Robust Exponential Stability of Uncertain Delayed Neural Networks With Stochastic Perturbation and Impulse Effects

Article *in* IEEE Transactions on Neural Networks and Learning Systems · June 2012

DOI: 10.1109/TNNLS.2012.2192135

Robust Exponential Stability of Uncertain Delayed Neural Networks With Stochastic Perturbation and Impulse Effects

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***Abstract*— This paper focuses on the hybrid effects of parameter uncertainty, stochastic perturbation, and impulses on global stability of delayed neural networks. By using the Ito formula, Lyapunov function, and Halanay inequality, we established several mean-square stability criteria from which we can estimate the feasible bounds of impulses, provided that parameter uncertainty and stochastic perturbations are well-constrained. Moreover, the present method can also be applied to general differential systems with stochastic perturbation and impulses.**

***Index Terms*— Delayed neural networks (DNN), exponential stability, impulse, mean-square stability, parameter uncertainty, stochastic perturbation.**

I. Introduction

IN THE past two decades, since the initiation of Hopfield neural network model [1], several types of recurrent neural networks have been extensively studied and successfully applied in many areas, such as combinatorial optimization, signal processing, pattern recognition, and nonlinear control [1]–[5]. In these applications, ensuring the global exponential/asymptotic stability of designed neural networks is very important. On the other hand, axonal signal transmission delays often occur in various neural networks, and may cause undesirable dynamic network behaviors, such as oscillation and instability. Therefore, there has been a growing research interest on analyzing stability problems for delayed neural networks, and a large number of studies have been available [6]–[17].

Manuscript received September 25, 2011; accepted March 21, 2012. Date of publication April 9, 2012; date of current version May 10, 2012. This work was supported in part by the Section III was made possible by NPRP Grant 4-1162-1-181 from the Qatar National Research Fund (a member of the Qatar Foundation). The authors acknowledge support from the National Natural Science Foundation of China under Grant 60974020, Grant 60972155, and Grant 10971240, and the Fundamental Research Funds for the Central Universities of China under Project CDJZR10 18 55 01 for other sections. The statements made herein are solely the responsibility of the authors.

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Digital Object Identifier 10.1109/TNNLS.2012.2192135

As is well known, a real system is usually affected by external disturbances. Generally speaking, there are two kinds of disturbances to be considered when one models neural networks: parameter uncertainty and stochastic perturbations, with the latter being unavailable in practice. Usually, there are two kinds of parameter uncertainty: time-varying, norm-bounded structures and interval structure. Both structures have been widely exploited in the problems of robust control and filtering of uncertain systems, including neural networks ([18] for the first kind and [20], [21] for the second). In real nervous systems, the synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. It has also been known that a neural network could be stabilized or destabilized by certain stochastic inputs [18]. Hence, the stability analysis problem for stochastic neural networks becomes increasingly significant, and some results related to this problem have recently been published [19]–[37].

Besides delay effects, impulse effects also exist in neural networks. For instance, in the implementation of electronic networks, the state of the network is subject to instantaneous perturbations and experiences abrupt change at certain instances, which may be caused by switching phenomenon, frequency change, or sudden noise. Moreover, many evolutionary processes, particularly some biological systems such as biological neural networks and bursting rhythm models in pathology, might exhibit impulse effects. Other examples include optimal control models in economics, frequency-modulated signal processing systems, and flying object motions. All these real-world systems and natural processes behave in a discrete or continuous style interlaced with instantaneous and abrupt changes. The mathematical foundations of impulsive systems are impulsive differential or difference equations, which are found in almost every field of applied science. Impulse effects have also been introduced to several neural network models, and the dynamics of these models with impulse effects have been thoroughly studied [38]–[45].

Recently, impulsive and stochastic effects have been taken into account when modeling neural networks. The dynamic analysis of delayed neural networks with impulsive and stochastic effects has been an attractive topic, and several stability criteria have been reported [46], [47]. For impulsive and stochastic Cohen-Grossberg neural networks with delays, Song and Wang [46] investigated the existence, uniqueness, and exponential p-stability of the equilibrium point by employing M-matrix theory and stochastic analysis technique. Wang et al. [47] generalized a few previously known results and removed some restrictions by using L-operator differential inequality, M-cone theory, and stochastic analysis technique. However, the results in [46] require that impulses must have the stabilizing effects. This condition is restrictive and might lead to conservatism. Meanwhile, although the authors in [47] overcome this weakness, the proposed results are difficult to verify. Moreover, neither [46] nor [47] addressed the parameter uncertainty problem.

Motivated by the shortcomings of the aforementioned research in this area, in this paper we study the robust exponential stability of delayed neural networks with parameter uncertainties, stochastic perturbation, and impulses, and establish several easily verified stability criteria to make the readers understand the complex effects of delays, uncertainties, stochastic perturbation, and impulses on the system's stability. In addition, from the main results in this paper, we can estimate the feasible upper bound of impulse when parameter uncertainties are norm-bounded. Specifically, this paper has three contributions. First, we formulate the generalized neural network model with time delays, parameter uncertainties, stochastic perturbations, and impulses. Second, we establish several easily verified stability criteria with the characterization of complex effects of time delays, parameter uncertainties, stochastic perturbations, and impulses on the stability of the reference networks. And third, we estimate the feasible upper bound of impulses. These contributions generalize and/or improve the existing results, including those in [46] and [47].

The rest of this paper is organized as follows. In the next section, we describe the problem to be considered and introduce the needed preliminaries. We then state the main results in Section III. In Section IV, we present two examples to verify our theoretical results. Finally, the conclusions are drawn in Section V.

II. Problem Descriptions and Preliminaries

The model we consider in this paper is composed of two subsystems given by

$$\begin{cases} du(t) = \big[-(C+\Delta C)u(t) + (A+\Delta A)f(u(t)) \\ \qquad + (B+\Delta B)f(u(t-\tau)) + \xi\big]dt \\ \qquad + [\Delta W_0 u(t) + \Delta W_1 u(t-\tau)]dW(t), \quad t \neq t_k & \text{(1a)} \\ \Delta u(t_k) = J_k(u(t_k-)), \quad t = t_k, \quad k = 1, 2, \ldots & \text{(1b)} \end{cases}$$

where impulse-free subsystem or continuous component (1a) is the continuous part of (1), which describes the continuous evolution processes of the neural networks, in which $u(t) = [u_1(t), \ldots, u_n(t)]^T \in R^n$ denotes the state vector of neurons, $C = \mathrm{diag}(c_1, \ldots, c_n) > 0$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $f(u) = [f_1(u_1), \ldots, , f_n(u_n)]^T \in R^n$, and $\xi = [\xi_1, \ldots, \xi_n]^n \in R^n$ denote the external inputs, and $\tau > 0$ denotes system delay. $\Delta C, \Delta A, \Delta B, \Delta W_0$ and ΔW_1 are time-varying matrices on $R^{n\times n}$ that denote the parameter uncertainties. $W(t) = [w_1(t), \ldots, , w_n(t)]^T$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and F_0 contains all P-null sets). The second part, (1b), is the discrete part of model (1), which describes the abrupt change of state at the moments of time t_k (called impulsive moments), in which $\{t_k\}$ satisfy $0 \leq t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$ is the impulse at moment t_k, and $J_k(u) \in C[R^n, R^n]$ is the jump operator.

