CS & IT ENGINEERING

DISCRETE MATHS
COMBINATORICS

Lecture No. 6

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TOPICS

01 Practice on Generating function

02 Sequence In Generating Function

3 Exercise

Extended binomial coefficient :

$$^{-n}C_k = (-1)^k \; {}^{n+k-1}C_k .$$

k → even → coefficient → +ve.

k → odd → coefficient → −ve

$$^{-11}C_2 = +$$

$$^{-11}C_3 = -$$

$$(a+b)^n = \sum_{i=0}^{n} {}^nC_i \times a^{n-i} \times b^i$$

$${}^{-n}C_K = (-1)^K \, {}^{n+K-1}C_K.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \cdots$$

$$\boxed{(1+(-x))^{-1} = (a+b)^n}$$

$$\frac{1}{1-x} = (1-x)^{-1} = \left(1+(-x)\right)^{-1} = {}^nC_0\, a^{n-0} b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2$$

$$(a+b)^n.$$

$$= \underbrace{{}^{-1}C_0 (1^{-1-0})(-x)^0}_{1} + \underbrace{{}^{-1}C_1 \, 1^{-1-1}(-x)^1}_{x}$$

$a=1 \quad b=-x. \quad n=-1$

$$-n_{C_k} = (-1)^k \, {}^{n+k-1}C_k$$

$${}^{-1}C_0 = 1 \qquad {}^{-1}C_2 = 1$$

$${}^{-1}C_1 = -1 \qquad {}^{-1}C_3 = -1$$

$${}^{-1}C_0 = {}^{-n}C_k$$

$n = 1 \quad k = 0$

$$(-1)^k \, {}^{n+k-1}C_k$$

$$(-1)^0 \cdot {}^{1+0-1}C_1$$

$$1 \cdot 1$$

$$\frac{1}{1-ax} = (1+(-ax))^{-1}$$

$$= \underbrace{{}^{-1}C_0\, 1^{-1-0}(-ax)^0}_{1} + \underbrace{{}^{-1}C_1}_{(-1)} (1)^{-1-1} \overbrace{(-ax)^{1}}^{ax} + \underset{+ve}{{}^{-1}C_2}(1)^{-1-2} \underset{+ve}{(-ax)^{2}}$$

$$1 + (-1)\cdot 1 \times (-1)(ax) + (ax)^2 + (ax)^3$$

$$\frac{1}{1-ax} = 1 + ax + (ax)^2 + (ax)^3 + (ax)^4 + \ldots\ldots$$

ax^2 ✗

$$\frac{1}{1-ax} = 1 + ax + (ax)^2 + (ax)^3 + (ax)^4 + \dots$$

$a = 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 .$$

$a = 3$

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \dots$$

$a = 2$

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \dots$$

$$\frac{1}{1+ax} = (1+ax)^{-1}$$

$$= {}^{-1}C_0 \, 1^{-1} (ax)^0 + {}^{-1}C_1 \, 1^{-1-1} (ax)^1$$

$$\frac{1}{1+ax} = 1 - ax + (ax)^2 - (ax)^3 + (ax)^4 - \ldots\ldots$$

$a = 1$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5$$

$a = 2$

$$\frac{1}{1+2x} = 1 - 2x + (2x)^2 - (2x)^3 + (2x)^4$$

How many ways to distribute 10 similar coins among 3 children such that each child gets atleast 2 coins & atmost 4 coins?

