ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-14

Calculus

By- Chetan Sir

Topics to be Covered

- APPLICATION OF INTEGRATIONS
- LENGTH OR PERIMETER OF CURVE
- SURFACE AREA OF REVOLUTION
- VOLUME OF SOLID OF REVOLUTION
- MULTIPLE INTEGRALS

MULTIPLE INTEGRALS

$= dV$

Triple Integration

It is denoted by $\iiint f(x,y,z)\,dx\,dy\,dz$ over volume R.

I :- $$\int_{x=a}^{x=b}\int_{y=c}^{y=d}\int_{z=e}^{z=f} f(x,y,z)\,dz\,dy\,dx$$

(Limits constant, order of integration is insignificant).

II :- $$\int_{x_1=a}^{x_2=b}\int_{y_1=f(x)}^{y_2=f(x)}\int_{z_1=f(x,y)}^{z_2=f(x,y)} f(x,y,z)\,dz\,dy\,dx$$

CHANGE OF ORDER OF INTEGRATION

First $Y \rightarrow$ then X

→ If variable limits are of y i.e. $f(x) \Rightarrow$ strip ∥ to y-axis

→ Now plot the limits & make strip ∥ to x-axis.

→ Put the limits first in X & then in y.

First x then Y

First $X \rightarrow$ then Y

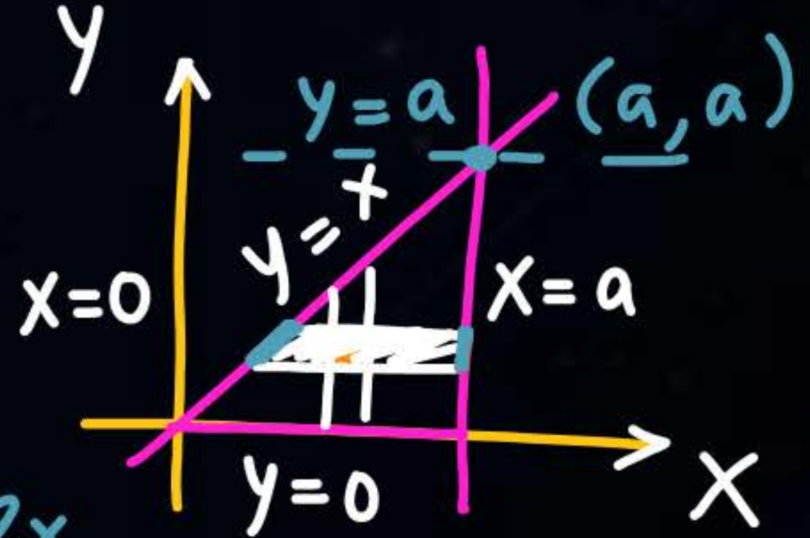
→ . . . . of x i.e. $f(y) \Rightarrow$ strip ∥ to x-axis

First $Y \rightarrow$ then X

Ex:- Change the order of integration $\int_0^a \int_0^x f(x,y)\,dx\,dy$

$y=0\,;\ y=x\ ,\ x=0\,;\ x=a$

$$\int_{y=0}^{y=a} \int_{x=y}^{x=a} f(x,y)\,dx\,dy$$



Ex:- Reverse the order of integration $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dx\,dy$

$Y \to X$

$X \to Y$

$y=x^2\,;\ y=2x\,;\ x=0\,;\ x=2$

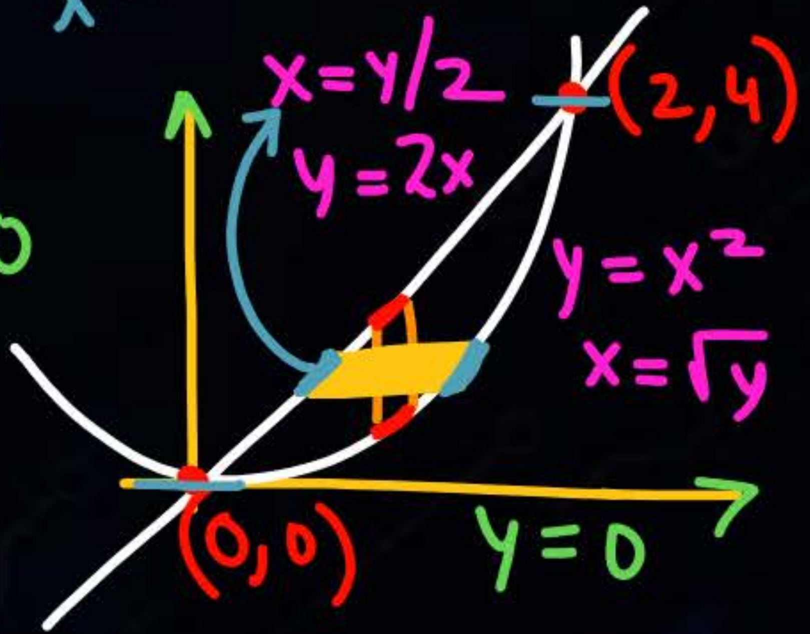
$$\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} f(x,y)\,dx\,dy$$

