CS & IT ENGINEERING

DISCRETE MATHS
SET THEORY

[redacted]

Lecture No. 04

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TOPICS TO BE COVERED

01 Basics of Functions

02 Terms in Functions

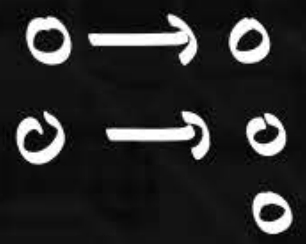
03 Number of Functions

04 Types of Functions

05 Various Examples in Functions

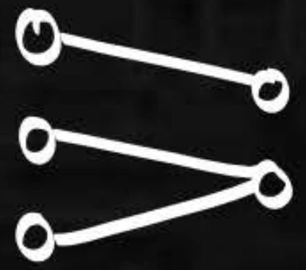
Functions

1:1



$|A| \leq |B|$

onto:



$|A| \geq |B|$

1:1 (injective)

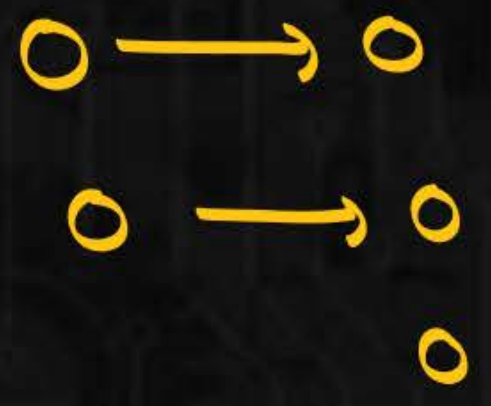
onto (surjective)

1:1 correspondance = 1:1 $\wedge$ onto

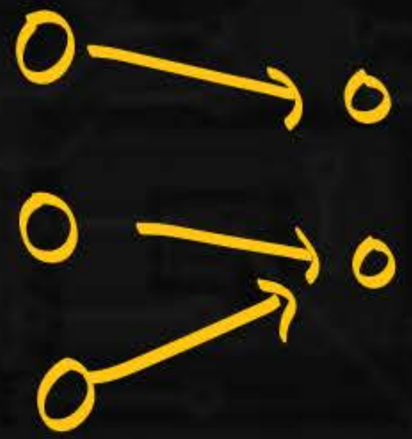
bijective = injective $\wedge$ surjective

$= |A| \leq |B| \wedge |A| \geq |B| \rightarrow (A = B)$

Functions

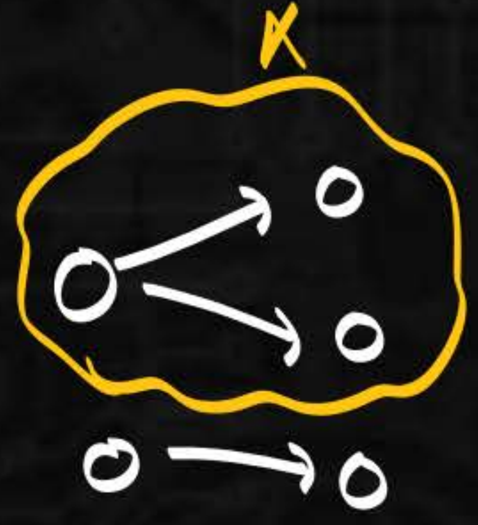


1:1 ✓
onto ✗

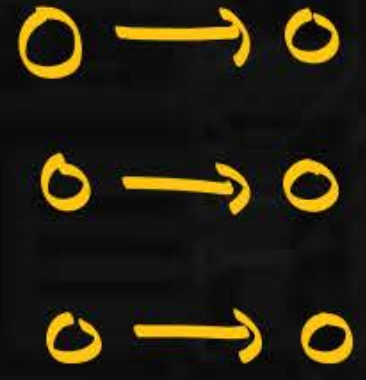


1:1 ✗
onto ✓

1:1 ✗
onto ✗
function ✓



not
Function.



1:1 ✓
onto ✓
1:1 c

Functions

$f(x) = x + 1$

$f : \mathbb{Z} \to \mathbb{Z}$

$f(a) = f(b) \to a = b$

$a+1 = b+1 \to a = b$

1:1

$f(x) = x + 1$

$\downarrow$

$y = x + 1$

$y - 1 = x$

onto ✓

$1 \longrightarrow 2$

$2 \longrightarrow 3$

$3 \longrightarrow 4$

$0 \longrightarrow 0$

$0 \longrightarrow 0$

Functions

$0 \to F$
$1 \to T$

$\{1, 2, 3\}$ $\{1\ 2\ 3\}$

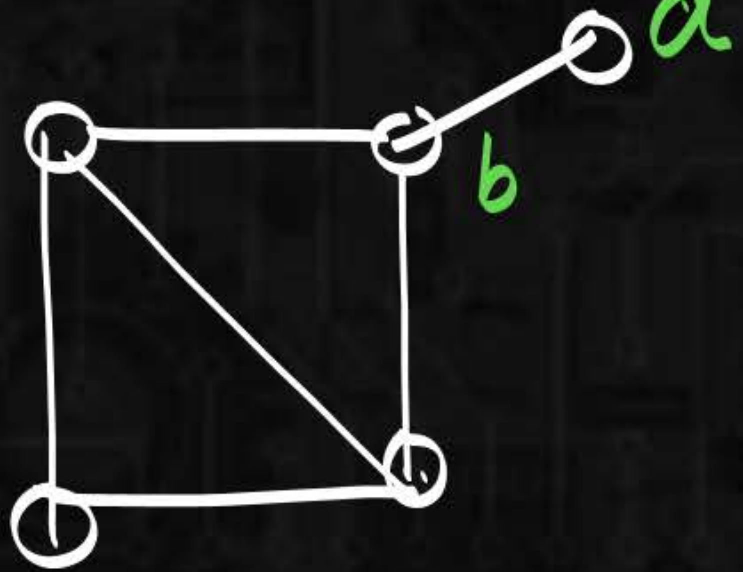
$$2^3 \begin{cases} \phi \to 000 \to FFF \\ \{1\} \to 100 \to TFF \\ \{2\} \to 010 \to FTF \\ \{3\} \to 001 \\ \{12\} \to 110 \\ \{13\} \to 101 \\ \{23\} \to 011 \\ \{123\} \to 111 \to TTT \end{cases} \quad 2^3$$

