ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-6

Calculus

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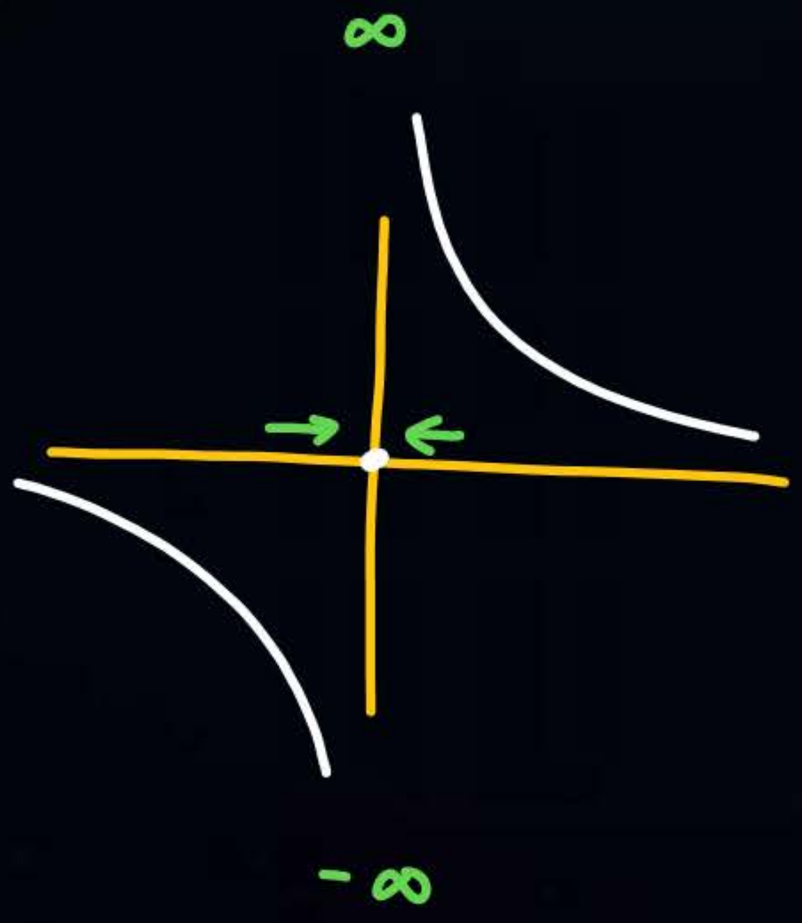
Topics to be Covered

- INCREASING- DECREASING FUNCTION
- MAXIMA AND MINIMA OF SINGLE VARIABLE FUNCTION
- MAXIMA AND MINIMA OF TWO VARIABLE FUNCTION
- LAGRANGE’S CONDITION FOR MAXIMA OR MINIMA

[DIFFERENTIABILITY]

The function $f(x) = \frac{1}{x}$ which is not continuous at x = 0 has no derivative at x = 0.



At $x = 0$

fn. is discontinuous

$\Rightarrow$ Non-differentiable.

DIFFERENTIABILITY

If a function f(x) is defined as:

$$f(x) = \frac{xe^{1/x}}{1+e^{1/x}}; x \neq 0$$

$$= 0 ; \qquad x = 0$$

$$\text{LHL} = \lim_{h\to 0} f(0-h) = \frac{(0-h)\, e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}} = \frac{-h\, e^{-\frac{1}{h}}}{1+e^{-1/h}} = 0$$

$$\text{RHL} = \lim_{h\to 0} f(0+h) = \frac{(0+h)\, e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}} = \frac{h\, e^{1/h}}{1+e^{1/h}} = \frac{h}{\underbrace{\frac{1}{e^{1/h}}}_{0}+1} = 0$$

∴ Fn. is continuous

LHL = RHL = Value = 0.

$$\text{LHD} = \frac{f(0)-f(0-h)}{0-(0-h)} = \lim_{h\to 0}\frac{0-\dfrac{-\cancel{h}e^{-1/h}}{1+e^{-1/h}}}{\cancel{h}} = 0$$

$$\text{R.H.D.} = \frac{f(0+h)-f(0)}{0+h-0} = \lim_{h\to 0}\frac{\dfrac{\cancel{h}}{\frac{1}{e^{1/h}}+1}-0}{\cancel{h}} = \frac{1}{0+1} = 1$$

Since LHD $\neq$ RHD $\therefore$ fn. is Non-diff. at $x=0$.

