ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-02

Vector Calculus

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Topics to be Covered

- VECTOR BASICS
- STRAIGHT LINES/3D PLANES
- GRADIENT (VECTOR DIFFERENTIATION)
- DIVERGENCE (VECTOR DIFFERENTIATION)
- CURL (VECTOR DIFFERENTIATION)
- LINE, SURFACE, VOLUME INTEGRAL (VECTOR INTEGRATION)
- GREEN, & STOKE'S THEOREM (VECTOR INTEGRATION)
- GAUSS DIVERGENCE THEOREM (VECTOR INTEGRATION)

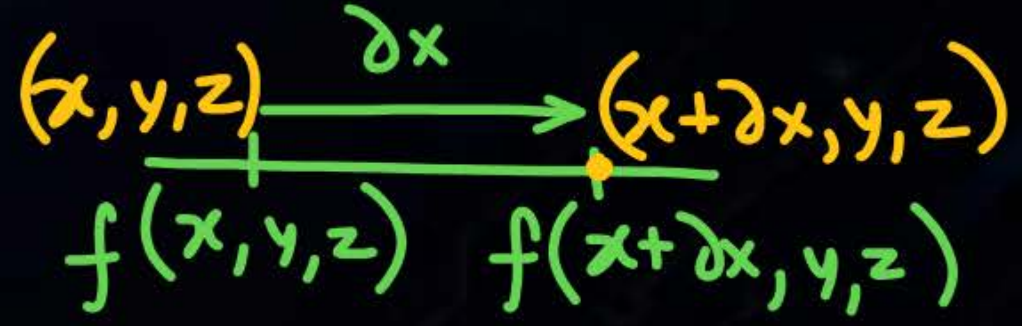
PARTIAL DERIVATIVES OF VECTORS

Let $\vec{r} = f(x, y, z)$ is a vector of 3 variables x, y, z.

$$y, z \left\{ \frac{\partial r}{\partial x} = \lim_{\partial x \to 0} \frac{f(x+\partial x, y, z) - f(x, y, z)}{(x+\partial x) - x} \right.$$

$$x, z \left\{ \frac{\partial r}{\partial y} = \lim_{\partial y \to 0} \frac{f(x, y+\delta y, z) - f(x, y, z)}{(y+\partial y) - y} \right.$$

$$\begin{matrix} x, y \\ \text{constant} \end{matrix} \left\{ \frac{\partial r}{\partial z} = \lim_{\partial z \to 0} \frac{f(x, y, z+\partial z) - f(x, y, z)}{(z+\partial z) - z} \right.$$



POINT FUNCTIONS

Scalar point fn :- To every point P in region R there corresponds a scalar $\phi(P)$

Explain :- $\phi = x^2yz$

$P(1,1,1)\ ;\ \phi(P) = 1$

$P(1,1,2)\ ;\ \phi(P) = 2$

POINT FUNCTIONS

Vector point fn :- To every point P in region R, there corresponds a vector f(P).

Example:- $x^2\,\hat{\imath} + y\,\hat{\jmath} + z^2\,\hat{k}$

$P(1,1,1) \qquad f(P) = \hat{\imath} + \hat{\jmath} + \hat{k}$

[OPERATORS]

$$\nabla(\text{del/Nabla}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

Ex:- Find $\nabla \phi$ where $\boxed{\phi = x^2yz}$ → Scalar point function

ϕ (Scalar) ↓ $\nabla\phi$ (Vector)

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2yz)$$

$$= \frac{\partial}{\partial x}(x^2yz)\hat{i} + \frac{\partial}{\partial y}(x^2yz)\hat{j} + \frac{\partial}{\partial z}(x^2yz)\hat{k}$$

$$= (2xyz)\hat{i} + (x^2z)\hat{j} + (x^2y)\hat{k}$$

[OPERATORS]

Laplace Operator: $\nabla^2 = \vec{\nabla}.\vec{\nabla} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

If f is a scalar pt fn. then $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

If $\vec{f}$ is a vector pt fn. then $\nabla^2 \vec{f} = \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2}$

Any fn. ϕ which satisfies laplace equation is called Harmonic function $\phi \rightarrow$ Harmonic function

Note :- Harmonic Function $\boxed{\nabla^2 \varphi = 0}$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

2-D laplace eqn.

