ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-04

Calculus

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Topics to be Covered

- FUNDAMENTALS OF CONTINUITY
- KINDS OF DISCONTINUITIES
- PROPERTIES OF CONTINUOUS FUNCTIONS
- CONTINUITY OF FUNCTION OF TWO VARIABLES
- DIFFERENTIABILITY

H.W. $\lim_{x \to 0} \dfrac{\sin 2x + a\sin x}{x^3} = b$ $\quad a = ?\ 2 \quad b = ?\ 1$

Infinite series limit

$$\log ab = \log a + \log b$$
$$\log \frac{a}{b} = \log a - \log b$$
$$\log y = a \Rightarrow y = e^a$$
$$e^{\log_e a} = a$$
$$\log x^y = y \log x$$

$$\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \cdots \frac{1}{n+n} \right] = \log 2$$

$$\lim_{n \to \infty} \left(\frac{n!}{n^n} \right)^{1/n}$$

$$\lim_{n \to \infty} \left[\int_{x=1}^{n} \frac{1}{x+n} \right] = \left[\log(x+n) \right]_1^n = \log 2n - \log(n+1)$$
$$= \lim_{n \to \infty} \log\left(\frac{2n}{n+1} \right) = \log\left[\frac{2}{1+\frac{1}{n}} \right]$$

$$\lim_{n \to \infty} \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n}{n} \right)^{1/n}$$

$$\log y = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{1}{n}\cdot\frac{2}{n}\cdot \dots \cdot \frac{n}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \dots \log\frac{n}{n}\right]$$

$$\log y = \lim_{n\to\infty} \frac{1}{n} \cdot \int_{x=1}^{n} \log\frac{x}{n}$$

Trick :-

1^∞

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a}[f(x)-1 \cdot g(x)]}$$

$$\lim_{x\to\infty} \frac{1}{f(x)^{g(x)}} = \frac{1}{\lim_{x\to\infty}[f(x)-1]\cdot g(x)}$$

Ex:- $\lim_{x\to 0} (1-\sin x)^{\frac{1}{\sin x}}$

$$= e^{(1-\sin x-1)\cdot\frac{1}{\sin x}}$$

$$= e^{-\sin x\cdot\frac{1}{\sin x}} = e^{-1}$$

Ex:- $\lim_{x\to 0} (\cos mx)^{\frac{1}{x^2}}$

$$= e^{\lim_{x\to 0}\frac{\cos mx-1}{x^2}} = e^{-m^2/2}$$

Ex:- $\lim_{x\to 0} (e^x)^{\frac{1}{x}}$

$$= e^{\lim_{x\to 0}(e^x-1)\cdot\frac{1}{x}} = e^1$$

Ex:- $\lim_{x\to 0} (a^x)^{\frac{1}{x}}$

$$= e^{\lim_{x\to 0}(a^x-1)\cdot\frac{1}{x}} = e^{\log_e a}$$

$$= a$$

$$\lim_{x\to\infty}\left[\frac{5x-4}{5x+6}\right]^{\frac{x+1}{7}}$$

$$e^{\lim_{x\to\infty}\left[\frac{5x-4}{5x+6}-1\right]\cdot\left(\frac{x+1}{7}\right)}$$

$$e^{\lim_{x\to\infty}\frac{-10}{(5x+6)}\cdot\frac{(x+1)}{7}}$$

$$e^{\lim_{x\to\infty}-\frac{10}{7}\left(\frac{x+1}{5x+6}\right)}$$

$$e^{\lim_{x\to\infty}-\frac{10}{7}\left(\frac{1+\cancel{1/x}^{\,0}}{5+\cancel{6/x}^{\,0}}\right)} = e^{-2/7}$$

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$$

$$= e^{\lim_{x\to 0}\left[\frac{\tan x}{x}-1\right]\cdot\frac{1}{x^2}}$$

$$= e^{\left\{\lim_{x\to 0}\frac{\tan x - x}{x^3}\right\}}$$ → Solve by EL Hospital

$$= e^{\frac{\cancel{x}+\frac{x^3}{3}-\cancel{x}}{x^3}}$$

$$= e^{+1/3}$$

[SANDWICH THEOREM]

