# ALL BRANCHES

# ENGINEERING MATHEMATICS

Lecture No.-9

Calculus

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# Topics to be Covered

- PARTIAL DIFFERENTIATION
- HOMOGENEOUS FUNCTION
- EULER'S THEOREM
- INTEGRATION
- DEFINITE INTEGRALS
- PROPERTY OF DEFINITE INTEGRALS

# [ MACLAURIN'S THEOREM ]

## Important Maclaurin's Expansion

i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

ii) $\tan x = 0 + x.1 + \frac{x^2}{2!}.(0) + \frac{x^3}{3!}.(2). + \cdots\cdots$

iii) $\sinh x = 0 + x.(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \cdots = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

iv) $\cosh x = 1 + x.(0). + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \cdots = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

v) $\sin^{-1} x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \cdots$

vi) $\cos^{-1} x = \frac{\pi}{2} + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots$

$f(x) = \sinh x = 0$

$f'(x) = \cosh x = 1$

$f''(x) = \sinh x = 0$

$f'''(x) = \cosh x = 1$

$f(x) = \cosh x = 1$

$f'(x) = \sinh x = 0$

$f''(x) = \cosh x = 1$

$f'''(x) = \sinh x = 0$

$f(x) = \sin^{-1} x = 0$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} = 1$

$f''(x) = -\dfrac{(-2x)}{2(1-x^2)^{3/2}} = \dfrac{x}{(1-x^2)^{3/2}} = 0$

$f'''(x) = \dfrac{(1-x^2)^{3/2} \cdot 1 - x\,\frac{3}{2}(1-x^2)^{1/2}(-2x)}{(1-x^2)^{\frac{3}{2}\times 2}} = 1$

$f(x) = \cos^{-1} x = \pi/2$

$f'(x) = -\dfrac{1}{\sqrt{1-x^2}} = -1$

$f''(x) = -\dfrac{x}{(1-x^2)^{3/2}} = 0$

Ex:- Taylor series expansion of $\sin x$ about $x = \pi/6$ [GATE]

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$f(x) = \frac{1}{2} + (x - \pi/6)\frac{\sqrt{3}}{2} + \frac{\left(x - \frac{\pi}{6}\right)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(x - \pi/6)^3}{3!}\left(-\frac{\sqrt{3}}{2}\right)$$

$f(x) = \sin x = 1/2$

$f'(x) = \cos x = \sqrt{3}/2$

$f''(x) = -\sin x = -1/2$

$f'''(x) = -\cos x = -\sqrt{3}/2$

$f^{iv}(x) = \sin x = 1/2$

$\sqrt{3}/2$

$-1/2$

$-\sqrt{3}/2$

Ex:- $f(x) = 2x^3 + 7x^2 + x - 1$ expand in terms of $(x-2)$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2)$$

$$f(x) = 2x^3 + 7x^2 + x - 1 = 45$$

$$f'(x) = 6x^2 + 14x + 1 = 53$$

$$f''(x) = 12x + 14 = 38$$

$$f'''(x) = 12$$

$$f(x) = 45 + (x-2)53 + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12)$$

$$f(x) = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

# Arithmetic - Geometric Progression (A.G.P.):-

A.P. $\Rightarrow a, a+d, a+2d \ldots\ldots a+(n-1)d$

$$\text{Sum} = \frac{n}{2}[a + a + (n-1)d]$$

G.P. $\Rightarrow a, ar, ar^2, ar^3 \ldots\ldots ar^{n-1}$

$r \rightarrow$ Common ratio

$$\text{Sum} = a\left[\frac{r^n - 1}{r-1}\right] \quad \text{if } r > 1$$

$$= a\left[\frac{1 - r^n}{1-r}\right] \quad \text{if } r < 1$$

If G.P. is infinite & $r < 1$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

Ex:- $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$

$$S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$$

Ex:- $S = 1 + \frac{4}{2} + \frac{7}{2^2} + \frac{10}{2^3} + \frac{13}{2^4} + \cdots\cdots = 8$

$\frac{1}{2} \cdot S = \frac{1}{2} + \frac{4}{2^2} + \frac{7}{2^3} + \frac{10}{2^4} + \frac{13}{2^5}$

$d = 3$

$r = 1/2 < 1$

$S - \frac{S}{2} = 1 + \frac{4-1}{2} + \frac{7-4}{2^2} + \frac{10-7}{2^3} + \frac{13-10}{2^4} + \cdots$

$\frac{S}{2} = 1 + \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \cdots$

$1 + \frac{3}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \infty\right]$

$\frac{S}{2} = 1 + \frac{3}{2}\left[\frac{1}{1 - 1/2}\right] = 1 + \frac{3}{2} \times 2 = 4$

$\boxed{S = 8}$

# Convergence & Divergence of Infinite Series :-

- If $\sum_{n=1}^{\infty} a_n = \text{finite}$, series is convergent.
- If $\sum_{n=1}^{\infty} a_n = \text{infinite}$, series is divergent.
- If $\sum_{n=1}^{\infty} a_n = $ Neither finite nor infinite, series is oscillatory.

