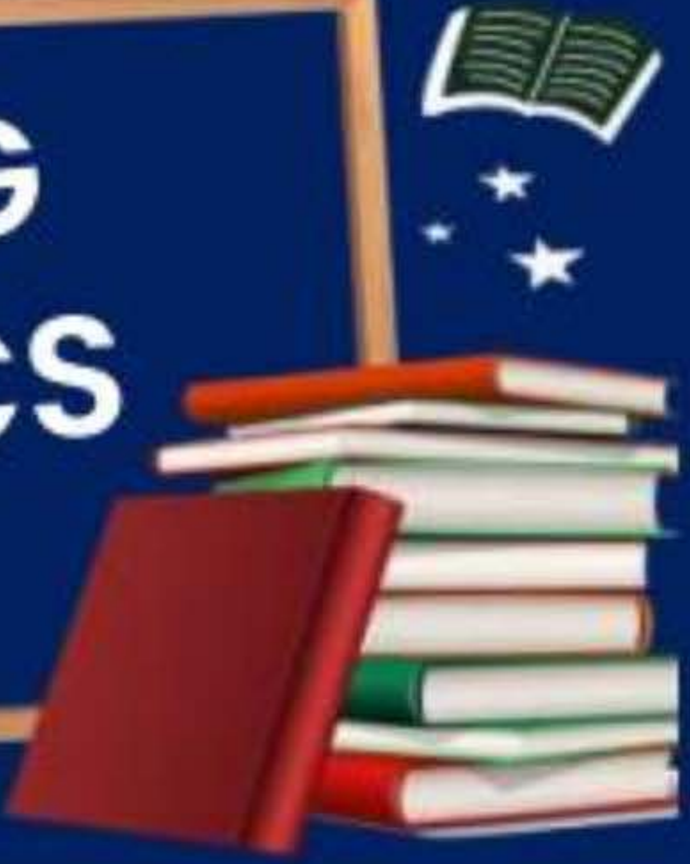
# ALL BRANCHES

# ENGINEERING MATHEMATICS



Lecture No.-4

Linear Algebra

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# Topics to be Covered

- Rank of Matrix
- Method to Finding Rank
- Elementary Transformations or E-operation
- Equivalent Matrix
- Properties of Rank
- Elementary Matrices
- Row Rank and Column Rank of a Matrix

Singular matrix ; $|A| = 0$

Non-singular matrix ; $|A| \neq 0$

Ex:-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ -1 & 2 & 0 \end{bmatrix}_{3\times 3}$$

No. of $3 \times 3$ minors $\rightarrow {}^3C_3 \cdot {}^3C_3 = 1$

No. of $2 \times 2$ minors $\rightarrow {}^3C_2 \cdot {}^3C_2 = 3 \times 3 = 9$

No of $1 \times 1$ minors $\rightarrow {}^3C_1 \cdot {}^3C_1 = 3 \times 3 = 9$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ -1 & 2 & 0 \end{vmatrix}$$

$1(0-14) + 0 - (14-15)$

$-14 + 1 = -13 \neq 0$

$\therefore$ Rank$(A) = 3$

Check :-

if any $3 \times 3$ minors ; $|A| \neq 0 \Rightarrow$ Rank$(A) = 3$

if all $3 \times 3$ minors ; $|A| = 0 \Rightarrow$ Rank$(A) < 3$

if any $2 \times 2$ minors; $|A| \neq 0 \Rightarrow$ Rank$(A) = 2$

if all $2 \times 2$ minors ; $|A| = 0 \Rightarrow$ Rank$(A) < 2$

if any $1 \times 1$ minors ; $|A| \neq 0 \Rightarrow$ Rank$(A) = 1$

if all $1 \times 1$ minors ; $|A| = 0 \Rightarrow$ Rank$(A) < 1$

## Find the rank of matrix :-

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{bmatrix}$

$\rightarrow$ Check 3X3 minors ; $|A|_{3\times3} = 0 \quad \therefore \rho(A) < 3$