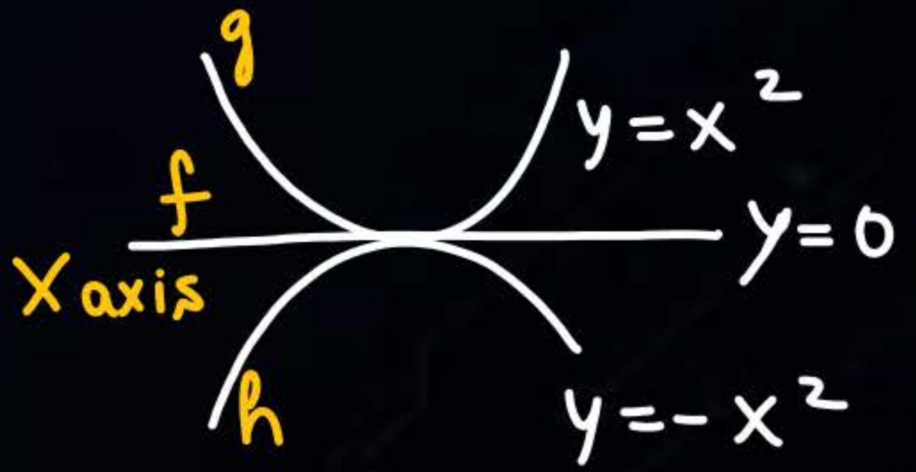
$f(x)$

$\rightarrow$ If $g(x) \le f(x) \le h(x)$

then $\underbrace{\lim_{x\to a} g(x)}_{t} \le \lim_{x\to a} f(x) \le \underbrace{\lim_{x\to a} h(x)}_{t}$

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = t$

then, $\lim_{x\to a} f(x) = t$



$$1+2+\cdots n = \frac{n(n+1)}{2} \quad , 1+3+5 \cdot\cdot n = n^2$$

$$1^2+2^2+\ldots n^2 = \frac{n(n+1)(2n+1)}{6} \quad , 2+4+6+\cdot n = n^2+n$$

$$1^3+2^3+\ldots n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\lim_{n\to\infty} \frac{[x]+[2x]+[3x]\ldots\ldots\ldots[nx]}{n^2}$$

$$\to x-1 < [x] \le x$$

$$\to 2x-1 < [2x] \le 2x$$

$$\to 3x-1 < [3x] \le 3x$$

$$\vdots$$

$$\to nx-1 < [nx] \le nx$$

$+$

$$x\frac{n(n+1)}{2} - n < N_r \le x\frac{n(n+1)}{2}$$

$$\frac{x\frac{n(n+1)}{2} - n}{n^2} < \frac{N_r}{n^2} \le \frac{x\frac{n(n+1)}{2}}{n^2}$$

$$\frac{n^2x}{2n^2} + \frac{nx}{2n^2} - \frac{n}{n^2} < \frac{N_r}{n^2} \le \frac{n^2x}{2n^2} + \frac{nx}{2n^2}$$

$$\frac{x}{2} + \cancel{\frac{x}{2n}}^{\,0} - \cancel{\frac{1}{n}}^{\,0} < \frac{N_r}{n^2} \le \frac{x}{2} + \cancel{\frac{x}{2n}}^{\,0}$$

$n\to\infty$

$$\frac{x}{2} < \lim_{n\to\infty}\frac{N_r}{n^2} \le \frac{x}{2}$$

$$\therefore \lim_{n\to\infty}\frac{N_r}{n^2} = \frac{x}{2}$$

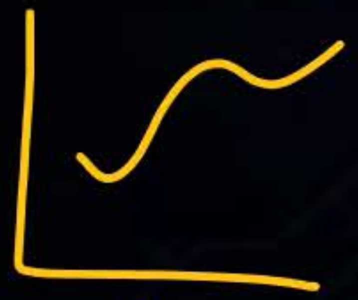
[FUNCTIONS OF TWO VARIABLES]

Let x and y are two independent variables. If a third variable z depends upon x and y, then z is called a function of two independent variables x, y, which is represented by the functional relation z = f(x, y)