$|x| = 30$

$f : x \to y \ (1:1c)$

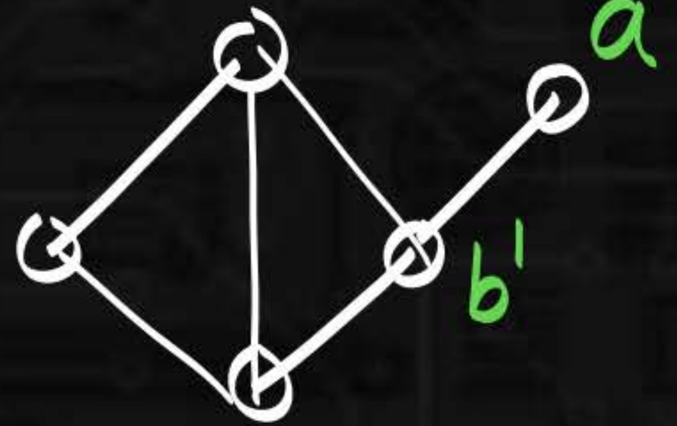
$|x| = |y| = 30$

Functions



$G_1 = (V_1, E_1)$

$G_1 = (V_1, E_1, \psi_1)$



$G_2 = (V_2, E_2)$

$G_2 = (V_2, E_2, \psi_2)$

$f : G_1 \to G_2$ → (1:1 c)

(1:1 c)

$f : V_1 \to V_2$ $\quad |V_1| = |V_2|$

$f : a \to a'$ $\quad f : E_1 \to E_2$

$f : b \to b'$

Functions

$f : A \to B$ $\quad |A| = |B| = n.$ $\quad$ Total 1:1 c functions = $n!$

(1:1 c)

3 ways

2 ways

1 way

$= 3 \cdot 2 \cdot 1 = 3!$

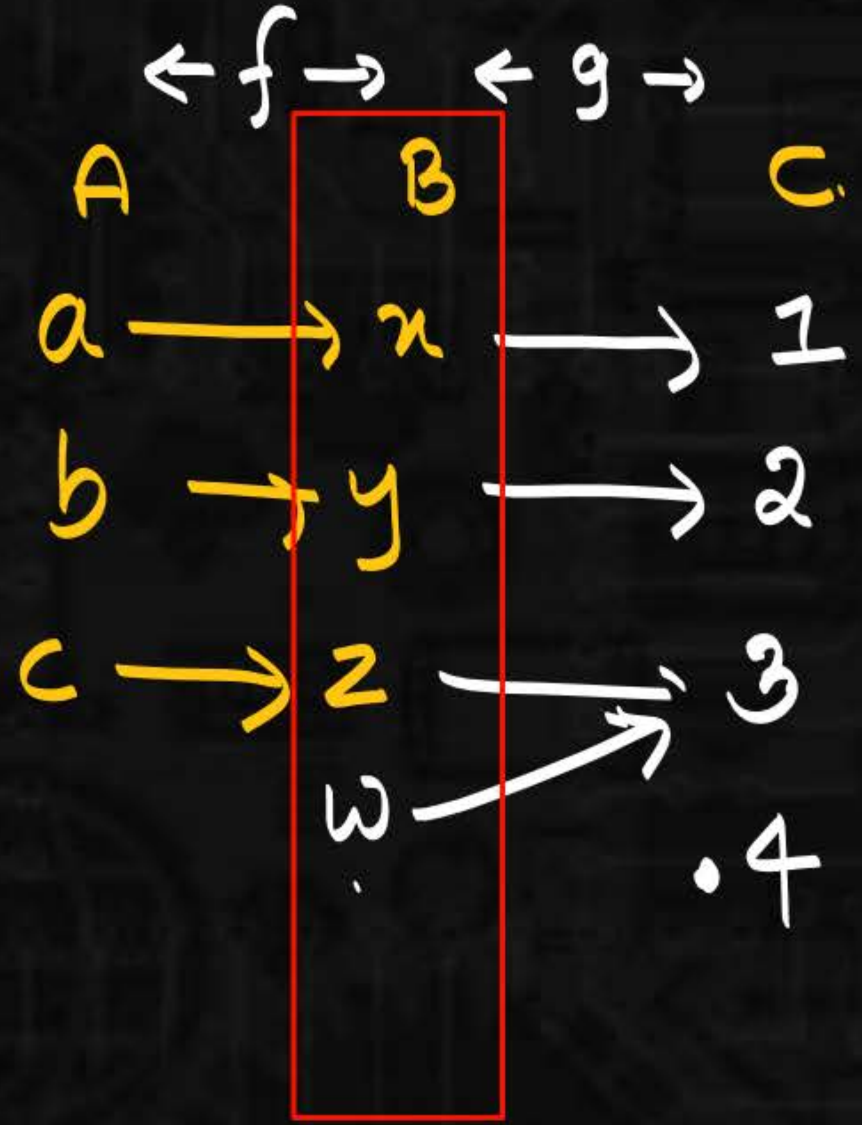
a	1
b	2
c	3.

Functions

Composition of functions :

