CS & IT ENGINEERING

DISCRETE MATHS
SET THEORY

Lecture No. 10

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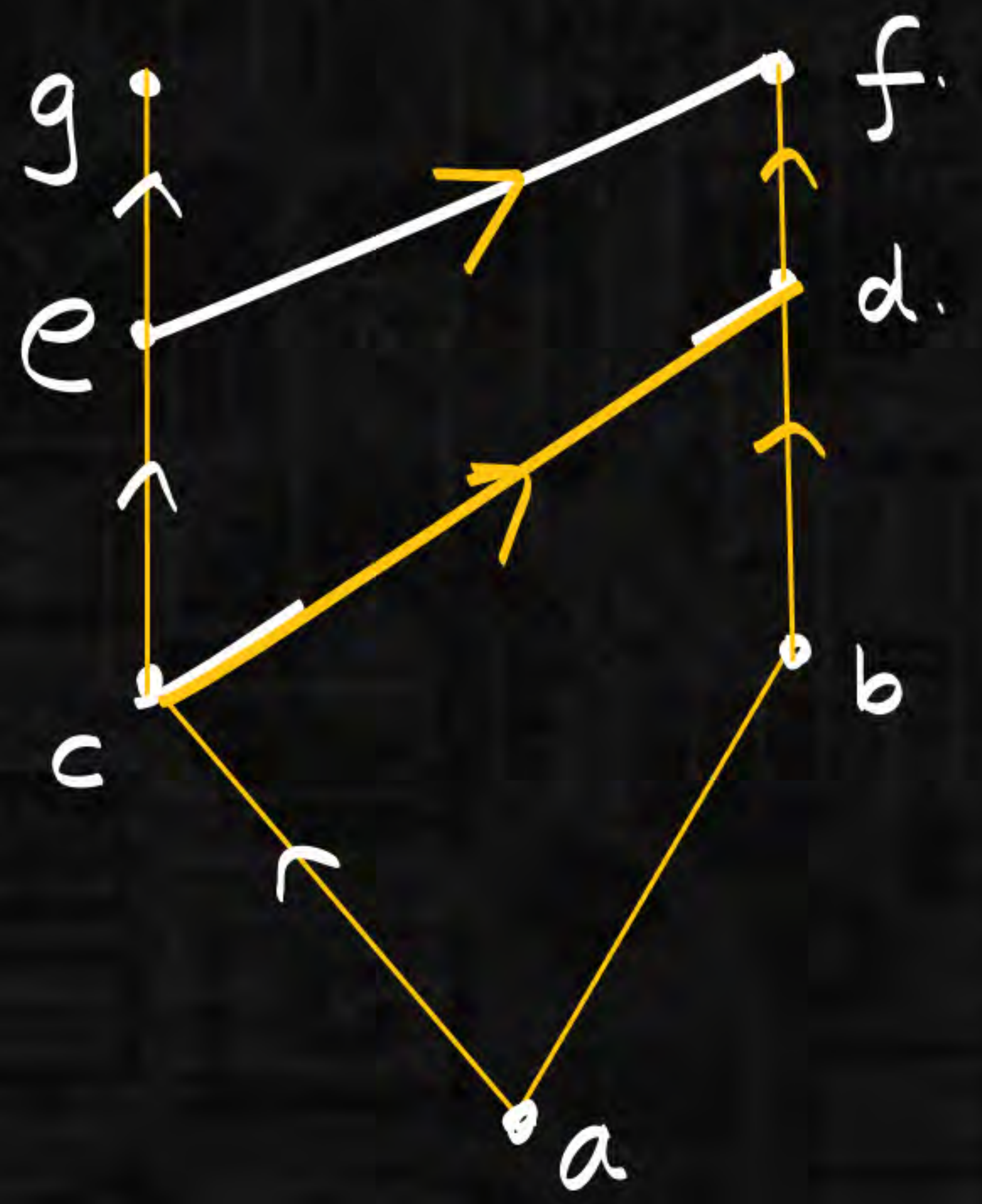
TOPICS

01 Partial order relation

02 Poset /toset

3 lattice



(A, R) poset

Greatest element (GE)

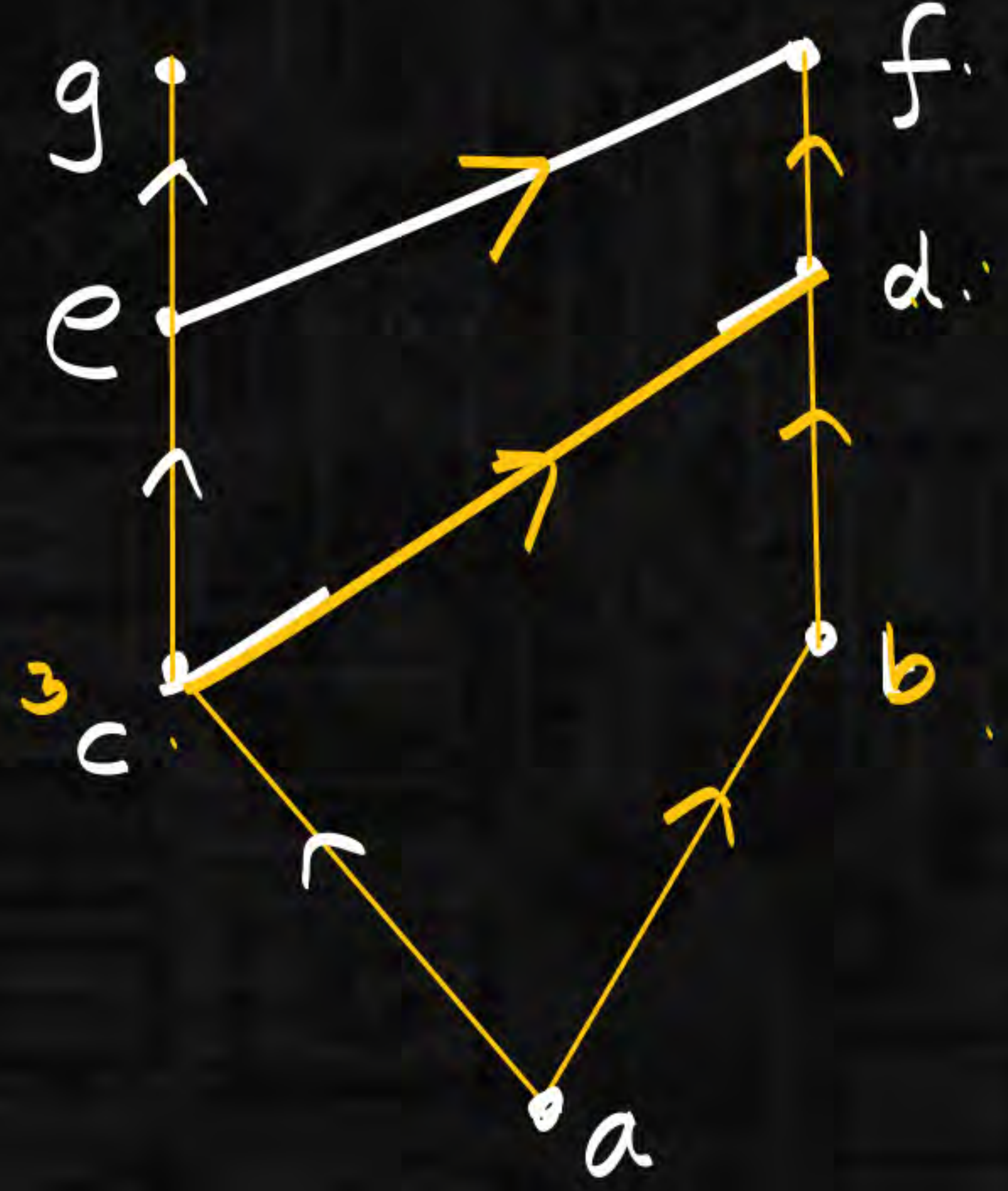
x is called Greatest element of (A, R)

all elements $\leq x \in A$

check for g : g is not GE.

