ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-5

Calculus

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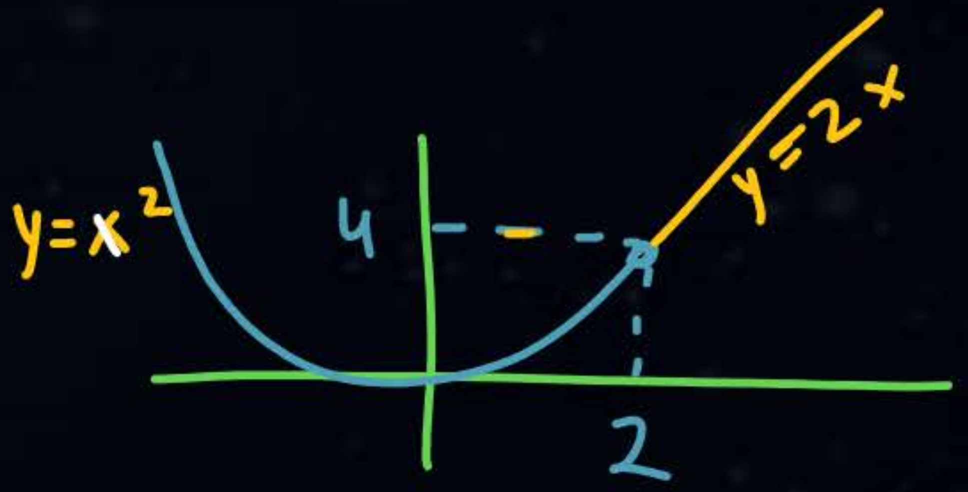
Topics to be Covered

- MEAN VALUE THEOREMS
- ROLLE'S THEOREM
- LAGRANGE MEAN VALUE THEOREM
- CAUCHY MEAN VALUE THEOREM

Ex:- $f(x) = \begin{cases} x^2 ; & x \le 2 \\ 2x ; & x > 2 \end{cases}$

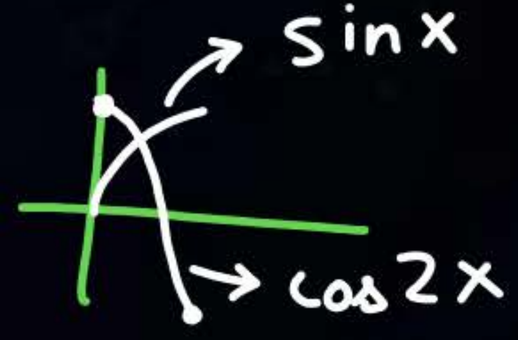
Check for continuity at $x = 2$



$f(2-h) = (2-h)^2 = 4$

$f(2+h) = 2(2+h) = 4$

$f(2) = 2^2 = 4$



Ex:- Check if the functions are continuous :-

i) $e^{-x} \sin x \Rightarrow C$

ii) $a^x(2x^2+6x+5) \Rightarrow C$

iii) $e^x \tan x \Rightarrow$ Discontinuous at $x = (2n+1)\frac{\pi}{2}$

iv) $\frac{e^x}{a^x} \Rightarrow$ Continuous

v) $\sin x + \cos 2x$ in $[0, \frac{\pi}{2}]$

$\downarrow \qquad \downarrow$

C $\quad$ C

Continuous

KINDS OF DISCONTINUITIES

(1) Removable Discontinuity

A function f(x) is said to have a discontinuity of removable kind at x = a if $\lim_{x \to a} f(x)$ exist but not equal to the value of function at x = a

$$f(a-h) = f(a+h) \neq f(a)$$

$$\underbrace{LHL = RHL}_{\text{Limit exist}} \neq \text{Value.}$$

KINDS OF DISCONTINUITIES

Removable Discontinuity

$$f(x) = x\sin\frac{1}{x}, x \neq 0$$
$$= 2 \qquad x = 0$$

$$\begin{cases} f(x) = x\sin\frac{1}{x} ; x \neq 0 \\ = 0 \quad ; x = 0 \end{cases}$$

$$\lim_{x \to 0^-} f(0-h) = (0-h)\sin\frac{1}{0-h} = -h\sin-\frac{1}{h} = h\sin\frac{1}{h} = 0 \times \text{Oscillatory Value}$$
$$= 0$$