Functions

$f: A \to B \quad g: B \to C \quad$ composition of function : $A \to C$



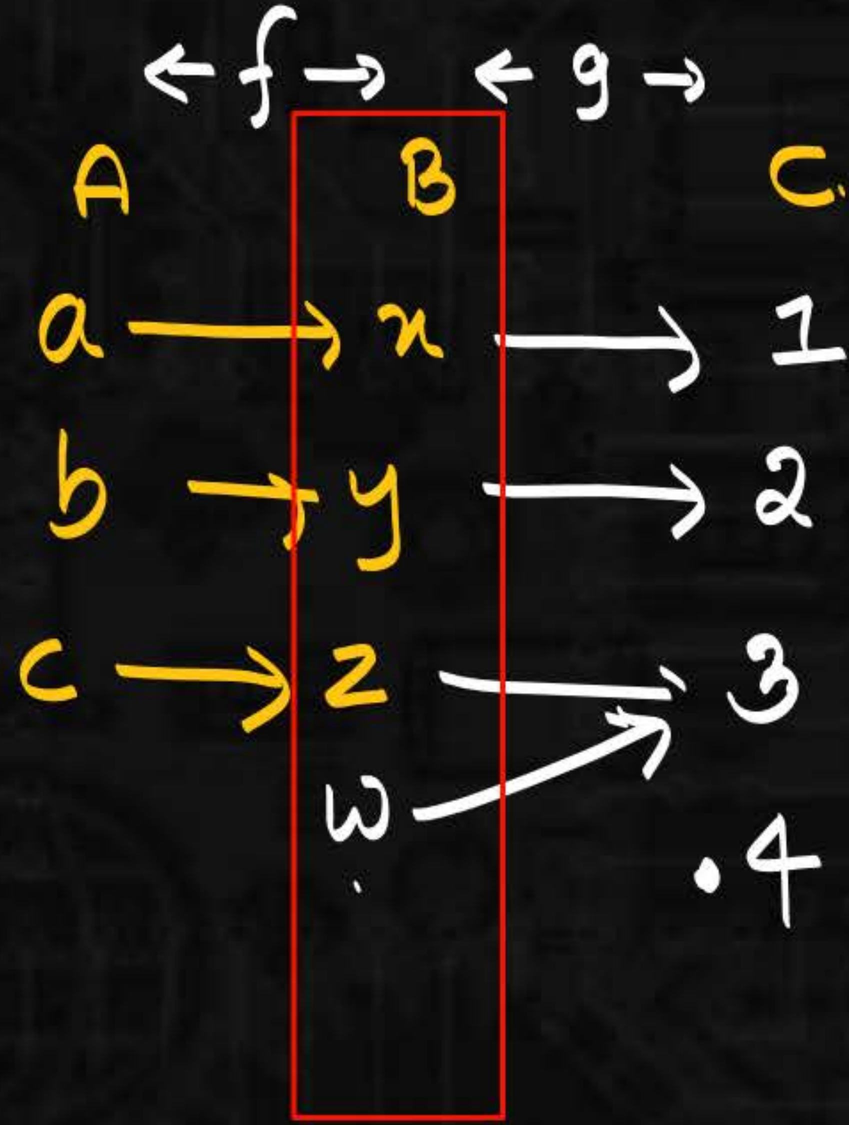
$f(a) = x$
$f(b) = y$
$f(c) = z$

$g(x) = 1$
$g(y) = 2$
$g(z) = 3$
$g(w) = 3$

$g(f(a)) = 1$
$g(f(b)) = 2$
$g(f(c)) = 3$

Functions

$f : A \to B \qquad g : B \to C$



$g(f(\ ))$

$\downarrow$

gof

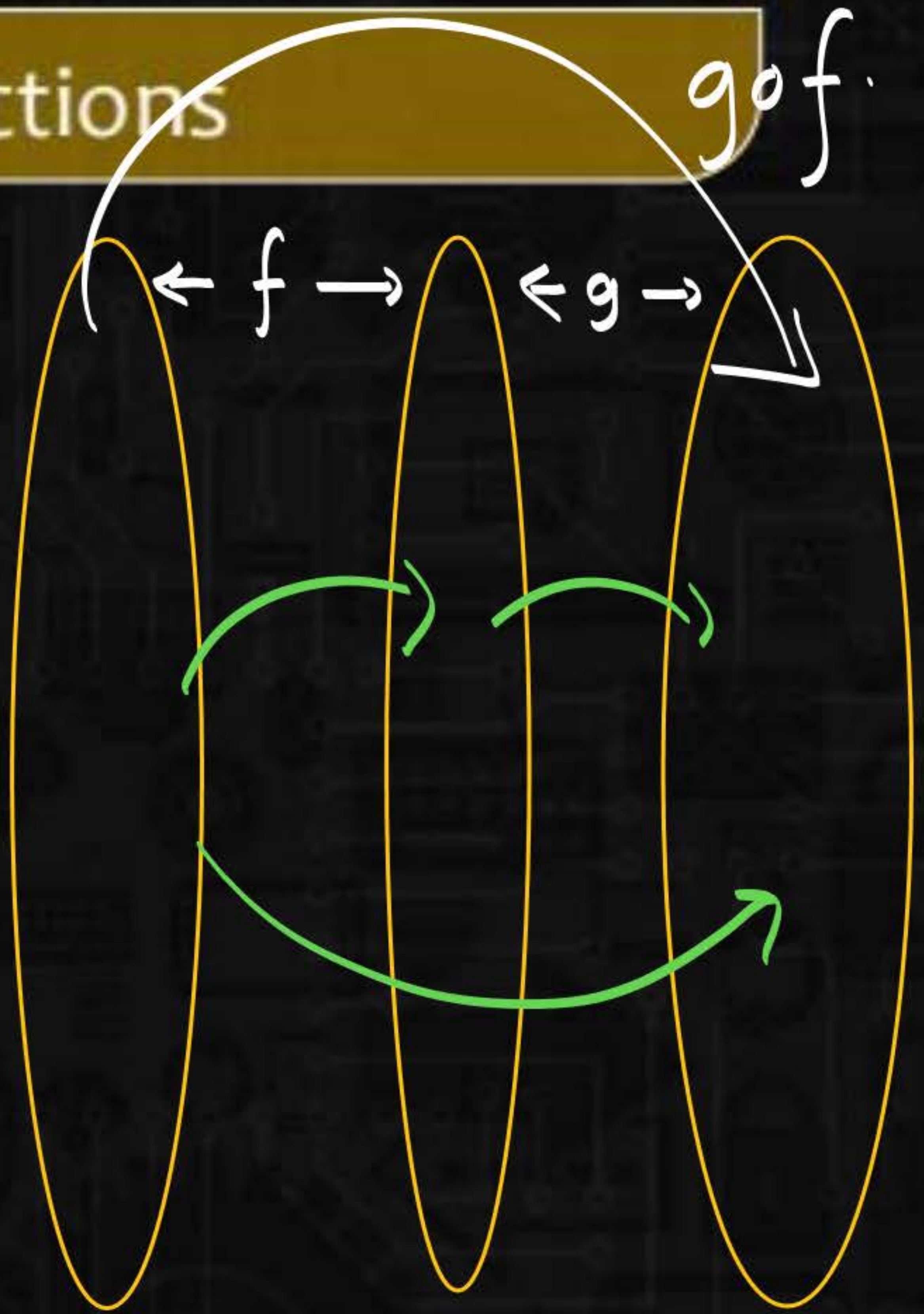
$gof : A \to C$

$g(f(a)) = 1$

$g(f(b)) = 2$

$g(f(c)) = 3$

Functions



$f : A \to B \quad g : B \to C.$

