CS & IT ENGINEERING

DISCRETE MATHS
COMBINATORICS

Lecture No. 09

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TOPICS

01 Homogenous equation

02 Non Homogenous equation

3 Exercise

Euler-totient-function: $\phi(n)$

no. of relative primes less than n.

Relative prime.
coprime

$\phi(4) = 2$

1 $\gcd(1,4) = 1$ ✓
2 $\gcd(2,4) = 2$ ✗
3 $\gcd(3,4) = 1$ ✓
4 $\gcd(4,4) = 4$ ✗

$\phi(3) = 2$

✓ ✓ ✗
1 2 3

$\gcd(1,3) = 1$ $\gcd(2,3) = 1$

$\gcd(a,a) = a$

$\gcd(a,b) = 1$

$\gcd(3,5) = 1$

3, 5 are relative prime.

$\phi(12) = 4$

1 ✓
2 ✗
3 ✗
4 ✗
5 ✓
6 ✗
7 ✓
8 ✗
9 ✗
10 ✗
11 ✓
12 ✗

$\phi(12) = \phi(3 \times 4)$

$= \boxed{\phi(3)} \times \phi(4)$

$\gcd(3,4) = 1$

2×2

$= 4$

$\phi(A \cdot B)$

$\phi(A) \cdot \phi(B)$

$\gcd(A, B) = 1$

$$\boxed{\begin{gathered}\phi(A \cdot B) = \phi(A) \cdot \phi(B) \\ \gcd(A, B) = 1\end{gathered}}$$

$\phi(7) = 6$

$\phi(P) = P-1$ { P is primeno }

1
2
3 gcd(_ , 7) = 1
4
5
6
7.

$\phi(49) =$ Total elements $-$ nonrelativeprime.

$= 49 - 7 = 49 - \left\{\frac{49}{7}\right\}$

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 … 28 … 49

gcd(7, 49) = 7 (nonrelative prime)

$\phi(49) = 49 - \frac{49}{7}$

$\phi(7^2) = 7^2 - \frac{7^2}{7}$

$$\boxed{\phi(p^a) = p^a - \frac{p^a}{p}}$$

$a \geq 0$

prime.

$$n = p_1^{a} \cdot p_2^{b} \cdot p_3^{c} \cdots \left\{ \begin{array}{l} a, b, c \ldots \geq 0 \\ p_1, p_2, p_3 \ldots \\ \text{prime no} \end{array} \right.$$

$$n = 2^{k} \cdot y \quad \left(\begin{array}{l} k \geq 0 \\ y \to \text{odd no} \end{array} \right)$$

$$4 = 2^{2} \cdot 3^{0} \cdot 5^{0} \cdot 7^{0} \cdots$$

$$5 = 2^{0} \cdot 3^{0} \cdot 5^{1} \cdots$$

$$48 = 2 \cdot 24 = 2^{4} \cdot 3^{1} \cdot 5^{0} \cdot 7^{0}$$

$$= 2 \cdot 2 \cdot 12$$

$$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$110 = 2 \cdot 55$$

$$= 2 \cdot 5 \cdot 11$$

$$= 2^{1} \cdot 3^{0} \cdot 5^{1} \cdot 7^{0} \cdot 11^{1}$$

$$n = p_1^a \cdot p_2^b \cdot p_3^c \cdots$$

$$\phi(n) = \phi\left(p_1^a \cdot p_2^b \cdot p_3^c \cdots\right)$$

$$= \phi(p_1^a) \times \phi(p_2^b) \times \phi(p_3^c) \times$$

$$= \left(p_1^a - \frac{p_1^a}{p_1}\right) \times \left(p_2^b - \frac{p_2^b}{p_2}\right) \times \left(p_3^c - \frac{p_3^c}{p_3}\right)$$

$$= p_1^a\left(\frac{p_1 - 1}{p_1}\right) p_2^b\left(\frac{p_2 - 1}{p_2}\right) p_3^c\left(\frac{p_3 - 1}{p_3}\right)$$

$$= \frac{p_1^a \cdot p_2^b \cdot p_3^c \cdots [p_1 - 1][p_2 - 1][p_3 - 1]}{p_1 \cdot p_2 \cdot p_3 \cdot}$$

$$\boxed{\phi(n) = \frac{n \cdot (p_1 - 1)(p_2 - 1)(p_3 - 1)\cdots}{p_1 \cdot p_2 \cdot p_3 \cdot}}$$

