ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-04

Vector Calculus

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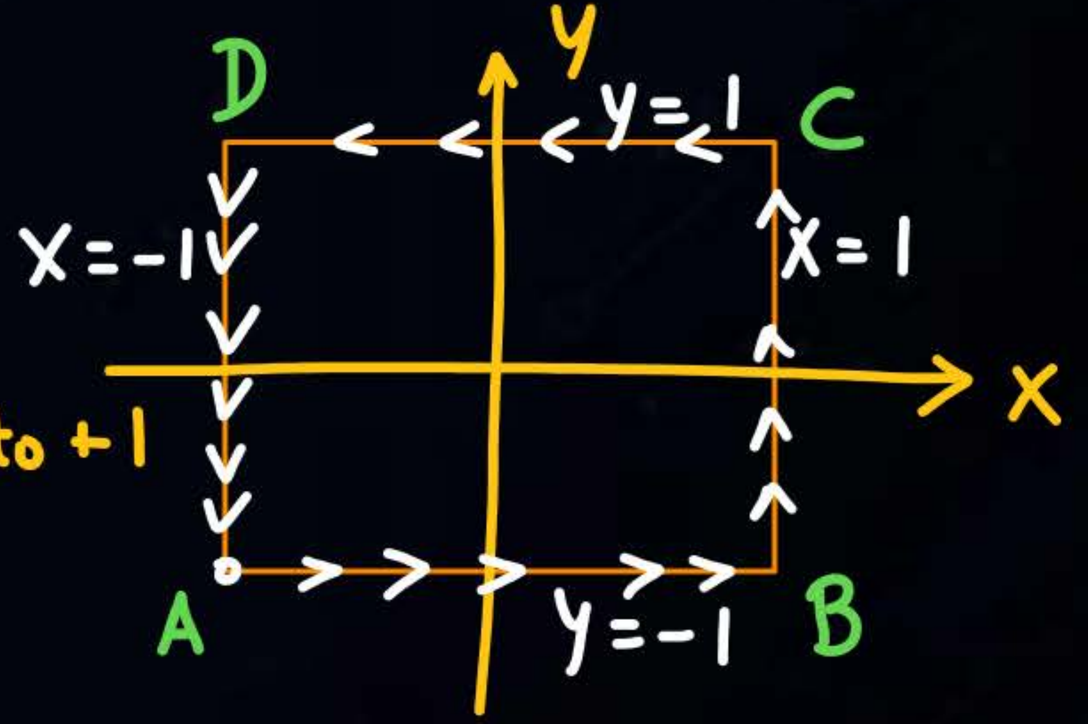
Topics to be Covered

- VECTOR BASICS
- STRAIGHT LINES/3D PLANES
- GRADIENT (VECTOR DIFFERENTIATION)
- DIVERGENCE (VECTOR DIFFERENTIATION)
- CURL (VECTOR DIFFERENTIATION)
- LINE, SURFACE, VOLUME INTEGRAL (VECTOR INTEGRATION)
- GREEN, & STOKE'S THEOREM (VECTOR INTEGRATION)
- GAUSS DIVERGENCE THEOREM (VECTOR INTEGRATION)

Q. $\int_C \vec{F}\cdot\overrightarrow{dr}$ where $\vec{F}=\frac{y\hat{\imath}-x\hat{\jmath}}{x^2+y^2}$ and C is square formed by line and $x=\pm 1$ & $y=\pm 1$,

$$\int\left(\frac{y\hat{i}-x\hat{j}}{x^2+y^2}\right)\cdot\left(dx\,\hat{i}+dy\,\hat{j}\right)$$



AB $\to x=-1$ to $+1$

$y=-1$

$dy=0$

$$AB \to \int_{x=-1}^{x=+1}\frac{y\,dx-x\,dy}{x^2+y^2}=\int_{-1}^{+1}\frac{-dx}{x^2+(-1)^2}$$

$$=-\left[\tan^{-1}x\right]_{-1}^{+1}=-\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=-\frac{\pi}{2}$$