Ex:- $f(x) = \begin{cases} x\tan^{-1}\dfrac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$$\text{L.H.D.} = \frac{f(0)-f(0-h)}{0-(0-h)} = \frac{0-(0-h)\tan^{-1}\left(\frac{1}{0-h}\right)}{h} = -\tan^{-1}\frac{1}{h} = -\pi/2$$

$$\text{R.H.D.} = \frac{f(0+h)-f(0)}{0+h-0} = \frac{(0+h)\tan^{-1}\frac{1}{0+h}-0}{0+h-0} = \tan^{-1}\frac{1}{h} = \pi/2$$

$\therefore$ fn. is non-diff. $\because$ LHD $\neq$ RHD.

but fn. is continuous at $x=0$. [LHL = RHL = Value]

$$LHL = (0-h)\tan^{-1}\frac{1}{0-h} = h\tan^{-1}\frac{1}{h} = 0\times\pi/2 = 0$$

$$RHL = (0+h)\tan^{-1}\frac{1}{0+h} = h\tan^{-1}\frac{1}{h} = 0\times\pi/2 = 0$$

$$\frac{h^{1/3}}{h^1} = \frac{1}{h^{1-1/3}}$$

Ex:- $f(x) = x^{1/3}$ Check at $x=0$

$$\left.\begin{aligned} LHL &= (0-h)^{1/3} = -h^{1/3} = 0 \\ RHL &= (0+h)^{1/3} = +h^{1/3} = 0 \end{aligned}\right\} \text{Continuous}$$

$$LHD = \frac{f(0)-f(0-h)}{0-(0-h)} = \frac{0-(-h)^{1/3}}{h} = \frac{1}{h^{2/3}} = +\infty$$

$$R.H.D. = \frac{f(0+h)-f(0)}{0+h-0} = \frac{(0+h)^{1/3}-0}{h} = \frac{1}{h^{2/3}} = +\infty$$

Vertical tangent

Fn. in Non-Diff. at $x=0$.

MEAN VALUE THEOREMS

ROLLE'S THEOREM :-

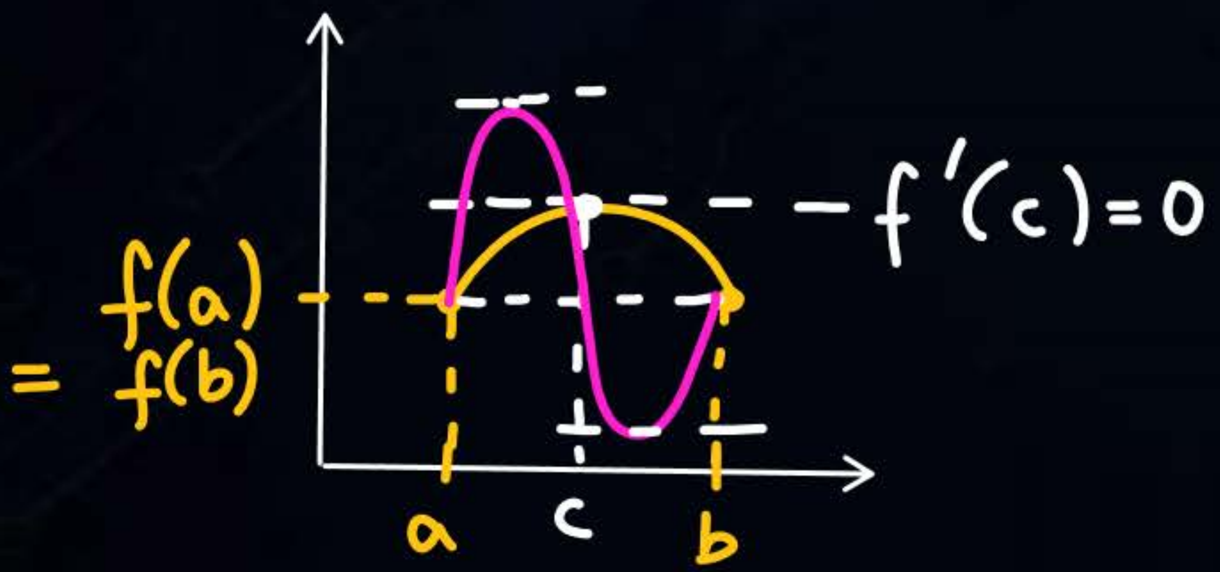
- $f(x)$ is continuous in closed interval $[a, b]$ $\quad a \leq x \leq b$
- $f(x)$ is differentiable in open interval (a, b) $\quad a < x < b$
- If $f(a) = f(b)$, then at least at some point (one point) $c \in (a, b)$
  $f'(c) = 0$