abcdefg $\leq g$

$a \leq g$ (T) $b \leq g$ (False)



(A, R) poset

Greatest element (GE)

x is called Greatest element of (A, R)

all elements $\leqslant x \in A$

check for f: [f is also not GE]

abcdefg $\leq$ f

$a \leq f$ (T) $c \leq f$ (T) $e \leq f$ (T)

$b \leq f$ (T) $d \leq f$ (T) $f \leq f$ (T)

$g \not\leq f$ (false)

$(a, b) \in R$ OR  OR aRb

$a \leq b$

aRb

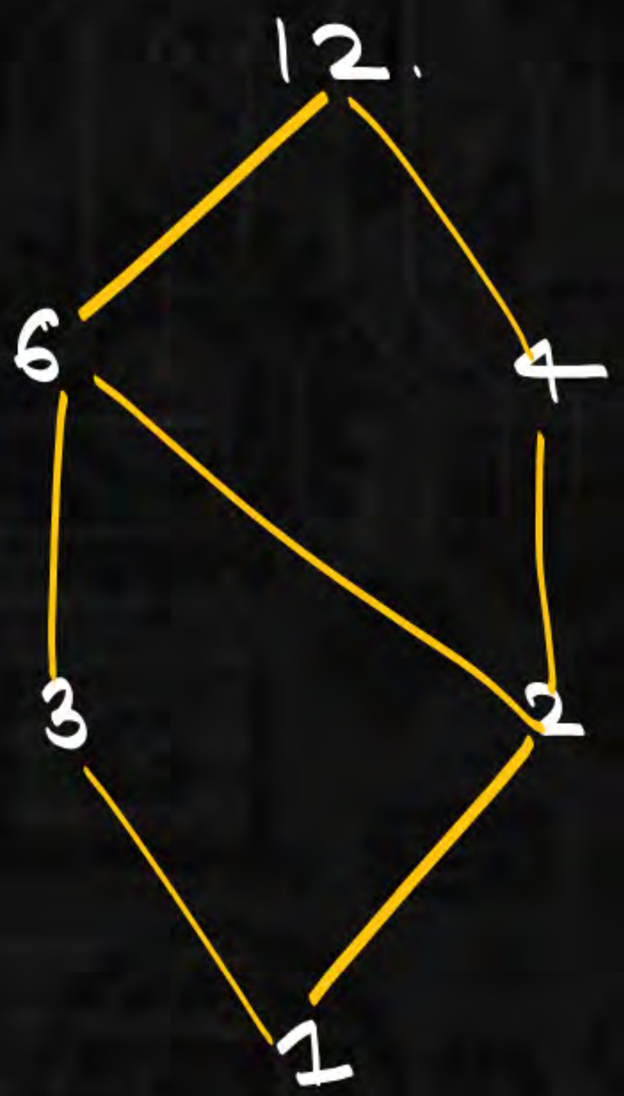
$a \leq b$

$\uparrow \qquad \uparrow$

a will be @ lower level

b will be @ higher level

$(D_{12}, |)$



Greatest element:

all elements $\leq x \in A$ $[12 \text{ is G.E}]$

Check for 12:

$1\ 2\ 3\ 4\ 6\ 12 \leq 12$

$1 \leq 12$ $\quad$ $3 \leq 12$

$2 \leq 12$ $\quad$ $4 \leq 12$

$6 \leq 12$

$12 \leq 12$

Thm: if Greatest element exist then it will be unique.

Assume: x_1, x_2 are G.E

x_1 & x_2 will be unique.

$x_1 = G.E$

$x_2 = G.E$

all elements $\leq x_1$ $\wedge$ all elements $\leq x_2$

$x_2 \leq x_1 \wedge x_1 \leq x_2$.

assumption is wrong.

no x_1, x_2 are diff

$$x_2 \leq x_1 \wedge x_1 \leq x_2 \longrightarrow x_1 = x_2$$

$\boxed{x_1 = x_2}$

$A = \{1, 2, 3\}$

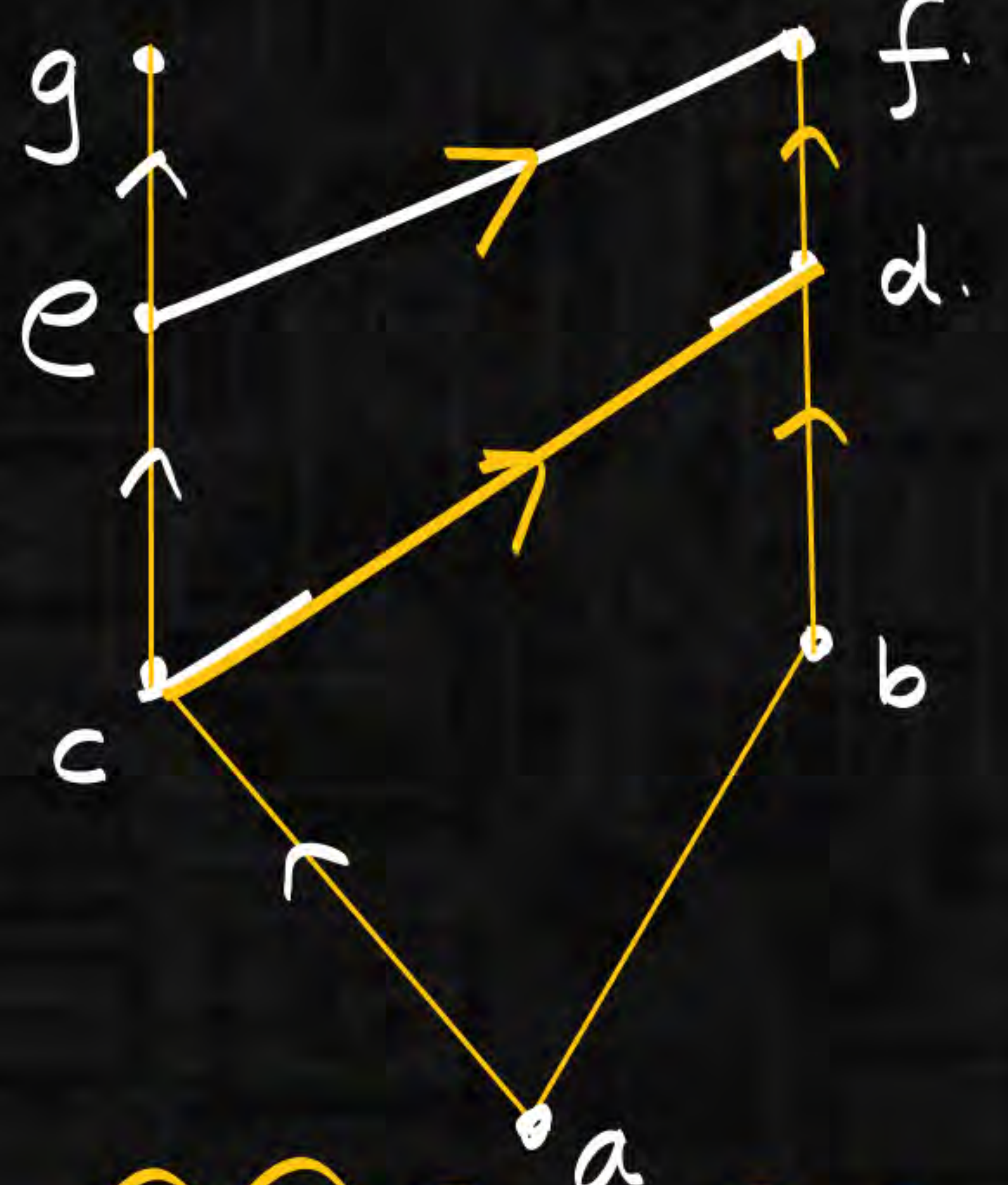
$(2^A, \subseteq)$

OR

$(P(A), \subseteq)$

G.E : {1, 2, 3}

$\boxed{\text{G.E : A.}}$



(A, R) poset

least element (minimum element)

x is called least element

$x \in A \leq$ all elements of A.

