ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-01

Vector Calculus

By- Chetan Sir

Topics to be Covered

- VECTOR BASICS
- STRAIGHT LINES/3D PLANES
- GRADIENT (VECTOR DIFFERENTIATION)
- DIVERGENCE (VECTOR DIFFERENTIATION)
- CURL (VECTOR DIFFERENTIATION)
- LINE, SURFACE, VOLUME INTEGRAL (VECTOR INTEGRATION)
- GREEN, & STOKE'S THEOREM (VECTOR INTEGRATION)
- GAUSS DIVERGENCE THEOREM (VECTOR INTEGRATION)

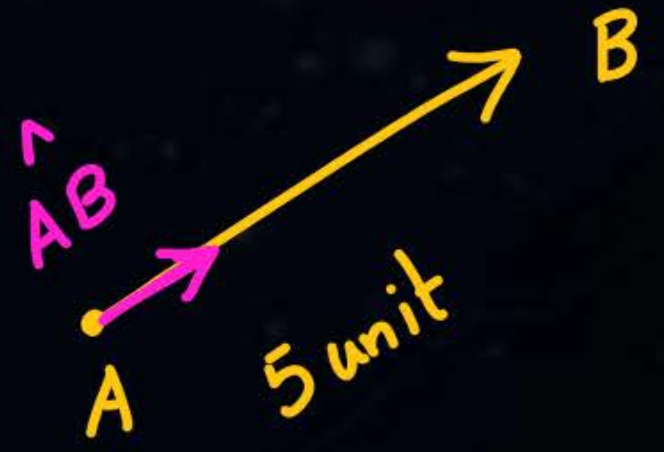
Vector Differentiation

Vector integration

VECTOR BASICS

$$\vec{AB} = \underbrace{|\vec{AB}|}_{\text{Magnitude}} \ \underbrace{\hat{AB}}_{\text{Dir}^n.}$$

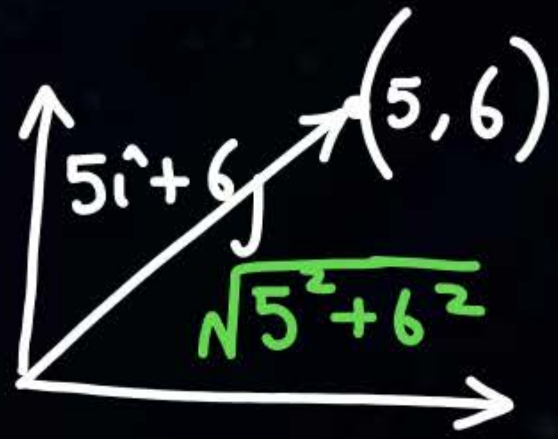
$5\hat{i}+6\hat{j}$



Unit vector

$$|\hat{a}| = 1$$

Ex:- $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$ $\quad |\vec{a}| = \sqrt{2^2+3^2+6^2} = 7$



$$\left\{ \hat{a} = \frac{\vec{a}}{|\vec{a}|} \right\} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

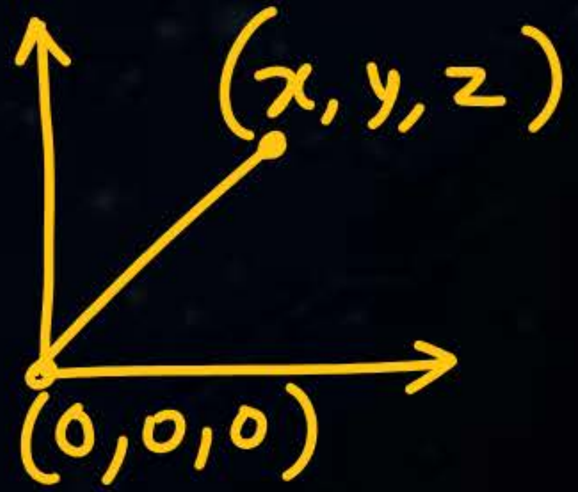
$$\sqrt{l^2+m^2+n^2} = 1$$

Direction Ratio's $\underset{}{\overset{a,\,b,\,c}{\langle 2, 3, 6 \rangle}}$

Direction Cosines $\overset{l,\,m,\,n}{\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle}$

Position vector $= x\hat{i} + y\hat{j} + z\hat{k}$

Vector joining point with origin.



Vector joining two points :-

$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$

$$\vec{AB} = \vec{OB} - \vec{OA} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

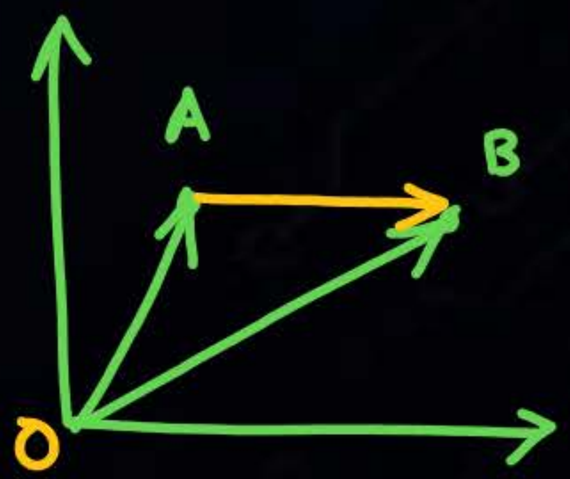
$A(1, 0, -1)$

$B(5, 2, 3)$

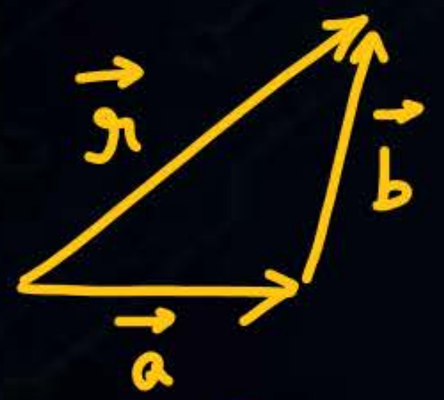
$\vec{AB} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

