# ALL BRANCHES

# ENGINEERING MATHEMATICS

Lecture No.-8

**Linear Algebra**

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# Topics to be Covered

- Eigen Values
- Eigen vectors
- Properties of Eigen Values
- Properties of Eigen Vectors
- Eigen Values of Special Matrices

# Eigen Value Problem

Consider a homogenous system with square matrix $A_{n\times n}$

$\boxed{AX = \lambda X}$ → Eigen value problem

$\boxed{AX - \lambda I X = 0}$ → Homogenous system with infinte solution.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$(A - \lambda I)X = 0$$

(Taking determinant both sides)

$$\boxed{|A - \lambda I| = 0}$$

→ Characterstic equation of A.

$\lambda$ → Eigen values / Characterstic roots

$X$ → Eigen vectors

# Eigen Value Problem

$AX = 0$

$\rho(A) = n$
$\Rightarrow$ Consistent unique/Trivial soln.
$\Rightarrow |A| \neq 0$

$\rho(A) < n$
$\Rightarrow$ Consistent infinte/non-trivial soln.
$\Rightarrow |A| = 0$

$(A - \lambda I)X = 0$

$\rho(A - \lambda I) = n$
$\Rightarrow |A - \lambda I| \neq 0$
$\Rightarrow$ Consistent unique trivial soln.

$\rho(A - \lambda I) < n$
$\Rightarrow |A - \lambda I| = 0$
$\Rightarrow$ Consistent infinite/non-trivial soln.

# Eigen Value Problem

1. $$(A-\lambda I)X = 0 \quad \text{-----} \quad (1)$$

$\Rightarrow$ Non zero solution $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}$ is called eigen vector/characterstic vector

2. If $X \to$ non-zero solution of (1) then $KX$ is also a solution of (1), where $K$ is any scalar

3. Non-zero solution $\Rightarrow$ Non-trivial soln. $\Rightarrow$ Infinite soln. $\Rightarrow$ L.D. vectors i.e. all are equivalent statements

4. $(n-r)$ gives no. of LI solution/LI eigen vector. of (1).

5. If 2 eigen values are same and there is only 1 LI soln. then there will be same eigen vector for both the eigen values.

# Eigen Value Problem

6. If 2 eigen values are same but there is 2 L I solution then there will be different eigen vector for each eigen values.

7. Roots of characterstic eqn. are Known as eigen values/character-stic values/eigen roots/characterstic roots/Latent roots.

8. Set consisting eigen vectors corresponding to different eigen values is L.I.

9. The set of eigen values of matrix A is called spectrum of matrix A and the largest eigen value is called spectral radius of A.

# [ Eigen values & vectors ]

Ex:- Find the eigen values/characterstic roots of A :-

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$$|(A - \lambda I)| = 0$$

$$|A - \lambda I| = \left| \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(2-\lambda) - (-8) = 0$$

$$\lambda^2 - 10\lambda + 16 + 8 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 6)(\lambda - 4) = 0$$

$\lambda = 4, 6 \longrightarrow$ Eigen values

- Spectrum of A $= (4, 6)$
- Spectral radius of A. $= 6$

# Eigen values & vectors

Ex :- (1) Find the eigen values & vectors of $A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$

$$|A - \lambda I| = 0 \qquad \left| \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 10 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\boxed{\lambda = 1, 6} \rightarrow \text{Eigen values}$$

I :- Find eigen vector corresponding to $\lambda = 1$.

# Eigen values & vectors

$$(A-\lambda I)\,X=0$$

$$\begin{bmatrix}2-\lambda & -1\\ -4 & 5-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Put $\lambda=1$

$$\begin{bmatrix}1 & -1\\ -4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Reduced to

$$\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$\Rightarrow \rho(A-\lambda I)=1<n(2)$

$\Rightarrow$ Consistent infinte soln.

$\Rightarrow$ No. of free variables $=n-r$
$=2-1=1\ (K)$

$$x_1-x_2=0$$

Let $x_1=K$
then $x_2=K$

$$\therefore\ X_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}K\\ K\end{bmatrix}=K\underbrace{\begin{bmatrix}1\\ 1\end{bmatrix}}_{1\ \text{LI soln.}}$$

$K\in R$

# Eigen values & vectors

II :- Find eigen vector corresponding to $\lambda = 6$

$$(A - \lambda I) X = 0$$

Put $\lambda = 6$

$$\begin{bmatrix} -4 & -1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Reduced to

$$\begin{bmatrix} -4 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rho(A - \lambda I) = 1 < n(2)$

$\Rightarrow$ No. of free variables

$= n - r$

$= 2 - 1 =$

$$-4x_1 - x_2 = 0$$

Let $x_1 = K$

then $x_2 = -4K$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ -4K \end{bmatrix}$$

$K \in R$

$$= \underbrace{K \begin{bmatrix} 1 \\ -4 \end{bmatrix}}_{1 \text{ L I soln.}}$$

$$A \begin{cases} \lambda_1 = 1 \longrightarrow X_1 = K \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 = 6 \longrightarrow X_2 = K \begin{bmatrix} 1 \\ -4 \end{bmatrix} \end{cases}$$

# Eigen values & vectors

$$S = (X_1, X_2) = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \qquad \rho(S) = 2$$

$\therefore$ $X_1$ and $X_2$ are L.I.