BC $y=-1$ to $+1$

$x=1$

$dx=0$

$$BC \to \int_{y=-1}^{y=+1}\frac{y\,dx-x\,dy}{x^2+y^2}=\int_{-1}^{+1}\frac{-dy}{1+y^2}=-\left[\tan^{-1}y\right]_{-1}^{+1}=-\frac{\pi}{2}$$

$$CD \rightarrow \int_{x=1}^{x=-1} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_{1}^{-1} \frac{dx}{x^2+1} = -\pi/2$$

$$DA \rightarrow \int_{y=1}^{y=-1} \frac{y\,dx - x\,dy}{x^2+y^2} = \int_{1}^{-1} \frac{dy}{1+y^2} = -\pi/2$$

CD $x = 1$ to -1

$y = 1$

$dy = 0$

DA $y = 1$ to -1

$x = -1$

$dx = 0$

$$\oint \vec{F}\cdot dr = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CD} \vec{F}\cdot\vec{dr} + \int_{DA} \vec{F}\cdot\vec{dr}$$

$$-\pi/2 \quad -\pi/2 \quad -\pi/2 \quad -\pi/2 \quad = -2\pi$$

Ex:- $\int_C xy\,dz + yz\,dx + zx\,dy$ where C is a curve joining the points $(1,1,1)$ to $(5,6,7)$

$$\int d(xyz) = [xyz]_{(1,1,1)}^{(5,6,7)}$$

$$= 5\times6\times7 - 1\times1\times1$$

$$= 209$$

$$d(xy) = x\,dy + y\,dx$$

Ex:- $\int_C x^2y\,dz + 2xyz\,dx + x^2z\,dy$ from $(1,1,1)$ to $(5,6,7)$

$$\int d(x^2yz) = [x^2yz]_{(1,1,1)}^{(5,6,7)}$$

Q. **Let $\nabla \cdot (f\vec{v}) = x^2y + y^2z + z^2x$, where f and v are scalar and vector respectively. If $\vec{v} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$, then $\vec{v}.\nabla f$ is**

$\operatorname{div}(f\vec{v}) = f \operatorname{div}\vec{v} + \vec{v} \operatorname{grad} f$

$\nabla . f\vec{v} = 0 + \vec{v} . \nabla f$

A. $x^2y + y^2z + z^2x$

B. $2xy + 2yz + 2zx$

C. $x + y + z$

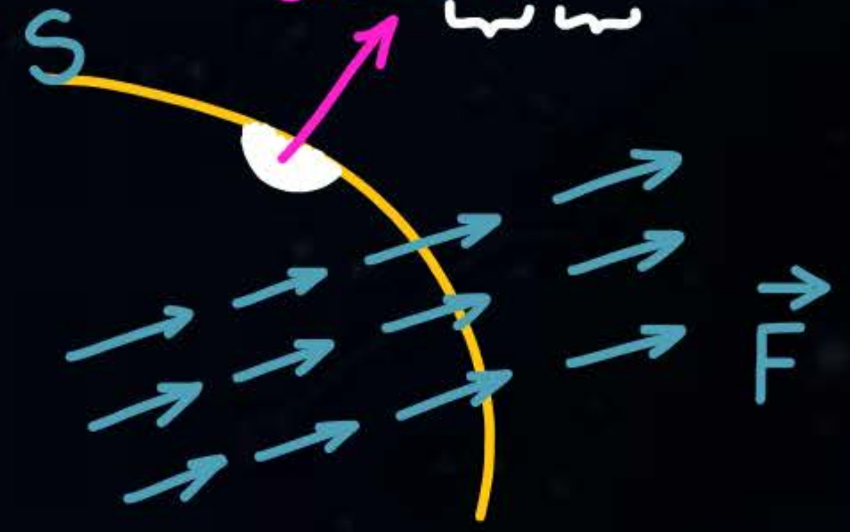
D. 0

SURFACE INTEGRAL

$$\hat{n} = \frac{\vec{dS}}{dS}$$

$$\vec{dS} = \underbrace{\hat{n}}\,\underbrace{dS}$$

let dS be surface area of element on surface S. Surface integral of $\vec{F}$ over surface S



