ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-9

Linear Algebra

By- Chetan Sir

Topics to be Covered

- Eigen Values
- Eigen vectors
- Properties of Eigen Values
- Properties of Eigen Vectors
- Eigen Values of Special Matrices

Properties of Eigen Values

1) Eigen values of symmetric/hermitian matrix are real.
2) Eigen values of skew-symmetric/skew-hermitian are 0 and purely imaginary.
3) Eigen values of orthogonal/unitary matrix are of unit modulus.

   For ex:- 1, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$
4) Eigen values of idempotent matrix are either 0 or 1.
5) Eigen values of involutary matrix are either -1 or 1.
6) If $(a+ib)$ is an eigen value then $(a-ib)$ is also an eigen value. Complex eigen roots will always exist in conjugate pairs.
7) Eigen values of A and A^T are same but eigen vectors may or may not be same.

Properties of Eigen Values

8) Any two characterstic/eigen vectors corresponding to two distinct eigen/characterstic roots of unitary or real symmetric matrix are orthogonal.

9) If leading minors of real symmetric matrix are positive then it's eigen values are also positive.

Ex:- $\begin{bmatrix} 10 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Leading minors $1 \times 1 \rightarrow |10| = 10\ (+)$

$2 \times 2 \rightarrow \begin{vmatrix} 10 & -1 \\ -1 & 2 \end{vmatrix} = 19\ (+)$

$3 \times 3 \rightarrow \begin{vmatrix} 10 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 19\ (+)$

→ Since leading minors are +ve $\therefore$ eigen values $\lambda_1, \lambda_2, \lambda_3$ will be positive

→ If $\lambda_1, \lambda_2, \lambda_3 \longrightarrow X_1, X_2, X_3$
then $X_1 . X_2^T = X_2 . X_3^T = X_1 . X_3^T = 0$

Properties of Eigen Values

10) When all entries of row and column add upto n, then n is an eigen value.

Ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ $\quad |A - \lambda I| = 0 \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{vmatrix} = 0$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda = 6, \ldots$

11) If A and B are of same order then A and B have same eigen values if A and B are similar, $B = P^{-1}AP$.

12) Eigen vectors corresponding to different eigen values are L.I.

13) Eigen vectors corresponding to repeated eigen values may or may not be L.I.

*** 14) If eigen values of A are $\lambda_1, \lambda_2, \ldots$

then the eigen values of

i) $A^2 \rightarrow \lambda_1^2, \lambda_2^2 \ldots$

ii) $A^3 \rightarrow \lambda_1^3, \lambda_2^3 \ldots$

iii) $KA \rightarrow K\lambda_1, K\lambda_2 \ldots$

iv) $A^{-1} \rightarrow \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots$

v) $\text{Adj}A \rightarrow |A|/\lambda_1, |A|/\lambda_2, \ldots$

vi) $A^n \rightarrow \lambda_1^n, \lambda_2^n, \ldots$

vii) $A + KI \rightarrow (\lambda_1 + K), (\lambda_2 + K) \ldots$

viii) $(A^3 + 5A^2 + I) \rightarrow (\lambda_1^3 + 5\lambda_1^2 + 1), (\lambda_2^3 + 5\lambda_2^2 + 1) \ldots$

$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ $|A| = 24$

$\rightarrow \lambda_1 = 4, \lambda_2 = 6$

$A^2 \rightarrow 16, 36$

$A^3 \rightarrow 4^3, 6^3 \rightarrow 64, 216$

$6A \rightarrow 6 \times 4, 6 \times 6$

$A^{-1} \rightarrow \frac{1}{4}, \frac{1}{6}$

$\text{Adj}\,A \rightarrow 24/4, 24/6$

15) Eigen values of A^{-1}, Adj A, A^m, KA will be different from the eigen values of A but eigen vectors are same.

16) Eigen values of identity, scalar, diagonal, LTM and UTM are leading diagonal elements.

*** ** Theorem :-

A) Sum of all eigen values = Trace of that matrix

$$\lambda_1 + \lambda_2 + \lambda_3 + \dots = a_{11} + a_{22} + a_{33} + \dots$$

(Sum of diagonal elements)

B) Product of eigen values = Determinant of matrix

$$\lambda_1 . \lambda_2 . \lambda_3 \dots = \Delta$$

If $|A| = 0$ (singular matrix), then at least 1 eigen value is 0.

If $|A| \neq 0$ (non-singular matrix), then all eigen values are non-zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (\ldots\ldots)\lambda - (|A|) = 0$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Number of terms in expansion of $|A|_{n\times n} = n!$

$1\times 1 \longrightarrow 1!$

$2\times 2 \longrightarrow 2!$

$3\times 3 \longrightarrow 3!$

$$ax^3 + bx^2 + cx + d = 0$$

Sum of the roots $= -b/a$

Product of the roots $= -d/a$

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

$$\lambda_1 \lambda_2 \lambda_3 = |A|$$

DIAGONALISATION

SIMILAR MATRICES :-

Let A and B are two matrices of same order then A is said to be similar to B if there exist an invertible matrix P such that $\boxed{B = P^{-1}AP}$ $\quad A \leftrightarrow B$

A matrix A is said to be diagonalisable if it is similar to diagonal matrix (D).

