Convolution & Laplace Transform

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Convolution

- Continuous Time

$$y(t) = h(t) * u(t) = \int_{\tau=0}^{\infty} h(t-\tau)u(\tau)d\tau$$

- Discrete Time

$$\int_{\tau=0}^{\infty} h(t-\tau)u(\tau)d\tau \approx \lim_{a\to 0}\sum_{k=0}^{\infty} h_a[n-ka]u(ka)a\,, \qquad \tau = ka$$

$$y[n] = h[n] * u[n] = \sum_{k=0}^{\infty} h[n-k]u[k]$$

Convolution

- 길이 n의 신호 u가 다음과 같이 정의

$$u[n] = \sum_{k=0}^{n} u[k]\delta[n-k] = u[0]\delta[n] + \cdots + u[n]\delta[0] = 0 + \cdots + u[n]$$

- LTI 시스템에서

$$impulse\ input\ \rightarrow impulse\ response$$

$$\delta[n] \rightarrow h[n]$$

$$\delta[n-k] \rightarrow h[n-k]\text{(time invariancy)}$$

$$u[k]\delta[n-k] \rightarrow u[k]h[n-k]\text{(homogeneity)}$$

$$\sum_{k=0}^{n} u[k]\delta[n-k] \rightarrow \sum_{k=0}^{n} u[k]h[n-k]\ \text{(additivity)}$$

$$u[n] \rightarrow h[n]u[0] + h[n-1]u[1] + \cdots + h[1]u[n-1] + h[0]u[n]$$

$$y[n] = h[n]u[0] + h[n-1]u[1] + \cdots + h[1]u[n-1] + h[0]u[n]$$

Convolution

- 따라서, 다음과 같이 convolution이 성립함을 알 수 있음
  - $y[0] = h[0]u[0]$
  - $y[1] = h[0]u[1] + h[1]u[0]$
  - $y[2] = h[0]u[2] + h[1]u[1] + h[2]u[0]$
  - $y[n] = h[0]u[n] + h[1]u[n-1] + \cdots + h[n-1]u[1] + h[n]u[0]$
  - 현재출력 = (첫 임펄스응답 * 현재입력) + (1지난 임펄스응답 * 현재-1입력) + …
    → convolution
- Impulse response $h[n]$을 알고 있다면, 입력 $u[n]$과 convolution하여 모든 출력 $y[n]$을 계산 가능 → 따라서 impulse response가 system model임

Laplace Transform

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

$$\mathcal{L}[y(t)] = \mathcal{L}[\,h(t) * u(t)\,] \quad \rightarrow \quad Y(s) = H(s)U(s)$$

$$\frac{output}{input} = \frac{Y(s)}{U(s)} = H(s), \qquad transfer\ function$$

- Transfer function을 알고 입력을 알고 있다면 출력을 계산할 수 있음
  - $y(t) = \mathcal{L}^{-1}[H(s)U(s)]$
  - Transfer function → impulse response의 라플라스 변환, 시스템 모델

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Example 1

- 다음 두 신호를 plot (T는 임의로 설정)

$$u(t = nT) = u[n] = \begin{cases} 1 & t < 2 \\ 0 & t \geq 2 \end{cases}$$

$$r(t = nT) = r[n] = \begin{cases} t & t \leq 1 \\ 0 & t > 1 \end{cases}$$

- 두 신호를 이용하여 다음 convolution을 각각 계산하고 plot (함수 conv 사용)
  1) $y_1 = u * u$
  2) $y_2 = u * r$
  3) $y_3 = r * u$
  4) $y_4 = r * r$

Example 2

- 다음 1차 선형미분방정식의 impulse respons와 transfer function 계산

$$2\frac{dy(t)}{dt} + y(t) = u(t)$$

- 다음 세 가지 방식으로 step response를 계산하여 비교
  1) Impulse response h(t)를 구한 후 step input과 convolution (함수 conv 사용)
  2) Transfer function H(s)를 구한 후 $\frac{1}{s}$를 곱한 뒤 $\mathcal{L}^{-1}$을 통해 계산 (함수 ilaplace, eval 사용)
  3) 함수 step을 사용하여 계산