$\phi(110) =$

$110 = 2 \cdot 55$

$= 2 \cdot 5 \cdot 11$

$$\phi(n) = n \cdot \frac{(p_1-1)(p_2-1)(p_3-1)}{p_1 \cdot p_2 \cdot p_3}$$

$$= \cancel{110}\,\frac{(2-1)(5-1)(11-1)}{\cancel{2 \cdot 5 \cdot 11}}$$

$$= 1 \cdot 4 \cdot 10 = \underline{\underline{40}}$$

$\phi(248)$

$\underline{\underline{120}}$

Recurrence relation

homogenous:

$a_{n+1} = 6a_{n+1} - 7a_{n-2} + f(n)$

$f(n) = 0$

$a_n = 2a_{n-1} + f(n)$

$f(n) = 0$

nonhomegenous.

$a_n = X a_{n-1} + Y a_{n-2} + f(n)$

$f(n) = 2^n$

$f(n) = n \cdot 2^n$

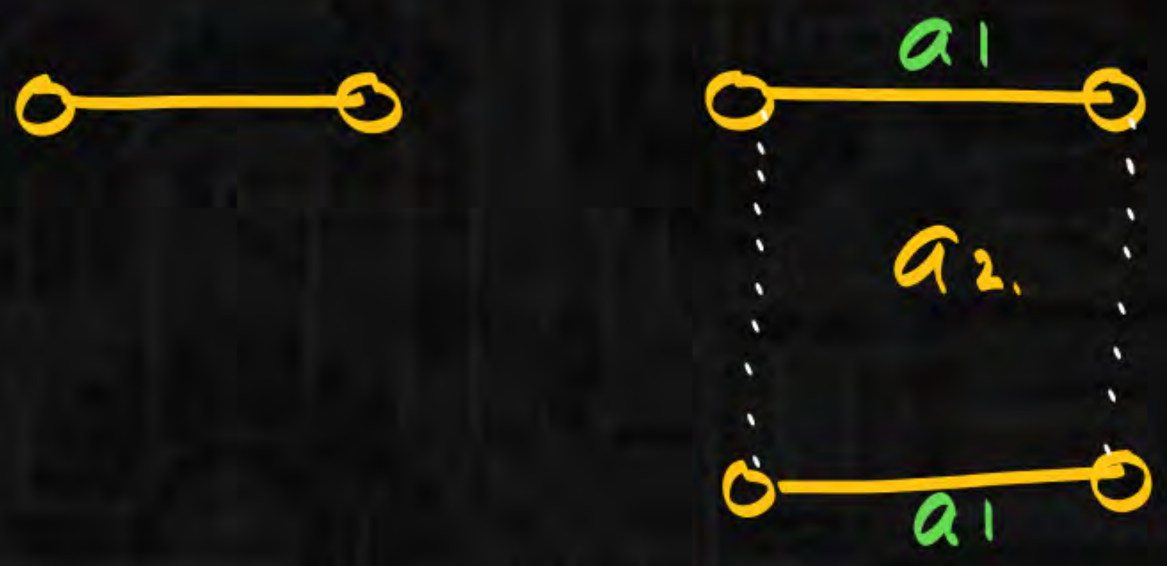
$\boxed{f(n) = 1}$

$f(n) = n$