Throughout this paper, we assume that for any $\varphi \in PC^b_{F_0}([-\tau, t_0], R^n)$ (i.e., the family of all bounded F_0-measurable), there exists at least one solution of (1), which is denoted by $u(t, t_0, \varphi)$, or, $u(t)$ if no confusion occurs. As the solution $u(t)$ is discontinuous at the moment t_k, by theory of impulsive differential equations, we assume, as usual, that the solution is right continuous at t_k, i.e., $u(t_k) = u(t_k^+)$.

Obviously, (1) includes several well-known neural network models as its special cases and, thus, it is the generalization of the models presented in [6], [8], [9], [18], [20], [23], [26], [31], [33], and [34].

If impulses do not occur, i.e., $J_k(u) \equiv 0$, (1) turns into the general stochastic neural networks with delay [18], [20], [23], [26], [31], [33], [34]

$$\begin{aligned} du\,(t) = \big[&-(C+\Delta C)u(t) + (A+\Delta A)f(u(t)) \\ &+(B+\Delta B)f(u(t-\tau)) + \xi\big]dt \\ &+[\Delta W_0 u(t) + \Delta W_1 u(t-\tau)]dW(t). \end{aligned} \tag{2}$$

If no stochastic perturbation occur, i.e., $\Delta W_0 = \Delta W_1 = 0$, then (2) reduces to the following model [6]:

$$\begin{aligned} \dot{u}\,(t) = &-(C+\Delta C)u(t) + (A+\Delta A)f(u(t)) \\ &+(B+\Delta B)f(u(t-\tau)) + \xi \end{aligned} \tag{3}$$

which contains the classic form of recurrent neural networks with delays [8]–[10]

$$\dot{u}\,(t) = -Cu(t) + Af(u(t)) + Bf(u(t-\tau)) + \xi. \tag{4}$$

In the sequel, we make the following assumptions.

(H1) The activation function $f_i(\cdot)$ is bounded, and satisfies Lipschitz condition, namely, there exist positive scalars $l_i (i = 1, 2, \ldots, n)$ such that $|f_i(\alpha) - f_i(\beta)| \leq l_i|\alpha - \beta|$, for any $\alpha, \beta \in R$. We denote $L = \mathrm{diag}(l_i, i = 1, \ldots, n)$.

(H2) The parameter uncertainties $\Delta C, \Delta A, \Delta B, \Delta W_0$, and ΔW_1 are of the form

$$\begin{aligned} &[\Delta C, \Delta A, \Delta B, \Delta W_0, \Delta W_1] \\ &\qquad = DF(t)\,[\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5] \end{aligned} \tag{5}$$

where $D, \Phi_i, i = 1, \ldots, 5$ are known real constant matrices with appropriate dimensions, and $F(t)$ is the time-varying uncertain matrix that satisfies

$$F^T(t)F(t) \leq I \tag{6}$$

where I denotes an identity matrix with appropriate dimension.

Remark 1: In (5), $F(t)$ is the uncertainty, and $D, \Phi_i, i = 1, \ldots, 5$ are known real matrices that characterize the structure of the uncertainty. There are several reasons for assuming that the system uncertainty has such structure. First, a linear interconnection of a nominal plant with the uncertainty $F(t)$ leads to a structure of the form (5) [52]. In addition, there are many physical systems in which the uncertainty can be modeled in this manner, e.g., systems satisfying "matching conditions" [53]. The parameter uncertainty structure as in (5) and (6) has been widely exploited in the uncertain neural networks (e.g., [6], [18], [20], [23], [26], [31], [33], [34] and the references therein).

It is known from stability theory of delay neural networks that the assumption (H1) implies the existence of equilibrium point of (3) (see [34]). Let $u^* = (u_1^*, \ldots, u_n^*)^T$ be an equilibrium point of (3). For the purpose of simplicity, we can shift the equilibrium u^* to the origin by the transformation $x(t) = u(t) - u^*$, which converts system (3) to the following:

$$\begin{aligned} dx(t) = {} & [-(C + \Delta C)x(t) + (A + \Delta \mathrm{A})g(x(t)) \\ & +(B + \Delta B)g(x(t - \tau))]dt \\ & +[\Delta W_0 x(t) + \Delta W_1 x(t - \tau)]dW(t). \end{aligned} \tag{7}$$

Under the following assumption on jump operator.

(H3) $J_k(u) = -\mathrm{diag}(\gamma_{k_1}, \ldots, \gamma_{k_n})[u(t) - u^*]$, which implies $J_k(u^*) \equiv 0$, (1) can be rewritten as the following form:

$$\begin{cases} dx(t) = [-(C + \Delta C)x(t) + (A + \Delta \mathrm{A})g(x(t)) \\ \quad +(B + \Delta B)g(x(t - \tau))]dt \\ \quad +[\Delta W_0 x(t) + \Delta W_1 x(t - \tau)]dW(t), \quad t \neq t_k \\ \Delta x(t_k) = -\mathrm{diag}(\gamma_{k_1}, \ldots, \gamma_{k_n})x(t_k^-), \quad t = t_k \end{cases} \tag{8}$$

where $g(x) = f(x + u^*) - f(u^*)$.

In the next section, we study the stochastic stability of (8). It is worth noting that the stability of equilibrium u^* of (1) is the same with that of the origin of (8).

To end this section, let us introduce the required definitions, lemmas, and notations.