$$\lim_{x \to 0^+} f(0+h) = (0+h)\sin\frac{1}{0+h} = h\sin\frac{1}{h} = 0 \times \text{Oscillatory value}$$
$$= 0$$

$$\underset{(0)}{LHL} = \underset{(0)}{RHL} \neq \underset{(2)}{\text{Value}}$$

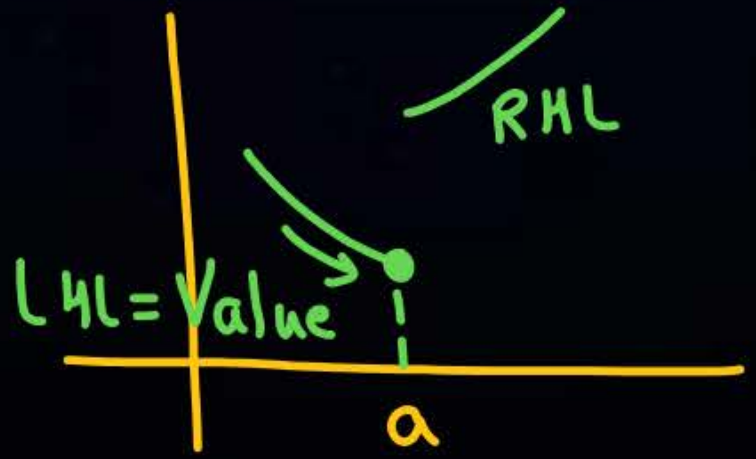
[KINDS OF DISCONTINUITIES]

(2) Discontinuity of First Kind/Jump Discontinuity

A function f(x) is said to have a discontinuity of first kind at x = a if both f(a – 0) and f(a + o) exist but are unequal. The point x = a is said the point of discontinuity from left or from right according to as follows

$f(a - 0) \neq f(a) = f(a + 0)$ or $f(a - 0) = f(a) \neq f(a + 0)$

It is also known as ordinary discontinuity.



[KINDS OF DISCONTINUITIES]

Discontinuity of First Kind/Jump Discontinuity

$$f(x) = [x],\ x \neq 0$$
$$\quad = 0\ x = 0$$

$$[x]\begin{cases} -2 ; & -2 \leq x < -1 \\ -1 ; & -1 \leq x < 0 \\ 0 ; & 0 \leq x < 1 \\ 1 ; & 1 \leq x < 2 \end{cases}$$

At integers point
function is discontinuous ;

At $x=0$

LHL $\neq$ RHL $=$ $f(0)$
(-1) (0) (0)

[KINDS OF DISCONTINUITIES]

3 Discontinuity of Second Kind

A function f(x) is said to have a discontinuity of second kind at $x = a$ if none of the limit $f(a - 0)$ and $f(a + 0)$ exist at $x = a$. The point $x = a$ is the point of discontinuity of second kind from left or right accordingly $f(a - 0)$ or $f(a + 0)$ does not exists.

Either LHL or RHL does not exist.
or both does not exist.

KINDS OF DISCONTINUITIES

Discontinuity of Second Kind

$$f(x) = \sin\frac{1}{x}, x \neq 0$$
$$= 0, \quad x = 0$$

$f(0) = 0$

$\sin(-\theta) = -\sin\theta$

$LHL = \lim_{h \to 0} f(0-h) = \sin\frac{1}{0-h} = \sin -\frac{1}{h} = -\sin\frac{1}{h} = \text{Oscillatory value}$

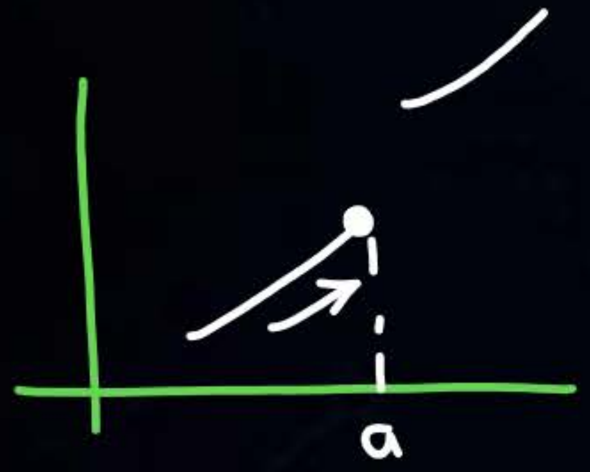
$RHL = \lim_{h \to 0} f(0+h) = \sin\frac{1}{0+h} = \sin\frac{1}{h} = \text{Oscillatory value}$

LHL & RHL both does not exist.