$\rightarrow$ Check 2x2 minors ; $|A|_{2\times2} = 0 \quad \therefore \rho(A) < 2$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 3 \\ 5 & 15 \end{vmatrix} = 0 \qquad \begin{vmatrix} 4 & 6 \\ 10 & 15 \end{vmatrix} = 0$

$\rightarrow$ Check 1X1 minors ; $|A|_{1\times1} \neq 0 \quad \therefore \boxed{\rho(A) = 1}$

**NOTE :-** When all rows/columns are proportional then Rank is always 1.

b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 0 & -9 \end{bmatrix}$

$|A|_{3\times3} = 0$

$\rightarrow$ Check 3X3 minors ; $|A|_{3\times3} = 0 \quad \therefore \rho(A) < 3$

$\rightarrow$ Check 2x2 minors ; $|A|_{2\times2} \neq 0 \quad \therefore \boxed{\rho(A) = 2}$

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \qquad \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} = -12 \neq 0$

Rank = 2

# [DEFINITION OF RANK]

A number r is said to be rank of a matrix A if

(i) There exist at least one square submatrix of order r which is non singular.

(ii) Every square submatrix of order (r + 1) is singular.

OR

The rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

* Rank → Order of largest non-zero minor (non-singular)

# DEFINITION OF RANK

H.W.

Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

## DEFINITION OF RANK

$4$ → Out of 4 columns
$C_3$ → Select 3 at a time

Find the rank of the matrix. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}_{3\times 4}$

No. of $4\times4$ minors $= 0$
No. of $3\times3$ minors $= {}^3C_3 \cdot {}^4C_3 = 1\times4 = 4$
No. of $2\times2$ minors $= {}^3C_2 \cdot {}^4C_2 = 3\times6 = 18$
No. of $1\times1$ minors $= {}^3C_1 \cdot {}^4C_1 = 3\times4 = 12$

i) Check all $3\times3$ minors

$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 0$ $\quad \begin{vmatrix} 1 & 2 & 4 \\ 1 & 4 & 8 \\ 3 & 6 & 12 \end{vmatrix} = 0$ $\quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 8 \\ 6 & 9 & 12 \end{vmatrix} = 0$ $\quad \begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 8 \\ 3 & 9 & 12 \end{vmatrix} = 0$

ii) Check all $2\times 2$ minors $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$ $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 0$

$\therefore |A|_{2\times 2} \neq 0 \Rightarrow \text{Rank}(A) = 2$

$$\boxed{\rho(A) = 2}$$

Ex :-

$$\begin{bmatrix} 1 & -1 & 7 & 9 \\ 5 & 2 & -5 & 8 \\ 0 & 3 & 4 & 7 \\ 6 & 1 & -3 & -2 \end{bmatrix}_{4\times 4}$$

No. of $4\times 4$ minors $= {}^4C_4 \cdot {}^4C_4 = 1$

No. of $3\times 3$ minors $= {}^4C_3 \cdot {}^4C_3 = 4\times 4 = 16$

No. of $2\times 2$ minors $= {}^4C_2 \cdot {}^4C_2 = 6\times 6 = 36$

No. of $1\times 1$ minors $= {}^4C_1 \cdot {}^4C_1 = 4\times 4 = 16$

# ELEMENTARY TRANSFORMATIONS OR E-OPERATION

1. $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

   Interchange any two rows/columns.

2. $R_i \rightarrow KR_i$ or $C_i \leftrightarrow KC_i$

   Multiplying any row/column by a number.

3. $R_i \rightarrow R_i + KR_j$ or $C_i \rightarrow C_i + KC_j$

These transformations will not change the rank.

# EQUIVALENT MATRICES

Two matrices A and B are said to be equivalent, A-B, if the matrix B can be obtained from the matrix A by applying the elementary transformation to A, and A can be obtained from B by applying the elementary transformation to B.