$$\boxed{f'(c) = 0}$$

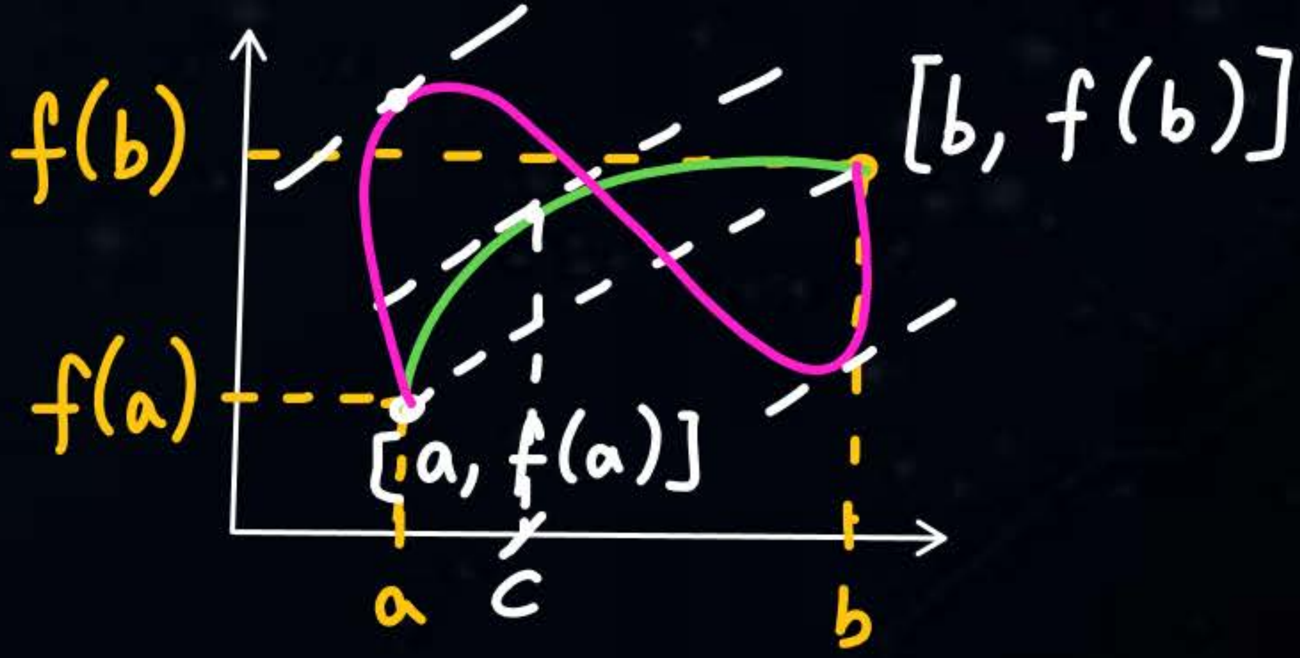
LAGRANGE'S MEAN VALUE THEOREM :-

- $f(x)$ is cont. in closed interval $[a, b]$ $\quad a \leq x \leq b$
- $f(x)$ is diff. in open interval (a, b) $\quad a < x < b$
- If $f(a) \neq f(b)$, then at least at some point (one point) $c \in (a, b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$



$f'(c)=0$



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In LMVT, if $f(b) = f(a)$ then $f'(c) = 0$ $\therefore$ Rolle's theorem is special case of L.M.V.T.

