ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-03

Numerical Methods

By- Chetan Sir

Topics to be Covered

NUMERICAL INTEGRATION

- TRAPEZOIDAL RULE
- SIMPSON'S 1/3RD RULE
- SIMPSON'S 3/8TH RULE

Numerical solution of a differential equation

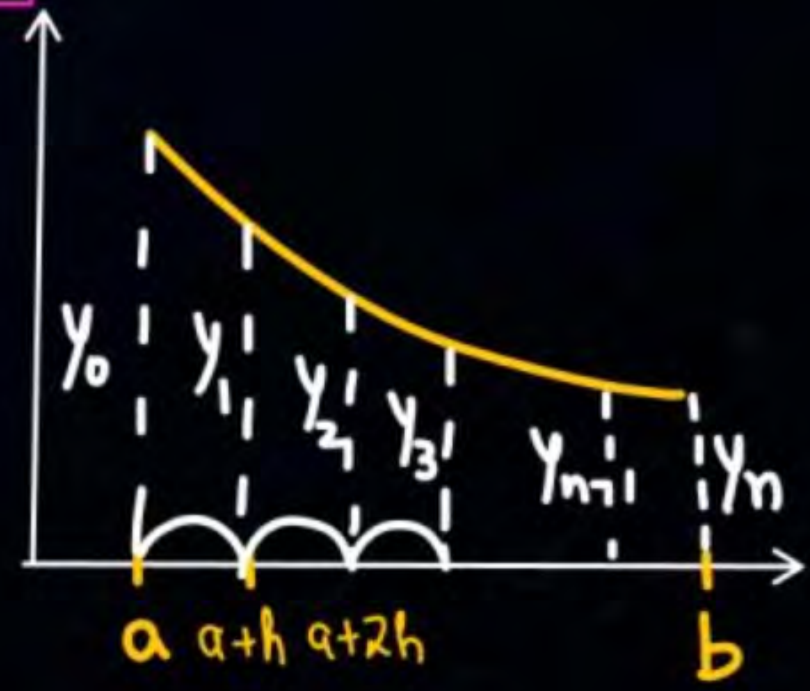
- TAYLOR'S METHOD
- PICARD'S METHOD
- EULER'S METHOD/FORWARD EULER/EXPLICIT EULER METHOD
- BACKWARD EULER/IMPLICIT EULAR METHOD
- RUNGE KUTTA METHOD

[NUMERICAL INTEGRATION]

$$b = a + nh$$
$$h = \frac{b-a}{n}$$

Trapezoidal Formula

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left\{(y_0+y_n)+2(y_1+y_2+y_3+\cdots y_{n-1})\right\}}$$



- This formula will give no error / is suitable for LINEAR function.

Cumulative error $\rightarrow \boxed{|E| < \frac{(b-a)}{12}h^2 M}$

$M \rightarrow \max(y_0'', y_1'', y_2'' \ldots y_{n-1}'')$

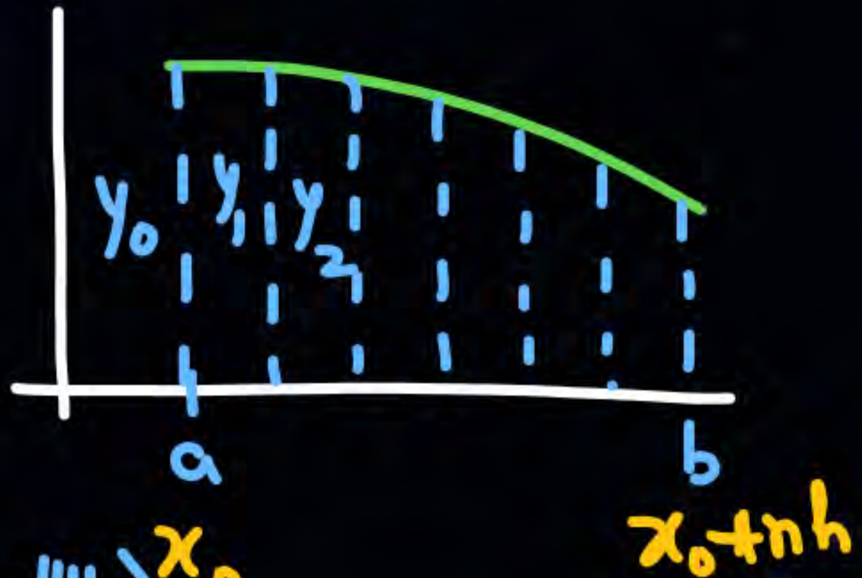
NUMERICAL INTEGRATION

SIMPSON'S 1/3 RULE

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left\{(y_0+y_n) + 4\overbrace{(y_1+y_3+\cdots y_{n-1})}^{\text{Odd ordinates}} + 2\overbrace{(y_2+y_4+\cdots y_{n-2})}^{\text{Even ordinates}}\right.}$$

→ It is only applicable when no. of steps is even (multiple of 2) or no. of ordinates are odd.