$a \leq$ all

Check for a:

$a \leq abcdefg$

$a \leq a$ ✓ $a \leq d$ ✓

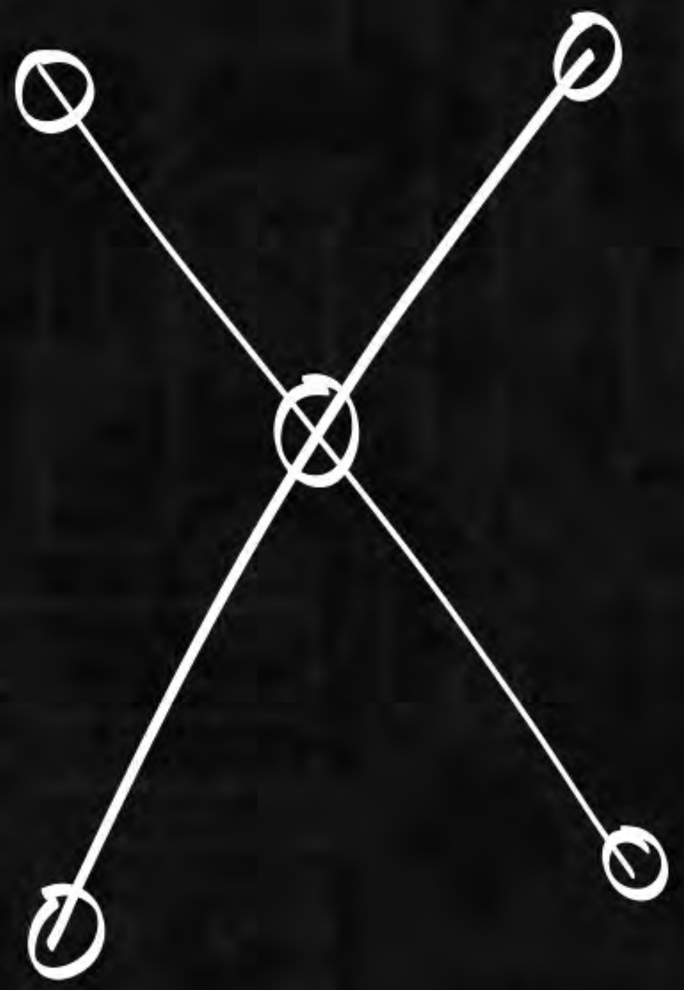
$a \leq b$ ✓ $a \leq e$ ✓

$a \leq c$ ✓ $a \leq f$

$a \leq g$

if LE exist it will be unique.

a is least element

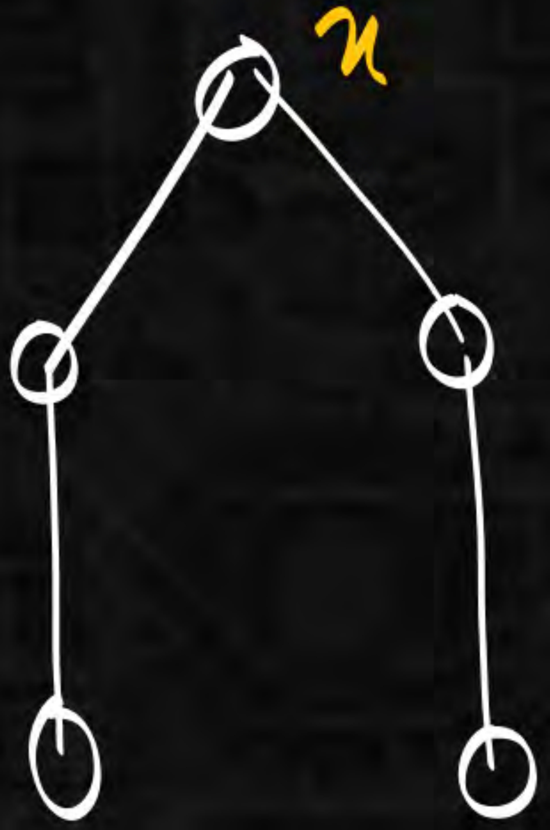


L.E: X

GE: X

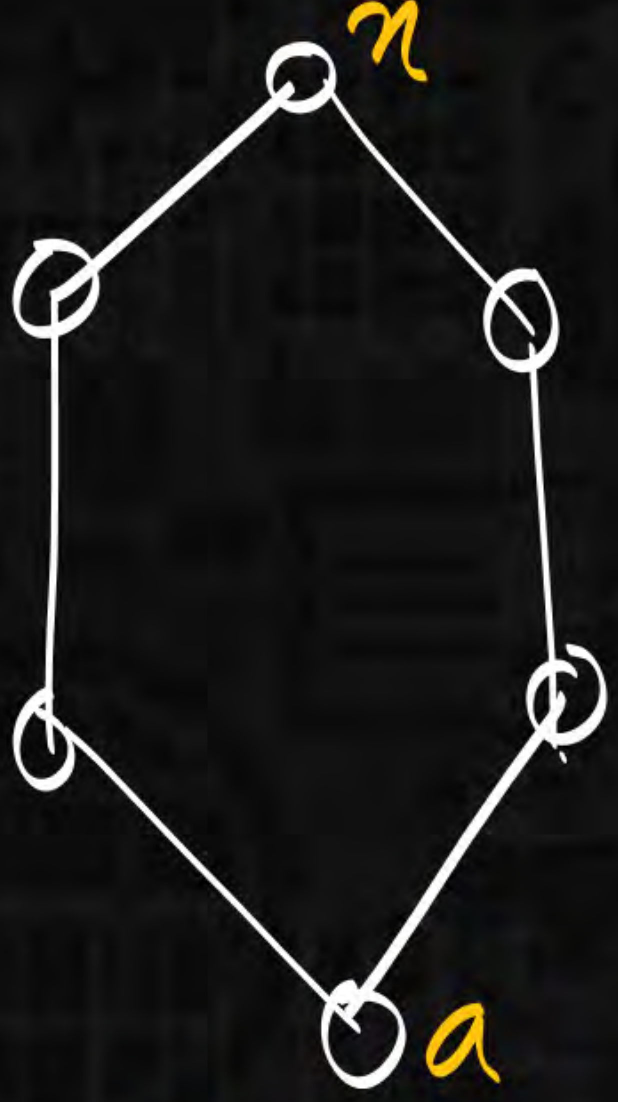
LE: a

GE: X.



LE: X

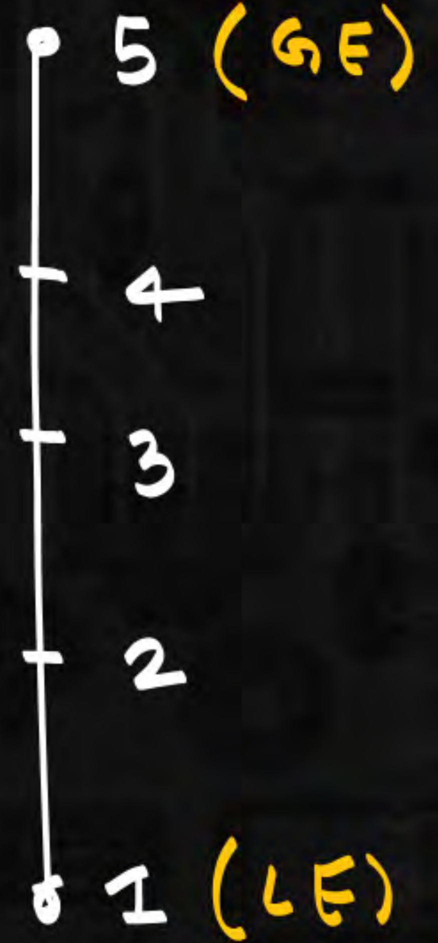
GE: x(exist)



LE: a

GE: x.

