ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-7

Calculus

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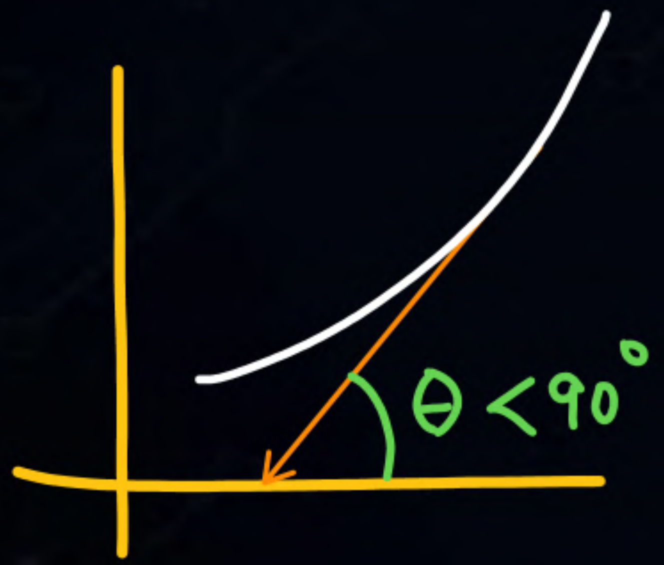
Topics to be Covered

- INCREASING- DECREASING FUNCTION
- MAXIMA AND MINIMA OF SINGLE VARIABLE FUNCTION
- MAXIMA AND MINIMA OF TWO VARIABLE FUNCTION
- LAGRANGE'S CONDITION FOR MAXIMA OR MINIMA

INCREASING- DECREASING FUNCTION



Slope = +ve

$$f'(x) > 0$$

$$\frac{dy}{dx} > 0$$

Strictly increasing

$$f'(x) \geq 0$$

Increasing fn.

Slope = -ve

$$f'(x) < 0$$

$$\frac{dy}{dx} < 0$$

Strictly decreasing

$$f'(x) \leq 0$$

Decreasing fn.

NOTE :- Monotonic functions are either S.I. or S.D.

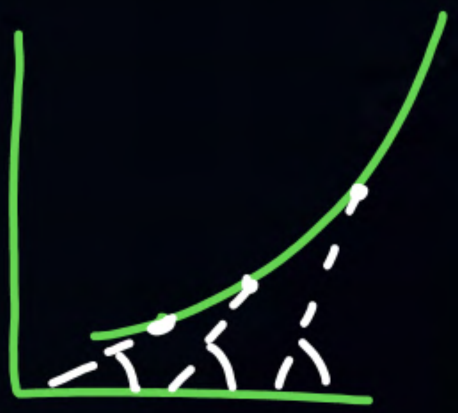
Monotonic functions are one-one and onto (bijective function).

Ex:- $f(x) = x^3 + x + 5$ in $[2, 5]$

$f'(x) = 3x^2 + 1$ $\quad f'(2) = 13$

$f'(x) > 0$ in $[2, 5]$ $\quad f'(5) = 76$

Hence $f(x)$ is S.I.



Slope ↑ing

$f'(x)$ ↑ing

$f'(x) > 0$

S.I.

Slope ↓ing

$f'(x)$ ↓ing

$f'(x) > 0$

S.I.