[KINDS OF DISCONTINUITIES]

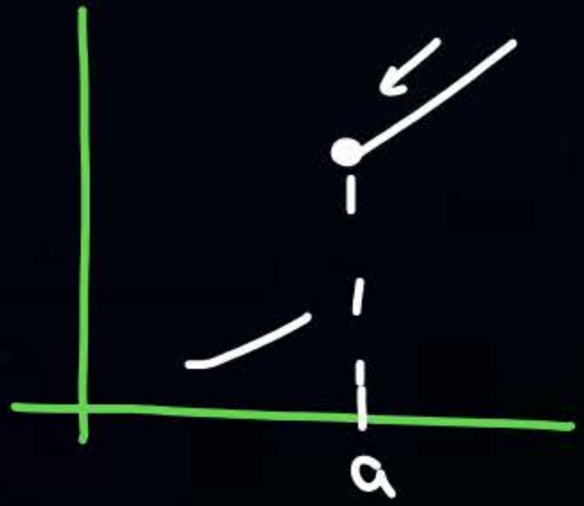
4 Mixed discontinuity

A function f(x) is said to have a discontinuity of mixed kind at x = a if f(x) has a discontinuity of second kind on one side of a and on the other side it has discontinuity of first kind or may be continuous.



1st Kind $f(a-h) = f(a) \neq f(a+h)$

2nd Kind $f(a+h)$ D.N.E.



1st Kind $f(a+h) = f(a) \neq f(a-h)$

2nd Kind $f(a-h)$ D.N.E

KINDS OF DISCONTINUITIES

Mixed discontinuity

$$f(x) = e^{1/x}\sin\frac{1}{x}, x \neq 0$$

$$0 \quad , x = 0$$

L.H.L. $\lim_{h\to 0} f(0-h) = e^{\frac{1}{0-h}} \cdot \sin\frac{1}{0-h} = e^{-\frac{1}{h}} \cdot \sin\left(-\frac{1}{h}\right) = \lim_{h\to 0} -\frac{\sin\frac{1}{h}}{e^{1/h}} = \frac{\text{Oscill..}}{\infty}$

$= 0$

R.H.L. $\lim_{h\to 0} f(0+h) = e^{\frac{1}{0+h}} \cdot \sin\frac{1}{0+h} = e^{\frac{1}{h}} \cdot \sin\frac{1}{h} = \infty \times (\text{Oscillatory value})$

$= \infty$

1st Kind	L.H.L $= f(0) = 0$ exist
2nd Kind	R.H.L. does not exist.

KINDS OF DISCONTINUITIES

5 Infinite discontinuity

A function f(x) at x = a is said to have discontinuity of infinite kind if f(a + 0) or f(a – 0) is ∞ or –∞.

Either or both LHL or RHL is $+\infty$ or $-\infty$.

At $x \to a$

L.H.L. = R.H.L = Value = Finite value	Limit exist (Continuous)
L.H.L. = R.H.L. = Oscillatory value	Limit D.N.E. (Discont.)
L.H.L. = R.H.L. = ∞ / $-\infty$	Limit exist (Discontinuous)

[KINDS OF DISCONTINUITIES]

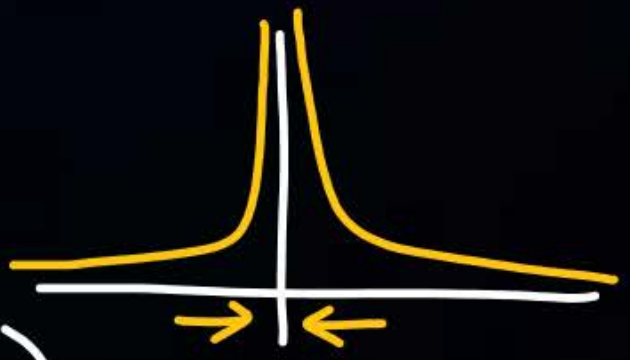
Infinite discontinuity

Ex: $f(x) = \dfrac{1}{x}$ at $x = 0$.