$(\{1, 2, 3, 4, 5\}, \leq)$



$(Z^+, \leq)$

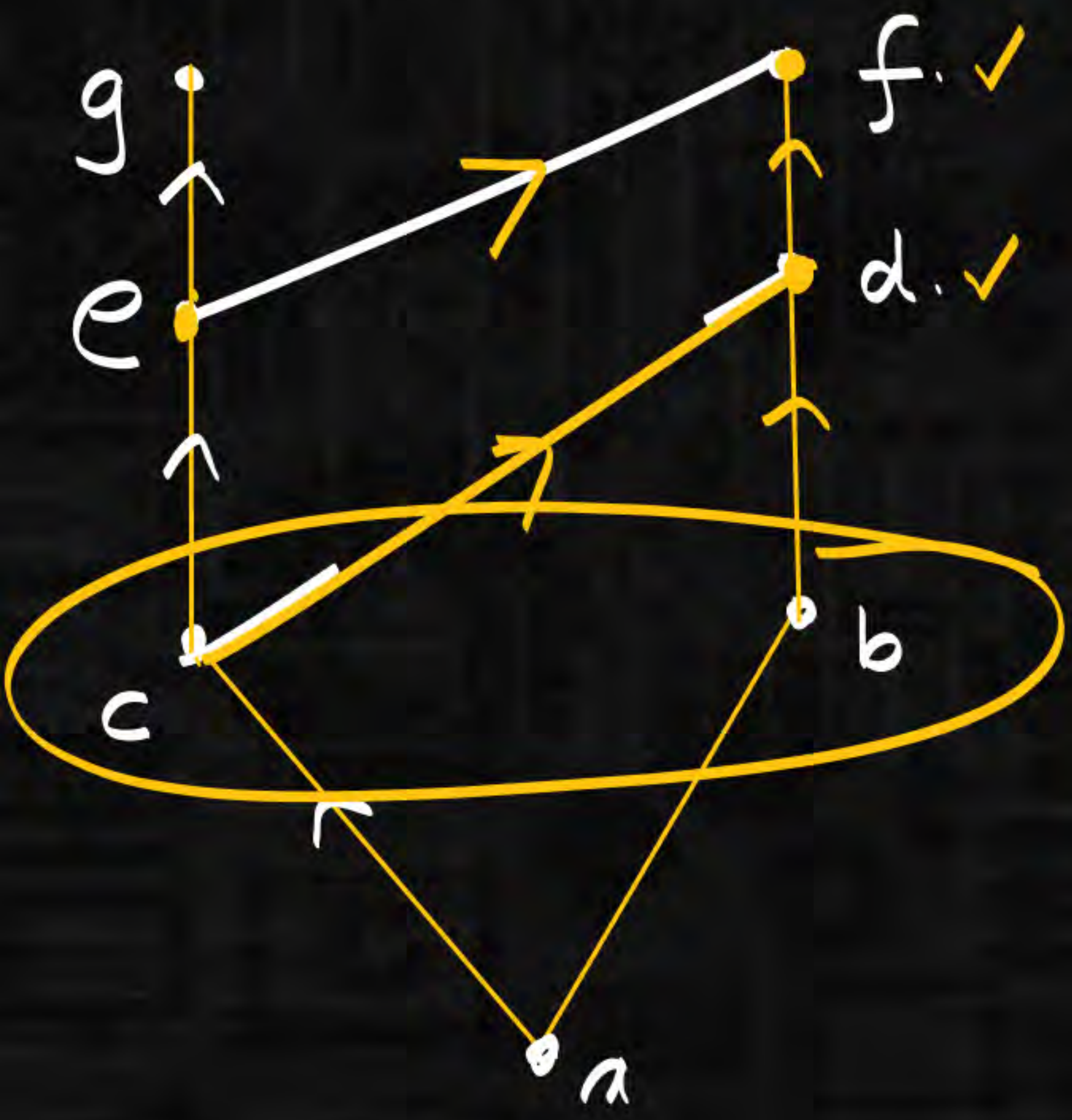
LE: 1.

GE: NA

$(Z, \leq)$

LE: NA

GE: NA.



(A, R) poset.

$B \subseteq A$

upper bound:

x is called upperbound of B.

$$\boxed{\text{all elements} \in B \leq x \in A.}$$

$B = \{b, c\}$

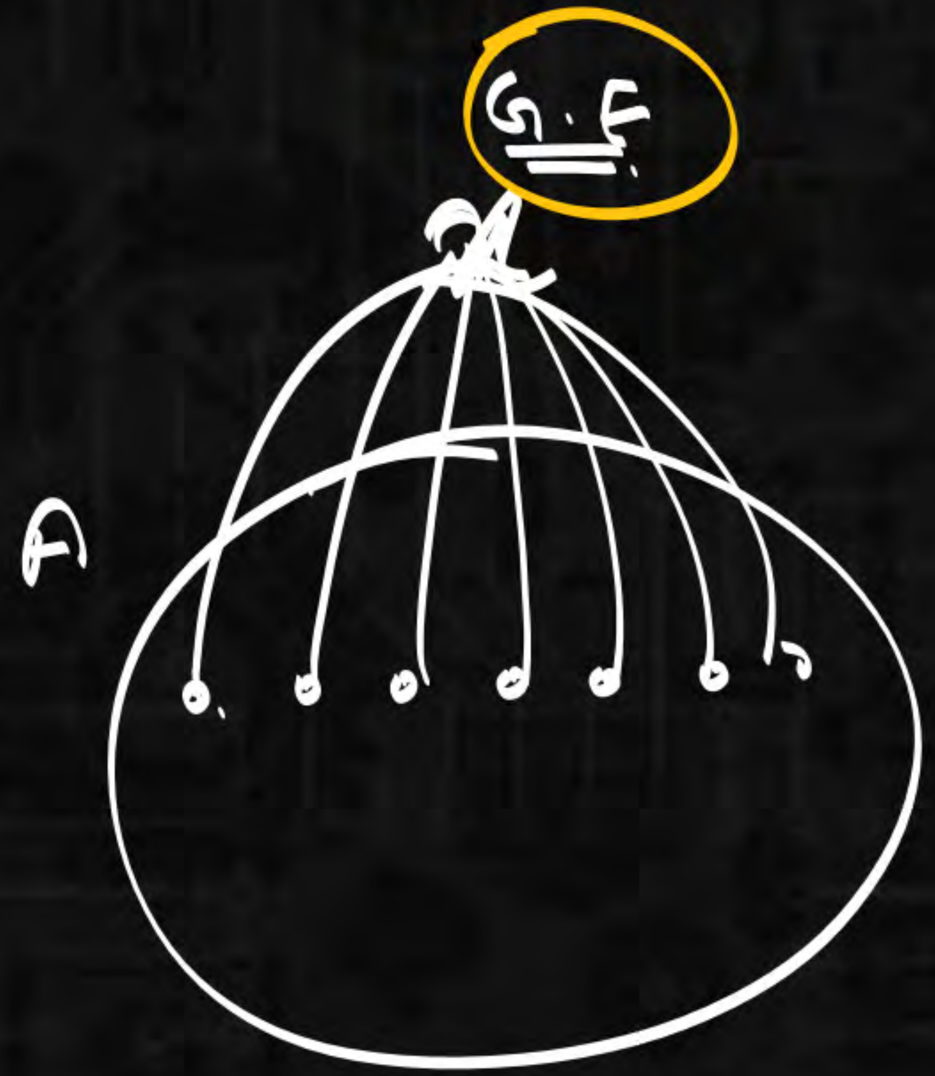
$bc \leq d$	$bc \leq f$
$b \leq d$	$b \leq f$
$c \leq d$	$c \leq f$

