ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-10

Calculus

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Topics to be Covered

- PARTIAL DIFFERENTIATION
- HOMOGENEOUS FUNCTION
- EULER'S THEOREM
- INTEGRATION
- DEFINITE INTEGRALS
- PROPERTY OF DEFINITE INTEGRALS

1. $f_{xy} = f_{yx}$ $\qquad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

2. $\dfrac{\partial^2 f}{\partial x^2} \neq \left(\dfrac{\partial f}{\partial x}\right)^2$ & $\dfrac{\partial^2 f}{\partial x \partial y} \neq \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial f}{\partial y}$

3. $\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 + 2\,\dfrac{\partial u}{\partial x} \cdot \dfrac{\partial u}{\partial y}$

4. $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + 2\,\dfrac{\partial^2 u}{\partial x \partial y}$

Ex:-

$f = \tan^{-1} y/x$

~~i)~~ $f_{yx} = f_{xy}$

~~ii)~~ $f_{xx} \neq (f_x)^2$

iii) $(f_x + f_y)^2$

iv) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f$

$= f_{xx} + f_{yy} + 2 f_{xy}$

$$\frac{\partial f}{\partial x} = \frac{y}{1+(y/x)^2}\left(-\frac{1}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} \neq \left(\frac{\partial f}{\partial x}\right)^2$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Ex:- If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Prove i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

ii) Find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan u\left[\sec^2 u - 1\right]$

$$x\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x^2+y^2}{x+y}\right)}}\left[\frac{(x+y)(2x) - (x^2+y^2).(1)}{(x+y)^2}\right]x$$

$$y\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x^2+y^2}{x+y}\right)}}\left[\frac{(x+y)(2y) - (x^2+y^2).(1)}{(x+y)^2}\right]y$$

HOMOGENEOUS FUNCTION

$$f(x,y) = a_0 y^n + a_1 x y^{n-1} + a_2 x^2 y^{n-2} + \dots\dots\dots a_n x^n$$

$$x^n\left[a_0\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + \dots\dots a_n\left(\frac{y}{x}\right)^0\right]$$

$$f(x,y) = x^n f(y/x)$$
$$= y^n f(x/y)$$

then f is homogenous fn. of degree of n.

$$f(x,y) \rightarrow f(Kx,Ky) = K^n f(x,y)$$

then fn. is homogenous of degree of n.

$$x^2+y^2 \longrightarrow (Kx)^2+(Ky)^2 = K^2(x^2+y^2)$$
$$K^{\boxed{2}} f(x,y)$$

Find Homogenous functions :-

1) $x^2+y^2 = x^2\left[1+(y/x)^2\right] = x^{2}\,\cancelto{0}{f(y/x)}$

2) $\dfrac{x^2+y^2}{x-y} = \dfrac{x^2\left[1+(y/x)^2\right]}{x\left[1-y/x\right]} = x^{1}\,\cancelto{0}{f(y/x)}$

3) $\dfrac{x^{1/3}+y^{1/3}}{x^{1/4}-y^{1/4}} = \dfrac{x^{1/3}\left[1+(y/x)^{1/3}\right]}{x^{1/4}\left[1-(y/x)^{1/4}\right]} = x^{1/12}\,\cancelto{0}{f(y/x)}$

4) $x^2y+x^3+xy^2+y^3\sin(y/x) = x^{3}\,\cancelto{0}{\left[\dfrac{y}{x}+1+\left(\dfrac{y}{x}\right)^2+\left(\dfrac{y}{x}\right)^3\sin\left(\dfrac{y}{x}\right)\right]}$

5) $\dfrac{1}{x^2}-\dfrac{y}{x^3}+\dfrac{3}{y^2}(\log x-\log y) = x^{-2}\,\cancelto{0}{\left[1-\dfrac{y}{x}+3\left(\dfrac{x}{y}\right)^2\log\left(\dfrac{x}{y}\right)\right]}$

6) $x^2y^2+y^4+x^4+x^3y+\sin(x^4) = x^4\left[\left(\dfrac{y}{x}\right)^2+\left(\dfrac{y}{x}\right)^4+1+\dfrac{y}{x}+\dfrac{\sin x^4}{x^4}\right]$

$\Rightarrow$ Non-homogenous fn.