Limit D.N.E. $\begin{cases} L.H.L. = f(0-h) = \dfrac{1}{0-h} = -\dfrac{1}{h} = -\infty \\ R.H.L. = f(0+h) = \dfrac{1}{0+h} = +\dfrac{1}{h} = +\infty \end{cases}$

Ex: $f(x) = \dfrac{1}{x^2}$ at $x=0$

$\left.\begin{array}{l} LHL = \dfrac{1}{(0-h)^2} = \dfrac{1}{h^2} = \infty \\ RHL = \dfrac{1}{(0+h)^2} = \dfrac{1}{h^2} = \infty \end{array}\right\}$ Limit exist (Discontinuous)



KINDS OF DISCONTINUITIES

Infinite discontinuity

Find all point of discontinuity of the following functions:

$$(i)\ f(x) = \frac{x^2 + 3x - 10}{x + 5}$$

At $x = -5$,

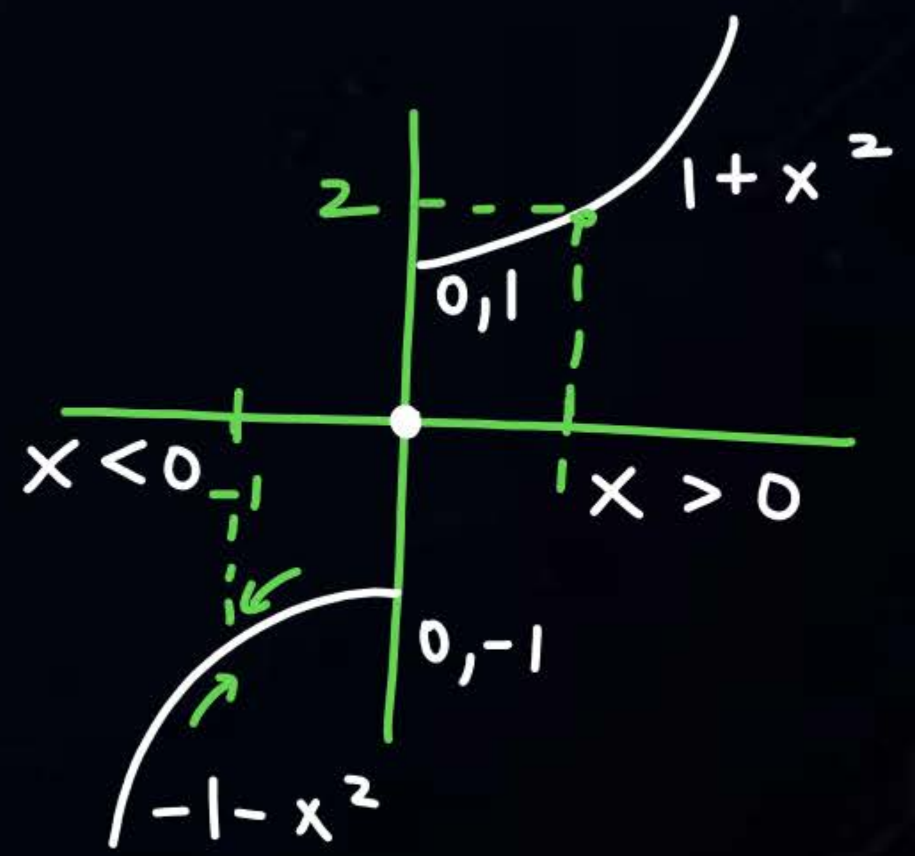
[KINDS OF DISCONTINUITIES]

Infinite discontinuity

A function f(x) is defined as follows:

$$f(x) = \begin{cases} -1 - x^2 \ \ when\, x < 0 \\ 0 \quad when \ \ x = 0 \\ 1 + x^2 \ \ when \ \ x > 0 \end{cases}$$

$y = -(1 + x^2)$



Find the points of discontinuity. $(x=0)$

$$\left.\begin{array}{l} LHL. = -1 \\ RHL = +1 \end{array}\right\} \text{limit D.N.E.}$$

Value = 0

(Discontinuous)

$$f(x)\begin{cases} \dfrac{\lambda \cos x}{\pi/2 - x} & ; x \neq \pi/2 \\ 1 & ; x = \pi/2 \end{cases}$$

