ALL BRANCHES

ENGINEERING MATHEMATICS

Lecture No.-11

Calculus

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Topics to be Covered

- PARTIAL DIFFERENTIATION
- HOMOGENEOUS FUNCTION
- EULER'S THEOREM
- INTEGRATION
- DEFINITE INTEGRALS
- PROPERTY OF DEFINITE INTEGRALS

Ex:-

$$u = \boxed{\sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}} \rightarrow \text{Homo. of degree } 0.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$= 0$$

$x \rightarrow Kx$

$y \rightarrow Ky$

$u(x,y) = K^0 u(x,y)$

Ex:-

$$u = \sin^{-1}\frac{x}{y} \Rightarrow \sin u = \frac{x}{y}$$

$$x u_x + y u_y = \frac{n\, f(u)}{f'(u)} = \frac{0 \sin u}{\cos u} = 0$$

Ex:-

$$u = \frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}, \quad v = \frac{x^3+y^3}{x+y}$$

$u \rightarrow$ Degree $3/2$

$v \rightarrow$ Degree 2

$W = u + v$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = (x u_x + y u_y) + (x v_x + y v_y)$$

$$= \frac{3}{2}\cdot u + 2v$$

$$\rightarrow \int \underbrace{e^{ax}}_{\mathrm{I}} \sin(bx+c) = \frac{e^{ax}\left(a\sin(bx+c) - b\cos(bx+c)\right)}{a^2+b^2}$$

$$\rightarrow \int \underbrace{e^{ax}}_{\mathrm{I}} \underbrace{\cos(bx+c)}_{\mathrm{II}} = \frac{e^{ax}\left(a\cos(bx+c) + b\sin(bx+c)\right)}{a^2+b^2}$$

INTEGRATION BY PARTS :-

$$\int \underbrace{f}_{\mathrm{I}} \cdot \underbrace{g}_{\mathrm{II}}\, dx = f\int g\, dx - \int f' \int g\, dx$$

I → Inv. trig.
L → Log
A → Alg.
T → Trig.
E → Exp.

Ex :- $\int \underset{\mathrm{I}}{(x+5)}\, \underset{\mathrm{II}}{e^x} = (x+5)\, e^x - \int 1 \cdot e^x\, dx$

$e^x(x+5-1) = (x+4)e^x$

Ex :- $\int x^3 \sin x = x^3(-\cos x) - \int 3\underbrace{x^2}_{\mathrm{I}}\underbrace{(-\cos x)}_{\mathrm{II}}\, dx \rightarrow x^2 \sin x - \int 2\underbrace{x}_{\mathrm{I}}\underbrace{\sin x}_{\mathrm{II}}$

$$\int e^{3x} \cdot \sin 5x \, dx = e^{3x} \cdot \frac{[3 \sin 5x - 5 \cos 5x]}{3^2 + 5^2}$$

$$\int e^{5x} \cos 2x \, dx = e^{5x} \frac{[5 \cos 2x + 2 \sin 2x]}{5^2 + 2^2}$$

TRICK :-

$\underbrace{A}_{\mathrm{I}} \cdot \underbrace{T}_{\mathrm{II}}$ $\underbrace{A}_{\mathrm{I}} \cdot \underbrace{E}_{\mathrm{II}}$

$$\int x^3 (\sin x) = (x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x) + c$$

$$\int \underbrace{(x^2 + 2x + 5)}_{\mathrm{I}} \underbrace{e^x}_{\mathrm{II}} dx = (x^2 + 2x + 5)(e^x) - (2x + 2)(e^x) + (2)(e^x) + c$$

Ex:- $\int \log x \cdot 1 dx = \log x \cdot x - \int \frac{1}{x} \cdot x \, dx$

$$\boxed{\int \log x \, dx = x \log x - x}$$

Substitution method :-

① $\int \frac{x}{x^2+5^2} dx = \frac{1}{2}\int \frac{1}{t} dt$

$x^2+5^2 = t$

$2x \, dx = dt$

$x dx = \frac{dt}{2}$

$= \frac{1}{2} \log t + c$

$\frac{1}{2} \log(x^2+5^2) + c$

② $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{dt}{t} = -\log(\cos x)+c$

