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The Fuzzy Robust Graph Coloring Problem

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Abstract. Fuzzy graph model can represent a complex, imprecise and uncertain problem, where classical graph model may fail. In this paper, we propose a fuzzy graph model to represent the examination scheduling problem of a university and introduce a genetic algorithm based method to find the robust solution of the scheduling problem that remains feasible and optimal or close to optimal for all scenarios of the input data. Fuzzy graph coloring method is used to compute the minimum number of days to schedule the examination. But problem arises if after the examination schedule is published, some students choose new courses in such a way that it makes the schedule invalid. We call this problem as fuzzy robust coloring problem (FRCP). We find the expression for robustness and based on its value, robust solution of the examination schedule is obtained. The concept of fuzzy probability of fuzzy event is used in the expression of robustness, which in turn, is used for fitness evaluation in genetic algorithm. Each chromosome in the genetic algorithm, used for FRCP, represents a coloring function. The validity of the coloring function is checked keeping the number of colors fixed. Fuzzy graphs with different number of nodes are used to show the effectiveness of the proposed method.

Keywords: Fuzzy graph, fuzzy probability, fuzzy event, fuzzy graph coloring, robustness.

1 Introduction

The Graph coloring problem (GCP) are used for a wide variety of applications, such as frequency assignment, time-table scheduling, register allocation, bandwidth allocation, circuit board testing, etc. For graph coloring problem, the decision environment is often characterized by the following facts:

1. Uncertainty is essential to human being at all levels of their interaction with the real world. So, the data are uncertain / inexact.

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2. Two adjacent vertices, assigned different colors, must remain feasible for all meaningful realizations of the data.

3. Bad optimal solutions, i.e., optimal solutions which become severely infeasible in the case of even relatively small changes in the nominal data, are not uncommon. Those solutions are not robust.

Here, we propose a method for graph coloring that would be able to handle those facts, mentioned above, which basically represents an uncertain environment. In this paper, examination scheduling problem of a university is considered. Here courses are represented by the nodes of a graph and every pair of incompatible courses is connected by an edge. The coloring of this graph provides a feasible schedule of the courses and computes the minimum number of time slots which is equal to the chromatic number of the graph. Problem arises if after the examination schedule is published, some students choose a new course that makes the schedule invalid. This type of uncertainty exists in real world. Uncertainty of data comes from two different sources: randomness and incompleteness. Fuzzy graph model is used when uncertainty is there in the description of the objects or in its relationships or in both.

Let x^* be an optimal solution of an examination scheduling problem. But after scheduling, the input data changes. The change may be caused due to many reasons, e.g., new preliminaries are to be considered, data is updated or there may be some disturbances. In that new situation, the previous optimal solution x^* may not be feasible or if feasible, may not be optimal. A solution is considered as robust if it is still feasible and optimal or nearly optimal if changes of the input data occur within a prescribed uncertainty set. It protects the decision maker against parameter ambiguity and stochastic uncertainty.

There are few works on graph coloring problems with robustness. However, there is no work, as per our knowledge, that has incorporated robustness and fuzzy graph coloring in an integrated manner for scheduling problem. Works on robustness in scheduling problem can be found in [9], [8], [4], [3]. Some works related to fuzzy graph coloring are there in [1], [6], [7], [5]. A work, somewhat similar to us, can be found in [9]. However, the authors have used crisp graph to represent the model. In their robust graph coloring problem (RGCP) at most c colors can be assigned to all the vertices of the graph, where c is the given number of available colors. RGCP imposes a penalty for each pair of nonadjacent vertices if they are equally colored. Accordingly, the objective of the RGCP is to minimize the rigidity level of the coloring, which is defined as the sum of such penalties. They have proposed a genetic algorithm to find the robust solution for the problem.

In this paper, we propose a method to find the robust solution of an examination scheduling problem of a university in uncertain environment. We call this problem as Fuzzy robust coloring problem (FRCP). Probability of fuzzy event is used to bridge the crisp world (probability) and real world (fuzzy). We use a genetic algorithm for FRCP, where each chromosome represents an ordered arrangement of nodes without repetition and is checked its validity keeping the number of colors fixed. Here, we propose a fitness function for the chromosome based on the concept of fuzzy probability of a fuzzy event. Roulette wheel selection, two point crossover and one point mutation are used in the genetic algorithm.