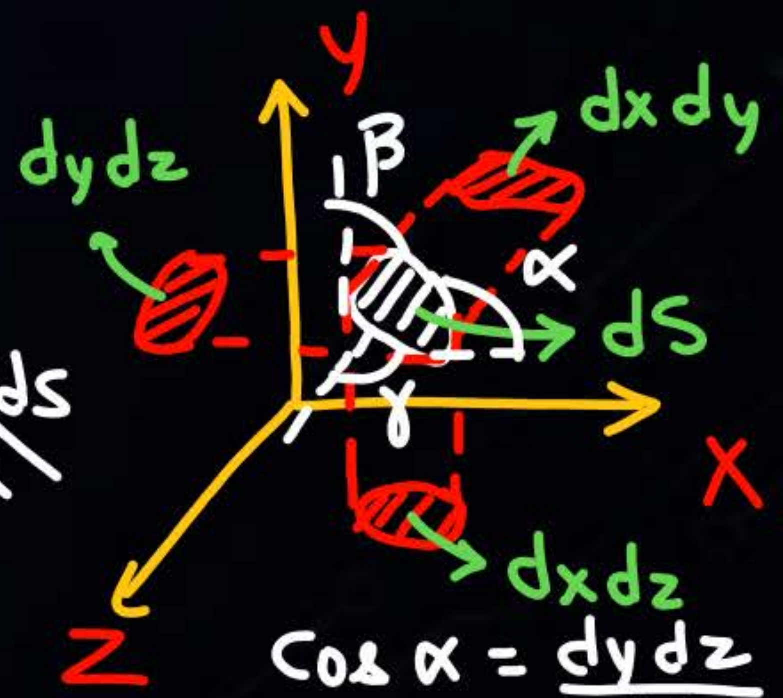
$$\boxed{S.I. = \iint_S \vec{F}\cdot\vec{dS} = \iint \vec{F}\,\hat{n}\,dS}$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\vec{dS} = \hat{n}\,dS = [\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}]\,dS$$

$$= \left[\frac{dy\,dz}{dS}\hat{i} + \frac{dx\,dz}{dS}\hat{j} + \frac{dx\,dy}{dS}\hat{k}\right] dS$$

$$\vec{dS} = \hat{n}\,dS = dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dx\,dy\,\hat{k}$$



$$\cos\alpha = \frac{dy\,dz}{dS}$$

$$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\hat{n}\,dS = \iint F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$$

Vector form | Cartesian form

$\iint_S \vec{F}\cdot\hat{n}\,dS$

- Projection on X-Y plane $= \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$
- Projection on Y-Z plane $= \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$
- Projection on X-Z plane $= \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

Ex:- $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\,\hat{k}$ & S is the surface of cylinder $x^2+y^2=16$ in the first octant b/w $z=0$ & $z=5$.