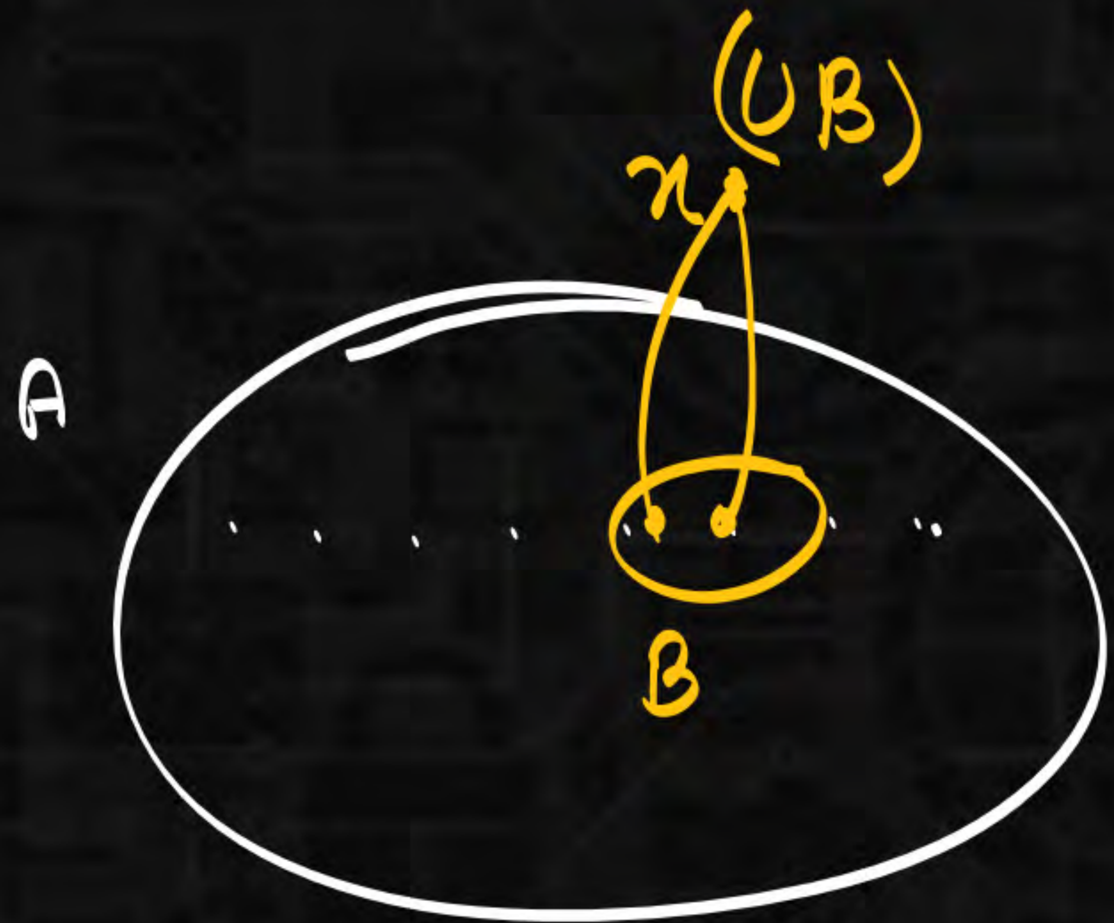
$bc \leq e$

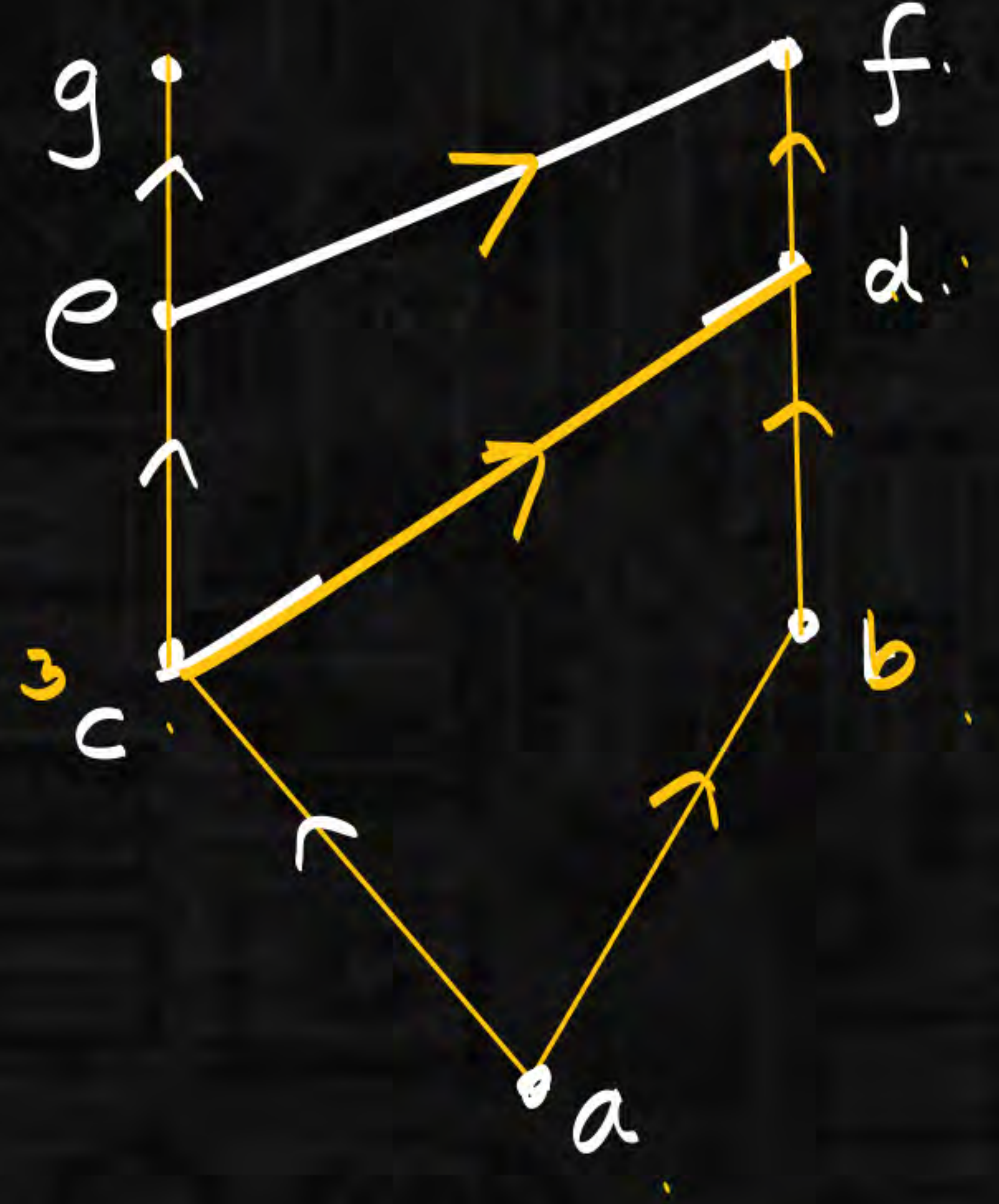
$c \leq e$ (T)

$b \leq e$ (false)

