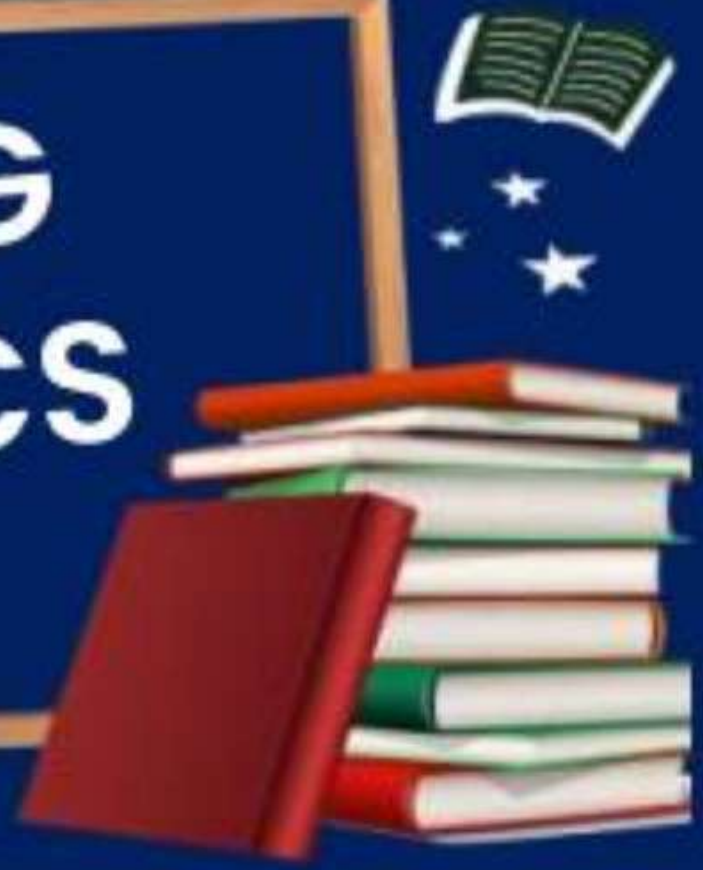
ALL BRANCHES

ENGINEERING MATHEMATICS



Lecture No.-4

Differential equations

By- Chetan Sir

Topics to be Covered

- DEFINITION & TYPES
- ORDER & DEGREE OF DE
- SOLUTION OF DE
- FORMATION OF DE
- WRONSKIAN & LD/LI SOLUTIONS
- METHODS OF SOLVING DE
- PARTIAL DIFFERENTIAL EQUATIONS

Methods of Solving D.E. :-

ODE

1) Observation Method
2) D.E. of first order & first degree
   - a) Variable separable mtd.
   - b) Homogenous D.E. mtd.
   - c) Linear D.E. mtd.
   - d) Bernoulli D.E. mtd.
3) Exact differential equations (Non exact D.E. → Exact D.E.)
4) L.D.E. of n^{th} order with
   - → constant coefficients
   - → variable coefficients

   } (C.F. + P.I.)
5) Methods for solving non-linear D.E.

PDE

6) Methods for solving P.D.E.

How to find particular integral :-

$$\boxed{f(D)\, y = Q} \quad \text{—} \; (1)$$

$y = \dfrac{1}{f(D)} . Q$, this y will satisfy eqn. (1) . It is Known as Particular Integral.

General formula :-

i) $$\frac{1}{D-\alpha} . Q = e^{\alpha x} \int e^{-\alpha x} . Q \, dx$$

ii) $$\frac{1}{D+\alpha} . Q = e^{-\alpha x} \int e^{\alpha x} . Q \, dx$$

Ex:-

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{4x}$$

$$(D^2+5D+6)\,y = e^{4x}$$

$$m^2+5m+6=0$$

$$(m+3)(m+2)=0$$

$$m=-2,\,-3$$

$$y = C.F. = c_1 e^{-2x} + c_2 e^{-3x}$$

$$P.I. = \frac{1}{D^2+5D+6}\cdot e^{4x} = \frac{1}{(D+3)(D+2)}\,e^{4x} = \frac{1}{D+3}\,e^{-2x}\int e^{2x}\cdot e^{4x}\,dx$$

$$= \frac{1}{D+3}\,e^{-2x}\cdot\frac{e^{6x}}{6} = \frac{1}{6}\,\frac{1}{D+3}\,e^{4x} = \frac{1}{6}\,e^{-3x}\int e^{3x}\cdot e^{4x}\,dx$$

$$= \frac{1}{6}\cdot e^{-3x}\cdot\frac{e^{7x}}{7} = \boxed{\frac{1}{42}\,e^{4x}} \rightarrow P.I.$$

$$y = C.F. + P.I. = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{4x}}{42}.$$