Soln :- $$\iint_S F\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

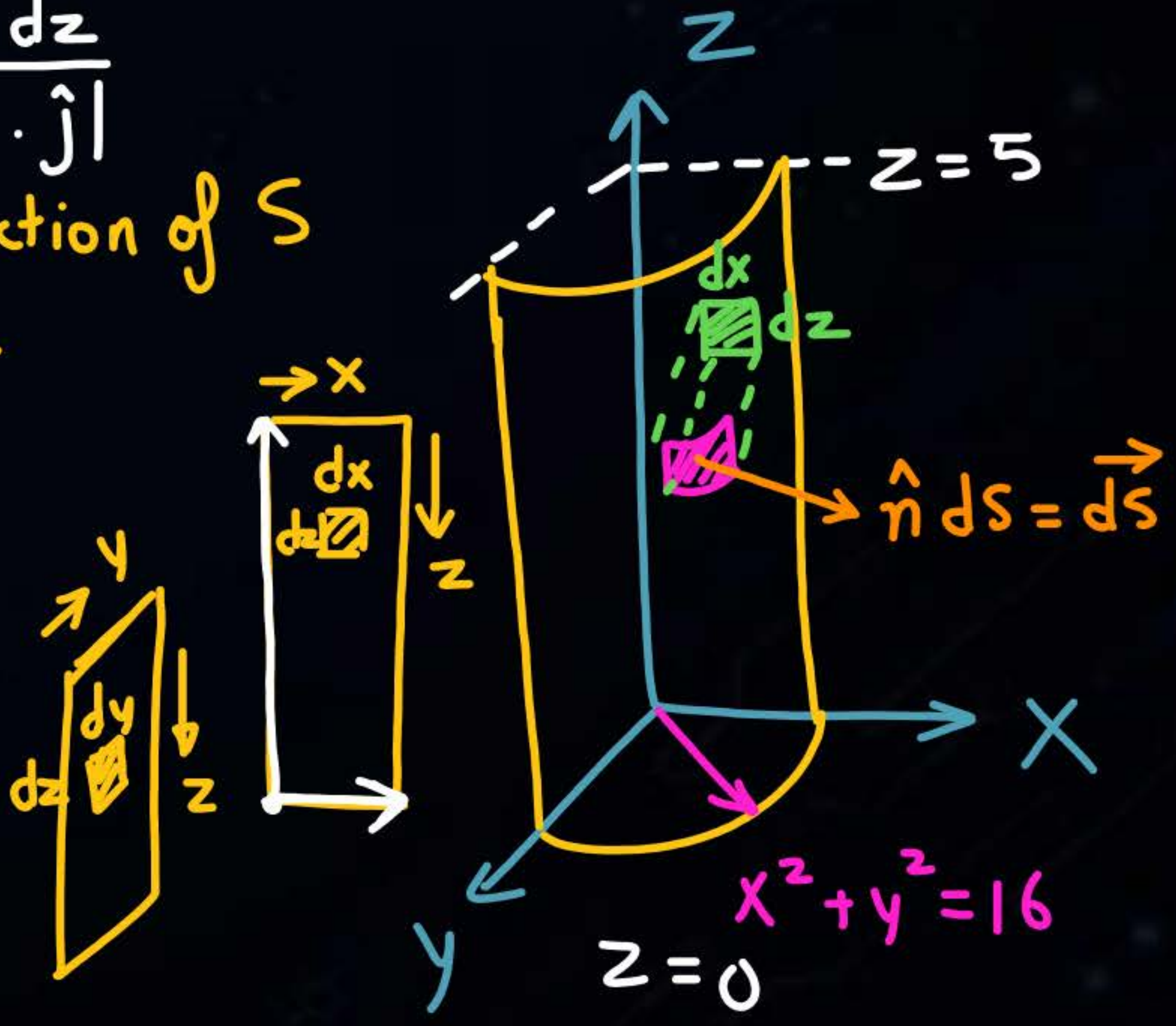
R → Region of projection of S on x-z plane.

$$\phi = x^2+y^2-16$$

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2x\,\hat{i} + 2y\,\hat{j}}{\sqrt{(2x)^2+(2y)^2}}$$

$$\boxed{\hat{n} = \frac{x\hat{i}+y\hat{j}}{4}}$$

$$\hat{n}\cdot\hat{j} = y/4$$



$$\int_{z=0}^{z=5}\int_{x=0}^{x=4}\left(z\hat{i}+x\hat{j}-3y^2z\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}}{\cancel{4}}\right)\cdot\frac{dx\,dz}{y/\cancel{4}}$$

$$\int\int\left(xz+xy\right)\frac{dx\,dz}{y}$$

$$\int\int\left(\frac{xz}{y}+x\right)dx\,dz$$

$$\int_{z=0}^{z=5}\int_{x=0}^{x=4}\frac{x\,z}{\sqrt{16-x^2}}+x\quad dx\,dz=90$$

Q. Find $\iint_S \vec{F}\cdot\hat{n}\,ds$ over the surface S: → closed surface by planes $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Soln :-

$$\oiint \vec{F}\cdot\hat{n}\,ds = \underbrace{\iint_{OABC} F\cdot\hat{n}\,ds}_{0} + \underbrace{\iint_{DEFG} F\cdot\hat{n}\,ds}_{1}$$

$$+\ \underbrace{\iint_{OAFE} F\cdot\hat{n}\,ds}_{0} + \underbrace{\iint_{BCDG} F\cdot\hat{n}\,ds}_{1}$$

$$+\ \underbrace{\iint_{OCDE} F\cdot\hat{n}\,ds}_{0} + \underbrace{\iint_{ABGF} F\cdot\hat{n}\,ds}_{1}$$

