ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-02

Differential equations

By- Chetan Sir

Topics to be Covered

- DEFINITION & TYPES
- ORDER & DEGREE OF DE
- SOLUTION OF DE
- FORMATION OF DE
- WRONSKIAN & LD/LI SOLUTIONS
- METHODS OF SOLVING DE
- PARTIAL DIFFERENTIAL EQUATIONS

WRONSKIAN & LD/LI SOLUTION

→ If y_1, y_2 are two solns. of 2^{nd} order D.E. $a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$ then Wronskian is defined as –

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

→ If y_1, y_2, y_3 are 3 solns. of 3^{rd} DE. $a_0 \frac{d^3y}{dx^3} + a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = 0$ then Wronskian is defined as –

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Linearly Dependent (LD)/Linearly independent (LI) Solution

→ let y_1, y_2, y_3 are solns. of D.E. if there exist a relation b/w y_1, y_2, y_3 such that $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$ where c_1, c_2 & c_3 are non-zero then they are linearly dependent.

In such case; $\boxed{W = 0}$

→ Let y_1, y_2, y_3 are solns. of D.E if there does not exist a relation b/w y_1, y_2, y_3 such that $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$ when $c_1, c_2, c_3 = 0$ then they are linearly independent.

In such case; $\boxed{W \neq 0}$

Ex:- Prove that 1, x, x^2 are linearly independent & form the DE

$y_1 = 1 \quad y_2 = x \quad y_3 = x^2 \quad \rightarrow 3^{rd}$ order D.E.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

$W \neq 0$ Hence given solutions are L.I.

Hence general soln :- $\boxed{y = c_1(1) + c_2(x) + c_3(x^2)}$

$y' = c_2 + 2c_3 x$

$y'' = 2c_3$

$y''' = 0$

$y''' = 0 \quad \boxed{\frac{d^3y}{dx^3} = 0} \rightarrow 3^{rd}$ order D.E.

Ex:-LI Solutions (e^x, xe^x, x^2e^x) form DE

Soln:- $W \neq 0$; then general soln is $\boxed{y = c_1 e^x + c_2(xe^x) + c_3(x^2e^x)}$

$$y' = c_1 e^x + c_2 e^x \cdot 1 + c_2 x e^x + c_3 e^x 2x + c_3 x^2 e^x$$

$$y' = y + c_2 e^x + 2c_3 x e^x$$

$$y'' = y' + \boxed{c_2 e^x + 2c_3 x e^x} + 2c_3 e^x \cdot 1$$

$$y'' = y' + (y' - y) + 2c_3 e^x$$

$$y''' = y'' + y'' - y' + \boxed{2c_3 e^x}$$

$$y''' = 2y'' - y' + (y'' - 2y' + y)$$

$$y''' = 3y'' - 3y' + y$$

Order = 3
Degree = 1
Linear

$$\boxed{y''' - 3y'' + 3y' - y = 0}$$

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

Methods of Solving D.E. :-

ODE

1) Observation Method

2) D.E. of first order & first degree

a) Variable separable mtd. b) Homogenous D.E. mtd.

b) Linear D.E. mtd. d) Bernoulli D.E. mtd.

3) Exact differential equations (Non exact D.E. → Exact D.E.)

4) L.D.E. of n^{th} order with
- → constant coefficients
- → variable coefficients

} (C.F. + P.I.)

5) Methods for solving non-linear D.E.

PDE

6) Methods for solving P.D.E.

METHODS OF SOLVING DE

1) Observation method

- $d(xy) = x\,dy + y\,dx$
- $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

Ex:- $\int \underbrace{x\,dy + y\,dx} - x^2\,dx = 0$

$\int d(xy) - x^2\,dx$

$xy - \frac{x^3}{3} + c = 0$

Ex:- $\int \frac{y\,dx - x\,dy}{y^2} - \frac{y^2}{y^2}dy = \frac{0}{y^2}$

$\int d\left(\frac{x}{y}\right) - 1\,dy = 0$

$\frac{x}{y} - y + c = 0$

2) DE of first order & first degree

(a) Variable separable method

Ex:- tan y sec²x dx + tanx sec² y dy = 0

$$\int \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\log(\tan x) + \log(\tan y) = \log c$$

$$\log(\tan x \tan y) = \log c$$

$$\tan x . \tan y = c$$

$$\frac{dy}{dx} + xy = 0$$

$$\int \frac{dy}{y} = -\int x\, dx$$

$$\log y = -\frac{x^2}{2} + c$$

$$\tan x = t$$

$$\sec^2 x\, dx = dt$$

Ex:- $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$$y - ay^2 = \frac{dy}{dx}(x+a)$$