The paper is organized as follows. Section 2 briefly reviews some basic definations associated with fuzzy graph coloring and probablity. In Section 3, we describe the examination scheduling problem by fuzzy graph model. We present an algorithm based on GA for FRCP in Section 4. The results of the proposed approach are presented in Section 5. Finally, we conclude in Section 6.

2 Preliminaries

In this section, we present some definitions related to fuzzy graph and fuzzy probability, which are associated with the proposed model.

Definition 1 [7]. Let V be a finite non empty set. A fuzzy graph G is defined by a pair of functions σ and μ, $G = (\sigma, \mu)$. σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ, i.e., σ: $V \longrightarrow [0, 1]$ and μ: $V \times V \longrightarrow [0, 1]$ such that

$$\mu\left(u, v\right) \leq \left(\sigma\left(u\right) \cap \sigma\left(v\right)\right), \forall u, \forall v \in V \tag{1}$$

A fuzzy graph is a generalization of crisp graph.

Definition 2 [6]. A fuzzy graph $G = (\sigma, \mu)$ is a complete fuzzy graph if

$$\mu\left(x, y\right) = \left(\sigma\left(u\right) \cap \sigma\left(v\right)\right), \forall u, \forall v \in \sigma \tag{2}$$

Definition 3 [7]. The complement of a fuzzy graph $G = (\sigma, \mu)$ is also a fuzzy graph and it is denoted as $\overline{G} = (\overline{\sigma}, \overline{\mu})$, where $\overline{\sigma} = \sigma$ and

$$\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v) \tag{3}$$

Definition 4 [10][11]. Let event $\widetilde{A}$ is a fuzzy event or a fuzzy set considered in the space $\Re^n$. All events in the space $\Re^n$ are mutually exclusive and the probability of each event is as follows.

$$P(\widetilde{A}) = \sum_{x \in \widetilde{A}} \mu_{\widetilde{A}}(x) P(x) \tag{4}$$

Here, $\mu_{\widetilde{A}}(x)$ is the membership function of the event $\widetilde{A}$.

Definition 5 [5]. A family $\mathit{\Gamma} = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ of fuzzy subsets on V is called a k- fuzzy coloring of $G = (\sigma, \mu)$ if

(a)$\gamma_1 \cup \gamma_2 \cup ... \cup \gamma_k = \sigma$

(b)$min\{\ \gamma_i(u), \gamma_j(u) \mid 1 \leq i \neq j \leq k\} = 0, \forall u \in V$

(c)For every strongly adjacent vertices u, v of G, $min\{\ \gamma_i(u), \gamma_i(v)\} = 0$ $(1 \leq i \leq k)$

Fuzzy chromatic number of fuzzy graph G is the least value of k for which the G has k-fuzzy coloring and is denoted by $\chi_F(G)$.