$x^2-2x=0$

$x(x-2)=0$

$x=0,2$

$y=0,4$

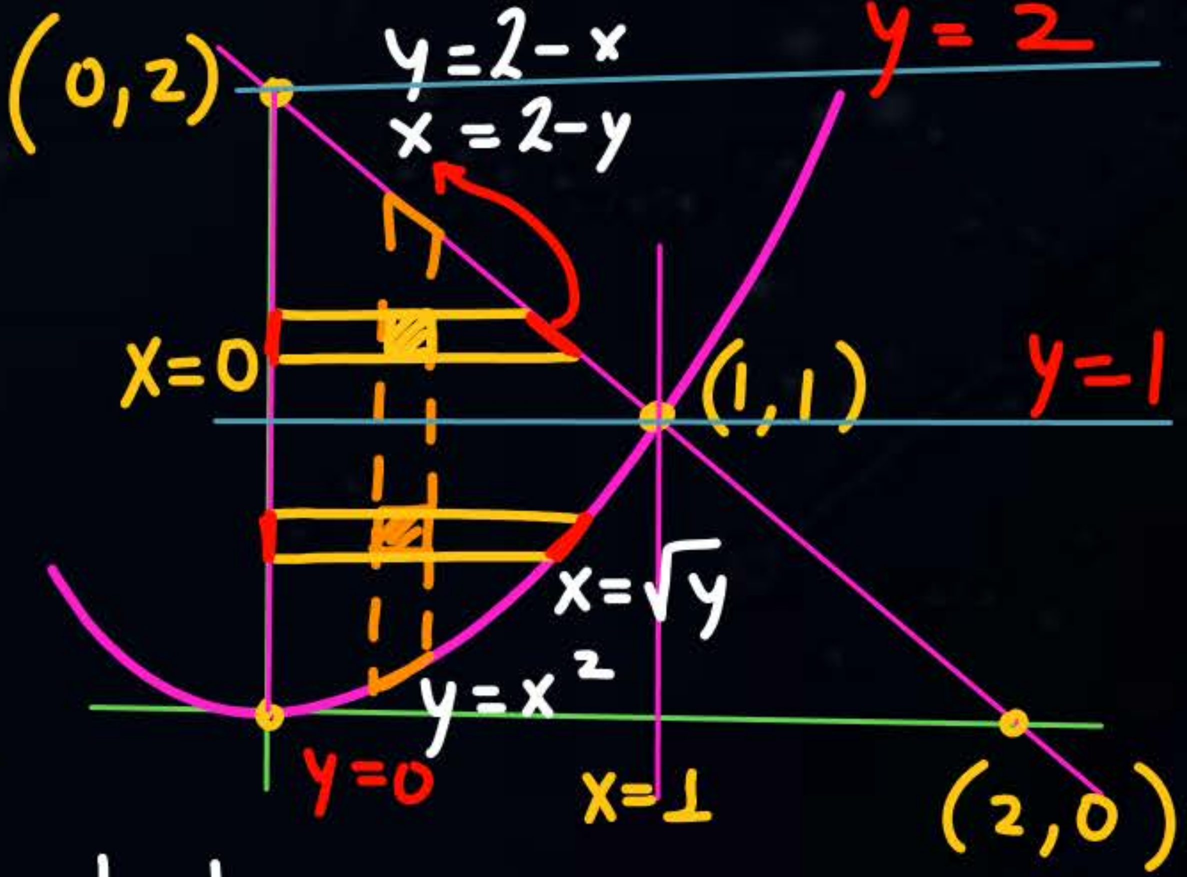


Ex:- Reverse order of integration:-

$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=2-x} xy\, dy\, dx$$

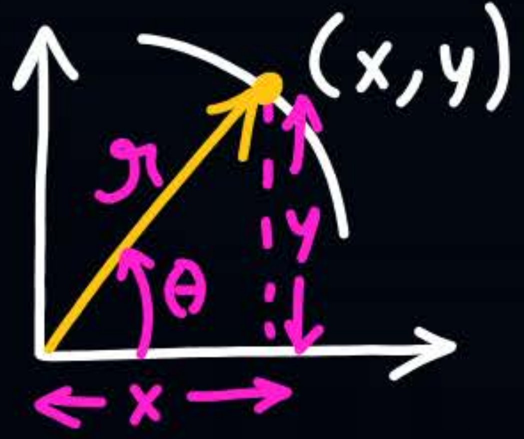
$y=x^2$; $y=2-x$; $x=0$; $x=1$

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{y=2}\int_{x=0}^{x=2-y} xy\, dx\, dy$$



Polar form

$(x, y) \rightarrow (r, \theta)$



$x = r\cos\theta \quad — 1)$

$y = r\sin\theta \quad — 2)$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2$

$r = \sqrt{x^2 + y^2} \quad - 3)$

$\rightarrow \quad x^2 + y^2 = a^2$

$\rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = a^2$

$r^2 = a \Rightarrow \boxed{r = a}$