Definition 1: For the neural network (1) and every $\varphi \in PC_{F_0}^b([-\tau, t_0], R^n)$, the trivial solution (equilibrium point u^*) is robustly, globally, and exponentially stable in the mean square if, for all admissible uncertainties satisfying (5) and (6), there exist positive constants $\alpha > 0$ and $u > 0$ such that every solution $u(t, t_0, \varphi)$ of (1) satisfies

$$\begin{aligned} & E\left[\left|u(t, t_0, \varphi) - u^*\right|^2\right] \\ & \qquad \leq u\left(\sup_{t_0 - \tau \leq s \leq t_0} E\left[|\varphi(s)|^2\right]\right) e^{-\alpha(t - t_0)}. \end{aligned}$$

Lemma 1 (Ito Formula [48]): Let $y(t)$ be an Ito process given by

$$dy(t) = udt + \upsilon dw(t).$$

Let $V(t, y) \in C^2[[0, +\infty) \times R, R]$ is an Ito process with second derivative respect to y, then

$$dV(t, y) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial y}dy + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(dy)^2$$

where $(dy)^2 = (dy) \cdot (dy)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dw(t) = dw(t) \cdot dt = 0, \;\; dw(t) \cdot dw(t) = dt.$$

Definition 2: The Dini's upper right-hand derivative of a continuous function $f : R \to R$ is defined by

$$D^+ f(t) \triangleq \limsup_{h \to 0+} \frac{f(t + h) - f(t)}{h}.$$

Lemma 2 (Halanay Inequality for Stochastic Systems): Under assumption (H1), let $x(t)$ be a solution of system (7) and assume that there exists a positive, continuous function $V(t, x(t))$ (for $t \geq t_0 - \tau$ and $x \in R^n$) for which there exist positive constants c_1, c_2, and $p > 1$, such that

$$c_1 |x(t)|^p \leq V(t, x(t)) \leq c_2 |x(t)|^p$$

and for some constants $0 \leq \beta < \alpha$

$$\begin{aligned} D^+ E(V(t, x(t))) \leq {} & -\alpha E(V(t, x(t))) \\ & +\beta E(V(t - \tau, x(t - \tau))). \end{aligned}$$

Then, for $t \geq t_0$

$$E\left(|x(t)|^p\right) \leq \frac{c_2}{c_1} E\left(\sup_{s \in [t_0 - \tau, t_0]} |\varphi(s)|^p\right) e^{-\upsilon(t - t_0)}$$

where $\upsilon \in (0, \alpha - \beta)$ is the unique positive solution of the equation $\upsilon = \alpha - \beta e^{\upsilon\tau}$.

Remark 2: This lemma is the generalization of Halanay inequality for delayed differential equations, which can be proved easily based on [49, Th. 7].

III. Main Results

Let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote, respectively, the minimal and maximal eigenvalues of a symmetric matrix P. We now state our main results that are used to establish criteria for exponential stability for the class of neural networks considered.

Theorem 1: Assume that (H1)–(H3) hold. If there exist positive diagonal matrices P, Q_1, Q_2 and positive scalars $\alpha, \beta(\beta < \alpha)$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2$, and σ such that the following conditions are satisfied:

1) $\Omega_1 + \alpha P \leq 0$;
2) $\Omega_2 - \beta P \leq 0$;
3) $D^T P D - \beta_1 I \leq 0$ and $(D^T P D)^2 - \beta_2 I \leq 0$;
4) $\tau \leq t_j - t_{j-1}, \ln \bar{d}_j / t_j - t_{j-1} \leq \sigma < \upsilon$, for any $j = 1, 2, \ldots$

where $\bar{d}_j = \max\{e^{\upsilon\tau}, \bar{\gamma}_j^2\}$ with $\bar{\gamma}_j = \max_{1 \leq i \leq n}\{|1 - \gamma_{ji}|\}$

$$\begin{aligned} \Omega_1 = {} & -PC - CP + PAQ_1^{-1}A^T P + LQ_1 L \\ & +PBQ_2^{-1}B^T P + \alpha_1^{-1} PDD^T P + \alpha_1 \Phi_1^T \Phi_1 \\ & +\alpha_2^{-1} PDD^T P + \alpha_2 \lambda_{\max}\left(\Phi_2^T \Phi_2\right) L^2 \\ & +\alpha_3^{-1} PDD^T P + \beta_1 \Phi_4^T \Phi_4 + \alpha_4 \Phi_4^T \Phi_4 \\ \Omega_2 = {} & LQ_2 L + \alpha_3 \lambda_{\max}\left(\Phi_3^T \Phi_3\right) L^2 + \alpha_4^{-1} \beta_2 \Phi_5^T \Phi_5 \end{aligned}$$

and $\upsilon \in (0, \alpha - \beta)$ is the unique positive solution of the equation $\upsilon = \alpha - \beta e^{\upsilon\tau}$. Then, the equilibrium u^* of system (1) is globally, robustly, and exponentially stable in the mean square for any admissible parameter uncertainty governed by (5) and (6).

Proof : Let us consider a Lyapunov function candidate for (8) as

$$V(x(t)) = x^T(t) P x(t) \tag{9}$$

where $P > 0$ is a positive diagonal matrix to be determined.

At the impulse moments, i.e., $t = t_k$, we have

$$\begin{aligned} V(x(t_k)) &= x^T(t_k)Px(t_k) \\ &= x^T(t_k^-)(I-R)^T P(I-R)x(t_k^-) \\ &\le \left[\max_{1\le j\le n}(1-\gamma_{kj})^2\right] V(x(t_k^-)). \end{aligned} \tag{10}$$

When $t \in [t_{k-1}, t_k)$, by applying Lemma 1 (Ito formula), the stochastic derivative of $V(x(t))$ along the solution of (8) can be obtained as follows:

$$\begin{aligned} dV(x(t)) = &\left\{ 2x^T(t)P\begin{bmatrix} -(C+\Delta C)x(t) + (A+\Delta A)g(x(t)) \\ +(B+\Delta B)g((x(t-\tau)) \end{bmatrix}\right. \\ &+ [\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]^T \\ &\left. \quad P\,[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)] \right\} dt \\ &+ 2x^T(t)P\,[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]\,dW(t) \\ = &\left\{ x^T(t)\,[-PC-CP]\,x(t) + 2x^T(t)PAg(x(t)) \right. \\ &+2x^T(t)PBg((x(t-\tau)) - 2x^T(t)P\Delta Cx(t) \\ &+2x^T(t)P\Delta Ag(x(t)) + 2x^T(t)P\Delta Bg(x(t-\tau)) \\ &+x^T(t)(\Delta W_0)^T P(\Delta W_0)x(t) \\ &+2x^T(t)(\Delta W_0)^T P(\Delta W)x(t-\tau) \\ &\left. +x^T(t-\tau)(\Delta W_1)^T P(\Delta W_1)x(t-\tau) \right\} dt \\ &+2x^T(t)\,[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]\,dW(t). \end{aligned} \tag{11}$$