$$\{2 \le x_i \le 4\}$$

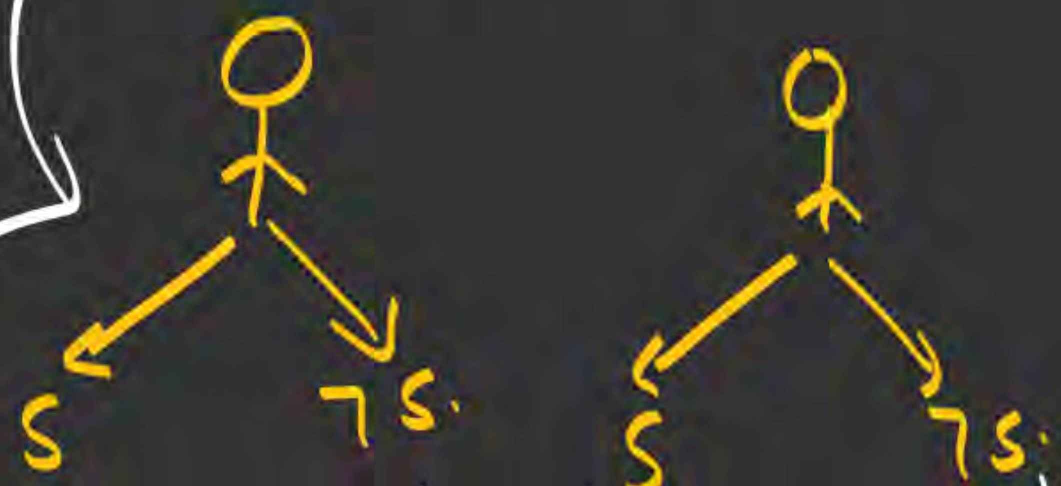
2	2	2
3	3	3
4	4	4

Giving 2 coins is represented as x^2.

Giving 3 coins is represented as $x3$

— 4 — - . . . x^4.

How many ways to select 6 out of 10? → Ans: $^{10}C_6$.



$$(x^1 + x^0) \times (x^1 + x^0)$$

Q → change. → Help: binomial.

(x^6)

$$(1+x) \times (1+x) \times (1+x) \ldots = (1+x)^{10}$$

coefficient of x^6.

atleast → 2. Total = 10.

atmost 4.

possible.

$$\text{Sol}^n :\rightarrow 2+4+4 = 10 \quad (2 \le x_i \le 4)$$

coefficient of x^{10}

$$(x^2+x^3+x^4) \times (x^2+x^3+x^4) \times (x^2+x^3+x^4)$$

$$x^2 \times x^4 \times x^4 = x^{2+4+4} = x^{10}$$

$$x^{\boxed{e_1+e_2+e_3}} = x^{\boxed{10}}$$

atl → 2

atmost → 4

$$e_1+e_2+e_3 = 10.$$

$$(x^2+x^3+x^4)\times(x^2+x^3+x^4)\times(x^2+x^3+x^4)$$

$$(x^2+x^3+x^4)^3$$

coefficient of x^{10}.

$$\begin{pmatrix} x^2 \\ + \\ x^3 \\ + \\ x^4 \end{pmatrix} \times \begin{pmatrix} x^2 \\ + \\ x^3 \\ + \\ x^4 \end{pmatrix} \times \begin{pmatrix} x^2 \\ + \\ x^3 \\ + \\ x^4 \end{pmatrix}$$

2 4 4.
4 2 4.
4 4 2
3 3 4.
3 4 3
4 3 3.

Ans: 6

atleast, atmost

$$(x^{2}+x^{3}+x^{4})^{3}$$

take x^2 common.

$$(x^2)^3\left[1+x+x^2\right]^3$$

$$x^6\left[\frac{1-x^3}{1-x}\right]^3 = x^6(1-x^3)^3(1-x)^{-3}$$

$$= x^6\left[1-3x^3+3x^6-x^9\right]\left[1-x\right]^{-3}$$

G.P:

$$1+x+x^2=\left(\frac{1-x^3}{1-x}\right)$$

1 2 1

1 3 3 1

+ − + −

$$(1-x^3)^3$$

$x^6[1-3x^3+3x^6-x^9][1-x]^{-3}$

$x^6[1-3x^3+3x^6-x^9]\left[{}^{-3}C_0\, 1^{-3}(-x)^0 + {}^{-3}C_1\, 1^{-3-1}(-x)^1 + {}^{-3}C_2\, 1^{-3-2}(-x)^2 + \ldots\ldots\right]$

coefficient of x^{10}

$[x^6 - 3x^9 + \underbrace{3x^{12} - x^{15}}_{\text{of no use}}][\ \ldots$

Ans: 6

$x^6 \times {}^{-3}C_4\, 1\,(-x)^4 - 3x^9\left({}^{-3}C_1\, 1\,(-x)^1\right)$

$(15-9)x^{10}$

$x^6 \times (-1)^4\ {}^{3+4-1}C_4\ 1 \times (-1)^4 x^4 - 3 \times x^9\left((-1)^1\ {}^{3+1-1}C_1\ 1\,(-1)\,x\right)$

$6x^{10}$

12 coins
among 4 students
atleast $\rightarrow$ 2
atmost $\rightarrow$ 5

2	2		
3	3		
4	4		
5	5		

$(x^2 + x^3 + x^4 + x^5) \times (x^2 + x^3 + x^4 + x^5) \times \; \times$

atleast. almost.