[GATE]

At $x = \pi/2$ function is continuous if $\lambda =$ ___

$$\lim_{x \to \pi/2} \frac{\lambda \cos x}{\pi/2 - x} \quad \left(\frac{0}{0}\right)$$

$$\frac{\lambda(-\sin x)}{0 - 1} = \lambda \sin x = \lambda \sin \pi/2 = \lambda$$

limit = Value

$\lambda = 1$

KINDS OF DISCONTINUITIES

Infinite discontinuity

A function f(x) is defined as follows:

$$f(x) = \frac{x^2}{a} - a \quad x < a$$

$$= 0, \qquad x = a$$

$$= a - \frac{a^2}{x}, x > a$$

Prove that the function f(x) is continuous at x = a.

KINDS OF DISCONTINUITIES

Infinite discontinuity

Find the points and kinds of discontinuity of the function defined by

$$f(x) = \begin{cases} x^2, & x \le 0 \\ 5x - 4, & 0 < x \le 1 \\ 4x^2 - 3x, & 1 < x < 2 \\ 3x + 4, & x \ge 2 \end{cases}$$

$x \to 0^- = 0$, $x \to 0^+ = -4$, $x(0) = 0$

[KINDS OF DISCONTINUITIES]

Infinite discontinuity

Discuss the continuity of the function f(x) at x = a:

$$f(x) = \begin{cases} \dfrac{1}{x-a}\text{cosec}\left(\dfrac{1}{x-a}\right), & x \neq a \\ = 0, & x = a \end{cases}$$

[KINDS OF DISCONTINUITIES]

Infinite discontinuity

Investigate the kind of discontinuity of the following function f(x) at x = 0:

$$f(x) = \tan^{-1}\left(\frac{1}{x}\right), \qquad x \neq a$$

f(0) = 0 at x = 0

KINDS OF DISCONTINUITIES

Infinite discontinuity

Discuss the continuity of the function at x = a:

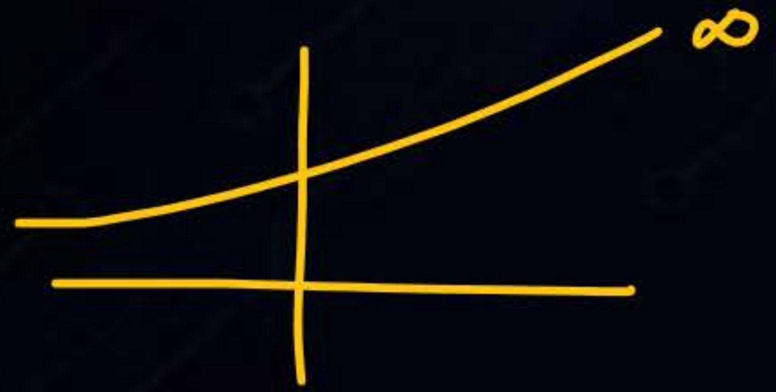
$$f(x) = \frac{1}{x-a} sin \frac{1}{x-a}; \quad x \neq a$$
$$= 0, \quad x = a$$

PROPERTIES OF CONTINUOUS FUNCTIONS

(i) A function which is continuous in a closed interval is also bounded in that interval.

(ii) A continuous function which has opposite signs at two points vanishes at least once between these points and vanishing point is called root of the function.

(iii) A continuous function $f(x)$ in the closed interval $[a, b]$ assumes at least once every value between $f(a)$ and $f(b)$, it being assumed that $f(a) \neq f(b)$.



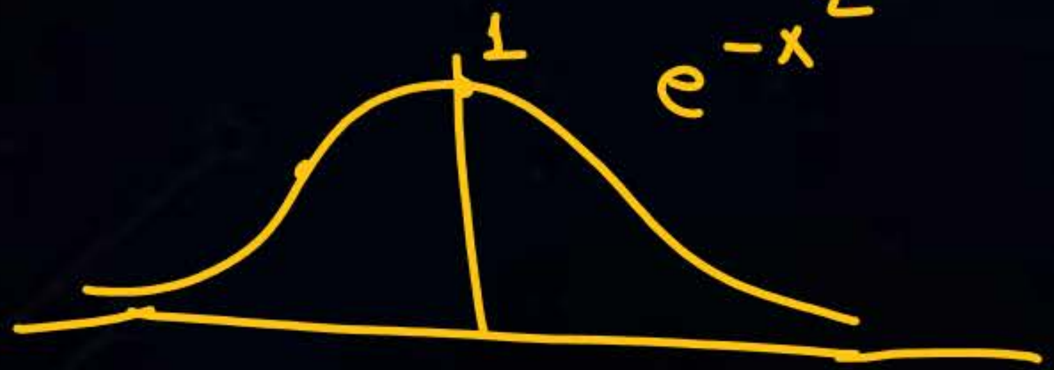
$a \leq x \leq b$

e^x

Unbounded.