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

3-D laplace eqn.

Q. Check whether it is harmonic function or not $\boxed{\phi = x^2 - y^2}$

$$\phi = x^2 - y^2$$

$$\frac{\partial \phi}{\partial x} = 2x \qquad \frac{\partial \phi}{\partial y} = -2y$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 \qquad \frac{\partial^2 \phi}{\partial y^2} = -2$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0$$

ϕ satisfies Laplace eqn. $\therefore$ $\phi \rightarrow$ Harmonic function.

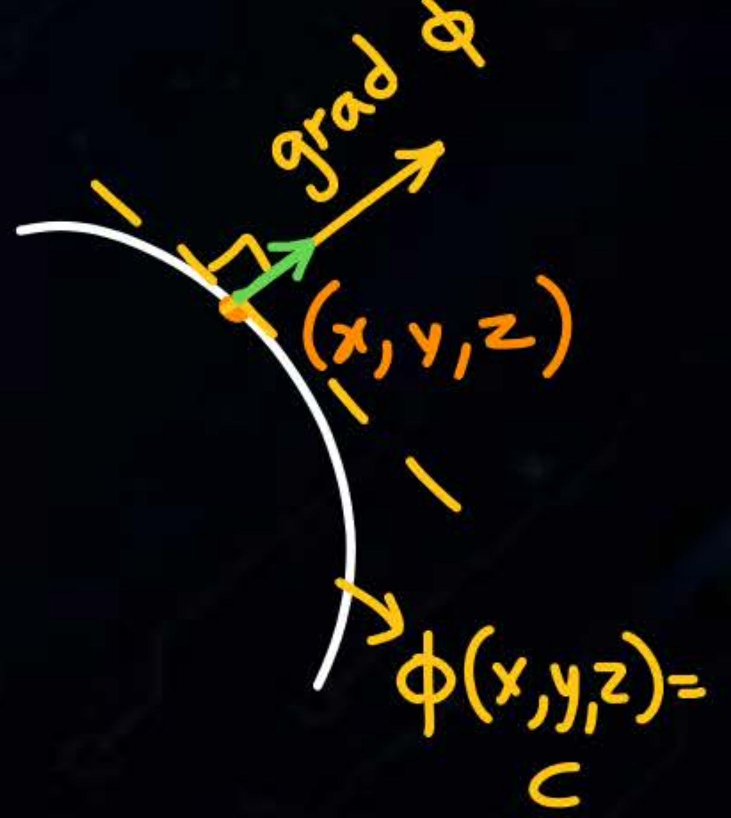
GRADIENT OF SCALAR POINT FUNCTION

If $\underbrace{\phi(x,y,z) = c}_{\text{Scalar pt. fn.}}$ then grad ϕ is a vector fn. is defined as :-

Normal

$$\boxed{\underset{(\vec{n})}{\text{grad}\,\phi} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}} \rightarrow \text{Vector fn.}$$

Physical significance — It represents a vector normal to the surface $\phi(x,y,z) = c$