Note that

$$\begin{aligned} &2x^T(t)\,PAg(x(t)) \\ &\le x^T(t)\,PAQ_1^{-1}A^TPx(t) + g^T(x(t))\,Q_1 g(x(t)) \\ &\le x^T(t)\,[PAQ_1^{-1}A^TP + LQ_1L]x(t) \end{aligned} \tag{12}$$

$$\begin{aligned} &2x^T(t)\,PBg(x(t-\tau)) \\ &\le x^T(t)\,PBQ_2^{-1}B^TPx(t) \\ &\quad +g^T(x(t-\tau))\,Q_2 g(x(t-\tau)) \\ &\le x^T(t)\,PBQ_2^{-1}B^TPx(t) \\ &\quad +x^T(t-\tau)\,LQ_2Lx(t-\tau) \end{aligned} \tag{13}$$

$$\begin{aligned} &-2x^T(t)\,P\Delta Cx(t) = -2x^T(t)\,PDF(t)\,\Phi_1 x(t) \\ &\le x^T(t)\,PD\alpha_1^{-1}D^TPx(t) \\ &\quad +\alpha_1 x^T(t)\,\Phi_1^T F^T(t)\,F(t)\,\Phi_1 x(t) \\ &\le x^T(t)\,[PDD^TP\alpha_1^{-1} + \alpha_1\Phi_1^T\Phi_1]x(t) \end{aligned} \tag{14}$$

$$\begin{aligned} &2x^T(t)\,P\Delta Ag(x(t)) = 2x^T(t)\,PDF(t)\,\Phi_2 g(x(t)) \\ &\le x^T(t)\,[\alpha_2^{-1}PDD^TP + \alpha_2\lambda_{\max}\left(\Phi_2^T\Phi_2\right)L^2]x(t) \end{aligned} \tag{15}$$

$$\begin{aligned} &2x^T(t)\,P\Delta Bg(x(t-\tau)) \le x^T(t)\,PD\alpha_3^{-1}D^TPx(t) \\ &\quad +\alpha_3 g^T(x(t-\tau))\,\Phi_3^T F^T(t)\,F(t)\,\Phi_3 g(x(t-\tau)) \\ &\le x^T(t)\,PD\alpha_3^{-1}D^TPx(t) \\ &\quad +\alpha_3\lambda_{\max}\left(\Phi_3^T\Phi_3\right)x^T(t-\tau)\,L^2x(t-\tau). \end{aligned} \tag{16}$$

Based on condition 3), one observes that the following inequalities hold:

$$\begin{aligned} &x^T(t)\,(\Delta W_0)^T\,P\,(\Delta W_0)\,x(t) \\ &\le x^T(t)\,[DF(t)\,\Phi_4]^T\,P\,[DF(t)\,\Phi_4]\,x(t) \\ &\le \beta_1 x^T(t)\,\Phi_4^T\Phi_4 x(t) \end{aligned} \tag{17}$$

$$\begin{aligned} &2x^T(t)(\Delta W_0)^T P(\Delta W_1)x(t-\tau) \\ &\le \alpha_4 x^T(t)\Phi_4^T F^T(t)F(t)\Phi_4 x(t) \\ &\quad +\alpha_4^{-1}x^T(t-\tau)\Phi_5^T F^T(t)D^TPDD^TPDF(t)\Phi_5 x(t-\tau) \\ &\le \alpha_4 x^T(t)\Phi_4^T\Phi_4 x(t) + \alpha_4^{-1}\beta_2 x^T(t-\tau)\Phi_5^T\Phi_5 x(t-\tau). \end{aligned} \tag{18}$$

and

$$\begin{aligned} &x^T(t-\tau)\Phi_5^T F^T(t)D^TPDF(t)\Phi_5 x(t-\tau) \\ &\quad \le \beta_1\cdot x^T(t-\tau)\Phi_5^T\Phi_5 x(t-\tau). \end{aligned} \tag{19}$$

Using (12)–(19), we obtain from (11) that

$$\begin{aligned} dV(x(t)) \le &\left\{ x^T(t)[-PC-CP+PAQ_1^{-1}A^TP+LQ_1L \right. \\ &+PBQ_2^{-1}B^TP + PDD^TP\alpha_1^{-1} \\ &+\alpha_1\Phi_1^T\Phi_1 + \alpha_2^{-1}PDD^TP \\ &+\alpha_2\lambda_{\max}(\Phi_2^T\Phi_2)L^2 + \alpha_3^{-1}PDD^TP + \beta_1\Phi_4^T\Phi_4 \\ &+\alpha_4\Phi_4^T\Phi_4]x(t) + x^T(t-\tau)\big[LQ_2L \\ &+\alpha_3\lambda_{\max}(\Phi_3^T\Phi_3)L^2 \\ &\left. +\alpha_4^{-1}\beta_2\Phi_5^T\Phi_5\big]x(t-\tau)\right\} dt \\ &+2x^T(t)[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]dW(t) \\ = &\left\{ x^T(t)\Omega_1 x(t) + x^T(t-\tau)\,\Omega_2 x(t-\tau)\right\} dt \\ &+2x^T(t)[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]dW(t). \end{aligned}$$

Therefore, based on conditions 1) and 2), we have

$$\begin{aligned} dV(x(t)) \le &[-\alpha V(x(t)) + \beta V(x(t-\tau))]dt \\ &+2x^T(t)[\Delta W_1 x(t) + \Delta W_1 x(t-\tau)]dW(t). \end{aligned} \tag{20}$$

Taking the mathematical expectation of both sides of (20), we have

$$\begin{aligned} &\frac{dE[V(x(t))]}{dt} \le -\alpha E[V(x(t))] + \beta E[V(x(t-\tau))] \\ &\quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots \end{aligned} \tag{21}$$

Note that

$$\lambda_{\min}(P)|x(t)|^2 \le V(x(t)) \le \lambda_{\max}(P)|x(t)|^2$$

which implies that

$$\lambda_{\min}(P)E\big[|x(t)|^2\big] \le E\big[V(x(t))\big] \le \lambda_{\max}(P)E\big[|x(t)|^2\big]. \tag{22}$$

Therefore, by Lemma 2, one observes, for $t \in [t_{k-1}, t_k)$

$$E\,[V(x(t))] \le E\left[\bar{V}(x(t_{k-1}))\right]e^{-\upsilon(t-t_{k-1})} \tag{23}$$

where $\bar{V}(x(t_{k-1})) = \sup_{t_{k-1}-\tau\le s\le t_{k-1}} V(x(s))$, and $\upsilon \in (0, \alpha-\beta)$ is the unique positive solution of the equation $\upsilon = \alpha - \beta e^{\upsilon t}$.

From (10), one observes that

$$E\,[V(x(t_k))] \le \bar{\gamma}_k^2 E\left[V(x(t_k^-))\right] \tag{24}$$

where $\bar{\gamma}_k = \max_{1\le j\le n}|1-\gamma_{kj}|$.

