ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-3

Linear Algebra

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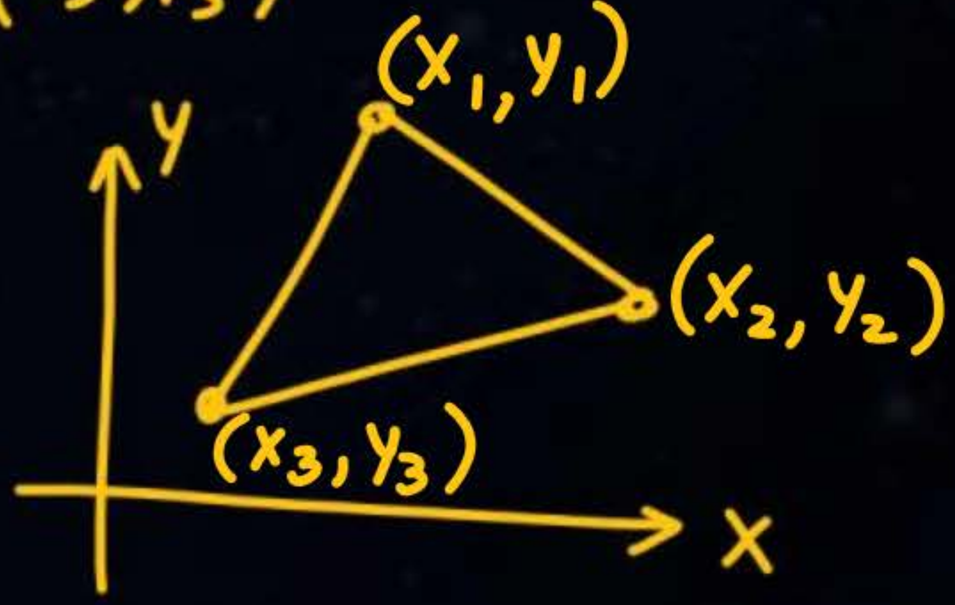
Topics to be Covered

SPECIAL TYPES OF MATRICES

Let triangle have vertices $(x_1, y_1)\ (x_2, y_2)\ (x_3, y_3)$

$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}\right|$$



Ex:- Triangle having corners $(3,0)\ (4,1)$ and $(10,3)$

Find its area = ?

$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} 1 & 3 & 0 \\ 1 & 4 & 1 \\ 1 & 10 & 3 \end{vmatrix}\right|$$

[INVERSE OF A MATRIX]

Definition: Let a be a square matrix of order n. If there is a matrix B such that

$$A.\mathrm{B} = \mathrm{B.A.} = \mathrm{I}, \text{ then}$$

B is called the inverse of the matrix A and denotes by A^{-1}. Thus, if A is square matrix of order n, then A^{-1} is also a square matrix of order n.

$$\boxed{A^{-1} = \frac{adjA}{|A|}}$$

A^{-1} exist if $|A| \neq 0$

$$A\cdot(\mathrm{Adj}\,A) = |A|\,I$$

$$A^{-1}A\,(\mathrm{Adj}\,A) = |A|\,A^{-1}I \quad \text{(Pre multiply by } A^{-1}\text{)}$$

$$I\,(\mathrm{Adj}\,A) = |A|\,A^{-1}$$

$$\boxed{A^{-1} = \frac{\mathrm{Adj}\,A}{|A|}}$$

Properties of inverse & Adjoint of Matrix :-

1. $(AB)^{-1} = B^{-1}A^{-1}$

   $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

   $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

2. $A(\mathrm{Adj}\,A) = (\mathrm{Adj}\,A).A = |A|\,I_n$

3. $\text{Adj}(\text{Adj} A) = |A|^{n-2} A$

4. $|A^{-1}| = \dfrac{1}{|A|}$

5. $|\text{Adj} A| = |A|^{n-1}$

6. $|\text{Adj}(\text{Adj} A)| = |A|^{(n-1)^2}$

7. $|\text{Adj}(\text{Adj}(\text{Adj} A))| = |A|^{(n-1)^3}$

$n \rightarrow$ Order of square matrix

NOTE :- Inverse of diagonal matrix is reciprocal of diagonals elements.