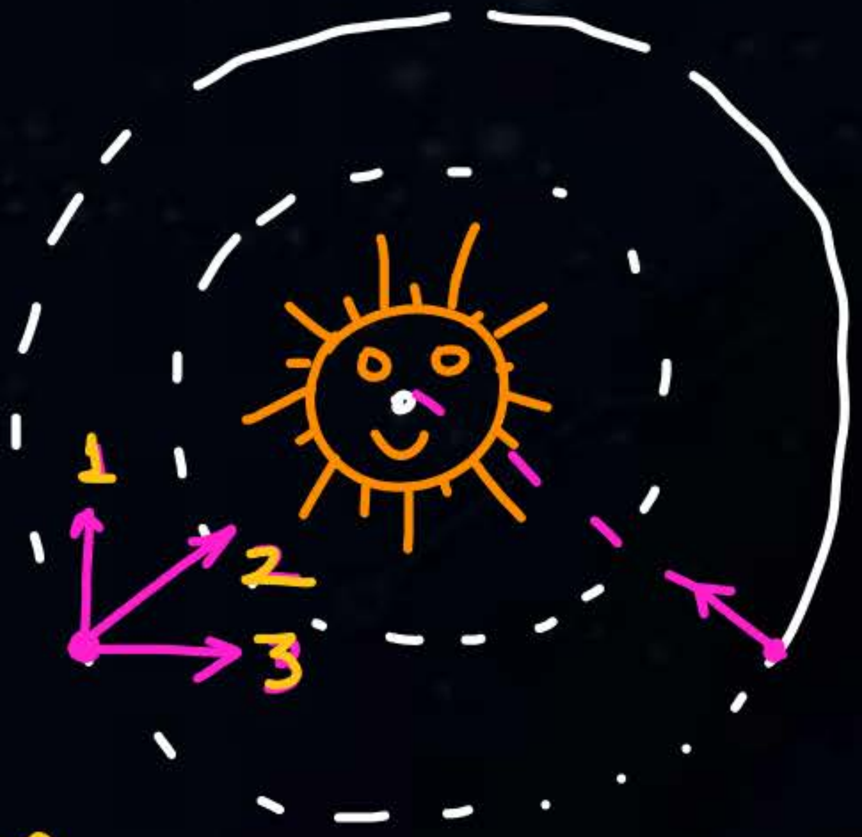
- Unit normal vector at any point of surface $\hat{n} = \dfrac{\overrightarrow{\text{grad}\,\phi}}{|\overrightarrow{\text{grad}\,\phi}|}$

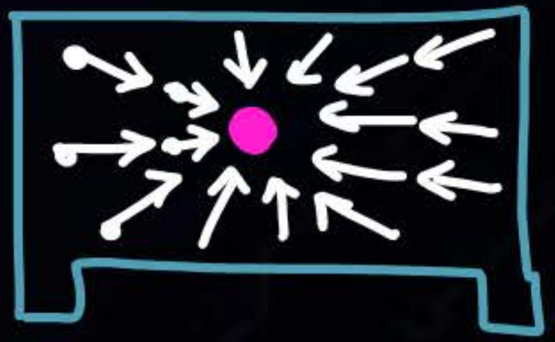
GRADIENT OF SCALAR POINT FUNCTION

→ It indicates direction of max. change (greatest increase)