$\rightarrow \left.\begin{array}{l} x = a\cos t \\ y = a\sin t \end{array}\right\} (t)$

Cartesian $\rightarrow \quad y = x^2$

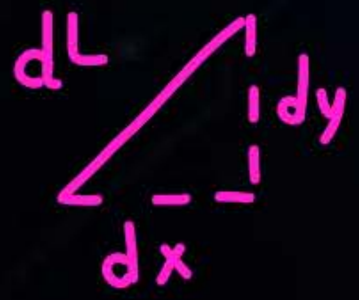
Polar $\rightarrow \quad r\sin\theta = (r\cos\theta)^2$

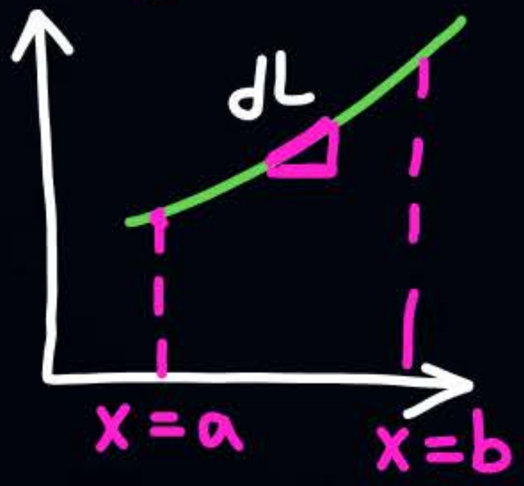
Para-metric $\rightarrow \left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} (t)$

APPLICATION OF INTEGRATIONS

Length or perimeter of curve → Process of finding the length of arc of curve.

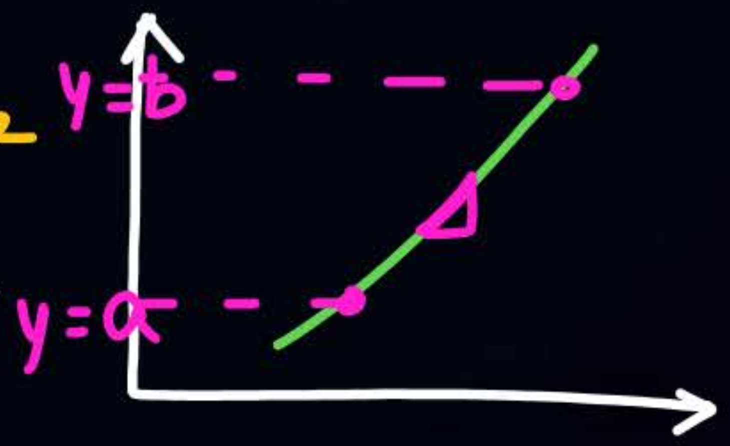
1) When eqn. of curve is in Cartesian form :-





$$dL^2 = dx^2 + dy^2$$

$$dL = \sqrt{dx^2 + dy^2}$$



$$\boxed{L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$$\boxed{L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

