ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-1

Linear Algebra

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Topics to be Covered

- DEFINITION OF MATRIX
- TYPES OF MATRICES
- PRODUCT OF MATRIX BY A SCALAR (OR CONSTANT)
- ADDITION AND SUBTRACTION OF MATRICES
- MULTIPLICATION OF MATRICES
- MINORS OF MATRIX
- COFACTORS OF MATRIX

Syllabus (13-15 Marks in GATE)

- *☐ Linear Algebra
- *☐ Calculus
- *☐ Probability
- ☐ Differential Equations (P.D.E.)
- ☐ Complex Variables
- ☐ Laplace Transform
- ☐ Fourier Series
- ☐ Numerical Methods

DEFINITION OF MATRIX

A set of m n objects or numbers (real or complex) arranged in a rectangular array of m rows and n columns, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m_1} & a_{m_2} & \cdots & a_{mn} \end{bmatrix}$$

$i < j$

$A = [a_{ij}]_{m \times n}$

$1 \le i \le m$

$1 \le j \le n$

is called matrix of order m×n.

$i > j$

Q. Form the matrix such that

$$A = [a_{ij}] \quad \text{where} \quad a_{ij} = i.j$$

$\boxed{1 \le i}$ $\quad$ $\boxed{j \le n}$

$\boxed{n \le 3}$

Soln :- For $1 \le i \quad ; \; j \le 3$

$$A = \begin{bmatrix} 1\times1 & 1\times2 & 1\times3 \\ 2\times1 & 2\times2 & 2\times3 \\ 3\times1 & 3\times2 & 3\times3 \\ \vdots & & \end{bmatrix}$$

$$A = \begin{bmatrix} 1\times1 & 1\times2 \\ 2\times1 & 2\times2 \\ 3\times1 & 3\times2 \\ \vdots & \end{bmatrix}$$

$$A = \begin{bmatrix} 1\times1 \\ 2\times1 \\ 3\times1 \\ \vdots \end{bmatrix}$$

TYPES OF MATRICES

1. Row Matrix : A matrix having one row and any number of columns is called a row matrix, or a row vector, e.g., $[a_{11}, a_{12}, \ldots a_{1n}]$ or $[a_1, a_1, \ldots a_n]$ is a row matrix of order n or matrix of order $1 \times n$

$$\begin{bmatrix} 1 & 5 & -1 & 2 \end{bmatrix}_{1\times 4}$$

2. **Column Matrix :** A matrix having one column and any number of rows is called a column matrix or a column vector.

e.g., $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ *or* $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ is a column matrix of order m or matrix of order m × 1

$\begin{bmatrix} 1 \\ 5 \\ 2 \\ 3 \end{bmatrix}_{4\times 1}$

3. **Null Matrix of Zero Matrix:** Any matrix in which all the elements are zero is called a zero matrix or Null Matrix i.e.

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A + O = O + A = A$

$O \rightarrow$ Additive identity

TRANSPOSE OF MATRIX:-

If $A = a_{ij}$
then $A^T = a_{ji}$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 6 \\ 5 & 4 & 2 \end{bmatrix}$$

$$Tr(A) = 3$$

$$A^T = A' = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 4 \\ 3 & 6 & 2 \end{bmatrix}$$

$$Tr(A^T) = 3$$

Transpose of any matrix is possible.

4. **Square Matrix :** A matrix in which the number of rows is equal to the number of columns is called a square matrix i.e. $A = (a_{ij})_{m \times n}$ is a square matrix if and only if m = n. A matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$ is a square matrix of order 3.

The elements a_{11}, a_{22}, a_{33} of the above square matrix are called its diagonal elements and the diagonal containing these elements is called the principal diagonal or leading diagonal or main diagonal.

Square matrix $\rightarrow 1\times1,\ 2\times2,\ 3\times3$

$$A = \begin{bmatrix} 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & -1 \\ 7 & 5 & 6 \\ 1 & 9 & 0 \end{bmatrix}$$

$a_{ij}\ ;\quad i = j$ [Principal diagonal elements]

$$Tr(A) = 1$$
$$Tr(B) = 1 + 4 = 5$$
$$Tr(C) = 1 + 5 + 0 = 6$$

Trace of Matrix: The sum of the diagonal elements of a square matrix is called trace of the matrix.

Properties of trace :-

i) $Tr(A+B) = Tr(A) + Tr(B)$

ii) $Tr(A) = Tr(A^T)$

iii) $Tr[(A+B)^T] = Tr(A^T) + Tr(B^T)$

5. **Diagonal Matrix:** A square matrix is called diagonal matrix if all its non diagonal elements are zero i.e. in general a matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$;

$a_{ij} \neq 0 \quad i = j$

$a_{ij} = 0 \quad i \neq j$

For example,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ is a diagonal matrix of order 3.

6. **Scalar Matrix :** If all the elements of a diagonal matrix of order n are equal, i.e., if $a_{ij} = k \ \forall \ i$, then the matrix is called a scalar matrix , i.e.,

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ is a scalar matrix of order 3'.}$$

$$A = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = 5I$$

7. **Unit or Identify Matrix :** A square matrix is called a unit matrix or identity matrix if all the diagonal elements are unity and non-diagonal element are zero. e.g.

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are identity matrices of order 3 × 3 and 2 × 2 respectively.

