ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-5

Differential equations

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Topics to be Covered

- DEFINITION & TYPES
- ORDER & DEGREE OF DE
- SOLUTION OF DE
- FORMATION OF DE
- WRONSKIAN & LD/LI SOLUTIONS
- METHODS OF SOLVING DE
- PARTIAL DIFFERENTIAL EQUATIONS

6) Methods of solving P.D.E

(More than 1 independent variable)

$z \to x, y$

$u \to x, y, z$

Order $\to$ Highest derivative

Degree $\to$ Exponent of highest derivative.

Ex:-

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} + f(x, y, c) = 0$$

PARTIAL DIFFERENTIAL EQUATIONS

Find order and degree of PDE

1. $$yz\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y} = xy$$

Order = 1
Degree = 1

$z \to f(x, y)$

2. $$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 1$$

Order = 2
Degree = 1

NON- LINAR PDE

1. Exponent of z > 1
2. Degree > 1
3. Exponent of any partial D.C. > 1
4. Product of z & its partial derivative should be present.
5. Product of partial derivatives should be present.

$z.\frac{\partial z}{\partial x}$ ✓ 4

$z \to f(x,y)$

$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$ ✓ 5

Linear PDE :- D.E. not following above 5 properties is linear P.D.E.

PDE

1. $x z_{xx} + (3x+4y)Z_{xy} + e^x Z_{yy} + 5XZ_x = x^2 + y^2$

$z \to f(x,y)$

$O = 2$
$D = 1$
(Linear)

2. $x^2 U_{xx} + y^2\, U_{yy} + Z^2\, U_{zz} = 3U$

$U \to f(x,y,z)$

$O = 2,\ D = 1$
(Linear)

3. $Z_{xx} + 4(Z_x)^2 + 5\, Z_y + 6Z = 9$

$O = 2,\ D = 1$
(Non-linear)
Property 3

[PDE]

4. $(x^2 - y^2)Z_x.Z_y - xy\,(Z_x + Z_y) - 1 = 0$ (Non-linear)
Property (5)

5. $(Z_x)^2.x + (Z_y)^2.y = z$ (Non-linear) Property (3)

6. $x\,Z_{XX} + (y + COSX)Z_{xy} + Z.Z_y = 5$ (Non-linear)
Property (4)

CLASSIFICATION OF 2nd ORDER LINAR P.D.E IN 2 VARIABLES

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial_x \partial y} + C\frac{\partial^2 u}{\partial y^2} = f(X, Y, U, U_X, U_Y)$$

$u \to f(x, y)$

$\left.\begin{matrix} O = 2 \\ D = 1 \end{matrix}\right\}$ Linear

→ $B^2 - 4AC < 0 \Rightarrow$ Elliptic

→ $B^2 - 4AC = 0 \Rightarrow$ Parabolic

→ $B^2 - 4AC > 0 \Rightarrow$ Hyperbolic

CLASSIFICATION OF 2nd ORDER LINAR P.D.E IN 2 VARIABLES

Ex. $Z_{xx} + Z_{yy} = 0$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$A z_{xx} + B z_{xy} + C z_{yy} = 0$$

$A = 1$, $B = 0$, $C = 1$

$B^2 - 4AC = 0^2 - 4(1)(1) = -4 < 0$

$\Rightarrow$ Elliptic

CLASSIFICATION OF 2nd ORDER LINAR P.D.E IN 2 VARIABLES

Ex. $Z_{xx} = Z_y$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$

$$A z_{xx} + B z_{xy} + C z_{yy} = f(\cdot \, \cdot \cdot)$$

$$A = 1, \; B = 0, \; C = 0$$

$$B^2 - 4AC = 0^2 - 4(1)(0) = 0$$

$\Rightarrow$ Parabolic

[CLASSIFICATION OF 2nd ORDER LINAR P.D.E IN 2 VARIABLES]

