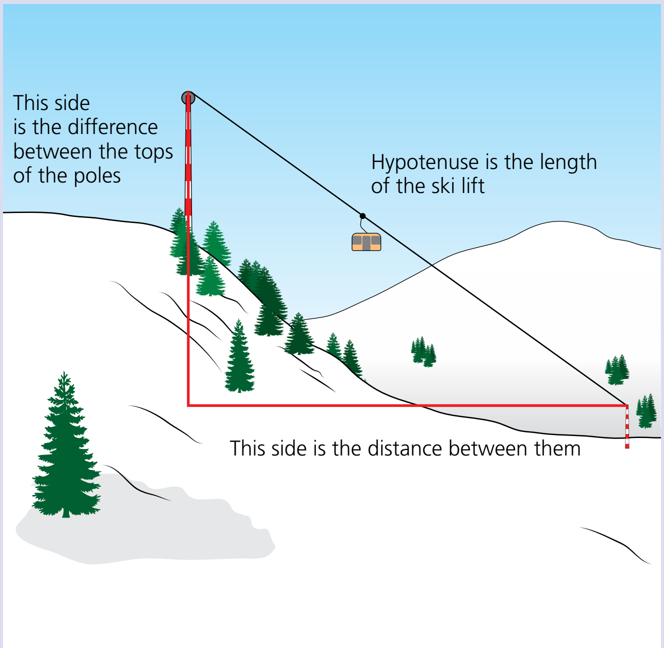


The picture above shows a ski lift with a large and a small pole exactly 370 metres away from each other, as the crow flies (see below). If the top of the small pole is 250 metres directly below the top of the large pole, how long it the ski lift?

▼ Links to: English; Language and literature; Geography

As the crow flies, is an idiom for the shortest distance between two points. Find more **idioms used to describe distance**.

Solution



Worded problems can be made easier by taking out all the unnecessary visual information and finding the triangle.

From the diagram above we can see that

$$c^2 = a^2 + b^2$$

Or $(\text{ski lift})^2 = (\text{height difference})^2 + (\text{distance})^2$

$$(\text{ski lift})^2 = 250^2 + 370^2$$

$$(\text{ski lift})^2 = 62500 + 136900$$

$$= 199400$$

$$\therefore \text{ski lift} = \sqrt{199400}$$

$$= 446.5422712\,\text{m}$$

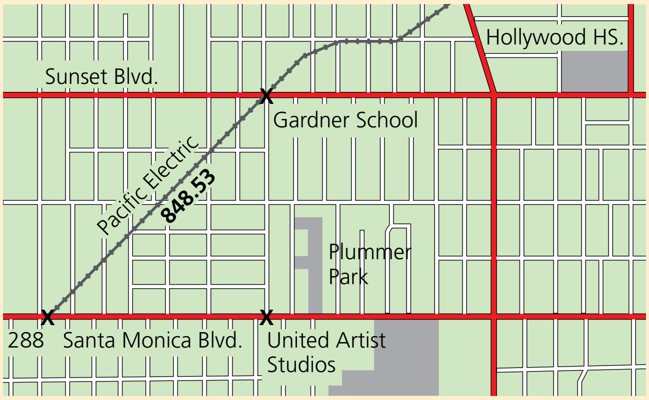
As no level of accuracy was mentioned in the question, an *appropriate* answer would be either 447 m (3 s.f.) or 450 m (same s.f. as the question values).

Example

How far is the United Artist Studios from 288 Santa Monica Boulevard if it is the same distance from the Gardner School?

Hint

Use the Pacific Electric railway line information.



■ Pacific Electric railway line map

Solution

The Gardner School, the Studios and 288 Santa Monica Boulevard form a right-angled triangle.

We know the two shorter sides are identical as the question tells us they are **equidistant**. This means that we can name each of the sides by the *same* letter, *a*.

So instead of *a* and *b*, Pythagoras' theorem will now look like

$a^2 + a^2 = c^2$

$a^2 + a^2 = (848.53)^2$ as the Pacific Electric railway is the hypotenuse value

$a^2 + a^2 = 720003.16$

$2a^2 = 720003.16$ by collecting like terms

$a^2 = 360001.58$ by halving

$a = \sqrt{360001.58}$ by square rooting

$a = 600.00\,\text{m}$

ACTIVITY: What has a drone got to do with right-angles?

ATL

- Critical-thinking skills: Gather and organize relevant information to formulate an argument

If we consider drones surveying the land from above, where could we see right-angled triangles occurring? How could Pythagoras' theorem help us to calculate important unknowns?

■ A drone surveying the land from above

Hint

Think of the area covered by a drone's camera.

If we change direction and think about the drones being used to survey upright objects, such as buildings, homes or geographical features, can we find right-angled triangles in these situations also? Consider how this makes you feel.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

How do trigonometric relationships work?

WHAT ARE THE TRIGONOMETRIC RATIOS?

Trigonometry, which comes from the Greek meaning 'triangle measurement' uses the property of similarity to find unknown sides or angles. The fact that side lengths of similar triangles are always in the same ratio has allowed mathematicians to name these ratios and devise uses for them.

These three ratios: **sine**, **cosine** and **tangent** each pair two of the three sides of a right-angles triangle, relative to an angle. The ratios are commonly abbreviated or shortened to sin, cos and tan. See if you can find them on your calculator.

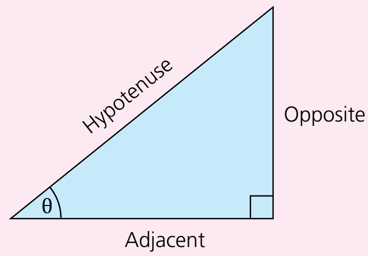
The other two sides of a right-angled triangle are labelled as either opposite or adjacent (touching) depending on/relative to the angle in question.

Links to: Design

Ask your Design teacher about 'innovating the adjacent possible'. What does it mean? How does it relate to the mathematical meaning of adjacent?

Links to: Language and literature; History

Research where the terms **sine**, **cosine** and **tangent** come from. **Investigate** the **history of trigonometry**.



Labelled triangle A

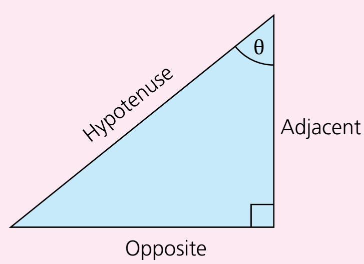
For a right-angled triangle, the sine, cosine and tangent of an angle θ are defined as

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Of course, if the angle moves to a different location, the opposite and adjacent will also move, relative to the angle. The hypotenuse will always be the longest side, opposite to the right angle.



Labelled triangle B

These ratios will always have the same value for any particular angle, no matter what the size of the triangle. This is due to the nature of **similar triangles**.

Until relatively recently, many countries did not allow calculators in examinations and students were required to look up the trigonometric ratios in a book of tables (log book).

As the use of technology to support our mathematics is encouraged in IB, how do we use a scientific calculator or graphical display calculator to find a sin, cos or tan?

Using a scientific or graphical display calculator

All scientific calculators and apps will have buttons for sin, cos and tan.

■ sin, cos and tan buttons

With some calculators, you press the button first and input the value of the angle second. For others, type the angle value first, then press the sin, cos or tan button.

Test yours now with sin 30° to see which order your calculator prefers. Whichever order gives an answer of 0.5 is the correct one.

Hint

Remember to check mode! If your calculator is not in degree mode (but in radian mode), then your answers will be incorrect for these questions.

ACTIVITY: Inquiry practice

Find the following values on your calculator:

sin 10° =	cos 10° =	tan 10° =
sin 20° =	cos 20° =	tan 20° =
sin 30° =	cos 30° =	tan 30° =
sin 40° =	cos 40° =	tan 40° =
sin 50° =	cos 50° =	tan 50° =

Do you notice any patterns? Can you extend this pattern?

◆ Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion B: Investigating patterns.

ACTIVITY: How can angles be manipulated?

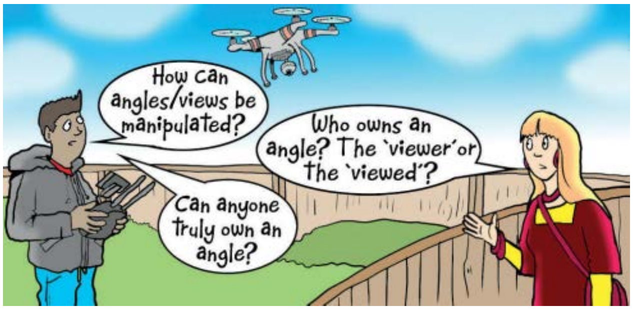
ATL

- Collaboration skills: Listen actively to other perspectives and ideas
- Critical-thinking skills: Gather and organize relevant information to formulate an argument

Two of your friends Robin and Sumaya live next door to one another. Robin has recently purchased a small camera drone and is excited to be using it in his garden. The drone can fly reasonably high and see over fences and trees. He does not fly it beyond the limits of his own garden. He feels that he has a right to fly anywhere in his own garden and that any angles extended by the drone in that space belong to him.

Sumaya lives next door to Robin and is worried about the drone's angles of view. She trusts him not to spy but she also feels that she should have the right to decide where Robin can fly if there is a chance he will invade the privacy of her family. The two cannot agree and have asked you to mediate.

In pairs, or small groups, consider these questions:



■ Can you 'own' an angle?

EXTENSION

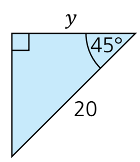
As trigonometric ratios are defined as the ratio of two lengths, what is their unit? Why? Prove this by showing an example or citing a source.

◆ Assessment opportunities

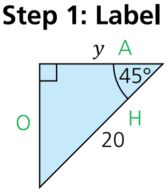
- In this activity you have practised skills that are assessed using Criterion D: Applying Mathematics in real-life contexts.

Can we calculate what we can't measure?

Example 1 Find missing side y



Step 1: Label sides

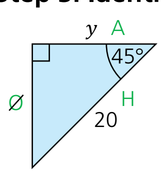


identify the hypotenuse, the side opposite the angle and the touching (adjacent) side

Step 2: Write the ratios

$S\frac{O}{H}\ C\frac{A}{H}\ T\frac{O}{A}$

Step 3: Identify which sides are given/needed



The hypotenuse is given (20)

We are looking for y (adj)

$S\frac{O}{H}\ C\frac{A}{H}\ T\frac{O}{A}$

This is the only one to contain A and H

Step 4: Write formula

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$$

Step 5: Substitute values

$$\cos 45 = \frac{y}{20}$$

Step 6: Re-arrange and **solve**

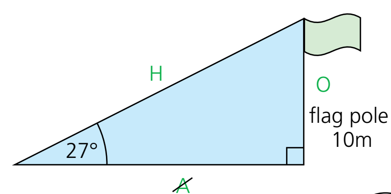
$$\cos 45 = \frac{y}{20}$$

so $y = 20\cos 45$ multiply x20

$y = 14.14$

Example 2

Find the hypotenuse for a right-angled triangle where an observer has an angle of elevation of 27° on a flagpole 10 metres high.



$\sin = \frac{O}{H}$ $S\frac{O}{H}\ C\frac{A}{H}\ T\frac{O}{A}$

$\sin 27° = \frac{10}{H}$

cross multiply

$H \sin 27 = 10$

$H = \frac{10}{\sin 27}$

$H = 22\text{m}$

Using trigonometric ratios

$S\frac{O}{H}\ C\frac{A}{H}\ T\frac{O}{A}$

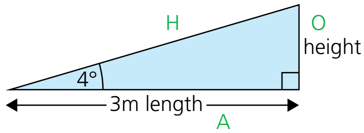
■ A tool for remembering the order of ratios

This is a tool which can be used to help you remember the order of the ratios and how to find them. By laying them out in this way, you can easily choose between the three and it reminds you how to find the **quotient**, i.e. which side is the numerator and which is the denominator.

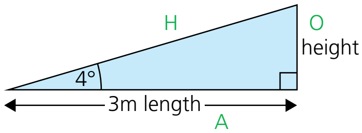
Example 3: worded problem

If a wheelchair ramp has an angle of elevation of 4° and a length of 3m, how high must it be?

First **construct** the triangle using the information provided in the question:



Now **label** sides



SOH CAH TOA

We have A and want O

SOH CAH TOA

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}}$$

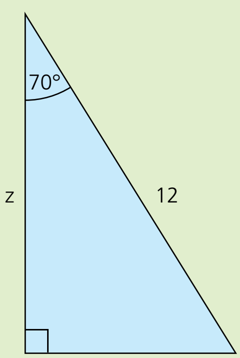
$$\tan 4 = \frac{O}{3}$$

$$= 3 \times \tan 4°$$

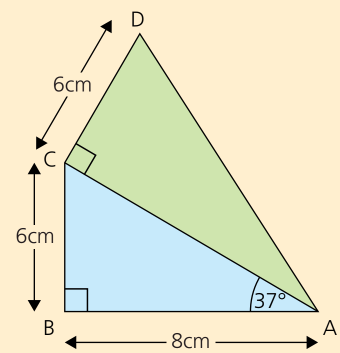
$$= 0.20978$$

The opposite side, or height, is 0.21m high

Find *z*.



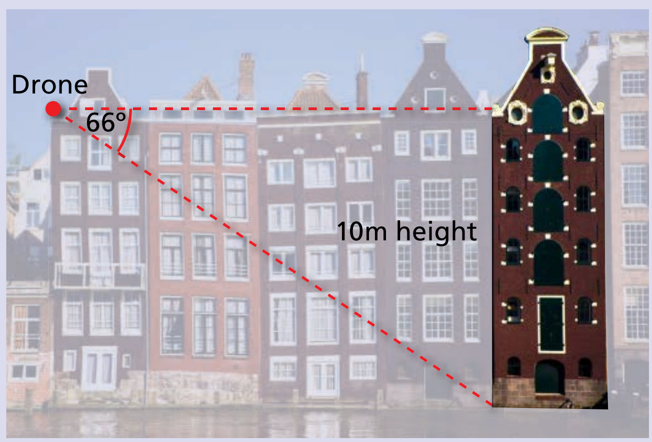
In the diagram below, find all the missing angles and sides using trigonometric ratios and Pythagoras' theorem.



Hint

Don't convert trigonometric ratios until the last step to avoid errors in rounding.

A tourist in Amsterdam wishes to take a photo of a typical canal house using their drone camera. The camera has an angle of depression of 66°.



How far back from the 10m high building should the drone be to photograph *the whole building* as clearly as possible?

How do I find a missing angle or side?

HOW CAN I USE 'SOHCAHTOA' TO FIND ANGLES?

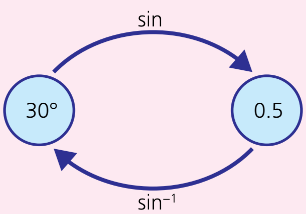
In algebra, you have learned rules for rearranging, transposing or 'manipulating' formulas to find what you are looking for. For example, you may wish to inverse a 'squared' in a Pythagoras problem by square rooting the other side. Trigonometric ratios also have an inverse function each. This allows us to 'undo' and isolate the angle to solve.

$\sin \to \sin^{-1}$ $\cos \to \cos^{-1}$ $\tan \to \tan^{-1}$

To work this out use the $\sin^{-1}$ on the calculator.

$\sin^{-1} 0.5 = 30°$

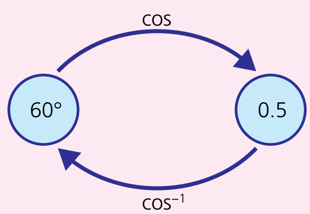
$\sin^{-1}$ is the inverse of sin. It is sometimes called arcsin.

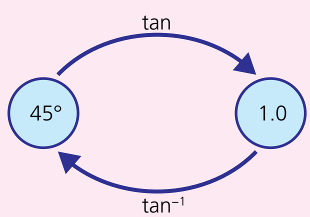


If I know the value of a sin in a given triangle, such as $\sin x = 0.5$, I need to isolate or solve for x to find the angle. Removing the sin, or moving it to the right-hand side, means I must use the inverse $\sin^{-1}$

$\sin^{-1} 0.5 = 30°$

Likewise,





PRACTICE EXERCISES

1 **Find the angles, correct to 1 d.p.**
   a $\sin^{-1} 0.35$
   b $\cos^{-1} 0.8760$
   c $\tan^{-1} 1.2$

2 **Find the angles, correct to the nearest degree**
   a $\sin^{-1} 0.5$
   b $\cos^{-1} 0.176$
   c $\tan^{-1} \frac{4}{5}$

Hint

Using a calculator or graphic display calculator

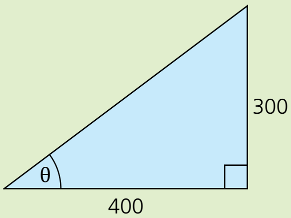
Most apps or calculators have inverse buttons. You may need to use the shift button to access the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions.

Ensure that your calculator is in degrees' mode and not radians, this is very important. You will learn more about radians later.

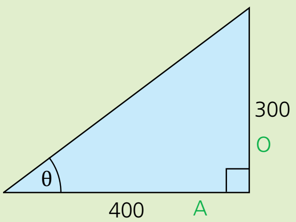
Remember that $\sin^{-1}$ is not a power, it is an inverse operation notation.

HOW DO WE USE THIS INVERSE IN SOLVING PROBLEMS?

Example 1



As before, **identify** the sides.



So we will use $\cancel{S^O_H}\cancel{C^A_H}T^O_A$

$$\tan = \frac{O}{A}$$

$$\tan\theta = \frac{300}{400}$$

$$\tan\theta = 0.75$$

But we're looking for θ

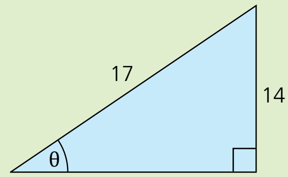
$$\boxed{\tan}\ \theta = 0.75$$

$\tan^{-1}$

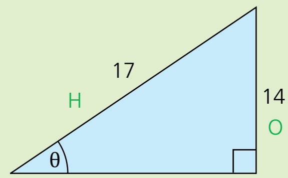
$$\theta = \tan^{-1} 0.75$$

$$\theta = 37°$$

Example 2



Label sides and choose $S^O_H C^A_H T^O_A$



$S^O_H \cancel{C^A_H}\cancel{T^O_A}$

$$\sin = \frac{O}{H}$$

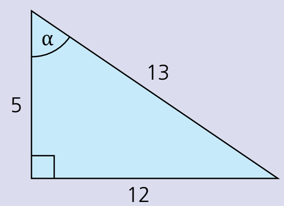
$$\sin\theta = \frac{14}{17}$$

$$\theta = \sin^{-1}\left(\frac{14}{17}\right)$$

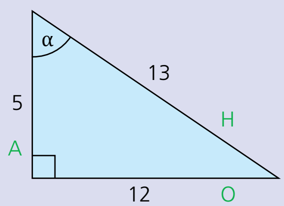
From calculator →

$$\theta = 55.4°$$

Find α



Label sides and choose S$\frac{O}{H}$C$\frac{A}{H}$T$\frac{O}{A}$



Unusually, we have all three sides, so we can choose *any* of the ratios.

$$\sin = \frac{O}{H}$$

$$\sin \alpha = \frac{12}{13}$$

$$\sin \alpha = 0.923$$

$$\alpha = \sin^{-1} 0.923$$

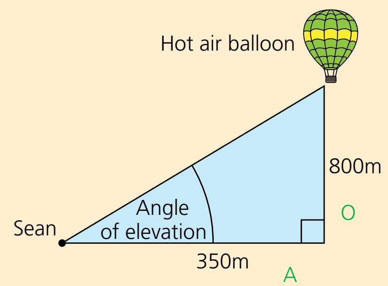
$$\alpha = 67.38013505\ldots$$

$$\therefore \alpha \approx 67°$$

A hot air balloon is flying at a height of 800 metres. If Sean is looking at the balloon 350 metres away from the balloon, what is the angle of elevation through which Sean is looking?

Solution

First **identify** the relevant information from the question and **construct** a triangle to label.



$\cancel{S}\frac{\cancel{O}}{H}\cancel{C}\frac{\cancel{A}}{H}\,T\frac{O}{A}$

$$\tan = \frac{O}{A}$$

$$\tan \theta = \frac{800}{350}$$

$$\theta = \tan^{-1}\left(\frac{800}{350}\right)$$

$$\theta \approx 66.4°$$

Therefore, Sean must be looking up at an angle of elevation of approximately 66°.

EXTENDED

If Lola is in the hot air balloon and looks down to see Sean, through which angle of depression must she be looking?

The angle of depression is the angle looking down from the balloon. By using the fact that the angles in a triangle add to 180° and the fact that we know the other angles are 67° and 90°, that means

angle of depression = 180° – 67° – 90°

Lola is looking down through (or has an angle of depression of) an angle of 23°.

ACTIVITY: Sundials

ATL

- Creative-thinking skills: Design new machines and technologies

■ Sundial

■ Sundial otel, Fethiye, Turkey

A sundial is a device used since ancient times to estimate the time of day using the shadows cast by the sun. Research how a **sundial works**.

Explain, using diagrams, how the angle of the sun affects the length of the shadows.

Now, using this information, collaborate with a partner to **create** your own unique sundial. Take a series of measurements at certain times. Use the information to **find** the angle of the sun at various times of the day.

◆ Assessment opportunities

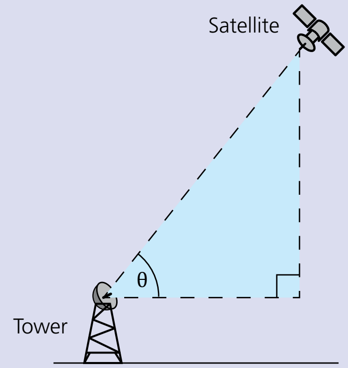
- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts. You might even choose to expand this activity to a Personal Project topic.

Where do geometric shapes occur around us?

Real-life problems: How do I find the hidden triangles?

It is very helpful to imagine the triangle 'hidden' in the diagram. First look for the lengths or measurements you have been given and mark them on the image. Remember to mark the right angle, both to show your communication and to self-check that you have correctly identified the relevant elements of the problem.

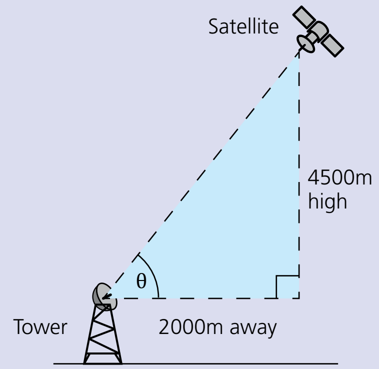
What if you have only words and no diagram? Don't let this worry you. **Identify** the dimensions/lengths in the question. **Sketch** a triangle and mark the right angle. **Match** the lengths (or angles) on the triangle. **Reread** the question to make sure you have correctly identified the information included. **Solve** as normal.



A tower has lost contact with a GPS satellite. The satellite is at 4500 metres over the Earth at a location 2000 metres from the tower.

At what angle should the tower point and scan to make contact again? Note the height of the tower is not included

Solution



$\tan = \frac{O}{A}$

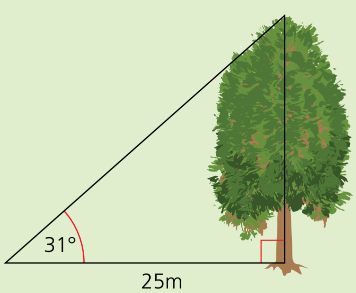
$\tan\theta = \frac{4500}{2000}$

$\theta = \tan^{-1}\left(\frac{4500}{2000}\right)$

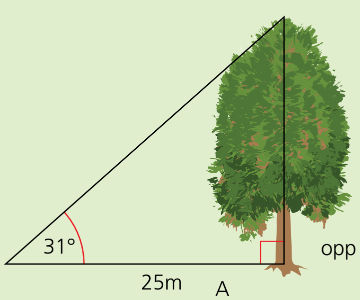
$\theta = 66°$

Example

Find the height of the tree.



Solution



$\tan = \frac{O}{A}$

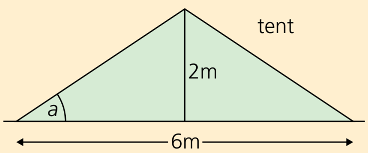
$\tan\theta = \frac{opp}{25}$

$opp = 25 \times \tan 31$

$opp = 15m$

The tree is 15 metres tall.

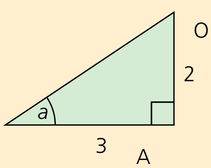
Worded problem: For a tent which is 2 m high and 6 m wide, what angle does the side of the tent make with the ground?



Solution

Find *a*.

Isolate triangle



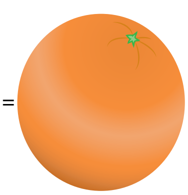
$\tan = \dfrac{O}{A}$

$\tan a = \dfrac{2}{3}$

$a = \tan^{-1}\dfrac{2}{3}$

$a = 33.7°$ to 3 s.f.

ACTIVITY: Can you get a tan in class?

$\dfrac{\text{sin (gerine)}}{\text{cos (gerine)}} =$ 

- What does this actually mean?

Using what you know about trigonometric relationships and **substitution**, show why $\dfrac{\sin}{\cos}$ will result in a tan?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

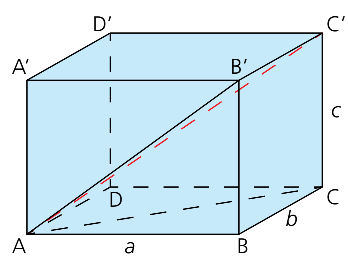
ACTIVITY: Will the relationships work in 3D models?

ATL

- Transfer skills: Apply skills and knowledge in unfamiliar situations; Inquire in different contexts to gain a different perspective

Imagine you are trying to find a diagonal in a box. Or you are trying to find the longest length in a room for a laser beam security system. Or are base jumping off a building and want to make sure you know where you will land. Or are trying to calculate if your skis will go into a box that is definitely too short but might work on the diagonal.

Will Pythagoras help you in three dimensions?

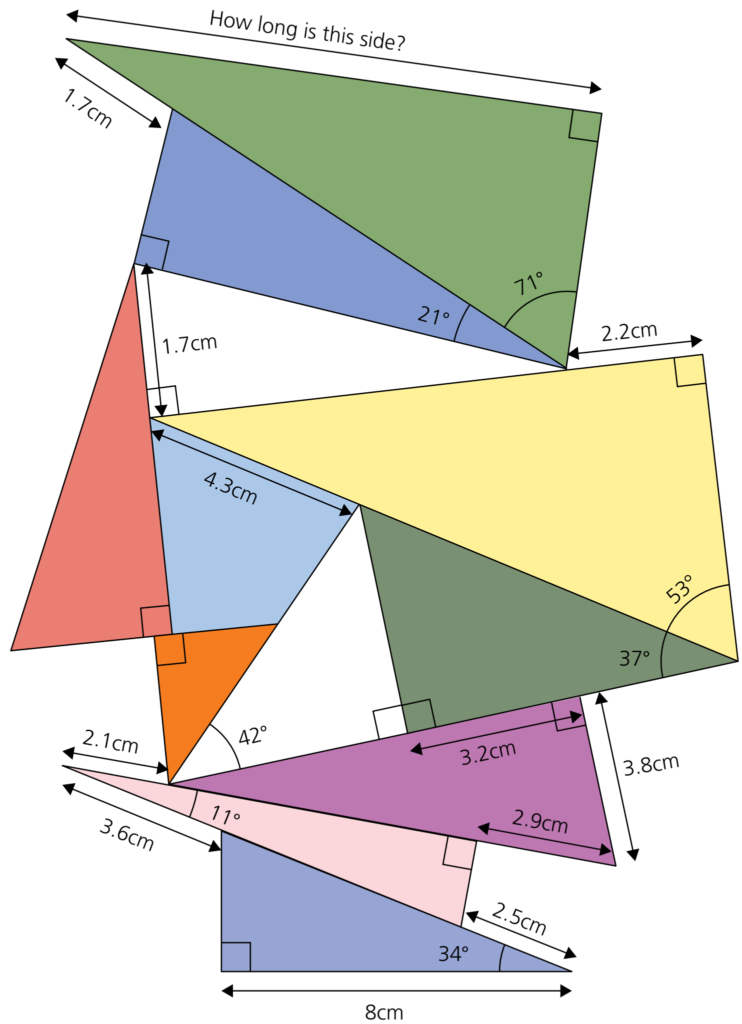


With a partner, **construct** a 3D model of a cuboid (or a 2D drawing to represent 3D space). **Determine** whether you could use Pythagoras' theorem to **find** the longest length (diagonal) inside the box if you know the dimensions of the box (cuboid).

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

ACTIVITY: Trigonometry pile up!



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EXTENDED

Verify whether this calculation is correct for the diagram above!

ATL

- Self-management shills: Demonstrating resillience.

$$\frac{\sin 71 \times \sqrt{\left\{\sin 53 \times \left(4.3 + \left[\left(\left[\sqrt{\left[\left[\left(\frac{8}{\cos 34} - 2.5 + 3.6\right)\cos 11\right] - 2.1 + 2.9\right]^2 - 3.8^2}\right] - 3.2\right)\frac{\cos 42}{\sin 37}\right]\right) - 2.2\right\}^2 + 1.7^2}}{\cos 21}$$

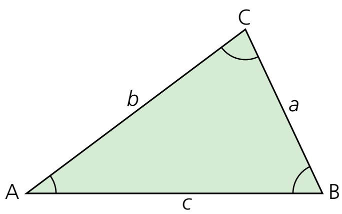
WHAT IS THE SINE RULE?

Now that you have mastered the lengths and sides of right-angled triangles of any size, we must begin to consider other non-right-angled triangles.

The sine rule

The first relationship which you can use to find unknowns is called the **sine rule**.

For any triangle ABC,



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Hint

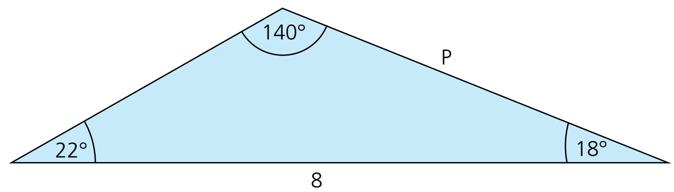
Notation

Remember that lowercase letters are used to denote sides and uppercase letters are given to angles. Note that the angles correspond in letters to their opposite sides.

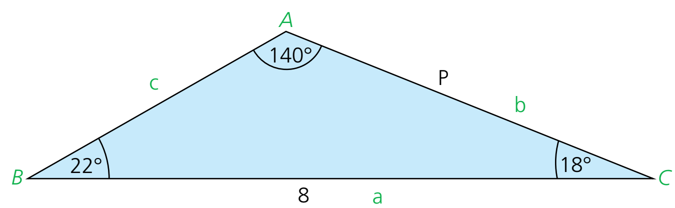
If you are given any two angles and one side of a triangle, you can use the sine rule to find the other sides. You can also use algebraic manipulation to find unknown sides. 2A + S or 2S + 1A problems. Often you will see these referred to as AAS (angle, angle, side) or SSA (side, side, angle) questions.

How much information do you need to solve a question using the sine rule?

Find P.



Solution



Write sine rule $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Identify sides and label.

Substitute $\quad \dfrac{8}{\sin 140} = \dfrac{P}{\sin 22} = \dfrac{c}{\sin 18}$

We don't need all three versions to solve.

So take the first equation

$$\frac{8}{\sin 140} = \frac{P}{\sin 22}$$

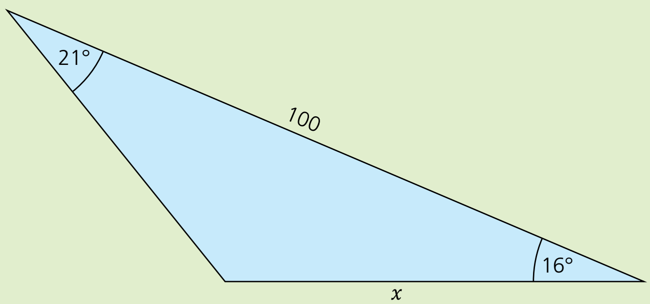
We want to isolate P so first cross-multiply

$$P \sin 140 = 8 \sin 22$$

$$P = \frac{8 \sin 22}{\sin 140} \longrightarrow \text{now plug into calculator}$$

$$P = 4.66$$

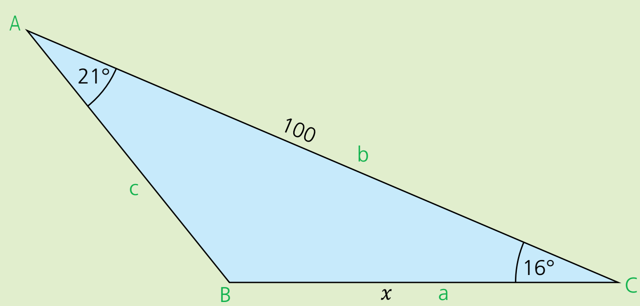
As you can see, we only need to identify the pairs of angles and sides we will use and solve from that single equation. Once you **identify** the correct pairs, all that is left to **solve** for the missing side or angle is to be careful and accurate in your algebraic manipulation.



Find x.

Solution

First **label** sides and angles.



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We are looking for x.

$$\frac{\sin A}{a} = \frac{\sin 143}{b}$$

$$\frac{\sin 21}{x} = \frac{\sin 143}{100}$$

We weren't given B but we know angles in a triangle add to 180°

so $B = 180 - 21 - 16$

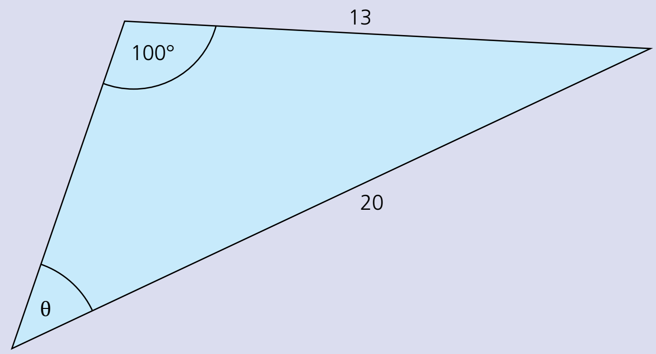
$B = 143°$

cross-multiply

$$x(\sin 143) = 100(\sin 21)$$

$$x = \frac{100 \sin 21}{\sin 143}$$

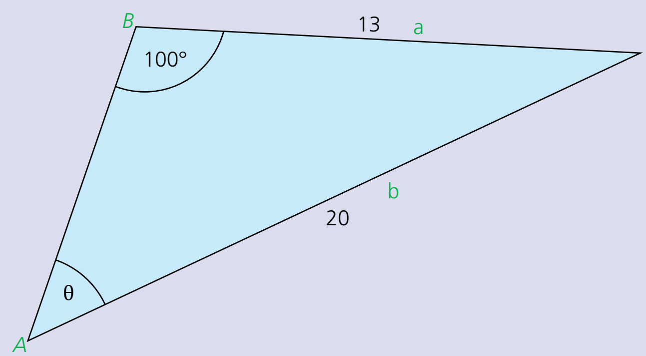
$$x = 59.5$$



Find θ.

Solution

Label sides and angles.



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{13} = \frac{\sin 100}{20}$$

The order is not important as we will cross-multiply

$$20 \sin \theta = 13 \sin 100 \quad \text{cross-multiply}$$

$$\sin \theta = \frac{13 \sin 100}{20}$$

$$\sin \theta = 0.6401250\ldots$$

To find the angle, use $\sin^{-1}$

$$\theta = \sin^{-1} 0.6401250\ldots$$

$$\theta \approx 39.8°$$

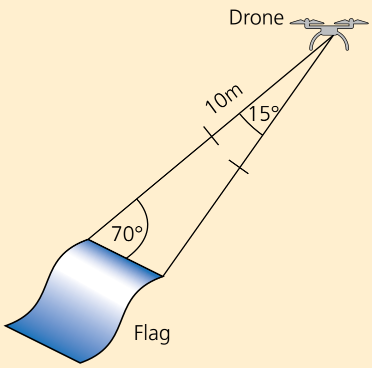
Hint

Don't forget the degree symbol when finding an angle.

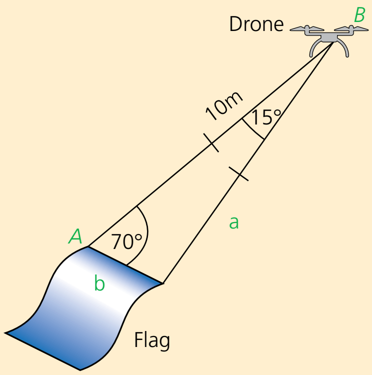
Drones are continuing to be used in new and varied ways all the time. Recently a protest using a drone and a flag disrupted an international football game and the game had to be stopped until the drone was brought down.

www.bbc.com/news/world-europe-29627615

Given the information in the diagram below, **find** the width of the flag being dragged by the drone.



Solution



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{10}{\sin 70} = \frac{b}{\sin 15}$$

$$b \sin 70 = 10 \sin 15$$

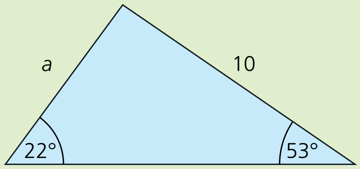
$$b = \frac{10 \sin 15}{\sin 70}$$

$$b = 2.75\,\text{m}$$

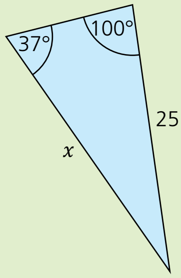
The flag was nearly 3 m wide!

PRACTICE QUESTIONS

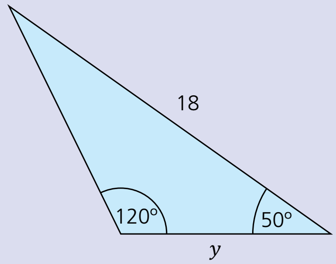
1 Find a.



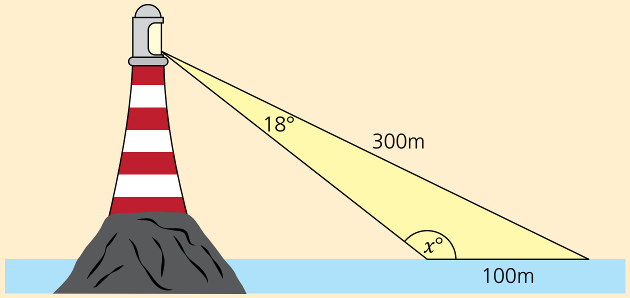
2 Find x.



3 Find y.



4 Solve for x.



ACTIVITY: Would the sine rule work on a right-angled triangle as well?

ATL

- Collaboration skills: Understanding mathematical notation

Investigate if it works on right-angled triangles by **constructing** your own.

Test to see if both the sine rule and SOHCAHTOA gives the same answer.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

WHAT IS MEANT BY THE COSINE RULE?

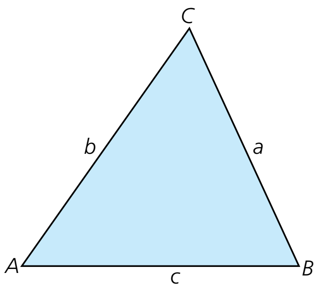
The cosine rule

In any triangle ABC with sides a, b and c units in length and opposite angles A, B and C respectively,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$



The cosine rule can be used to solve problems involving triangles given

- two sides and the included angle, or
- three sides

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

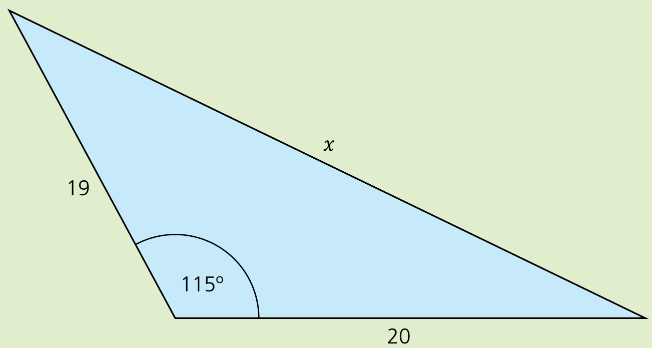
are useful rearrangements of the cosine rule. They can be used to find a missing angle if we are given all three side lengths of a triangle to find a missing angle.

This time we will use the rule for situations where the question mentions all three sides and an angle, no matter which one of those four is missing. Remember that both versions will be given to you in an examination or assessment. Your job is to choose which one to use and when, based on the context.

APPLYING THE COSINE RULE TO FIND SIDES OR ANGLES

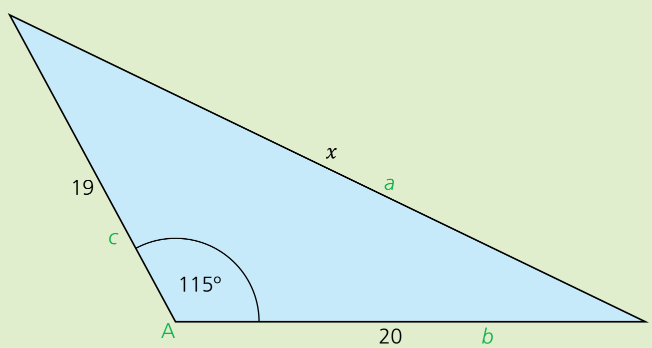
Example

Find x, using the cosine rule.



Solution

Label the sides.



We choose the first version as we are looking for a missing side.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = (20)^2 + (19)^2 - 2(19)(20) \cos 115°$$

$$a^2 = 400 + 361 - (760)(-0.4226)$$

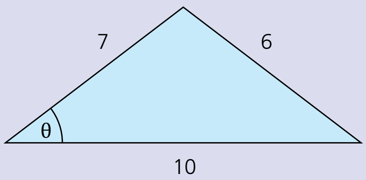
$$a^2 = 761 + 321.19$$

$$a^2 = 1082.19$$

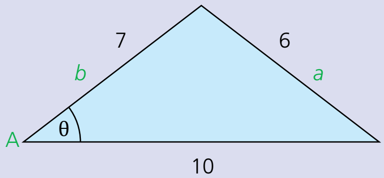
$$a = \sqrt{1082.19}$$

$$a = 32.9 \text{ to 3 s.f.}$$

Find θ.



Solution



Now we are looking for an *included* angle. So we can use this version of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

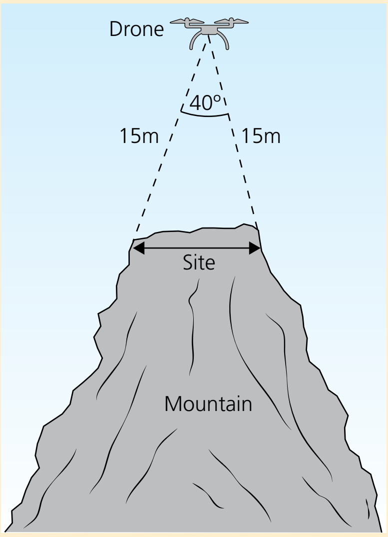
$$\cos A = \frac{7^2 + 10^2 - 6^2}{2(7)(10)}$$

$$\cos A = \frac{113}{140}$$

$$\cos A = 0.807$$

$$A = \cos^{-1} 0.807$$

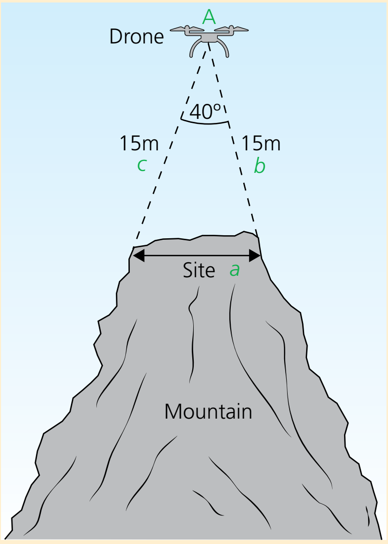
$$A \approx 36°$$



A drone is photographing fossils on an inaccessible mountaintop. The camera has a viewing angle of 40° and is positioned as follows to get the best picture of the site.

How wide is the site, given the information provided?

Solution



We are looking for the size or width of the site, the side a.

So $a^2 = b^2 + c^2 - 2bc \cos A$

$$a^2 = 15^2 + 15^2 - 2(15)(15) \cos 40°$$

$$a^2 = 225 + 225 - 344.7199994$$

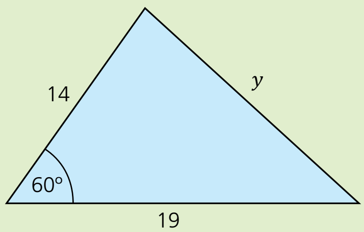
$$a^2 = 105.28$$

$$a = \sqrt{105.28}$$

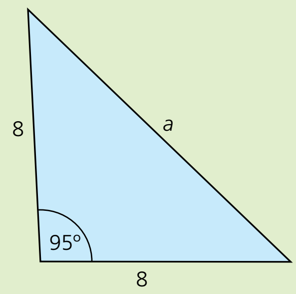
$$a = 10.26\text{m}$$

PRACTICE QUESTIONS

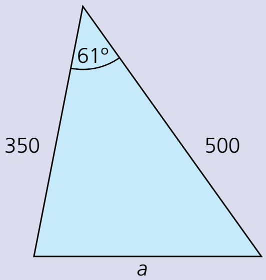
1 **Find y.**



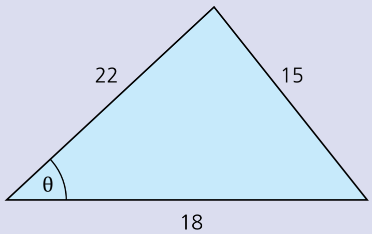
2 **Find a.**



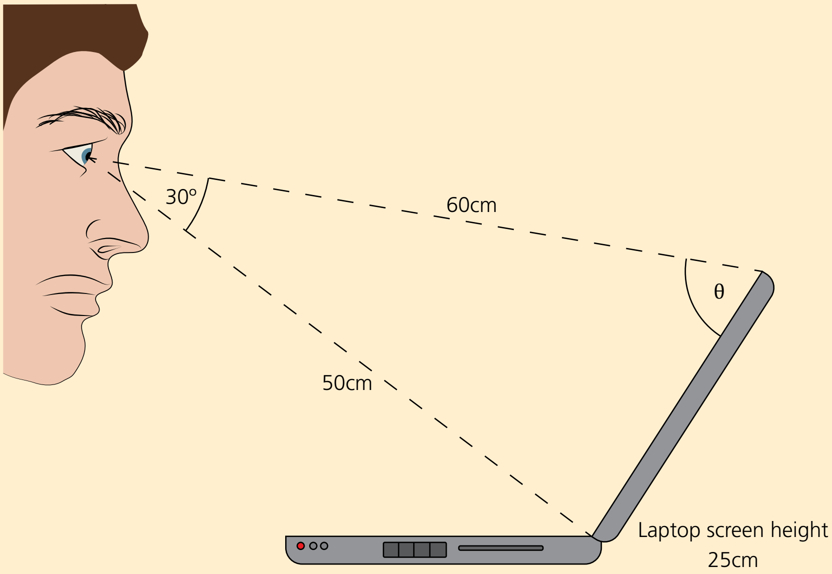
3 **Find a.**



4 **Find θ.**



5 Given the information in the diagram, what is the value of angle θ?



ACTIVITY: Tangent rule

ATL

- Information literacy skills: Access information to be informed and to inform others

Is there also a tangent rule? Carry out research to find out if a tangent rule exists. If so, what is it? If not, why doesn't it exist?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

EXTENDED

Can you show how to get from

$a^2 = b^2 + c^2 - 2bc \cos A$

to

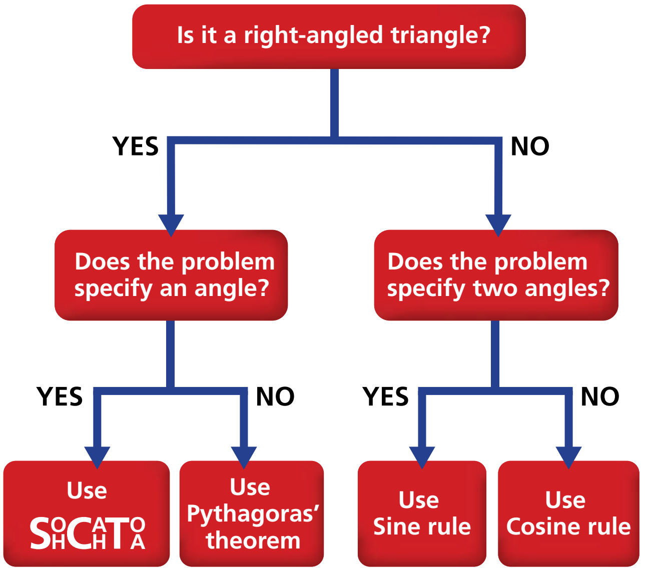
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

without using any numbers?

ACTIVITY: The best relationship

ATL

- Information literacy skills: Present information in a variety of formats and platforms



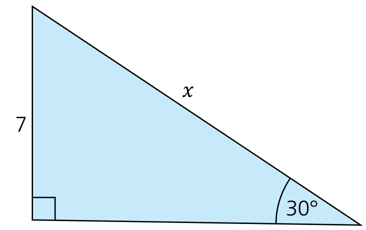
Follow the flow chart above to find the best relationship. Look at the problem and see the properties specified (specified means included or indicated, mentioned in the question, either known or unknown).

Assessment opportunities

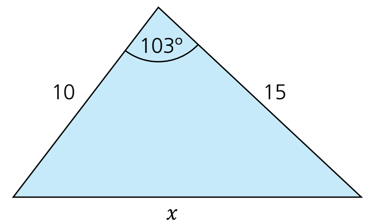
- In this activity you have practised skills that are assessed using Criterion A: Knowing and Understanding.

Use the flow chart to decide which method to use to find x in the following problems:

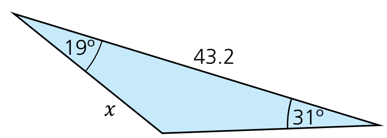
1



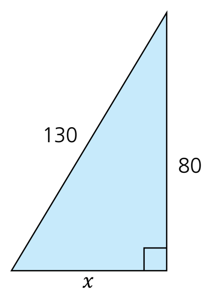
2



3



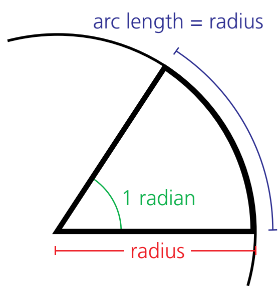
4



Now that you know which method to use, **solve** each of the problems above.

EXTENDED: Using radians

A radian can be found by taking the radius and wrapping it on the circle, i.e. placing around part of the circumference of the circle. The angle **subtended** by this arc is equal to 1 radian. One radian is defined as the angle **subtended** at the centre of a circle by an arc that is equal in length to the radius of the circle.



The circumference is found by:

$c = \pi \times d$

or

$c = 2 \times \pi \times r$

So a full circle by definition has 360° or, in a different (but equal) measure 2π radians.

Hint

If you are going to calculate and operate using radians, you must ensure that your calculator is in the right mode. Check you manual or ask your teacher how to change the mode from DEG to RAD and vice versa. You will also find tutorial videos online to assist you.

EXTENDED ACTIVITY: How to convert angles to radians?

ATL

- Communication skills: Understand and use mathematical notation; Organize and depict information logically

Find the relationship between radians and degrees. Justify the accuracy you have chosen.

Show the relationship in action by converting:

a **60° to radians**
b **173° to radians**
c **π radians to degrees**
d **1.4 radians to degrees**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A Knowing and understanding and Criterion C: Communicating.

Why would we need radians?

Radians are the **SI unit** for angles which means they are the accepted standard internationally. They are used in many trigonometric functions, see Chapter 9. You will also see them in much more detail in the Diploma Programme and when working with waves in Physics.

HOW CAN WE USE THIS KNOWLEDGE?

Here are two case studies for creative uses for drones and innovative spatial relationships.

http://motherboard.vice.com/blog/a-vast-network-of-delivery-drones-will-one-day-move-our-stuff

Matternet Is Developing a Vast Network of Delivery Drones to Move Our Stuff

Motherboard: So you want to do something good with drones, for once?

Andreas Raptopoulos: Yeah, exactly, I see a lot of opportunity to do good. Our particular goal is to do transportation in places that are not easily accessible. We think we stumbled upon something that can be the next paradigm for micro-transportation. We started thinking: 'how can we serve places that are not connected by roads?' Say, many Sub Saharan African countries, many South American countries, where you need to deliver medicine, you need to deliver vaccines, you need to move blood samples for HIV—and there's just no road to allow you to do it reliably.

We started thinking, 'can we use drones to help us do it?' So we created a concept—as you know, small UAVs today are very, very capable. They move very reliably, and navigate by GPS and do missions that allow you to carry small loads. Our threshold right now is 2 kilograms, which is about 4 pounds. But the problem they have now is a battery life which doesn't allow them to travel for long distances. So we created a system that basically allows us to counteract this disadvantage. We use small landing stations that do automatic battery swaps that allow one of those vehicles to land switch batteries and go out again.'

Andreas Raptopoulos is the founder and CEO of Matternet, building a network of unmanned aerial vehicles to transport medicine and goods in places with poor road infrastructure.'

www.independent.co.uk/news/world/ex-nasa-man-to-plant-one-billion-trees-a-year-using-drones-10160588.html

Ex-Nasa man to plant one billion trees a year using drones

'BioCarbon Engineering wants to use drones for good, using the technology to seed up to one billion trees a year, all without having to set foot on the ground … First, drones flies above an area and report on its potential for restoration, then they descend to two or three metres above ground and fire out pods containing seeds that are pre-germinated and covered in a nutritious hydrogel.

'With two operators manning multiple drones, he thinks it should be possible to plant up to 36,000 trees a day, and at around 15% of the cost of traditional methods.'

■ Drones used for delivering and crop spraying

ACTIVITY: Creative uses for drones

Read the two case studies and consider these questions:

- **Can you think of other similar situations where drones could be of service?**
- **Can you think of potential problems with this idea?**

! Take action

- Listen to the following radio show: **www.npr.org/sections/money/2014/05/30/317074394/drone-wars-who-owns-the-air**
- Consider the following questions:
  - Different cultures have different expectations of privacy – who has the right to say what can and can't be viewed? How can we determine what is for the common good and what is potentially damaging to individuals, cultures and/or the environment?
  - How does innovation challenge our personal and cultural expectations of spaces?
  - Are we really thinking about triangles at all?
- **Mathematical debate:** The government of your local region is considering how they will regulate drones, now and for the future. They are considering a set of regulations, including minimum flying height, distance from buildings and maximum possible angle view. Your teacher will provide the regulations to the class.
- The class will be divided into those who think the regulations are sufficient to protect the public and those who do not. You must choose whether to support these regulations or whether you disagree with them. You must make your case for your side and use mathematical knowledge and understanding to support their claims.
- **Report:** Alternatively, write a report to the government making recommendations to any changes or additions they should make to the laws. **Verify** these suggestions by giving mathematical arguments or examples.
- Supporting resource: **http://telecom.economictimes.indiatimes.com/news/banned-china-made-drones-with-hd-cameras-fly-off-shelves-near-mumbai-police-chiefs-office/51181884**
- **Reflection:** Now that we have seen many of the situations where drones can be used; height, distance or angles can be calculated or found ...

N = Need to know
What else do you need to know or find out about this idea or proposition? What additional information would help you to evaluate things?

E = Excited
What excites you about this idea or proposition? What's the upside?

W = Worrisome
What do you find worrisome about this idea or proposition? What's the downside?

S = Suggestion for moving forward
What is your current stance or opinion on the idea or proposition? How might you move forward in your evaluation of this idea or proposition?

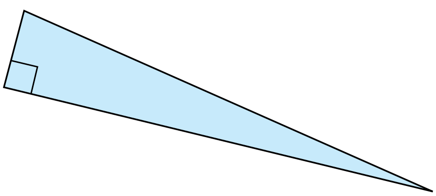
SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

1 **State** Pythagoras' theorem.

2 **Label** the hypotenuse on this triangle.

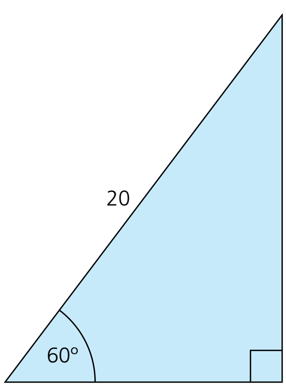


3 **Label** the sides of this triangle according to $a^2 + b^2 = c^2$.

4 How can you tell which side is 'adjacent'?

5 What do the letters in SOHCAHTOA stand for?

6 **Label** the sides correctly.



7 **State** the *inverse* operation:

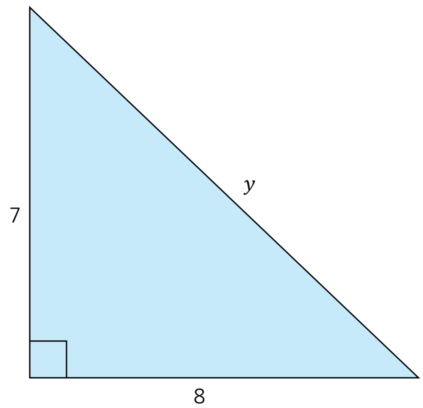
a sin

b cos

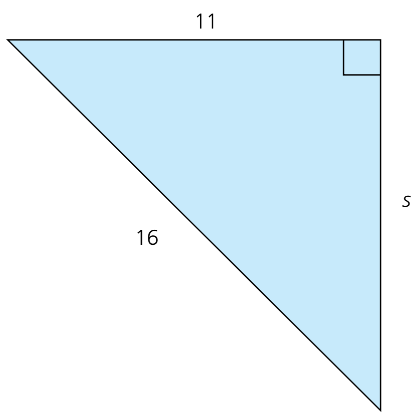
c tan.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

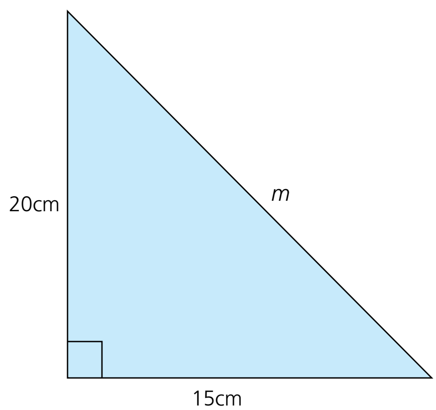
8 **Find** the length of side y.



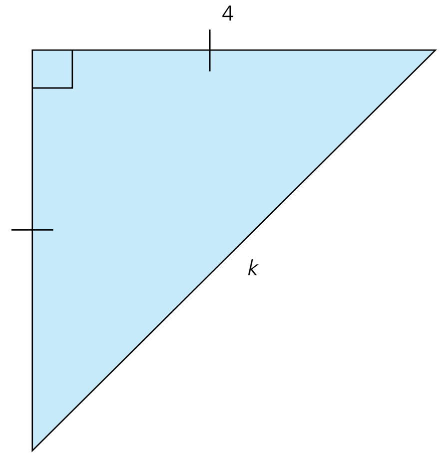
9 **Find** the length of side s.



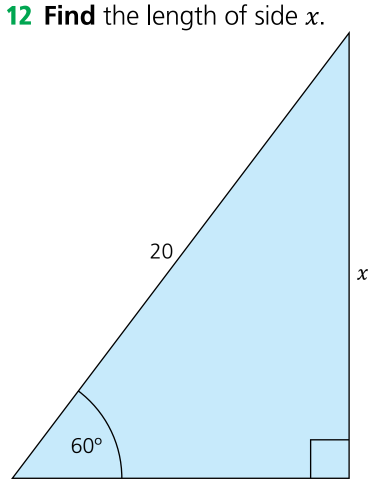
10 **Find** the length of side *m*. Give the answer correct to 2 s.f.



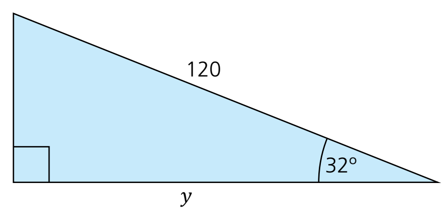
11 **Find** the length of side *k*. Give your answer correct to 1 d.p.



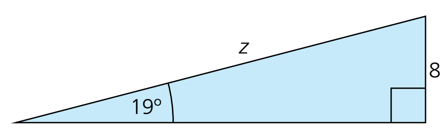
12 **Find** the length of side x.



13 **Find** the length of side y.



14 **Find** the length of side *z*.

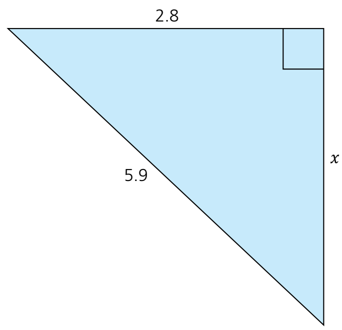


15 **State** the sine rule.

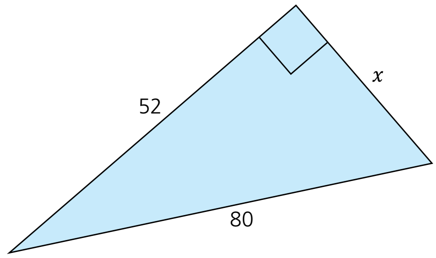
16 **State** the cosine rule

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

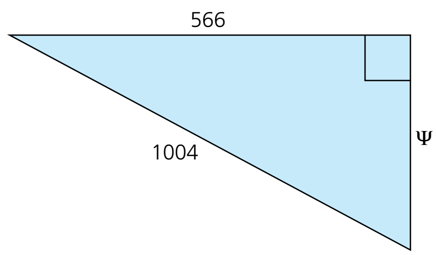
17 **Find** the length of side x.



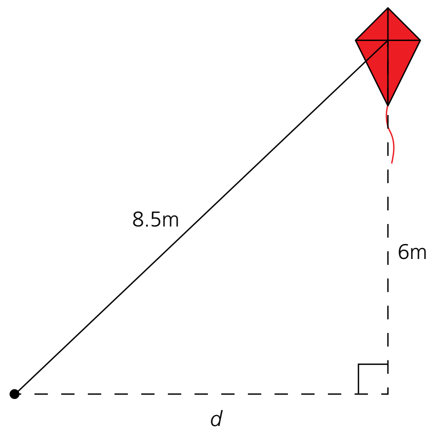
18 **Find** the length of side x.



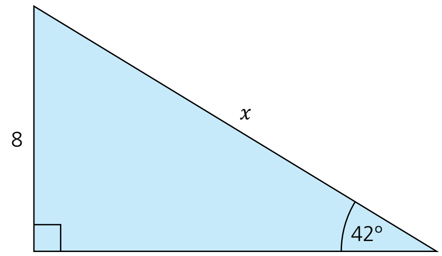
19 **Find** the length of side Ψ.



20 **Find** the length of side d.

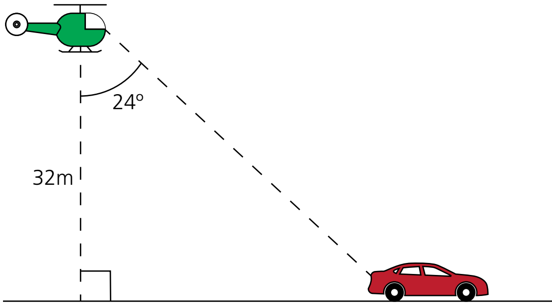


21 **Find** x.

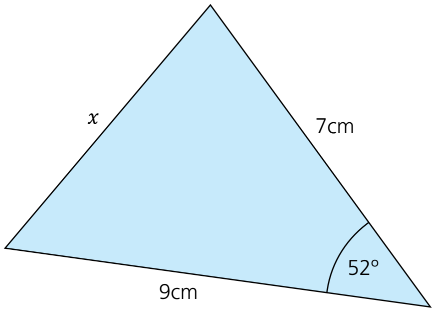


22 A drone is flying at a height of 0.75m looking in a sitting room window 3m away. What angle of depression does the drone currently have on those dimensions?

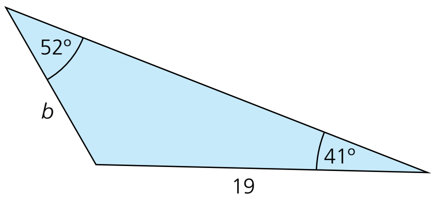
23 A helicopter is filming a car chase for a movie, flying at a height of 32 metres. If it must stay at an angle of 24° as shown on the diagram for the camera's best shot, how far behind the car must it remain?



24 **Find** x.



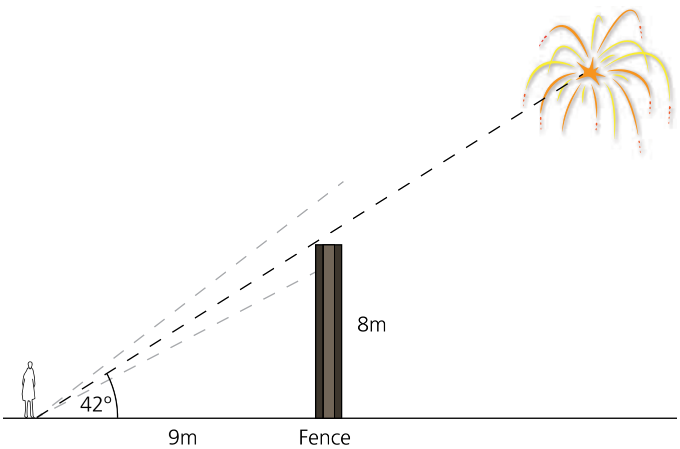
25 **Find** the length b.



THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

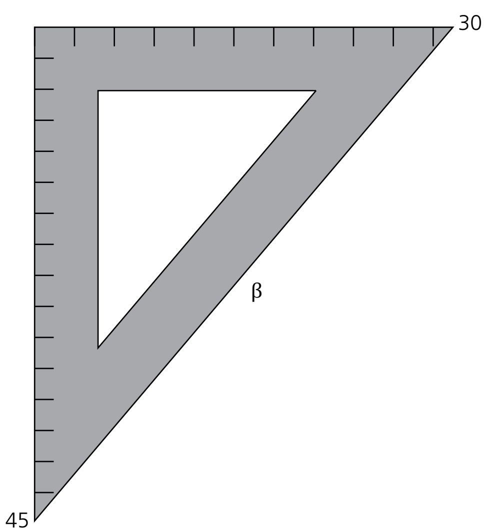
26 Computer and tablet screen 'sizes' advertised are actually the length of the diagonal. What is the screen 'size' of a tablet with length of 25cm and height of 12cm?

27 Darryl lives near Sydney Bridge in Australia and can partially see the New Year's Fireworks from his back garden. If he stands 9 metres back from his 8m fence, will he be able to see over the fence if he is looking up through an angle of elevation of 42°?

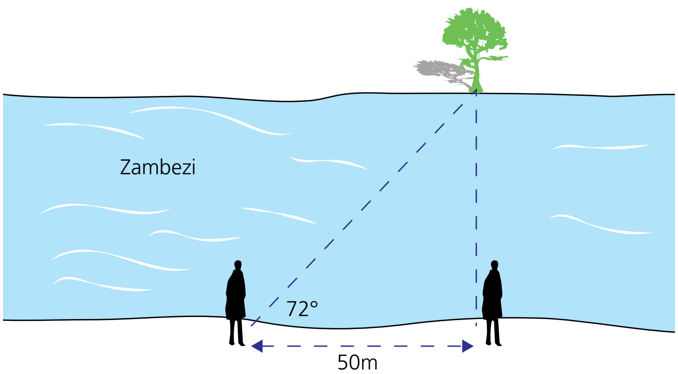


28 A security camera on the top of a building 12m high can see to the end of a garden 30m long. What is the diagonal range of the camera??

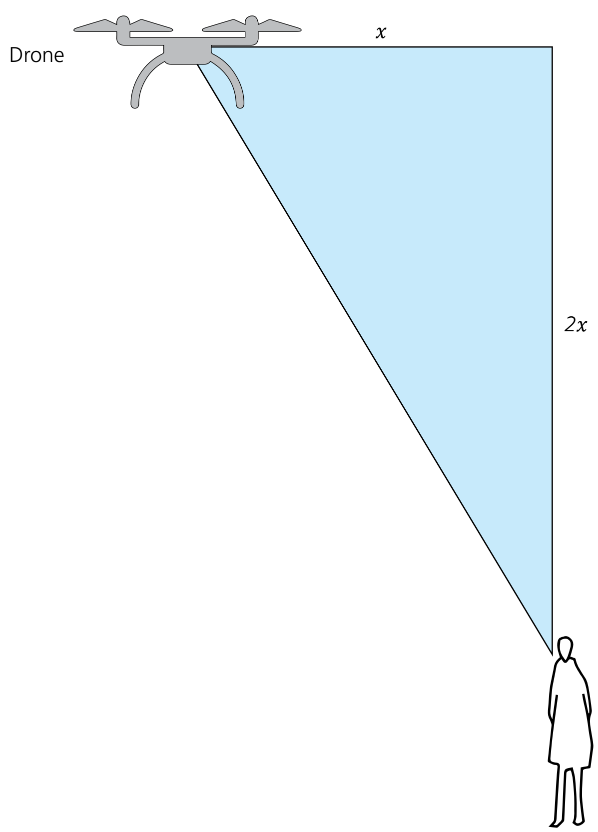
29 **Find** the length of side β.



30 Robert sees a tree across the Zambezi River in Zambia and estimates it to be at an angle of 72° from him. He then walks 50 metres until he is directly opposite the tree. What is the width of the river?



