ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-2

CALCULUS

By- Chetan Sir

Topics to be Covered

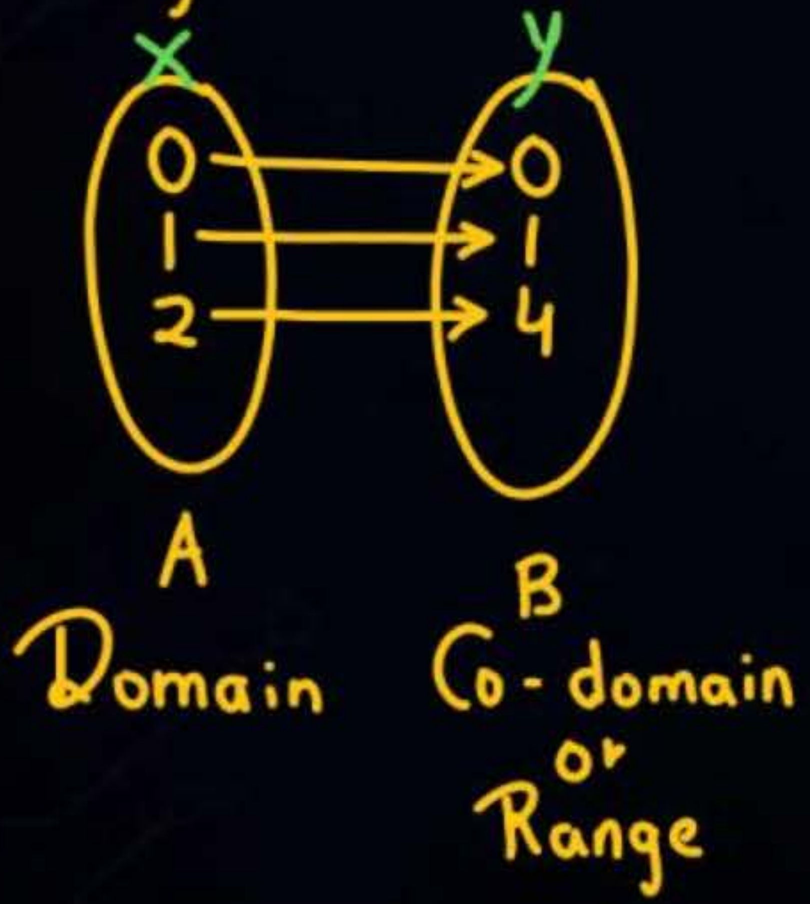
- LIMIT OF A FUNCTION
- THEOREMS ON LIMITS
- IMPORTANT RESULTS ON LIMITS
- IMPORTANT EXPANSIONS OF FUNCTION
- INDETERMINATE FORMS
- L-HOSPITAL RULE

[FUNCTIONS]

A relation R from set A to B is said to be a function (f) if every element of set A has one and only one image in set B:

$f(x) = x^2$

Input $\xrightarrow{f}$ Output



i) one-one and into

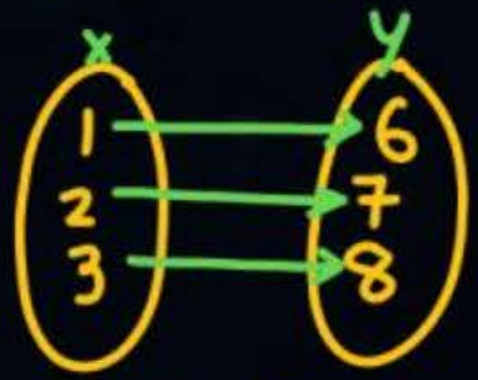
ii) one-one and onto

iii) many-one and into

iv) many-one and onto.

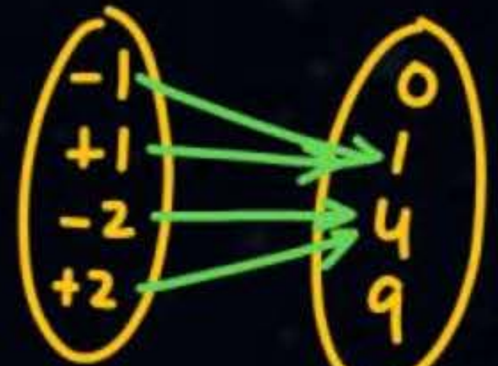
[TYPES OF FUNCTIONS]

$y = x + 5$



One - one

$y = x^2$



many - one

I). One-One function (Injective):

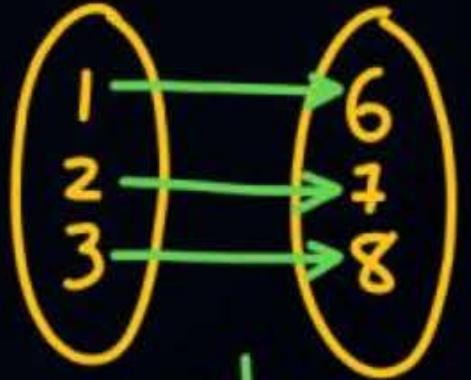
Image of $A \xrightarrow{in} B$ is unique.

$y = x+5$

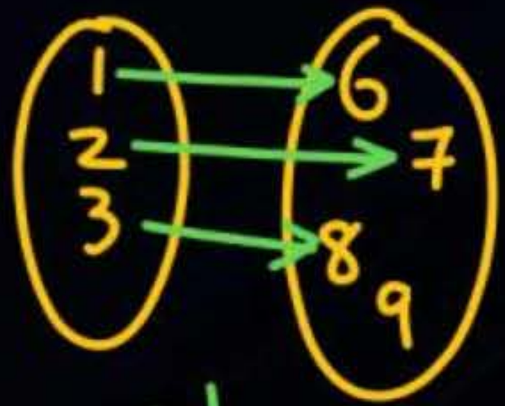
II). Onto function (Surjective):

If the entire range is covered
↳ ONTO (Surjective)

If the entire range is not covered/mapping.
↳ INTO



onto
one-one
(Bijective)



into
one-one
(Not bijective)

III). Bijective function:

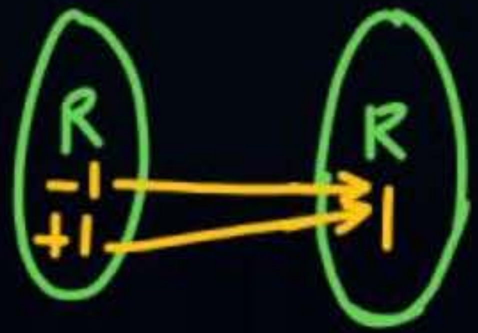
↳ one-one and onto both

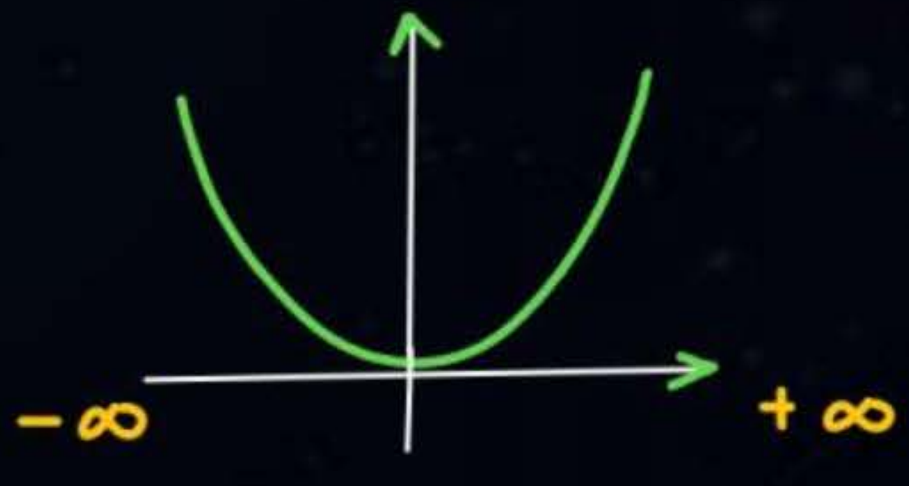
DOMAIN AND RANGE OF FUNCTIONS

Ex:- Is x^2 is bijective?

$x \in R$

$y \in R$





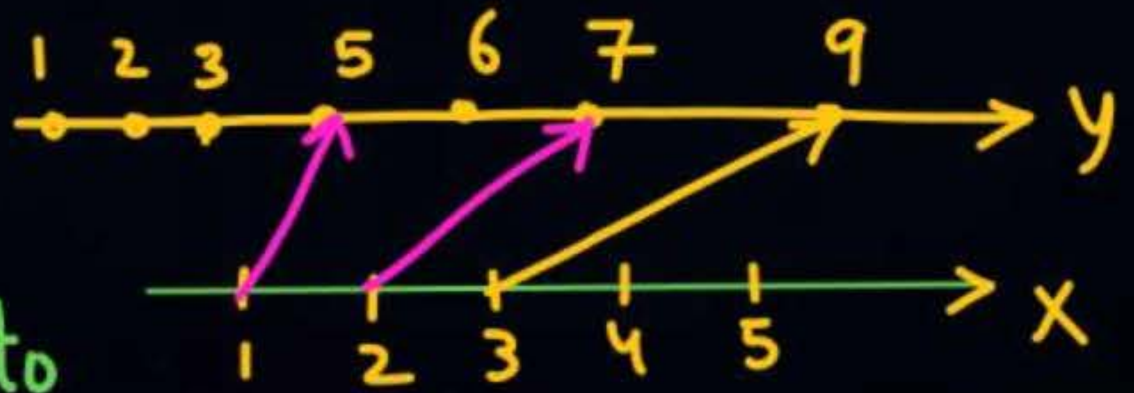
many one & into

$\therefore$ it is not bijective.

