ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-6

Linear Algebra

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Topics to be Covered

- System of Linear Equations
- Augmented Matrix
- Non-Homogenous Linear Equations
- Solution of Non-Homogenous Linear Equations
- Homogenous Linear Equations
- Solution of Homogenous Linear Equations

VECTOR SPACE

Ordered pair :- (a, b) is an ordered pair in which position of a, b are fixed.

Ordered triplet :- (a, b, c) is an ordered triplet in which position of a, b and c are fixed.

Ordered n-tuple :- Set of ordered n-numbers $\{x_1, x_2, x_3 \ldots\ldots x_n\}$ in which their positions are fixed.

$\vec{X} = \hat{i} + 2\hat{j}$

$3\hat{i} - 5\hat{j}$

$\vec{X} = [1 \quad 2]$ $[3, -5]$

$\vec{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$\hat{i} + 2\hat{j} - 3\hat{k}$

$[1 \quad 2 \quad -3]$

$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

Ordered n-tuple

$[1 \quad -1 \quad 2 \quad 5 \ldots]$

LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

→ If $\vec{X}_1, \vec{X}_2, \vec{X}_3 \dots \vec{X}_n$ are n-vectors such that they can be expressed as a linear combination where $K_1, K_2, K_3 \dots$ are all not 0 simultaneously.

$$K_1 X_1 + K_2 X_2 + K_3 X_3 + \cdots K_n X_n = 0$$

(Relation exist between them)

Then these set of vectors $X_1, X_2, X_3 \dots$ are L.D.

→ If $\vec{X}_1, \vec{X}_2, \vec{X}_3 \dots \vec{X}_n$ are n-vectors such that they can be expressed as $K_1 X_1 + K_2 X_2 + \cdots K_n X_n = 0$ where $K_1, K_2 \dots K_n$ are all 0 simultaneously.

(No relation exist b/w them)

Then these set of vectors $X_1, X_2, X_3 \dots$ are L.I.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 15 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 0 \\ 2 & 10 & -1 \\ 4 & 20 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & -3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ -7 & 1 & -6 \\ 1 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 & 3 \\ 1 & 7 & 5 \\ 5 & 9 & 8 \end{bmatrix}$$

	$R_2 = 3R_1$, $R_3 = 5R_1$	$5C_1 = C_2$	$R_1 = R_2$	$C_1 = C_3$	$C_1 + C_2 = C_3$	$R_1 + R_2 = R_3$
No. of relations	2	1	1	1	1	1
Nullity	2	1	1	1	1	1
Rank	1	2	2	2	2	2

Order of square matrix = Rank + Nullity

Rank → L I rows/columns

Nullity → L D rows/columns

2-Dimension space :-

Ex:- $\vec{X}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \vec{X}_2 = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$\vec{X}_2 = K \vec{X}_1 \quad \begin{bmatrix} 5 \\ 10 \end{bmatrix} = K \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

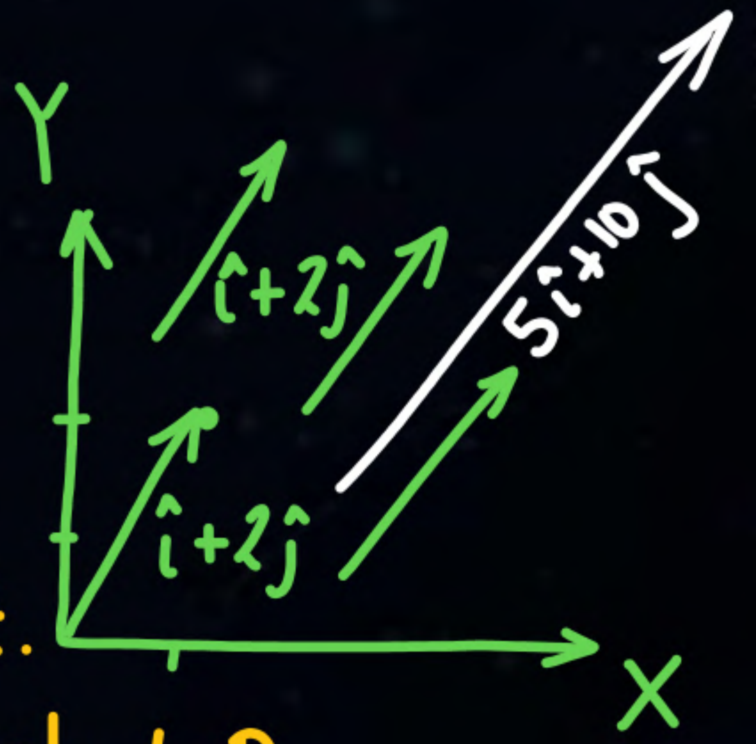
$\therefore K = 5$

NOTE :-

If $\vec{X}_1 = K\vec{X}_2$, then

- $\vec{X}_1, \vec{X}_2$ are co-linear.
- $\vec{X}_1, \vec{X}_2$ are L. Dependent.
- These $\vec{X}_1, \vec{X}_2$ spans only 1-D
- $A = [x_1 \ \ x_2] = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$

  $\rho(A) = 1 < 2 \Rightarrow$ L.D.



