ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-03

Vector Calculus

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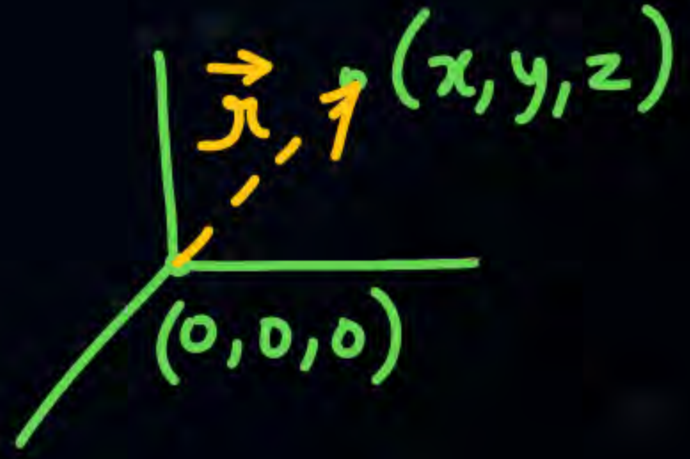
Topics to be Covered

- VECTOR BASICS
- STRAIGHT LINES/3D PLANES
- GRADIENT (VECTOR DIFFERENTIATION)
- DIVERGENCE (VECTOR DIFFERENTIATION)
- CURL (VECTOR DIFFERENTIATION)
- LINE, SURFACE, VOLUME INTEGRAL (VECTOR INTEGRATION)
- GREEN, & STOKE'S THEOREM (VECTOR INTEGRATION)
- GAUSS DIVERGENCE THEOREM (VECTOR INTEGRATION)

CURL OF VECTOR POINT FUNCTION

Vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Scalar $r^2 = x^2 + y^2 + z^2$



$\text{grad } r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$

$\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$

$2r\frac{\partial r}{\partial x} = 2x$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

$\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$\boxed{\text{grad } r = \frac{\vec{r}}{r} = \hat{r}}$

$\boxed{\text{Grad } r = \frac{\vec{r}}{r} = \hat{r}}$

$\boxed{\text{Div } \vec{r} = 3}$

$\boxed{\text{Curl } \vec{r} = 0}$

$\text{div } \vec{r} = 1 + 1 + 1 = 3$

$\text{Curl } \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$

$\boxed{\text{grad}[f(r)] = f'(r) \cdot \frac{\vec{r}}{r} = f'(r).\hat{r}}$

Q.

$$\text{grad } r = \frac{\vec{r}}{r} = \hat{r}$$

- **grad (log r) =** $\frac{1}{r} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

- **grad (r²) =** $2r \cdot \frac{\vec{r}}{r} = 2\vec{r}$

- **grad (r log r) =** $\left[r \cdot \frac{1}{r} + 1 \cdot \log r\right] \cdot \frac{\vec{r}}{r} = \left[\frac{1+\log r}{r}\right]\vec{r}$

Q.

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

- $\text{grad}\,(\vec{a}\cdot\vec{r}) = \vec{a}$

$$\vec{a}\cdot\vec{r} = a_1 x + a_2 y + a_3 z$$
$$\text{grad}\,\vec{a}\cdot\vec{r} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

$$\vec{r}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$
$$= (a_3 y - a_2 z)\hat{i} - (a_3 x - a_1 z)\hat{j} + (a_2 x - a_1 y)\hat{k}$$

- $\text{div}\,\vec{a} = 0$

NOTE:- Div & curl of constant vector is always $\underline{0.}$

$\rightarrow$ $\text{div}\,(\vec{r}\times\vec{a}) = 0$

$\rightarrow$ $\text{curl}\,(\vec{r}\times\vec{a}) = -2\vec{a}$

$$[-a_1 - (a_1)]\hat{i} - [a_2 - (-a_2)]\hat{j} \; [-a_3 - (a_3)]\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