2) When the eqn. is in polar form:-

$$\boxed{L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta} = \boxed{\int_{r_1}^{r_2} \sqrt{1 + \left(r\frac{dr}{d\theta}\right)^2}\, dr}$$

3) When the eqn. is in parametric form:-

$(x, y) \rightarrow f(t)$

$$\boxed{L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt}$$

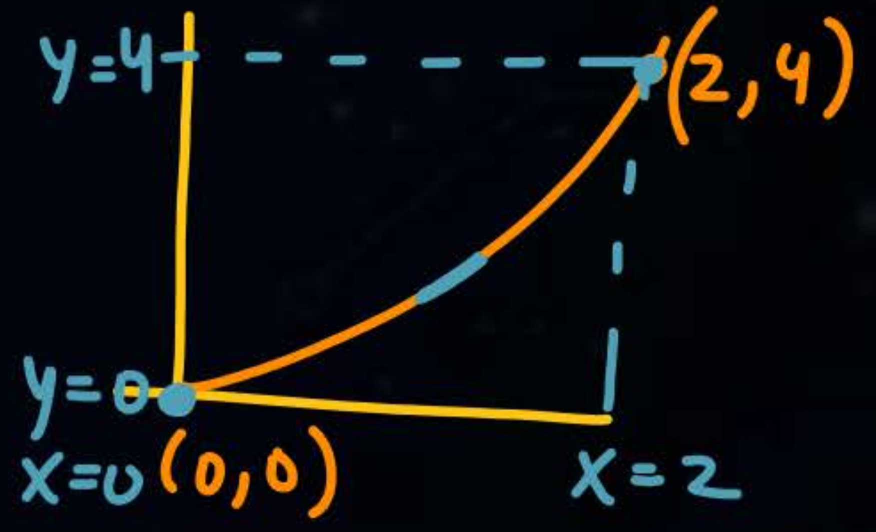
Find the length of arc of parabola $y=x^2$ b/w $(0,0)$ & $(2,4)$

Soln :-

Cartesian

$$L = \int_{x=0}^{x=2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{1+(2x)^2}\, dx$$

$$L = \int_{y=0}^{y=4} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$



Parametric

$$L = \int_{t=0}^{t=2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt = \int_0^2 \sqrt{1^2+(2t)^2}\, dt$$