31 Heidi is flying her drone twice as high as it is far away from her horizontally. The length of the hypotenuse is $\sqrt{45}$. How high and how far away is the drone flying?

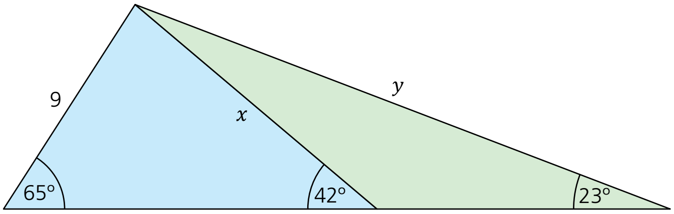


32 Police in Tokyo have developed a drone-catching drone to prevent the skies around airports and other secure locations. During a display, shown below, a photograph was taken as the capture drone (with the net) was 4 metres above and 2 metres behind the rogue drone.

a How far away is the capture drone with the net away from the smaller one (on the diagonal)?

b If the net is 1-metre-long, how far is the bottom of the net from the smaller drone?

33 **Find** the values of x and y.



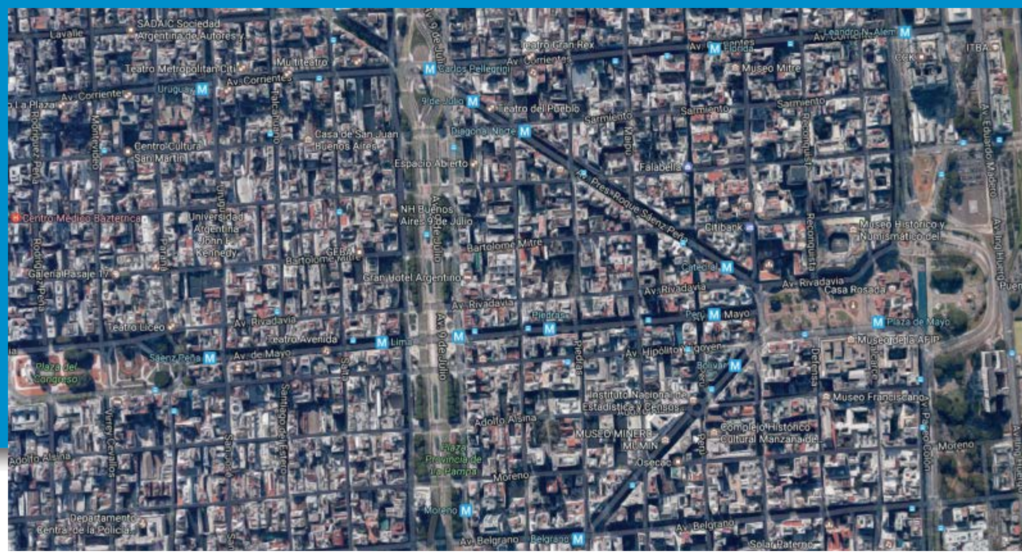
■ Drone used in a survey being carried out for prehistoric human fossils at the Pinilla del Valle site, Spain

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: How can we calculate unknown angles and sides? How do trigonometric relationships work?					
Conceptual: Where do geometric shapes occur around us? How do their relationships give us insight into the unknown? How do I find a missing angle or side? What can we measure? What can't we measure? Can we calculate what we can't measure?					
Debatable: Can anyone 'own' an angle? How much should models affect our understanding of the real world?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Information literacy skills					
Critical-thinking skills					
Collaboration skills					
Creative-thinking skills					
Transfer skills					
Communication skills					
Learner profile attribute	How did you demonstrate your skills of being principled in this chapter?				
Principled					

5 How can we move in space?

Applying mathematical **logic** to **spatial dimensions** can open **personal, cultural and social entrepreneurship** opportunities.



Buenos Aires, Argentina

CONSIDER THESE QUESTIONS:

Factual: How do we find the length of a line segment? How can we find distances between points on a map? What are vectors?

Conceptual: How do we describe points, lines, line segments and vectors? How can logic help us to solve problems?

Debatable: Can there be bad applications of geometry? Is the mathematical ability to find location in opposition to personal and cultural norms?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how to apply linear equation understanding in the new context of coordinate geometry calculations for technological innovations.
- **Explore** what information we can find and use from groups of points on a planar space.
- **Take action** by collaborating with younger years to enrich their learning experiences and to receive constructive feedback to ways to use algebra in everyday situations and special occasions.

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Collaboration skills
- Critical-thinking skills
- Creative-thinking skills
- Transfer skills
- Media literacy skills

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

We will reflect on this learner profile attribute …

- Communicator – we express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

PRIOR KNOWLEDGE

You will already know:

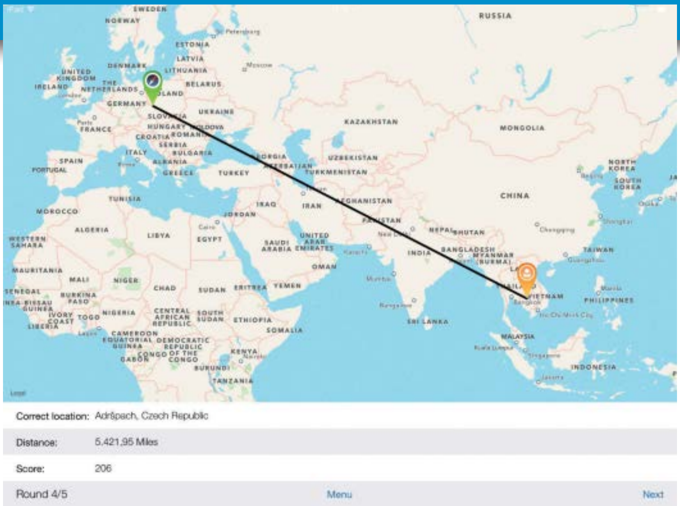
- how to plot and read points on a Cartesian plane
- how to plot linear equations on a Cartesian plane
- the correct communication for constructing, labelling and notation of graphs
- how to find the length of a hypotenuse.

GEOGAME – AN INQUIRY INTO LOGIC IN SPATIAL DIMENSIONS

Have you heard of Geogame? It's is a trivia game that tests a player's cultural, geographic and general knowledge. The user is presented with a panoramic photo that can be moved in any direction and which can be zoomed in or out. This location could be anywhere on the earth and uses Google Maps and Images, with some obvious clues and faces blurred out.

You must use your knowledge to identify the location by placing a pin on a map as quickly as possible. Points are awarded based on how close you were to the actual location. Bonuses are given for quicker times but for the moment we will not consider this.

In the picture you can see the player has guessed a location in Vietnam but the correct location was in the Czech Republic. The player was a distance of 5421.95 miles away from the correct location. Based on the length of this distance, i.e. between the guessed and the actual location, the player has scored only 206 points.



■ Geogame

DISCUSS

How have the creators of this game used geometry for fun?

How are these distances calculated? How can we use these properties for similar, or different, purposes? Are they really straight lines if we are talking about the surface of the earth?

In this chapter we will look at geometric and spatial relationships, such as lengths, midpoints, gradients and vectors, and find out how they are being used all around us.

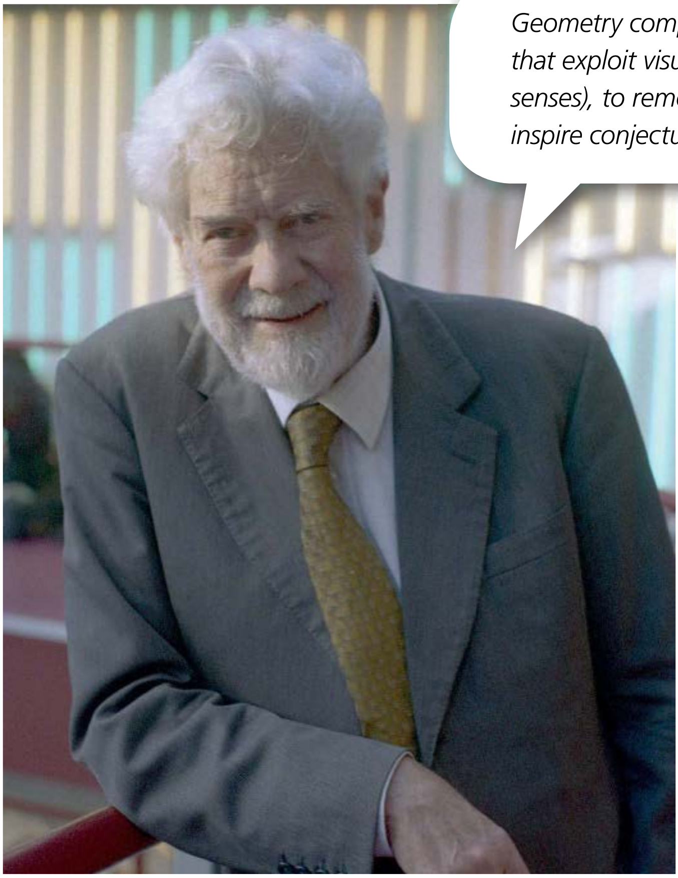
KEYWORDS

Cartesian grid	slanted
magnitude	vector
quadrant	

WHAT IS A POINT IN SPACE?

In the previous chapter, we concentrated on finding angles and distances in trigonometric (triangle) relationships. Now we will turn our attention to the shapes and measures of geometry, particularly on flat surfaces or planes.

As we know from Chapter 3, a Cartesian grid is a tool used to plot points, lines, to represent relationships and for equations. We also use a Cartesian grid to find the distances or to describe movements between points and understand their relationships.



Geometry comprises those branches of mathematics that exploit visual intuition, (the most dominant of our senses), to remember theorems, understand proofs, inspire conjecture, perceive reality and give insight.

■ Sir Christopher Zeeman (1925–2016)

THINK–PAIR–SHARE

- **What is meant by Prof. Zeeman's statement?**
- **Why would Prof. Zeeman say that vision is the most dominant of our senses? Do you agree?**
- **How can we exploit our vision using mathematics to bring about new insights?**
- **What is a point on a grid? How would you define a point in space? What makes it lie where it does?**

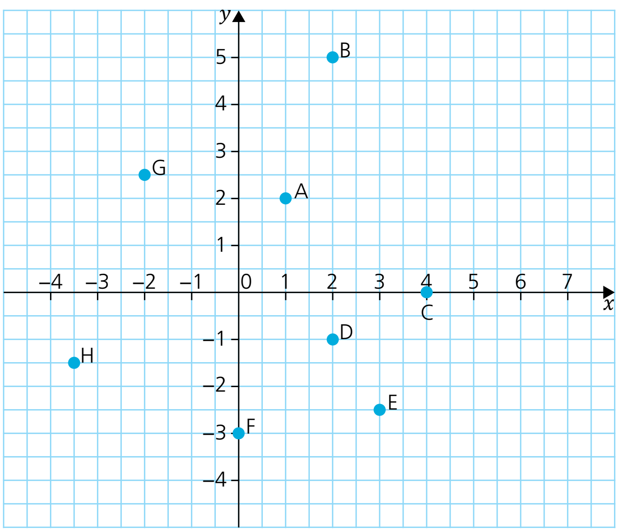
ACTIVITY: Revision of points on a Cartesian plane

ATL

- Communication skills: Understand and use mathematical notation

How well do you use appropriate mathematical notation and change between representations?

Using the grid below, **identify** which letter corresponds to which point and **state** which quadrant that point is in.



Copy and complete the table with your answers.

Letter	Point	Quadrant (in words or Roman numerals)
A		
B		
C		
D		
E		
F		
G		
H		

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating (i, ii and iii).

Take action

ATL

- Creative-thinking skills: Create original works and ideas; use existing works and ideas in new ways
- Collaboration skills: Giving and receiving meaningful feedback

! How did you learn about coordinate points when you were younger? Can you recall how you learnt? How did you practice them?

! Now it's time to share and communicate your knowledge to the younger years' students. Using your knowledge of plotting points, **create** an activity for a class learning coordinates for the first time. **Create** a join-the-dots activity on a Cartesian grid for the students to practice their skills. Choose a grid size and numbers that are appropriate to their level.

! **Instructions**

- Make sure that your dots connect to each other to make a recognizable shape or message.
- Make it fun for the students to practice and learn.
- Include dots in each quadrant (so the students practice their negative numbers also).
- Your teacher can collect your worksheets to share with, and challenge, a younger class. They may wish to give feedback on how they enjoyed your activity or how you might have improved upon it. It could make a nice classroom display too.

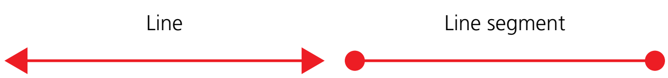
How do we find the length of a line segment?

DISCUSS

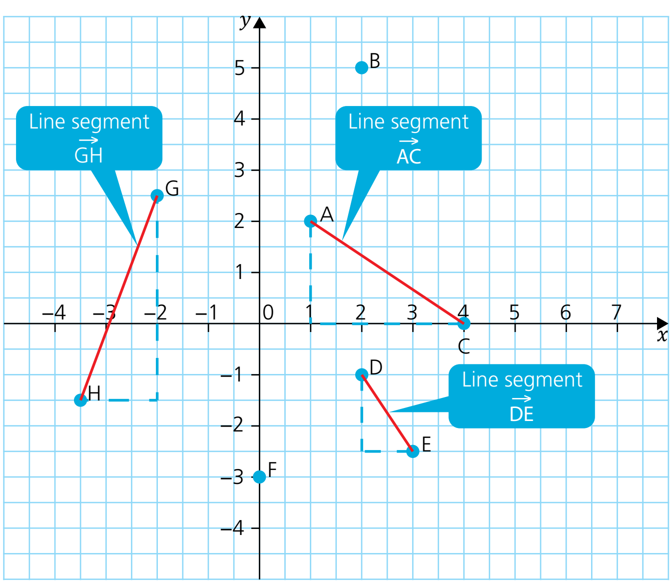
How is a line segment different to a line? Use mathematical terms to distinguish between them.

CONNECTING THE DOTS IN GEOMETRY

When we connect two points on a Cartesian plane, it forms a **line segment**. This line segment has a length and a direction in space. The gradient or slope of the line segment is also a feature that is useful to know.

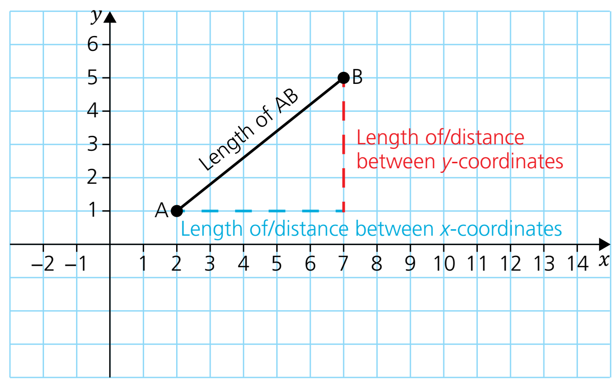


Why do we graph or plot points? What are we looking for? What parts of geometry can be used to find information about locations? How can these be used? Take a look at the grid below.



■ Examples of line segments

One of the first things that we can tell is relevant is the length of a line segment. How can we find the length of a line segment? If the line segment is horizontal or vertical, is it easy to find the length as we can simply count the boxes or read off the values on the x- or y-axis. But how do we find the length when the line segment makes an angle with either axis?



■ Distance between points of line segments

Consider the grid above – does it remind you of what we learnt in the previous chapter? If we break a line segment down into its coordinates, we see the familiar shape of a right-angled triangle emerge, with the line segment acting as the hypotenuse.

So in this example:

the length along the x-axis, or horizontal, is 5 units (from 2 to 7)

the length along the y-axis, or vertical, is 4 units (from 1 to 5)

so we can make use of Pythagoras' theorem to find the length of the line segment $\overrightarrow{AB}$ (which we will call simply AB from now on, by convention) by

$$AB^2 = 4^2 + 5^2$$

$$AB^2 = 16 + 25$$

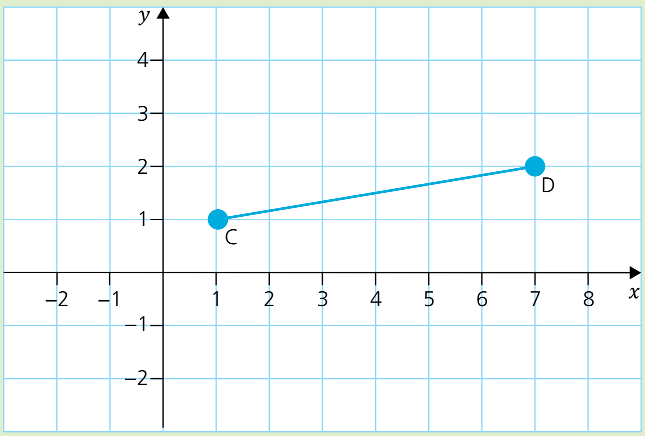
$$AB^2 = 41$$

$$AB = \sqrt{41}$$

This is the exact value of the length of AB

($AB \approx 6.4$ is the approximate value)

Find the length of CD.



■ Line segment CD

Solution

The distance between the x-coordinates of both points is (7 – 1) = 6

The distance between the y-coordinates of both points is (2 – 1) = 1

So by Pythagoras' theorem

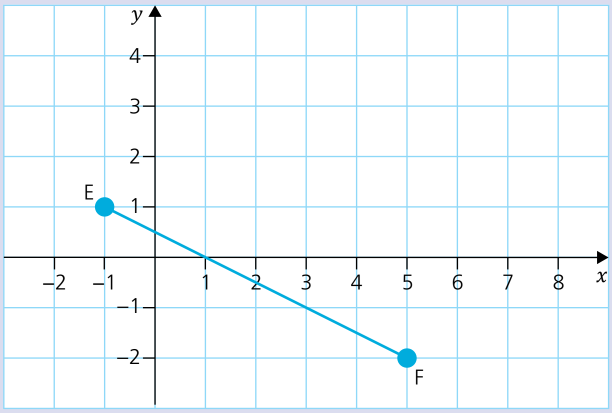
$(CD)^2 = (7 - 1)^2 + (2 - 1)^2$

$(CD)^2 = 36 + 1$

$\therefore\ CD = \sqrt{37}$

$CD \approx 6.1$

Find the distance between the points E (-1, 1) and F (5, -2).



■ Line segment EF

As you can see from the image above, we can use mathematical logic to make a connection between the distance between points E and F and the length of line segment EF.

Once again, to find the length of EF we can use Pythagoras' theorem by:

- The distance between the x-coordinates of both points is from -1 to 5 (5 – –1) or a total of 6 boxes
- The distance between the y-coordinates of both points is from 1 to –2 (–2 – 1) or a total of 3 boxes.

According to Pythagoras' theorem

$(EF)^2 = (5 - -1)^2 + (-2 - 1)^2$

$(EF)^2 = (6)^2 + (-3)^2$

$(EF)^2 = 36 + 9$

$(EF)^2 = 45$

$\therefore\ EF = \sqrt{45}$

$EF \approx 6.7$

Hint

If you are dealing with negative coordinates, make sure you handle the distances between them carefully, paying particular attention to the signs.

CONNECTING PYTHAGORAS' THEOREM TO THE FORMULA FOR DISTANCE BETWEEN TWO POINTS

As we know from Pythagoras' theorem:

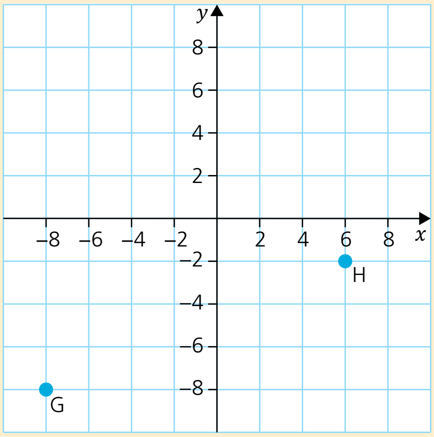
$$(GH)^2 = (\text{difference in } x \text{ values})^2 + (\text{difference in } y \text{ values})^2$$

So rearranging the formula before substitution would give:

$$GH = \sqrt{(\text{difference in } x \text{ values})^2 + (\text{difference in } y \text{ values})^2}$$

Notice how the index/power has moved to the other side as a square root. This is a transposition or rearrangement we have made in each of the examples above, even though it occurs at a later stage in the calculation. In the next example, we will use this alternative arrangement to find the length.

Carla is playing a simplified version of the Geogame. She guesses that a location is at G (−8, −8) but the location is actually at H (6, −2). How far away from each other are they?



■ Carla's choice (G). The location (H).

$$GH = \sqrt{(\text{difference in } x \text{ values})^2 + (\text{difference in } y \text{ values})^2}$$

The difference in x values is from −8 to 6 (6 − −8) or 14

The difference in y values is from −8 to −2 (−2 − −8) or 6

So

$$GH = \sqrt{(6 - -8)^2 + (-2 - -8)^2}$$

$$GH = \sqrt{(14)^2 + (6)^2}$$

$$GH = \sqrt{(196 + 36)}$$

$$GH = \sqrt{232}$$

$$\therefore GH \approx 15.2$$

This means Carla was 15.2 units away from the actual location.

Extended

This rearrangement to find the length of a line segment is given by the formula

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

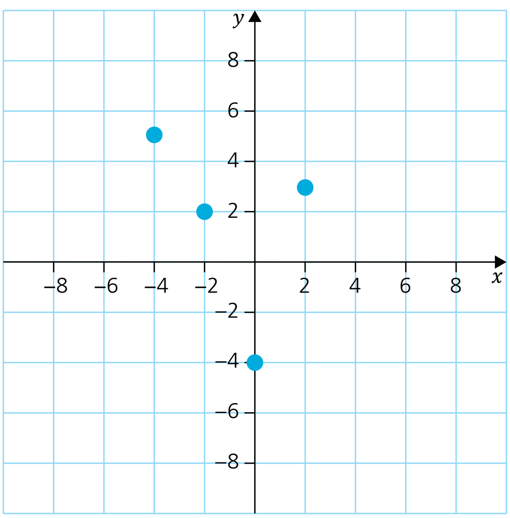
Where (x_1, y_1) and (x_2, y_2) are the coordinates of either end of the line segment. You may wish to use this formula, or any other method which serves you well (and gives correct answers) to find the length of a line segment and/or the distance between two points.

PRACTICE QUESTIONS

1 Identify if these line segments are horizontal, vertical or slanted:

- **a** **IJ where I = (0, 11) and J = (2, 11)**
- **b** **KL where K = (2, 5) and L = (−6, −9)**
- **c** **MN where M = (9, 4) and N = (9, 14).**

2 Find the distances between each of these points.



3 Find the length of the line segments between:

- **a** **(1, 1) and (5, 9)**
- **b** **(3, 3) and (7, 1)**
- **c** **(5, 3) and (6, 7)**
- **d** **(−2, −4) and (0, 0)**
- **e** **(0, 7) and (8, 5)**
- **f** **(−1, 3) and (−9, −3).**

4 Find the distance between these points:

- **a** **(5, 3) to (7, 4)**
- **b** **(13, −1) to (−7, −5)**
- **c** **(5, 7) to (10, 15)**

How can we find distances between points on a map?

Links to: Geography

Remember that coordinates on a grid are numbered positive and negative to indicate location. Coordinates on the Earth's surface are indicated by longitude and latitude. You may have already learned this in map reading in Geography.

ACTIVITY: Taxi!

ATL

- Critical-thinking skills: Analyse mathematical concepts and synthesize their logical implications to create new understanding
- Media literacy skills: Locate, organize, analyse, evaluate, synthesize and ethically use information from a variety of sources and media

The taxicab industry has been revolutionized and disrupted by this technology. Apps such as Uber, Didi Dache, Ola, TaxiPixi and Hailo are making use of this geometry and GPS to locate, track and empower drivers and customers. This has been a controversial matter and not without a number of complicated ethical and political considerations.

Research some of the **implications of this technological advance**, **both positive and negative** to better understand how geometry can affect lives and livelihoods.

ACTIVITY: Distances on a real-life Cartesian grid

ATL

- Transfer skills: Apply skills and knowledge in unfamiliar situations

Salt Lake City, Utah, USA is an excellent example of a city that has been planned and mapped out in a grid formation. Each block measures 200m in length.

Aerial photograph and map of Salt Lake City

Josh is at location E and is waiting for a taxicab to collect him. He uses an app which traces the available taxis at locations A, B, C and D. The distance between Josh and the taxis is calculated as the crow flies, or the shortest possible direct distance.

Find out how far each taxi is away from Josh and therefore which one the app will automatically assign to him. Make sure that you:

- **identify the relevant elements of the situation**
- **select the correct mathematical strategies to model the situation**
- **apply the strategies to reach a correct solution.**

Once you have found a **mathematical** justification for the best taxi to send to Josh, make sure you then:

- **justify the degree of accuracy of your answer and**
- **justify whether the solution makes sense in the context of the situation.**

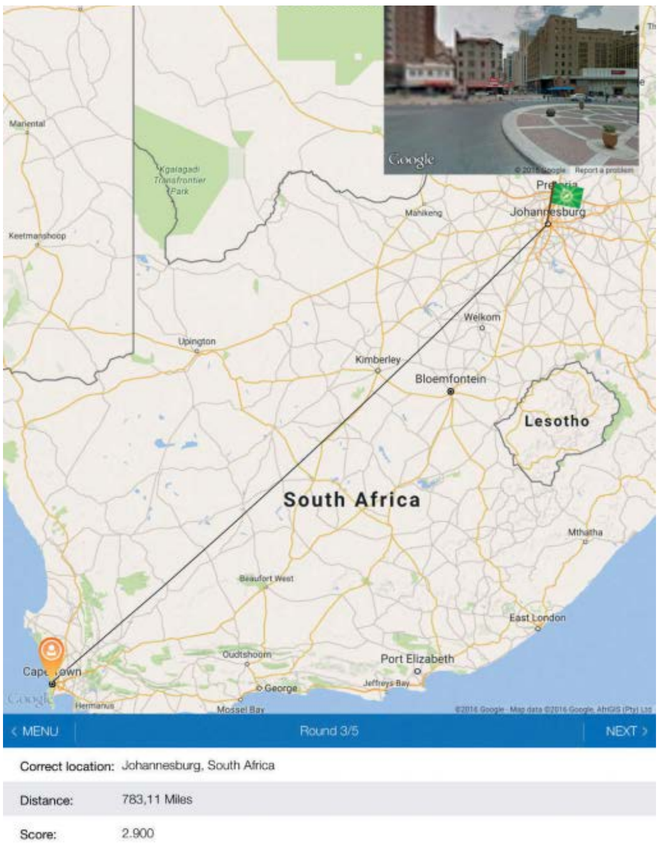
Make sure to use clear and concise lines of reasoning to communicate your thinking.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real life contexts.

ACTIVITY: How could we relate Cartesian coordinates to longitude and latitude through the Geogame?

Kiranjit is playing Geogame and she estimates a photograph's location to be Cape Town. You can see the image she was given in the top right-hand corner and the location she chose in the bottom left-hand side of the map below.



The correct location of the photograph was in fact Johannesburg, South Africa so Kiranjit was in the right country but in the wrong city. She looked up the locations of each city and found their geographical coordinates were:

- **Cape Town: (33.9249° S, 18.4241° E)**
- **Johannesburg: (26.2041° S, 28.0473° E).**

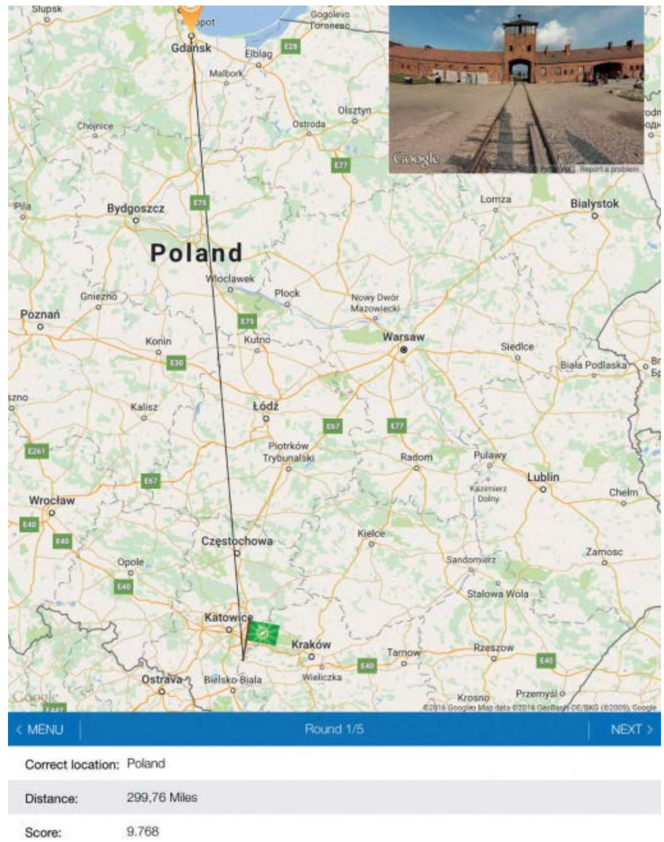
By rounding them and by treating them as Cartesian coordinates, she found the lengths of the line segments (from guess to correct location). She then created the following formula:

> Length of line segment × 65 = distance
> (the distance the game said she was wrong by)

ATL

- Media literacy skills: Locate, organize, analyse, evaluate, synthesize and ethically use information from a variety of sources and media (including digital social media and online networks)

In the next round, she recognized the historical site of Auschwitz in Poland but incorrectly guessed the location as Gdansk rather than Brzezinka.



Using information from both rounds of the game, **investigate** how accurate Kiranjit's formula is.

Assessment opportunities

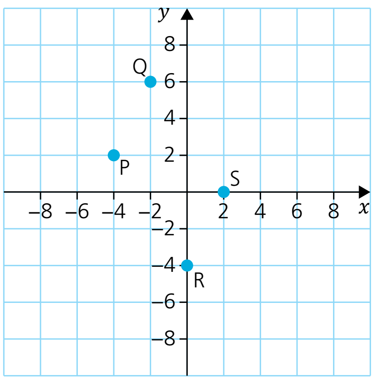
- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

Links to: History

Discover the relevance of Auschwitz in European 20th century History. **Discuss** whether you think it is right or appropriate for Auschwitz to feature in a game.

And now back to the Cartesian grid:

ACTIVITY: Using line segment characteristics to find out other information



Information from line segments

- **Identify the coordinates of P, Q, R and S.**
- **Find the exact lengths of each side of the quadrilateral (formed by) PQRS.**
- **Give the lengths of each side correct to 2 s.f.**
- **What else can you deduce about the shape PQRS?**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding.

PRACTICE QUESTIONS

1. **Given the points T(2, 2) and U(−2, 2) can you find another point which would form a triangle?**
2. **Can you find a point V that would form an isosceles triangle?**
3. **Can you find more points that would form an isosceles triangle with the points TU?**
4. **What do these points have in common?**
5. **Where would you need to place a point V to form an equilateral triangle TUV?**

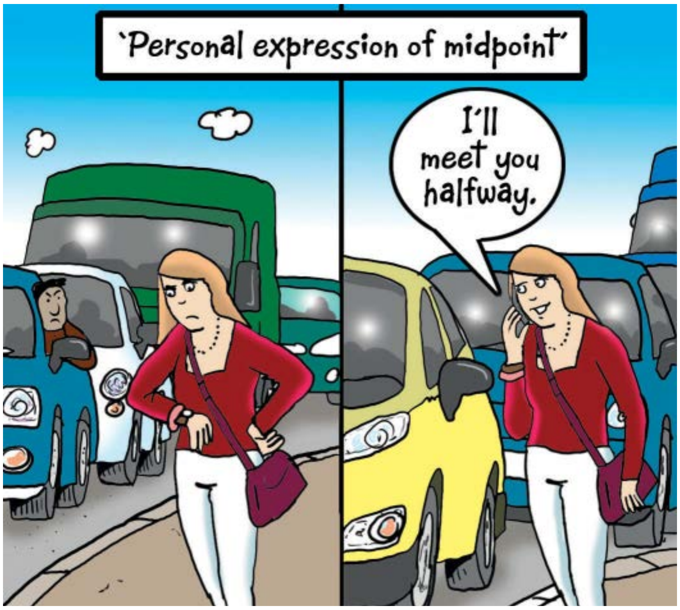
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding.

FINDING THE HALFWAY OR MIDPOINT

Why would we need to find a midpoint? In geometry, it can often be important to **bisect** a line or to find the midpoint in order to solve another problem such as area or perimeter.

Thinking of our global context, why would we be interested in midpoints for personal, cultural or entrepreneurial reasons?



Personal expression of midpoint

Links to: History; Physical and health education

Cultural expression of midpoint: In ancient warfare, armies would position themselves across from one another and strategize battle tactics. Soldiers would often meet in the middle to engage in combat and try to gain ground over the No Man's Land in the middle.

In modern times, a small remnant of this type of thinking remains in sports events and similar to events in which a neutral venue may have to be chosen between two home venues, so neither side has an advantage.

Sometimes we might wish to split a journey or to meet up with a friend and halfway seems like a good compromise. Websites such as **www.geomidpoint.com/meet or www.whatshalfway.com** can calculate the midpoint along a journey to help you 'Find great places to meet or stop between two points'.

ACTIVITY: Midpoints

Plot each of these sets of points and **find** the mid-point:

1. (1, 1) and (5, 9)
2. (3, 3) and (7, 1)
3. (5, 3) and (6, 7)
4. (–2, –4) and (0, 0)
5. (0, 7) and (8, 5)
6. (–1, 3) and (–9, –3)

Hint

You can save paper by using the same grid and by using different colours to plot each pair of points and to connect them to form line segments.

PRACTICE QUESTIONS

1. Can you **determine** the midpoint without plotting or measuring for:
   - **a** larger numbers like (64, 105) and (168, 213)
   - **b** or negative values (–2, –4) and (–100, –24)
   - **c** a combination of both negative and large numbers (15, –3) and (–13, 107)
   - **d** as well as decimals (9, 2.5) and (0, –3)?
2. Revisit the general rule that you came up with earlier. Now **describe** in words how you can tell what the midpoint is by looking at the x and y values.

WHAT MAKES YOU SAY THAT?

1. What's going on with the x- and y-coordinate values from points to get to the midpoint?
2. What do you see that makes you say that? Can you formulate a simply worded general rule?

Links to: Economics; Business management; Physics

Entrepreneurial expression of midpoint: A recent development in stock market trading is the rise of very high speed trading. This kind of trading, called 'high frequency trading', is where thousands of shares are sold by computers. Some experts estimate that 70% of daily trades now employ this kind of exchange.

Companies around the world have spent millions of dollars making faster computers, some claim that their computers can calculate a trade in 500 microseconds, a thousand times faster than the human eye can blink. But no matter how fast these processors get, they will always run into the universal speed limit, the speed of light. We know from Einstein that nothing in the universe can travel faster than light. Planet Earth is so big that it takes light 67 milliseconds to travel from one end to another; hundreds of times slower than the fastest computers can calculate a trade.

The problem arises when traders are at two different parts of the planets and it takes time for information to travel from one point to another. If one location such as London can react faster to the change than people in New York can, then they may get higher prices for their stock and have an advantage.

A team of entrepreneurs have come up with the solution to set up computers with automatic trading instructions at points precisely between the two locations. So, for New York and London, there would be a computer server tuned to the financial markets floating in the middle of the Atlantic Ocean. It would have an advantage over New York when something happens in London, and an advantage on something happening on London, when something started happening in New York.

The team identified all of the **midpoints** between 52 of the world's largest financial markets, marking places that would be ideal for their remote trading servers, thereby creating the blueprint for a huge worldwide network.

■ Midpoints between the largest financial markets

PRACTICE QUESTIONS

Using the midpoint formula, or otherwise, **find** the midpoint of these pairs of points:

a (3, 3) and (2, 6)
b (5, 8) and (3, 7)
c (4, 6) and (4, 2)
d (6, 8) and (1, 0)
e (0, 1) and (–2, –3)
f (4, –1) and (3, –3)
g (11, –2) and (–1, –13)
h (–0.5, 1) and (0.5, –1)

Finding the midpoint

The mathematical formula (rule) to find the mid-point

The midpoint can be found by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

for points (x_1, y_1) and (x_2, y_2).

EXTENDED

How can we divide a line segment in three equal parts?

Can you find points which break a line segment in the ratio of 3:2?

WHAT ELSE CAN WE LEARN FROM LINE SEGMENTS?

ACTIVITY: Creating curves from lines

Draw the following line segments P_xQ_x by **plotting** points P_x and Q_x.

Connect points

P_1 (0, 15) to Q_1 (1, 0)

P_2 (0, 14) to Q_2 (2, 0)

P_3 (0, 13) to Q_3 (3, 0)

Describe any patterns you see in the construction of line segments P_xQ_x

What will points P_4 and Q_4 be?

Verify these points by **plotting** line segment P_4Q_4 and checking visually if it follows the pattern of the other line segments.

Describe the pattern(s) as a general rule.

Prove, or **verify** and **justify** this general rule by finding and **plotting** other line segments for P_xQ_x.

Complete the pattern by **plotting** each line segment possible following this pattern for $1 \leq x \leq 15$, and $1 \leq y \leq 15$.

EXTENDED

The pattern of line segments creates what is called a Bézier curve. For more about Bézier, see the Mathematician Profile box in Chapter 8.

http://mathythings.com/arts.htm

Look at some string art creating Bézier curves through geometry **https://plus.maths.org/content/bridges-string-art-and-bezier-curves**

◆ Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns.

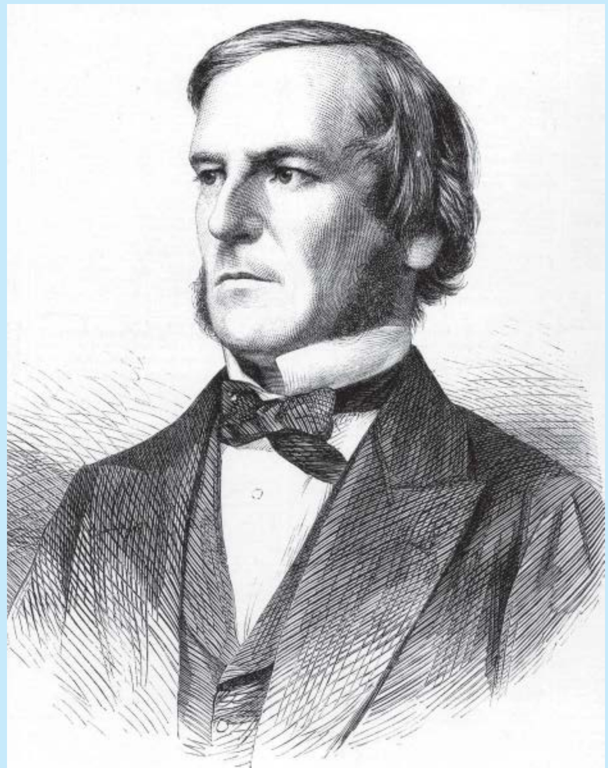
▼ Links to: Arts; Design

Explore this website on Bézier curves and the work of artist, Picasso

https://jeremykun.com/2013/05/11/bezier-curves-and-picasso/

MEET A MATHEMATICIAN: MARY EVEREST BOOLE (1832–1916) AND GEORGE BOOLE (1816–1864)

Learner Profile: Communicator



■ Mary Everest Boole and George Boole

A mathematician who worked a great deal with Bézier curves, even though they weren't called that at the time (curve stitching) was Mary Everest Boole (1832–1916). Mary Everest Boole was a largely self-educated mathematician who contributed greatly, not only to her husband's work, but also in improving mathematical education communication. She was a progressive educationalist who wanted to encourage joy in mathematics through play, for example through **curve stitching**. Her book *Philosophy and Fun of Algebra* explained algebra and logic to children in interesting ways, starting with a fable, and including bits of history throughout.

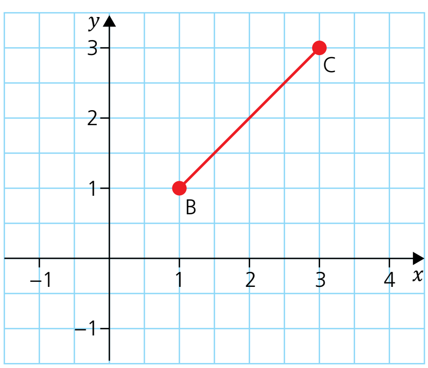
George Boole (1816–1864), her husband, is often credited as laying the foundations for the Information Age by his invention of a system of Boolean logic. Boolean algebra is a branch of algebra in which variables are communicated as either true or false, 1 or 0. Using AND, OR, NOT and other operations allows us to describe logical relations in the same way that algebra describes number relations. Search engines use Boolean Algebra to find and to refine search results.

Their daughters were also highly involved or influential in mathematics and interdisciplinary fields such as four-dimensional geometry and chemistry. Many of their grandchildren were equally impressive – find out more about them.

WHAT ELSE CAN WE TELL FROM A LINE SEGMENT?

Another property that lines and line segments share is the property of gradient. As you know from Chapter 3, $y = mx + c$ where m = gradient.

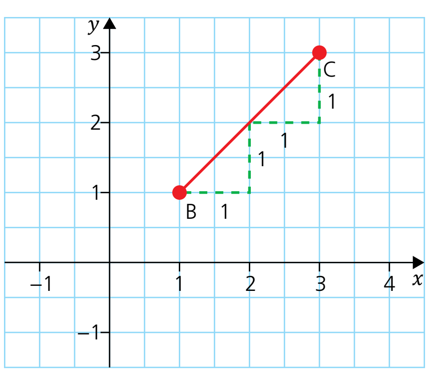
Take an example line segment BC, in the grid below:



■ Line segment BC

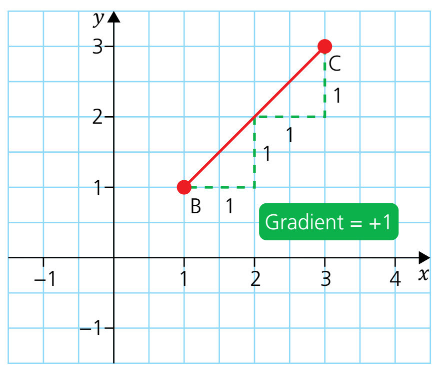
We know the gradient, or slope, tells us how quickly the line increases or decreases with relation to the x-axis. You often hear it described as $\frac{\text{rise}}{\text{run}}$ or the $\frac{\text{change in } y}{\text{change in } x}$

If we look at how the y-value (or dependent) changes relative to the x-value (or independent), we can see:



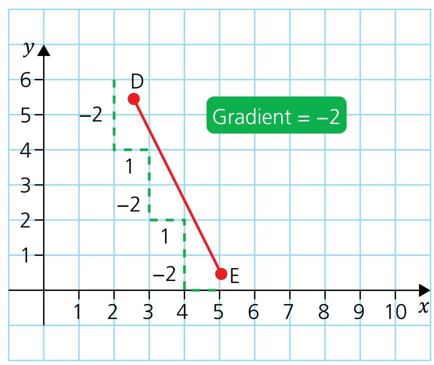
■ Find changes in values

Which tells us the line segment has a gradient of +1



■ Gradient = +1

Similarly, in the grid below, we can see that the line segment DE goes down two boxes on the y-axis for every step across the x-axis.



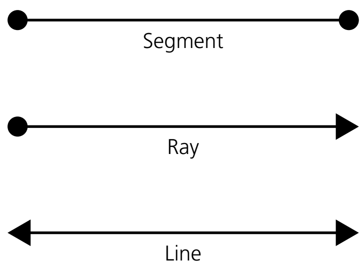
■ Line segment CE

Therefore, we can tell that for DE $m = -2$.

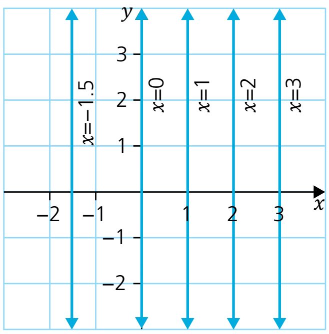
WHAT OTHER TYPES OF LINES CAN WE HAVE IN THIS SPACE?

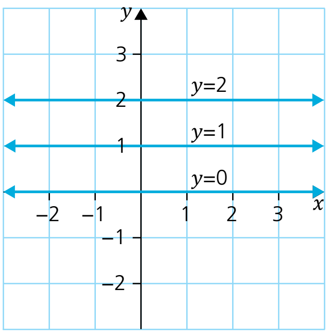
What is the difference between a line and a line segment? A line continues on without ending in both directions whereas a line segment is a section of a line with end points. So, what is in between these two types?

A ray is between a line and a line segment because one side of the line has an end, a finite point or terminus while the other side extends to infinity. Can you link this to the same word in everyday situations or other subjects?



Segments, rays and lines





Horizontal and vertical lines are easily recognized because one of the variables will not change, i.e. only a constant will remain. For example, in the diagram you will see vertical line $x = 1$. This means that all points on this line will have an x-coordinate of 1, for example (1, 3), (1, 10) (1, –2) are all points on the line. Likewise, a horizontal line will have a constant y-value.

If we look at any of these lines, we will notice that all vertical lines, as well as all horizontal lines, are parallel to each other. What is meant by parallel?

SPECIAL LINES

While all lines are significant, there are some which have particular characteristics relating to the space they occupy.

A view of bullet trains lined up at a train station in Japan – an example of parallel rays

ACTIVITY: Naming vertical and horizontal lines

ATL

- Transfer skills: Apply skills and knowledge in unfamiliar situations
- Collaboration skills: Help others to succeed

Adelaide, Australia

The image above shows an aerial view of part of the city of Adelaide, Australia. Your task is to **superimpose** a Cartesian grid on the city of Adelaide so you can correctly label each street with an equation of a line to represent it. In pairs, you must decide how and where to draw the axes so you can give the equations for as many lines (streets) as you can.

This is a timed activity – each pair has 15 minutes to label as many streets as possible with the correct equations. You and your partner must collaborate to divide the work in a way to get as many as possible in the time.

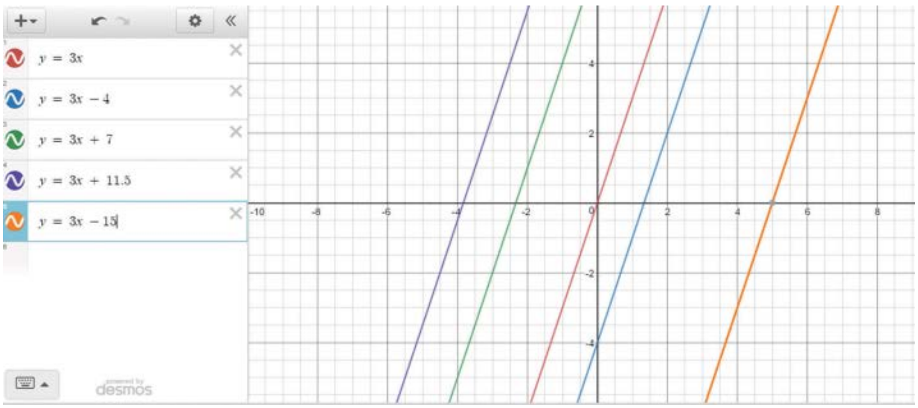
You must also communicate to decide what to do with any lines which are neither horizontal nor vertical, time permitting. You may wish to use other equipment in this task, such as a ruler or calculator.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts.

PARALLEL AND PERPENDICULAR LINES

The diagram below shows five different lines, each in the form $y = mx + c$:



■ Lines in the form $y = mx + c$

Notice that when the lines all have the same gradient of 3, the lines will never intersect, get closer (converge) or get further away from one another (diverge). The only thing which keeps these lines apart from one another is the different constant (y-intercept) they all have.

PRACTICE EXERCISES

Identify the pairs of parallel lines in the following groups of 3:

1 $y = 5x + 14$

2 $y = \frac{10x}{2} + 2$

3 $y = -2x$

4 $y = \frac{1}{2}x - 2$

5 $2x + y = 4$

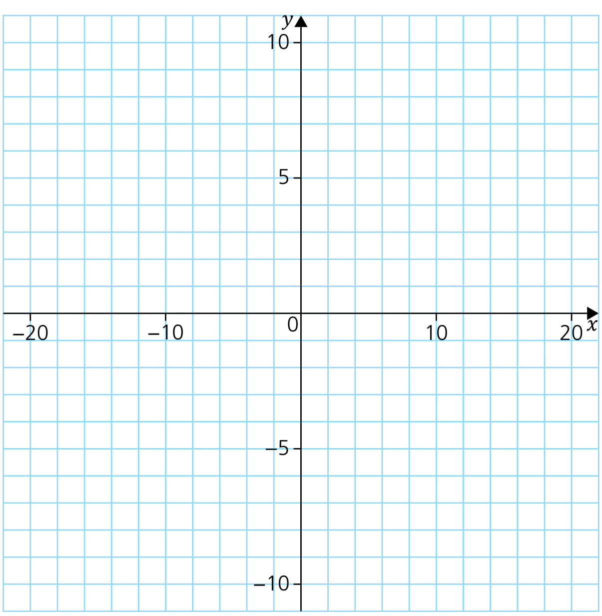
6 $y = 0.5x + 200$

7 $y = 6x$

8 $6x + y = 4$

9 $6x - y = 7$

ACTIVITY: Parallel lines and bounded regions



■ A Cartesian plane

1 **Using a grid such as this one, plot the lines:**
   a $y = 2x + 1$
   b $y = -x - 3$
2 **Identify the point of intersection of these simultaneous equations.**
3 **State the y-intercepts for both equations.**
4 **Draw three lines which are parallel to $y = 2x + 1$.**
5 **Draw four lines which are parallel to $y = -x - 3$.**
6 **Label each line with the appropriate equation.**

Several of the lines intersect and create a **bounded** region. **Create** a colourful pattern by shading in each of these bounded regions a different colour.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

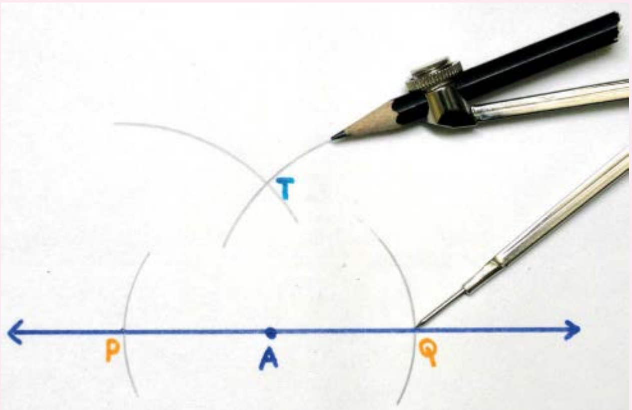
HOW DO WE CONSTRUCT PERPENDICULAR LINES?

Perpendicular lines

A perpendicular line is one which is at 90° to another line. There are two tools which can help us construct perpendicular lines by hand – a compass and a set square.

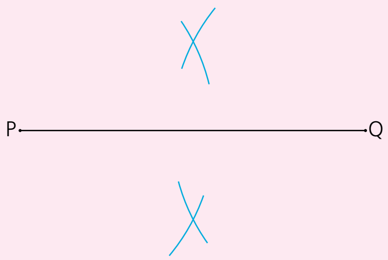
Method 1: Constructing a perpendicular line with a compass

A compass is a tool for marking out equidistant points on a circular path. Widen the distance between the pointer and the pencil to make a large radius. Place the compass on the line and make marks above and below the line.



■ Construction with a compass

Do the same action from a different point on the other side of the line. If the radius has been chosen to be large enough, they will cross to form an intersection point on either side of the line.



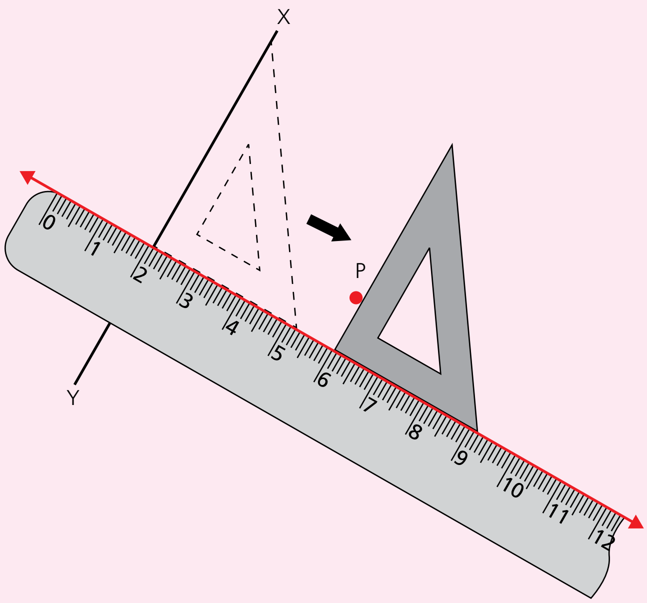
■ Intersection points

Use a ruler to connect the points of intersection and draw the line between them. This newly drawn line is perpendicular to the original line.

Visit **www.wikihow.com/Construct-a-Perpendicular-Line-to-a-Given-Line-Through-Point-on-the-Line** for more detail on this method.

Method 2: Constructing a perpendicular line with a set square

To draw a perpendicular line to a given line, place a ruler along the original line. Place the set square on its side and place on top of, adjacent to, the ruler. Now you will be able to move the set square along the ruler. The set square is now at right angles to the original line. Draw the perpendicular line using the upright side of the set square.



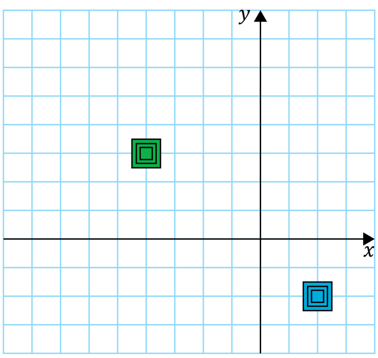
■ Construction with a set square

What are vectors?

A line segment has a length – a starting point and an ending point. If the line segment has a length *and* a direction associated with it, or if it describes a movement, then we call this a vector. We can use vectors to describe movements through space, such as airplanes or drones flying through three-dimensional paths.

Example

Describe the transformation that moves the blue box to the green box as a vector.



■ Transformation as a vector

If we look at the path that moves the first box to the position of the second box, we notice it moved five boxes up. It also moved six boxes to the left or in the negative direction.

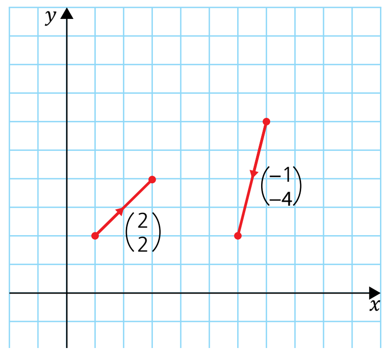
Remember in a vector, just like coordinates, the x-term will come first. In this case the x term will be in the top part of the vector and the y term in the bottom.

So this transformation can be described as the vector: $\begin{pmatrix} -6 \\ 5 \end{pmatrix}$.

Vectors

Vectors can be written in the form $\begin{pmatrix} a \\ b \end{pmatrix}$ or $a\mathbf{i} + b\mathbf{j}$

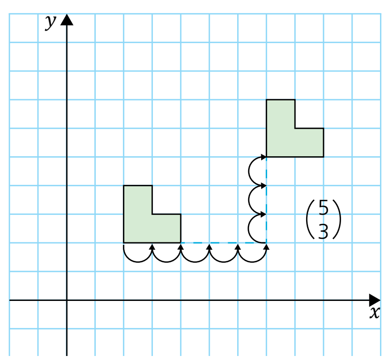
where a and b represent the x- and y-values of the movement or direction.



■ The form of vectors

Where $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ represents 2 in the positive direction along the x-axis and 2 in the positive direction along the y-axis. You could also describe this as '2 to the right, 2 up'.

Likewise $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ indicates a movement in the negative direction along the x-axis of 1 and along the y-axis 4 boxes in the negative direction. This can also be described as '1 to the left, 4 down'.



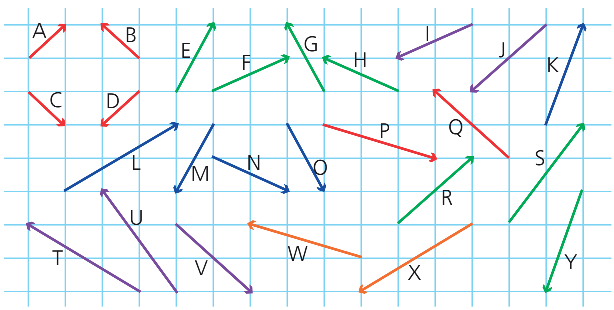
■ Movement on a grid

In the grid above you see that the L-shape has moved five boxes to the right and three boxes up. If the L-shape was moving in the opposite direction, i.e. returning back to its original position, what do you think the vector for that would be?

ACTIVITY: Name that vector!

ATL

- Communication skills: Understand and use mathematical notation



Vectors

Links to: Physics

Want more practice? Try **www.physicsclassroom.com/Physics-Interactives/Vectors-and-Projectiles/Name-That-Vector/Name-That-Vector-Interactive**

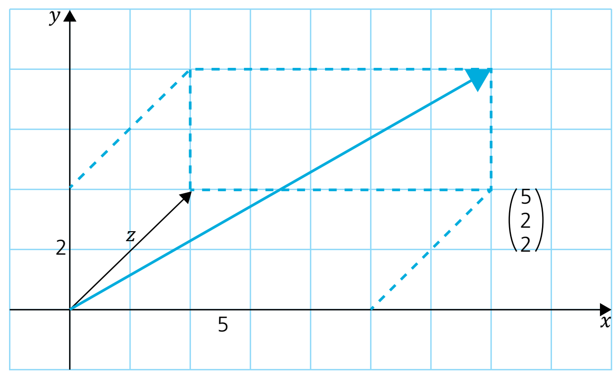
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

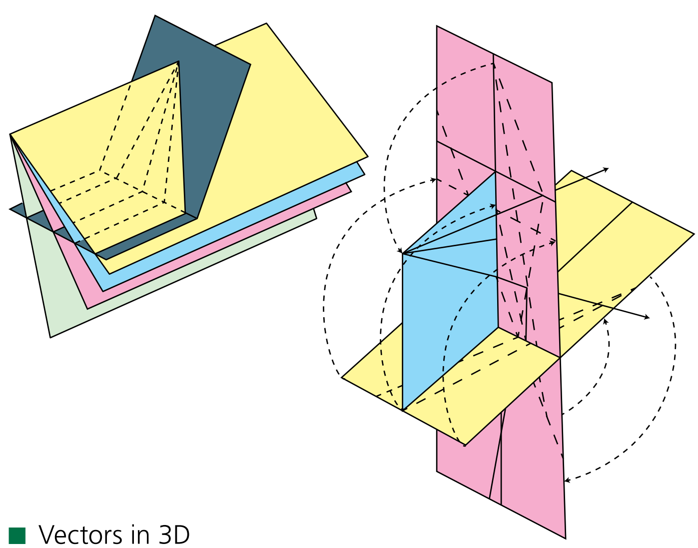
As **vectors** can describe movement, they are a useful tool to describe **translations**. If an object has moved, a vector can indicate how far it travelled and in which direction. This is because a vector has a **magnitude** (length) and a direction indicated by an arrow. Without the direction, it would be just a line segment.

EXTENDED: VECTORS IN 3D

If we add a third row to the vector, we can use this to indicate a third dimension. This dimension is difficult to draw in 2D but it represents the direction coming out of, or going behind, the page. Vectors are also incredibly important in computer animation and digital communications.



Vectors in a third dimension



Vectors in 3D

THINK–PAIR–SHARE

Now that you have considered a vector in three-dimensional space, what other geometric rules can you visualize in three dimensions? Will a midpoint be relevant in three dimensions? Length? Angles and gradients?

ACTIVITY: Pre-assessment task

ATL

- Transfer skills: Inquire in different contexts to gain a different perspective

■ Barcelona, Spain

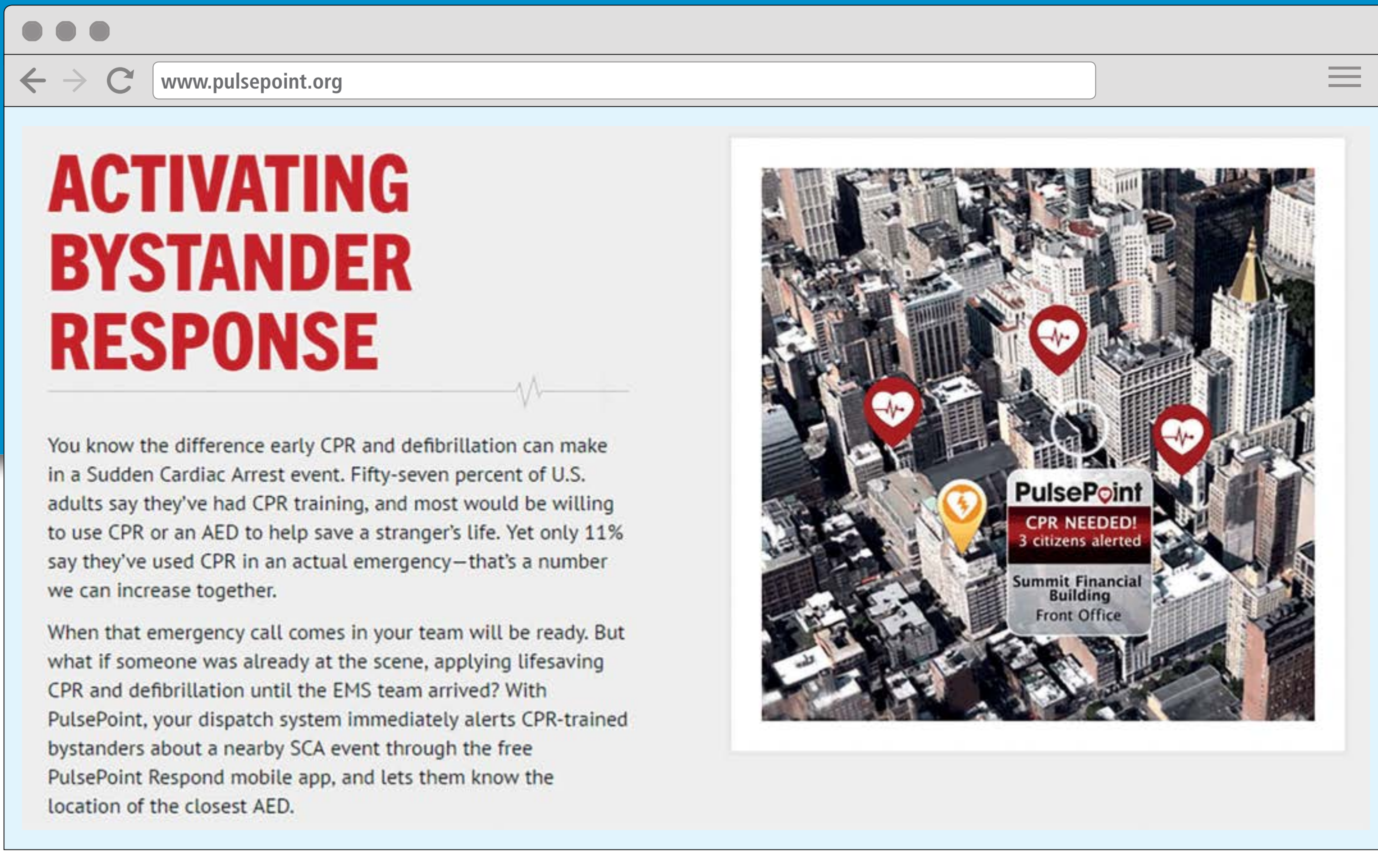
Barcelona is another good example of a city mapped out in a grid BUT the orientation doesn't match the right direction in space to fit easily on a typical x–y grid. Can you think of a solution to this problem?

You may wish to draw a grid on top of the image, in order to be able to use the skills you have learned to answer the following questions. Use your solution/grid to:

1. **Find the distance between the Hospital Clinic and Luz de Gas.**
2. **Find the midpoint between the Hospital Clinic and Luz de Gas.**
3. **Find the gradient of the Av. Diagonal.**
4. **Find the gradient of the other transversal street (a street which cuts across the grid formation).**
5. **Calculate the perpendicular distance from the Hospital Clinic de Barcelona to *both* transversal streets.**
6. **According to your new grid, find the equation which describes the Carrer d'Arago.**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real life contexts.


www.pulsepoint.org

ACTIVATING BYSTANDER RESPONSE

You know the difference early CPR and defibrillation can make in a Sudden Cardiac Arrest event. Fifty-seven percent of U.S. adults say they've had CPR training, and most would be willing to use CPR or an AED to help save a stranger's life. Yet only 11% say they've used CPR in an actual emergency—that's a number we can increase together.

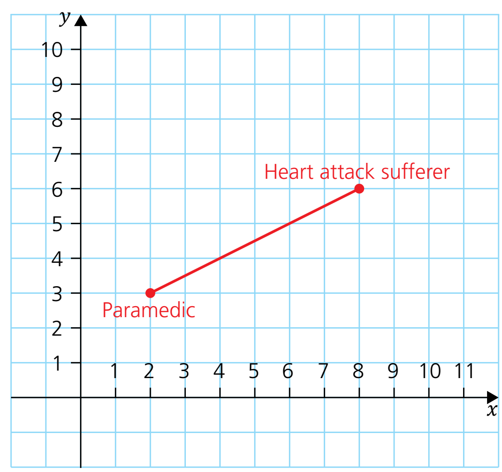
When that emergency call comes in your team will be ready. But what if someone was already at the scene, applying lifesaving CPR and defibrillation until the EMS team arrived? With PulsePoint, your dispatch system immediately alerts CPR-trained bystanders about a nearby SCA event through the free PulsePoint Respond mobile app, and lets them know the location of the closest AED.

WHAT OTHER USES CAN THIS GEOMETRY HAVE IN TECHNOLOGICAL INNOVATIONS?

Pulsepoint is an app which connects emergency services with someone in need of assistance in a local area.

It allows cities to become more connected so that any nearby trained responders can help while the emergency services are on their way. In these life-threatening situations, distance can be a danger, so location and direction are incredibly important.

Again, we can see the impact geometry will have on this real-life context if we represent the distances on a Cartesian grid:



■ Geometry saving lives

ACTIVITY: Research

There are many other applications of these ideas which have been enhanced by **technology and GPS**. For example, orienteering has been a popular outdoor activity for a long time but technological advances have enabled variations on this such as **geocaching, geohashing, benchmarking** and many others.

Carry out research to learn more about what these activities are and how they work. Consider, and reflect on how these uses of geometrical ideas can be a form of personal expression.

A SUMMATIVE TASK TO TRY

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D

Task 1: Mobile app

You have been tasked by an Innovation team to create an idea for a mobile app (an application or program which can be used on a handheld device) using GPS. The app must use geometric principles such as those you have learned in this chapter to make use of GPS technology.

The purpose of the app may be to:

- Fulfil a personal or individual user's needs.
- Address or benefit a social or cultural issue.
- Create a financial opportunity which will help others.

In your proposal, you must:

- **Explain** how your app will use location to address a need (or needs).
- **Identify** what geometric strategies or calculations will be used by the programmers.
- **Justify** how accurate (or true to real life) the mathematics will be to the real-world situation.

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: How do we find the length of a line segment? How can we find distances between points on a map? What are vectors?					
Conceptual: How do we describe points, lines, line segments and vectors? How can logic help us to solve problems?					
Debatable: Can there be bad applications of geometry? Is the mathematical ability to find location in opposition to personal and cultural norms?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Communication skills					
Creative-thinking skills					
Collaboration skills					
Transfer skills					
Critical-thinking skills					
Media literacy skills					
Learner profile attribute	Reflect on the importance of being a good communicator for our learning in this chapter.				
Communicator					

6 How well do data reflect reality?

We must take care to ask the right questions and to measure the correct data to understand **relationships** so we can use information to **make the world a better and fairer place**.

CONSIDER THESE QUESTIONS:

Factual: What are data? How do we collect data? How is our understanding and use of data changing? How can we use data to develop ourselves and the world around us?

Conceptual: How can we find out what collected data tells us? How can change be represented fairly to all? How can we better understand the relationships which cause change and those which are correlated? How are our relationships affected by data?

Debatable: Just because something can be counted, does that mean it should be? Can we overcome bias in data? How personal is the interpretation of data? How well does data reflect reality?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

'The universe is made of energy, matter and information but information is actually what makes the universe interesting. People tend to think of it as messages or something ethereal but information is physical; it's the arrangement of physical things. When you shuffle a deck of cards, you're not changing the mass but changing the way in which they are arranged – you're actually changing the information' – Cesar Hidalgo, Why information grows, RSA Journal *Issue 2, 2015*

IN THIS CHAPTER, WE WILL …

- **Find out** how to source, sort, process and evaluate data.
- **Explore** how much statistics can (or can't) tell us about reality.
- **Take action** by addressing our own, and other people's, misperceptions about what data are actually telling us.

These Approaches to Learning (ATL) skills will be useful …

- Critical-thinking skills
- Communication skills
- Affective skills
- Media literacy skills
- Transfer skills

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

You	Will	See	Lots	Of	Tables
In	This	Chapter	As	Tables	Are
A	Useful	Tool	For	Organizing	And
Representing	Data				

We will reflect on this learner profile attributes …

- Caring – we show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.
- Open-minded – we critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.

PRIOR KNOWLEDGE

You will already know how to:

- find three types of average – mean, mode and median
- collect simple data, individually or in groups, as tally charts and frequency tables
- create, read and interpret bar charts, pictograms, pie charts and line graphs.