$g \circ f : A \to C.$

Functions

$$f: Z \to Z \qquad g: Z \to Z$$

$$f(x) = 2x+3 \qquad g(x) = x^2$$

$$g(a) = a^2$$

$$g \circ f = g(f(x)) = g(2x+3)$$

$$g \circ f = (2x+3)^2$$

$$g \circ f \neq f \circ g$$

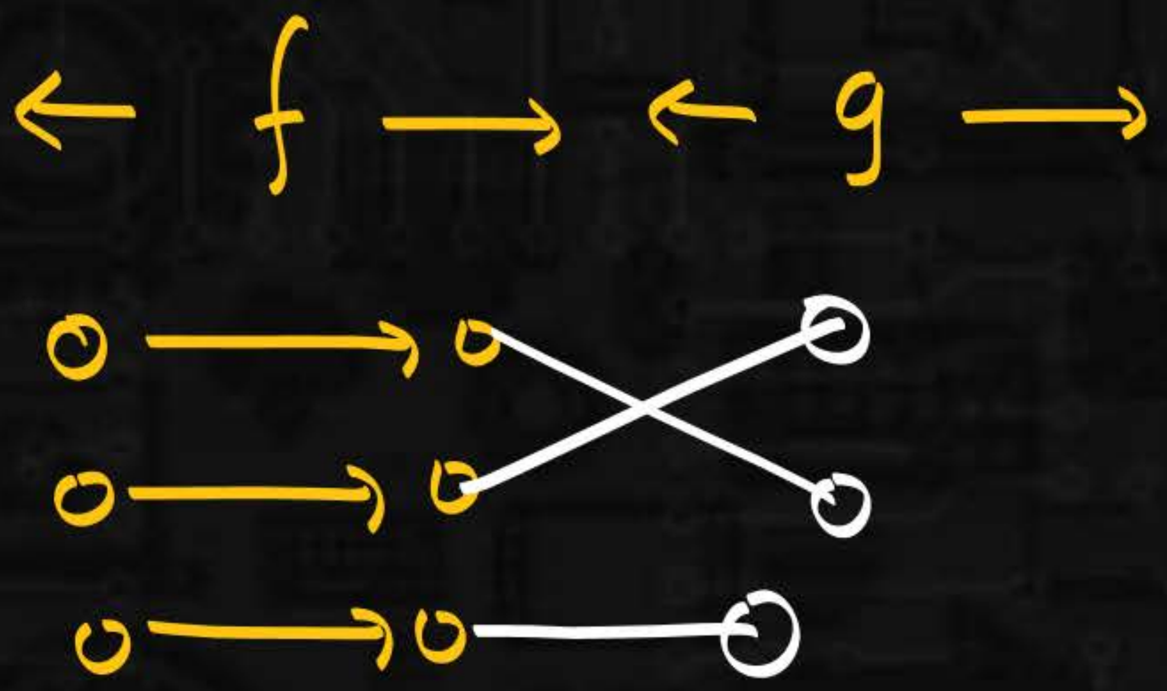
$$f \circ g = f(g(x))$$

$$= f(x^2)$$

$$= 2x^2+3$$

Functions

1) if f & g are 1:1 c then $g \circ f$ is 1:1 c (True)



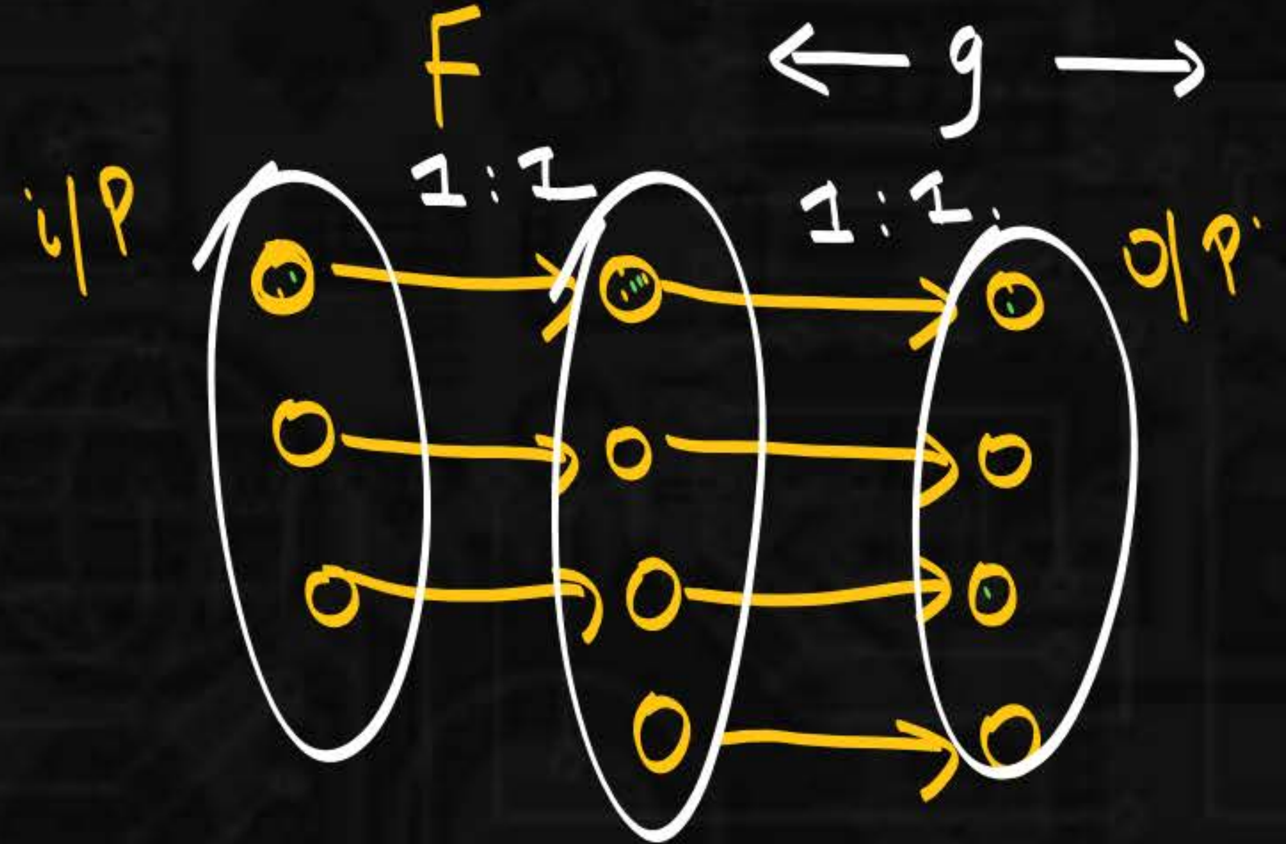
$g \circ f \rightarrow$ 1:1.

