ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-8

Calculus

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Topics to be Covered

- EXPANSION OF FUNCTIONS
- TAYLOR'S THEOREM
- MACLAURIN'S THEOREM
- CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

$$rt - s^2 = 4(-6y) - 2^2$$
$$= -24y - 4$$

At $(0,0)$; $rt - s^2 = -24\times 0 - 4 < 0 \Rightarrow$ Neither max. nor min.

At $\left(\frac{1}{6}, -\frac{1}{3}\right)$; $rt - s^2 = -24\left(-\frac{1}{3}\right) - 4 = 4 > 0$

At $\left(\frac{1}{6}, -\frac{1}{3}\right) \rightarrow r = 4$ (+ve, minima)

Ex:- $f(x,y) = x^3 + y^3 - 3axy$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - 3ay = 0 \\ \frac{\partial f}{\partial y} &= 3y^2 - 3ax = 0 \end{aligned}\right\}$$

$x^2 = ay$
$y^2 = ax$

$r = \frac{\partial^2 f}{\partial x^2} = 6x$ $\quad$ $t = \frac{\partial^2 f}{\partial y^2} = 6y$ $\quad$ $s = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = -3a$

$y(y^3 - a^3) = 0$

$\therefore y = 0, a$

$\therefore x = 0, a$ Critical points $(0,0)$ and (a,a)

$rt - s^2 = (6x)(6y) - (-3a)^2 = 36xy - 9a^2$

At $(0,0)$; $rt - s^2 = 36.0.0. - 9a^2 = -9a^2 < 0$ (Neither max. nor minima)

At (a,a) ; $rt - s^2 = 36.a.a. - 9a^2 = 27a^2 > 0$

$\hookrightarrow r = 6x = 6a$

If $a > 0$; $r > 0 \Rightarrow$ Minima at (a,a)

$f(x,y) = a^3 + a^3 - 3a^3 = \boxed{-a^3}$ −ve

If $a < 0$; $r < 0 \Rightarrow$ Maxima at (a,a)

$f(x,y) = a^3 + a^3 - 3a^3 = \boxed{-a^3}$ +ve

At $(0,0)$ no extreme point

Ex:- $f(x,y) = x^2y - 3xy + 2y + x$ has

A) No local extremum

B) One local max. & no. local min.

C) " " min. & " " max.

D) One " min & one " max.

[GATE]

$x = 1, 2$

$y = 1, -1$

p ✓ $f_x = \frac{\partial f}{\partial x} = 2xy - 3y + 1 = 0$

At $x=1$; $y=1$

At $x=2$; $y=-1$

s $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x - 3$

q ✓ $f_y = \frac{\partial f}{\partial y} = x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0$

$x = 1, 2$

r $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y$

t $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0$

$rt - s^2 = (2y)(0) - s^2$

$= -\boxed{s^2} \rightarrow +ve \quad \therefore rt - s^2 < 0$

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:-

Let $f(x,y,z)$ be a fn. whose max. & min. is tobe found & $\phi(x,y,z) = c$ is the given relation. $\rightarrow$ ④

$$F = f + \lambda\phi$$

Now $dF = 0$

$$dF = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy + \frac{\partial f}{\partial z}.dz + \lambda\left(\frac{\partial \phi}{\partial x}.dx + \frac{\partial \phi}{\partial y}.dy + \frac{\partial \phi}{\partial z}.dz\right)$$

$$\cancelto{0}{\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)}.dx + \cancelto{0}{\left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)}dy + \cancelto{0}{\left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)}.dz = 0$$

① ② ③

$x^2 + y^2 = 0$

$2x\,dx + 2y\,dy = 0$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Solving ①, ②, ③ & ④ find λ, x, y, z

Ex: Find min. value of $x^2+y^2+z^2$, given that $ax+by+cz=p$

$$f = x^2+y^2+z^2$$
$$\phi = ax+by+cz-p$$

Lagrange's fn. $F = f + \lambda \phi$

$$(2x+\lambda a)dx + (2y+\lambda b)dy + (2z+\lambda c)dz = 0$$

$$2x+\lambda a = 0 \quad \ldots i) \quad \times x$$
$$2y+\lambda b = 0 \quad \ldots ii) \quad \times y$$
Add
$$2z+\lambda c = 0 \quad \ldots iii) \quad \times z$$

$$2(x^2+y^2+z^2) + \lambda(ax+by+cz) = 0$$
$$2f + \lambda p = 0 \Rightarrow \boxed{\lambda = -2f/p}$$

Then $x = -\dfrac{\lambda a}{2} = -\left(-\dfrac{2f}{p}\right)\cdot\dfrac{a}{2} = \boxed{\dfrac{af}{p} = x}$ $\boxed{y = \dfrac{bf}{p}}$ $\boxed{z = \dfrac{cf}{p}}$

$$f = x^2+y^2+z^2 = \frac{a^2f^2}{p^2}+\frac{b^2f^2}{p^2}+\frac{c^2f^2}{p^2}$$

$$\cancel{f} = \frac{(a^2+b^2+c^2)f^{\cancel{2}}}{p^2}$$

$$\boxed{f = \frac{p^2}{a^2+b^2+c^2}}$$

Either this is the maximum or minimum value of f at $\left(\frac{af}{p}, \frac{bf}{p}, \frac{cf}{p}\right)$

→ Problem statement:-

1) Given quantity Q (maximise/minimise)

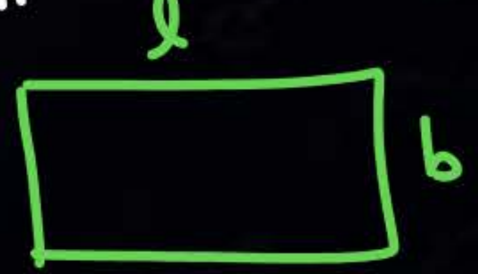
2) Given relation / value.