$= \log(\sec x)+c$

$\cos x = t$

$-\sin x \, dx = dt$

③ $\int e^{x^2+5x+2}(2x+5)\,dx = \int e^t\,dt = e^t + c$

$= e^{x^2+5x+2}$

④ $\int \frac{\cos x}{1+(\sin x)^2}\,dx = \int \frac{dt}{1+t^2}$

$\sin x = t$
$\cos x\,dx = dt$

$= \tan^{-1} t + c = \tan^{-1}(\sin x) + c$

$$\boxed{\begin{aligned} &\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)} + c \\ &\int \frac{f'(x)}{1+[f(x)]^2}\,dx = \tan^{-1}[f(x)] + c \\ &\int \frac{f'(x)}{f(x)}\,dx = \log[f(x)] + c \end{aligned}}$$

$\int \frac{f'(x)\,dx}{\sqrt{1-[f(x)]^2}} = \sin^{-1}[f(x)] + c$

IMPROPER INTEGRAL :-

If either range of integration is infinite or $f(x)$ is unbounded in the range of integration.

Type I :

$$\int_0^{\infty} \frac{dx}{x+3} \qquad \int_{-\infty}^{\infty} x^2\,dx \qquad \int_{-\infty}^{0} \frac{dx}{\sqrt{x+2}}$$

Type II :-

$$\int_{-1}^{+1} \frac{1}{x^{2/3}}\,dx = \left[\frac{x^{-2/3+1}}{-2/3+1}\right]_{-1}^{+1} = 3\left[x^{1/3}\right]_{-1}^{+1} = 3\left[1-(-1)\right] = \boxed{6}$$

$$\int_{-3}^{0} \frac{dx}{2x+5}$$

NOTE :- Improper integrals may converge or it may diverge.

Fundamental theorem of Integral Calculus :-

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$ Leibnitz theorem.

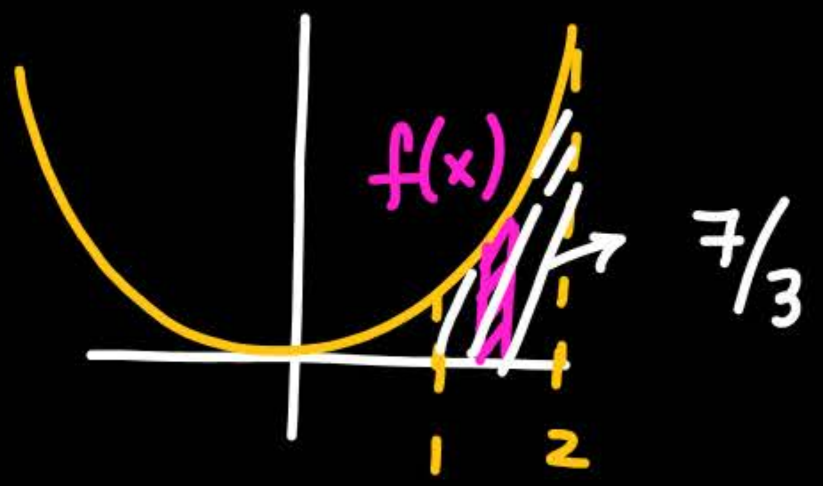
DEFINITE INTEGRAL AS A LIMIT OF SUM :-

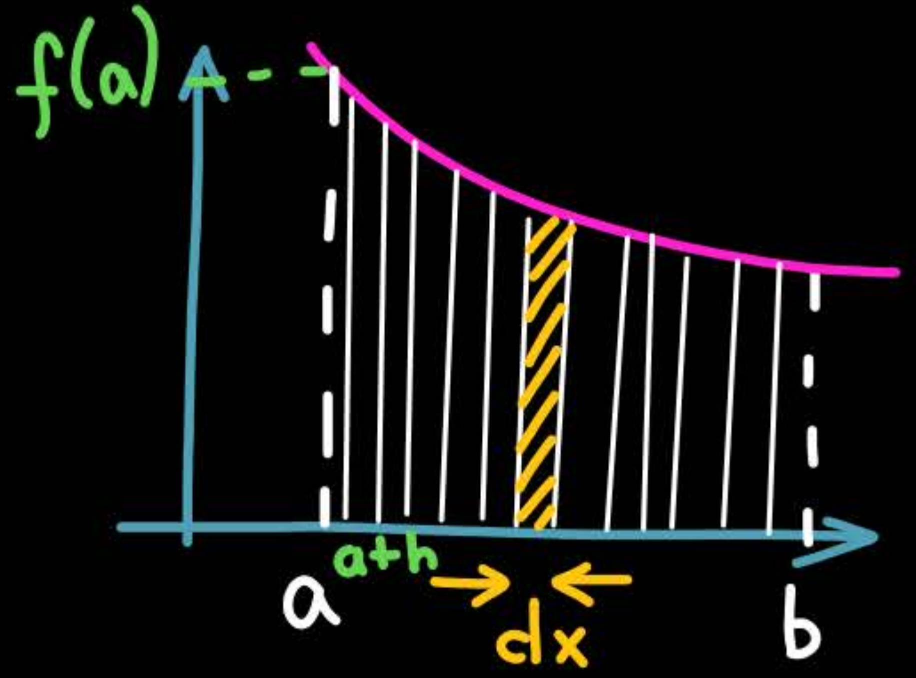
$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} \left[h\,f(a) + h\,f(a+h) + h\,f(a+2h) + \dots\; h\,f(a+(n-1)h) \right]$$