KEY WORDS

anomalies	deviation
bias	outliers
causation	privacy
correlation	sensitive

THINK–PUZZLE–EXPLORE

Just because we *can* measure something, doesn't necessarily mean that we *should*. Just because we can find a statistical measure, it doesn't mean it will tell us anything. While computers are excellent generators, collectors and processors of data, it takes human reasoning and interpretation to draw conclusions from the data. **Our ultimate goal in data handling is to find the signal in the noise.**

- **What could this phrase mean? What is meant by 'noise' and why would that be in data?**
- **What do you think you know about this topic?**
- **What questions or puzzles do you have?**
- **How can you explore this topic further?**

LEARNER PROFILE: CARING

We often consider the Learner Profile attribute of **Caring** as a reminder that we should take care of others, however, in this chapter we will look at it somewhat differently and will be using our ability to care for mathematics itself.

It is important that we care *for* numbers and that we listen carefully to hear what the numbers are trying to tell us. We should be caring of the data we choose to investigate and what we do with them. We have a commitment to handle these data respectfully, carefully and correctly. We aim to avoid assumptions, laziness or shortcuts. This caring for the statistics will help us to try to avoid bias and might even stop us from drawing unfair conclusions.

Think of it as 'you' investing in 'future you' because mathematics will, in return, care for you by helping *you* as a lifelong learner. It will help you become statistically literate and insightful into what numbers actually tell you – not what others tell you that they tell you!

Spoiler: Sometimes you can find connections in data that tell you absolutely nothing. You will find out more about these **spurious correlations** later in this chapter.

What are data?

WHY DO WE COLLECT DATA? WHY IS INFORMATION IMPORTANT?

Hint

Individual points of data are called a ***datum****. The plural for datum is data, so make sure when you say data, you follow it with a plural form of a verb; for example, 'the data* ***are*** *highly accurate', 'irrelevant data* ***were*** *collected', the data* ***tell*** *us that the trend is increasing.*

Links to: Individuals and societies

Humans have been recording data since civilization began – archaeologists and historians have discovered vast evidence of humankind's interest in information from the movement of the stars to ancient tax records such as the Rosetta Stone (195 BCE). We also see the importance of the recording of family histories in religious books; for example, the Holy Bible, the Qur'an and the Torah, as well as descriptions of large-scale data collection in the form of a **census**. These are all examples of datasets in different forms.

ACTIVITY: Which is which?

ATL

- Communication skills: Understand and use mathematical notation

State whether each number or set of numbers is a datum or are data

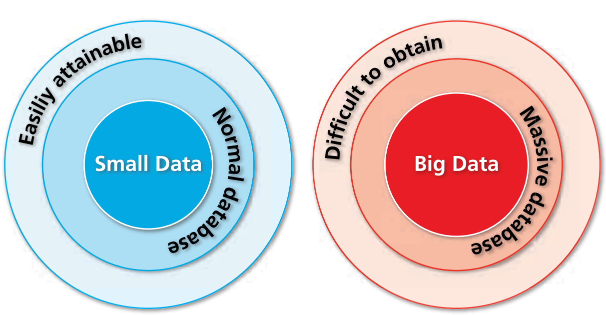
3, 8, 2, 4, 3	:	datum / data
$100 000 000	:	datum / data
8, 2, 1, 7, 2, 7	:	datum / data
Green, red, green	:	datum / data
Blonde hair	:	datum / data
2nd, 3rd, 3rd, 1st	:	datum / data
1.23×10^{-3} kg	:	datum / data
00100101	:	datum / data

In pairs, discuss how you knew which was which. Explain the difference between datum and data.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding

HOW IS OUR UNDERSTANDING AND USE OF DATA CHANGING?



Small data vs Big Data

Most of your previous experience with data handling will have been with small datasets. Maybe you were adding a list of 10 numbers or calculating the mean heights of your fellow students. These datasets are easier to record and to process. Calculating the mean height of every class in every school in the world would be a much bigger, and more complex, task. There has always been a huge amount of information available all around us but technology has now made it possible to record and to store more data than ever before. Sources of Big Data include:

Lifelogging: lots of people use their smartphones to track all kinds of personal information, such as location, physical activity and finances.

Sports: fans and broadcasters can access a huge amount of statistics drawn from data both during and after the sports have taken place. Computers can track the distance run by each player during a football game, shots on target of each puck in hockey, even the pressure on individual wheels in a Grand Prix race.

Internet: Click activity, viewing time, likes per image, reblogs and cookie history are all examples of data generated by internet activity. 'Data brokers' can keep track of this type of information and sell this to marketing companies so that they can target specific markets for better success.

Health: Historically, insurance companies were among the first to share data to discover patterns. Now information can be decoded from DNA and computer analytics allow us to find hidden information. These big datasets can be used to find cures, to prevent or cause mutations or to transfer characteristics from one situation to another.

SO WHO USES THIS DATA?

This tracking of Big Data is already affecting our real and virtual lives. The advertisement and search results we are shown are determined by Big Data and algorithms that use them. Personalized medicine is making advances due to developments in computing power on datasets. Citizen Science uses Big Data to gather huge amounts of data from individuals who want contribute to science experiments but who otherwise wouldn't have had the chance, for example Pollen Counts or Galaxy counting. These types of experiments were not possible on this scale before.

When Forbes Magazine published its list of the most promising jobs of 2016 and beyond, **Data Scientist** came first for salary and job satisfaction, topping several best paid jobs lists over the past few years. This and other articles drew attention to the rise of the Data Scientist (Statistician). The role of Data Scientist is a new interpretation of the job of Statistician and is constantly developing due to ever larger and larger datasets. Over the last few years, companies and organizations have been collecting a vast amount of data on products, services, groups and individuals. They hire data scientists to help them to make sense of the data. Data scientists can sift through or drill down into the data to find patterns, identify **correlations** and even find possible **causations**. Data-handling has become a very highly sought-after skill not only in the role of Data Scientist but as a big part of other roles such as Marketing, Sales, Research, Science, Law and Journalism. Take a look at this infographic to find out what a data scientist does: **https://goo.gl/0Dy6Tq**

ACTIVITY: Data overload

ATL

- Critical-thinking skills: Evaluate evidence and arguments
- Information literacy skills: Finding, interpreting, judging and creating information

'More data will be produced in the next year than in the history of civilization.'

Investigate this claim. What could be meant by it? What kinds of data are all around us? Why are we as humans creating so much data nowadays? Is anyone collecting it? If so, how are they using it? Should we be concerned?

Write a short article summarizing your findings and your answers to the questions posed above.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

How do we collect data?

TYPES OF DATA

1 **Discrete vs Continuous**

Discrete data are described as data that can only take certain values and values in between two consecutive discrete values are not possible. Examples of discrete data include clothes sizes and shoe sizes which have fixed values, given that decimal values are not available. Counting the numbers of people is also an example of discrete data as you cannot count a fifth of a person. Names are discrete because there are no values possible between two names like Xena and Yanni.

Continuous data are not restricted in the same way – they can be any possible values over a continuous range. Height of students, currency rates or broadband speeds are examples of continuous data. This type of data is continuous as there are an infinite number of possibilities between any two continuous data points.

2 **Secondary vs Primary**

Primary data are data which are collected by the researcher/person who is processing the data. If you collect the data yourself, then you are using primary data. Collecting data can be done by surveys, experimentation or measurement. Primary data are more time consuming and expensive to collect but you can choose and trust your own sampling.

Secondary data are data which you get from another source that you wish to process. You can find the data from a variety of sources, either online or in books but it is important to remember you might also need to evaluate the source. Secondary data are easier and quicker to obtain but may be outdated or unreliable. Critical thinking skills will help you to recognize unstated assumptions and bias before or after you choose to process the data.

3 **Quantitative vs Qualitative (also known as numerical and categorical)**

Quantitative data are data which have a numeric value – much of the data we will handle will be quantitative in nature. Examples of quantitative data include weight, speeds, test grades and insurance cost. This type of data tends to be easier to analyse using typical processing methods such as averages.

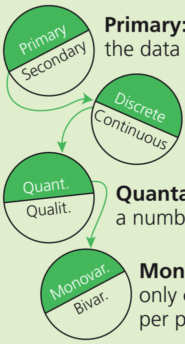
Qualitative data have no numeric values but are descriptive in nature. Examples of qualitative data include eye colour, as eye colour does not have a number but a category (blues, greens, etc.) or description. Similarly, countries, nationalities, flavours and species are all examples of qualitative data.

4 **Monovariate vs Bivariate data**

Monovariate data are data that all describe the same variable. These data are usually in the form of a list or table. Mono comes from the Greek word 'monos', meaning alone. This tells us that the data are all of the same type, alone from other types of data. Examples of monovariate data include birthdates, sunset times or finger lengths as these data are not linked.

If you are investigating a relationship between two sets of data, this is called **Bivariate** data. Bi- in front of a word indicates two. Bivariate data are two sets of data which are being analysed for possible relationships. Examples of bivariate data are the number of words in an essay and the time spent researching, grades vs tv watching time, the wages earned dependent on gender of the employee or the number of arrests related to ethnicity.

Asking your teachers their age (in years).



Primary: You are collecting the data

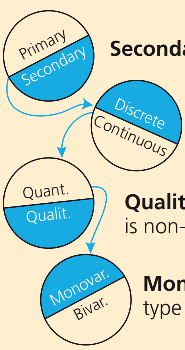
Discrete: Only whole years allowed

Quantative: Age is a number

Monovariate: You are only collecting one datum per person (one question)

∴ The data you collect will be primary, discrete, quantative, monovariate data

Online database of car colours sold in 2017.



Secondary: online source

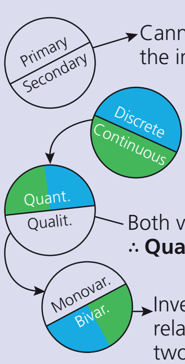
Discrete: Only certain colours allowed

Qualitative : car colour is non-numeric

Monovariate: Only one type of data collected

∴ Car colours sold = Secondary, discrete, qualitative, monovariate data

Data mapping the hours of sunshine vs volume of water in a reservoir.



Cannot tell from the information provided

Hours of sunshine: **Discrete** (hours)

Volume of water: **Continuous**

Both values have a number ∴ **Quantative**

Investigating a relationship between two values so **Bivariate** data

■ Describing the data mathematically

ACTIVITY: Describing data

ATL

- Communication skills: Organize and depict information logically

For each of the data types listed below **describe**, in as much detail as you can, what type of data they are (discrete/continuous, primary/secondary, quantitative/qualitative)

1. **IQ test scores**
2. **Plant growth**
3. **Number of friends you have**
4. **Contact lens colour**
5. **Number of characters in a novel**
6. **Hair colour**
7. **Hair length**
8. **Political party**
9. **Number of votes in an election**
10. **Baby names**
11. **Number of TV channels**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating

HOW IS OUR UNDERSTANDING AND USE OF DATA CHANGING?

In recent years there have been several instances in which polling (or surveying) companies predicted an outcome to an election and got it very wrong! These companies were publicly embarrassed and some people lost faith in them. Some experts thought that they may have asked the *wrong* people, others thought they hadn't asked *enough* people. Another theory was that people were not willing to tell the data collectors the full truth about how they intended to vote, as this can be very personal to some people.

Links to: Citizenship and International mindedness

Remember, it's really important to cast your vote whenever you get a chance, even if your choice doesn't end up being the winning one. It means that your voice is heard and that you have had a chance to express your preference. Whether it is a class election, a local government vote or a country wide election, it is a responsibility and a right to inform yourself and to express your opinion.

DISCUSS

Can reducing personal data to a number or a category be fair or unfair? If you or your data was recorded as only one datum, such as skin colour, ethnicity or IQ how would it make you feel? What is the nature of data and its reflection of reality?

DISCUSSING DATA COLLECTION

www.bbc.co.uk/programmes/p03ymsh3

Listen to the podcast above and consider the following questions. Discuss your opinions as a group.

Why might data be sensitive?

Why is consent (permission) to use data important?

How can data mislead us?

www.youtube.com/watch?time_continue=20&v=2eBmZ0FidcY

Data collection can be automated by technology but we will first look at **manual data collection**.

Sampling

It can be difficult to accurately get data on a whole population, so often a sample is chosen to represent the entire group. Television viewing figures are calculated in this way by having a number of recording devices in a sample of homes to represent the entire country's viewing habits.

Creating a sample from a population

- Benefits of sampling: Sampling makes data collection easier, faster and can be quite reliable depending on the sample size and selection process.
- Disadvantages of sampling: Sampling does not, and cannot, give you the full picture. The results may be affected by who you select and how you select the sample.

There are various methods for sampling:

- **Random** – you do not control who you ask and how you find them. Everyone in a population has an equal chance of being recorded.
- **Systematic** – you use a strategy to choose the sample by using a rule such as every fifth^{h} person will be sampled.
- **Convenience** – you use whatever data are already available (this is possibly the least desirable method but most convenient, hence the name!).
- **Stratified** – this is where you break your population into groups or strata (meaning levels). Then you have a required number from each group to represent the total population groups. Examples might include insuring you have more Grade 6s than Grade 12s in a survey because they are a larger class. Other strata might include sampling different numbers of males and females, or different age groups, based on the overall proportions of the population.

! Take action:

- ! The article '**Is there a sexist data crisis?**' on the BBC website explores how women have historically been left out of data. Read the full article and carry out some initial research to decide whether you think this claim is accurate.
- ! In pairs, groups or alone develop a short list of guidelines for survey designers to help them be more aware of the problem. Include some helpful suggestions or reflection questions they might ask themselves to avoid the problems outlined in the article.

◆ Assessment opportunities

- ◆ In this activity you have practised skills that are assessed using Criterion C: Communicating.

ACTIVITY: TV habits

ATL

- Critical-thinking skills: Test generalizations and conclusions

Áine carried out a survey to collect data on TV watching habits. Her first data collection took place at a local shopping centre on a Tuesday morning from 11a.m. to 1p.m. Her second collection took place at the same shopping centre on Saturday afternoon at 3p.m. to 4p.m. She used random sampling and asked any willing participants who passed her way.

- First collection

Number of TV shows you watch per day	Tally chart	Frequency
0 – 2	卌 卌 |	
3 – 5	卌 |||	
More than 5	卌 卌 卌 卌 ||	

- Second collection

Number of TV shows you watch per day	Tally chart	Frequency
0 – 2	卌 卌	
3 – 5	卌 卌 卌 卌 卌 卌 卌 卌 ||||	
More than 5	|||	

Áine makes the following statement: 'People who shop midweek watch more TV and are therefore lazier than the people who shop at the weekend'.

Evaluate Áine's comment. Make sure that in your answer you

- **Copy the tables and complete the Frequency column for each data collection.**
- **Discuss the sampling and population of her survey**
- **Comment on the data she has collected**
- **Comment on the fairness and/or bias in the survey**
- **Discuss the language used by from Áine in her conclusion**
- **Suggest improvements to her investigation.**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts.

DIFFERENT TABLES FOR DATA REPRESENTATION

Remember that there are different types of tables also.

a **Frequency table** of students' ages in detention one week

Age (in years)	Frequency
13	2
14	3
15	3
16	10
17	1
18	1

When calculating the mean of a frequency table, multiply the value of the first column (age) by the frequency and then divide by the frequency.

$$\bar{x} = \frac{\sum fx}{\sum f}$$

where $\bar{x}$ is the mean.

We do this because the numbers have been tallied to be put in the table rather than in a long list, which we would usually add individually.

b **Grouped frequency table** of the same data

Age (in years)	Frequency
$13 \le a \le 14$	5
$15 \le a \le 16$	13
$17 \le a \le 18$	2

A grouped frequency table uses groups or intervals of values to represent the collected data. This can be especially useful if there is a large range of data.

We find the mean value from a grouped frequency table in the same way as the frequency table with the added step of first finding the **mid-interval value**. Once you know the midpoint of the interval, multiply *this* by the frequency, as before and divide the answer by the frequency.

c **Cumulative frequency table**

Cumulative frequency is a running total of the frequencies, where we add each new frequency to the previous total.

Age (in years)	Frequency	Cumulative frequency
13	2	2
14	3	5
15	3	8
16	10	18
17	1	19
18	1	20

The final frequency should always equal the total sample size. You will learn more about cumulative frequency graphs, and what we can find from them later, in this chapter.

We can process data, i.e. perform mathematical operations on them, to find out what the data tell us. These statistical processes often use formulae and the numbers can be substituted or 'plugged in' to give us an answer.

Statistics is the science of getting the most out of data and interpreting it sensibly and reliably.

It involves a high level of mathematical skill but also benefits from 'joined-up thinking' to be accurate and sensible in the conclusions. We can find patterns or interesting individual points in data, from which we make observations.

How can we find what the collected data tells us?

WHAT ARE 'MEASURES OF SPREADS'?

In your previous mathematical and statistical studies, you will already have studied how to find three types of average: mean, mode and median.

Review of measures of spread (averages)

Mode

You should already know that:

- Mode is the most common datum appearing in a dataset.
- The data do not have to be ordered to find the mode, but sometimes it helps

But you might not have known that:

- Sometimes a set of data might not have a mode. This may be because a particular set of data did not have a most common or frequently occurring number or datum
- Sometimes a set may have more than one mode. If the data has two modes, then we call it **bimodal data**.

Median

You should already know that:

- The median is the middle number (or halfway between the two middle numbers) in a set of data
- The data must be ordered to find the median value.

But you might not have known that:

- the median is a measure of spread which is often used in the real-world contexts of analysing house prices or salary scales. House prices and salary scales can vary hugely from the top to the bottom (or the range) but tracking the median prices over time can tell you if the middle value is overall increasing or decreasing. This measure can be useful to tell the overall trend of the market. The mean price or modal prices cannot give you the same information.

Why would the mean house price be less useful as an indicator of growth? What if an extremely expensive house was sold? How does that affect the mean? Why doesn't it have the same effect/**skew** on the median?

Mean

You should already know that:

- the mean is found by dividing the sum of the data by the number of datum points
- this is the mathematical term which is most often confused with average, in everyday language.

But you might not have known that:

- A mean is often remembered by the following memory aid: 'Don't be mean! Add them up and share them out'.
- 'The sum of the numbers' is often represented by the mathematical symbol Σ (sigma)
- The mean does not always have to include all the data provided. Sometimes it is more useful to identify the mean and see whether it is changing over time. For example, a mean of all goals scored in a season might be less useful then the mean number of goals scored in the last five games. If a team's mean was higher over the last five games than the season mean, this might show a recent improvement. These are known as 'moving averages' in everyday language or better yet, 'moving means' in mathematical terms.

Hey diddle diddle,
The median's the middle;
You add and divide for the mean.
the mode is the one that appears the most,
And the range is the difference between.

Links to: Language and literature

'A rising tide raises all boats' is an idiom in English. It is taken to mean that when things, such as the economy, are improving then it has a positive effect on everyone. How can you link this to the idea of using the median as an indicator of growth or decline?

A group of seven students in a class were asked how many siblings they had. They gave the following answers

0, 0, 1, 1, 1, 3, 6

Find the mode, median and mean of the data. What else do you observe or notice in this data?

Solution

Mode: By looking at the short list, we can easily recognize that there are more 1s than any other number so ...

The mode/modal value is 1.

Median: The numbers are already given in order (ordered data) so there is no need to rearrange them. By crossing off a number from the front and a number from the back of the list, it is possible to work down to find the remaining middle number.

The median is 1.

Mean: To find the mean we must find the sum of the numbers and divide them by 7.

$0 + 0 + 1 + 1 + 1 + 3 + 6 = 12$

$$\frac{12}{7} = 1.7$$

The mean is 1.7, correct to 1 d.p.

Observations:

- There are two "only children" in the group.
- One person has a large family with seven children (six siblings).
- The most common number of siblings is 1; therefore, the most common number of children per family is 2.

The **range** gives the distance from the highest to the lowest. This can be found by subtracting them.

DISCUSS

Look at this image of a car park. How does it resemble a bar chart? In what ways is it different to a bar chart? What else might it resemble?

Another class was surveyed to find their weekly pocket money/allowance. The following amounts were recorded, in Euro:

10, 5, 55, 20, 10, 15, 10, 18, 0, 0, 10

Find the mode, median, mean and range of this data. Give any observations that you notice.

Solution

First order the data

0, 0, 5, 10, 10, 10, 10, 15, 18, 20, 55

Mode: 10

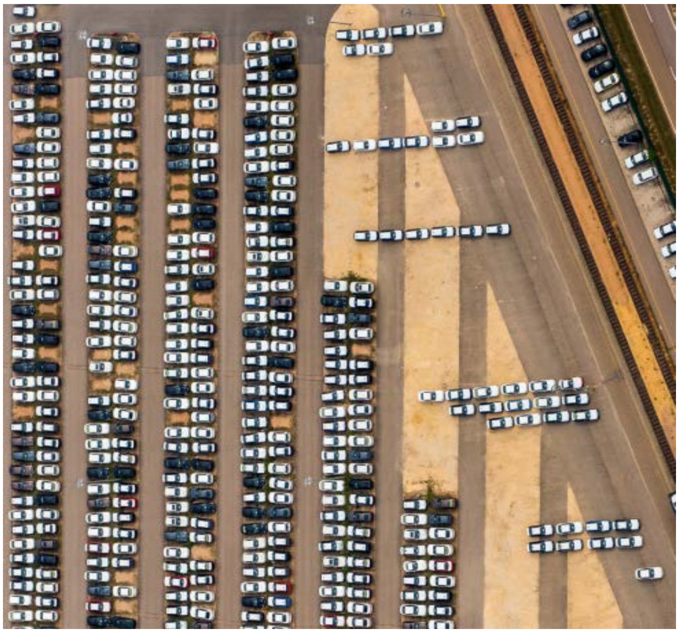
Median: 10

Mean: $\frac{153}{11} = 13.9$

Range: 55

Observations:

- One student receives a very big allowance, which we can identify as an **outlier** or **anomaly** and two students receive no allowance/pocket money at all.
- Another parent trying to decide how much to give their kid might well choose €10 per week as this is the mode and the median and because €13.90 would be an awkward figure to pay each week.



ATTENTION REGULATION

A recent app developer has noted that people were becoming concerned about the time they spent on their phones and how often they checked them. The developers thought it would be useful for people to see the data on the number of times they checked their phone per day. This way people may be surprised how often they look at their phone or might identify if there were patterns or correlations – maybe they check their phone more on a weekend, if they are bored or if some important online event or conversation is happening.

The app creators hope that people will use it to be more mindful of their phone use and encourage us to use our phones less and to tune into what is happening all around us. They do this without judgment and simply present the data and allow us to draw our own conclusions.

Epiphanie has downloaded the app mentioned above and wants to check her phone use over the next fortnight (two weeks). After two weeks, her data shows the following phone checks:

41, 27, 11, 22, 67, 69, 54, 0, 11, 59, 102, 12, 7, 77

What can she tell?

Solution

First, *order* the data:

0, 7, 11, 11, 12, 22, 27, 41, 54, 59, 67, 69, 77, 102

Mode: 11

Median: 34 (halfway between 27 and 41)

Mean: $\frac{559}{14} = 39.93$, correct to 4 s.f.

Range: 102

Observations:

- Epiphanie is analysing the data and sees that there was one day on which she did not check the phone at all (but then remembers it was lost).
- She sees that she checks her phone 40 times a day, as a mean, which she finds to be surprising and a little worrying.
- She also notices that she checks it a lot more on the weekend than on a school day.

Reflection: If you did this, what would your data tell you about your phone use?

ACTIVITY: FANTASY OR FACT?

ATL

- Information literacy skills: Collect, record and verify data
- Critical-thinking skills: Interpret data; Draw reasonable conclusions or generalizations

Cora has been watching a popular fantasy TV series called ‘Shame of Gnomes’, a tale of battles, dragons and zombies. Her parents are concerned that it is too gory and violent. She is interested to find out more about the number of characters who are killed off in every episode and what that can tell you about the show overall.

The casualties per episode are recorded as follows:

0, 3, 2, 52, 1, 0, 2000, 15, 202, 2

Calculate all relevant statistics you can from the dataset.

What else can you tell Cora or her parents so they can make an *informed* decision?

What doesn’t the data tell you, which may be relevant?

Can you suggest other things Cora may wish to measure?

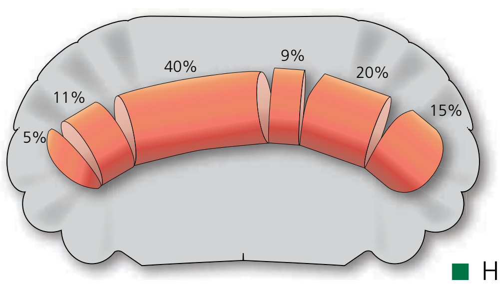
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

How is our representation of data changing?

You will already be familiar with bar charts, pie charts, line graphs, pictograms and scatter graphs. It is very important with all of these types of graphs and charts to ensure that you

- always give the title
- clearly indicate axes, if appropriate
- distinguish between different variables, using colours where appropriate
- include a key/legend if the data has been simplified or rounded
- do not use '3D' versions of graphs and charts if they do not add anything useful or important to the data represented.



Hot dog pie chart

WHAT ARE THE INNOVATIONS IN DATA REPRESENTATION?

Programmes such as Microsoft Excel™ and online applications have made generating representations very easy. While the skill of hand-drawing these representations might become less important, knowing how to show the data effectively and what it tells you becomes **more** important.

ACTIVITY: Poster time

ATL

- Transfer skills: Compare conceptual understanding across multiple subject groups and disciplines

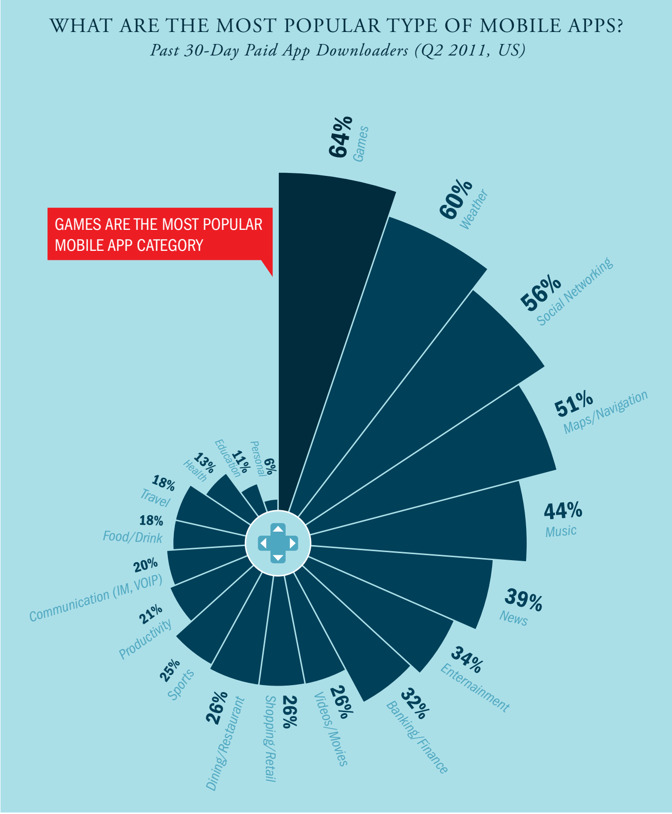
Search for **information visualization** types, click **Images**. Choose five visualization types and match each one to a subject you are studying.

Create an example for each type to show how it could be used to represent information in that subject. Be sure to make them clear and colourful.

You may need to do some research to find out how the visualizations are constructed before you begin them. Remember that many of the visualizations will require mathematical calculation to draw them accurately. It is also possible to construct them as animated infographics if you are feeling like a risk-taker!

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

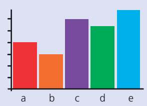
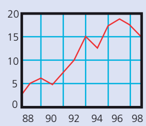
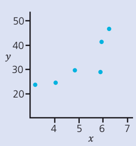
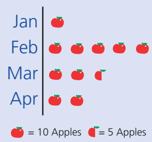


ACTIVITY: Representations of data

ATL

- Information literacy skills: Present information in a variety of formats and platforms

Complete the following table to consolidate what you know about when and how to use these representations:

Name	Image	What you should use them to represent	When you shouldn't use them
Pie charts		Parts of a whole such as composition of the atmosphere or preferences Discrete variables Qualitative variables	To show changes over time To show bivariate data (the relationship between 2 variables)
Bar charts			
Line graphs			
Scatter graphs			
Pictograms			
Stem and Leaf			

ACTIVITY: Compare and contrast

ATL

- Media literacy skills: Understand the impact of media representations and modes of presentation

The diagram at the bottom of page 146 shows the most popular apps downloaded by category. **Compare and contrast** this data representation to a traditional pie chart.

In what ways is it similar to a pie chart? In what ways is it different to a pie chart? How do you think the sizes of the relevant sections are calculated?

Assessment opportunities

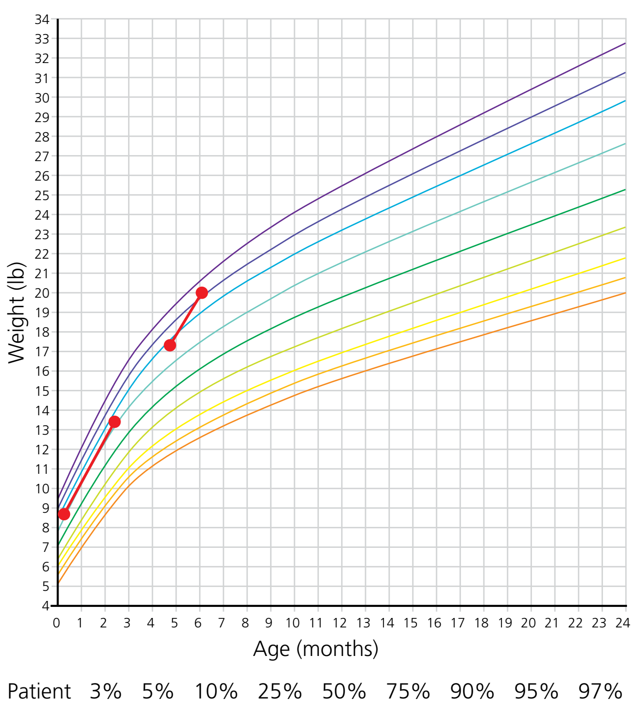
- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts.

How personal is the interpretation of data?

Statistics are often used to give feedback to people based on their comparison with others. A particular example is the use of percentiles with the development of babies.

When a baby or young child visits the doctor, their height and weight are usually measured. The parent(s) or carer(s) of the baby are then told where this places the child compared to a large sample of children of the same age.

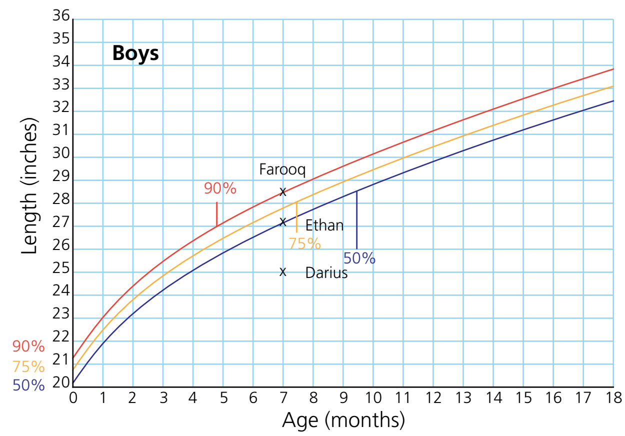
Age vs weight (girls)



Case study: Ellie was taken to the paediatrician for her 6 months' check-up. Her weight was given as 20lb (pounds). From the **graph** you can see that this places her at the 95th percentile for children of her age and gender. That means that 95% of other babies of that age weigh less than her and 5% of children her age weigh more. It can be a useful marker or indicator of development.

PRACTICE QUESTIONS

Read the graph showing three babies' heights marked: Darius, Ethan and Farooq.



1. What is Darius' height (length)?
2. How tall is Ethan?
3. How tall is Farooq?
4. What percentile would be indicated for each child?
5. What percentage of children would be taller than Ethan? How many would be shorter more?
6. What percentage of children would be taller than Farooq? How many would be shorter?
7. How might Darius' parent interpret this information?
8. What can the percentile tell you about the child's development?
9. Should any of these parents worry about the information from the graph?
10. Why do you think the heights are plotted on different graphs for boys and girls?
11. How accurate do you think these graphs are?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion D: Applying mathematics in real-life contexts.

How can we use data to develop ourselves and the world around us?

WHY DO WE DO 'CUMULATIVE' FREQUENCY?

As mathematicians and people, we accept that there will always be outliers. A group will have a spread and while we care for each individual datum, it can be tricky when those outliers are throwing our interpretation off. For example, a famous basketballer's salary might be so huge that it artificially inflates the average wages for his or her ethnic group or age group. This might lead us to make false assumptions about that group. So while we record and handle *all* the information, one useful measure is to look at the middle 50% of the data to see what that tell us. Where is the half of the population that is spread around the mean?

Quartiles are the values at which the variable can be divided to separate the population into quarters.

Three such values exist

1. First quartile (Q_1) or Lower quartile (LQ) – this value is between the first and second quarter of the recorded data
2. Second quartile – this value lies between the second and third quarter of the data. This value is also known as the **median**.
3. Third quartile (Q_3) or Upper quartile (UQ) - this separates the third quarter and the final quarter of the population.

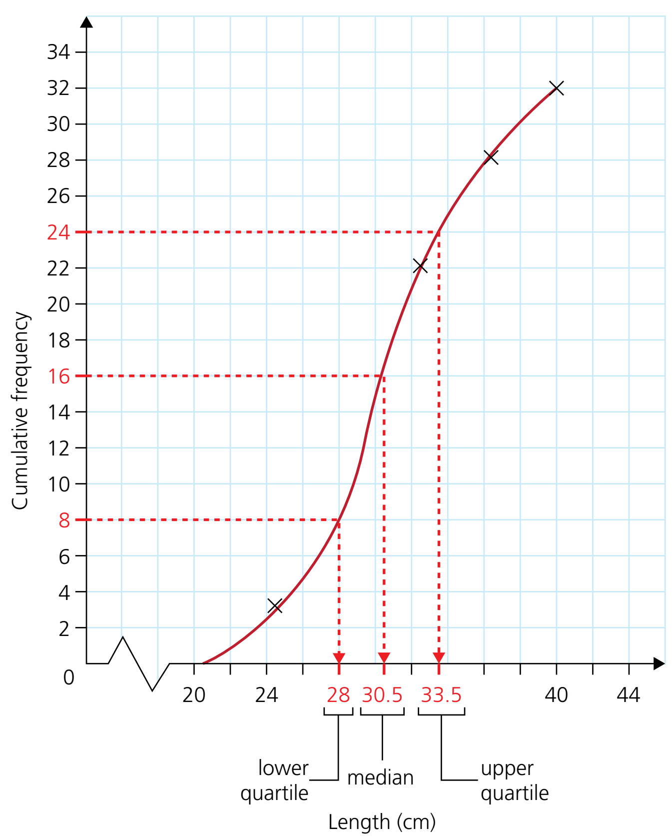
If we isolate the bottom 25% (or below the first quartile) as being very low and the uppermost or top 25% (above the third quartile) as being very high, then we can focus on that middle group. Therefore, the people between the first and third quartiles tells us a lot about that same population. This measure is called the **interquartile range** or

$IQR = Q_3 - Q_1$ or $IQR = UQ - LQ$

ACTIVITY: Cumulative frequency

We have already seen cumulative frequency tables earlier in this chapter, as well as in previous Middle Years Programme 1–3 books. Complete the following cumulative frequency table to show the number of patients with chicken pox admitted to hospital in one week.

Day	Frequency of new chicken pox cases	Cumulative frequency
1	1	
2	3	
3	7	
4	9	
5	5	
6	1	
7	2	



MEET A MATHEMATICIAN: HANS ROSLING (1948 – 2017)

Learner Profile: Open-mindedness

> *'In Hans Rosling's hands, data sings. Global trends in health and economics come to vivid life. And the big picture of global development—with some surprisingly good news—snaps into sharp focus.'*

With his innovative use of data visualizations and his enthusiasm for his subject, Hans was a very interesting speaker on Statistics. His lectures titles include 'How not to be ignorant about the world' and '200 Countries, 200 Years, 4 Minutes' and a TV series called 'The joy of stats'. In these Hans used his knowledge gained from having spent two decades studying outbreaks of disease in remote rural areas across Africa as both a doctor and statistician. He remained open-minded to what the data tells us and finds results that often challenge our thinking on the progress being made in tackling some of our greatest challenges.

For more about Hans Rosling and his work, watch his TED talks, his TV shows or visit his development organization's website **gapminder.org**. He challenges people's preconceptions by asking them to rethink development and to take his Ignorance Test. Sadly he passed away in 2017 but left an incredible legacy of hope and optimism in using data handling to tell us the truth.

Listen to a tribute to his life here: **www.bbc.co.uk/programmes/p04sbvlt**

THINK–PUZZLE–EXPLORE

- **What do you know about the phenomenon of fact-checking?**
- **Why do you think it happens?**
- **Have a look at the work of Fact Check Africa to find out more.**

! Take action

Open mindedness

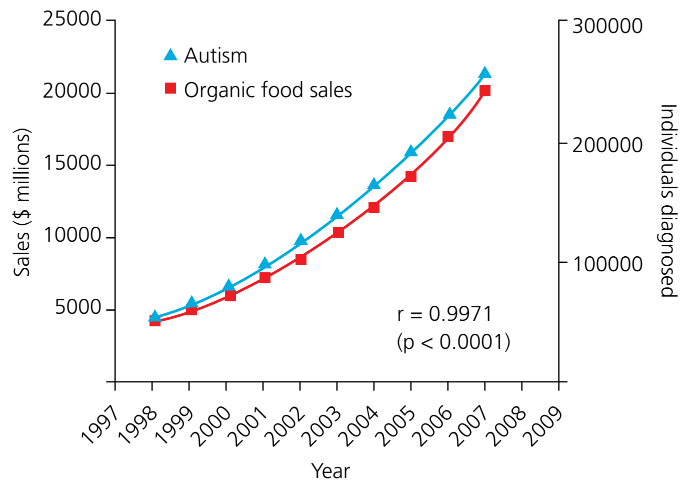
! Complete the following multiple choice questions. See if you can do better than the chimpanzees:

www.bbc.com/news/magazine-24836917

! Listen to Hans discussing the quiz and his findings here: **www.bbc.co.uk/programmes/p02rzjy0**

! Reflect: What does this quiz tell you about your own open-mindedness? Do you feel better equipped to find the truth inside the data?

HOW CAN WE BETTER UNDERSTAND THE RELATIONSHIPS WHICH CAUSE CHANGE AND THOSE WHICH ARE CORRELATED?

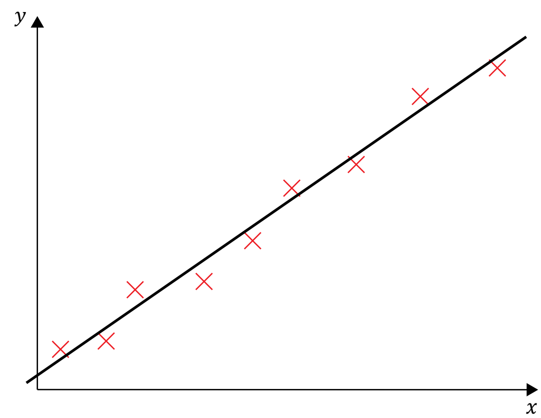


- Does this graph show correlation or causation?

The image shows an often-shared graph used to explain the difference between causation and correlation. The graph shows the increase in both autism and organic food sales for the years 1998 to 2007. Looking at the graph, people could make the quick conclusion that because the numbers both increase at similar rates, that there must therefore be a link between them. Is there an increase in autism because more people are buying more organic food? Can organic food cause autism?

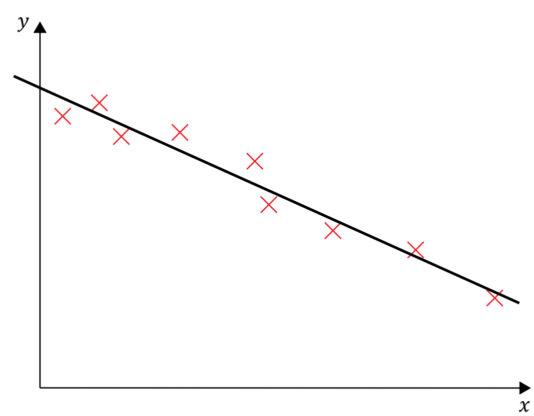
Correlation is the process or measure of finding out a relationship or connection between two things, as in the example shown above. It does not prove a link that one causes the other. Causation is the proof that one variable has affected the other one, i.e. that there is a cause and effect happening. Mathematicians will use correlation regularly to measure the relationships within data and to identify possible causations, while scientists will investigate causation by and through experimentation.

Positive correlation:



- The gradient of the line of best fit is positive.
- As the x-values increase, the y-values also increase. The variables are **directly proportional**.
- The stronger the correlation, the nearer the points are to the line.
- Example of positive correlation – the **more** you read, the **greater** your vocabulary.

Negative correlation:

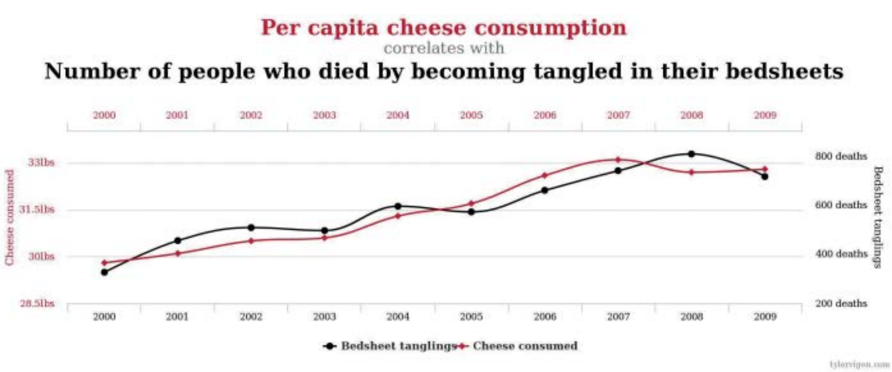


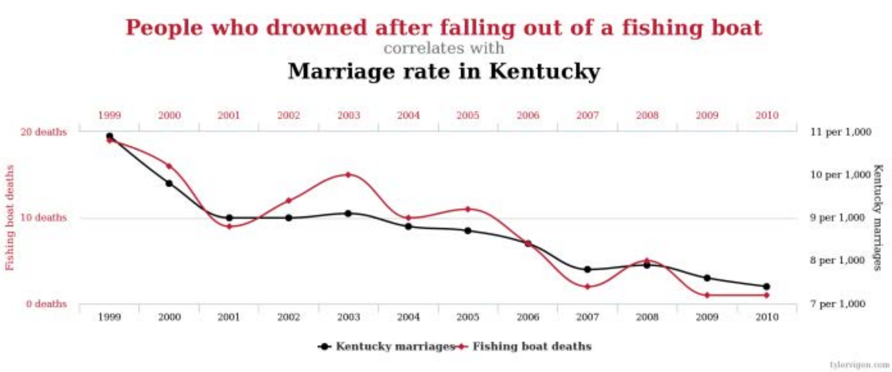
- The gradient of the line of best fit is negative.
- As x-values increase, the y-values decrease. The variables are **inversely proportional**.
- The stronger the correlation, the nearer the points are to the negatively sloping line.
- Example of negative correlation – the **older** you get, the **lower** your maximum heart rate becomes.

SPURIOUS CORRELATIONS

Tyler Vigen curates a website called **www.tylervigen.com/spurious-correlations** on which he shows correlations that seem to be related, but common sense tells us are not caused by one another. Tyler describes the project:

> *'Spurious Correlations was a project I put together as a fun way to look at correlations and to think about data. Empirical research is interesting, and I love to wonder about how variables work together. The charts on this site aren't meant to imply causation nor are they meant to create a distrust for research or even correlative data. Rather, I hope this project fosters interest in statistics and numerical research.*
>
> *'I'm not a math or statistics researcher (and there are better ways to calculate correlation than I do here), but I do have a love for science and discovery and that's all anyone should need.'*





Links to: Language and literature

What does the word 'spurious' mean? What connotations does it have? Does it change our relationship with the data?

Why not try to find your own spurious or dodgy correlations? **http://tylervigen.com/discover**

- Does eating cheese cause the tangling?
- What does this image tell us?

What Lipstick Tells Us About the Economy

Designer coats and crystal glasses can be hard sells in a tough economy. But lipstick is another story. At least that's the thinking behind the 'lipstick index,' termed by Leonard Lauder, chairman emeritus of cosmetics company Estée Lauder, in 2001 to explain the surge in lipstick sales during that recession.

Lately, the lipstick theory hasn't been holding up. Lipstick sales – on the decline since 2007 – continued to fall in 2010, according to the latest data on the product released by market research firm Mintel. That throws lipstick out the window in the hunt for measures of recessionary spending in the Great Recession. I caught up with Lauder today to ask him what gives with the bomb in lipstick sales and whether that upends his theory.

Lauder's response? Nail polish is the new lipstick. And lipstick wasn't really the point. 'We have long observed the concept of small luxuries, things that can get you through hard times and good ones. And they become more important during harder times. The biggest surge in movie attendance came during the 1930s during the Depression,' says Lauder. That trend is similar to the uptick in lipstick sales during the 2001 recession and other colorful cosmetics like nail polish in this one.

Source: **http://business.time.com/2011/09/14/what-lipstick-tells-us-about-the-economy/**

HOW DO WE FIND STANDARD DEVIATION?

We must take care to ask the right questions and measure the correct data to understand **relationships** so we can use information to **make the world a better and fairer place**.

Another measure by which you can judge a set of numbers is standard deviation. This tells you how far the numbers are away from the average (mean) on average. It is a measure of how spread out the numbers are.

A low standard deviation tells you that the numbers mostly stay close to the mean and are not spread out much comparatively speaking.

A high standard deviation tells you that the numbers vary greatly from the mean and are more spread out.

The larger the standard deviation, the greater the variability of the dataset.

Converse: State the opposite of this rule:

The smaller …

So how do we calculate the standard deviation?

The formula is given as:

$$\sigma = \sqrt{\frac{\sum(x - \bar{x})^2}{N}}$$

where

σ = standard deviation

x = each individual number

$\bar{x}$ = the mean of the numbers

N = the number of numbers (the size of the dataset)

Procedure Pro – But how does it work?

Step 1:	**Find** the mean	$\bar{x}$
Step 2:	**Subtract** the mean from each individual number,	$x - \bar{x}$
Step 3:	**Square** the differences (this will also get rid of any neg. or minus signs)	$(x - \bar{x})^2$
Step 4:	**Add** them together/find the sum	$\sum(x - \bar{x})^2$
Step 5:	**Divide** by how many numbers you have	$\frac{\sum(x - \bar{x})^2}{N}$
Step 6:	**Square** root (as this is the inverse operation to the square root)	$\sqrt{\frac{\sum(x - \bar{x})^2}{N}}$

The final answer is σ, the standard deviation

Hint

Using a table to lay out each steps can be efficient and really helpful for tracking any errors.

Let's see it in operation with some real numbers.

EXAMPLE QUESTION

Find the standard deviation for the following six numbers:

9, 10, 4, 6, 0, 1

Solution

Step 1: Find the mean.

$$\bar{x} = \frac{9 + 10 + 4 + 6 + 0 + 1}{6} = 5$$

Step 2: **Subtract** the mean from each individual x

Draw a table, with a row for each x.

x	$\bar{x}$	$x - \bar{x}$
9	5	4
10	5	5
4	5	–1
6	5	1
0	5	–5
1	5	–4

Step 3: **Square** the differences.

Add a new column so you can easily square each difference.

x	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
9	5	4	16
10	5	5	25
4	5	–1	1
6	5	1	1
0	5	–5	25
1	5	–4	16

Step 4: **Find** the sum.

This time you don't need an extra column because you are adding up the last column only.

x	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
9	5	4	16
10	5	5	25
4	5	–1	1
6	5	1	1
0	5	–5	25
1	5	–4	16
$\sum(x - \bar{x})^2 =$			84

Step 5: **Divide** by how many numbers you have

In this case $N = 6$ from the dataset.

$$\bar{x} = \frac{\sum(x - \bar{x})^2}{N} = \frac{84}{6} = 14$$

Step 6: **Square root** this number.

As the numbers in the dataset were whole numbers, it would be appropriate to give the answer correct to 1 or 2 d.p. usually.

$$\sqrt{\frac{\sum(x - \bar{x})^2}{N}} = \sqrt{14} = 3.7$$

What does this tell you about the numbers?

Real-life context: Anti-bullying campaign.

On the first week of a new term at Santo Andreo High School, there was a high number of bullying incidents reported:

Monday	6
Tuesday	3
Wednesday	11
Thursday	3
Friday	7

What was the mean and standard deviation for the bullying incidents that week?

Solution

Mean $\bar{x} = \dfrac{6 + 3 + 11 + 3 + 7}{5}$

$= 6$ incidents per day

Using the table method:

x	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
6	6	0	0
3	6	–3	9
11	6	5	25
3	6	–3	9
7	6	1	1
$\sum(x - \bar{x})^2 =$			44

$$\frac{\sum(x - \bar{x})^2}{N} = \frac{44}{5} = 8.8$$

$$\sqrt{\frac{\sum(x - \bar{x})^2}{N}} = \sqrt{8.8}$$

$\therefore \sigma \approx 3$

This tells us that there is a high number of incidents per day (mean = 6) and that the standard deviation indicates that many of the incidents reported were ± 3 from the mean (i.e. between 3 and 9). It was also relevant to note that all the incidents came from the same year group (grade). These figures shocked both the students and the teachers and they decided to act.

The students consulted the **Bystander Revolution website** for solutions to possible tensions that might be causing the bullying. They organized their own **discussions** and workshop afternoon, using the **Weekly Stand** actions.

The week following the intervention (workshop), the following data for bullying incidents were recorded:

Monday	2
Tuesday	0
Wednesday	0
Thursday	1
Friday	5

Find the mean and standard deviation. What observations can you make from this week's data, compared to the previous week's?

Solution

Mean

$\bar{x} = \dfrac{2 + 0 + 0 + 1 + 5}{5}$

$= 1.6$ incidents per day

Using the table:

x	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$
2	1.6	0.4	0.16
0	1.6	–1.6	2.56
0	1.6	–1.6	2.56
1	1.6	–0.6	0.36
5	1.6	3.4	11.56
$\sum(x - \bar{x})^2 =$			17.2

$$\frac{\sum(x - \bar{x})^2}{N} = \frac{17.2}{5} = 3.44$$

$$\sqrt{\frac{\sum(x - \bar{x})^2}{N}} = \sqrt{3.44}$$

$\therefore \sigma \approx 1.9$

Observations:

Both the mean incidents of bullying and the standard deviation have decreased. This shows that there has been a reduction in the number of incidents reported and the number of incidents are closer to one another than previously.

This could mean that the bullying has decreased. It might also mean that the *reporting* of bullying has decreased. The statistics cannot tell you which is the case.

Another observation is that Friday seems to be an outlier with the highest number of incidents reported that week. This may be an anomaly or might indicate that the positive effect of the session is wearing off. Again, we cannot say for sure without further investigation.

■ You can use online calculators to find standard deviation

Using technology is an efficient short-cut to finding the $\bar{x}$ (mean) and σ (standard deviation). Using a GDC, you can input the list of numbers, press some buttons and be given the measures of spread. Ask your teacher for instructions on how to input the values and to output the statistics.

So why do we need to learn how to do it manually? While technology can give you instant answers, what you understand from the answers may be very superficial if you don't know where the measures came from. Carrying it out yourself helps you to dive deeper into the meaning and reality of the operations.

PRACTICE QUESTIONS

Santo Andreo High School continues to monitor and record bullying incidents over the whole half-term of 7 weeks. This includes the two weeks that you have already been given.

Here is the data collected:

Week	Mon	Tue	Wed	Thurs	Fri
1	6	3	11	3	7
2	2	0	0	1	5
3	0	1	1	0	4
4	3	1	1	0	0
5	0	0	0	2	0
6	0	2	0	10	3
7	4	5	1	0	0

Level 1–2

1 **Find** the mean and modal number of bullying incidents for the entire dataset.

2 **Explain** why the median might not be useful for analysis

Level 3–4

3 **Find** the standard deviation of the dataset.

Present your work in a logical and organized manner.

Level 5–6

4 **Explain** what that data suggests

5 **Verify** your answers using technology (GDC, app or online tool)

Level 7–8

6 Aoife suggests that it is not accurate to include the week before the workshop in the calculations. She thinks it affects the result and shouldn't be applied to decide whether the workshop was effective or not.

Recalculate the standard deviation for weeks 2 to 7.

7 **Draw** any conclusions you can from this data.

8 **Explain** whether these findings make sense in the context of this real-life situation.

◆ Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts.

SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

1 **State** the difference between datum and data.

2 **Describe** this data in one word:
- a Data you collected yourself
- b Data collected by someone else you do not know
- c Data that has no "in between" values
- d Data that is descriptive

3 You want to record the weight of every student in your class.

Describe how you would gather and organize this information.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

4 **Explain** what is meant by Big Data.

5 **Describe** this data in two words:
- a Students' height collected by your teacher
- b Nationality of students in your school
- c Weights for check-in luggage on airplane flights
- d Attendance at assemblies

6 You want to record the weight of every student in your class.
- a **Describe** how you would gather and organize this information.
- b **Draw** the chart or table you would use to collect this data.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

7 **Explain** why data, and datasets, are becoming larger and larger.

8 **Describe** this data as fully as possible
- a Religious denomination recorded by the National Census
- b Number of teachers vs number of students in Canadian schools
- c Grains of sand on beaches

9 You want to record the weight and height of every student in your class.
- a **Describe** how you would gather and organize this information.
- b **Draw** the charts or tables you would use to collect the data.
- c **Identify** possible sources of concern students might have about you collecting this personal data.

10 **Evaluate** this statement:

Students learning in a second language achieve higher mathematics grades than native speakers.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

11 **Discuss** how data can be Small, Big and/or Complex.

12 **Identify** a dataset that would yield (give) the following types of data:
- a Quantitative discrete data:
- b Secondary quantitative continuous data
- c Primary discrete data
- d Bivariate qualitative data

13 You want to record the weight and height of every student in your class.
- a **Describe** how you would gather and organize this information.
- b **Draw** the charts or tables you would use to collect the data.
- c **Identify** possible sources of concern students might have about you collecting this personal data. How can you ensure that the data would be as free from bias and fair as possible?

14 **Evaluate** this statement:
'Students in our school almost all achieve above average for the school.'

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion D: Applying mathematics in real-life contexts

The following data show 14 people's personal information, including their age, gender and number of friends on a social media site.

Name	Number of friends	Male	Female	Age < 30	Age ≤ 30
Ailbhe	1052		X	X	
Brooke	250		X		X
Christina	153		X	X	
Danielle	77		X		X
Eileen	89		X		X
Fidelis	25		X		X
Greta	558		X	X	
Hendrick	669	X			X
Iain	997	X		X	
Jahdae	2523	X		X	
Kieran	47	X		X	
Liam	93	X			X
Mike	107	X			X
Naaz	930	X		X	

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D TO LEVEL 1–2

15 **Find** the modal number of friends from the data.
16 **Find** the median number of friends.
17 **Find** the mean number of friends.
18 **Find** the range of friends.
19 **State** one thing that appears to you to be fair about this data.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D TO LEVEL 3–4

20 **Find** the mode, median and mean number of friends for the males on the list.
21 **Find** the mode, median and mean number of friends for the females on the list.
22 What observations can you make from these statistics?
23 **State** one thing which might be unfair about the data.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D TO LEVEL 5–6

24 **Find** the mean number of friends for the younger (<30 years old) group
25 **Find** the mean number of friends for the older (>30 years old) group
26 **Identify** any outliers in the data
27 What observations can you make for this group?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D TO LEVEL 7–8

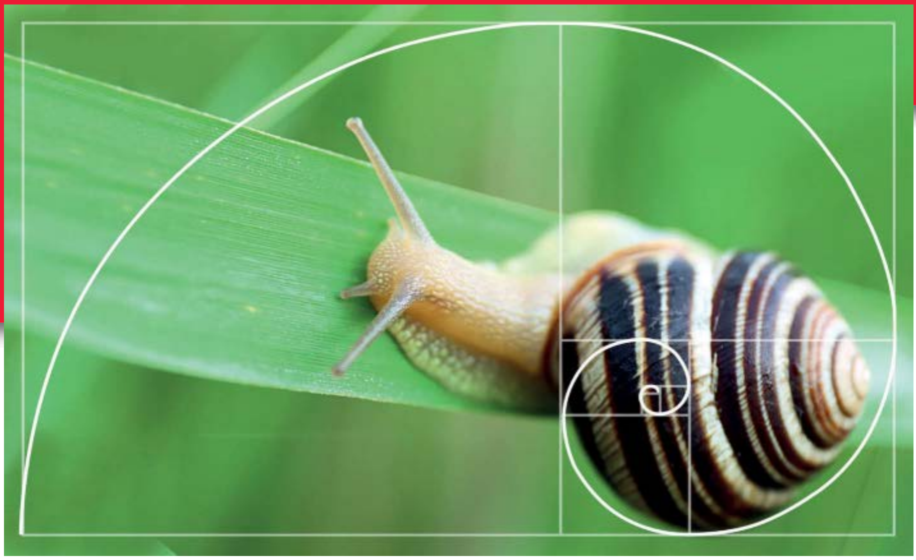
28 Can you **identify** any outlier(s) to the previous relationships?
29 **Calculate** the standard deviation for the younger and the older groups.
30 What observations can you make from these results?
31 What conclusion would you draw if asked about the effect of gender or age on the number of social media friends?

Reflection

Use this table to reflect on your own learning in this chapter.						
Questions we asked	Answers we found	Any further questions now?				
Factual: What are data? How do we collect data? How is our understanding and use of data changing? How can we use data to develop ourselves and the world around us?						
Conceptual: How can we find out what collected data tells us? How can change be represented fairly to all? How can we better understand the relationships which cause change and those which are correlated? How are our relationships affected by data?						
Debatable: Just because something can be counted, does that mean it should be? Can we overcome bias in data? How personal is the interpretation of data? How well does data reflect reality?						
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?				
		Novice	Learner	Practitioner	Expert	
Communication skills						
Media literacy skills						
Creative-thinking skills						
Learner profile attribute(s)	Reflect on the importance of being open minded and caring for your learning in this chapter?					
Open-minded						
Caring						

7 Making the world a fairer and more equal place?

The differences between **quantities** can be represented by **inequalities**, which allows us to solve and **logically** address **inequality** in Mathematics and in life.



CONSIDER THESE QUESTIONS:

Factual: How do we represent inequalities? How do we solve inequalities? What do linear inequalities look like on a Cartesian plane? How do we find general rules for sequences? What are geometric sequences? What other types of sequences are there?

Conceptual: What is an inequality? What quantities are associated with linear equalities? What is a number pattern? Where can we find number sequences or patterns in the world around us?

Debatable: Is inequality always a bad thing? Does life follow sequences exactly?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** more about how number patterns can be hidden around us.
- **Explore** the idea that mathematics can represent both equality and inequality and how we can transfer those ideas to other disciplines.
- **Take action** by discussing our responsibility to recognize and address inequality, when we see it.

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Affective skills
- Transfer skills
- Information literacy skills

We will reflect on this learner profile attribute …

- Caring – we show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.

Number sequences can help us to understand and predict and can be seen in the world around us

Assessment opportunities in this chapter:

- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

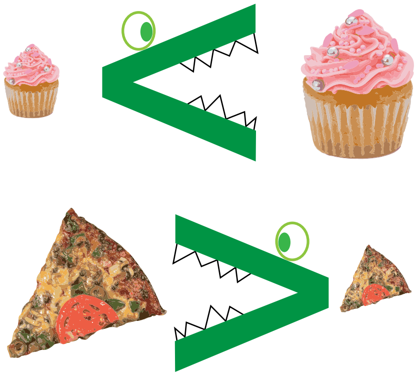
PRIOR KNOWLEDGE

You will already know:

- the meanings of, and symbols for, important numbers sets, **N**, **Z**, **Q** and **R**
- how to represent the concept of *greater than, less than* in symbols
- how to solve equalities and equations to find solutions for lines
- how to represent linear equations graphically.

KEY WORDS

common difference	series
inequality	strict
interval	term
sequence	



Inequalities

DISCUSS

What is a sequence? What is a pattern? Can you think of any patterns that exist around you? Why do humans care so much about patterns?

Links to: Language

Discuss what is meant by equality. Do ideas of inequality and equality change from culture to culture? How is equality related to fairness?

Take action

- Why should we care about inequality? Is inequality ever a good thing? Give some examples.
- Where do we see instances of inequality around us? Why do these instances exist? What do we do to make things more equal? Can we do more?

What is an inequality?

As we already know, quantity has to do with an amount or a number. We understand when one number is greater or smaller than another and what that means. We say expressions are equal if they have the same quantity or total and we use this property of numbers to solve equations.

We are familiar with the symbols which we use to talk about equality:

$=$ (equal to)

$\approx$ (approximately, or nearly, equal to)

$\equiv$ (identical to)

THINK–PAIR–SHARE

- **What is the difference between these symbols?**
- **Can you give an example of each of these being used?**

What is the difference between equality and inequality?

In Mathematics, an inequality shows the relation between two expressions, or two quantities, which are **not** equal.

This might be as simple as showing that they are not equal to one another:

$6 + 0.5 \neq 10$

We know that if we add 6 and 0.5, that it will never equal 10.

But we can also show in which way the inequality is directed (i.e. which is bigger or smaller than the other) by using different symbols.

$6 + 0.5 < 10$

This tells us that whatever the sum of 6 and 0.5 actually is, it is less than 10. It doesn't tell us by how much (it isn't an equation) but it tells us that that an imbalance exists.

If we used the opposite symbol > greater than to express this relationship, we would also have to reverse the expressions.

$10 > 6 + 0.5.$

Spoiler alert: this reversing of the inequality symbol and the expressions will be important later in **solving** inequalities.

We can also describe when the **cardinal number** of a set is larger or smaller than another set, e.g. the number of mosquitos on the planet (set) **is greater than** the number of human beings on the planet (set) represented as:

Σ Mosquitos $>$ Σ Humans.

For numbers that are totally disproportionate or of different orders of magnitude (where one is much much bigger or much much smaller), we can use the even more specific symbols

$\ll$ (much much less than)

and

$\gg$ (much much greater than)

For example, the number of humans on the planet is smaller (much, much less) than the number of stars in the Universe:

humans $\ll$ stars.

GREATER THAN OR EQUAL TO …

Other inequality symbols that you will be familiar with include:

$\leq$ (greater than or equal to),

$\geq$ (less than or equal to)

For an expression like

$x \geq 4$

this tells us that x can hold any value above **and including** 4.

PRACTICE QUESTIONS

Explain what is mean by the inequalities

a people applying for jobs in Media $\gg$ the number of jobs available

b $7 \leq x \leq 21$

c $x \geq 2.5$, $x \in N$

STRICT INEQUALITIES

An inequality is **strict** if no equality conditions for the expression(s) are possible.

$x \geq \pi$ x can be any value greater than, and including, π (pi)

$x > \pi$ x can be any value greater than pi, but **strictly cannot** be equal to π (pi)

ACTIVITY: Using strict inequality symbols

ATL

- Communication skills: Using mathematical notation correctly

Determine which symbol you think should be written between these sets to show the inequality correctly. **Select** from among the strict inequality symbols:

$<$, $\ll$, $>$, $\gg$

The number of female politicians in your country	The number of male politicians in your country
Grains of sand on a beach	People sitting on a beach
Girls studying HL Mathematics in Diploma Worldwide	Boys studying HL Mathematics in Diploma worldwide
Planets in the solar system	Days in the week
Smartphones in the world	People in the world
Fossil fuels in the ground today	Fossils fuels in the ground 100 years ago
Access to fresh water in Iceland	Access to fresh water in Jordan
The speed of a formula one car	The speed of a jet engine
The speed of a tortoise	The speed of light
Gun ownership per capita in the US	Gun ownership per capita in Switzerland
The population of Indonesia	The population of India
Languages spoken	Number of man-made satellites orbiting the earth now
Number of animals that have been in outer space	Number of humans that have been in outer space
Number of men that have been in outer space	Number of women that have been in outer space
Mean wage of college/university graduates	Mean wage of non-graduates
A celebrity's followers on Instagram	The number of people that celebrity is following on Instagram
The set of prime numbers	The set of integers

Verifying your answers

Solutions: ask your teacher or carry out research to check how accurate your estimations were (how far from reality were your guesses?)

Did using strict inequalities alone cause any problems with your choices?

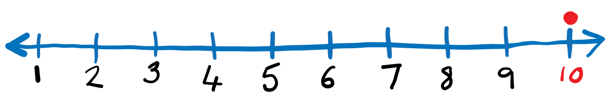
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion Ci, ii: Communicating, and Criterion D: Applying mathematics in real-life contexts.

How do we represent inequalities?

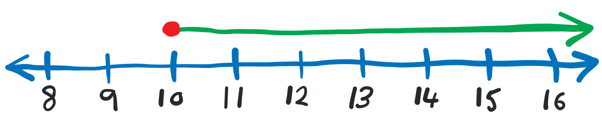
How do we represent this?

If you were asked to show a number on a number line, such as $x = 10$, you would probably draw and mark it like this:



■ $x = 10$

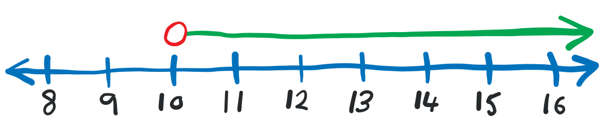
An inequality, by definition, has a range of values possible for an answer. If $x \geq 10$, then x could be equal to 19, 17.3, 11, 10.12222, 10 and so on …



■ $x \geq 10$

As 10 is included in the possible solutions, we colour in the circle to show it is included in the interval. The line will continue to infinity as the only condition is that it is larger than 10 (no end is specified).

If we had been given a **strict** inequality instead, such as $x > 10$, we could not have included $x = 10$ and the interval would look like this:



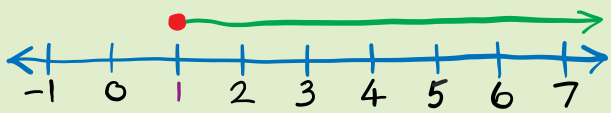
■ $x > 10$

Example questions

Represent the following inequalities on a number line.

$x \geq 1$

Solution



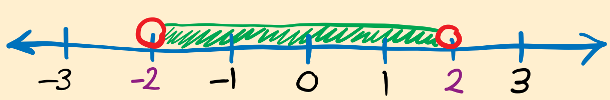
$x \leq 0.25$

Solution

$-2 < x < 2$

Solution

As there are now two conditions for the inequality, there will be a beginning and an end to the interval. The x-values must be larger than (but not equal to) −2 and they must be smaller than (and not equal to) 2.



Notice the use of the open circles to show strict inequality.

PRACTICE EXERCISES

Represent each of these inequalities on a number line.

1 $x < 4$

2 $x \leq 4$

3 $x > 0$

4 $x \leq 3.5$

5 $x < \pi$

6 $-3 \leq x \leq 4$

7 $2 < x < 12$

8 $-0.5 < x \leq 9.5$

9 $-4 < x \leq 9.5$, where $x \in \mathbf{N}$

In the previous question, the addition of $x \in \mathbf{N}$ meant that there were certain discrete values for the solutions. This means that the solutions must only be the natural numbers (whole positive numbers) between -4 and 9.5. These can be listed as:

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

These would be marked on a number line as individual dots on each value.

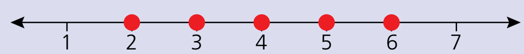
PRACTICE QUESTIONS

1 Can you list *all* of the solutions for the following?

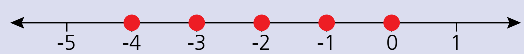
$\{x : x \in \mathbf{Z},\ -2 \leq x \leq 3\}$

Describe the inequalities

2 a

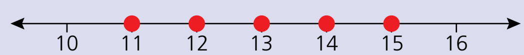


b



c

d



3 Why would it be difficult to give the solutions for this inequality?

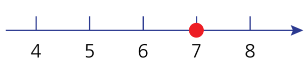
$\{x : x \in \mathbf{R},\ 1 \leq x \leq 3\}$

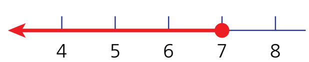
How do we solve inequalities?

If we are trying to solve an inequality with only one unknown, we can solve this in a similar way to solving a simple equation. Then we could represent it on a number line as before.

Example 1

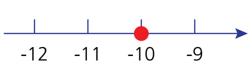
Equation	Inequality
$3x + 7 = 28$	$3x + 7 \leq 28$
$3x = 28 - 7$	$3x \leq 28 - 7$
$3x = 21$	$3x \leq 21$
$x = 7$	$x \leq 7$

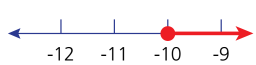




Example 2

$\frac{x+4}{3} = -2$	$\frac{x+4}{3} > -2$
$x + 4 = -6$	$x + 4 > -6$
$x = -6 - 4$	$x > -6 - 4$
$x = -10$	$x > -10$





Example 3

Solve

$$2\left(\frac{x-3}{4}\right) \geq 9$$

$$\frac{x-3}{2} \geq 9$$

$$x - 3 \geq 18$$

$$x \geq 21$$

Worded example

Jacqualane wants to go to a 'Coding summer camp' that costs 1000 dollars. She applies for scholarships to add to her savings of 100 dollars. How many scholarships must she win, if they are each for \$250, to be sure that she can go to the camp? Express and solve this problem as an inequality

Let's call the number of scholarships = s

Each scholarship is worth 250 $\quad 250 \times s$

So, she needs $250s + 100$ to be greater than, or equal to, 1000 to make sure she can go.

$$250s + 100 \geq 1000$$

$$250s \geq 900$$

$$s \geq \frac{900}{250}$$

$$s \geq 3.6$$

Therefore, Jacqualane must win 3.6 scholarships or more to go to the summer camp. Realistically that means that she has to win four scholarships, as these are *discrete* values.