$$2\sqrt{t^2+\left(\frac{1}{2}\right)^2}\, dt$$

$$\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\} (t)$$

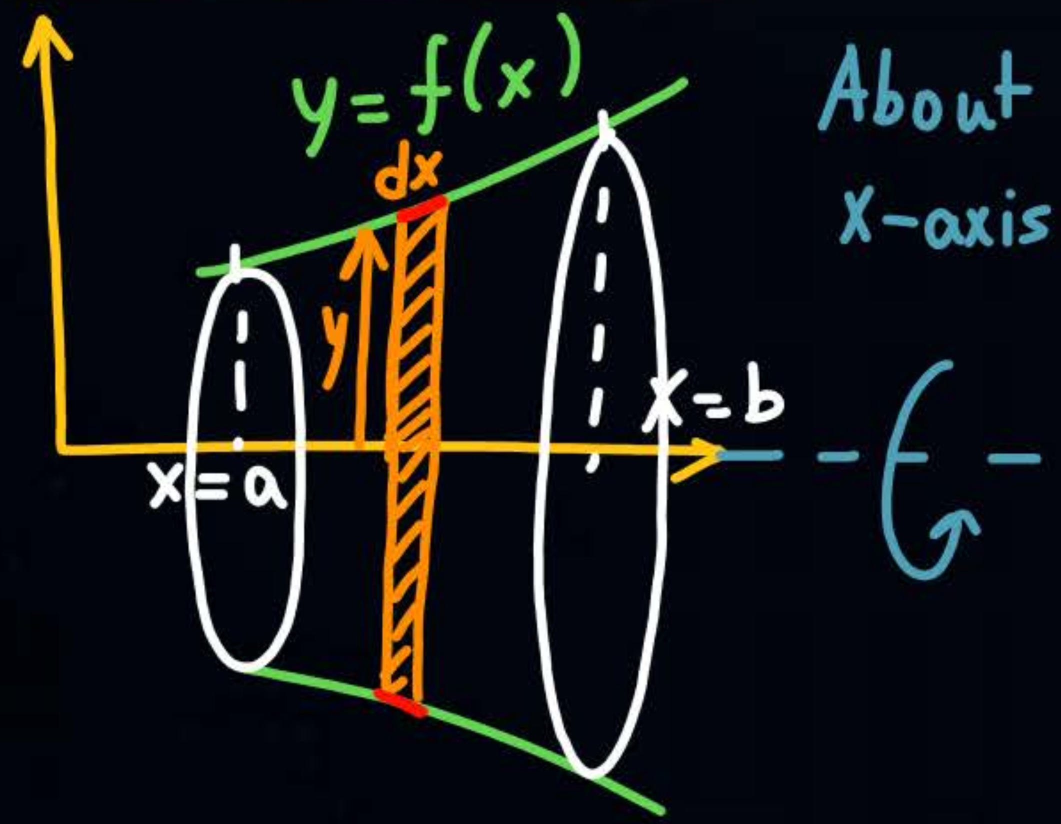
$t \to 0 \quad (0,0)$

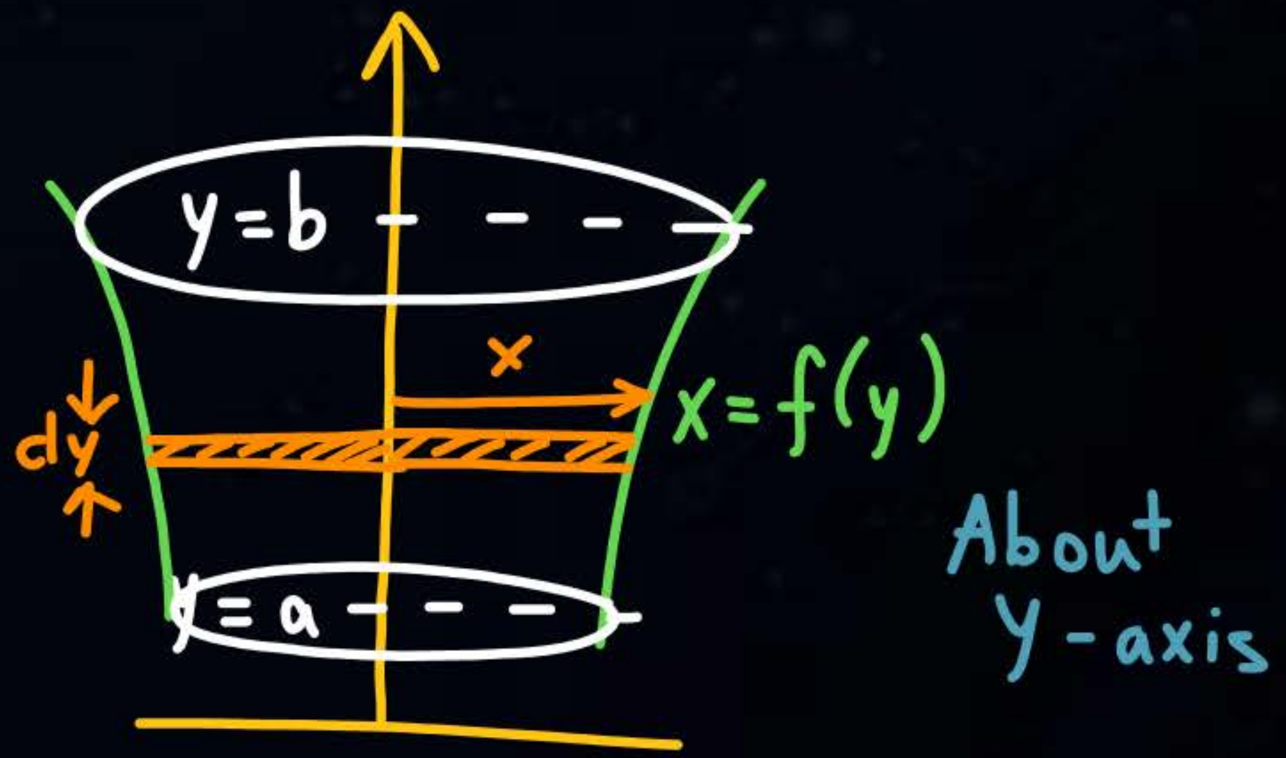
$t \to 2 \quad (2,4)$

$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

Volume of Solid of revolutions





$$\boxed{V = \int_{x=a}^{x=b} \pi y^2 \, dx}$$

About X-axis

$$\boxed{V = \int_{y=a}^{y=b} \pi x^2 \, dy}$$

About Y-axis

Surface area of revolution

Surface of solid generated by $y = f(x)$ & X-axis b/w $x = a$ & $x = b$

$$\boxed{S.A. = \int_{x_1}^{x_2} 2\pi y \, dL}$$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