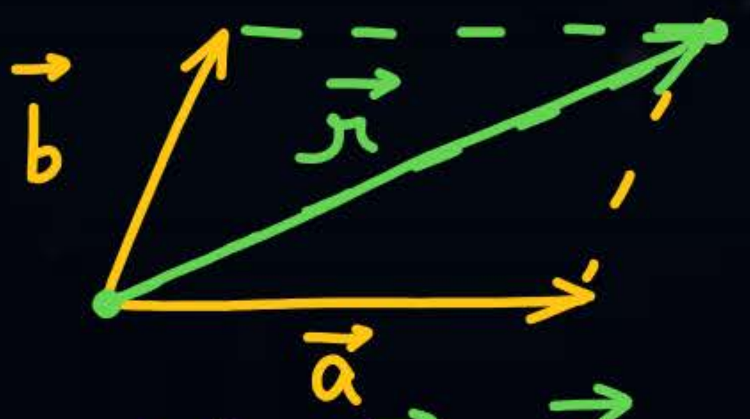


ADDITION OR SUBSTRACTION OF VECTORS



$$\vec{r} = \vec{a} + \vec{b}$$

(Triangle law)



$$\vec{r} = \vec{a} + \vec{b}$$

(Parallelogram law)

$$\vec{r} = \vec{a} + \vec{b} + \dots \vec{h}$$

Parallel vectors :-

$$\vec{a} = \lambda \vec{b}$$

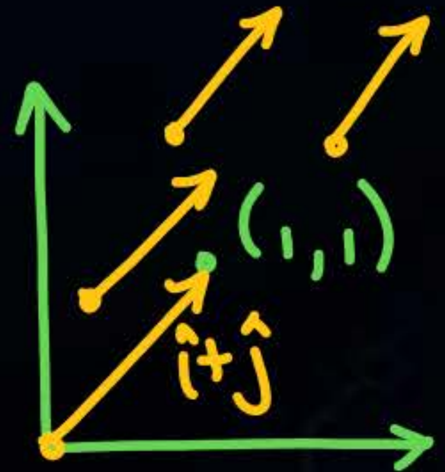
∴ then $\vec{a}$, $\vec{b}$ are parallel

$\hat{i} + \hat{j}$

$5\hat{i} + 5\hat{j}$

$5\hat{i} \longrightarrow$

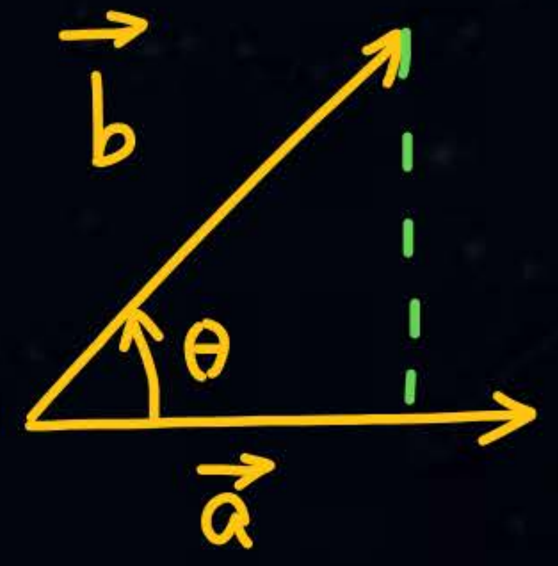
$-5\hat{i} \longleftarrow$



DOT PRODUCT & CROSS PRODUCT

$$\boxed{\vec{a}\cdot\vec{b}\cdot=|\vec{a}||\vec{b}|\cos\theta}$$

Scalar quantity

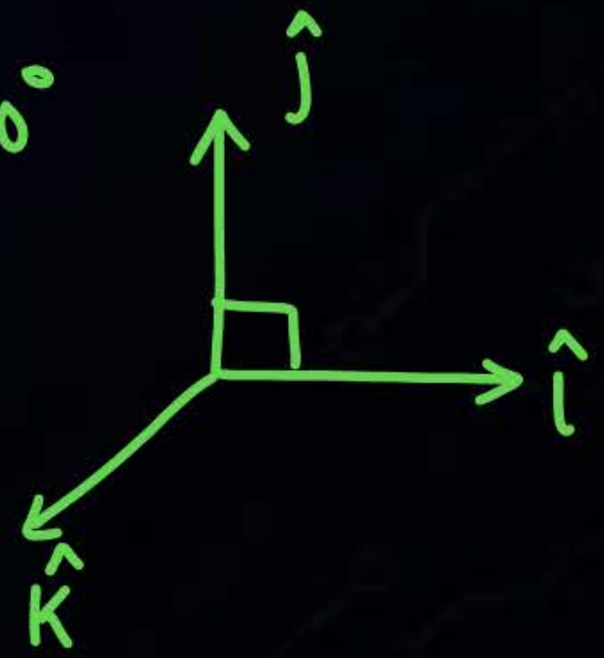


$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
- $\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1$
- $\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{k}=\hat{k}\cdot\hat{\imath}=0$

$\hat{i}\cdot\hat{i}=|\hat{i}||\hat{i}|\cos 0^\circ$

$1.1.1=1$



* Two perpendicular vectors dot product is <u>0</u>.

$(2\hat{i}+3j+6\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = 2+3+6=11$

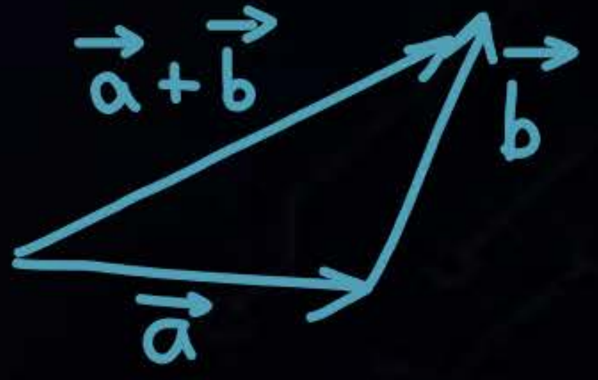
$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Angle between 2 vectors

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Triangle inequality :-

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$



Triangle inequality

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

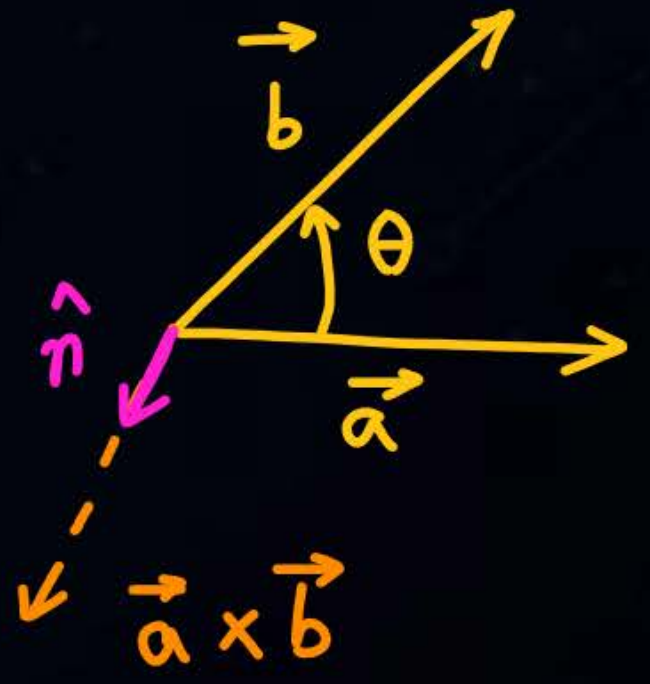
DOT PRODUCT & CROSS PRODUCT

Cross Product :- (Vector quality)

$$\boxed{\vec{a} \times \vec{b} = \underbrace{|a||b|\sin\theta}_{\text{Magnitude}} \times \hat{n}}$$