$f(x) \rightarrow +\infty / -\infty \Rightarrow$ Unbounded

$f(x) \rightarrow$ Finite values & continuous (Bounded)



e^{-x^2}

$x \in R$

Values $\rightarrow$ (0 to 1)

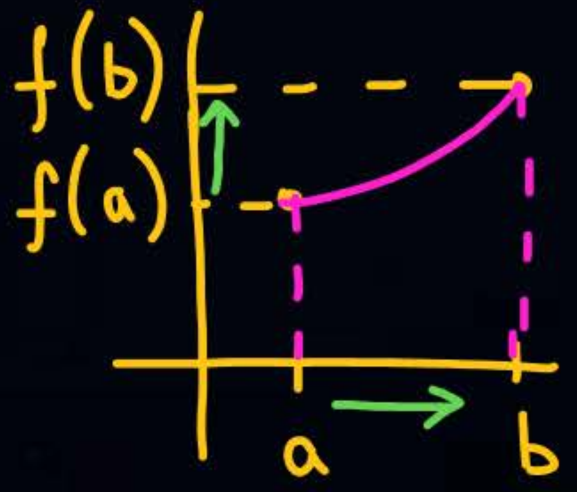
BOLZANO THEOREM:-

$x = a, \; f(a) = +$

$x = b, \; f(b) = -$

At least there is one point c b/w a & b where $f(x=c) = 0$

$$f(x) = 0$$



$a < x < b$

(Bounded)

PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ be defined for the interval $(0, 1)$ as follows:

$= 0$ for $x = 0$

$f(x) = 1 - x$ for $0 < x < 1$

$= 1$ for $x = 1$

Show that $f(x)$ is not continuous at the points $x = 0$ and $x = 1$ although $f(x)$ assumes once and only once every value between $f(0)$ and $f(1)$.

[CONTINUITY OF FUNCTION OF TWO VARIABLES]

Show that the function

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, (x, y) \neq 0$$

$= 0 \; (x, y) = (0, 0)$

is continuous at the origin.