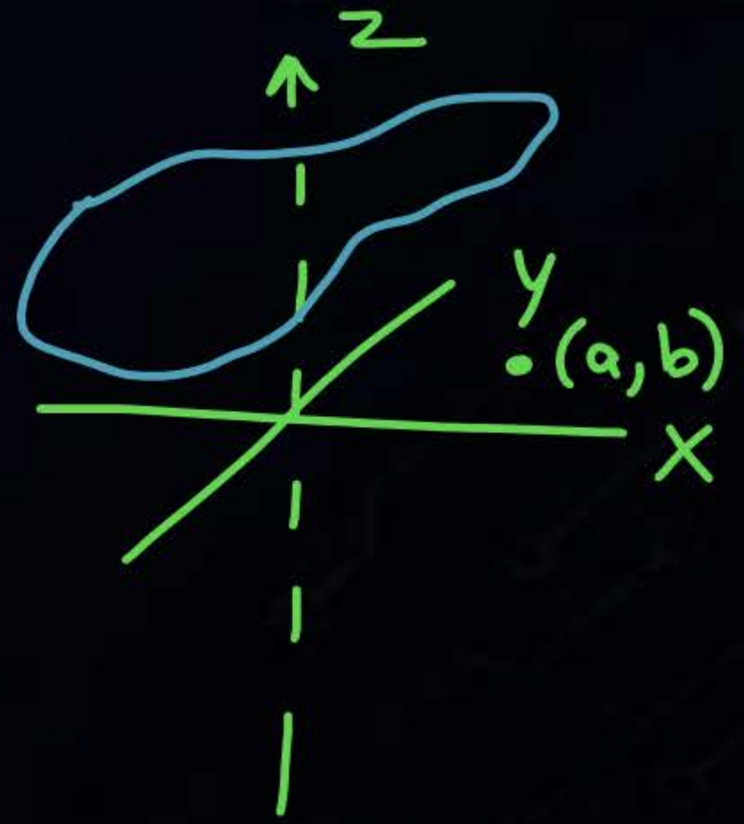
y dependent on x.

$z \rightarrow f(\underbrace{x, y}_{\text{Independent variables}}) \Rightarrow$ Surface / Region

$y = f(x)$

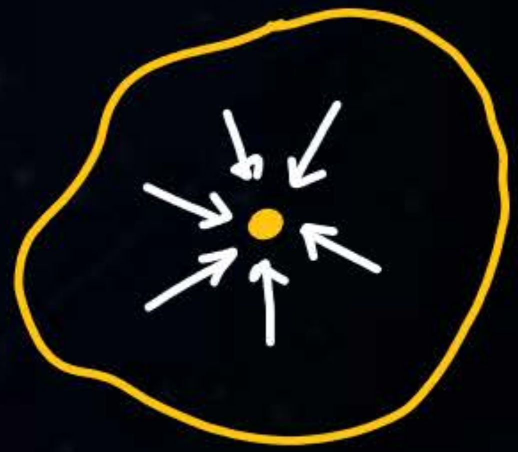


Curve



LIMIT OF A FUNCTION OF TWO VARIABLES

Limit of 2 variable function will exist if function approaches same value from all the multiple paths.



Type I :-

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y)$$

$\rightarrow \lim_{y \to b} \left[\lim_{x \to a} f(x, y) \right] = l_1$

$\rightarrow \lim_{x \to a} \left[\lim_{y \to b} f(x, y) \right] = l_2$

If $l_1 = l_2$ then limit $f(x,y)$ exists as $(x,y) \to (a,b)$

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$$

$\rightarrow$ Along X-axis $\rightarrow f_1$

$\rightarrow$ Along Y-axis $\rightarrow f_2$

$\rightarrow$ Along $y = mx \rightarrow f_3$

$\rightarrow$ Along $y = mx^2 \rightarrow f_4$

$\vdots$

$\rightarrow$ Along $y = x - mx^2 \rightarrow f_5$

If $f_1 = f_2 = f_3 \dots$ limit exists.