Ex. $$(1-x^2)Z_{xx} - 2xy\,Z_{xy} + (1-y^2)Z_{yy} = (2z - X\,Z_x - 3x^2y\,Z_y)$$

$$A = (1-x^2)\;;\; B = -2xy\;;\; C = 1-y^2$$

$$B^2 - 4AC = (-2xy)^2 - 4(1-x^2)(1-y^2)$$

$$= 4x^2y^2 - [4x^2y^2 - 4x^2 - 4y^2 + 4]$$

$$= -4 + 4x^2 + 4y^2 = 4(-1 + x^2 + y^2)$$

$\Rightarrow x^2+y^2 > 1$ $\quad B^2-4AC > 0 \Rightarrow$ Hyperbolic

$\Rightarrow x^2+y^2 = 1$ $\quad B^2-4AC = 0 \Rightarrow$ Parabolic

$\Rightarrow x^2+y^2 < 1$ $\quad B^2-4AC < 0 \Rightarrow$ Elliptic

[ONE DI~~A~~MENSIONAL WAVE EQUATION] [Hyperbolic]

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

$y \to x, t$

Displacement

$y = f(x, t)$

Boundary conditions :- $(x=0, l)$

$$y(0,t) = 0 \quad ; \quad y(l,t) = 0$$

Initial conditions :- $(t=0)$

$$y(x,0) = f(x) \quad , \quad \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0$$

Soln. of wave eqn.

$$y = X\,T$$

Let this be the soln.

X → fn. of x

T → fn. of t

ONE DIAMENSIONAL WAVE EQUATION

Using separation of variables;

$$XT'' = c^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K \text{ (Let)}$$

Case I :- $K = 0$

$$\frac{X''}{X} = 0 \qquad\qquad \frac{1}{c^2}\frac{T''}{T} = 0$$

$$\frac{\partial^2 X}{\partial x^2} = 0 \qquad\qquad \frac{\partial^2 T}{\partial t^2} = 0$$

$$\frac{\partial X}{\partial x} = c_1 \qquad\qquad \frac{\partial T}{\partial t} = c_3$$

$$\{X = c_1 x + c_2\} \qquad\qquad \{T = c_3 t + c_4\}$$

$$y = XT$$

$$y = (c_1 x + c_2)(c_3 t + c_4) \rightarrow \text{Soln 1)}$$

ONE DIAMENSIONAL WAVE EQUATION

Case II :- $K = p^2 > 0$

$$\frac{X''}{X} = p^2 \qquad \Big| \qquad \frac{1}{c^2}\frac{T''}{T} = p^2$$

$$X'' - p^2 X = 0 \qquad \Big| \qquad T'' - c^2 p^2 T = 0$$

$$\frac{\partial^2 X}{\partial x^2} - p^2 X = 0 \qquad \Big| \qquad \frac{\partial^2 T}{\partial t^2} - c^2 p^2 T = 0$$

$$(D^2 - p^2) X = 0 \qquad \Big| \qquad (D^2 - c^2 p^2) T = 0$$

$$m = \pm p \qquad \Big| \qquad m = \pm cp$$

$$X = C_1 e^{px} + C_2 e^{-px} \qquad \Big| \qquad T = C_3 e^{cpt} + C_4 e^{-cpt}$$

$$y = XT$$

$$y = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 e^{cpt} + C_4 e^{-cpt}\right)$$

— Soln 2)

ONE DIAMENSIONAL WAVE EQUATION

Case III :- $K = -p^2 < 0$

$$\frac{X''}{X} = -p^2 \qquad \frac{1}{c^2}\frac{T''}{T} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} + p^2 X = 0 \qquad \frac{\partial^2 T}{\partial t^2} + c^2 p^2 T = 0$$

$$(D^2 + p^2)X = 0 \qquad (D^2 + c^2p^2)T = 0$$

$$m = 0 \pm ip \qquad m = 0 \pm cp\,i$$

$$X = e^{0x}[C_1 \cos px + C_2 \sin px] \qquad T = e^{0t}[C_3 \cos cpt + C_4 \sin cpt]$$

$$y = XT$$

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{--- Soln 3)}$$

Soln. 3 will be suitable since it contains periodic terms similar to physical nature of problem.