Ex:- $f(x) = 2x + 3$

$\{x, y \in N\}$

$\Rightarrow$ one-one & into

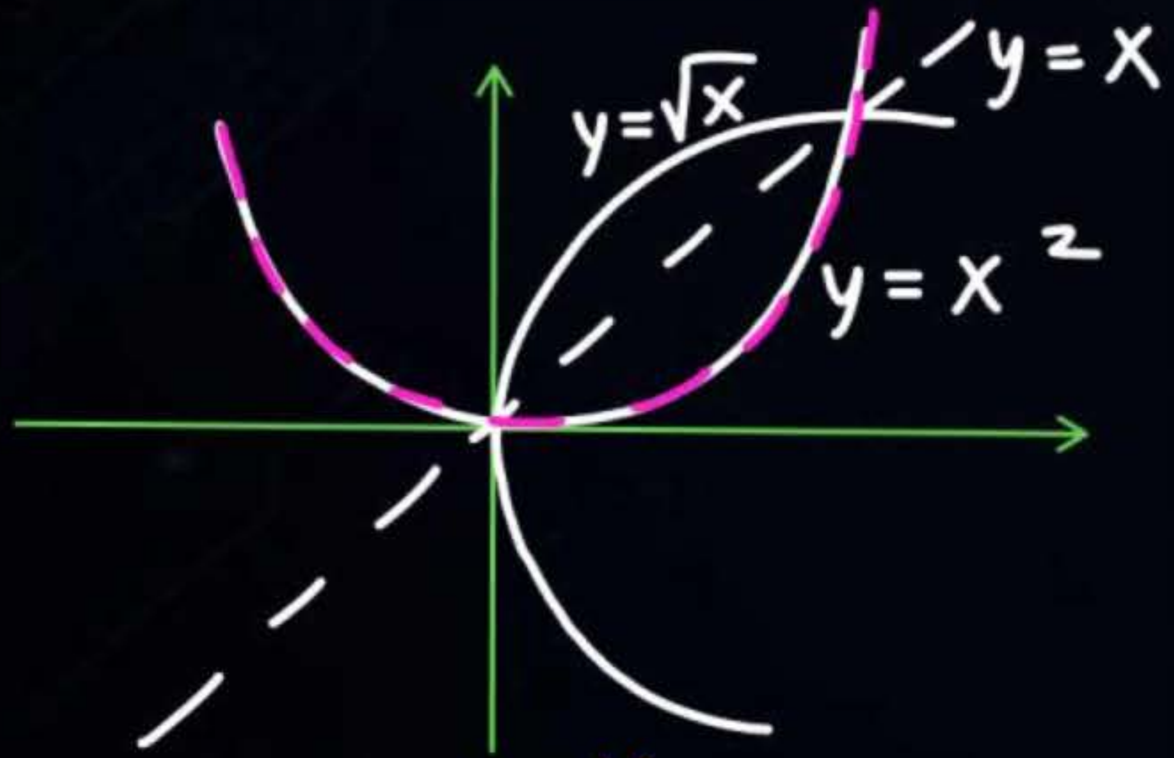


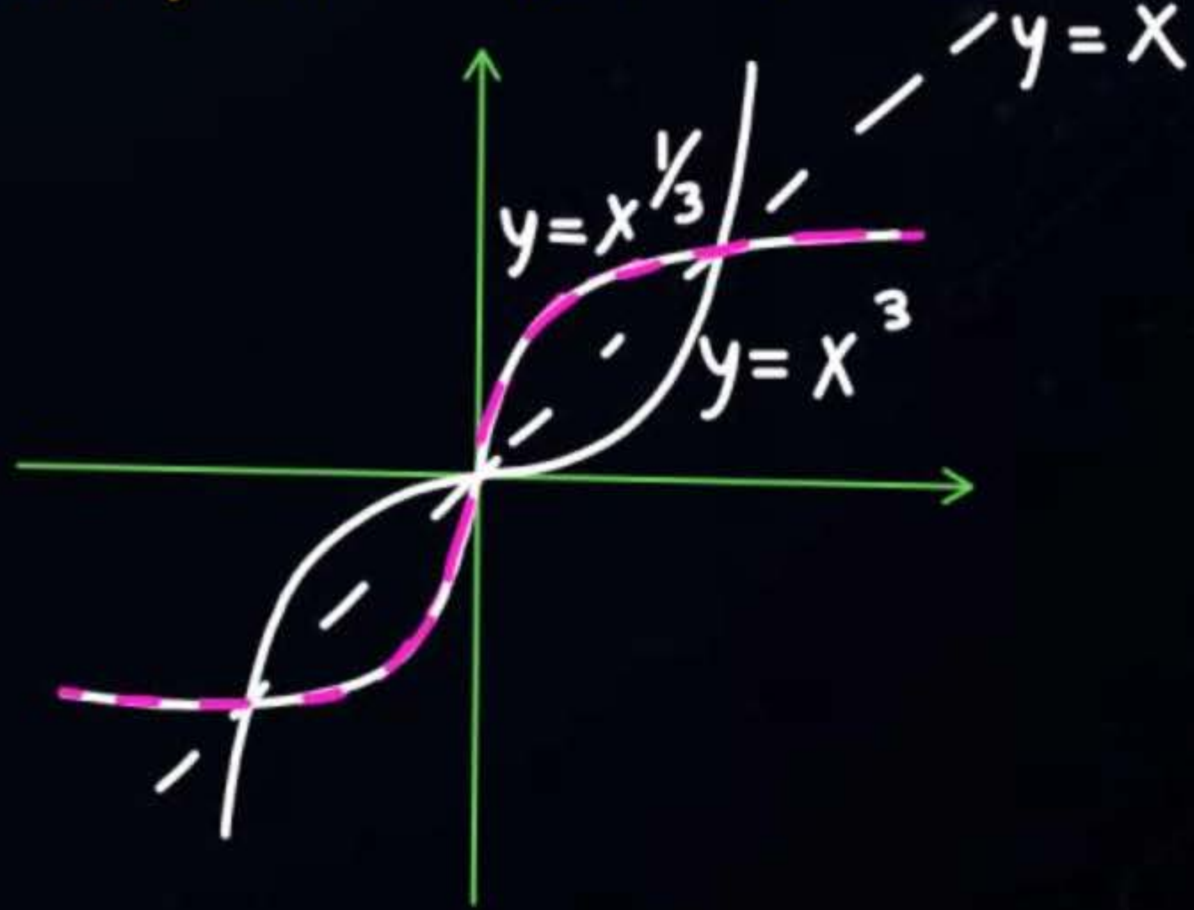
GRAPH OF FUNCTIONS

- FUNCTIONS
  - Algebraic
    - Polynomial
    - Rational
    - Irrational
    - Piecewise
      - Modulus
      - Signum
      - Greatest integer function
      - Fractional part function
      - Least integer function
  - Transcendental
    - Trigonometric
    - Exponential
    - Logarithmic/Inverse of exponential
    - Inverse trigonometric curves

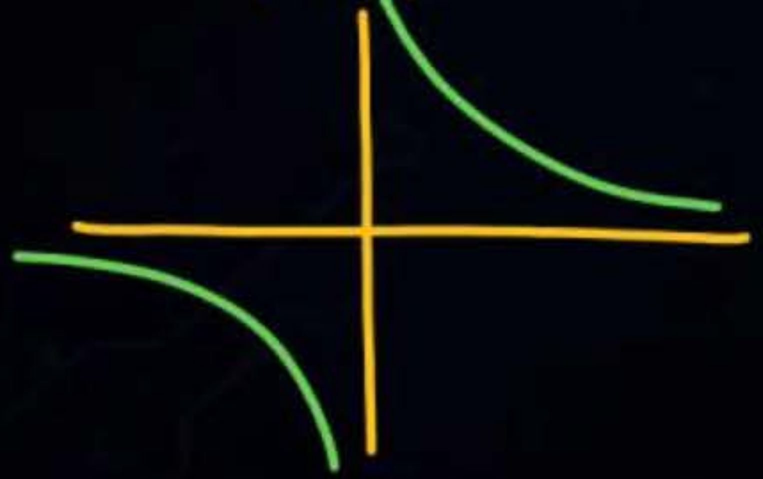
GRAPH OF FUNCTIONS

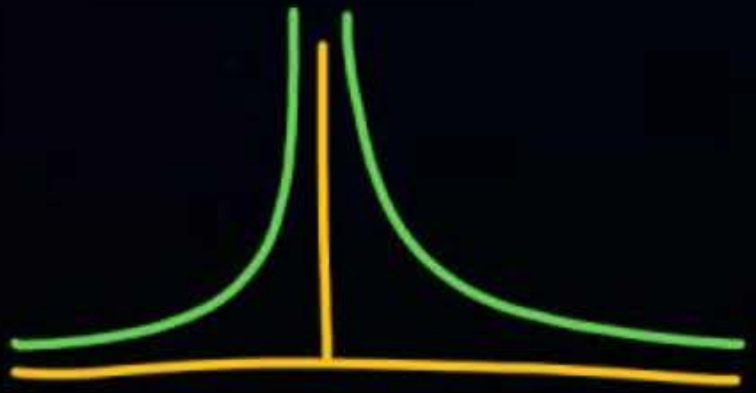
1) Polynomials :- $f(x) = x, x^2, x^3, x^2+5, x^3+3x+6.$





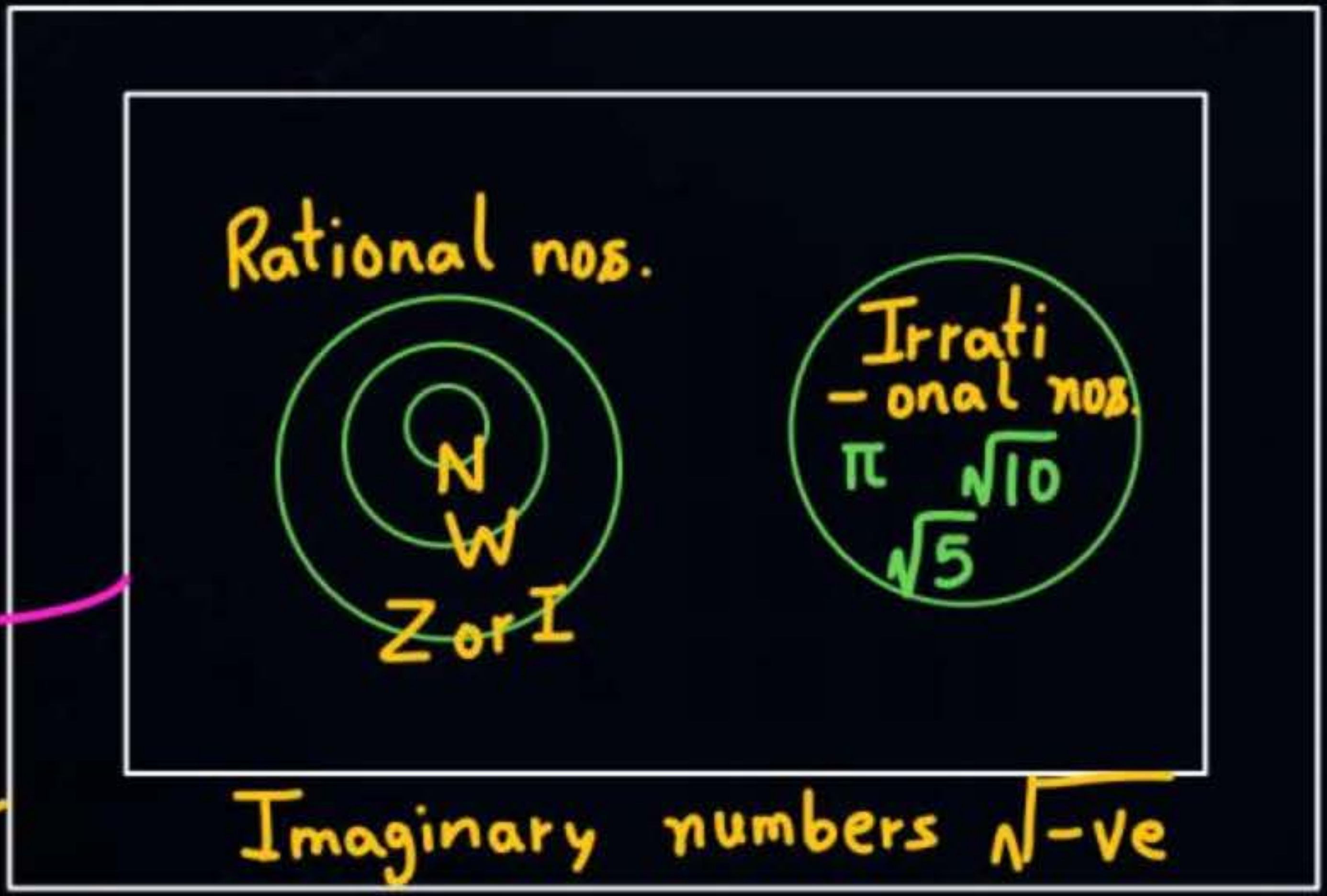
$y = 1/x$





$y = \frac{1}{x^2}$

GRAPH OF FUNCTIONS



Real numbers

Complex numbers
↓
a + ib
Real Imaginary

$N \rightarrow$ Natural numbers $(1, 2, 3, \ldots)$
$W \rightarrow$ Whole numbers $(0, 1, 2, 3 \ldots)$
$Z, I \rightarrow$ Integers $(-5, -4 \ldots 0, 1, 2, 3 \ldots)$

Non-terminating
(Non-recurring)
Irrational numbers → any no. which cannot be expressed in form of fraction.
Ex:- Some roots, π, constants

→ Terminating
→ Non-terminating & recurring.
Rational numbers → any no. which can be expressed as fraction.

