On the Distributivity of Fuzzy Implications Over Nilpotent or Strict Triangular Conorms

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On the Distributivity of Fuzzy Implications Over Nilpotent or Strict Triangular Conorms

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***Abstract*—Recently, many works have appeared in this very journal dealing with the distributivity of fuzzy implications over t-norms and t-conorms. These equations have a very important role to play in efficient inferencing in approximate reasoning, especially fuzzy control systems. Of all the four equations considered, the equation $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$, when S_1, S_2 are both t-conorms and I is an R-implication obtained from a strict t-norm, was not solved. In this paper, we characterize functions I that satisfy the previous functional equation when S_1, S_2 are either both strict or nilpotent t-conorms. Using the obtained characterizations, we show that the previous equation does not hold when S_1, S_2 are either both strict or nilpotent t-conorms, and I is a continuous fuzzy implication. Moreover, the previous equation does not hold when I is an R-implication obtained from a strict t-norm, and S_1, S_2 are both strict t-conorms, while it holds for an R-implication I obtained from a strict t-norm T if and only if the t-conorms $S_1 = S_2$ are Φ-conjugate to the Łukasiewicz t-conorm for some increasing bijection φ of the unit interval, which is also a multiplicative generator of T.**

***Index Terms*—Combs methods, functional equations, fuzzy implication, R-implication, t-conorm, t-norm.**

I. Introduction

DISTRIBUTIVITY of fuzzy implication operations over different fuzzy logic connectives has been studied in the recent past by many authors. This interest, perhaps, was kick started by Combs and Andrews in [13], wherein they exploit the following classical tautology:

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$$

in their inference mechanism toward reduction in the complexity of fuzzy "If–Then" rules. They refer to the left-hand side of this equivalence as an intersection rule configuration (IRC) and to its right-hand side as a union rule configuration (URC). Subsequently, there were many discussions [14]–[16], [24], most of them pointing out the need for a theoretical investigation required for employing such equations, as concluded by Dick and Kandel [16], "Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications" or by Mendel and Liang [24], "We think that what this all means is that we have to look past the mathematics of IRC⇔URC and inquire whether what we are doing when we replace IRC by URC makes sense." It was Trillas and Alsina [32] who were the first to investigate the generalized version of the previous law, viz.,

$$I(T(x, y), z) = S(I(x, z), I(y, z)), \quad x, y, z \in [0, 1] \qquad (1)$$

where T, S are a t-norm and a t-conorm, respectively, generalizing the $\wedge, \vee$ operators, respectively, and I is a fuzzy implication. From their investigations of (1) for the three main families of fuzzy implications, viz., S-implications, R-implications, and QL-implications, it was shown that in the case of R-implications obtained from left-continuous t-norms and S-implications, (1) holds if and only if $T = \min$ and $S = \max$. Also along the previous lines, Balasubramaniam and Rao [10] considered the following dual equations of (1):

$$I(S(x, y), z) = T(I(x, z), I(y, z)) \qquad (2)$$

$$I(x, T_1(y, z)) = T_2(I(x, y), I(x, z)) \qquad (3)$$

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z)) \qquad (4)$$

where again, T, T_1, T_2 and S, S_1, S_2 are t-norms and t-conorms, respectively, and I is a fuzzy implication. Similarly, it was shown that when I is either an R-implication obtained from a left-continuous t-norm or an S-implication, in almost all the cases, the distributivity holds only when $T = T_1 = T_2 = \min$ and $S = S_1 = S_2 = \max$, while (4) for the case when I is an R-implication obtained from a strict t-norm was left unsolved (cf. [10, Th. 4]). This forms the main motivation of this paper.

Meanwhile, Baczyński in [2] and [3] considered the functional equation (3), both independently and along with other equations, and characterized fuzzy implications I in the case when $T_1 = T_2$ is a strict t-norm. Some partial studies regarding distributivity of fuzzy implications over maximum and minimum were presented by Bustince *et al.* in [12]. It may be worth recalling that (3) with $T_1 = T_2$ is one of the characterizing properties of A-implications proposed by Türksen *et al.* in [33].

As we mentioned earlier, the previous equations (1)–(4) have an important role to play in inference invariant rule reduction in fuzzy inference systems (see also [8], [9], and [30]). The very fact that about half-a-dozen works have appeared in this very journal dealing with these distributive equations is a pointer to the importance of these equations. That more recent works dealing with distributivity of fuzzy implications over uninorms (see [27] and [28]) have appeared is an indication of the sustained interest in the previous equations.

This paper differs from the previous works that have appeared in this journal on these equations, in that, we attempt to solve the problem in a more general setting, by characterizing functions I that satisfy the functional equation (4) when S_1, S_2 are either both strict or nilpotent t-conorms. Then, using these

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characterizations, we also investigate the conditions under which (4) holds when I is an R-implication obtained from a strict t-norm.

The paper is organized as follows. In Section II, we give some results concerning basic fuzzy logic connectives and functional equations that will be employed extensively in the sequel. In Section III, we study (4), when I is a binary operation on $[0,1]$, while S_1, S_2 are both strict t-conorms. Based on the obtained characterization, we show that there exists no continuous solution I for (4) that is a fuzzy implication. Subsequently, we obtain the characterization of noncontinuous fuzzy implications I that are solutions for (4). In Section IV, we mimic the approach taken in Section III, except that in this case, S_1, S_2 are both nilpotent t-conorms. Again in this case, the results parallel those of Section III. In Section V, we study (4) in the case when I is an R-implication obtained from a strict t-norm T. We show that (4) does not hold when S_1, S_2 are strict t-conorms, while using one of the characterization obtained in the previous section, we show that it holds if and only if the t-conorms $S_1 = S_2$ are Φ-conjugate to the Łukasiewicz t-conorm for some increasing bijection φ, which is a multiplicative generator of the strict t-norm T.

II. Preliminaries

A. Basic Fuzzy Logic Connectives

First, we recall some basic notations and results that will be useful in the sequel. We start with the notation of conjugacy (see [21, p. 156]). By Φ, we denote the family of all increasing bijections $\varphi\colon [0,1] \to [0,1]$. We say that functions $f, g\colon [0,1]^n \to [0,1]$ are Φ-conjugate, if there exists a $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}\left(f(\varphi(x_1), \ldots, \varphi(x_n))\right)$$

for all $x_1, \ldots, x_n \in [0,1]$. If F is an associative binary operation on $[a,b]$ with neutral element e, then the power notation $x_F^{[n]}$, where $n \in \mathbb{N}_0$, is defined by

$$x_F^{[n]} := \begin{cases} e, & \text{if } n = 0 \\ x, & \text{if } n = 1 \\ F\big(x, x_F^{[n-1]}\big), & \text{if } n > 1. \end{cases}$$

Definition 1 (see [20], [29]):

1) An associative, commutative, and increasing operation $T\colon [0,1]^2 \to [0,1]$ is called a t-norm if it has the neutral element 1.
2) An associative, commutative, and increasing operation $S\colon [0,1]^2 \to [0,1]$ is called a t-conorm if it has the neutral element 0.

Definition 2 [20, Definitions 2.9 and 2.13]: A t-norm T (t-conorm S, respectively) is said to be

1) Archimedean, if for every $x, y \in (0,1)$, there is an $n \in \mathbb{N}$ such that $x_T^{[n]} < y$ ($x_S^{[n]} > y$, respectively);
2) strict, if T (S, respectively) is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ ($S(x,y) < S(x,z)$, respectively) whenever $x > 0$ ($x < 1$, respectively) and $y < z$;
3) nilpotent, if T (S, respectively) is continuous and if for each $x \in (0,1)$, there exists $n \in \mathbb{N}$ such that $x_T^{[n]} = 0$ ($x_S^{[n]} = 1$, respectively).

Remark 1:

1) For a continuous t-conorm S, the Archimedean property is given by the simpler condition (cf. [19, Prop. 5.1.2])

$$S(x,x) > x, \qquad x \in (0,1).$$

2) If a t-conorm S is continuous and Archimedean, then S is nilpotent if and only if there exists some nilpotent element of S, which is equivalent to the existence of some $x, y \in (0,1)$ such that $S(x,y) = 1$ (see [20, Th. 2.18]).
3) If a t-conorm S is strict or nilpotent, then it is Archimedean. Conversely, every continuous and Archimedean t-conorm is either strict or nilpotent (cf. [20, Th. 2.18]).

We shall use the following characterizations of continuous Archimedean t-conorms.

Theorem 1 ([23], cf. [20, Corollary 5.5]): For a function $S\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) S is a continuous Archimedean t-conorm.
2) S has a continuous additive generator, i.e., there exists a continuous, strictly increasing function $s\colon [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$S(x,y) = s^{(-1)}(s(x) + s(y)), \quad x, y \in [0,1] \qquad (5)$$

where $s^{(-1)}$ is the pseudoinverse of s given by

$$s^{(-1)}(x) = \begin{cases} s^{-1}(x), & \text{if } x \in [0, s(1)] \\ 1, & \text{if } x \in (s(1), \infty]. \end{cases}$$

Remark 2:

1) A representation of a t-conorm S as earlier can be written without explicitly using of the pseudo-inverse in the following way:

$$S(x,y) = s^{-1}(\min(s(x) + s(y), s(1))) \qquad (6)$$

for $x, y \in [0,1]$.
2) S is a strict t-conorm if and only if each continuous additive generator s of S satisfies $s(1) = \infty$.
3) S is a nilpotent t-conorm if and only if each continuous additive generator s of S satisfies $s(1) < \infty$.

Next, two characterizations of strict t-norms and nilpotent t-conorms are well known in literature and can be easily obtained from the general characterizations of continuous Archimedean t-norms and t-conorms (see [20, Sec. 5.2]).