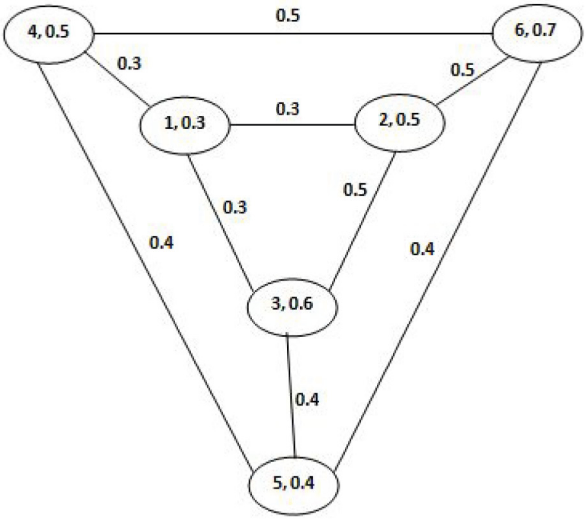

Fig. 1. Fuzzy graph model of an examination schedule

3 The Examination Scheduling Problem

Let us consider an example of examination scheduling problem, where the examination of 6 courses of a particular program at a university has to be scheduled. Two courses sharing at least one student cannot be scheduled on the same day. Taking into account the course incompatibilities, a fuzzy graph $G = (\mu, \sigma)$, shown in Fig 1, is constructed, where a node / vertex represents a course and an edge $\{i, j\}$ is included in the edge set when the courses i and j share at least one student. Here, σ and μ are respectively the membership values of the vertices and edges. Membership value of a node measures the degree to which an event occurs. Let the membership values of the vertices of the fuzzy graph are $\sigma(1)$ =0.3, $\sigma(2)$ = 0.5, $\sigma(3)$ = 0.6, $\sigma(4)$ = 0.5, $\sigma(5)$= 0.4 and $\sigma(6)$= 0.7. We use a complete fuzzy graph to represent the problem and find the membership values of the edges using (2).

In order to minimize the duration of examinations, a minimal coloring problem is considered. From the Definition 5, the fuzzy chromatic number of the fuzzy graph is computed as $\chi_F(G) = 3$ and the following coloring γ_1^0, in this sense, is optimal.

$$\gamma_1^0(u_1) = \begin{cases} 0.3 & i = 1 \\ 0.4 & i = 5 \\ 0, & \text{otherwise} \end{cases} \qquad \gamma_1^0(u_2) = \begin{cases} 0.5 & i = 2 \\ 0.5 & i = 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1^0(u_3) = \begin{cases} 0.6 & i = 3 \\ 0.7 & i = 6 \\ 0, & \text{otherwise.} \end{cases}$$

Table 1. Probability and membership values of edges of $\overline{G}$

$\{i,j\} \in \overline{E}$	pr_{ij}	$\mu_{i,j}$
1-5	0.0506	0.3
1-6	0.0127	0.3
2-4	0.4557	0.5
2-5	0.3038	0.4
3-4	0.1519	0.5
3-6	0.0253	0.6

We measure the probability of the fuzzy event [11], [10] and use it as robustness value of the model. The fuzzy probability that a complementary edge will be added to the graph is computed. Using (3), we also compute the membership values of all complementary edges of the fuzzy graph, shown in Table 1. The robustness of the coloring can be measured as the fuzzy probability of such coloring after one random complementary edge has been added to the edge set. We assume that the probability of the complementary edge $\{i,j\}$ be proportional to the number of students registered in courses i and j. Let $n = 50$ be the total number of students who has chosen two out of the six courses and n_i be the number of students registered in course i. In this example, we consider $n_1 = 5$, $n_2 = 30$, n_3=10, n_4=30, n_5=20 and $n_6 = 5$.

Given the coloring γ_1^0, the probability that the coloring remains valid, assuming statistical independence, can be computed as follows.

$$(1 - pr_{15} \cdot \mu_{1,5})\,(1 - pr_{24} \cdot \mu_{2,4})\,(1 - pr_{36} \cdot \mu_{3,6})=0.9477$$

Let us consider the following coloring γ_1^1,

$$\gamma_1^1(u_1) = \begin{cases} 0.3 & i = 1 \\ 0.7 & i = 6 \\ 0, & \text{otherwise} \end{cases} \qquad \gamma_1^1(u_2) = \begin{cases} 0.5 & i = 2 \\ 0.4 & i = 5 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1^1(u_3) = \begin{cases} 0.6 & i = 3 \\ 0.5 & i = 4 \\ 0, & \text{otherwise.} \end{cases}$$

Then the corresponding fuzzy probability is

$$(1 - pr_{16} \cdot \mu_{1,6})\,(1 - pr_{25} \cdot \mu_{2,5})\,(1 - pr_{34} \cdot \mu_{3,4})=0.80866$$

If the number of days to schedule the examination is not 3 as before, say 4, the fuzzy probability increases choosing the following coloring γ_1^2.