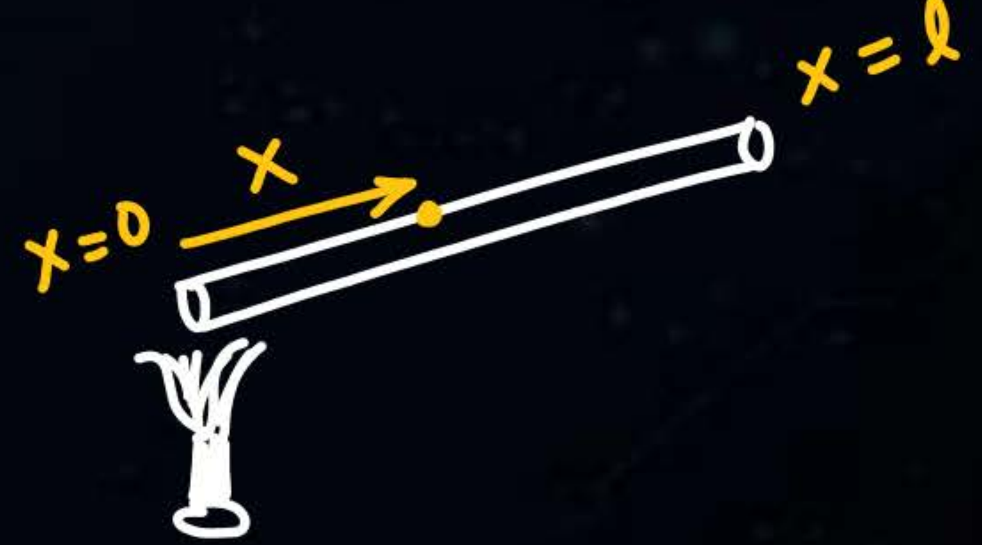
[ONE DI~~A~~MENSIONAL HEAT EQUATION]

Consider a flow of heat in uniform rod

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

(Parabolic)

(Heat)
$u \to f(x, t)$



Let the soln. of 1-D heat eqn.

$$U = X\,T$$

fn. of x fn. of t

$$X T' = c^2 X'' T$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = K \text{ (say)}$$

ONE DI~~A~~MENSIONAL HEAT EQUATION

Case I :- $K=0$; $u=(c_1x+c_2)c_3$ Soln 1

Case II :- $K=p^2>0$; $u=(c_1e^{px}+c_2e^{-px})(c_3e^{c^2p^2t})$ Soln 2

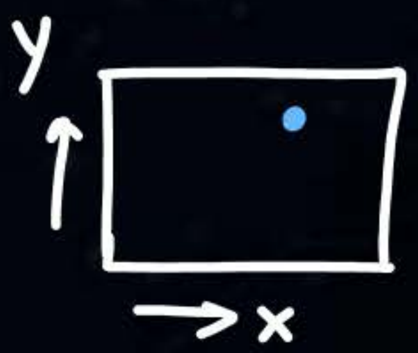
Case III :- $K=-p^2<0$; $u=(c_1\cos px+c_2\sin px)(c_3e^{-c^2p^2t})$ Soln 3

Soln 3 will be suitable similar to physical nature of problem.

TWO DI~~A~~MENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Flow of heat in metal plate in transient state.



For steady state, $\frac{\partial u}{\partial t} = 0 \Rightarrow$ Laplace equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

(Elliptic)

Let the soln. of this eqn. $u = XY$

X → fn. of x, Y → fn. of y

$$X''Y + Y''X = 0$$

LAPLACE EQUATIONS

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

Case I :- $K=0$ $\quad u=(C_1 x + C_2)(C_3 y + C_4) \quad -1)$

Case II :- $K=p^2>0$ $\quad u=(C_1 e^{px}+C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad -2)$

Case III :- $K=-p^2<0$ $\quad u=(C_1 \cos px + C_2 \sin px)(C_3 e^{py}+C_4 e^{-py}) \quad -3)$

Solution 3 will be suitable.