Ex:- $f(x) = \dfrac{e^x}{1+e^x}$

$$f'(x) = \frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} > 0$$

Hence $f(x)$ is S.I. $[f'(x) > 0]$

Ex:- $\sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

$$f'(x) = 4\sin^3 x \cdot \cos x - 4\cos^3 x \sin x$$

$$= 4\sin x \cos x \left(\sin^2 x - \cos^2 x\right)$$

$$= 2\sin 2x\,(-\cos 2x)$$

$$f'(x) = -\sin 4x$$

$[\because \cos^2 x - \sin^2 x = \cos 2x]$

$[\because \sin 2x = 2\sin x \cos x]$

$[\because \sin 4x = 2\sin 2x \cos 2x]$

$$f'(x) = -\sin 4x = 0$$

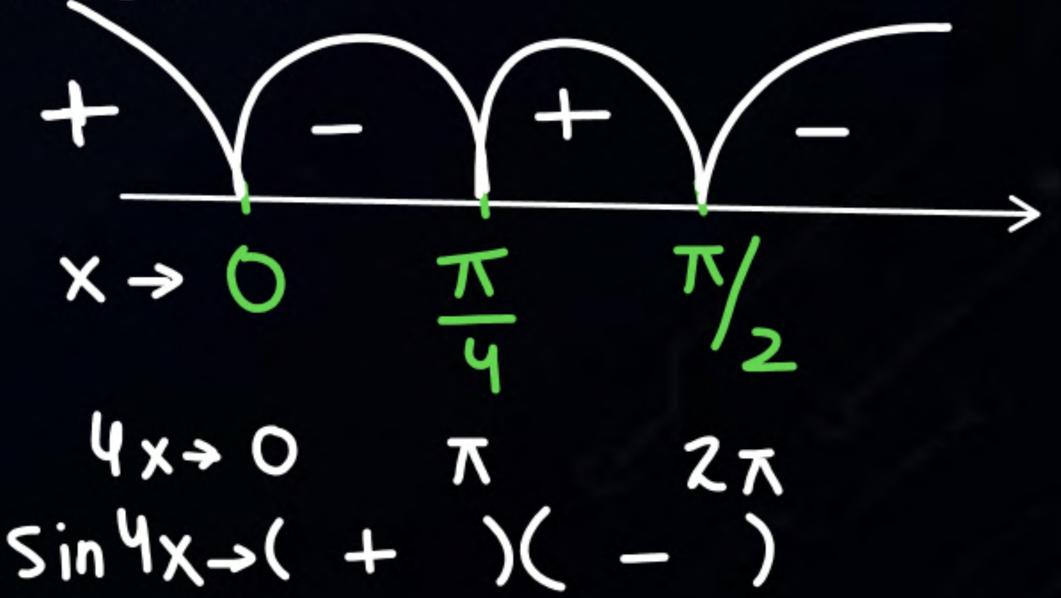
$$\sin 4x = 0$$

$$\sin 4x = 0$$

$$4x = 0, \pi, 2\pi, 3\pi$$

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$$

$f'(x) = -\sin 4x$



$x \in \left(0, \frac{\pi}{4}\right)$ $f(x)$ is S.D.

$f'(x) < 0$

$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ $f(x)$ is S.I.

$f'(x) > 0$

Ex:- $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

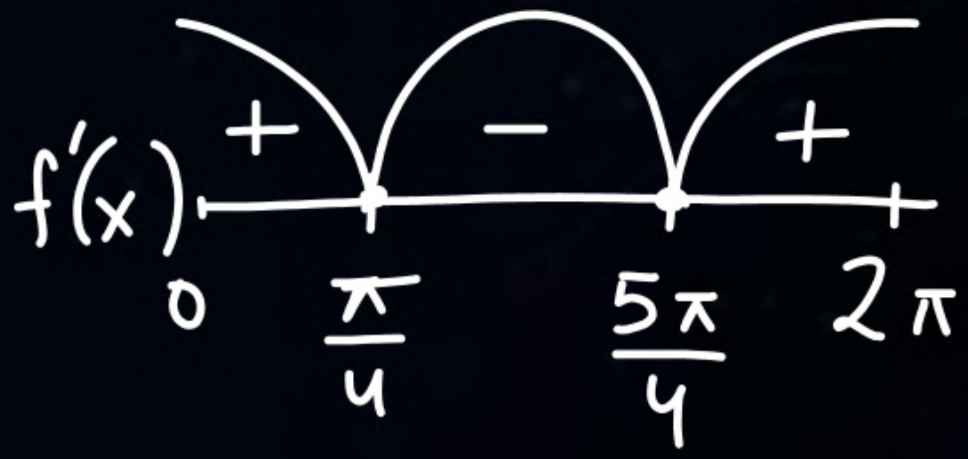
$$f'(x) = \cos x - \sin x$$

$$f'(x) = 0 = \cos x - \sin x$$

$$\sin x = \cos x$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$



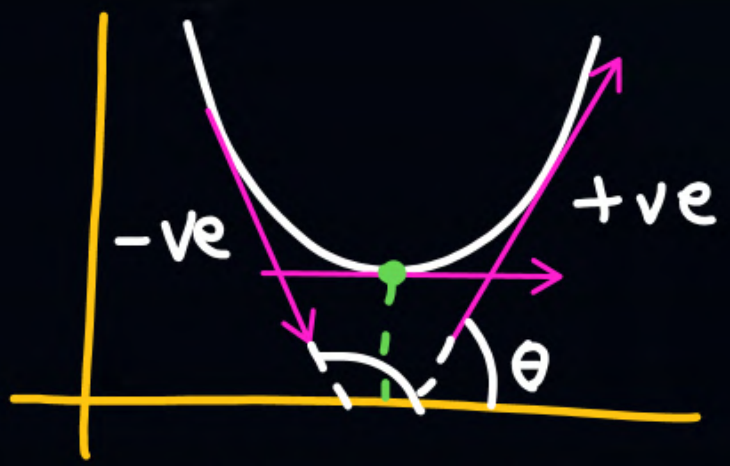
$f(x)$ is S.I. in $x \in \left(0, \frac{\pi}{4}\right)$ $f'(x) > 0$

