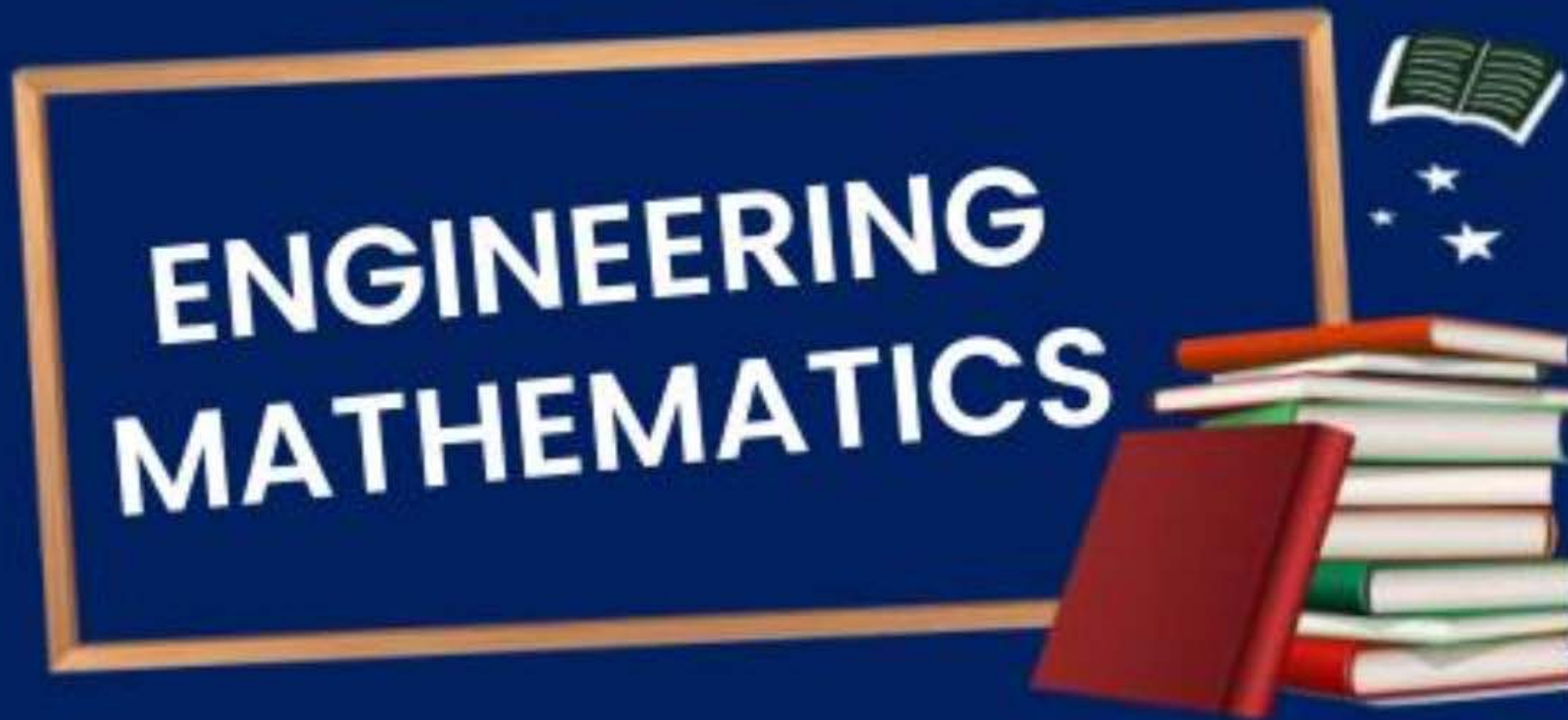




ALL BRANCHES



Lecture No.-2

Linear Algebra



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Topics to be Covered

PROPERTIES OF DETERMINANTS

ADJOINT OF MATRIX

INVERSE OF A MATRIX

CONJUGATE OF A MATRIX

CONJUGATE TRANSPOSE OF A MATRIX

Minimum number of multiplications :-



$A_{3 \times 5}$

$B_{5 \times 2}$

$C_{2 \times 3}$

$$(AB)C \Rightarrow (AB)_{3 \times 2} C_{2 \times 3} \Rightarrow 3 \times 5 \times 2 + 3 \times 2 \times 3 \\ 30 + 18 \Rightarrow \boxed{48}$$

$$(BC)A \Rightarrow (BC)_{5 \times 3} A_{3 \times 5} \Rightarrow 5 \times 2 \times 3 + 5 \times 3 \times 5$$

$$(CA)B \Rightarrow (CA)_{2 \times 5} B_{5 \times 2} \Rightarrow 2 \times 3 \times 5 + 2 \times 5 \times 2 \\ 30 + 20 \Rightarrow 50$$

Minimum no. of multiplications
 $(AB)C = 48$

[MINORS OF MATRIX]

The determinant value of the square matrix obtained from the original matrix of any order by the omission of the rows and columns is called a minor of a matrix. For example

If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 7 & 1 \\ 8 & 2 & 7 & 3 \end{bmatrix}_{3 \times 4}$

3×3 minor $\Rightarrow \Delta$
 2×2 minors $\Rightarrow M_{11}, M_{12} \dots$
 1×1 minors $\Rightarrow a_{ij}$

is a matrix of order 3×4 .