$\hat{n}$ or $\vec{a} \times \vec{b} \perp \vec{a}$

$\hat{n}$ or $\vec{a} \times \vec{b} \perp \vec{b}$

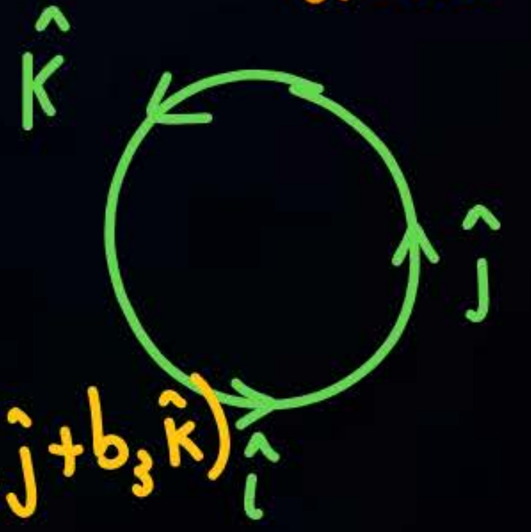


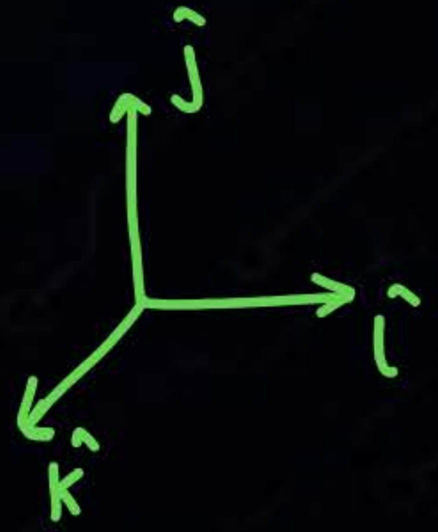
$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$ $\quad \hat{j} \times \hat{k} = \hat{i}$ $\quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$ $\quad \hat{k} \times \hat{j} = -\hat{i}$ $\quad \hat{i} \times \hat{k} = -\hat{j}$





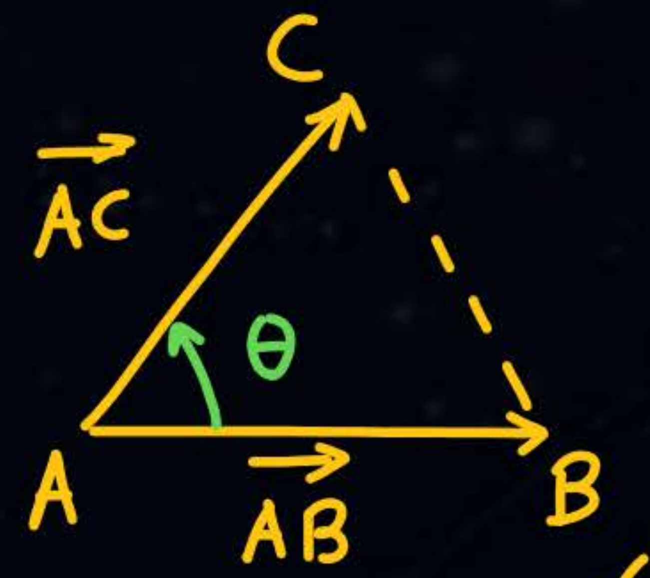
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

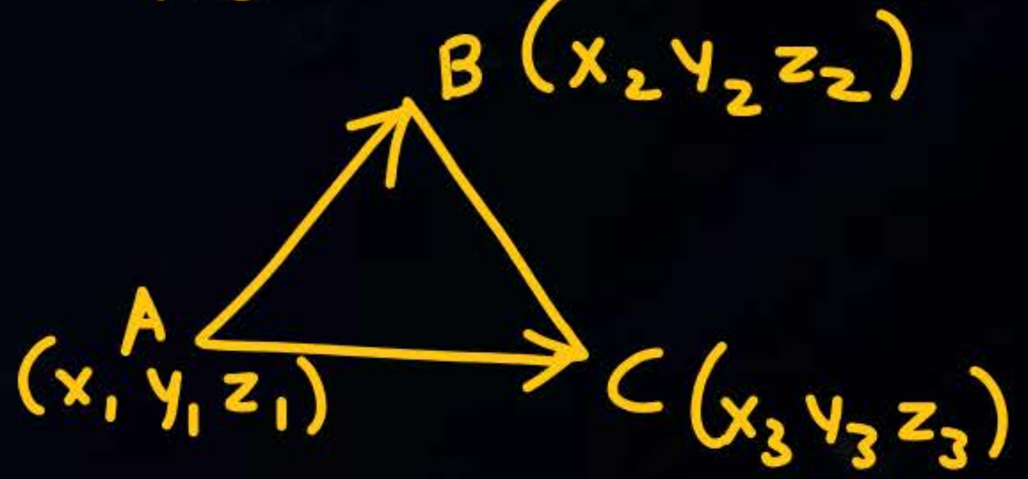
[DOT PRODUCT & CROSS PRODUCT]

- Area of triangle :-

$$\frac{1}{2}\left|\vec{AB}\times\vec{AC}\right| = \frac{1}{2}\left|\vec{AB}\right|\left|\vec{AC}\right|\sin\theta$$

$$= \frac{1}{2}\left|\,\left|\vec{OB}-\vec{OA}\right| \times \left|\vec{OC}-\vec{OA}\right|\,\right|$$