3) Express Q in single variable.

4) $\dfrac{dQ}{d(\underset{\text{var.}}{\text{Single}})} = 0$

5) Find the given point at which Q is max./min.

Ex:- A rectangular park of given perimeter P (40 m). Find the length & breadth for its area to be maximum



i) $A = lb$

ii) $P = 2(l+b)$ → Given

iii) $A = l\left(\frac{P}{2} - l\right)$ $\rightarrow$ A is only fn. of l.

iv) $\frac{dA}{dl} = 0$

$$A = \frac{P}{2}l - l^2$$

$$\frac{dA}{dl} = \frac{P}{2} - 2l = 0$$

$$= l = P/4$$

$$\frac{d^2A}{dl^2} = -2 < 0 \text{ (Maxima)}$$

$l = \frac{P}{4}$, $b = \frac{P}{4}$ $\quad l = 10\,m$, $b = 10\,m$

$$P = 2(l + b)$$

$$P = 2\left(\frac{P}{4} + b\right)$$

$$\frac{P}{4} = b$$

$\therefore$ Max. $A = lb = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16} = \frac{40^2}{16} = 100\,m^2$

INFINITE SERIES :-

Algebric

Ex :- x^2, x^3+5x+6

Transcendental

- Trig. → $\sin x$, $\cos x$
- Exp. → e^x, a^{-x}
- Log → $\log(1+x)$, $\log x$
- Inv. trig. → $\sin^{-1}x$, . .

EXPANSION OF FUNCTIONS

We can expand transcendental fn. in ascending powers of $(x-a)$.

Imp. expansions:-

1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \dots \infty$

2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty$

3) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \infty$

4) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$

5) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

6) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

TAYLOR'S THEOREM

$$x \to a+h$$
$$h \to x-a$$

$f(x)$ can be expanded in the neighbourhood of $x = a$ as follows:-

$$f(\underbrace{a+h}_{a+x-a}) = \frac{f(a)}{0!} + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots \frac{h^n}{n!}f^n(a)$$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Drawbacks:-

1) $f(x)$ or any of its derivatives, become infinite
2) " " " " " " , becomes discontinuous
3) When $n \to \infty$, the remainder of that term should tend to non-zero.

Ex:- Expand $\ln(1+x)$ about $x=0$ using Taylor series:-

$$f(x) = f(0) + (x-0)f'(0) + \frac{(x-0)^2}{2!}f''(0) + \frac{(x-0)^3}{3!}f'''(0) + \dots$$

$$= 0 + x.1 + \frac{x^2}{2!}.(-1) + \frac{x^3}{3!}.(2)$$

$$f(x) = \ln(1+x) = 0$$

$$f'(x) = \frac{1}{1+x} = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} = -1$$

$$f'''(x) = +\frac{2}{(1+x)^3} = 2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\frac{1}{x}$$

$$-\frac{1}{x^2}$$

$$+\frac{2}{x^3}$$

$$-\frac{6}{x^4}$$

$$+\frac{24}{x^5}$$

Ex:- Expand $\ln(1+x)$ about $x=1$.

$$f(x) = f(1) + (x-1)\cdot f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$$= \ln 2 + (x-1)\cdot\frac{1}{2} + \frac{(x-1)^2}{2!}\left(-\frac{1}{4}\right) + \frac{(x-1)^3}{3!}\left(\frac{1}{4}\right) + \cdots$$

$f(1) = \ln 2$

$f'(1) = 1/2$

$f''(1) = -1/4$

$f'''(1) = 1/4$

Ex:- Find coefficient of $(x-3)^4$ in expansion of e^x.

$$e^x = f(3) + (x-3) f'(3) + \frac{(x-3)^2}{2!} f''(3) + \frac{(x-3)^3}{3!} f'''(3) + \frac{(x-3)^4}{4!} f''''(3)$$

$$\frac{e^3}{24}$$

Coefficient of $(x-3)^4 = \frac{1}{4!} f^{iv}(3)$

$f(x)\ e^x$

$f^{iv}(x) = e^x$

$$= \frac{1}{24}\cdot e^x = \frac{1}{24}\cdot e^3$$

Ex:- Find third term in expansion of e^x at $x = a$.

$$3^{rd} \text{ term} = \frac{(x-a)^2}{2!} f''(x)$$

$$\boxed{= \frac{(x-a)^2}{2} \cdot e^a}$$

MACLAURIN'S THEOREM

It is special case of Taylor's series in which $f(x)$ is expanded about $x=0$ as follows:-

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

MACLAURIN'S THEOREM

Important Maclaurin's Expansion

i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

ii) $\tan x = 0 + x.1 + \frac{x^2}{2!}.(0) + \frac{x^3}{3!}.(2). + \dots\dots\dots\dots$

iii) $\sinh x$

iv) $\cosh x$

v) $\sin^{-1} x$

vi) $\cos^{-1} x$

Thank you

GW Soldiers !