$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$

Ex :- $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3\times3}$

$n \rightarrow 3$

Cofactor matrix $= \begin{bmatrix} -1 & 0 & -1 \\ +1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$

i) $\text{Adj} A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

ii) $A^{-1} = \dfrac{\text{Adj} A}{|A|} = \dfrac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$|A| = 1(0-1) - 0 + 1(-1-0)$

$= -1 - 0 - 1 = -2$

iii) $|A^{-1}| = 1/|A| = 1/-2 = -\frac{1}{2}$

iv) $|\text{Adj } A| = |A|^{n-1} = (-2)^{3-1} = (-2)^2 = 4$

v) $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2} = (-2)^{(3-1)^2} = (-2)^4 = 16$

vi) $|A^2| = |A|^2 = (-2)^2 = 4$

vii) $|A^3| = |A|^3 = (-2)^3 = -8$

Shortcut for 2 X 2 matrix :-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

SARRUS RULE for Δ :-

$$\begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 4 \\ 2 & 3 & 6 \end{vmatrix}$$

$$\begin{matrix} 1 & 1 & 0 & 1 & 1 \\ 5 & 0 & 4 & 5 & 0 \\ 2 & 3 & 6 & 2 & 3 \end{matrix}$$

$$\Delta = (0+8+0) - (0+12+30)$$

$$8 - 42$$

$$\Delta = -34$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{3\times3} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \qquad \text{Cofactor matrix} = \begin{bmatrix} -1 & 0 & -1 \\ +1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

i) $\text{Adj } A = \begin{bmatrix} -1 & +1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$$|A| = 1(0-1) + 0 + 1(-1-0)$$
$$|A| \quad -1+0-1 = -2$$

ii) $A^{-1} = \dfrac{\text{adj } A}{|A|} = -\dfrac{1}{2}\begin{bmatrix} -1 & +1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

iii) $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{-2}$

iv) $|\text{Adj } A| = |A|^{n-1} = (-2)^{3-1} = (-2)^2 = 4$

v) $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2} = (-2)^{(3-1)^2} = (-2)^4 = 16$

vi) $|A^2| = |A|^2 = (-2)^2 = 4$

vii) $|A^3| = |A|^3 = (-2)^3 = -8$

$$A = \begin{bmatrix} 1 & 6 \\ -2 & 5 \end{bmatrix} \qquad \begin{vmatrix} + & - \\ - & + \end{vmatrix} \qquad \text{Cof. matrix} = \begin{bmatrix} 5 & 2 \\ -6 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{17}\begin{bmatrix} 5 & -6 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -6 \\ +2 & 1 \end{bmatrix}$$

$N \rightarrow$ 1, 2, 3 . . . .

$W \rightarrow$ 0, 1, 2, 3, ...

Z or $I \rightarrow$ −3, −2, −1, 0, 1, 2, 3, 4 . . .

$R \rightarrow$

Complex → Real + Imaginary

↳ Root of -ve number

$\sqrt{-25}$

$\sqrt{-1}\,\sqrt{25} = 5i$

CONJUGATE OF A MATRIX

Let A be any m × n matrix having complex numbers as its elements. The matrix of order m × n which is obtained from A by replacing each element of A by its conjugate is called the conjugate of A denoted by $\bar{A}$. Thus if

$A = [a_{ij}]_{m\times n}, \bar{A} = [a_{ij}]_{m\times n}$ where $\bar{a}_{ij}$ is the conjugate of a_{ij}.

