ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-13

Calculus

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Topics to be Covered

- APPLICATION OF INTEGRATIONS
- LENGTH OR PERIMETER OF CURVE
- SURFACE AREA OF REVOLUTION
- VOLUME OF SOLID OF REVOLUTION
- MULTIPLE INTEGRALS

LEIBNITZ RULE OF DIFFERENTIATION UNDER SIGN OF INTEGRATION :-

$$\boxed{\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = \frac{d\psi}{dx}\cdot f(\psi_x) - \frac{d\phi}{dx} f(\phi_x)}$$

Ex :-

$$\frac{d}{dx}\left[\int_{x}^{x^2} t^2\,dt\right] = \left[\frac{t^3}{3}\right]_x^{x^2} = \frac{d}{dx}\left(\frac{x^6 - x^3}{3}\right) = \frac{6x^5 - 3x^2}{3} = 2x^5 - x^2$$

$f(t) = t^2$

By Leibnitz Rule;

$$2x\cdot(x^2)^2 - 1\cdot(x)^2 = 2x^5 - x^2$$

Ex:-

$$\frac{d}{dx}\left[\int_{x^2+5}^{x^3}(t^2+1)\,dt\right] = 3x^2\left[(x^3)^2+1\right] - 2x\left[(x^2+5)^2+1\right]$$

Ex:-

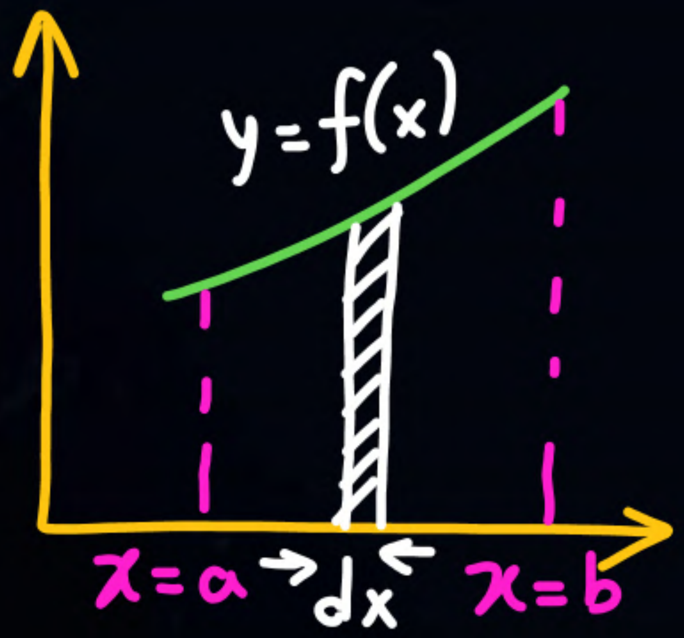
$$\phi = \int_{1/x}^{\sqrt{x}}(\sin t^2)\,dt \; ; \text{ find } \phi'(1) = \_\_\_\_ .$$

Ex:-

$$\phi = \int_{1/x}^{x^2}(\sin t^2)\,dt \; ; \text{ find } \phi'(1) = \underline{3\sin 1} .$$

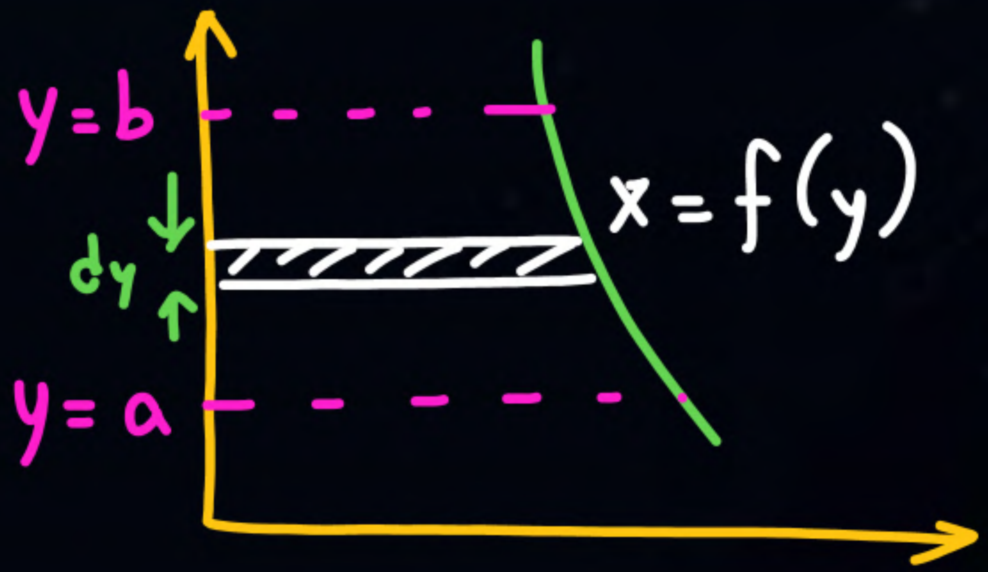
[APPLICATION OF INTEGRATIONS]