In the sequel, we shall prove that the following claim is true

$$E\left[V(x(t))\right] \le \left(\prod_{j=1}^{k-1} \bar{d}_j\right) E[\bar{V}(x(t_0))]e^{-\upsilon(t-t_0)}$$
$$t \in [t_{k-1}, t_k), k = 2, 3, \ldots \tag{25}$$

where $\bar{d}_j = \max\{\bar{\gamma}_j^2, e^{\upsilon\tau}\}$.

To show that claim (25) holds true, the mathematical induction method is used. When $t \in [t_0, t_1)$, we have from (23)

$$E[V(x(t))] \le E[\bar{V}(x(t_0))]e^{-\upsilon(t-t_0)}.$$

Therefore

$$\begin{aligned} E\left[V(x(t_1))\right] &\le \bar{\gamma}_1^2 E\left[V(x(t_1^-))\right] \\ &\le \bar{\gamma}_1^2 E[\bar{V}(x(t_0))]e^{-\upsilon(t_1-t_0)} \end{aligned} \tag{26}$$

and

$$\begin{aligned} E[\bar{V}(x(t_1))] &= \sup_{t_1-\tau\le s\le t_1} E[V(x(s))] \\ &= \max\left\{\sup_{t_1-\tau\le s< t_1} E[V(x(s))], E[V(x(t_1))]\right\} \\ &\le \max\left\{E\left[\bar{V}(x(t_0))\right]e^{-\upsilon(t_1-t_0)}e^{\upsilon\tau}\right. \\ &\qquad \left.\bar{\gamma}_1^2 E\left[V(x(t_0))\right]e^{-\upsilon(t_1-t_0)}\right\} \\ &= E\left[\bar{V}(x(t_0))\right]e^{-\upsilon(t_1-t_0)} \cdot \max\left\{e^{\upsilon\tau}, \bar{\gamma}_1^2\right\} \\ &= \bar{d}_1 E\left[\bar{V}(x(t_0))\right] \cdot e^{-\upsilon(t_1-t_0)} \end{aligned} \tag{27}$$

where $\bar{d}_1 = \max\{e^{\upsilon\tau}, \bar{\gamma}_1^2\}$.

For $t \in [t_1, t_2)$, i.e., $k = 2$, one observes from (23) and (27) that

$$\begin{aligned} E[V(x(t))] &\le E[\bar{V}(x(t_1))]e^{-\upsilon(t-t_1)} \\ &\le \bar{d}_1^2 E[\bar{V}(x(t_0))] \cdot e^{-\upsilon(t-t_0)}. \end{aligned} \tag{28}$$

This implies that claim (25) holds true for $k = 2$.

Let us assume that claim (25) also holds true for $k = 2, 3, \ldots, \bar{k}(\bar{k} \ge 2)$

$$E[V(x(t))] \le \left(\prod_{j=1}^{k-1} \bar{d}_j\right) E[\bar{V}(x(t_0))]e^{-\upsilon(t-t_0)} \tag{29}$$

for any $t \in [t_{k-1}, t_k), k = 2, 3, \ldots, \bar{k}$.

Then, we have

$$\begin{aligned} E[V(x(t_{\bar{k}}))] &\le \bar{\gamma}_{\bar{k}}^2 E[V(x(t_{\bar{k}}^-))] \\ &\le \bar{\gamma}_{\bar{k}}^2 E[\bar{V}(x(t_0))] \cdot \left(\prod_{j=1}^{\bar{k}-1} \bar{d}_j\right) e^{-\upsilon\left(t_{\bar{k}}-t_0\right)} \end{aligned} \tag{30}$$

and

$$\begin{aligned} E\left[\bar{V}(x(t_{\bar{k}}))\right] &= \sup_{t_{\bar{k}}-\tau\le s\le t_{\bar{k}}} E\left[V(x(s))\right] \\ &= \max\left\{\sup_{t_{\bar{k}}-\tau\le s< t_{\bar{k}}} E[V(x(s))], E[V(x(t_{\bar{k}}))]\right\} \\ &\le \max\left\{\left(\prod_{j=1}^{\bar{k}-1} \bar{d}_j\right) E\left[\bar{V}(x(t_0))\right]e^{-\upsilon\left(t_{\bar{k}}-t_0\right)}e^{\upsilon\tau}\right. \\ &\qquad \left.\bar{\gamma}_{\bar{k}}^2 E\left[\bar{V}(x(t_0))\right]\right\} \\ &\le \left(\prod_{j=1}^{\bar{k}-1} \bar{d}_j\right) E\left[\bar{V}(x(t_0))\right]e^{-\upsilon\left(t_{\bar{k}}-t_0\right)} \cdot \max\{e^{\upsilon\tau}, \bar{\gamma}_{\bar{k}}^2\} \\ &= \left(\prod_{j=1}^{\bar{k}} \bar{d}_j\right) E\left[\bar{V}(x(t_0))\right]e^{-\upsilon\left(t_{\bar{k}}-t_0\right)}. \end{aligned} \tag{31}$$

Therefore, from (23) and (31), one obtains that, for any $t \in [t_{\bar{k}}, t_{\bar{k}+1})$

$$\begin{aligned} E\left[V(x(t))\right] &\le E\left[\bar{V}(x(t_{\bar{k}}))\right]e^{-\upsilon\left(t-t_{\bar{k}}\right)} \\ &\le \left(\prod_{j=1}^{\bar{k}} \bar{d}_j\right) \cdot E[\bar{V}(x(t_0))]e^{-\upsilon(t-t_0)}. \end{aligned}$$

This implies that claim (25) holds true for $k = \bar{k} + 1$. Hence, by mathematical induction, claim (25) holds true for any $k = 2, 3, \ldots$

By (25) together with (22), one obtains

$$\begin{aligned} &E[|x(t)|^2] \\ &\quad \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} E[|\bar{\varphi}|^2]e^{-\upsilon(t-t_0)} \left(\prod_{j=1}^{k-1} \bar{d}_j\right), \quad t > t_0. \end{aligned} \tag{32}$$

By condition (iv) of the theorem, we also have

$$\begin{aligned} e^{-\upsilon(t-t_0)} \cdot \prod_{j=1}^{k-1} \bar{d}_j &= \exp\left\{-\upsilon(t-t_0) + \sum_{j=1}^{k-1} \ln\bar{d}_j\right\} \\ &= \exp\left\{-\upsilon\,(t-t_{k-1}) + \sum_{j=1}^{k-1} \ln\left[\bar{d}_j - \upsilon(t_j - t_{j-1})\right]\right\} \\ &\le \exp\left\{-(\upsilon-\sigma)(t-t_0)\right\}. \end{aligned}$$

This implies that

$$E\left[|x(t)|^2\right] \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} E\left[|\varphi|^2\right] e^{-(\upsilon-\sigma)(t-t_0)}, \quad t > t_0. \tag{33}$$

By Definition 1, the origin of system (8) is globally, robustly, and exponentially stable in mean square and, therefore, the equilibrium point $u^*$of (1) is globally, robustly, and exponentially stable in mean square. The proof is thus complete.