$f(x)$ is S.D. in $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ $f'(x) < 0$

$f(x)$ is S.I. in $x \in \left(\frac{5\pi}{4}, 2\pi\right)$ $f'(x) > 0$

MAXIMA AND MINIMA OF FUNCTION

Concave fn.



MINIMA

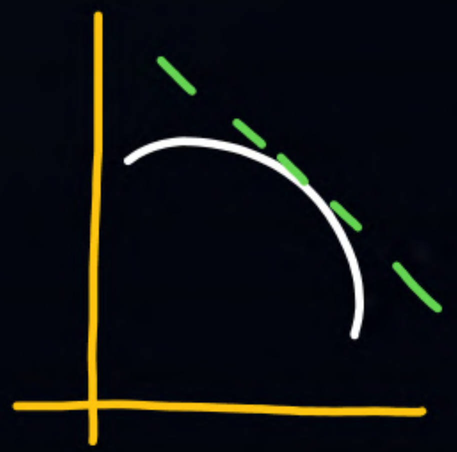
At pt. of minima;

$f'(x) = 0$

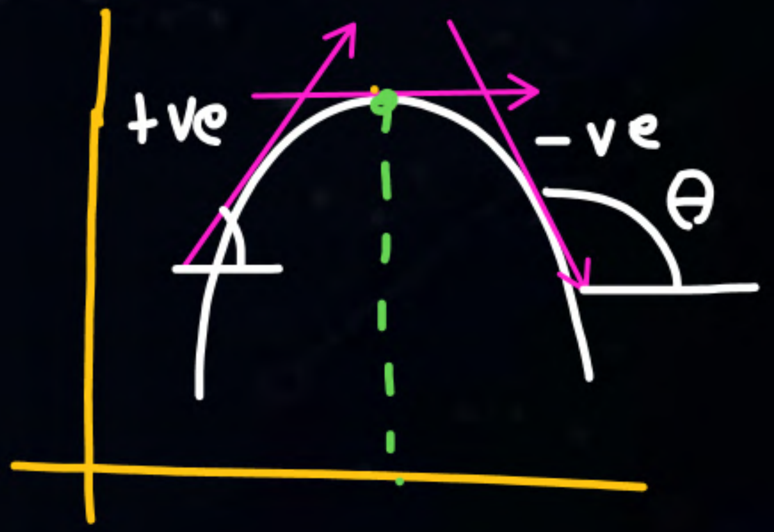
Slope → −ve → +ve

$f''(x) > 0$

Slope is increasing.



Convex fn.