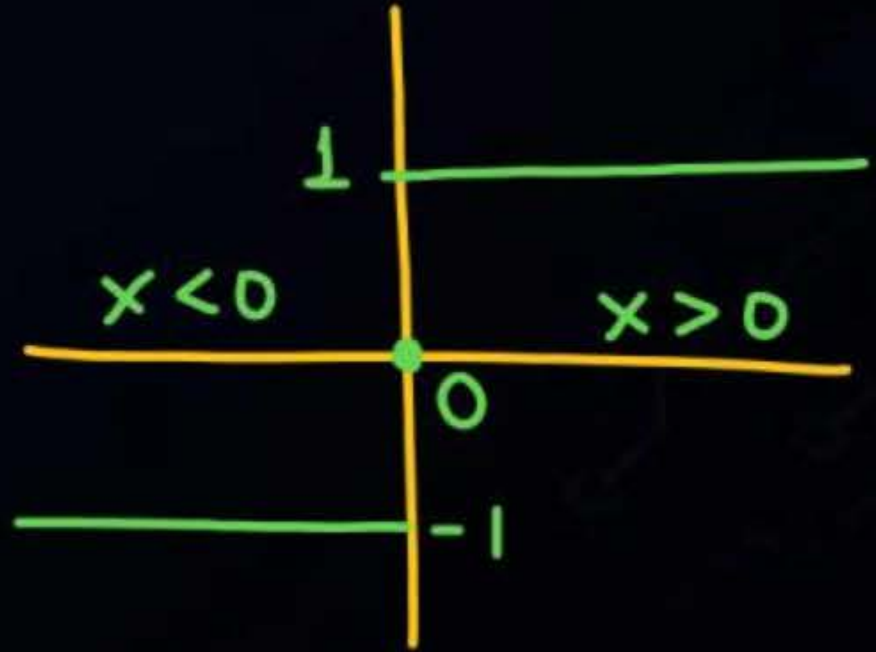
GRAPH OF FUNCTIONS

2) Modulus function :-

$$f(x) = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

3) Signum function :-

$$f(x) = \frac{|x|}{|x} = \begin{cases} 1 & ; x > 0 \\ 0 & ; x = 0 \\ -1 & ; x < 0 \end{cases}$$



GRAPH OF FUNCTIONS

4) Greatest integer function :-

$$x = \underbrace{[x]}_{\text{G.I.}} + \underbrace{\{x\}}_{\text{Fractional part}}$$

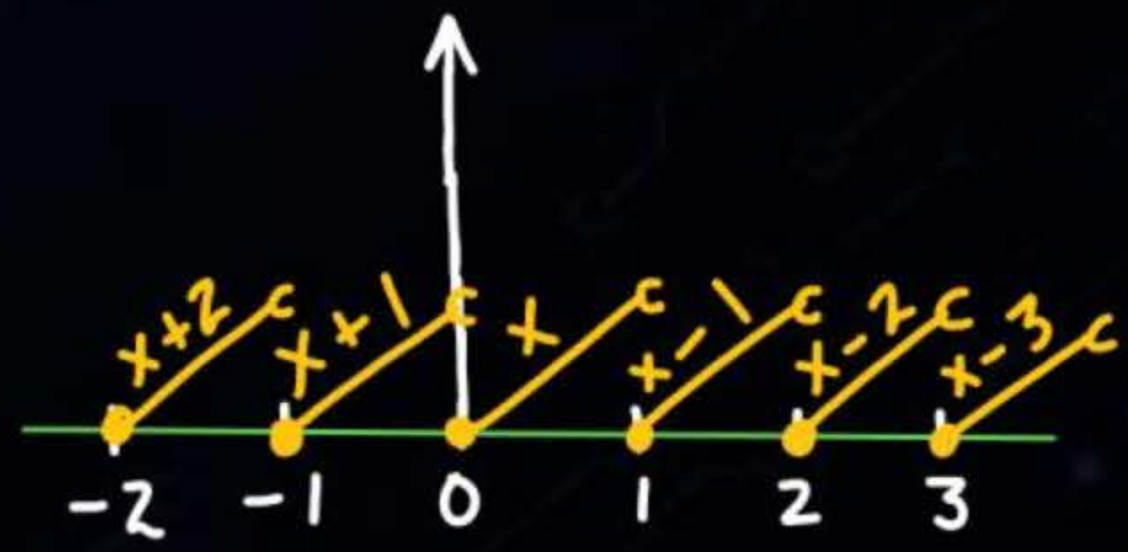
$$f(x) = [x] = \begin{cases} -2 & ; -2 \le x < -1 \\ -1 & ; -1 \le x < 0 \\ 0 & ; 0 \le x < 1 \\ 1 & ; 1 \le x < 2 \end{cases}$$

5) Fractional part :-

$$f(x) = \{x\} = \begin{cases} x+2 & ; -2 \le x < -1 \\ x+1 & ; -1 \le x < 0 \\ x & ; 0 \le x < 1 \\ x-1 & ; 1 \le x < 2 \\ x-2 & ; 2 \le x < 3 \end{cases}$$

$$\downarrow$$

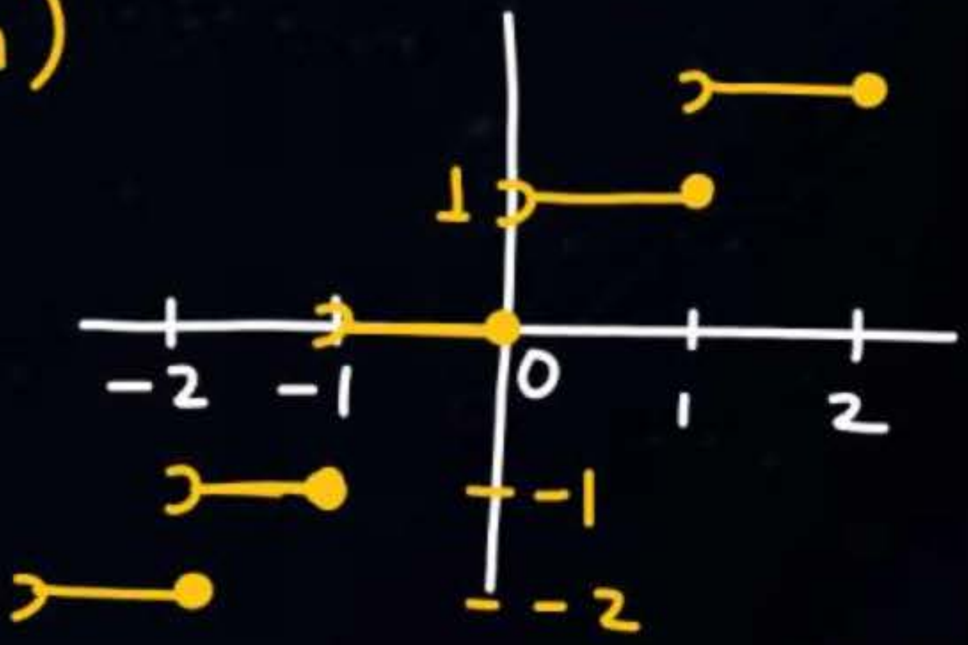
$$0 \le \{x\} < 1$$



GRAPH OF FUNCTIONS

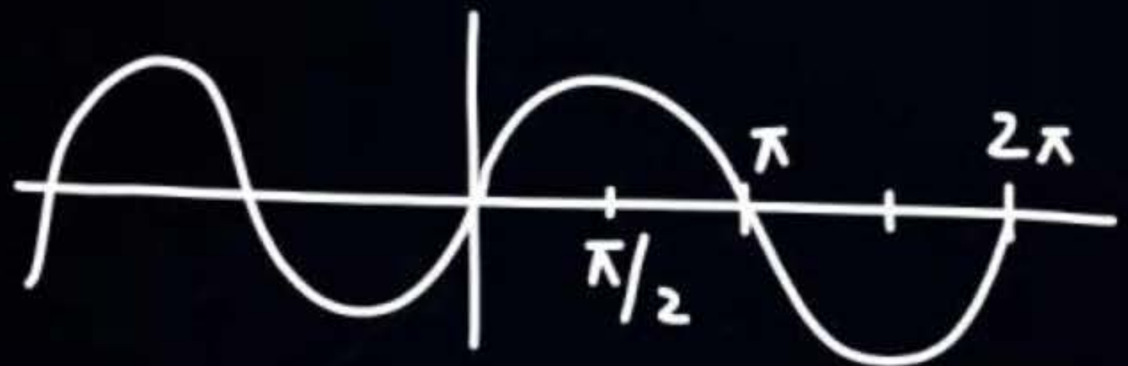
6) Least integer function :- (Ceiling function)

$$f(x) = \lceil x \rceil = \begin{cases} -1 ; & -2 < x \le -1 \\ 0 ; & -1 < x \le 0 \\ 1 ; & 0 < x \le 1 \\ 2 ; & 1 < x \le 2 \end{cases}$$



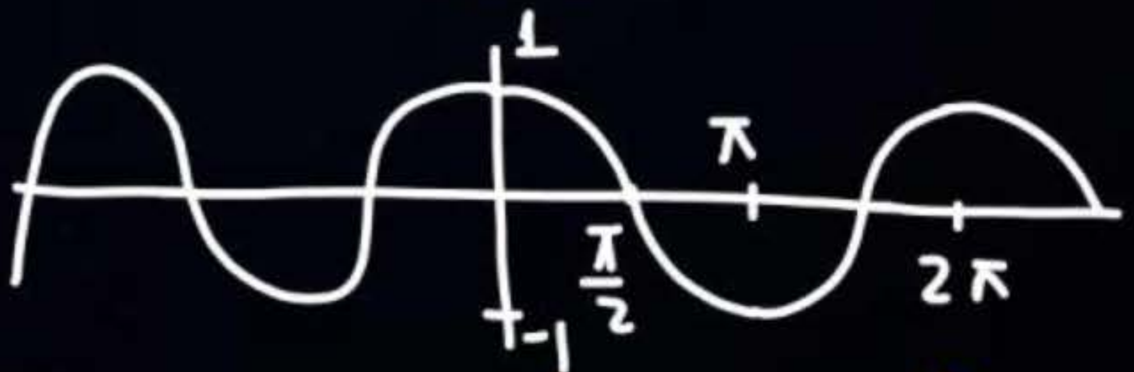
7) Trigonometric functions :-

sin x



2π → One cycle

cos x

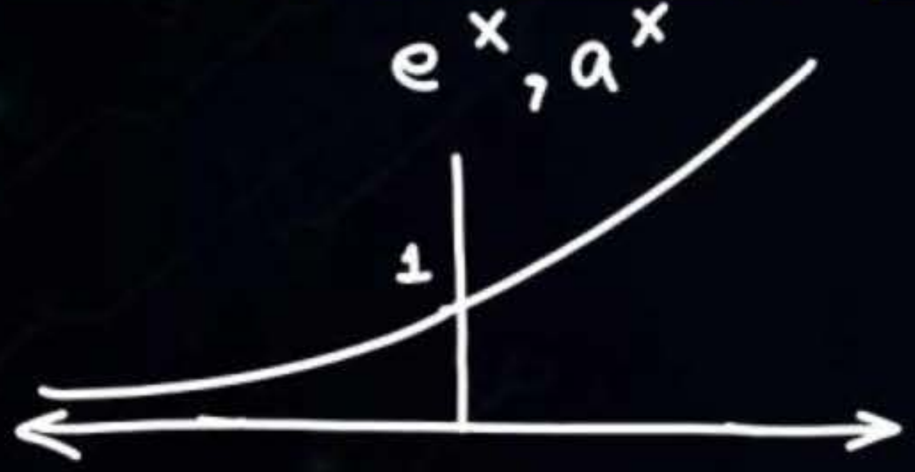


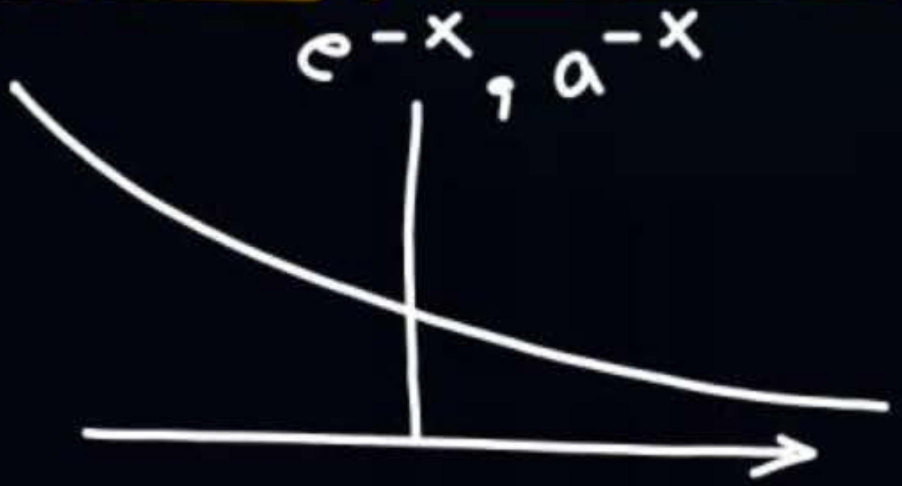
2π → One cycle

sin x, cos x → oscillates (-1, 1)