$$\gamma_1^2(u_1) = \begin{cases} 0.3 & i = 1 \\ 0.4 & i = 5 \\ 0, & \text{otherwise} \end{cases} \qquad \gamma_1^2(u_2) = \begin{cases} 0.6 & i = 3 \\ 0.7 & i = 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma_1^2(u_3) = \begin{cases} 0.6 & i = 2 \\ 0, & \text{otherwise} \end{cases} \quad \gamma_1^2(u_4) = \begin{cases} 0.5 & i = 4 \\ 0, & \text{otherwise} \end{cases}$$

The fuzzy probability that this coloring γ_1^2 remains valid is

$(1 - pr_{15} \cdot \mu_{1,5})\,(1 - pr_{36} \cdot \mu_{3,6})$=0.96987, which is more than the previous two colorings.

4 Proposed Genetic Algorithm for FRCP

Genetic Algorithm [2] is a probabilistic heuristic search process based on the concepts of natural genetics. Given an examination scheduling problem, proposed genetic algorithm finds the robust solution for this problem. Below we describe the chromosome encoding, fitness function, coloring function and genetic operators.

4.1 Encoding

Each node in the graph is represented by a unique integer value. A chromosome represents an order of all nodes without repetition. Thus, the chromosome length is same as the number of vertices in the graph. We have to find a coloring function for each chromosome and check its validity keeping the number of colors fixed.

4.2 Fitness Function

The value of the fitness function is a measure of the quality of a solution. More is the fitness, better the solution is. For FRCP, the fitness of i^{th} chromosome, given below in (5), is calculated using the penalty values and the membership values of the complimentary edges. $R(C_i)$ represents the rigidity level of the coloring function C of the i_{th} chromosome. Thus, it is a minimization problem.

$$R(C_i) = \sum_{\{i,j\} \in \overline{E}, C(i)=C(j)} p_{ij} \cdot \mu_{i,j} \tag{5}$$

Here, $\mu_{i,j}$ and p_{ij} are respectively the membership value and the penalty value of the edge (i, j). The penalty values for the complimentary edges and membership values of the vertices are generated randomly with uniform distribution in the interval [0, 1].

4.3 Greedy Algorithm

Given the adjacency matrix of a graph and an optimal order of nodes *optorder*, greedy algorithm, presented below in Algorithm 1, construct a coloring C:$\rightarrow$ $\{1, 2, ..., c\}$, such that for every edge (i, j), $C(i) \neq C(j)$.

Algorithm 1. Greedy Algorithm

begin
for i=1 to n **do**
    set color(i) = 0
end for
Set color($optorder(1)$) = 1
for j=2 to n **do**
    Choose a color k >0 for the node $optorder(j)$ that differs from its neighbor's colors
end for
end

Let us consider the chromosome 4 1 3 2 6 5, which is an ordered arrangement of the nodes for the fuzzy graph shown in Fig. 1. We take this order as the optimal order. First, node 4 is colored with color 1.The next node in the optimal order is node 1, which is adjacent to node 4, cannot be colored with color 1. So, node 1 is assigned color 2. Similarly, we traverse through the optimal ordering one by one and assign colors to each node so that every node has a color different from the colors of its neighboring nodes. We finally get a coloring C as $C(1) = 2$, $C(2) = 3$, $C(3) = 1$, $C(4) = 1$, $C(5) = 3$, and $C(6) = 2$.

4.4 Genetic Operators

We use Roulette wheel selection. A two point crossover is used in the proposed approach, which is described in Fig. 2. Two parents $P1$ and $P2$ are selected from the mating pool and the following operation is performed to obtain two children $C1$ and $C2$.

Step 1. Choose two random points $cp1$ and $cp2$ such that $cp1 < cp2$.
Step 2. Copy the nodes from $P2$ into $C1$ starting from position $cp1$ to the position $cp2$.
Step 3. Traverse the nodes of $P1$ one by one from 1 to n, if a node of $P1$ is not already present in $C1$ then copy it into $C1$ at the first available position from 1 to n.
Step 4. Copy the nodes from $P1$ into $C2$ starting from position $cp1$ to the position $cp2$.
Step 5. Traverse the nodes of $P2$ one by one from 1 to n, if a node of $P2$ is not already present in $C2$ then copy it into $C1$ at the first available position from 1 to n.