→ Max rate of change = $|\text{grad}\ \phi|$

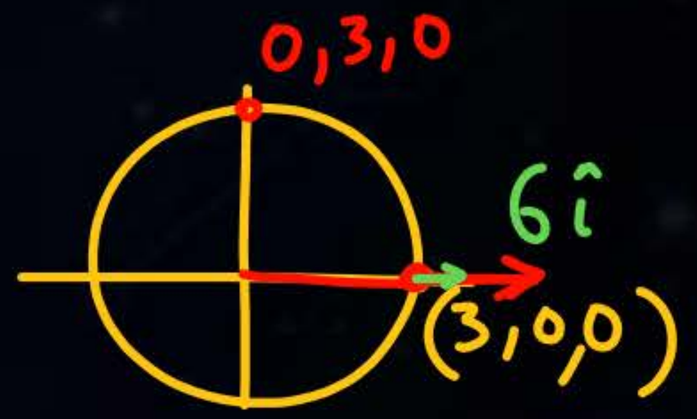
(Increase in value) = $|\nabla \phi|$





Q. Find normal vector to the surface $x^2 + y^2 + z^2 = 9$ at (3, 0, 0).

$$\phi\left(x^2+y^2+z^2-9\right) = 0$$



$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

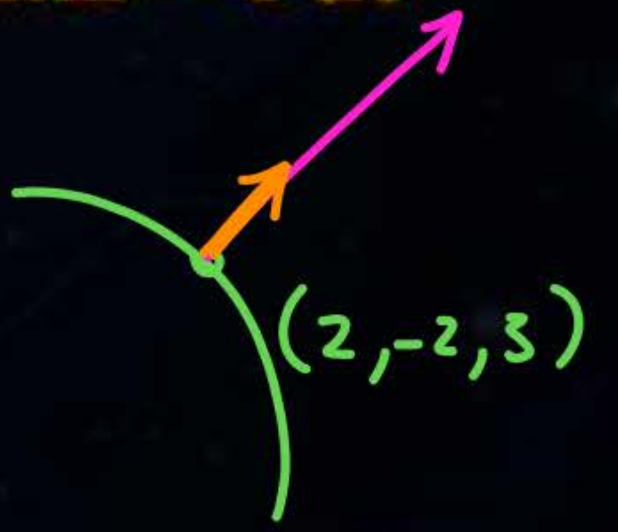
$$\nabla\phi = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

(3,0,0)

$$\nabla\phi = 6\hat{i}$$

Unit normal $= \dfrac{6\hat{i}}{6} = \hat{i}$

Q. **Find a unit normal vector to a level surface $x^2y + 2xz = 4$ at point (2, -2, 3)**



$$\phi(x^2y + 2xz - 4) = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Normal to surface

$$\nabla\phi = (2xy + 2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}$$

$$[2(2)(-2) + 2(3)]\hat{i} + (2^2)\hat{j} + 2(2)\hat{k} \quad (2,-2,3)$$

$$\boxed{-2\hat{i} + 4\hat{j} + 4\hat{k}} \rightarrow \text{Normal}$$

Unit normal at (2,-2,3)

$$= \frac{\nabla\phi}{|\vec{\nabla}\phi|} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{(-2)^2 + 4^2 + 4^2}} = -\frac{\hat{i}}{3} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Q. What is the greatest rate of increase of $\underbrace{u = xyz^2}$ at the point (1, 0, 3)?