$(x^2 + x^3 + x^4 + x^5)^4 \rightarrow$ children

Coefficient of x^{12}.

coefficient of x^{12}.

$$(x^2+x^3+x^4+x^5)^4.$$

take x^2 common.

$$(x^2)^4\left[1+x+x^2+x^3\right]^4 = x^8\left[\frac{1-x^4}{1-x}\right]^4 = x^8\,\frac{(1-x^4)^4}{(1-x)^4}$$

$$= x^8(1-x^4)^4(1-x)^{-4}$$

$x^8(1-x^4)^4(1-x)^{-4}$ $\quad x^{12}$

$x^8\left(1-4x^4+\underbrace{6x^8\ldots\ldots}_{\text{no use}}\right)(1-x)^{-4}$

$\left(x^8-4x^{12}+\ldots\ldots\right)(1-x)^{-4}$

$$\begin{array}{ccccc} & 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ + & - & + & - & + \end{array}$$

$x^8\left({}^{-4}C_4 1 (-x)^4\right)-4x^{12}=31x^{12}$

$x^8(1-x^4)^4(1-x)^{-4}$ coefficient of x^{12}

$x^8[1-4x^4+\underbrace{6x^8-4x^{12}+x^{16}}_{\text{no use}}][1-x]^{-4}$ $(1-x^4)^4$

$1 \quad -4x^4+6x^8-4x^{12}+1x^{16}$

$[x^8-4x^{12}+6x^{16} \ldots\ldots]$

$(1-x)^{-4}$

$x^8\left(-4_{C_4} 1 \cdot (-x)^4\right) - 4x^{12}$

$\left(-4_{C_0}(1)^{-4-0}(-x)^0 + -4_{C_1}(1)^{-4-1}(-x)^1 \right.$

$\vdots$

$\left. -4_{C_4}(1)^{-4-4}(-x)^{+4} \right)$

15 coins.

among 4.

atleast → 2.

atmost → 6.

coefficient of x^{15}.

$$(x^2+x^3+x^4+x^5+x^6)^4$$

$$(x^2)^4\left[1+x+x^2+x^3+x^4\right)^4$$

$$x^8\left(\frac{1-x^5}{1-x}\right)^4 = x^8(1-x^5)^4(1-x)^{-4}$$

$$= x^8(1-4x^5+6x^{10}\ldots\ldots(1-x)^{-4}$$

$$= (x^8-4x^{13}+\ldots)(1-x)^{-4}$$

Ans: 80

$$x^8\left(-{}^4C_7(-x)^7\right) - 4x^{13}\left(-{}^4C_2\,1\,(-x)^2\right)$$

$$\left({}^{10}C_3\,x^{15} - 4\times{}^5C_2\,x^{15}\right)$$

x^{12}

$$(x^3 + x^4 + x^5 + x^6 \cdots\cdots)^3$$

$$(x^3)^3 (1 + x + x^2 + \cdots)^3$$

$$x^9 \left(\frac{1}{1-x}\right)^3 \qquad x^9 (1-x)^{-3}$$

$$x^9 \left({}^{-3}C_3 (-x)^3 \right)$$

$$(-1)^3 \; {}^{3+3-1}C_3 (-1)^3 x^3$$

Ans: 10

Express each of the sums in closed form

$$\sum_{k=0}^{n} \binom{n}{k} 5^k$$

$$\sum_{i=0}^{n} \binom{n}{i} x^i$$

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} x^j$$

$$\sum_{i=0}^{m} \binom{m}{i} p^{m-i} q^{2i}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{2^i}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} 5^{n-i} 2^i$$

$$\sum_{k=0}^{n} \binom{n}{k} 5^k = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 5^k = (1+5)^n = 6^n$$

$$\sum_{i=0}^{n} \binom{n}{i} x^i = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} x^i = (1+x)^n$$

$$\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} x^j = \sum_{j=0}^{2n} \binom{2n}{j} 1^{2n-j} (-x)^j = (1-x)^{2n}$$