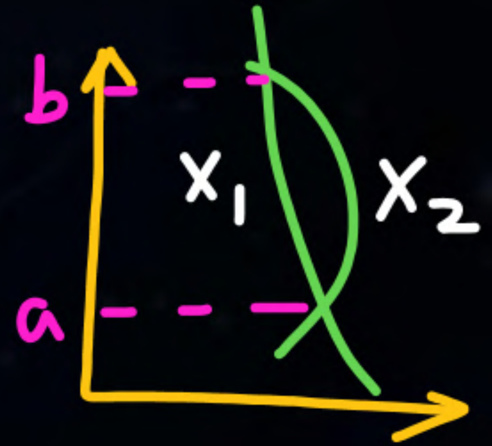
AREA UNDER CURVES:-



Area w.r.t. x-axis

Area b/w Curve & x-axis $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$





Area w.r.t. y-axis

Area b/w curve & y-axis $\displaystyle\int_{y=a}^{y=b} f(y)\,dy$

APPLICATION OF INTEGRATIONS

Area b/w 2 curves ; $= \int_{x_1}^{x_2} (y_2 - y_1)\, dx$ $= \int_{y_1}^{y_2} (x_2 - x_1)\, dy$

Ex:- Find the area b/w $y = x$ and $y = x^2$.