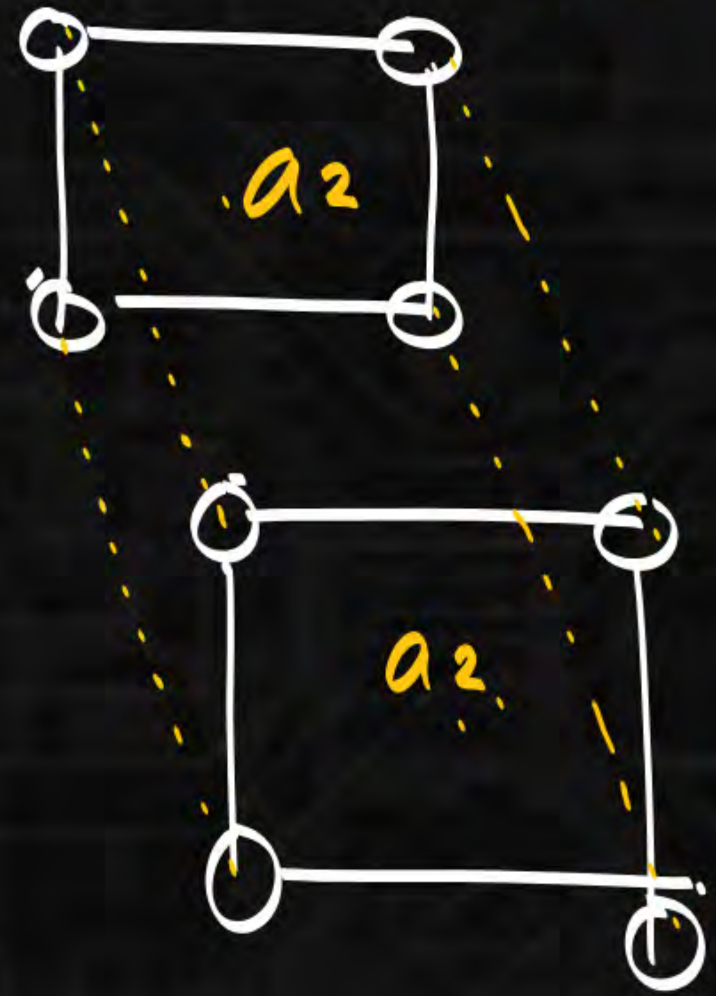
$f(n) = n^2$

f

$$a_3 = 2a_2 + 2^2$$



$$a_2 = 2a_1 + 2^1$$



a_3

$$a_2 = 2a_1 + 2^1$$

$$a_3 = 2a_2 + 2^2$$

$$a_{n+1} = 2a_n + 2^n$$

$$f(n) = 2^n$$

$$\boxed{a_n - 3a_{n-1} = 5(7^n)} \quad a_0 = 2.$$

Total $sol^n = a_n = a_n^H + a_n^P$

$$a_n - 3a_{n-1} = 0$$

$$x - 3 = 0$$

$$x = 3$$

Roots:

$$\boxed{a_n^H = 3^n C}$$

Particular $solt^n$

$$\boxed{a_n^P = A(7^n)}$$

$$\frac{A(7^n)}{7^{n-1}} - 3A\frac{(\cancel{7^{n-1}})}{\cancel{7^{n-1}}} = \frac{5(7^n)}{7^{n-1}}$$

$$\frac{A(\cancel{7^n})}{\frac{\cancel{7^n}}{7}} - 3A = \frac{5(\cancel{7^n})}{\frac{\cancel{7^n}}{7}}$$