Ex:- Verify Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

i) $f(x)$ is continuous in $[-\sqrt{2}, \sqrt{2}]$

ii) $f(x)$ is diff. in $(-\sqrt{2}, \sqrt{2})$

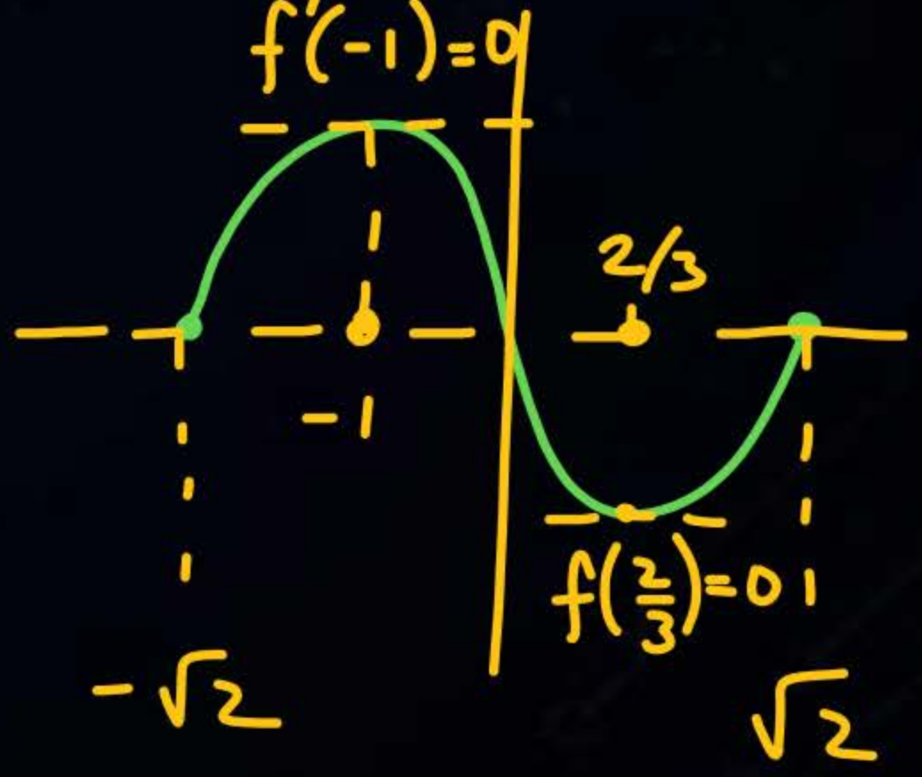
ROLLE'S THEOREM

$$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2 = 0$$
$$f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2 = 0$$
$$\therefore f(\sqrt{2}) = f(-\sqrt{2})$$

$$f'(x) = 6x^2 + 2x - 4$$
$$\therefore f'(c) = 6c^2 + 2c - 4 = 0$$
$$3c^2 + c - 2 = 0$$
$$3c^2 + 3c - 2c - 2 = 0$$
$$3c(c+1) - 2(c+1) = 0$$
$$(3c-2)(c+1) = 0$$
$$c = -1, 2/3 \in (-\sqrt{2}, \sqrt{2})$$

$-\sqrt{2}, \sqrt{2}$

$-1.414, 1.414$



Since both points are b/w $-\sqrt{2}$ and $+\sqrt{2}$ $\therefore$ Rolle's theo. is satisfied.