Soln:- $x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

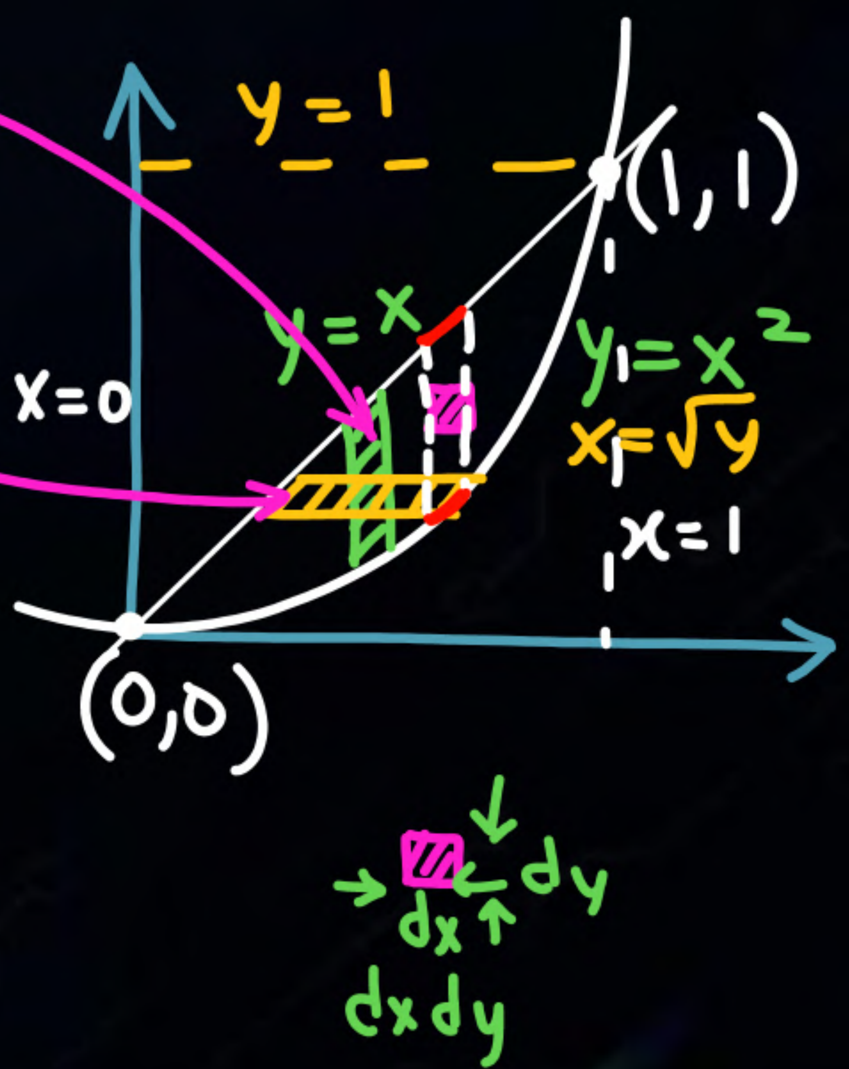
$x = 0, 1$

$y = 0, 1$

Area w.r.t. x-axis $= \int_{x=0}^{x=1} (x - x^2)\, dx = 1/6$

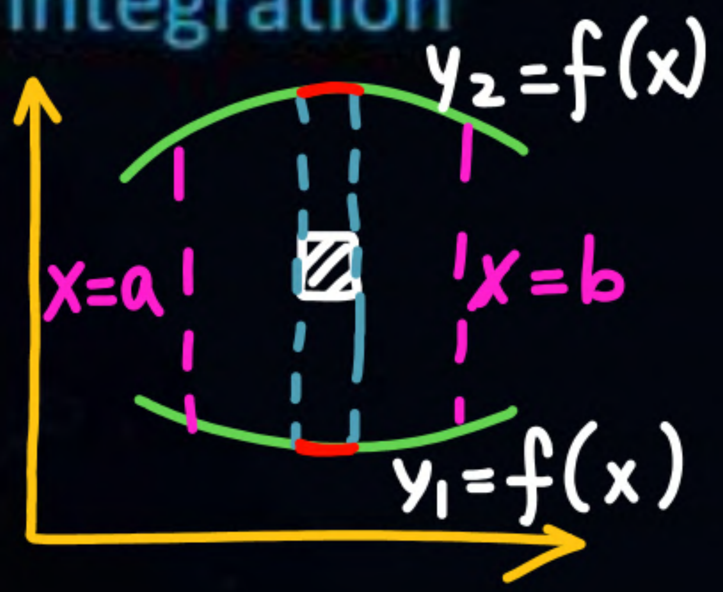
Area w.r.t. y-axis $= \int_{y=0}^{y=1} (\sqrt{y} - y)\, dy = 1/6$

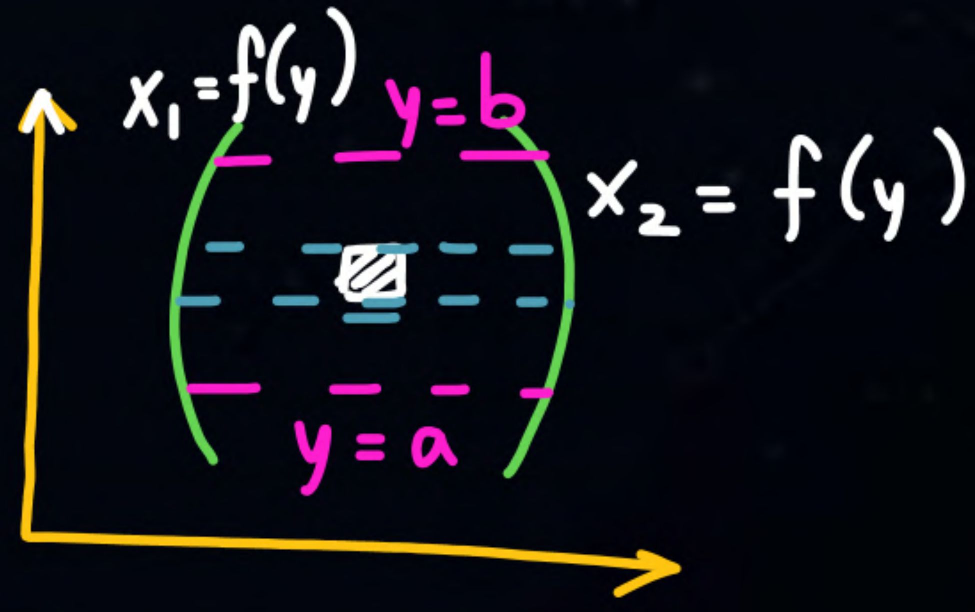
Area $= \int\int dx\, dy = \boxed{\int_{x=0}^{x=1} \boxed{\int_{y=x^2}^{y=x} dy}\, dx} = \int_0^1 (x - x^2)\, dx$