Then minors of A are $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 8 & 2 & 7 \end{vmatrix}; \begin{vmatrix} 0 & 3 \\ 8 & 2 \end{vmatrix} \& \begin{vmatrix} 7 & 1 \\ 7 & 3 \end{vmatrix}$ etc.

$$3 \times 3 \rightarrow [C_1 \ C_2 \ C_3] \quad [C_1 \ C_2 \ C_4] \quad [C_2 \ C_3 \ C_4] \quad [C_1 \ C_3 \ C_4]$$



$$\text{Total no. of } 3 \times 3 \text{ matrices} = {}^3C_3 \cdot {}^4C_3 = 1 \times 4 = 4$$

$$\text{Total no. of } 2 \times 2 \text{ matrices} = {}^3C_2 \cdot {}^4C_2 = 3 \times 6 = 18$$

$$\text{Total no. of } 1 \times 1 \text{ matrices} = {}^3C_1 \cdot {}^4C_1 = 3 \times 4 = 12$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex:-

$$\begin{vmatrix} 1 & 0 & 3 \\ 7 & 1 & 0 \\ 5 & -1 & 2 \end{vmatrix}_{3 \times 3} \Rightarrow$$

$$M_{11} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$$

$$M_{21} = \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14$$

$$M_{22}$$

$$M_{32} =$$

$$M_{13} = \begin{vmatrix} 7 & 1 \\ 5 & -1 \end{vmatrix} = -12$$

$$M_{23}$$

$$A_{32} = ?$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

[COFACTORS OF A MATRIX]

The cofactors of the element a_{ij} is defined as

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

where $|M_{ij}|$ is the determinant obtained by deleting the i^{th} row j^{th} column from given matrix.

$$\begin{vmatrix} 1,1 & 1,2 \\ 2,1 & 2,2 \end{vmatrix} \Rightarrow \begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

$$\begin{vmatrix} 1,1 & 1,2 & 1,3 \\ 2,1 & 2,2 & 2,3 \\ 3,1 & 3,2 & 3,3 \end{vmatrix} \Rightarrow \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{aligned} \Delta &= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} \\ &= 1 \times 2 + 0 \times -14 + 3 \times -12 \end{aligned}$$

$$\Delta = -34$$

$$A_{11} = (-1)^{1+1} |M_{11}| = + (2) = 2$$

$$A_{12} = (-1)^{1+2} |M_{12}| = - (14) = -14$$

$$A_{13} = (-1)^{1+3} |M_{13}| = + (-12) = -12$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} = -(-21) = 21$$

[PROPERTIES OF DETERMINANTS]

- ✓ 1. $|A'| = |A|$ $|A^T| = |A|$
- ✓ 2. If we interchange any two rows or columns then sign of determinants change.
- ✓ 3. If any 2 rows or any 2 columns in a determinant are identical (or proportional) then the value of the determinant is zero.
- ✓ 4. Multiplying a determinant by K means multiplying the elements of only one row (or one column) by K.

e.g. $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

then $2|A| = 2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 8 - 12 = -4 = 2(-2)$

[PROPERTIES OF DETERMINANTS]

5. ✓ If elements of a row in a determinant can be expressed as the sum of two or more elements then the given determinant can be expressed as the sum of 2 or more determinants.

e.g.
$$|A| = \begin{vmatrix} a+c & b+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix} + \begin{vmatrix} c & d \\ e & f \end{vmatrix}$$

6. * ✓ If we apply the operations like $R_i \rightarrow R_i + KR_j$ or $C_i \rightarrow C_i + KC_j$, the value of the determinant remains unchanged.

$$\begin{matrix} R_1 \rightarrow R_1 + 4R_2 \\ \Delta & \Delta \end{matrix}$$

$$\begin{matrix} C_2 \rightarrow C_2 - 2C_1 \\ \Delta & \Delta \end{matrix}$$

$$\begin{matrix} R_2 \rightarrow R_2 - R_3 + 3R_1 \\ \Delta & \Delta \end{matrix}$$

$$\begin{matrix} R_1 \rightarrow 5R_1 - R_3 \\ \Delta & 5\Delta \end{matrix}$$

$$\begin{matrix} C_2 \rightarrow 4C_2 - 6C_1 \\ \Delta & 4\Delta \end{matrix}$$

PROPERTIES OF DETERMINANTS

7. Thus, the determinant: is the sum of products of elements of any row and their corresponding cofactors. i.e.,
 $|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$
 where i represents rows. This expansion is called the expansion by cofactors.