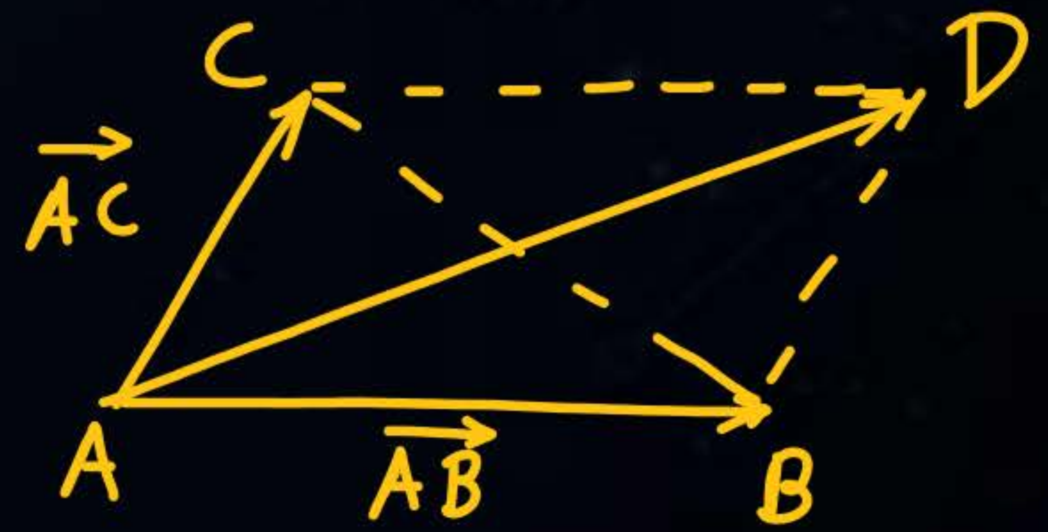
DOT PRODUCT & CROSS PRODUCT

- Area of parallelogram

$$= |\vec{AC} \times \vec{AB}|$$

$$= \frac{1}{2} |\vec{AD} \times \vec{BC}|$$

$$= \frac{1}{2} (\text{Product of diagonals})$$

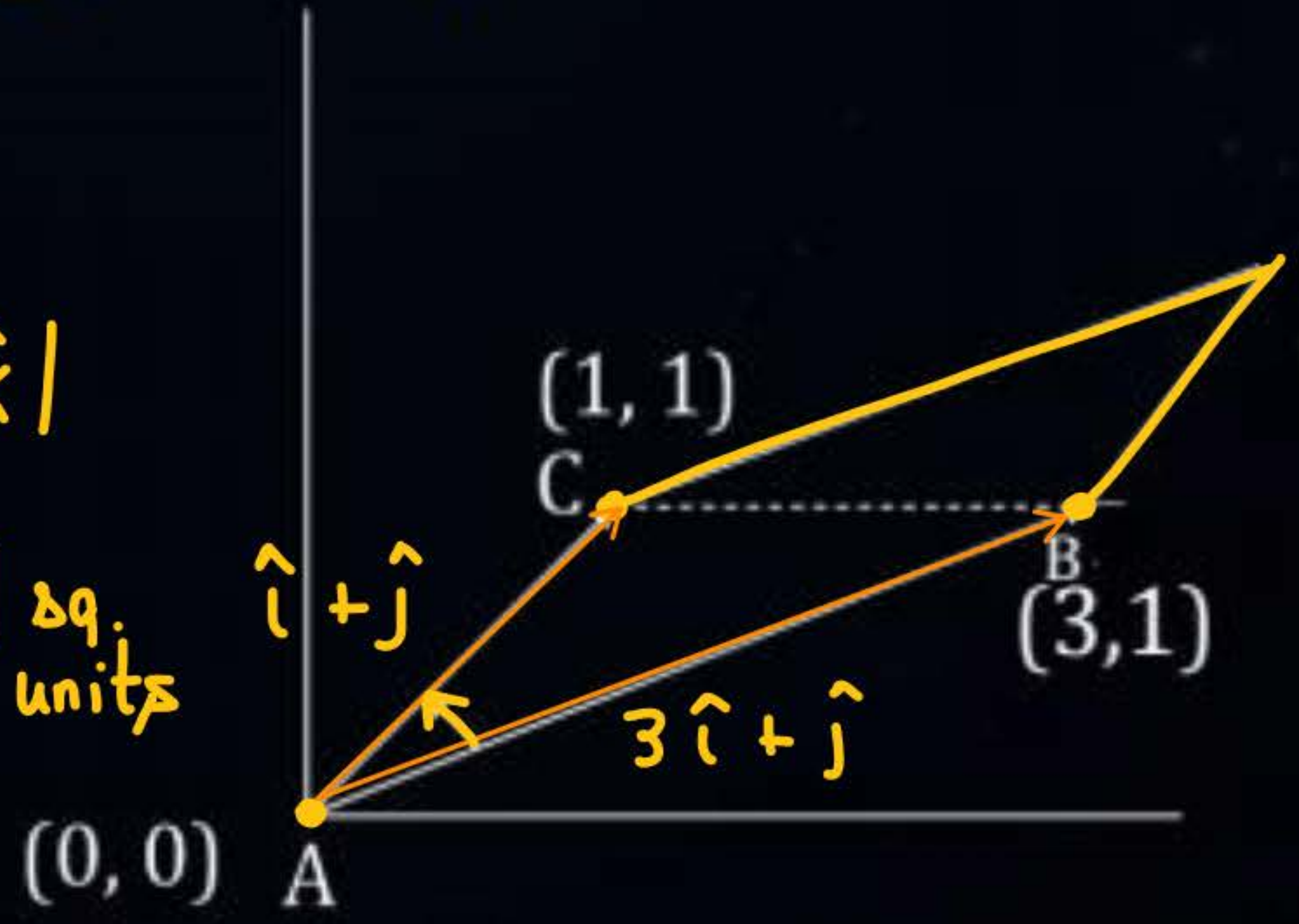


Q. Find Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

$$\underset{\Delta}{\text{Area}} = \frac{1}{2}\left|(\hat{i}+\hat{j}) \times (3\hat{i}+\hat{j})\right|$$

$$= \frac{1}{2}\left|\hat{k} - 3\hat{k}\right| = \frac{1}{2}\left|-2\hat{k}\right|$$

$$= \frac{2}{2} = 1 \text{ sq. units}$$



$$\underset{(\parallel gm)}{\text{Area}} = 2 \text{ sq units}$$

SCALAR TRIPLE PRODUCT

Lagrange's identity :-

$$\boxed{|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$$

$|\vec{a}|^2|\vec{b}|^2\sin^2\theta\, \cancel{\hat{n}^2}^{\,1} = |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$\hat{a}\cdot\hat{a} = |\hat{a}||\hat{a}|\cos 0$