Can you see any other types of 'inequalities' or unfairness in this scenario?

Inquiry challenge

Suri has found the following inequality in a textbook and is wondering how to **solve** it.

$$-2 \leq x + 4 \leq 12, \qquad x \in \mathbf{Z}$$

Find out how to handle an inequality like this and **solve** it.

List all the elements of the solution set.

Represent the solutions on a number line.

WHAT HAPPENS IF I NEED TO MOVE A NEGATIVE SIGN WHEN SOLVING?

In a spoiler alert mentioned previously, we said 'this reversing of the inequality symbol and the expressions will be important later in **solving** inequalities'.

Now it's time to see what that means:

Solve

$3 - x > 2$

$-x > 2 - 3$

$-x > -1$

BUT as usual with algebraic solving, we are looking for x, not $-x$.

So

$x < 1$

Did you notice how the sign has changed direction with the operation of dividing by –1?

Always remember to reverse the signs when moving a negative sign to the other side of the inequality.

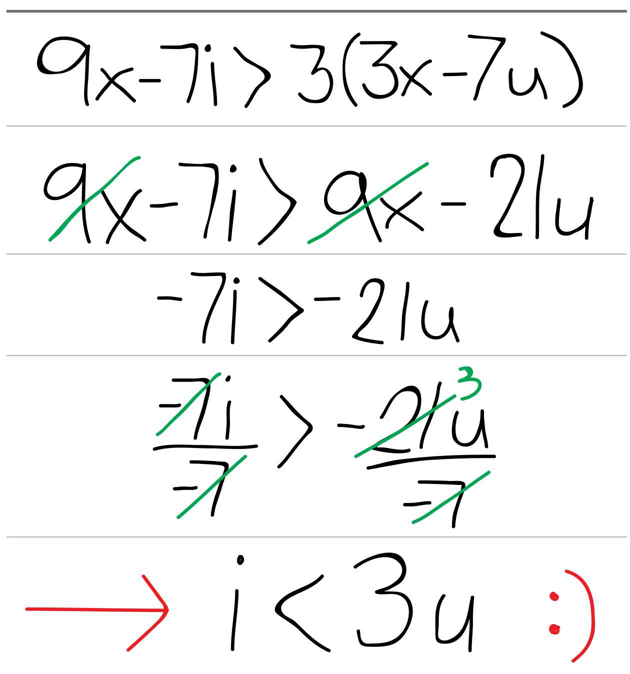
DISCUSS

Why would this be? Can you explain how the two statements $-x > -1$ and $x < 1$ logically relate to one another? You can use a number line or verbal argument, or both, to explain your reasoning.

PRACTICE QUESTIONS

Solve these inequalities and represent them on a number line.

1. $5x + 1 > 6$
2. $x - 3 < 10$
3. $x + 2 \geq 9$
4. $3(x + 4) < 11$
5. $9x - 1 > 4x - 6$
6. $2x - 2 \leq 2$
7. $6 - x < 2x + 2.5$
8. $-2(x - 5) > -x - 2$
9. $6(4x + 11) \geq 78$



www.mathfunny.com

■ i <3 u

USING LOGIC WITH INEQUALITIES

If Kyle is older than Manuel ($K > M$) and Manuel is twice the age of Oliver ($M = 2O$), what statement can you make about Kyle's and Oliver's ages?

A group of four friends are competing to participate in as many Service Learning activities as possible in a month.

At the end of the month Barry didn't take part in the most activities. Either Charlotte or Dora came third, Adam didn't win but he did come ahead of Dora. Who participated in the fewest activities? Who did the most?

Solution

How do we solve this type of logic problem? What do we know?

Every time we know something which is not true, we mark it with X and for anything true we make with a O.

We take each statement in turn to see what it tells us:

1 Barry didn't take part in the most $B \neq 1$

So $B > 1$

Friend	1st	2nd	3rd	4th
Adam				
Barry	X			
Charlotte				
Dora				

2 If either Charlotte or Dora came third, then $B \neq 3$

and $A \neq 3$

Friend	1st	2nd	3rd	4th
Adam			X	
Barry	X		X	
Charlotte				
Dora				

3 Adam didn't win so $A \neq 1$

Friend	1st	2nd	3rd	4th
Adam	X		X	
Barry	X		X	
Charlotte				
Dora				

4 But he did come ahead of Dora A < D

Which means A ≠ 4

Friend	1st	2nd	3rd	4th
Adam	X		X	X
Barry	X		X	
Charlotte				
Dora	X			

and D ≠ 1

5 Logically this tells us that Adam must have come second.

Friend	1st	2nd	3rd	4th
Adam	X	O	X	X
Barry	X	X	X	
Charlotte		X		
Dora	X	X		

6 Now we see that only Charlotte could have come first, which means Dora came third.

Friend	1st	2nd	3rd	4th
Adam	X	O	X	X
Barry	X	X	X	O
Charlotte	O	X	X	X
Dora	X	X	O	X

Solution: 1st – Charlotte

2nd – Adam

3rd – Dora

4th – Barry

PRACTICE EXERCISES

Now, try this logical approach to equality and inequality to solve this problem:

A number of 'real-life superheroes' have set up shop in Paradise City, hoping to clean up the streets. Match each person to his superhero identity, and determine the year that he started and the neighborhood which he patrols.

1. **'Prism Shield' began 1 year after** Tim Trevino.
2. **'Max Fusion' is** Ned Nielsen.
3. **'Max Fusion' began sometime after** Matt Minkle.
4. **'Max Fusion' began 1 year before** Tim Trevino
5. **'Green Avenger' began his crusade in 2009.**

		Superheroes				Real names			
		Green Avenger	Max Fusion	Prism Shield	Ultra Hex	Danny Doyle	Matt Minkle	Ned Nielsen	Tim Trevino
Years	2007								
	2008								
	2009								
	2010								
Real names	Danny Doyle								
	Matt Minkle								
	Ned Nielsen								
	Tim Trevino								

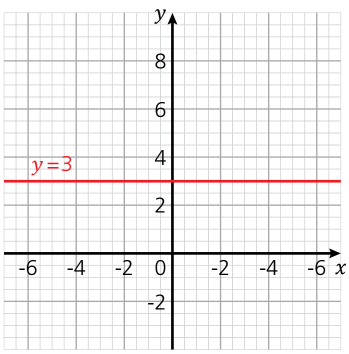
This puzzle, and many more besides, can be found on the website www.logic-puzzles.org/ where you can practice your skills of logic and deduction.

Reflection: Do you notice any inequalities in the set above? Can you make any statements based on the set?

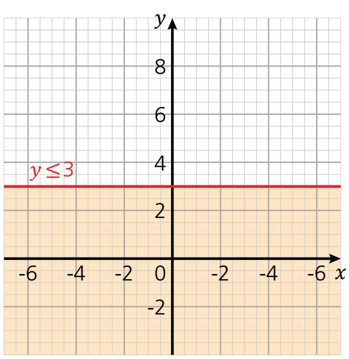
What quantities are associated with linear equalities?

In Chapter 3, we looked at linear equations and how they are represented on a Cartesian plane. Now we are going to consider the spaces above and below the line. A line cuts the entire Cartesian plane into two regions called half-planes, one above (>) and one below (<) the line. These regions are defined by the use of the inequality symbols <, ≤, >, ≥ depending on whether the line is included in the region or not.

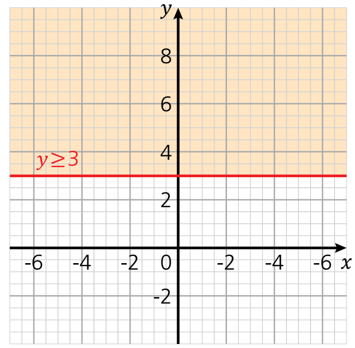
Let's consider the horizontal line y = 3.



The region y ≤ 3 is represented by:



The region above the line y = 3 is described as y ≥ 3. We shade this region to show it in this way:



How could we tell by the inequality symbol whether the line is included?

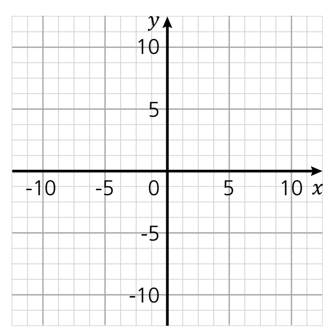
ACTIVITY: Shade the regions

■ ATL

- Communication skills: Use and interpret a range of discipline-specific terms and symbols

On a grid like the one below, represent the following inequalities:

a $y \geq 7$
b $y \leq -5$
c $x \geq 6$
d $x \leq -2$
e $y \geq x$



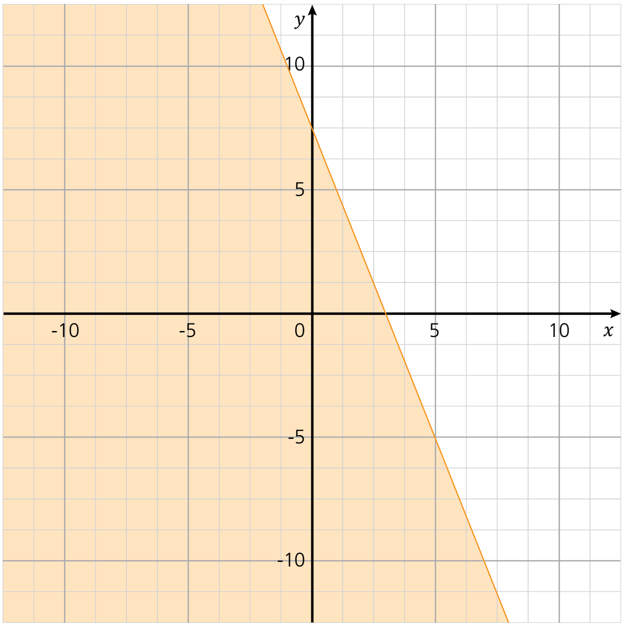
Hint

Using different colours to shade the regions might help to keep them distinct.

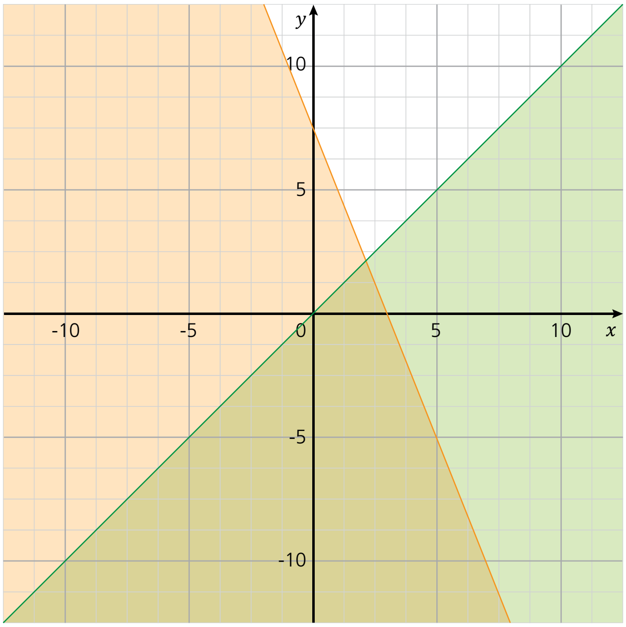
◆ Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

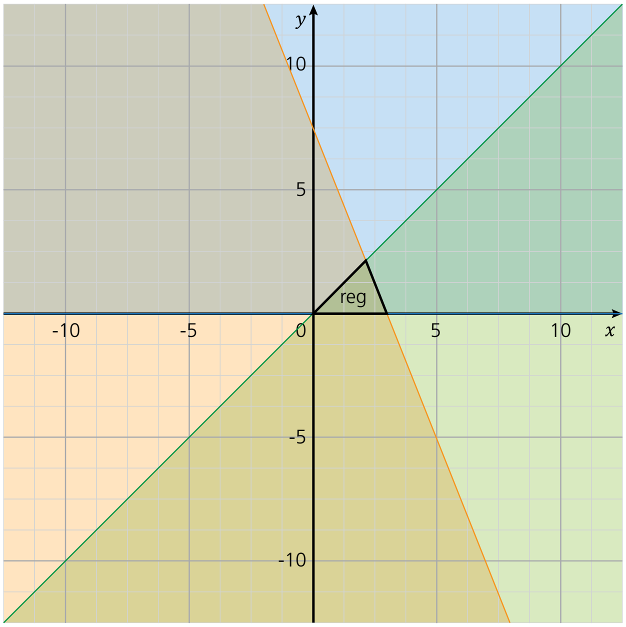
For the inequality $y \leq -3x + 7$ we shade the region below the line for $y = -3x + 7$



If we add another inequality $y \leq x$ to the same graph we see that there is an overlap in the shaded regions for both inequalities.



If we add a third inequality, such as y > 0, we can see a **bounded** region

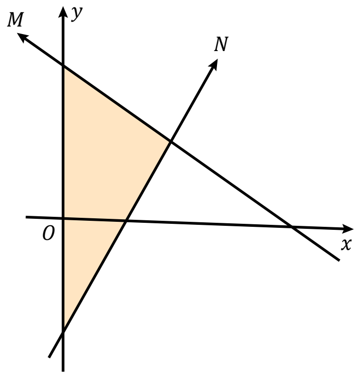


PRACTICE EXERCISES

Write down the three inequalities that define the shaded region in the diagram.

The equation of the line M is $2x + y = 10$

The equation of the line N is $4x - y = 8$



REAL-LIFE SCENARIO

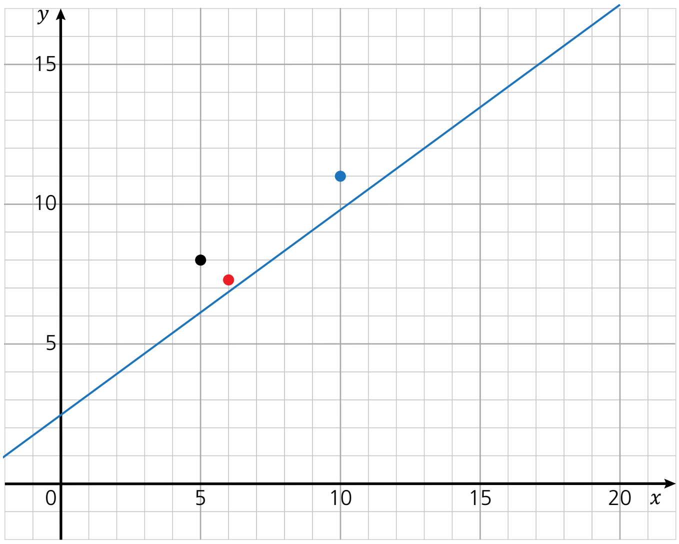
The equation describing a currency conversion at a bank is as follows:

$$y = 0.732x + 2.50$$

Where x and y are different currencies, the exchange rate 0.732 and the fee that the bank charges is 2.50.

If you exchange money at this bank you will get exactly this rate and nothing else. The values for any possible amount you may wish to exchange are shown by the continuous line on the graph below.

Imagine you changed money at three other locations, the amounts are marked as coordinates on the graph.



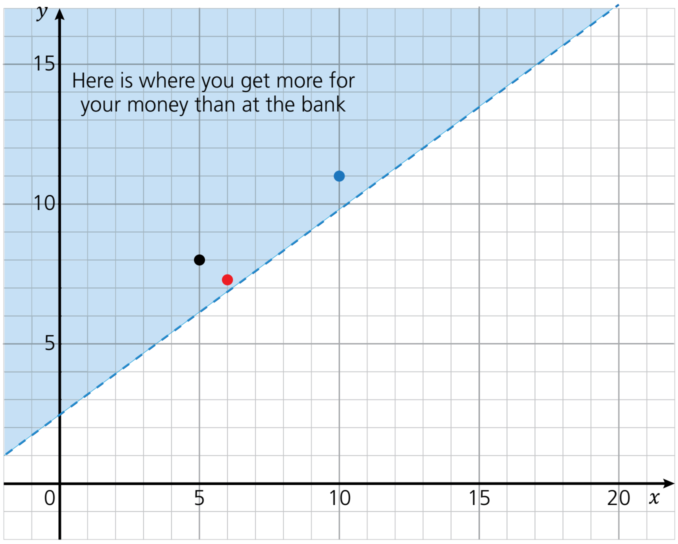
■ Money exchange rates

As you can see, all of the points are above the line, so you got more than the bank would have given you. Anywhere you get a better deal will be a point on the half-plane above the line. Anywhere you get a worse deal (less currency, y, for your money) will be in the half-plane below the line.

So the inequality

$$y > 0.732x + 2.50$$

shows you the region where you get 'a good deal'.



■ Where you get a good deal

What can you say about the region $y < 0.732x + 2.50$?

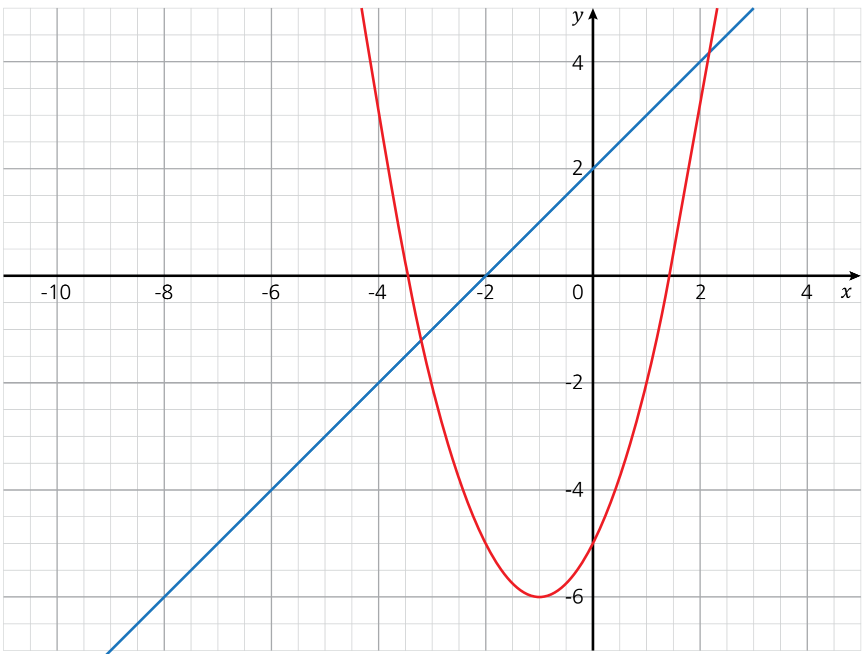
EXTENDED

Zein has been given the following inequalities to represent on a Cartesian grid. As there are only two inequalities, he doesn't think that logically there will be a closed region. He thinks this because in all the previous examples, we needed three inequalities to get a closed region.

$y \geq x^2 + 2x - 5$

$y \leq x + 2$

Zein graphed them as follows:



Has he correctly graphed the inequalities? Can you give Zein some advice to improve his mathematical communication and accuracy?

Consider the region defined by the inequalities. Why is it closed? What is different to previous examples? Describe the differences in detail, using clear and logical lines of reasoning.

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

What is a number pattern?

Patterns are defined in MYP Mathematics as a set of numbers or objects that follow a specific order or rule. In this chapter our main focus is discovering, checking and describing number patterns correctly. These questions are ideal practice to develop your understanding of Criterion B: Investigating patterns to achieve the highest possible levels of achievement. Let's look at some of the most important differences between the levels.

Criterion B: Investigating patterns

Maximum: 8

At the end of year 5, students should be able to:

i. select and apply mathematical problem-solving techniques to discover complex patterns
ii. describe patterns as general rules consistent with findings
iii. prove, or verify and justify, general rules.

Achievement level	Level descriptor
0	The student does not reach a standard described by any of the descriptors below.
1–2	The student is able to: i. **apply**, with teacher support, mathematical problem-solving techniques to discover simple patterns ii. **state** predictions consistent with patterns.
3–4	The student is able to: i. **apply** mathematical problem-solving techniques to discover simple patterns ii. **suggest** general rules consistent with findings.
5–6	The student is able to: i. **select** and **apply** mathematical problem-solving techniques to discover complex patterns ii. **describe** patterns as general rules consistent with findings iii. **verify** the validity of these general rules.
7–8	The student is able to: i. **select** and **apply** mathematical problem-solving techniques to discover complex patterns ii. **describe** patterns as general rules consistent with correct findings iii. **prove**, or **verify** and **justify**, these general rules.

In many of the upcoming activities and questions you will be presented with a sequence of numbers or pictures and your job will be to discover patterns, either simple or more complex. As you become more confident you will move from making simple predictions (Level 1–2) to suggesting some general rules that match the numbers you have been given (Level 3–4).

To 'level up' to Level 5–6, you must describe your patterns as general rules, which in this chapter will often be an algebraic description of the n^{th} term (or u_n). You must also start to think about proving that your general rules make sense by testing them with other numbers.

To demonstrate that you are capable of achieving a Level 7–8 in pattern investigation, you must test your general rule, or nth term, in a more structured and formal way. This will often involve predicting which numbers will appear in a sequence outside of those provided. The definition for the command term 'verify' indicates this: 'provide evidence that validates the result.'

Your result is the general rule and the evidence is whether your general rule works to correctly predict other numbers in the sequence.

UNDERSTANDING SEQUENCES

A sequence is an ordered list, where one term follows another. All of the terms follow some kind of rule. Without a rule, it would be a random list. The sequence could be made of…

1. Numbers: set of numbers such as {5, 7, 9, 11, ….}
2. Pictures: a sequence of visual images
3. Table values: a sequence of terms in a table

Some sequences are **arithmetic**: where the numbers in the sequence increase or decrease by a constant amount.

for example 4, 7, 10, 13, 16, …each term changes by +3

or 101, 91, 81, 71, … each term changes by -10

HOW DO WE FIND GENERAL RULES FOR SEQUENCES?

There are several methods for discovering a general rule to describe the sequence. We will focus on two, an informal method and a formal method.

The informal method: Bring it back …

Procedure pro:

Step 1

How much ↑ or ↓? $\underbrace{4, 7}_{+3}, \underbrace{10}_{+3}, \underbrace{13}_{+3}$

So the first part of the general rule is $+3n$.

Step 2

Image there was a term zero. Find the term by bringing it backwards by the same amount back by the same amount.

(+1), $\underbrace{4, 7}_{+3}, \underbrace{10}_{+3}, \underbrace{13}_{+3}$

The term before 4, where $n = 1$, would have been +1.

Step 3

Put them together

$3n + 1$

You can test this by substituting $n = 1$ to see if the result is the same as the first term, if $n = 2$ also gives the right result for the second term and $n = 3$ gives the third then your general rule is correct.

The general rule, or u_n, also known as the n^{th} term, is useful because now you can find any term you want on the sequence by substituting for n.

Worked example

For the sequence 101, 91, 81, 71… find the u_n and hence find the 10^{th} term (u_{10}).

$\underbrace{101, 91}_{-10}, \underbrace{81}_{-10}, \underbrace{71}_{-10}$ ….

$-10n$

Work backwards one term

$\underbrace{111 \quad 101}_{\text{go up 10}}$, 91, 81, 71

$\therefore u_n = -10n + 111$

So the 10^{th} term

$u_{10} = -10(10) + 111$

$u_{10} = -100 + 111$

$u_{10} = 11$

Check this by "quickly counting backwards" method.

The formal method: Using a formula to find the u_n

One advantage of this method is that this is the method used in later mathematics courses (DP) and this formula will be provided for you in examinations.

You need to be confident in your algebraic skills for this method because you will be substituting values and expanding brackets to get a final answer. If you feel that you need more practice in these skills, refer to Chapter 2 of this book.

The n^{th} term of an arithmetic sequence:	$u_n = u_1 + (n-1)d$

Where u_1 is the first term of the sequence, n is the variable and d is the common difference.

What is the common difference in an arithmetic sequence?

An arithmetic sequence increases or decreases by a **common difference**.

The common difference (d) is found by subtracting a term by the preceding term.

$d =$ (any term) – (previous term), for example

4, 7, 10, 13, 16, …	$d = (7-4)$	or	$d = (13-10)$
	$d = 3$		$d = 3$
or 101, 91, 81, 71, …	$d = (91-101)$	or	$d = (81-91)$
	$d = -10$		$d = -10$

We already know from the informal method that we would like to be able to describe any arithmetic sequence with a general rule, so that we can find any term in the sequence. We use the letter n to represent the general number or term.

Finding the **general rule** for an arithmetic sequence – n^{th} term – u_n

Example 1

4, 7, 10, 13

Common difference = 3

1^{st} term	$u_1 = 4$
Formula	$u_n = u_1 = (n-1)d$
So	$u_n = 4 + (n-1)3$
Expand	$u_n = 4 + n(3) - 1(3)$
	$u_n = 4 + 3n - 3$
	$u_n = 3n + 1$

The same general rule as before.

Example 2

101, 91, 81, 71 …

$u_n = u_1 + (n-1)d$

$u_1 = 101 \qquad d = -10$

$u_n = 101 + (n-1)(-10)$

$u_n = 101 - 10n + 10$

$u_n = -10n + 111$

Also the same n^{th} term, or general rule, as before.

PRACTICE QUESTIONS

Using whichever method you prefer, for each sequence in the table **find**:

- **the next two terms u_4 and u_5**
- **the n^{th} term, u_n**
- **the 10^{th} term, u_{10}.**

Sequence
2, 5, 8, …
7, 9, 11, …
20, 50, 80, …
21, 18, 15, …
2.5, 3, 3.5, …
0.2, 0.6, 1.0, …

ACTIVITY: Making connections

ATL

- Transfer skills: Inquire in different contexts to gain a different perspective

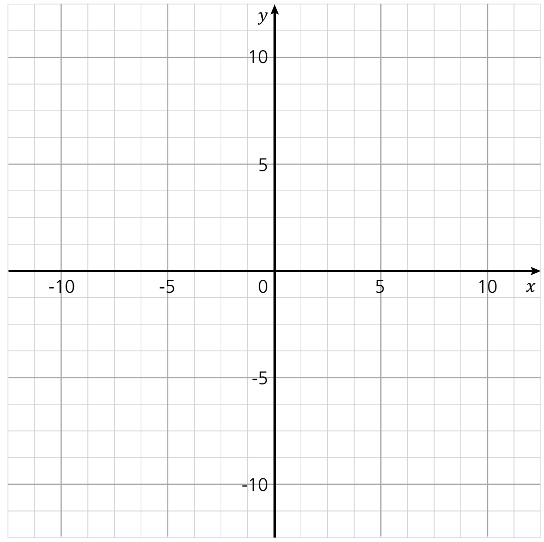
The sequence 2, −2, −6, −10, −14, … has been placed on a table to show the relationship between the position and the term *n*.

n	1	2	3	4	5	6	7	8
Sequence	2	−2	−6	−10	−14	−18	−22	−26

Using the table, or otherwise:

- **Find u_6**
- **Find u_7**
- **Find a general rule to describe the pattern u_n**
- **Using the n^{th} term rule, or otherwise, find u_8**
- **Find u_{17}**

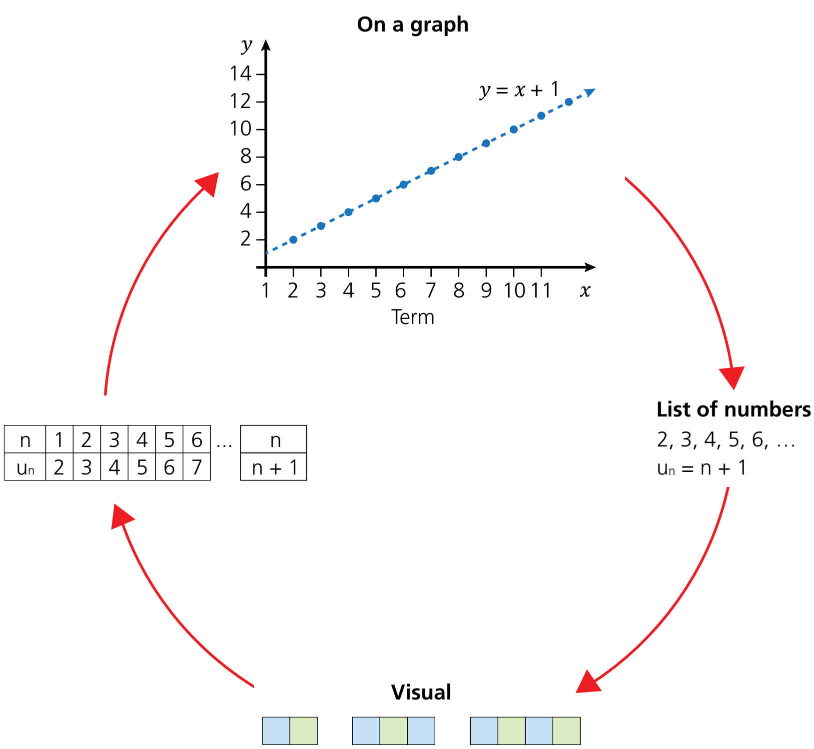
Place the terms from the table onto a grid like the one shown here, where the x-coordinate represents the *n* (term position) and the y-coordinate the number (term value).



What does the sequence resemble? What connections can you make to other representations? How could this help you to find the u_n (n^{th} term)?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

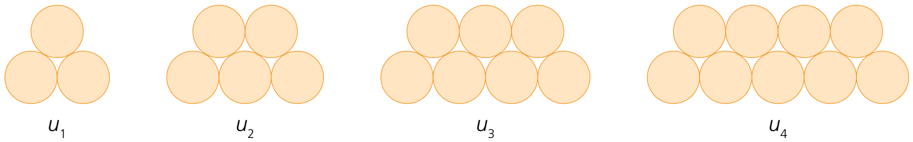


- How are these representations – graph, number sequence, table, pictures – all related?

FINDING NUMBER PATTERNS IN IMAGES

We can also identify number patterns in visual sequences

Example



The first picture (term), or u_1, has a total of 3 circles $\quad u_1 = 3$

The second picture (term), or u_2, has a total of 5 circles $\quad u_2 = 5$

The third picture (term), or u_3, has a total of 7 circles $\quad u_3 = 7$

The fourth picture (term), or u_4, has a total of 9 circles $\quad u_4 = 9$

So these images can be transformed into a number sequence:

3, 5, 7, 9….

What are the next two terms?

$u_5 = ?$

$u_6 = ?$

Solution

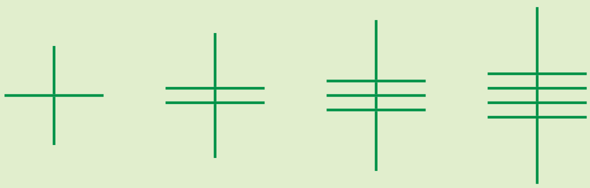
A new circle is added on each row to form the next term. Therefore, the common difference is +2, the numbers are 'going up in 2s'

So $\quad u_5 = 9 + 2 = 11$

and $\quad u_6 = 11 + 2 = 13$

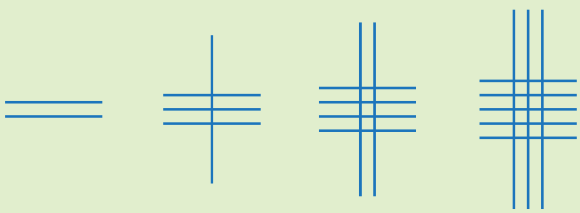
or $\quad u_n = 2n + 1$

PRACTICE QUESTIONS



1 **Convert** these images into a sequence of numbers.

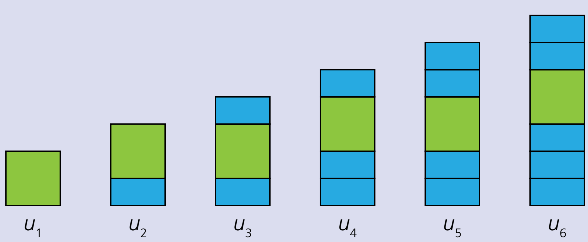
2 **Draw** the next two terms.



3 **Convert** these images into a sequence of numbers.

4 **Find** u_n

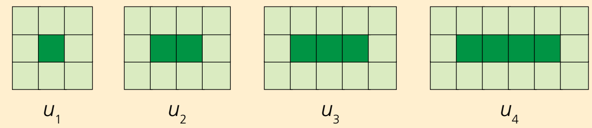
5 The following image shows green boxes that are one full box and blue boxes which are half-boxes



a **Draw** the next three terms in the sequence.

b **Find** u_n

c Do you notice any other patterns in the sequence?



6 What pattern or patterns do you see in the images above?

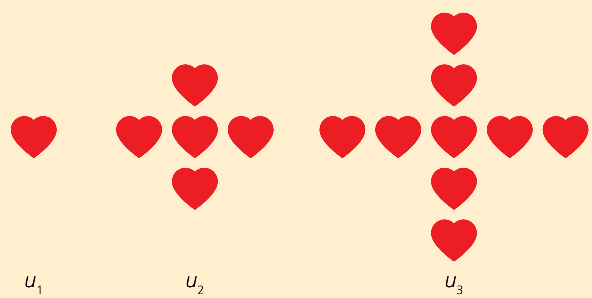
7 **Draw** the next three terms

8 **Determine** u_n for the

a light green boxes

b dark green boxes

c total boxes.



9 How many hearts in the next term?

10 How many hearts in u_5?

11 **Find** the n^{th} term/ u_n

12 Use this general rule to find u_7

13 **Prove** this by drawing all terms from u_4 to u_7 to **verify** your rule.

14 Which term will be the first to have more than 100 hearts?

Where can we find number sequences in the world around us?

REAL-WORLD PROBLEMS

■ Counting coins

Example 1

■ Making savings

Elizabeth was born on the 1 of January 2014 and her father decides to open savings account for her straight away. He starts the account with an initial deposit (money lodged into an account) of £20. He puts an additional £5 into the account each month after that one, which she can withdraw on her 18th birthday. Find the following:

a u_1 – the amount at the end of the first month
b u_3 – the amount at the end of the third month
c u_{12} – the amount at her first birthday
d The common difference
e A general rule for u_n so Elizabeth can work out the total at any given time
f How much Elizabeth will have on her 18th birthday?
g What month of which year will the total exceed (pass) £500?

Should be

a $u_1 = £20$ — as this is the starting amount (1st term).

b $u_3 = £20 + £5 + £5$
$u_3 = £30$

c $u_{12} = £75$ — this can be found by counting up in 5's, listing the terms or applying the formula

d The common difference is £5, as this is the amount the sequence (account) increases by each term (month).

e General rule: $u_n = 5n + 15$ — this can be found using either method.

f 18th birthday $= 18 \times 12 = 216$th month

$$u_{216} = 5(216) + 15$$
$$= 1095$$

Elizabeth will have £1095 dollars on her 18th birthday.

g Which term (n = ?) will be ≥ 500?

$$5n + 15 \geq 500$$
$$5n \geq 500 - 15$$
$$5n \geq 485$$
$$n \geq \frac{485}{5}$$
$$n \geq 97$$

This is a whole number so the account will equal £500 on her 97th month (not long after her 8th birthday). The next month it will be greater than £500.

Example 2

A class of 32 students is having trouble with self-regulation/ organization skills and keeps forgetting to do their home learning. The teacher decides that the fairest thing is for everyone to complete an after-school detention each time they forget their work.

The first week everyone forgets. The following week, four students remember their work. The next week after that, four **more** remember. This pattern continues.

Solve the following problems:

- **a** **Find** the first three terms of the sequence
- **b** **Find** the common difference
- **c** **How** many students are in detention on the 5th week?
- **d** **Find** u_n
- **e** **Find** u_8
- **f** **Find** u_{10}
- **g** **Comment on** the accuracy of your answer above.
- **h** When will the entire class remember their work?

Solutions

- **a** 32, 28, 24 the first three terms are found by counting down from the total number of students, decreasing by 4 each week.
- **b** Common difference $d = -4$
- **c** 5th week 32, 28, 24, 20, 16 there are 16 students in detention on Week 5.
- **d** $u_n = -4n + 36$
- **e** $u_8 = -4(8) + 36$
  $= -32 + 36$
  $= 4$ students
- **f** $u_{10} = -4n + 36$
  $= -4(10) + 36$
  $= -4$ students
- **g** Clearly there is no such thing as negative or minus students so this tells us that all the students have already remembered their work. The answer is mathematically correct but does not make sense in the context of the problem.
- **h** This question is asking which term = 0
  $-4n + 36 = 0$
  $-4n = -36$
  $n = \frac{-36}{-4}$
  $n = 9$
  On the 9th week there will be no students in detention, therefore they must all have remembered their work.

Practice problems

Psoriasis is a skin condition that can be helped by exposure to UVB in sunlight or by specialised lamps. Benji is instructed to sit in direct sunlight, but has to build up his tolerance. On the first day he begins with 5 minutes and must increase each subsequent day by 30 seconds.

Find the answers to the following questions:

- **a** How long will Benji sit for each of the first three days?
- **b** Write these answers as a sequence
- **c** **Find** u_4
- **d** **Find** u_n
- **e** **Find** u_{10}
- **f** **Find** u_{12}
- **g** Sun protection is recommended for most skin types after 15 minutes in direct sunlight. On which day might this advice be relevant to Benji?

Problem 2

A doctor prescribes medication to a patient of 1000 mg per day while in hospital. The patient leaves hospital and must begin to reduce their dose every day. The doctor recommends reducing the dose by one pill of 50 mg every day.

If the dosage of Day 1 is 1000 mg

- **a** **Find** the dosage on Day 3.
- **b** **Find** u_6
- **c** **Find** u_n
- **d** **Find** when the dosage is half the amount it was in the hospital?
- **e** **Find out** when the patient will stop taking the medication.
- **f** Why do you think this approach/ sequence might be necessary?

What are geometric sequences?

Not all sequences follow the same patterns as arithmetic sequences.

Sequences which involve doubling, halving or any other multiplication factor from term to term are called **geometric sequences**.

Examples of geometric sequences include:

2, 4, 8, 16, 32, …	where each term is multiplied by 2 to find the next
1, 10, 100, 1000, 10 000, …	Where each term is multiplied by 10 to find the next
200, 100, 50, 25, …	where each term is multiplied by 0.5 to find the next
3, –6, 12, –24 …	where each term is multiplied by –2 to find the next.

The multiplication factor is known as the common ratio.

The common ratio is found by $\text{common ratio} = \dfrac{\text{any term}}{\text{previous term}}$

Try this definition with all the examples earlier to see if you find the same common ratio.

Practice problems

Find the common ratios for the following geometric sequences:

a 3, 30, 300, 3000, …

b 8, 4, 2, 1, …

c $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

d 2, –8, 32, –128, …

We will deal with geometric sequences very briefly in this chapter because you will see a lot of them in Diploma Mathematics or in later study.

Just as with arithmetic sequences, there is a formula to find u_n for a geometric sequence.

The n^{th} term of an geometric sequence:	$u_n = u_1\, r^{n-1}$

where r = common ratio and all other letters remain as before.

Worked example 1

Find the u_n, and u_6 for the sequence

2, 4, 8, 16, 32, …

$r = 2$

$u_1 = 2$

$u_n = u_1\, r^{n-1}$

$u_n = 2(2^{n-1})$

Do NOT multiply 2 × 2 here

$u_6 = 2(2^{n-1}) \quad n = 6$

$u_6 = 2(2^{6-1})$

$u_6 = 2(2^5)$

$u_6 = 2(32)$

$u_6 = 64$

Example 2

Find u_n and u_6 for the following sequence

200, 100, 50, 25, …

$r = 0.5$

$u_1 = 200$

$u_n = u_1\, r^{n-1}$

$u_n = 200(0.5^{n-1})$

$u_6 = 200(0.5^{n-1}) \qquad n = 6$

$= 200(0.5^{6-1})$

$= 200(0.5^5)$

$= 200(0.03125)$

$= 6.25$

THINK–PAIR–SHARE

With a partner, discuss why you think it is so important **not** to multiply the numbers as indicated in examples 1 and 2. What would it affect if you did? Can you show this? What rule could you use to remind you of this?

Share your thoughts with the rest of the class.

PRACTICE QUESTIONS

Find u_n and u_6 for each of the following sequences

a **1, 10, 100, 1000, 10000, …**
b **3, −6, 12, −18**
c **3, 30, 300, 3000, …**
d **8, 4, 2, 1, ….**
e $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
f **2, −8, 32, −128, …**

What is the difference between a sequence and a series?

A **sequence** is set of ordered numbers such as {3, 6, 9, 12, …} whereas a series is the summative of the whole set, ∑{3, 6, 9, 12, …} which is 3 + 6 + 9 + 12 + …

Series are the sums of sequences

Some sequences are finite (have a start and an ending) and some sequences are infinite (have a start but continue on to infinity.

SEE–THINK–WONDER

Here are the formulas for the sum of arithmetic and geometric sequences; you will use them a lot in Diploma Mathematics.

The sum of n terms of an arithmetic sequence:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$$

The sum of n terms of a finite geometric sequence:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r},\ r \neq 1$$

The sum of an infinite geometric sequence:

$$S_\infty = \frac{u_1}{1 - r},\ |r| < 1$$

What do you think when you see these formulas? What do they make you think about summing up series? How can an infinite series be added up? Why is it only for geometric sequences that finite/infinite are mentioned? Do you recognize any other symbols which we have used in this chapter?

ACTIVITY: For the risk-takers

ATL

- Affective skills: Resilience Practise ‘bouncing back’ after adversity, mistakes and failures; Practise ‘failing well’

Are you feeling brave? Could you apply the formula above for the sum of an infinite geometric sequence to try to find the sum of the series:

$$2 + 1 + 0.5 + 0.25 + \ldots\ldots?$$

Remember to show your working, even the incorrect attempts, to get credit for your risk-taking!

What other types of sequences are there?

FIBONACCI NUMBERS

Let's look at one of the most famous sequences, Fibonacci numbers.

One of the most amazing, and therefore famous, sequences in Mathematics is called the Fibonacci sequence.

The Fibonacci sequence begins as follows:

1, 1, 2, 3, 5, 8, 13, 21…

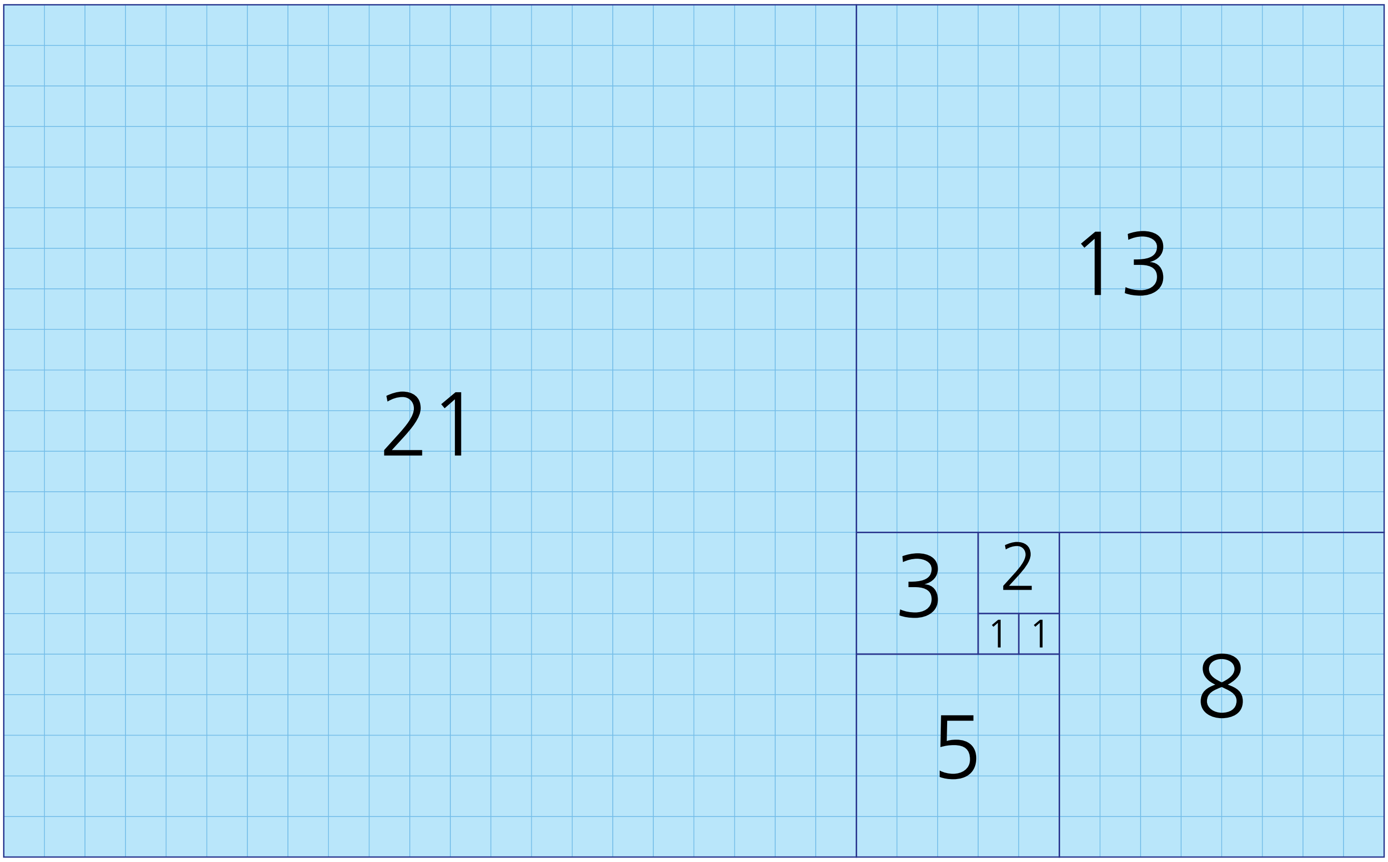
The relationship between each term depends on what directly preceded it (what came before). Can you **identify** the pattern?

Find the next two terms in the sequence.

Now imagine that instead of numbers, they were represented as square tiles of that length (as shown in the diagram below).

What would the next term in the sequence look like on the image? And the next one?

Could you redraw the image above and add the two new terms that you have just discovered? How would you **describe** this pattern, either the visual or the numerical pattern?



A tiling pattern with tiles whose side lengths correspond to Fibonacci numbers

ACTIVITY: Finding the Fibonaccis

ATL

- Information literacy skills: Finding, interpreting, judging and creating information

We say that **Fibonacci numbers are all around us**. Using your research skills, find out what we mean by this statement. How do these numbers above relate to the phenomena you discover?

■ Sunflowers

■ Plants as an example of the Fibonacci sequence

■ Stars in a galaxy are examples of Fibonacci numbers

The images here are examples of the Fibonacci sequence in nature and the universe.

Do you know any more?

EXTENDED

1. Who was Fibonacci? Why is he credited with the discovery of these numbers if it was known to Indian Mathematicians much earlier?
2. What are 'negafibonaccis'?
3. You can see in these images, some examples of the Fibonacci numbers occurring in nature. Can you find any examples of these numbers occurring elsewhere in real-life contexts?

Assessment opportunities

- This activity can be assessed using Criterion C: Communicating; and Criterion D: Applying mathematics in real-life situations.

EXTENDED

Verify if the rule would still remain true if the L-shape rotated by 90° clockwise.

SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding and Criterion B: Investigating patterns.

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION B

Using a 10 × 10 grid like this …

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Consider the first shape L_1,

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Consider the second shape L_2,

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

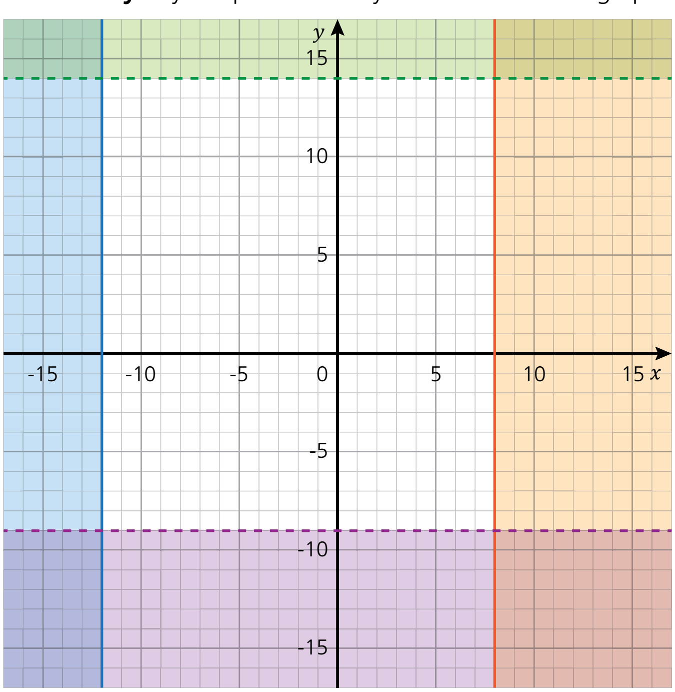
and the third shape L_3.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1 **Find** the values of L_1, L_2 and L_3.
2 **Predict** the next two terms in this sequence.
3 **Verify** these terms.
4 **Describe** the pattern of the totals created by moving the L-shape.
5 **Find** a general rule to predict the total of any L-shape.
6 **Verify** the rule.
7 Can you **prove** where this rule comes from, either algebraically or otherwise?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

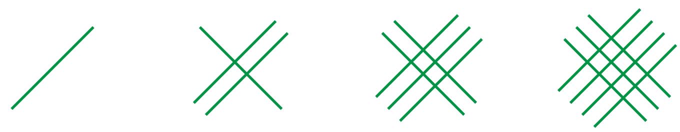
1 **Identify** any inequalities that you can find on the graph



Hint

Watch out for dotted lines and solid lines when choosing your inequality symbol.

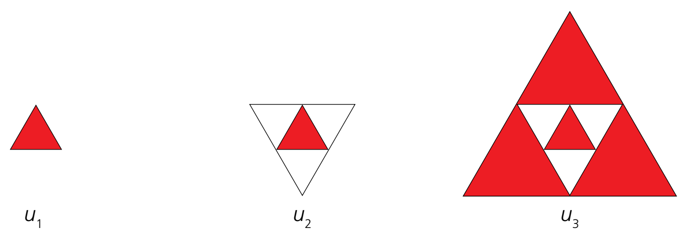
2 For the sequence:



Draw u_5 and u_6

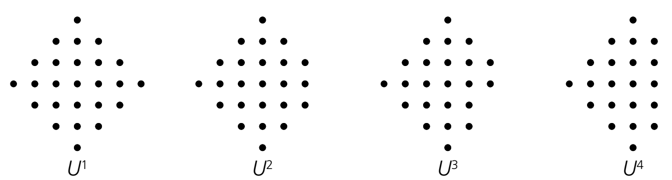
THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3 – 4

3 **Find** u_n for the pattern above.
4 **Find** the 10th term in the series, numerically and visually.
5 **Draw** the next three terms in the sequence.



THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

Disappearing dots…



6 **Find** the next two terms, both visually and numerically.
7 **Find** u_n
8 Which term will have no dots in the pattern?
9 Which is the first term to have number of dots < 0?
10 Can you think of a creative way to represent this?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

11 A Paralympic athlete has been fitted with a prosthetic leg. The doctor has advised that to build up the supporting muscles, the athlete must start by wearing the prosthetic leg for an hour on the first day and then increase that amount each day by 15 minutes.

- **a** **Find** the first five terms of the sequence.
- **b** **Find** u_n
- **c** **Find** the day when the athlete would be able to wear the leg for the entire 24 hours.

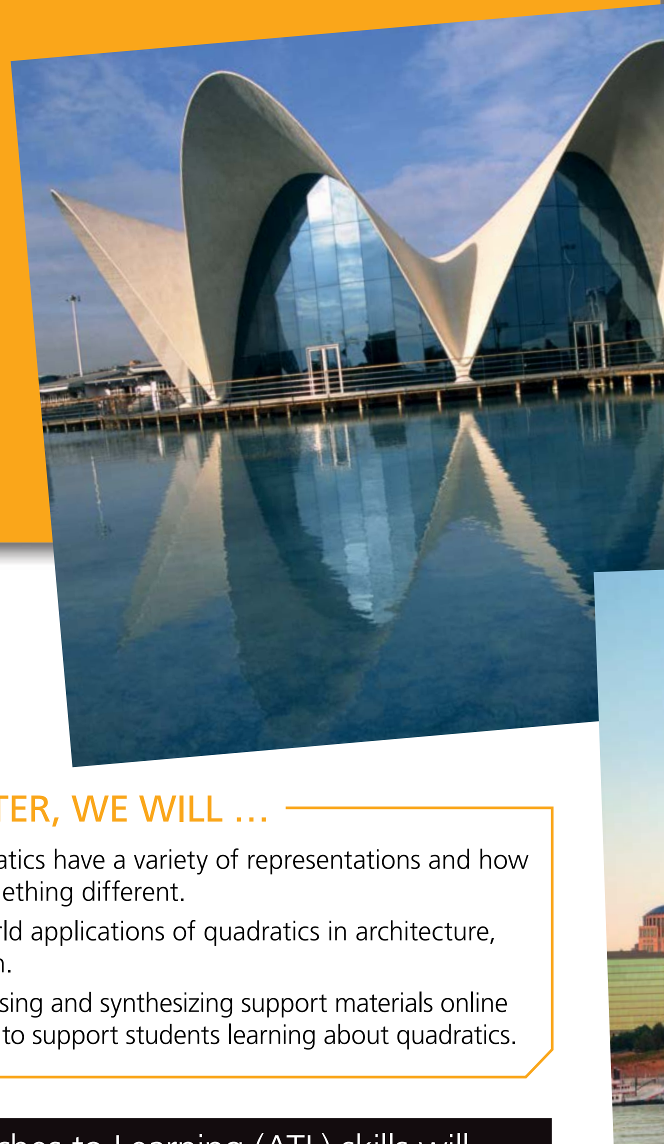
12 The athlete follows the instructions for the first day. He is overeager and after that ignores the doctor's warnings and doubles the length of time each day. By day five he is exhausted, why is this? What else can you say/find out about the athlete's decision?

Support your answer with mathematical evidence.

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: How do we represent inequalities? How do we solve inequalities? What do linear Inequalities look like on a Cartesian plane? How do we find general rules for sequences? What are geometric sequences? What other types of sequences are there?					
Conceptual: What is an inequality? What quantities are associated with linear equalities? What is a number pattern? Where can we find number sequences or patterns in the world around us?					
Debatable: Is inequality always a bad thing? Does life follow sequences exactly?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Communication skills					
Transfer skills					
Affective skills					
Information literacy skills					
Learner profile attribute	Reflect on the importance of being caring for your learning in this chapter.				
Caring					

8 How many forms has a quadratic?



Representing relationships visually and algebraically can allow us to find and optimize 'best case scenarios' and sustainable solutions.

CONSIDER THESE QUESTIONS:

Factual: How do we graph quadratic functions? How do we solve or find the roots of a quadratic equation? What relationships can we model using quadratic equations?

Conceptual: What's more 'powerful' than a linear equation? Where can we see the representation of quadratic equations and functions around us? What do these relationships in different forms tell us? What is the quadratic function telling us to do with an input?

Debatable: Can mathematics give us solutions to problems of sustainability or are they suggestions? Why should we optimize production and use of materials using quadratics? How can we collaborate more to improve communication?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how quadratics have a variety of representations and how each one tells us something different.
- **Explore** the real-world applications of quadratics in architecture, projectiles and design.
- **Take action** by accessing and synthesizing support materials online to make a top ten list to support students learning about quadratics.

These Approaches to Learning (ATL) skills will be useful …

- Communication skills
- Reflection skills
- Information literacy skills

We will reflect on this learner profile attribute …

- Open-minded – we critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.
- Principled – we act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

DISCUSS

The photographs above show examples of a parabola in architecture. Can you think of any examples near where you live?

Some important features of the quadratic function or parabola are shown.

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

THINK–PAIR–SHARE

- **Imagine an astronaut moving along the surface of the moon. Their steps look very different to walking on Earth. How would you describe the movement?**
- **What actions or movements look like this on Earth?**
- **What are projectiles and how do they move?**

PRIOR KNOWLEDGE

You will already know:

- how to recognize quadratic expressions
- how to factorize quadratic expressions
- how to solve simple equations.

KEY WORDS

expression	maximum	quadratic
intercept	minimum	vertex

What's more 'powerful' than a linear equation?

Algebra models relationships between variables. An equation or function expresses how one variable changes another, it gives an output when you choose an input.

Let's start with linear equations, which we have already met in Chapter 3. We saw two forms of linear equations. Both of these forms contain the variables x and y, although they are represented in a different order.

$$y = mx + c \qquad ax + by = c$$

coefficients constants

But not all relationships are straight lines. Many relationships are actually curved lines (yes, curves are a type of line). If we are observing a relationship, it may appear to be linear or increasing at first, but over time it may reach a peak (or trough) and start to go the other way. We know then that the equation is not a linear one as a line continues on forever, by definition.

WHAT IS A QUADRATIC EQUATION?

The first thing you will notice about a quadratic expression is that the highest power in any given term will be $\square^2$. Most of the time we will be discussing x^2 but other letters are also possible. In this chapter, we will learn how to draw these equations by hand, using technology, and how to identify and find (by solving) important features of the graph.

When we last saw the term quadratic, we were factorizing quadratic expressions into brackets and removing common factors.

$$y = ax^2 + bx + c$$

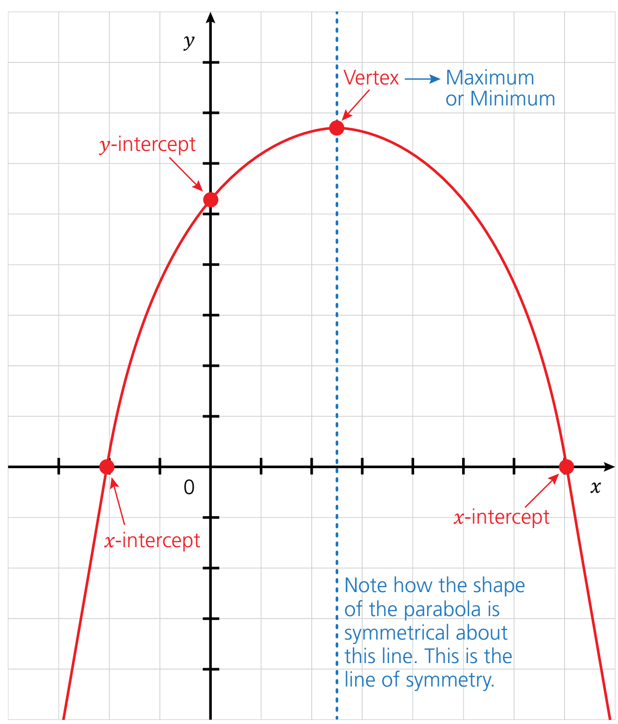
coefficients constant

A quadratic expression or equation can be recognized by the ax^2 term, this must be a non-zero coefficient, i.e. $a \neq 0$.

Other important things to know about quadratic equations and functions:

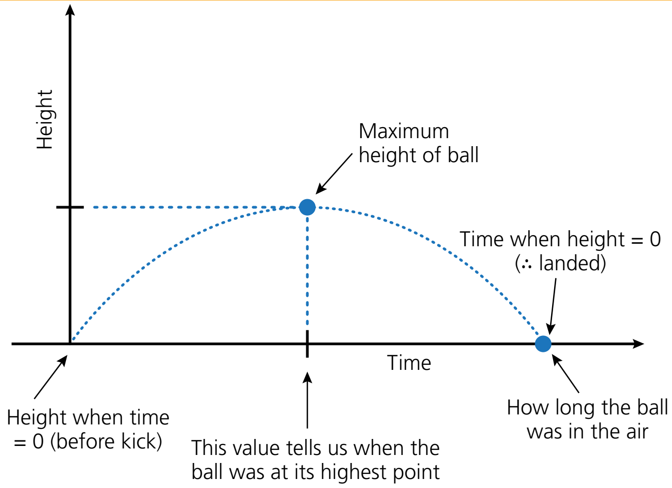
1. The highest power in any term must be the x^2. Higher terms such as x^3 or x^5 have a different form.
2. The x-term can be equal to zero (b = 0), which means it can still be a quadratic without an x-term but it must still have an x^2
3. The constant value gives the same information as it did for linear equations, which means it tells you the y-intercept. So, the quadratic crosses the y-axis at $(0, c)$.
4. The constant value may also be equal to 0.

A quadratic function will reach a maximum or minimum and behave symmetrically. The shape is called a parabola and you can recognize it in many features of architecture in the images throughout this chapter.



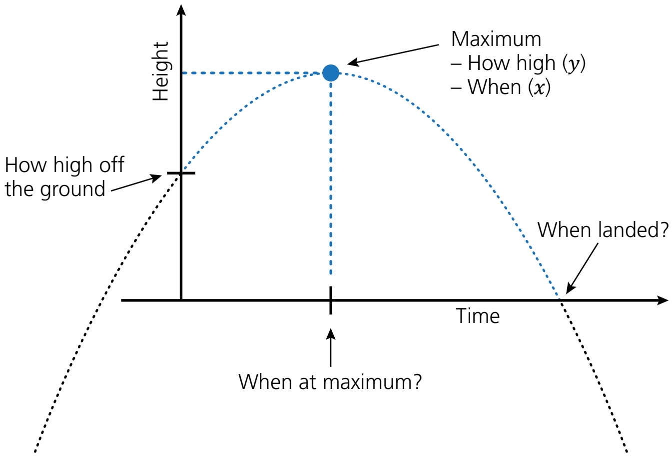
■ Key features of the quadratic function

If we see that a phenomenon follows a quadratic relationship or if we notice that measured data are showing the same characteristics as a quadratic equation, we can use this modelling to make predictions. Some important uses include finding a maximum (peak) or minimum (trough) to **optimize** production or to **identify** ranges.

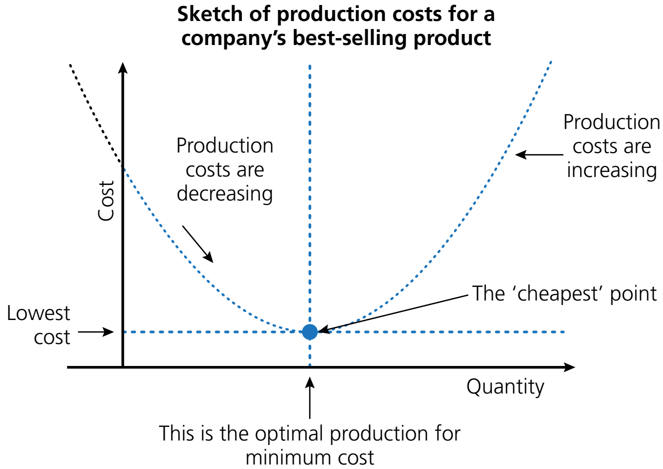


■ Sketch of a projectile

Let's look at some real-world examples. A projectile is an object which has been projected, or thrown from the ground. As we know from Physics, gravity will cause a projectile to be pulled towards the centre of the Earth and cause it to return to the ground.



■ **Example 1:** A projectile fired upwards from a height



■ **Example 2:** Finding the best case or optimizing

ACTIVITY: Identifying equations

■ ATL

■ Reflection: Consider the process of learning; choosing and using ATL skills

Which of these equations are linear, quadratic or other?

$$2x + y = 4$$

$$x^2 + 3x + 1 = 0$$

$$y = 4x - 1$$

$$y = 3x^3$$

$$y = 8$$

$$4x^3 + 3x^2 + 2x - 1 = y$$

$$y = x^2 + x$$

Can you guess what the 'other' equation or function above might be called?

Quadratic equations can be very useful to find best case scenarios like this one. As they make more and more of the product, the costs go down to a certain point. After a certain number of products however, the costs start to increase again. Maybe because they have to hire more people or purchase more machinery. Whatever the reason, the turning point (maximum or minimum) tells you the optimum solution. We will learn more about optimization later in this chapter, with particular reference to helping us make sustainable choices.

■ Another real-world example of a parabola. Would you know how to write the equation for this arc?

What is the quadratic function telling us to do with an input?

WHAT IS THE RELATIONSHIP?

Example 1

Let's begin with an equation.

$y = x^2 + 1$

What does this mean? Let's choose an arbitrary (randomly chosen) value for x and find the y and analyse what this tells us.

$$y = x^2 + 1$$

$x = 1$ (we can choose any number here but 1 is easy to substitute.)

$\therefore\ y = 1^2 + 1$

$y = 2$

This tells us a lot about the relationship:

The $x + y$ values

$x = 1$

$y = 2$

It tells us (1, 2) is a point on the graph

And we now know that an input of 1 gives an output of 2

$1 \rightarrow 2$

The rule for this equation is square the number and add 1

What would we get if $x = 2$?

Or the input was 3?

How do we generate more points to plot the graph using the function, or rule?

Example 2

$y = x^2 + 5$

Rule: square the number, then add 5

If $x = 1$ $y = 1^2 + 5$

$y = 1 + 5$

$y = 6$

If $x = 2$ $y = 2^2 + 5$

$y = 4 + 5$

$y = 9$

If $x = 5$ $y = 5^2 + 5$

$y = 25 + 5$

$y = 30$

The values (1, 6), (2, 9) and (5, 30) all appear on the graph although you would probably need more points to draw it accurately.

Example 3

$y = 3x^2$

Rule: square the number, then multiply by 3

If $x = 1$ $y = 3(1)^2$

$y = 3(1)$

$y = 3$

If $x = 10$ $y = 3(10)^2$

$y = 3(100)$

$y = 300$

If $x = 4$ $y = 3(4)^2$

$y = 3(16)$

$y = 48$

If $x = -2$ $y = 3(-2)^2$

$y = 3(4)$

$y = 12$

Example 4

$$y = \frac{x^2}{2}$$

$$x = 2 \quad y = \frac{2^2}{2}$$

$$y = \frac{4}{2}$$

$$y = 2$$

$$x = 3 \quad y = \frac{3^2}{2}$$

$$y = \frac{9}{2}$$

$$y = 4.5$$

$$x = 10 \quad y = \frac{10^2}{2}$$

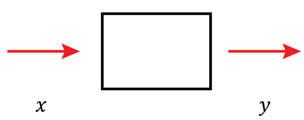
$$y = \frac{100}{2}$$

$$y = 50$$

Rule: multiply the number by itself (square it) then half the number.

ACTIVITY: What's in the black box?

Your teacher will draw the following diagram on the board to play the game. You may use a calculator for the larger numbers.



Your teacher will then ask for input numbers (x) and will give you the output (y).

For example, Ellie volunteers the number 2 and the teacher answers with the number 5. Ellie thinks the rule is to square it and add one ($x^2 + 1$) but wants to wait until the next number is given before she guesses.

Round 1: The winner is the first person to correctly guess the rule. Bonus points for giving in algebraic form rather than in words.

Round 2: now that everyone understands the game, one student must go outside and the class decides on a rule together. Everyone writes it down and it is wiped from the board. When the student returns, they can ask any student for an output by giving them the number. Refer to the notes if you need to, to make sure you're giving them the right answer.

Round 3: Challenge your teacher! See if your teacher is a risk-taker and will go outside and let the class choose. Make sure to make it challenging (but not impossible) because they can give you any number as an input and you must be able to calculate it.

Sometimes when you are playing this game the volunteer who goes outside is stopped by passing teachers or students and asked if they have been kicked out of class! Tell them that you are learning quadratic functions through inquiry and invite them to play too. And remind them to be more open-minded ☺

Were there any input numbers that helped you see the patterns easier? What were those numbers, if you saw any? Why would this be?

Links to: Sciences

What does the phrase 'black box' mean in Physics? Where does it come from? How is it used now?

How do we graph quadratic functions?

Gateway Arch, St. Louis, USA

In the 'guess the rule' game you chose input variables at random to try and find out what the equation was. Now let's look at this process in a different, and more strategic and organised, way. We will take the equation, substitute in values that help us to plot the equation and find the output. If we choose a logical sequential set of values, we can use tables like this to help us.

x	y
−2	
−1	
0	
1	
2	

OR

x	−2	−1	0	1	2
y					

Generating values using a table to graph a quadratic function

You can see from the tables above; we are going to input the following values for x

−2, −1, 0, 1, 2 OR $\{-2 \leq x \leq 2, x \in \mathbf{Z}\}$

into the equation to find the y-values. We can then represent this as a graph. We could have chosen any numbers, including decimals and we could have plotted them also. Remember the equation included all possible points for any given input, not just these five possible points.

Example 1

Using the table right find points and thus graph the following equation:

$y = x^2 + 2$

If you are quick and accurate at substituting into equations, you can use this table.

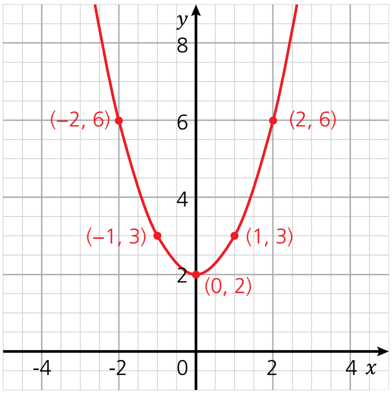
x	y
−2	6
−1	3
0	2
1	3
2	6

You can break it down by adding the two extra columns in the table below.

x	x^2	+2	$= y$
−2	$(-2)^2 = 4$	+2	6
−1	$(-1)^2 = 1$	+2	3
0	$(0)^2 = 0$	+2	2
1	$(1)^2 = 1$	+2	3
2	$(2)^2 = 4$	+2	6

Notice the symmetry of the y-values.