CONJUGATE OF A MATRIX

NOTE: If A real matrix, then $\bar{A} = A$

e.g. If $A = \begin{bmatrix} 1+2i & i \\ 3 & 2-3i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 1-2i & -i \\ 3 & 2+3i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1-2i & -i \\ 3 & 2+3i \end{bmatrix}$$

(Real matrix) $A = \begin{bmatrix} 5 & 0 \\ -1 & -2 \end{bmatrix} \longrightarrow \bar{A} = \begin{bmatrix} 5 & 0 \\ -1 & -2 \end{bmatrix}$

CONJUGATE TRANSPOSE OF A MATRIX

The conjugate of the transpose of a matrix A is called the conjugate transpose of A and denoted by A*. Thus if A = $[a_{ij}]$, then

$$A^* = (\bar{A'}) = (\bar{A})' = [\bar{a}_{ji}]$$

$$A^* = (\bar{A})' = \overline{(A')} = A^{\theta}$$

Clearly the conjugate of the transpose is the same as the transpose of the conjugate.

Transjugate

$\swarrow$ $\searrow$

Transpose Conjugate

$$\boxed{\begin{aligned} (A+B)^* &= A^* + B^* \\ (AB)^* &= B^* A^* \\ (A^*)^* &= A \end{aligned}}$$

CONJUGATE TRANSPOSE OF A MATRIX

NOTE: (i) $(A^*)^* = A$

(ii) If A is real matrix, then $A^* = A'$

e.g. If $A \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $A' \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7-8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$;

$$A^* = (\bar{A}') = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

SPECIAL TYPES OF MATRICES

IDEMPOTENT MATRIX

A square matrix A is said to be idempotent if $\boxed{A^2 = A}$.

- If A → idempotent, then (I − A) is idempotent.
- If A and B are idempotent, then (A + B) is idempotent when AB = O

E.g: $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$A.A = A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

INVOLUTORY MATRIX

A square matrix A is said to be involutory matrix if $\boxed{A^2 = I}$ (unit matrix).

$\boxed{A = A^{-1}}$

E.g: $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

NILPOTENT MATRIX

A square matrix A such that $A^k = O$ where k is the least positive integer, is called the nilpotent matrix of index k.

Ex :- $A \neq 0$

$A^2 \neq 0$

$A^3 \neq 0$

$A^4 = 0$

then A is nilpotent of index 4.

E.g: $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$$A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

A is nilpotent matrix of index 2.

Square matrices

Real Matrices	Complex Matrices
→ Symmetric matrix $A = A^T$	→ Hermitian matrix $A = A^*$
→ Skew-symmetric matrix $A = -A^T$	→ Skew-Hermitian matrix $A = -A^*$
→ Orthogonal matrix $AA^T = I$	→ Unitary matrix $AA^* = I$

SYMMETRIC MATRIX

$\boxed{A = A^T}$

$A \to a_{ij}$

$A^T \to a_{ji}$

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be symmetric if $A = A'$

i.e. $\boxed{a_{ij} = a_{ji}}$ for all values of i and j

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$, then

3×3

$n = 3$

$A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix} = A,$

- Symmetric matrix is symmetrical about diagonal.

Let $A_{n\times n}$ be any matrix,

$\rightarrow$ $A + A^T$ is always symmetric

$\rightarrow$ AA^T is always symmetric

- Maximum number of unique elements

$= n + \frac{n^2 - n}{2} = \frac{n(n+1)}{2}$

$2\times 2 \to 3 \quad 3\times 3 \to 6 \quad 4\times 4 \to 10$

SKEW-SYMMETRIC MATRIX

$A = a_{ij}$ $\quad A^T = a_{ji}$

$\boxed{A = -A^T}$

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be skew-symmetric if $A = -A'$

i.e. $\boxed{a_{ij} = -a_{ji}}$ $\forall$ i & j, e.g. If $A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 6 \\ -4 & -6 & 0 \end{bmatrix}$, then

$A' = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & -6 \\ 4 & 6 & 0 \end{bmatrix} = -A.$

→ Sum of all elements = 0

→ Diagonal elements are always 0.