Functions

$f : A \to \underline{B}$ $\quad g : \underline{\underline{B}} \to C$ $\qquad gof : A \to C$

$\hookrightarrow 1:1 \qquad \hookrightarrow 1:1. \qquad \hookrightarrow 1:1.$

if f & g are 1:1 then gof is 1:1. (True)



Functions

$f: Z \to Z \qquad g: Z \to Z$

$f(x) = x+1 \qquad g(x) = x+2$

$\to 1:1 \qquad \to 1:1$

$gof = g(f(x))$

$= g(x+1)$

$= (x+1)+2 = x+3.$

$\qquad x+1 \qquad x+2.$

$1 \longrightarrow 2 \longrightarrow 4$

$2 \longrightarrow 3 \longrightarrow 5$

$3 \longrightarrow 4 \longrightarrow 6$

$1 \longrightarrow 4$

$2 \longrightarrow 5$

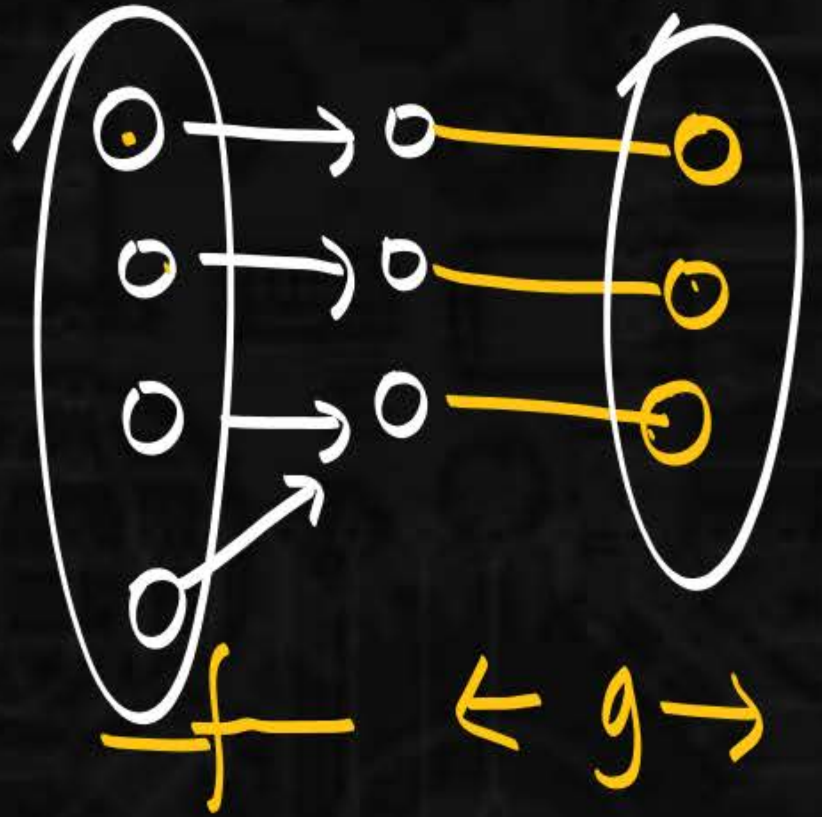
$3 \longrightarrow 6$

Functions

if f & g are onto then gof is onto (True)

$\leftarrow f \rightarrow \leftarrow g \rightarrow$

onto   onto



$f: \mathbb{Z} \to \mathbb{Z}$

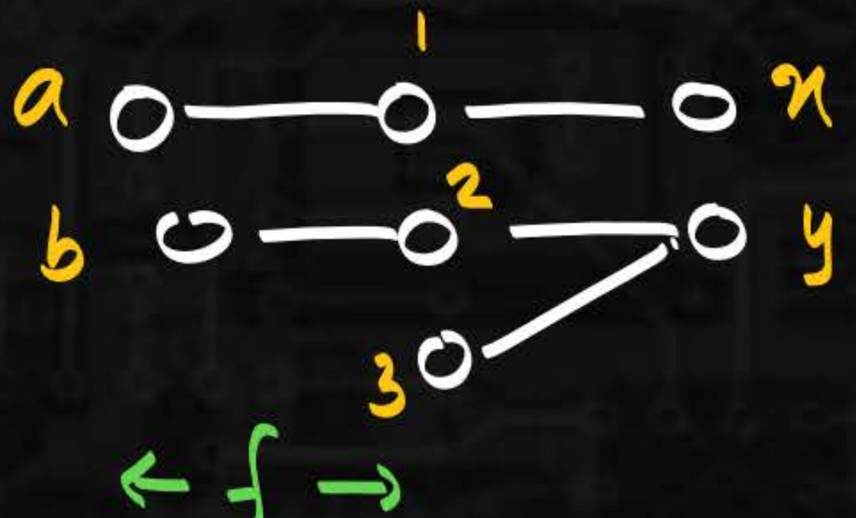
$g: \mathbb{Z} \to \mathbb{Z}.$

$f(x) = x+1$ ✓

$f(x): x+3$ ✓

Functions

if gof is onto then f is onto (false)

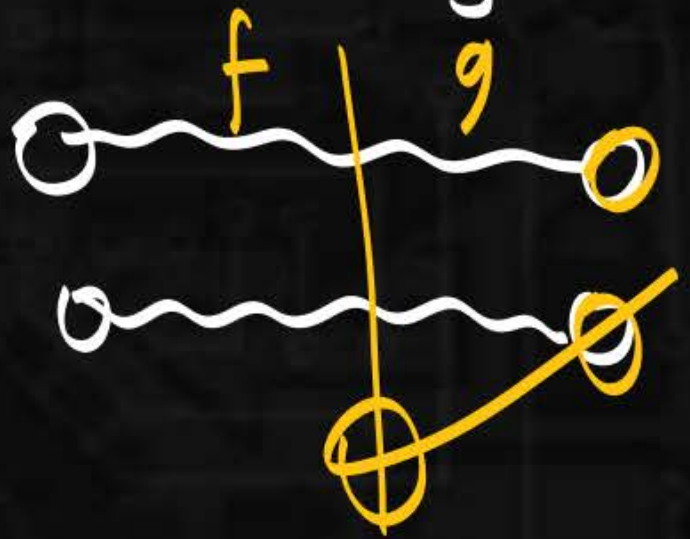


gof: → R.S Full.

a — x

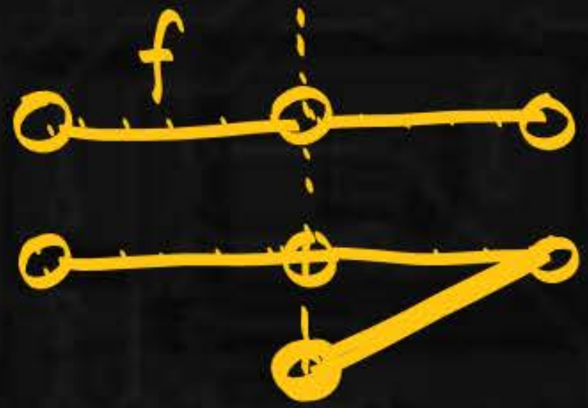
b — y.

if gof is onto then g is onto. (True)



