CS & IT ENGINEERING

DISCRETE MATHS
SET THEORY

Lecture No. 6

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TOPICS TO BE COVERED

01 Basics of relations

02 Types of relations

03 Number Of relations

Relations

demand diagonal $\rightarrow 2^{n^2-n}$

Reflexive $\forall a \in A\ (a,a) \in R.$

Symmetric $\forall a \forall b\ [(a,b) \in R \rightarrow (b,a) \in R] \rightarrow 2^{n}.\quad 2^{\frac{n^2-n}{2}}$

↳ allows same element

↳ demands flipping.

IRR $\forall a \in A\ (a,a) \notin R.$

(hates same element) $2^{n^2-n}.$

Relations

Antisymmetric. → does not allows flipping.
→ allows same element.

$$\forall a \forall b \left[(a,b) \in R \wedge (b,a) \in R \rightarrow a = b \right]$$

$R_1 = \{ \ \}$ ✓

$$\underbrace{\underbrace{\underbrace{(a,b) \in R}_{F} \wedge \underbrace{(b,a) \in R}_{F}}_{F} \rightarrow a = b}_{\text{True}}$$

$R_2 = \{(1, 1)\}$ Anti ✓

$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

$\underbrace{(1,1) \in R}_{T} \wedge \underbrace{(1,1) \in R}_{T} \rightarrow 1 = 1$

$a = 1$
$b = 1$

$T \wedge T \rightarrow T$

Relations

$R_3 = \{(1,2)(2,1)\}$ Anti ✗.

$(a,b) \in R \wedge (b,a) \in R \longrightarrow a=b.$

$\underbrace{(1,2) \in R}_{T} \wedge \underbrace{(2,1) \in R}_{T} \longrightarrow 1=2. \rightarrow F$

$a=1$
$b=2.$

$\underbrace{\underbrace{(1,2)\in R \wedge (2,1)\in R}_{T} \rightarrow F}_{\text{False}}$

$R_4 = \{(1,2)\}$ Anti ✓

$(a,b) \in R \wedge (b,a) \in R \longrightarrow a=b.$

$\underbrace{\underbrace{(1,2) \in R \wedge F}_{F} \longrightarrow}_{\text{True}}$

Relations

$R_5 = \{ \}$ ✓

$R_6 = \{(2\,2)(1\,1)\}$ ✓

$R_7 = \{(2\,3)(1\,1)(2\,2)(3\,3)(1\,3)\}$ ✓

$R_8 = \{(2\,3)\}$ ✓

$R_9 = \{(1\,3)\}$ ✓

$R_{10} = \{(1\,1)(2\,2)\boxed{(1\,2)(2\,1)}\}$ ✓ ✓ ✗.

not Anti

Relations

Anti → allows same element.
↳ does not allows flipping.

$R_1 = \{(1\,2)\}$ ✓

$R_2 = \{(2\,1)\}$ ✓

$R_3 = \{\ \}$ ✓

present

$\boxed{(1\,2)\quad(2\,1)}$

present absent

Relations

$A = \{1, 2, 3\}$.

$$\text{boxes} = \frac{n^2 - n}{2}$$

$$A \times A = \{ \underbrace{(11)(22)(33)}_{n} \boxed{(12)(21)} \boxed{(13)(31)} \boxed{(23)(32)} \}$$

$n^2 - n$

n^2

$$2^n \times 3^{\frac{n^2-n}{2}}$$

Relations

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

→ 2 choices

2^n

Total $= n^2$

Diagon $= n$

non diagonal $= n^2 - n$

boxes $= \frac{n^2-n}{2}$

→ 3 choices

$$\left\{ 2^n \cdot 3^{\frac{n^2-n}{2}} \right.$$

Relations

$A = \{1, 2, 3\}$.

what is smallest size of Antisymmetric $\rightarrow$ 0

largest size of Antisymmetric $\rightarrow \left\{ n + \frac{n^2 - n}{2} \right\}$

Relations

Total no. of Anti

$2^n \cdot 3^{\frac{n^2-n}{2}}$

$\{\ \}$ ← smallest size.

$(1\,1)$

$(2\,2)$

$(3\,3)$

$(1\,2)\,(1\,1)$

⋮

$\{\underbrace{(1\,1)(2\,2)(3\,3)}_{\leftarrow\ n\ \rightarrow}\underbrace{(1\,2)(1\,3)(2\,3)}_{\frac{n^2-n}{2}}\}$ ← largest size.