Ex: Verify L.M.V.T. in the interval [0, 4] for function :-

$$f(x) = (x-1)(x-2)(x-3)$$

Soln:-

$$(x-1)(x^2-5x+6)$$

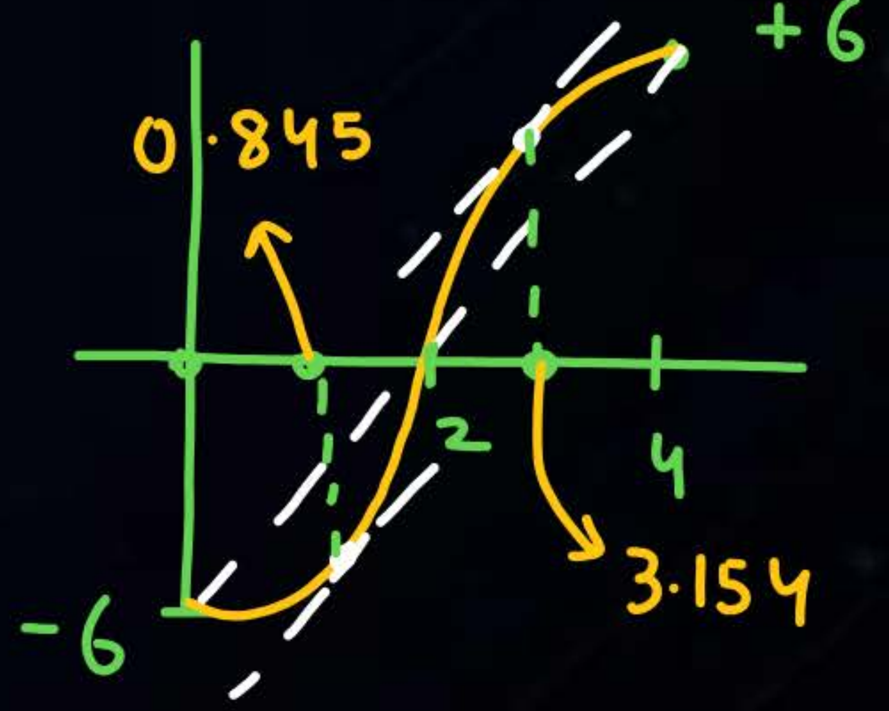
$$x^3-5x^2+6x-x^2+5x-6$$

$$f(x) = x^3-6x^2+11x-6$$

i) $f(x)$ is cont. in $[0,4]$

ii) $f(x)$ is diff. in $(0,4)$

$f(0) = -6 \quad f(4) = 6$



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3x^2-12x+11 = \frac{f(4)-f(0)}{4-0} = \frac{6-(-6)}{4-0} = 3$$

$$3c^2-12c+11-3 = 0$$

On solving eqn $\frac{-(-12) \pm \sqrt{12 - 4\times(+3)\times 8}}{2\times 3} = 2 \pm \frac{2\sqrt{2}}{3} = \begin{matrix} 3.154 \\ 0.854 \end{matrix} \in [0,4]$

CAUCHY MEAN VALUE THEOREM

→ If f and g are two fn. defined in a and b.