Max. rate of increase $= |\nabla \phi|$

$$\nabla \phi = (yz^2)\hat{i} + (xz^2)\hat{j} + (2xyz)\hat{k}$$

$$= 0\hat{i} + 9\hat{j} + 0\hat{k}$$

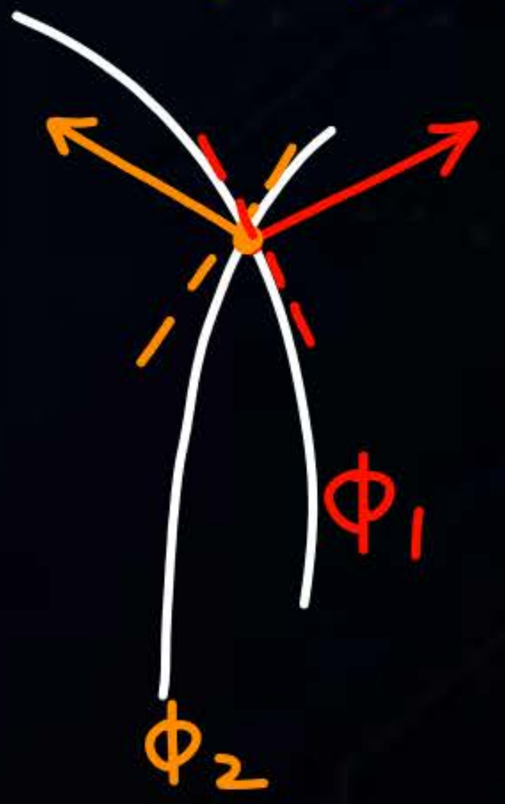
$$|\nabla \phi| = |9\hat{j}| = 9$$

GRADIENT OF SCALAR POINT FUNCTION

- Angle between two surface :- = Angle b/w their normal at pt. of intersection

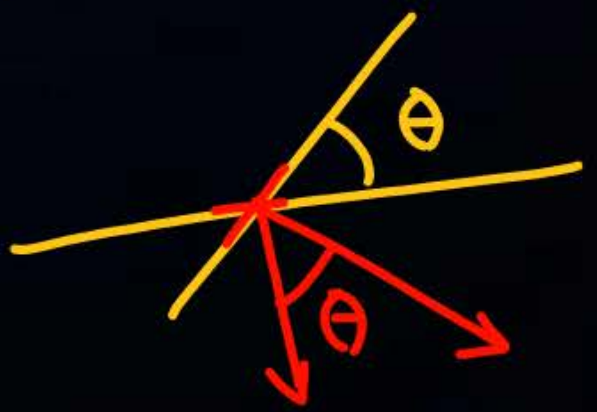
Let ϕ_1 & ϕ_2 be two surfaces :-

$\phi_1 \downarrow \nabla\phi_1$ $\quad$ $\phi_2 \downarrow \nabla\phi_2$



$$\theta = \cos^{-1}\left(\frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|}\right)$$

If $\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$, then the surface are orthogonal.



Q. Find the angle between $S_1 \rightarrow x^2 + y^2 + z^2 = 9$ and $S_2 \rightarrow x^2 + y^2 - z = 3$ at the point of intersection $(2, -1, 2)$

$$\nabla \phi_1 = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$$

$(2, -1, 2)$

$$4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\nabla \phi_2 = (2x)\hat{i} + (2y)\hat{j} - \hat{k}$$

$(2, -1, 2)$

$$4\hat{i} - 2\hat{j} - \hat{k}$$

$$\theta = \cos^{-1}\left(\frac{16 + 4 - 4}{\sqrt{4^2 + (-2)^2 + 4^2}\sqrt{4^2 + (-2)^2 + (-1)^2}}\right) = 54.41^\circ$$

GRADIENT OF SCALAR POINT FUNCTION

$\theta \uparrow \cos\theta \downarrow$

- Directional derivative of grad ϕ :-

Component of grad ϕ in the direction of vector $\vec{a}$.

$$\boxed{D.D. = \overrightarrow{grad}\,\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \overrightarrow{grad}\,\phi \cdot \hat{a} = \vec{\nabla}\phi \cdot \hat{a}}$$

$$D.D. = |\overrightarrow{grad}\,\phi| \cdot \cos\theta$$

$$\vec{a} \cdot \overrightarrow{grad}\,\phi = |\vec{a}| \underbrace{|\overrightarrow{grad}\,\phi| \cos\theta}_{D.D.}$$

$$D.D. = \overrightarrow{grad}\,\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

NOTE :- D.D. is maximum in the dir^n. of normal to the surface. $(\vec{a} = \vec{\nabla}\phi)$

$$\text{Max D.D.} = \vec{\nabla}\phi \cdot \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{|\vec{\nabla}\phi|^2}{|\vec{\nabla}\phi|} = |\nabla\phi|$$

$$\boxed{\text{Max D.D.} = |\nabla\phi|}$$

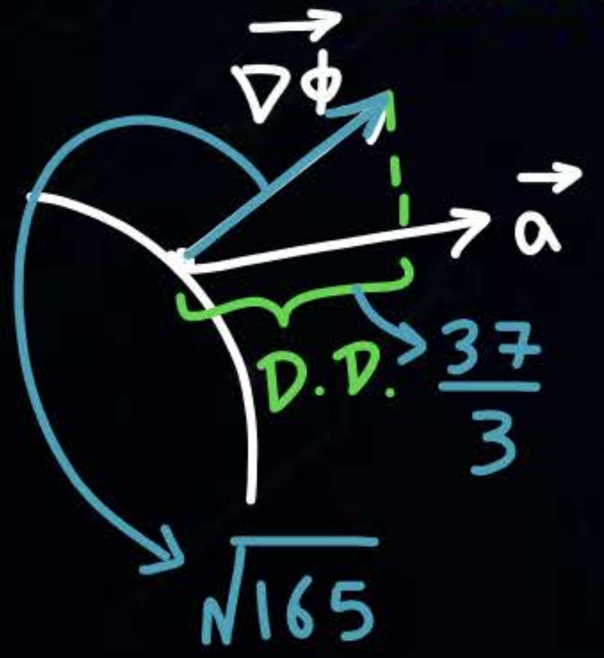
For max D.D. $\Rightarrow \boxed{\theta = 0^\circ}$