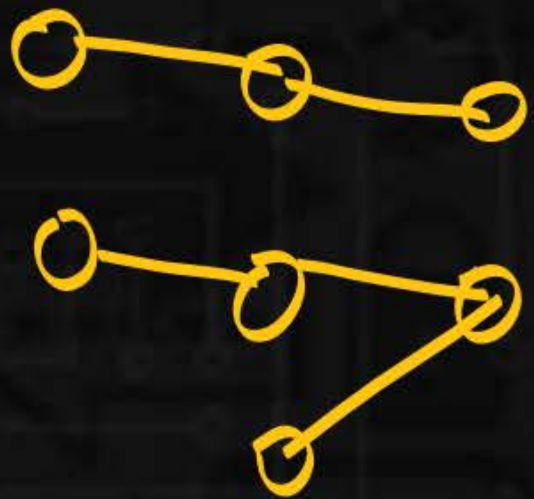
Functions

if $g \circ f$ is 1:1 then f is 1:1. (True)

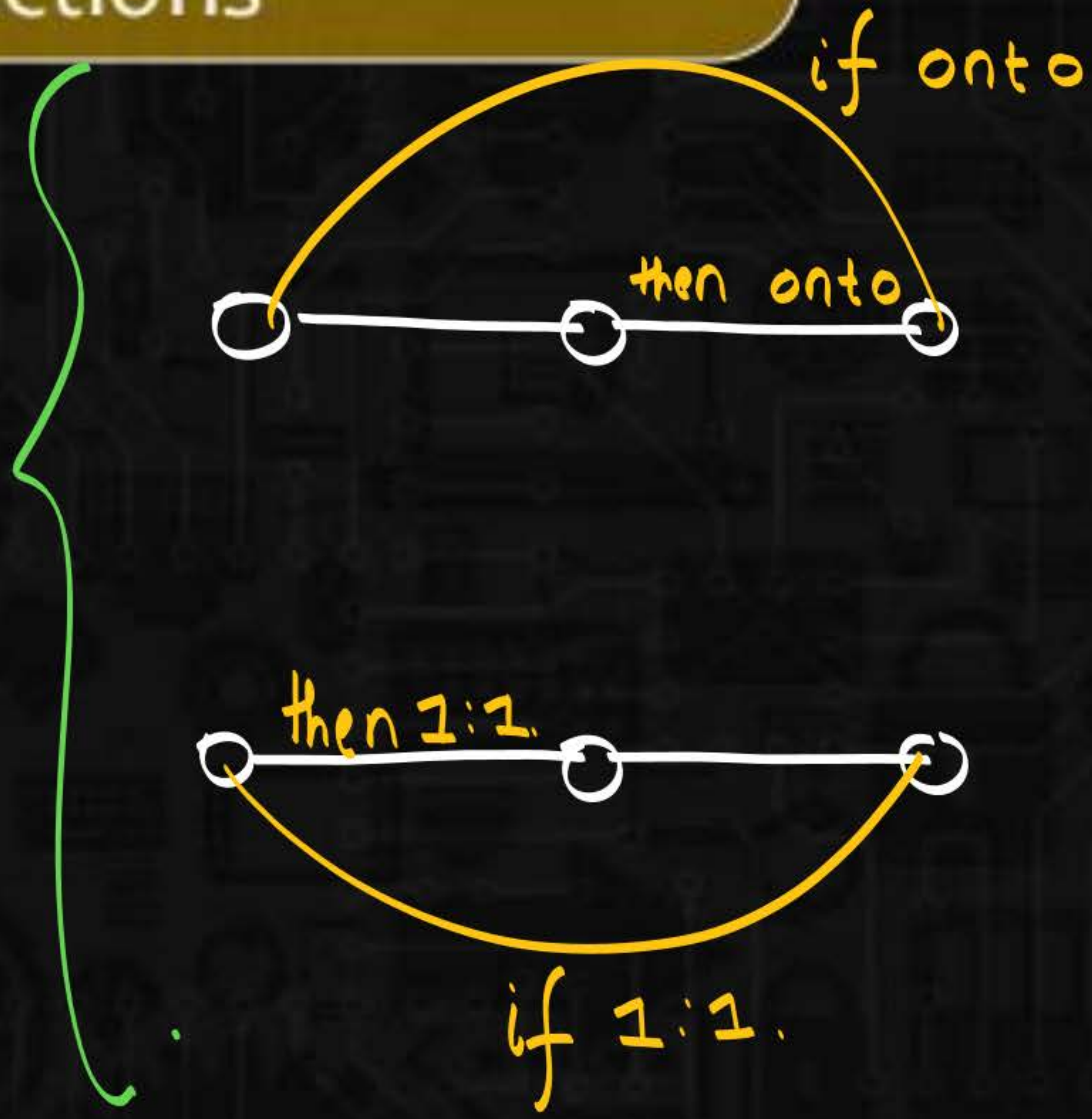


1:1 ✓

if $g \circ f$ is 1:1 then g is 1:1. (false)



Functions



Functions

Inverse function.

$f : A \to B$ (1:1 c)

$a \to x$ $f(a) = x$.

$b \to y$ $f(b) = y$

$c \to z$. $f(c) = z$

Invertible

$f^{-1} : B \to A$.

$a \leftarrow x$ $f^{-1}(x) = a$.

$b \leftarrow y$ $f^{-1}(y) = b$

$c \leftarrow z$ $f^{-1}(z) = c$.

Functions

1 : 1 ✗

$f : A \to B$

$a \longrightarrow x$

$b \longrightarrow y$

z

$f^{-1} : B \to A$

x — a

y — b

z

✗

onto

$a \longrightarrow x$

$b \longrightarrow x$

onto ✓

$f^{-1} : B \to A$

$x \to a$

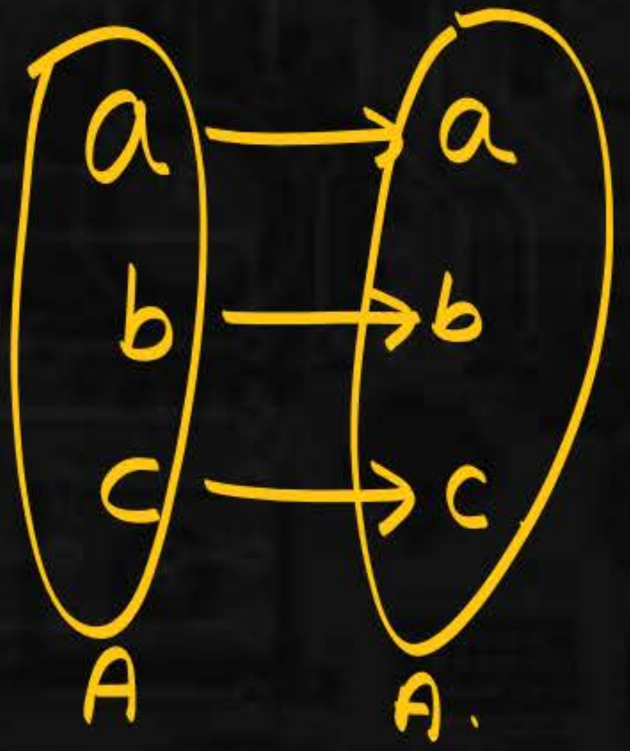
$x \to b$

✗

Functions

$i_A : A \to A$

↳ identity function.