Theorem 2: For a function $T\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) T is a strict t-norm.
2) T is Φ-conjugate with the product t-norm, i.e., there exists $\varphi \in \Phi$, which is uniquely determined up to a positive constant exponent, such that

$$T(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y)), \qquad x, y \in [0,1]. \qquad (7)$$

Theorem 3: For a function $S\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) S is a nilpotent t-conorm.
2) S is Φ-conjugate with the Łukasiewicz t-conorm, i.e., there exists $\varphi \in \Phi$, which is uniquely determined, such that for all $x, y \in [0,1]$, we have

$$S(x,y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1)). \tag{8}$$

In the literature, we can find several diverse definitions of fuzzy implications [12], [18]. In this paper, we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens (see [18, Def. 1.15]).

Definition 3: A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable.} \tag{I1}$$

$$I \text{ is increasing in the second variable.} \tag{I2}$$

$$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0. \tag{I3}$$

From the previous definition, we can deduce that, for each fuzzy implication, $I(0,x) = I(x,1) = 1$ for $x \in [0,1]$. Moreover, I also satisfies the normality condition

$$I(0,1) = 1 \tag{NC}$$

and consequently, every fuzzy implication restricted to the set $\{0,1\}^2$ coincides with the classical implication.

There are many important methods for generating fuzzy implications (see [17]–[19]). In this paper, we need only one family—R-implications.

Definition 4: A function $I\colon [0,1]^2 \to [0,1]$ is called an R-implication if there exist a t-norm T such that

$$I(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}, \quad x,y \in [0,1]. \tag{9}$$

If I is generated from a t-norm T by (9), then we will sometimes write I_T.

It is very important to note that the name "R-implication" is a short version of "residual implication," and I_T is also called as "the residuum of T." This class of implications is related to a residuation concept from the intuitionistic logic. In fact, it has been shown that in this context, this definition is proper only for left-continuous t-norms.

Proposition 1 (cf. [19, Proposition 5.4.2 and Corollary 5.4.1]): For a t-norm T, the following statements are equivalent.

1) T is left-continuous.
2) T and I_T form an adjoint pair, i.e., they satisfy

$$T(x,t) \le y \iff I_T(x,y) \ge t, \qquad x,y,t \in [0,1].$$

3) The supremum in (9) is the maximum, i.e.,

$$I_T(x,y) = \max\{t \in [0,1] \mid T(x,t) \le y\}$$

where the right side exists for all $x, y \in [0,1]$.

The following characterization of R-implications generated from left-continuous t-norms is also well known in the literature.

Theorem 4 ([25], cf. [18, Th. 1.14]): For a function $I\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) I is an R-implication generated from a left-continuous t-norm.
2) I is right-continuous with respect to the second variable, and it satisfies (I2), the exchange principle

$$I(x, I(y,z)) = I(y, I(x,z)), \qquad x,y,z \in [0,1] \tag{EP}$$

and the ordering property

$$x \le y \iff I(x,y) = 1, \qquad x,y \in [0,1]. \tag{OP}$$

It should be noted that each R-implication I satisfies the left neutrality property

$$I(1,y) = y, \qquad y \in [0,1]. \tag{NP}$$

Further, if T is a strict t-norm, then we have the following representation from [26, Th. 6.1.2] (see also [4, Th. 19]).

Theorem 5: If I is an R-implication generated from a strict t-norm T, then I is Φ-conjugate to the Goguen implication, i.e., there exists $\varphi \in \Phi$, which is uniquely determined up to a positive constant exponent, such that

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \varphi^{-1}\left(\dfrac{\varphi(y)}{\varphi(x)}\right), & \text{otherwise} \end{cases} \tag{10}$$

for all $x, y \in [0,1]$.

We would like to underline that the increasing bijection φ earlier can be seen as a multiplicative generator of T in (7). The proof that φ is uniquely determined up to a positive constant exponent has been presented by Baczyński and Drewniak (see [6, Th. 6]).

B. Some Results Pertaining to Functional Equations

Here, we present some results related to the additive and multiplicative Cauchy functional equations:

$$f(x+y) = f(x) + f(y) \tag{11}$$

$$f(xy) = f(x)f(y) \tag{12}$$

which are crucial in the proofs of the main theorems.

Theorem 6 ([1], cf. [22, Th. 5.2.1]): For a continuous function $f\colon \mathbb{R} \to \mathbb{R}$, the following statements are equivalent.

1) f satisfies the additive Cauchy functional equation (11) for all $x, y \in \mathbb{R}$.
2) There exists a unique constant $c \in \mathbb{R}$ such that

$$f(x) = cx \tag{13}$$

for all $x \in \mathbb{R}$.

Theorem 7 ([22, Th. 13.5.3]): Let $A \subset \mathbb{R}$ be an interval such that $0 \in clA$, where clA denotes the closure of the set A and let $B = A + A = \{a_1 + a_2 \mid a_1 \in A,\ a_2 \in A\}$. If a function $f\colon B \to \mathbb{R}$ satisfies the additive Cauchy functional equation (11) for all $x, y \in A$, then f can be uniquely extended onto $\mathbb{R}$ to an additive function g such that $g(x) = f(x)$ for all $x \in B$.

By virtue of the previous theorems, we get the following new results.

Proposition 2: For a function $f\colon [0,\infty] \to [0,\infty]$, the following statements are equivalent.

1) f satisfies the additive Cauchy functional equation (11) for all $x, y \in [0,\infty]$.

2) Either $f = \infty$, or $f = 0$, or

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{if } x > 0, \end{cases} \qquad x \in [0, \infty] \tag{14}$$

or

$$f(x) = \begin{cases} 0, & \text{if } x < \infty, \\ \infty, & \text{if } x = \infty, \end{cases} \qquad x \in [0, \infty] \tag{15}$$

or there exists a unique constant $c \in (0, \infty)$ such that f admits the representation (13) for all $x \in [0, \infty]$.

Proof: 2) $\Longrightarrow$ 1) It is a direct calculation that all the previous functions satisfy (11).

1) $\Longrightarrow$ 2) Let $f: [0, \infty] \to [0, \infty]$ satisfy (11). Setting $x = y = 0$ in (11), we get $f(0) = f(0) + f(0)$, so $f(0) = 0$, or $f(0) = \infty$. If $f(0) = \infty$, then for any $x \in [0, \infty]$, we get

$$f(x) = f(x + 0) = f(x) + f(0) = f(x) + \infty = \infty$$

and thus, we obtain the first possible solution $f = \infty$.

Now, setting $x = y = \infty$ in (11), we get $f(\infty) = f(\infty) + f(\infty)$. Therefore, $f(\infty) = 0$ or $f(\infty) = \infty$. If $f(\infty) = 0$, then for any $x \in [0, \infty]$, we have

$$0 = f(\infty) = f(x + \infty) = f(x) + f(\infty) = f(x) + 0 = f(x)$$

and thus, we obtain the second possible solution $f = 0$.

Let us assume that $f \neq \infty$ and $f \neq 0$. Considering the alternate cases earlier, we get $f(0) = 0$ and $f(\infty) = \infty$. Define a set

$$Z = \{x \in (0, \infty) \mid f(x) = \infty\}.$$

If $Z \neq \emptyset$, then $Z = (0, \infty)$. Indeed, let us fix some real $x_0 \in Z$ and take any $x \in (0, \infty)$. If $x \geq x_0$, then we get

$$\begin{aligned} f(x) &= f((x - x_0) + x_0) = f(x - x_0) + f(x_0) \\ &= f(x - x_0) + \infty = \infty. \end{aligned}$$

If $x < x_0$, then there exists a natural $k \geq 2$ such that $kx > x_0$. From the previous point, we get that $f(kx) = \infty$. Further, by the induction, we obtain

$$\begin{aligned} \infty &= f(kx) = f(x + (k-1)x) = f(x) + f((k-1)x) \\ &= \overbrace{f(x) + \cdots + f(x)}^{k\text{-times}} = kf(x) \end{aligned}$$

and thus, $f(x) = \infty$. This implies, with the assumptions $f(0) = 0$ and $f(\infty) = \infty$, that if $Z \neq \emptyset$, then we obtain the third possible solution (14).

On the other hand, if $Z = \emptyset$, then $f(x) \in [0, \infty)$ for $x \in [0, \infty)$. By Theorem 7, for $A = B = [0, \infty)$, f can be uniquely extended to an additive function $g: \mathbb{R} \to \mathbb{R}$, such that $g(x) = f(x)$ for all $x \in [0, \infty)$. Consequently, g is bounded below on the set $[0, \infty)$. Further, $-g$ is an additive function bounded above on $[0, \infty)$. Since any additive function is convex, by virtue of theorem of Bernstein–Doetsch (see [11] or [22, Coro. 6.4.1]), $-g$ is continuous, and hence, g is continuous. Now, by Theorem 6, there exists a unique constant $c \in \mathbb{R}$ such that $g(x) = cx$ for every $x \in \mathbb{R}$, i.e., $f(x) = cx$ for every $x \in [0, \infty)$. Since the domain and the range of f are nonnegative, we see that $c \geq 0$. If $c = 0$, then we get the fourth possible solution (15), because of our assumption $f(\infty) = \infty$. If $c > 0$, then we obtain the last possible solution (13), since $c \cdot \infty = \infty = f(\infty)$. ■

Corollary 1: For a continuous function $f: [0, \infty] \to [0, \infty]$, the following statements are equivalent.

1) f satisfies the additive Cauchy functional equation (11) for all $x, y \in [0, \infty]$.
2) Either $f = \infty$, or $f = 0$, or there exists a unique constant $c \in (0, \infty)$ such that f admits the representation (13) for all $x \in [0, \infty]$.

