ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-3

Calculus

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Topics to be Covered

- SANDWICH THEOREM
- FUNCTIONS OF TWO VARIABLES
- LIMIT OF A FUNCTION OF TWO VARIABLES
- ALGEBRA OF LIMITS
- REPEATED LIMITS

Ex:- Examine the limit at $x = 1/2$ for

$$f(x) = \begin{cases} \frac{1}{2} - x & ; 0 < x < 1/2 \\ 1/2 & ; x = 1/2 \\ 3/2 - x & ; 1/2 < x < 1 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \frac{1}{2} - \left(\frac{1}{2} - h\right)$$

$$= 0$$

$$\text{R.H.L.} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$\lim_{h \to 0} \frac{3}{2} - \left(\frac{1}{2} + h\right)$$

$$= 1$$

$$\underset{0}{\text{LHL}} \neq \underset{1}{\text{RHL}} \neq \underset{1/2}{\text{Value}}$$ (Limit DNE, discontinuous at $x = 1/2$)

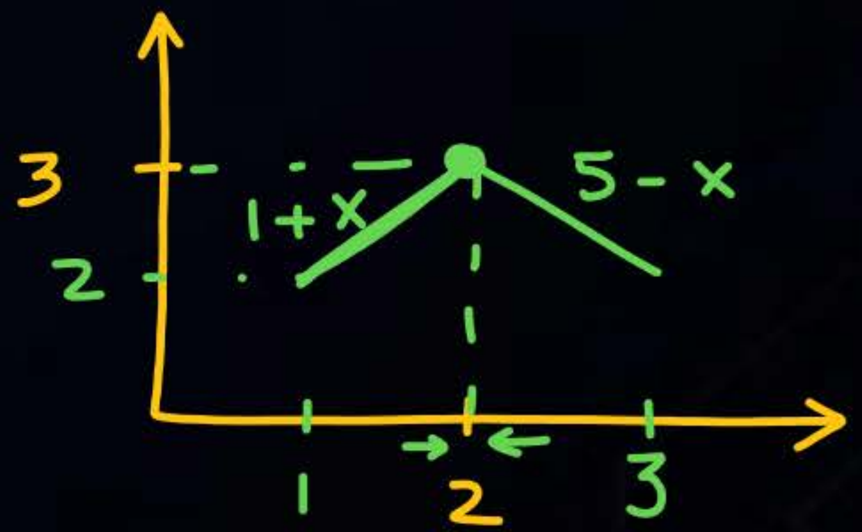
Ex:- Find $\lim_{x \to 2} f(x)$ $\quad f(x) = \begin{cases} 1+x ; & x \le 2 \\ 5-x ; & x > 2 \end{cases}$

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = 3$$

$$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = 3$$

$$\text{Value at } x=2 = 3$$

$$\underbrace{\text{LHL} = \text{RHL}}_{\text{Limit exist}} = \text{Value} \quad (\text{Continuous})$$



THEOREMS ON LIMITS

Theorem 1:

The limit of a function if exists, is unique, (Uniqueness Theorem)

Theorem 2:

If $\lim\limits_{x \to a} f(x) = l_1, \lim\limits_{x \to a} g(x) = l_2$ then

(i) $\lim\limits_{x \to a} [f(x) \pm g(x)] = l_1 \pm l_2$

(ii) $\lim\limits_{x \to a} [f(x) . g(x)] = l_1 . l_2$

(iii) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$, (where $l_2 \neq 0$)

THEOREMS ON LIMITS

Theorem 3:

If $\lim\limits_{x\to a} f(x) = l$, then $\lim\limits_{x\to a} |f(x)| = |l|$

Theorem 4:

If $\lim\limits_{x\to a} f(x) = l$, then $\lim\limits_{x\to a} e^{f(x)} = e^{l}$

Ex: $\lim\limits_{x\to 2} f(x) = 3$

$\rightarrow \lim\limits_{x\to 2} e^{f(x)} = e^{3}$

$\rightarrow \lim\limits_{x\to 2} \log f(x) = \log 3$

$\rightarrow \lim\limits_{x\to 2} |f(x)| = |3| = 3$

IMPORTANT RESULTS ON LIMITS

1. $\lim_{x \to 0} (1+x)^{1/x} = e$

2. $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)^{x} = e$

3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

→ radian

$\lim_{x \to 0} \frac{\cos x}{1} = 1$

$\sin x = x$
(When x is very small).
$\tan x = x$

Let $y = (1+x)^{1/x}$

$\log y = \lim_{x \to 0} \frac{\log(1+x)}{x} \quad \left(\frac{0}{0}\right)$

$\log y = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$

$y = e^1$

Shortcut :-

1^{∞}

$$\boxed{\begin{aligned} \lim_{x\to 0}\left(1+\frac{a}{b}x\right)^{\frac{c}{d}x} &= e^{\frac{ac}{bd}} \\ \lim_{x\to \infty}\left(1+\frac{a}{bx}\right)^{\frac{c}{d}x} &= e^{\frac{ac}{bd}} \end{aligned}}$$

Ex :-

$$\lim_{x\to 0}\left(1-\frac{2}{3}x\right)^{1/x} = e^{-\frac{2}{3}\times 1} = e^{-2/3}$$

$$\lim_{x\to \infty}\left(1-\frac{9}{5x}\right)^{-2x} = e^{-\frac{9}{5}\times -2} = e^{18/5}$$

$$\lim_{x\to \infty}\boxed{\left(1-\frac{3}{x}\right)^{-2x}}\boxed{\left(1-\frac{3}{2x}\right)^{-\frac{4}{5}x}} = e^{-3\times -2}\cdot e^{-\frac{3}{2}\times -\frac{4}{5}}$$

$$e^{6}\cdot e^{\frac{12}{10}}$$

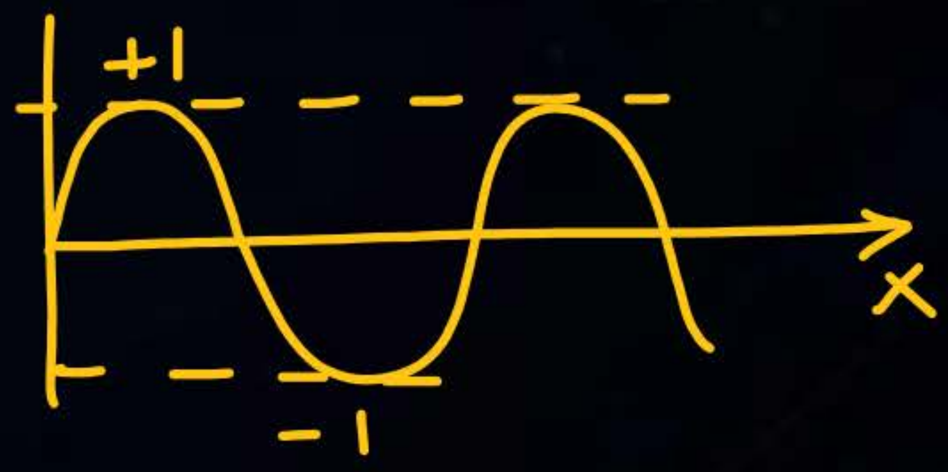
$$= e^{36/5}$$

IMPORTANT RESULTS ON LIMITS

$$\lim_{x\to\infty}\frac{\sin x}{x}=\frac{(-1,+1)}{x}$$
$$=0$$

4. $\lim\limits_{x\to 0}\cos x = 1$

5. $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$ $x \to$ radian $\frac{\sec^2 x}{1}=1$



6. $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$, n is integer

$$x\to 0 \quad \frac{n x^{n-1}-0}{1-0}=na^{n-1}$$

7. $\lim\limits_{x\to 0}\sin\frac{1}{x} = \lim\limits_{x\to 0}\cos\frac{1}{x}$ = oscillatory value, so limit does not exist.
(between -1 to +1)

8. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$\left(\frac{0}{0}\right) \lim_{x \to 0} \frac{e^x}{1} = e^0 = 1$

9. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

$\left(\frac{0}{0}\right) \lim_{x \to 0} \frac{a^x \log_e a}{1} = \log_e a$

8) $\lim_{x \to 0} \frac{1 - \cos mx}{x^2} = m^2/2$

9) $\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$

$\lim_{x \to 0} \frac{1 - \cos mx}{x^2} \quad \left(\frac{0}{0}\right)$

$\lim_{x \to 0} \frac{-(-m \sin mx)}{2x} \quad \left(\frac{0}{0}\right)$

$\lim_{x \to 0} \frac{m^2 \cos mx}{2} = \frac{m^2}{2}$

[IMPORTANT EXPANSIONS OF FUNCTION]

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \infty$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \infty$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \infty$

4. $\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots \infty$

INDETERMINATE FORMS :-

Such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 0^0, $\infty - \infty$.

The limiting value of indeterminate form is K/a its true value.

Most std. indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can be found out by using L Hospital's Rule.

→ $\lim_{x \to a} \frac{F(x)}{g(x)}$ if $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then differentiate N_r & D_r again & until we get a determinate true value.

TRICK :- How to solve limit question :-

→ Direct substitution

→ L Hospital's Rule

→ When variable is in exponent (power) then take log

→ Standard results

→ Series expansion

→ $x \to \infty$, (Reduce to $\frac{1}{x} \to 0$ form) [Divide by biggest power]

→ Rationalization

→ Basic approach

$$L.H.L. = \lim_{h \to 0} f(a-h)$$

$$R.H.L. = \lim_{h \to 0} f(a+h)$$

Ex:- Rationalization

$$\lim_{n\to\infty} \sqrt{n^2+n} - \sqrt{n^2+1} \times \frac{\sqrt{n^2+n}+\sqrt{n^2+1}}{\sqrt{n^2+n}+\sqrt{n^2+1}}$$

$$\lim_{n\to\infty} \frac{(n^2+n)-(n^2+1)}{\sqrt{n^2+n}+\sqrt{n^2+1}} = \frac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}}$$

$$= \frac{1-1/n}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+\sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}}} = \frac{1-\cancel{1/n}^{\,0}}{\sqrt{1+\cancel{1/n}^{\,0}}+\sqrt{1+\cancel{1/n^2}^{\,0}}} = \frac{1}{1+1} = \frac{1}{2}$$

Ex:-

$$\lim_{x\to 0} \frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3} = \frac{\left(\cancel{1}+\cancel{x}+\cancel{\frac{x^2}{2!}}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-\left(\cancel{1}+\cancel{x}+\cancel{\frac{x^2}{2}}\right)}{x^3}$$

$$\lim_{x \to 0} \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5} + \ldots$$

$$= \frac{1}{6}$$

Ex:- $\lim_{x \to 0} \frac{x - |x|}{x}$

$$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{h \to 0} f(0-h) = \frac{(0-h) - |0-h|}{0-h}$$

$$= \frac{-h - h}{-h} = \frac{-2h}{-h} = 2$$

$$\text{R.H.L.} = \lim_{h \to 0} f(0+h) = \frac{0+h - |0+h|}{0+h}$$

$$= \frac{h - h}{h} = 0$$

limit DNE

$\lim_{x\to 0}$ $\lim_{x\to\infty} \dfrac{2x^3+5x^2+x}{5x^2+6x+7} \left(\dfrac{\infty}{\infty}\right) = \infty, 0$ $\quad \dfrac{6x^2+10x+1}{10x+6} \to \dfrac{12x+10}{10} \to \infty$