- $\text{Curl}\,\vec{a} = 0$

VECTOR IDENTITIES:-

1. $\text{grad}(uv) = u \,\text{grad}\, v + v \,\text{grad}\, u$.
2. $\text{div}(u\vec{a}) = u \,\text{div}\, \vec{a} + \vec{a} \,\text{grad}\, u$
3. $\text{div}(\vec{a}\times\vec{b}) = \vec{b} \,\text{curl}\, \vec{a} - \vec{a} \,\text{curl}\, \vec{b}$
4. $\text{curl}(u\vec{a}) = u \,\text{curl}\, \vec{a} + (\text{grad}\, u)\times\vec{a}$
5. $\text{div grad}\, f = \vec{\nabla}\cdot\vec{\nabla f} = \nabla^2 f$
6. $\text{div curl}\, \vec{f} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{f}) = 0$
7. $\text{curl grad}\, f = \vec{\nabla}\times\vec{\nabla f} = 0$
8. $\text{grad div}\, \vec{f} = \text{curl curl}\, \vec{f} + \nabla^2\vec{f}$
9. $\text{curl curl}\, \vec{f} = \text{grad div}\, \vec{f} - \nabla^2\vec{f}$

$f \to f(r)$

$$\boxed{\begin{aligned} \text{div}\, \vec{f} &= \vec{\nabla}\cdot\vec{f} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 |f| \end{aligned}}$$

$$\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

Ex:- i) Find div curl $\vec{f}$; $\vec{f} = x^2y\,\hat{i} + y^2\,\hat{j} + 2zy\,\hat{k}$

$= 0$

Ex:- ii) Find curl grad f ; $f = x^2yz$

$= 0$

$$\text{curl}\ \underbrace{\left(2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}\right)}_{\text{grad } f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz & x^2z & x^2y \end{vmatrix} = (x^2 - x^2)\,\hat{i} - (2xy - 2xy)\,\hat{j} + (2xz - 2xz)\,\hat{k}$$

$= 0$

Q. $f = x^2 - y^2$, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Harmonic

Find div(grad f) ?

$$\vec{\nabla} \cdot \vec{\nabla f} = \nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 - 2 = 0$$

Ex:- $f = ax^2 y - y^3$

$\text{div}(\text{grad} f) = 2ay - 6y$

Q. Value of $\nabla \cdot (\vec{\nabla} \times \vec{v})$ where $\vec{v} = (2yz)\hat{\imath} + (3xz)\hat{\jmath} + (4xy)\hat{k}$

div curl $v = 0$

Ex:-

$$\left.\begin{aligned} \vec{A} &\rightarrow f(r) \\ |\vec{A}| &= 4r^3 \end{aligned}\right\} \text{Given}$$

Find div $\vec{A} = \vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2|\vec{A}|\right) = \frac{1}{r^2}\frac{\partial}{\partial r}.r^2(4r^3)$