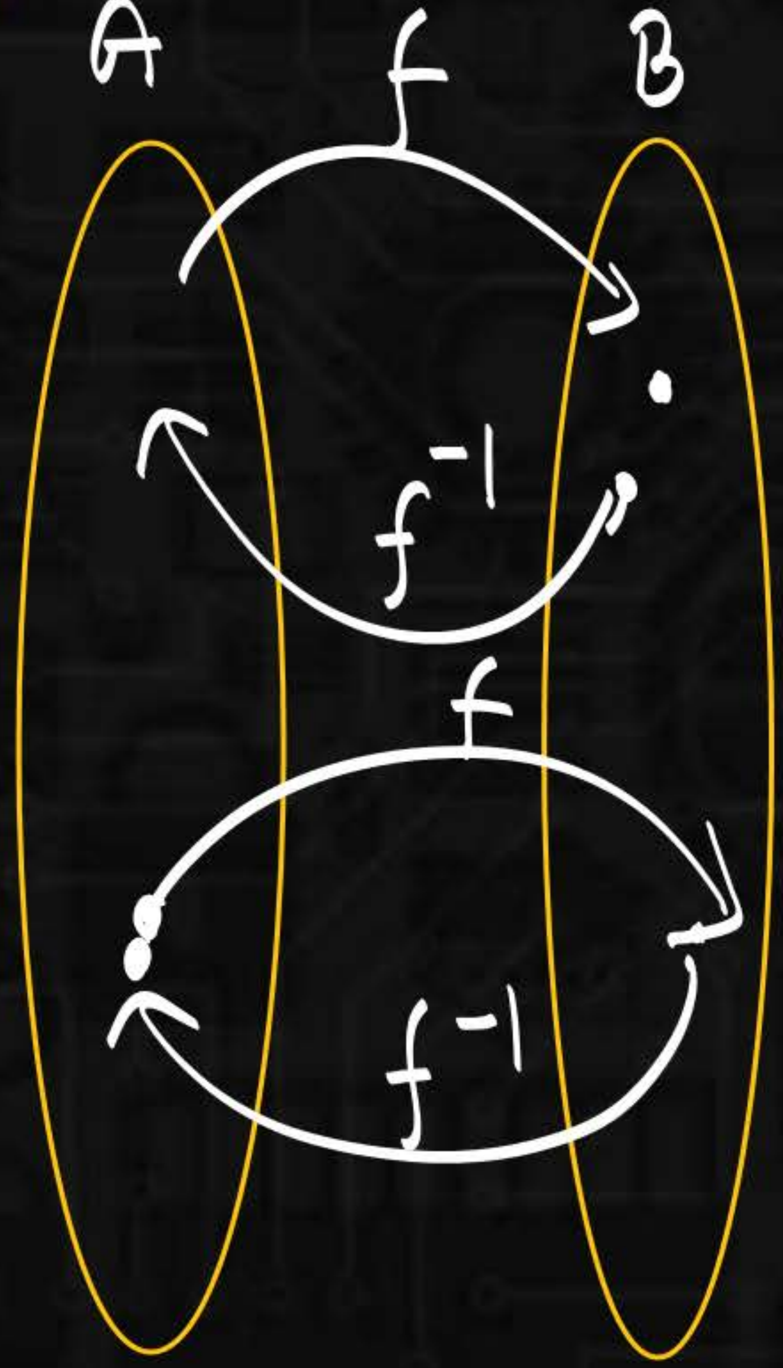


Functions

$f : A \to B$ (1:1 c)

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$



$i_B : B \to B$

$i_A : A \to A$

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 4$, $g(x) = ax + b$, and $h(x) = 2x + 1$, where a and b are real numbers. Suppose that $(f \circ g \circ h)(x) = 6x + 5$. Find $(h \circ g \circ f)(x)$.

$$
\begin{aligned}
(f \circ g \circ h)(x) &= f[g(2x+1)] = f[a(2x+1)+b] \\
&= f[a(2x+1)+b] \\
&= f(2ax + a + b) \\
&= 3(2ax + a + b) - 4 \\
&= 6ax + 3a + 3b - 4 \\
&= 6x + 5.
\end{aligned}
$$
15.

From this it follows that $a = 1$ and $b = 2$.
Therefore

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are defined by

$$f(x) = x^2 + x - 1 \qquad \text{and} \qquad g(x) = 3x + 2.$$

Express, in simplest terms, $(f \circ g)(x) - (g \circ f)(x)$.

10. Note that

$$
\begin{aligned}
(f \circ g)(x) &= f(3x+2) \\
&= (3x+2)^2 + (3x+2) - 1 \\
&= 9x^2 + 15x + 5 \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
(g \circ h)(x) &= g(x^2 + x - 1) \\
&= 3(x^2 + x - 1) + 2 \\
&= 3x^2 + 3x - 1 \qquad \text{and so}
\end{aligned}
$$

14.

$$
\begin{aligned}
(f \circ g)(x) - (g \circ f)(x) &= (9x^2 + 15x + 5) \\
&\quad - (3x^2 + 3x - 1) \\
&= 6x^2 + 12x + 6 \\
&= 6(x+1)^2.
\end{aligned}
$$

(a) [BB] $A = \{x \in \mathsf{R} \mid x \neq 4\}$, $f(x) = 1 + \dfrac{1}{x-4}$

1

If $f(x_1) = f(x_2)$, then $1 + \frac{1}{x_1 - 4} = 1 + \frac{1}{x_2 - 4}$, $\frac{1}{x_1 - 4} = \frac{1}{x_2 - 4}$ and $x_1 - 4 = x_2 - 4$. Thus $x_1 = x_2$ and f is one-to-one.

rng $f = B = \{y \in \mathsf{R} \mid y \neq 1\}$ and $f^{-1}(x) = 4 + \frac{1}{x-1}$.