GRADIENT OF SCALAR POINT FUNCTION

Maximum value of D.D = $|\text{grad}\ \phi|$ $(\theta = 0^\circ)$

Minimum value of D.D = 0 $(\theta = 90^\circ)$

Q. Find D.D of $f(x, y, z) = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}$



$$D.D. = \vec{\nabla}\phi \cdot \hat{a}$$

$$= \left[(2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}\right] \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$= \left[(2(1)(-2)(-1) + 4(-1)^2)\hat{i} + (1^2(-1))\hat{j} + (1^2(-2) + 8(1)(-1))\hat{k}\right] \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$(8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$$

$$\frac{16 + 1 + 20}{3} = \boxed{\frac{37}{3}}$$

$$\text{Max D.D.} = |\vec{\nabla}\phi|$$

$$= \sqrt{8^2 + (-1)^2 + (-10)^2}$$

$$= \sqrt{165}$$

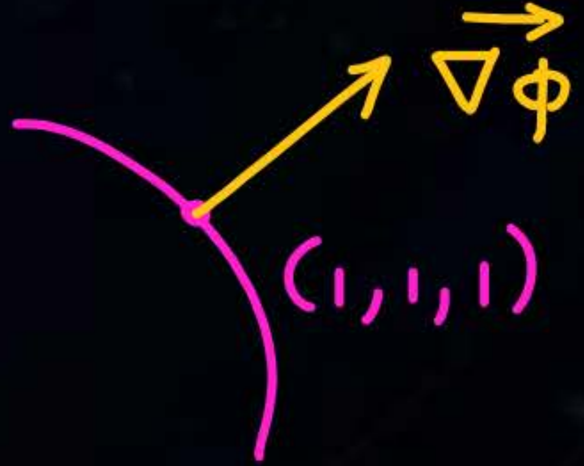
Q. (i) What direction from point (1, 1, 1) is the D.D of $f = x^2 - 2y^2 + 4z^2$ maximum ?

(ii) Also find the value of maximum directional derivative.

$$f = x^2 - 2y^2 + 4z^2$$

$$\vec{\nabla} f = (2x)\hat{i} - (4y)\hat{j} + (8z)\hat{k}$$

$$= \boxed{2\hat{i} - 4\hat{j} + 8\hat{k}} \quad (1,1,1)$$

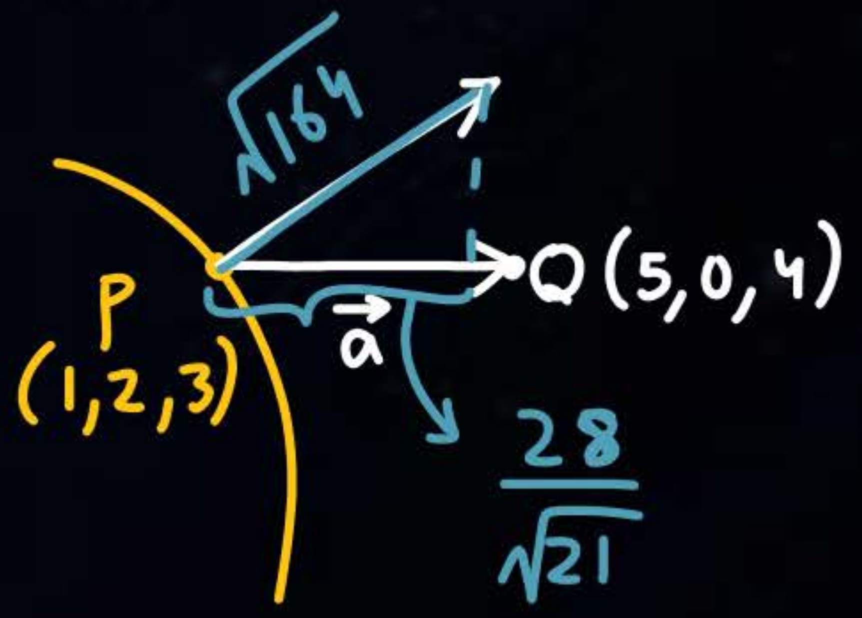


$$\text{Max. D.D.} = |\vec{\nabla}\phi| = \sqrt{2^2 + (-4)^2 + 8^2} = \sqrt{84}$$