$$n = \frac{b-a}{h}$$

$$\left.\begin{matrix} h \to 0 \\ n \to \infty \end{matrix}\right\}$$ Limit of sum

Ex:- Area under the curve $f(x)=x^2$ from $x=1$ to $x=2$





$$\int_{x=1}^{x=2} f(x)\,dx = \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8-1}{3}$$

$$= 7/3$$

$$h = \frac{b-a}{n}$$

If $n \to \infty$, $h \to 0$

$h \to 0$

$n \to \infty$

Fundamental Properties of Definite Integral:-

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex:- $\int_1^5 f(x)\,dx = -\int_5^1 f(x)\,dx$

$\downarrow$

$a \qquad -a$

2. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dx$

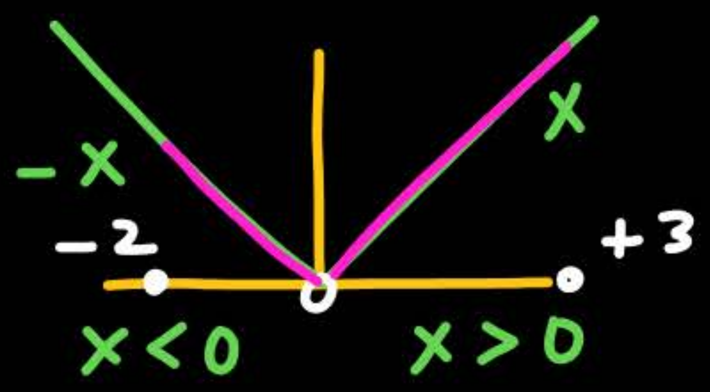
3) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

$f(x)$ is piecewise continuous.

(Ex:- Modulus, G.I.F., Fractional part)

Ex:- $\displaystyle\int_{-2}^{+3} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_{0}^{3} x\, dx$

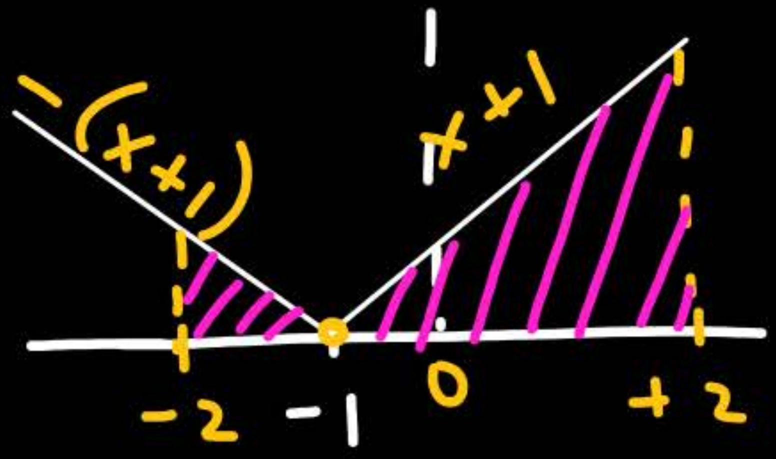
$\displaystyle = -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{3}$