→ This formula is suitable/gives no error for 'Quadratic function'.



$$\boxed{|E| < \frac{(b-a)}{180} h^4 M}$$

$M \rightarrow \max(y_0'''', y_1'''', \ldots y_{2n-2}'''')$

NUMERICAL INTEGRATION

SIMPSON'S 3/8th RULE

$$\int_a^b f(x)\,dx = \frac{3}{8}h\left\{(y_0+y_n) + 3\overbrace{(y_1+y_2+\dots y_{n-2}+y_{n-1})}^{\text{Remaining ordinates}} + 2\overbrace{(y_3+y_6+\dots y_{n-3})}^{\text{Multiple of 3}}\right.$$

→ This formula is applicable when subintervals is a multiple of 3.
(no. of steps)

→ This formula is suitable for 'CUBIC' function.

$$\boxed{|E| \le \frac{b-a}{80}h^4 M}$$

$M \to \max(y_0^{iv}, y_1^{iv}, \dots)$

Q.

Evaluate $\boxed{\int_0^6 \frac{1}{1+x^2}\,dx}$ step size; h = 1

1). By trapezoidal method

2). By Simpson's 1/3 rd Rule

3). By Simpson's 3/8 rd Rule

$$f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	$1/17$	$1/26$	$1/37$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1) Trapezoidal Rule :- (Steps → Multiple of 1)

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{1}{2}\left\{\left(1+\frac{1}{37}\right)+2\left(0.5+0.2+0.1+\frac{1}{17}+\frac{1}{26}\right)\right\}$$

$$= 1.410798581$$

2) Simpson's $\frac{1}{3}$rd rule :- (Steps → (Multiple of 2)

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{1}{3}\left\{\left(\overset{y_0}{1}+\overset{y_6}{\frac{1}{37}}\right)+4\left(\overset{y_1}{0.5}+\overset{y_3}{0.1}+\overset{y_5}{\frac{1}{26}}\right)+2\left(\overset{y_2}{0.2}+\overset{y_4}{\frac{1}{17}}\right)\right\}$$

$$= 1.366173413$$

3) Simpson $\frac{3}{8}$th rule :- (Steps → Multiple of 3)

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3}{8}\left\{\left(\overset{y_0}{1} + \overset{y_6}{\frac{1}{37}}\right) + 2\left(\overset{y_3}{0.1}\right) + 3\left(\overset{y_1}{0.5} + \overset{y_2}{0.2} + \overset{y_4}{\frac{1}{17}} + \overset{y_5}{\frac{1}{26}}\right)\right\}$$

$$= 1.357080836$$

4) Actual integration :- $\int_0^6 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^6 = \tan^{-1}6 - \tan^{-1}0$

$$= 80.56^\circ$$

$$= 1.405647649$$

Actual error = Integration − Numerical integration (formula)

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

(1) Taylor's method

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Initial condition

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$\boxed{y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \cdots}$$

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

(ii) Picard's method of successive approximation

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y_0)\, dx$$

$$\boxed{y_{i+1} = y_0 + \int_{x_0}^{x} f(x,y_i)\, dx}$$

Q. Apply picard's method to solve D.E. $\frac{dy}{dx} = y - x$; $y(0) = 2$; $x_0 = 0, y_0 = 2$

$$\frac{dy}{dx} = y - x = f(x,y)$$

1st approximation; $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 2 + \int_0^x (2-x)\,dx$

$$y_1 = 2 + 2x - x^2/2$$

2nd approximation; $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = 2 + \int_0^x 2 + 2x - \frac{x^2}{2} - x$

$$y_2 = 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$$

Now for exact solution;

$$\frac{dy}{dx} - y = -x \qquad \text{I.F.} = e^{\int -1\,dx} = e^{-x}$$

$$y \cdot e^{-x} = \int -x \cdot e^{-x}$$

$$y e^{-x} = -(-x-1)e^{-x} + c$$

$$y e^{-x} = (x+1)\, e^{-x} + c$$

$$y = (x+1) + ce^{x} \qquad x_0 = 0\;;\; y_0 = 2$$

$$y = (x+1) + 1\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \qquad \text{we get } c = 1$$

$$= \underbrace{2 + 2x + \frac{x^2}{2}}_{} + \frac{x^3}{6}$$

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

(iii) Euler's Method/Forward Euler method/ Explicit Euler method

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

**

$$\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)} \qquad n = 0, 1, 2, 3 \ldots$$