Case I :- When $Q = e^{ax}$, then

$D \to a$

$$P.I. = y = \frac{1}{f(D)} \cdot Q \Rightarrow \frac{1}{f(a)} \cdot Q \quad \text{provided } f(a) \neq 0$$

if $f(a) = 0$; then $P.I. = \dfrac{x}{f'(D)} \cdot e^{ax} = \dfrac{x}{f'(a)} \cdot e^{ax}$; $f'(a) \neq 0$

if $f'(a) = 0$; then $P.I. = \dfrac{x^2}{f''(D)} \cdot e^{ax} = \dfrac{x^2}{f''(a)} \cdot e^{ax}$; $f''(a) \neq 0$

$\vdots$

Ex :- $(D^2 + 5D + 6)\, y = e^{4x}$

$$y = \frac{1}{f(D)} \cdot Q = \frac{1}{D^2 + 5D + 6} \cdot e^{4x} = \frac{1}{4^2 + 5(4) + 6} \cdot e^{4x} = \boxed{\frac{1}{42} e^{4x}}$$

Ex :- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$(D-1)^2 = 0$

$D = 1, 1$

$(D^2 - 2D + 1)y = e^x$

$P.I. = \frac{1}{D^2 - 2D + 1} \cdot e^x = \frac{1}{1^2 - 2(1) + 1} \cdot e^x$; $f(a) = 0$

$P.I. = \frac{x}{2D - 2} \cdot e^x = \frac{x}{2(1) - 2} \cdot e^x$; $f'(a) = 0$

$f(D) = D^2 - 2D + 1$

$f'(D) = 2D - 2$

$f''(D) = 2$

$P.I. = \frac{x^2}{2} \cdot e^x = \frac{x^2}{2} \cdot e^x$

$y = C.F. + P.I. = (c_1 + c_2 x)e^x + \frac{x^2}{2} \cdot e^x$

Ex :- $(D^2 + 2D + 1)y = \cosh x \rightarrow \frac{e^x + e^{-x}}{2}$

Ex :- $(\quad '' \quad)y = e^{4x} + e^{-x}$

Case II :- If $Q = \sin ax$ or $\cos ax$, then $D^2 \to -a^2$

$$P.I. = y = \frac{1}{f(D^2)} \cdot Q = \frac{1}{f(-a^2)} \cdot Q \quad ; \text{provided } f(-a^2) \neq 0$$

if $f(-a^2) = 0 \Rightarrow$ $$y = \frac{x}{f'(D^2)} \cdot Q = \frac{x}{f'(-a^2)} \cdot Q \quad ; \text{provided } f'(-a^2) \neq 0$$

if $f'(-a^2) = 0 \Rightarrow$ $$y = \frac{x^2}{f''(D^2)} \cdot Q = \frac{x^2}{f''(-a^2)} \cdot Q \quad ; \text{provided } f''(-a^2) \neq 0$$

$\vdots$

Ex:- $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10 \sin x$$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1, 2$

$$y = \frac{1}{D^2 - 3D + 2} \cdot 10 \sin x = 10 \cdot \frac{1}{-1^2 - 3D + 2} \sin x$$

$$= 10 \frac{1}{1 - 3D} \sin x$$

$D^2 \to -a^2$

$\to -1^2 \to -1$

$$10 \cdot \frac{1}{1-3D} \times \frac{1+3D}{1+3D} \sin x = 10 \frac{(1+3D)}{1-9D^2} \cdot \sin x$$

$$= \cancel{10} \frac{(1+3D)}{\cancel{1-9(-1^2)}} \sin x = (1+3D) \sin x = \sin x + 3\, D(\sin x)$$