$n + \frac{n^2-n}{2}$

Relations

1 1	1 2	1 3
2 1	2 2	2 3
3 1	3 2	3 3

$$n + \frac{n^2 - n}{2}$$

$$\frac{2n + n^2 - n}{2}$$

$$\frac{n(n+1)}{2}.$$

Relations

Symmetric:

$\{\ \}$ ← smallest size = 0

largest size $= n^2$.

$\{(11)(22)(33)(12)(21)(13)(31)(23)(32)\}$

Relations

Reflexive:

smallest = n

(11) (22) (33)

largest = n^2.

(11) (22) (33) (12) (21) (13) (31) (23) (32)

IRR

{ }

smallest = 0

(12) (21) (13) (31) (23) (32)

largest

$n^2 - n$

Relations

Asymmetric :
→ does not allows same element.
→ does not allows flipping.

$$\forall a \forall b\left((a,b) \in R \rightarrow (b,a) \notin R\right)$$

$R_1 = \{\ \}$ ✓

$(a,b) \in R \rightarrow (b,a) \notin R$

$\underbrace{F \rightarrow}_{\text{True}}$

$R_2 = \{(1,1)\}$ Asy ✗

$(a,b) \in R \rightarrow (b,a) \notin R$

$\underbrace{\underbrace{(1,1) \in R}_{T} \rightarrow \underbrace{(1,1) \notin R}_{F}}_{\text{False}}$

Relations

$R_3 = \{(1\,2)\}$ Asy ✓

$(a, b) \in R \rightarrow (b, a) \notin R.$

$\underbrace{(1, 2) \in R}_{T.} \rightarrow \underbrace{(2, 1) \notin R}_{T.}$

if (a, b) belongs to R then (b, a) should not belongs to R.

if (a b) is present then (b, a) must be absent

Relations

(1, 1)	(1, 2)	(1, 3)
(2, 1)	(2, 2)	(2, 3)
(3, 1)	(3, 2)	(3, 3)

→ 1 choice absent.

$$\frac{n^2-n}{2}.$$

3

$$\boxed{(1\,2)\quad(2\,1)}$$

(1 2) → present

(2 1) → present

(2 1) → absent.

Symmetric

→ allows same element

→ demands flipping

$2^n \cdot 2^{\frac{n^2-n}{2}}$

$(a,b) \in R \rightarrow (b,a) \in R$

Antisymmetric

→ allows same element.

→ no flipping.

$2^n \cdot 3^{\frac{n^2-n}{2}}$

$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

Asymmetric.

→ no same elements.

→ no flipping.

$3^{\frac{n^2-n}{2}}$

$(a,b) \in R \rightarrow (b,a) \notin R$.

Relations

$A = \{1, 2, 3, 4\}$ (GATE-98)

$R = \{(1\,1)\ (2\,1)\ (2\,2)\ (2\,3)\ (2\,4)\ (3\,1)\ (3\,2)\ (3\,3)\ (3\,4)\}$

Ref → (4, 4) is absent no Ref.

Irr → hates same X.

Sym → X

Anti X

Asy X

Relations

(GATE-09)

xy ✓ xz zx zy $A = \{x, y, z\}$

Sym ✗

Anti ✗

Relations

	Ref	IRR	Sym	Anti	Asy
	2^{n^2-n}	2^{n^2-n}	$2^n \cdot 2^{\frac{n^2-n}{2}}$	$2^n \cdot 3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$
Ref	×	0	$1^n \cdot 2^{\frac{n^2-n}{2}}$	$1^n \cdot 3^{\frac{n^2-n}{2}}$	0
IRR		×	$1^n \cdot 2^{n^2-n/2}$	$1^n \cdot 3^{n^2-n/2}$	$1^n \cdot 3^{n^2-n/2}$
Sym			×	2^n	1
Anti				×	$3^{n^2-n/2}$
Asy					×

diagonal absent

no flipping

Relations

Reflexive ∧ Irreflexive.

$\{(11)(22)(33)\ldots\ldots\}$

Ref

$\{\underline{11\ 22\ 33}\}$

Symm.

$\{\boxed{11\ 22\ 33}\ (12)(21)\cdots\}$

11	12	13
21	22	23
31	32	33

Relations

$A \to$ non empty set.

$A = \{1, 2, 3\}$.

$R_1 \to$ Reflexive relation on A.

$R_2 \to$ reflexive —— " ——.

$R_1 = \{\boxed{11 \quad 22 \quad 33}\ (\ )\ (\ ) \ldots\}$

$R_2 = \{\boxed{11 \quad 22 \quad 33}\ (\ )\ (\ ) \ldots\}$

$R_1 \cup R_2 = \{\boxed{}\}$ Reflexive.