Theorem 8 ([22, Th. 13.6.2]): Fix a real $a > 0$. Let $A = [0, a]$, and let $H \subset \mathbb{R}^2$ be the set

$$H = \{(x, y) \in \mathbb{R}^2 \mid x \in A,\ y \in A, \text{ and } x + y \in A\}. \tag{16}$$

If $f: A \to \mathbb{R}$ is a function satisfying (11) on H, then there exists a unique additive function $g: \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in A$. Moreover, the closed interval $[0, a]$ may be replaced by any one of these intervals $(0, a)$, $[0, a)$, and $(0, a]$.

Proposition 3: Fix real $a, b > 0$. For a function $f: [0, a] \to [0, b]$, the following statements are equivalent.

1) f satisfies the functional equation

$$f(\min(x + y, a)) = \min(f(x) + f(y), b) \tag{17}$$

for all $x, y \in [0, a]$.
2) Either $f = b$, or $f = 0$, or

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ b, & \text{if } x > 0, \end{cases} \qquad x \in [0, a] \tag{18}$$

or there exists a unique constant $c \in [b/a, \infty)$ such that

$$f(x) = \min(cx, b), \qquad x \in [0, a]. \tag{19}$$

Proof: 2) $\Longrightarrow$ 1) It is obvious that $f = 0$ and $f = b$ satisfy (17). Let f have the form (18). If $x = y = 0$, then the left side of (17) is equal to $f(\min(0 + 0, a)) = f(0) = 0$ and the right side of (17) is $\min(f(0) + f(0), b) = \min(0 + 0, b) = 0$. If $x \neq 0$ or $y \neq 0$, then the both sides of (17) are equal to b.

Finally, if f has the form (19) with some $c \in [b/a, \infty)$, then the left side of (17) is equal to

$$\begin{aligned} f(\min(x + y, a)) &= \min(c \cdot \min(x + y, a), b) \\ &= \min(c(x + y), ca, b) \\ &= \min(c(x + y), b), \qquad x, y \in [0, a] \end{aligned}$$

since $ca \geq b$. Now, the right side of (17) is equal to

$$\begin{aligned} \min(f(x) + f(y), b) &= \min(\min(cx, b) + \min(cy, b), b) \\ &= \min(cx + cy, cx + b, cy + b, b + b, b) \\ &= \min(c(x + y), b), \qquad x, y \in [0, a] \end{aligned}$$

which ends the proof in this direction.

1) $\Longrightarrow$ 2) Let f satisfy (17). Setting $x = y = 0$ in (17), we get

$$f(0) = \min(f(0) + f(0), b). \tag{20}$$

If $f(0) = b$, then for any $x \in [0, a]$, we have

$$\begin{aligned} f(x) &= f(\min(x + 0, a)) = \min(f(x) + f(0), b) \\ &= \min(f(x) + b, b) = b \end{aligned}$$

and thus, we obtain the first possible solution $f = b$.

Now, let us substitute $x = y = a$ in (17). We have

$$f(a) = \min(f(a) + f(a), b). \tag{21}$$

If $f(a) = f(a) + f(a)$, then $f(a) = 0$ and for every $x \in [0, a]$, we get

$$\begin{aligned} 0 = f(a) &= f(\min(a + x, a)) = \min(f(a) + f(x), b) \\ &= \min(f(x), b) \end{aligned}$$

and therefore, $f(x) = 0$, and we obtain the second possible solution $f = 0$.

Let us assume that $f \neq 0$ and $f \neq b$. Considering the alternate cases in (20) and (21), we get that $f(0) = 0$ and $f(a) = b$. Let us define

$$x_0 = \inf\{x \in [0, a] \mid f(x) = b\}.$$

First, we will show that if $x_0 < a$, then

$$f(x) = b, \qquad x \in (x_0, a]. \tag{22}$$

Indeed, let us take any $x \in (x_0, a]$. From the definition of the element x_0, there exists $x_1 \in (x_0, x]$ such that $f(x_1) = b$. Further

$$\begin{aligned} f(x) &= f(x_1 + (x - x_1)) = f(\min(x_1 + (x - x_1), a)) \\ &= \min(f(x_1) + f(x - x_1), b) = \min(b + f(x - x_1), b) \\ &= b. \end{aligned}$$

Thus, if $x_0 = 0$, then we get the third possible solution (18).

Let us assume, that $x_0 > 0$. Now we show, that for any $x, y \in [0, x_0)$ such that $x + y \in [0, x_0)$, the function f is additive, and therefore, it satisfies (11). Suppose that this does not hold, i.e., there exist $x_1, y_1 \in [0, x_0)$ such that $x_1 + y_1 \in [0, x_0)$ and $f(x_1 + y_1) \neq f(x_1) + f(y_1)$. Setting $x = x_1$ and $y = y_1$ in (17), we get

$$\begin{aligned} f(x_1 + y_1) &= f(\min(x_1 + y_1, a)) \\ &= \min(f(x_1) + f(y_1), b) = b. \end{aligned}$$

However, $x_1 + y_1 < x_0$, which is a contradiction to the definition of x_0. We proved that f satisfies the additive Cauchy functional equation (11) on the set H defined by (16) for $A = [0, x_0)$. By Theorem 8, the function f can be uniquely extended to an additive function $g: \mathbb{R} \to \mathbb{R}$, such that $g(x) = f(x)$ for all $x \in [0, x_0)$. Consequently, g is bounded on $[0, x_0)$, and by virtue of theorem of Bernstein–Doetsch (see [11] or [22, Th. 6.4.2]), g is continuous. Because of Theorem 6, there exists a unique constant $c \in \mathbb{R}$ such that $g(x) = cx$ for every $x \in \mathbb{R}$, i.e., $f(x) = cx$ for every $x \in [0, x_0)$. Since the domain and the range of f are nonnegative, we get that $c \geq 0$. Moreover

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^-} cx = cx_0 \leq b$$

and consequently, $c \in [0, b/x_0]$. If we assume that $c \in [0, b/x_0)$, then we get

$$\begin{aligned} f(x_0) &= f\left(\frac{x_0}{2} + \frac{x_0}{2}\right) = f\left(\min\left(\frac{x_0}{2} + \frac{x_0}{2}, a\right)\right) \\ &= \min\left(f\left(\frac{x_0}{2}\right) + f\left(\frac{x_0}{2}\right), b\right) \\ &= \min\left(c\frac{x_0}{2} + c\frac{x_0}{2}, b\right) = \min(cx_0, b) \\ &= cx_0 < b \end{aligned}$$

since $c < b/x_0$. Hence, if $x_0 = a$, then we get a contradiction to our assumption $f(a) = b$. If $x_0 \in (0, a)$, then there exists $x_1 \in (0, x_0)$ such that $cx_0 + cx_1 < b$. Setting $x = x_0$ and $y = x_1$ in (17), we get, by (22), that

$$\begin{aligned} b &= f(\min(x_0 + x_1, a)) = \min(f(x_0) + f(x_1), b) \\ &= \min(cx_0 + cx_1, b) = cx_0 + cx_1 \end{aligned}$$

which contradicts the previous assumption.

Consequently, we showed that if $f \neq 0$ and $f \neq b$ and $x_0 \in (0, a]$, then there exists a unique $c = b/x_0 \geq b/a$ such that

$$f(x) = \begin{cases} cx, & \text{if } x \leq x_0, \\ b, & \text{if } x > x_0, \end{cases} \qquad x \in [0, a].$$

Easy calculations show that for $x \in [0, a]$, we have

$$\begin{aligned} f(x) &= \begin{cases} cx, & \text{if } \frac{b}{x_0} x \leq b \\ b, & \text{if } \frac{b}{x_0} x > b \end{cases} = \begin{cases} cx, & \text{if } cx \leq b \\ b, & \text{if } cx > b \end{cases} \\ &= \min(cx, b) \end{aligned}$$

i.e., f has the last possible representation (19). ■

Corollary 2: Fix real $a, b > 0$. For a continuous function $f: [0, a] \to [0, b]$, the following statements are equivalent.

1) f satisfies the functional equation (17) for all $x, y \in [0, a]$.
2) Either $f = 0$, or $f = b$, or there exists a unique constant $c \in [b/a, \infty)$ such that f has the form (19).

Theorem 9 ([22, Th. 13.1.6]): Let D be one of the sets $(0, 1)$, $[0, 1)$, $[0, \infty)$, $(0, \infty)$, $\mathbb{R}$. For a continuous function $f: D \to \mathbb{R}$, the following statements are equivalent.

1) f satisfies the multiplicative Cauchy functional equation (12) for all $x, y \in D$.
2) Either $f = 0$, or $f = 1$, or f has one of the following forms:

$$f(x) = |x|^c, \qquad x \in D$$
$$f(x) = |x|^c \operatorname{sgn}(x), \qquad x \in D$$

with a certain $c \in \mathbb{R}$. If $0 \in D$, then $c > 0$.

Corollary 3: For an increasing bijection $h: [0, 1] \to [0, 1]$, the following statements are equivalent.

1) h satisfies the multiplicative Cauchy functional equation (12) for all $x, y \in [0, 1]$.
2) There exists a unique constant $c \in (0, \infty)$, such that $h(x) = x^c$ for all $x \in [0, 1]$.

Corollary 4: For an increasing bijection $h: [0, 1] \to [0, 1]$, the following statements are equivalent.

1) h satisfies the functional equation

$$h\left(\frac{y}{x}\right) = \frac{h(y)}{h(x)}, \qquad x, y \in [0,1], \qquad x > y. \tag{23}$$

2) There exists a unique constant $c \in (0,\infty)$, such that $h(x) = x^c$ for all $x \in [0,1]$.

