Branch : CSE/IT **Batch : Hinglish**

Discrete Mathematics

Combinatorics

DPP-05

[MCQ]

1. In how many ways can 3000 identical envelopes be divided, in packages of 25, among four student groups so that each group gets at least 150, but not more than 1000, of the envelopes?

(a) $\binom{99}{96} - 4\binom{64}{61} + 6\binom{29}{26}$

(b) $\binom{99}{96} - 4\binom{62}{61} + 6\binom{29}{26}$

(c) $\binom{99}{96} - 4\binom{62}{61} + 5\binom{29}{26}$

(d) None

[NAT]

2. Determine the coefficient of x^{15} in

$f(x) = (x^2 + x^3 + x^4 + \cdots)^4$.

[NAT]

3. In how many ways can a police captain distribute 24 rifle shells to four police officers so that each officer gets at least three shells, but not more than eight?

[MCQ]

4. Determine the sequence generated by the following generating functions:

f(x) = 1/(1 + 3x)

(a) $1, 3, 3^2, 3^3, \ldots$

(b) $1, 3, 3^2, -3^3, \ldots$

(c) $1, -3, 3^2, 3^3, \ldots$

(d) $1, -3, 3^2, -3^3, \ldots$

[NAT]

5. In how many ways can two dozen identical robots be assigned to four assembly lines with at least three robots assigned to each line?

[MCQ]

6. Find a recurrence relation, with initial condition, that uniquely determines the following geometric progressions:

7, 14/5, 28/25, 56/125

(a) $a_n = 5a_{n-1}$, $n \geq 1$, $a_0 = 2$

(b) $a_n = -3a_{n-1}$, $n \geq 1$, $a_0 = 6$

(c) $a_n = (2/5)a_{n-1}$, $n \geq 1$, $a_0 = 7$

(d) None of these

[MSQ]

7. If a_n, $n \geq 0$, is the unique solution of the recurrence relation $a_{n+1} - da_n = 0$, and $a_3 = 153/149$, $a_5 = 1377/2401$, What is d?

(a) d = 3/7
(b) d = 4/7
(c) d = –3/7
(d) d = –4/7

[MCQ]

8. If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, and $a_3 = 37$ satisfy the recurrence relation $a_{n+2} + ba_{n+1} + ca_{n=0}$, where $n \geq 0$ and b, c are constants, determine values for b, c respectively

(a) b = –4, c = –21

(b) b = –21, c = –4

(c) b = 4, c = 21

(d) None of these

[MCQ]

9. Determine the constants b and c if $a_n = c_1 + c_2(7^n)$, $n \geq 0$, is the general solution of the relation $a_{n+2} + ba_{n+1} + ca_{n=0}$, $n \geq 0$.

(a) b = 7, c = –8
(b) b = –8, c = 7
(c) b = –8, c = 8
(d) None of these

Answer Key

1. **(a)**
2. **(120)**
3. **(125)**
4. **(d)**
5. **(455)**
6. **(c)**
7. **(a, c)**
8. **(a)**
9. **(b)**

Hints and Solutions

1. (a)

Consider each package of 25 envelopes as one unit. Then the answer to the problem is the coefficient of x^{120} in $\left(x^6+x^7+\ldots+x^{38}+x^{40}\right)^4$

$=x^{24}\left(1+x+\ldots+x^{34}\right)^4$. This is the same as the coefficient of x^{96} in $\left[\left(1-x^{35}\right)/\left(1-x\right)\right]^4$

$$=\left(1-x^{35}\right)^4(1-x)^{-4}=\left[1-4x^{35}+6x^{70}-\ldots+x^{140}\right]$$

$$\left[\binom{-4}{0}+\ldots+\binom{-4}{26}(-x)^{26}+\ldots+\binom{-4}{61}(-x)^{61}+\ldots+\binom{-4}{96}(-x)^{96}+\ldots\right].$$