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

(iv) Backward Euler method/ Implicit Euler method

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$$\boxed{y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})} \qquad n = 0, 1, 2, 3 \ldots$$

Q. Solve the initial value problem :-

$\boxed{\frac{dy}{dx} = x + y}$; $y(0) = 1, x\varepsilon[0, 1]$ by forward Euler :-

$x_0 = 0$
$y_0 = 1$

1).By forward Euler :- (Step Size $h = 0.1$)

x	0	0.1	0.2	0.3 1
y	1	1.1	1.22	1.362 1.5282 3.1875

① $y_1 = y_0 + h f(x_0, y_0)$

$y_1 = 1 + 0.1(x_0 + y_0) = 1 + 0.1(0+1) = 1.1$

② $y_2 = y_1 + h f(x_1, y_1) = 1.1 + 0.1(x_1 + y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$

2) By Backward Euler's :-

① $y_1 = y_0 + h f(x_1, y_1)$

$$y_1 = 1 + 0.1(x_1 + y_1) = 1 + 0.1(0.1 + y_1) = 1 + 0.01 + 0.1y_1$$

$$y_1 - 0.1y_1 = 1.01$$

$$y_1 = 1.01/0.9 = 1.122$$

② $y_2 = y_1 + h f(x_2, y_2)$

$$y_2 = 1.122 + 0.1(0.2 + y_2)$$

$$y_2 = 1.2688$$

x	y
0	1
0.1	1.122
0.2	1.2688
0.3	1.43
⋮	
1	

Euler Method

$$\boxed{y_1 = y_0 + h f(x_0, y_0)}$$

Simple, but least accurate

Terms included upto h.

Hence, truncation error is of order h^2.

NOTE :-

$$y_{n+1} = y_n + h f(x_n, y_n)$$
$$= y_n + h(-3y_n + 2)$$
$$y_{n+1} = \underbrace{(1-3h)} y_n + 2h$$

Taylor method

$$\boxed{y_1 = y_0 + h y_0'} + \frac{h^2}{2!} y_0'' + \dots$$

More accurate

$$-1 < 1-3h < +1$$

$$\boxed{0 < h < \frac{2}{3}}$$

; $\frac{dy}{dx} = -3y + 2$

Euler method is stable only when $|1-3h| < 1$

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

(5) Runge – Kutta Methods:

(a).Runge-kutta method of order 1 (Euler's method)

$$y_1 = y_0 + K_1 \qquad K_1 = h f(x_0, y_0)$$
$$= y_0 + h f(x_0, y_0)$$

Truncation error is of order h^2.

(b).Runge-kutta method of order 2 (modified Euler method)

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2) \qquad K_1 = h f(x_0, y_0)$$
$$K_2 = h f(x_0 + h, y_0 + K_1)$$

Truncation error is of order h^3. (Terms including h^2)

[NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION]

(5) Runge – Kutta Methods:

(c) Runge-kutta method of order 3

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

$$K_1 = h f(x_0, y_0)$$

$$K_2 = h f(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = h f(x_0 + h, y_0 + K')$$

$$\hookrightarrow K' = h f(x_0 + h, y_0 + K_1)$$

Truncation error is of order h^4. (Terms including h^3)

(5) Runge – Kutta Methods:

(d) Runge-kutta method of order 4 (Classical runge kutta method)

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = h f(x_0, y_0)$$

$$K_2 = h f(x_0 + h/2, y_0 + K_1/2)$$

$$K_3 = h f(x_0 + h/2, y_0 + K_2/2)$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

Truncation error is of order h^5 (Terms including h^4)

Q. $\boxed{\dfrac{dy}{dx} = x + y^2}; y(0) = 1$

$x_0 = 0$
$y_0 = 1$

x	y
0	1
0.1	1.1165
0.2	

$h = 0.1$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = 0.1\,(x_0 + y_0^2) = 0.1(0 + 1^2) = \boxed{0.1}$$

$$k_2 = h\, f(x_0 + h/2,\ y_0 + k_1/2) = 0.1\, f(0.05, 1.05) = 0.1\,(0.05 + 1.05^2) = \boxed{0.11525}$$

$$k_3 = h\, f(x_0 + h/2,\ y_0 + k_2/2) = 0.1\, f(0.05,\ 1 + 0.05763) = 0.1(0.05 + 1.05763^2)$$
$$= \boxed{0.11686}$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3) = 0.1\, f(0.1,\ 1.11686) = 0.1(0.1 + 1.11686^2) = \boxed{0.13474}$$

$$y_1 = 1 + \frac{1}{6}(0.1 + 2 \times 0.11525 + 2 \times 0.11686 + 0.13474) = 1.1165$$

$$y_2 = 1.27356$$

Thank you

GW

Soldiers !