DIFFERENTIABILITY

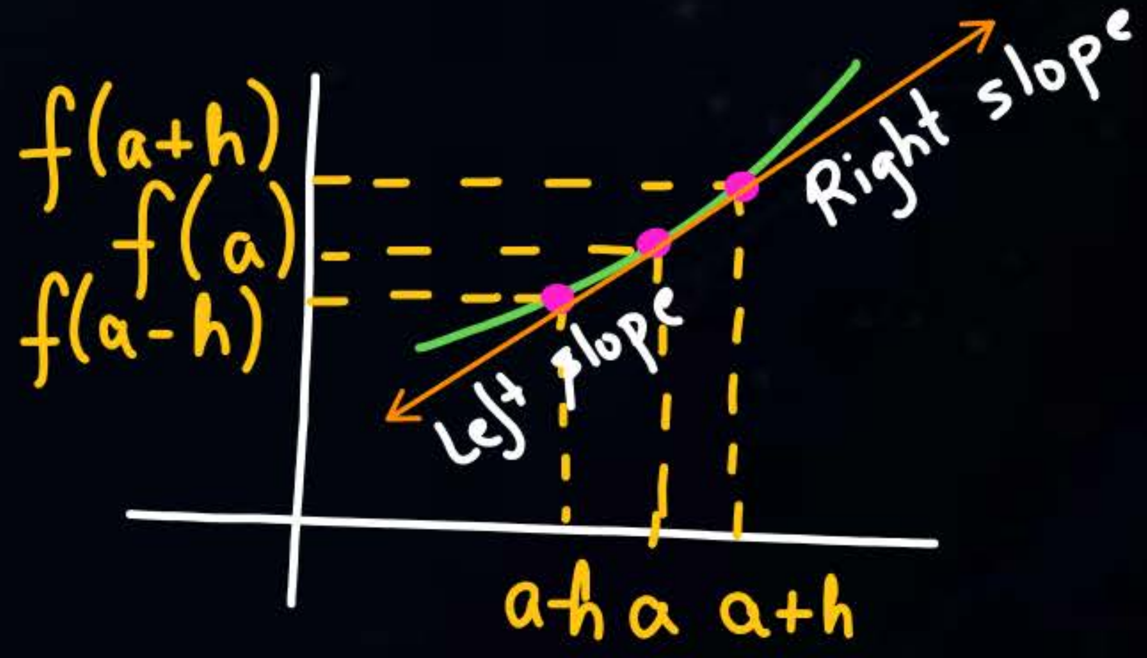
Differentiability

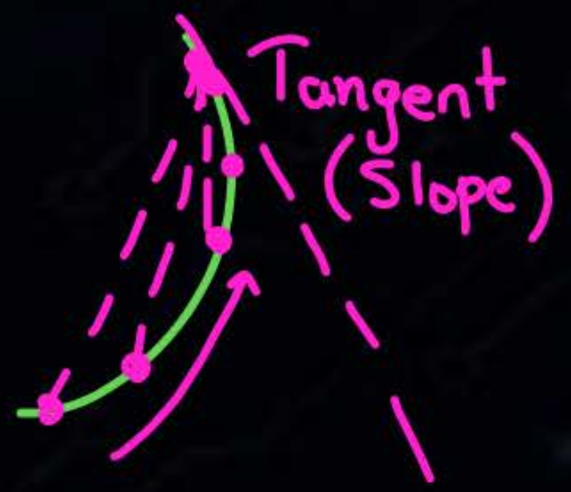
$$L.H.D. = \lim_{h \to 0} \frac{f(a) - f(a-h)}{a-(a-h)}$$

$$R.H.D. = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h-a}$$

If L.H.D. = R.H.D. the function is differentiable at $x=0$.

(Slope is continuous)





DIFFERENTIABILITY

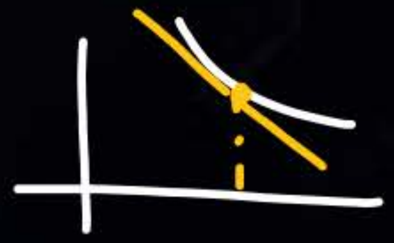
A necessary Condition for the Existence of a Finite Derivative

→ If LHD = RHD , then slope is continuous at $x = a$. (Differentiable)

NOTE :-

- All differentiable functions are continuous.
  Differentiability ⇒ Continuity ⇒ Limit exists
- If limit exist (LHL = RHL) , then fn. may or may not be continuous.
- If fn. is continuous then it may or may not be differentiable.

Continuous → Derivable (x^2, Polynomials)