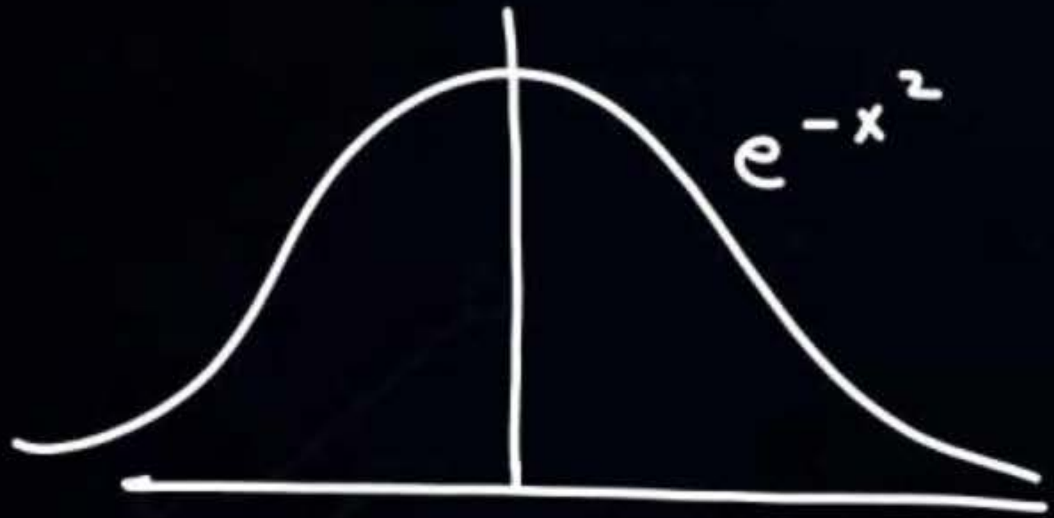
GRAPH OF FUNCTIONS

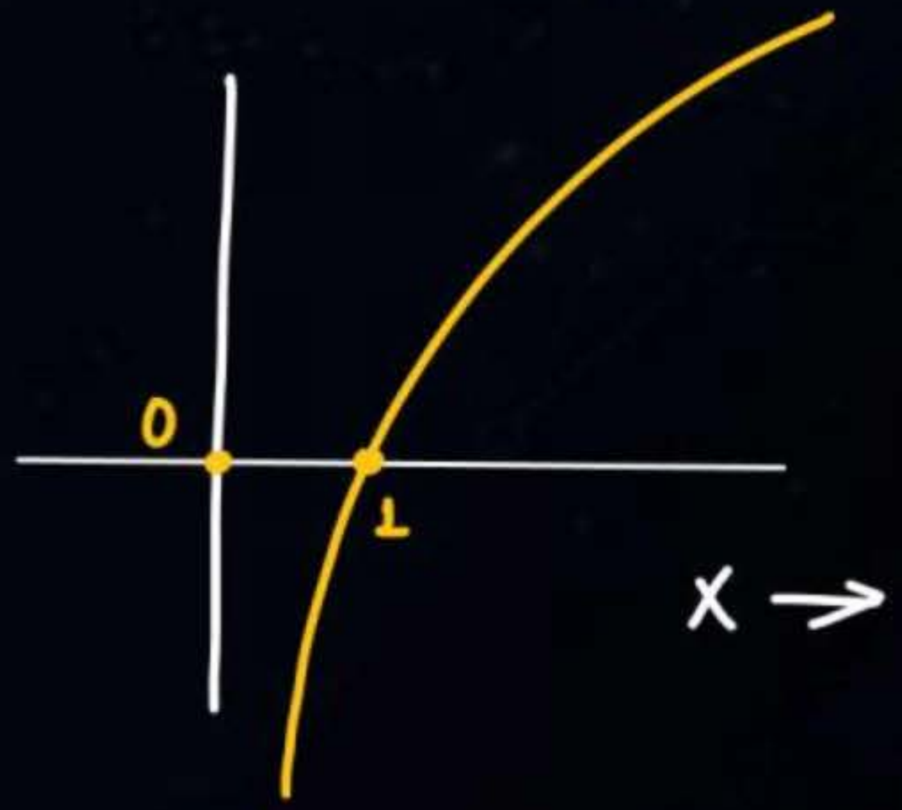
8) Exponential / Logarithmic functions :-