$= 1$

$\hat{a}\cdot\hat{a} = |\hat{a}|^2 = 1$

SCALAR TRIPLE PRODUCT

Scalar triple product

$\vec{a} = a_1, a_2, a_3$

$\vec{b} = b_1, b_2, b_3$

$\vec{c} = c_1, c_2, c_3$

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

SCALAR TRIPLE PRODUCT

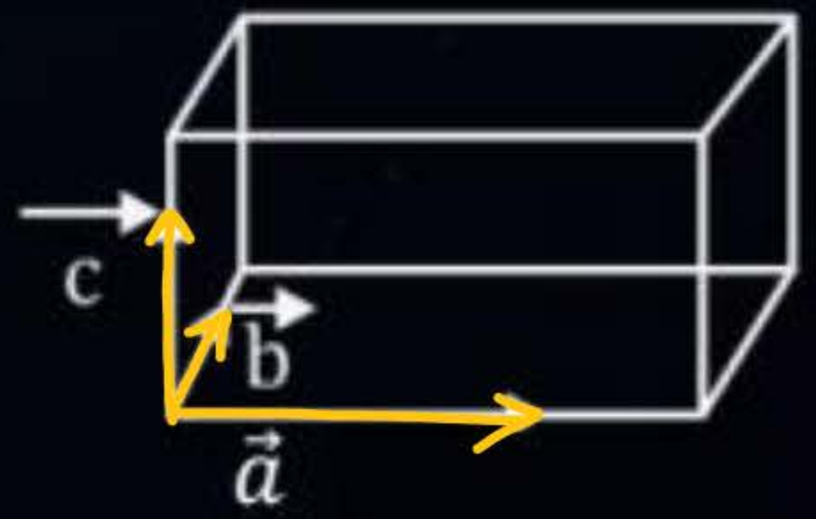
- $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$
- $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{b}\,\vec{a}\,\vec{c}]$
- If coplanar vector, then $[\vec{a}, \vec{b}, \vec{c}] = 0$
- If the vector are same then also, $[\vec{a}, \vec{b}, \vec{a}] = 0$
- Volume of parallelopiped = $[\vec{a}, \vec{b}, \vec{c}]$ where $\vec{a}, \vec{b}, \vec{c}$ are sides at parallelopiped

Q. Find vol. of parallelopiped where edges are

$\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$

$\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$\vec{c} = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$



$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & -2 \end{vmatrix} = |-35| = 35 \text{ cubic units}$$

$$\vec{c}\cdot(\vec{a}\times\vec{b}) = -35$$

$$\vec{b}\cdot(\vec{c}\times\vec{a}) = -35$$

VECTOR TRIPLE PRODUCT

$$\text{V.T.P} = \vec{a} \times (\vec{b} \times \vec{c}) = \underbrace{(\vec{a} \cdot \vec{c})}\vec{b} - \underbrace{(\vec{a} \cdot \vec{b})}\vec{c}$$

Find V. T. P. of $\vec{a}, \vec{b}, \vec{c}$

$$a \times (\vec{b} \times \vec{c})$$

$$= (\vec{a} . \vec{c})\,\vec{b} - (\vec{a} . \vec{b}) . \vec{c}$$

DIRECTION COSINES & RATIOS

$a\hat{i} + b\hat{j} + c\hat{k} \Rightarrow \dfrac{a\hat{i}+b\hat{j}+c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

Dir^n ratios $\langle a, b, c \rangle$

$a^2+b^2+c^2 = |\vec{a}|^2$

Dir^n cosines $\langle l, m, n \rangle$

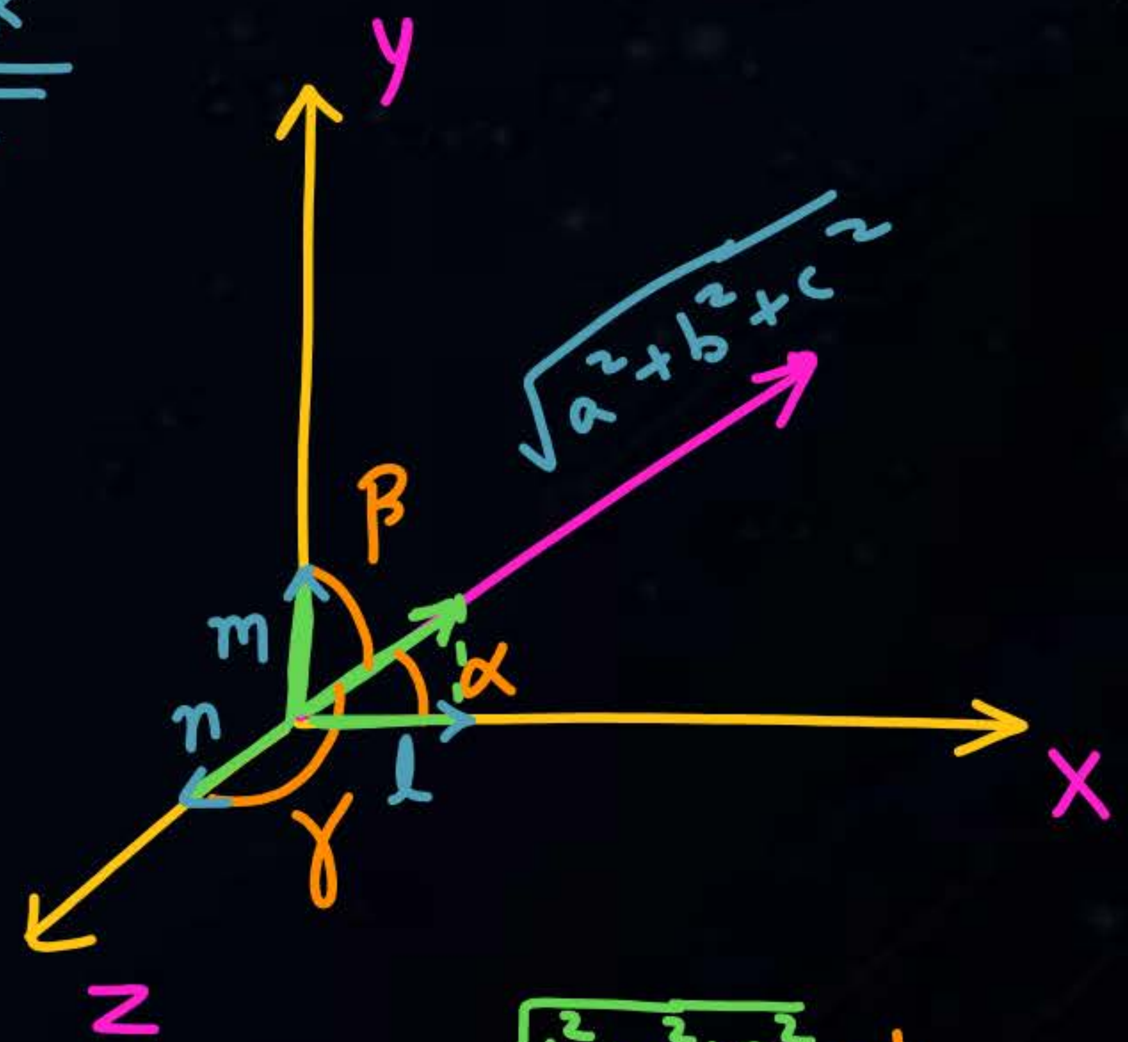
(D.R.s of unit vector) $l^2+m^2+n^2 = 1$