8. [BB] Let f, g and $h\colon \mathsf{R} \to \mathsf{R}$ be defined by

$$f(x) = x + 2, \quad g(x) = \frac{1}{x^2+1}, \quad h(x) = 3.$$

Compute $g \circ f(x)$, $f \circ g(x)$, $h \circ g \circ f(x)$, $g \circ h \circ f(x)$, $g \circ f^{-1} \circ f(x)$, and $f^{-1} \circ g \circ f(x)$.

$g \circ f(x) = \dfrac{1}{(x+2)^2+1} = \dfrac{1}{x^2+4x+5}$; $\quad f \circ g(x) = \dfrac{1}{x^2+1} + 2 = \dfrac{2x^2+3}{x^2+1}$;

$h \circ g \circ f(x) = 3$; $g \circ h \circ f(x) = g(3) = \frac{1}{10}$.

Since $f^{-1} \circ f(x) = x$, we have $g \circ f^{-1} \circ f(x) = g(x) = \dfrac{1}{x^2+1}$.

Since $f^{-1}(x) = x - 2$, we have

$$f^{-1} \circ g \circ f(x) = f^{-1}\left(\frac{1}{(x+2)^2+1}\right) = \frac{1}{(x+2)^2+1} - 2 = \frac{-2(x+2)^2-1}{(x+2)^2+1}.$$

16. [BB] Let $A = \{x \in \mathsf{R} \mid x \neq 2\}$ and $B = \{x \in \mathsf{R} \mid x \neq 1\}$. Define $f\colon A \to B$ and $g\colon B \to A$ by

$$f(x) = \frac{x}{x-2}, \qquad g(x) = \frac{2x}{x-1}.$$

(a) Find $(f \circ g)(x)$.

(b) Are f and g inverses? Explain.

(a) For $x \in B$, $(f \circ g)(x) = f(\frac{2x}{x-1}) = \dfrac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = x$.

(b) For $x \in A$, $(g \circ f)(x) = g(\frac{x}{x-2}) = \dfrac{2(\frac{x}{x-2})}{\frac{x}{x-2} - 1} = x$ and so, by Proposition 3.2.7, f and g are inverses.

. **a)** For $A = \{1, 2, 3, 4, \ldots, 7\}$, how many bijective functions $f\colon A \to A$ satisfy $f(1) \neq 1$?

(a) **There are 7! bijective functions on A – of these, 6! satisfy $f(1) = 1$. Hence there are $7! - 6! = 6(6!)$ bijective functions $f: A \to A$ where $f(1) \neq 1$.**

3. Let $f, g\colon \mathbf{R} \to \mathbf{R}$, where $g(x) = 1 - x + x^2$ and $f(x) = ax + b$. If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.

$9x^2 - 9x + 3 = g(f(x)) = 1 - (ax+b) + (ax+b)^2 = a^2x^2 + (2ab - a)x + (b^2 - b + 1)$. By comparing coefficients on like powers of x, $a = 3, b = -1$ or $a = -3, b = 2$.

6. Let $f, g\colon \mathbf{R} \to \mathbf{R}$ where $f(x) = ax + b$ and $g(x) = cx + d$ for all $x \in \mathbf{R}$, with a, b, c, d real constants. What relationship(s) must be satisfied by a, b, c, d if $(f \circ g)(x) = (g \circ f)(x)$ for all $x \in \mathbf{R}$?

$(f \circ g)(x) = f(cx + d) = a(cx + d) + b$
$(g \circ f)(x) = g(ax + b) = c(ax + b) + d$
$(f \circ g)(x) = (g \circ f)(x) \iff acx + ad + b = acx + bc + d \iff ad + b = bc + d$

10. For each of the following functions $f: \mathbf{R} \to \mathbf{R}$, determine whether f is invertible, and, if so, determine f^{-1}.

a) $f = \{(x, y) | 2x + 3y = 7\}$

b) $f = \{(x, y) | ax + by = c, b \neq 0\}$

c) $f = \{(x, y) | y = x^3\}$

d) $f = \{(x, y) | y = x^4 + x\}$

(a) $f^{-1} = \{(x,y) | 2y + 3x = 7\}$ (b) $f^{-1} = \{(x,y) | ay + bx = c,\ b \neq 0,\ a \neq 0\}$
(c) $f^{-1} = \{(x,y) | y = x^{1/3}\} = \{(x,y) | x = y^3\}$
(d) Here $f(0) = f(-1) = 0$, so f is not one-to-one, and consequently f is not invertible.

12. If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$, and $f: A \to B$ where $f = \{(1, 2), (2, 6), (3, 6), (4, 8), (5, 6), (6, 8), (7, 12)\}$, determine the preimage of B_1 under f in each of the following cases.

a) $B_1 = \{2\}$ **b)** $B_1 = \{6\}$

c) $B_1 = \{6, 8\}$ **d)** $B_1 = \{6, 8, 10\}$

e) $B_1 = \{6, 8, 10, 12\}$ **f)** $B_1 = \{10, 12\}$

(a) $f^{-1}(\{2\}) = \{a \in A | f(a) \in \{2\}\} = \{a \in A | f(a) = 2\} = \{1\}$
(b) $f^{-1}(\{6\}) = \{a \in A | f(a) \in \{6\}\} = \{a \in A | f(a) = 6\} = \{2, 3, 5\}$
(c) $f^{-1}(\{6, 8\}) = \{a \in A | f(a) \in \{6, 8\}\} = \{a \in A | f(a) = 6 \text{ or } f(a) = 8\} = \{2, 3, 4, 5, 6\}$, because $f(2) = f(3) = f(5) = 6$ and $f(4) = f(6) = 8$.
(d) $f^{-1}(\{6, 8, 10\}) = \{2, 3, 4, 5, 6\} = f^{-1}(\{6, 8\})$ since $f^{-1}(\{10\}) = \emptyset$.
(e) $f^{-1}(\{6, 8, 10, 12\}) = \{2, 3, 4, 5, 6, 7\}$
(f) $f^{-1}(\{10, 12\}) = \{7\}$

15. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$. How many functions $f: A \to B$ are such that $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$?

Since $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$ there are three choices for each of $f(1)$ and $f(2)$ – namely, 6, 7 or 8. Furthermore $3, 4, 5 \notin f^{-1}(\{6, 7, 8\})$ so $3, 4, 5 \in f^{-1}(\{9, 10, 11, 12\})$ and we have four choices for each of $f(3)$, $f(4)$, and $f(5)$. Therefore, it follows by the rule of product that there are $3^2 \cdot 4^3 = 576$ functions $f: A \to B$ where $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$.

22. If $|A| = |B| = 5$, how many functions $f: A \to B$ are invertible?

there are $5!$ invertible functions $f : A \longrightarrow B$.


THANK YOU GW
SOLDIERS !