MAXIMA

At point of maxima

$f'(x) = 0$

Slope → +ve → −ve

$f''(x) < 0$

Slope is decreasing

MAXIMA AND MINIMA OF FUNCTION

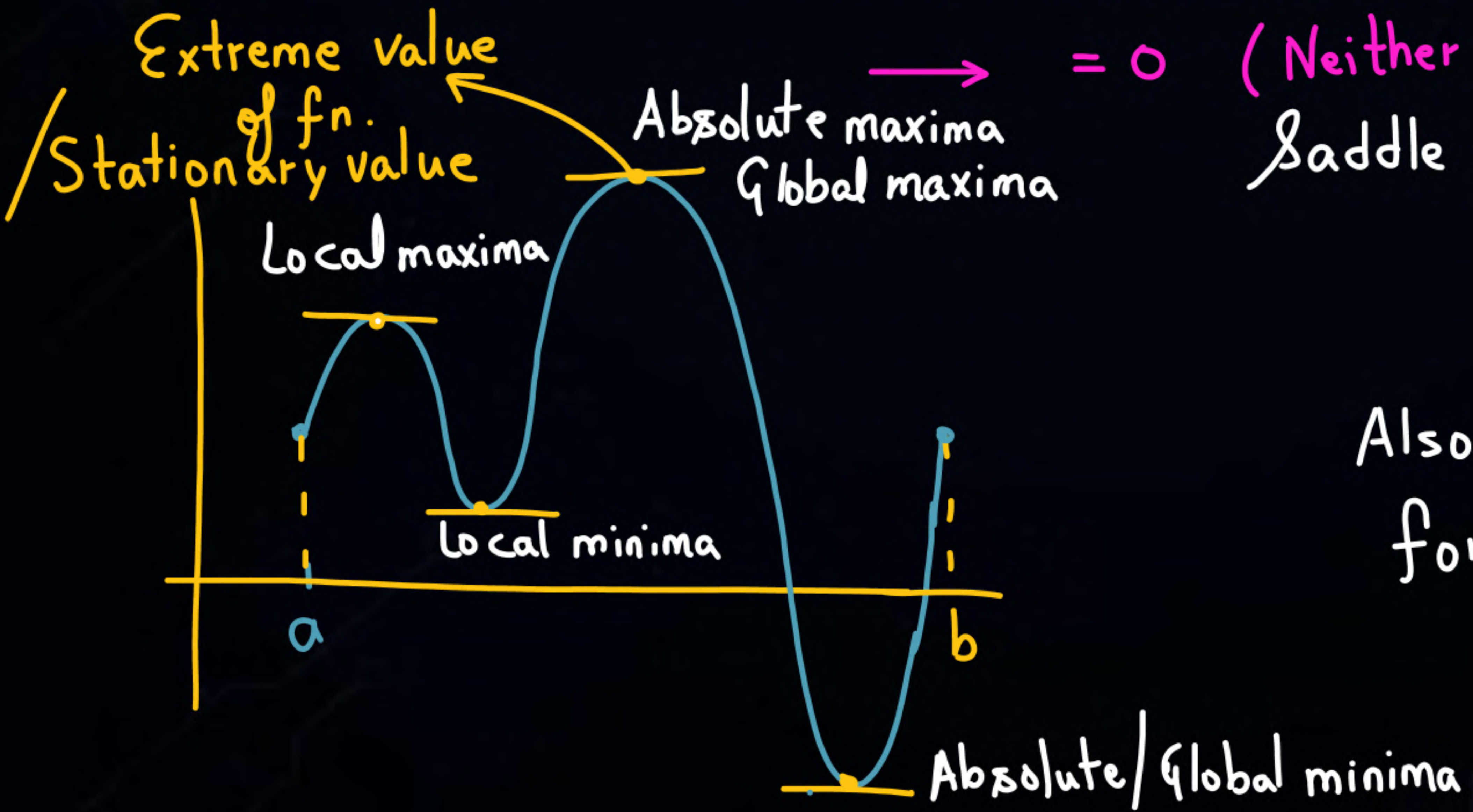
Rule :- i) $f'(x) = 0$; find critical/turning points extreme/stationary

ii) Check $f''(x)$ $\rightarrow$ > 0 (Minima)

$\rightarrow$ < 0 (Maxima)

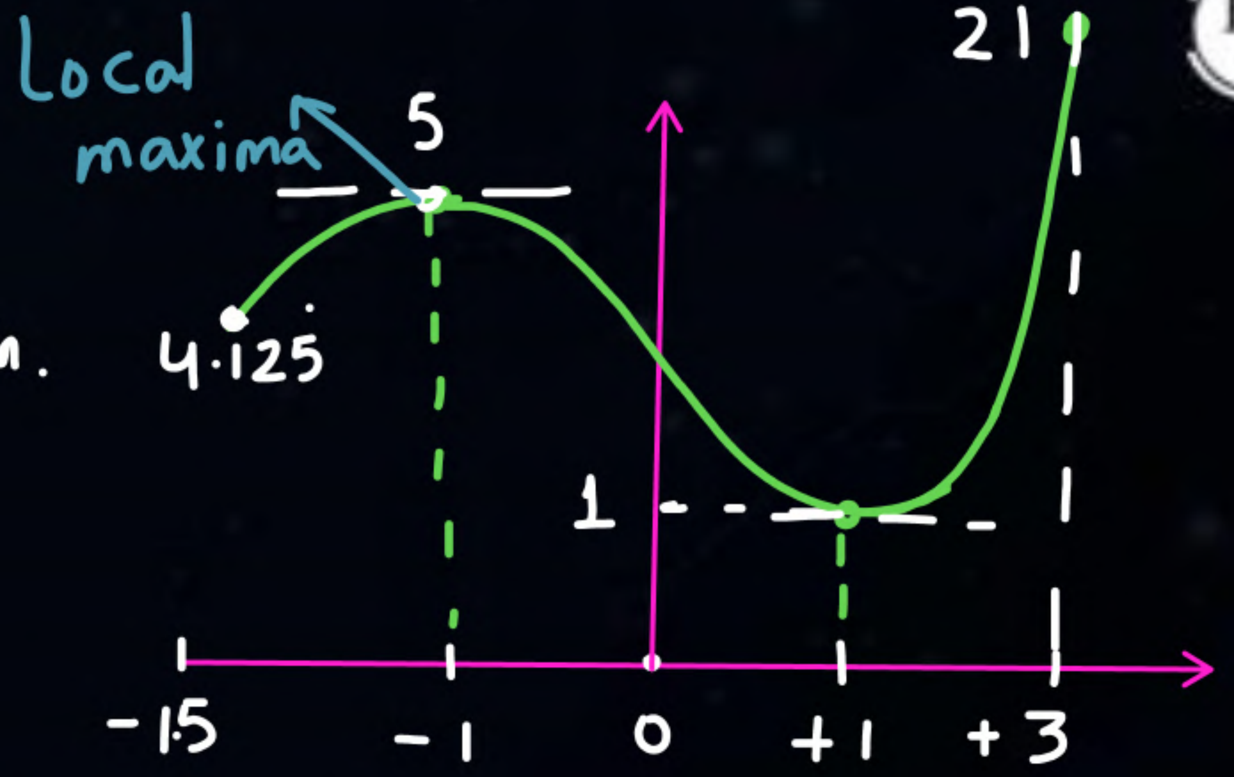
$\rightarrow$ $= 0$ (Neither maxima nor minima)