**NOTE:-** Necessary condition for convergence of series :- $\lim_{n \to \infty} a_n = 0$

Ex:- $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\cdots$

$r = 2/3$

$a = 1$

$S_\infty = \frac{a}{1-r} = \frac{1}{1 - 2/3} = \boxed{3}$ $\therefore$ Series is convergent.

Ex: $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \log\frac{5}{4} + \cdots$

$$= \log\left(\frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{5}{\cancel{4}}\cdots\cdots\frac{n+1}{\cancel{n}}\right)$$

$$= \lim_{n\to\infty} \log(n+1) = \infty$$ $\therefore$ Series is divergent.

Ex:- Series $a_n = (-1)^n$

$$= -1 + 1 - 1 + 1 - 1 + 1 \qquad = 0 \qquad n \text{ is even}$$

$$= -1 + 1 - 1 + 1 - 1 \qquad = -1 \qquad n \text{ is odd}$$

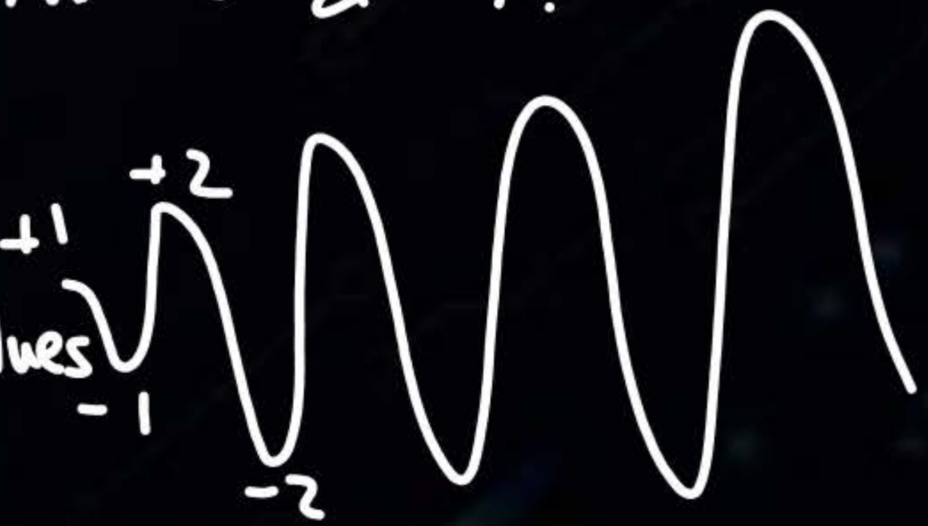
This series is oscillating finitely b/w 0 & -1.

Ex: Series $a_n = (-1)^n \cdot n$

| Terms | Sum |
|---|---|
| 1 → | -1 |
| 2 → | 1 |
| 3 → | -2 |
| 4 → | 2 |

$$= -1 + 2 - 3 + 4 - 5 + 6 - \cdots = \text{diff. values for diff. values of } n.$$

This series is oscillating infinitely



# METHODS TO FIND CONVERGENCE OF INFINITE SERIES:-

1) Comparison Test:- $(a_n < b_n)$ $\Sigma a_n$ and $\Sigma b_n$

→ If $\Sigma b_n$ converges, then $\Sigma a_n$ also converges.

→ If $\Sigma a_n$ diverges, then $\Sigma b_n$ also diverges.

2) Ratio Test:-

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \begin{cases} <1, & \text{series is convergent} \\ >1, & \text{series is divergent} \\ =1, & \text{case fails.} \end{cases}$$

3) Cauchy Root Test:-

$$\lim_{n\to\infty}(a_n)^{1/n} = \begin{cases} <1, & \Sigma a_n \text{ is convergent} \\ >1, & \Sigma a_n \text{ is divergent} \\ =1, & \text{case fails} \end{cases}$$

Ex:-

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots \frac{1}{n^n} + \cdots \infty$$

$$a_n = \frac{1}{n^n} < b_n = \frac{1}{2^n}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

$$S(b_n)_\infty = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1$$

$$a_n < b_n$$

Since $\sum b_n$ converges $\therefore$ $\sum a_n$ also converges

Ex:-

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$$

$$a_n = \frac{1}{2^n + 3^n} < b_n = \frac{1}{3^n}$$

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$$

$$S(b_n)_\infty = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

Since $\sum b_n$ converges then $\sum a_n$ also converges.