→ Δ of odd order skew-symmetric matrix is 0.

→ $A_{n \times n}$ is any matrix, $A - A^T$ is always skew-symmetric

[SKEW-SYMMETRIC MATRIX]

NOTE:

1. If $A_{n \times n}$ such that A is skew symmetric and n is odd then $|A| = 0$ (always).
2. Every square matrix can be uniquely expressed as the sum of the symmetrical and skew-symmetrical matrices i.e. $A = \left(\frac{A+A'}{2}\right) + \left(\frac{A-A'}{2}\right) = P + Q$ where P is symmetric and Q is skew symmetric.
3. Symmetric matrix is symmetrical about leading diagonal.

SKEW-SYMMETRIC MATRIX

E.g: Show that every diagonal element of a skew-symmetric matrix is necessarily zero.

Solution: Let A be any skew-symmetric matrix i.e.

$$a_{ij} = -a_{ji} \ \forall \text{ i and j} \qquad \ldots(i)$$

For diagonal elements of a matrix, we can put i = j in (i)

i.e., $$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$$

$\Rightarrow$ every element in the principal diagonal is necessarily zero.

SKEW-SYMMETRIC MATRIX

E.g: Write the following matrix as the sum of a symmetric and a skew-symmetric matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

If A is any square matrix, then symmetric and skew-symmetric matrices are $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ respectively and A can be written as

SKEW-SYMMETRIC MATRIX

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$

$$\Rightarrow \quad A + A' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix} \quad \ldots(i)$$

SKEW-SYMMETRIC MATRIX

and $A - A' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$ ...(ii)

Adding (i) and (ii), we get

$$2A = (A + A') + (A - A') = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$$

SKEW-SYMMETRIC MATRIX

$$\Rightarrow A = \frac{1}{2}\overbrace{\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 6 \\ 10 & 6 & 0 \end{bmatrix}}^{A+A^T} + \frac{1}{2}\overbrace{\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}}^{A-A^T} = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 0 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{Skew-symmetric}}$$

A = symmetric matrix + skew – symmetric matrix.

ORTHOGONAL MATRIX

$\boxed{AA^T = I}$ $\boxed{A^T = A^{-1}}$

A square matrix A is called an orthogonal matrix if the product of matrix A with its transpose matrix A' is an identity matrix, i.e. AA' = I

E.g: Show that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal.

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$R_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ $R_2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$ $|R_1| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$

ORTHOGONAL MATRIX

$$\therefore AA' = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

Hence A is an orthogonal matrix.

Properties :-

- $\rightarrow R_1 . R_2^T = 0$ $\quad \rightarrow C_1 . C_2^T = 0$
- $\rightarrow R_2 . R_3^T = 0$ $\quad \rightarrow C_2 . C_3^T = 0$
- $\rightarrow R_1 . R_3^T = 0$ $\quad \rightarrow C_1 . C_3^T = 0$
- $\rightarrow |R_1| = |R_2| = |R_3| = |C_1| = |C_2| = |C_3| = 1$

If X_1 and X_2 are 2 vectors, then they are orthogonal if

$$X_1 . X_2^T = 0$$

$R_1 . R_2^T$

$$\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}2/3\\ 1/3\\ -2/3\end{bmatrix} = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

[ORTHOGONAL MATRIX]

NOTE:

1. The value of determinant of an orthogonal matrix is either 1 or –1.

   Proof : Let A be any orthogonal matrix i.e.