NOTE: The equivalent matrices have the same rank, i.e., the rank of matrix remains uncharged by E-transformation.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -1 & 6 \end{bmatrix}} \xrightarrow{R_2 \to 5R_2} \underset{B}{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 5 & 15 \\ 0 & -1 & 6 \end{bmatrix}} \qquad A \sim B$$

# [METHODS OF FINDING RANK]

1. Echelon form Method (Triangular form)
2. Normal form Method (Canonical form)

ECHELON FORM :-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

a) Row Echelon form (zeroes move - Bottom left)

b) Column Echelon form (zeroes move - Top Right)

Rank → No. of non zero rows in Row echelon form
→ No. of non zero columns in Column echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 6 & 3 & 3 \end{bmatrix}$$

# [ECHELON FORM / TRIANGULAR FORM]

A matrix A is said to be in echelon form if

(a) All the zero rows occurs below non zero rows i.e. all the zero rows occurs at the bottom of matrix. (optional)

(b) No. of zeros before the first non-zero element in a row is less than no. of such zeros in the next row. (Compulsory)

$$\begin{bmatrix} 2 & 1 & 4 & 1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 2 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$$

No. of zero

$$\begin{array}{c} \begin{bmatrix} \boxed{2} & 1 & 4 & 1 \\ 0 & \boxed{1} & 0 & 7 \\ 0 & 0 & \boxed{4} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 4 \end{array} \checkmark$$

$$\begin{bmatrix} 2 & 4 & 1 & 2 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ 1 \\ 4 \end{array} \qquad \begin{bmatrix} 2 & 4 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 0 \\ 3 \\ 4 \end{array}$$

a ✓    b ✓ ✓      a ✓    b ✓ ✓

# [ECHELON FORM / TRIANGULAR FORM]

row

Reduce the matrix in echelon form and hence find its rank.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix}$$

How to convert into echelon form :-

i) Fix $a_{11}$ & makes 0s below it using E-row operations.

ii) Fix $a_{22}$ & " " " " " " "

iii) Fix $a_{33}$ & " " " " " " "

until a and b are satisfied.

$$A = \begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ -1 & 1 & -3 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & -1 & 3 & 3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & \boxed{2} & -4 & 4 \\ 0 & -1 & 2 & 1 \\ 0 & -2 & 4 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 + R_2/2} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_4 \to R_4 - 2R_3}$$

(Row echlon form)

a ✓ b ✓

No. of non-zero rows $= \rho(A)$

$\therefore \rho(A) = 3$

No. of zero rows = 1

$\therefore$ Nullity = 0

# ECHELON FORM / TRIANGULAR FORM

H.W.

Find the rank of a matrix. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$

## [ECHELON FORM / TRIANGULAR FORM]

H.W.

Find the rank of the following matrix. $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

# ECHELON FORM / TRIANGULAR FORM

Find the rank of the following matrix. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}_{3 \times 3}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = 2R_1$ ✓

$R_3 = -3R_1$ ✓

a ✓

b ✓

Row echelon form

No. of non. zero rows = 1

$\therefore \rho(A) = 1$

No. of zero rows = 2

$\therefore$ Nullity = 2

# NORMAL FORM OF A MATRIX (CANONICAL FORM)

By performing E-transformations any matrix can be reduced to following form :-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} I_2 & 0 \end{bmatrix} \quad \begin{bmatrix} I_3 \end{bmatrix}$$

How to convert into normal form :-

i) Make $a_{11}$ unity & fix it.
ii) Make 0s below it and to right side of it
iii) Make $a_{22}$ unity & fix it.
iv) Make 0s below it and to right side of it unit it reduces to

# NORMAL FORM OF A MATRIX (CANONICAL FORM)

Find the rank of matrix. $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ by reducing it to normal form.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -5 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[C_4 \to C_4 - 3C_1]{\substack{C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -5 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -5 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[\text{\& Row operations \& column operations}]{\text{Make } a_{22}=1} \cdots\text{Proceed}\cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1 2 3 4 5 6

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Order of identity matrix = 3

$\therefore \rho(A) = 3$

# [NORMAL FORM OF A MATRIX (CANONICAL FORM)]

H. W.

Express the following matrix into normal form and find its rank:

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Rank =
- → Order of largest non-zero minor.
- → No. of non-zero rows in Row Echelon form
- → Order of Identity matrix in normal form

# Thank you

# GW Soldiers !