$$\boxed{a_n = -\frac{27}{4} \cdot 3^n + \frac{35}{4} \cdot 7^n}$$

$$7A - 3A = 5 \cdot 7$$

$$4A = 35$$

$$A = \frac{35}{4}$$

$$\boxed{a_n^P = \frac{35}{4}(7^n)}$$

$$\boxed{a_n = a_n^H + a_n^P}$$

$$a_n = 3^n C + \frac{35}{4} \cdot 7^n$$

$$C = -\frac{27}{4}$$

$n = 0$

$$a_0 = 3^0 C + \frac{35}{4} \cdot 7^0$$

$$2 = C + \frac{35}{4}$$

$a_{n+1} = 2a_n + \underbrace{2^n}$ $\quad a_0 = 0$

$a_n = a_n^H + a_n^P$

Root: 3 $\quad f(n) = 5(7^n) \quad A(7^n)$

Root: 3 $\quad f(n) = 3^n \quad An(3^n)$

$a_{n+1} = 2a_n$

$x = 2$

Root: 2

$a_n^H = 2^n c.$

$a_n^P = An(2^n)$

$\dfrac{A(n+1)\cdot \cancel{2}^{n+1}}{\cancel{2^n}} = \dfrac{2(An\,\cancel{2^n})}{\cancel{2^n}} + \dfrac{\cancel{2^n}}{\cancel{2^n}}$

$2A(n+1) = 2An + 1.$

$\cancel{2An} + 2A = \cancel{2An} + 1$

$2A = 1$

$A = 1/2$

$a_n^P = \frac{1}{2}n(2^n)$

$a_n = a_n^H + \frac{1}{2}n(2^n)$

$\boxed{a_n = 2^n c + \frac{1}{2}n(2^n)}$

$0 = 2^0 C + \frac{1}{2}\cdot 0(2^0)$

$c = 0$

$a_n = \frac{1}{2}n(2^n)$

$a_n = n\cdot 2^{n-1}$

$$a_n - 3a_{n-1} = 5(3^n) \qquad a_0 = 2.$$

Homogenous:

$a_n - 3a_{n-1} = 0$

$x - 3 = 0$

Root : 3

$a_n^H = 3^n c.$

$a_n^P = An(3^n)$

$An(3^n) - 3A(n-1)3^{n-1} = 5 \cdot 3^n$

$\div$ by 3^n

$$\frac{An(\cancel{3^n})}{\cancel{3^n}} - \frac{\cancel{3}A(n-1)\cancel{3^{n-1}}}{\cancel{3^n}} = \frac{5 \cdot \cancel{3^n}}{\cancel{3^n}}$$

$An - A(n-1) = 5$

$An - An + A = 5$

$A = 5$

Root : 3 $\quad f(n) = 5(3^n)$

$\hookrightarrow a_n^P = \boxed{An(3^n)}$

$a_n^P = 5n(3^n)$

$\boxed{a_n = 3^n \cdot 2 + 5n \cdot 3^n}$

$a_n = a_n^H + a_n^P$

$a_n = 3^n c + 5n(3^n)$

$n = 0$

$2 = 3^0 c + 5 \cdot 0(3^n)$

$c = 2$

$f(n)$	a_n^{p}
5	A
$5(7^n)$	$A(7^n)$
$5(3^n)$	$An(3^n)$
$5(3^n)$	$An^2(3^n)$

Root: 3

Root: 3

Root: 3, 3

$a_n^{H} = 3^n c_1 + 3^n \cdot n \cdot c_2$

$f(n)$	a_n^{p}
n	$An+b$
n^2	An^2+bn+c
$n(7^n)$	$An+b(7^n)$

$$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$$

Root: 3, 3.

$$a_n^H = 3^n c_1 + n \cdot 3^n c_2.$$

$$a_n^P = A(2^n) + Bn^2(3^n)$$

3. If a_n, $n \geq 0$, is the unique solution of the recurrence relation $a_{n+1} - da_n = 0$, and $a_3 = 153/49$, $a_5 = 1377/2401$, what is d?

3. $a_{n+1} - da_n = 0$, $n \geq 0$, so $a_n = d^n a_0$. $153/49 = a_3 = d^3 a_0$, $1377/2401 = a_5 = d^5 a_0 \Longrightarrow$ $a_5/a_3 = d^2 = 9/49$ and $d = \pm 3/7$.