$\lim_{x\to 0}$ $\lim_{x\to\infty} \dfrac{2x^2+6x-1}{3x^2-5x+7} \left(\dfrac{\infty}{\infty}\right) = \dfrac{2}{3}, -\dfrac{1}{7}$ $\quad \dfrac{4x+6}{6x-5} \to \dfrac{4}{6}$

$\lim_{x\to 0}$ $\lim_{x\to\infty} \dfrac{x+5}{2x^2+5x+6} \left(\dfrac{\infty}{\infty}\right) = 0, \dfrac{5}{6}$ $\quad \dfrac{1}{4x+5} \to 0$

	$\lim_{x\to\infty}$	$\lim_{x\to 0}$
i) Degree $N_r > D_r$	∞	depend on coeff.
ii) " $N_r = D_r$	a/b	depend on coeff.
iii) $N_r < D_r$	0	depend on coeff.

E.g.

Evaluate $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$ $\left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2} \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{2\sec x \cdot \sec x \tan x - (-\sin x)}{6x}$$

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x} \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{4\sec^2 x \tan x \cdot \tan x + 2\sec^2 x \cdot \sec^2 x + \cos x}{6} = \frac{2+1}{6} = \frac{1}{2}$$

Product Rule.

$$\frac{d}{dx}(u \cdot v) = u v' + v u'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v u' - u v'}{v^2}$$

E.g.

Evaluate $\lim_{n\to\infty}\left[\frac{2n^3}{(2n^2+3)}+\frac{(1-5n^2)}{(5n+1)}\right]$

$n \to \frac{1}{x}$

$n \to \infty,\ x \to 0$

$$\lim_{x\to 0} \frac{\frac{2}{x^3}}{\frac{2}{x^2}+3} + \frac{1-\frac{5}{x^2}}{\frac{5}{x}+1}$$

$$= \frac{2/x^3}{\frac{(2+3x^2)}{x^2}} + \frac{(x^2-5)/x^2}{(x+5)/x} = \frac{2}{x(2+3x^2)} + \frac{x^2-5}{x(x+5)}$$

$$= \frac{2x^2+10x+(2x+3x^3)(x^2-5)}{x(2+3x^2)\,x(x+5)} = \frac{2x^2+\cancel{10x}+2x^3-\cancel{10x}+3x^5-15x^3}{x^2(2x+10+3x^3+15x^2)}$$

$$\frac{2x^2-13x^3+3x^5}{x^2(3x^3+15x^2+2x+10)} = \frac{2-13x+3x^3}{3x^3+15x^2+2x+10} = \frac{2}{10} = \frac{1}{5}$$

E.g. $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3} = b$

$\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n} \right]$

$\lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{1/n}$

E.g.

Find $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ $\left(\frac{0}{0}\right)$

$$= \frac{\frac{1}{2\sqrt{1+x}}\cdot 1 - \frac{1}{2\sqrt{1-x}}(-1)}{1} = \frac{\frac{1}{2}+\frac{1}{2}}{1} = 1$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right)$$

$$= \frac{1}{2}x^{1/2-1}$$

$$= \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}\cdot 1$$

E.g.

Find the values of a and b in order that $\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3}$ may be $\left(\frac{0}{0}\right)$ equal to 1.

$$\lim_{x\to 0} \frac{(1+a\cos x).1+x(-a\sin x)-b\cos x}{3x^2} = \frac{1+a+0-b}{3x^2} \quad \left(\frac{0}{0}\right)$$

$\therefore a-b+1=0$ ....(1)

$$\lim_{x\to 0} \frac{-a\sin x+1(-a\sin x)+(-a\cos x)x+b\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{-a\cos x-a\cos x+(-a\cos x).1+x(a\sin x)+b\cos x}{6} = \frac{b-3a}{6} = 1$$

$\therefore b-3a=6$ ....(2)

$a-\cancel{b}=-1$

$\cancel{b}-3a=6$

$-2a=5$ $\quad a=-\frac{5}{2}, b=-\frac{3}{2}$

Thank you

GW Soldiers !