Now it's time to represent these points graphically.

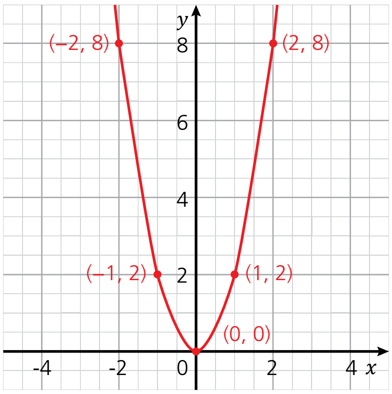


Try to draw the curve very carefully, paying careful attention to the turning point, keeping it smooth rather than pointed.

Example 2

$y = 2x^2$

x	x^2	× 2	$= y$
−2	$(-2)^2 = 4$	2	8
−1	$(-1)^2 = 1$	2	2
0	$(0)^2 = 0$	2	0
1	$(1)^2 = 1$	2	2
2	$(2)^2 = 4$	2	8



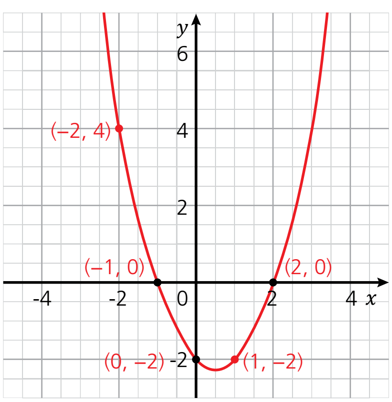
Notice how the y-intercept is at the origin (0, 0) as the equation had no constant term.

Example 3

Sketch the following equation

$$y = x^2 - x - 2$$

x	x^2	$-x$	-2	$= y$
−2	$(-2)^2 = 4$	$-(-2)$	−2	4
−1	$(-1)^2 = 1$	$-(-1)$	−2	0
0	$(0)^2 = 0$	−0	−2	−2
1	$(1)^2 = 1$	−1	−2	−2
2	$(2)^2 = 4$	−2	−2	0



This time notice how the minimum wasn't one of the points you plotted. Looking at the last column in the table, you can see that it must have turned between the points (0, −2) and (1, −2) because the curve is no longer decreasing. In these cases, you can estimate where the minimum would be when drawing. Take care to draw the minimum carefully when you are drawing free hand.

Hint

If you like to be accurate, substitute the mid-value to find the exact position of the maximum or minimum. In this case it would be at $x = 0.5$. The y-value can be found by

$y = (0.5)^2 - (0.5) - 2$

$\therefore y = -2.25$

The minimum occurs at (0.5, −2.25)

PRACTICE EXERCISES

Plot the following equations manually (by hand, using graph paper and pencil)

1 $y = x^2 + 3$

2 $y = x^2 - 3$

3 $y = x^2 + 5$

4 $y = x^2 - 1$

5 $y = x^2 + x$

6 $y = x^2 + 3x$

7 $y = x^2 - 4x$

8 $y = x^2 + 4x - 1$

9 $y = 2x^2 + 3$

10 $y = 3x^2 + 2x + 1$

11 $y = 4x^2 + x$

12 $y = -x^2 + 7$

13 $y = -x^2 + 0.5$

14 $y = 5x^2 - 7+ 3$

15 $y = (x + 3)(x + 4)$

Be warned! you will start to see parabolas or quadratic shapes all around you, especially in architecture and design.

What do these relationships tell us?

HOW CAN WE CHANGE THE SHAPE OF THE 'ARCH'?

Now that you understand how to draw quadratic graphs and what the important points tell us, we can make use of technology to speed up the process. A graphing display calculator is an important tool, particularly in Diploma Programme Mathematics and is allowed in many of the examinations. If you do not have access to a GDC yet, you can also use online or mobile apps such as Desmos as we did in Chapter 3 with linear functions.

Whether you are using a GDC or an app, you can find any values on the graph using a cursor. By clicking the graph (on the apps) or moving the cursor (on the GDCs) you can see the coordinates for the turning point, intercepts or any other points you wish.

There will also be a table of values, like the one you used on the last page. We can consider that the table of values is 'hidden' behind the graph, the calculator is set to default to whole numbers or integers but these are not the only points on the parabola. As we have learned earlier, every point on the parabola has a value and changing the settings will allow you to see decimal values on there too.

ACTIVITY: Playing with parabolas

ATL

- Information literacy skills: Understand and use technology systems

In this formative activity we will change the parameters of the equation, i.e. the 'numbers' in each term, either coefficients or constants. Each change we make will have an effect on the appearance of the shape. By changing one parameter or value at a time, notice how the shape or location changes.

$$y = ax^2 + bx + c$$

parameters

Remember when you change these values, they can be

- **positive or negative**
- **an integer or decimal/fraction**
- **zero or non-zero.**

Using technology, graph several quadratic equations, each time changing a, b or c. By playing with possible values, see how many answers to the following questions you can discover....

What do you do to make the parabola wider?

... to make it narrower? ... to flip it upside down?

How do you move the axis of symmetry away from the y-axis?

... to the left? ... to the right?

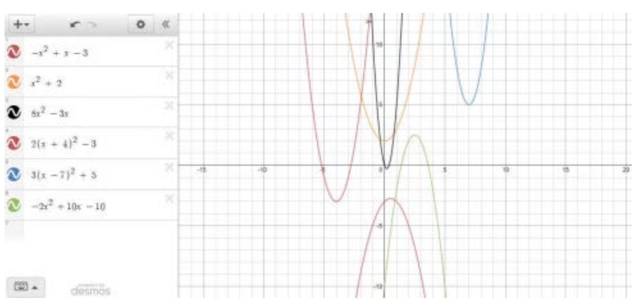
How can you move it up, along the y-axis?

... or down, along the y-axis?

Make it so it doesn't have any x-intercepts?

In your own words, **summarize** what you learned from the activity.

Describe the relationships between the changes to the different parts of the equation and the transformations you observed.



Assessment opportunities

- This activity can be assessed using Criterion C: Communicating.

In the previous activity we looked at the changes or **transformations** which happened when we changed the terms in a quadratic equation. Let's look at a different representation of quadratic equations and see what happens when we make similar changes.

ACTIVITY: Formal investigation of quadratic transformations

ATL

- Information literacy: Process data and report results

One form of expressing a quadratic equation in general terms is like this:

$y = k(x - a)^2 + b$

What effect does varying a, b and k have on the graph? How are these numbers significant? Can you describe what you have found in words, algebraically and/or graphically?

Remember this is a Criterion B: Investigating patterns assessment. In your working, you must show what you changed when and include all graphical representations. You must also **demonstrate** that you:

i. **selected and applied mathematical problem-solving techniques to discover complex patterns**
ii. **described patterns as general rules consistent with your findings**
iii. **proved, or verified and justified, your general rules.**

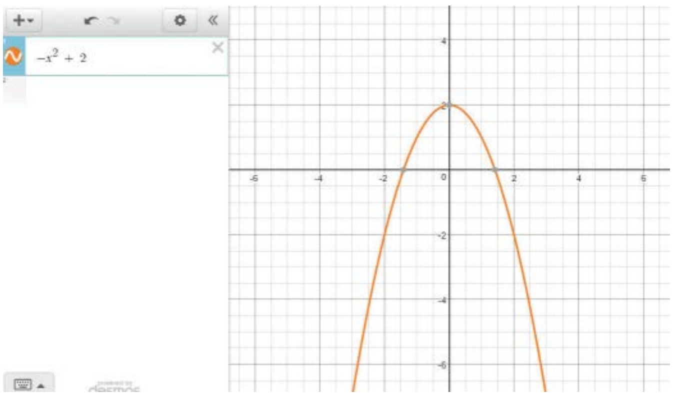
Assessment opportunities

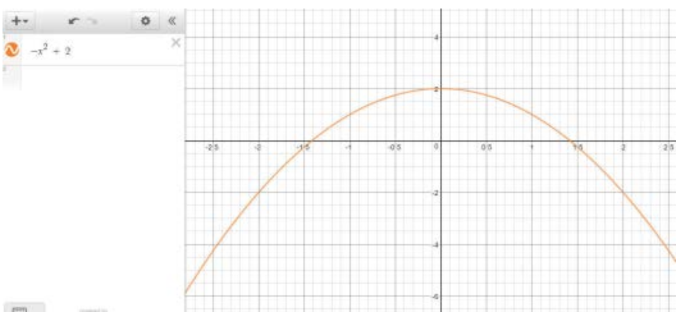
- This activity can be assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

WHAT ELSE COULD APPEAR TO CHANGE THE SHAPE?

The zoom effect

As we know from our MYP studies, so much depends on context; what you see depends on where you are looking. **Compare** the following two graphs:





These are in fact the same equation in both images. **Verify** this by checking the equation on both graphs. Changing the axes or the window view will *appear* to change the shape of the function. We can check that the shape hasn't really changed by using the cursor or table to check the points. Always make sure to check the axes and to reframe the window settings by zooming in or out, to get the most complete picture.

How do we solve or find the roots of a quadratic equation?

HOW CAN WE USE FACTORS TO SOLVE QUADRATIC EQUATIONS?

Solving quadratics by factorization

In Chapter 3, we spent a lot of time factorizing quadratic *expressions*. We can now make use of this to solve quadratic *equations*. Can you **state** the difference between an expression and an equation?

EQUATION OR EXPRESSION?

$3x+4$ | $y=mx+c$ | $15xy$

$4t^2+4t-6$ | $7y^2=700$

$4(x+3)(x-1)+3x^2-3$

We have seen in graphs and images that the result of an *equation* with x^2 as the highest power is a parabola which is symmetrical and has a maximum or minimum point. The shape will also have a y-intercept and it may have x-intercepts.

So how does factorizing help us here?

Example 1

As we have seen before in Chapter 3, you can find the x-intercepts when $y = 0$

$$x^2 + 8x + 12 = 0$$

By factorizing correctly, we know that

$(x + 6)(x + 2) = 0$ remember to check the x-terms

Now what?

The Null Factor Law tells us that if $x \times y = 0$, then either x or y must be zero.

Which means that either

$$(x + 6) = 0 \text{ or } (x + 2) = 0$$

$$x + 6 = 0 \text{ or } x + 2 = 0$$

$$\therefore x = -6 \qquad x = -2$$

By factorizing the equation and making both brackets equal to zero, we have found that the parabola cuts the x-axis at $x = -6$ and at $x = -2$

Example 2

Solve

$$x^2 + 2x - 24 = 0$$

$$(x \quad)(x \quad) = 0$$

try factors of -24 $\quad (x - 12)(x + 2) = 0$

test the x's $\quad (x - 12)(x + 2) = 0$

$$-12x$$

$$2x \qquad = -10x \quad ✗$$

These were not the correct factors. Try again

$$(x - 4)(x + 6) = 0$$

$$-4x$$

$$+6x \qquad = 2x$$

$$x - 4 = 0 \qquad x + 6 = 0$$

$$x = 4 \qquad x = -6$$

Example 3

If the coefficients get larger, the number of possible combinations increase. Be patient and logical and try the combinations of factors that you think will get you to the coefficient of the x-term

Solve

$12x^2 + 25x - 7 = 0$

There are lots of factors of 12 to try here 12 × 1, 2 × 6, 3 × 4 but fortunately 7 is a prime.

$$(6x + 7)(2x - 1) = 0$$

$$+14x$$

$$-6x \quad \Rightarrow +8x$$

try again

$$(4x + 1)(3x - 7) = 0$$

$$3x$$

$$-28x \quad \Rightarrow -25x$$

Nearly! The sign is wrong but the number is right. Swap the signs.

$$(4x - 1)(3x + 7) = 0$$

$$4x - 1 = 0 \qquad 3x + 7 = 0$$

$$4x = 1 \qquad 3x = -7$$

$$x = \frac{1}{4} \qquad x = -\frac{7}{3}$$

$$x = 0.25 \qquad x = -2.33$$

Example 4

Solve

$3x^2 + 27x + 60 = 0$

These terms have a common factor of 3. To refresh your knowledge of taking out a common factor outside a bracket, read Chapter 2.

$3(x^2 + 9x + 20) = 0$

Now we can factorize this more easily than the previous representation.

$3(x + 4)(x + 5) = 0$

We know that $3 \neq 0$. So once again the brackets must be equal to zero, from the Null Factor Law.

$$(x + 4) = 0 \qquad (x + 5) = 0$$

$$x = -4 \qquad x = -5$$

PRACTICE QUESTIONS

Factorize and solve these equations

1. $x^2 + 5x + 6 = 0$
2. $x^2 + 11x + 18 = 0$
3. $x^2 + 15x + 50 = 0$
4. $x^2 + 10x + 24 = 0$
5. $x^2 + 11x + 28 = 0$
6. $x^2 + 15x + 56 = 0$
7. $x^2 - 15x + 44 = 0$
8. $x^2 - 5x + 6 = 0$
9. $x^2 - 5x - 14 = 0$
10. $x^2 + 7x - 60 = 0$
11. $x^2 - 19x - 20 = 0$
12. $x^2 - 2x - 99 = 0$
13. $60x^2 + 32x + 4 = 0$
14. $100x^2 - 25x - 6 = 0$
15. $24x^2 + 32x + 8 = 0$
16. $24x^2 - 32x + 8 = 0$
17. $100x^2 + 496x - 20 = 0$
18. $10x^2 - 98x - 20 = 0$

Hint

Taking out a common factor from all terms, if they have one in common, makes the factorizing easier.

PRACTICE QUESTIONS:

Factorize and solve

1 $2x^2 + 22x + 60 = 0$
2 $2x^2 + 32x + 128 = 0$
3 $3x^2 - 3x - 18 = 0$
4 $5x^2 + 55x + 50 = 0$
5 $14x^2 + 35x + 14 = 0$
6 $27x^2 - 117x + 36 = 0$
7 $24x^2 - 32x + 8 = 0$
8 $100x^2 + 496x - 20 = 0$
9 $10x^2 - 98x - 20 = 0$

Does that last group of questions look familiar?

EXTENDED

If the equation has a $-x^2$ term, take out a common factor of -1 before factorizing

$$-x^2 + 4x - 3 = 0$$
$$\downarrow$$

take out a common factor of –1

$$-1(x^2 - 4x + 3) = 0$$
$$-1(x - 3)(x - 1) = 0$$

factorize as normal

$x - 3 = 0$ or $x - 1 = 0$

$x = 3$ $\quad x = 1$

DISCUSS

Have you ever seen this sign before $\pm$? What does it mean?

Can you guess what $\mp$ might be? **Research** to find out what the meaning is and how the two symbols are different.

SOLVING QUADRATICS BY COMPLETING THE SQUARE

Not all quadratic equations can be solved by factorizing.

For example, it's not possible to find whole number factors to put in brackets for this equation:

$$x^2 + 4x + 2 = 0$$

Not sure? Test it by using the same method as before.

There are other ways to rearrange this expression that makes the equation possible to solve.

How do I **complete the square**?

Completing the square is a technique that allows us to solve for many equations which are not possible to put in brackets.

We know from factorizing in Chapter 3 that perfect square trinomials expand like this:

$$(x + a)^2 = x^2 + 2ax + a^2$$

For example, $(x + 5)^2 = x^2 + 10x + 25$ Notice how the x-term is (2×5) and the constant is 5^2

Example 1

We can use this knowledge to help us to solve an equation such as

$$x^2 + 4x + 2 = 0$$

$$x^2 + 4x + 2 = 0$$

This is almost a perfect square trinomial.

It would have been if it were

$$x^2 + 4x + 4$$

How did we find that?

$x^2 + 4x$ divide 4 by 2, square the result $= (4 \div 2)^2$

$$x^2 + 4x + 4$$

but we have $x^2 + 4x + 2 = 0$

We could do this without changing the volume

$$x^2 + 4x + 4 - 2 = 0$$

so $\quad (x^2 + 4x + 4) - 2 = 0$

and $\quad (x + 2)^2 - 2 = 0$

start solving $\quad (x + 2)^2 = 2$

$$x + 2 = \pm\sqrt{2}$$

$$x = \pm\sqrt{2} - 2$$

$\sqrt{2} = 1.41 \qquad x = 1.41 - 2 \qquad x = -1.41 - 2$

$\qquad\qquad x = -0.59 \qquad x = -3.41$

So essentially we rearrange the equation to make a part of it possible to factorize and then balance the remainder. This way we are not changing the overall value of numbers in the question.

Let's look at that again.

Another way to remember is to halve the x and put in brackets, then subtract the square from the constant term.

www.youtube.com/watch?v=MnxWXarD1Tw

www.youtube.com/watch?v=-ehmSJTSBuk

Example 2

$$x^2 + 10x + 11 = 0$$

halve the x $\qquad$ subtract the square

$$(x + 5)^2 + 11 - 25 = 0$$

$$(x + 5)^2 - 14 = 0$$

$$(x + 5)^2 = 14$$

$$x + 5 = \pm 14$$

$x = \pm\sqrt{14} - 5 \qquad$ in *exact* form

For *approximate*, decimal answers find $\sqrt{13}$ on a calculator.

$$x = \pm 3.7 - 5$$

$x = +3.7 - 5 \qquad x = -3.7 - 5$

$x = -1.3 \qquad x = -8.7$

Hint

A question involving completing the square may ask you to leave the answer in exact form. This means you should leave the versions with radicals/root form. To revise exact form, see Chapter 1.

PRACTICE EXERCISES

Solve these equations by completing the square:

1. $x^2 + 4x - 2 = 0$
2. $x^2 + 8x + 4 = 0$
3. $x^2 + 20x + 40 = 0$
4. $x^2 + 16x - 9 = 0$
5. $x^2 - 14x + 40 = 0$
6. $x^2 - 3x + 1 = 0$

This method might not always work to **solve** as you can't find the square roots to negative numbers. But what can be done if an equation can't be solved by factorizing OR completing the square?

WHAT IS THE MIDNIGHT FORMULA?

One of the most well-known of all the equations in mathematics is this:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $ax^2 + bx + c = 0$

This equation, or formula, can also help you to **solve** for x. We have seen earlier that if $y = 0$, the x values are the roots or x-intercepts of the equation. Generations of mathematics students have learned this formula by heart and many adults will still recognize it (and some shudder) to this day. In Germany, the formula is called the *Mitternachtsformel*, or midnight formula, because students traditionally had to learn it so well that if they were woken from sleep at midnight, they would be able to recite it perfectly. Even asleep. What is it called in your country?

Let's look at the formula in action. Communication is really important here. Lay out your lines of reasoning clearly. This will make it easier to correct any mistakes, should you have them. It will also gain higher achievement levels in Criterion C assessments.

You no longer have to memorize the formula (unless you want to) as the formula is available to you in class and will be provided to you in IB examinations.

Hint

Make sure that your square root symbol stretches all the way across, so that it is above the last number in the calculation. Likewise, make sure that the fraction is also clearly all the way across. Otherwise, this would be an example of poor communication of the calculation.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2}}{2(1)} - 4(1)(6) \quad ✗$$

Example 1

Solve

$$x^2 + 5x + 4 = 0$$

Step 1: Identify a, b, c

$$x^2 + 5x + 4 = 0$$

$a = 1$ $\quad b = +5$ $\quad c = +4$

because $\quad x^2 = 1x^2$

Step 2: Write down the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 3: Substitute

$a = 1$

$b = 5$ $\qquad x = \frac{-5 \pm \sqrt{5^2 - 4(1)(4)}}{2(1)}$

$c = 4$

Step 3a: Double check your substitutions (especially signs)

Step 4: Simplify, showing working

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 16}}{2}$$

$$x = \frac{-5 \pm \sqrt{9}}{2}$$

Step 5: How do we handle the $\pm$?

We know this symbol means plus or minus. We know that this is because square roots have two possible answers (see Chapter 1). So, we proceed with both calculations.

$$x = \frac{-5 \pm \sqrt{9}}{2}$$

plus $\qquad$ minus

$x = \frac{-5 + 3}{2}$ $\qquad x = \frac{-5 - 3}{2}$

$x = \frac{-2}{2}$ $\qquad x = \frac{-8}{2}$

$x = -1$ $\qquad x = -4$

PRACTICE QUESTIONS

1 $x^2 - 3x + 2 = 0$

2 $x^2 - 6x + 9 = 0$

3 $x^2 - 8x + 7 = 0$

4 $x^2 - 2x - 15 = 0$

5 $x^2 - 6x - 16 = 0$

6 $x^2 + 21x + 76 = 0$

7 $2x^2 - 7x + 3 = 0$

8 $3x^2 + 4x - 15 = 0$

9 $x^2 + 4x - 15 = 0$

Example 2

Solve

$x^2 - 5x + 6 = 0$

Step 1: Identify $a = 1 \quad b = -5 \quad c = 6$

Step 2: State formula $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Step 3: Substitute $\quad x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$

Step 4: $\quad x = \dfrac{5 \pm \sqrt{25 - 24}}{2}$

$$x = \frac{5 \pm \sqrt{1}}{2}$$

$+$: $x = \dfrac{5+1}{2}$, $x = \dfrac{6}{2}$, $x = 3$

$-$: $x = \dfrac{5-1}{2}$, $x = \dfrac{4}{2}$, $x = 2$

! Take action

! There are hundreds of videos online of students singing, rapping, performing or animating ways to remember the quadratic formula. Create a top ten list of links to help you and your fellow students to remember what it's for and how to use it.

! Here are some questions to consider:

- What are you basing your decisions on?
- How did you choose the top ten?
- Does it inspire you to make one of your own?

Example 3

$x^2 - 2x - 15 = 0 \qquad a = 1 \quad b = -2 \quad c = -15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-15)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

$$x = \frac{2 \pm \sqrt{64}}{2}$$

$+$: $x = \dfrac{2+8}{2}$, $x = \dfrac{10}{2}$, $x = 5$

$-$: $x = \dfrac{2-8}{2}$, $x = \dfrac{-6}{2}$, $x = -3$

Watch out for this common error:

$x^2 - 3x - 10 \qquad a = 1 \quad b = -3 \quad c = -10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{-3^2 - 4(1)(-10)}}{2(1)}$$

Can you see the mistake?

The student forgot to use brackets and now will have the wrong sign *twice*. This will definitely lead to the wrong answer.

$$x = \frac{-3 \pm \sqrt{-3^2 - 4(1)(-10)}}{2(1)} \times$$

Should be

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)}$$

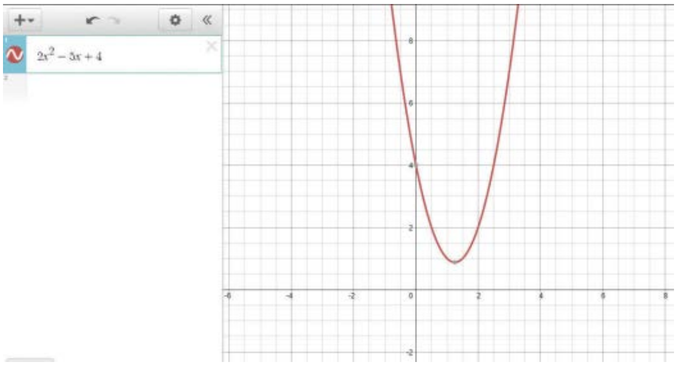
Solutions can also have decimal values. Remember to give the decimals to an appropriate decimal place, usually 1 or 2 d.p.

How can we collaborate to improve communication?

A team of students are collaborating on some graphing and solving work for quadratics. Each student receives a selection of equations that they must map on a giant Cartesian plane on the classroom wall. As principled learners, and because it is for display, they want to make sure that everything is correct before they plot.

Student 1:

So-Shan has been given the equation $y = 2x^2 - 5x + 4$ to graph. She plots it on the GDC/App and is given the following:



She decided to find the roots using the quadratic formula but quickly runs into a problem. By applying the formula and showing your working, find out what So-Shan's worry is.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What does it mean? **Use** the graph to support your answer.

Student 2:

Andy's equation is $y = x^2 - 5x + 6.25$. He decides not to use technology and wants to find the roots first, as well as generate x and y values manually. When he applies the quadratic formula, however, he finds that he gets the same root twice. He repeats the calculation twice more but keeps getting the same answer. Has Andy done something wrong? **Verify** his working by finding the roots and graph the equation to show your results.

Student 3:

Irina is given the equation $y = x^2 - 12x + 20$. She decides to complete the work as efficiently as she can and rearranges the equations in the following ways:

First:

$y = x^2 - 12x + 20$

$y = (x - 2)(x - 10)$

Secondly:

$y = x^2 - 12x + 20$

$y = x^2 - 12x + 36 - 16$

$y = (x - 6)(x - 6) - 16$

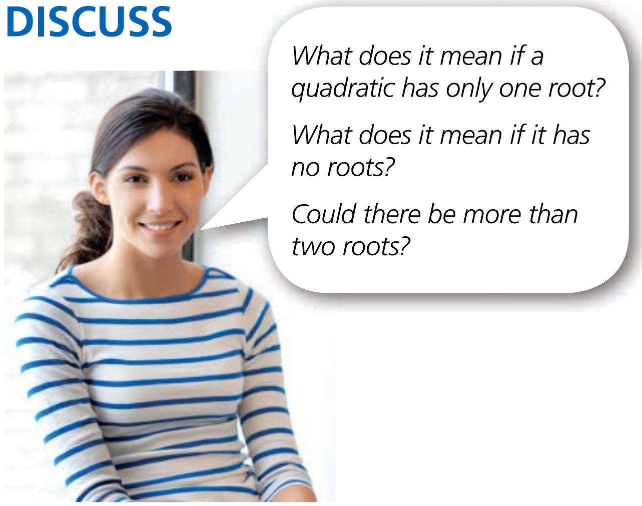
$y = (x - 6)^2 - 16$

She then goes to the wall display, charts the points (2, 0), (10, 0) and (6, –16). She creates a parabola by joining them. How did she know these points? What do you call the processes/calculations she carried out?

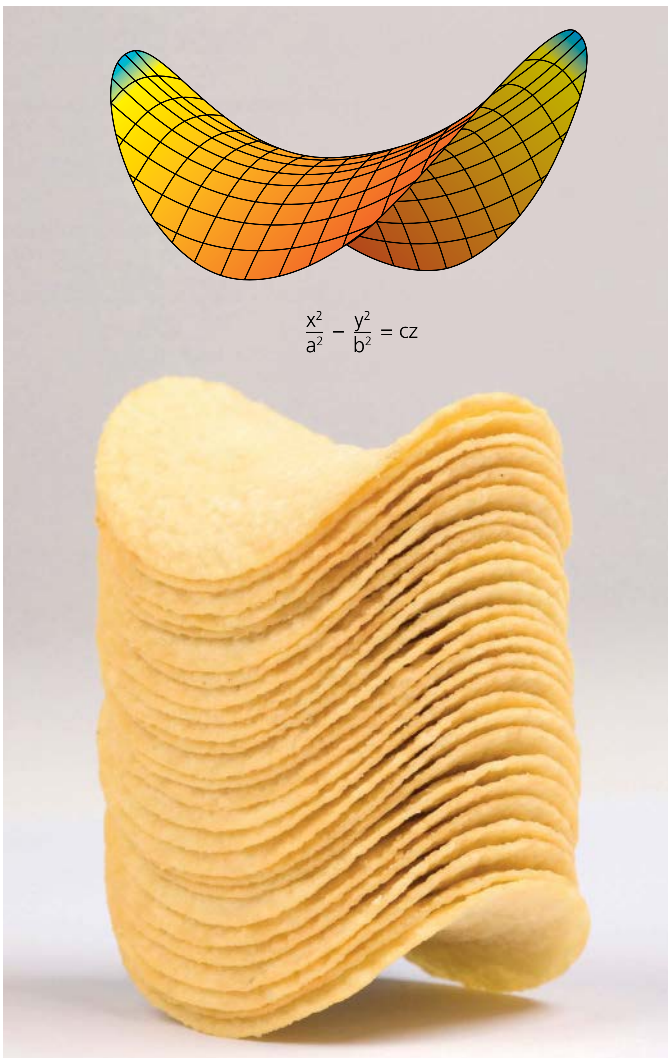
Take action

- Why not try the same activity for your classroom? Pick the largest available space or free wall in your classroom and superimpose as many quadratic equations as you can. Use different colours and both x^2 as well as $-x^2$ terms to create an intricate artistic pattern.

DISCUSS



With your partner, discuss your answers to these questions. What would you tell this student?



Pringles are examples of hyperbolic paraboloids can you see the parabolas in a Pringle?. Can you see the parabolas in the Gardens by the bay photo?

Gardens by the bay, Singapore

ACTIVITY: A bad day for Gareth

ATL

- Communication: Give and receive meaningful feedback

Gareth is having a bad day and everything seems to be going wrong with his calculation. He knows he is going wrong and asks you to look at his work and correct him so he can learn from his mistakes. How many can you find? Can you **explain** to him what he has done wrong?

$3x^2 + 5x - 2 = 0 \qquad a = -3 \quad b = 5 \quad c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{5 \pm \sqrt{5^2 - 4(-3)(-2)}}{2(-3)}$$

$$x = -\frac{5 \pm \sqrt{25 - 39}}{6}$$

$$x = \frac{-5 \pm}{6}\sqrt{64}$$

$$x = -5\frac{\pm 8}{6}$$

$$x = -5\frac{+8}{6} \qquad\qquad x = \frac{-5-8}{6}$$

$$\frac{3}{6} \qquad\qquad x = \frac{2}{6}$$

$$x = 0.33333\ldots$$

$$\underline{\underline{0.5}} \qquad\qquad x = \underline{\underline{0.4}}$$

Assessment opportunities

- This activity can be assessed using Criterion C: Communicating.

What do these relationships in different forms tell us?

WHY DO WE LEARN MULTIPLE WAYS TO SOLVE OR REPRESENT THE SAME FUNCTION?

Summary of the methods for solving quadratic equations:

1 First we **factorized** the quadratic equations …
- put into two brackets (or took out a common factor first)
- then used the Null Factor law to make each bracket equal to zero and solved for x

In general terms: we wrote in the factored form $y = k(x - a)(x - b)$ where k is a common term.

Why was this form useful? Because in this form, a and b represent the roots or solution of the equation when equal to 0. The role of k is less obvious. What happens when you change that common factor outside the brackets?

2 Next **we completed the square**:
- removed any common factor by placing outside the bracket
- then halving the x-term, and putting into $\left(x - \frac{b}{2}\right)^2$ and subtracting the square and rearranged to **solve** for x.

In general terms, this gave us the form (known as vertex form)

$y = k(x - a)^2 + b$ where k is a common term.

Why was this useful? Because this form shows us the *vertex* – minimum or maximum, in the form of (a, b). What happens if you change the k here?

3 Lastly we used the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $y = ax^2 + bx + c$, we substituted the values for a, b and c and solved.

In general terms, this came in the form $= ax^2 + bx + c$, as seen above.

Why was this useful? This form gives us the values for a, b and c so we can apply the formula and find the roots or solutions.

Hint

The letters a and b mean different things in each instance. Don't let that confuse you or catch you out. More important to remember is the structure of the various representations and what each number means when it appears in the various locations.

REFLECTION

Which method do you find easiest to remember? Which do you find fastest to carry out? Why? In which method do you make the fewest mistakes?

Each different representation of the quadratic provides a different insight into characteristics of the quadratic equation.

PRACTICE EXERCISES

Review each method by **solving** the equations below. **Use** the information from the three solutions to **draw** a rough sketch of the quadratic functions. Remember to make the function (or y) = 0 to **solve** the equation.

a $y = x^2 - 7x + 12$

Factor form:

Completing the square/ Vertex form:

Using the quadratic formula:

b $y = 21x^2 + x - 2$

Factor form:

Completing the square/ Vertex form:

Using the quadratic formula:

c $y = 15x^2 + 18x + 3$

Factor form:

Completing the square/ Vertex form:

Using the quadratic formula:

d $y = -x^2 + 2x + 15$

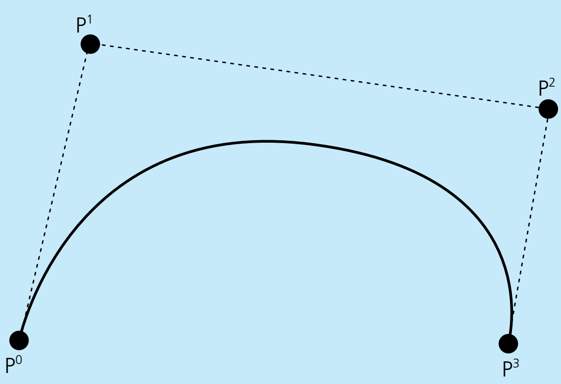
Factor form:

Completing the square/ Vertex form:

Using the quadratic formula:

MEET AN INTERDISCIPLINARY MATHEMATICIAN: PIERRE ETIENNE BEZIER (1910–1999)

Learner Profile attribute: Open-minded



Bézier curve

Pierre Étienne Bézier (September 1, 1910 – November 25, 1999) was a French engineer and one of the founders of the fields of solid, geometric and physical modelling as well as in the field of representing curves.

He dedicated his entire working life and career to the French automobile manufacturing company Renault. As a designer and engineer, he was open-minded to the power of new technologies with 'old' mathematics and did not stay stuck in old patterns of thinking about manufacturing. Instead, he became interested in computer design when it was just beginning and he became a leader in the transformation of design and manufacturing, through mathematics and computing tools, into computer-aided design and three-dimensional modelling.

Bézier patented and popularized the Bézier curves and Bézier surfaces that are now used in most computer-aided design and computer graphics systems. His academic publications came very late in life and his principled loyalty to his employer was repaid by their full support for his innovating work.

Why should we optimize production and use of materials using quadratics?

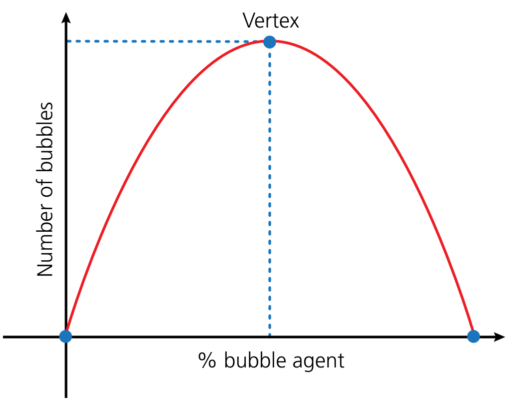
Algebraic functions are a way to represent relationships – these relationships can tell us something about the real world. If we can model this relationship, we can make predictions and decide on the 'best case scenario' for a given context.

When you find a turning point, either a maximum or a minimum that tells you something very specific about the relationship. It may be the highest point of an arrow being flown through the air or the minimum amount of materials wasted in the production of a computer chip. Solutions of this nature are called **optimizing** because you are finding the optimum solution for your problem.

Example 1

Imagine a bubble mix for a children's toy. The amount of the foaming agent in the water clearly affects the number of bubbles. Not enough agent and the bubbles will not form well, too much and it is too soapy and the bubbles won't last.

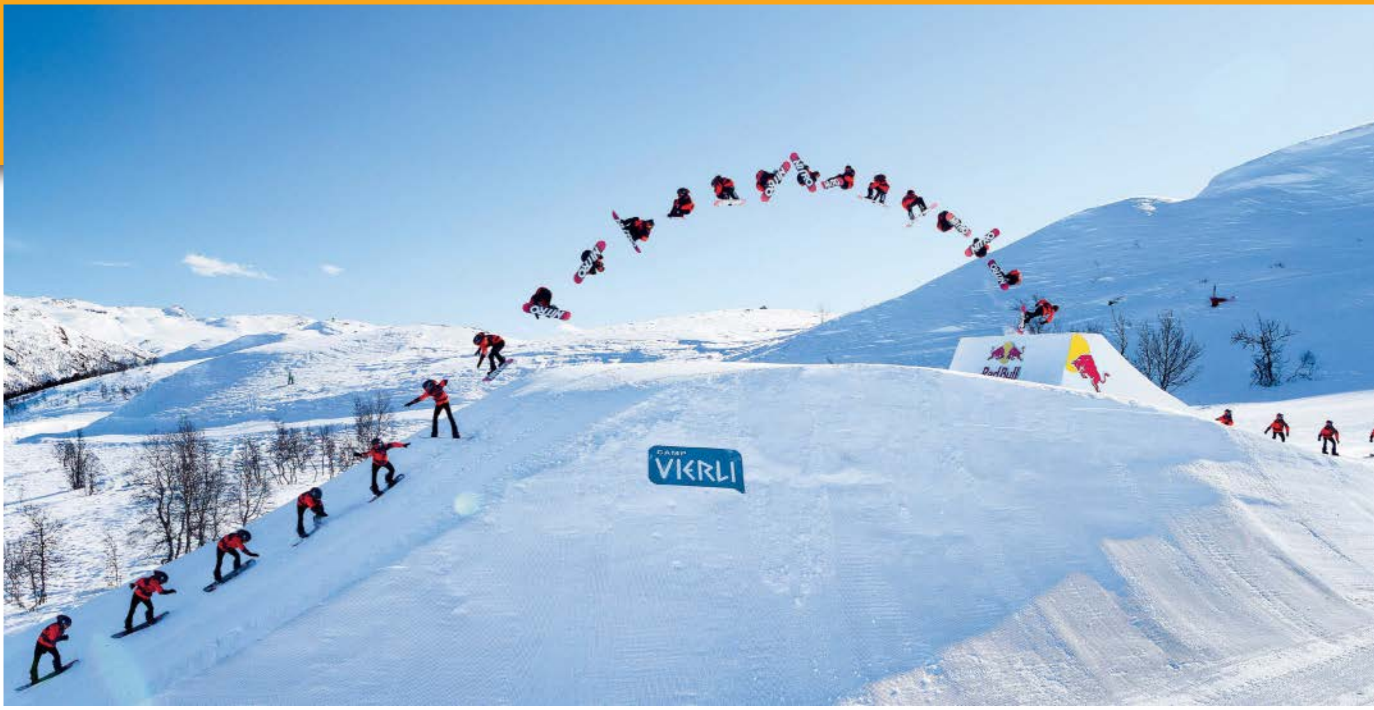
Imagine the relationship was a quadratic one like this:



■ The vertex on an arc

By finding the vertex, either graphically or algebraically we can tell the best quantity of bubble agent to add to get maximum bubbles. But why is this helpful in real life?

Let's consider our global context for this unit: Globalization and sustainability and more specifically Consumption, conservation, natural resources and public goods. If we want to conserve our resources, minimise our use of natural resources and materials, reduce our consumption or maximise our efficiencies then optimizing the solutions can help us.



Marcus Kleveland's snowboard jump

Example: Optimizing a projectile

http://board-channel.com/13-year-old-snowboarder-lands-triple-cork-marcus-kleveland-2013/

Marcus Kleveland, a 13-year-old snowboarder, leaves the end of the ramp in the picture at high speed to complete a complicated 'triple cork'. His height at any point (or the function) is given by

$y = -5x^2 + 20x$ where y = height and x = time

Use the information provided to **find** as much as you can about Marcus' triple cork and optimize the equation to find when he will be at his highest point.

Solution

First, factorizing

Let $-5x^2 + 20x = 0$

take out common factor of $-5x$ $\quad -5x(x - 4) = 0$

Null Factor Law

$-5x = 0 \qquad x - 4 = 0$

$x = 0$ s $\qquad x = 4$ s

So, Marcus is on the ground (zero height y-intercept) at 0 s and again at 4 s.

Now complete the square to get vertex form:

$-5x^2 + 20x = 0$

Common factor -5 $\quad -5[x^2 - 4x] = 0$

$-5[(x - 2)^2 - 4] = 0$

So we know vertex = (2, 20)

Which tells us that after 2 seconds, Marcus' height is 20 metres.

Solve

$-5[(x - 2)^2 - 4] = 0 \qquad \div 5$

$(x - 2)^2 - 4 = 0 \qquad + 4$

$(x - 2)^2 = 4 \qquad \sqrt{\ }$

$x - 2 = \pm\sqrt{4}$

$x - 2 = \pm 2 + 2$

$x = \pm 2 + 2$

$x = -2 + 2 \qquad x = +2 + 2$

$x = 0$ s $\qquad x = 4$ s

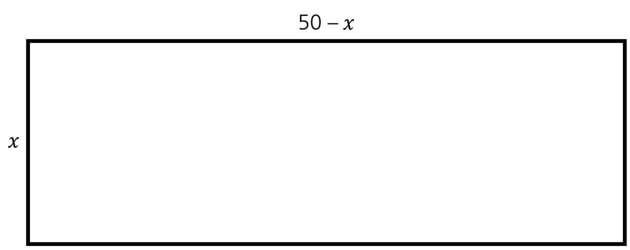
These are the same values as previously, so we know we are correct. The quadratic formula will give the same values.

SOME SUMMATIVE PROBLEMS TO TRY

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION D: APPLYING MATHEMATICS IN REAL-LIFE CONTEXTS

A farmer wishes to build a pen (enclosure) to protect his chickens from predatory foxes. He wants the chicken to have as much freedom and room to move as possible. The fencing panels come in 1 metre panels (i.e. no decimal values possible, integers only) and he can only afford 100 panels.

The farmer can vary the lengths of the sides of the pen, which will affect the area. Planning restrictions say that the pen must be rectangular and must be in the form provided below because all 4 sides must add to 100m.



■ Plan for the pen

i. **Identify** relevant elements of authentic real-life situations.

Explain why as the length of the pen increases, the width decreases.

Explain why the sides of the pen have been labelled as x and $(50 - x)$

Show why the area (y) is represented by the function $y = -x^2 + 50x$

ii. Select appropriate mathematical strategies when solving authentic real-life situations.

The farmer wants to know what areas he would be able to find for different lengths. By choosing different lengths (values for x), find the corresponding areas. You may use a table to show your results

iii. **Apply** the selected mathematical strategies successfully to reach a solution.

Represent the quadratic function on a graph to show how area changes with length

Find the optimum dimensions (both length and width) of the pen for the farmer

iv. **Justify** the degree of accuracy of a solution

How realistic is this problem? Would your answer help the farmer? How close to the 'real' value can you get with a function?

v. **Justify** whether a solution makes sense in the context of the authentic real-life situation.

How does this problem relate to other situations where the resources are limited?

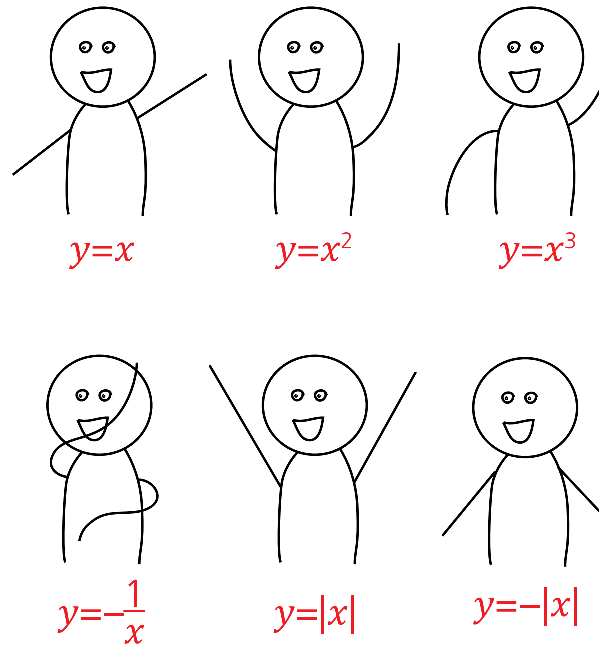
In what other similar situations could you use this strategy?

Reflection

<table>
<tr><th colspan="6">Use this table to reflect on your own learning in this chapter.</th></tr>
<tr><td>Questions we asked</td><td>Answers we found</td><td colspan="4">Any further questions now?</td></tr>
<tr><td>Factual: How do we graph quadratic functions? How do we solve or find the roots of a quadratic equation? What relationships can we model using quadratic equations?</td><td></td><td colspan="4"></td></tr>
<tr><td>Conceptual: What’s more ‘powerful’ than a linear equation? Where can we see the representation of quadratic equations and functions around us? What do we mean by solving equations? What do these relationships in different forms tell us? What is the quadratic function telling us to do with an input?</td><td></td><td colspan="4"></td></tr>
<tr><td>Debatable: Can mathematics give us solutions to problems of sustainability or are they suggestions? Why should we optimize production and use of materials using quadratics? How can we collaborate more to improve communication?</td><td></td><td colspan="4"></td></tr>
<tr><td rowspan="2">Approaches to Learning you used in this chapter:</td><td>Description – what new skills did you learn?</td><td colspan="4">How well did you master the skills?</td></tr>
<tr><td></td><td>Novice</td><td>Learner</td><td>Practitioner</td><td>Expert</td></tr>
<tr><td>Communication skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Media literacy skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Creative-thinking skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Learner profile attribute(s)</td><td colspan="5">Reflect on the importance of being open-minded and principled for your learning in this chapter.</td></tr>
<tr><td>Open-minded</td><td colspan="5"></td></tr>
<tr><td>Principled</td><td colspan="5"></td></tr>
</table>

Form | Generalization | Identities and relationships

9 How do functions function?



Relationships can be **identified** by **generalizing** data into various models and **forms**, which allows us to solve and predict these real-world relationships.

CONSIDER THESE QUESTIONS:

Factual: What is a function? How do we use functions to find values? How can functions be reversed or combined? How and why do we develop the form of functions to improve their accuracy?

Conceptual: Why do we use functions to generalize? How can we use them as predictions? What is meant by exponential growth or decay?

Debatable: How far should we trust functions or models to tell us about the real world? Can multiple functions be correct for a single situation?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

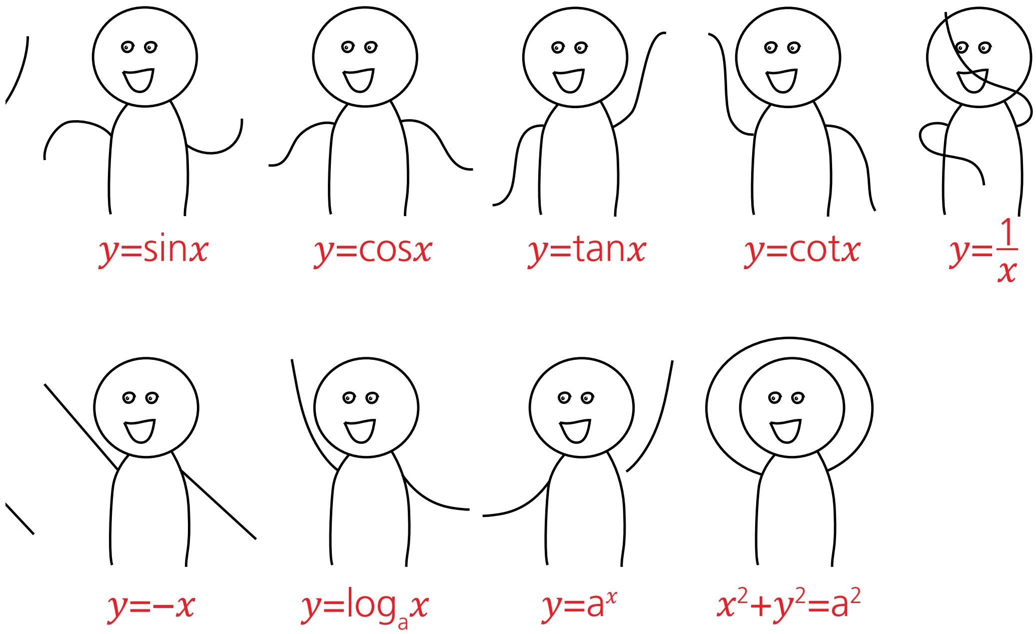
- **Find out** how functions work and how they occur in real-world numbers and visual examples.
- **Explore** the nature of a function from a variety of perspectives.
- **Take action** through both investigation and debate, focused on the rise of online 'clicktivism', a thoroughly modern phenomenon.

These Approaches to Learning (ATL) skills will be useful …

- Reflection skills
- Communication skills
- Organization skills
- Information literacy skills

We will reflect on this learner profile attribute …

- Inquirer – We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.



Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-world contexts

'A new equation, showing how our happiness depends not only on what happens to us but also how this compares to other people, has been developed by researchers. The team developed an equation to predict happiness in 2014, highlighting the importance of expectations, and the new updated equation also takes into account other people's fortunes.' – **www.sciencedaily.com/releases/2016/06/160614083116.htm**

PRIOR KNOWLEDGE

You will already know:

- what linear equations are
- how to solve equations, linear and quadratic
- how to apply the order of operations rules, when substituting
- how to represent equations in a variety of ways
- how to plot data onto graphs so they can be analysed.

KEY WORDS

composite	function
corresponding	interpolate
exponential	inverse
extrapolate	

DISCUSS

We hear a lot about models in reports and news items. Scientists, and data scientists try to find patterns in data, then find the best match to model that data. Why are we, as humans, always looking for order in (apparent) chaos?

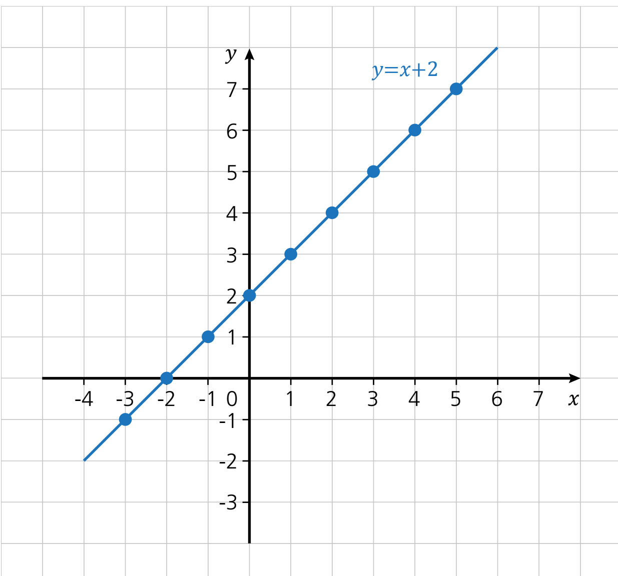
SEE–THINK–WONDER

'The brain is nothing more than a pattern processing system' – Bruce Hood, psychologist.

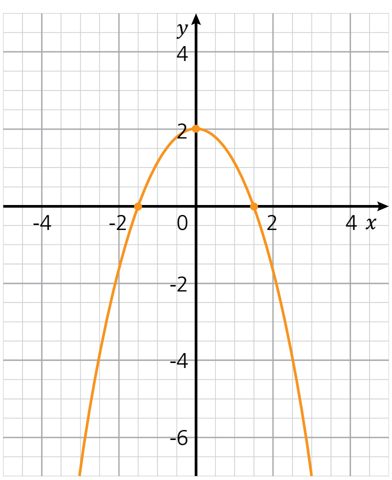
Why do you think he said this? What do you think it means? **Discuss**, with a partner, examples of where we try to find patterns as humans – both within and outside of mathematics if you can.

What is a function?

There are many ways to understand the meaning and purposes of functions.



■ Example of a linear function



■ Example of a quadratic function $f(x) = -x^2 + 2$

ACTIVITY: Evaluating propositions

ATL

- Reflection skills: Considering the process of learning; choosing and using ATL skills: Identify strengths and weaknesses of personal learning strategies (self-assessment)

A function is a complex thing as it can be seen in many different ways. As a result, there are many definitions to explain what a function is. Which is best?

Consider the following definitions:

a **Mapping: A function can be understood as a mapping from one set of numbers to another, where each x-value has a corresponding y-value.**

b **Machine: A function gives you instructions how to calculate from one set of numbers to another.**

c **Algebraic: A function is a rule which receives an input and produces only one output for that value.**

d **Graphical: A function describes change or variation between two sets of values, (the independent and the dependent variables).**

Given the four definitions, **state** which definition makes the most sense to you and why? What appealed to you about this one over the others?

How is the version you have chosen connected to how you learn? Reflecting on your choice, what does this tell you about your own strengths in mathematics?

Which one looks the *least* appealing or understandable to you? Why do you think that might be? What does this tell you about any possible weaknesses in your mathematical knowledge? How could you combat this?

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

In both linear and quadratic practice problems we found the output (y-value or dependent variable), if we were given the input (x-value or independent variable). We used the functions to find values by substitution, to find y, or solving, to find x.

MOVING BETWEEN REPRESENTATIONS

Function notation also sometimes uses f(x) = to indicate a function. Until now, we have been more familiar with the y = version.

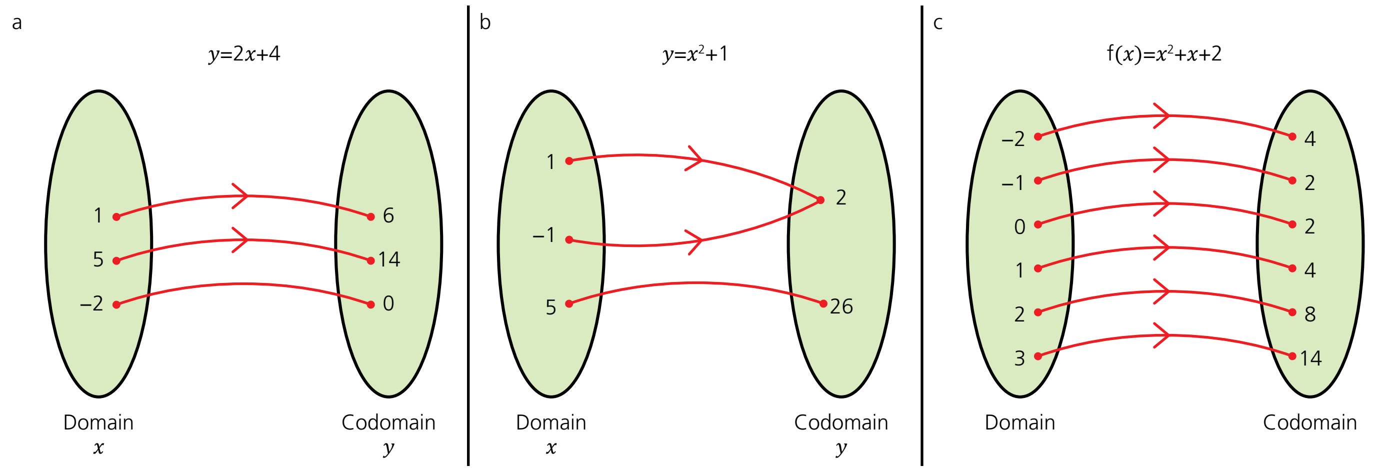
> **Hint**
> This is read 'f of x', meaning a *function of x*.
> Watch out! It is NOT f multiplied by x.

We will change between them both, depending on the question or context. It is important to be able to move between these representations flexibly and confidently.

Let's look at the various forms of functions for each of the definitions in the Activity: Evaluating Propositions. You will see that we have used the same colour coding. How does each one work and how are they connected?

Function as a mapping

We can visualize the relationships between these sets of numbers as a **mapping** between them.



Note that there can be more than one input or x-value mapping onto the same output value, see image c above. An output can have one or more inputs but an input can only have one output.

In this example of a quadratic function, we see that each value for x maps onto a corresponding value for y, in the codomain. Although each input is unique, the mapping shows the same number as an output for more than one input. How could this diagram be simplified?

THINK–PAIR–SHARE

Based on these images of mappings in this section, how would you define *domain*? *Codomain?*

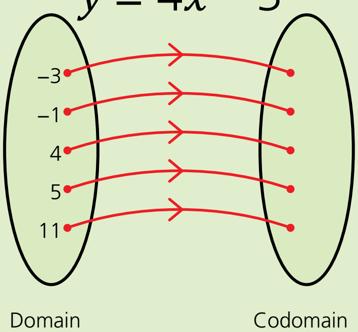
Could the number of elements in the codomain ever be greater than the number of elements in the domain? Could infinity be a number in the codomain?

Consider these ideas and share your definitions with your partner. Come to an agreement on your joint definition, then share with the class.

PRACTICE QUESTIONS:

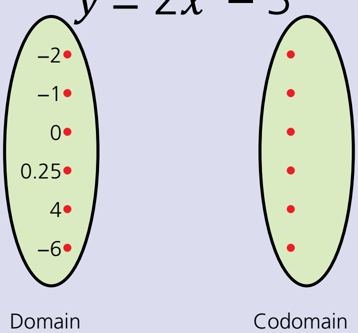
1 **Complete** the following mapping for

$y = 4x - 3$



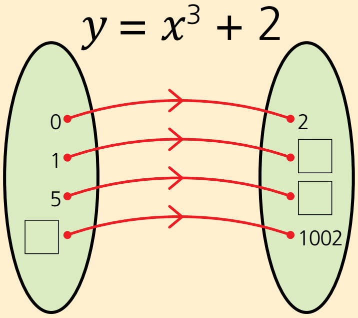
2 **Complete** the following mapping for the quadratic function

$y = 2x^2 - 3$



3 **Find** the missing elements (values) in the set for the function

$y = x^3 + 2$

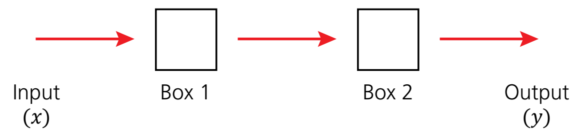


Function as a machine

HOW CAN WE REPRESENT THE FUNCTION MACHINE?

We know that we can consider a function to be a set of couples, none of which have the same input (although they may have the same output)

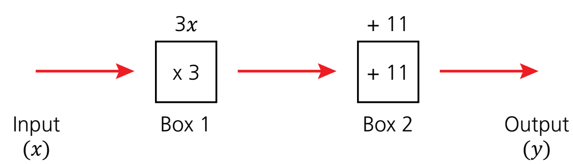
The couples (2, 1), (3, 4), (2, 2) and (3, −1) cannot be a function as the input occurs twice for both $x = 2$ and $x = 3$



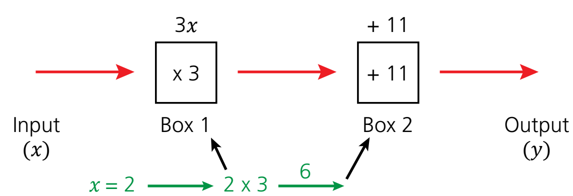
■ A function machine with a box for each step of the substitution

Example 1

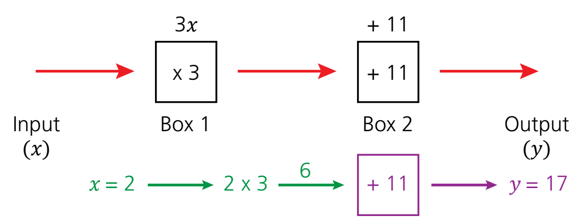
You have been given the function $y = 3x + 11$ to represent. This requires **two** operations – first multiplying by three and then adding 11. The *order of operations* tells you that you must multiply before you add.



For $x = 2$, we use this an input into the machine.



The first 'box' will multiply the number by 3, which gives us an output of 6 to go into the second box.



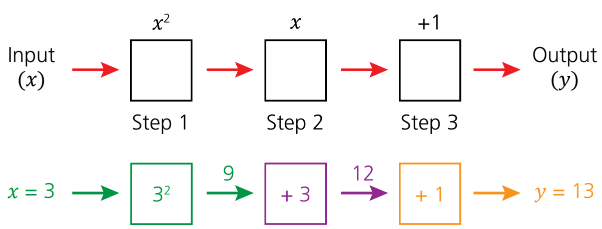
As the second box always adds 11, the final output of the machine is 17. This gives us the values $x = 2$, $y = 17$, or (2, 17).

Remember the ‘guess the rule’ game, where you had to step outside the room? The guesser gave an input and by looking at what was given in return, they had to guess the rule, or function? These function machines are the same idea – except we see what is happening in the boxes and we don’t have to guess. You could play this game again now, with more complex functions but don’t make it too hard! Keep it to a maximum of two steps or it will get far too complicated.

Example 2

A function such as $f(x) = x^2 + x + 1$ would require three boxes in the function machine.

So for $x = 3$, we find



Each time you repeat the function machine for a different value, you are finding a set of values (x, y) that can be plotted on a plane. The more values you substitute, the more points you will need to plot on a graph. The more points you plot, the clearer the form or shape of the function becomes.

While this is a clear and correct way to find corresponding values or coordinate points, it has **limitations**. It is time consuming, a little simplistic and many people begin to calculate these two or three step-equations in their head or on a calculator quickly.

Functions in algebraic form

How do we find the numbers in the domain or the codomain?

We already know from linear equations and functions that for $x = 0$, $x = 1$ and $x = 2$ into the function

$$y = -2x + 3$$

$$y = -2x + 3$$

$x = 0$ $\quad y = -2(0) + 3$

$y = 0 + 3$

$y = 3$

$\rightarrow x = 0 \quad y = 3$

And for the next two values.

$$y = -2x + 3$$

$(x = 0)$, $y = -2(0) + 3$ $\quad$ $(x = 1)$, $y = -2(1) + 3$ $\quad$ $(x = 2)$, $y = -2(2) + 3$

$y = 0 + 3$ $\quad$ $y = -2 + 3$ $\quad$ $y = -4 + 3$

$y = 3$ $\quad$ $y = 1$ $\quad$ $y = -1$

$\rightarrow x = 0, y = 3$ $\quad$ $\rightarrow x = 1, y = 1$ $\quad$ $\rightarrow x = 2, y = -1$

GENERATING VALUES USING A TABLE

A table showing corresponding x and y values (or the start and the end of the function machine/ the input and the output) is a quick way to show the *corresponding* points and allows you to identify any patterns in the dependent variables.

Example

Using a table to find points on the function $f(x) = x^2 - 5$

There may be multiple steps to substituting into a function. If it helps, you can build those steps into the table to remind you of each one. In this example, the subtraction of 5 has been added to the table so it can easily be taken from the value of x^2.

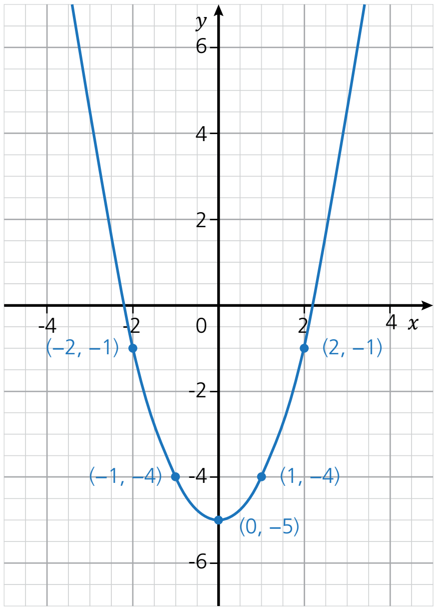
x	−2	−1	0	1	2
x^2	4	1	0	1	4
−5	−5	−5	−5	−5	−5
y	−1	−4	−5	−4	−1

So we see the couples, or points,

$x = -2, y = -1$ $x = -1, y = -4$ $x = 0, y = -5$ $x = 1, y = -4$ $x = 2, y = -1$

(−2, −1) (−1, −4) (0, −5) (1, −4) (2, −1)

Remember, these points aren't the only ones that would appear on a graph and they aren't the only ones on the function. Look at the plotted table and the corresponding graph below. Can you name any other points which satisfy the equation and must therefore be on the function?



For linear or quadratic functions a few points are enough to allow you to draw the line clearly. With other functions, the shape may be less obvious and may require more points. With enough points, you can identify the shape by eye and the GDC can display every possible set of values for that function. The uses for these functions will be investigated later in the chapter.

Or we can change the orientation to work this way:

For the function $f(x) = \frac{1}{x^2}$

x	x^2	$\frac{1}{x^2}$	y or $f(x)$ in decimal form
1	1	$\frac{1}{1}$	1
2	4	$\frac{1}{4}$	0.25
3	9	$\frac{1}{9}$	0.11
4	16	$\frac{1}{16}$	0.0625
5	25	$\frac{1}{25}$	0.04

*NB: Communication: using integers above – in Chapter 3 you learned that picking –2 to 2 as values for x (or domain) comes mostly from habit or as a convention around the origin. We could pick any rational values and these would also appear on the graph of the function. In fact, the tables in the GDC can show any required values, depending on their settings.

Functions in graphical form

If you include more and more values into a **mapping**, a **machine** or a list of **algebraic terms (points or couples)**, the set will get bigger and more complex and soon it will stop being useful to see the relationships. Is there are another way we could represent these sets of points clearly?

Equations and functions express relationships between quantities. The relationship shown by a function can be represented visually - by a line or points in space. These lines can be straight or curved lines. The shape and connectedness of the graph depend on how the variables relate to one another, how one changes as the other one changes. A function can also be represented symbolically using algebra but a graph can make patterns obvious, particularly to visual learners.

Hint

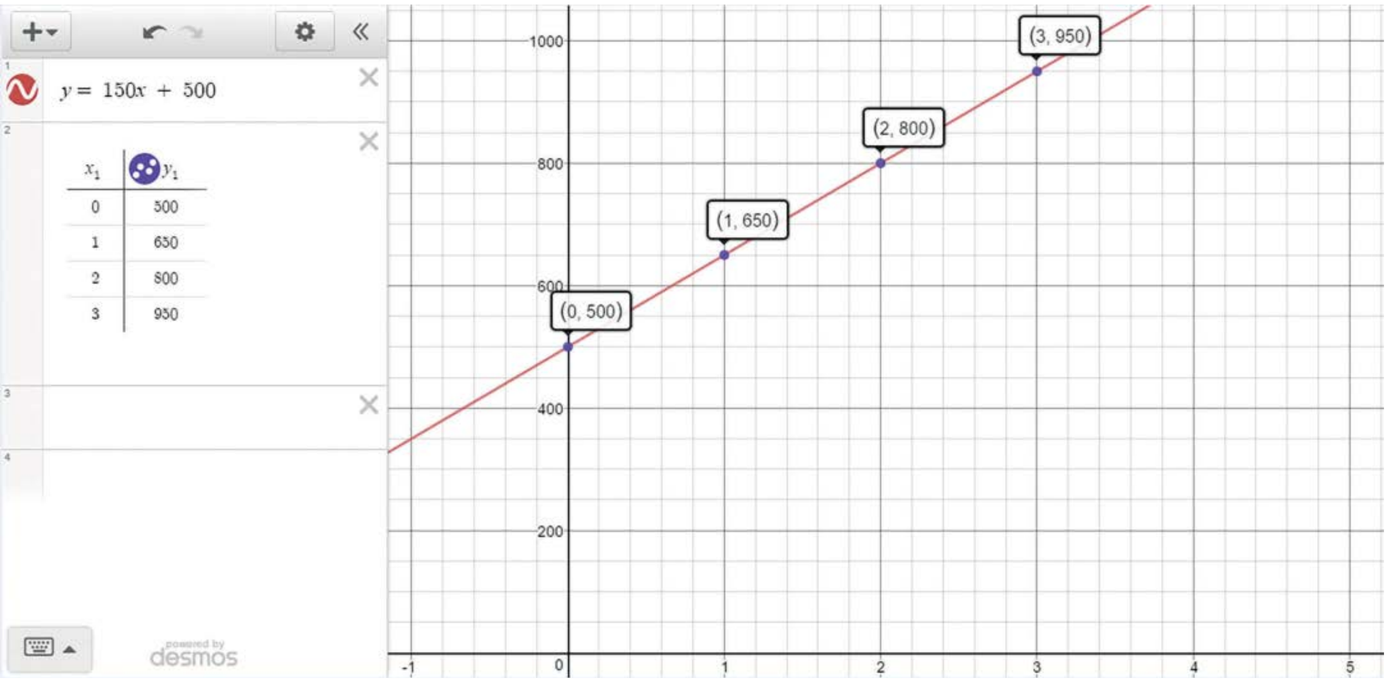
Often the equations involved in a real-world context will be more complicated. The method remains exactly the same, however, for substitution and solving.

- The coefficients might have many decimals places. Make sure to use a calculator to check your multiplications and divisions.
- There may be negative coefficients or constant. Make sure you know your rules regarding multiplications and division of signs.
- If a line has been found by technology, it will most likely contain an exponent. Remember the E^ or e^ in a number means '10 to the power of'. Be careful not to leave this in a final answer as this is not considered good mathematical communication.
- You will be given the input and asked to find the output or vice versa. The first type of question is substitution, the second is solving for an equation. This could be worded in many ways – make sure you are very clear on which variable is dependent and which is independent.

Real-world example

Brianna signs a contract with an app developer to create graphics for their newest game. They will pay her 500 dollars as a signing bonus and then 150 dollars for each level of the game completed.

The function which describes this relationship can be described as points, as well as a graph below.



How can we use the graph to find out how much she will earn if she completed five levels this month? Or if she completes only two and a half?

In mathematical terms, we say this is f(5) = ?　　　f(2.5) = ?

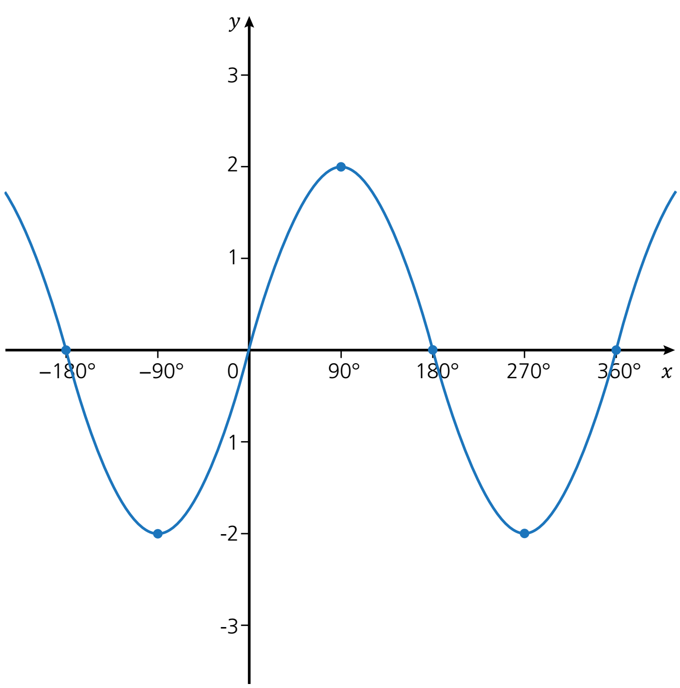
What other information will this graph tell you?