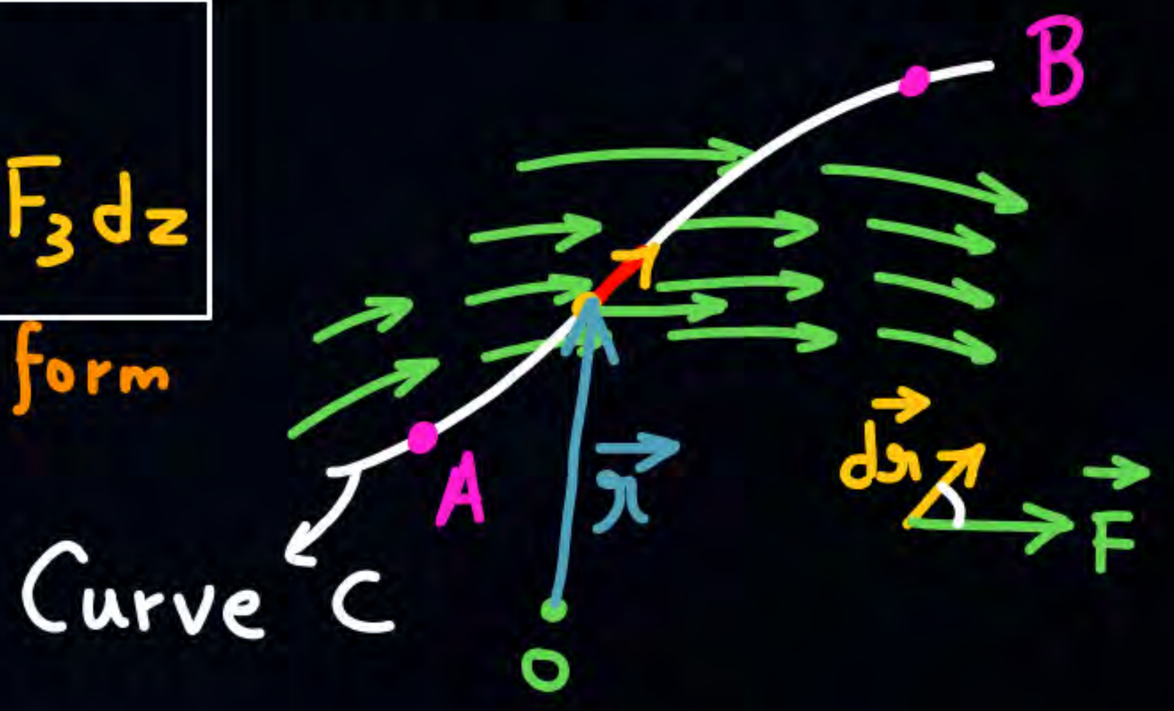
$= \frac{1}{r^2}.4(5r^4) = 20r^2$

[VECTOR INTEGRATION]

Line integral :- Any integral which is to be evaluated along a curve.

$$\boxed{L.I. = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B F_1\,dx + F_2\,dy + F_3\,dz}$$

Vector form   Cartesian form



$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

Work done by a force $(\vec{F})$ in moving particle from A to B along curve C is $\left\{ W = \int_A^B \vec{F}\cdot d\vec{r} \right\}$

[VECTOR IMETEGRATOPN]

Simple closed curve

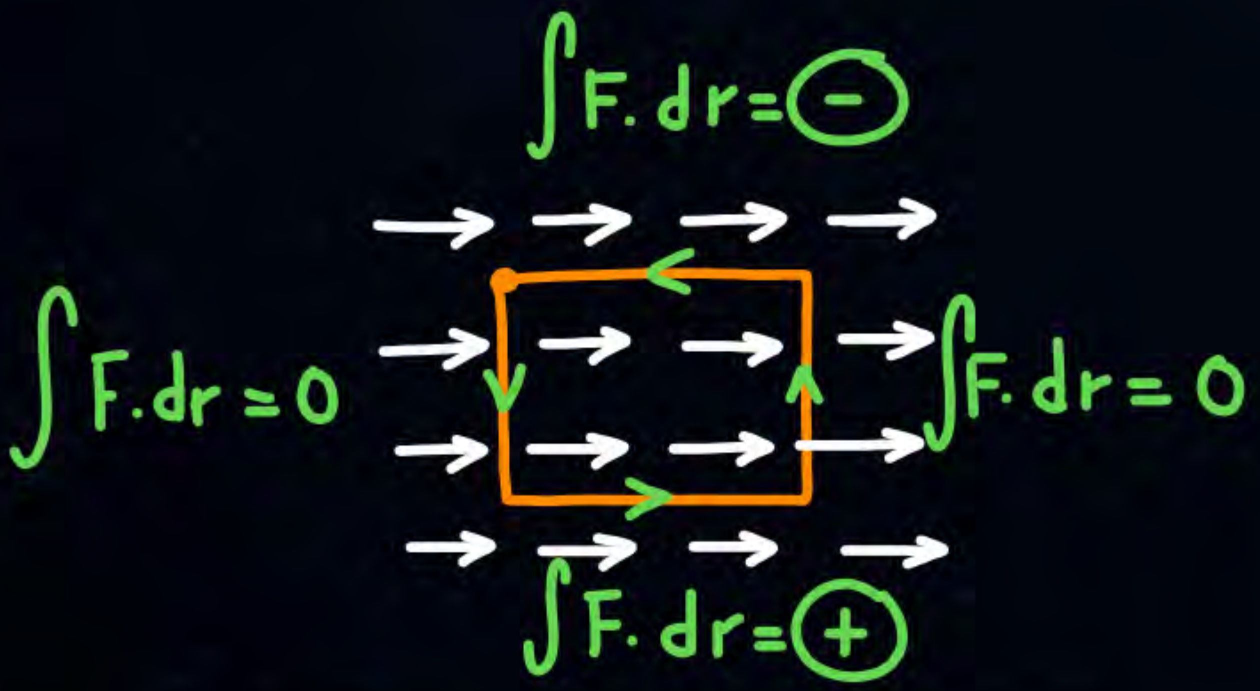
[A curve which do not intersect itself]