$$= \sin x + 3 \cos x$$

$$y = C.F. + P.I. = C_1 e^{x} + C_2 e^{2x} + \sin x + 3 \cos x$$

Ex:- $$\frac{d^3y}{dx^3} + a^2 \frac{dy}{dx} = \sin ax$$

$$(D^3 + a^2 D)\, y = \sin ax$$

$$m^3 + a^2 m = 0$$

$$m(m^2 + a^2) = 0$$

$$m(m + ia)(m - ia) = 0$$

$$m = 0,\ 0 \pm ia$$

$$P.I. = y = \frac{1}{D(D^2 + a^2)} \cdot \sin ax = \frac{1}{D(\cancel{-a^2 + a^2}) \to 0} \cdot \sin ax$$

$$\text{if } f(-a^2) = 0 \Rightarrow P.I. = \frac{x}{3D^2 + a^2} \sin ax = \frac{x}{3(-a^2) + a^2} \cdot \sin ax = -\frac{x}{2a^2} \sin ax$$

$$y = C_1 e^{0x} + e^{0x}\left[C_2 \cos x + C_3 \sin x\right] - \frac{x}{2a^2} \cdot \sin ax$$

METHODS OF SOLVING DE

Case III:- When $Q = x^m$, m is + ve integer

$$y = P.I. = \frac{1}{f(D)} \cdot x^m = [f(D)]^{-1} x^m$$

Step I :- Take lowest term common so that first terms becomes unity.
& it will take form of $1+\phi(D)$ / $1-\phi(D)$

Step 2 :- Write this form in numerator & expand.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots +$$

Step 3 :- Expand till ascending power of D gives 0.

$$D^m(x^m) = m! \qquad D^{m+1}(x^m) = 0$$

METHODS OF SOLVING DE

Ex:- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$

$(D^3 + 2D^2 + D)y = x^2$

$m^3 + 2m + m = 0$

$m(m^2 + 2m + 1) = 0$

$m(m+1)^2 = 0$

$m = 0, -1, -1$

$$P.I. = \frac{1}{D + 2D^2 + D^3} \cdot x^2 = \frac{1}{D[1 + 2D + D^2]} \cdot x^2 = \frac{1}{D}[1 + \underbrace{2D + D^2}_{\phi(D)}]^{-1} \cdot x^2$$

$$\frac{1}{D}\left[1 - (2D + D^2) + (2D + D^2)^2 - (2D + D^2)^3 + \cdots\cdots\right]x^2$$

$$\frac{1}{D}\left[1 - 2D - D^2 + 4D^2 + D^4 + 4D^3 - \cdots\cdots\right]x^2$$

$$\frac{1}{D}\left[x^2 - 2D(x^2) + 3D^2(x^2) + 0\right] = \frac{1}{D}\left[x^2 - 4x + 6\right]$$

$$= \frac{x^3}{3} - 4\frac{x^2}{2} + 6x$$

METHODS OF SOLVING DE

Ex:- $y'' + 4y' + 4y = x^2 + x$

$$P.I. = \frac{1}{D^2+4D+4}\cdot(x^2+x) = \frac{1}{(D+2)^2}(x^2+x) = \frac{1}{4\left(1+\frac{D}{2}\right)^2}\cdot(x^2+x)$$

$$\frac{1}{4}\left[1+\underbrace{\frac{D}{2}}_{x}\right]^{-2}(x^2+x) = \frac{1}{4}\left[1-2\left(\frac{D}{2}\right)+3\left(\frac{D}{2}\right)^2-\dots\right](x^2+x)$$

$$= \frac{1}{4}\left[x^2+x - D(x^2+x) + \frac{3}{4}D^2(x^2+x) + 0\dots\right]$$

$$= \frac{1}{4}\left[x^2+x-2x-1 + \frac{3}{4}\times 2\right]$$

$$= \frac{1}{4}\left[x^2-x+\frac{1}{2}\right]$$

METHODS OF SOLVING DE

$D \rightarrow D + \alpha$

Case IV:- If $Q = e^{\alpha x}$ V, V is function of x

$$y = \frac{1}{f(D)} \cdot e^{\alpha x} \cdot V = e^{\alpha x} \frac{1}{f(D+\alpha)} \cdot V$$