Similarly, we can, expand the determinant as the sum of the product of elements of any column and the corresponding cofactors of the elements of the same column. That is,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

[PROPERTIES OF DETERMINANTS]

8. This expansion has a very important property that sum of products of elements of the i^{th} row and the corresponding cofactors of the elements of the i_1^{th} row is zero, that is,

$a_{i1}A_{i_11} + a_{i2}A_{i_12} + a_{i3}A_{i_13} = 0$, in the case of the determinant of general 3×3 matrix where $i \neq i_1$. Thus, if we choose $i = 1, i_1 = 3$

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$$

$$= a_{11}A_{31} + a_{21}A_{32} + a_{31}A_{33} = 0$$

(PROPERTIES OF DETERMINANTS)

$$a_{11} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}(a_{12}a_{23} - a_{13}a_{22}) - a_{12}(a_{11}a_{23} - a_{13}a_{21}) + a_{13}(a_{11}a_{22} - a_{12}a_{21})$$

$$= a_{11}a_{12}a_{23} - a_{11}a_{13}a_{22} - a_{12}a_{11}a_{23} + a_{12}a_{13}a_{21} + a_{13}(a_{11}a_{22}) - a_{13}a_{12}a_{21} = 0$$

The same result holds good for the column expansion. That is $a_{1j}A_{1j_1} + a_{2j}A_{2j_1} + a_{3j}A_{3j_1} = 0$ where $j \neq j_1$

$$9. |A B C D| = |A||B||C||D|$$

$$|A^n| = |A|. |A| \dots n \text{ times} = |A|^n$$

[DETERMINANT OF A MATRIX]

→ Determinant is defined only for square matrix.

→ Δ = Sum of products of any row/column ^{elements} \times Corresponding

\circ = Sum of products of any row/column ^{elements} \times ^{cofactors} Non-corresponding cofactors

②

$$\begin{vmatrix} 1 & 0 & 5 \\ -1 & 2 & 3 \\ 3 & 6 & 7 \end{vmatrix} \begin{matrix} \searrow \\ \nearrow \end{matrix} \begin{vmatrix} 3 & 6 & 7 \\ -1 & 2 & 3 \\ 1 & 0 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 6 & 3 & 7 \\ 2 & -1 & 3 \\ 0 & 1 & 5 \end{vmatrix}$$

$$1(14-18) + 5(-6-6)$$

$$-4 - 60$$

$$\Delta = -64$$

$$\Delta = 64$$

$$\Delta = -64$$

③

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 5 & 4 \end{vmatrix}$$

(Identical)

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 0 & -6 \end{vmatrix}$$

(Proportional)

$$\Delta = 0$$

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$$

④

a) $\begin{vmatrix} 5 & 0 & 10 \\ 4 & 8 & -4 \\ 3 & 6 & 9 \end{vmatrix} = 5 \times 4 \times 3 \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}$

b) $\begin{vmatrix} 5 & 0 & 0 \\ 10 & -5 & -10 \\ 15 & 20 & 10 \end{vmatrix} = 5^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 4 & 2 \end{vmatrix} = 5^3 (-2 - (-8)) = 5^3 \times 6$

c) $[A]_{3 \times 3} \xrightarrow[\text{by } 4]{\text{Multiply}} \begin{bmatrix} 4a_{11} & 4a_{12} & 4a_{13} \\ 4a_{21} & 4a_{22} & 4a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{bmatrix} = 4^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 4^3 \times 5 = 320$

$\Delta = 5$

$$\begin{array}{ccc}
 [A]_{n \times n} & \xrightarrow[\text{by } K]{\text{Multiply}} & [KA]_{n \times n} \\
 (\Delta) & \longrightarrow & K^n \Delta
 \end{array}
 \quad \left\{ |KA| \rightarrow K^n |A| \right\}$$

$A_{n \times n}$

⑤

$$\begin{aligned}
 \begin{vmatrix} 1 & 1+x & 1+x^2 \\ 1 & 1+y & 1+y^2 \\ 1 & 1+z & 1+z^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1+x^2 \\ 1 & 1 & 1+y^2 \\ 1 & 1 & 1+z^2 \end{vmatrix} + \begin{vmatrix} 1 & x & 1+x^2 \\ 1 & y & 1+y^2 \\ 1 & z & 1+z^2 \end{vmatrix} \\
 &= 0 + \begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \\
 &= 0 + 0 + \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \\
 &= (x-y)(y-z)(z-x)
 \end{aligned}$$