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{1}{2^i} = \sum_{i=0}^{m} \binom{m}{i} 1^{m-i} \left(-\frac{1}{2}\right)^i$$

$$= \left(1 - \frac{1}{2}\right)^m = \frac{1}{2^m}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} 5^{n-i} 2^i = \sum_{i=0}^{n} \binom{n}{i} 5^{n-i} (-2)^i = (5-2)^n = 3^n$$

Find the coefficient of x^{16} in the expansion of $\left(2x^2 - \frac{x}{2}\right)^{12}$.

$$\binom{12}{k}(2x^2)^{12-k}\left(-\frac{x}{2}\right)^k = \binom{12}{k}2^{12-k}\left(-\frac{1}{2}\right)^k x^{24-k}.$$

We want $24 - k = 16$; thus, $k = 8$. The coefficient is $\binom{12}{8}2^4\left(-\frac{1}{2}\right)^8 = \frac{1}{16}\binom{12}{8}$ $\frac{495}{16}$.

Find generating functions for the following sequences.

a) $\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, \ldots, \binom{8}{8}$

b) $\binom{8}{1}, 2\binom{8}{2}, 3\binom{8}{3}, \ldots, 8\binom{8}{8}$

c) $1, -1, 1, -1, 1, -1, \ldots$

d) $0, 0, 0, 6, -6, 6, -6, 6, \ldots$

e) $1, 0, 1, 0, 1, 0, 1, \ldots$

f) $0, 0, 1, a, a^2, a^3, \ldots, a \neq 0$

X

(a) $(1+x)^8$ (b) $8(1+x)^7$ (c) $(1+x)^{-1}$
(d) $6x^3/(1+x)$ (e) $(1-x^2)^{-1}$ (f) $x^2/(1-ax)$

2. Determine the sequence generated by each of the following generating functions.

a) $f(x) = (2x-3)^3$

b) $f(x) = x^4/(1-x)$

c) $f(x) = x^3/(1-x^2)$

d) $f(x) = 1/(1+3x)$

e) $f(x) = 1/(3-x)$

f) $f(x) = 1/(1-x) + 3x^7 - 11$

X

(a) $-27, 54, -36, 8, 0, 0, 0, \ldots$ (b) $0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots$

(c) $f(x) = x^3/(1-x^2) = x^3[1 + x^2 + x^4 + x^6 + \ldots] = x^3 + x^5 + x^7 + x^9 + \ldots$, so $f(x)$ generates the sequence $0, 0, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$

(d) $f(x) = 1/(1+3x) = 1 + (-3x) + (-3x)^2 + (-3x)^3 + \ldots$, so $f(x)$ generates the sequence $1, -3, 3^2, -3^3, \ldots$

(e) $f(x) = 1/(3-x) = (1/3)[1/(1-(x/3))] = (1/3)[1 + (x/3) + (x/3)^2 + (x/3)^3 + \ldots]$, so $f(x)$ generates the sequence $1/3, (1/3)^2, (1/3)^3, (1/3)^4, \ldots$

(f) $f(x) = 1/(1-x) + 3x^7 - 11 = (1 + x + x^2 + x^3 + \ldots) + 3x^7 - 11$, so $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots$, where $a_0 = -10$, $a_7 = 4$, and $a_i = 1$ for all $i \neq 0, 7$.