Square → Diagonal → Scalar → Unit/Identity

$AI = A$ or $IA = A$

$I \rightarrow$ Multiplicative identity.

$$AI = IA = A$$

8. **Upper triangular Matrix :** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ whenever $i > j$. Thus, in an upper triangular matrix all the elements below the principal diagonal are zero. For example.

$i < j$; $a_{ij} \neq 0$
$i > j$; $a_{ij} = 0$

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

are 4×4 and 3×3 upper triangular matrices respectively.

9. **Lower Triangular Matrix :** A square matrix of $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ whenever $i < j$. Thus in a lower triangular matrix all the elements above the principal diagonal are zero. For example.

$i < j \; ; \; a_{ij} = 0$

$i > j \; ; \; a_{ij} \neq 0$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 5 & 4 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

are 4 × 4 and 3 × 3 lower triangular matrices respectively.

10. **Sub matrix :** A matrix obtained from a given matrix, $A = (a_{ij})_{m \times n}$ by deleting some rows or column or both is called a sub matrix of A. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 1 \\ 7 & 8 & 0 & 2 \\ 1 & 7 & 2 & 3 \end{bmatrix} \text{ then the matrix } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 7 & 8 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 5 \\ 8 & 0 \\ 7 & 2 \end{bmatrix}$$

are sub matrices of A.

11. Equal Matrices:

Two matrices are said to be equal if :

(i) They are of the same order.

(ii) The elements in the corresponding positions are equal.

Thus if $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Then A = B

$$\begin{bmatrix} x & x+y \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find x and y   Solve $x = 1, y = -1$

PRODUCT OF MATRIX BY A SCALAR (OR CONSTANT)

Let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$ and k is a constant, then their product is matrix $kA = [ka_{ij}]_{m \times n}$.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 5 & 6 \\ -9 & 7 & 3 \end{bmatrix}$$

$$4A = \begin{bmatrix} 0 & 4 & -4 \\ 8 & 20 & 24 \\ -36 & 28 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} = 5\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

ADDITION AND SUBTRACTION OF MATRICES

Properties of Matrix Addition

(i) Matrix addition is $\boxed{\text{commutative}}$:

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be matrices of the same order $m\times n$ then $A + B = B + A$.

(ii) Matrix addition is $\boxed{\text{associative}}$:

Let A, B, C can be the matrices of the same order,

Then $(A + B) + C = A + (B + C)$

(iii) Cancellation law for matrix addition :

Let A, B, C be the matrices of the same order, then

$A + B = A + C$ holds if and only if $B = C$.

MULTIPLICATION OF MATRICES

The product AB of two matrices A and B is possible only when the number of columns in A is equal to the number of rows in B. Such matrices are said to be conformable for multiplication.

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 6 & 3 \end{bmatrix}_{\substack{2\times 3 \\ m\times n}} \quad B = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}_{\substack{3\times 2 \\ n\times p}}$$

$$AB = \begin{bmatrix} 1\times 6 + -2\times 1 + 5\times 0 & 1\times -1 + -2\times 2 + 5\times 3 \\ 0\times 6 + 6\times 1 + 3\times 0 & 0\times -1 + 6\times 2 + 3\times 3 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 21 \end{bmatrix}_{2\times 2}$$

Order of $AB \rightarrow m \times p$

Total no. of multiplications to obtain one element $= n$

Total no. of additions to obtain one element $= n-1$

Total no. of multiplications for $AB = mnp$

Total no. of additions for $AB = m(n-1)p$

Ex:- Find no. of multiplications and additions in

$A \rightarrow 3\times 2 \quad B \rightarrow 2\times 4 \quad C \rightarrow 4\times 2$

i) $AB = A_{3\times 2} B_{2\times 4} = 3\times 2\times 4 = 24 \;,\; 3(2-1)4 = 12$

ii) $BC = B_{2\times 4} C_{4\times 2} = 2\times 4\times 2 = 16 \;,\; 2(4-1)2 = 12$

iii) $(AB)C = (AB)_{3\times 4} C_{4\times 2} = 3\times 4\times 2 = 24 \;,\; 3(4-1)2 = 18$

[Properties of Matrix Multiplication]

may or may not be

1. Multiplication of matrices is not commutative i.e. $AB \neq BA$

2. Multiplication of matrices is associative, i.e. $A(BC) = (AB)C$

3. Matrix multiplication is distributive with respect to addition

   i.e. $A(B + C) = AB + AC$ $\qquad$ $A(B \times C) \neq AB \times AC$

4. Multiplication with Identity Matrix :

   If A be n×n matrix and I_n is a unit matrix of order n, then

   $AI_n = I_nA = A$

5. $A^m \cdot A^n = A^{m+n}$

6. Zero divisor ; $AB = 0$

if $A = 0, B \neq 0$; $AB = 0$ ✓

if $B = 0$; $A \neq 0$; $AB = 0$ ✓

if $A \neq 0, B \neq 0$; $AB \begin{cases} \to 0 \\ \to \neq 0 \end{cases}$ (this is a possibility)

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$A \neq 0$ $\quad B \neq 0$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 0+0 \\ 1-1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Minimum number of multiplications :-

$A_{3\times 2}$ $B_{2\times 5}$ $C_{5\times 3}$

$(AB)C$

$(BC)A$

$(CA)B$

Thank you

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Soldiers !