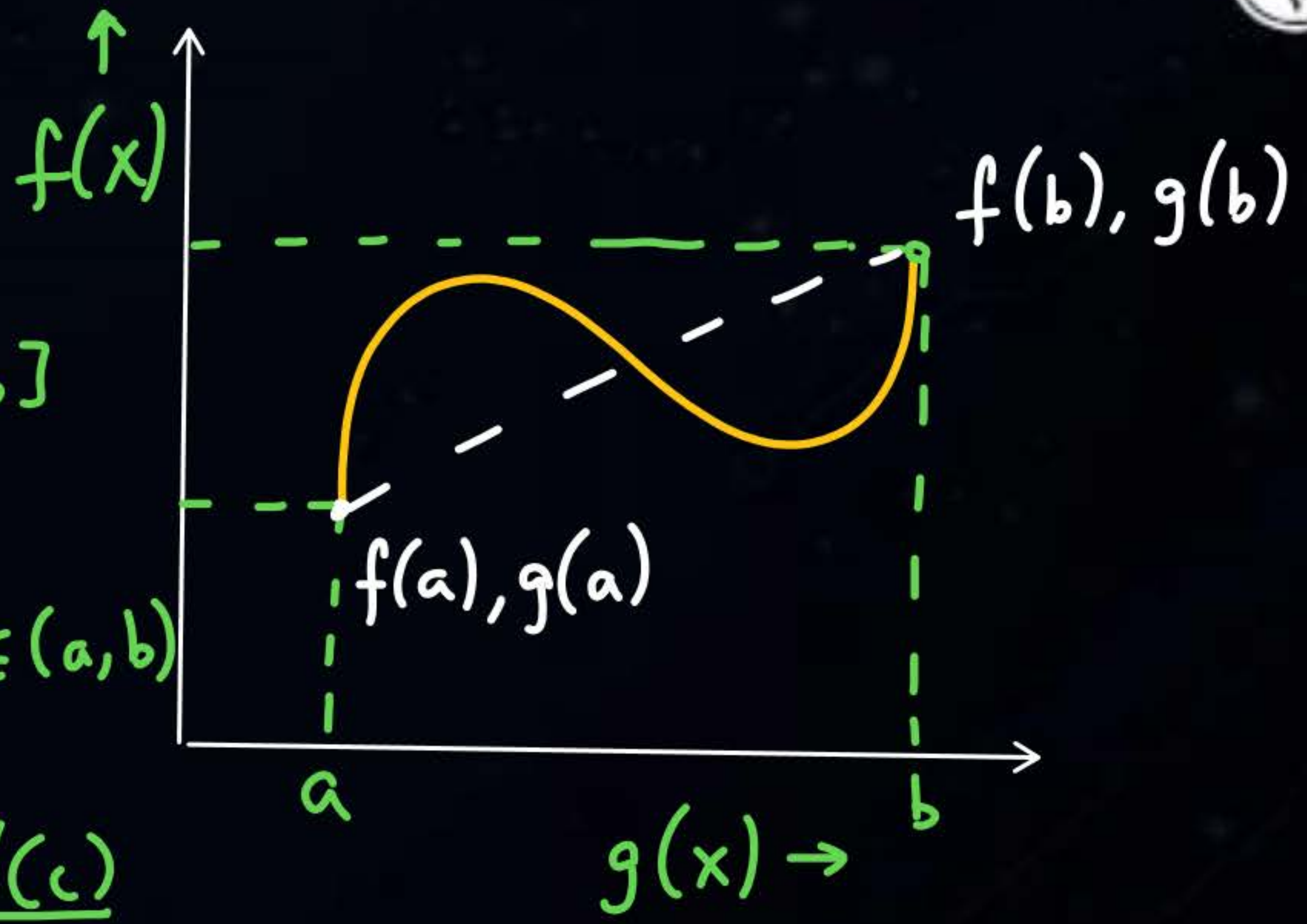
i) f and g are continuous in $[a, b]$

ii) f and g are diff. in (a, b)

iii) then at least at some point $c \in (a, b)$ (one point)

such that $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

also $g'(x) \neq 0$



Ex :- Verify Cauchy Mean Value theorem for x^2 and x^3 in $[1, 2]$.

Soln :- $f(x) = x^2$ $g(x) = x^3$

i) f and g are cont. in $[1, 2]$

ii) f and g are diff in $(1, 2)$

iii) $$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{4 - 1}{8 - 1} = \frac{2c}{3c^2}$$

$$3(3c^2) = 7(2c)$$

$$9c^2 - 14c = 0$$

$$c(9c - 14) = 0$$

$\therefore$ $c = 0$ ✗

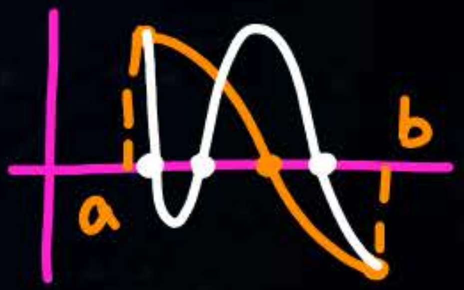
$c = 14/9$ ✓

Since $c = 14/9$, $c \in [1, 2]$

$\therefore$ C.M.V.T. is verified.

BOLZANO THEOREM :-

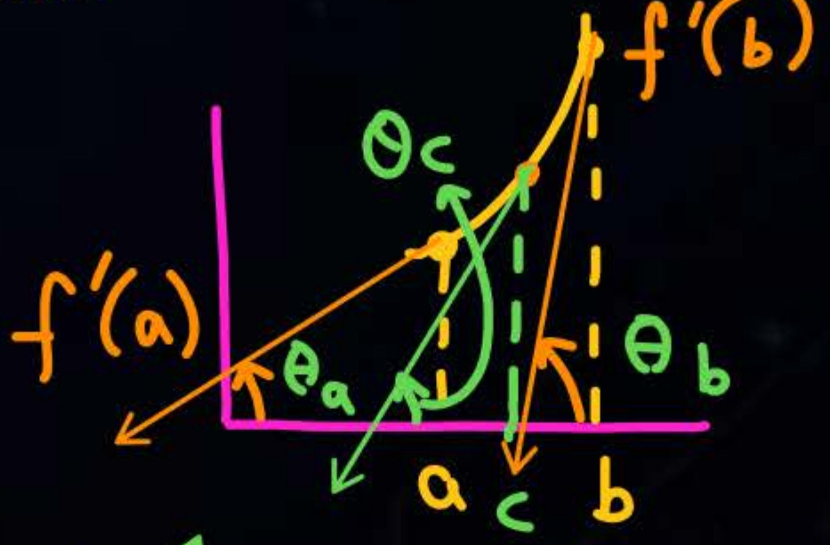
If fn. is cont. & diff. in a domain then at two points $x = a$, $x = b$ if values of function has opposite sign then there exist at least one root $c \in (a,b)$ such that $f(c) = 0$