Ex:-

$$1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots$$

$$a_n = \frac{2^n}{n!} \qquad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Ratiotest $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$

$\Sigma a_n$ $\therefore$ Series is convergent.

Ex:-

$$\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \cdots$$

$a_n = \frac{n^3}{3^n}$ Cauchy root test $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left[\frac{n^3}{3^n}\right]^{1/n} = \lim_{n\to\infty}\frac{n^{3/n \to 0}}{3} = \frac{1}{3} < 1$

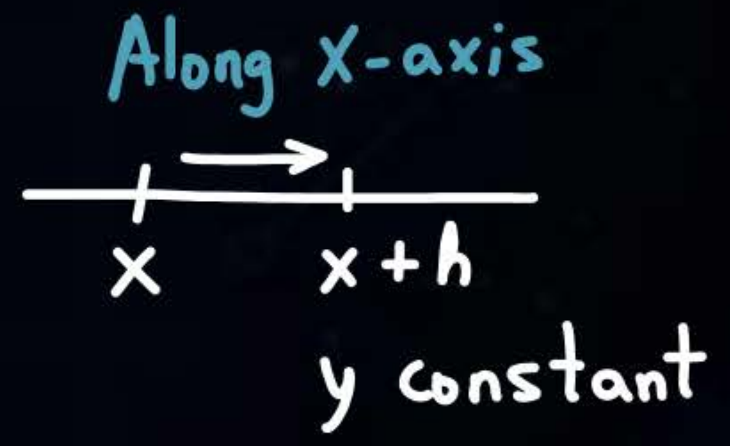
By Cauchy root test; $\Sigma a_n$ (series) is convergent.

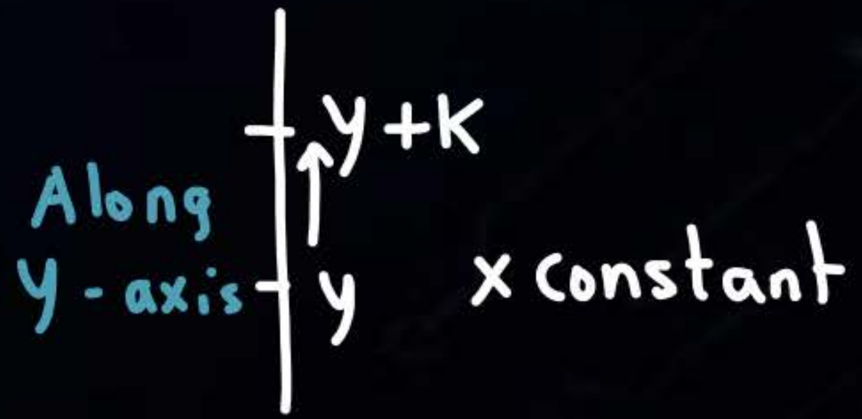
# PARTIAL DIFFERENTIATION

$$Z \rightarrow f(x, y)$$

y is constant $\left\{ \dfrac{\partial f}{\partial x} = \dfrac{f(x+h) - f(x)}{x+h-x} \right.$

x is constant $\left\{ \dfrac{\partial f}{\partial y} = \dfrac{f(y+k) - f(y)}{y+K-y} \right.$



1. $f_{xy} = f_{yx}$ $\qquad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

2. $\dfrac{\partial^2 f}{\partial x^2} \neq \left(\dfrac{\partial f}{\partial x}\right)^2$ & $\dfrac{\partial^2 f}{\partial x \partial y} \neq \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial f}{\partial y}$

3. $\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + 2\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial u}{\partial y}$

4. $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + 2\dfrac{\partial^2 u}{\partial x \partial y}$

Ex:-

$$f = \tan^{-1} y/x$$

i) $f_{yx} = f_{xy}$

ii) $f_{xx} \neq (f_x)^2$

iii) $(f_x + f_y)^2$

iv) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f$

$= f_{xx} + f_{yy} + 2 f_{xy}$

$$\frac{\partial f}{\partial x} = \frac{y}{1+(y/x)^2}\left(-\frac{1}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

# Thank you

# GW Soldiers !