Remark 3: This theorem characterizes the hybrid effects of parameter uncertainties, stochastic perturbation, impulses, and time delay on the global exponential stability of (1). Conditions 1), 2), and 4) are to ensure the exponential stability with exponential convergence rate υ for the impulse-free stochastic neural network system (2). In condition 4), $\bar{d}_j$

depends on both the exponential stability property of impulse-free subsystem and impulse strength. Condition 4) combines the exponential convergence rate υ of impulse-free subsystem, impulse strength γ_{kj}, and impulse interval $t_j - t_{j-1}$ and characterizes the relationship among them.

Remark 4 (LMI Form): It is worth noting that conditions 1)–4) in this theorem can be formulated as a set of linear matrix inequalities (LMI) with respect to the variables $P, Q_1, Q_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, and β_1. Specifically, condition 4) itself is an LMI, condition 2) is identical to the LMI of the form (here, we take $\beta_2 = \beta_1^2$)

$$\begin{bmatrix} LQ_2L + \alpha_3\lambda_{\max}(\Phi_3^T\Phi_3)L^2 - \beta P & \beta_1\Phi_5^T \\ \beta_1\Phi_5^T & -\alpha_4 I \end{bmatrix} \le 0 \quad (34)$$

and condition 1) can be rewritten as the following LMI:

$$\begin{bmatrix} \Gamma_{11} & PA & PB & PD & PD & PD \\ & -Q_1 & & & & \\ & & -Q_2 & & & \\ & & & -\alpha_1 I & & \\ & & & & -\alpha_2 I & \\ & & & & & -\alpha_3 I \end{bmatrix} \le 0 \quad (35)$$

where

$$\Gamma_{11} = -PC - CP + LQ_1L + \alpha_1\Phi_1^T\Phi_1 + \alpha_2\lambda_{\max}\left(\Phi_2^T\Phi_2\right)L^2 + (\beta_1+\alpha_4)\,\Phi_4^T\Phi_4 + \alpha P.$$

Remark 5: From this theorem, we can obtain several stability criteria for the special case of (1). For example, conditions 1)–3) ensure the global, robust, and exponential stability for impulse-free (2).

Taking $\Phi_i = 0 (i = 1-5)$ in Theorem 1, Theorem 1 is suitable to determine the global and exponential stability of the following neural networks with impulse:

$$\begin{cases} \dot{u}(t) = -Cu(t) + Af(u(t)) + Bf(u(t-\tau)) + \xi . t \neq t_k \\ \Delta u(t_k) = J_k(u(t_k^-)), \quad t = t_k. \end{cases} \quad (36)$$

For computational simplicity, we derive the following corollaries based on Theorem 1.

Corollary 1: Assume that (H1)–(H3) hold true. If $\beta < \alpha$ and condition (iv) is satisfied, then the equilibrium is globally, robustly, and exponentially stable in the mean square for all admissible parameter uncertainties, where

$$\alpha = 2\underline{c} - \left[2\sqrt{a}\bar{l} + 2\sqrt{\eta_1 d} + 2\sqrt{\eta_2 d}\bar{l} + b + d + \eta_4 + d\eta_4\right]$$
$$\beta = \bar{l}^2(1+\eta_3) + d^2\eta_5$$

in which

$$\underline{c} = \min_{1\le i\le n}(c_i), \quad b = \lambda_{\max}(BB^T),$$
$$a = \lambda_{\max}(AA^T), \quad d = \lambda_{\max}(DD^T),$$
$$\eta_1 = \lambda_{\max}(\Phi_1^T\Phi_1), \quad \eta_2 = \lambda_{\max}(\Phi_2^T\Phi_2),$$
$$\eta_3 = \lambda_{\max}(\Phi_3^T\Phi_3), \quad \eta_4 = \lambda_{\max}(\Phi_4^T\Phi_4),$$
$$\eta_5 = \lambda_{\max}(\Phi_5^T\Phi_5), \quad \bar{l} = \max_{1\le i\le n}\{l_i\}.$$

Proof: Let $P = Q_2 = I, Q_1 = \bar{l}^{-1}\sqrt{a}I, \alpha_1 = \sqrt{d/\eta_1}, \alpha_2 = \bar{l}^{-1}\sqrt{d/\eta_2}, \quad \alpha_3 = \alpha_4 = 1$ and $\beta_1 = d$. Obviously, condition 4) is satisfied. It is also easy to show that conditions 1)–2) are both satisfied and, therefore, we omit it here.

Corollary 2: Assume that (H1)–(H3) hold. If $\beta < \alpha$, $t_k - t_{k-1} \ge T \ge \tau$ and $\upsilon - 1/T\ln\bar{d} > 0$, where $\bar{d} = \sup_{1\le j<\infty}\{\bar{d}_j\}, \upsilon \in (0, \beta-\alpha)$ is the unique positive solution of $\upsilon = \alpha - \beta e^{\upsilon\tau}$, and α, β are defined in Corollary 1. Then, the equilibrium u^* is globally, robustly, and exponentially stable in the mean square.

Proof: Since $t_k - t_{k-1} \ge T$, for $t \in [t_{k-1}, t_k)$, we have $t - t_0 \ge t_{k-1} - t_0 \ge (k-1)T$, which implies that $k - 1 \le 1/T(t - t_0)$.

Moreover

$$e^{-\upsilon(t-t_0)} \cdot \prod_{j=1}^{k-1}\bar{d}_j \le \exp\left\{-\upsilon(t-t_0) + (k-1)\ln\bar{d}\right\}$$
$$= \exp\left\{-\left(\upsilon - \frac{1}{T}\ln\bar{d}\right)(t-t_0)\right\}.$$

Similar to the proof of Theorem 1, one can conclude the proof of Corollary 2.

Remark 6: Corollary 2 allows us to estimate the admissible interval of impulse rate $\gamma_{ji} (i = 1, \ldots, n; j = 1, 2, \ldots)$ by estimating the upper bound of $\bar{d}$. In fact, since $\bar{d} < e^{\upsilon T}$, with

$$\bar{d} = \sup_j \bar{d}_j = \sup_j \max\left\{e^{\upsilon\tau}, \bar{\gamma}_j^2\right\}$$
$$= \sup_j \max\left\{e^{\upsilon\tau}, \max_{1\le i\le n}(1-\gamma_{ji})^2\right\}$$

one easily obtains that

$$1 - e^{\frac{1}{2}\upsilon T} \le \gamma_{ji} \le 1 + e^{\frac{1}{2}\upsilon T}. \quad (37)$$

IV. Numerical Examples

In order to show the effectiveness of the theoretical results, we present two numerical examples in this section.