INTERMEDIATE VALUE THEOREM :-

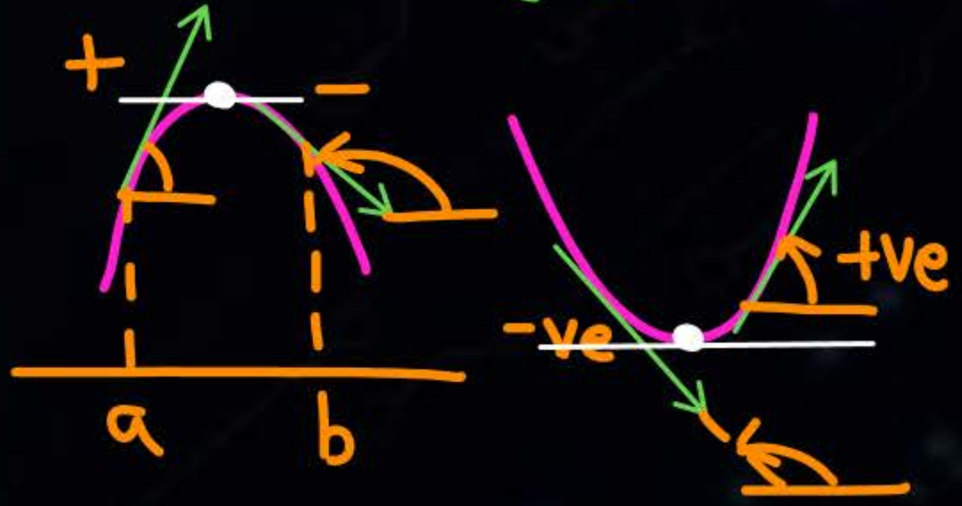
If $f(x)$ is continuous & diff. such $f'(a) \neq f'(b)$ then there exist at least one point $c \in (a,b)$ such that

$$f'(a) < f'(c) < f'(b) \qquad \theta_a < \theta_c < \theta_b$$



DARBOUX THEOREM :-

If $f(x)$ is finitely differentiable in $[a,b]$ and $f'(a)$ and $f'(b)$ have opposite sign, then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.



Descartes Rule of Sign Change:-

$f(x)$ is polynomial/function:-

→ No. of sign changes in $f(x)$ = No. of max. +ve real roots

→ No. of sign changes in $f(-x)$ = No. of max. -ve real roots

→ If sign change = 1 (Actual no. of +/- roots).

→ Complex roots exists in conjugate pair.

Ex:-

$$f(x) = x^3 \underbrace{-}_{1} 2x^2 \underbrace{-}_{0} x \underbrace{+}_{1} 2 = 0$$

No. of max. +ve real roots = 2

$$f(-x) = -x^3 \underbrace{-}_{0} 2x^2 \underbrace{+}_{1} x \underbrace{+}_{0} 2 = 0$$

No. of max. -ve real roots = 1

	+ve roots	-ve roots	Complex roots	Total roots
Possibility 1)	2	1	0	3
Possibility 2)	0	1	2	3

Ex:-

$$f(x) = x^2 + 5x + 6$$

$0 \quad 0$

Max. +ve real roots = 0

$$f(-x) = x^2 - 5x + 6$$

$1 \quad 1$

Max. -ve real roots = 2

	+	−	C	Total
1	0	2	0	2
2	0	0	2	2

Ex:-

$$f(x) = x^5 - 3x^4 + 3x^3 - 9x^2 - 4x + 12 = 0$$

$1 \quad 1 \quad 1 \quad 0 \quad 1$

Max +ve real roots = 4

$$f(-x) = -x^5 - 3x^4 - 3x^3 - 9x^2 + 4x + 12 = 0$$

$0 \quad 0 \quad 0 \quad 1 \quad 0$

Max -ve real roots = 1

	+	−	C	Total
1	4	1	0	5
2	2	1	2	5
3	0	1	4	5

[GATE]

$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x - a_0 = 0$

with all positive coefficients :-

[2 Mark]

× A) no real roots

× B) no -ve real roots

× C) odd no. of real roots (2, 4)

✓ D) at least one + and one - real root.

No. of sign changes in $f(x) = 1$ (Max +ve real roots)

$f(-x) = a_4x^4 - a_3x^3 + a_2x^2 - a_1x - a_0 = 0$

1 1 1 0

No. of sign changes in $f(-x) = 3$

	+ (real)	- (real)	C	Total
1	1	3	0	4
2	1	1	2	4

Thank you

GW

Soldiers !