1. Solve the following recurrence relations. (No final answer should involve complex numbers.)

a) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$

b) $2a_{n+2} - 11a_{n+1} + 5a_n = 0$, $n \geq 0$, $a_0 = 2$, $a_1 = -8$

c) $a_{n+2} + a_n = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 3$

d) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5$, $a_1 = 12$

1. (a) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$.
Let $a_n = cr^n$, $c, r \neq 0$. Then the characteristic equation is $r^2 - 5r - 6 = 0 = (r-6)(r+1)$, so $r = -1, 6$ are the characteristic roots.
$a_n = A(-1)^n + B(6)^n$
$1 = a_0 = A + B$
$3 = a_1 = -A + 6B$, so $B = 4/7$ and $A = 3/7$.
$a_n = (3/7)(-1)^n + (4/7)(6)^n$, $n \geq 0$.

(b) $a_n = 4(1/2)^n - 2(5)^n$, $n \geq 0$.

(c) $a_{n+2} + a_n = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 3$.
With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2 + 1 = 0$ yields the characteristic roots $\pm i$. Hence $a_n = A(i)^n + B(-i)^n = A(\cos(\pi/2) + i\sin(\pi/2))^n + B(\cos(\pi/2) + i\sin(-\pi/2))^n = C\cos(n\pi/2) + D\sin(n\pi/2)$.
$0 = a_0 = C$, $3 = a_1 = D\sin(\pi/2) = D$, so $a_n = 3\sin(n\pi/2)$, $n \geq 0$.

(d) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5$, $a_1 = 12$.
Let $a_n = cr^n$, $c, r \neq 0$. Then $r^2 - 6r + 9 = 0 = (r-3)^2$, so the characteristic roots are 3,3 and $a_n = A(3^n) + Bn(3^n)$.
$5 = a_0 = A$; $12 = a_1 = 3A + 3B = 15 + 3B$, $B = -1$.
$a_n = 5(3^n) - n(3^n) = (5-n)(3^n)$, $n \geq 0$.

3. If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, and $a_3 = 37$ satisfy the recurrence relation $a_{n+2} + ba_{n+1} + ca_n = 0$, where $n \geq 0$ and b, c are constants, determine b, c and solve for a_n.

3. $(n = 0):$ $a_2 + ba_1 + ca_0 = 0 = 4 + b(1) + c(0)$, so $b = -4$.
$(n = 1):$ $a_3 - 4a_2 + ca_1 = 0 = 37 - 4(4) + c$, so $c = -21$.
$a_{n+2} - 4a_{n+1} - 21a_n = 0$
$r^2 - 4r - 21 = 0 = (r-7)(r+3)$, $r = 7, -3$
$a_n = A(7)^n + B(-3)^n$
$0 = a_0 = A + B \Longrightarrow B = -A$
$1 = a_1 = 7A - 3B = 10A$, so $A = 1/10$, $B = -1/10$ and $a_n = (1/10)[(7)^n - (-3)^n]$, $n \geq 0$.

9. For $n \geq 0$, let a_n count the number of ways a sequence of 1's and 2's will sum to n. For example, $a_3 = 3$ because (1) 1, 1, 1; (2) 1, 2; and (3) 2, 1 sum to 3. Find and solve a recurrence relation for a_n.

9. $a_n = a_{n-1} + a_{n-2}$, $n \geq 0$, $a_0 = a_1 = 1$

(Append '+1') (Append '+2')

$a_n = A[(1+\sqrt{5})/2]^n + B[(1-\sqrt{5})/2]^n$
$1 = a_0 = A + B$; $1 = a_1 = A(1+\sqrt{5})/2 + B(1-\sqrt{5})/2$ or
$2 = (A+B) + \sqrt{5}(A-B) = 1 + \sqrt{5}(A-B)$ and $A - B = 1/\sqrt{5}$.
$1 = A + B$, $1/\sqrt{5} = A - B \implies A = (1+\sqrt{5})/2\sqrt{5}$, $B = (\sqrt{5}-1)/2\sqrt{5}$ and $a_n = (1/\sqrt{5})[((1+\sqrt{5})/2)^{n+1} - ((1-\sqrt{5})/2)^{n+1}]$, $n \geq 0$.