⑥ $\left| \begin{array}{ccc} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{array} \right| \xrightarrow[\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1}]{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1}} \left| \begin{array}{ccc} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{array} \right| = (y-x)(z-x) \left| \begin{array}{ccc} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{array} \right|$

$$(y-x)(z-x) \left| \begin{array}{cc} 1 & y+x \\ 1 & z+x \end{array} \right|$$

$$\Rightarrow (y-x)(z-x)(z-y)$$

$$\Rightarrow (x-y)(y-z)(z-x)$$

$$\begin{cases} R_1 \rightarrow R_1 + 2R_2 \\ R_3 \rightarrow R_3 - 2R_1 \end{cases}$$

- This should not be done otherwise Δ will change.
- Never row & column operations are applied simultaneously.

$$\begin{array}{c}
 \left| \begin{array}{ccc|c}
 \boxed{1} & 2 & 3 & \\
 2 & 5 & 0 & \\
 3 & 2 & 1 &
 \end{array} \right| \xrightarrow[\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}]{\text{}} \left| \begin{array}{ccc|c}
 1 & 2 & 3 & \\
 0 & \boxed{1} & -6 & \\
 0 & -4 & -8 &
 \end{array} \right| \xrightarrow{R_3 \rightarrow R_3 + 4R_2} \left| \begin{array}{ccc|c}
 1 & 2 & 3 & \\
 0 & 1 & -6 & \\
 0 & 0 & -32 &
 \end{array} \right| \searrow \Delta = -32
 \end{array}$$

NOTE:- For diagonal, scalar, identity, UTM, LTM :-
 $\Delta = \text{Product of diagonal elements}$

Ex:-

$$A = \begin{vmatrix} 0 & 0 & 0 & -5 \\ 1 & 0 & 2 & -3 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xrightarrow{\text{swap}} \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xrightarrow{\text{swap}} \begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$\Delta = -15$
 $\Delta = 15$
 $\Delta = 1 \times 1 \times 3 \times -5 = -15$

⑨ $|A^3| = |A|^3 = (-15)^3 = -3375$

[ADJOINT OF MATRIX]

Let $A = [a_{ij}]_{n \times n}$ be a square matrix and A_{ij} is the cofactor of a_{ij} . Then the transpose of the matrix B of cofactors of the elements of A is known as the adjoint of A and is denoted by $\text{Adj } A$ i.e. $\text{Adj } A = B'$ where $B = A_{ij} =$ cofactor matrix of A .

Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Cofactor matrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

[ADJOINT OF MATRIX]

Then $adj A = \text{transpose of } \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\text{Cofactor matrix}} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where A_{ij} denotes the cofactors of the corresponding element a_{ij} .

$$Adj A = [Cofactor matrix]^T$$

[RESULTS ON ADJOINT OF A MATRIX]

If $A = [a_{ij}]_{n \times n}$ be a square matrix of order n , then

$$A. (Adj A) = |A|. I_n = (adj A). A.$$

where I_n is the identity matrix of the same order as A .

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

INVERSE OF A MATRIX

Definition: Let A be a square matrix of order n . If there is a matrix B such that

$$A \cdot B = B \cdot A = I, \text{ then}$$

B is called the inverse of the matrix A and denoted by A^{-1} . Thus, if A is square matrix of order n , then A^{-1} is also a square matrix of order n .

$$A^{-1} = \frac{\text{adj}A}{|A|}$$

$$A(\text{adj } A) = |A|I_n$$

Multiply by A^{-1} ; $A^{-1}A(\text{adj } A) = |A|A^{-1}I_n$

$$I_n(\text{adj } A) = |A|A^{-1}$$

$$(\text{adj } A) = |A|A^{-1}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Properties of inverse & Adjoint:-

$$1. (AB)^{-1} = B^{-1}A^{-1}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

$$2. A (\text{Adj } A) = |A| I_n$$

$$3. \text{Adj}(\text{Adj } A) = |A|^{n-2} A$$

$$4. |A^{-1}| = \frac{1}{|A|}$$

$$5. |\text{Adj } A| = |A|^{n-1}$$

$$6. |\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$$

$$7. |\text{Adj}(\text{Adj}(\text{Adj } A))| = |A|^{(n-1)^3}$$

$n \rightarrow$ Order of square matrix.

$$\begin{vmatrix} 1 & 1 & 2 \\ 5 & 0 & 6 \\ 7 & 8 & 3 \end{vmatrix}$$

SARRUS RULE for Δ .

Ex:-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 1 & 2 \\ 5 & 0 & 6 \\ 7 & 8 & 3 \end{array}$$

$$(0 + 42 + 80) - (0 + 48 + 15)$$

$$\Delta =$$

i) $\text{Adj } A$

ii) A^{-1}

iii) $|A^{-1}|$

iv) $|\text{Adj } A|$

v) $|\text{Adj}(\text{Adj } A)|$

vi) $|A^2|$

vii) $|A^3|$

Thank you

GW
Soldiers!