Surface of solid generated by $x = f(y)$ & y-axis b/w $y = a$ & $y = b$

$$\boxed{S.A. = \int_{y_1}^{y_2} 2\pi x \, dL}$$

$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$$

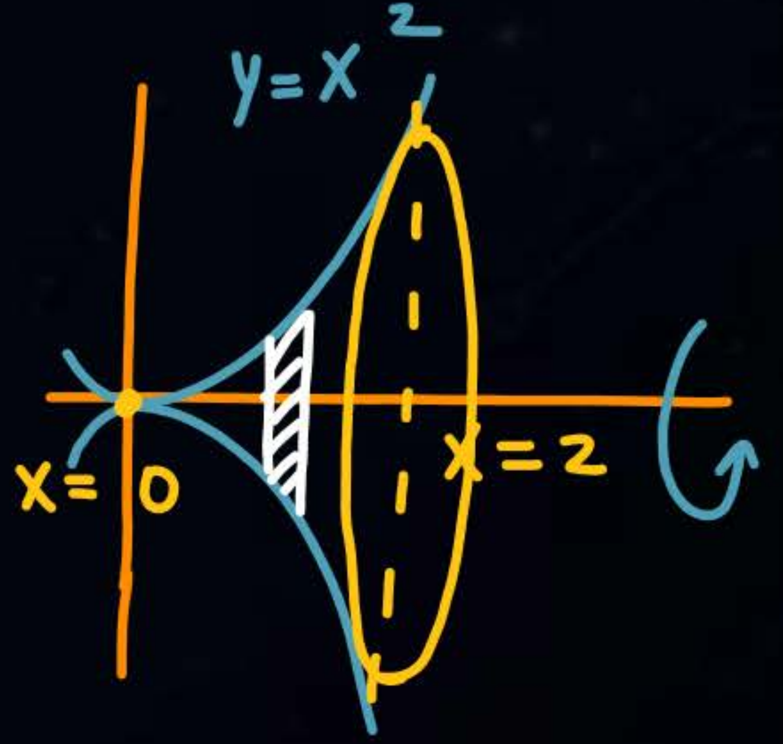
Ex:- Find vol. generated by $y=x^2$ about X-axis b/w $x=0$ & $x=2$.

Soln:-

$$\int_{x=0}^{x=2} \pi y^2 \, dx$$

$$\Rightarrow \int_0^2 \pi (x^2)^2 \, dx = \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{32}{5}\pi$$