31. Solve the recurrence relation $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$, where $n \geq 0$ and $a_0 = 4$, $a_1 = 13$.

32. Determine the constants b and c if $a_n = c_1 + c_2(7^n)$, $n \geq 0$, is the general solution of the relation $a_{n+2} + ba_{n+1} + ca_n = 0$, $n \geq 0$.

31. Let $b_n = a_n^2$, $b_0 = 16$, $b_1 = 169$.
This yields the linear relation $b_{n+2} - 5b_{n+1} + 4b_n = 0$ with characteristic roots $r = 4, 1$, so $b_n = A(1)^n + B(4)^n$.
$b_0 = 16$, $b_1 = 169 \implies A = -35$, $B = 51$ and $b_n = 51(4)^n - 35$. Hence $a_n = \sqrt{51(4)^n - 35}$, $n \geq 0$.

32. $a_n = c_1 + c_2(7)^n$, $n \geq 0$, is the solution of $a_{n+2} + ba_{n+1} + ca_n = 0$, so $r^2 + br + c = 0$ is the characteristic equation and $(r-1)(r-7) = (r^2 - 8r + 7) = r^2 + br + c$. Consequently, $b = -8$ and $c = 7$.

5. Solve the following recurrence relations.

a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \quad n \geq 0, \quad a_0 = 0, \quad a_1 = 1$

b) $a_{n+2} + 4a_{n+1} + 4a_n = 7, \quad n \geq 0, \quad a_0 = 1, \quad a_1 = 2$

6. Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$, where $n \geq 0$ and $a_0 = 1$, $a_1 = 4$.

7. Find the general solution for the recurrence relation $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 3 + 5n, n \geq 0$.

5. (a) $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$.
With $a_n = cr^n$, $c, r \neq 0$, the characteristic equation $r^2 + 3r + 2 = 0 = (r+2)(r+1)$ yields the characteristic roots $r = -1, -2$.
Hence $a_n^{(h)} = A(-1)^n + B(-2)^n$, while $a_n^{(p)} = C(3)^n$.
$C(3)^{n+2} + 3C(3)^{n+1} + 2C(3)^n = 3^n \Longrightarrow 9C + 9C + 2C = 1 \Longrightarrow C = 1/20$.
$a_n = A(-1)^n + B(-2)^n + (1/20)(3)^n$
$0 = a_0 = A + B + (1/20)$
$1 = a_1 = -A - 2B + (3/20)$
Hence $1 = a_0 + a_1 = -B + (4/20)$ and $B = -4/5$. Then $A = -B - (1/20) = 3/4$.
$a_n = (3/4)(-1)^n + (-4/5)(-2)^n + (1/20)(3)^n$, $n \geq 0$

(b) $a_n = (2/9)(-2)^n - (5/6)(n)(-2)^n + (7/9)$, $n \geq 0$

6. $a_{n+2} - 6a_{n+1} + 9a_n = 3(2)^n + 7(3)^n$, $n \geq 0$, $a_0 = 1$, $a_1 = 4$.
$a_n^{(h)} = A(3)^n + Bn(3)^n$ $\quad a_n^{(p)} = C(2)^n + Dn^2(3)^n$.
Substituting $a_n^{(p)}$ into the given recurrence relation, by comparison of coefficients we find that $C = 3$, $D = 7/18$.
$a_n = A(3)^n + Bn(3)^n + 3(2)^n + (7/18)n^2(3)^n$
$1 = a_0, 4 = a_1 \Longrightarrow A = -1, B = 17/18$, so
$a_n = (-2)(3)^n + (17/18)n(3)^n + (7/18)n^2(3)^n + 3(2)^n$, $n \geq 0$.