$$\frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

$$\int \frac{dx}{x+a} = \int \frac{dy}{y} + \int \frac{a}{1-ay}\,dy$$

$$\log(x+a) = \log y + \cancel{a}\,\frac{\log(1-ay)}{-\cancel{a}} + \log c$$

$$\log(x+a) = \log y + \log(1-ay)^{-1} + \log c$$

$$\boxed{x+a = \frac{cy}{1-ay}}$$

$$\frac{1}{y(1-ay)} = \frac{\overset{1}{A}}{y} + \frac{\overset{a}{B}}{1-ay}$$

$$\frac{1}{y(1-ay)} = \frac{A(1-ay)+By}{y(1-ay)}$$

$$1 = A + y(-aA+B)$$

$$\boxed{A=1}$$

$$-aA+B=0$$

$$-a+B=0$$

$$\boxed{B=a}$$

2) DE of first order & firs degree

(b) Homogenous DE method

Let $y/x = v$

✓ $y = vx \rightarrow$ Diff. w.r.t. x

✓ $\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$

Let $x/y = v$

✓ $x = vy \rightarrow$ Diff. w.r.t. y

✓ $\frac{dx}{dy} = v.1 + y\frac{dv}{dy}$

Homogenous function $\rightarrow$ $x, y \rightarrow (x, vx) \rightarrow (v, x)$ $(x, y) \rightarrow (vy, y) \rightarrow (v, y)$

Variable separable

then solve & put value of v.

2) DE of first order & first degree

(b) Homogenous DE method

Homogenous $\longrightarrow \dfrac{x(1+y/x)}{x(1-y/x)} = x^{\boxed{0}} f(y/x)$

Ex:- $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

$y = vx \quad ; \quad \dfrac{dy}{dx} = v.1 + x\dfrac{dv}{dx}$

$v + x\dfrac{dv}{dx} = \dfrac{x+vx}{x-vx}$

$x\dfrac{dv}{dx} = \dfrac{1+v}{1-v} - v$

$x\dfrac{dv}{dx} = \dfrac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$

$\displaystyle\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$

$\displaystyle\int \frac{1}{1+v^2} - \int \frac{2v}{2(1+v^2)} = \int \frac{dx}{x}$

$\tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log x + c$

$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(1+\left(\frac{y}{x}\right)^2\right) = \log x + c$

2) DE of first order & firs degree

(b) Homogenous DE method

Homogenous fn. → $y^0 f\left(\frac{x}{y}\right)$

Ex:-$(1 + e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

$x = vy \qquad \frac{dx}{dy} = v.1 + y\frac{dv}{dy}$

$(1+e^v)dx + e^v(1-v)dy = 0$

$\frac{dx}{dy} = -\frac{e^v(1-v)}{1+e^v}$

$v + y\frac{dv}{dy} = \frac{-e^v+ve^v}{1+e^v}$

$y\frac{dv}{dy} = \frac{-e^v+\cancel{ve^v}-v-\cancel{ve^v}}{1+e^v}$

$\int\frac{(1+e^v)}{v+e^v}dv = -\int\frac{dy}{y}$

$\log_e(v+e^v) = -\log_e y + \log c$

$\log_e y(v+e^v) = \log_e c$

$y\left(\frac{x}{y}+e^{x/y}\right) = c$

$x + ye^{x/y} = c$

2) DE of first order & firs degree

(c) Linear D.E method

$$\frac{dy}{dx} + Py = Q$$

$$P, Q \rightarrow f(x)$$

Find I.F. $= e^{\int P\,dx}$

$$\int e^{\int P dx}\frac{dy}{dx} + e^{\int P dx}.Py = \int Q\, e^{\int P dx}$$

$$y.\, e^{\int P dx} = \int Q\,(\text{I.F.})$$

$$y.(\text{I.F.}) = \int Q(\text{I.F.})$$

$$\frac{dx}{dy} + Px = Q$$

$$P, Q \rightarrow f(y)$$

I.F. $= e^{\int P\,dy}$

$$\int e^{\int P dy}\frac{dx}{dy} + e^{\int P dy}Px = \int Q\, e^{\int P dy}$$

$$x.\, e^{\int P dy} = \int Q\,(\text{I.F.})$$

$$x(\text{I.F.}) = \int Q(\text{I.F.})$$

2) DE of first order & firs degree

(c) Linear D.E method

Ex:- $\frac{dy}{dx}+\frac{y}{x}=x^2$

$$\frac{dy}{dx}+Py=Q \qquad P=1/x \;;\; Q=x^2$$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int \frac{1}{x}dx}=e^{\log_e x}=x$$