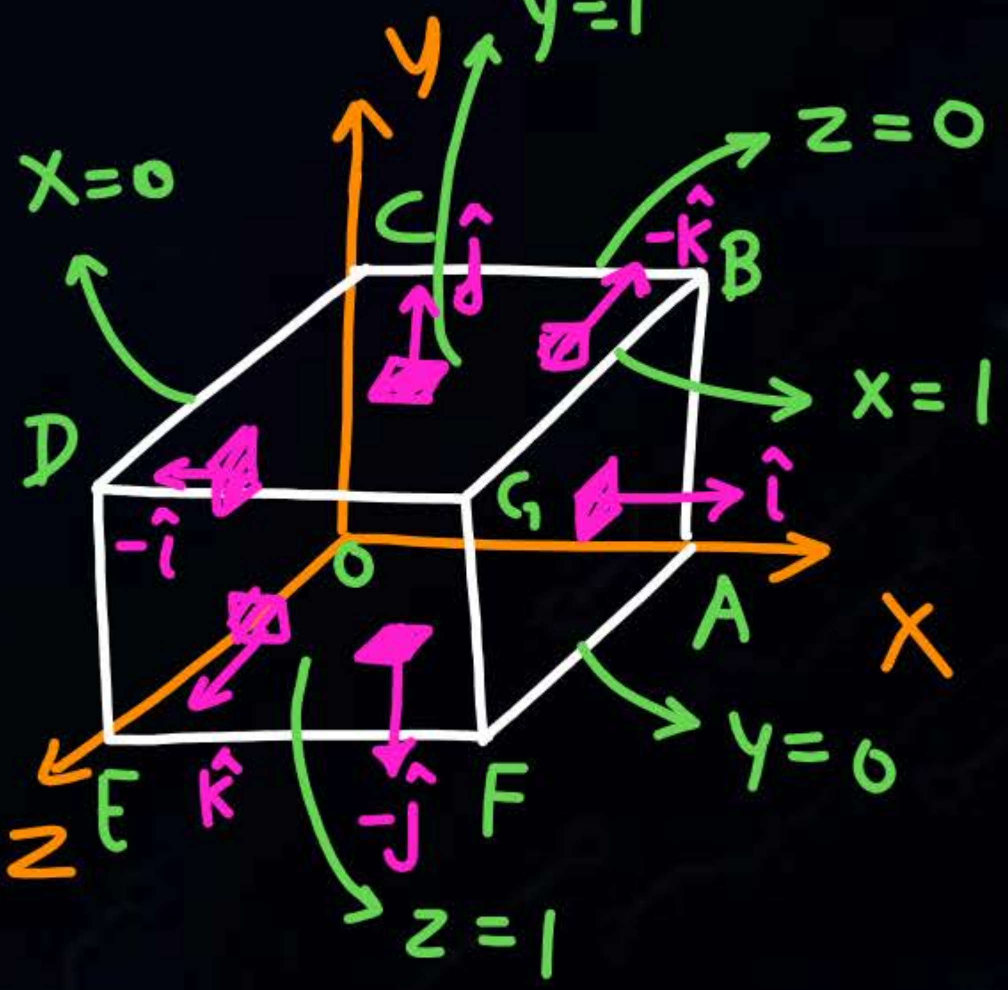
$$\oiint \vec{F}\cdot\hat{n}\,ds = 0+0+0+1+1+1 = 3$$

Apply Gauss Div. theorem:-

$$\oiint \vec{F}\cdot\hat{n}\,dS = \iiint \operatorname{div}\vec{F}\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (1+1+1)\,dx\,dy\,dz$$