EULER'S THEOREM

Let $f(x,y)$ be a homogenous function, then

THEOREM 1 :-

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

n → Degree of homo. fn

$$\boxed{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$$

→ $f(x_1, x_2 \ldots x_n)$ be a homo. fn. of degree n.

then $x_1 f_{x_1} + x_2 f_{x_2} + x_3 f_{x_3} + \cdots x_n f_{x_n} = nf$

Ex:- $f(x,y) = \dfrac{x^{1/3} + y^{1/3}}{x^{1/4} - y^{1/4}} = x^{1/12} f(y/x)$

Find i) $x f_x + y f_y = nf = \dfrac{1}{12}\left[\dfrac{x^{1/3} + y^{1/3}}{x^{1/4} - y^{1/4}}\right]$

ii) $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1) f = \dfrac{1}{12}\left(-\dfrac{11}{12}\right)\left[\dfrac{x^{1/3}+y^{1/3}}{x^{1/4}-y^{1/4}}\right]$

Ex:- $u = ayz + bxz + cxy$

Find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

$= nu = 2u = 2(ayz + bxz + cxy)$

Theorem 2 :- If $V = f(u)$ is a homogenous fn. of degree n.

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \boxed{\frac{n\,f(u)}{f'(u)}}} \rightarrow \phi(u)$$

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u) - 1]}$$

$$\left.\begin{matrix}\text{Trig}\\ \text{Inv.}\\ \text{Exp.}\\ \text{log}\end{matrix}\right\}\left[\begin{matrix}\text{Homogenous}\\ \text{fn.}\end{matrix}\right] \rightarrow \text{then apply theorem 2.}$$

Ex:- $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$V = \underset{f(u)}{\sin u} = \left(\frac{x^2+y^2}{x+y}\right)$ is a homogenous fn. of degree 1.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)} = \frac{1 \times \sin u}{\cos u} = \tan u \rightarrow \phi(u)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1]$$

$$= \tan u[\sec^2 u - 1]$$

Ex:- $f = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \Rightarrow \tan f = \frac{x^3+y^3}{x-y}$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{2 \times \tan f}{\sec^2 f} = 2\sin f\cos f = \sin 2f$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = \sin 2f[2\cos 2f - 1]$$

Ex:-

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$$

Total Differential Coefficient :-

- $f \rightarrow x, y \longrightarrow t$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$f = x^2 + y^2$

$x = t$

$y = t^2$

- $f \rightarrow x, y \rightarrow r, \theta$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

- $f \rightarrow x, y \longrightarrow x$ alone

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = \partial f/\partial x + \partial f/\partial y \cdot dy/dx$$

- $u \to x, y, z \to t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

- $u \to x, y, z \to r, \theta, \phi$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \phi}$$

Let $f(x, y)$ be a implicit function:-

$$f(x, y) = c$$

$$df = 0$$

$$\frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 0$$

If we are unable to separate y & x then it is implicit function, otherwise explicit fn.

$$\boxed{\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}}$$

Ex:- $f(x,y) = c$

$f \to x^3 + x^2y - 5$

$x^3 + x^2 y = 5$

Find $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(3x^2 + 2xy)}{x^2}$

$(3x^2 + 2xy)\,dx + (x^2)\,dy = 0$

$$\frac{dy}{dx} = -\frac{(3x^2 + 2xy)}{x^2}$$

Ex:- $x^2 + y^2 + z^2 = c$

If $df = 0$; $\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz = 0$

→ Find dy/dx :-

i) $x^y + y^x = c$

→ If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$ find du/dx

JACOBIAN :-

If u, v are functions of x, y then $J(u, v)$

$$\frac{\partial(u,v)}{\partial(x,y)} = J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

If u, v, w are functions of x, y, z then $J(u, v, w)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

FUNCTIONAL DEPENDENCE :-

If functions are dependent and there exist a relationship between them , then $\boxed{J(u, v\ldots) = 0}$ otherwize functions are independent (no relation b/w them).

Ex :- $u = x + 2y + z$; $v = x - 2y + 3z$; $w = 2xy - xz + 4yz - 2z^2$

Find the relationship b/w u, v and w.

$$J(u,v,w) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix} \xrightarrow[C_3 - C_1]{C_2 - 2C_1} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & (2x-4y+6z) & -x+2y-3z \end{vmatrix}$$

$$-4(-x+2y-3z)-2(2x-4y+6z)$$

$$\cancel{4x}-\cancel{8y}+\cancel{12z}-\cancel{4x}+\cancel{8y}-\cancel{12z}=0$$

If $J=0$, then u, v, w are functionally dependent.

$$u+v=2x+4z$$

$$u-v=4y-2z$$

$$(u+v)(u-v)=2(x+2z)\,2(2y-z)$$

$$=4[2xy-xz+4yz-2z^2]$$

$$=4w$$

$$\boxed{u^2-v^2=4w}$$

INTEGRATION

Integration is anti-derivative.

$$\int f(x)\,dx = F(x) + c$$

<u>Some standard functions</u> :-

1) $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

2) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

3) $\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + c$

4) $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$

Thank you

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Soldiers !