$$\int x\frac{dy}{dx}+\frac{y}{x}x=\int x^2\cdot x$$

$$\int d(yx)=\int x^3$$

$$y\cdot x=\frac{x^4}{4}+c$$

$$y(\text{I.F.})=\int Q(\text{I.F.})$$

$$y\cdot x=\int x^2\cdot x$$

$$y\cdot x=\frac{x^4}{4}+c$$

2) DE of first order & firs degree

(c) Linear D.E method

$P = \frac{2}{x}$, $Q = e^x$

Ex:- $\frac{dy}{dx} + \frac{2y}{x} = e^x$

$\frac{dy}{dx} + Py = Q$

I.F. $= e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{\ln_e x^2} = x^2$

$y \cdot x^2 = \int \overset{\text{II}}{e^x} \overset{\text{I}}{(x^2)}$

$y \cdot x^2 = (x^2)(e^x) - (2x)(e^x) + (2)(e^x) + c$

$y = e^x(1 - 2/x + 2/x^2) + c/x^2$

2) DE of first order & firs degree

(c) Linear D.E method

Ex:- $\frac{dy}{dx} + \frac{y}{x} = x$ *with the condition. that* $y(1) = 1$, *is* [GATE]

(a) $Y = \frac{2}{3x^2} + \frac{y}{x}$ (b) $Y = \frac{x}{2} + \frac{1}{2x}$

(c) $Y = 2/3 + x/3$ (d) $Y = \frac{2}{3x} + \frac{x^2}{3}$

$$\frac{dy}{dx} + \frac{y}{x} = x$$

$$I.F. = x$$

$$y \cdot x = \int x \cdot x$$

$$y \cdot x = \frac{x^3}{3} + c$$

At $x = 1$; $y = 1$

$$1 \cdot 1 = \frac{1^3}{3} + c$$

$$\boxed{c = 2/3}$$

$$\boxed{y = \frac{x^2}{3} + \frac{2}{3x}}$$

TRICK :-

P	I.F.
$\frac{1}{x}$	$x^1 \to x$
$-\frac{1}{x}$	$x^{-1} \to 1/x$
$\frac{2}{x}$	$x^2 \to x^2$
$-\frac{2}{x}$	$x^{-2} \to 1/x^2$
$\frac{3}{x}$	$x^3 \to x^3$
$-\frac{3}{x}$	$x^{-3} \to 1/x^3$

2) DE of first order & firs degree

(d) Bernoulli DE method (reducible to L DE method)

$$\frac{dy}{dx} + Py = Qy^n \; ; \; n > 0$$

Divide by y^n

$$\frac{1}{y^n} \cdot \frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

$$\frac{1}{y^{n-1}} = z$$

$$-\frac{(n-1)}{y^n} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$(y, x) \dashrightarrow (z, x)$

$$\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q$$

Reducible to L.D.E.

$$\boxed{\frac{dz}{dx} + P(1-n)\, z = Q(1-n)}$$

$$z \cdot (\text{I.F}) = \int Q(1-n)\, \text{I.F.}$$

$$\text{I.F.} = e^{\int P(1-n)\, dx}$$

Ex:- $\frac{dy}{dx}+\frac{y}{x}=x^2y^3$

$$\frac{1}{y^3}\frac{dy}{dx}+\frac{1}{xy^2}=x^2$$

$$\frac{1}{-2}\frac{dz}{dx}+\frac{1}{x}.z=x^2$$

$$(z,x)\rightarrow \frac{dz}{dx}-\frac{2}{x}z=-2x^2$$

$$\downarrow P \qquad \downarrow Q$$

$$z.\,x^{-2}=\int -2\cancel{x^2}.\cancel{x^{-2}}$$

$$z/x^2=-2x+c$$

$$\frac{1}{y^2}=z$$

$$-\frac{2}{y^3}\frac{dy}{dx}=\frac{dz}{dx}$$

$$\frac{1}{x^2y^2}=-2x+c$$

METHODS OF SOLVING DE

Ex:- $\frac{dy}{dx} + \frac{y}{x} \cdot (\log y) = \frac{y(\log y)^2}{x}$

Divide $y(\log y)^2$

Thank you

GW

Soldiers !