Proof: 1) $\Longrightarrow$ 2) Let $x, y \in [0,1]$ and $x > y$ be fixed. Define $z = y/x$. We see that $z \in [0,1)$. Setting $y = zx$ in (23), we get the multiplicative Cauchy functional equation

$$h(zx) = h(z)h(x), \qquad x, z \in [0,1], \qquad x > 0, \qquad z < 1.$$

For $x = 0$ or $z = 1$, the function h also satisfies previous equation, since $h(0) = 0$ and $h(1) = 1$. By virtue of Corollary 3, we get the thesis.

2) $\Longrightarrow$ 1) This implication is obvious. ∎

III. On the Equation (4) When S_1, S_2 Are Strict T-Conorms

Our main goal in this section is to present the representations of some classes of fuzzy implications that satisfy (4) when S_1, S_2 are strict t-conorms. Within this context, we firstly describe the general solutions of (4) when S_1, S_2 are strict t-conorms. It should be noted that the general solutions of the distributive equation

$$F(x, G(y,z)) = G(F(x,z), F(y,z))$$

where F is continuous and G is assumed to be continuous, strictly increasing and associative were presented by Aczél (see [1, Th. 6, p. 319]). Our results can be seen as a generalization of the previous result without any assumptions on the function F and less assumptions on the function G.

Theorem 10: Let S_1, S_2 be strict t-conorms. For a function $I\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and for every fixed $x \in [0,1]$, the vertical section $I(x,\cdot)$ has one of the following forms:

$$I(x,y) = 0, \qquad y \in [0,1] \tag{24}$$

$$I(x,y) = 1, \qquad y \in [0,1] \tag{25}$$

$$I(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ 1, & \text{if } y > 0, \end{cases} \qquad y \in [0,1] \tag{26}$$

$$I(x,y) = \begin{cases} 0, & \text{if } y < 1, \\ 1, & \text{if } y = 1, \end{cases} \qquad y \in [0,1] \tag{27}$$

$$I(x,y) = s_2^{-1}\left(c_x \cdot s_1(y)\right), \qquad y \in [0,1] \tag{28}$$

with a certain $c_x \in (0,\infty)$ that is uniquely determined up to a positive multiplicative constant, depending on constants for s_1 and s_2.

Proof: 2) $\Longrightarrow$ 1) Let t-conorms S_1, S_2 have the representation (5) with some continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ such that $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$. By Theorem 1 and part 2) of Remark 2, the functions S_1, S_2 are strict t-conorms. Let us fix arbitrarily $x \in [0,1]$. We consider five cases.

If $I(x,y) = 0$ for all $y \in [0,1]$, then the left side of (4) is $I(x, S_1(y,z)) = 0$ and the right side of (4) is $S_2(I(x,y), I(x,z)) = S_2(0,0) = 0$ for all $y, z \in [0,1]$.

If $I(x,y) = 1$ for all $y \in [0,1]$, then the left side of (4) is $I(x, S_1(y,z)) = 1$, and the right side of (4) is $S_2(I(x,y), I(x,z)) = S_2(1,1) = 1$ for all $y, z \in [0,1]$.

Let $I(x,y)$ have the form (26) for all $y \in [0,1]$. Fix arbitrarily $y, z \in [0,1]$. If $y = 0$, then the left side of (4) is $I(x, S_1(0,z)) = I(x,z)$ and the right side of (4) is $S_2(I(x,0), I(x,z)) = S_2(0, I(x,z)) = I(x,z)$. Analogously, if $z = 0$, then both sides of (4) are equal to $I(x,y)$. If $y > 0$ and $z > 0$, then $S_1(y,z) > S_1(0,0) = 0$ since S_1 is strict. Now, the left side of (4) is $I(x, S_1(y,z)) = 1$, and the right side of (4) is $S_2(I(x,y), I(x,z)) = S_2(1,1) = 1$.

Let I have the form (27) for all $y \in [0,1]$. Fix arbitrarily $y, z \in [0,1]$. If $y = 1$, then the left side of (4) is $I(x, S_1(1,z)) = I(x,1) = 1$, and the right side of (4) is $S_2(I(x,1), I(x,z)) = S_2(1, I(x,z)) = 1$. Analogously, if $z = 1$, then both sides of (4) are equal to 1. If $y < 1$ and $z < 1$, then $S_1(y,z) < S_1(1,1) = 1$ since S_1 is strict. Now, the left side of (4) is $I(x, S_1(y,z)) = 0$, and the right side of (4) is $S_2(I(x,y), I(x,z)) = S_2(0,0) = 0$.

Let I have the form (28) for all $y \in [0,1]$. Fix arbitrarily $y, z \in [0,1]$. If $y, z \in [0,1)$, then we have

$$\begin{aligned} I(x, S_1(y,z)) &= I(x, s_1^{-1}(s_1(y) + s_1(z))) \\ &= s_2^{-1}\left(c_x(s_1(y) + s_1(z))\right) \\ S_2(I(x,y), I(x,z)) &= S_2\left(s_2^{-1}\left(c_x s_1(y)\right), s_2^{-1}\left(c_x s_1(z)\right)\right) \\ &= s_2^{-1}\left(s_2 \circ s_2^{-1}\left(c_x s_1(y)\right)\right. \\ &\qquad \left. + s_2 \circ s_2^{-1}\left(c_x s_1(z)\right)\right) \\ &= s_2^{-1}\left(c_x s_1(y) + c_x s_1(z)\right) \\ &= s_2^{-1}\left(c_x(s_1(y) + s_1(z))\right) \\ &= I(x, S_1(y,z)) \end{aligned}$$

since $s_1(y) < \infty$ and $s_1(z) < \infty$. If $y = 1$ or $z = 1$, then

$$I(x, S_1(y,z)) = S_2(I(x,y), I(x,z)) = 1.$$

Finally, let us assume that, for some $x \in [0,1]$, the vertical section is given by (28). We know, by Theorem 1, that s_1, s_2 are uniquely determined up to positive multiplicative constants. We show that, in this case, the constant c_x in (28) depends on previous constants. To prove this, let $a, b \in (0,\infty)$ be fixed and assume that $s_1'(x) = a s_1(x)$ and $s_2'(x) = b s_2(x)$ for $x \in [0,1]$. By Theorem 1, functions s_1' and s_2' are also continuous additive generators of S_1 and S_2, respectively. Let us define $c_x' := (b/a)c_x$.

For all $y \in [0,1]$, we get

$$\begin{aligned} s_2'^{-1}\left(c_x' \cdot s_1'(y)\right) &= s_2'^{-1}\left(\frac{b}{a} c_x a s_1(y)\right) \\ &= s_2'^{-1}\left(b c_x s_1(y)\right) \\ &= s_2^{-1}\left(\frac{b c_x s_1(y)}{b}\right) \\ &= s_2^{-1}\left(c_x s_1(y)\right) \end{aligned}$$

i.e., the vertical section for s_1, s_2, and c_x is the same as that of s_1', s_2', and c_x'.

$1) \Longrightarrow 2)$ Let us assume that functions S_1, S_2, and I are the solutions of the functional equation (4) satisfying the required properties. From Theorem 1 and part 2) of Remark 2, the t-conorms S_1 and S_2 admit the representation (5) for some continuous additive generators $s_1, s_2\colon [0,1] \to [0,\infty]$ such that $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$. Moreover, both generators are uniquely determined up to positive multiplicative constants. Now, (4) becomes

$$I\left(x, s_1^{-1}(s_1(y) + s_1(z))\right) = s_2^{-1}(s_2(I(x,y)) + s_2(I(x,z))) \tag{29}$$

for all $x, y, z \in [0,1]$. Let $x \in [0,1]$ be arbitrary but fixed. Define a function $I_x\colon [0,1] \to [0,1]$ by the formula

$$I_x(y) = I(x,y), \qquad y \in [0,1].$$

By routine substitutions, $h_x = s_2 \circ I_x \circ s_1^{-1}$, $u = s_1(y)$, $v = s_1(z)$, for $y, z \in [0,1]$, from (29), we obtain the additive Cauchy functional equation

$$h_x(u+v) = h_x(u) + h_x(v), \qquad u, v \in [0,\infty]$$

where $h_x\colon [0,\infty] \to [0,\infty]$. By Proposition 2, we get either $h_x = \infty$, $h_x = 0$, or

$$h_x(u) = \begin{cases} 0, & \text{if } u = 0 \\ \infty, & \text{if } u > 0 \end{cases}$$

for $u \in [0,\infty]$, or

$$h_x(u) = \begin{cases} 0, & \text{if } u < \infty \\ \infty, & \text{if } u = \infty \end{cases}$$

for $u \in [0,\infty]$, or there exists a constant $c_x \in (0,\infty)$ such that $h_x(u) = c_x \cdot u$, for $u \in [0,\infty]$.

Because of the definition of the function h_x, we get either $I_x = 1$, or $I_x = 0$, or

$$I_x(y) = \begin{cases} 0, & \text{if } y = 0 \\ 1, & \text{if } y > 0 \end{cases}$$

for $y \in [0,1]$, or

$$I_x(y) = \begin{cases} 0, & \text{if } y < 1 \\ 1, & \text{if } y = 1 \end{cases}$$

for $y \in [0,1]$, or $I_x(y) = s_2^{-1}\left(c_x \cdot s_1(y)\right)$ for $y \in [0,1]$ and with $c_x \in (0,\infty)$.