Saddle point $\left\{ \begin{matrix} f''(x) = 0 \text{ but} \\ f'''(x) \neq 0 \end{matrix} \right\}$



Also check the boundary points for absolute max. or min.

MAXIMA AND MINIMA OF FUNCTION



Ex:- $f(x) = x^3 - 3x + 3$ in $[-1.5, 3]$

Find absolute max./min. & local max./min.

Soln:-

$$f'(x) = 3x^2 - 3 = 0$$

$$3(x-1)(x+1) = 0$$

$$x = 1, -1$$

$$f''(x) = 6x$$

At $f''(-1) = -6 < 0$ (Maxima) $\Rightarrow$ Slope $\downarrow$ing

At $f''(+1) = +6 > 0$ (Minima) $\Rightarrow$ Slope $\uparrow$ing

✓ $f(-1.5) = (-1.5)^3 - 3(-1.5) + 3 = 4.125$

Turning points $\begin{cases} f(-1) = (-1)^3 - 3(-1) + 3 = 5 \\ f(+1) = 1^3 - 3(1) + 3 = 1 \end{cases}$

✓ $f(3) = 3^3 - 3(3) + 3 = 21$

Abs. max exist at $x=3$ $f(3)=1$

Abs. min exist at $x=1$ $f(1)=1$

Ex:- $f(x) = x^2 - x - 2$

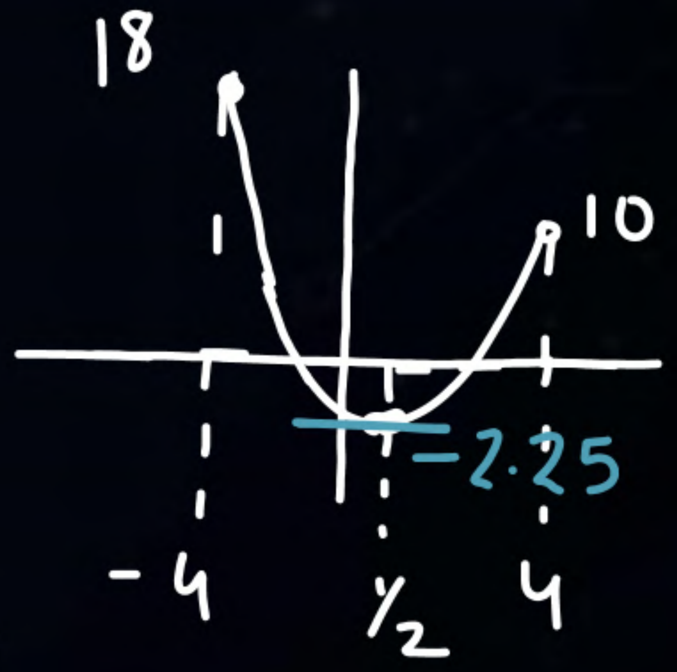
The max. of value of $f(x)$ in closed int. $[-4, 4]$ is