Consequently the answer is

$$\binom{-4}{96}(-1)^{96}-4\binom{-4}{61}(-1)^{61}+6\binom{-4}{26}(-1)^{26}$$

$$=\binom{99}{96}-4\binom{64}{61}+6\binom{29}{26}.$$

2. (120)

Since $(x^2+x^3+x^4+\ldots)=x^2(1+x+x^2+\ldots)=x^2/(1-x)$, the coefficient of x^{15} in f(x) is the coefficient of x^{15} in $(x^2/(1-x))^4=x^8/(1-x)^4$. Hence the coefficient sought is that of x^7 in $(1-x)^{-4}$, namely,

$$\binom{-4}{7}(-1)^7=(-1)^7\binom{4+7-1}{4}(-1)^7=\binom{10}{7}=120.$$

3. (125)

The choices for the number of shells each officer receives are given by $x^3+x^4+\cdots+x^8$. There are four officers, so the resulting generating function is

$$f(x)=(x^3+x^4+\cdots+x^8)^4$$

We seek the coefficient of x^{24} in $f(x)$. With $(x^3+x^4+\cdots+x^8)^4=x^{12}(1+x+x^2+\cdots+x^5)^4=x^{12}\left((1-x^6)/(1-x)\right)^4$, the answer is the coefficient of x^{12} in $(1-x^6)^4$. $(1-x)^{-4}=$ $\left[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\binom{4}{3}x^{18}+x^{24}\right]\left[\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+\cdots\right]$, which is $\left[\binom{-4}{12}(-1)^{12}-\binom{4}{1}\binom{-4}{6}(-1)^6+\binom{4}{2}\binom{-4}{0}\right]=\left[\binom{15}{12}-\binom{4}{1}\binom{9}{6}+\binom{4}{2}\right]=125.$

4. (d)

$f(x)=1/(1+3x)=1+(-3x)+(-3x)^2+(-3x)^3+\ldots$, so f(x) generates the sequence $1, -3, 3^2, -3^3, \ldots$

5. (455)

$(x^3+x^4+\cdots)^4=x^{12}(1+x+x^2+\cdots)^4=x^{12}(1-x)^{-4}$. The coefficient of x^{12} in $(1-x)^{-4}$ is $\binom{-4}{12}(-1)^{12}=(-1)^{12}\binom{4+12-1}{12}(-1)^{12}=\binom{15}{12}$.

6. (c)

$a_n=(2/5)a_{n-1}$, $n\geq 1$, $a_0=7$

7. (a, c)

$a_{n+1}-da_n=0$, $n\geq 0$, so $a_n=d^na_0$ $153/49=a_3=d^3a_0$, $1377/2401=a_5=d^5a_0\Rightarrow a_5/a_3=d^2=9/49$ and $d=\pm 3/7$

8. (a)

$(n=0)$; $a_2+ba_1+ca_0=0=4+b(1)+c(0)$, so $b=-4$.

$(n=1)$; $a_3+4a_2+ca_1=0=37-4(4)+c$, so $c=-21$.

$a_{n+2}-4a_{n+1}-21a_n=0$

$r^2 - 4r - 21 = 0 = (r - 7)(r + 3)$, $r = 7, -3$

$a_n = A(7)^n + B(-3)^n$

$0 = a_0 = A + B \Rightarrow B = -A$

$1 = a_1 = 7A - 3B = 10A$, so $A = 1/10$, $B = -1/10$ and $a_n = (1/10)[(7)^n - (-3)^n]$, $n \geq 0$.

9. (b)

$a_n = c_1 + c_2(7)^n, n \geq 0$, is the solution of $a_{n+2} + ba_{n+1} + ca_n = 0$, so $r^2 + br + c = 0$ is the characteristic equation and $(r-1)(r-7) = (r^2 - 8r + 7) = r^2 + br + c$.

Consequently, $b = -8$ and $c = 7$