Another key feature of a graph is the range. The **range** is where the y-values are mapped out, the extent to which the y-axis is used. We know from statistics that range is from the highest to the lowest value in our data. In this situation that means from the highest value on the y-axis to the lowest value. We can see this as the 'height' of the function on the graph.

Similarly, the **domain** is the full set of independent values. For most of the functions we have seen so far, e.g. linear and quadratic, this stretches to infinity in both directions for the x-values. We say that

$$-\infty \leq x \leq \infty \qquad \text{or better yet, } \times \in \mathbf{R}$$

If we look at this example of a trigonometric function, we can see that the range is $-2 \leq y \leq 2$ because the maximum value is at $y = 2$ and the minimum value is at $y = -2$. You will see more trigonometric functions in Chapter 10.



ACTIVITY: Mapping your mind

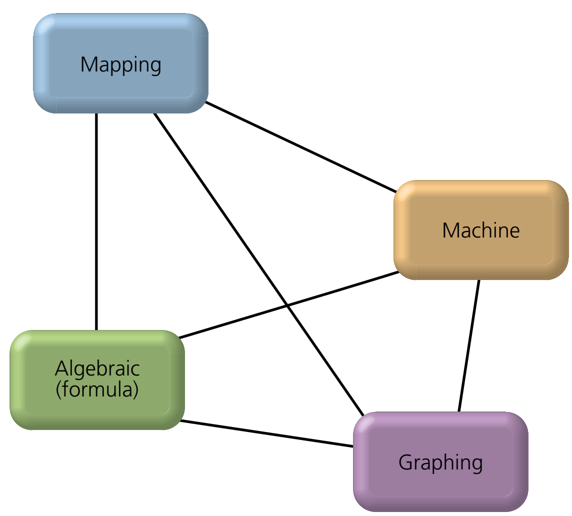
ATL

- Communication skills: Make effective summary notes for studying

For the following **network**, write along the connections (or **paths**) between forms:

- **Something both forms have in common**
- **Something that is different between the forms**

You might wish to write them in different colours or on different sides of the line to distinguish between them.



Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion C: Communicating.

How do we use functions to find values?

In the previous examples, we saw that f(x) can be considered as an alternative notation for y =.

We also learnt that finding a value for y, given x= 2 can be written as f(2).

Let's look at some examples

For f(x) = 10x – 6

Find

a f(6) **b** f(1) **c** f(10)

Solution

Find f(6) when

a $x = 6$

$y = 10(6) - 6$

$y = 54$

$\therefore$ f(6) = 54

b f(1) $x = 1$

$y = 10(1) - 6$

$y = 4$

$\therefore$ f(1) = 4

c f(10) = 10(10) – 6

$y = 100 - 6$

$y = 94$

$\therefore$ f(10) = 94

If f(x) = $x^2 + x - 3$

Find

a f(2) **b** f(–2) **c** f(0.5)

Solution

a f(2) $y = x^2 + x - 3$

$y = (2)^2 + (2) - 3$

$y = 4 + 2 - 3$

$y = 3$

b f(–2) $y = (-2)^2 + (-2) - 3$

$y = 4 - 2 - 3$

$y = -1$

c f(0.5) $y = (0.5)^2 + (0.5) - 3$

$y = 0.25 + 0.5 - 3$

$y = -2.25$

If f(x) = 6x – 3

Solve the following:

a f(x) = 27 **b** f(x) = –9 **c** f(x) = –3

Solution

a f(x) = 27

$\therefore 6x - 3 = 27$

$6x = 27 + 3$

$x = \frac{30}{6}$

$x = 5$

b f(x) = –9

$\therefore 6x - 3 = -9$

$6x = -9 + 3$

$6x = -6$

$x = -1$

c f(x) = –3

$\therefore 6x - 3 = -3$

$6x = 0$

$x = 0$

PRACTICE QUESTIONS

Now it's time to practice them.

1 For $f(x) = 4x + 3$

Find f(3), f(0), and f(–1)

2 For $f(x) = 2x^2 + x$

Find f(3), f(0), and f(–1)

3 For the functions

a $f(x) = 100x - 7$

b $f(x) = x^2 - 4$

Solve each when

i $f(x) = 3$

ii $f(x) = 0$

iii $f(x) = -1$

Unfamiliar

4 Andrew says $y = x^2 - 2$ will go through the point (4, 2).
Is he right? **Prove** your answer

Assessment opportunities

- In this activity you have practiced skills that are assessed using Criterion A: Knowing and understanding.

Hint

If you have more than one function and you want to avoid getting confused between two f(x)'s, why not try g(x) as the second one?

What could you use to keep three different functions distinct (different to one another)?

HOW DO WE INCREASE THE POWER?

Introducing higher power functions: Now we will begin to consider functions that we are less familiar with. We will calculate these initially manually but increasingly with technology, as the functions get more complicated.

ACTIVITY: Inquiry

ATL

- Organization: Use appropriate strategies for organizing complex information

For the function $y = x^3$

Find

f(1)	f(3)	f(0)
f(2)	f(–1)	f(–3.5)

- **Find** the values.
- **Draw** and complete an appropriate table.
- **Plot** these points.
- **Connect the dots – as smoothly as possible.**
- **Describe** the shape formed by the cubic function, i.e. a variable to the power of three.
- **Extend your investigation, where will you go next? How can you modify this function to see if the characteristics are shared by all x^3 graphs in general?**
- **What about other, higher or larger, powers?**

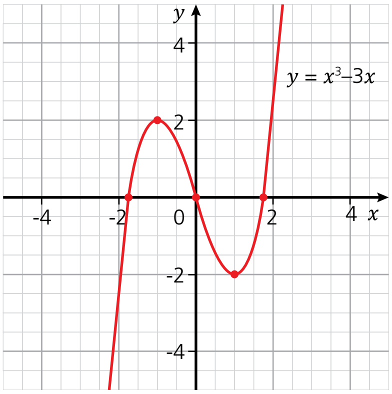
Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

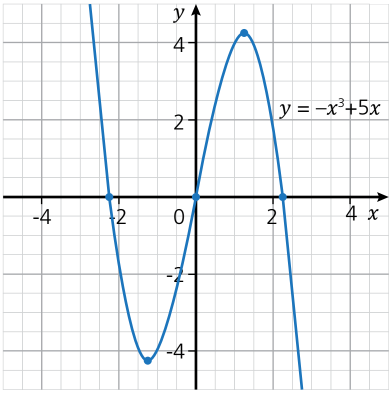
THE CUBIC FUNCTION

The graph of any cubic function can be described as $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

It can also be described as $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$



In this example: $a = 1$, $b = 0$ (no x^2 term), $c = -3$, $d = 0$ (no constant)



Positive coefficient vs negative coefficient

COLOUR–SYMBOL–IMAGE

Does the curve remind you of anything? How could you remember which orientation or direction it will have?

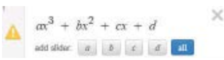
Choose a colour, a symbol and an image (maybe an animal?) to help you recall the difference between a positive and negative cubic function.

ACTIVITY: Sliding on the cubic curve

ATL

- Information literacy skills: Understand and use technology systems

Using DESMOS's sliders function,



investigate what happens when you change *a*, b, *c* and/or *d* by playing the animation.

Describe, using mathematical language and/or notation what is happening to the function when these parameters are changed.

PRACTICE EXERCISE

Draw the graph of $y = 2x^3 + x^2 - 3x + 1$

Using the graph, table or otherwise, **find**

f(2) f(5) f(–2)

Assessment opportunities

- This activity can be assessed using Criterion C: Communicating.

ACTIVITY: Research the drama

ATL

- Critical thinking skills: Recognize unstated assumptions and bias

Conflicting views

We saw in Leibniz's bio that he is credited alongside Newton with the discovery of calculus. The two did not openly collaborate on this discovery. Research the history of this 'simultaneous discovery' and the rumours and speculation that surrounds this story. What was the perspective from each side? What is our current understanding?

MEET A MATHEMATICIAN: GOTTFRIED LEIBNIZ (1646–1716)

Learner Profile: Inquirer

Gottfried Leibniz

Leibniz is considered to be one of the greatest mathematicians of all time. Leibnitz was a German polymath, which means he was interested in many areas of knowledge, from mathematics, medicine, history to philosophy. His first job was as a secretary in an alchemical society, about which he knew very little but presented himself as very knowledgeable. He was an early believer of 'fake it til you make it'. He certainly made it because before long as, along with Newton, he was credited with the discovery of calculus, where rates of change in functions are explored. The notation he invented has been widely used ever since. He led a vivid and colourful life, which rewards further reading, including controversy and battles and was a life-long inquirer into as many branches of knowledge as he could master.

Take action

- Why not have a historical debate – with one side arguing on the side of Newton, while the other supports Leibniz's greatness? Who deserves more credit? Who truly discovered calculus?

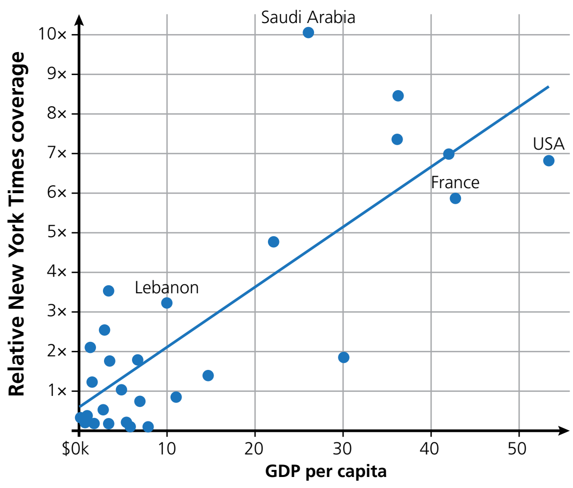
Why do we use functions to generalize?

HOW CAN WE USE THEM AS PREDICTIONS?

One of the most important reasons for learning about functions is so we can use them as modelling tools.

A model is defined as 'a depiction of a real-life event using expressions, equations or graphs'. Modelling is when we take the information from the data given and we fit it to the best, most appropriate, matching function. We can use this model to make generalizations for other real-life situations. These predictions or 'best guesses' are limited in their accuracy by how closely the data matches the function. The better the match between data and the model, the more reliable your prediction will be. We will see more about using multiple models to predict outcomes in Chapter 11, Probability.

Example: The following diagram shows data points relating to coverage given in the New York Times to a terrorist attack related to the Gross Domestic Product (an approximate marker for a country's wealth) of that same country.



■ Coverage in the New York Times

What can we say in general about the coverage in the newspaper given to countries with a low GDP? And a high one?

If we know the equation for the line of best fit, we could predict how much coverage the New York Times will give to a terrorist attack in a country based on their GDP.

How could the editors and writers of the New York Times use this information to reflect upon? What would you say to them, if you could?

HOW CAN WE USE THEM TO INTERPRET?

One basic property we can see instantly from a function is whether it is increasing or decreasing with relation to each other at given points. In a linear function, this rate of increase or decrease is called the slope or gradient.

After increasing or decreasing, it tells us how quickly it is doing that over time? Is it rising rapidly, then levelling off? Is it increasing, then decreasing before it repeats the same pattern?

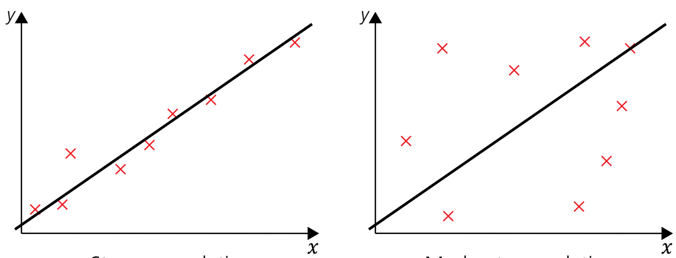
CORRELATION

Correlation is a measure of how close our data is to the function chosen to model it. In chapter 6 we learned that correlation is the process or measure of finding out a relationship or connection between two things. We saw the difference between positive and negative correlation.

The curve can extend beyond what we now see. This might be a zooming out or a continuation of the trend. We might want to continue off the page, beyond the graph paper or outside the current visual frame on a GDC or app.

If we have a set of data, we look for the closest function that would help us to model that data. We look at the direction of change from term to term, the rate of change as well as overall trend. **Regression** is the process that helps us not only pick the right function but also tell us how well it matches that function

An example of a positive correlation: 'The more you study, the better the grades you will achieve.' That may be true (that is, correlated) but how strong is that relationship? What about those outliers – people who don't seem to study but do very well? Or people who work very hard but who freeze on the test? How do they effect the 'trueness' of the relationship?



Strong correlation
r = 0.92

Moderate correlation
r = 0.71

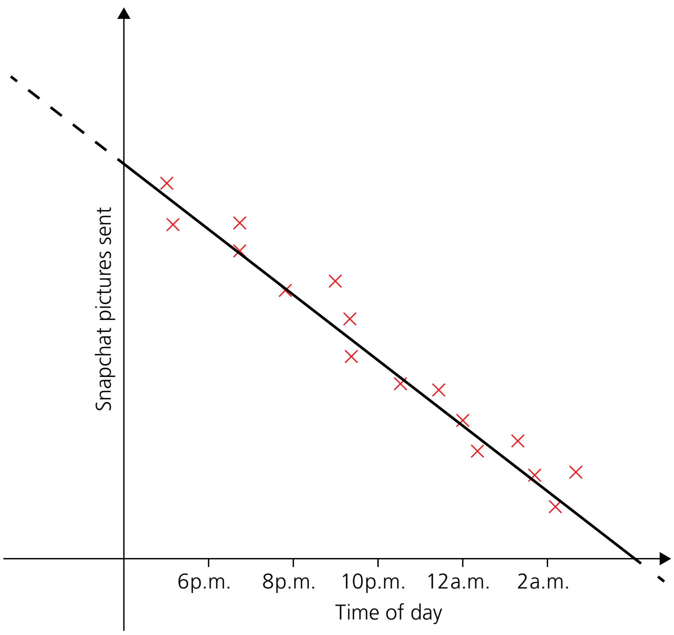
Classes A and B both collected data on the numbers of hours they studied for a particular test and the marks they received. We can see the difference in correlation visually, that Class A's results were much more regular and Class B's results showed more variation.

You will study more about correlation and correlation coefficients later but for now it is enough to know that the closer a correlation coefficient is to 1 or -1, then the closer the data is to matching the function or model. In the example given above Class A had a correlation coefficient r = 0.92, while Class B had a much lower correlation coefficient of 0.71.

Technology-enabled lines of best fit can give you r (the correlation coefficient) very quickly to help you decide which is the closest function to the data, although sometimes programs such as Microsoft Excel will only give you r^2.

As we know how to find the square root of a number, this is not a problem which will stop you finding and using the value of r to evaluate the model.

In the example taken from Chapter 3 below, we see from the graph that the number of Snapchat pictures sent is negatively correlated with the time at night, i.e. the later it gets, the fewer pictures are sent. This appears to be a linear relationship.



But what if that isn't the whole story? If this model were extrapolated (extended outside the current data set), then eventually there would be no pictures sent. A few hours later only a negative number of pictures would be sent, and so on. Clearly this does not make sense in real-world contexts. If we gathered more data, we may see the trend line changing. This may require a change to a different function which better models the behaviour of the data.

CASE STUDY: FINDING THE LINE OF BEST FIT

You should already know from your previous learning in mathematics and sciences how to draw a line of best fit 'by eye'. In this case study, we will take a set of bivariate data and practice some function fitting.

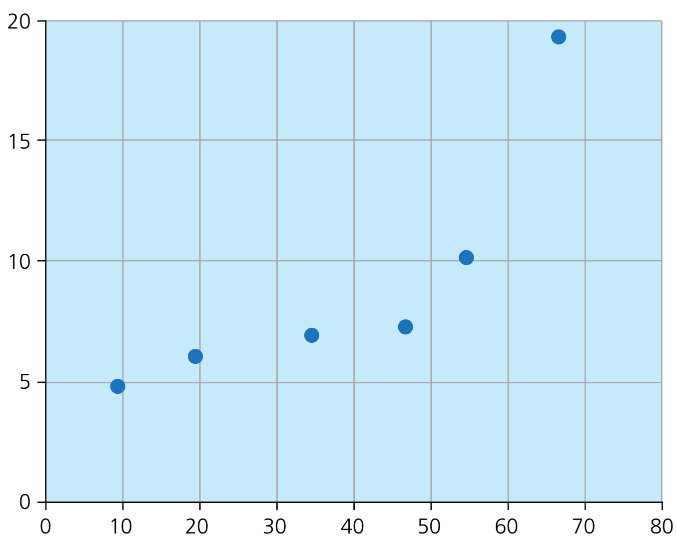
Aim: Using technology to find an appropriate model.

First, we start with bi-variate data

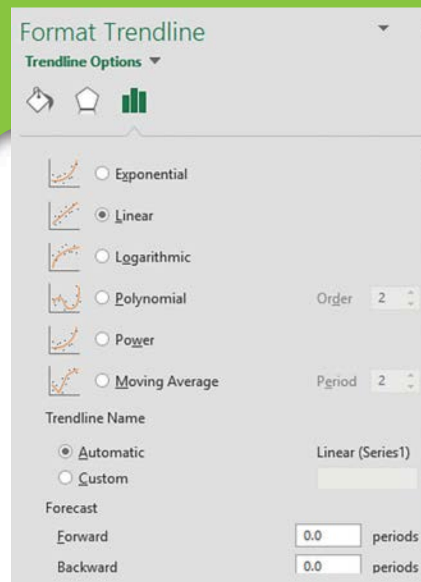
x	y
10	4.5
20	5.9
35	6.772
47	7.011
55	9.999
67	19

Next, we plot these points on a scatter graph using the software of your choice (we have used Microsoft Excel™). It is also possible to carry out this regression technique on a GDC.

We can see that the trend is definitely increasing, we can also say that there is a positive correlation between the data but we cannot say how strong that correlation is yet, nor have we identified the function that would work best for this data.



■ The plotted points from the data provided



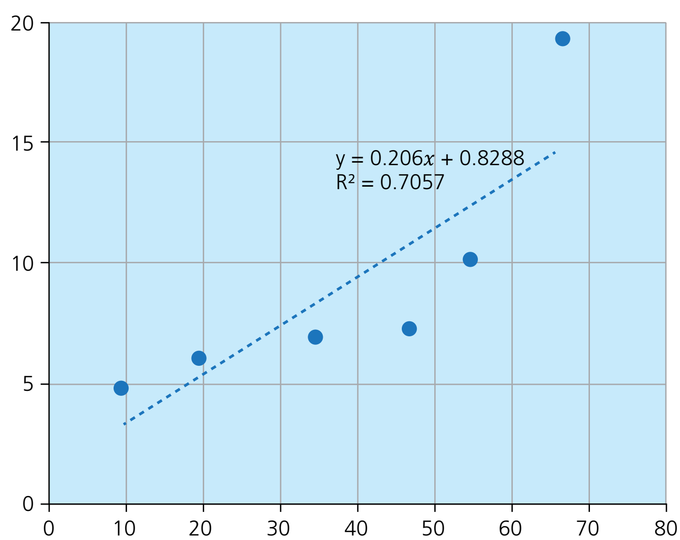
We want to add a line of best fit, which can also be known as a trendline.

Is a straight line really the 'best fit'?

Often technology platforms allow us to fit a variety of 'trendlines' or models so we can match the one which is closest to the data. From the image below, you can see that only interpolation has taken place (that is, the dotted line). It is also possible to extrapolate by choosing to 'forecast' the data backwards and forwards.

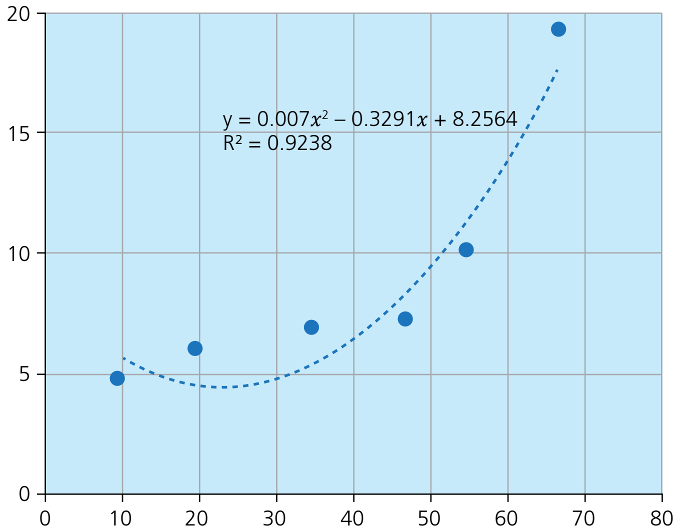
If we are not happy that a correlation coefficient of 0.84 is a strong enough correlation, then we may wish to try another function.

Note we found the correlation coefficient r by $\sqrt{0.7057} = 0.8401$



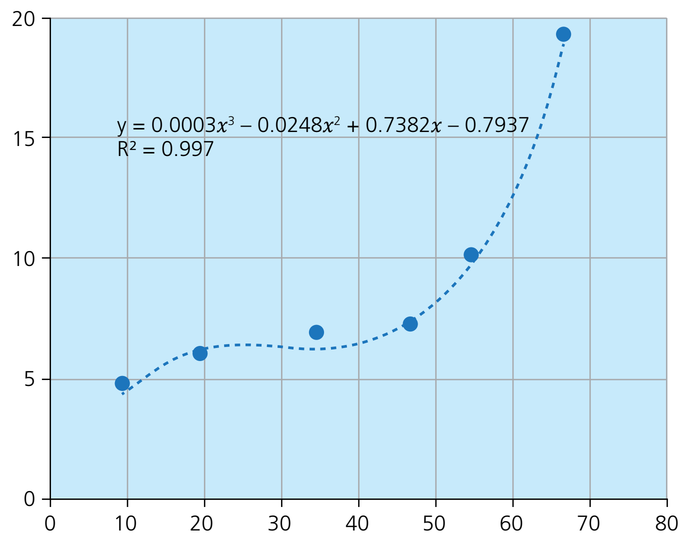
■ The linear function

Next we will try a quadratic (or 2nd power) function to find the line of best fit.



Do you think this shape is closer to the data? Are there any obvious outliers or anomalies? The correlation coefficient is improved to 0.96.

Now onto a cubic fit or trendline:



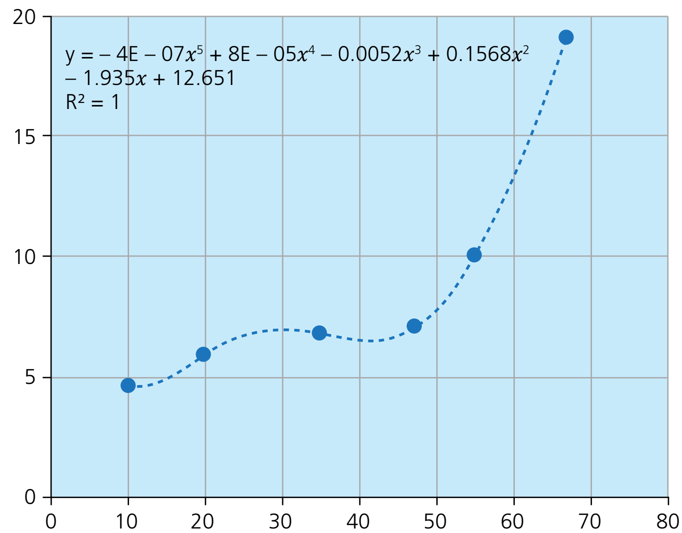
It appears that a cubic function is getting closer to the true nature of the data.

QUINTIC – OR x TO THE POWER OF 5

The next reasonable attempt might be to input the power of four but let's try one higher than that to see if it fits even better.

How would we input this function in the trendline options? What is the polynomial?

Quintic



What does *E* represent here? What is 8*E*-05 as a normal number?

For a simplified version of the function

$$f(x) = -0.0000004x^5 - 0.00008x^4 - 0.005x^3 + 0.2x^2 - 2x + 10$$

Find f(40) Find f(70)

Comment on the possible continuation of the graph.

What could change this interpretation significantly?

Models are a useful tool for finding the function that best fits the data in front of you. If you get more data or if you exclude some obvious anomalies or errors, then the best fitting function may have to change. The more information we have, the closer our model is to reality.

Let's recall our debatable questions for this unit - How far should we trust functions or models to tell us about the real world? Can multiple functions be correct for a single situation?

How far should we trust functions or models to tell us about the real world?

HOW CAN SOMETHING 'GO VIRAL'?

Exponential functions are so called because the variable is in the exponent (or power)

a^x is an exponential expression

$y = a^x$ or $f(x) = a^x$ is an exponential equation or function

where a is the base and x is the power or index or exponent.

Exponential growth and the legend of Paal Paysam

"Exponential Growth is an immensely powerful concept. To help us grasp it better let us use an ancient Indian chess legend as an example.

The legend goes that the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna himself.

The king was a big chess enthusiast and had the habit of challenging wise visitors to a game of chess. One day a traveling sage was challenged by the king. To motivate his opponent, the king offered any reward that the sage could name. The sage modestly asked just for a few grains of rice in the following manner: the king was to put a single grain of rice on the first chess square and double it on every consequent one.

Having lost the game and being a man of his word the king ordered a bag of rice to be brought to the chess board. Then he started placing rice grains according to the arrangement: 1 grain on the first square, 2 on the second, 4 on the third, 8 on the fourth and so on:

Following the exponential growth of the rice payment the king quickly realized that he was unable to fulfil his promise because on the twentieth square the king would have had to put 1,000,000 grains of rice. On the fortieth square the king would have had to put 1,000,000,000 grains of rice. And, finally on the sixty fourth square the king would have had to put more than 18,000,000,000,000,000,000 grains of rice which is equal to about 210 billion tons and is allegedly sufficient to cover the whole territory of India with a meter thick layer of rice. At ten grains of rice per square inch, the above amount requires rice fields covering twice the surface area of the Earth, oceans included.

It was at that point that the lord Krishna revealed his true identity to the king and told him that he doesn't have to pay the debt immediately but can do so over time. That is why to this day visiting pilgrims are still feasting on Paal Paysam and the king's debt to lord Krishna is still being repaid."

Taken from: **www.singularitysymposium.com/exponential-growth.html**

We have seen this type of growth when talking about geometric sequences in chapter 7. So what is the shape or form of an exponential function?

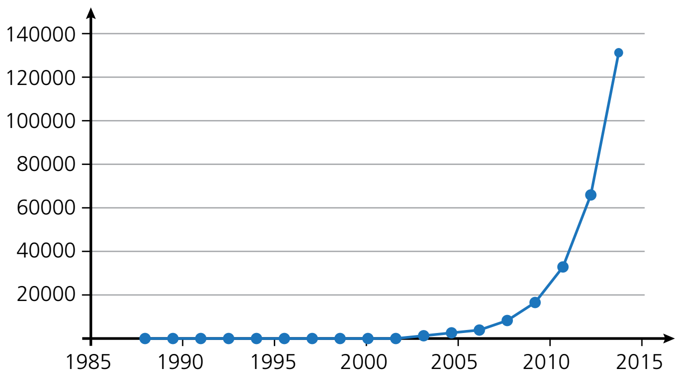
Can you identify any mathematical errors or inaccuracies in this tale?

Why might these be present in the story?

Real-world example: Moore's Law

Moore's Law is actually not a law but a prediction from Gordon Moore (co-founder of Intel) in 1965 that computing power would double, for the same price, every 18 months. For much of the time since then, his prediction has seemed to be exactly correct with computer chip sizes halving and power doubling in the following way:

If we draw a graph of Moore's Law since 1988, it looks something like this:

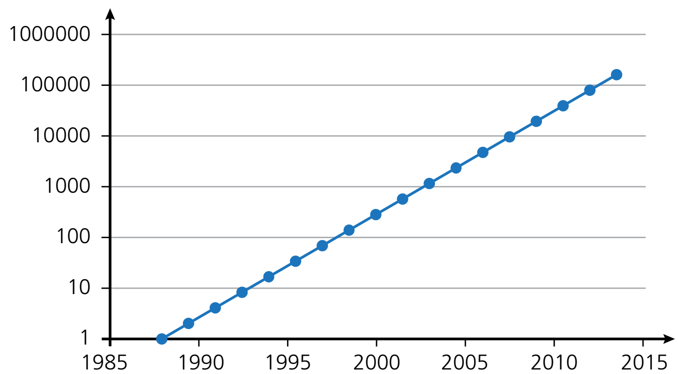


■ Exponential function showing Moore's Law

In other words, a computer now is around 130 000 times more poweful than in 1988. We call this an **exponential** function.

This function is also known as the 'doubling graph'. It appears from the shape of the curve as if nothing much happened for a very long time and then suddenly shot up. In fact, computing power was doubling all the time, getting twice as large each time but the final values are so huge that the initial ones look alike. This is a problem with the scale of the y-axis.

If we use a logarithmic scale, which means that each increase on the y-axis is actually 10× larger than the previous one, the y-axis will step up from 1 to 10, then 100, then 1000 and so on. Each increment will increase by an order of magnitude – then we see a very different shape to the same relationship.



■ A logarithmic scale showing Moore's Law

If you use a logarithmic scale, and the graph looks like a line, then it is exponential.

PRACTICE QUESTIONS

1 For $y = 2^x$

  a **Show** this function on a linear scale.

  b **Show** this function on a logarithmic scale.

2 Given $f(x) = 3^x$

  a **Find** f(2)

  b **Find** f(3)

  c **Find** f(5)

  d **Describe** the relationship as fully as you can.

3 For the function $f(x) = 10^x$

  a **Find** f(1)

  b **Find** f(0)

  c What can you say about f(0) for any exponential curve?

  d **Find** f(–1)

  e **Find** f(–2)

  f **State** what you notice about these negative values

  g Would this be the same for any exponential function?

CHANGING THE PARAMETERS OF AN EXPONENTIAL EQUATION

What does changing the coefficient do to the shape? How could you **describe** this as mathematically as possible?

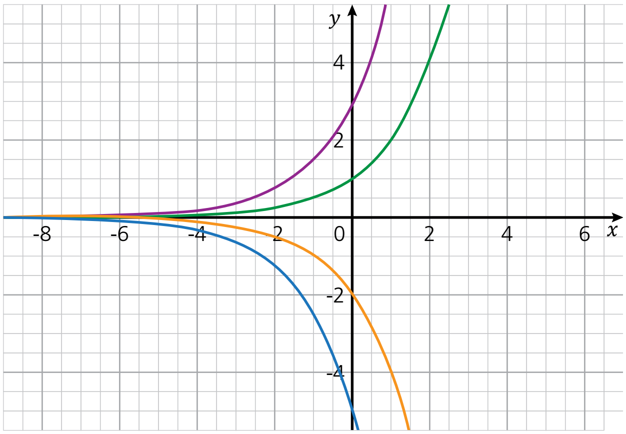
$y = 2^x$

$y = 2(2^x)$

$y = -3(2^x)$

$y = -8(2^x)$

How can you tell that all of these functions are exponential? How will changing the coefficient affect their shape?



Using technology, or otherwise, match each of the functions above to the correct curve.

Describe how changing the coefficient changes the shape, and the nature of the curve.

Two of the curves on on the graph above showed exponential growth while the other two showed exponential decay or decline. These are important concepts in the natural world.

Bath, England

Joost has taken this photograph in Bath in England. He thinks it shows an exponential curve. Can you see a potential curve in the line of the rooftops? Or the chimney stacks? How could you check to see if the relationship actually models an exponential function or not?

Flags at the European Commission

Joost is getting worried because he is starting to see functions everywhere. He thinks that the flagpoles outside the European Commission in Brussels also follow a function. Can you help him determine if they do, and which one it might be? How would you **apply** mathematics in the real-world problem?

Communication

Remember: when using a calculator, use the button ^ to input an exponent. Be careful never to use this symbol in your answer as this is calculator shorthand and will not be considered good mathematical communication for Criterion C. Other calculators may have an x^y button to indicate indices or exponents.

What is meant by exponential growth or decay?

Exponential growth or decline can be seen in many natural phenomena including:

- bacteria growth
- radioactive decay
- population growth.

Links to: Language and literature

The term exponentially is often used in reports, news items, articles, claims by companies, for example: 'sales of our products have grown exponentially', 'ticket prices are changing hands at exponential prices'. Often what they mean is simply rapidly increasing or decreasing. What exact conditions would need to be happening to product sales or ticket prices to make these statements precise and mathematically correct?

Link to: Sciences

http://wormshack.ua.edu/lab-video-page.html

Using the video and your knowledge of *cell division*, describe how cell division takes place. How many cells are there after each stage of meiosis?

Stage 1: Stage 2: Stage 3:

How many cells would there be after stage 4? Stage 10? After 100 divisions? And is this what truly happens biologically?

ACTIVITY: Baby, I love you logarithmically

ATL

- Communication skills: Use a variety of speaking techniques to communicate with a variety of audiences

Kyle Evans, a mathematician and folk singer, discusses the song that made him fall in love with mathematics.

www.youtube.com/watch?v=vz1RzRyH5Oc

Watch the video of Kyle discussing and singing his song about love as a mathematical function. What did you think?

Kyle claims that describing love as a linear or exponential function is sad and unrealistic. In his own song, he claims that a logarithmic function would be a better description of love.

Research task: What is a logarithmic function? Why would that shape of the function be more romantic than either of the others he mentions?

EXTENDED

Write a reply to Kyle's song that shows the function of love through a break-up. How would that go?

Assessment opportunities

In this activity you have practiced skills that are assessed using Criterion C: Communicating.

MEET AN APPLIED MATHEMATICIAN: SHARAD GOEL

Learner profile: Inquirer

■ Sharad Goel

Sharad Goel is an assistant professor at Stanford University in the United States, he is interested in studying the underlying mathematics of social phenomenon. He is an excellent example of someone who is using mathematical and computational techniques in an interdisciplinary way to inquire deeper into real-world phenomenon and interests. These include police discrimination, swing voting, filter bubbles, do-not-track, and media bias. Let's look at an extract of an interview he gave for nature.com on a viral phenomenon in 2015.

Read this article: **www.nature.com/news/the-mathematics-behind-internet-virality-1.17046**

DISCUSS

Where does the phrase 'going viral' come from? What might that have to do with natural phenomena, as mentioned before? How might it link to the exponential functions we are looking at?

! Take action

! Find out how mathematics is being used to investigate viral phenomena and other online digital relationships. What role is modelling playing for large social media companies? What kind of data do they track? What can they tell about you by modelling?

EXTENSION

How can functions be reversed or combined?

Inverse functions

■ Explaining inverse functions

Why would we need to inverse a function? If we choose to reverse or undo a function, we need to work backwards. We go from output to input.

Procedure pro:

- Write the function in its current form (use y and x to make it easier).
- As we are reversing the variables, switch or interchange x and y without changing any of the other parameters/coefficients and constants.
- Now rearrange.
- This is the inverse function.

We will only look at a very few examples as this is something you will cover in much greater depth later.

Example 1

Find f^{-1}

if $\quad y = 3x - 2$

$x = 3y - 2$

Bring y to LHS

$3y - 2 = x$

$3y = x + 2$

$$y = \frac{x + 2}{3}$$

Example 2

Find f^{-1} $\quad f(x) = 2x^3 + 1$

$$y = 2x^3 + 1$$

$$x = 2y^3 + 1$$

$$2y^3 + 1 = x$$

$$2y^3 = x - 1$$

$$y^3 = \frac{x-1}{2}$$

$$y = \sqrt[3]{\frac{x-1}{2}}$$

Predict: what would happen if you find the function of a number and then put it into the inverse? What should come out? How can you test this prediction?

Composite functions

Sometimes there might be more than one function, one after another. A **composite function** is one which uses the output from one function as the input for a second function.

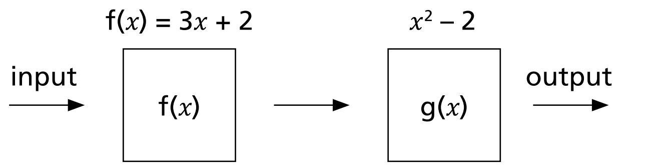
Let's say we have two functions, we will need to name them different letters to avoid confusion.

$f(x) = 3x + 2$

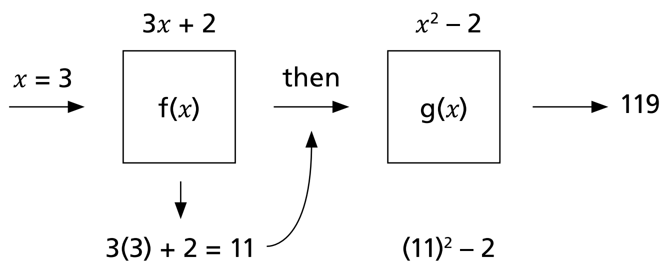
$g(x) = x - 2$

Here we will put a number into $f(x)$, then plug the answer into $g(x)$

A useful way to visualize this is to recall our functions as machines from earlier.



so for $x = 3$



$$\therefore\ g(f(3)) = (3x + 2)^2 - 2$$

$$= 119$$

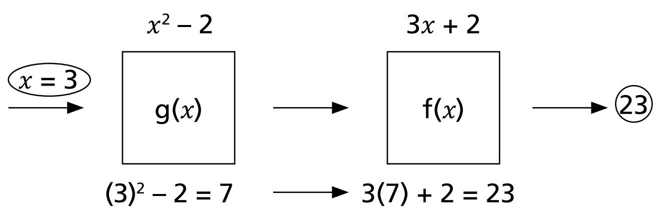
The input or x-value or independent variable has now been 'through' both functions, first f, then g.

We call this

$g(f(x))$ or g o f, both pronounced as 'g of f'.
So $g(f(3)) = 119$ or $g \circ f(3) = 119$

If we did it in a different order, that is, put the variable through g first then f:

then …



This is the value of $f(g(3)) = 23$

Extreme challenge:

$f(x) = 2x - 1$

$g(x) = x^2 + 2$

$h(x) = 3x$

$i(x) = x^3$

Using machines, or otherwise:

Find $f(g(h(i(x))))$

and **find** $i(h(g(f(x))))$

How and why do we develop the form of functions to improve their accuracy?

! Take action

! The rise and rise of online petitions.

Find an online petition platform. Identify a cause you believe in and would like to see grow. Chart the growth of popularity for the cause over time.

What will you use for your independent and dependent variables or values for x and y?

When did they achieve important milestones? Or did you take your 'readings' in a different way? Can you see a relationship emerging? Can you find a line of best fit (linear or non-linear/curved) for the data?

Some petitions really speak to people and their desire to make the world a better place through action. Similar to videos, memes, games and apps, petitions have the potential to go 'viral' in digital media circles. Can you find examples of petitions that were extremely popular, successful or even 'went viral'?

ACTIVITY: Finding information from a model

ATL

Information literacy skills: Understand and use technology systems

Using a function to find missing values

www.change.org

Chip in £3 or more to help win this campaign

Petitioning the Department for Environment Food & Rural Affairs: save the red squirrel

Rita –

Take the next step by sponsoring this petition on Change.org to give it more exposure. Your contribution of £3 or more will feature the petition to dozens of like-minded change makers and drive it toward victory.

The more you give, the more people will be shown this campaign and asked to add their names in support.

The Change.org Team

Select an amount:

£3	Shows petition to 21 potential supporters
£10	Shows petition to 71 potential supporters
£25	Shows petition to 178 potential supporters
£50	Shows petition to 356 potential supporters

£ Other amount

Explore the *relationship* between the amount of money you can donate and the number of people the petition will be shown to, according to change.org

Based on your calculations, if you donated the following *amounts*, how many *people* would be shown the petition:

- **5 pounds**
- **20 pounds**
- **No donation**
- $x = 2.50$
- $x = 18.50$
- $x = 40.00$
- f(35)=
- f(100) =
- f(1000) =
- f(–10) =

You should try to:

- **Give as much detail as you can, using mathematical communication both visual and numerical**
- **Identify which variable is which**
- **Comment on the accuracy of your solution**
- **Comment on whether your explanation makes sense in the context of an online petition**
- **Suggest how the organization might have come up with these figures.**

Assessment opportunities

- In this activity you have practised skills that are assessed using Criterion A: Knowing and understanding, and Criterion D: Applying mathematics in real-life contexts.

EXTENDED

Does the cost of genome sequencing follow a useful function for prediction? Research if it is linear, exponential, polynomial…?

Genome sequencing is the process of decoding an organism's DNA. A whole genome sequence involves mapping every A, T, C and G that make up 3 billion base pairs of DNA. Until very recently, this process was extremely expensive and out of the reach of individuals and most researchers. This has been changing over time. Your task is to look at the large set of data and try to find relationships that match the change in cost over time (or sections of that time period).

Date	Cost per genome
Sep-01	$95 263 072
Mar-02	$70 175 437
Sep-02	$61 448 422
Mar-03	$53 751 684
Oct-03	$40 157 554
Jan-04	$28 780 376
Apr-04	$20 442 576
Jul-04	$19 934 346
Oct-04	$18 519 312
Jan-05	$17 534 970
Apr-05	$16 159 699
Jul-05	$16 180 224
Oct-05	$13 801 124
Jan-06	$12 585 659
Apr-06	$11 732 535
Jul-06	$11 455 315
Oct-06	$10 474 556
Jan-07	$9 408 739

Date	Cost per genome
Apr-07	$9 047 003
Jul-07	$8 927 342
Oct-07	$7 147 571
Jan-08	$3 063 820
Apr-08	$1 352 982
Jul-08	$752 080
Oct-08	$342 502
Jan-09	$232 735
Apr-09	$154 714
Jul-09	$108 065
Oct-09	$70 333
Jan-10	$46 774
Apr-10	$31 512
Jul-10	$31 125
Oct-10	$29 092
Jan-11	$20 963
Apr-11	$16 712
Jul-11	$10 497

Date	Cost per genome
Oct-11	$7 743
Jan-12	$7 666
Apr-12	$5 901
Jul-12	$5 985
Oct-12	$6 618
Jan-13	$5 671
Apr-13	$5 826
Jul-13	$5 550
Oct-13	$5 096
Jan-14	$4 008
Apr-14	$4 920
Jul-14	$4 905
Oct-14	$5 731
Jan-15	$3 970
Apr-15	$4 211
Jul-15	$1 363
Oct-15	$1 245

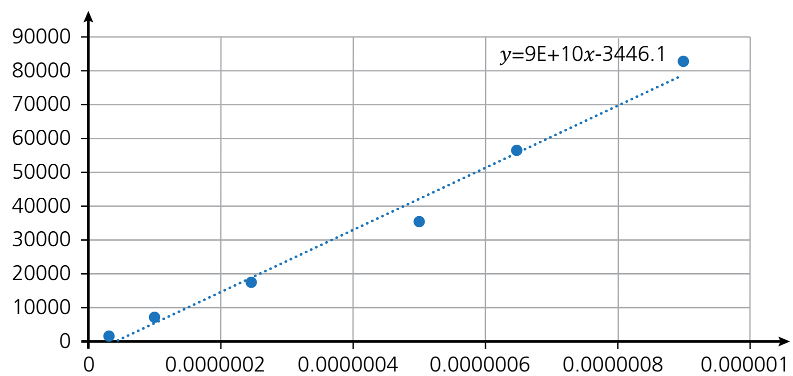
■ Table showing the cost of genome sequencing over time from September 2001 to October 2015

SOME SUMMATIVE PROBLEMS TO TRY

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding.

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

1 **Name** the function used as the line of best fit on this graph.



THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

2 **Comment** on correlation of the data to the function.

3 **Find** two values of the data.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

4 **State** the linear function shown on the graph.

5 **Use** that linear function to find:

- a f(0.0000003)
- b f(0.0000011)

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

6 For the previous question, which value was an extrapolation and which was an interpolation?

7 If y = 36000, **calculate** the independent variable.

Reflection

<table>
<tr><th colspan="6">Use this table to reflect on your own learning in this chapter.</th></tr>
<tr><td>Questions we asked</td><td>Answers we found</td><td colspan="4">Any further questions now?</td></tr>
<tr><td>Factual: What is a function? How do we use functions to find values? How can functions be reversed or combined? How and why do we develop the form of functions to improve their accuracy?</td><td></td><td colspan="4"></td></tr>
<tr><td>Conceptual: Why do we use functions to generalize? How can we use them as predictions? What is meant by exponential growth or decay?</td><td></td><td colspan="4"></td></tr>
<tr><td>Debatable: How far should we trust functions or models to tell us about the real world? Can multiple functions be correct for a single situation?</td><td></td><td colspan="4"></td></tr>
<tr><td rowspan="2">Approaches to Learning you used in this chapter:</td><td rowspan="2">Description – what new skills did you learn?</td><td colspan="4">How well did you master the skills?</td></tr>
<tr><td>Novice</td><td>Learner</td><td>Practitioner</td><td>Expert</td></tr>
<tr><td>Reflection skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Communication skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Organization skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Information literacy skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Learner profile attribute</td><td colspan="5">How did you demonstrate your skills as an inquirer in this chapter?</td></tr>
<tr><td>Inquirer</td><td colspan="5"></td></tr>
</table>

10 'What do I get by learning these things?'

Statements about the **spaces and shapes** around us can be **justified** to show they are **invariant** through **space and time**.

CONSIDER THESE QUESTIONS:

Factual: What is a circle made of? What are circle theorems? How can we graph them?

Conceptual: How do we prove theorems by justifying them? What are trigonometric identities and why are they interesting? Can a circle wave? Can you see circles in real-life contexts?

Debatable: Can a theorem tell us a 'universal truth' or is that impossible?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how mathematical logic and proofs can improve your general reasoning and arguing skills.
- **Explore** trigonometric identities and how they can be used to make difficult expressions disappear!
- **Take action** by incorporating mathematical and design thinking to create logos and images.

These Approaches to Learning (ATL) skills will be useful …

- Reflection skills
- Communication skills
- Critical-thinking skills
- Research skills
- Media literacy skills
- Transfer skills

We will reflect on this learner profile attribute …

- Thinker – using critical thinking skills to analyse and take responsible action on complex problems; exercising initiative in making reasoned, ethical decisions.

■ Circles can be found in the world around us

◆ Assessment opportunities in this chapter:

- ◆ **Criterion A:** Knowing and understanding
- ◆ **Criterion B:** Investigating patterns
- ◆ **Criterion C:** Communicating
- ◆ **Criterion D:** Applying mathematics in real-life contexts

PRIOR KNOWLEDGE

You will already know:

- the names and meanings of the parts of a circle
- the formula for area and circumference of a circle and how to apply them
- how to use a ruler, protractor and compass to draw shapes
- how to recognize and name the four quadrants of a circle
- the meaning of, and the difference between congruent and similar (equiangular) shapes.

KEY WORDS

axiom	secant
chord	tangent
proof	theorem

'Pure mathematics is, in its way, the poetry of logical ideas.' – Albert Einstein

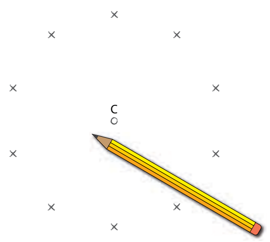
Invariance is the property of an object which remains unchanged under certain transformations, for example that a triangle will contain 180° degrees no matter how it is drawn, or that a certain angle will measure the same no matter how or where its shape is rotated or enlarged.

The properties of shapes, angles and circles are a rich field for developing theorems and mathematicians have investigated them throughout history. The process of discovering these hidden rules develops your thinking skills and increases your non-numeric reasoning. While you may never see circle theorems again once you leave school, the ability to identify similarities and invariances in life will add to your skill set **immeasurably**.

Look beyond the content of what you are learning to the logical thinking beneath it. Questioning assumptions and proving what you say is true are transferrable skills to many other areas. You are acquiring an organized approach to complex problem solving and developing as a thinker as a result.

What is a circle made of?

Using a ruler, mark ten points away from point C in any direction.

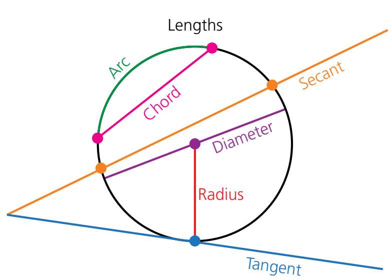


Mark another ten. And another ten. If we keep going, we start to see the familiar shape of a circle.

What is a faster method to draw this?

A circle is the visual representation of the relationship that shows all the possible points at a fixed distance from a certain point. That distance from the fixed points is called the radius. The circumference is the total length of the representation of that relationship (all the points connected together form the circumference).

First let's look at lengths we can find in, around or through a circle



■ Parts of a circle

One length which does not appear above is the circumference.

ACTIVITY: Definitions

■ ATL

Communication skills: Use and interpret a range of discipline-specific terms and symbols

What is the difference between …

1 **a diameter and a radius?**
2 **a chord and a diameter?**
3 **a chord and a secant?**
4 **a secant and a tangent?**
5 **a tangent and a radius?**
6 **an arc and a chord?**
7 **an arc and a radius?**
8 **a secant and a diameter?**

Select the correct term or terms that matches the description.

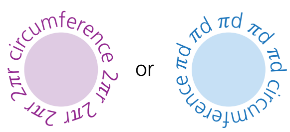
- **Is a line segment:** ______________________
- **Is a line:** ______________________
- **Is a curved line:** ______________________
- **Is a line that intersects (cuts through) the circle:** ___
- **Is a line that 'touches' the circle only:** __________
- **'Touches' (or crosses) the centre of the circle:** _____
- **Does not 'touch' the centre of the circle:** _________

Important things to remember about the circumference:

- It is a length so it is a one-dimensional measurement.
- For any other shape it would be called the perimeter.
- It can be found from the formula:

Circumference = $\pi \times$ diameter
or circumference = $2\pi \times$ radius

Why are there two formulae?



$$c = \pi d \text{ and } d = 2r$$

$$c = \pi d$$

$$= \pi(2r)$$

$$\therefore c = 2\pi r$$

WHAT ABOUT AREA?

Circles can be found in the real world.

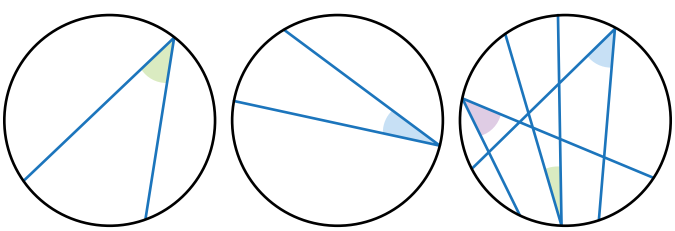
Is this measurement an area, a circumference, an arc or neither?

- The length of a bowling alley.
- The width of a bowling ball.
- Shadow of a planet crossing in front of the sun.
- Hat size for a baseball team.
- Regions on a dartboard.
- Size of a pizza.
- Leftover crusts of an eaten pizza.
- The distance of a round-the world voyage.

- How many circles do you see in this image? Do you see any other circle parts as mentioned above?

Where can we find angles in a circle?

If we draw lines within a circle, we can see angles formed by those lines. There are many ways we can draw two lines to form an angle inside the circle.



- Angles inscribed by the same circle

DISCUSS

Look at diagram. All of the angles are inscribed by the same circle. How would you explain the term **'inscribed'** based on that information?

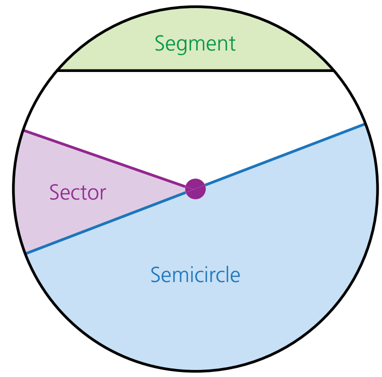
ACTIVITY: Inside the circle

ATL

Communication skills: Use and interpret a range of discipline-specific terms and symbols

Define each of the three sections of a circle shown below:

- **Semicircle**
- **Sector**
- **Segment**



What makes each one unique (different from the others)?

When does a segment become a semicircle?

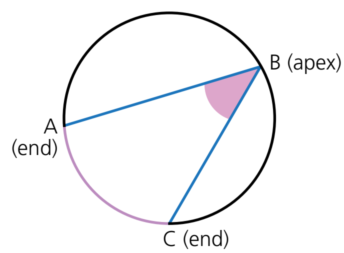
Can a sector > semicircle? Justify your answer with a diagram

Can a sector = semicircle?

Does the orientation of the circle affect these quantities?

Assessment opportunities

This activity can be assessed with Criterion A: Knowing and understanding, and Criterion C: Communicating.



- Angles within a circle

The diagram above shows the angle ∢ ABC, where points A and C are called end points and B is called the apex point. The angle ∢ ABC is subtended by the arc AC (shown in pink). You can consider this as the angle standing on the arc.

A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, 'What do I get by learning these things?' So Euclid called a slave and said 'Give him three pence, since he must make a gain out of what he learns.' – Stobaeus, Extracts.

Why do you think the student asked this question? Do you think Euclid was happy with this question? How does his response answer this student? What do you think of his response? Do you think we should learn things that don't immediately pay us back? Should we be paid to learn?

Do you notice anything else about the quotation that orients it in time? What differences do you note about learning and living in these times which separates it from now?

MEET A MATHEMATICIAN: EUCLID (C. 330 TO 260 BCE)

Learner Profile: Thinker

Although we know little about Euclid's life, it is fair to say that his name is famous through his work. He was a mathematician and teacher and his great work 'The Elements' is considered one of the most widely used textbooks of all time as it has been used for more than 2000 years. He showed through his work how we can move from the known to the unknown in a series of logical steps. Euclid used clear lines of logical thinking to organize, demonstrate and prove all the known geometry at the time.

He is known as the Father of Geometry but could easily be called the longest running (and maybe best) teacher of all time. Consider just how many people might have read and studied his texts in that time, over two millennia. 'The Elements' is a masterpiece of logic and proof, in thirteen volumes.

While we accept that Euclid alone did not prove all the results in The Elements, the organization and communication of the material is due to him. Euclidean geometry is the perfect expression of a Thinker - – using critical thinking skills to analyse complex problems, assumptions and the nature of invariant (unchanging) truth.

The Greek philosopher Proclus records that when Ptolemy I asked if there was an easier way to study geometry than The Elements, Euclid replied, 'Sire, there is no *royal* road to geometry.'

What do you think this statement meant?

Other Euclidian quotations:

'The laws of nature are but the mathematical thoughts of God.' – Euclid

'If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.' – Albert Einstein

'I tell you that I accept God simply. But you must note this: If God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid.' – Ivan, in The Brothers Karamazov*, by Fyodor Dostoyevsky (1821–1881)*

'Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions.' – Eric Temple Bell

'Let no one come to our school, who has not first learnt the elements of Euclid.' – Notice posted on school doors by Greek philosophers

Quotations taken from **www.mathopenref.com/euclid.html**

What are circle theorems?

WHAT IS CIRCLE GEOMETRY?

We are going to inquire into five circle theorems in this section, together and independently. Some proofs you will be shown and some you will discover for yourself, with support if you need it. This will give you a chance to practice your reasoning skills and develop an appreciation for the logical power and elegance of mathematical geometry. We already know what a circle is, so let's look at what a theorem means.

What is a theorem?

An **axiom** is a statement accepted already proven angles in a straight line add to 180° is an example of an axiom.

A **theorem** is a statement that can be shown to be true through the use of axioms and logical argument.

HOW DO WE PROVE CIRCLE THEOREMS BY JUSTIFYING THEM?

A theorem is a statement demonstrated to be true. This statement is justified and given in general terms so that the theorem can be used as a fact (axiom) in other proofs or problem-solving. These circle theorems are interconnected in that we use some to prove others and they are invariable (or unchanging) for all similar cases no matter where you are in time or space.

How do we prove them?

There are several types or methods for **proofs**.

Proofs can be done by **induction**, **contradiction**, indirectly and many other methods – you will learn how to categorize and use a variety of methods in Diploma Mathematics. For the moment it is important to know that a proof is the use of a sequence of logical steps to obtain the required result in a formal way.

ACTIVITY: Guided inquiry into angles on an arc

ATL

- Critical-thinking skills: Gather and organize relevant information to formulate an argument

Construct a circle sufficiently large to draw and measure angles inside it. Indicate an arc from points D and F, **label** the points and show the arc clearly.

Draw an angle ∢ DEF subtended by the arc DF, i.e. any angle which has D and F as the end points.

You can choose the apex point to be anywhere and label it E.

Measure the angle, as accurately as you can.

Construct another circle and repeat this for a different angle on the same arc.

Repeat for a third circle. What do you notice? Can you state your findings as a general rule?

Test your general rule by drawing a fourth circle and measuring a different angle subtended by the same arc. Does your rule still seem to be true?

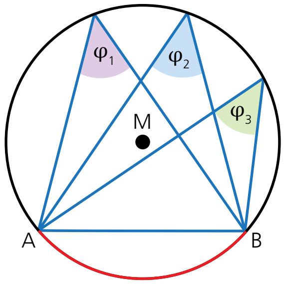
What is your conclusion? How could you test this further? What, if anything, could be affecting the accuracy of your answer?

Assessment opportunities

- This activity can be assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

Theorem 1: 'Angles subtended by the same arc' theorem

This theorem states that wherever you draw an angle on an arc, i.e. for angles subtended by the same arc, to the circumference then the angles will always be equal to one another.



Angles subtended by AB

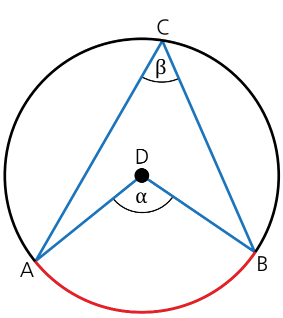
The diagram shows us that for all angles subtended by (constructed on) the arc AB,

$$\varphi_1 = \varphi_2 = \varphi_3 = \dots$$

If you were drawing and measuring carefully in the activity above, you will have seen that the angles were approximately equal. If you had continued drawing and measuring for *all* possible angles, this would be an example of a **proof by exhaustion**. ('Proof by exhaustion' requires that there must be a finite number of cases.) How long would it have taken to do that? Would you have been exhausted?

Another way of proving that a theorem is by justifying it in general terms, known as **proof by induction**. This tells us that the case is true and invariant for all similar conditions. Let's work through the second theorem together.

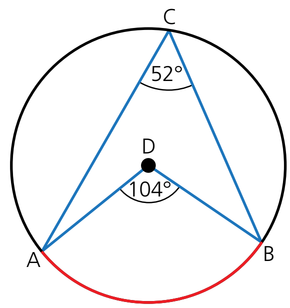
Theorem 2: 'Central angle theorem'



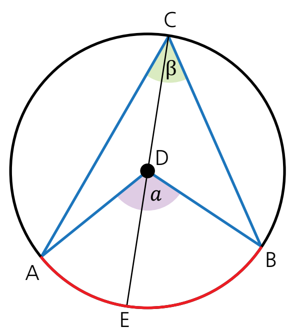
Consider a circle with an angle β constructed on or subtended by an arc AB. Now consider another angle α on the same arc but with an apex at the centre of the circle.

This theorem states that: The angle at the centre is always **twice** that of any angle subtended by the same arc.

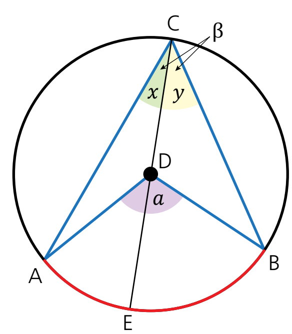
For example:



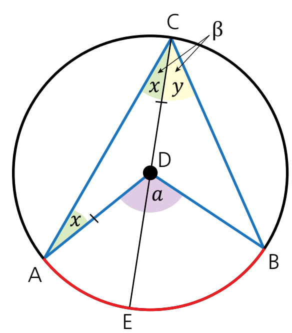
This is a statement which needs to be *proven* to be considered a theorem.



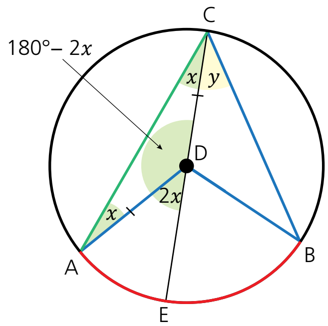
Draw a line connecting the apex of angle β through the centre.

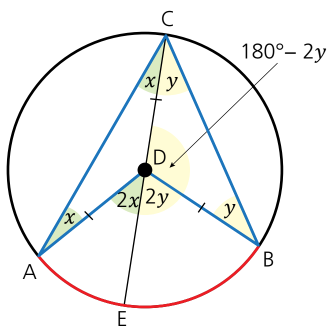


Let's name the two new angles formed at C by this line x and y.



The triangle ACD formed by this line is isosceles because two sides (AD + CD) are the same, both are radii and both base angle = x, base angles in an isosceles are equal.





This means that the other angle $\sphericalangle$ ADC must be 180° $-x-x$ … angles in Δ = 180°

and the exterior angle $2x$ … angles on a straight line = 180°

Now Δ BCD

$\sphericalangle$ CBD = $\sphericalangle$ BCD … isosceles Δ due to radii

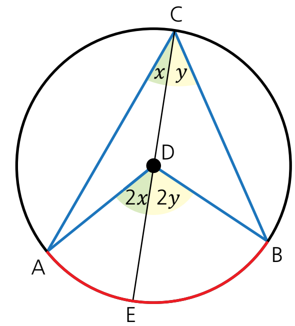
$\therefore$ $\sphericalangle$ CDB = 180° $- y - y$ angles in a Δ = 180°

and $\sphericalangle$ EDB at centre = $2y$

$\sphericalangle$ ACB = $x + y$

Hint

When calculating or solving for angles, it is considered good communication to say which theorem applies. This allows the reader to see what axioms you have used to find the missing angles, or to make your assumptions.



$\sphericalangle$ ADB at centre on same arc

$= 2x + 2y$

$= 2(x + y)$

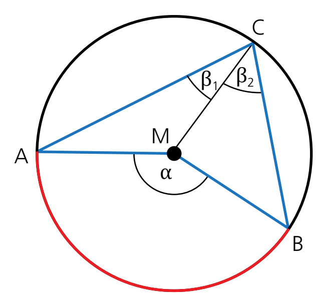
$\therefore$ angle at centre always = 2(angle on circumference)

So, that is the theorem proven. We used a variety of axioms (or rules that we already knew) to show how this would be true no matter the values for α or β, x or y. Mathematics is often praised for its elegance and simplicity, while appearing very complex. Here is an example of where complicated relationships of unknown quantities can be simplified and rationalized to extend to any situation involving a central angle and another angle on the same arc.

Nerd alert: Any Star Trek fans? Does this visual representation of the theorem remind you of anything?

USING THE THEOREM TO SOLVE PROBLEMS

Practice question:



In the diagram above, the angle β_1 = 30° and β_2 = 42°

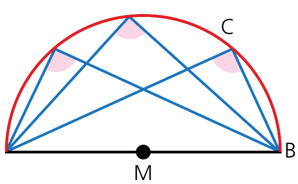
Find α

ACTIVITY: Semicircles

ATL

- Critical thinking skills: Analyse complex concepts and projects into their constituent parts and synthesize them to create new understanding.

Before we continue with the third theorem, let's consider the following diagram, where AB is a diameter of the circle and the three angles drawn are all subtended by the arc AB:



State the second theorem, either in your own words or using the text above.

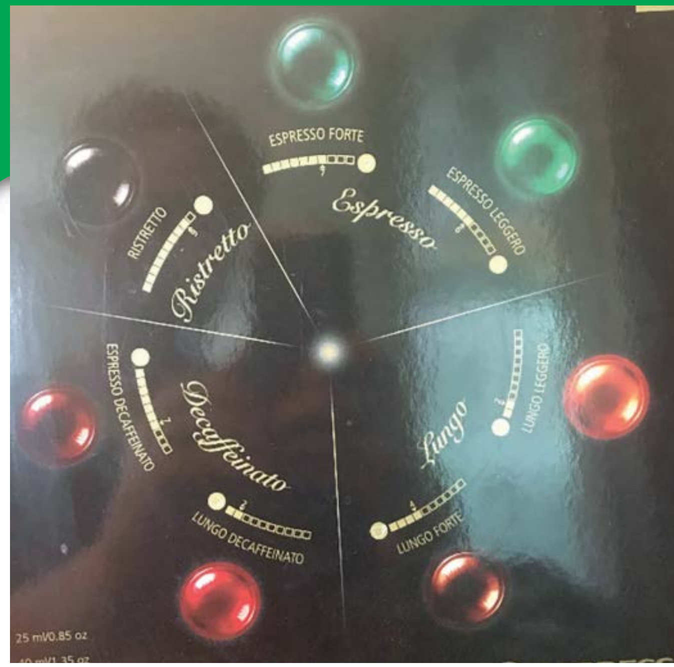
How does it relate to this diagram? What relationships can you see?

What angles can you see at M? What value (or quantity) do they have?

What does that mean (or what can you deduce) about the value of the other angles marked in blue? What must they be equal to?

Are they all equal? Why? Which theorem tells you that?

Measure the angles using a protractor. Does this confirm or contradict your findings?



- What is the relationship between the angles in this image?

Theorem 3: Angles subtended by a diameter

JUSTIFY your answer

In the previous activity, all the pink angles had to be 90° because they were subtended on the same arc as the diameter. The diameter is a straight line angle (180°). Therefore, each angle must be half that of the central angle of 180°, from the second theorem. This gives us the rule that

any angle on a diameter must be a right angle

This is the third theorem.

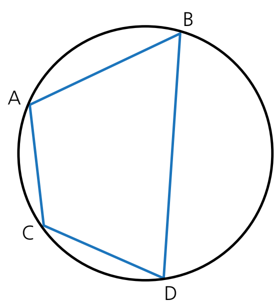
1. **Self-check: Did you correctly identify the rule in the activity above?**
2. **Gita says this theorem should state that all angles on the same arc as a diameter equal 90°. Luke says the theorem should actually be that the angle inscribed in a semicircle is a right angle.**
   a. **Who is right?**
   b. **How can you prove this?**

Assessment opportunities

- This activity can be assessed using Criterion A: Knowing and undertanding and Criterion C: Communicating.

WHAT IS A CYCLIC QUADRILATERAL?

As we know a quadrilateral is any shape with four sides, either regular or **irregular**. If the quadrilateral is inscribed by the circle, we say it is a cyclic quadrilateral.

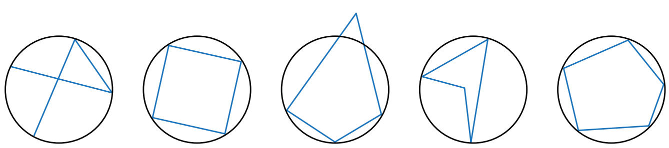


■ Cyclic quadrilateral ABCD

Are these images

a cyclic quadrilaterals? or

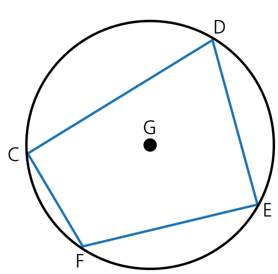
b non-cyclic quadrilaterals? **c** neither?



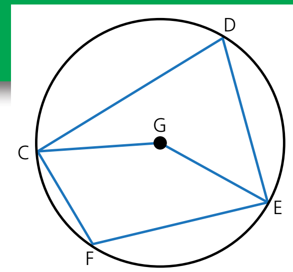
Theorem 4: Opposite angles in a cyclic quadrilateral add to 180°

For the fourth theorem, we will look at a proof by induction where we use rules or axioms we already know to **prove** that this statement is true. Look carefully at how this information is communicated to progress through the proof.

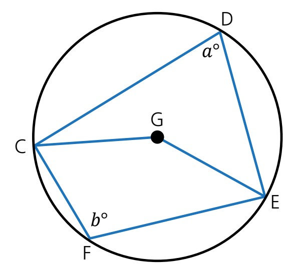
Let's start with a cyclic quadrilateral CDEF with a centre G:



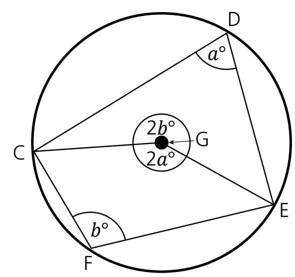
■ Cyclic quadrilateral CDEF



■ If we connect points C and E to the centre G, we will form two radii and create two angles at the centre

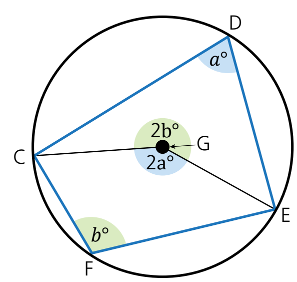


■ Next we mark the angles a and b at both apex D ∢ CDE and F ∢ CFE



■ Lastly, we can mark the angles at the centre as 2a and 2b (because of the relationship to the angles standing on the same arc).

So, let's **prove** that the opposite angles a and b must always add up to 180°



Consider a cyclic quadrilateral.