$a, e > 1$





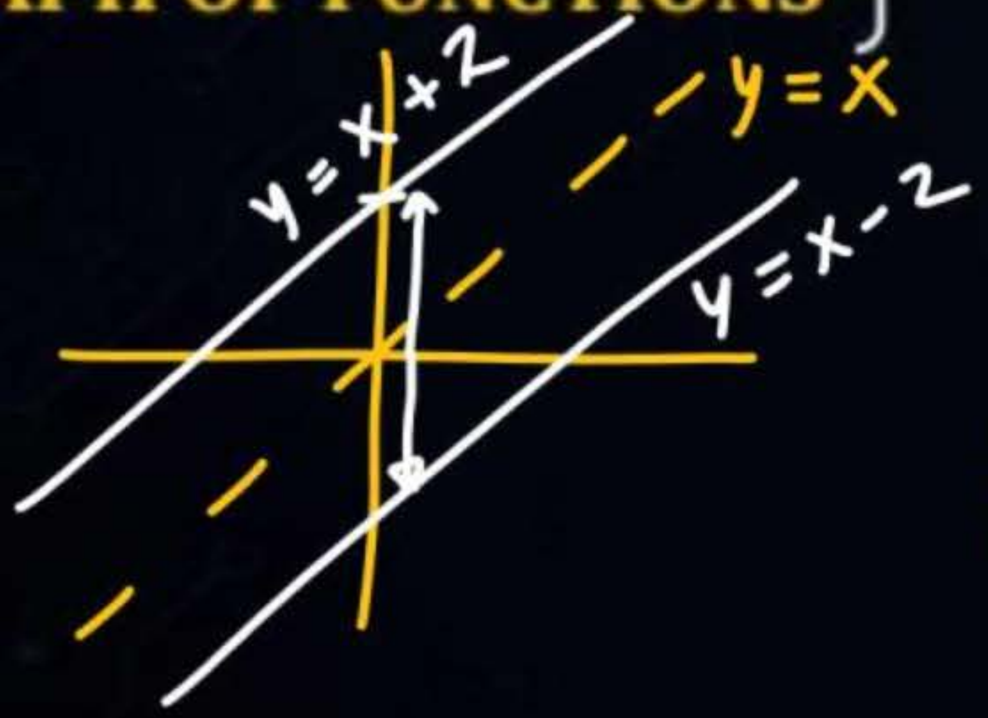
$\log_{10} 0 = -\infty$

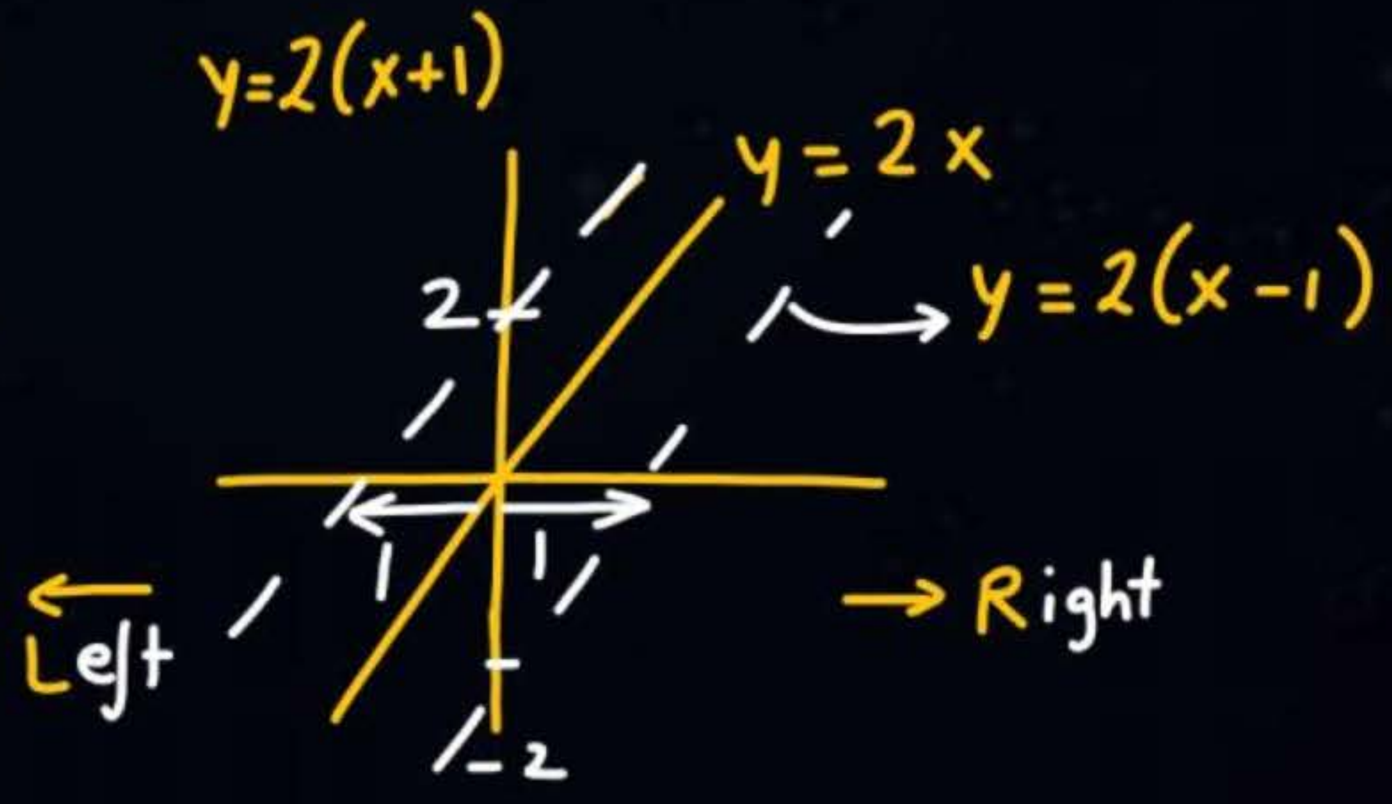
$\log_{10} 1 = 0$

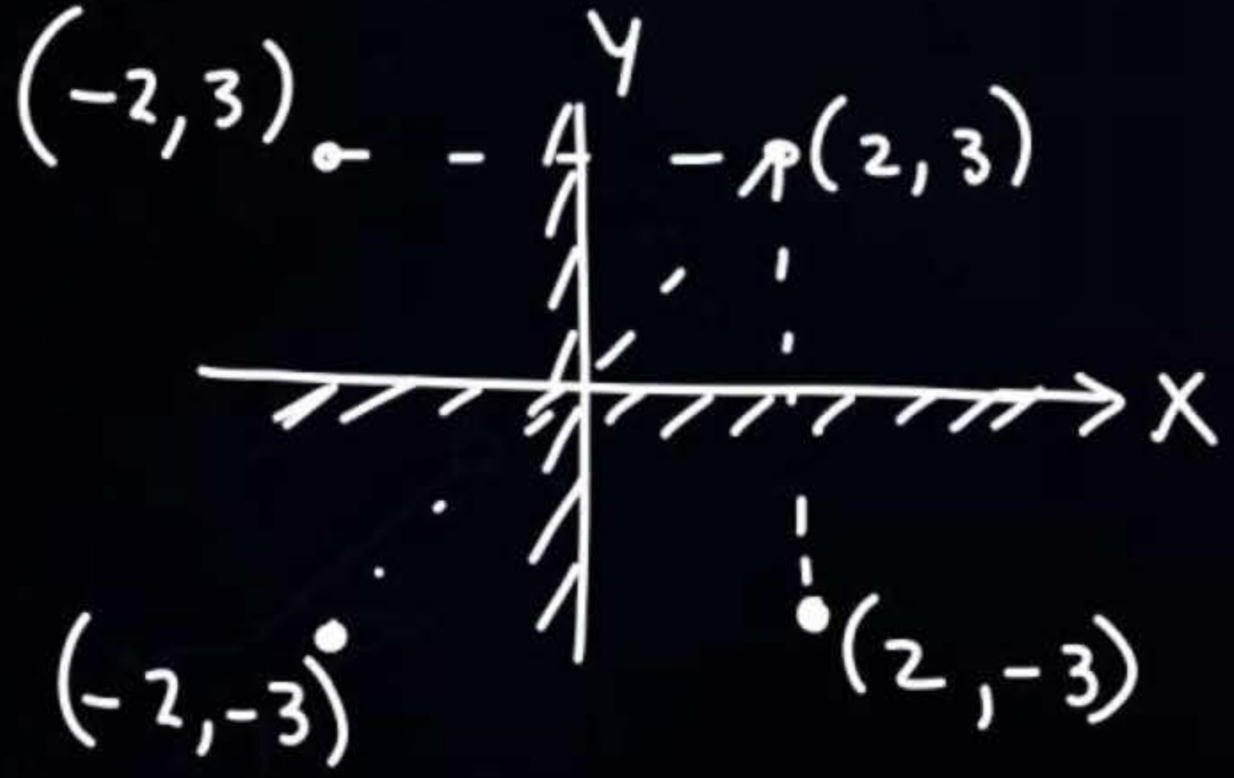
$\log_{10} 10 = 1$

$\log_{10} 100 = 2$

GRAPH OF FUNCTIONS







$-f(x)$

Mirror image in X axis → $y \to -y$

" " " Y axis → $x \to -x$

" " " origin → $x \to -x$
$y \to -y$

[GRAPH TRANSFORMATION]

1. (i) $f(x) + a \rightarrow$ shift the graph of $f(x)$ upward by a units.

   (ii) $f(x) - a \rightarrow$ shift the graph of $f(x)$ downward by a units.

2. (i) $f(x + a) \rightarrow$ shift the graph of $f(x)$ leftward by a units.

   (ii) $f(x - a) \rightarrow$ shift the graph of $f(x)$ rightward by a units.

3. (i) $af(x) \rightarrow$ stretch the graph of $f(x)$, a times along y axis.

   (ii) $\frac{1}{a}f(x) \rightarrow$ shrink the graph of $f(x)$, a times along y axis.

GRAPH TRANSFORMATION

4. (i) $f(ax) \rightarrow$ stretch the graph of $f(x)$, a times along x axis.

   (ii) $f\left(\frac{x}{a}\right) \rightarrow$ stretch the graph of $f(x)$, a times along x axis.

5. (i) $f(-x) \rightarrow$ Take the mirror image of $f(x)$, about x axis.

   (ii) $-f(x) \rightarrow$ Take the mirror image of $f(x)$ about x axis.

   (iii) $-f(-x) \rightarrow$ First take the mirror image about y axis and take the mirror image of new graph about x axis.

GRAPH TRANSFORMATION

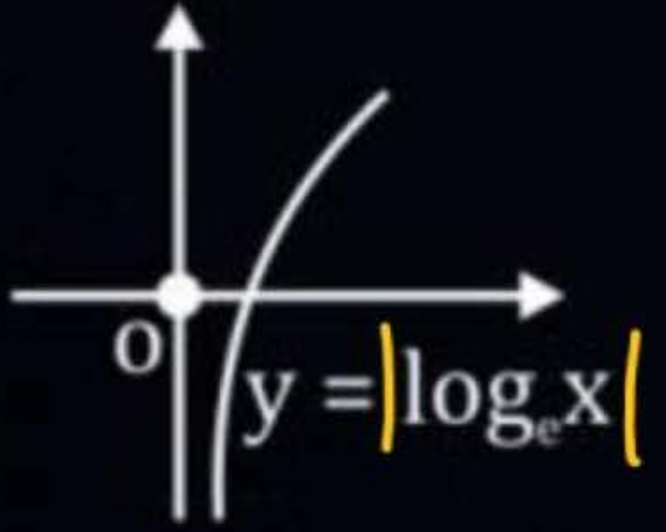
6. (i) $|f(x)| \rightarrow$ Take the mirror image about x axis, of that portion of graph, which lies below x axis. While graph that lies above x axis remains as it is.

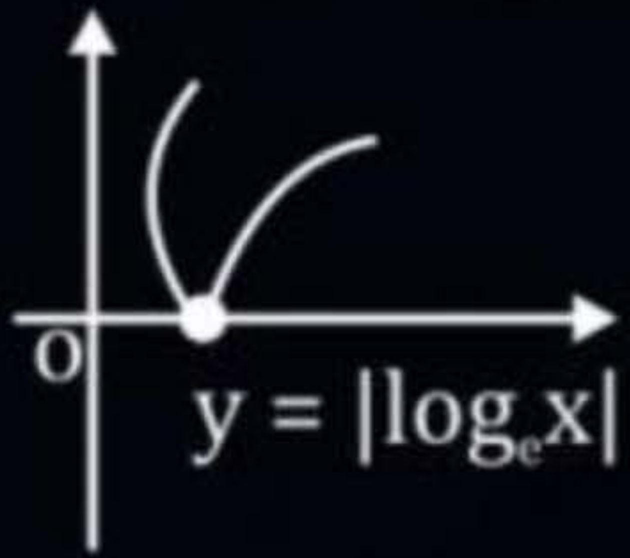
   (ii) $f(|x|) \rightarrow$ First unit that portion of graph which lies in the left side of y axis, and then take the mirror image about y axis of the remaining portion of the graph.

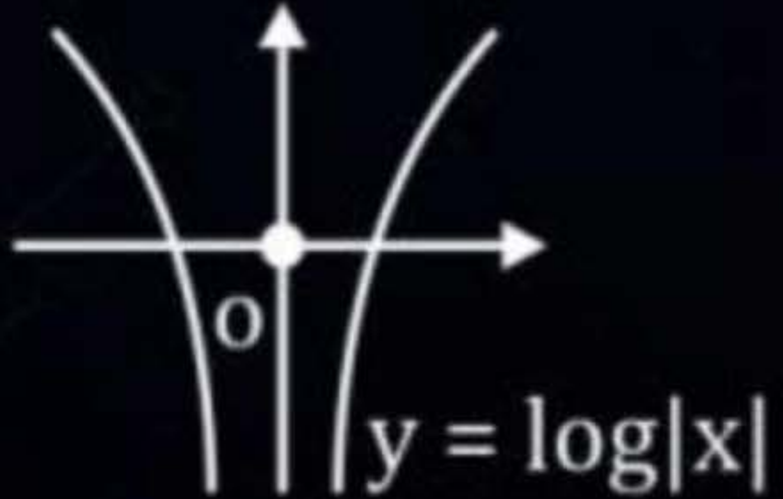
GRAPH TRANSFORMATION

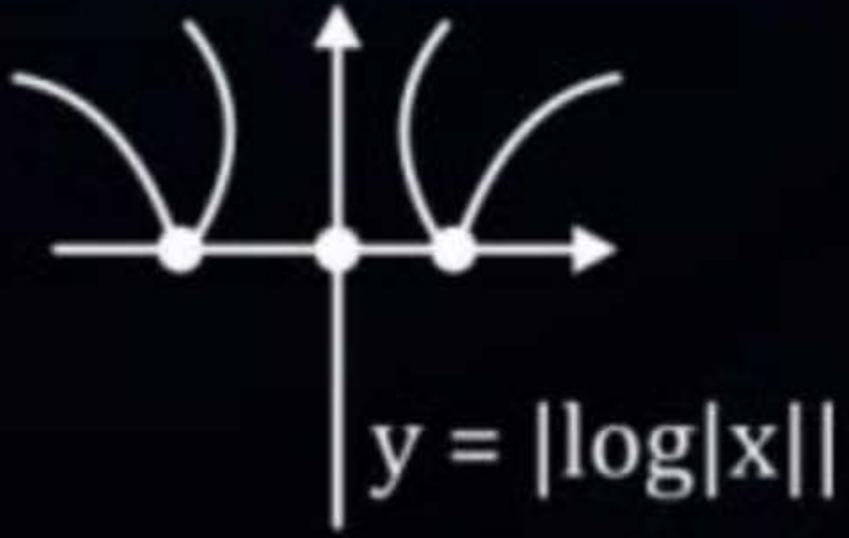
(iii) $|f(|x|)| \rightarrow$ First form the graph of $|f(x)|$ using part (i) and then from the graph of $|f(x)|$ using part (ii).

$x < 1$
$\log x \rightarrow -ve$
$x > 1$
$\log x \rightarrow +ve$









Q.

The curve given by the equation $x^2 + y^2 = 3axy$ is [GATE]

(a) Symmetrical about x-axis

(b) Symmetrical about y-axis

(c) Symmetrical about the line $y = x$ ✓

(d) Tangential to $x = y = a/3$

NOTE :-

→ If fn. is symmetrical about

i) X axis $(y \to -y)$

Ex :- $y^2 = x$

ii) y-axis $(x \to -x)$

Ex :- $y = x^2$

iii) Origin $\begin{pmatrix} x \to -x \\ y \to -y \end{pmatrix}$

$x = y^2$

$x = (-y)^2$

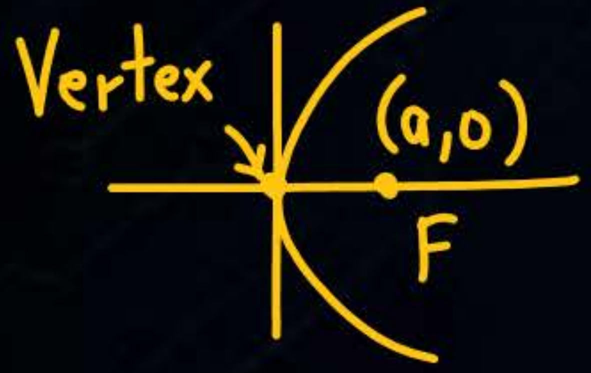
$x^2 + y^2 = 3axy$

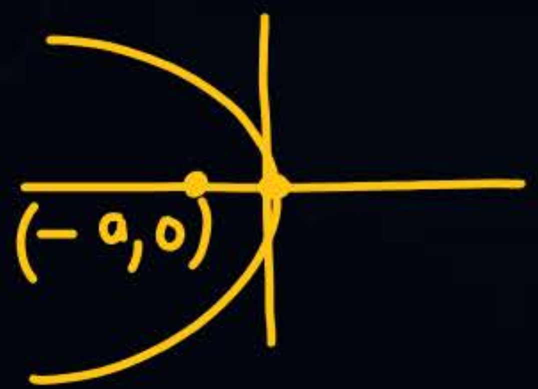
$y \leftrightarrow x$

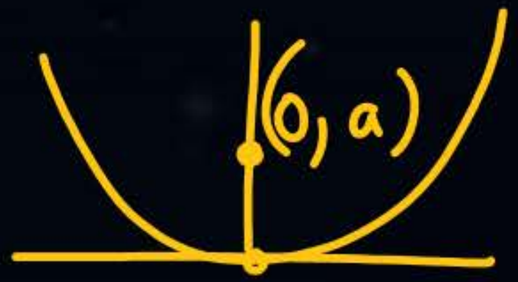
$y^2 + x^2 = 3ayx$

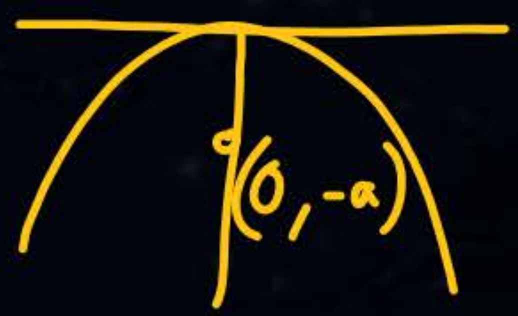
$\therefore$ Symm. about $y = x$ and origin