5. **a)** Find the coefficient of x^7 in

$$(1 + x + x^2 + x^3 + \cdots)^{15}.$$

b) Find the coefficient of x^7 in

$$(1 + x + x^2 + x^3 + \cdots)^n \text{ for } n \in \mathbf{Z}^+.$$

6. Find the coefficient of x^{50} in $(x^7 + x^8 + x^9 + \cdots)^6$.

7. Find the coefficient of x^{20} in $(x^2 + x^3 + x^4 + x^5 + x^6)^5$.

5. (a) $\binom{-15}{7}(-1)^7 = (-1)^7\binom{15+7-1}{7}(-1)^7 = \binom{21}{7}$

(b) $\binom{-n}{7}(-1)^7 = (-1)^7\binom{n+7-1}{7}(-1)^7 = \binom{n+6}{7}$

6. $\binom{-6}{8}(-1)^8 = (-1)^8\binom{6+8-1}{8}(-1)^8 = \binom{13}{8}$

7. $\binom{-5}{10}(-1)^{10} - \binom{5}{1}\binom{-5}{5}(-1)^5 + \binom{5}{2}\binom{-5}{0} = \binom{14}{10} - \binom{5}{1}\binom{9}{5} + \binom{5}{2}$

10. In how many ways can two dozen identical robots be assigned to four assembly lines with (a) at least three robots assigned to each line? (b) at least three, but no more than nine, robots assigned to each line?

11. In how many ways can 3000 identical envelopes be divided, in packages of 25, among four student groups so that each group gets at least 150, but not more than 1000, of the envelopes?

12. Two cases of soft drinks, 24 bottles of one type and 24 of another, are distributed among five surveyors who are conducting taste tests. In how many ways can the 48 bottles be distributed so that each surveyor gets (a) at least two bottles of each type? (b) at least two bottles of one particular type and at least three of the other?

13. If a fair die is rolled 12 times, what is the probability that the sum of the rolls is 30?

14. Carol is collecting money from her cousins to have a party for her aunt. If eight of the cousins promise to give \$2, \$3, \$4, or \$5 each, and two others each give \$5 or \$10, what is the probability that Carol will collect exactly \$40?

(a) $(x^3 + x^4 + \dots)^4 = x^{12}(1 + x + x^2 + \dots)^4 = x^{12}(1-x)^{-4}$. The coefficient of x^{12} in $(1-x)^{-4}$ is $\binom{-4}{12}(-1)^{12} = (-1)^{12}\binom{4+12-1}{12}(-1)^{12} = \binom{15}{12}$.

B)

$\binom{15}{12} - 4\binom{8}{5}$.

11

Consider each package of 25 envelopes as one unit. Then the answer to the problem is the coefficient of x^{120} in $(x^6+x^7+\ldots+x^{39}+x^{40})^4 = x^{24}(1+x+\ldots+x^{34})^4$. This is the same as the coefficient of x^{96} in $[(1-x^{35})/(1-x)]^4 = (1-x^{35})^4(1-x)^{-4} = [1-4x^{35}+6x^{70}-\ldots+x^{140}][\binom{-4}{0}+\ldots+\binom{-4}{26}(-x)^{26}+\ldots+\binom{-4}{61}(-x)^{61}+\ldots+\binom{-4}{96}(-x)^{96}+\ldots]$.

Consequently the answer is $\binom{-4}{96}(-1)^{96}-4\binom{-4}{61}(-1)^{61}+6\binom{-4}{26}(-1)^{26} = \binom{99}{96}-4\binom{64}{61}+6\binom{29}{26}$.

12 a)

$\binom{18}{14}^2$ ways.

(b) The coefficient of x^{24} in $(x^3+x^4+\ldots)^5$ is $\binom{13}{9}$ and the answer is $\binom{18}{14}\binom{13}{9}$.

13. $(x+x^2+x^3+x^4+x^5+x^6)^{12} = x^{12}[(1-x^6)/(1-x)]^{12} = x^{12}((1-x)^6)^{12}[\binom{-12}{0}+\binom{-12}{1}(-x)+\binom{-12}{2}(-x)^2+\ldots]$. The numerator of the answer is the coefficient of x^{18} in $(1-x^6)^{12}[\binom{12}{0}+\binom{-12}{1}(-x)+\ldots] = [1-\binom{12}{1}x^6+\binom{12}{2}x^{12}-\binom{12}{3}x^{18}+\ldots+x^{72}][\binom{-12}{0}+\binom{-12}{1}(-x)+\ldots]$ and this equals $\binom{-12}{18}(-1)^{18}-\binom{12}{1}\binom{-12}{12}(-1)^{12}+\binom{12}{2}\binom{-12}{6}(-1)^6-\binom{12}{3}\binom{-12}{0} = \binom{29}{18}-\binom{12}{1}\binom{23}{12}+\binom{12}{2}\binom{17}{6}-\binom{12}{3}$. The final answer is obtained by dividing the last result by 6^{12}, the size of the sample space.