We show that in the last case, the constant c_x is uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2. Let $s_1'(x) = a s_1(x)$ and $s_2'(x) = b s_2(x)$ for all $x \in [0,1]$ and some $a, b \in (0,\infty)$. Further, let c_x' be a constant in (28) for s_1', s_2'. If we assume that

$$s_2^{-1}\left(c_x \cdot s_1(y)\right) = s_2'^{-1}\left(c_x' \cdot s_1'(y)\right)$$

then we get

$$s_2^{-1}\left(c_x \cdot s_1(y)\right) = s_2^{-1}\left(\frac{c_x' \cdot s_1'(y)}{b}\right)$$

and therefore,

$$c_x \cdot s_1(y) = \frac{c_x' \cdot a \cdot s_1(y)}{b}$$

and thus, when $y \neq 0$, we get

$$c_x' = \frac{b}{a} c_x.$$

∎

Remark 3: From the previous proof, it follows that if we assume that $S_1 = S_2$ and for some $x \in [0,1]$, the vertical section $I(x,\cdot)$ has the form (28), then the constant c_x is uniquely determined.

Since we are interested in finding solutions of (4) in the fuzzy logic context, we can easily obtain an infinite number of solutions that are fuzzy implications. It should be noted that, with this assumption, the vertical section (24) is not possible, while for $x = 0$, the vertical section should be (25). Also, a fuzzy implication is decreasing in the first variable while it is increasing in the second one.

Example 1: If S_1, S_2 are both strict t-conorms, then the greatest solution that is a fuzzy implication is the greatest fuzzy implication [5]:

$$I_1(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

The vertical sections are the following: For $x \in [0,1)$, this is (25), and for $x = 1$, this is (26).

Example 2: If S_1, S_2 are both strict t-conorms, then the least solution that is a fuzzy implication is the least fuzzy implication [5]:

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The vertical sections are the following: For $x = 0$, this is (25), and for $x \in (0,1]$, this is (27).

A. *Continuous Solutions for I in (4) With Strict T-Conorms*

From the previous result, we are in a position to describe the continuous solutions I of (4).

Theorem 11 (cf. [1]): Let S_1, S_2 be strict t-conorms. For a continuous function $I\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$, which are uniquely determined up

to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and either $I = 0$, or $I = 1$, or there exists a continuous function $c: [0,1] \to (0,\infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2, such that I has the form

$$I(x,y) = s_2^{-1}\left(c(x) \cdot s_1(y)\right), \qquad x,y \in [0,1]. \quad (30)$$

Proof: 2) $\Longrightarrow$ 1) It is obvious that all functions I described in 2) are continuous. By previous general solution, they satisfy our functional equation (4) with strict t-conorms S_1, S_2 generated from s_1 and s_2, respectively.

1) $\Longrightarrow$ 2) From Theorem 10, we know what are the possible vertical sections for the fixed $x \in [0,1]$. Since I is continuous, for every $x \in [0,1]$, the vertical sections are also continuous, and consequently, the vertical sections (26) and (27) are not possible.

Let us assume that there exists some $x_0 \in [0,1]$ such that $I(x_0, y) = 0$ for all $y \in [0,1]$, i.e., the vertical section for x_0 is (24). In particular, $I(x_0, 1) = 0$, but for the other possible vertical sections, we always have $I(x,1) = 1$; therefore, the only possibility in this case is $I = 0$.

Analogously, let us assume that there exists some $x_0 \in [0,1]$ such that $I(x_0, y) = 1$ for all $y \in [0,1]$ i.e., the vertical section for x_0 is (25). In particular, $I(x_0, 0) = 1$, but for the other possible vertical sections, we always have $I(x,0) = 0$; therefore, the only possibility in this case is $I = 1$.

Finally, assume that, for all $x \in [0,1]$, the vertical sections $I_x \neq 0$ and $I_x \neq 1$. This implies that the vertical section is (28). Therefore, there exists a function $c: [0,1] \to (0,\infty)$ such that I has the form (30). This function c is continuous since for any fixed $y \in (0,1)$, it is a composition of continuous functions

$$c(x) = \frac{s_2(I(x,y))}{s_1(y)}, \qquad x \in [0,1].$$

From the previous formula, one can immediately obtain, that the function c is uniquely determined up to a positive multiplicative constant, depending on constants for s_1 and s_2. ■

Example 3: If we assume that $S_1 = S_2$ and the function $c = 1$ in (30), then our solution is trivial

$$I(x,y) = s^{-1}\left(c(x) \cdot s(y)\right) = s^{-1}\left(s(y)\right) = y, \quad x,y \in [0,1].$$

This function is not a fuzzy implication.

Since (4) is the generalization of a tautology from the classical logic involving Boolean implication, it is reasonable to expect that the solution I of (4) is also a fuzzy implication, but from Theorem 11, we obtain the following result.

Corollary 5: If S_1, S_2 are strict t-conorms, then there are no continuous solutions I of (4) that satisfy (I3).

Proof: Let a continuous function I satisfy (I3) and (4) with some strict t-conorms S_1, S_2 with continuous additive generators s_1, s_2, respectively. Then, I has the form (30) with a continuous function $c: [0,1] \to (0,\infty)$, but in this case, we get

$$I(0,0) = s_2^{-1}\left(c(0) \cdot s_1(0)\right) = s_2^{-1}\left(c(0) \cdot 0\right) = s_2^{-1}(0) = 0$$

and therefore, I does not satisfy the first condition in (I3). ■

B. Noncontinuous Solutions of (4) With Strict T-Conorms

From Corollary 5, it is obvious that we need to look for solutions that are not continuous at the point $(0,0)$, and we explore this case now.

Theorem 12: Let S_1, S_2 be strict t-conorms and let a function $I: [0,1]^2 \to [0,1]$ be continuous except at the point $(0,0)$, which satisfies (I3) and (NC). Then, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2: [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and a continuous function $c: [0,1] \to (0,\infty]$ with $c(x) < \infty$ for $x \in (0,1]$, $c(0) = \infty$ that is uniquely determined up to a positive multiplicative constant, depending on constants for s_1 and s_2, such that I has the form

$$I(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \\ s_2^{-1}\left(c(x) \cdot s_1(y)\right), & \text{otherwise} \end{cases} \quad (31)$$

for all $x, y \in [0,1]$.

Proof: 2) $\Longrightarrow$ 1) It is obvious that S_1, S_2 are strict t-conorms. Moreover, the function I defined by (31) is continuous except at the point $(0,0)$ and satisfies (I3) and (NC), since

$$I(0,0) = 1$$
$$I(x,1) = s_2^{-1}\left(c(x) \cdot s_1(1)\right) = s_2^{-1}(\infty) = 1, \qquad x \in [0,1]$$
$$I(0,x) = s_2^{-1}\left(c(0) \cdot s_1(x)\right) = s_2^{-1}(\infty) = 1, \qquad x \in (0,1].$$

By our previous general solution, they satisfy our functional equation (4) with strict t-conorms S_1, S_2 generated from s_1 and s_2, respectively.

1) $\Longrightarrow$ 2) Let us assume that the functions I and S_1, S_2 are the solutions of (4) satisfying the required properties. From Theorem 10, there exist continuous and strictly increasing functions $s_1, s_2: [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively.

Let $x \in (0,1]$ be arbitrary but fixed. Again from Theorem 10, we get either $I_x = 1$, or $I_x = 0$, or $I_x(y) = s_2^{-1}\left(c(x) \cdot s_1(y)\right)$ for all $y \in [0,1]$ and with $c(x) \in (0,\infty)$.

From the continuity of the function I and the assumptions of I, as shown in the proof of [2, Th. 5], the first two cases are not possible. Indeed, if we take $y = 1$, then there are only two possibilities, for any $x \in (0,1]$, either $I_x(1) = 0$, or $I_x(1) = 1$. However, $I_1(1) = I(1,1) = 1$ and from the continuity of I on the first variable (for $x > 0$ and $y = 1$), we get $I_x(1) = 1$ for every $x \in (0,1]$, so $I_x \neq 0$ for every $x \in (0,1]$. On the other hand, taking $y = 0$, we also obtain two possibilities, for any $x \in (0,1]$, either $I_x(0) = 0$, or $I_x(0) = 1$, but $I_1(0) = I(1,0) = 0$, and from the continuity of I on the first variable (for $x > 0$ and $y = 0$), we get $I_x(1) = 0$ for every $x \in (0,1]$; therefore, $I_x \neq 1$ for every $x \in (0,1]$. We proved that there exists a function

$c\colon (0,1] \to (0,\infty)$ such that I has the form (30). This function c is continuous, since for any fixed $y \in (0,1)$, it is a composition of continuous functions:

$$c(x) = \frac{s_2(I(x,y))}{s_1(y)}, \qquad x \in (0,1]. \tag{32}$$

If $x = 0$, then using similar steps as in the proof of Theorem 10, we obtain the additive Cauchy functional equation

$$h_0(u+v) = h_0(u) + h_0(v), \qquad u,v \in (0,1] \tag{33}$$

where the function $h_0\colon (0,\infty] \to [0,\infty]$ defined by the formula $h_0 = s_2 \circ I_0 \circ s_1^{-1}$ is continuous (here, $I_0(y) = I(0,y)$ for all $y \in [0,1]$). Corollary 2 implies that $h_0 = \infty$, or $h_0 = 0$, or there exists $c(0) \in (0,\infty)$ such that $h_0(u) = c(0) \cdot u$ for $u \in (0,1]$. However

$$h_0(1) = s_2(I_0(s_1^{-1}(1))) = s_2(I(0,1)) = s_2(1) = \infty$$

and therefore, $h_0 \neq 0$, and the solution $h_0 = \infty$ implies $c(0) = \infty$. Therefore, we get

$$I(0,y) = I_0(y) = s_2^{-1}\left(c(0) \cdot s_1(y)\right), \qquad y \in (0,1] \tag{34}$$

and some $c(0) \in (0,\infty]$. We show that $c(0) = \infty$. From (29), substituting $x = 0$ and $z = 0$, it follows that, for all $y \in [0,1]$, we have

$$I\left(0, s_1^{-1}(s_1(y) + s_1(0))\right) = s_2^{-1}(s_2(I(0,y)) + s_2(I(0,0))).$$

Since $s_1(0) = 0$, $s_1(1) = \infty$, and $I(0,0) = 1$, we get

$$I(0,y) = 1, \qquad y \in [0,1].$$

Let $y \in (0,1)$ be fixed. By (34), we get

$$1 = s_2^{-1}\left(c(0) \cdot s_1(y)\right)$$

thus, $c(0) \cdot s_1(y) = s_2(1) = \infty$. Since $y \in (0,1)$, we obtain that $c(0) = \infty$.