LHL = RHL

LIMIT OF A FUNCTION OF TWO VARIABLES

Ex:- $$\lim_{\substack{x \to 3 \\ y \to 2}} \left[\frac{x^2 + y^2}{3xy} \right]$$

$$\Rightarrow \lim_{y \to 2} \left[\lim_{x \to 3} \frac{3^2 + y^2}{3(3)y} \right]$$

$$\lim_{y \to 2} \left[\frac{9 + y^2}{9y} \right]$$

$$\frac{9 + 4}{9 \times 2} = \frac{13}{18}$$

$$\lim_{x \to 3} \left[\lim_{y \to 2} \frac{x^2 + 4}{3x(2)} \right]$$

$$\lim_{x \to 3} \left[\frac{x^2 + 4}{6x} \right]$$

$$\frac{9 + 4}{6 \times 3} = \frac{13}{18}$$

$$\therefore \lim_{\substack{x \to 3 \\ y \to 2}} \left(\frac{x^2 + y^2}{3xy} \right) = \frac{13}{18}$$

Algebra of Limits

Let f(x, y) and g(x, y) be two functions defined on some neighbourhood of a point (a, b) such that the f(x, y) = l, lim g(x, y) = m, as (x, y) → (a, b), then

(i) $\lim_{(x\cdot y)\to(a\cdot b)}[f(x,y) \pm g(x,y)] = \lim_{(x\cdot y)\to(a\cdot b)} f(x,y) \pm \lim_{(x\cdot y)\to(a\cdot b)} g(x,y) = l \pm m$

Algebra of Limits

(ii) $\lim\limits_{(x\cdot y)\to(a\cdot b)}\{f(x,y)\cdot g(x,y)\} = \lim\limits_{(x\cdot y)\to(a\cdot b)} f(x,y)\cdot \lim\limits_{(x\cdot y)\to(a\cdot b)} g(x,y) = l\cdot m$

(iii) $\lim\limits_{(x\cdot y)\to(a\cdot b)}\left\{\dfrac{f(x,y)}{g(x,y)}\right\} = \dfrac{\lim\limits_{(x\cdot y)\to(a\cdot b)} f(x,y)}{\lim\limits_{(x\cdot y)\to(a\cdot b)} g(x,y)} = \dfrac{l}{m}, provided, m \neq 0$

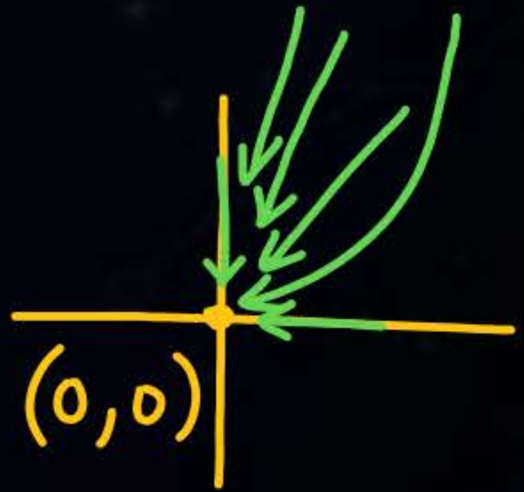
Algebra of Limits

Evaluate the limit of $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, as (x, y) → (0, 0).

→ Along x axis ; $y=0$ $f(x,0) = \frac{x^2-0}{x^2+0} = 1$

→ Along y axis; $x=0$ $f(0,y) = \frac{0-y^2}{0+y^2} = -1$

Limit does not exist.



Algebra of Limits

Evaluate the limit of $f(x,y) = \frac{2xy}{x^2+y^2}, (x,y) \neq (0,0)$ as $(x, y) \to (0, 0)$.

→ Along x axis $f(x,0) = \frac{2x.0}{x^2+0^2} = 0$

→ Along y axis $f(0,y) = \frac{2.0.y}{0^2+y^2} = 0$

→ Along $y=mx$ $f(x,mx) = \frac{2x.mx}{x^2+(mx)^2} = \frac{2\cancel{x^2}m}{\cancel{x^2}(1+m^2)} = \frac{2m}{1+m^2}$

Limit is dependent on m, hence it can take diff values as $(x,y) \to (0,0)$.