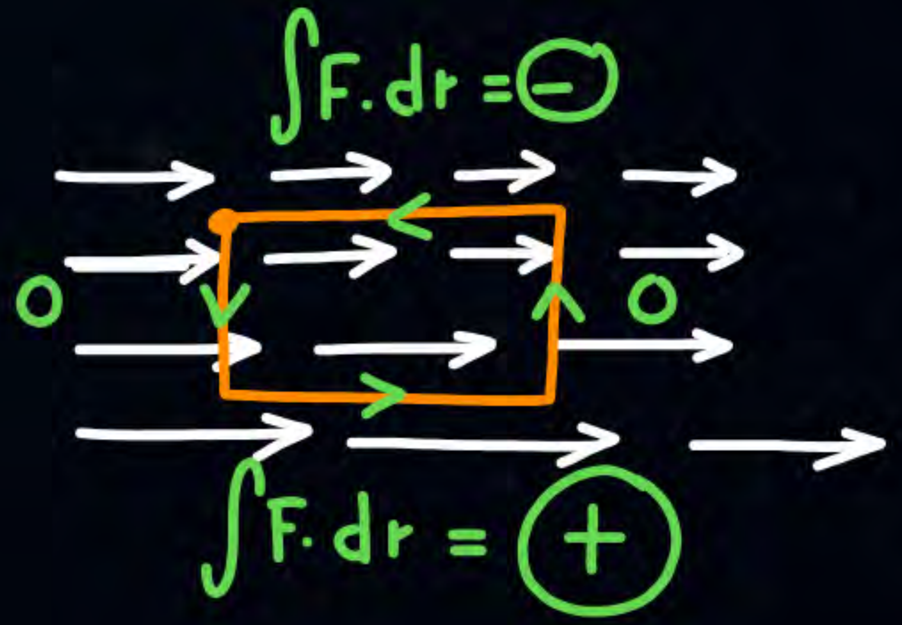
Closed curve but not simple

[A curve which intersect itself anywhere]

CIRCULATION:- The L.I. of a vector pt fn. $\vec{F}$ along a simple closed curve is called as circulation of $\vec{F}$ along C.

$$\oint \vec{F}\cdot d\vec{r}$$





$\oint F.dr = 0$

curl $\vec{F} = 0$

Irrotational field

Conservative field

$\oint F.dr \neq 0$

curl $\vec{F} \neq 0$

Rotational field

Non-conservative field

→ If L.I. (Work done) is path independent

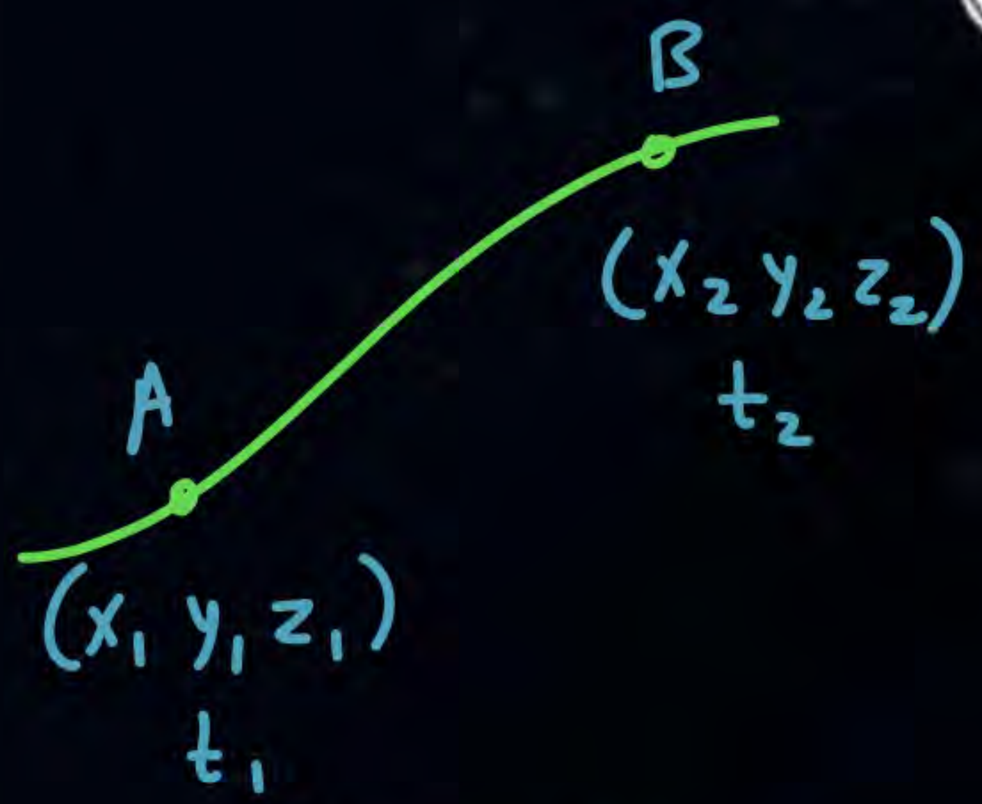
i.e. L.I. is same in moving from A to B along any path.