MULTIPLE INTEGRALS

Double integration





First in Y → then in X

$$\int_{x=a}^{x=b} \boxed{\int_{y_1=f(x)}^{y_2=f(x)} f(x,y)\,dy}\,dx$$

First in X → then in Y

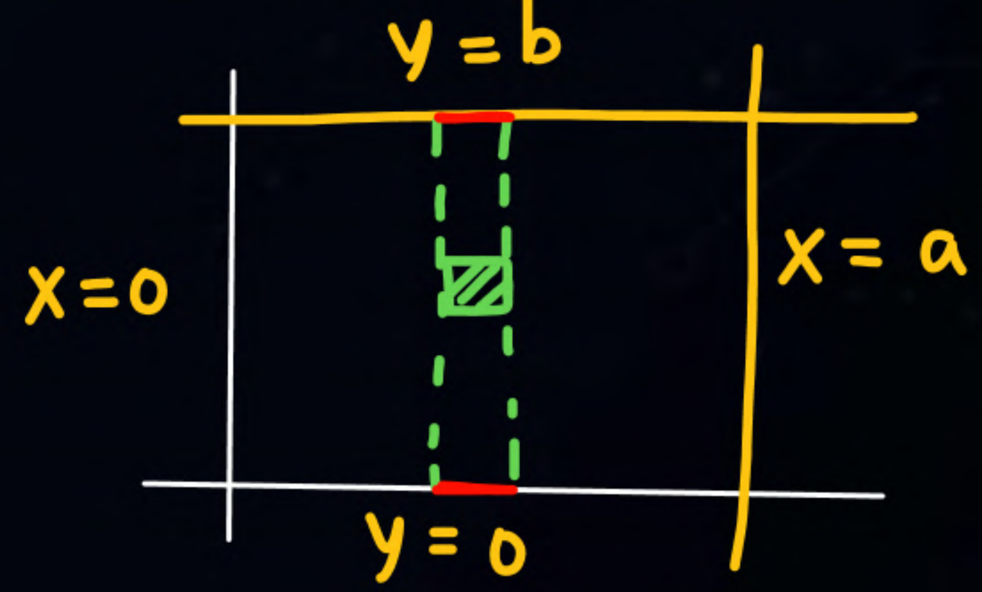
$$\int_{y=a}^{y=b} \boxed{\int_{x_1=f(y)}^{x_2=f(y)} f(x,y)\,dx}\,dy$$

I) $\displaystyle\int_{y_1=c}^{y_2=d}\int_{x_1=a}^{x_2=b} f(x,y)\,dx\,dy$

II) $\displaystyle\int_{x_1=a}^{x_2=b}\int_{y_1=f(x)}^{y_2=f(x)} f(x,y)\,dy\,dx$

III) $\displaystyle\int_{y_1=a}^{y_2=b}\int_{x_1=f(y)}^{x_2=f(y)} f(x,y)\,dx\,dy$

Ex:- Find the area b/w X, Y axis $x=a$ & $y=b$.

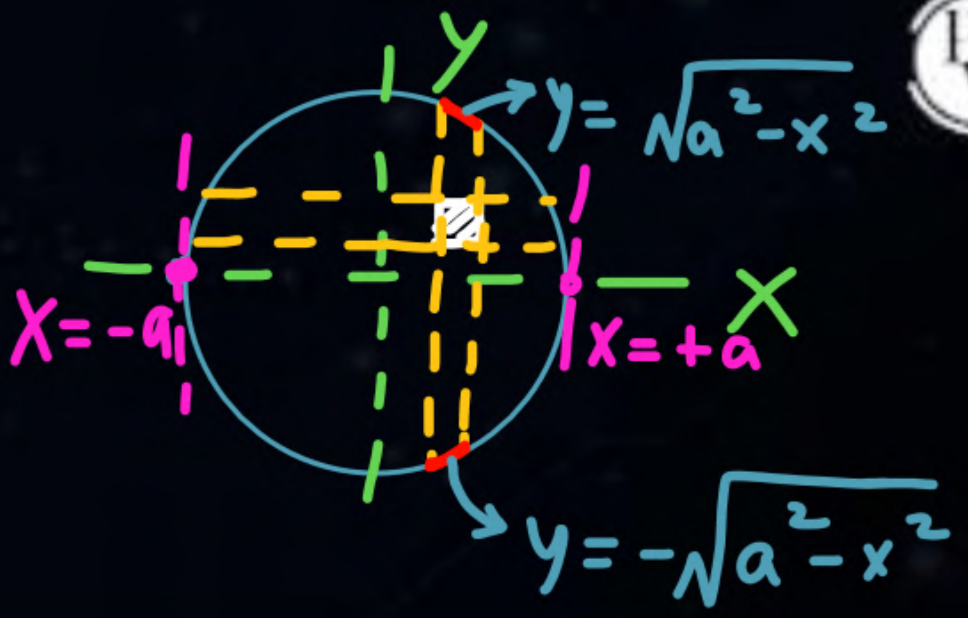


$$\int_{x=0}^{x=a}\boxed{\int_{y=0}^{y=b} dy}\,dx$$

$$\int_{x=0}^{x=a} dx \int_{y=0}^{y=b} dy = (a-0)(b-0) = ab$$

Ex:- Find the area of circle :- $x^2 + \cancel{y^2 = a^2}$

- $y = \pm\sqrt{a^2 - x^2}$
- $x = \pm a$



First Y → then X ;

$$\text{Area} = \int_{x=-a}^{x=+a} \int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} dy\ dx$$