(Dependent on m)

Limit does not exist

Algebra of Limits

Evaluate the limit of the following function as (x, y) → (0, 0)

(i) $f(x,y) = \dfrac{x^3y^3}{x^2+y^2}$

→ Along x-axis $f(x,0) = 0$

→ Along y-axis $f(0,y) = 0$

→ Along $y = mx$ $\lim_{x\to 0} f(x, mx) = \dfrac{x^3(mx)^3}{x^2+(mx)^2} = \dfrac{m^3x^6}{x^2(1+m^2)} = \lim_{x\to 0}\dfrac{m^3x^4}{(1+m^2)} = 0$

→ Along $y = mx^2$ $f(x, mx^2) = 0$

limit exist at $(0,0) = 0$

Limit is independent of m.

Algebra of Limits

Evaluate the limit of the following function as $(x, y) \to (0, 0)$

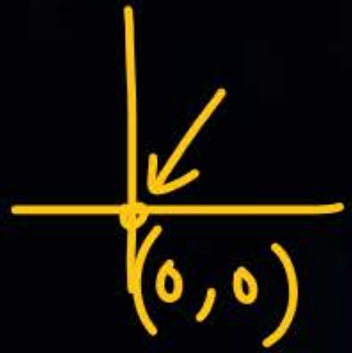
(ii) $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2}$

$\rightarrow$ Along X axis $(y=0) \rightarrow \dfrac{x^3}{x^2} = x = 0$

$\rightarrow$ Along y axis $(x=0) \rightarrow -\dfrac{y^3}{y^2} = -y = 0$

$\rightarrow$ Along $y = mx$

$$\lim_{x\to 0} \frac{x^3 - (mx)^3}{x^2 + (mx)^2} = \frac{x^3(1-m^3)}{x^2(1-m^2)} = x\,f(m) = 0$$



limit is independent of m.

$\therefore$ limit exist & $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Algebra of Limits

Evaluate the limit of the function

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, when\ (x,y) \to (0,0).$$

Algebra of Limits

Evaluate the limit of the function

$$\lim_{(x \cdot y) \to (0,0)} \frac{2xy^2}{x^2 + y^4}.$$

Algebra of Limits

Evaluate the limit of the function

Check along $y = x - mx^3$

$$f(x,y) = \frac{x^3 + y^3}{x - y}, (x,y) \neq (0,0) \text{ at origin.}$$

[FUNDAMENTALS OF CONTINUITY]

Continuity

A function y = f(x) is said to be continuous if the graph of the function is a continuous curve. On the other hand if a curve is broken at some point say x = a, we say that the function is not continuous or discontinuous.

(1) $f(x)$ exists at $x = a$

(2) $\lim_{x \to a} f(x)$ exists at $x = a$ [LHL = RHL]

(3) $\lim_{x \to a} f(x) = f(a)$ $[\underbrace{\text{LHL} = \text{RHL}}_{\text{Limit}} = \text{Value}]$

Limit = Value

[FUNDAMENTALS OF CONTINUITY]

Definition of Continuity

A function f(x) is said to be continuous at x = a if and only if the following three conditions are satisfied

(i) f(x) exists; that is f(x) is defined at x = a

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

FUNDAMENTALS OF CONTINUITY

Jump of a Function at a Point

Let f(x) be a function for which the two limits $f(\alpha + 0)$ and $f(\alpha - 0)$ at $x = \alpha$ both exists, where

$$f(\alpha + 0) = \lim_{h \to 0} f(\alpha + h) \text{ and } f(\alpha - 0) = \lim_{h \to 0} f(\alpha - h)$$

Then their non-negative differences $|f(\alpha + 0) - f(\alpha - 0)|$ is called the jump of the function at $x = \alpha$.