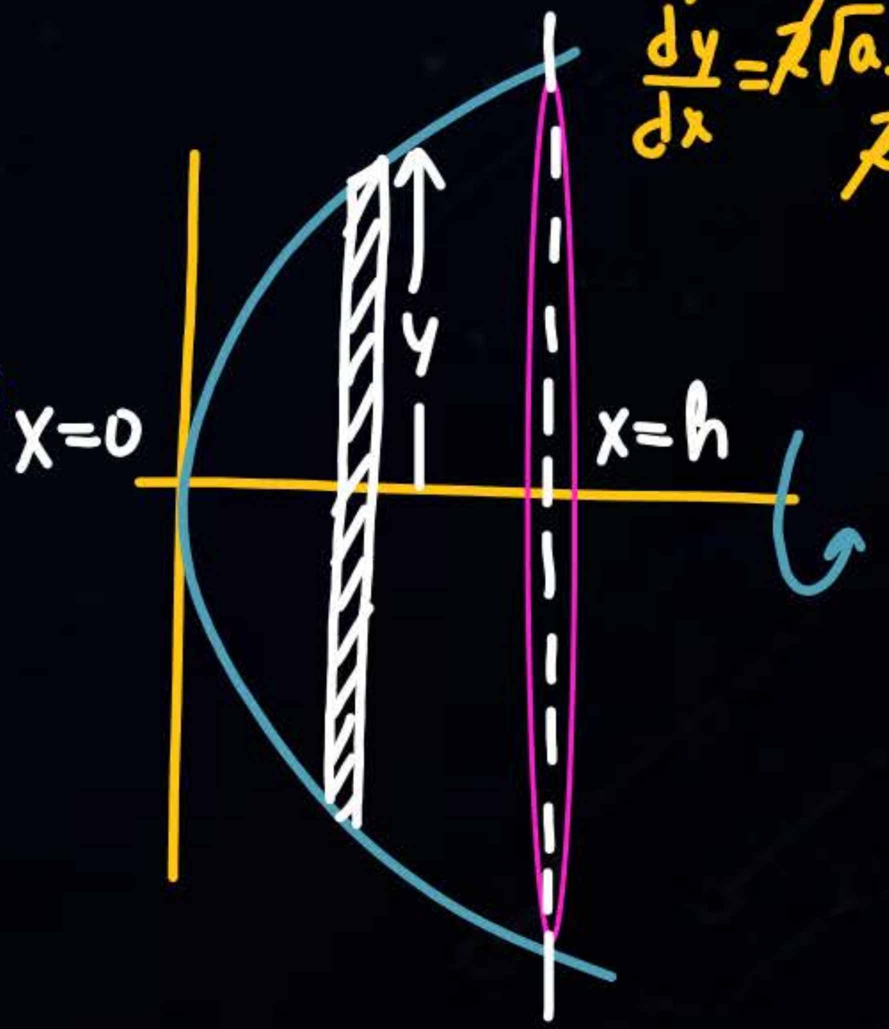
Ex:- Find the curved surface area of solid generated by revolution of $y^2 = 4ax$ about X-axis & ordinate $x = h$.

Soln:- Surface area $= \int_{x=0}^{x=h} 2\pi y \, dL$

$$\int_{x=0}^{x=h} 2\pi\sqrt{4ax}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx = 4\pi\sqrt{a}\int_0^h \sqrt{x}\sqrt{\left(1+\frac{a}{x}\right)}\,dx$$

$y = \sqrt{4ax}$

$\frac{dy}{dx} = \cancel{2}\sqrt{a}\,\frac{1}{\cancel{2}\sqrt{x}}$



$$= 4\pi\sqrt{a}\int_0^h \sqrt{x+a}\,dx = 4\pi\sqrt{a}\left[\frac{(x+a)^{3/2}}{3/2}\right]_0^h$$