Basic Functions :-



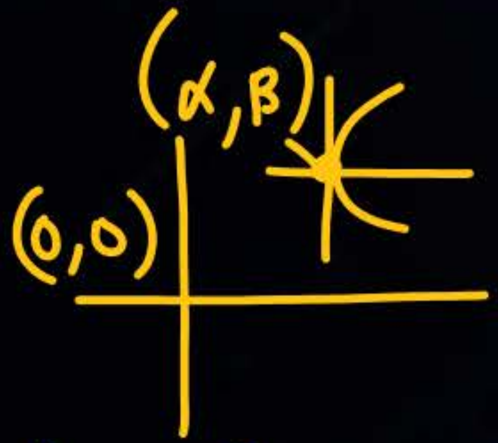




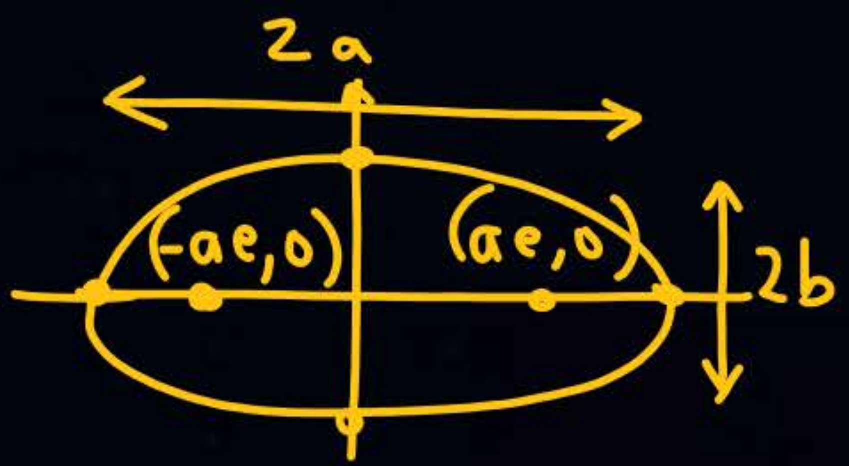


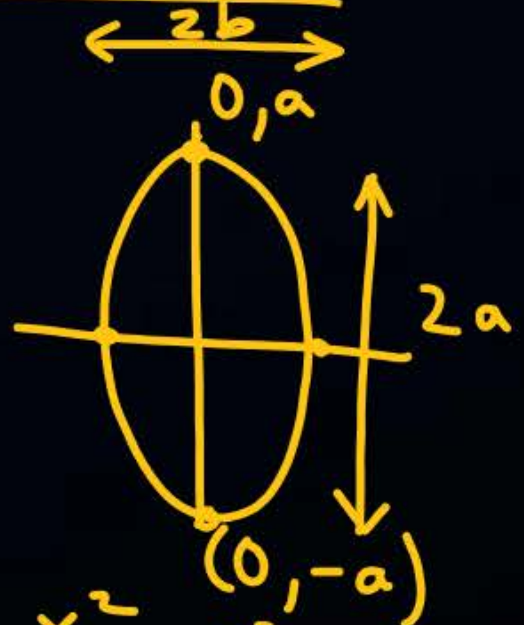
$y^2 = 4ax$ $\quad$ $y^2 = -4ax$ $\quad$ $x^2 = 4ay$ $\quad$ $x^2 = -4ay$

PARABOLAS



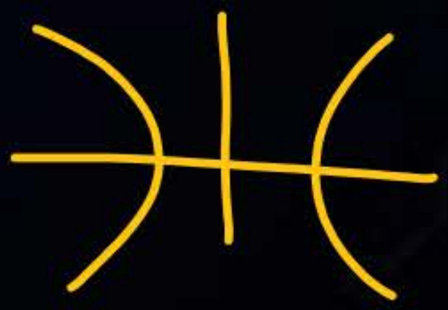
$(y-\beta)^2 = 4a(x-\alpha)$

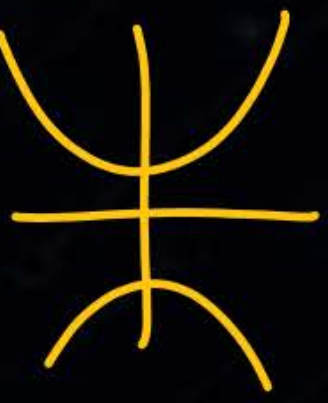




$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad$ $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

ELLIPSE





$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad$ $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$

HYPERBOLA

Classification of second degree conic section :- Discriminant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$f(x,y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c$$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\Delta \neq 0 \begin{cases} 1)\ h = 0,\ a = b \Rightarrow \text{Circle} \\ 2)\ h^2 - ab < 0,\ (e<1) \Rightarrow \text{Ellipse} \\ 3)\ h^2 - ab = 0,\ (e=1) \Rightarrow \text{Parabola} \\ 4)\ h^2 - ab > 0,\ (e>1) \Rightarrow \text{Hyperbola} \\ 5)\ h^2 - ab > 0,\ a+b=0,\ (e>1) \Rightarrow \text{Rectangular hyperbola} \end{cases}$$

Circle :-

$$(x-a)^2 + (y-b)^2 = r^2$$

Centre (a, b)

Radius r

$x^2 - y^2 = a^2$

Asymptote are $\perp$

(1) $3x^2 + 7y^2 + 10xy + 5x + 9y + 6 = 0$

$\Rightarrow a = 3,\ b = 7,\ h = 5$

$10\cancel{xy} = 2h\cancel{xy}$

$h = 5$

$h^2 - ab = 5^2 - 3.7 = 4 > 0$

Hyperbola Since $a+b \neq 0$

$\therefore$ it is not rectangular hyperbola.

(2) $2x^2 - y^2 + 6xy = 0$

$\Rightarrow a = 2,\ b = -1,\ h = 3$

$h^2 - ab = 3^2 - 2(-1) = 11 > 0$

Hyperbola.

(3) $2x^2 - 2y^2 + 8xy = 0$

$\Rightarrow a = 2,\ b = -2,\ h = 4$; $h^2 - ab = 16 - 2(-2) = 20 > 0$

$a + b = 0$ $\therefore$ Rectangular hyperbola

(4) $5x^2 + 5y^2 = 60$

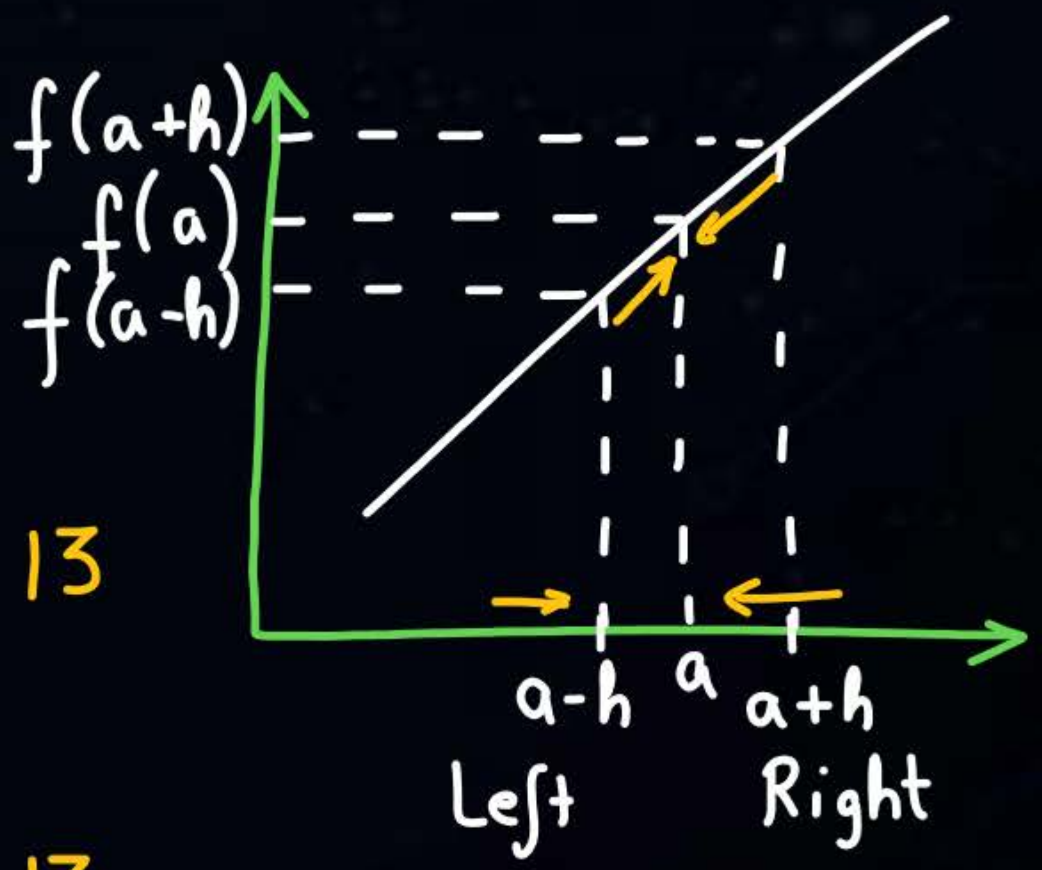
$a = 5,\ b = 5,\ h = 0$

$h^2 - ab = 0^2 - 5\times5 = -25 < 0 \Rightarrow$ Circle

(5) $2x^2 + 8y^2 + 4xy = 0$ $\quad 2^2 - 2(8) = -12 < 0 \Rightarrow$ Ellipse

LIMIT OF A FUNCTION

$f(x)$ is defined



$$\text{L.H.L.} = f(a-h)$$

$$= \lim_{h \to 0} f(a-h) = 2(5-h)+3 = 13$$

$$\text{R.H.L.} = f(a+h)$$

$$= \lim_{h \to 0} f(a+h) = 2(5+h)+3 = 13$$

If L.H.L. = R.H.L. [limit exist]

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 13$$

$f(x) = 2x+3$

Value at $x = 5$

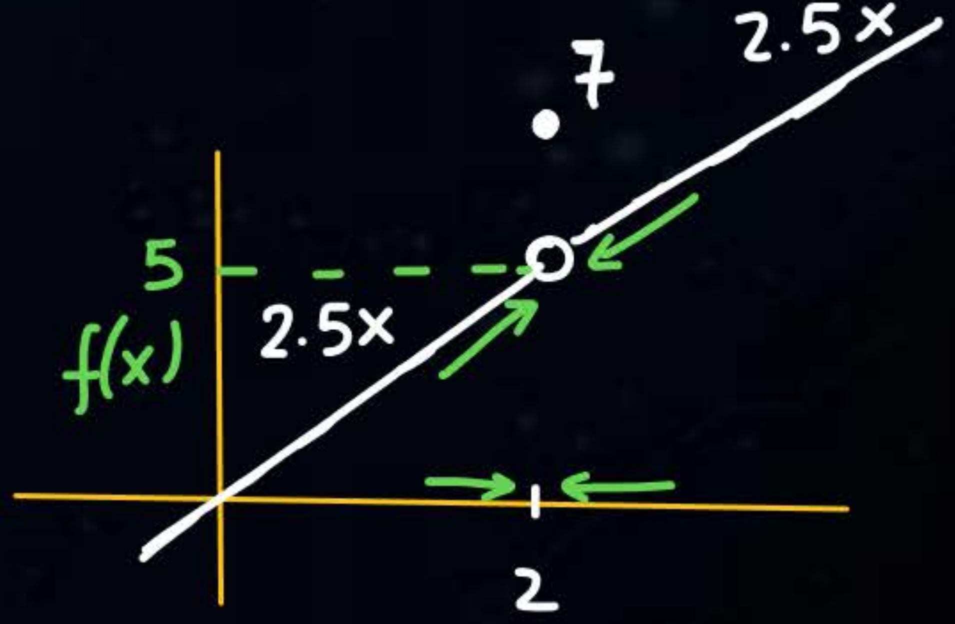
$f(5) = 13$

$h \to$ infinitely small.