$R_1 \cap R_2 = \{\boxed{} \ldots\}$ Reflexive.

Relations

	Union ($\cup$)	Inter- ($\cap$)
R1, R2 reflexive	✓	✓
R1, R2 IRR	✓	✓
R1, R2 symm	✓	✓
Anti	✗	✓
ASY	✗	✓

$R_1 = \{(1\,2)\}$ $R_2 = \{(2\,1)\}$

Anti ✓ Anti ✓

$R_1 \cup R_2 = \{(1\,2)(2\,1)\}$ not Anti

$R_1 \cap R_2 = \{\ \}$

check for

reflexive, symmetric, antisymmetric, or transitive.

$\mathcal{R} \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$ where $a \mathcal{R} b$ if $a|b$ (read "a divides b,"

b) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $a \mathcal{R} b$ if $a|b$.

c) For a given universe $\mathcal{U}$ and a fixed subset C of $\mathcal{U}$, define $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ as follows: For $A, B \subseteq \mathcal{U}$ we have $A \mathcal{R} B$ if $A \cap C = B \cap C$.

d) On the set A of all lines in $\mathbf{R}^2$, define the relation $\mathcal{R}$ for two lines ℓ_1, ℓ_2 by $\ell_1 \mathcal{R} \ell_2$ if ℓ_1 is perpendicular to ℓ_2.

e) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $x \mathcal{R} y$ if $x + y$ is odd.

f) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $x \mathcal{R} y$ if $x - y$ is even.

g) Let T be the set of all triangles in $\mathbf{R}^2$. Define $\mathcal{R}$ on T by $t_1 \mathcal{R} t_2$ if t_1 and t_2 have an angle of the same measure.

h) $\mathcal{R}$ is the relation on $\mathbf{Z} \times \mathbf{Z}$ where $(a, b)\mathcal{R}(c, d)$ if $a \leq c$. [*Note*: $\mathcal{R} \subseteq (\mathbf{Z} \times \mathbf{Z}) \times (\mathbf{Z} \times \mathbf{Z})$.]

(a) reflexive, antisymmetric, transitive
(b) transitive
(c) reflexive, symmetric, transitive
(d) symmetric
(e) (odd): symmetric
(f) (even): reflexive, symmetric, transitive
(g) reflexive, symmetric
(h) reflexive, transitive

a) If $\mathcal{R}$ is a relation on A and $|\mathcal{R}| \geq n$, then $\mathcal{R}$ is reflexive.

b) If $\mathcal{R}_1, \mathcal{R}_2$ are relations on A and $\mathcal{R}_2 \supseteq \mathcal{R}_1$, then $\mathcal{R}_1$ reflexive (symmetric, antisymmetric, transitive) $\Rightarrow \mathcal{R}_2$ reflexive (symmetric, antisymmetric, transitive).

c) If $\mathcal{R}_1, \mathcal{R}_2$ are relations on A and $\mathcal{R}_2 \supseteq \mathcal{R}_1$, then $\mathcal{R}_2$ reflexive (symmetric, antisymmetric, transitive) $\Rightarrow \mathcal{R}_1$ reflexive (symmetric, antisymmetric, transitive).

(a) **False:** Let $A = \{1,2\}$ and $\mathcal{R} = \{(1,2),(2,1)\}$.

(b)
- (i) **Reflexive:** True
- (ii) **Symmetric:** False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(1,2)\}$.
- (iii) **Antisymmetric & Transitive:** False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}$, $\mathcal{R}_2 = \{(1,2),(2,1)\}$.

(c)
- (i) **Reflexive:** False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(2,2)\}$.
- (ii) **Symmetric:** False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}, \mathcal{R}_2 = \{(1,2),(2,1)\}$.
- (iii) **Antisymmetric:** True
- (iv) **Transitive:** False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2),(2,1)\}, \mathcal{R}_2 = \{(1,1),(1,2),(2,1),(2,2)\}$.

10. If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, and antisymmetric.

(a) 2^{12} (b) $(2^4)(2^6) = 2^{10}$ (c) 2^6

(d) 2^{11} (e) $(2^4)(2^5) = 2^9$ (f) $2^4 \cdot 3^6$

(g) $2^4 \cdot 3^3$ (h) (2^4) (i) 1

17. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. How many symmetric relations on A contain exactly (a) four ordered pairs? (b) five or-

(a) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{0}\binom{21}{2}$

(b) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{1}\binom{21}{2}$


THANK YOU GW
SOLDIERS !