$$= 4\pi\sqrt{a}\,\frac{2}{3}\left[(h+a)^{3/2} - a^{3/2}\right]$$

$$S.A. = \frac{8\pi}{3}\sqrt{a}\left[(a+h)^{3/2} - a^{3/2}\right]$$

Ex:- S.A. obtained by revolving $y = 2x$ for $x \in [0, 2]$ about y-axis

A) $2\pi\sqrt{5}$

B) $4\pi\sqrt{5}$

C) $2\sqrt{5}\pi$

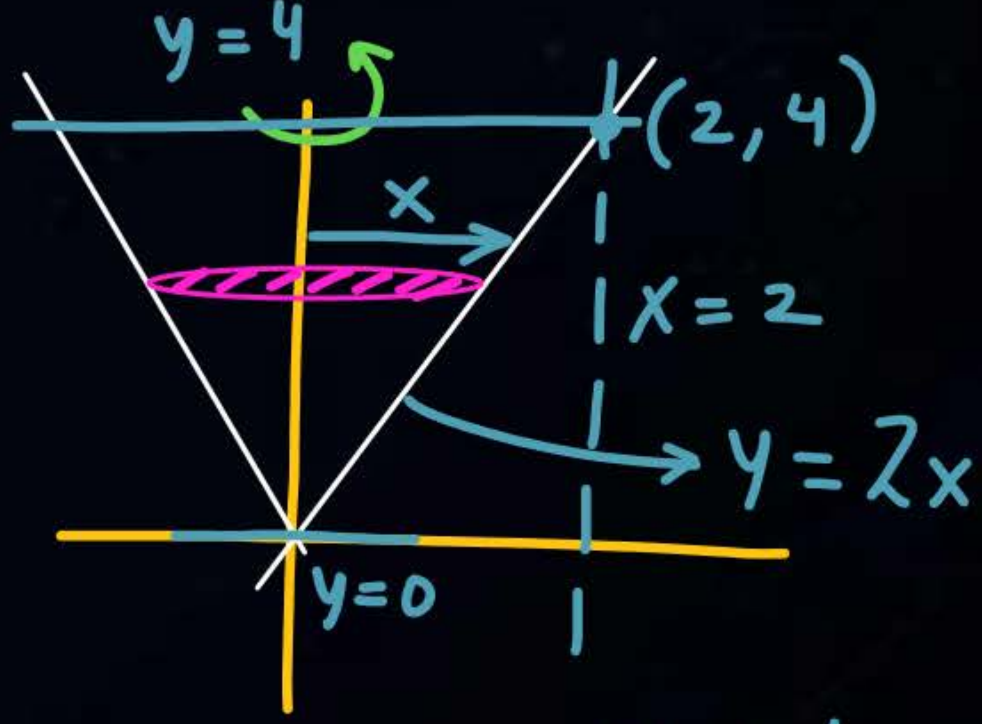
D) $4\sqrt{5}\pi$

$$\int_{y=0}^{y=4} 2\pi x \, dL$$

$$\int_0^4 \cancel{2}\pi\left(\frac{y}{\cancel{2}}\right)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy$$

$$\int_0^2 \pi y \sqrt{1+\left(\frac{1}{2}\right)^2}\cdot dy$$

$$= 4\pi\sqrt{5}$$



$x = y/2$

$dx = dy/2$

$\frac{dx}{dy} = \frac{1}{2}$

S.A. of cone $= \pi r l$

$l = \sqrt{r^2 + h^2}$

Change of Variable (Jacobian's Rule)

1. Change cartesian cordinates (2-D) $\longrightarrow$ Polar cordinates

$$(x, y) \longrightarrow (r, \theta)$$

$x = r\cos\theta$

$y = r\sin\theta$

$$dx\,dy = |J|\,dr\,d\theta$$

(x, y) (r, θ)

$dx\,dy \rightarrow r\,dr\,d\theta$

$dx\,dy \neq dr\,d\theta$

$$\boxed{I = \iint_R f(x,y)\,\boxed{dx\,dy} = \iint_R f(r,\theta)\,\boxed{r\,dr\,d\theta}}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \boxed{r}$$

Change of Variable (Jacobian's Rule)

2. Change 3-D cartesian cordinates → Spherical cordinates

$$P(x,y,z) \longrightarrow P(r,\theta,\phi)$$

$$I = \iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\theta,\phi)\,|J|\,dr\,d\theta\,d\phi$$

$$= \iiint f(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2\sin\theta$$

$$dx\,dy\,dz \rightarrow \boxed{r^2\sin\theta}\,dr\,d\theta\,d\phi$$

Change of Variable (Jacobian's Rule)

3. Change 3-D cartesian cordinates $\longrightarrow$ Cylindrical cordinates

$$P(x, y, z) \longrightarrow P(\rho, \phi, z)$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\boxed{\iiint f(x, y, z)\, dx\, dy\, dz = \iiint_R f(\rho, \phi, z)\, \boxed{\rho}\, d\rho\, d\phi\, dz}$$

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$$

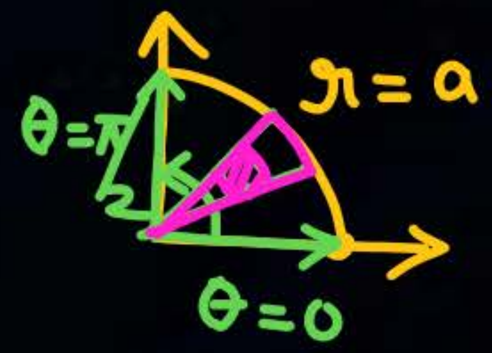
Area & Volume in Different Co-ordinate System.

1. Cartesian system; (2-D, 3-D)

$$\text{Area} = \iint_A dx\,dy \qquad \text{Vol.} = \iiint_V dx\,dy\,dz$$

2. Polar system; (2-D)

$$\text{Area} = \int_{\theta_1}^{\theta_2}\int_0^{r} r\,dr\,d\theta = \int_{\theta_1}^{\theta_2} \frac{r^2}{2}\,d\theta$$



3. Spherical cordinate; system (3-D)

$$\text{Vol.} = \iiint r^2 \sin\theta\,dr\,d\theta\,d\phi$$

4. Cylindrical cordinate; System (3-D)

$$\text{Vol.} = \iiint \rho\,d\rho\,d\phi\,dz$$

Thank you

GW Soldiers !