Ex:- $\vec{X}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{X}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

$\vec{X}_2 \neq K \vec{X}_1 \quad \begin{bmatrix} 5 \\ -6 \end{bmatrix} = K \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$K = 5 \ , \ K = -2$

- $\vec{X}_1, \vec{X}_2$ are not co-linear
- $\vec{X}_1, \vec{X}_2$ are L. Independent.
- These spans $\vec{X}_1, \vec{X}_2$ spans 2-D.
- $A = \begin{bmatrix} 1 & 5 \\ 3 & -6 \end{bmatrix}$ $\quad \rho(A) = 2 =$ No. of vectors.

NOTE :- If $A = [X_1 \;\; X_2 \;\; X_3 \;\; \ldots X_n]$ and each vector is of n-dimension.

① $\rho(A) <$ No. of vectors $\Rightarrow |A| = 0$

$\Rightarrow$ Then the set of vectors are L. D.

② $\rho(A) =$ No. of vectors $\Rightarrow |A| \neq 0$

$\Rightarrow$ Then the set of vectors are L. I.

NOTE :- Dimension of each vector $>$ No. of vectors

$\Rightarrow$ Then vectors are L. I.

Dimension of each vector $<$ No. of vectors

$\Rightarrow$ Then vectors are L. D.

Considering 3-D vectors

$\vec{X_1} \rightarrow$ 1-D

$\vec{X_1}, \vec{X_2}$ —
- $\rightarrow \vec{X_1} = K\vec{X_2}$ (1-D)
- $\rightarrow \vec{X_1} \neq K\vec{X_2}$ (2-D)

$\vec{X_1}, \vec{X_2}, \vec{X_3}$ —
- $\rightarrow \rho(A) = 1$ (1-D)
- $\rightarrow \rho(A) = 2$ (2-D)
- $\rightarrow \rho(A) = 3$ (3-D)

3-D Space :-

Ex :- Check sets of vectors are LD or LI.

i) $\vec{X}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$ $\vec{X}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$A = [X_1 \ X_2 \ X_3]$

$= \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $|A|_{3\times 3} \neq 0$

$\therefore \rho(A) = \text{No. of vectors} = 3 \ \{\because \text{L.I.}\}$

$K_1 X_1 + K_2 X_2 + K_3 X_3 = 0$ {when K_1, K_2, K_3 are 0}

ii) $\vec{X}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ $\vec{X}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{X}_4 = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix}$

$A = [X_1 \ X_2 \ X_3 \ X_4]$ $A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\therefore \rho(A) < \text{No. of vectors}$ $\Rightarrow$ Set of vectors are L.D.

$3 < 4$

$$K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4 = 0$$

$$K_1\begin{bmatrix}1\\2\\4\end{bmatrix} + K_2\begin{bmatrix}2\\-1\\3\end{bmatrix} + K_3\begin{bmatrix}0\\1\\2\end{bmatrix} + K_4\begin{bmatrix}-3\\7\\2\end{bmatrix} = 0$$

$$\begin{aligned} K_1 + 2K_2 - 3K_4 &= 0 \\ 2K_1 - K_2 + K_3 + 7K_4 &= 0 \\ 4K_1 + 3K_2 + 2K_3 + 2K_4 &= 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} K_1 + 2K_2 - 3K_4 &= 0 \\ -5K_2 + K_3 + 13K_4 &= 0 \\ K_3 + K_4 &= 0 \end{aligned}$$

$\neq 0$

$$K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4 = 0$$

where K_1, K_2, K_3 and K_4 are not all 0 simultaneously

Let $K_4 = t$

$\Rightarrow K_3 = -t \quad \Rightarrow -5K_2 - t + 13t = 0 \Rightarrow K_2 = 12/5\, t$

$K_1 + 2\left(\frac{12}{5}t\right) - 3(t) = 0 \Rightarrow K_1 = -\frac{9}{5}t$

$$\begin{aligned} K_1 &= -9/5\, t \\ K_2 &= 12/5\, t \\ K_3 &= -t \\ K_4 &= t \end{aligned}$$