Example 1: Consider the stochastic neural network model (1) with the following parameters:

$$\begin{cases} dx(t) = [-(C+\Delta C)x(t) \\ \quad +(B+\Delta B)f(x(t-\tau))]dt \\ \quad +[\Delta W_0 x(t) + \Delta W_1 x(t-\tau)]dW(t), \qquad t \neq t_k \\ \Delta x(t_k) = -\mathrm{diag}\left(\gamma_{k_1}, \gamma_{k_2}, \cdots, \gamma_{k_n}\right)x\left(t_k^-\right), \\ \qquad\qquad t = t_k, \quad k = 1, 2, \ldots \end{cases} \quad (38)$$

with

$$C = 4I, D = \mathrm{diag}(0.1, 0.5, 0.3), \Phi_1 = 0.6I,$$
$$\Phi_3 = 0.2I, \Phi_4 = \Phi_5 = 0.2I, F(t) = 0.6I$$
$$f_i(\alpha) = 0.5\tanh(\alpha), i = 1, 2, 3.$$
$$B = \begin{bmatrix} 0.3 & -0.8 & 0.5 \\ -0.1 & 0.6 & 0.1 \\ 0.6 & 0.4 & -0.3 \end{bmatrix}$$

where I denotes the identity matrix with appreciate dimension. By solving LMIs (34) and (35) and Condition 3) in Theorem 1,

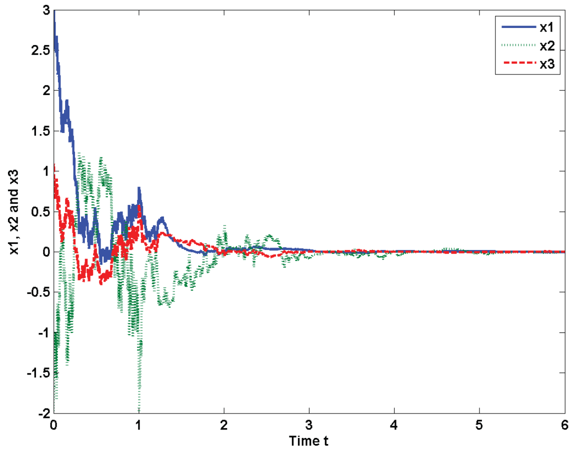

Fig. 1. Time–response curves of impulse-free version of (38) with initial function $x(\theta) = [3, -2, 1]^T$, $\theta \in [-1, 0]$.

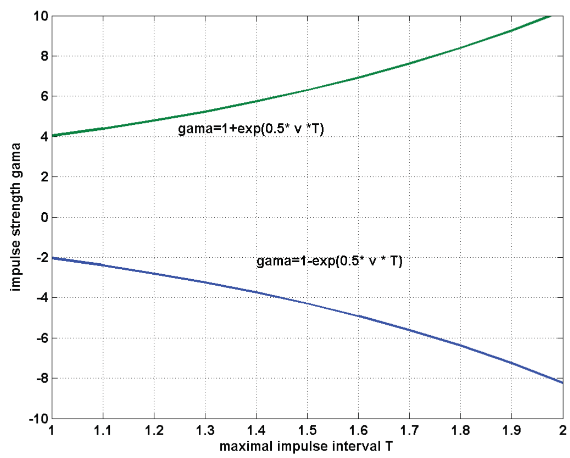

Fig. 2. Relationship between the impulse strength and impulse interval. The region between two curves is the feasible region of (T, γ_{ki}).

we obtain that $\alpha = 6.44$ and $\beta = 0.5$ when

$$P = \begin{bmatrix} 1.625969 & 0.274455 & 0.078403 \\ 0.274455 & 1.479031 & -0.153170 \\ 0.078403 & -0.153170 & 1.581636 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 1.189058 & 0.297209 & 0.113081 \\ 0.297209 & 0.880583 & -0.069861 \\ 0.113081 & -0.069861 & 1.182447 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1.664753 & 0.099701 & -0.067295 \\ 0.099701 & 2.214071 & -0.412016 \\ -0.067295 & -0.412016 & 1.502604 \end{bmatrix}$$

$$\alpha_1 = 1.06421021742691, \quad \alpha_2 = 1.94254018041615$$

$$\alpha_3 = 1.88286360430972, \quad \alpha_4 = 1.51954215574645$$

$$\beta_1 = 1.23064946440599.$$

From Theorem 1 and its corollaries, and Remark 6, we can obtain the following observations.

Observation 1: The origin of impulse-free version of (38) is globally robust and exponentially stable in mean square for any delay. The time–response curves of impulse-free model are shown in Fig. 1 where $\tau = 1$.

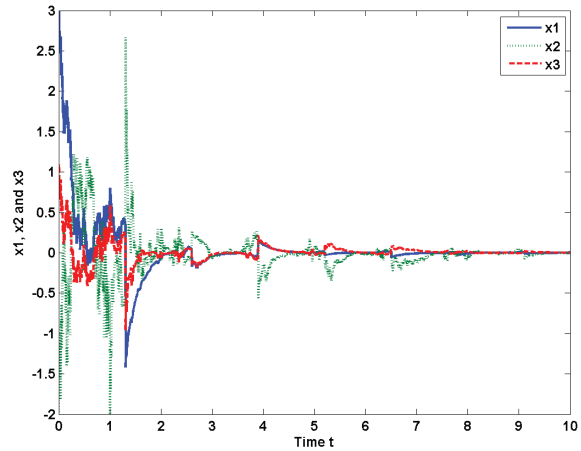

Fig. 3. Time–response curves of impulsive model (38) with $t_{j+1} - t_j \equiv T = 1.3$, $\tau = 1$, $\gamma_{ki} \equiv 5$ and initial function $x(\theta) = [3, -2, 1]^T$, $\theta \in [-1, 0]$.

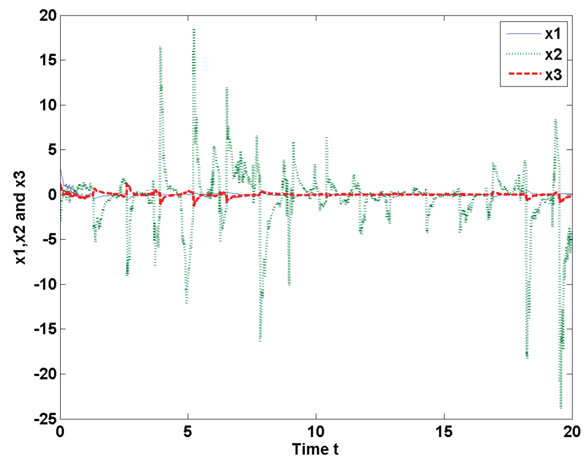

Fig. 4. Time–response curves of impulsive model (38) with $t_{j+1} - t_j \equiv T = 1.3$, $\tau = 1$, $\gamma_{ki} \equiv 10$ and initial function $x(\theta) = [3, -2, 1]^T$, $\theta \in [-1, 0]$. Large impulses lead to unstable dynamics.