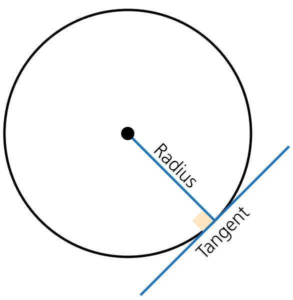
∢ CDE = a°	∢ CFE = b°
∴ ∢ CGE (obtuse) = 2a°	… angle at centre theorem
and ∢ CGE (reflex) 2b°	… angle at centre theorem
$2a + 2b = 360°$	… angles around a point
$a + b = 180°$	divide by 2

∴ opposite angles in cyclic quad = 180°

ACTIVITY: Tangent to a circle theorem

ATL

- Media literacy skills: Locate, organize, analyse, evaluate, synthesize and ethically use information from a variety of sources and media



The fifth theorem states that the angle between a tangent and a radius is 90°.

The proof for this theorem is an example of a proof by contradiction.

1 **Research** the proof or proofs for this theorem by searching online. Suggested search terms: proof tangent to a circle theorem. You may wish to watch several different versions as different explanations may cater to different learning styles.

2 **Summarize** the proof in your own words and draw examples in your notes.

A proof by contradiction shows that if a starting statement is shown to be impossible or contradictory, then you have proven the opposite must be true.

Links to: Language acquisition

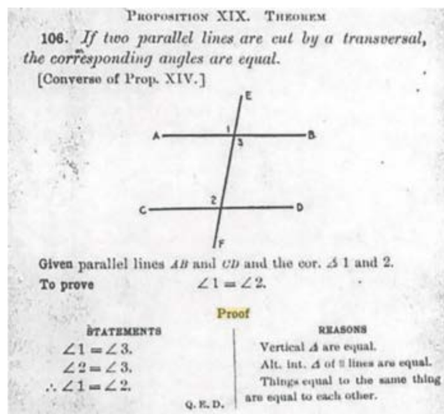
A proof by contradiction is also known as *reductio ad absurdum*. Where does this come from? What does it mean? What impression does it give? Does it indicate a specific time or place to you? Would you use it in the future, now that you know it?

Assessment opportunities

- This activity can be assessed using Criterion C: Communicating.

Really want to impress someone with your communication? Use Q.E.D. to show that your proof is complete. Q.E.D. is short for quod erat demonstrandum, which means 'which is what had to be proved'. This shows that the proof is done and you have proven what you were supposed to and signals the end of the logical argument.

Example



PROPOSITION XIX. THEOREM

106. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

[Converse of Prop. XIV.]

Given parallel lines *AB* and *CD* and the cor. ∠s 1 and 2.

To prove $\angle 1 = \angle 2$.

Proof

STATEMENTS	REASONS
$\angle 1 = \angle 3$.	Vertical ∠s are equal.
$\angle 2 = \angle 3$.	Alt. int. ∠s of ∥ lines are equal.
$\therefore \angle 1 = \angle 2$.	Things equal to the same thing are equal to each other.

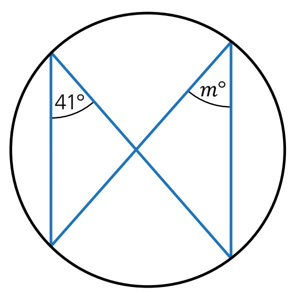
Q. E. D.

■ Quad Erat Demonstrandum

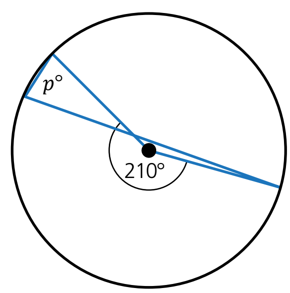
Practice questions:

Use the circle theorems to **find** the missing angles.

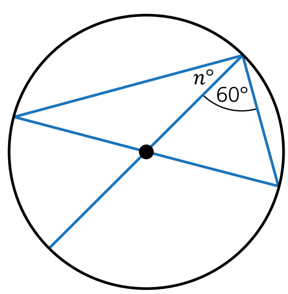
1 Level 1–2



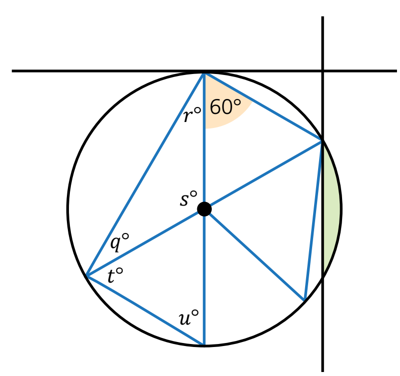
3 Level 5–6



2 Level 3–4



4 Level 7–8



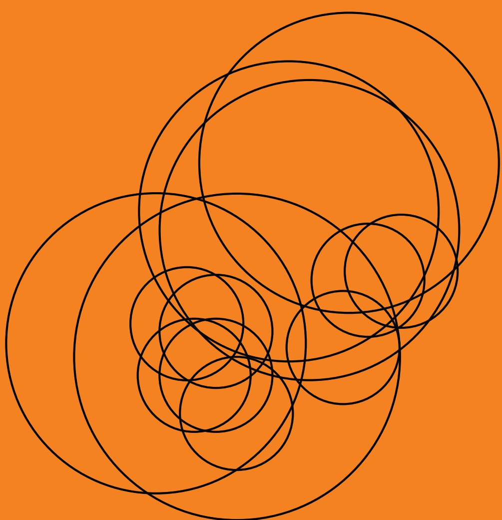
Can you see circles in real-life contexts?

Links to: Design

ATL

- Transfer skills: Combine knowledge, understanding and skills to create products or solutions

Take a look at the **Twitter** logo. The Twitter logo is actually constructed using only circles.



Can you see the Twitter logo in these circle constructions?

Look at this GIF: **https://twitter.com/amazinvids/status/711220282263007233/photo/1** Can you see it now?

Want to try it for yourself? **www.awwwards.com/reconstruct-the-twitter-icon-using-circle-shapes.html**

Many logos are constructed from circles or circle components such as arcs or sectors. Examples include:

- **Audi**
- **Beats**
- **Google**
- **Captain America**
- **xBox**
- **the Olympic Games**

How many of these logos do you recognize? What does it tell you about their effectiveness?

Use your knowledge of circle construction and **mensuration** (measurement) to construct an interesting and attractive logo. You should have a final version which showcases your design as well as a final mathematical 'blueprint' which clearly demonstrates your circle knowledge, measurements and construction.

How did you use knowledge from both mathematics and design to approach this task?

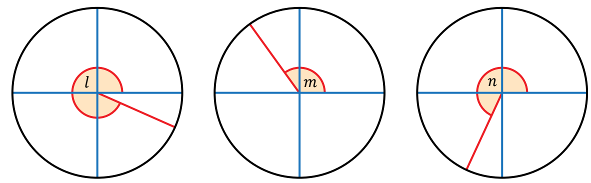
Assessment opportunities

- The final product and supporting documentation can be assessed using Criterion A: Knowing and understanding.
- The project can also be developed further and assessed using the Interdisciplinary Criteria.

Can a circle wave?

PRACTICE QUESTIONS

1 **Using a protractor, find the measures of angles *l*, *m* and *n*.**



2 **Using a compass and protractor, construct the following angles inside a circle:**

a **45°**

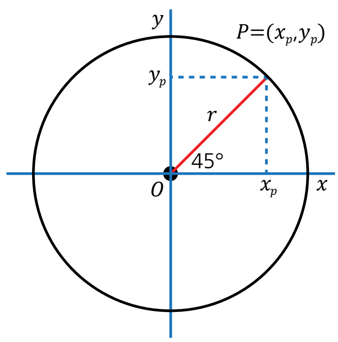
b **135°**

c **215°**

d **300°**

HOW DO WE FIND TRIGONOMETRIC FUNCTIONS IN CIRCLES?

As we know from previous chapters, trigonometry means triangle measurements which includes ratios, sine, cosine and tangent as well as Pythagoras' theorem. So where are triangles inside the circle?



By **convention**, we consider the positive x-axis to be at 0°. This means that a line on the positive y-axis would measure 90° on the negative y-axis would make 270° and so on.

Does that mean we measure increasing angles in a circle in a *clockwise* or *anticlockwise* direction? Does it matter?

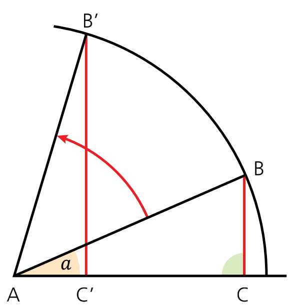
When we looked at trigonometric ratios previously, we *mostly* focused on angles, for example θ where $0 \leq \theta \leq 90$.

Of course we can have angles greater than 90° in a triangle. In triangles with sin and cos rule we applied them to angles < 180°. The angles in the triangles were never > 180°. Why was that?

We can see from the circle that angles formed by triangles in the circle could extend to 360°, as the radius rotates through the four quadrants. This means that logically sin, cos and tan *must* also exist up to 360°. Would it exceed 360°?

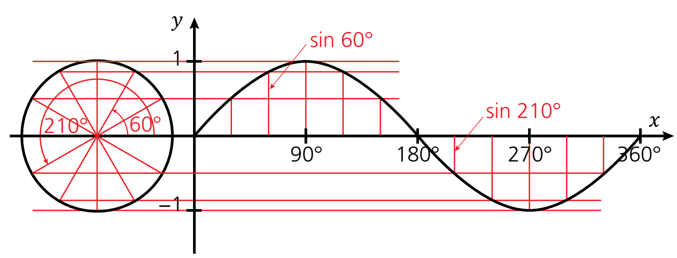
Sine: As we know, the sin = opp/hyp and this fraction will always give a decimal less than or equal to 1. This limit exists because the hypotenuse will always be greater than the opposite in a right-angled triangle.

How can the orientation of an angle in space affect the value of the relationships?



■ Movement through a circle

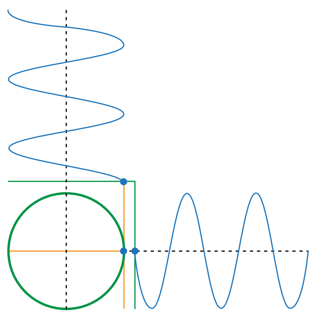
Consider the movement shown by this image. The radius is moving from B to B′. The radius, which is the hypotenuse of the triangle doesn't change. The opposite, or the red line in the triangles, is getting longer. This tells us that the sin values are increasing until they get to 1. What will happen when the radius moves into the second quadrant II?



■ The relationship between the angle and the sine is represented by the function

As the radius continues to move towards the negative y-axis (or $x = 0$), the opposite is getting smaller and smaller. This means it is moving back towards zero. The process repeats as it moves into the third quadrant (but now the results are negative and will not exceed –1).

It can often be easier to understand a moving graph rather than a static one. Why not see the function being generated by the moving angle here?: **http://31.media.tumblr.com/d4ff7401491c6cdc57abba47c1fc717b/tumblr_nazdxrbz0f1r068tfo6_250.gif**



Sin and cos functions generated by a circle on different axes

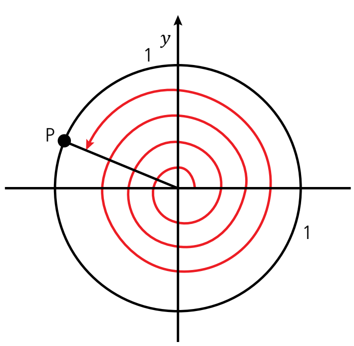
To see both the sine and cosine functions generated at the same time: **http://distractify.com/geek/2015/01/31/math-can-be-fun-1197822537**

So we can see that the trigonometric functions for sine and cosine have similarities but some important differences. They both have a range r, $-1 \leq y \leq 1$, or the y-values are between –1 and 1.

We have also been considering the domain for both functions between 0° and 360°, $0 \leq x \leq 360$

BUT

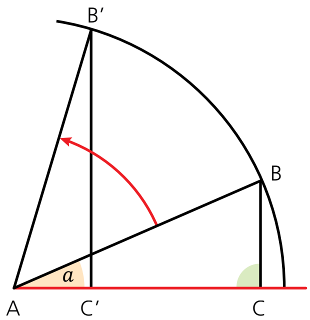
we *could* measure the angles above 360° by continuing around the circle in the same direction.



ACTIVITY: Cosine ratio

ATL

- Critical-thinking skills: Use models and simulations to explore complex systems and issues



Look at the diagram above to help answer these questions.

- **What is the formula for finding the cosine of an angle?**
- **What is happening to length of the adjacent as the angle increases?**
- **When will the adjacent be at its smallest value?**
- **When is it at its largest?**
- **What does that tell you about the angle as it increases towards 90°?**
- **What happens when it goes beyond 90°?**

Now, choose 20 angles (φ) between 0° and 360°. **Using** a GDC or calculator app **find** the cos φ for each of the angles you have chosen.

You now have a set of bivariate data to represent as a function. Which is the independent variable? Which is the dependent variable?

Sketch this data as a graph. What does the relationship look like?

Assessment opportunities

- This activity can be assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

What will this mean for the values of sin and cos? Is the orientation in space any different if the angle is 30° or 390°? What do you predict?

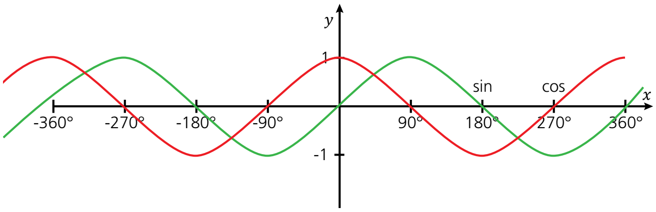
$\sin(360° + \alpha) = \sin \alpha$

$\cos(360° + \alpha) = \cos \alpha$

What do these two equations tell you?

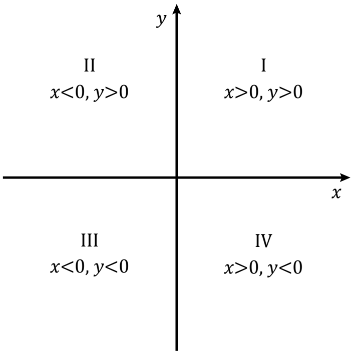
How could you **verify** if they are correct?

Give several examples, using your GDC

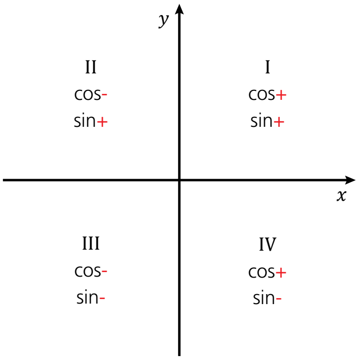


■ Sine and cosine functions

We know from plotting points and lines in chapter 3 that in each quadrant:



Now we see from the relationships above that:



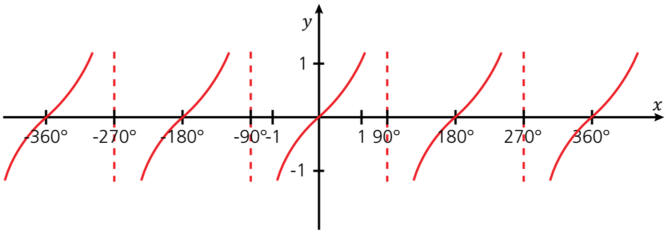
But what about tan?

We haven't looked at tan yet, for a good reason. As sin and cos are both limited by the maximum length of hypotenuse, they will have similar properties. Tangent isn't limited in the same way. Why not? How can you tell this from the formula?

When we look at the relationship of increasing angles for tan, we are looking at the changes in both the opposite and the adjacent.

Complete the following sentences by choosing the correct words:

As the angle increases, the opposite side increases/decreases. At the same time, the adjacent side increases/decreases. This means that the tan of that angle will increase/decrease



■ Tan function for $-360° \leq x \leq 360°$

Notice how different the function appears to sin and cos.

What other differences or similarities can you find between the three trigonometric functions?

Hint

If you are having trouble visualizing where these values come from, use the earlier link to the gif, where you will find the tan function generated lower on the same page. Alternatively search for tan function circle gif to find a similar illustration.

The values of tan extend from negative infinity to positive infinity. What does that mean? Where does it come from?

What are trigonometric identities and why are they interesting?

There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics … … or finding a new way to explain … an old mathematical structure. – William P. Thurston

WHAT'S THE DIFFERENCE BETWEEN AN IDENTITY AND AN EQUATION?

As we know from earlier algebra, equations come from expressions being equal. This allows us to **solve** to find the values of unknowns in that case; for example

$$2x^2 + 3 = 5$$

An identity is an equation which is always true, no matter the values. We can use the symbol ≡ to show that an expression is identical (or identically equal to) something; for example

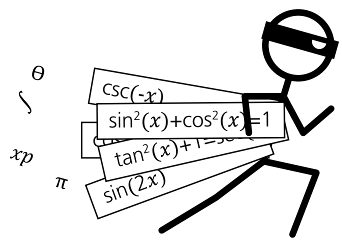
Σ angles in a triangle $\equiv 180°$	the sum of the angles is identical (always equal) to 180°

Note: this symbol is not always used. Often you will see only the equals sign. If you do decide to use it, make sure you are communicating it correctly.

Links to: Language

How does this idea of mathematical *identity* connect to our usual use of the term '*identical*'?

Protect yourself from...



IDENTITY THEFT

There are a vast number of trigonometric identities but we will concentrate on only the following skills:

Using simple trigonometric identities to simplify expressions and solve equations where $0° \leq \theta \leq 360°$

Note: Simple trigonometric identities expected are:

$$\sin^2(x) + \cos^2(x) = 1$$

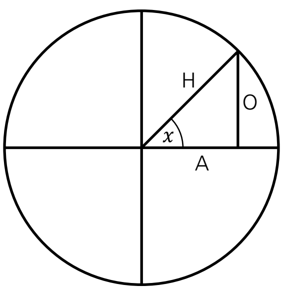
$$\tan(x) = \sin(x)/\cos(x)$$

It can be *mathematically* enjoyable to see how one side of an expression can turn into the other by clever substitution and drawing upon other rules and facts. You will rely on these skills more in Diploma Programme and beyond.

In examinations you often see questions with the command term 'show that'. These questions are asking you to **prove** how this equation has come to be with the information you are given. It's a good technique to start with one side only (often the more complicated looking side) and then make any substitutions or cancellations necessary to get you to the final 'other side' (usually the simpler looking side).

HOW DID THESE IDENTITIES ORIGINATE? HOW DO WE KNOW THEY ARE TRUE?

Demonstrating identity 1



$\sin^2 x + \cos^2 x = 1$ $\qquad$ $\sin^2 x = (\sin x)(\sin x)$

$\sin x = \dfrac{O}{H}$ $\qquad$ $\cos x = \dfrac{A}{H}$ $\qquad$ from $\qquad S^{O}_{H}C^{A}_{H}T^{O}_{A}$

Substitute these into the original

$$\sin^2 x + \cos^2 x = \left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2$$

$$\sin^2 x + \cos^2 x = \frac{O^2}{H^2} + \frac{A^2}{H^2} \qquad \text{add the fraction}$$

$$\sin^2 x + \cos^2 x = \frac{O^2 + A^2}{H^2} \qquad \text{from Pythagoras}$$

$$\therefore \sin^2 x + \cos^2 x = \frac{H^2}{H^2} \qquad a^2 = b^2 + c^2$$

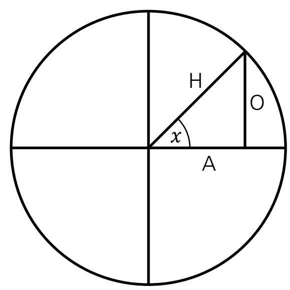
$$\sin^2 x + \cos^2 x = 1 \qquad H^2 = O^2 + A^2$$

Now we have shown how the LHS can be manipulated into the different (but equivalent) number 1, by using substitution and simplification. We relied on our knowledge from Pythagoras' theorem as well as how to add fractions from Primary school. This means that if you see $\sin^2(x)$ + *cos*$^2(x)$ anywhere in an expression in future you can combine them and replace them with a 1! This simpler form may help you to solve more complex problems.

ANOTHER IDENTITY

TAN $x = \dfrac{\text{SIN } x}{\text{COS } x}$

Why not try this one yourself before you read further? What could you substitute into the sin and cos? What other forms do we know for sin and cos? How could $S^{O}_{H}C^{A}_{H}T^{O}_{A}$ help you?



$$\tan x = \frac{\sin x}{\cos x} \longleftarrow \cos x = \frac{O}{H}$$

$$\cos x = \frac{A}{H}$$

$$= \frac{O}{H} \Big/ \frac{A}{H}$$

$$= \frac{O}{\cancel{H}} \Big/ \frac{A}{\cancel{H}}$$

$= \dfrac{O}{A}$ which *is* tan x. ✓✓✓ $\qquad$ cancel /H

So how can trigonometric identities help us?

Example

Expand and **simplify** $(\cos A + \sin A)^2$

Solution

Expand $(\cos A + \sin A)(\cos A + \sin A)$

Any method $\cos A \cos A + \cos A \sin A + \sin A \cos A + \sin A \sin A$

$\cos^2 A + \cos A \sin A + \cos A \sin A + \sin^2 A$

$+ 2 \cos A \sin A$

$\cos^2 A + \sin^2 A = 1$

$1 + 2 \cos A \sin A$

$\therefore (\cos A + \sin A)^2 = 1 + 2 \cos A \sin A$

Simplify $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}$

$= \dfrac{(\cos x)(\cos x) + (\sin x)(\sin x)}{\sin x \cos x}$ as a single fraction

$= \dfrac{\cos^2 x + \sin^2 x}{\sin x \cos x}$

$= \dfrac{1}{\sin x \cos x}$

We cannot simplify any further so this is the expression in its simplest form.

Show that $\sin x - \sin x \cos^2 x = \sin^3 x$

$\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x = 1 - \sin^2 x$

$= \sin x - \sin x (1 - \sin^2 x)$

$= \cancel{\sin x} - \cancel{\sin x} + \sin x \sin^2 x$

$= \sin^3 x$

$\therefore \sin x - \sin x \cos^2 x = \sin^3 x.$

PRACTICE PROBLEMS

1 **Simplify** $\cos Q \cos Q + 1 + \sin^2 Q$

2 **Simplify** $\dfrac{\tan x}{\sin x}$

3 **Simplify** $\cos^2 R(1 - \sin^2 R)$

a in terms of cos R (only)

b in terms of sin R (only)

Study shows that the brain reacts to beautiful mathematics in the same way as great art

The brain responds to mathematical beauty in the same way that it does to music and art, according to new research conducted by a team from University College London.

After analysing the reactions of 15 mathematicians to formulae through brain imaging, researchers found that the brain reacts similarly to seeing a beautiful equation as it does to magnificent art or music.

'When one looks at a formula rated as beautiful it activates the emotional brain - the medial orbito-frontal cortex - like looking at a great painting or listening to a piece of music,' said Professor Semir Zeki, lead author of the paper.

Professor Zeki, from the Wellcome Laboratory of Neuriobiology at UCL, added: 'To many of us mathematical formulae appear dry and inaccessible but to a mathematician an equation can embody the quintescence of beauty.'

One mathematician reported feeling 'a shiver of appreciation' when seeing a beautiful equation. Another said that viewing an equation was similar to 'hearing a beautiful piece of music, or seeing a particularly appealing painting'.

As part of the study, the mathematicians also ranked 60 different formulae as either 'beautiful', 'ugly', or 'indifferent'. According to this ranking the most beautiful formula is Euler's identity, which was deemed so aesthetically pleasing that it was compared to one of Hamlet's soliloquies.

Euler's identity is expressed as $e^{i\pi} + 1 = 0$ and is notable for combining the fields of geometry and algebra by using five fundamental mathmatical constants and three of the basic arithmetic operations. The latter trio are addition, multiplication and exponation, and the former quintet are e and π (both are transcendental numbers), i (the 'imaginary number), 0 and 1.

And for non-mathematicians hoping for a more accessible example, Pythagoras' theorem was also ranked highly. This formula ($a^2 + b^2 = c^2$) is used to work out the sides of a right-angled triangle and is often expressed as the statement 'the square of the hypotenuse is equal to the sum of the squares on the other two sides'.

Perhaps it is for this reason that the philosopher and mathematician Bertrand Russell declared that the discipline was 'capable of an artistic excellence as great as that of any music, perhaps greater':

'[Mathematics] gives in absolute perfection that combination, characteristic of great art, of godlike freedom, with the sense of inevitable destiny; because, in fact, it constructs an ideal world where everything is perfect but true,' wrote Russel in his 1967 autobiography.

Source **www.independent.co.uk/life-style/gadgets-and-tech/study-shows-that-the-brain-reacts-to-beautiful-mathematics-in-the-same-way-as-great-art-9126589.html**

ACTIVITY: Proofs

ATL

- Critical thinking: Analyse complex concepts and projects into their constituent parts and synthesize them to create new understanding

Now you will be presented with a theorem but in the form of a statement because it has not yet been proven. Your task is to prove it, or find the rule in general terms.

Statement: Two *tangents* to the same *circle* coming from *a common point* are of equal length or are *equidistant* from the circle.

Problem: How can you turn that statement into a theorem?

Task:

1. **Explain what each of the *mathematical terms* in the statement means.**
2. **Investigate this statement by constructing examples.**
3. **What conclusions can you draw? Prove, or verify and justify, your findings. Show your working, and reasoning, clearly.**

Links to: Physics

Look at the diagrams you have constructed for this activity. Imagine that the tangents were light rays or beams. Why would this theorem be relevant in Physics? What does it remind you of?

EXTENDED

Statement: A *perpendicular* to a *chord* in a circle *bisects* the chord.

Problem: How can you turn that statement into a theorem?

Task:

1. **Explain** what each of the mathematical terms in the statement means.
2. **Investigate** this statement by constructing examples.
3. What conclusions can you draw? **Prove**, or **verify** and **justify**, your findings. Show your working, and reasoning, clearly.

Assessment opportunities

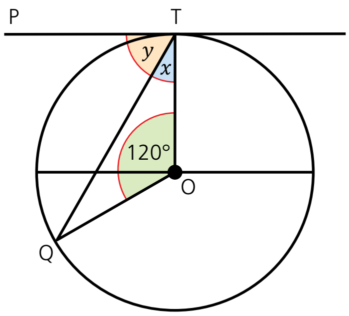
- In this activity you have practiced skills that are assessed using Criterion B: Investigating patterns, and Criterion C: Communicating.

SOME SUMMATIVE PROBLEMS TO TRY

We often hear that mathematics consists mainly of 'proving theorems.' Is a writer's job mainly that of 'writing sentences?' A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration; and proof, far from being the core of discovery is more often than not a way of making sure that our minds are not playing tricks.
– Gian-Carlo Rota

Use these problems to apply and extend your learning in this chapter. The problems are designed so that you can evaluate your learning at different levels of achievement in Criterion A: Knowing and understanding.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2

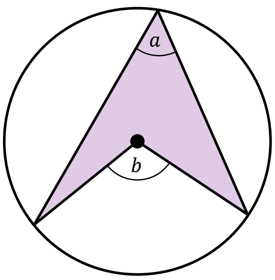


1 Which theorem is used to solve this problem:
- Angle at the centre is twice the angle at the circumference
- Tangent to a circle forms a right angle with the radius
- Opposite angles in a cyclic quadrilateral add to 180°

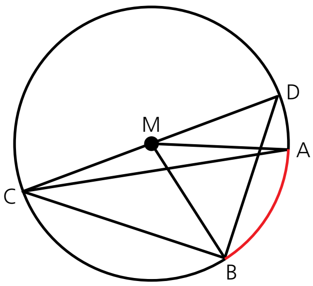
■ Cardiff, Wales, UK

2 **Identify** as many parts of a circle as you can in the photo above?

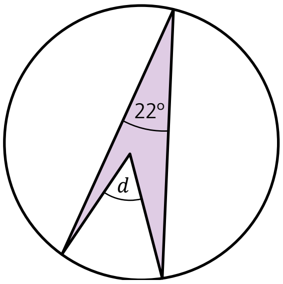
3 Which theorem is shown in this diagram?



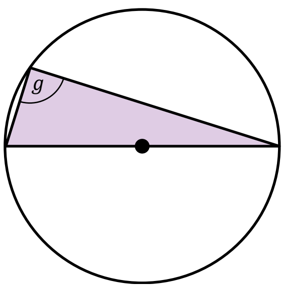
4 How many angles are standing on the red arc AB? **State** their names



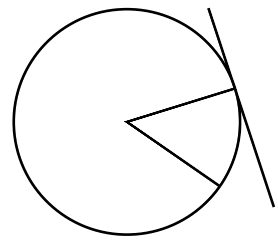
5 Find angle *d*

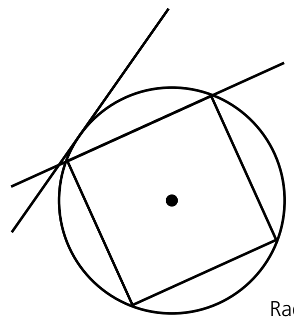


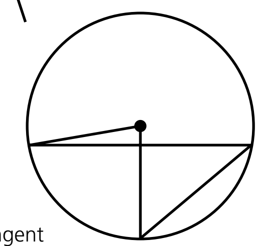
6 Find angle *g*



7 Match the theorem to the relevant diagram.







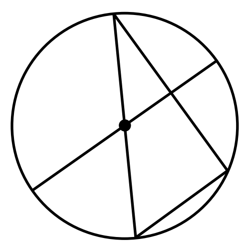
Angles in a cyclic quadrilateral = 180°

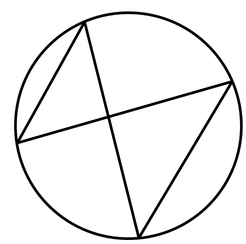
Central angles theorem

Same segment/arc

Radius perpendicular to the tangent

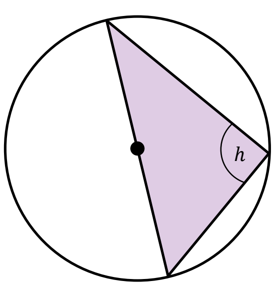
Angles on diameter = 90°



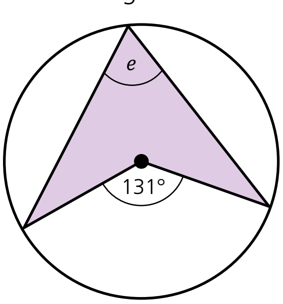


THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

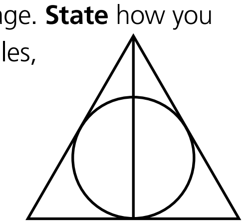
8 Find *h*



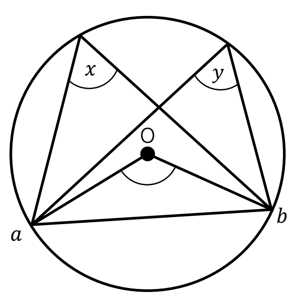
9 Find angle *e*



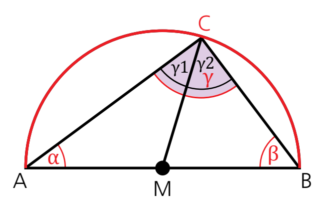
10 Label the value of any angles you know in the image. **State** how you know these angles, using theorems if relevant.



11 Describe what you see in the diagram and what you can **state** to be true.



12 If $\alpha = 38°$



Find

a $\gamma_1 =$

b $\gamma_2 =$

c $\gamma =$

d $\beta =$

For each calculation, **state** how you deduced/found the value

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

13 a **Name** two right angles

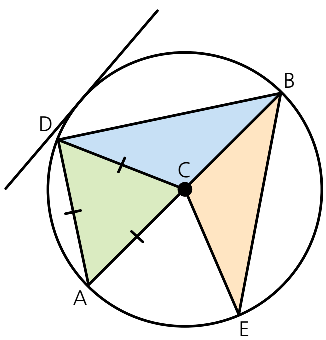
b **Name** three radii in the image

c **Name** 3 pairs of equal angles

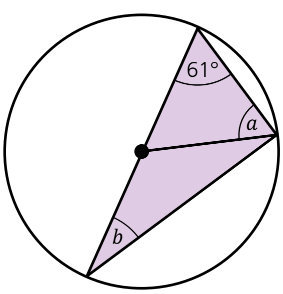
d **Name** an equilateral triangle

e ∢ DBC = ?

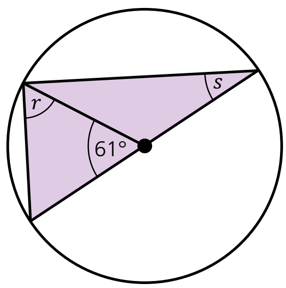
f Which theorem connects ∢ DCA to ∢ DBA?



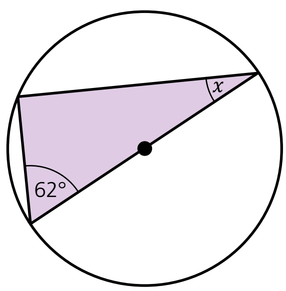
14 **Find** angles a and b



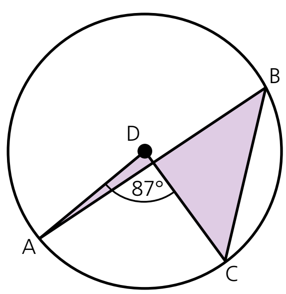
15 **Find** angles r and s



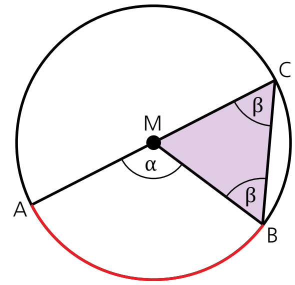
16 **Find** angle x



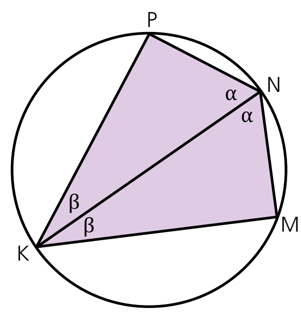
17 **Find** ∢ ABC



18 From the circle in the image, **write** an equation in terms of α and β



19 From the circle in the image, **write** an equation in terms of α and β in its simplest form.

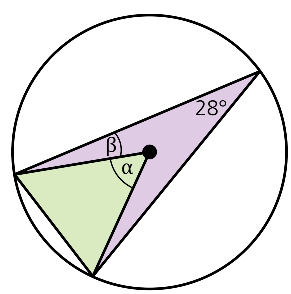


20 Simplify $\dfrac{\sin^2\theta}{1-\sin^2\theta}$

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

21 **Find** α

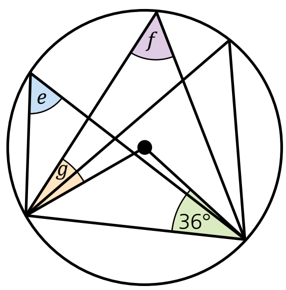
22 **Find** β



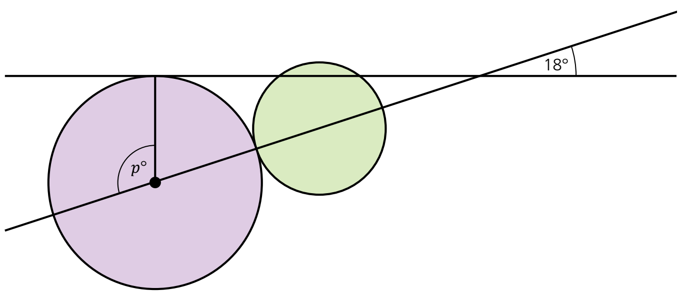
23 Find *e*

24 Find *f*

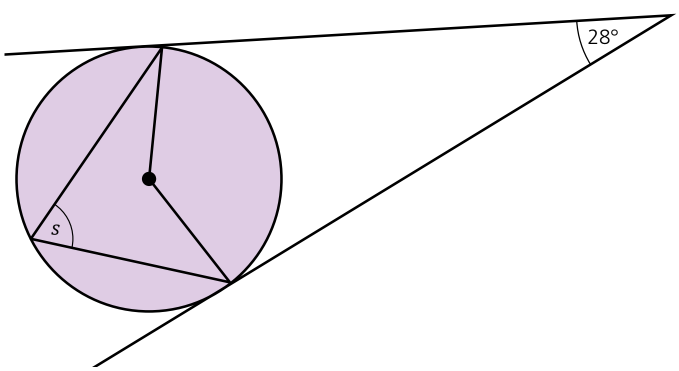
25 Find *g*



26 Find *p*



27 Find the value of *s*, explaining your reasoning.

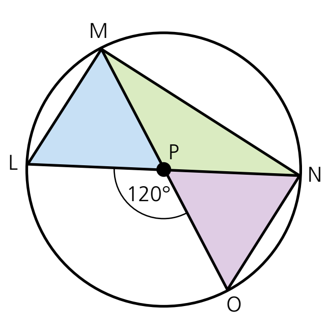


28 a Show that

Δ LMP and Δ NOP are isosceles

b Find ∢ MNP

c Find ∢ PNO



29 Show that $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

Reflection

Use this table to reflect on your own learning in this chapter.					
Questions we asked	Answers we found	Any further questions now?			
Factual: What is a circle made of? What are circle theorems? How can we graph them?					
Conceptual: How do we prove theorems by justifying them? What are trigonometric identities and why are they interesting? Can a circle wave? Can you see circles in real-life contexts?					
Debatable: Can a theorem tell us a 'universal truth' or is that impossible?					
Approaches to Learning you used in this chapter:	Description – what new skills did you learn?	How well did you master the skills?			
		Novice	Learner	Practitioner	Expert
Reflection skills					
Communication skills					
Critical-thinking skills					
Research skills					
Media literacy skills					
Transfer skills					
Learner profile attribute	How did you demonstrate your skills as a thinker in this chapter?				
Thinker					

11 The only sure thing?

An individual's **understanding** of **risk** and chance is highly **dependent** on both **logic** and **their personal experience**.

CONSIDER THESE QUESTIONS:

Factual: What is probability? How do we estimate and calculate probabilities? What is the difference between experimental and theoretical probabilities? What is a conditional probability? How can probabilities change? What is the purpose of a probability tree? What are systematic listing strategies? What is uncertainty? How do we handle multiple events? How are expected probabilities different from observation? Does probability change with sample size?

Conceptual: What is risk? When does 'unlikely' become 'impossible'? If probability is logical, why do we have coincidences? Why is measurement important and how is it linked to probability? When does a risk taker become too risky/unsafe?

Debatable: Is gambling a harmless diversion? Is uncertainty a bad or a good thing? Does perception of risk have a personal or cultural aspect?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how to assess and understand the uncertainty of life.
- **Explore** what we consider good risk-taking and bad risk-taking and how our perceptions can affect our belief in probabilities.
- **Take action** by celebrating the coincidences and randomness in life.

These Approaches to Learning (ATL) skills will be useful …

- Reflection skills
- Collaboration skills
- Critical-thinking skills
- Information literacy skills

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

We will reflect on this learner profile attribute …

- Risk-taker – we approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change.
- Balanced – We understand the importance of balancing different aspects of our lives – intellectual, physical, emotional - to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live.

'*When we measure anything, we use a* unit of measurement. *In most practical circumstances units are, in essence, the* result of agreement.

'Standards of measure *have evolved over time.*' – *from* Nuffield Key Ideas for Teaching Mathematics *9–19*

THINK–PAIR–SHARE

Read the quotation above. What do you think it means? What is measurement? Why do you think agreement is necessary to measure? Why would measurements need a standard? How have measurements evolved over time? Discuss these ideas with a partner and share your thoughts with the class.

PRIOR KNOWLEDGE

You will already know:

- that probability exists and how to estimate simple probabilities
- how to convert between various forms of probability – decimal, fractions and percentages
- what an 'event' is in probability
- how many playing cards in a pack and how many sides on regular die. (Singular form of dice)

KEY WORDS

certain	impossible
chance	risk
coincidence	theoretical
empirical	unlikely

What is probability?

Probability is a measure of how likely something is to happen. Probability puts a numeric, or quantitative measure on the chance of this particular event occurring. This ratio between the favourable events compared to the total possible events gives an indication of how likely our chosen or favourable event is to occur.

While statistics is concerned with the collection, analysis and interpretation of quantitative data, probability uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events.

As you work through this chapter, you will further develop your critical-thinking skills which will help you to differentiate between what happens in theory (probability) and what is observed (statistics) as well as seeing how the two are connected.

MEET A MATHEMATICIAN: TIMANDRA HARKNESS

Learner Profile: Risk-taker

■ Timandra Harkness, presenter, writer and comedian

Twitter: @timandraharkness

Timandra is an example of someone who understands mathematics and also how to make it relevant, interesting and even funny to wide audiences. She is a risk-taker, which she demonstrates by a variety of projects including stand-up, popular mathematics books and TV, radio and social media presenting.

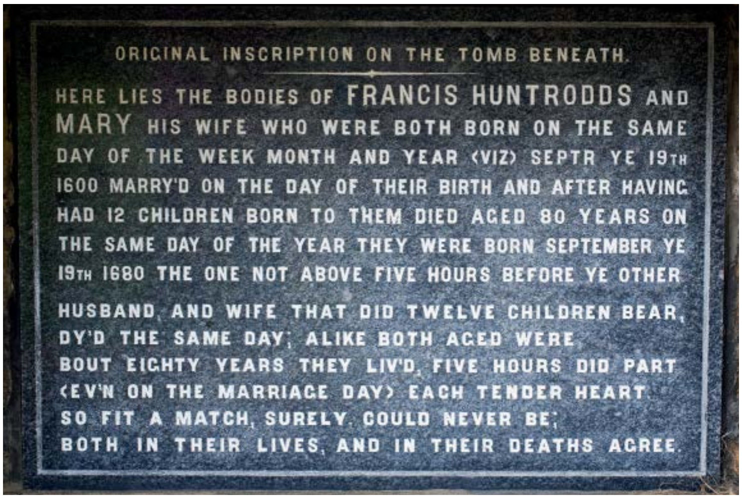
In her own words:

> *'I present BBC Radio 4 series, **Future Proofing**. I've also presented documentaries, **Data, Data Everywhere**, **Personality Politics & The Singularity**, and I'm resident reporter on Radio 4's social psychology series **The Human Zoo**, where I'm more an exhibit than a zookeeper.*
>
> *'Since winning the* Independent *newspaper's column-writing competition I've written for many print and online publications. My book,* Big Data: Does Size Matter? *is published by Bloomsbury Sigma in June 2016.'*
>
> *'I speak and host events for Cheltenham Science Festival, the institute of ideas, & the Wellcome Collection, among many others. My natural habitat is among dissenting adults in public.'*
>
> *'**Brainsex**, my solo comedy show on neuroscience and gender, toured 2012–15. Among my previous science comedy, **Your Days Are Numbered: the maths of death** was a surprise Edinburgh Fringe hit. Written & performed with Matt Parker, it then toured the UK and Australia.'*
>
> *'I have approximately 69.44% of an Open University degree in Mathematics & Statistics. By 2017 I'll be able to give you the confidence intervals on that. Though by then it should be 100%.'*
>
> *'In my spare time I enjoy opera, motorcycles, and talking to strangers.'*
>
> *'For speaking and writing enquiries, please contact my agent **Toby Mundy**.'*

www.statslife.org.uk/history-of-stats-science/1783-huntrodds-day-celebrating-coincidence-chance-and-randomness

Huntrodds' Day: celebrating coincidence, chance and randomness

'Friday, September 19, has been christened Huntrodds' Day by coincidence and chance enthusiasts David Spiegelhalter, Michael Blastland and Timandra Harkness. The aim of the day is to celebrate the extraordinary coincidences that occur in our lives.



'The best illustration of this is the story of Mr and Mrs Huntrodds, the couple who inspired the day. David Spiegelhalter came across the story while on holiday in Whitby, North Yorkshire. Francis and Mary Huntrodds were both born on the same day – September 19th 1600 – and were later married on their birthday and then sadly died on their 80th birthday in 1680.

'The Huntrodds had three extraordinary coincidences in their life events. But what are the odds of a couple sharing a birthday? Timandra explains the likelihood like this:

> *'A married person has a 1 in 365.25 chance (including leap years) of sharing their birthday with their partner. With approximately 11.6 million married couples in the UK, around 31,000 married couples in the UK would share a birthday.'*

'Over 30 000 couples isn't very unique sounding. How about couples who were born on the same day like Mr and Mrs Huntrodds?

> *'The commonest age gap between British spouses is one year. Two years is the average (mean), but nearly 1 in 10 UK marriages are between people born in the same year. So 3000 or so UK married couples would share one birthday cake, with the same number of candles.'*

'A more modern husband and wife story of coincidence came to light a few years ago when Mr and Mrs Pulsford of Pagham, West Sussex celebrated their 80th birthday. Again they were both born on the same day of the same year but also celebrated their eightieth birthday on 08/08/08. Or Mr and Mrs Wheeler, who discovered a childhood photo of them both playing on the same sandy beach – 11 years before they first met as adults.

'Everybody loves quirky coincidences and extraordinary events, even statisticians. The reason statisticians love them is that it gives them the opportunity to employ their knowledge and techniques to try and work out a logical explanation for why events that seem amazing and unlikely keep happening.

'David Hand has devoted a whole book to the subject called, *The Improbability Principle*, released earlier this year. In the book he investigates instances like the American author who, while browsing bookshops in Paris, discovers a book she was the previous owner of as a child. Or the man who got struck by lightning seven times in his lifetime. In the book, David explains why these astounding events happen through a set of five laws based on the mathematics of probability.

'However, the organisers of Huntrodds' Day don't want our wonder of chance happenings to be dampened by the statistical explanations. Instead, for one day of the year they want us to celebrate and talk about them. Then the statisticians can worry about explaining them for the other 364 days.

'The organisers are encouraging everyone to throw a Huntrodds' Party. The rules are simple – you must have 23 people in attendance, as this is the number needed for a more than 50% chance that two will share a birthday. Also, conversation has to be aimed at discovering coincidences among the guests (which basically gives attendees license to be nosey about each other).'

! Take action

- ! Why not plan your own Huntrodd's party at school? What other **improbabilities** do you know of? How could we celebrate them?
- ! Alternatively, do you have a crazy coincidence story of your own? You could collate the best stories from your class, family or school and submit them to a crowd-sourced collection on **http://cambridgecoincidences.org/**

THERE'S NOTHING CERTAIN IN LIFE BUT DEATH AND TAXES

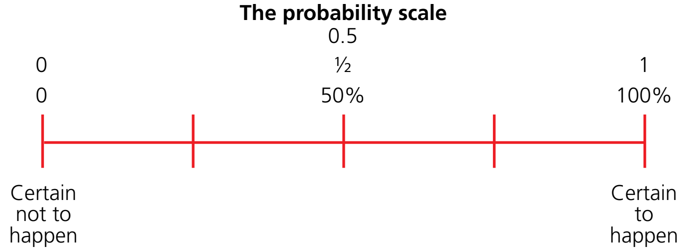
Sometimes you might feel that probability questions are a little morbid or sad because they focus on working out whether or not something bad will happen. This isn't done to be depressing. We use these types of facts because there are very few examples in life where something is certain, absolutely definitely going to happen, with no exceptions. Lots of things in life are *almost* certain (or on the other side, highly unlikely) but the Huntrodds are perfect examples that unlikely events do occur. People do win the lottery or get hit by lightning more than once in their lives.

Remember that probability can be represented in a variety of forms, which are all interchangeable and represent the same risk or chance.

For example, the chance of

- Fraction
- Decimal
- Percentage
- Ratio
- 1 in a …

The probability scale



> ### Links to: Languages
> We often say 'the impossible has happened' when we should more correctly say 'the highly improbable/unlikely has happened'. Saying 'the impossible has happened' is an exaggeration to show just how unlikely the event was. We often refer to people "beating the odds" which is also a reference to probability.

When we are handling multiple probabilities at once or a variety of possible combinations, it is considered easier to work in decimals. This may be because it is easier to multiply quickly or enter into the calculator. In an examination, decimals, fractions and percentages will often be equally recognized as valid answers. Worded answers, and often ratios, should be avoided to ensure you get maximum communication marks unless you are specifically asked for them.

ACTIVITY: What do you think the chances are?

ATL

- Reflection skills: Consider personal learning strategies

Qualify (put in words) and **quantify** (give a number value) for each of the risks below, in your opinion.

State how likely you think they are between impossible (=0) and definite/certain (=1).

- **Probability that a human will live to 1 000 000 years old.**
- **Probability that you know your name.**
- **Probability that a student chosen at random from your class is a girl.**
- **Probability that the most popular fruit is apple.**
- **Probability that we will die within the next 100 years.**
- **Probability that the author of this book is an alien.**
- **Probability that you have more than three eyes.**
- **Probability that the author of the book is called Tallulah.**
- **Probability that your computer has been hacked in the last three years.**
- **Probability that your computer will be hacked in the next five years.**
- **Probability that the sun will set in the west tonight.**
- **Probability that a dating app finds your true love.**

Remember when choosing both the words and the numbers to represent these probabilities that you must **justify** how you came to, and why you chose, these values.

How did you represent the number between 0 and 1? Why did you choose this form?

Think about your results. **Discuss** them with a partner. How many results did you agree on? How many did you disagree on? By how much were you different? Why might this be? Is there a way we could **verify** who is right and who is wrong?

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion C: Communicating.

From your previous study into probability, you will already know how to calculate a probability from given information.

ACTIVITY: Revision exercise

■ An unbiased 12-sided die

The image above shows a 12-sided die.

What is the probability that you will roll a 7?

What is the probability that you will roll a 9?

What is P(roll a 20)?

What is P(a number greater than 4)?

Find P(x), where $x < 10$.

Find P(x), where $x \leq 10$.

How can probabilities change?

PROBABILITY – SO MUCH MORE THAN DICE, SPINNERS, CARDS AND COINS ...

Why are so many questions about probability depending on examples of tossing coins, rolling a die or picking cards from a deck?

Many of our real-world systems are open systems that are complex and are dependent on many variables. The probability of an event occurring, such as an earthquake or a lightning strike, are extremely difficult to predict with great accuracy.

Coins, spinners and dice should give fair, unbiased and equal probability of each of their respective outcomes. Coins should give 50 : 50 chance of heads vs tails, each numbered side of a die should have a 1 in 6 chance of occurring and so on. While this works to illustrate probability on a simpler level, you can take these ideas and apply them to more complex situations.

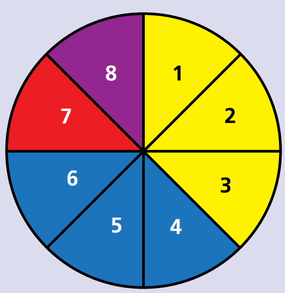
PRACTICE QUESTIONS

1 For a normal pack of playing cards, **find** the probability that a card chosen at random ...

- a is red
- b is a queen
- c is not a queen
- d is a card.

2 **Find**

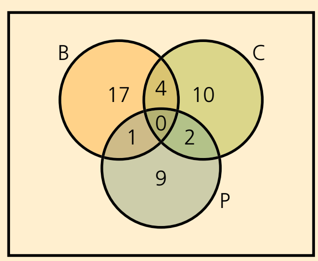
- a P(blue)
- b P(purple)
- c P(yellow or red)
- d P(green)
- e P(yellow and a 2).



Hint

Struggling to understand what probability has to do with uncertainty? Or what uncertainty is in our everyday lives? Check out these animations: https://understandinguncertainty.org/view/animations

3 This Venn diagram shows student subject choices for Diploma Sciences in a school.



- a What is the probability that a student is taking Biology?
- b What is the probability that a student is taking Chemistry *only*?
- c What is the probability that a student is taking three Sciences?

At this school, students who want to study medicine must take both Biology and Chemistry.

- d What is the probability that a student chosen at random is taking both those classes?
- e Can you **identify**, from the Venn diagram, the probability of a chosen student being a future medicine student? Why or why not?

COMPLEMENTARY EVENTS

The probability of something happening and the probability of something **not** happening must be 1. Why is this? Explain, in your own words, how you know this to be true.

Events which are complementary are ones which cannot both happen, that is, the chances of winning the lottery is complementary to not winning the lottery.

P(something happening) = 1 – P(something not happening)

Example

For the set of positive integers for $1 \leq x \leq 10$, find the following probabilities for a single number chosen at random:

a P(even)
b P(not even)
c P(7)
d P(any number *but* 7)

Solution

a P (even) = 2,4,6,8,10

$\therefore \dfrac{5 \text{ events}}{10 \text{ events}}$

P(even) = 0.5 or $\dfrac{1}{2}$

b P(not even) = 1 – P(even)

= 1 – 0.5

= 0.5

c P(7) = $\dfrac{1}{10}$ = 0.1

d *P*(any number but 7) =1,2,3,4,5,6,8,9,10

$= \dfrac{9 \text{ events}}{10 \text{ possible events}}$

= 0.9

PRACTICE QUESTIONS

What are the complementary probabilities to the ones you worked out before for the 12-sided die?

What is the probability that you will not roll a 7?

What is the probability that you will not roll a 9?

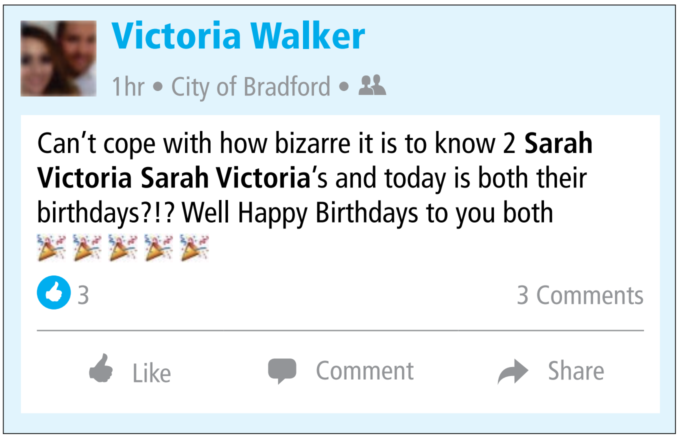
What is P(not 20)?

What is P(a number not greater than 4)?

Find P(x), where $x \geq 10$?

Find P(x), where $x > 10$?

Does perception of risk have a personal or cultural aspect?



The image above shows a post from someone who can't believe she has two friends with the same name and same birthday.

WHAT IS RISK EXACTLY? WHEN DOES A TAKING A RISK BECOME JUST TOO RISKY OR UNSAFE?

What is uncertainty? Randomness?

People searching for meaning often look for patterns to apply to randomness. We have said before that the brain is a pattern processing machine and that humans seem to be programmed to look for order in chaos.

A good example of this 'looking for patterns' occurs when people are trying to predict the next number in a game such as a roulette wheel. They look at the numbers which have been before and make statements such as 'we haven't had a red for a while' or '13 hasn't come up for ages' and they try to play according to these predictions. We know that each number is equally probable, if the game is fair, so why do we try to make these insights?

Perception of risk: our understanding of risk can be affected by many factors, some of which might be misleading. As with the roulette wheel example, we might only look at a small set of **recent** data and draw a conclusion which might not match to the actual probability.

For example, there is a huge set of data about flight safety and probability. A statistic often quoted is that you are more likely to die in the drive to the airport than on the plane itself. This probability is used to underline that air travel is considered one of the safest forms of transport compared to others, and not that driving to the airport is unsafe. This is calculated and compared to deaths in other forms of transport.

The perception of the safety of air travel can be affected by the concept of recency. If someone is afraid of flying, they may make statements about the risks involved based on their perception (their own personal bias or view) of what has happened recently:

- **Positive** recency:

  There have been a lot of plane crashes recently, so I don't want to fly. It's not safe.
- **Negative** recency:

  There hasn't been a plane crash recently, so we must be due one soon – so I don't want to fly.

This type of reasoning leads to rash decisions and is indicative of an incorrect understanding of probability. They are not thinking about the actual likelihoods of being in a plane crash and their ideas of risk have a personal interpretation not a strictly mathematical one.

Disclaimer: While the risk of crashes due to human error and computer failure have been consistently low over the past few years, sadly the same cannot be said for incidents of terrorism or conflict.

How might this have had a cultural impact on how people perceive (view) airline safety?

Doctors and surgeons deal with risk every day. They need to weigh the probability of a fatal complication, a side effect or a negative outcome with the patient's level of severity as well as the cost. In fact, a true understanding of probability and how it applies in the real world may actually advantage certain different people for different reasons

WHAT makes you say that?

Any medical operation carries a risk of infection. From life-saving operations to routine ones, the probability of contracting something during an operation is never zero.

What factors increase a risk? How high would the risk/probability need to be before you decided not to have the surgery done? How do you balance risk against reward?

What makes you say that? **Discuss** with a partner or with a group, what makes a risk worth taking? Could you put a numerical value on it?

ACTIVITY: How can these people be successful through the application of probability?

ATL

- Critical thinking skills: Consider ideas from multiple perspectives

How can understanding probability affect these different roles?

a **Patient**
b **Sailor**
c **Sportsperson**
d **Investor**
e **Insurer**
f **Lawyer**
g **Game developer**
h **Oil-rig engineer**
i **Chef**
j **Marketing manager**

Assessment opportunities

- In this activity you have practised skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

Probability of rain is now a regular feature of weather forecasts. Meteorologists give an estimated probability of how likely it is to rain at any given time, based on complex weather modelling.

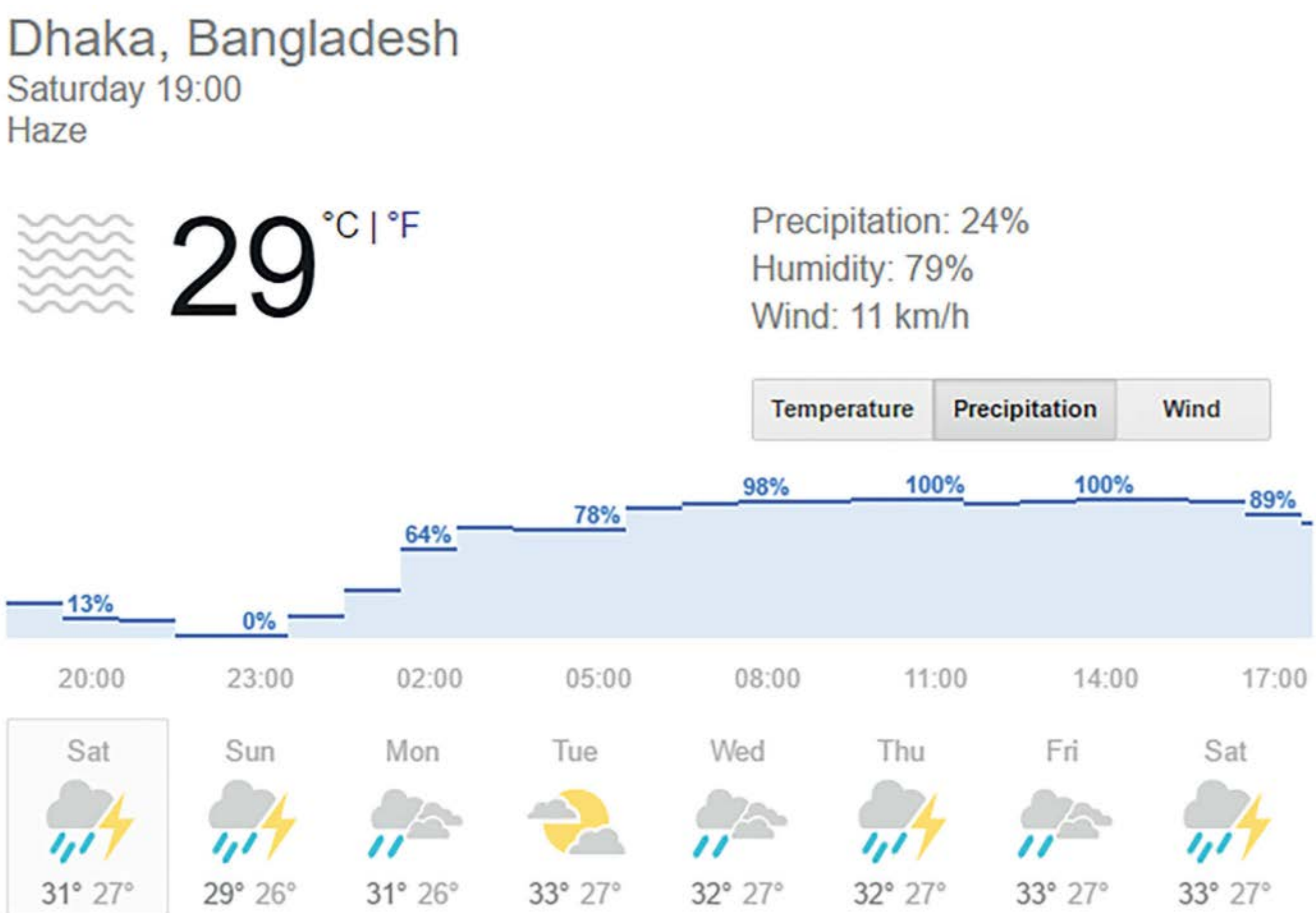
Is a 30% chance of rain a good reason to cancel an outing? What about 50%? Or 70%? Can you quantify (pick an exact number) when you would cancel and when you would take the risk and go ahead? What does that tell you about yourself?

ACTIVITY: Weather facts or fiction?

ATL

- Critical thinking skills: Recognize unstated assumptions and bias

The following shows a real-life weather forecast for Dhaka, Bangladesh.



Mirza understands that a forecast is a prediction, based on models of weather patterns which take many variables into account. What Mirza wants to know which of the numbers appearing on the weather forecast are probabilities and which are not.

Help Mirza understand the figures by explaining as many of the **quantitative values** given and what they mean. **Explain** which are probabilities and which or not, justifying your answer.

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion C: Communicating and Criterion D: Applying mathematics in real-life contexts.

ACTIVITY: Personal choices with probabilities in mind

Read the extract and answer the following questions (on the right)

Links to: Sciences

What are chromosomes and why do they divide? What role does randomness play in evolution?

Has the risk increased or decreased?

Convert the probabilities quoted into fractions, decimals and percentages.

Draw a number line and mark the probabilities on it.

Comment on what you observe.

What other factors should be taken into account when planning for a family at any given age?

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

'A woman's chances of giving birth to a child with Down syndrome increase with age because older eggs have a greater risk of improper chromosome division.

'At 20 years old, she has a 1 in 1500 chance and by age 35, a woman's risk of conceiving a child with Down syndrome is about 1 in 350. By age 40, the risk is about 1 in 100, and by age 45, the risk is about 1 in 30. However, most children with Down syndrome are born to women under age 35 because younger women have far more babies.' – The Mayo Clinic, **mayoclinic.org**

ACTIVITY: The Birthday paradox

ATL

- Collaboration skills: Delegate and share responsibility for decision-making

Remember Victoria's crazy coincidence of having two friends with the same names who share the same birthday?

The Birthday paradox is just one example of probabilities yielding some surprising results! What is the Birthday Paradox exactly?

In pairs or small groups, **discuss** what you think of the following statements – some of which are true and some of which are false.

- **In a sample set of 23 people, there is 0.5 (or 50–50) chance that two will have the same birthday.**
- **Everyone has a 1 in 365 chance of having a birthday or a certain day, therefore the chances that two people have the same birthday are tiny.**
- **With 47 people in the same room, the chance of a common birthday increases to 75%**
- **The probability that someone has a different birthday to yours is $\frac{364}{365}$**
- **With 47 people in the same room, the chance of a common birthday rises to 95%**
- **It is impossible that two people in a group of 10 would have the same exact birthday.**
- **The birthday paradox doesn't exist.**
- **In a group of 50 people, you will *always* find a common birthday.**
- **We can be absolutely certain that two people will share a birthday in a group of 367 people.**

Decide how you will investigate them as a group and determine the best way to test each statement. Collaborate to work efficiently as a group and decide on actions together.

Make a group presentation describing which statements are correct, which are incorrect and which may be inconclusive based on your knowledge of the Birthday paradox.

Assessment opportunities

- In this activity you have practised skills that can be assessed using Criterion A: Knowing and understanding and Criterion C: Communicating.

How do we handle multiple events?

LEICESTER LUCK

In this example it will help if you are familiar with football or soccer, particularly the Premier League in England. The 2015/16 was a special season, described as a *fairy tale* by fans and media alike. Why?

An unexpected winner of the season, Leicester City Football club surprised everyone by winning the league, despite not having the same wealth as the other clubs, less famous players, having had very poor seasons previously and almost being relegated (moved down into a lower league).

Every report after their win mentioned mathematical probability and how they 'beat the odds' and left the experts scratching their heads. The odds indicate how likely or unlikely the 'experts' think it will be for them to win. These odds are essentially predictions of success. Leicester city was given a 1 in 5000 chance to win at the beginning of the season. What is this $\frac{1}{5000}$ chance as a percentage?

One such expert, Danny Finklestein, from The Think Tank was surprised more than most. The Think Tank is a group of experts who use a model to try and predict who is likely to win the overall competition. The group calculates the probability of winning or losing each match by comparing measures such as shots on target, when and from where.

These probabilities are then worked into a mathematical model. The model calculates the various combinations of wins, losses and draws over the season.

A simplified version would look of a team's possible outcome would look like

$$P(\text{win}) \times P(\text{win}) \times P(\text{loss}) \times P(\text{draw}) \times \ldots$$

and so on for *all probable* combinations of games for that season

The model then runs over 100 000 different simulations of possible combinations to work out the likely winners based on the statistical analysis. In none of the simulations were Leicester City predicted to be the winners.

The model did calculate a 1.1% chance of a surprising team winning the league (how many simulations showed this?) – one of the lowest chances in history. But none of those surprising teams included Leicester City!

Listen to the source podcast here: **www.bbc.co.uk/programmes/p03tlnzl** or search **BBC**, **More or Less**, **Leicester fluke**.

What are the chances of Leicester winning again this season, if they had the same odds?

To find the probability of two events both happening, we multiply their individual probabilities. Does this make the chances of the two events happening bigger or smaller? Does that make sense in real life?

P(A **and** B happening) = P(A) × P(B)

P(A **or** B happening) = P(A) + P(B)   if they are mutually exclusive

Mutually exclusive – what does this phrase mean? How can it be applied when talking about possible events?

For example, you're invited to a party but it's on at the same time and day as your sister's graduation. The events are mutually exclusive. Choosing a card from a deck at random also has mutually exclusive outcomes; for example, P(hearts) and P(diamonds) are mutually exclusive because a card cannot be both at the same time.

P(queen) and P(hearts) are *not* mutually exclusive. Why?

Example

A fair, unbiased, coin is tossed three times.

a What is the probability that the coin landed on heads all three times? P(H and H and H)

b What is the probability that the coin landed on heads once only?

Solution:

a P(head) $= \frac{1}{2}$

P(H,H,H)$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

b P(heads once) = P(H,T,T) $= \frac{1}{8}$

P(T,H,T) $= \frac{1}{8}$

P(T,T,H) $= \frac{1}{8}$

$\therefore$ P(landing on heads once only) $= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

PRACTICE QUESTIONS

1 **State** whether the following probabilities are mutually exclusive or not and **explain** why:

- **a** P(rolling a 6), P(rolling an odd number)
- **b** P(winning the basketball game), P(drawing the basketball game)
- **c** P(being female), P(being a mathematician)
- **d** P(studying geography), P(studying mathematics)

2 In a group of 20 athletes, 14 females and 6 males, there is an equal chance of anyone being tested for banned substances.

- **a** **Find** P(female tested)
- **b** **Find** the complementary event, explain what this tells you
- **c** **Find** the probability that two athletes tested would both be female
- **d** **Find** the probability that one female and one male athlete are tested

3 A manufacturer has discovered from testing that three of the working parts of a drill each have a 2% chance of being faulty (not working).

- **a** What is the probability that two parts of the drill will be faulty?
- **b** What is the probability that all three parts of the drill will fail?
- **c** What is the probability that the drill works without any problems?

What are systematic listing strategies?

THE FUNDAMENTAL PRINCIPLE OF COUNTING

A phone manufacturer produces a new phone. Each phone has three possible memory sizes (8GB, 16GB and 32GB) and in four different possible colours (silver, white, black and rose gold).

How many possible combinations of phones are there?

We can list the possible combinations to ensure all combinations are included. One strategic way to do this would be to start with one memory size and vary the colours before moving on to the next size.

> (8, silver), (8, white), (8, black), (8, rose gold), (16, silver), (16, white), (16, black), (16, rose gold), (32, silver), (32, white), (32, black), (32, rose gold)

There is a total of 12 combinations. The combinations can be found by multiplying the number of memory sizes (3) by the number of colours (4), that is, $3 \times 4 = 12$

So, we can make a general rule for a number of possibilities, x, with a second set of possibilities, y, that gives us a total of xy outcomes. This set of xy outcomes is the sample space.

The sample space is the total number of possible outcomes for an event. In this example, an event would be a possible phone combination.

Example

Assuming that each phone is equally popular …

What is the probability of a phone purchased at random being a silver one?

Solution

P(silver phone) $= \frac{1}{4}$ there are four colours in total

What is the probability of a phone purchased being a silver 16GB phone?

Solution

P(16, silver) $= \frac{1}{12}$ this was 1 outcome in the sample space of 12

What is the probability of two phones purchased straight after one another both being rose gold phones?

Solution

P(rose gold) $= \frac{1}{4}$

To find two events both happening, we multiply their individual probabilities

$$\text{P(rose)} \times \text{P(rose)} = \frac{1}{4} \times \frac{1}{4}$$
$$= \frac{1}{16}$$

There is a $\frac{1}{16}$ chance that two successive phones sold will both be rose gold.

What is the probability of two phones purchased straight after one another both being rose gold 32GB phones?

Solution

P(32, rose gold) $= \frac{1}{12}$

$$\text{P(32, rose gold) AND P(32, rose gold)} = \frac{1}{12} \times \frac{1}{12}$$
$$= \frac{1}{144}$$

There is a $\frac{1}{144}$ or 0.006 944 444 or 0.69444% chance of two successive phones both being 32GB rose gold phones.

How are expected probabilities different from observation?