Q. Find D.D of $f = x^2 - y^2 + 2z^2$ at $P(1, 2, 3)$ in direction at line PQ where Q is $(5, 0, 4)$.

$$\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{\nabla}\phi = (2x)\hat{i} - 2y\,\hat{j} + 4z\,\hat{k}_{\,(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}$$



$$\text{D.D.} = \vec{\nabla}\phi \cdot \hat{a} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4^2 + (-2)^2 + 1^2}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \boxed{\frac{28}{\sqrt{21}}}$$

$$\text{Max D.D.} = |\vec{\nabla}\phi|$$
$$= \sqrt{2^2 + (-4)^2 + 12^2}$$
$$= \sqrt{164}$$

DIVERGENCE OF VECTOR POINT FUNCTION

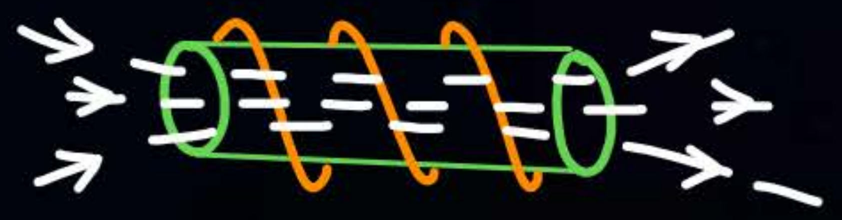
If $\vec{f} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then divergence is defined as:-

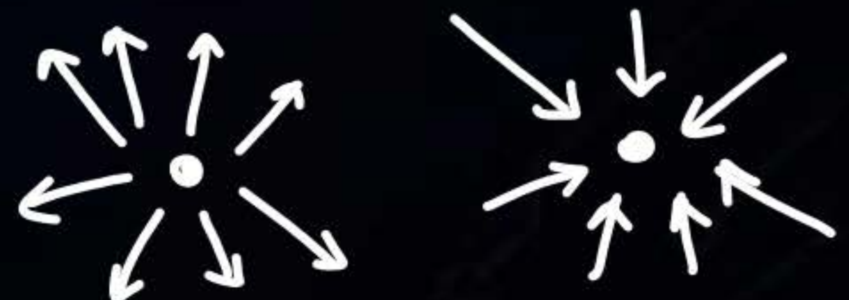
$f(x,y,z)$

$$\vec{\nabla}\cdot\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\boxed{\text{div}\,\vec{f} = \vec{\nabla}\cdot\vec{f} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}} \rightarrow \text{Scalar}$$

Physical significance:- It is a measurement of rate of fluid, originating at a point per unit volume.





Source (+ve divergence) Sink (-ve divergence)

Note:- If div $\vec{F} = 0$, then $\vec{F}$ is solenoidal vector

DIVERGENCE OF VECTOR POINT FUNCTION

Consider a fluid motion having $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then

$\boxed{\text{if div } \vec{v} = 0\text{, then incompressible fluid}}$

Q. Find div. $\vec{F}$ if $\vec{F} = \underbrace{x^2}_{F_1}\hat{\imath} + \underbrace{x^2y^2}_{F_2}\hat{\jmath} + \underbrace{z^2}_{F_3}\hat{k}$ at $(1, 1, 1)$.