For mutation two points or positions are selected randomly from a chromosome and the nodes in those positions are exchanged. It is illustrated in Fig. 3.

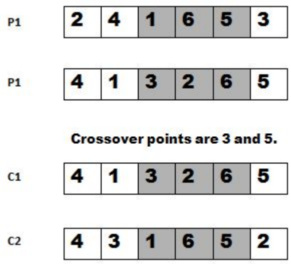

Fig. 2. Illustration of crossover operator

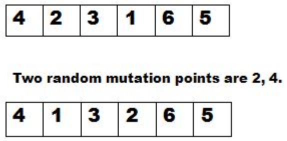

Fig. 3. Illustration of mutation operator

Table 2. Results with proposed GA and binary programming

no. of vertices	no. of colors	Rigidity (binary programming)	Rigidity (proposed)
6	3	0.69892	0.69892
6	4	0.50345	0.50263
8	4	1.4256	1.327433
10	3	1.9991562	1.999152
10	4	1.89543	1.900034
12	4	2.897612	2.897625
12	5	2.56734	2.6274512

5 Implementation and Results

In our experiment, number of generations, population size, crossover probability and mutation probability are respectively 30, 20, 0.6 and 0.1. The results of the proposed approach on a set of graphs with number of nodes 6, 8, 10 and 12 are shown in Table 2. The results for same set of graphs using binary programming with the same penalty and membership values are also shown in Table 2.

Our results are more or less similar to those of binary programming. It shows the effectiveness of the proposed approach on GA. Here, we consider binary programming as we did not find, best of our knowledge, any existing algorithm for robustness on fuzzy graph. The results show that for a particular graph when the number of colors increases the rigidity level of the coloring function decreases. The rigidity level is a measure of penalty when the coloring function becomes invalid on addition of any complimentary edge. The lower is the value of rigidity, the probability of the coloring function to be invalid is less.

6 Conclusions

In this work, we present the fuzzy robust graph coloring model that is simple, flexible, and easy to communicate to decision maker. With this model, we address the examination scheduling problem under uncertainty based on genetic algorithm, which produces robust solutions. Uncertainty in data comes from two different sources: randomness and vagueness. This model handles both type of uncertainty in decision making and bridges fuzzy graph coloring with robust optimization. Our model is capable of solving many real world optimization problems in uncertain environment.

References

1. Blue, M., Bush, B., Puckett, J.: Unified approach to fuzzy graph problems. Fuzzy Sets and Systems 125(3), 355–368 (2002)
2. Goldberg, D.E., et al.: Genetic algorithms in search, optimization, and machine learning, vol. 412. Addison-Wesley, Reading (1989)
3. Hasuike, T.: Robust shortest path problem based on a confidence interval in fuzzy bicriteria decision making. Information Sciences 221, 520–533 (2013)
4. Lim, A., Wang, F.: Robust graph coloring for uncertain supply chain management. In: Proceedings of the 38th Annual Hawaii International Conference on System Sciences, HICSS 2005, p. 81b. IEEE (2005)
5. Muñoz, S., Teresa Ortuño, M., Ramırez, J., Yáñez, J.: Coloring fuzzy graphs. Omega 33(3), 211–221 (2005)
6. Sunitha, M.: Studies on fuzzy graphs. Ph.D. thesis, Cochin University of Science and Technology (2001)
7. Sunitha, M., Vijayakumar, A.: Complement of a fuzzy graph. Indian Journal of Pure and Applied Mathematics 33(9), 1451–1464 (2002)
8. Wang, F., Xu, Z.: Metaheuristics for robust graph coloring. Journal of Heuristics 19(4), 529–548 (2013)
9. Yáñez, J., Ramırez, J.: The robust coloring problem. European Journal of Operational Research 148(3), 546–558 (2003)
10. Zadeh, L.A.: Fuzzy sets versus probability. Proceedings of the IEEE 68(3), 421 (1980)
11. Zadeh, L.A.: Probability measures of fuzzy events. Journal of Mathematical Analysis and Applications 23(2), 421–427 (1968)