Continuous → Not derivable
$|x|$, $|\sin x|$

→ Corner
→ Cusp

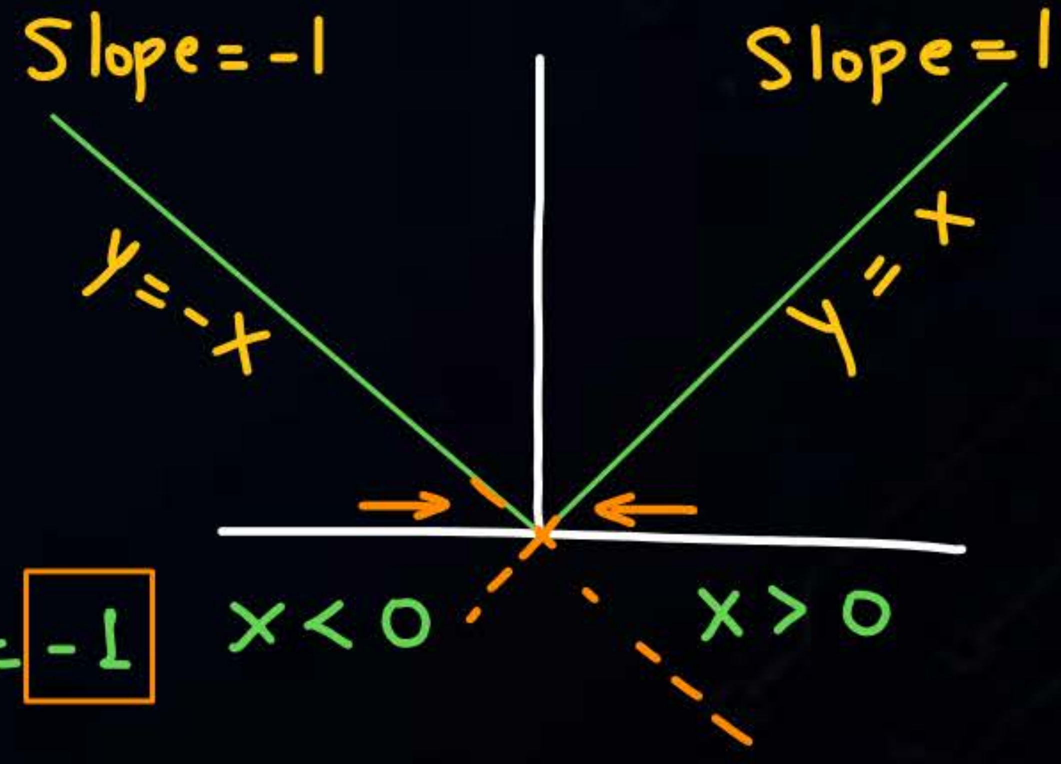
DIFFERENTIABILITY

Show that the function f(x) = |x| is continuous but not differentiable at x = 0.

LHL = RHL = Value = 0

$$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

At $x = 0$; $f(x) = |x|$ is continuous.



$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(0) - f(0-h)}{0-(0-h)} = \frac{0 - \overset{|0-h|}{[-(0-h)]}}{h} = \boxed{-1}$$

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{0+h-0} = \frac{(0+h) - 0}{h} = \boxed{1}$$

Hence at $x=0$; $|x|$ is non differentiable.

$f(x) = |\sin x|$

$\sin x$ → Limit exists, Continuous, Differentiable

$|\sin x|$ → Limit exists, Continuous, Not differentiable.

DIFFERENTIABILITY

Show that the function f(x) defined by

$$f(x) = x \sin(1/x), \quad x \neq 0$$
$$= 0, \qquad x = 0$$

is continuous but not differentiable.

$$\begin{cases} \text{L.H.L.} = \lim_{h \to 0} f(0-h) \\ \text{R.H.L.} = \lim_{h \to 0} f(0+h) \end{cases}$$

$$\begin{cases} \text{L.H.D.} = \lim_{h \to 0} \dfrac{f(0) - f(0-h)}{0-(0-h)} \\ \text{R.H.D.} = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{0+h-0} \end{cases}$$

[DIFFERENTIABILITY]

The function $f(x) = \frac{1}{x}$ which is not continuous at $x = 0$ has no derivative at $x = 0$.

DIFFERENTIABILITY

If a function f(x) is defined as:

$$f(x) = \frac{xe^{1/x}}{1+e^{1/x}}; x \neq 0$$

$$= 0 ; \qquad x = 0$$

Ex:- $f(x)=\begin{cases} e^x & ; x<1 \\ \ln x + ax^2+bx & ; x \geq 1 \end{cases}$

i) $f(x)$ is not diff. at $x=1$ for any value of a and b.

ii) " " diff. at $x=1$ for unique values of a and b.

iii) $f(x)$ is diff. at $x=1$ for all values of a and b such that $a+b=e$.

iv) " " " " " " " " a and b.

Soln:-

$LHL = e^1$

$RHL = \ln 1 + a(1)^2 + b(1)$

Differentiable $\Rightarrow$ Continuous

$LHL = RHL$

$e = 0 + a + b$

$\boxed{a+b=e}$ —— (1)

L.H.D. → Differentiation

$f'(x) = e^x$

R.H.D.

$f'(x) = \frac{1}{x} + 2ax + b$

At $x = 1$ L.H.D. = R.H.D.

$e^x = \frac{1}{x} + 2ax + b$

$\boxed{e = 1 + 2a + b}$ — 2

$a + \cancel{b} = 1 + 2a + \cancel{b}$

$\boxed{a = -1}$, $\boxed{b = e + 1}$ unique values

Thank you

GW

Soldiers !