[METHODS OF SOLVING DE]

$p \rightarrow \frac{dy}{dx}$

5) Methods for solving non-linear D.E

D.E. of first order & higher degree

Power of $p \neq 1$

D.E. is of the form :- $f(x, y, p) = 0$

$\Downarrow$

Solution:- $F(x, y, c) = 0$

Case I:- Equation solvable for p

Ex:- $p^2 + 1 = x^2$

$$\left(\frac{dy}{dx}\right)^2 + 1 = x^2$$

$O = 1$
$D = 2$ } Non-linear O.D.E.

$$p = \pm\sqrt{x^2 - 1}$$

$$\frac{dy}{dx} = \pm\sqrt{x^2 - 1}$$

$$\int dy = \pm\int\sqrt{x^2 - 1}\, dx$$

$$y = \pm\left[\frac{x}{2}\sqrt{x^2 - 1} + \frac{1}{2}\log\left(x + \sqrt{x^2 - 1}\right)\right] + c$$

Case II:- Equation solvable for Y

$$y = (x-a)\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

Ex:- Solve $y = (x-a)p - p^2$

Diff. w.r.t. x

$$p = 1.p + (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}(x-a-2p) = 0$$

$$\therefore \int \frac{dp}{dx} = \int 0$$

$$\boxed{p = c}$$

Put the value of p in i)

$$\boxed{y = (x-a)c - c^2}$$

[DE OF FIRST ORDER & HIGER DEGREE]

Case III:- Equation solvable for X

$$p \to \frac{dy}{dx}$$

$$\frac{1}{p} \to \frac{dx}{dy}$$

Ex:-Solve $y = 2px + yp^2$

$$y = 2 \cdot \frac{c}{y} \cdot x + y\left(\frac{c}{y}\right)^2 \quad \text{Soln.}$$

$$2px = y - yp^2$$

$$2x = \frac{y}{p} - yp$$

Diff. w.r.t. y

$$\frac{2}{p} = 1 \cdot \frac{1}{p} - \frac{y}{p^2} \cdot \frac{dp}{dy} - 1 \cdot p - y \cdot 1 \frac{dp}{dy}$$

$y \cdot \frac{1}{p}$

$$\frac{2}{p} - \frac{1}{p} + p = -y\frac{dp}{dy}\left(\frac{1}{p^2} + 1\right)$$

$\frac{1}{p} + p$

$\frac{1+p^2}{p}$

$$\cancel{\left(\frac{1+p^2}{p}\right)} = -y\frac{dp}{dy}\cancel{\left(\frac{1+p^2}{p^2}\right)}$$

$$\int \frac{dy}{y} = -\int \frac{dp}{p}$$

$$\log y = -\log p + \log c$$

$$\boxed{p = c/y}$$

CLAIRAUT'S EQUATION

is of the form $y = x\frac{dy}{dx} + \emptyset\left(\frac{dy}{dx}\right)$ [O.D.E.] $y = xp + \phi(p)$

Diff. w.r.t. x ; $\cancel{p} = 1.\cancel{p} + x\,1.\frac{dp}{dx} + \frac{d\phi}{dp}.\frac{dp}{dx}$

$$\frac{dp}{dx}\left(x + \frac{d\phi}{dp}\right) = 0$$

$$\int \frac{dp}{dx} = \int 0$$

$$\boxed{p = c}$$

Hence the complete soln. of Clairaut's eqn. is obtained by replacing p = constant.

$$\boxed{y = xc + \phi(c)}$$

CLAIRAUT'S EQUATION

Ex:-Solve $p = \tan(px - y)$

$$\tan^{-1}p = px - y$$

$$y = xp - \tan^{-1}p \quad \longrightarrow \text{Clairaut's eqn.}$$

$$y = xc - \tan^{-1}c$$

CLAIRAUT'S EQUATION

Ex:-Solve $p^2(x^2-a^2)-2xy\,p+y^2-b^2=0$

$$y^2-2xpy+p^2x^2-ap^2-b^2=0$$

$\Rightarrow$ Clairaut's eqn. [P.D.E.]:-

$z\to(x,y)$ $\quad p\to\frac{\partial z}{\partial x}$ $\quad q\to\frac{\partial z}{\partial y}$

$z=xp+yq+f(p,q)=0 \longrightarrow$ Clairaut's P.D.E.

$z=xa+yb+f(a,b)=0$

Soln. $\boxed{\begin{matrix}p=a\\ q=b\end{matrix}}$

Ex:-

$z=px+qy+(p^2+q)^4$

$z=ax+by+(a^2+b)^4$

Thank you

GW Soldiers !