- $\boxed{\text{Diagonal matrix}(D) = P^{-1}AP}$ (D is similar to A).

$A_{n\times n}$ & if there exist n linearly independent vectors then we can find invertible matrix $P = [X_1\ X_2 \ldots]$ (Modal matrix) such that $P^{-1}AP$ is in diagonal form.

DIAGONALISATION

If GM = AM then it is diagonalisable
If GM < AM " " " non-diagonalisable.

$$A = PDP^{-1}$$

Powers of A :- $\boxed{A^n = PD^nP^{-1}}$

If A and B are similar
→ the matrices A and B have same
- eigen values
- characterstic eqn.
- characterstic polynomial.

Ex:- Reduce the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ into diagonal form. Find A^4.

→ $|A - \lambda I| = 0$

Eigen values are 3, 1, 1

Eigenvectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

DIAGONALISATION

Now $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P^{-1} = \dfrac{\text{adj } P}{|P|} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

So, $D = P^{-1} A P = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{D}$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ A and D are similar

Find $A^4 = P D^4 P^{-1}$

$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\dfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 82 & 80 & 80 \\ 80 & 82 & 80 \\ 0 & 0 & 2 \end{bmatrix}$

[DIAGONALISATION]

Necessary condition :-

i) $\rho(A) = n. \Rightarrow |A| \neq 0$

Sufficient condition. $\rightarrow$ ii) n L.I. eigen vectors of A should be there, then only A can be diagonalized

[CAYLEY HAMILTON THEOREM]

" Every square matrix satisfies its own characterstic equation.

$\lambda \longrightarrow A$ $\qquad |A - \lambda I| = 0$

Applications of C.H.T. :-

i) Finding A^{-1}

ii) Finding higher powers of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Ex:- Find A^{-1}

Evaluate matrix $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$ Characterstic eqn.

[CAYLEY HAMILTON THEOREM]

By applying C.H.T. :-

$$A^3 - 11A^2 - 4A + I = 0$$

Evaluation of A^{-1} :-

$$A^2 - 11A - 4I + A^{-1} = 0$$

(Multiply both sides by A^{-1})

$$A^{-1} = 11A + 4I - A^2$$

$$A^{-1} = 11\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Evaluation of poly. B :-

$$-4A^2 + A^2 \qquad A \qquad A$$

$$A^5(A^3 - 11A^2 - 4A + I) + A^4 - 11A^3 - 3A^2 + 2A + I$$

$$A^5\overbrace{(A^3 - 11A^2 - 4I + I)}^{0} + A\overbrace{(A^3 - 11A^2 - 4A + I)}^{0} + A^2 + A + I$$

$$= A^2 + A + I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

CAYLEY HAMILTON THEOREM

Ex:- Find A^8 $\quad A = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 5 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 20 = 0$$

$$\lambda^2 - 3\lambda + 2 - 20 = 0$$

Characterstic eqn. :- $\lambda^2 - 3\lambda - 18 = 0$

$$A^2 - 3A - 18I = 0$$

$$A^2 = 3A + 18I \quad \ldots \quad 1)$$ [Squaring]

$$A^4 = (3A + 18I)^2 = 9A^2 + 324I + 108A$$

$$= 9(3A + 18I) + 324I + 108A$$

$$A^4 = 135A + 486I \quad — 2)$$ [Squaring]

$$A^8 = 18225A^2 + 236196I + 131220A$$

CAYLEY HAMILTON THEOREM

$$18225(3A+18I)+236196\,I+131220\,A$$

$$\boxed{A^8 = 185895\,A + 564246\,I}$$

Higher powers of A are expressed in linear form of A

Ex:- $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ Express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as linear polynomial of A.

Q18.

Consider the 5×5 matrix

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 15-\lambda & \cdot & \cdot & \cdot & \cdot \\ 15-\lambda & \cdot & \cdot & \cdot & \cdot \\ 15-\lambda & \cdot & \cdot & \cdot & \cdot \\ 15-\lambda & \cdot & \cdot & \cdot & \cdot \\ 15-\lambda & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

It is given that A has only one real eigen value. Then the real eigen value of A is

5 eigen roots

1 real roots

$\underbrace{a \pm ib \quad c \pm id}_{\text{4 complex roots}}$

[2 Marks]

(a) -25
(b) 0
(c) 15 ✓
(d) 25

Thank you

GW Soldiers !