$$3.1.1.1 = 3$$



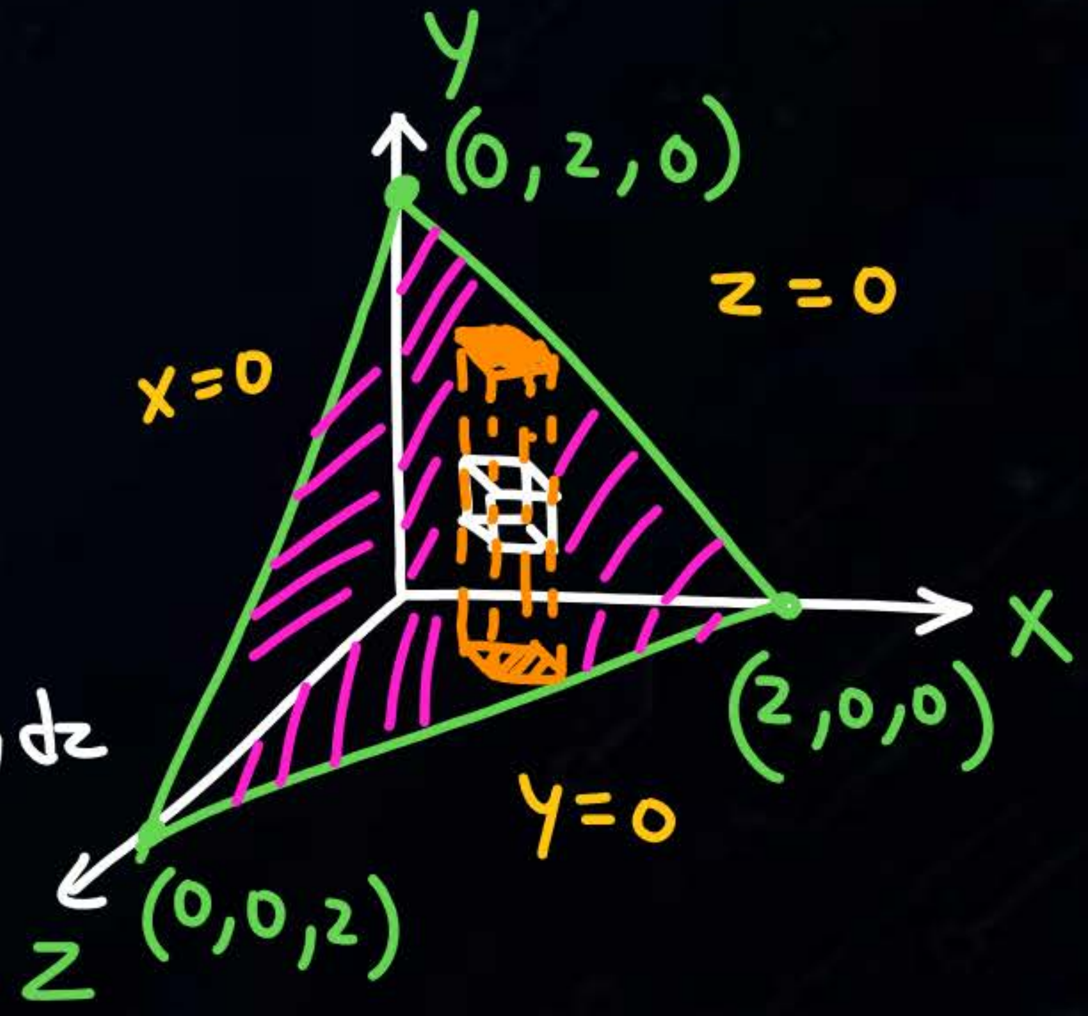
VOLUME INTEGRAL

$$\iiint_V f(x,y,z)\, dV \quad \text{or} \quad \iiint_V f(x,y,z)\, dx\, dy\, dz$$

Q. f = xyz, find $\iiint_V f\,dv$ where V is the volume enclosed by $x = 0$, $y = 0$ $z = 0$ and $\cancel{x} + \cancel{y} + z = 2$

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1$$

$$\int_{z=0}^{z=2}\int_{x=0}^{x=2-z}\int_{y=0}^{y=2-z-x} xyz\;dy\;dx\;dz$$



$\int$ → Length $\int dx$

$\int$ → Area $\int f\,dx$

$\iint$ → Area $\iint dx\,dy$

$\iint$ → Vol $\iint f\,dx\,dy$

$\iiint$ → Vol. $\iiint dx\,dy\,dz$

$\iiint$ → Vol. $\iiint f\,dV$

Line Integral
- → Open curve
- → Closed curve — it forms an open surface

Closed L.I. → Open S.I.
(Stoke's Theorem)

Surface Integral
- → Open surface
- → Closed surface — it forms a closed volume

Closed S.I. → Close V.I.
(Gauss-divergence theorem)

[GAUSS – DIVERGENCE THEOREM :-]

$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$, suppose V is the volume bounded by closed piecewise smooth surface S. Suppose F(z, y, z) is a vector fn. Which is continuous and has continuous partial derivatives in V. Then

(Vector notation) when closed surface is there.