LEFT HAND LIMIT

Ex:-

$$f(x) = \begin{cases} 2.5x & ; x \neq 2 \\ 7 & ; x = 2 \end{cases}$$



Find the limit at $x = 2$

$$\lim_{x \to 2} f(x)$$

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = 5$$

$$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = 5$$

$\text{L.H.L.} = \text{R.H.L.} = 5$ [limit exist]

$\text{L.H.L.} = \text{R.H.L.} \neq \text{Value}$ [Discontinuous at $x = 2$]

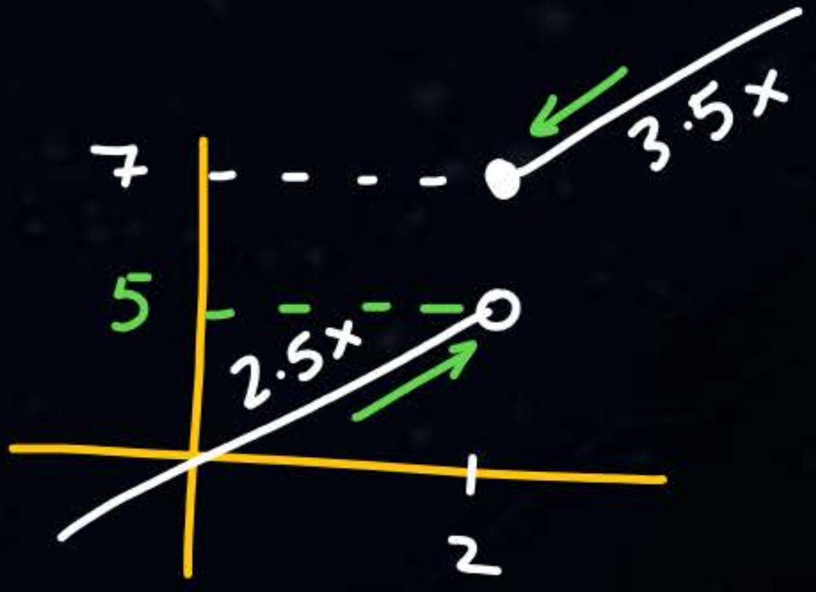
Value at $x = 2$

$f(2) = 7$

[RIGHT HAND LIMIT]

Ex :-

$$f(x) = \begin{cases} 2.5x & ; x < 2 \\ 7 & ; x = 2 \\ 3.5x & ; x > 2 \end{cases}$$



Soln :-

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = 2.5x = 5$$

$$x \longrightarrow 2^- \; ; \; f(x) \longrightarrow 5$$

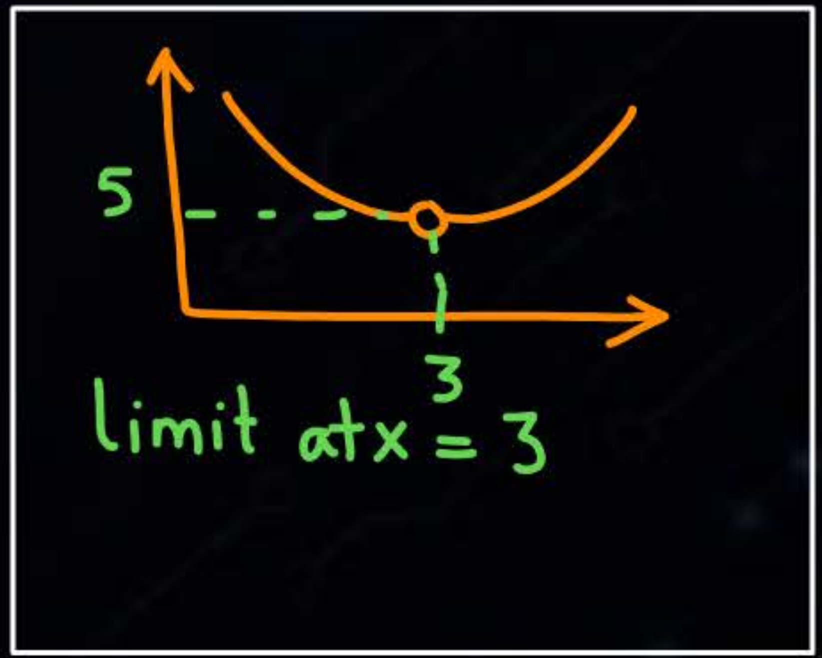
$$\text{LHL} \neq \text{RHL} = \text{Value}$$

$$\text{R.H.L} = \lim_{x \to 2^+} f(x) = 3.5x = 7$$

$$x \longrightarrow 2^+ \; ; \; f(x) = 7$$

Value at $x = 2$; $f(2) = 7$

Since LHL $\neq$ RHL [limit does not exist]



LIMITS

f is defined in the neighbourhood of $x = a$

$$\boxed{\lim_{x \to a} f(x) = l}$$

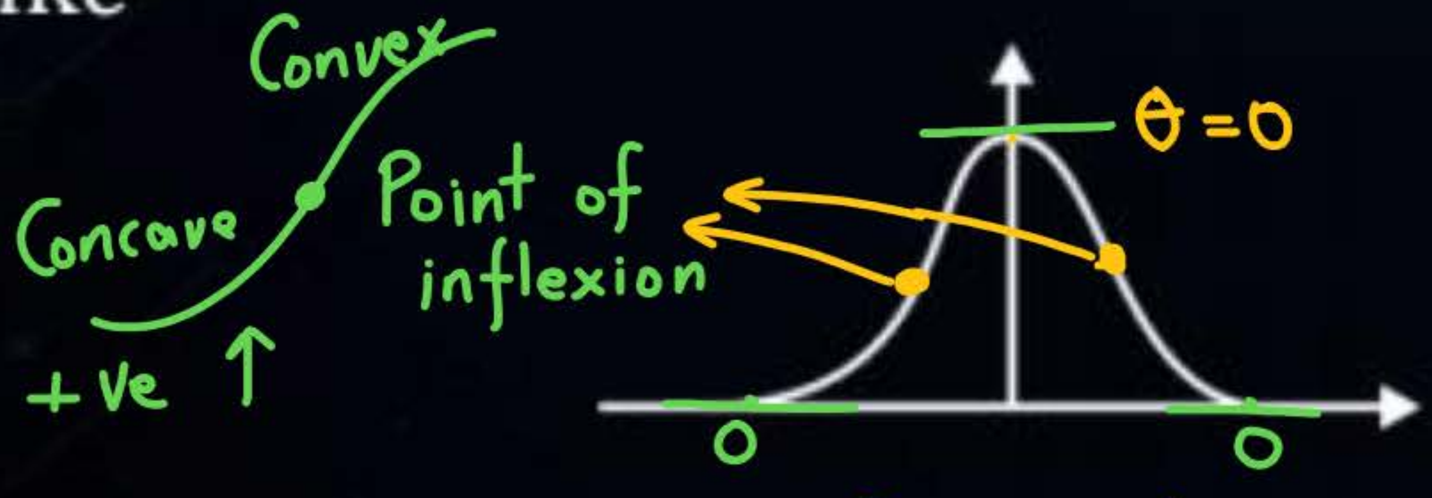
L.H.L :- $\lim_{h \to 0} f(a-h) = l_1$

R.H.L :- $\lim_{h \to 0} f(a+h) = l_2$

limit of $f(x)$ at $x = a$ will exist if LHL = RHL

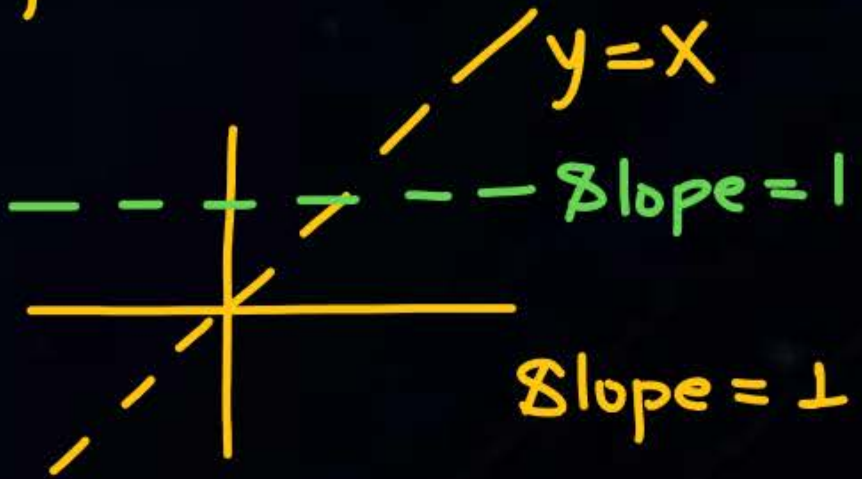
Q.

The derivative of the symmetric function drawn in given figure will look like

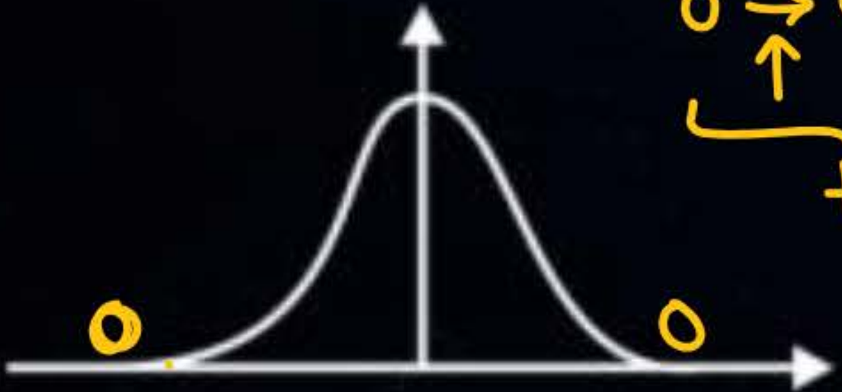


$y = f(x)$

$\frac{dy}{dx} = f'(x)$



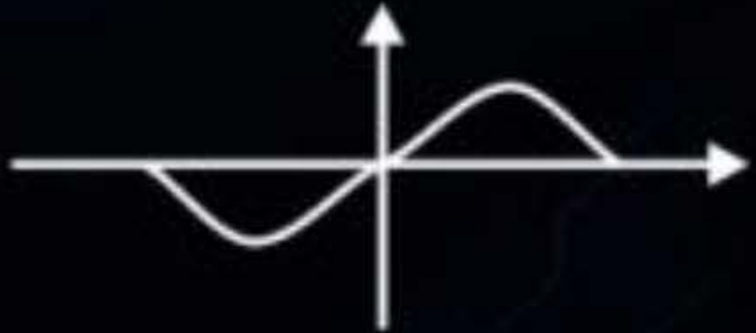
(a)



(b)

(c) ✓

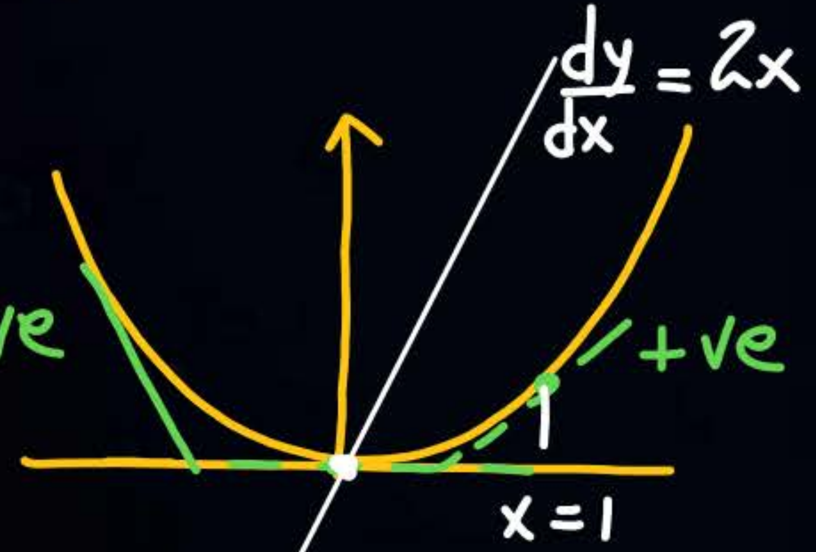
(d)



Ex:-

$y = x^2$

$\frac{dy}{dx} = 2x = \text{Slope}$



At $x = 1$ $\quad \frac{dy}{dx} = 2 = \tan\theta$

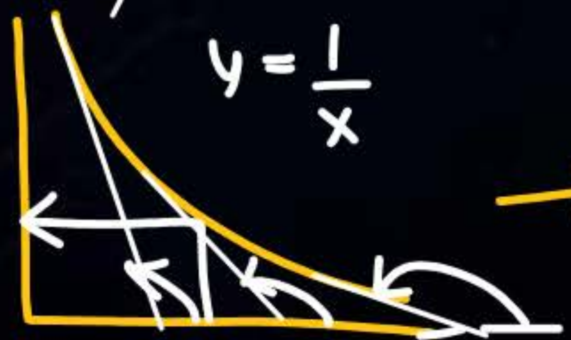
Slope = $\tan\theta$.