$$\int_{x_1}^{x_2} \vec{F}\cdot\vec{dr}$$

$$\int_{y_1}^{y_2} \vec{F}\cdot\vec{dr}$$

$$\int_{z_1}^{z_2} \vec{F}\cdot\vec{dr}$$

$$\int_{t_1}^{t_2} \vec{F}(t)\cdot dt$$



$x_1 \to x_2$
$y_1 \to y_2$
$z_1 \to z_2$
$t_1 \to t_2$

$y = 2x$
$dy = 2\,dx$

$$\int_{x=0}^{x=1} 3x(2x)\,dx - (2x)^2\,2dx$$

$$= \left[6\,\frac{x^3}{3} - 8\,\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

curl $\vec{F} \neq 0$

Q. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evalute $\int_C \vec{F}.\overrightarrow{dr}$ where C is given as y = $2x^2$ from (0, 0) to (1,2)

Soln :-

$$\int_{(0,0)}^{(1,2)} F.dr = (3xy\,\hat{i} - y^2\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int 3xy\,dx - y^2 dy$$

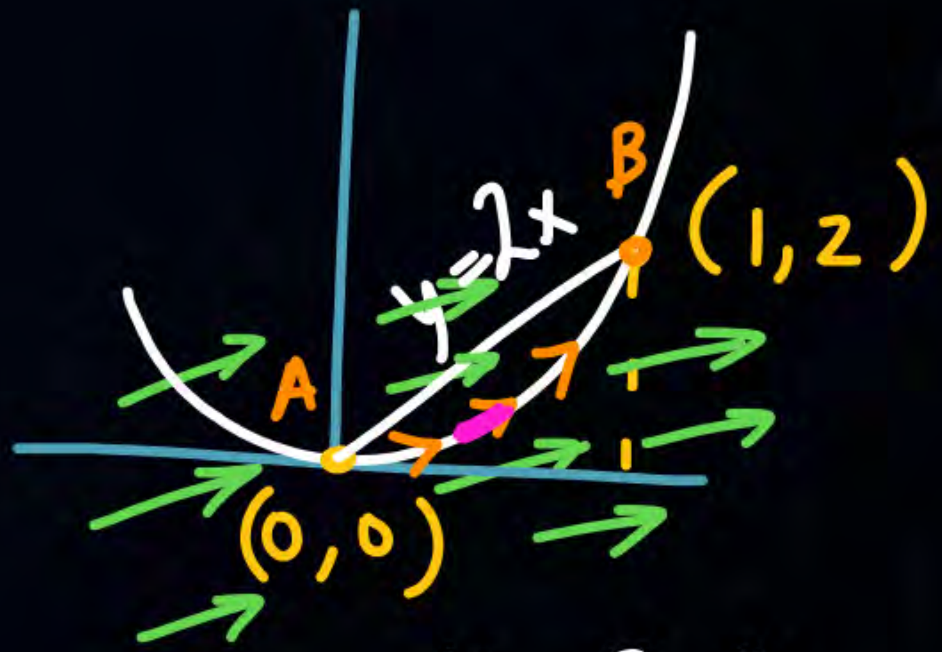
$$= \int_{x=0}^{x=1} 3x(2x^2)\,dx - (2x^2)^2(4x\,dx)$$