Ex:- $\displaystyle\int_{-2}^{+2} |x+1|\, dx =$

$\displaystyle\int_{-2}^{-1} -(x+1)\, dx + \int_{-1}^{2} (x+1)\, dx$

$\displaystyle -\left[\frac{x^2}{2}+x\right]_{-2}^{-1} + \left[\frac{x^2}{2}+x\right]_{-1}^{2}$



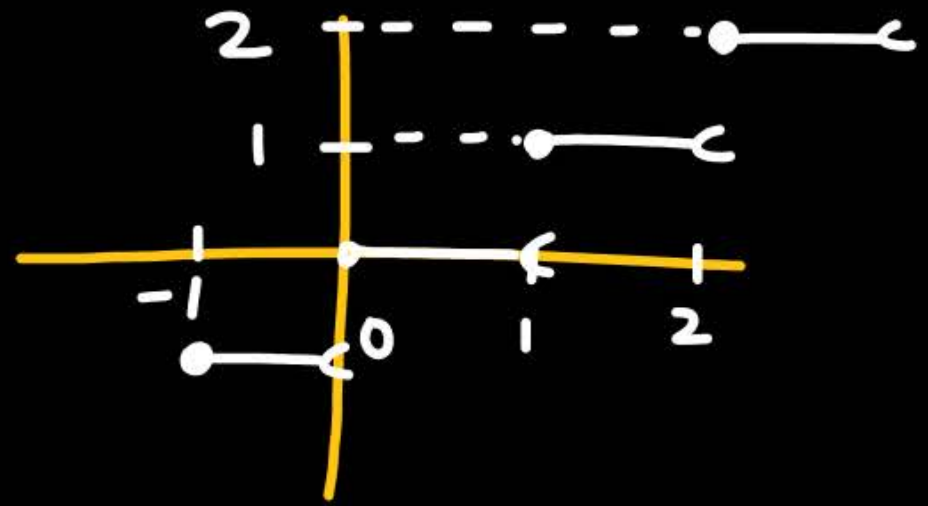
$(x+1)$ $\quad x+1>0$

$-(x+1)$ $\quad x+1<0$

$+x+1$ $\quad x>-1$

$-(x+1)$ $\quad x<-1$

Ex:- $\int_{-1}^{+3}[x]\,dx = \int_{-1}^{0} -1\,dx + \int_{0}^{1} 0\,dx + \int_{1}^{2} 1\,dx + \int_{2}^{3} 2\,dx$



Ex:- $\int_{0}^{1}[4x]\,dx =$

$$\int_{0}^{1/4} 0\,dx + \int_{1/4}^{2/4} 1\,dx + \int_{2/4}^{3/4} 2\,dx + \int_{3/4}^{1} 3\,dx$$

$x \rightarrow$ 0 $\quad$ 1/4 $\quad$ 2/4 $\quad$ 3/4 $\quad$ 1

$4x \rightarrow$ 0 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

0 $\quad$ 1 $\quad$ 2 $\quad$ 3

Ex:- $\int_{-3}^{-1} |2x+5|\,dx$

Ex:- $\int_{-3}^{+3} [4x-2]$

** 4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} = \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}$$

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} = \int_0^{\pi/2} 1\,dx$$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\boxed{I = \pi/4}$$

4. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Ex:- $I = \int_1^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}\,.dx = \int_1^5 \frac{\sqrt{6-x}}{\sqrt{6-x}+\sqrt{6-(6-x)}}\,dx$

$$I = \int_1^5 \frac{\sqrt{6-x}}{\sqrt{6-x}+\sqrt{x}}\,dx$$

$$2I = \int_1^5 \frac{\sqrt{x}+\sqrt{6-x}}{\sqrt{x}+\sqrt{6-x}}\,.dx = [x]_1^5$$

$$2I = 5-1$$

$$I = \frac{5-1}{2} = \boxed{2}$$

TRICK :- 4.

$$\boxed{\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \frac{b-a}{2}}$$

Ex :-
$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} = \frac{\pi/2 - 0}{2} = \pi/4$$

Ex :-
$$\int_4^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{11-x}}\,dx = \frac{7-4}{2} = 3/2$$

Ex:- $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)$

$\tan(A-B)$

$= \dfrac{\tan A - \tan B}{1+\tan A \cdot \tan B}$

Ex:- $\int \dfrac{3}{9+\sin^2\theta}\, d\theta$

Ex:- $\int_{0.25}^{1.25} (x-[x])\, dx$

Thank you

GW Soldiers !