First X → then Y ;

$$\text{Area} = \int_{y=-a}^{y=+a} \int_{x=-\sqrt{a^2-y^2}}^{x=+\sqrt{a^2-y^2}} dx\ dy$$

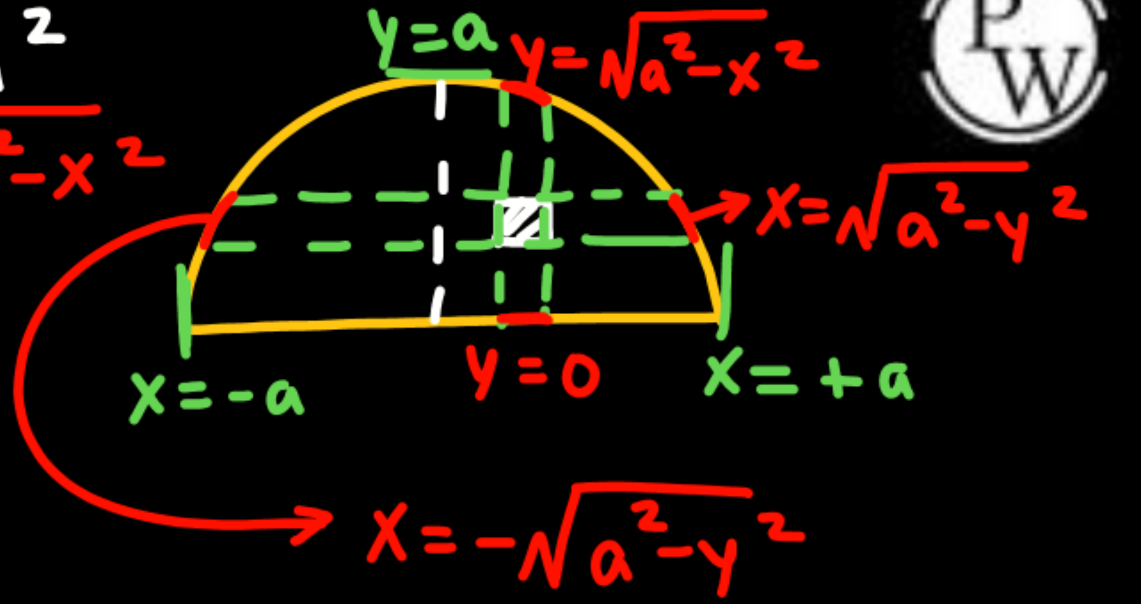
First in Y → then in X $\displaystyle\int_{x=-a}^{x=+a}\int_{y=0}^{y=\sqrt{a^2-x^2}} dy\; dx$

First in X → then in Y $\displaystyle\int_{y=0}^{y=a}\int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} dx\; dy$

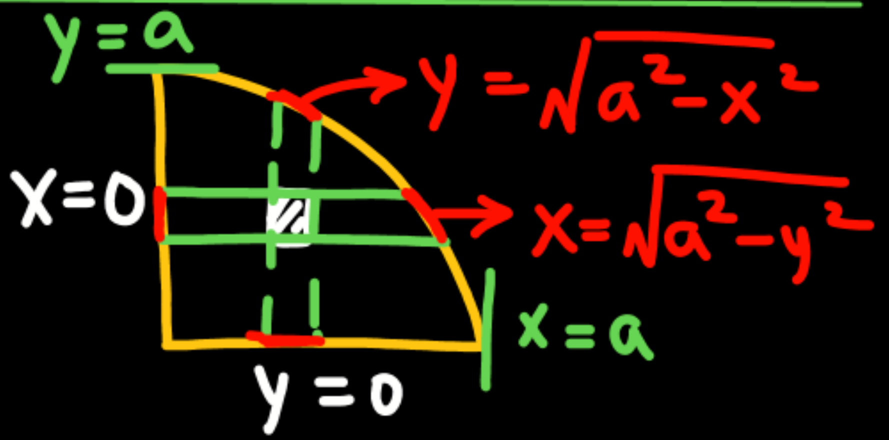
$x^2+\cancel{y^2}=a^2$

$y=\sqrt{a^2-x^2}$

$y=a$



First in Y → then in X $\displaystyle\int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} dy\; dx$