FUNDAMENTALS OF CONTINUITY

Fundamental Theorems on continuity

Theorem: If f(x) and g(x) are continuous at $x = \alpha$, then the functions

(i) $f(x) + g(x)$ (ii) $f(x) - g(x)$ (iii) $f(x)g(x)$

(iv) $\frac{f(x)}{g(x)}, g(x) \neq 0$ *and* (v) $f(g(x))$ are also continuous at $x = 0$

→ Every constant function $\{x = 5$ or $x = K\}$

→ All polynomials Ex :- $x, x^2, x - x^3, x^2 + 2x + 5$

→ $\sin x, \cos x, e^x, e^{-x}, a^x, a^{-x}$

These are continuous for all $x \in R$.

FUNDAMENTALS OF CONTINUITY

Discontinuous Functions

A function f(x) is said to be discontinuous at x = a if we have any of the following cases:

(i) $\lim_{x \to a} f(x)$ does not exist

(ii) $\lim_{x \to a} f(x) \neq f(a)$

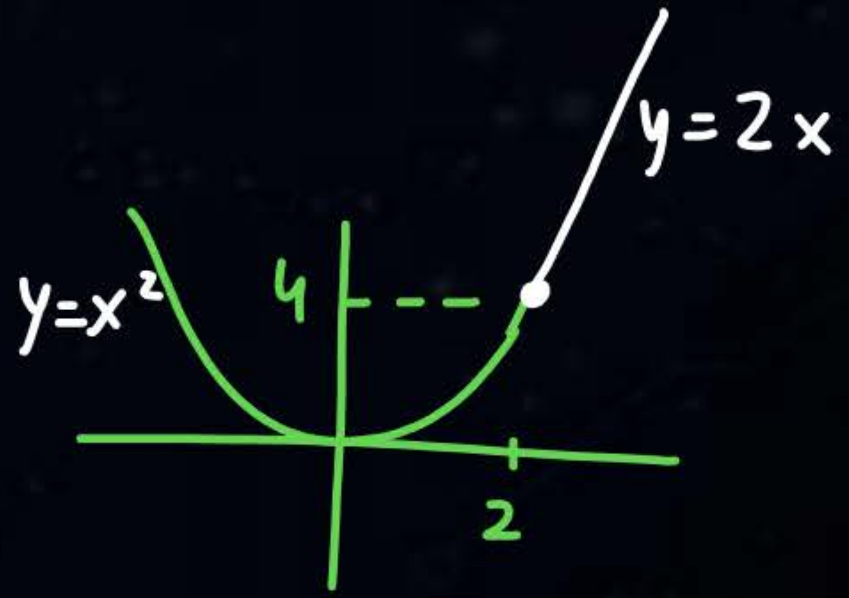
(iii) f(a) is undefined

Ex:- $f(x) = \begin{cases} x^2 & ; x \le 2 \\ 2x & ; x > 2. \end{cases}$

At $x = 2$.

$$\begin{cases} f(2-h) = (2-h)^2 = 4 \\ f(2+h) = 2(2+h) = 4 \end{cases}$$

$$f(2) = 2^2 = 4$$



Ex:- Check if functions are continuous:-

i) $e^{-x} \sin x \rightarrow C$

ii) $a^x(2x^2+6x+5) \rightarrow C$

iii) $e^x \tan x \rightarrow$ but discontinuous at $(2n+1).\pi/2$

iv) $\frac{e^x}{a^x} \rightarrow C$

[KINDS OF DISCONTINUITIES]

(1) Removable Discontinuity

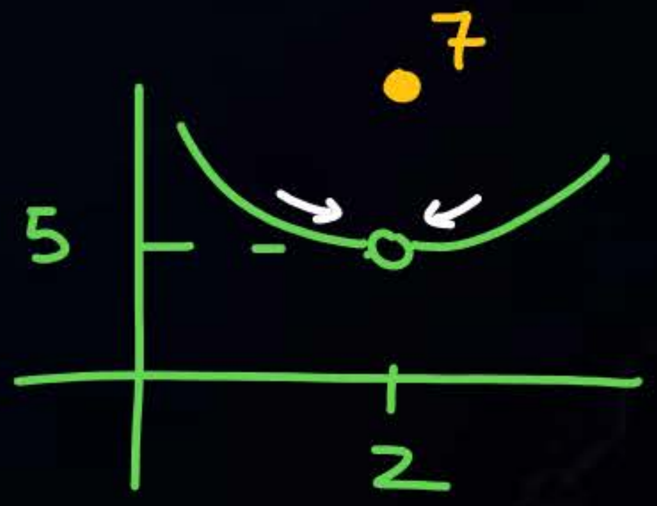
A function f(x) is said to have a discontinuity of removable kind at x = a if $\lim_{x \to a} f(x)$ exist but not equal to the value of function at x = a