$\theta \to 0 - 90°$

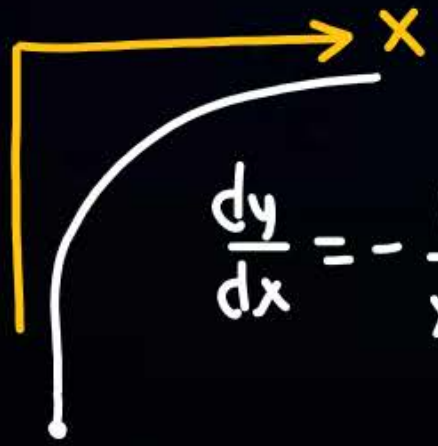
$\tan\theta = +ve$

+ve slope
(0 − 90°)

Ex:-



Function



$\frac{dy}{dx} = -\frac{1}{x^2}$

$\xrightarrow{+}$

$\theta \to 90° - 180°$

$\tan\theta = -ve$

$-\infty \;—\; -1 \to 0$

−ve slope
(90° − 180°)

Q.

If $g(x) = 1 - x$ and $h(x) = \frac{x}{x-1}$ then $\frac{g(h(x))}{h(g(x))}$ is

(a) $\frac{h(x)}{g(x)}$

(b) $\frac{-1}{x}$

(c) $\frac{g(x)}{h(x)}$

(d) $\frac{x}{(1-x)^2}$

Composite function

$g(x) = 1 - x$

$g[h(x)] = 1 - h(x) = 1 - \frac{x}{x-1} = \frac{x - 1 - \cancel{x}}{x-1}$

$g \circ h\, x$

$h(x) = \frac{x}{x-1}$

$h[g(x)] = \frac{g(x)}{g(x)-1} = \frac{1-x}{1-x-1} = \frac{1-x}{-x}$

$h \circ g\, x$

$= \frac{\frac{-1}{x-1}}{\frac{1-x}{-x}} = \frac{\frac{x}{x-1}}{1-x} = \frac{h(x)}{g(x)}$

$$f(x) = |x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$$g(x) = 1 - x^3 \quad ; \quad x \to (-\infty, \infty)$$

$$g \circ f\ x = g(f(x)) = \begin{cases} 1 - [f(x)]^3 ; & x \geq 0 \\ 1 - [f(x)]^3 ; & x < 0 \end{cases}$$

$$\boxed{f \circ g = \begin{cases} 1 - x^3 & ; x \geq 0 \\ -(1 - x^3) & ; x < 0 \end{cases} \quad f[g(x)]}$$

Range = ?

$$= \begin{cases} 1 - x^3 & ; x \geq 0 \\ 1 - (-x)^3 & ; x < 0 \end{cases}$$

$$g \circ f = \begin{cases} 1 - x^3 & ; x \geq 0 \\ 1 + x^3 & ; x < 0 \end{cases}$$

Q.

If for non-zero x, af (x) + bf $\left(\frac{1}{x}\right) = \frac{1}{x} - 25$ where $a \neq b$ $then$ $\int_1^2 f(x)dx$ is

(a) $\frac{1}{a^2-b^2}\left[a(ln2-25)+\frac{47b}{2}\right]$

(b) $\frac{1}{a^2-b^2}\left[a(2ln2-25)+\frac{47b}{2}\right]$

(c) $\frac{1}{a^2-b^2}\left[a(2ln2-25)+\frac{47b}{2}\right]$

(d) $\frac{1}{a^2-b^2}\left[a(\ln 2-25)-\frac{47b}{2}\right]$

$a\left[a f(x) + b f\left(\frac{1}{x}\right) = \frac{1}{x} - 25\right]$ — 1

Replace $x \to \frac{1}{x}$.

$-b\left[a f\left(\frac{1}{x}\right) + b f(x) = x - 25\right]$ — 2

1 $\times a$ − 2 $\times b$

$(a^2-b^2) f(x) = a\left(\frac{1}{x} - 25\right) - b(x-25)$

$f(x) = \frac{a\frac{1}{x} - 25a - bx + 25b}{a^2-b^2}$

Q.

A function $f : N^+ \to N^+$, defined on the set of positive integers N^+, satisfies the following properties

$f(n) = f\left(\frac{n}{2}\right)$ if n is even

$f(n) = f(n + 5)$ if n is odd

Let $R = \{i \mid \exists j : f(j = i)\}$ be the set of distinct value that f takes. The maximum possible size of R is ______ 2 distinct values

$\longrightarrow f(\underset{x}{2}) = f(\underset{x}{1})$; $f(\underset{x}{4}) = f(\underset{x}{2})$; $f(\underset{x}{6}) = f(\underset{x}{3})$; $f(\underset{x}{8}) = f(\underset{x}{4})$; $f(\underset{y}{10}) = f(\underset{y}{5})$; .. $f(14) = f(7)$

$\rightarrow f(\underset{x}{1}) = f(\underset{x}{6})$; $f(\underset{x}{3}) = f(\underset{x}{8})$; $f(\underset{y}{5}) = f(\underset{y}{10})$

$\left.\begin{array}{l} x \leftarrow f(1) \checkmark \\ y \leftarrow f(5) \checkmark \end{array}\right\}$ Only 2 distinct elements

2, 4, 6, 8, 10, .. 14

1 2 3 4 5 . . . 7

Q.

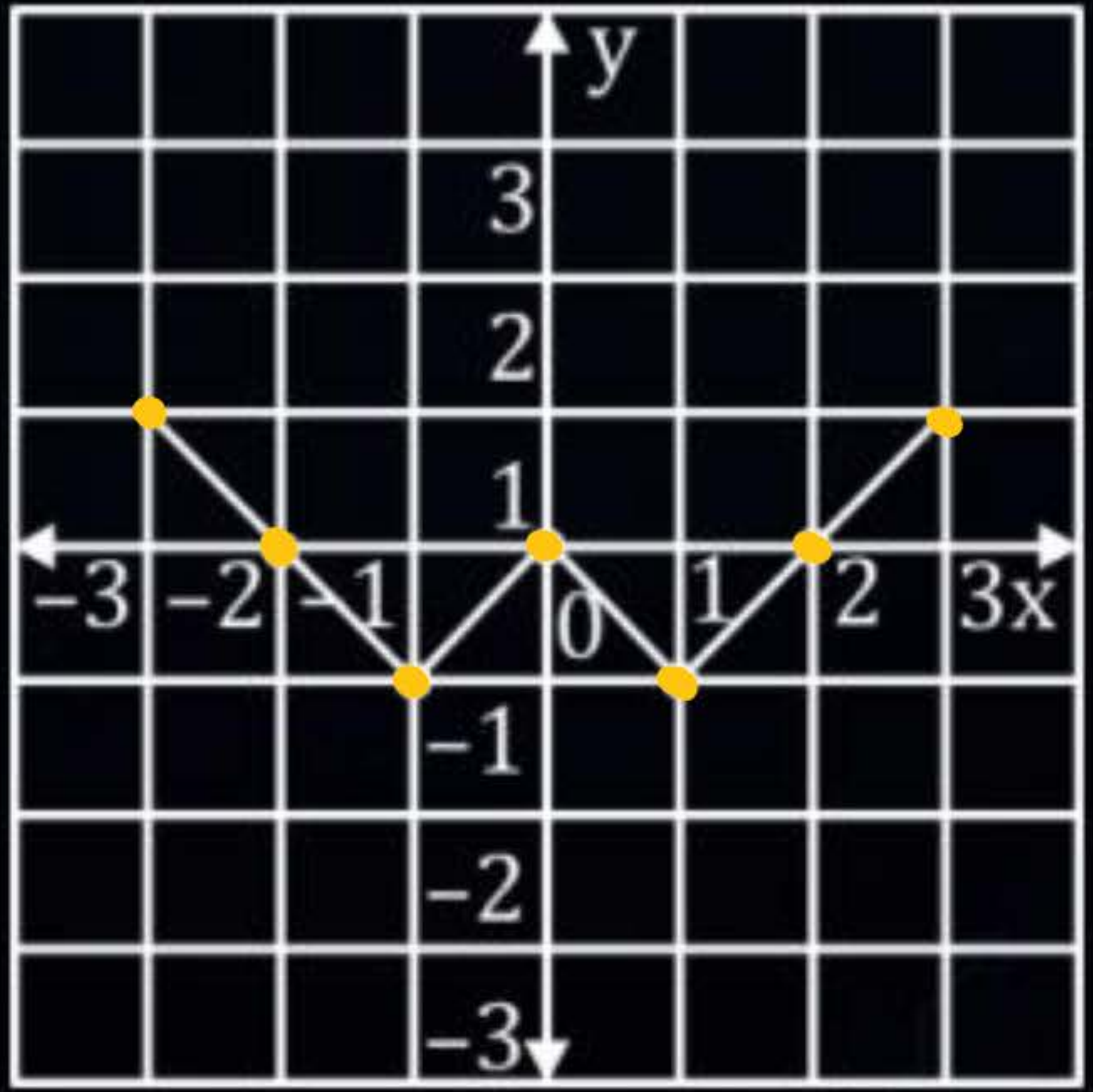
Which of the following function describe the graph shown in the below figure?

(a) $y = ||x| + 1| - 2$

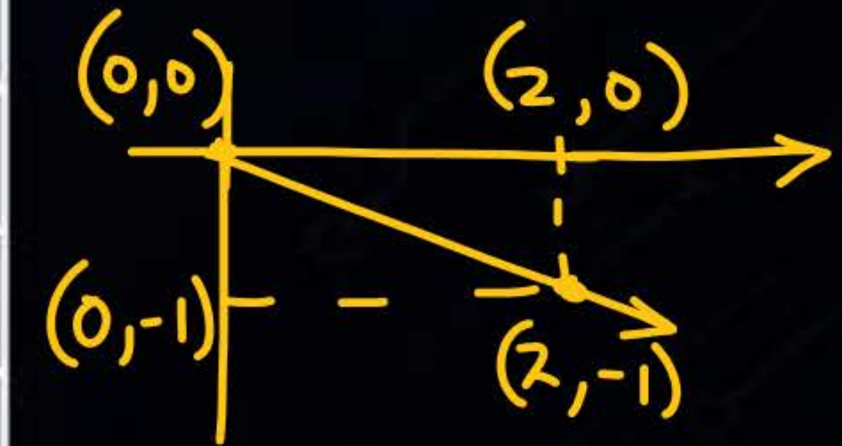
(b) $y = ||x| - 1| - 1$ ✓

(c) $y = ||x| + 1| - 1$

(d) $y = ||x - 1| - 1|$



x:	–2	–1	0	1	2
y:	0	–1	0	–1	0



Q. $ax^3 + bx^2 + cx + d$ is a polynomial on real x over real coefficients a, b, c, d wherein $a \neq 0$. Which of the following statements is true?

(a) a, b, c, d can be chosen to ensure that all roots are complex.

(b) no choice of coefficients can make all roots identical.

(c) c alone cannot ensure that all roots are real.

(d) d can be chosen to ensure that $x = 0$ is a root for any given set a, b, c.

$ax^3 + bx^2 + cx + d = 0$

3 roots → 0 complex roots, 3 real roots
→ 2 complex roots $(a \pm ib)$, 1 real roots

DESCARTES RULE :-

Thank you

GW Soldiers !