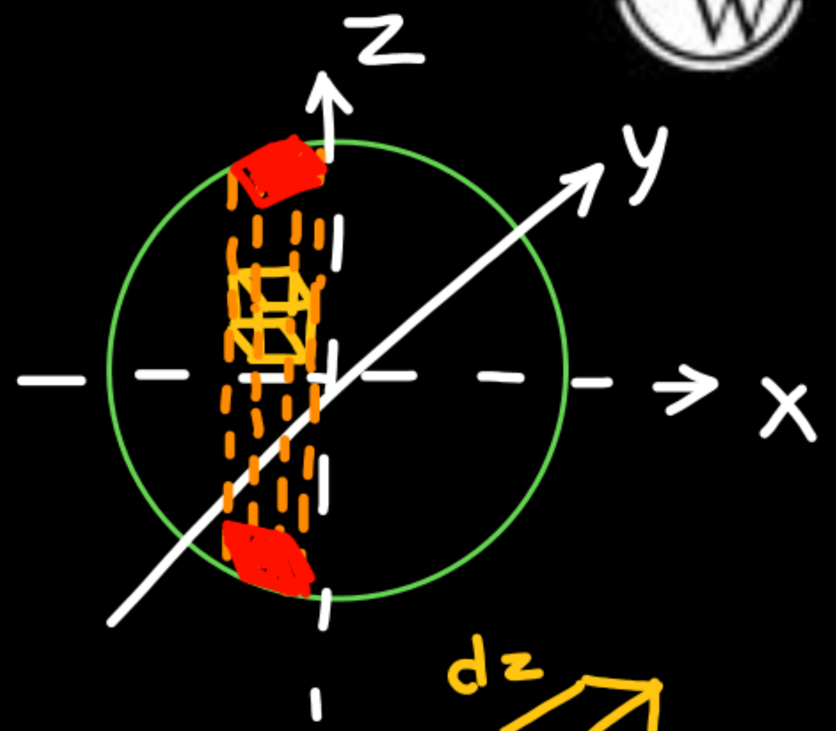


First in X → then in Y $\displaystyle\int_{y=0}^{y=a}\int_{x=0}^{x=\sqrt{a^2-y^2}} dx\; dy$

$x^2+y^2=a^2$

Ex:- Find the volume of sphere $x^2+y^2+z^2=a^2$

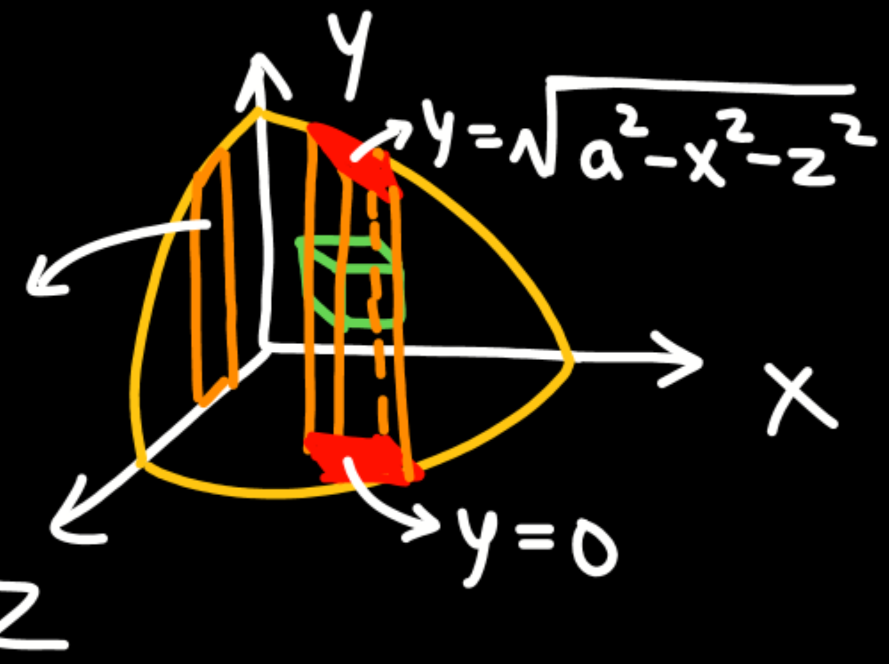
$$\int_{x=-a}^{x=+a}\int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2-y^2}}^{z=+\sqrt{a^2-x^2-y^2}} dz\;dy\;dx$$



Ex:-

$$\int_{z=0}^{z=a}\int_{x=0}^{x=\sqrt{a^2-z^2}}\int_{y=0}^{y=\sqrt{a^2-x^2-z^2}} dy\;dx\;dz$$

Planes	
xy	$z=0$
yz	$x=0$
zx	$y=0$



Ex:- Evaluate $\iint xy\,dx\,dy$ over the region in positive quadrant $x+y \le 1$