A) 18

B) 0

C) −18

D) 5



$f'(x) = 2x - 1 = 0$

$\boxed{x = 1/2} \rightarrow$ Turning point

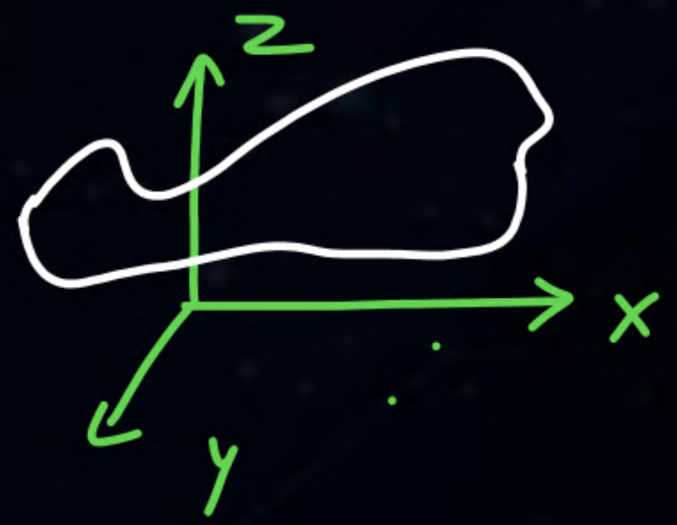
$f''(x) = 2 > 0$ (Minima) $\Rightarrow$ Slope is ↑ing.

$f(-4) = (-4)^2 - (-4) - 2 = 18 \Rightarrow$ Absolute max.

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = -2.25 \Rightarrow$ Absolute min.

$f(4) = (4)^2 - (4) - 2 = 10$

MAXIMA AND MINIMA OF TWO VARIABLE FUNCTION



$z \rightarrow f(\underbrace{x, y}_{\text{Independent variables}}) \longrightarrow$ Region/Surface

$$p = \frac{\partial f}{\partial x} \qquad q = \frac{\partial f}{\partial y}$$

$$r = \frac{\partial^2 f}{\partial x^2} \qquad t = \frac{\partial^2 f}{\partial y^2} \qquad s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Rule :- 1) Find critical points using $\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$

& we have obtained (a, b)

LAGRANGE'S CONDITION FOR MAXIMA OR MINIMA

2) Use Lagrange's conditions:

- $rt - s^2 > 0 \rightarrow$
  - $r > 0$ (Minima)
  - $r < 0$ (Maxima)
- $rt - s^2 < 0 \Rightarrow$ Neither maxima nor minima (Saddle point / Point of inflexion)
- $rt - s^2 = 0 \Rightarrow$ then case fails.

Ex:- $f(x,y) = 2x^2 + 2xy - y^3$ has [GATE]

A) only one stationary pt. $(0,0)$

B) 2 turning points at $(0,0)$ and $(1/6, -1/3)$ ✓

c) 2 " " " $(0,0)$ and $(1,-1)$

D) no turning point

$$f = 2x^2 + 2xy - y^3$$

$$p = \frac{\partial f}{\partial x} = 4x + 2y \qquad r = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 4 \qquad s = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2$$

$$q = \frac{\partial f}{\partial y} = 2x - 3y^2 \qquad t = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = -6y$$

i) $4x + 2y = 0 \Rightarrow x = -\frac{2y}{4}$

$$2x - 3y^2 = 0$$

$$2\left(-\frac{y}{2}\right) - 3y^2 = 0$$

$$-y - 3y^2 = 0$$

$$-y(1 + 3y) = 0$$

$$y = 0 \quad , \quad -1/3$$

$$\downarrow \qquad\qquad \downarrow$$

$$x = 0 \qquad x = 1/6$$

Stationary points $(0,0)$ & $(1/6, -1/3)$

$$rt - s^2 = 4(-6y) - 2^2$$
$$= -24y - 4$$

At $(0,0)$; $rt - s^2 = -24\times 0 - 4 < 0 \Rightarrow$ Neither max. nor min.

At $\left(\frac{1}{6}, -\frac{1}{3}\right)$; $rt - s^2 = -24\left(-\frac{1}{3}\right) - 4 = 4 > 0$

At $\left(\frac{1}{6}, -\frac{1}{3}\right) \rightarrow r = 4$ (+ve, minima)

Ex:- $f(x,y) = x^3 + y^3 - 3axy$

Thank you

GW Soldiers !