Ex:-

$$\vec{X}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{X}_2 = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \\ 0 & 2 \end{bmatrix} = [X_1 \; X_2] \qquad |A|_{2\times 2} \neq 0 \;\therefore\; \rho(A) = 2$$

$\rho(A) =$ No. of vectors $= 2$ $\therefore$ these vectors are L.I.

$$\underset{0}{K_1} X_1 + \underset{0}{K_2} X_2 = 0$$ (No relation b/w them)

Dimension of each vector (3) $>$ No. of vectors 2

Ex:- i) $\vec{X}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} b \\ c \end{bmatrix}$ $\vec{X}_3 = \begin{bmatrix} \; \end{bmatrix}$ .. $\vec{X}_4$

Dim. of each vector = 2
No. of vectors = 4

Set of vectors are L.D.
$4 > 2$

ii) $\vec{X}_1 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} g \\ h \\ e \\ i \end{bmatrix}$

Dim. of each vectors = 4
No. of vectors = 2

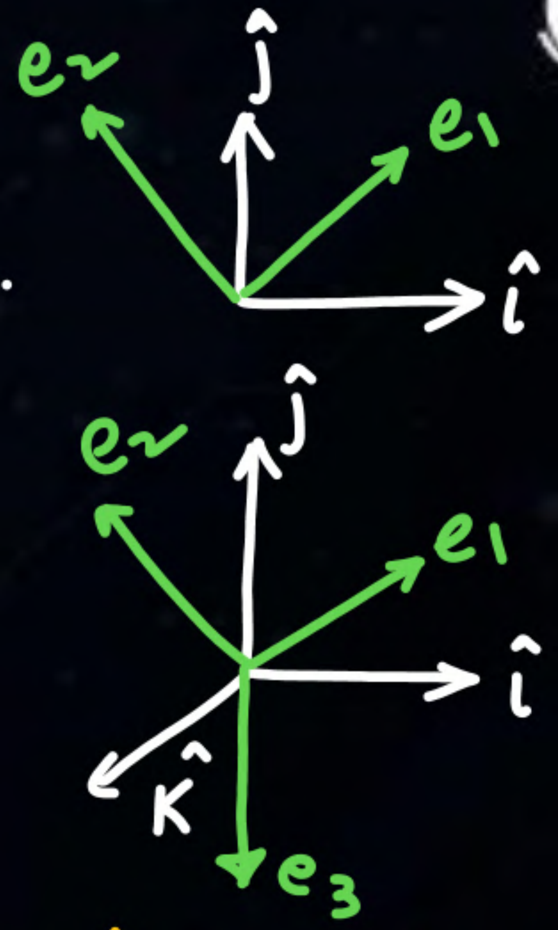
Set of vectors are L.I.
$2 < 4$

iii) $\vec{X}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\vec{X}_2$ $\vec{X}_3$ $\vec{X}_4$ $\vec{X}_5$

Dim. of each vector = 3
No. of vectors = 5

Set of vectors are L.D.
$5 > 3$

[BASIS]



Vector Space (V) :- The set of vectors spans a region.

ex:- R, R^2, R^3

Basis :- The set of vectors forms a basis (B) if

i) Those set of vectors are L.I.

ii) Those set of vectors span vector space V. (any vector V can be expressed as a linear combination of vectors in B).

Ex:-

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 3$

$\rho(A) = n = 3$

i) This set of vectors are L.I.

ii) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex: $\vec{X} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{X} = K_1 e_1 + K_2 e_2 + K_3 e_3$$

$$\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} 5 &= K_1 \\ -1 &= K_2 \\ 2 &= K_3 \end{aligned}$$

$$\vec{X} = 5e_1 - 1e_2 + 2e_3$$

Ex:- $e_1 = (1\ 0\ 2)$ $e_2 = (0\ 1\ 0)$ $e_3 = (-2\ 0\ 1)$ form an orthogonal basis of R^3 then express $u = (4, 3, -3)$ in terms of e_1, e_2, e_3.

$B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $\rho(B) = 3$ (Set of vectors in B are LI)

$\vec{u} = K_1 e_1 + K_2 e_2 + K_3 e_3$ $\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$$K_1 - 2K_3 = 4$$
$$K_2 = 3$$
$$2K_1 + K_3 = -3$$

On solving $K_1 = -\frac{2}{5}$, $K_2 = 3$, $K_3 = -\frac{11}{5}$

$$u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$$

$e_1, e_2, e_3 \rightarrow$ orthogonal basis

$e_1 \perp e_2 \perp e_3$

e_1, e_2, e_3 are L.I.

& they span 3-D.

Thank you

GW Soldiers !