Limit exist $\quad \lim_{x \to 0} f(x) \neq f(a)$

LHL = RHL = 5

Value = 7

Limit ≠ Value



[KINDS OF DISCONTINUITIES]

Removable Discontinuity

$$f(x) = x\sin\frac{1}{x}, x \neq 0$$
$$= 2 \qquad x = 0$$

$\lim_{h\to 0} f(0-h) = (0-h)\sin\frac{1}{0-h} = -h\sin\left(-\frac{1}{h}\right) = h\sin\frac{1}{h} \to 0 \times \text{Oscillatory value} = 0$

$\lim_{h\to 0} f(0+h) = (0+h)\sin\frac{1}{0+h} = h\sin\frac{1}{h} \to 0 \times \text{Oscillatory value} = 0$

$\left.\begin{array}{l}\text{Limit exist } (LHL = RHL = 0) \\ \text{Value} = 2\end{array}\right\}$ Discontinuous

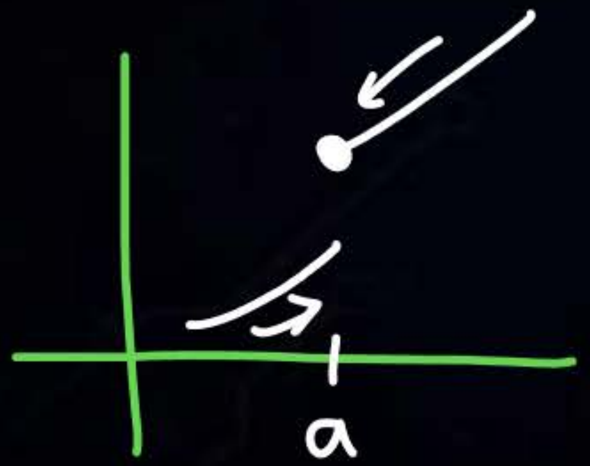
[KINDS OF DISCONTINUITIES]

Discontinuity of First Kind/Jump Discontinuity

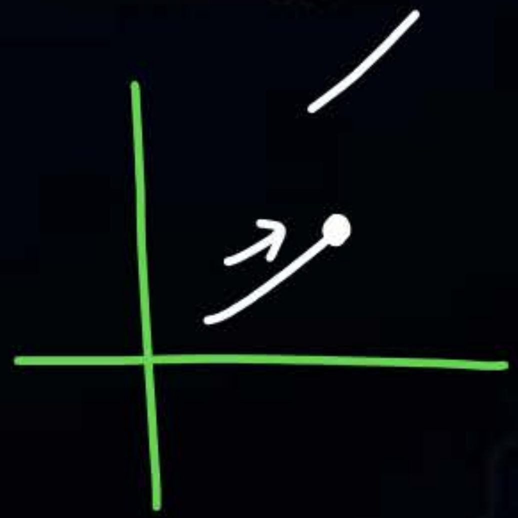
A function f(x) is said to have a discontinuity of first kind at $x = a$ if both $f(a - 0)$ and $f(a + o)$ exist but are unequal. The point $x = a$ is said the point of discontinuity from left or from right according to as follows

$f(a - 0) \neq f(a) = f(a + 0)$ or $f(a - 0) = f(a) \neq f(a + 0)$

It is also known as ordinary discontinuity.



LHL $\neq$ RHL = Value



Value = LHL $\neq$ RHL

[KINDS OF DISCONTINUITIES]

Discontinuity of First Kind/Jump Discontinuity

$$f(x) = [x], \quad x \neq 0$$
$$\quad\quad\;\; = 0 \quad x = 0$$

Thank you

GW

Soldiers !