Observation 2: Based on Remark 6, the feasible bounds of impulse strength depend on the maximal impulse interval T. For this example, we estimate that $\upsilon = 2.14954$. Therefore, by Remark 6, we plot the feasible region of (T, γ_{ki}), shown in Fig. 2. For any value of (T, γ_{ki}) in this region, the origin of impulse (38) is globally robust and exponentially stable in mean square. Fig. 3 presents the time–response curves of (38) with $t_{j+1} - t_j \equiv T = 1.3$, $\tau = 1$, and $\gamma_{ki} \equiv 5$, which are close to the upper limit of impulse stability region. As expected, the system response is exponentially stable. However, too large impulses will drive the system to unstable dynamics, as shown in Fig. 4.

Example 2: Consider the quadruple-tank process presented in Fig. 5. Assuming perfect flow and cylindrical tanks, the non-dimensional differential equations representing the mass balances in this quadruple-tank process are modified as follows [51]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + B_0 u(t - \tau_2) + B_1 u(t - \tau_3) \quad (39)$$

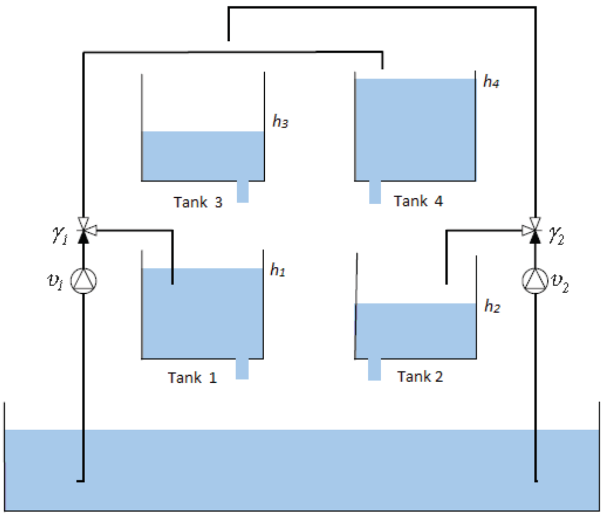

Fig. 5. Schematic representation of the quadruple-tank process (from [51]).

where

$$A_0 = \begin{bmatrix} -0.021 & 0 & 0 & 0 \\ 0 & -0.021 & 0 & 0 \\ 0 & 0 & -0.424 & 0 \\ 0 & 0 & 0 & -0.424 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.1113\gamma_1 & 0 & 0 & 0 \\ 0 & 0.1042\gamma_2 & 0 & 0 \end{bmatrix}^T$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0.1113\,(1-\gamma_1) \\ 0 & 0 & 0.1042\,(1-\gamma_2) & 0 \end{bmatrix}^T.$$

In the sequel, for simplicity, we assume that $\tau_1 = 0$, $\tau = 2$, $\gamma_1 = 0.333$, $\gamma_2 = 0.307$, and $u\,(t-\tau) = Kf\,(x\,(t-\tau))$ with

$$K = \begin{bmatrix} -0.1603 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}$$

and

$$f\,(x\,(t-\tau)) = [f_1\,(x_1\,(t-\tau))\,, \cdots, f_4\,(x_4\,(t-\tau))]^T$$
$$f_i\,(x_i) = 0.01\,[|x_i+1| - |x_i-1|]\,.$$

Thus, this system is further written as the form of

$$\dot{x}\,(t) = -\,(C+\Delta C)\,x\,(t) + Bf\,(x\,(t-\tau)) \tag{40}$$

where $C = -A_0$, $\Delta C = -A_1$, $B = (B_0 + B_1)\,K$. Moreover, it is assumed that this system is subject to impulsive perturbation described by

$$\Delta x\,(t_k) = -\gamma_4 x\left(t_k^-\right), t_k = 4k, \quad k = 1, 2, \ldots$$

Hence, the model of this quadruple-tank process is exactly the form of (1) without stochastic perturbation, namely

$$\begin{cases} \dot{x}\,(t) = -\,(C+\Delta C)\,x\,(t) + Bf\,(x\,(t-\tau))\,, t \neq 4k \\ \Delta x\,(t) = \gamma\,x\left(t^-\right), \quad t = 4k, \quad k = 1, 2, \ldots. \end{cases} \tag{41}$$

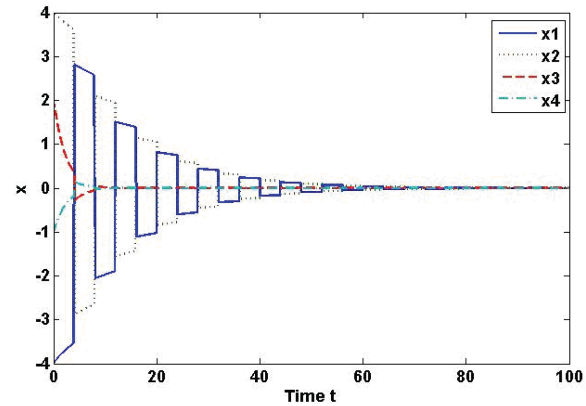

Fig. 6. Time–response curves of quadruple-tank systems with impulse effects.

Selecting $D = 0.1I$, $F\,(t) = I$, $\Phi_1 = -10 * A_1$, one observes that the conditions 1)–3) in Theorem 1 are feasible when $\alpha = 0.04$, $\beta = 0.0001$, which implies that $\upsilon = 0.0399$. Note that $t_j - t_{j-1} \equiv 4$. It is easy to show that condition 4) in Theorem 1 also holds provided that $-0.0831 < \gamma < 2.0831$. Namely, the considered system is globally and exponentially stable. Fig. 6 shows the trajectories of the system, from the initial condition $x\,(\theta) = [-4.04.06.0-5.0]^T$.

V. Conclusion

Different from most of the existing publications dealing with the stochastic stability problem of continuous-time delayed neural networks, this paper has derived new criteria on stochastic stability with the impulse effects. We have characterized the destabilizing effects of impulse and then estimated the feasible bound of impulses. The main result shows that when the impulse-free DNN with stochastic perturbation is globally and exponentially stable in mean square, the impulsive DNN with stochastic perturbation may preserve its stability property as long as the impulses are not beyond a specified bound.

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