UB's of $\{b, c\}$ is $\{d, f\}$


G.F
x
A


(UB)
x
A
B



$B_1 = \{e, b\}$ UB's ? Ans: f

$B_2 = \{c, d\}$ UB's <u>Ans</u>: $\{d, f\}$

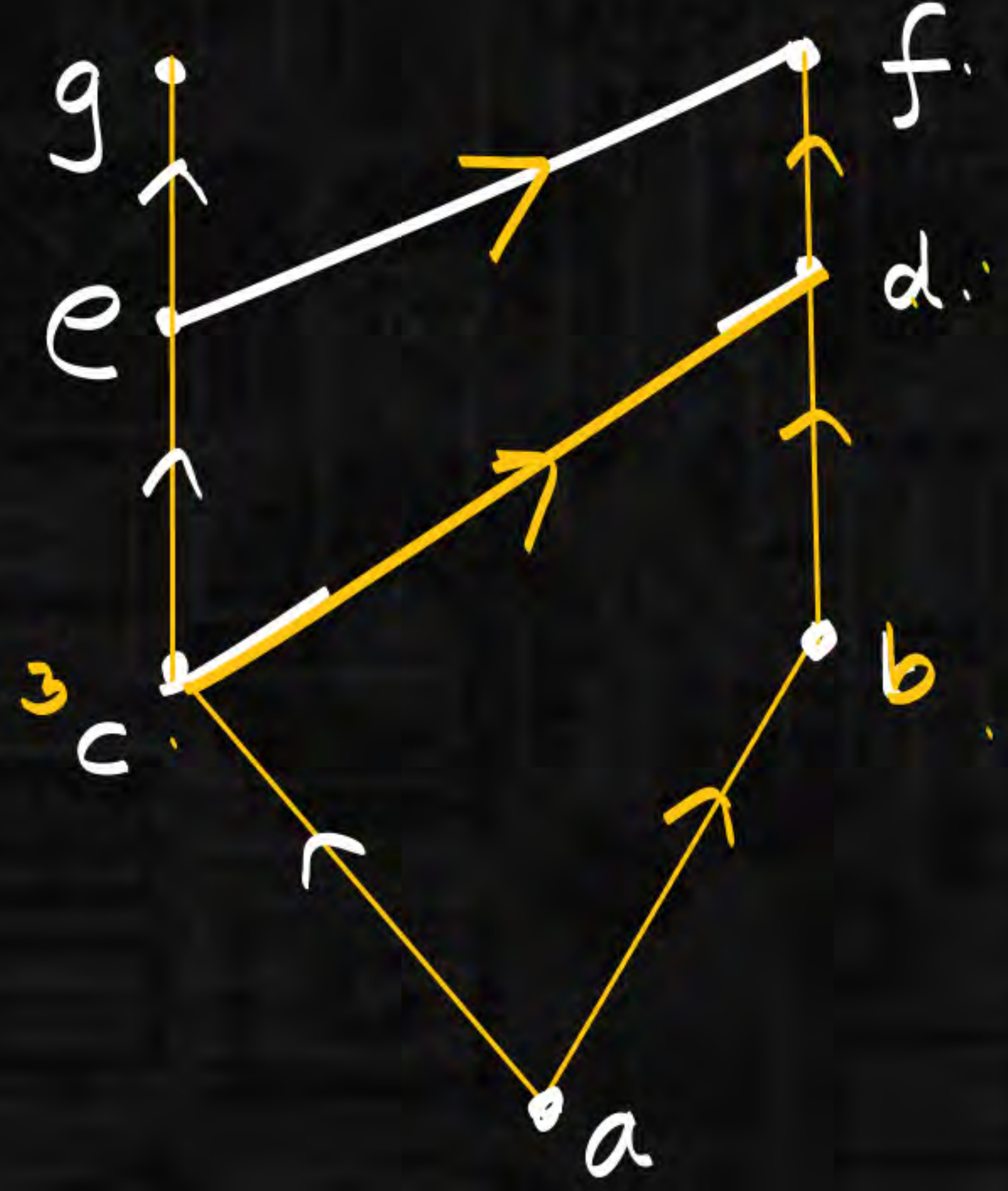
$B_3 = \{a, g\}$ UB's Ans: g

$B_4 = \{b, g\}$ UB's $\rightarrow$ N.A

$a\,g \leq c$

$a \leq c$ (T)

$g \leq c$ (F)

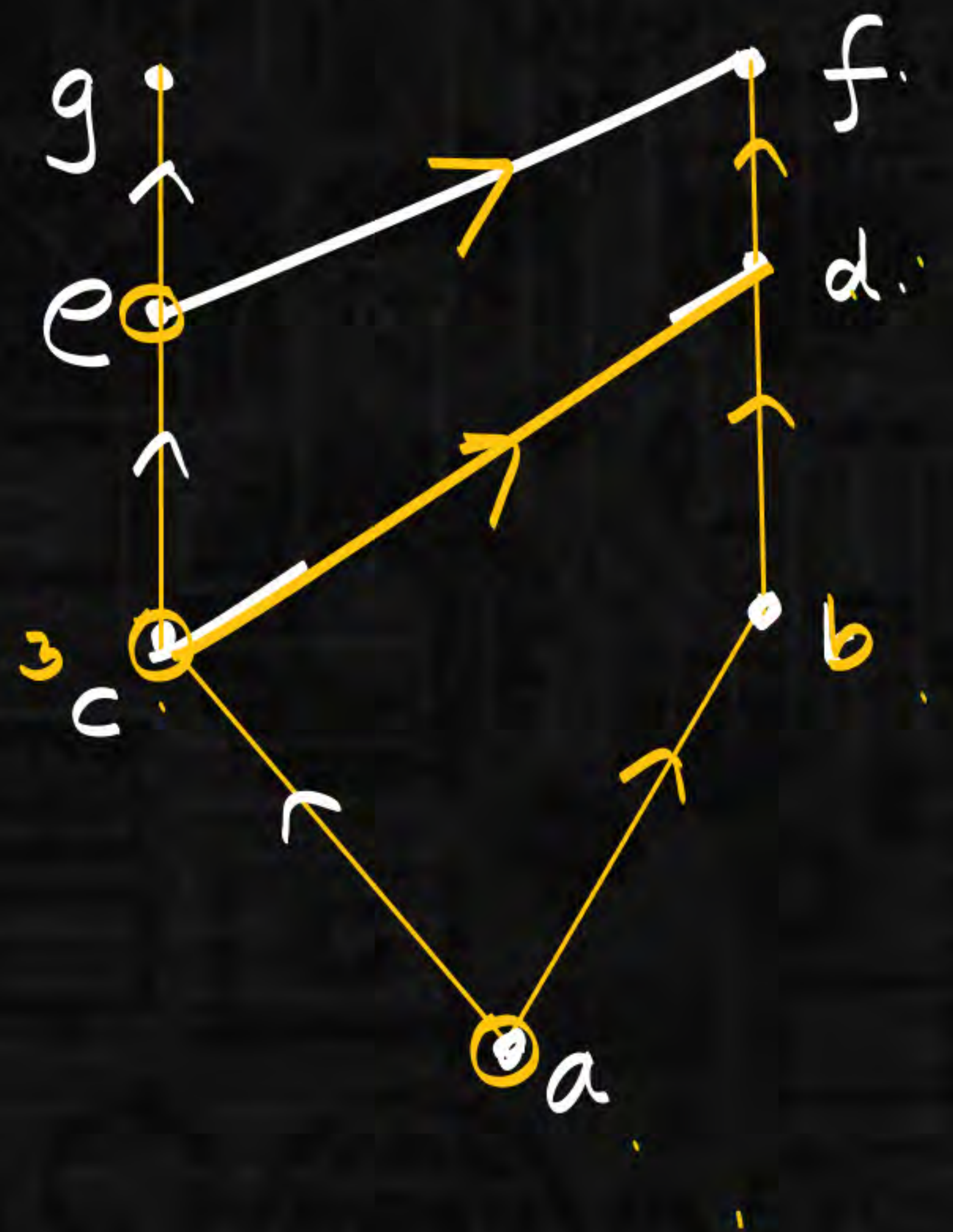


$B = \{a, b\}$ what will be upperbound's?

Ans: $\{b, f, d\}$

allelements $\in B \leq x \in A$

$ab \leq d$	$ab \leq f$	$ab \leq b$
$a \leq d$ ✓	$a \leq f$ ✓	$a \leq b$ (T)
$b \leq d$ ✓	$b \leq f$	$b \leq b$ (T)



(A, R) poset

$B \subseteq A$.