   $A'A = I \Rightarrow |A'A| = |I| \Rightarrow |A'| \cdot |A| = 1$ $\quad (\because |I| = I)$

   $\Rightarrow |A| \cdot |A| = 1$ $\quad (\because |A'| = A)$

   $\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

2. If A and B are orthogonal matrices then AB and BA are also orthogonal.

[HERMITIAN MATRIX]

$A \to a_{ij}$ $\overline{A^T} = \overline{a_{ji}}$

A square matrix X is called Hermitian if $\boxed{A = A^*}$. Thus, a square matrix A = $[a_{ij}]$ is Hermitian if $\boxed{a_{ij} = \bar{a}_{ji}}$ ∀ i and j

$A = \overline{(A^T)}$

E.g: The matrix $A = \begin{bmatrix} 1 & 2+i & 3+2i \\ 2-i & 3 & -3i \\ 3-2i & 3i & -2 \end{bmatrix}$ is Hermitian.

$a_{12} = \bar{a}_{21}$ $a_{13} = \bar{a}_{31}$ $a_{23} = \bar{a}_{32}$

→ $A + A^*$ is always Hermitian matrix

→ Diagonal elements are always real.

SKEW-HERMITIAN MATRIX

A square matrix A is said to be skew-Hermitian if $\boxed{A^* = -A}$. Thus, a square matrix $A = [a_{ij}]$ is skew-Hermitian if $\boxed{a_{ij} = -\bar{a}_{ji}}$ $\forall$ i and j

E.g: The matrix $A = \begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix}$ is skew-Hermitian.

$a_{12} = -\overline{a_{21}}$ $\quad a_{13} = -\overline{a_{31}}$ $\quad a_{23} = -\overline{a_{32}}$

→ $A - A^*$ is always skew-hermitian matrix.

→ Diagonal elements are always 0 or imaginary.

SKEW-HERMITIAN MATRIX

For $A'\begin{bmatrix} i & -2+i & -3+2i \\ 2+i & 3i & -3i \\ 3+2i & -3i & 0 \end{bmatrix}$ and

$$A^{*} = \bar{A}' = \begin{bmatrix} -i & -2-i & -3-2i \\ 2-i & -3i & 3i \\ 3-2i & 3i & 0 \end{bmatrix} =$$

$$-\begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix} = -A$$

[SKEW-HERMITIAN MATRIX]

NOTE:

1. If A and B are two matrices such that these are conformable for addition then $(\overline{A+B}) = \bar{A} + \bar{B}$
2. If A and B are two matrices conformable for multiplication, then $(AB) = \bar{A}.\bar{B}$
3. If A and B are any two matrices conformable for addition, then $(A+B)^* = A^* + B^*$
4. If A and B are any two matrices conformable for multiplication, then $(AB)^* = B^* A^*$
5. The diagonal elements of a Hermitian matrix are necessarily real.

SKEW-HERMITIAN MATRIX

6. The diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.
7. Every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix. $A = \left(\frac{A+A^*}{2}\right) + \left(\frac{A-A^*}{2}\right) = P + Q$ where P & Q are Hermitian and skew-Hermitian matrices respectively.
8. Every square matrix can be uniquely expressed as P + iQ where P and Q are Hermitian.

$$A = \left(\frac{A + A^*}{2}\right) + i\left(\frac{A - A^*}{2i}\right) = P + iQ$$

[SKEW-HERMITIAN MATRIX]

9. Every Hermitian matrix A can be uniquely expressed as P + iQ form, where P and Q are real symmetric and real skew-symmetric. Let A be a Hermitian matrix i.e. $A^* = A$. Now we can write

$$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$$

where $$P = \frac{1}{2}(A + \bar{A}) \text{ and } Q = \frac{1}{2i}(A - \bar{A})$$

Here P and Q are real symmetric and real skew-symmetric matrices respectively.

UNITARY MATRIX

A square matrix A is said to be unitary matrix if $A^* A = I = AA^*$

$$\boxed{AA^* = I}$$

NOTE:

If the matrix A is real then $\bar{A} = A, A^* = A'$, so we can write $A'A = I = AA'$. Thus a unitary matrix on a field of real numbers is also an orthogonal matrix.