Finally, we must prove the existence of the following limit $\lim_{x\to 0^+} c(x) = c(0)$. To this end, we fix arbitrarily $y \in (0,1)$. From the continuity, $\lim_{x\to 0^+} I(x,y) = I(0,y)$. Moreover, s_2 is continuous, and therefore

$$\lim_{x\to 0^+} c(x) = \lim_{x\to 0^+} \frac{s_2(I(x,y))}{s_1(y)} = \frac{s_2(\lim_{x\to 0^+} I(x,y))}{s_1(y)}$$
$$= \frac{s_2(I(0,y))}{s_1(y)} = \frac{s_2(1)}{s_1(y)} = \frac{\infty}{s_1(y)} = \infty = c(0)$$

and c is a continuous function. ■

Remark 4: The function I in the previous theorem can also be written in the form

$$I(x,y) = s_2^{-1}\left(c(x) \cdot s_1(y)\right), \qquad x,y \in [0,1]$$

with the convention that $0 \times \infty = \infty \times 0 = \infty$.

From the previous proof, we see that a function I given by (31) with a continuous function c satisfies conditions (I3). Additionally, by the increasing nature of continuous generators s_1 and s_2, we get that I is increasing with respect to the second variable. Unfortunately, we can say nothing about its monotonicity with respect to the first one. The next result solves this by showing some necessary and sufficient conditions.

Corollary 6: If S_1, S_2 are strict t-conorms and I is a fuzzy implication that is continuous except at the point $(0,0)$, then the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x,y,z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$ and $s_1(1) = s_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and a continuous, decreasing function $c\colon [0,1] \to (0,\infty]$ with $c(x) < \infty$ for $x \in (0,1]$, $c(0) = \infty$, uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2, such that I has the form (31) for all $x,y \in [0,1]$.

We would like to underline the main difference between Theorem 12 and the previous result. In Corollary 6, we have the assumption that I is a fuzzy implication in the sense of Definition 3.

Example 4: One specific example is the function $c(x) = 1/x$ for all $x \in [0,1]$, with the assumption that $1/0 = \infty$. In this case, the solution I is the following:

$$I(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \\ s_2^{-1}\left(\dfrac{1}{x} \cdot s_1(y)\right), & \text{otherwise} \end{cases}$$

for all $x,y \in [0,1]$. In the special case when $s_1 = s_2$, i.e., $S_1 = S_2$, we obtain the function from the Yager's class of g-generated fuzzy implications (see [34, p. 202]).

IV. On the Equation (4) When S_1, S_2 Are Nilpotent T-Conorms

In this section, our main goal is to present the characterizations of the classes of fuzzy implications that satisfy (4) when S_1, S_2 are both nilpotent t-conorms, but we first describe the general solutions of (4) when S_1, S_2 are nilpotent t-conorms. From this result, we again show that there are no continuous fuzzy implications I that are solutions for (4) for nilpotent t-conorms and, hence, proceed to investigate noncontinuous solutions for I obeying (4).

Theorem 13: Let S_1, S_2 be nilpotent t-conorms. For a function $I\colon [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x,y,z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$, $s_1(1) < \infty$ and $s_2(1) < \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and for every fixed $x \in [0,1]$, the vertical section $I(x,\cdot)$ has one of the following forms:

$$I(x,y) = 0, \qquad y \in [0,1] \tag{35}$$

$$I(x,y) = 1, \qquad y \in [0,1] \tag{36}$$

$$I(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ 1, & \text{if } y > 0, \end{cases} \qquad y \in [0,1] \tag{37}$$

$$I(x,y) = s_2^{-1}\left(\min(c_x \cdot s_1(y), s_2(1))\right), \quad y \in [0,1] \tag{38}$$

with a certain $c_x \in [s_2(1)/s_1(1), \infty)$ uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2.

Proof: 2) $\Longrightarrow$ 1) Let t-conorms S_1, S_2 have the representation (5) with some continuous and strictly increasing functions $s_1, s_2\colon [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0, s_1(1) < \infty$, and $s_2(1) < \infty$. By Theorem 1 and part 3) of Remark 2, the functions S_1, S_2 are nilpotent t-conorms. Let us fix arbitrarily $x \in [0,1]$. We consider four cases.

If $I(x,y) = 0$ for all $y \in [0,1]$, then the left side of (4) is equal to $I(x, S_1(y,z)) = 0$, and the right is equal to $S_2(I(x,y), I(x,z)) = S_2(0,0) = 0$ for all $y, z \in [0,1]$.

If $I(x,y) = 1$ for all $y \in [0,1]$, then the left side of (4) is equal to $I(x, S_1(y,z)) = 1$, and the right is equal to $S_2(I(x,y), I(x,z)) = S_2(1,1) = 1$ for all $y, z \in [0,1]$.

Let $I(x,y)$ have the form (37) for all $y \in [0,1]$. Fix arbitrarily $y, z \in [0,1]$. If $y = 0$, then the left side of (4) is equal to $I(x, S_1(0,z)) = I(x,z)$, and the right is equal to $S_2(I(x,0), I(x,z)) = S_2(0, I(x,z)) = I(x,z)$. Analogously, if $z = 0$, then both sides of (4) are equal to $I(x,y)$. If $y > 0$ and $z > 0$, then $S_1(y,z) \geq \min(S_1(y,y), S_1(z,z)) > \min(y,z) > 0$ since S_1 is nilpotent, i.e., continuous and Archimedean. Now, the left side of (4) is equal to $I(x, S_1(y,z)) = 1$, and the right is equal to $S_2(I(x,y), I(x,z)) = S_2(1,1) = 1$.

If I has the form (38) for all $y \in [0,1]$ with some $c_x \in [s_2(1)/s_1(1), \infty)$, then one can check, similar to the proof of Theorem 10, that the triple of functions S_1, S_2, I satisfies the functional equation (4).

Finally, let us assume that, for some $x \in [0,1]$, the vertical section is given by (38). We know, by Theorem 1, that continuous additive generators s_1, s_2 are unique up to a positive multiplicative constant. We show that, in this case, the constant c_x in (38) depends on previous constant. To prove this, let $a, b \in (0,\infty)$ be fixed and assume that $s_1'(x) = a s_1(x)$ and $s_2'(x) = b s_2(x)$ for all $x \in [0,1]$. By Theorem 1, functions s_1' and s_2' are also continuous additive generators of t-conorms S_1 and S_2, respectively. Let us define $c_x' := (b/a)c_x$. For all $y \in [0,1]$, we get

$$\begin{aligned}
&s_2'^{-1}\left(\min(c_x' \cdot s_1'(y), s_2'(1))\right) \\
&= s_2'^{-1}\left(\min\left(\frac{b}{a} c_x \cdot a s_1(y), b s_2(1)\right)\right) \\
&= s_2'^{-1}\left(b \cdot \min(c_x \cdot s_1(y), s_2(1))\right) \\
&= s_2^{-1}\left(\frac{b \cdot \min(c_x \cdot s_1(y), s_2(1))}{b}\right) \\
&= s_2^{-1}\left(\min(c_x \cdot s_1(y), s_2(1))\right)
\end{aligned}$$

i.e., the vertical section for s_1, s_2, and c_x is the same as for s_1', s_2', and c_x'.

1) $\Longrightarrow$ 2) Let us assume that functions S_1, S_2, and I are the solutions of the functional equation (4) satisfying the required properties. Then, from Theorem 1 and part 3) of Remark 2, the t-conorms S_1 and S_2 admit the representation (5) for some continuous additive generators $s_1, s_2\colon [0,1] \to [0,\infty]$ such that $s_1(0) = s_2(0) = 0$, $s_1(1) < \infty$, and $s_2(1) < \infty$. Moreover, both generators are uniquely determined up to positive multiplicative constants. Now, (4) becomes, for all $x, y, z \in [0,1]$

$$\begin{aligned}
&I\left(x, s_1^{-1}(\min(s_1(y) + s_1(z), s_1(1)))\right) \\
&\quad = s_2^{-1}(\min(s_2(I(x,y)) + s_2(I(x,z)), s_2(1))).
\end{aligned} \tag{39}$$

Fix arbitrarily $x \in [0,1]$ and define a function $I_x\colon [0,1] \to [0,1]$ by the formula

$$I_x(y) = I(x,y), \qquad y \in [0,1].$$

By routine substitutions, $h_x = s_2 \circ I_x \circ s_1^{-1}$, $u = s_1(y)$, $v = s_1(z)$ for $y, z \in [0,1]$, from (39), we obtain the following functional equation, for $u, v \in [0, s_1(1)]$:

$$h_x(\min(u+v, s_1(1))) = \min(h_x(u) + h_x(v), s_2(1))$$

where the function $h_x\colon [0, s_1(1)] \to [0, s_2(1)]$. By Proposition 3, we get either $h_x = s_2(1)$, $h_x = 0$, or

$$h_x(u) = \begin{cases} 0, & \text{if } u = 0 \\ s_2(1), & \text{if } u > 0 \end{cases}$$

for $u \in [0, s_1(1)]$, or there exists a constant $c_x \in [s_2(1)/s_1(1), \infty)$ such that $h_x(u) = \min(c_x \cdot u, s_2(1))$, for $u \in [0, s_1(1)]$.