14. $(x^2+x^3+x^4+x^5)^8(x^5+x^{10})^2 = x^{26}(1+x+x^2+x^3)^8(1+x^5)^2$, so we need the coefficient of x^{14} in $[(1-x^4)/(1-x)]^8(1+2x^5+x^{10}) = (1-x^4)^8(1-x)^{-8}(1+2x^5+x^{10}) = [1+\binom{8}{1}(-x^4)+\binom{8}{2}(-x^4)^2+\ldots+(-x^4)^8][\binom{-8}{0}+\binom{-8}{1}(-x)+\binom{-8}{2}(-x)^2+\ldots](1+2x^5+x^{10})$.

This coefficient is $[\binom{-8}{14}(-1)^{14}+2\binom{-8}{9}(-1)^9+\binom{-8}{4}(-1)^4]-\binom{8}{1}[\binom{-8}{10}(-1)^{10}+2\binom{-8}{5}(-1)^5+\binom{-8}{0}]+\binom{8}{2}[\binom{-8}{6}(-1)^6+2\binom{-8}{1}(-1)]-\binom{8}{3}[\binom{-8}{2}(-1)^2] = [\binom{21}{14}+2\binom{16}{9}+\binom{11}{4}]-\binom{8}{1}[\binom{17}{10}+2\binom{12}{5}+\binom{7}{0}]+\binom{8}{2}[\binom{13}{6}+2\binom{8}{1}]-\binom{8}{3}\binom{9}{2}$. This result is then divided by $(4^8)(2^2)$, the size of the sample space, in order to determine the probability.

1. Find the generating function for each of the following sequences.

a) 7, 8, 9, 10, . . .

b) $1, a, a^2, a^3, a^4, \ldots, \quad a \in \mathbf{R}$

c) $1, (1+a), (1+a)^2, (1+a)^3, \ldots, \quad a \in \mathbf{R}$

d) $2, 1+a, 1+a^2, 1+a^3, \ldots, \quad a \in \mathbf{R}$

2. Find the coefficient of x^{83} in

$$f(x) = (x^5+x^8+x^{11}+x^{14}+x^{17})^{10}.$$

1.

(a) $6/(1-x)+1/(1-x)^2$ (b) $1/(1-ax)$
(c) $1/[1-(1+a)x]$ (d) $1/(1-x)+1/(1-ax)$

2. Let $f(x)=(x^5+x^8+x^{11}+x^{14}+x^{17})^{10}=x^{50}(1+x^3+x^6+x^9+x^{12})^{10}$. The coefficient of x^{83} in $f(x)$ equals the coefficient of x^{33} in $((1-x^{15})/(1-x^3))^{10}=(1-x^{15})^{10}(1-x^3)^{-10}=[1-\binom{10}{1}x^{15}+\binom{10}{2}x^{30}-\ldots+x^{150}]\cdot[\binom{-10}{0}+\binom{-10}{1}(-x^3)+\binom{-10}{2}(-x^3)^2+\ldots]$. This coefficient is $\binom{-10}{11}(-1)^{11}-\binom{10}{1}\binom{-10}{6}(-1)^6+\binom{10}{2}\binom{-10}{1}(-1)^1=\binom{20}{11}-\binom{10}{1}\binom{15}{6}+\binom{10}{2}\binom{10}{1}$.

3. Sergeant Bueti must distribute 40 bullets (20 for rifles and 20 for handguns) among four police officers so that each officer gets at least two, but no more than seven, bullets of each type. In how many ways can he do this?

3. The generating function for each type of bullet is $(x^2+x^3+\ldots+x^7)^4=x^8(1+x+x^2+\ldots+x^5)^4$. The coefficient of x^{12} in $(1-x^6)^4(1-x)^{-4}=[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\ldots][\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+\ldots]$ is $\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}=\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}$. By the rule of product the answer is $[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}]^2$.


THANK YOU GW
SOLDIERS !