NOTE :-

$$|AA^*| = |I|$$
$$|A|\,|\bar{A}'| = 1$$
$$|A|\,|\bar{A}| = 1$$
$$|A|\,|A| = 1$$
$$\boxed{|A| = 1}$$

$$|A| = |A^T|$$
$$|A| = |\bar{A}|$$

UNITARY MATRIX

E.g: Show that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Solution : If $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ then $\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

So that $A^* = (\bar{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

UNITARY MATRIX

So that $A^* = (\bar{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$\therefore A^* A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is unitary matrix.

[UNITARY MATRIX]

Theorem: The modulus of the determinant of a unitary matrix is unity.

Proof: Let A is unitary matrix i.e.

$A * A = I \Rightarrow |A * A| = |I|$ [taking determinant on both sides]

$\Rightarrow |A^*||A| = 1 \Rightarrow |(\bar{A})'||A| = 1$ $[\because |\bar{A}'| = |\bar{A}|]$

$\Rightarrow |(\bar{A})||A| = 1 \Rightarrow |A|^2 = 1$ $[\because |\bar{A}| = |A|]$

UNITARY MATRIX

$\Rightarrow |A| = 1$

$\Rightarrow$ The modulus of the determinant of unitary matrix is unity.

E.g: Express $\begin{bmatrix} -2+3i & 1-i & 2+1 \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$ as the sum of a Hermitian and a Skew-Hermitian matrix.

UNITARY MATRIX

If A is any square matrix, then we can write

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

Where $\frac{1}{2}(A + A^*)$ is a Hermitian matrix and $\frac{1}{2}(A - A^*)$ is a Skew-Hermitian matrix.

UNITARY MATRIX

Let $A = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$

Then $\bar{A} = \begin{bmatrix} -2-3i & 1+i & 2-i \\ 3 & 4+5i & 5 \\ 1 & 1-i & -2-2i \end{bmatrix}$

$\therefore (\bar{A})' = A^* = \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$

UNITARY MATRIX

Now $\frac{1}{2}(A+A^*)=\frac{1}{2}\begin{bmatrix}-4 & 4-i & 3+i\\ 4+i & 8 & 6-i\\ 3-i & 6+i & -4\end{bmatrix}=\begin{bmatrix}-2 & 2-\frac{1}{2}i & \frac{3}{2}+\frac{1}{2}i\\ 2+\frac{1}{2}i & 4 & 3-\frac{1}{2}i\\ \frac{3}{2}-\frac{1}{2}i & 3+\frac{1}{2}i & -2\end{bmatrix}$

which is Hermitian matrix.

Again $\frac{1}{2}(A-A^*)=\frac{1}{2}\begin{bmatrix}6i & -2-i & 1+i\\ 2-i & -10i & 4+i\\ -1+i & -4+i & 4i\end{bmatrix}=\begin{bmatrix}3i & -1-\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i\\ 1-\frac{1}{2}i & -5i & 2+\frac{1}{2}i\\ -\frac{1}{2}+\frac{1}{2}i & -2+\frac{1}{2}i & 2i\end{bmatrix}$

UNITARY MATRIX

which is Skew-Hermitian matrix.

Thus $A = \begin{bmatrix} -2 & 2-\frac{1}{2}i & \frac{3}{2}+\frac{1}{2}i \\ 2+\frac{1}{2}i & 4 & 3-\frac{1}{2}i \\ \frac{3}{2}-\frac{1}{2}i & 3+\frac{1}{2}i & -2 \end{bmatrix} + \begin{bmatrix} 3i & -1-\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ 1-\frac{1}{2}i & -5i & 2+\frac{1}{2}i \\ -\frac{1}{2}+\frac{1}{2}i & -2+\frac{1}{2}i & 2i \end{bmatrix}$

where the first matrix on the R.H.S. is Hermitian and the second matrix is skew-Hermitian.

Thank you

GW Soldiers !