$$\int_0^1 6x^3 - 16x^5\,dx = \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1$$

$$= \boxed{-\frac{7}{6}}$$

$x = t$

$y = 2t^2$



$y = 2x^2$

$dy = 4x\,dx$

Q. Find $\int_c 3xy\ dx - y^2\ dy$ along the curve $y = 2t^2$; $x = t$ from $t = 0$ to $t = 2$.

$$\int_{t=0}^{t=2} 3xy\,dx - y^2\,dy$$

$$= \int_0^1 3\,t\,2t^2\,dt - (2t^2)^2\,4t\,dt$$

$$= \int_0^1 6t^3 - 16t^5\,dt = -7/6$$

$$\left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^2 = -440/3$$

$x = t$

$y = 2t^2$

$t \to 0 \to 2$

$(0,0) \to (2,8)$

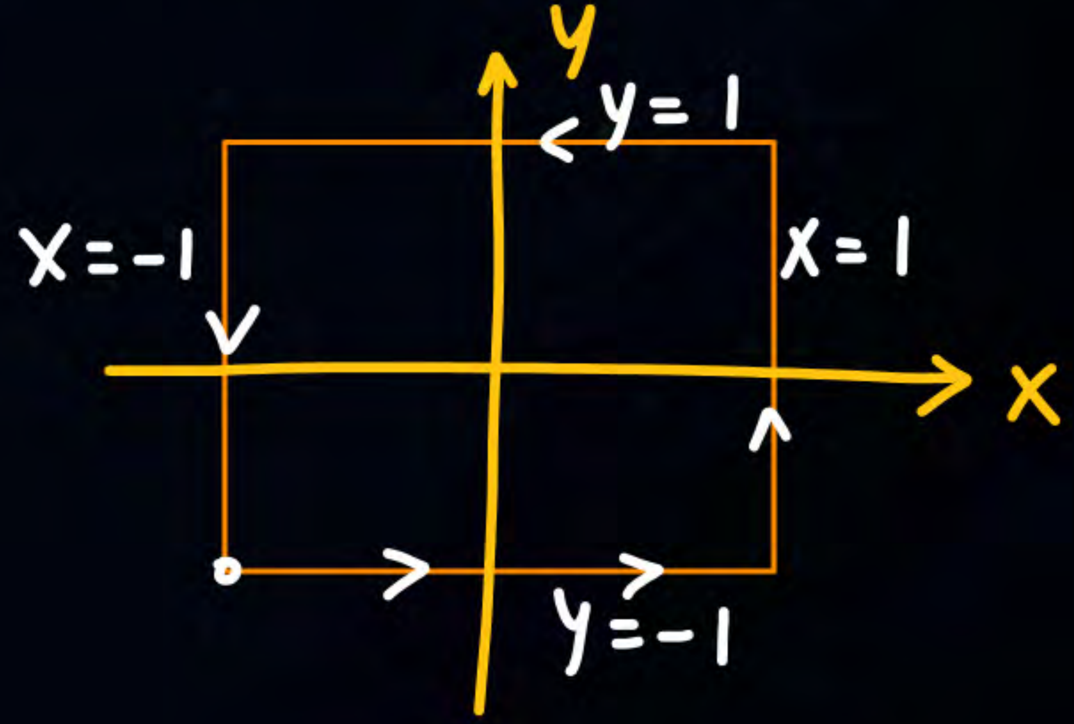
$dx = dt$

$dy = 4t\,dt$

$t \to 0 \to 1$

$(0,0)\quad(1,2)$

Q. $\int_C \vec{F}\cdot\overrightarrow{dr}$ where $\vec{F} = \frac{y\hat{\imath}-x\hat{\jmath}}{x^2+y^2}$ and C is square formed by line and $x = \pm 1$ & $y = \pm 1$,



Thank you

GW Soldiers !