$$F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$$

$$\text{div}.\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2x + 2x^2y + 2z \quad (1,1,1)$$

$$\text{div}\,\vec{F} = 2 + 2 + 2 = 6$$

Q. **Find the constant 'a' so that the vector $\vec{V} = \underbrace{(x+3y)}_{F_1}\hat{\imath} + \underbrace{(y-2z)}_{F_2}\hat{\jmath} + \underbrace{(x+az)}_{F_3}\hat{k}$ is solenoidal?**

$$\text{div } V = 0 = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1 + 1 + a = 0$$

$$\boxed{a = -2}$$

CURL OF VECTOR POINT FUNCTION

If $\vec{f}$ is a vector fn. $\vec{f} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$; then

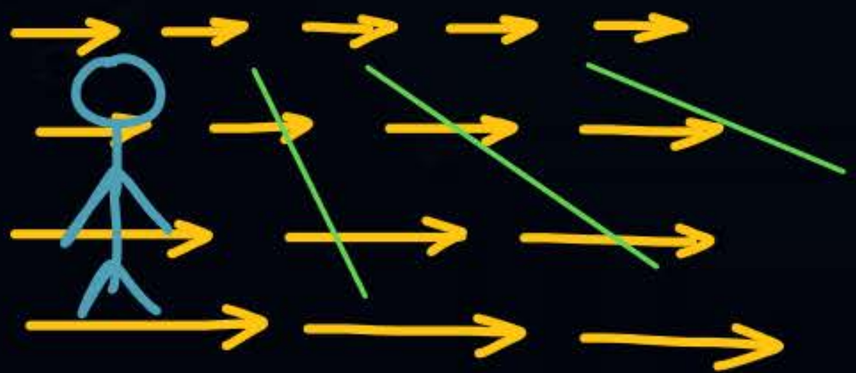
$$\text{curl } \vec{f} = \vec{\nabla} \times \vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \underbrace{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)}_{\substack{\text{About x-axis} \\ (\parallel \text{ to y-z plane})}}\hat{i} + \underbrace{\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)}_{\substack{\text{About y-axis} \\ (\parallel \text{ to x-z plane})}}\hat{j} + \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\substack{\text{About z-axis} \\ (\parallel \text{ to x-y plane})}}\hat{k}$$

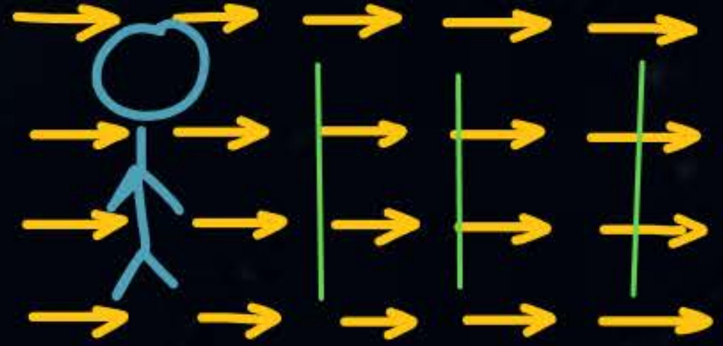
$F_1 \quad F_2 \quad F_3$

$x \quad y \quad z$



Rotational field

curl $\vec{F} \neq 0$



Irrotational field

Curl $\vec{F} = 0$

Irrotational field :-

$$\boxed{\text{curl } \vec{F} = 0}$$

CURL OF VECTOR POINT FUNCTION

Note:- If curl F = 0; then irrotational vector

$$\boxed{\text{Curl } \vec{v} = 2\vec{w}}$$

If a rigid body is in motion then curl of its linear velocity is twice of its angular velocity.

$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

Q. Find curl $\vec{F}$ if $\vec{F} = \underbrace{x^2y\hat{i}}_{F_1} - \underbrace{2xz\hat{j}}_{F_2} + \underbrace{2yz\hat{k}}_{F_3}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$(2z+2x)\hat{i} - (0-0)\hat{j} + (-2z-x^2)\hat{k}$$

$$= (2z+2x)\hat{i} - (2z+x^2)\hat{k}$$

Gradient	Scalar	Vector
Divergence	Vector	Scalar
Curl	Vector	Vector

Thank you

GW Soldiers !