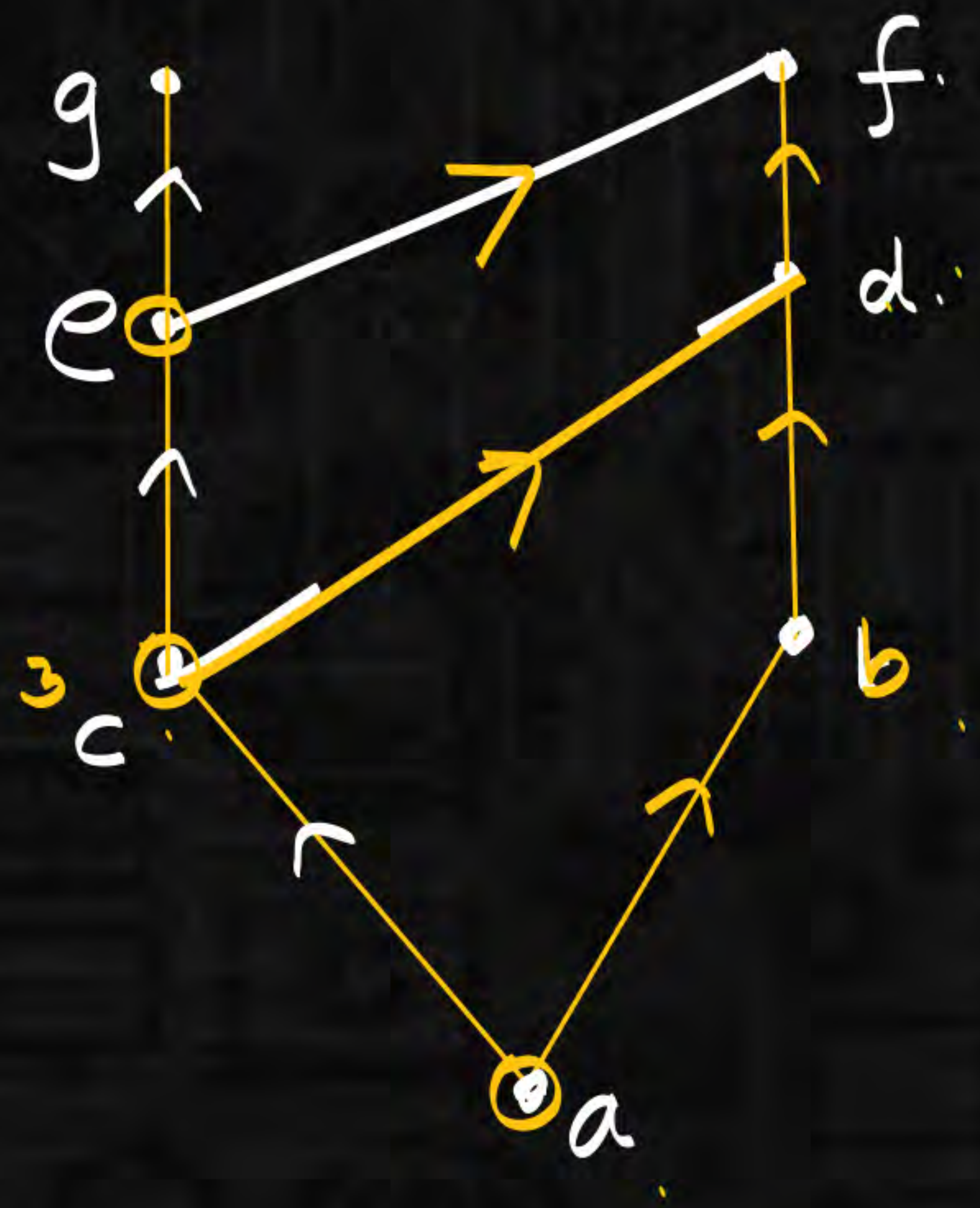
lower bound.

x is called lower bound.

$\boxed{x \in A \leq \text{all elements of } B}$

$B = \{e, f\}$ LB's of B is $\{e, c, a\}$.

$e \leq ef \mid c \leq ef \mid a \leq ef$

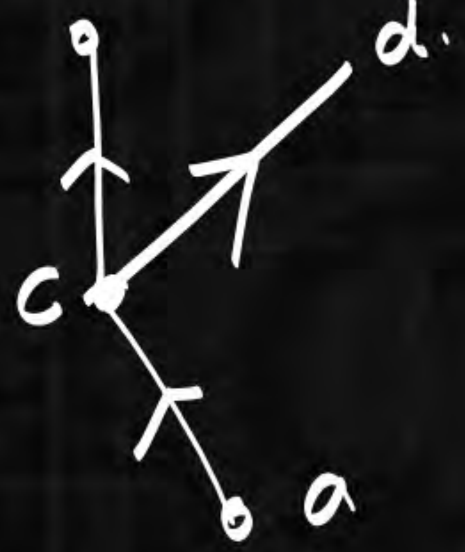


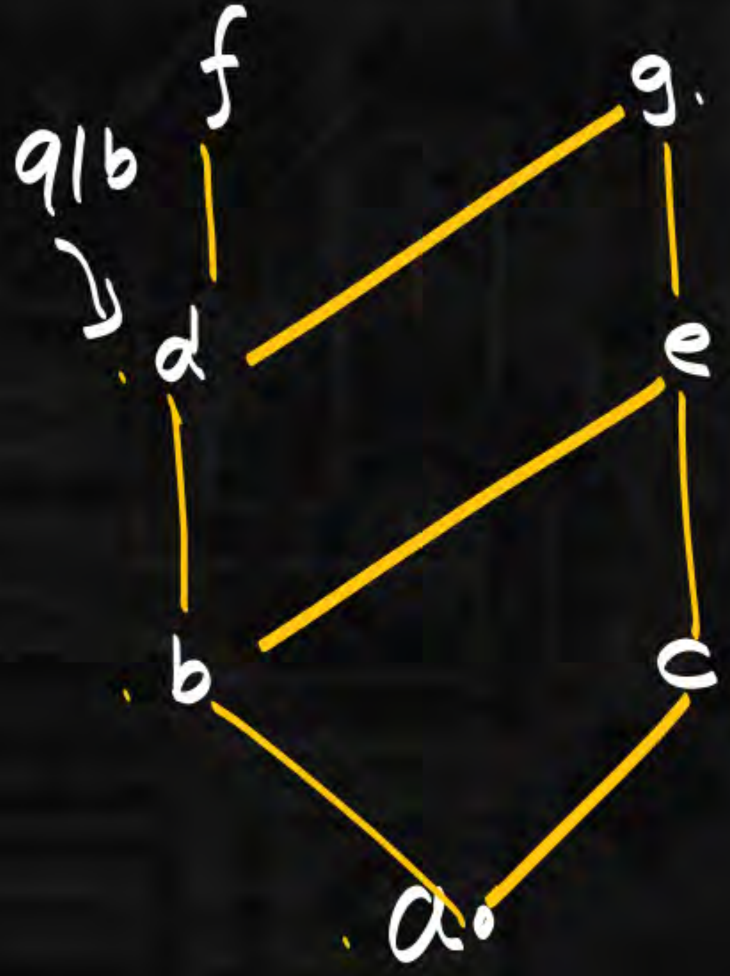
$\boxed{x \in A \leq \text{all elements} \in B.}$

$B_1 = \{g.f\}$ $\quad$ $B_2 = \{e\,b\}$ $\quad$ $B_3 = \{g, d\}$

LB $\quad$ LB $\quad$

Ans: $\{e, c, a\}$ $\quad$ Ans: a. $\quad$ Ans: $\{c, a\}$.





GLB: (~~Greatest~~ lower bound)

$x \in A \leq$ all elements of B

glb will be unique.

all lower bounds of B $\leq x \in A$

$B = \{f, g\}$

lower bound: $\{d, b, a\}$

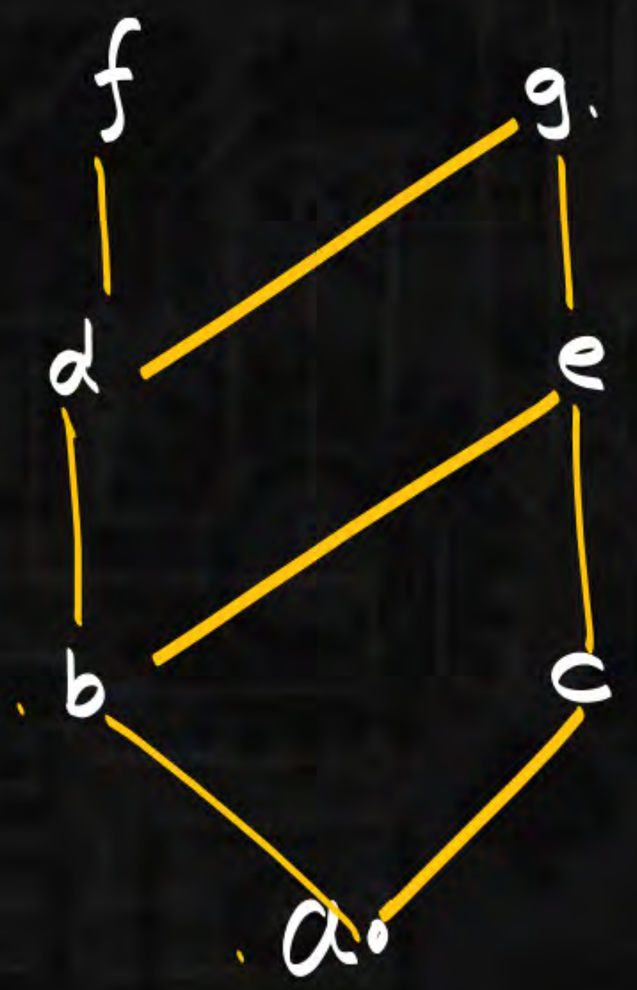
$d\,b\,a \leq d$ $\quad$ $d\,b\,a \leq b$ $\quad$ $d\,b\,a \leq a$.