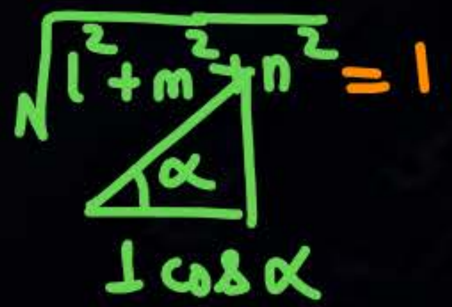
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$l = a/\sqrt{a^2+b^2+c^2} = \cos\alpha$

$m = b/\sqrt{a^2+b^2+c^2} = \cos\beta$

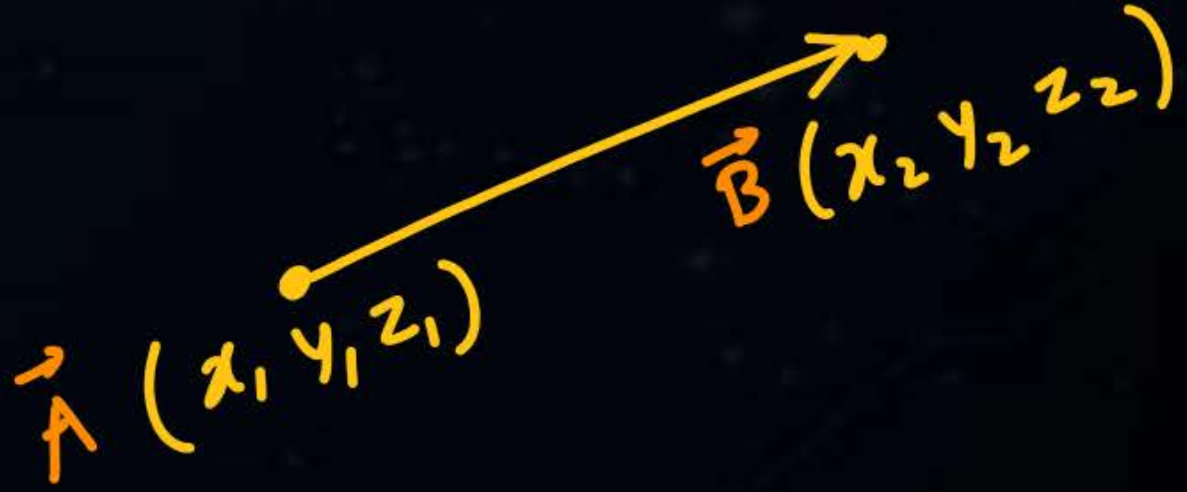
$n = c/\sqrt{a^2+b^2+c^2} = \cos\gamma$





DIRECTION COSINES & RATIOS

$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



DRs : $\langle x_2 - x_1 ,\ y_2 - y_1 ,\ z_2 - z_1 \rangle$

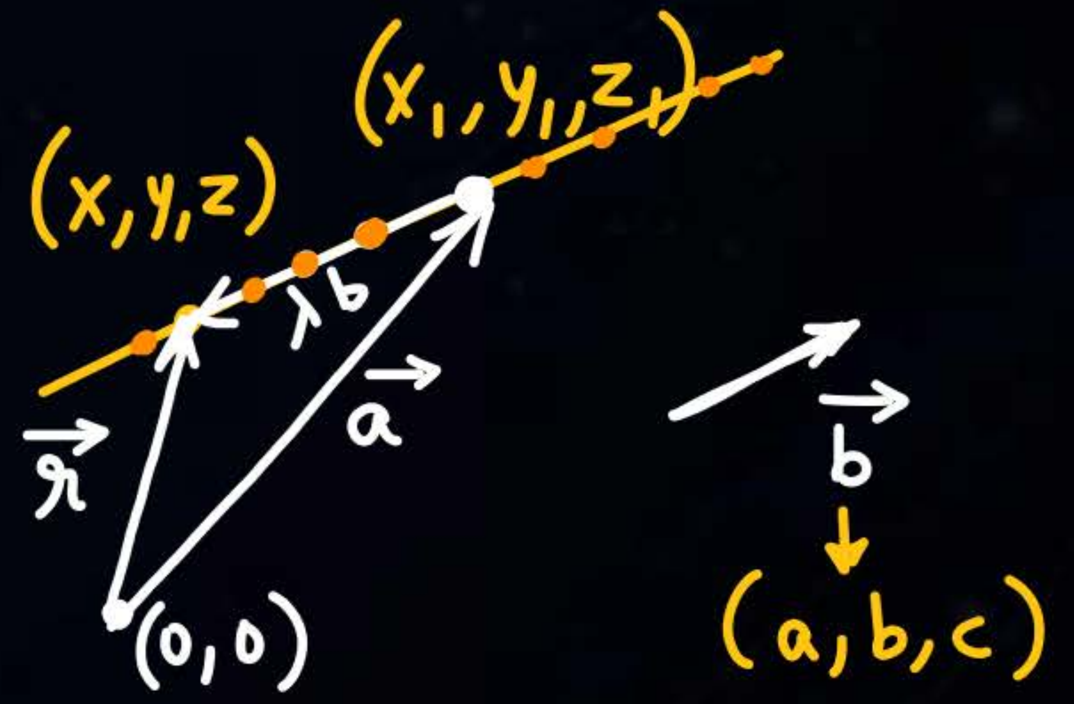
DCs : $\left\langle \dfrac{x_2 - x_1}{|\vec{AB}|} ,\ \dfrac{y_2 - y_1}{|\vec{AB}|} ,\ \dfrac{z_2 - z_1}{|\vec{AB}|} \right\rangle$

[STRAIGHT LINES IN 3-D]

Passing through point (x_1, y_1, z_1) & having Direction cosines (a, b, c)

Cartesian form $\boxed{\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}}$

Vector form ; $\boxed{\vec{r} = \underset{\text{Pt}}{\vec{a}} + \lambda \underset{\text{D.R.}}{\vec{b}}}$



Ex:- Find eqn. of line passing through $\overset{\vec{a}}{(1,2,3)}$ having D.R's $(4,5,6)$ $\vec{b}$

Vector form ; $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(4\hat{i} + 5\hat{j} + 6\hat{k})$

✓ $(x\hat{i} + y\hat{j} + z\hat{k}) = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(4\hat{i} + 5\hat{j} + 6\hat{k})$