Ex:- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{2x}.x$

$$(D^2+5D+6)\,y = e^{2x}.x$$

$$y = \frac{1}{D^2+5D+6}.e^{2x}.x = e^{2x}.\frac{1}{(D+2)^2+5(D+2)+6}.x$$

$$= e^{2x}.\boxed{\frac{1}{D^2+9D+20}.x} = \frac{e^{2x}}{20}\frac{1}{\left(1+\frac{9D}{20}+\frac{D^2}{20}\right)}.x$$

$$\frac{e^{2x}}{20}\Big(1+\underbrace{\frac{9D}{20}+\frac{D^2}{20}}_{\phi(D)}\Big)^{-1}x = \frac{e^{2x}}{20}\left(1-\left(\frac{9D}{20}+\frac{D^2}{20}\right)+\left(\frac{9D}{20}+\frac{D^2}{20}\right)^2-\quad\right)x$$

$$= \frac{e^{2x}}{20}\left(x-\frac{9}{20}.1\right)$$

METHODS OF SOLVING DE

$e^x \cdot V$

Ex:- $\frac{d^3y}{dx^3} - \frac{3d^2y}{dx^2} + \frac{3dy}{dx} - y = xe^x + e^x = e^x(x+1)$

$(D^3 - 3D^2 + 3D - 1)y = e^x(x+1)$

$$P.I. = \frac{1}{(D-1)^3} \cdot e^x \cdot (x+1) = e^x \frac{1}{(D+1-1)^3}(x+1)$$

$$= e^x \frac{1}{D^3}(x+1) = e^x \iiint x+1 = e^x\left[\frac{x^4}{24} + \frac{x^3}{6}\right]$$

Case V:- If $Q = xV$, V is fn of x

$$\boxed{y = P.I. = \frac{1}{f(D)} \cdot xV = x\frac{1}{f(D)} \cdot V - \frac{f'(D)}{[f(D)]^2} \cdot V}$$

Ex:- $(D^2+5D+6)y = x.e^x$

$$P.I. = \frac{1}{D^2+5D+6} \cdot xe^x = x\frac{1}{D^2+5D+6} \cdot e^x - \frac{2D+5}{(D^2+5D+6)^2} \cdot e^{1x}$$

$$x\frac{1}{1^2+5(1)+6} \cdot e^x - \frac{2(1)+5}{(1^2+5(1)+6)^2} \cdot e^x$$

$$x \cdot \frac{e^x}{12} - \frac{7}{144}e^x = e^x\left[\frac{x}{12} - \frac{7}{144}\right]$$

METHODS OF SOLVING DE

Linear n^{th} order DE with variable coefficients: (Euler Cauchy form)

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots \cdots \; a_n y = Q$$

$$\left(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots\cdots \; a_n\right) y = Q$$

Put $x = e^z$ or $z = \log x$

Rules :-

- $xD \rightarrow D'$
- $x^2 D^2 \rightarrow D'(D'-1)$
- $x^3 D^3 \rightarrow D'(D'-1)(D'-2)$

$\vdots$

- $x^n D^n \rightarrow D'(D'-1)(D'-2)\cdots(D'-n+1)$

$D \rightarrow d/dx$, $D' \rightarrow d/dz$

$D^2 \rightarrow d^2/dx^2$, $(D')^2 \rightarrow d^2/dz^2$

METHODS OF SOLVING DE

Ex:- $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$

$(x^2D^2+xD-1)\,y=0$ (Variable coefficients)

$\{D'(D'-1)+D'-1\}\,y=0$ (Constant coefficients)

$\{(D')^2-D'+D'-1\}\,y=0$

$\{(D')^2-1\}\,y=0$

$m^2-1=0 \Rightarrow m=1,-1$

$y=C_1e^{z}+C_2e^{-z} = C_1e^{\log_e x}+C_2e^{-\log_e x}$

$= C_1x+\frac{C_2}{x}$

METHODS OF SOLVING DE

Ex:- $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

$$(x^2D^2 - 3xD + 4)y = 2x^2$$

$$(D'(D'-1) - 3D' + 4)y = 2(e^z)^2$$

$$((D')^2 - D' - 3D' + 4)y = 2e^{2z}$$

$$((D')^2 - 4D' + 4)y = 2e^{2z}$$

$$m^2 - 4m + 4 = 0$$

$$\frac{1}{D'^2 - 4D' + 4} \cdot 2e^{2z}$$

Thank you

GW Soldiers !