*

$$\boxed{\begin{aligned} \oiint_S \vec{F}\cdot\hat{n}\, ds &= \iiint_V \operatorname{div} F\, dV \\ &= \iiint \vec{\nabla}\cdot\vec{F}\, dv \end{aligned}}$$

Vector form

GAUSS – DIVERGENCE THEOREM :-

$$\boxed{\iint_S F_1 dy\, dz + F_2 dx\, dz + F_3 dx\, dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz}$$

Cartesian form

Q. If S is closed surface, enclosing volume V, then find $\int_S \vec{r}\cdot\hat{n}\,ds$

$$\int_S \vec{r}\cdot\hat{n}\,dS = \int_V \text{div}\,\vec{r}\,dV$$

$$= 3\int_V dV$$

$$= 3V$$

$$= 3\left(\frac{4}{3}\pi 3^3\right)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

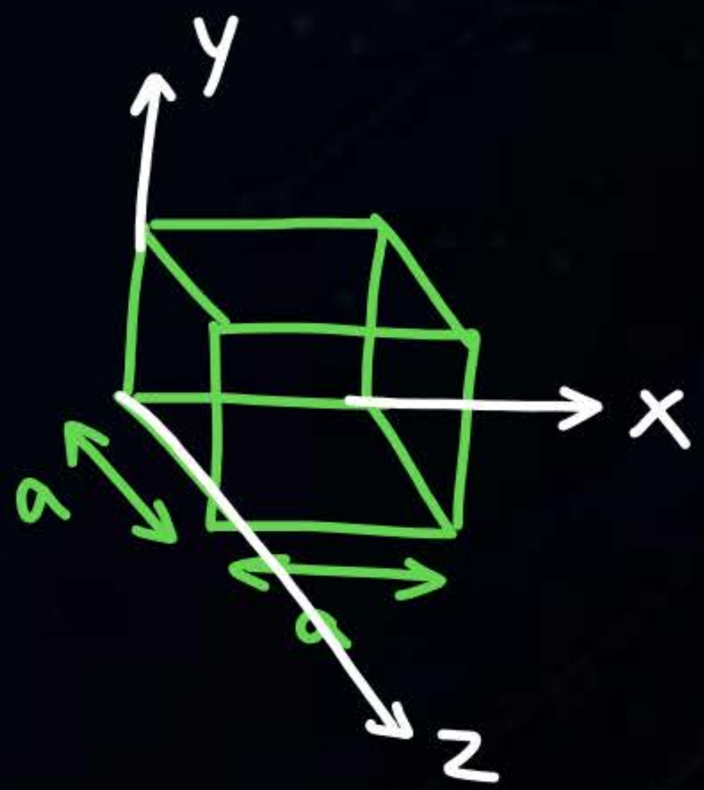
$$1+1+1=3$$

$$x^2+y^2+z^2=9$$

$$= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi 3^3$$

Q. $\int_S [\underbrace{(x^3 - yz)}\hat{\imath} \underbrace{- (2x^2y)}\hat{\jmath} + \underbrace{2}\hat{k}] \cdot \hat{n}\, ds$ **where S is the surface of cube bounded by planes x=0; x=a; y=0; y=a & z=0; z=a**

Apply GDT



$$\int_S \vec{F} \cdot \hat{n}\, dS = \iiint \text{div}\, \vec{F} \cdot dV$$

$$= \iiint (3x^2 - 2x^2)\, dV$$

$$\int_0^a \int_0^a \int_0^a x^2\, dx\, dy\, dz$$

$$\left[\frac{x^3}{3}\right]_0^a [y]_0^a [z]_0^a = \boxed{\frac{a^5}{3}}$$

Thank you

GW

Soldiers !