$x = 1 + 4\lambda$
$y = 2 + 5\lambda$
$z = 3 + 6\lambda$

✓ $\left\{ \dfrac{x-1}{4} = \dfrac{y-2}{5} = \dfrac{z-3}{6} = \lambda \right\}$

STRAIGHT LINES IN 3-D

Passing through points (x_1, y_1, z_1) & (x_2, y_2, z_2)

Cartesian form
$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$$

Vector form
$$\vec{r} = \vec{a}_1 + \lambda(\vec{a}_2 - \vec{a}_1)$$

Ex :-

$$\frac{x-4}{5} = \frac{y-5}{5} = \frac{z-6}{3}$$

$$\frac{x+1}{5} = \frac{y-0}{5} = \frac{z-3}{3}$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \lambda(5\hat{i} + 5\hat{j} + 3\hat{k})$$

Q. Find point of intersection of lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

And $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$

General point on line L $\underbrace{(\overset{x}{2\lambda+1},\ \overset{y}{3\lambda+2},\ \overset{z}{4\lambda+3})}$

$$\frac{2\lambda+1-4}{5}=\frac{3\lambda+2-1}{2}=\frac{4\lambda+3}{1}$$

$\lambda=-1$ $\qquad$ $\lambda=-1$

$\lambda=-1$

$(-1,-1,-1)$

Point of intersection $(-1,-1,-1)$

If lines are skew, i.e., they do not intersect, then unique value at λ does not exist

If diff. values of λ, then lines do not intersect.

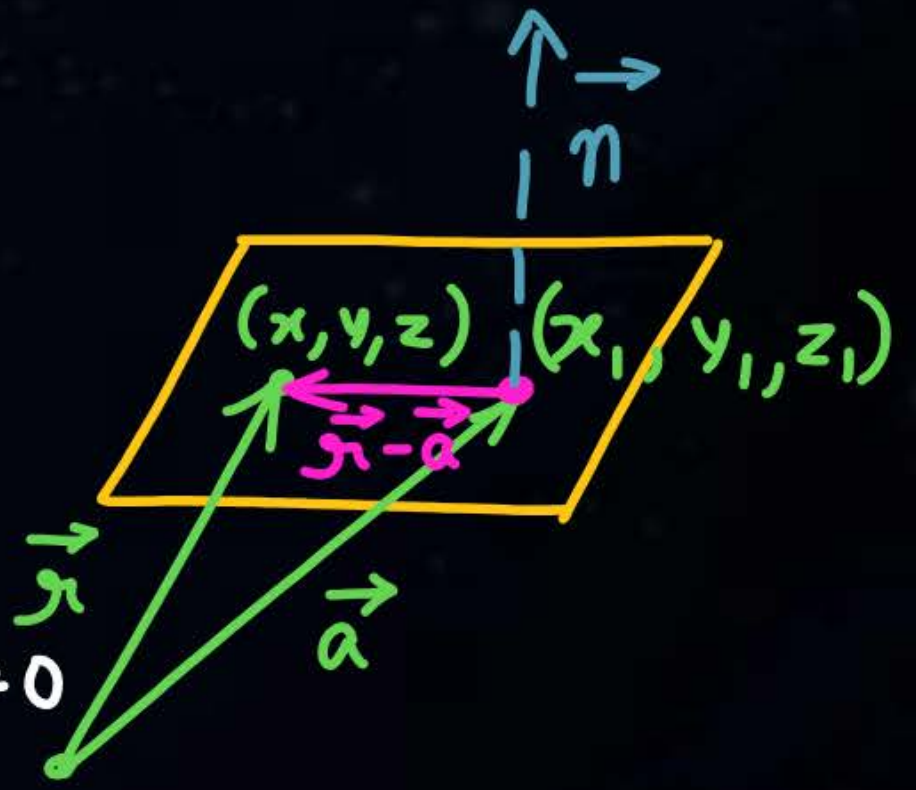
3D PLANES

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

DR of normal to plane $\langle a, b, c \rangle$

DCs of normal plane $\langle l, m, n \rangle$



$$\boxed{(\vec{r} - \vec{a}) \cdot \vec{n} = 0} \quad \text{(Vector form)}$$

$$\left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$ax + by + cz = ax_1 + by_1 + cz_1$$

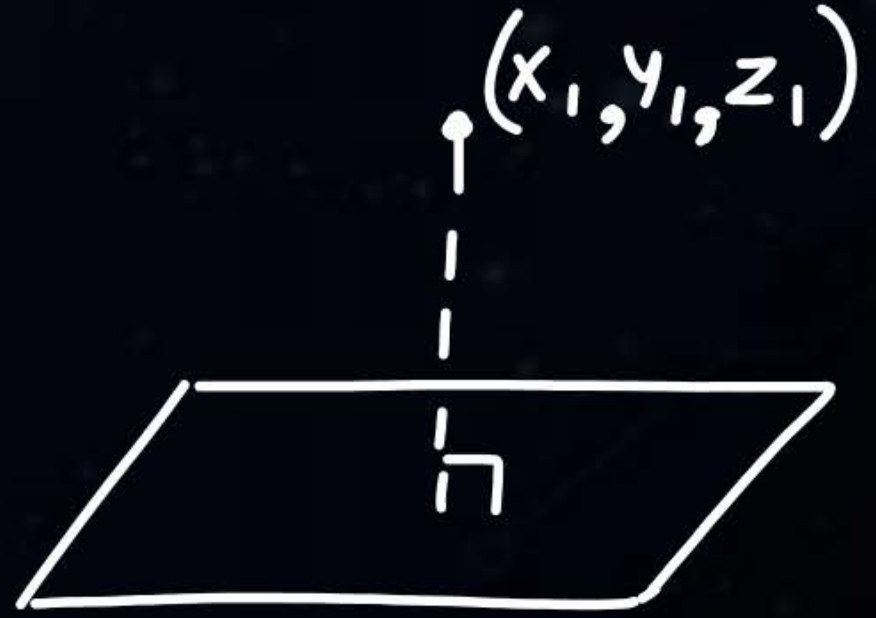
$$\boxed{ax + by + cz = d} \quad \text{(Cartesian form)}$$

$$\frac{a}{\sqrt{a^2+b^2+c}}x + \frac{b}{\sqrt{a^2+b^2+c}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{d}{\sqrt{a^2+b^2+c^2}} \rightarrow \text{Distance of plane from origin}$$

$$\boxed{lx + my + nz = p}$$

Distance of point from plane.

$$\boxed{D = \left|\frac{ax_1 + by_1 + (cz_1 + d)}{\sqrt{a^2 + b^2 + c^2}}\right|}$$