Because of the definition of the function h_x we get either $I_x = 1$, $I_x = 0$, or

$$I_x(y) = \begin{cases} 0, & \text{if } y = 0 \\ 1, & \text{if } y > 0 \end{cases}$$

for $y \in [0,1]$, or $I_x(y) = s_2^{-1}\left(\min(c_x \cdot s_1(y), s_2(1))\right)$ for $y \in [0,1]$ and with $c_x \in [s_2(1)/s_1(1), \infty)$.

We show that, in the last case, the constant c_x is uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2. Let $s_1'(x) = a s_1(x)$ and $s_2'(x) = b s_2(x)$ for all $x \in [0,1]$ and some $a, b \in (0,\infty)$. Further, let c_x' be a constant in (38) for s_1', s_2'. If we assume that

$$s_2^{-1}\left(\min(c_x \cdot s_1(y), s_2(1))\right) = s_2'^{-1}\left(\min(c_x' \cdot s_1'(y), s_2'(1))\right)$$

then we get

$$\begin{aligned}
&s_2^{-1}\left(\min(c_x \cdot s_1(y), s_2(1))\right) \\
&\quad = s_2^{-1}\left(\frac{\min(c_x' \cdot a s_1(y), b s_2(1))}{b}\right)
\end{aligned}$$

and therefore

$$\min(c_x \cdot s_1(y), s_2(1)) = \min\left(\frac{c_x' \cdot a s_1(y)}{b}, s_2(1)\right)$$

and thus, whenever $c_x \cdot s_1(y) < s_2(1)$, we have

$$c_x \cdot s_1(y) = \frac{c_x' \cdot a \cdot s_1(y)}{b}.$$

Therefore, if $y \neq 0$, then we get $c_x' = (b/a)c_x$. ■

Remark 5: From the previous proof, it follows that if we assume that $S_1 = S_2$ and for some $x \in [0,1]$, the vertical section $I(x,\cdot)$ has the form (38), then the constant c_x is uniquely determined.

We can easily obtain an infinite number of solutions that are fuzzy implications. It should be noted that, with this assumption, the vertical section for $x = 0$ should be (36).

Example 5: If S_1, S_2 are both nilpotent t-conorms, then the greatest solution of (4), which is a fuzzy implication, is the greatest fuzzy implication I_1.

Example 6: If S_1, S_2 are both nilpotent t-conorms, then the least solution of (4), which is a fuzzy implication, is the following:

$$I(x,y) = \begin{cases} 1, & \text{if } x = 1 \\ s_2^{-1}\left(\min\left(\frac{s_2(1)}{s_1(1)} \cdot s_1(y), s_2(1)\right)\right), & \text{if } x < 1. \end{cases}$$

In the special case, when $s_1 = s_2$, i.e., $S_1 = S_2$, then we obtain the following fuzzy implication:

$$I(x,y) = \begin{cases} 1, & \text{if } x = 1 \\ y, & \text{if } x < 1 \end{cases}$$

which is also the least (S, N)-implication (see [7, Ex. 1.5]).

A. Continuous Solutions of (4) With Nilpotent T-Conorms

Similar to the proofs of Theorems 11 and 13, we can deduce the following result.

Theorem 14: Let S_1, S_2 be nilpotent t-conorms. For a continuous function $I: [0,1]^2 \to [0,1]$, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2: [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$, $s_1(1) < \infty$ and $s_2(1) < \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and either $I = 0$, or $I = 1$, or there exists a continuous function $c: [0,1] \to [s_2(1)/s_1(1), \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2, such that I has the form
$$I(x,y) = s_2^{-1}\left(\min(c(x) \cdot s_1(y), s_2(1))\right) \tag{40}$$
for all $x, y \in [0,1]$.

Corollary 7: If S_1, S_2 are nilpotent t-conorms, then there are no continuous solutions I of (4) that satisfy (I3).

Proof: Let a continuous function I satisfy (I3) and (4) with some nilpotent t-conorms S_1, S_2 with continuous additive generators s_1, s_2, respectively. Then, I has the form (40) with a continuous function $c: [0,1] \to [s_2(1)/s_1(1), \infty)$, but in this case, we get

$$\begin{aligned} I(0,0) &= s_2^{-1}\left(\min\left(c(0) \cdot s_1(0), s_2(1)\right)\right) \\ &= s_2^{-1}\left(\min(0, s_2(1))\right) = 0 \end{aligned}$$

and therefore, I does not satisfy the first condition in (I3). ∎

B. Noncontinuous Solutions of (4) With Nilpotent T-Conorms

From Corollary 7, it is obvious that we need to look for solutions that are not continuous at the point $(0,0)$. Using similar methods as earlier, we can prove the following fact.

Theorem 15: Let S_1, S_2 be nilpotent t-conorms and let a function $I: [0,1]^2 \to [0,1]$ be continuous except at the point $(0,0)$, which satisfies (I3) and (NC). Then, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2: [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$, $s_1(1) < \infty$ and $s_2(1) < \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and a continuous function $c: [0,1] \to [s_2(1)/s_1(1), \infty]$ with $c(x) < \infty$ for $x \in (0,1]$, $c(0) = \infty$, uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2, such that I has the form
$$I(x,y) = \begin{cases} 1, & \text{if } x{=}y{=}0 \\ s_2^{-1}\left(\min\left(c(x)s_1(y), s_2(1)\right)\right), & \text{otherwise} \end{cases} \tag{41}$$
for $x, y \in [0,1]$.

Remark 6: The function I in the previous theorem can also be written in the form

$$I(x,y) = s_2^{-1}\left(\min\left(c(x)s_1(y), s_2(1)\right)\right), \qquad x, y \in [0,1]$$

with the convention that $0 \times \infty = \infty \times 0 = \infty$.

It can easily be verified that a function I given by the formula (41) with a continuous function c satisfies conditions (I3). Additionally, by the increasing nature of continuous generators s_1 and s_2, we get that I is increasing with respect to the second variable, but we can say nothing about its monotonicity with respect to the first one. The next result solves this by showing some necessary and sufficient conditions.

Corollary 8: Let S_1, S_2 be nilpotent t-conorms and I be a fuzzy implication that is continuous except at the point $(0,0)$. Then, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) There exist continuous and strictly increasing functions $s_1, s_2: [0,1] \to [0,\infty]$ with $s_1(0) = s_2(0) = 0$, $s_1(1) < \infty$ and $s_2(1) < \infty$, which are uniquely determined up to positive multiplicative constants, such that S_1, S_2 admit the representation (5) with s_1, s_2, respectively, and a continuous decreasing function $c: [0,1] \to [s_2(1)/s_1(1), \infty]$ with $c(x) < \infty$ for $x \in (0,1]$, $c(0) = \infty$, uniquely determined up to a positive multiplicative constant depending on constants for s_1 and s_2, such that I has the form (41) for all $x, y \in [0,1]$.

Here again, we would like to underline the main difference between Theorem 15 and the previous result. In Corollary 8, we have the assumption that I is a fuzzy implication in the sense of Definition 3.

Example 7: One specific example, when $s_2(1) \leq s_1(1)$, is again the function $c(x) = 1/x$ for all $x \in [0,1]$, with the assumption that $1/0 = \infty$. In this case, the solution I is the following:

$$I(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \\ s_2^{-1}\left(\min\left(\frac{1}{x}\cdot s_1(y), s_2(1)\right)\right), & \text{otherwise} \end{cases}$$
$$= s_2^{(-1)}\left(\frac{1}{x}\cdot s_1(y)\right)$$

for all $x, y \in [0,1]$. In the special case when $s_1 = s_2$, i.e., $S_1 = S_2$, we obtain the function from the Yager's class of g-generated fuzzy implications (see [34, p. 202]). Quite evidently, there are other candidates for the function c, viz., $c(x) = 1 + 1/x$ or $c(x) = 1/x^2$ for $x \in [0,1]$.

V. On (4) When I is an R-Implication

In this section, we discuss the distributive equation (4) when I is an R-implication obtained from a continuous Archimedean t-norm T. The case when T is a nilpotent t-norm has been investigated by Balasubramaniam and Rao in [10]. The result from this paper can be written in the following form.

Theorem 16 [10, Th. 4]: Let S_1, S_2 be t-conorms. For an R-implication I obtained from a nilpotent t-norm, the following statements are equivalent.

1) The triple of functions S_1, S_2, I satisfies the functional equation (4) for all $x, y, z \in [0,1]$.
2) $S_1 = S_2 = \max$.

In [10], we can find the following sentence "... the authors have a strong feeling that it holds for the case when the R-implication is obtained from a strict t-norm ...." We will show in this section that this is not true, i.e., for an R-implication generated from a strict t-norm, there exist other solutions than maximum.

We start our presentation with some connections between solutions S_1, S_2 of (4) and the properties of R-implications.

Lemma 1: Let S_1, S_2 be t-conorms and $I: [0,1]^2 \to [0,1]$ be a function that satisfies the left neutrality property (NP). If a triple of functions S_1, S_2, I satisfies the functional equation (4), then $S_1 = S_2$.

Proof: Let I satisfy (NP). Putting $x = 1$ in (4), we get

$$I(1, S_1(y,z)) = S_2(I(1,y), I(1,z)), \qquad y, z \in [0,1]$$

and thus

$$S_1(y,z) = S_2(y,z), \qquad y, z \in [0,1].$$

Hence, $S_1 = S_2$. ■

Note that the previous result is true for any binary operations S_1 and S_2.

Lemma 2: Let S_1, S_2 be continuous Archimedean t-conorms, and let $I: [0,1]^2 \to [0,1]$ be a function that satisfies the ordering property (OP). If S_1, S_2, I satisfy the functional equation (4), then S_2 is a nilpotent t-conorm.