PRACTICE QUESTIONS

1 What are the chances of a phone chosen at random being a black phone?

2 What are the chances of a phone chosen at random being a yellow phone?

3 What is the probability of a purchased phone being a white, 16GB phone?

4 What is the probability that two successive phones purchased are both white?

5 What is the probability that the first phone is white and the second phone is 32GB?

6 What is the probability that the first and second phones have the same colour or same memory size?

7 What is the probability that the first and second phones have the same colour and same memory size?

WHY IS MEASUREMENT IMPORTANT?

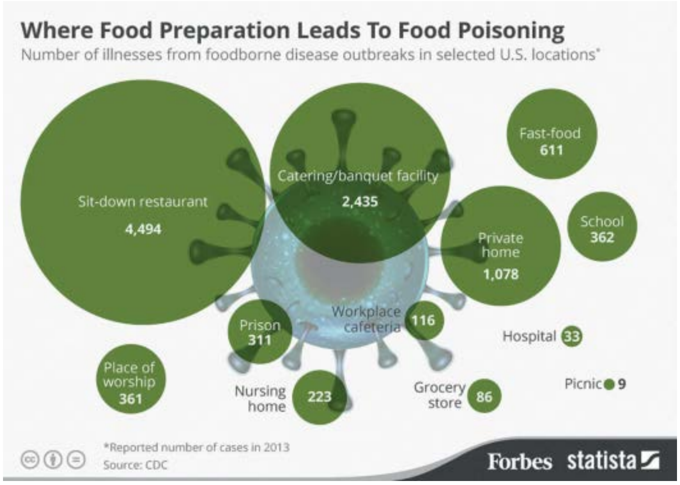
Measurement allows us to prove or disprove our predictions. In science, it allows us to refine our hypothesis and inquire into phenomena to discover new facts, information or laws.

Often we use measurement of data, and in turn our interpretation of the probability to make decisions. In a company, this may be a decision to develop or market a new product based on the probability of success. In a school, it may be a student deciding which subjects to take, based on past data and the probability of future success in those subjects.

Hint

Remember that measurement is only ever approximate. Calculations are exact.

Measurement is only ever approximate; no measurement is fully exact. Calculations, however, can be exact. Measurement is often used to make decisions for the future, based on what happened in the past and the probability that it will happen that way again.



The image above shows the number of reported food poisoning cases in the USA broken down by location for the year 2013. Given the statistics recorded, calculate the probabilities (if you got food poisoning), that it was…

a At a school

b P(prison)

c P(not at a sit-down restaurant)

d P(either at home or on a picnic)

e If the number of reported cases rose in 2014 by 10%, what would you expect the number of cases for each category to change to? Why?

f If they number of cases in one category changed significantly, e.g. the number of cases coming from a single school, what might that mean?

g How would it affect the probabilities at the other locations? Does it make them less or more probable?

What is uncertainty?

IF EVERYBODY LOOKED THE SAME, WE'D GET TIRED OF LOOKING AT EACH OTHER …

Uncertainty makes things compelling. If everything were predictable and probabilities were fixed, life would be less colourful. Sports games are exciting because we don't know the future and what could or will occur in the remaining minutes. Musical performances are exciting because an artiste can bring their own interpretations into pieces, which we didn't see coming, and this expectation of seeing something new and unexpected draws us into the art. If we used the data of past events to calculate the odds for future events only, there would be no point in having the game. You could just declare a winner. How do bookies calculate the odds of a team winning, drawing or losing? Past events only?

WHAT IS THE DIFFERENCE BETWEEN EXPERIMENTAL AND THEORETICAL PROBABILITIES?

Theoretical probability is defined as:

$$\text{P(obtaining our chosen event)} = \frac{\text{number of ways that can occur}}{\text{total possibilities}}$$

This is assuming the event is unbiased.

BUT

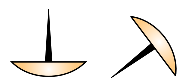
experimental probability is defined as:

$$\text{P(chosen event)} = \frac{\text{the number of times the chosen event occurred}}{\text{how many times the experiment was repeated}}$$

ACTIVITY: Pushpin or thumbtack toss

ATL

- Critical thinking skills: Test generalizations and conclusions



The image above shows two thumbtacks, one which has landed on the back and one which is on its side.

1. **If you threw 10 thumbtacks in the air, what would you predict the theoretical probability would be of the number of tacks to land on the back? What would you predict for not landing on the back, i.e. P(side)?**
2. **Now carry out this event and record the number of actual occurrences of landing on the back. What is the *experimental* probability for P(back)?**
3. **What would you predict the probability of P(back) for 20 thumbtacks landing after being thrown in the air?**
4. **Now carry out this event and record the number of actual occurrences of landing on the back. What is the *experimental* probability for P(back)? Has it changed?**
5. **Repeat for *theoretical* probabilities and *experimental* probabilities for 30 thumbtacks, 50 thumbtacks and 100 thumbtacks.**
6. **What observations can you make about the change in probabilities for increasing population (thumbtack numbers) sizes?**

Assessment opportunities

- In this activity you have practiced skills which can be assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

DOES PROBABILITY CHANGE WITH SAMPLE SIZE?

Why do we sample? Sampling gives us an insight into a population, when we cannot measure the whole thing. We use the sample to draw inferences about the whole group or groups based on what we see in front of us. Making a judgment about the population based on the sample of data available, through measurement, outside the observed samples.

It's like looking at the world through a window with ripples in the glass, up close its hard to see because the distortion from the ripples but if you step back you can see a lot more because the distortion, or blocking effect, is less (just as when the samples are large).

It is important to understand that empirical unbiased (or fair) systems *tend* towards theoretical probabilities when you increase the sample size. To be able to see this, you must be able to compare frequencies you record with those you predicted (that is, compare theoretical with experimental)

PRACTICE QUESTIONS:

■ At the fair

Anne and Luke were at a school fair and noticed that there were some excellent prizes at the raffle. There were 100 prizes, from small to large, and there were only 1200 tickets. They liked their chances and decided to apply mathematical logic to make use of theoretical or expected probabilities.

1 What is the probability that they might win a prize from a single purchase of a ticket?

2 What is the probability that they might win one prize, if they bought two tickets?

3 What is the probability that they might win two prizes, if they bought two tickets?

4 What is the probability that they might win three prizes, if they bought two tickets?

5 **Calculate** how many prizes they might expect to win if they purchased 600 tickets?

6 How many prizes might they expect to win if they bought 300 tickets?

7 If Anne purchased 12 tickets and Luke purchased 10, how much better are Anne's chances than Luke's?

8 They both want to win some prizes, so they decide to buy tickets in multiples of 12. Suggest why they come up with that number particularly.

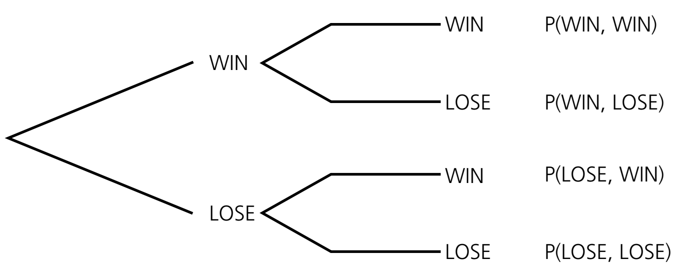
9 They buy three dozen tickets and win four prizes. **Comment** on the difference between the theoretical probability versus the experimental probability.

10 Were they "lucky" or "unlucky"? **Explain** why.

How do we estimate and calculate probabilities?

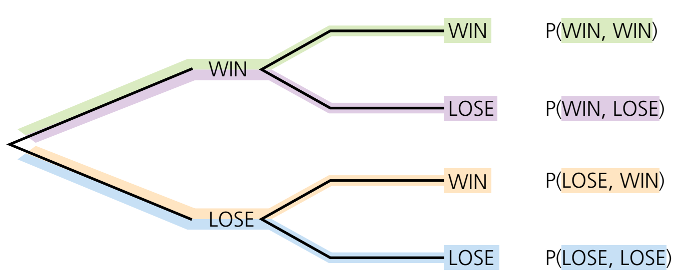
WHAT IS THE PURPOSE OF A PROBABILITY TREE?

A tree diagram is a good way to represent a small number of outcomes, over repeated events. This shows all possible outcomes and each branch shows one possible combination.



■ Tree diagram

The tree diagram above shows the possible outcomes of two games, with a team either winning or losing for both events.



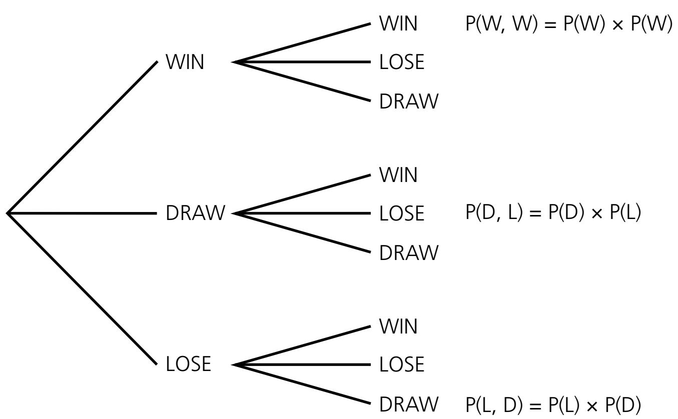
■ Finding probabilities on a tree diagram

You already know that P(A and B) = P(A) × P(B).

This means that we multiply each probability along each branch to find the total probability of that branch. The outcome for all the branches must add to 1.

Why would this be?

We can have more than two probabilities for a single event but that makes the tree more complicated, with more branches, each one of which would need to be multiplied.



■ Many probabilities make tree diagrams complicated

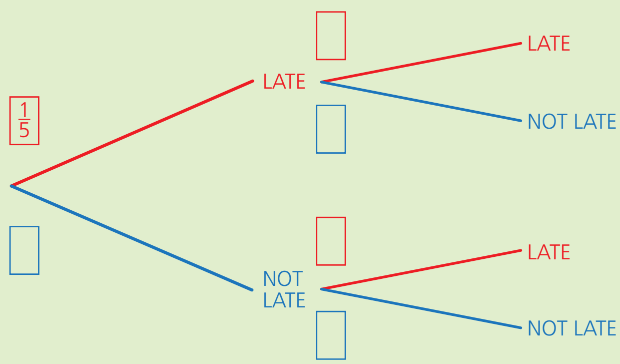
PRACTICE QUESTIONS

Arriving late to school

Tavis has calculated that he is late to school usually once a week. He therefore estimates that his

P(late) = $\frac{1}{5}$ or 0.2 and

P(not late) = $\frac{4}{5}$ or 0.8



1 What is the probability that Tavis is late on two successive days?

2 What is the probability that Tavis is not late on either day?

3 What is the probability that Tavis is late on three successive days?

Now we must add a third event to show the third day at school and all the possible outcomes.

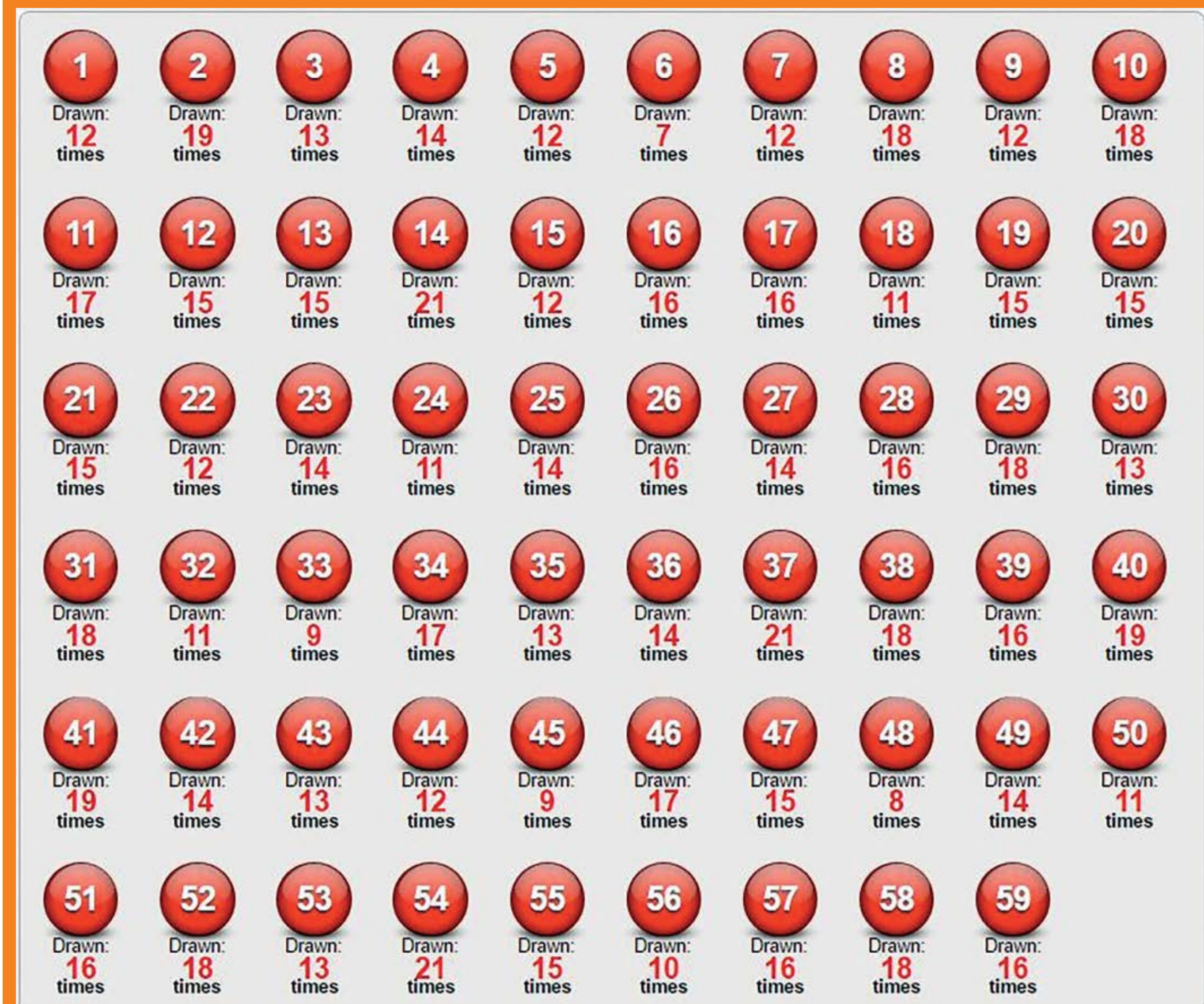
4 A tattoo parlour has recorded an infection rate of 1 in 1000 cases. If two people have an equal probability of becoming infected, **show** all possible outcomes and the probabilities for each.

Links to: Learner profile

Real-world application of Mathematics – read about the mathematician who cracked the code of the scratch card lottery. The most honourable, or principled, part is what he did when he found it out! **www.wired.com/2011/01/ff_lottery/**

Chances of winning the lottery – find the winner animation **http://understandinguncertainty.org/files/animations/WallyLottery/IsotypeZoom.swf**

In this animation, a happy smiley has won the lottery. The other 16, 777 215 sad smileys all lost. Can you find the winner?



■ The frequency of each number in the history of a National Lottery

SEE–THINK–WONDER

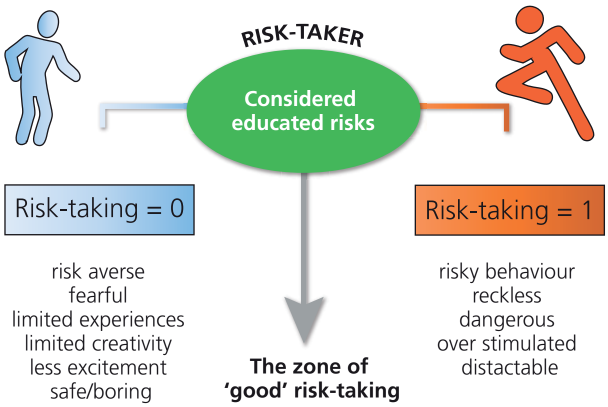
In the standard 6 ball lottery with 49 numbers, the chance of matching all 6 is comparable at 1 in 13 983 816. What does that probability look like in reality? What is the probability of winning the lottery twice? What is the probability of not winning the lottery?

ACTIVITY: Developing your risk-taker profile

ATL

- Reflection skills: Identify strengths and weaknesses of personality

Consider the following diagram which shows represents a spread of personalities from risk-averse (someone who doesn't like to take risks) to highly risky (some who takes risks without considering the consequences) on a spectrum.



It's time to reflect honestly on yourself. Where do you put yourself on this scale or spectrum now? What makes you say that? How does it make you feel?

Where would you like to be, to develop yourself as a person? To the left or to the right? How can you achieve this?

EXTENDED

Take a risk now! Share your honest thoughts with a partner, either a friend or an acquaintance in class. This may be uncomfortable as you are revealing how you think of yourself. They may agree with your analysis or they might have something interesting to share with you, which might challenge how you see yourself. Honest communication with ourselves and others is a good tool for self-development.

EXTENDED

What is a conditional probability?

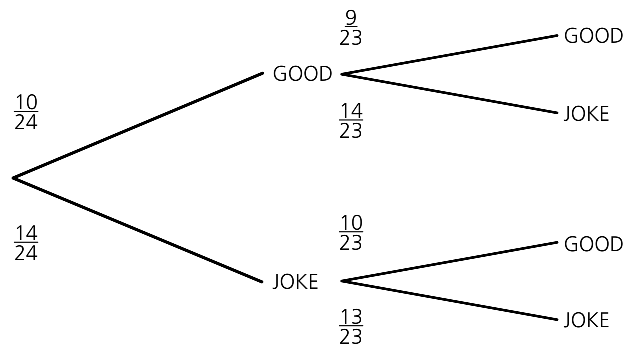
Previously, we used tree diagrams to show multiple events where the outcomes were independent of one another. For example, whether Tavis was late on day one had no effect on his lateness on day two. One person getting a tattoo becoming infected didn't increase or decrease the chances for a second person. (assuming the equipment was clean)

There are other events which can be conditional or dependent, which means that the outcome of the first event has an effect on the subsequent events. Often this dependency comes from the fact that something has not been *replaced* in the original set.

Example

A party game consists of a 'lucky dip' where kids can choose a gift but it may be a good gift or a joke gift. The kids don't give back the gift once they have it, so the total number of gifts decreases and either the good gifts or the joke gifts also decrease. This in turn has an effect on the probabilities for each on the second turn, depending on what was chosen first.

If there were 10 good gifts and 14 joke gifts, the probability tree would look like this:



Notice how the denominator changes because one gift has been taken and **not replaced**. The numerator also changes, but differently for each outcome of event one, depending on which gift was chosen first.

From the example given, **find** the probability of:

a Getting two good gifts.

b Getting two joke gifts.

c Getting one of each (in any order).

ACTIVITY: Can mathematics prove a curse?

ATL

- Critical-thinking skills - Read critically and for comprehension

The Curse of the London Olympics

In the three years following the London Olympics, the French media began to report a curious, and tragic phenomenon – the curse of the London Olympics. Eighteen athletes who had participated had died.

Research task: Find out more about this strange, and sad, phenomenon. How does it compare to the theoretical or empirical probabilities based on actual statistics?

Reflection: Was it really a curse? While tragic, is it so unusual for such a thing to happen? Why do you think French media reported it, while the British media did not?

It is important to remember that probabilities are an estimate. They are subject to variation, just as statistics is (with outliers and occasional freakish results). We know from statistics, sciences and life in general that not all variation is explainable.

EXTENSION: COMBINATIONS VS PERMUTATIONS

ATL

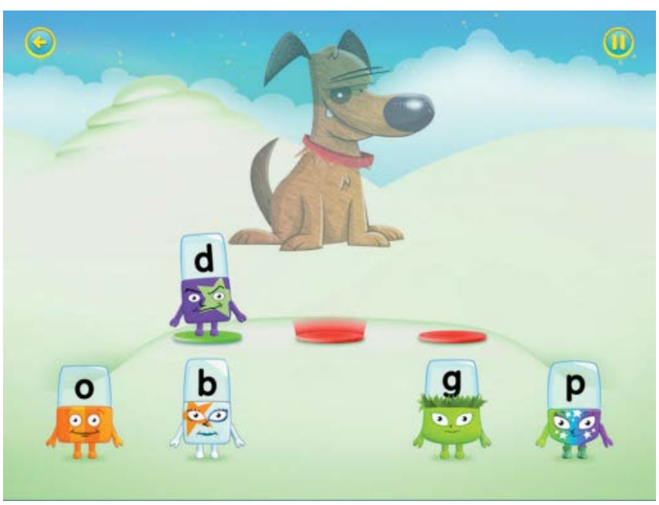
- Critical-thinking skills: Identify trends and forecast possibilities

Alphablocks

A spelling game on a touchscreen tablet

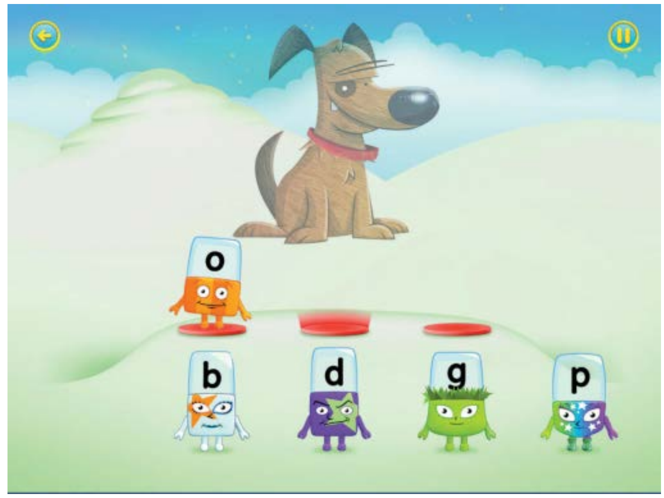
Rangin is babysitting a small child who cannot read but loves playing this spelling game anyway. The aim of the game is drag the correct letters to the correct places to spell a given word.

The baby drags any random letter to a place and the game leaves it there, whether right or wrong, until all three places are filled.



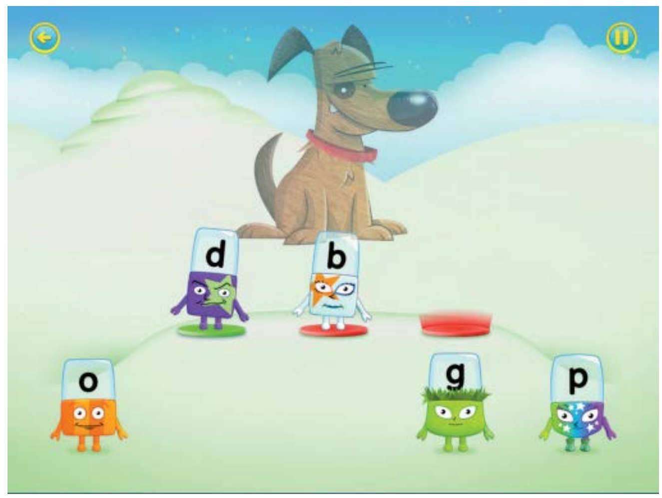
Guessing the first letter correctly

What is the probability that the baby (who has no idea what they are doing) gets the first letter correct?



Guessing the first letter incorrectly

What is the probability that they get it wrong?



Still guessing

Rangin watches her get it wrong over and over again and wonders how likely it is that the baby might get it right, completely by chance.

How could you calculate that probability, where no letters are replaced *and* they have to come in a correct order? Make sure to communicate your working clearly and concisely, using lines of reasoning and logic to come to a final answer.

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion C: Communicating, and Criterion D: Applying mathematics in real-life contexts.

ACTIVITY: Fact-checking

ATL

- Information literacy skills: Use critical-thinking skills to analyse and interpret media communications

Your task is to verify the claims made by a make up company that a pack containing 10 eyeshadows could combine or permutate to give 1001 different ways to wear them. **Using** the box method, or otherwise, **find** out if this number is accurate.

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

Earlier in the chapter, we learned about the process of counting all possible outcomes, the fundamental principle of counting. Let's look at outcomes, when the arrangement does and doesn't matter.

The colours red, blue and green can be arranged in the following ways:

RBG, RGB, BRG, BGR, GRB, RBR

There are six possible arrangements, each of which have the same colours in them but in a different order. To calculate the number of possible arrangements, or permutations, we consider how many choices we have for the first place, then for the second (now that one is gone) and so on …

3	x	2	x	1
One of 3 colours possible		2 choices remaining		Only 1 option left

If we were to make a three-letter code with any letter of the alphabet, used only once, we would have the following number of arrangements:

26 x 25 x 24

= 15,600

To calculate the number of permutations possible for any set of numbers we can use the general rule

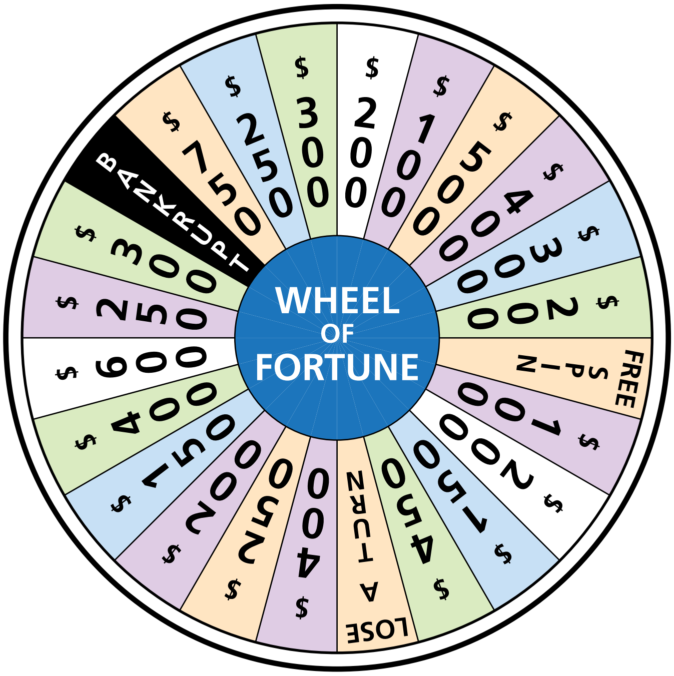
n x n -1 x n - 2 for *n* boxes.

The number of arrangements (or permutations) of n different objects is $n!$, where

$$n! = n\,(n-1)\,(n-2)\ \ldots\ 3 \times 2 \times 1.$$

SOME SUMMATIVE PROBLEMS TO TRY

THIS PROBLEM CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 1–2



1 From this wheel of fortune, **calculate**

a the probability of winning \$200 on one spin

b the probability of winning more than \$500 on one spin

c the probability of going 'bankrupt' on one spin.

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 3–4

2 You are going to be randomly assigned a whole number between 1 and 100.

What is the probability that you get

a an even number?

b an odd number?

c the number 400?

d 6.7?

3 Huw has a 500 Vietnamese dong coin which he flips three times. Instead of heads, the coin shows the denomination 500 and instead of tails the coin shows the state crest on the reverse.

a **Find**

i P(500, 500, crest)

ii P(500, crest, crest)

iii P(crest, crest, crest).

b Huw flips the dong three times and gets the crest every time. He suspects his dong is biased – what would you say to this hypothesis? How could you test it?

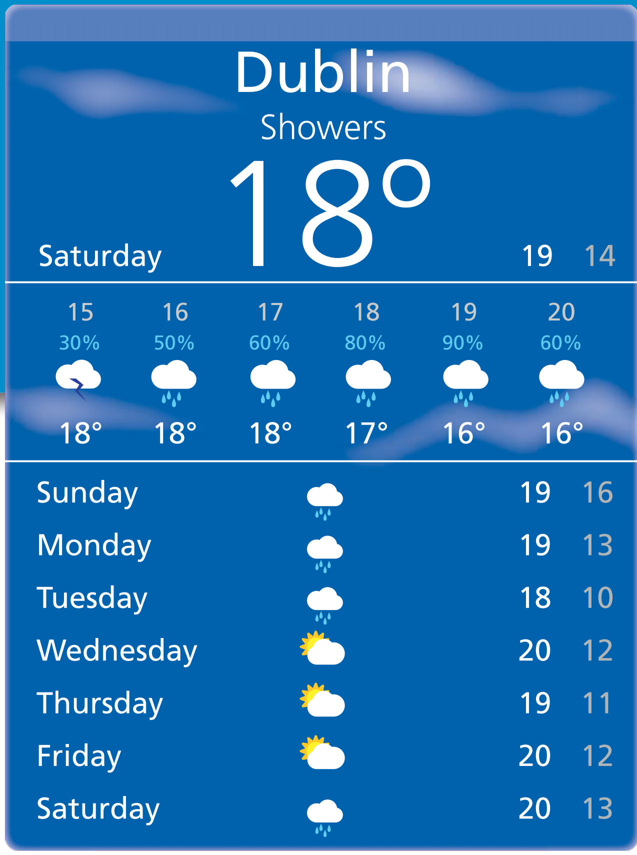
4 In the final stages of a fashion reality TV show, the contestants are assigned at random to a mentor. If there are four contestants – Jose, Francesca, Angelica and Davide and four mentors – Georg, Rain (male), Rainbow (female) and Isaac.

a **Find** the probability that Jose is paired with Georg.

b **Find** the probability that Francesca does not get paired with Rain.

c **Find** the probability that Angelica is not paired with a woman.

d **Find** the probability that all pairings are of the same gender.

5 

- **a** What is P(rain) at 4p.m.?
- **b** What is the probability of rain at 7p.m.?
- **c** What is the probability of it *not* raining at 7p.m.?
- **d** **Write down** all the probabilities shown in *two* other forms.
- **e** **Describe** how the probability of rain changes throughout the day.

6 Bermuda brunch: Fairways Hotel introduces a Sunday special –a free brunch to anyone who rolls three dice and gets three of a kind.

- **a** If you were there one Sunday for brunch, what is the probability you might win a free meal?
- **b** What if you were at a table of ten people, what is the probability of someone at the table winning?
- **c** What is the probability of two people at the table winning?
- **d** What is the probability of all ten people winning a free brunch?

7 A fair die is rolled four times.
What is the probability that …

- **a** exactly one of the rolls is a three?
- **b** at least one of the rolls is a three?
- **c** all four rolls are three?
- **d** none of the rolls are three?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 5–6

8 If two fair dice are rolled, what are the odds that …

- **a** the product of the numbers is odd?
- **b** the difference of the numbers is greater than 3?
- **c** the difference of the numbers is greater than or equal to 3?
- **d** the quotient of the number is an integer?

9 The website immunize.org gives the chances of contracting the disease polio from a live vaccine as 'about once in every 2.4 million doses'.

- **a** **Show** this probability as a percentage, a decimal and as a fraction.
- **b** **Discuss** what this probability tells you about the risk associated with this particular vaccine.
- **c** What are the chances of two people both contracting the disease?

10 You are going to be randomly assigned a number between 1 and 100.
What is the probability that you get

- **a** a square number?
- **b** the sum of two squares?
- **c** the difference of two squares?

11 A licence plate in Northern Ireland begins with three letters, followed by some numbers.
Calculate how many *three-letter combinations* can be made from the alphabet if

- **a** each letter can be reused?
- **b** each letter can be used only once?

THESE PROBLEMS CAN BE USED TO EVALUATE YOUR LEARNING IN CRITERION A TO LEVEL 7–8

12 You are going to be randomly assigned a number between 1 and 100.
What is the probability that you get

- **a** the sum of two cubes?
- **b** a triangular number?
- **c** a hexagonal number?

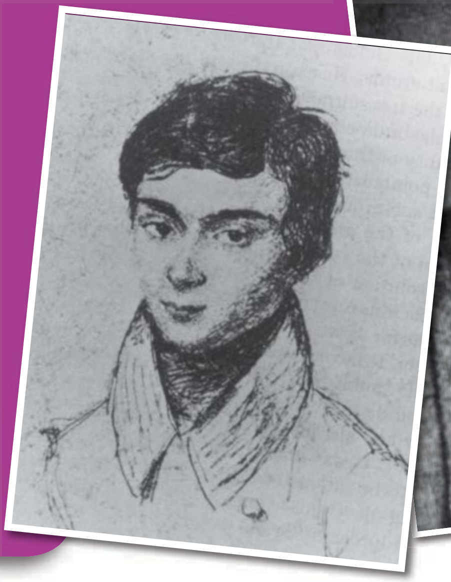
13 There are six runners in a race. How many ways could the top three places be filled?

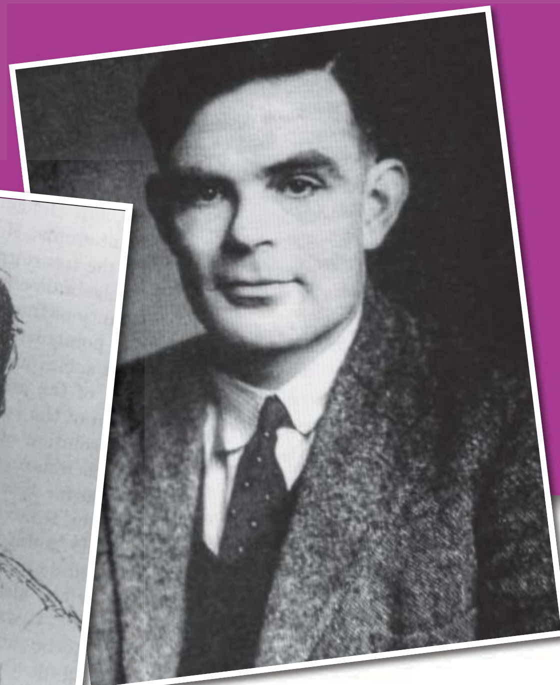
14 **Explain** the mathematical notation $n!$ and how it relates to probability from the material we have studied.

Reflection

<table>
<tr><th colspan="6">Use this table to reflect on your own learning in this chapter.</th></tr>
<tr><td>Questions we asked</td><td>Answers we found</td><td colspan="4">Any further questions now?</td></tr>
<tr><td>Factual: What is probability? How do we estimate and calculate probabilities? What is the difference between experimental and theoretical probabilities? What is a conditional probability? How can probabilities change? What is the purpose of a probability tree? What are systematic listing strategies? What is uncertainty? How do we handle multiple events? How are expected probabilities different from observation? Does probability change with sample size?</td><td></td><td colspan="4"></td></tr>
<tr><td>Conceptual: What is risk? When does ‘unlikely’ become ‘impossible’? If probability is logical, why do we have coincidences? Why is measurement important and how is it linked to probability? When does a risk taker become too risky/unsafe?</td><td></td><td colspan="4"></td></tr>
<tr><td>Debatable: Is gambling a harmless diversion? Is uncertainty a bad or a good thing? Does perception of risk have a personal or cultural aspect?</td><td></td><td colspan="4"></td></tr>
<tr><td rowspan="2">Approaches to Learning you used in this chapter:</td><td rowspan="2">Description – what new skills did you learn?</td><td colspan="4">How well did you master the skills?</td></tr>
<tr><td>Novice</td><td>Learner</td><td>Practitioner</td><td>Expert</td></tr>
<tr><td>Communication skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Reflection skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Information literacy skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Critical-thinking skills</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Learner profile attribute(s)</td><td colspan="5">How did you demonstrate your skills as a balanced person and as a risk-taker in this chapter?</td></tr>
<tr><td>Risk-taker</td><td colspan="5"></td></tr>
<tr><td>Balanced</td><td colspan="5"></td></tr>
</table>

12 Am I ready?





Your future **relationship** with mathematics will be determined by your understanding of both **traditional and innovative systems**.

CONSIDER THESE QUESTIONS:

Factual: What have you learned so far? What else do you need to know? How is the knowledge you have learned connected to the aims of MYP Mathematics? How will it prepare you for Diploma Mathematics?

Conceptual: How can relationships (like trigonometric functions) transform? How has your own relationship with mathematics transformed?

Debatable: To what extent will you be able to apply all of the mathematics you've learned in future learning? Can you say you have mastered these topics?

Now **share and compare** your thoughts and ideas with your partner, or with the whole class.

IN THIS CHAPTER, WE WILL …

- **Find out** how much you've learned and where you need to revise or refresh.
- **Explore** the most appropriate course in Diploma Programme Mathematics for you.
- **Take action** by considering ethical, moral and social implications of mathematics.
- **Practise** 'slow judgment' rather than 'no judgment' as a good preparation for Theory of Knowledge (TOK).

These Approaches to Learning (ATL) skills will be useful …

- Self-management skills (affective)
- Critical-thinking skills
- Transfer skills

We will reflect on this learner profile attribute …

- Knowledgeable – using and developing conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

SEARCH

The pictures here show some famous mathematicians. What do you think makes a good mathematician? Do they all share the same qualities?

Research the work of other mathematicians to find out more. You might want to look at **Évariste Galois**, a young French mathematician. You might also find the film **X + Y** interesting as it follows a young mathematician at the International Mathematics Olympiad.

DISCUSS

Reflect on the aims of MYP Mathematics – have you achieved those aims?

Think back to when you started MYP Mathematics or started using this book. What was your opinion of Mathematics. Has your opinion changed?

Are you ready to continue your learning beyond MYP?

Assessment opportunities in this chapter:

- **Criterion A:** Knowing and understanding
- **Criterion B:** Investigating patterns
- **Criterion C:** Communicating
- **Criterion D:** Applying mathematics in real-life contexts

How is the knowledge you have learned connected to the aims of MYP Mathematics?

1. ENJOY MATHEMATICS, DEVELOP CURIOSITY, BEGIN TO APPRECIATE ITS ELEGANCE AND POWER

ACTIVITY: Mathematics and football

In 2016, before a crucial game against Spain, the Romanian team changed their usual player numbers to mathematics questions on the backs of their shirts.

The article below explains why the Romanian football organisation decided to try this to raise awareness and interest in mathematics.

Task scenario: Imagine that Liverpool Football Club thinks this is a great idea and want to explore the possibility of doing something similar for a friendly tournament. They would like their problems to all be *simultaneous equations questions*, one of which would be displayed on the front of the jersey and the other equation on the back.

www.theguardian.com/football/2016/mar/27/romania-players-change-squad-numbers-for-maths-equations-before-spain-game

Your task is to write three examples of pairs of simultaneous equations questions, which when solved would give the player number (x) and age (y) of the following players:

Name	Player number	Age
Daniel Sturridge	15	27
Nathaniel Clyne	2	25
Phillippe Coutinho	10	24

Make sure to include the working out to show them how the problem is solved too!

EXTENSION

To really demonstrate your knowledge, try and include at least one question with a simultaneous linear and non-linear (quadratic) function.

◆ Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion A: Knowing and understanding, and Criterion C: Communicating.

This activity revises material from Chapter 3: Linear and simultaneous equations.

2. DEVELOP AN UNDERSTANDING OF THE PRINCIPLES AND NATURE OF MATHEMATICS

ACTIVITY: Trigonometry

This activity revises material from Chapter 9: trigonometric functions, including $y = \sin x$, $y = \cos x$ and $y = \tan x$. You should be familiar with the **domain** and **range** for each. We also know from other functions that it is possible to change the shape or location of a function by **transformation**.

Plot each of these trigonometric functions, in turn.

Hint

Make sure to have your window settings correct to see from -360 $\leq x \leq$ 360. Sketch what you can see on the screen. Identify the range and domain for each function.

Plot the following functions:

$y = \sin x$ $\quad$ $y = \sin(x + 30°)$

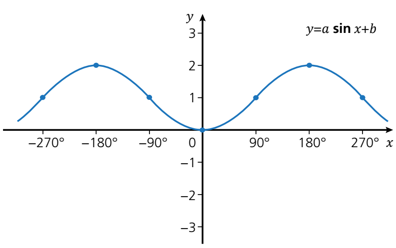
$y = 3 \sin x$ $\quad$ $y = \sin x + 4$

Describe, using correct mathematical notation and vocabulary, how each function has been transformed from:

a **The original function of $y = \sin x$**

b **The function immediately preceding it**

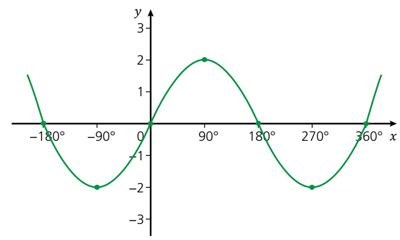
Activity questions:



1 **True or false?**

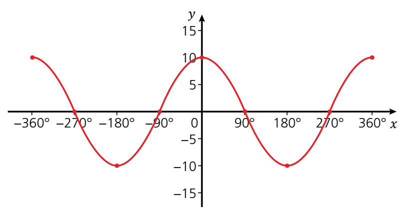
- **This is a trigonometric function.**
- **The range is $0 \leq y \leq 1$**
- **The range is $0 \leq y \leq 2$**
- **The 'wave' has moved down 1 on y - axis.**
- **The function has shifted left.**
- **The function has moved to the right 90°**

2



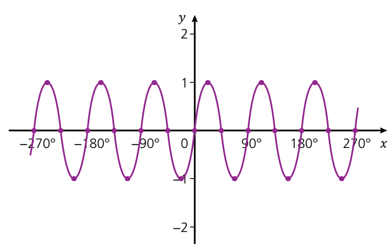
a **Describe the transformation from $\sin x$ to this graph.**

b **Hence, or otherwise, give the correct function $f(x) = ?$**



3 **How has the function of $f(x) = \cos x$ been transformed into this function? State the new function.**

4 **The following image shows a transformed sin function.**



a **What feature of the function has changed?**

b **What should it be for $\sin x$?**

c **Have any other transformations occurred?**

d **How can you tell?**

5 **'Invariance can be achieved by combinations of transformations'.**
What does this phrase mean? What additional transformations could you carry out?

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion A: Knowing and understanding and Criterion C: Communicating

3. COMMUNICATE CLEARLY AND CONFIDENTLY IN A VARIETY OF CONTEXTS

ACTIVITY: Geek chic

Mathematics can be elegant and powerful and often tells you something very important when used appropriately. It can also be silly and amusing, when ironically applied. The t-shirts above are examples of 'geek chic' for a small but growing market for funny and clever uses of algebra.

Create an original or personal design for a clothing item. You can use expansion, factorisation, solving, powers/exponents to modify the letters. You may use lines of reasoning to communicate the message in a subtle way.

Not feeling inspired? Why not check out this store for more ideas:

www.scienceteecher.com/home.php

www.presentindicative.com/collections/mathematics

◆ Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

This activity revises material from Chapter 2: Algebra.

4. DEVELOP LOGICAL, CRITICAL AND CREATIVE THINKING AND TO DEVELOP CONFIDENCE, PERSEVERANCE, AND INDEPENDENCE IN MATHEMATICAL THINKING AND PROBLEM-SOLVING

ACTIVITY: Years of life lost

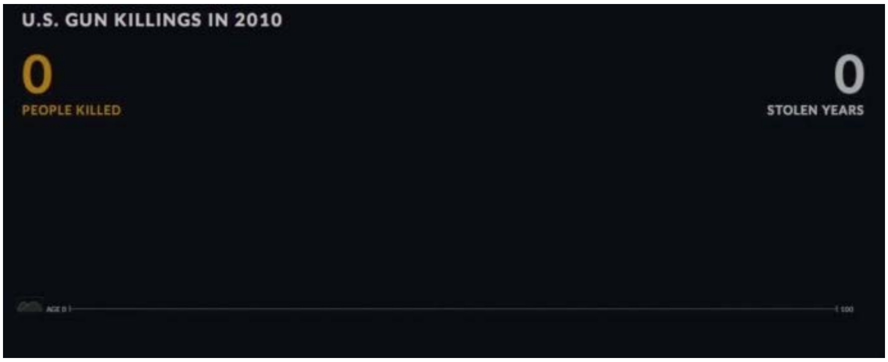
The following screenshots come from a video which analyses the age of gun murder victims and uses mathematical modelling to calculate the number of 'years stolen', i.e. years the victim may have lived longer had they not been shot.

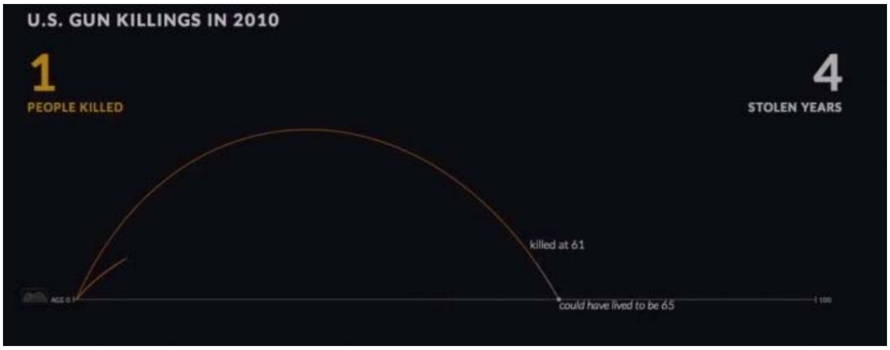
The video has been designed to make you think, and to make you feel. Your task is to be as objective and to critique the mathematics contained in it, not the sentiment or political outlook.

The first person's life to be graphed is a victim who was killed at 61 and they estimate he might have lived to be 65.

What shape or function does the animated life remind you of? Why do you think the function of a person's life has been drawn in this way?

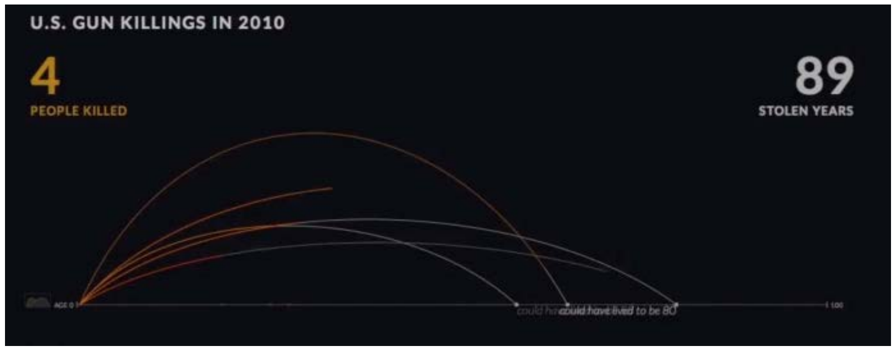
How does seeing the information shown in this way make you feel?





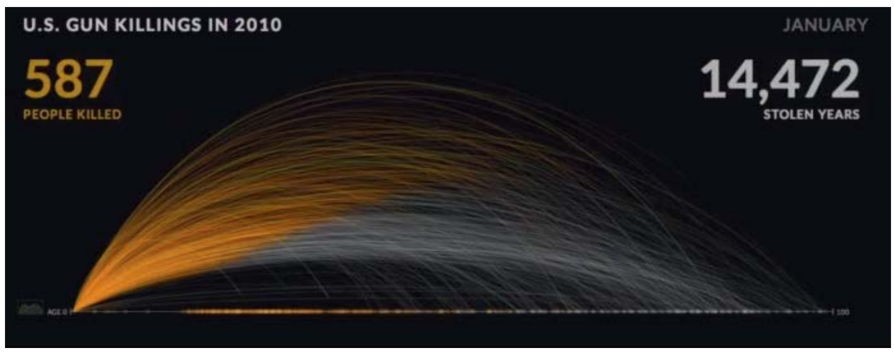
A few seconds later, we can see figures on the first four victims of 2010. What questions do you have about the shapes of the models? Could there be a mathematical reason or relevance for drawing them in this way?

What do the x- intercepts tell you?



What observations can you make about the predicted life span? How could these predicted ages of death be verified?

What kinds of questions could you ask about the claims made in this video? What data or calculations would you ask to see?



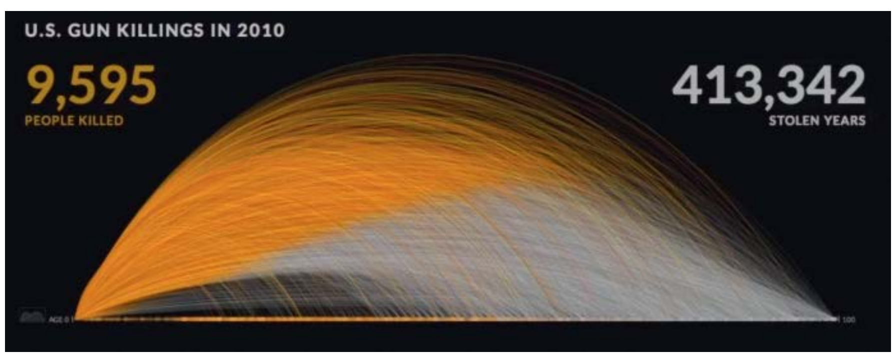
On reflection, what in this short video did you find

- **a** **Impressive?**
- **b** **Dubious?**
- **c** **Powerful?**

What is your own opinion on the matter?

What role does mathematics have in this real-life context?

This activity revises material from Chapter 8: Quadratic functions.



Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion D: Applying mathematics in real-life contexts

5. DEVELOP POWERS OF GENERALIZATION AND ABSTRACTION

ACTIVITY: The sieve of Eratosthenes

Inquiry questions:

- **How many primes are there below 20?**
- **How many primes are there below 40?**
- **How many primes are there below 60?**
- **Describe this pattern as a relationship or general rule.**
- **Verify this relationship or general rule.**

Here is one possible method to visualise numbers in order to see patterns: Construct a 100 square to show all the possible primes below 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Shade in all the multiples of the first prime number 2 to eliminate these numbers (as they cannot be primes if they are divisible by 2).

Continue with this technique to see a pattern emerging.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Now you can use these remainder numbers to count the number of primes below 20, 40 and 60. It also allows you to see the changes in how many primes are in each row. This information should inspire you to describe these patterns and allow you to verify if this works for higher rows.

Extend this information for much larger number squares. What do you predict the pattern will look like in the larger squares (e.g. 20 × 20 or 30 × 30 boxes)? What is the largest prime known to us?

Will this general rule or technique work for triangular or hexagonal numbers?

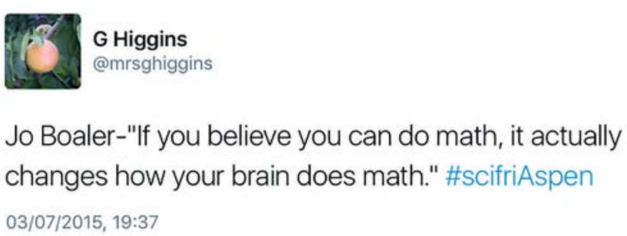
This activity revises material from Chapter 1: Numbers and number sets.

◆ Assessment opportunities

- ◆ In this activity you have practiced skills that can be assessed using Criterion B: Investigating patterns

6. APPLY AND TRANSFER SKILLS TO A WIDE RANGE OF REAL-LIFE SITUATIONS, OTHER AREAS OF KNOWLEDGE AND FUTURE DEVELOPMENTS

ACTIVITY: How many tweets?



The above image is a 'tweet' which is one form of social media communication on a platform called Twitter. A tweet has a current limit of 140 characters and will not be published or 'tweeted' if it is longer than this. A user can share their thoughts or information up to 1000 times a day.

You have been asked by a social media developer to **determine** what is the **maximum** number of possible tweets if:

a **You can use each symbol or character (like letters, numbers, etc.) only once.**

b **You can use each symbol or character repeatedly.**

You have been given no other additional information.

You must remember to:

- **apply the selected mathematical strategies successfully to reach a solution**
- **explain the degree of accuracy of a solution**
- **describe whether a solution makes sense in the context of the authentic real-life situation.**

◆ Assessment opportunities

- ◆ In this activity you have practiced skills that can be assessed using Criterion D: Applying Mathematics in real-life contexts.

This activity revises material from Chapter 11: Probability

7. APPRECIATE HOW DEVELOPMENTS IN TECHNOLOGY AND MATHEMATICS HAVE INFLUENCED EACH OTHER

ACTIVITY: From polygons to parabolas

How is the movie industry influencing advances in mathematical modelling?

Read the following article on www.theverge.com/2013/3/7/4074956/pixar-senior-scientist-derose-explains-how-math-makes-movies-games

In this article Pixar's senior scientist, Tony deRose explains how mathematics is used, and developed, to make popular movies and games.

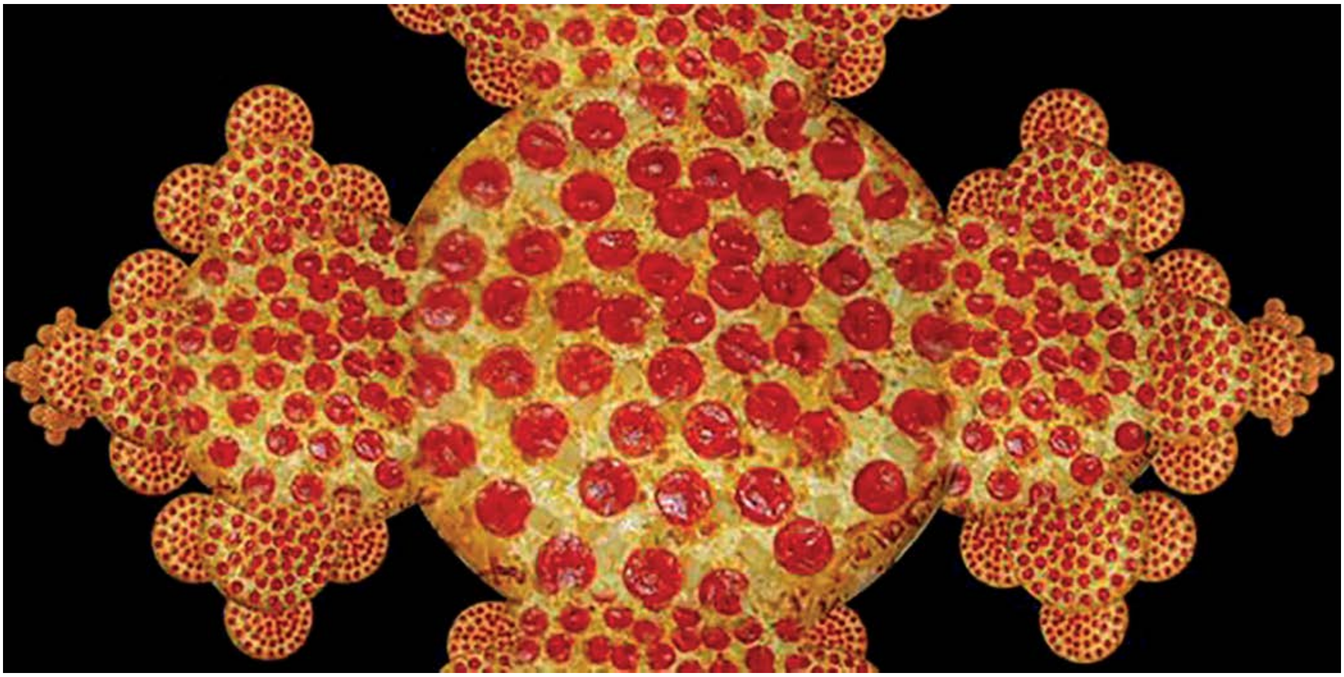
Research questions:

- **How is mathematics used in Pixar (and other) animated movies?**
- **What are fractals?**
- **What role do *they* play in animated movies?**
- **Where do they appear in the movie 'Frozen'?**

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion C: Communicating and Criterion D: Applying mathematics in real life contexts

This activity revises material from Chapter 5: Plane geometry



What is fractal pizza?

8. APPRECIATE THE MORAL, SOCIAL AND ETHICAL IMPLICATIONS ARISING FROM THE WORK OF MATHEMATICIANS AND THE APPLICATIONS OF MATHEMATICS

ACTIVITY: Using mathematics to reveal identities or locations

Case 1: Identifying Banksy

A popular graffiti artist, Banksy has very carefully preserved his identity from the art world. This had led to a huge amount of speculation as to who the artist could be and how they could be found.

Read the following articles about the use of mathematics to try to identify the artist:

www.cbsnews.com/news/scientists-use-math-to-hunt-for-identity-of-banksy/

9. APPRECIATE THE INTERNATIONAL DIMENSION IN MATHEMATICS THROUGH AN AWARENESS OF THE UNIVERSALITY OF MATHEMATICS AND ITS MULTICULTURAL AND HISTORICAL PERSPECTIVES

ACTIVITY: Billion Euro-o-gram

'For that kind of money, we could build eight hundred new schools or fifty stadiums!'

It can be difficult to imagine extremely large numbers in real terms. Search online for the **Billion Euro-o-gram** to see how one billion euros can be spent. A useful mathematical trick can be to calculate equivalent amounts by division or multiplication.

What is the most expensive *thing* ever? Your task is to **research** **the top five most expensive items, constructions or discoveries EVER**. Is it the Grand Mosque in Mecca? The International Space Station? Artificial islands? Nuclear power plants? The pyramids of Giza?

www.theguardian.com/artanddesign/2016/mar/05/banksy-unmasked-scientists-use-maths-and-criminology-to-map-artists-identity

How was mathematics used to reveal patterns in this scenario? How certain are the researchers that they have correctly identified the artist? What moral, social or ethical implications does this kind of work have? What implications does it have for privacy? For solving crimes?

The research indicates alternate uses for this type of mathematical investigation. Does this change your opinion on its potential use? How could it affect you, in your life?

Case 2: The missing flight

Read the following article:

http://fivethirtyeight.com/features/how-statisticians-could-help-find-flight-370/

This is a much longer, and more complex, article. You must first summarize the important points and state how *Probability* has been used in this case.

Are there moral, social and ethical implications of this kind of mathematical investigation? Are they the same or different to the first case?

How do you think this technique could be used in other real-world contexts? Would you allow, encourage or ban this? Why?

Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion D: Applying mathematics in real-life contexts

This activity revises material from Chapter 6: Data collection and Chapter 11: Probability

Next you must **calculate** these enormous amounts in terms of other, smaller units, to illustrate **equivalence**. Use this information to represent it as a data visualisation (infographic or pictogram, or similar).
Visit **informationisbeautiful.net** for interesting ways to visualize data.

Links to: History

How could we calculate the cost of building the pyramids of Giza? Could we ever estimate the cost of Stonehenge? Or other ancient wonders of the world? How is it possible to adjust historical works into modern prices?

Assessment opportunities

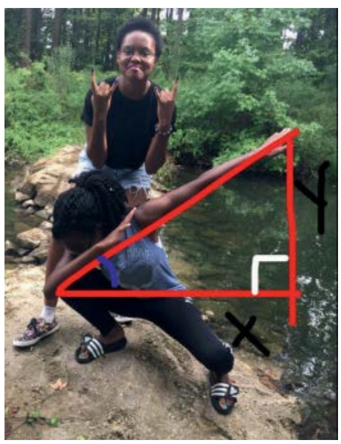
- In this activity you have practised skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

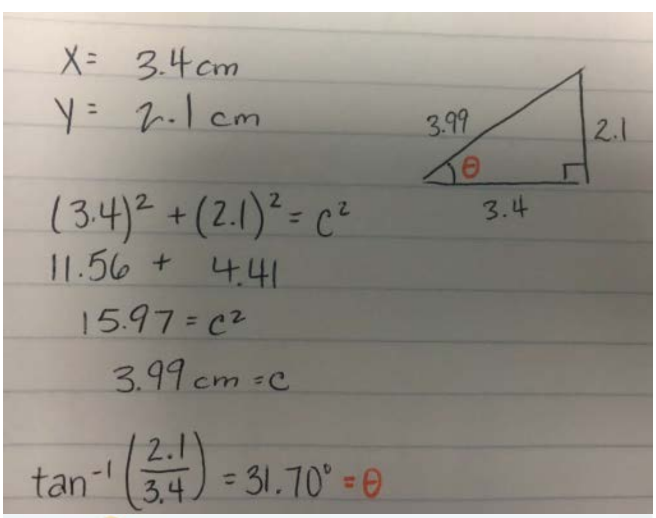
This activity revises material from Chapter 6: Data collection.

10. APPRECIATE THE CONTRIBUTION OF MATHEMATICS TO OTHER AREAS OF KNOWLEDGE

ACTIVITY: Mathematics of 'The Dab'

A recent internet phenomenon involved sports and music stars performing a particular dance move called 'The Dab'. As with most viral phenomena, it developed over time and people put their own personal creativity into their own expressions.





Anicca @13adh13 · 2 Sep 2016
I calculated the angle of my dab.
How's your Friday going? pic.twitter.com/GRfCB3fQyZ

Read more about reaction to Anicca's tweet here:
https://twitter.com/i/moments/772126982675042305

Discuss, with a partner or as a class, the various reactions and feedback Annica – @13adh13 received for her calculations. Notice the number of likes and retweets. What do you think of her 'communication'? What about her 'applying mathematics to real-life contexts'? How accurate do you think her answer might be?

This activity revises material from Chapter 4: Trigonometric ratios.

Assessment opportunities

- In this activity you have practised skills that can be assessed using Criterion D: Applying mathematics in real-life contexts.

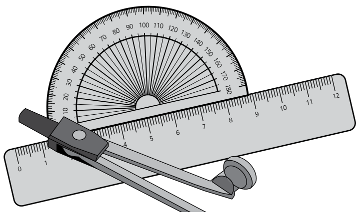
Anicca @13adh13 Follow

To everyone in my mentions criticizing the superfluous calculation of c:

I DID THIS FOR F U N 🙃 Technically NONE of it was necessary.

11. DEVELOP THE KNOWLEDGE, SKILLS AND ATTITUDES NECESSARY TO PURSUE FURTHER STUDIES IN MATHEMATICS

ACTIVITY: A beginner's guide to careful construction



For several years now, you have been using construction tools such as a compass, a protractor and rulers to draw and measure shapes. Students often don't realise what the tools are each for and may use them incorrectly, or not at all.

Your task for this activity is to take a basic geometry set and write a beginner's guide to how to use each item correctly. You may have to do some research if you haven't used them much recently.

Make sure that your guide shows how to:

- **Bisect angles**
- **Bisect lines**
- **Circumscribe a polygon**
- **Find and measure a circumradius**
- **Draw parallel and perpendicular lines correctly**

Your guide could be an infographic, a flipboard, a poster or any other format.

◆ Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion A: Knowing and understanding and Criterion C: Communicating

This activity revises material from Chapter 5: Plane geometry.

12. DEVELOP THE ABILITY TO REFLECT CRITICALLY UPON THEIR OWN WORK AND THE WORK OF OTHERS

ACTIVITY: Word cloud

Create a word cloud of mathematical topics and subjects, concepts and contexts you have learned and used in MYP Mathematics. Make sure to have the ones you enjoyed in larger font and the ones you didn't love so much in smaller font.

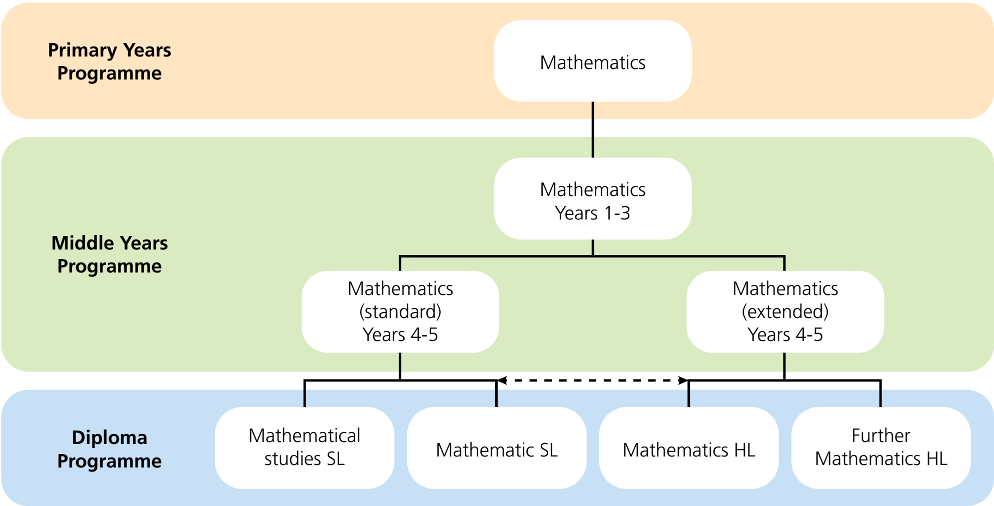
This activity revises material from all chapters 1–12.

◆ Assessment opportunities

- In this activity you have practiced skills that can be assessed using Criterion C: Communicating.

How will it prepare you for Diploma Mathematics?

The aims of all mathematics courses in Diploma Programme group 5 are to enable students to:	The aims of MYP mathematics are to encourage and enable students to:
1 **enjoy mathematics**, and **develop** an **appreciation of the elegance and power** of mathematics	• **enjoy mathematics**, **develop** curiosity and begin to **appreciate its elegance and power**
2 **develop an understanding of the principles and nature of mathematics**	• **develop an understanding of the principles and nature of mathematics**
3 **communicate clearly and confidently in a variety of contexts**	• **communicate clearly and confidently in a variety of contexts**
4 **develop logical, critical and creative thinking**, and patience and **persistence in problem-solving**	• **develop logical, critical and creative thinking** • **develop confidence, perseverance, and independence in mathematical thinking and problem-solving**
5 employ and refine their **powers of abstraction and generalization**	• develop powers of **generalization** and **abstraction**
6 **apply and transfer skills** to alternative **situations, to other areas of knowledge and to future developments**	• **apply and transfer skills** to a wide range of **real-life situations**, other areas of **knowledge and future developments**
7 **appreciate how developments in technology and mathematics have influenced each other**	• appreciate how developments in technology and mathematics have influenced each other
8 **appreciate the moral, social and ethical implications arising from the work of mathematicians and the applications of mathematics**	• **appreciate the moral, social and ethical implications arising from the work of mathematicians and the applications of mathematics**
9 **appreciate the international dimension in mathematics through an awareness of the universality of mathematics and its multicultural and historical perspectives**	• **appreciate the international dimension in mathematics through an awareness of the universality of mathematics and its multicultural and historical perspectives**
10 **appreciate the contribution of mathematics to other** disciplines, and as a particular '**area of knowledge**' in the TOK course.	• **appreciate the contribution of mathematics to other areas of knowledge**
	• develop the knowledge, skills and attitudes necessary to pursue further studies in mathematics
	• develop the ability to reflect critically upon their own work and the work of others.



Glossary

arbitrary values Values chosen for substitution chosen by judgment and not following any specific pattern or rule

arithmetic sequences Sequences which have a common difference, either positive or negative

bimodal data A set of data which has two modes, or most common numbers

bounded Enclosed or defined

cardinal number How many of something there are

coefficient A number used to multiply, or is indicated in front of, a variable

composite function A function which is applied after the initial application of another functions, one function after another

contradiction A type of proof which established the truth of a statement by assuming the opposite proposition is true and showing that this leads to an impossibility or contradiction

Cosine The ratio, or relationship, between the adjacent and the hypotenuse expressed as a quotient

directly proportional As one variable increases, the other variable also increases; the quotient of the variables gives a constant

domain The set of x-values which are inputs to a function, the values of x which exist for a function

equidistant Equal distance apart from two or more points or objects

exact form The most correct form of a number, often in a surd or radical form, without rounding or estimation

expansion To multiply to remove brackets from an expression

extend To make something longer

extrapolate To extend the range of values and assume certain behaviour outside the dataset, given recent or observed trends

geometric sequences Sequences which have a common ratio

gradient The rate of change, or incline, of a line

graph A visual representation of a relationship

higher orders of powers Increasing the value of a power or exponent

highest common factor Also known as greatest common factor is the highest factor that two numbers share with one another

immeasurably Without measure

indirectly As one variable increases, the other variable decreases; the product of the variables gives a constant

interpolate To use the values within a dataset to judge or find intermediate, or previously unknown, values

interquartile range The value of the difference between the upper quartile and the lower quartile in a set of data

limitations Limiting rules or usefulness, restrictions

line of best fit A line of a graph which best fits or represents the overall trend of the data

line segment The a part of a line which has a beginning and an end point

linear equations An equation which makes a straight line when represented graphically, an equation where the highest power in any term is the order of 1

logarithms A number to which the base must be raised to give the answer

magnitude Size or value of length, when referring to a vector

mensuration Measurement

mid-interval value The midpoint in a range of data, often used in grouped frequency data

network A group or system of interconnected things, including people

non-terminating decimal A decimal number which continues to infinity

optimize To find the best, or optimal, solution for a given situation

origin The point at which the x and y-axes intersect, also (0,0)

parameters A constant or variable term in a function that determines the specific form of the function but not its general nature

paths A route between any two points on a network

proof by exhaustion Proving by testing a proposition for all possible (exhaustive) cases

proof by induction A proof in which every step must be justified or explained by the use of an axiom

proofs A deductive or reasoned argument for a mathematical statement

quadratic expressions Expressions where the highest power in any given term is to the power of 2

quotient A term which involves or includes a division

range The value which is given by subtracting the lowest value from the highest value in a dataset, the set of values which exist as outputs or for y on a function

regression A process or measure of the relation between variables

SI unit A unit which follows the *Système International* for naming and defining units

similar Objects which have the same shape but their sizes may be different

simultaneous equations A set of equations which have a common solution

sine The ratio, or relationship, between the opposite and the hypotenuse expressed as a quotient

skew The measure of asymmetry of probability or statistics around the mean

slope Gradient

standard form Scientific notation, where a number is written as a value between one and ten followed by ten to the appropriate power

subtended Standing on.

superimpose To place something on top of another object

surd form A number with a root or radical form

tangent A line which touches but does not cross within a circle, also the ratio, or relationship, between the opposite and the adjacent expressed as a quotient

translations Isometric movement from one location to another

transpose To exchange, to switch or to move terms algebraically between sides of an equation

trigonometry The study of triangle measurement

x-intercept Where a line or function crosses the x-axis, i.e. where $y = 0$

y-intercept Where a line or function crosses the y-axis, i.e. where $x = 0$

Acknowledgements

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