$d \leq d$ ✓

$b \leq d$ ✓

$a \leq d$ ✓

d is glb of $\{f, g\}$



$B = \{f, d\}$

glb.

LB: $\{d, b, a\}$

d is glb.

$B_2 = \{f, c\}$

LB's $= \{a\}$

glb $= \{a\}$

$(D_{12}, |)$

$glb(2, 3) = 1$

$glb(2, 6) = 2$

$glb(3, 6) = 3$

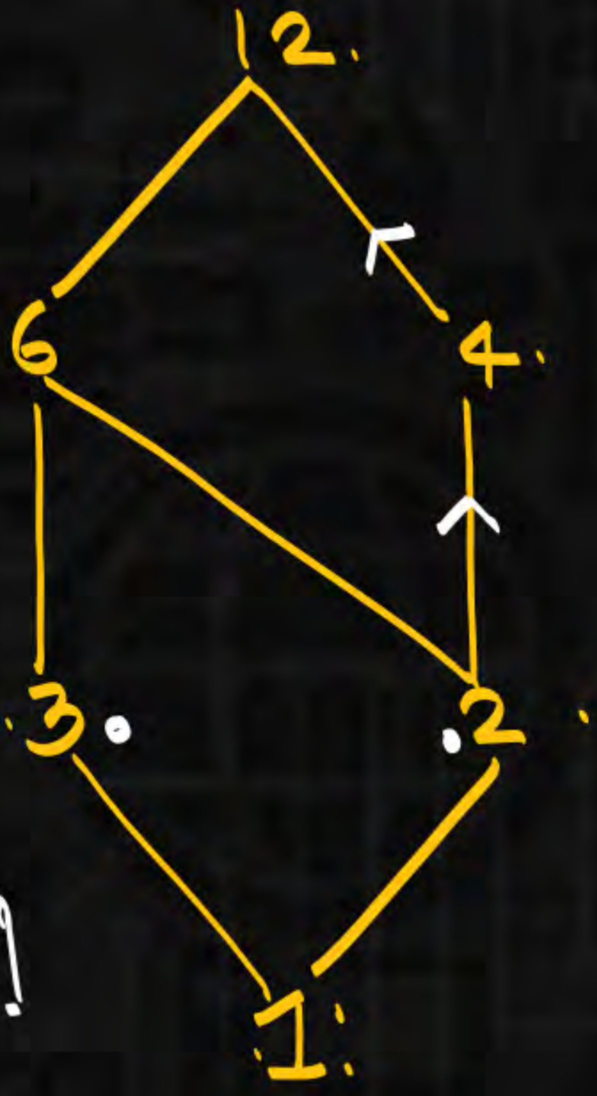
$glb(2, 12) = 2$

$\underline{\underline{lb}}$

$2 \leq 2$	$1 \leq 2$
$2 \leq 12$	$1 \leq 12$

$lb: \{1, 2\}$

$glb \to \{2\}$



LUB: (least upper bound)

all elements $\leq x \in A$.

$x \in A \leq$ all UB's of B.

$B = \{a, c\}$

UB's of $\{a, c\} \rightarrow \{c, e, g\}$

lub: $\{c\}$

Consequently, the relation $\mathcal{R}$ is a partial order for $\mathbf{Z}$. But it is *not* a total order. For example, $2, 3 \in \mathbf{Z}$ and we have neither $2\mathcal{R}3$ nor $3\mathcal{R}2$, because neither -1 nor 1, respectively, is a nonnegative even integer.

(a) For all $(a,b) \in A$, $a = a$ and $b \le b$, so $(a,b)\mathcal{R}(a,b)$ and the relation is reflexive. If $(a,b),(c,d) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b)$, then if $a \ne c$ we find that

$(a,b)\mathcal{R}(c,d) \Rightarrow a < c$, and

$(c,d)\mathcal{R}(a,b) \Rightarrow c < a$,

and we obtain $a < a$. Hence we have $a = c$.

And now we find that

$(a,b)\mathcal{R}(c,d) \Rightarrow b \le d$, and

$(c,d)\mathcal{R}(a,b) \Rightarrow d \le b$,

so $b = d$. Therefore, $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b) \Rightarrow (a,b) = (c,d)$, so the relation is antisymmetric. Finally, consider $(a,b),(c,d),(e,f) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then

(i) $a < c$, or (ii) $a = c$ and $b \le d$; and

(i)$'$ $c < e$, or (ii)$'$ $c = e$ and $d \le f$.

Consequently,

(i)$''$ $a < e$ or (ii)$''$ $a = e$ and $b \le f$ — so, $(a,b)\mathcal{R}(e,f)$ and the relation is transitive.

The preceding shows that $\mathcal{R}$ is a partial order on A.

b) & c) There is only one minimal element — namely, (0,0). This is also the least element for this partial order.

The element (1,1) is the only maximal element for the partial order. It is also the greatest element.

d) This partial order is a total order. We find here that

$$(0,0)\mathcal{R}(0,1)\mathcal{R}(1,0)\mathcal{R}(1,1).$$

(a) a (b) a (c) c (d) e (e) z (f) e (g) v
$(A, \mathcal{R})$ is a lattice with z the greatest (and only maximal) element and a the least (and only minimal) element.

19. Define the relation $\mathcal{R}$ on the set $\mathbf{Z}$ by $a \,\mathcal{R}\, b$ if $a - b$ is a nonnegative even integer. Verify that $\mathcal{R}$ defines a partial order for $\mathbf{Z}$. Is this partial order a total order?

20. For $X = \{0, 1\}$, let $A = X \times X$. Define the relation $\mathcal{R}$ on A by $(a, b) \,\mathcal{R}\, (c, d)$ if (i) $a < c$; or (ii) $a = c$ and $b \leq d$. (a) Prove that $\mathcal{R}$ is a partial order for A. (b) Determine all minimal and maximal elements for this partial order. (c) Is there a least element? Is there a greatest element? (d) Is this partial order a total order?

For each $a \in \mathbf{Z}$ it follows that $a\mathcal{R}a$ because $a - a = 0$, an even nonnegative integer. Hence $\mathcal{R}$ is *reflexive*. If $a, b, c \in \mathbf{Z}$ with $a\mathcal{R}b$ and $b\mathcal{R}c$ then

$$a - b = 2m, \text{ for some } m \in \mathbf{N}$$
$$b - c = 2n, \text{ for some } n \in \mathbf{N},$$

and $a - c = (a - b) + (b - c) = 2(m + n)$, where $m + n \in \mathbf{N}$. Therefore, $a\mathcal{R}c$ and $\mathcal{R}$ is *transitive*. Finally, suppose that $a\mathcal{R}b$ and $b\mathcal{R}a$ for some $a, b \in \mathbf{Z}$. Then $a - b$ and $b - a$ are both nonnegative integers. Since this can only occur for $a - b = b - a$, we find that $[a\mathcal{R}b \wedge b\mathcal{R}a] \Rightarrow a = b$, so $\mathcal{R}$ is *antisymmetric*.


THANK YOU GW
SOLDIERS !