7. Here the characteristic equation is $r^3 - 3r^2 + 3r - 1 = 0 = (r-1)^3$, so $r = 1, 1, 1$ and
$a_n^{(h)} = A + Bn + Cn^2$, $a_n^{(p)} = Dn^3 + En^4$.
$D(n+3)^3 + E(n+3)^4 - 3D(n+2)^3 - 3E(n+2)^4 + 3D(n+1)^3 + 3E(n+1)^4 - Dn^3 - En^4 = 3 + 5n \Longrightarrow D = -3/4$, $E = 5/24$.
$a_n = A + Bn + Cn^2 - (3/4)n^3 + (5/24)n^4$, $n \geq 0$.

10. The general solution of the recurrence relation $a_{n+2} + b_1 a_{n+1} + b_2 a_n = b_3 n + b_4$, $n \geq 0$, with b_i constant for $1 \leq i \leq 4$, is $c_1 2^n + c_2 3^n + n - 7$. Find b_i for each $1 \leq i \leq 4$.

11. Solve the following recurrence relations.

a) $a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n, \quad n \geq 0, \quad a_0 = a_1 = 1$

b) $a_n^2 - 2a_{n-1} = 0, \quad n \geq 1, \quad a_0 = 2$ (Let $b_n = \log_2 a_n$, $n \geq 0$.)

$a_{n+2} + b_1 a_{n+1} + b_2 a_n = b_3 n + b_4$
$a_n = c_1 2^n + c_2 3^n + n - 7$
$r^2 + b_1 r + b_2 = (r-2)(r-3) = r^2 - 5r + 6 \Longrightarrow b_1 = -5,\ b_2 = 6$

$a_n^{(p)} = n - 7$
$[(n+2) - 7] - 5[(n+1) - 7] + 6(n-7) = b_3 n + b_4 \Longrightarrow b_3 = 2,\ b_4 = -17.$

(a) Let $a_n^2 = b_n, n \geq 0$
$b_{n+2} - 5b_{n+1} + 6b_n = 7n$
$b_n^{(h)} = A(3^n) + B(2^n),\ b_n^{(p)} = Cn + D$
$C(n+2) + D - 5[C(n+1) + D] + 6(Cn + D) = 7n \Longrightarrow C = 7/2,\ D = 21/4$
$b_n = A(3^n) + B(2^n) + (7n/2) + (21/4)$
$b_0 = a_0^2 = 1,\ b_1 = a_1^2 = 1$
$1 = b_0 = A + B + 21/4$
$1 = b_1 = 3A + 2B + 7/2 + 21/4$
$3A + 2B = -31/3$
$2A + 2B = -34/4$
$A = 3/4,\ B = -5$
$a_n = [(3/4)(3)^n - 5(2)^n + (7n/2) + (21/4)]^{1/2},\ n \geq 0$
(b) $a_n^2 - 2a_{n-1} = 0,\ n \geq 1,\ a_0 = 2$
$a_n^2 = 2a_{n-1}$
$\log_2 a_n^2 = \log_2(2a_{n-1}) = \log_2 2 + \log_2 a_{n-1}$
$2\log_2 a_n = 1 + \log_2 a_{n-1}$
Let $b_n = \log_2 a_n$.
The solution of the recurrence relation $2b_n = 1 + b_{n-1}$ is $b_n = A(1/2)^n + 1$.
$b_0 = \log_2 a_0 = \log_2 2 = 1$, so $1 = b_0 = A + 1$ and $A = 0$.
Consequently, $b_n = 1,\ n \geq 0$, and $a_n = 2,\ n \geq 0$.


THANK YOU GW
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