Proof: Since S_1 is a continuous Archimedean t-conorm, from part 1) of Remark 1, for every $y \in (0,1)$, we have $S_1(y,y) > y$. Let us fix any $y \in (0,1)$ and take some $x \in (y, S_1(y,y))$. By (OP), we get $I(x,y) = y_0 < 1$, whereas from (4), we obtain

$$1 = I(x, S_1(y,y)) = S_2(I(x,y), I(x,y)) = S_2(y_0, y_0).$$

Hence, by part 2) of Remark 1, the t-conorm S_2 is nilpotent. ■

Since an R-implication I generated from left-continuous t-norm satisfies (NP) and (OP), from previous Lemma 1, we have $S_1 = S_2$ in (4), and hence, it suffices to consider the following functional equation:

$$I(x, S(y,z)) = S(I(x,y), I(x,z)), \qquad x, y, z \in [0,1]. \tag{42}$$

Further, from Lemma 2, we get the following.

Corollary 9: Let S be a continuous Archimedean t-conorm, and let I be an R-implication generated from some left-continuous t-norm. If the couple of functions S, I satisfies the functional equation (42), then S is nilpotent.

Corollary 10: An R-implication I obtained from a left-continuous t-norm does not satisfy (4), when S_1 and S_2 are both strict t-conorms.

From previous investigations, it follows that we should consider the situation when S is a nilpotent t-conorm. As a result, we obtain the following theorem.

Theorem 17: For a nilpotent t-conorm S and an R-implication I generated from a strict t-norm, the following statements are equivalent.

1) The couple of functions S, I satisfies the functional equation (42) for all $x, y, z \in [0,1]$.
2) There exist $\varphi \in \Phi$, which is uniquely determined, such that S admits the representation (8) with φ and I admits the representation (10) with φ.

Proof: 2) $\Longrightarrow$ 1) Assume that there exists a $\varphi \in \Phi$, such that S admits the representation (8) with φ and I admits the representation (10) with φ, i.e.,

$$S(x,y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1)), \qquad x, y \in [0,1]$$

and

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), & \text{if } x > y, \end{cases} \qquad x, y \in [0,1].$$

We will show that I and S satisfy (42). Let us take any $x, y, z \in [0,1]$. The left side of (42) is equal to

$$\begin{aligned} &I(x, S(y,z)) \\ &= I(x, \varphi^{-1}(\min(\varphi(y) + \varphi(z), 1))) \\ &= \begin{cases} 1, & \text{if } x \le \varphi^{-1}(\min(\varphi(y)+\varphi(z), 1)) \\ \varphi^{-1}\left(\frac{\min(\varphi(y)+\varphi(z),1)}{\varphi(x)}\right), & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } \varphi(x) \le \min(\varphi(y)+\varphi(z), 1) \\ \varphi^{-1}\left(\min\left(\frac{\varphi(y)+\varphi(z)}{\varphi(x)}, \frac{1}{\varphi(x)}\right)\right), & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } \varphi(x) \le \varphi(y)+\varphi(z) \\ \varphi^{-1}\left(\frac{\varphi(y)+\varphi(z)}{\varphi(x)}\right), & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, the right side of (42) is equal to

$$
\begin{aligned}
&S(I(x,y), I(x,z))\\
&= \begin{cases} S(1,1), & \text{if } x \le y \text{ and } x \le z \\ S\left(\varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), 1\right), & \text{if } x > y \text{ and } x \le z \\ S\left(1, \varphi^{-1}\left(\frac{\varphi(z)}{\varphi(x)}\right)\right), & \text{if } x \le y \text{ and } x > z \\ S\left(\varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), \varphi^{-1}\left(\frac{\varphi(z)}{\varphi(x)}\right)\right), & \text{if } x>y \text{ and } x>z \end{cases}\\
&= \begin{cases} 1, & \text{if } x \le y \text{ or } x \le z \\ S\left(\varphi^{-1}\left(\frac{\varphi(y)}{\varphi(x)}\right), \varphi^{-1}\left(\frac{\varphi(z)}{\varphi(x)}\right)\right), & \text{if } x>y \text{ and } x>z \end{cases}\\
&= \begin{cases} 1, & \text{if } x \le y \text{ or } x \le z \\ \varphi^{-1}\left(\min\left(\frac{\varphi(y)+\varphi(z)}{\varphi(x)}, 1\right)\right), & \text{if } x > y \text{ and } x > z \end{cases}\\
&= \begin{cases} 1, & \text{if } x \le y \text{ or } x \le z \\ 1, & \text{if } x > y \text{ and } x > z \text{ and } \varphi(x) \le \varphi(y) + \varphi(z) \\ \varphi^{-1}\left(\frac{\varphi(y)+\varphi(z)}{\varphi(x)}\right), & \text{otherwise} \end{cases}\\
&= \begin{cases} 1, & \text{if } \varphi(x) \le \varphi(y) + \varphi(z) \\ \varphi^{-1}\left(\frac{\varphi(y)+\varphi(z)}{\varphi(x)}\right), & \text{otherwise} \end{cases}
\end{aligned}
$$

which ends the proof in this direction.

1) $\Longrightarrow$ 2) Let S be a nilpotent t-conorm and I be an R-implication generated from a strict t-norm. Because of Theorem 3, there exists a unique $\varphi \in \Phi$ such that S has the form (8). In fact, the increasing bijection φ can be seen as a continuous generator of S. Further, by virtue of Theorem 5, there exists $\psi \in \Phi$, uniquely determined up to a positive constant exponent, such that

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \psi^{-1}\left(\frac{\psi(y)}{\psi(x)}\right), & \text{if } x > y \end{cases} \qquad x, y \in [0,1]. \tag{43}$$

It is obvious that I is a fuzzy implication that is continuous except at the point $(0,0)$. Therefore, if functions S, I satisfy the functional equation (42), then from Corollary 8, there exists a continuous decreasing function $c\colon [0,1] \to [1,\infty]$, with $c(x) < \infty$ for $x \in (0,1]$ and $c(0) = \infty$, such that

$$I(x,y) = \begin{cases} 1, & \text{if } x = y = 0 \\ \varphi^{-1}\left(\min\left(c(x) \cdot \varphi(y), 1\right)\right), & \text{otherwise} \end{cases} \tag{44}$$

for $x, y \in [0,1]$.

Let us take any $x \in (0,1]$. From (43), for any $y < x$, we have $I(x,y) < 1$, whereas from (44), we have that $I(x,y) = \varphi^{-1}(c(x) \cdot \varphi(y))$. Hence, from the continuity of I (for $x > 0$), we get

$$1 = I(x,x) = \lim_{y \to x^-} I(x,y) = \lim_{y \to x^-} \varphi^{-1}(c(x) \cdot \varphi(y))$$

and therefore, $c(x) \cdot \varphi(x) = 1$ for $x \in (0,1]$. This implies that

$$c(x) = \frac{1}{\varphi(x)} < \infty \tag{45}$$

for $x \in (0,1]$. Observe that c is a continuous decreasing function from $(0,1]$ to $[1,\infty)$. Moreover, this formula can be considered also for $x = 0$, since $\infty = c(0) = 1/\varphi(0) = 1/0$, i.e., c is well defined. Now, comparing (43) with (44) and setting (45) in (44), we obtain the functional equation

$$\varphi^{-1}\left(\frac{1}{\varphi(x)} \cdot \varphi(y)\right) = \psi^{-1}\left(\frac{\psi(y)}{\psi(x)}\right)$$

for $x, y \in [0,1]$ and $x > y$. By substitutions, $h_x = \varphi \circ \psi^{-1}$, $u = \psi(x)$, and $v = \psi(y)$, we obtain the functional equation

$$h\left(\frac{v}{u}\right) = \frac{h(v)}{h(u)}, \qquad u, v \in [0,1], \qquad u > v.$$

From Corollary 4, we get that there exists a unique constant $c \in (0,\infty)$ such that $h(u) = u^c$. By the definition of h, we get $\varphi \circ \psi^{-1}(u) = u^c$, thus $\varphi(x) = (\psi(x))^c$ for all $x \in [0,1]$. Since the increasing bijection ψ is uniquely determined up to a positive constant exponent, we get that I admits the representation (10) also with φ. ■

Example 8: Taking $\varphi = \mathrm{id}_{[0,1]}$, we obtain the interesting example that the Łukasiewicz t-conorm and the Goguen implication satisfy the distributive equation (42).

VI. CONCLUSION

Recently, in [10] and [32], the authors have studied the distributivity of R- and S-implications over t-norms and t-conorms. But the distributive equation (4) for R-implications obtained from strict t-norms was not solved. In this paper, we have characterized a function I that satisfies the functional equation (4), when S_1, S_2 are either both strict or nilpotent t-conorms. Using the previous characterizations, we have shown that for an R-implication I generated from a strict t-norm T, we have the following.

1) Equation (4) does not hold when t-conorms S_1, S_2 are strict.
2) Equation (4) holds if and only if t-conorms $S_1 = S_2$ are Φ-conjugate with the Łukasiewicz t-conorm for some increasing bijection φ, which is a multiplicative generator of the strict t-norm T.

It is established that in the cases when I is an S-implication or an R-implication, most of the equations (1)–(3) hold only when the t-norms and t-conorms are either $\min$ or $\max$. That the generalization (4) has more solutions in the case of R-implications obtained from strict t-norms is bound to have positive implications in applications, especially in the new research area of inference invariant rule reduction.

Also as part of characterizing (4), we have obtained a more general class of fuzzy implications [see (31)] that contains the Yager's class [34] as a special case.

In our future works, we will try to concentrate on some cases that are not considered in this paper, for example, when S_1 is a strict t-conorm and S_2 is a nilpotent t-conorm, and *vice versa*.

Also, the situation when S_1 and S_2 are continuous, t-conorms is still unsolved.

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