A) $1/6$

B) $1/12$

C) $1/18$

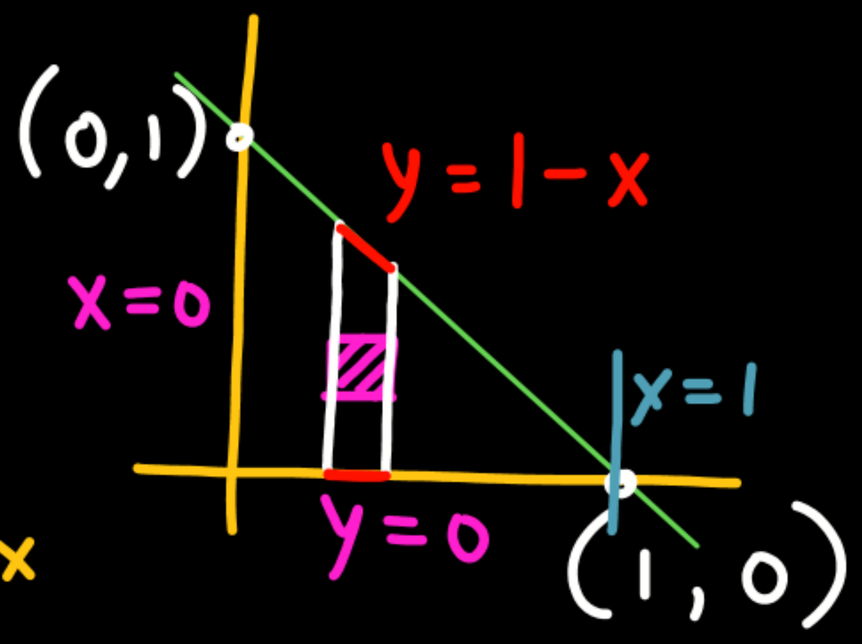
D) $1/24$ ✓

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} xy\,dy\,dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx = x\int_0^1 \frac{(1-x)^2}{2}\,dx$$

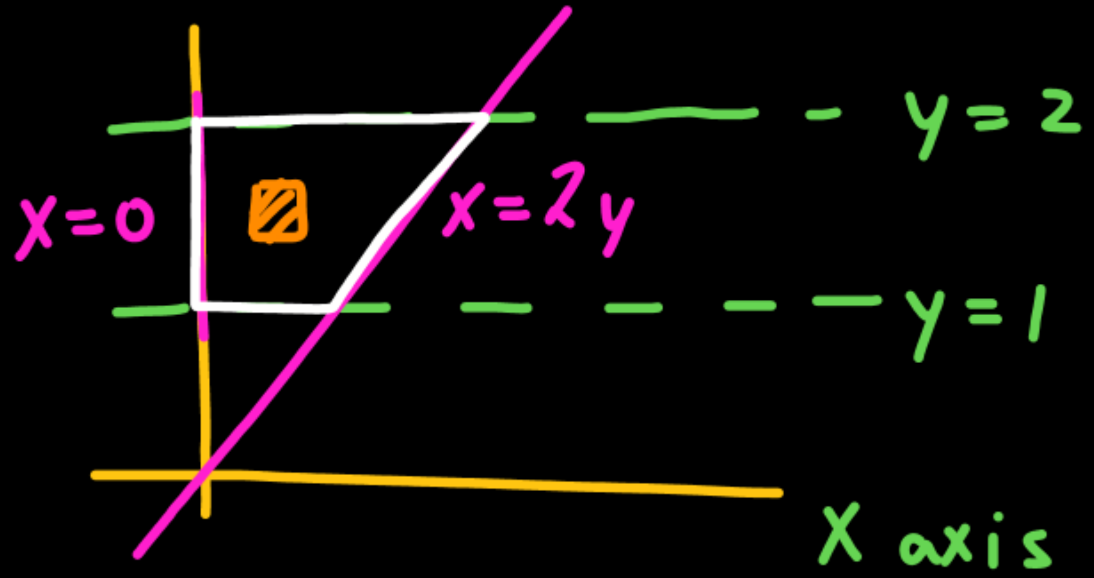
$$= \frac{1}{2}\int_0^1 x + x^3 - 2x^2 = \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - 2\frac{x^3}{3}\right]_0^1$$

$$\frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3} - 0\right] = \frac{1}{24}$$