Ex:- (2) Find eigen values & vectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda = 1, 1, 1$$

(A.M. of '$\lambda = 1$' $= 3$)

$$(A - \lambda I)X = 0$$

Put $\lambda = 1$ $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Consistent infinite

$$\rho(A - \lambda I) = 2 < n(3)$$

$\Rightarrow$ No. of free variables $= n - r = 3 - 2 = 1$

# Eigen values & vectors

$$x_2 + x_3 = 0$$
$$x_3 = 0$$

Let $x_1 = K$ and $x_2 = x_3 = 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = \underbrace{K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\substack{1 \text{ LI soln} \\ 1 \text{ LI Eigen vector.}}} \quad K \in R$$

$(\therefore$ GM of '$\lambda = 1$' $= 1$ $)$

(3) Find eigen vectors / values of :-

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(5-\lambda)(5-\lambda) = 0$$

$$\lambda = 5, 5, 5$$

(A.M. of '$\lambda = 5$' $= 3$)

# Eigen values & vectors

$$(A - \lambda I)X = 0$$

Put $\lambda = 5$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A - \lambda I) = 0 < n(3)$

$\Rightarrow$ No. of free variables $= n - r$

$= 3 - 0 = 3$

No eqn. from this problem.

Let $x_1 = K_1$, $x_2 = K_2$, $x_3 = K_3$

$$X = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} = K_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + K_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

No. of LI eigen vectors $= 3$

$(\because$ GM of '$\lambda = 5$' $= 3)$

# Eigen values & vectors

(4) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$\lambda = -3, -3, 5$

I :- $\lambda = -3$ (AM = 2)

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{Put } \lambda = -3)$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\{-\lambda + \lambda^2 - 12\} - 2(-2\lambda - 6) - 3(-4 - (-1+\lambda)) = 0$$

$$\cancel{2\lambda} - \cancel{2\lambda^2} + \cancel{24} + \cancel{\lambda^2} - \lambda^3 + \cancel{12\lambda} + \cancel{4\lambda} + \cancel{12} + \cancel{12} - \cancel{3} + \cancel{3\lambda} = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$(\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\lambda = -3, -3, 5$$

$\rho(A-\lambda I) = 1 < n(3)$

$\Rightarrow$ No. of free variables $= n - r = 3 - 1 = 2$

$x_1 + 2x_2 - 3x_3 = 0$

Let $x_2 = K_1$, $x_3 = K_2$

then $x_1 = -2K_1 + 3K_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2K_1 + 3K_2 \\ K_1 \\ K_2 \end{bmatrix} = \underbrace{K_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}_{\text{2 LI soln/eigen vector}}$$

(AM of '$-3$' $= 2$) (GM of '$-3$' $= 2$)

II :- $\lambda = 5$ (AM $= 1$)

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

Let $x_3 = K$ then $x = K$, $x_2 = -2K$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ -2K \\ K \end{bmatrix} = \underbrace{K \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\text{1 LI soln/eigen vector}}$$

# Properties of Eigen Values

- Algebraic multiplicity (A.M.) $\longrightarrow$ No. of times a particular eigen value repeats itself.
- Geometric multiplicity (G.M.) $\longrightarrow$ No. of distinct L.I. eigen vectors for a particular eigen value.

$\lambda = 5, 5, 5$ $(AM = 3)$

- $\longrightarrow n - r = 1 \; (K)$ $\quad GM = 1$
- $\longrightarrow n - r = 2 \; (K_1, K_2)$ $\quad GM = 2$
- $\longrightarrow n - r = 3 \; (K_1, K_2, K_3)$ $\quad GM = 3$

$\lambda = -3, -3 \; (AM = 2)$

- $\longrightarrow n - r = 1 \; (K)$ $\quad GM = 1$
- $\longrightarrow n - r = 2 \; (K_1, K_2)$ $\quad GM = 2$

$5 \; (AM = 1) \longrightarrow n - r = 1 \; (K)$ $\quad GM = 1$

$$\boxed{GM \leq AM}$$

Thank you

GW

Soldiers !