$ax + by + cz + d = 0$

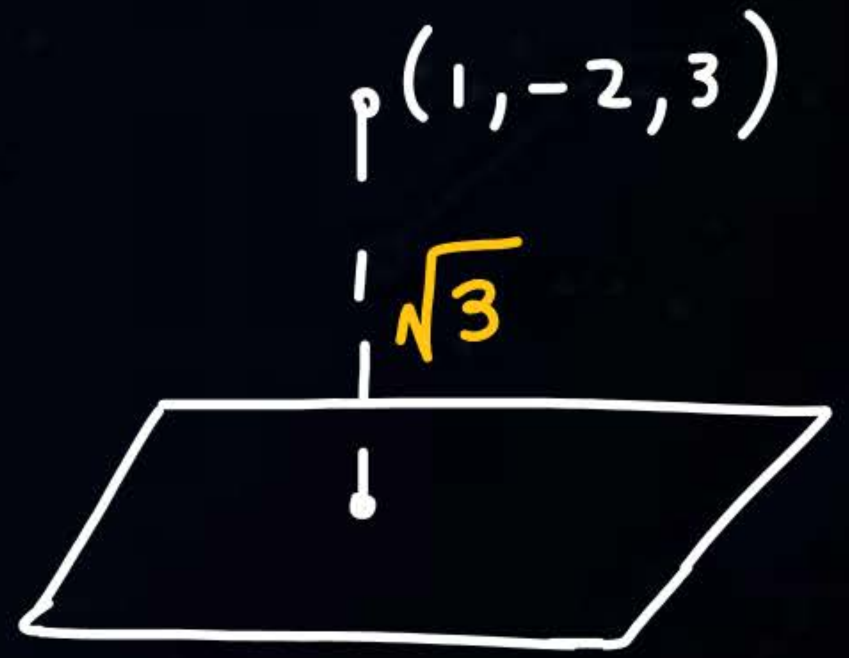
$\vec{n} = \langle a, b, c \rangle$

Q. Find distance of point (1, – 2, 3) from plane x –y + z = 5

$$\text{Distance} = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}} \right|$$

$$= \left| \frac{1 + 2 + 3 - 5}{\sqrt{1^2+(-1)^2+1^2}} \right|$$

$$= \frac{1}{\sqrt{3}}$$



$x - y + z - 5 = 0$

$\vec{n} = \langle 1, -1, 1 \rangle$

Q. A plane is passing through a pt. $\underbrace{(1, 2, 3)}$ and direction ratio of normal vector to the plane are $\underbrace{(2, -1, 3)}_{\vec{n}}$. Find equation of plane.

$\vec{a}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(\vec{r} - \vec{a}) \cdot \hat{n} = 0$

$$\boxed{[(x-1)\hat{i} + (y-2)\hat{j} + (z-3)\hat{k}] \cdot [2\hat{i} - \hat{j} + 3\hat{k}] = 0}$$

$$2(x-1) - (y-2) + 3(z-3) = 0$$

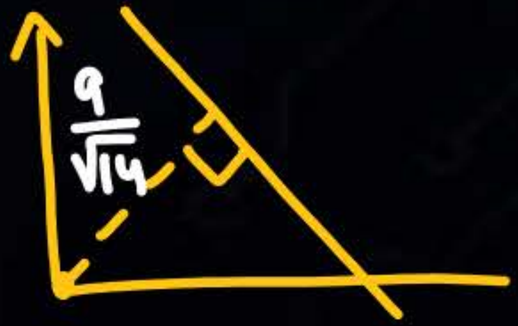
$\sqrt{2^2 + (-1)^2 + 3^2}$

$\sqrt{14}$

$$2x - y + 3z = 2 - 2 + 9$$

$$\boxed{2x - y + 3z = 9}$$

$\langle 2, -1, 3 \rangle$



$$\boxed{\frac{2}{\sqrt{14}}x - \frac{y}{\sqrt{14}} + \frac{3}{\sqrt{14}}z = \frac{9}{\sqrt{14}}}$$

→ Dist. of plane from origin.

Q. Find length & foot of perpendicular drawn from point (1, 1, 2) to plane $\vec{r}.(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + 5 = 0$

$\vec{n} = \langle 2, -2, 4 \rangle$

Normal line $\quad \dfrac{x-1}{2} = \dfrac{y-1}{-2} = \dfrac{z-2}{4} = \lambda$

Any pt. on this line $(2\lambda+1, -2\lambda+1, 4\lambda+2)$

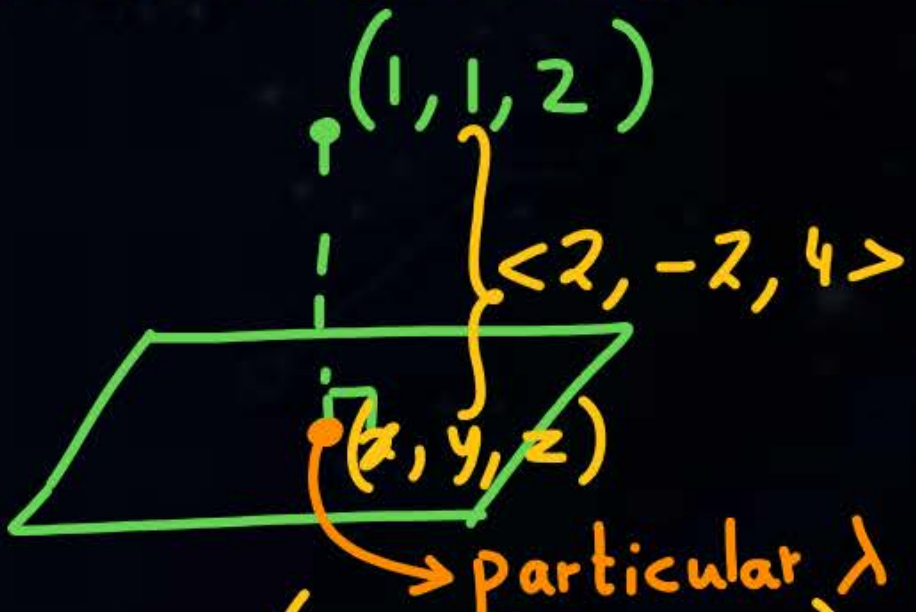
$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + 5 = 0$

$2x - 2y + 4z + 5 = 0$

$2(2\lambda+1) - 2(-2\lambda+1) + 4(4\lambda+2) + 5 = 0$

$\lambda = -13/24$

Point $\left(-1/12, 25/12, -1/6\right)$



$\left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$

$$\text{Dist.} = \sqrt{\left(1+\frac{1}{12}\right)^2 + \left(1-\frac{25}{12}\right)^2 + \left(2+\frac{1}{6}\right)^2}$$

$$= \frac{13\sqrt{6}}{12}$$

Thank you

GW

Soldiers !