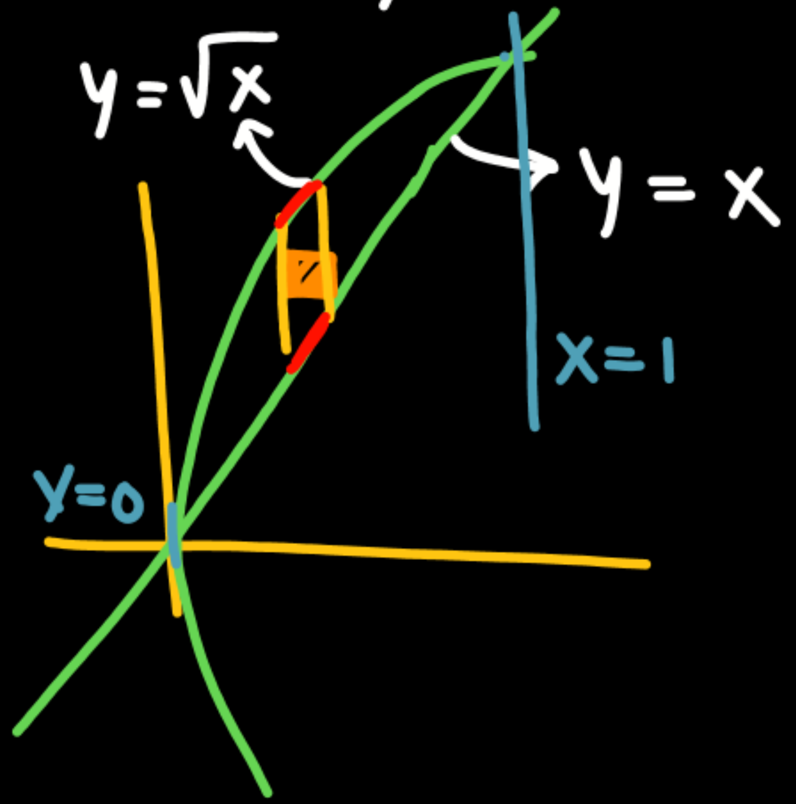
Ex:- Evaluate $\iint y\,dy\,dx$ b/w $y=1$, $y=2$, y-axis & $y=x/2$

$$\int_{y=1}^{y=2}\int_{x=0}^{x=2y} y\,dx\,dy = \int_{x=0}^{x=2y}\int_{y=1}^{y=2} y\,dy\,dx = \frac{14}{3}$$



Ex:- $\iint xy(x+y)\,dx\,dy$ over the area b/w $y^2=x$ & $y=x$.

$$\int_{x=0}^{x=1}\int_{y=x}^{y=\sqrt{x}} xy(x+y)\,dy\,dx = \frac{3}{56}$$



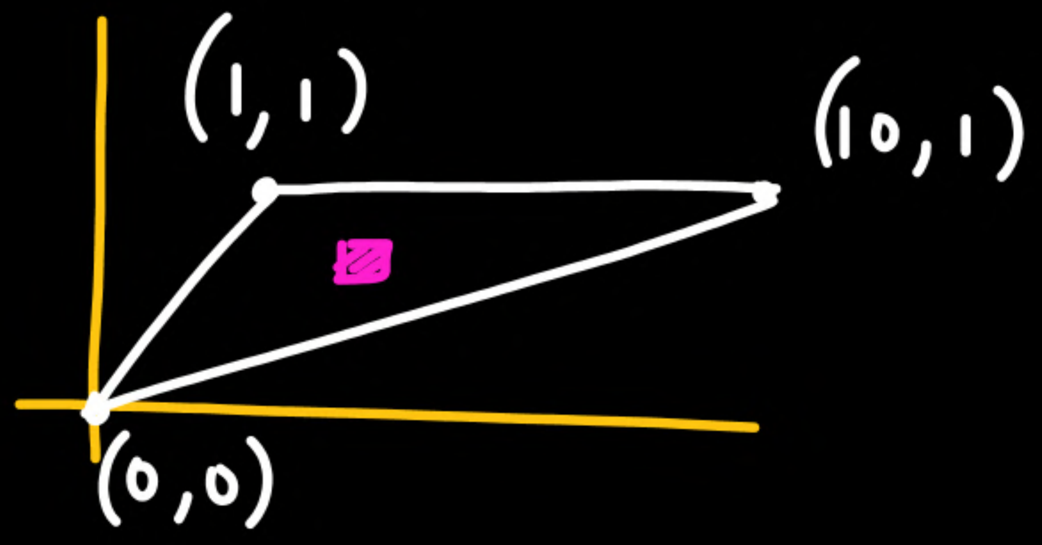
Ex:- $\iint_A xy\, dy\, dx$ where A is domain bounded by X-axis, ordinate $x=2a$ & curve $x^2=4ay$. $= a^4/3$

★ Ex:- $\int_0^z \int_0^x \int_0^{x+y} e^{(x+y+z)}\, dx\, dy\, dz$.

Ex:-

$$\iint_S \sqrt{xy - y^2} = 6$$

$S \rightarrow$ is a triangle with vertices $(0,0), (10,1)$ & $(1,1)$



Thank you

GW Soldiers !