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Synchronization of Memristor-Based Coupling Recurrent Neural Networks With Time-Varying Delays and Impulses

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***Abstract*— Synchronization of an array of linearly coupled memristor-based recurrent neural networks with impulses and time-varying delays is investigated in this brief. Based on the Lyapunov function method, an extended Halanay differential inequality and a new delay impulsive differential inequality, some sufficient conditions are derived, which depend on impulsive and coupling delays to guarantee the exponential synchronization of the memristor-based recurrent neural networks. Impulses with and without delay and time-varying delay are considered for modeling the coupled neural networks simultaneously, which renders more practical significance of our current research. Finally, numerical simulations are given to verify the effectiveness of the theoretical results.**

***Index Terms*— Impulse, memristor, recurrent neural networks, synchronization, time-varying delay.**

I. Introduction

In 1971, based on physical symmetry arguments, Chua [1] conceived and predicted that besides the resistor, capacitor, and inductor, there should be the fourth fundamental two-terminal circuit element called memristor, which is a contraction of memory resistor. Chua [1] also mathematically demonstrated that his hypothetical device would represent a relationship between flux and charge similar to what a nonlinear resistor provides between voltage and current. In 2008, Strukov *et al.* [2] proudly announced their realization of a memristor prototype. In their prototype, memristor is a two-terminal element with variable resistance called memristance, which depends on how much electric charge has been passed through in a particular direction. In other words, memristor has the distinctive ability to memorize the passed quantity of electric charge. Due to this feature, we can replace the resistor with memristor to build a new model of neural networks to emulate the human brain, we can also further apply memristor on the design of the next generation computer, such as the powerful brain-like neural computer [2]–[5].

In recent years, dynamic analysis of memristor-based recurrent neural networks has attracted increasing attention [6]–[12]. Guo *et al.* [6] investigated globally exponential dissipation and stabilization of memristor-based recurrent neural networks with time-varying delays. In [11], global exponential stability of a class of memristor-based recurrent neural networks with time-varying delays was studied by constructing proper Lyapunov functions and using the differential inclusion theory. In [12], exponential synchronization was investigated for memristor-based recurrent neural networks with time delays. However, a few authors have investigated the exponential synchronization of memristor-based coupling delayed recurrent neural networks with impulses.

Impulsive effects exist in neural networks. For instance, in the implementation of electronic networks, the state of the network is subject to instantaneous perturbations and experiences abrupt changes at certain instances, which may be caused by switching phenomenon, frequency change, or sudden noise. Although there are many results concerning the stability of impulsive systems, the synchronization problem of dynamical networks with impulsive effects has received little attention. Impulsive control is effective in dealing with dynamical systems [13]–[24]. However, impulsive controllers in [13]–[24] did not consider time-delay effects. Abrupt changes of the state in one subsystem cannot be received by its neighbors simultaneously, which implies that it is necessary to consider impulsive effects with time delays. In [25], impulse-induced exponential stability was studied for recurrent delayed neural networks. In [26], synchronization of TS fuzzy complex dynamical networks with time-varying impulsive delays and stochastic effects were investigated. To the best of our knowledge, a few published papers have considered the synchronization of memristor-based coupling recurrent neural networks (MRNNs) with delayed impulses. Motivated by the aforementioned discussions, we investigate the synchronization of MRNN with time-varying delayed impulses.

Notations: $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are used to denote the maximum and minimum eigenvalues of a real matrix. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix. Let $\mathbb{N}^+ = \{1, 2, \ldots\}$. The superscript T denotes the matrix or vector transposition. E_n is the $n \times n$ identity matrix and $\mathbf{1}_N$ denote the identity vector.

II. Problem Formulation and Preliminaries

In this section, we establish the mathematical model of the memristor-based recurrent neural networks, followed with some assumptions, definitions, and lemmas. According to [6] and [27], we provide a mathematical model of the memductance as follows:

$$\omega(v(t)) = \begin{cases} \omega', & v(s)\downarrow, \quad s \in (t-\delta, t] \\ \omega'', & v(s)\uparrow, \quad s \in (t-\delta, t] \\ \lim\limits_{s\to t^-} \omega(v(s)), & v(s)\to, \quad s \in (t-\delta, t] \end{cases} \tag{1}$$

where $\downarrow$ represents a decrease, $\uparrow$ represents an increase, $\to$ represents unchanges, and δ is a sufficiently small positive constant. $\lim_{s\to t^-}\omega(v(s))$ is either equal to ω' or ω'', which means that the memductance keeps the voltage value. Obviously, the memductance function may be discontinuous.

Remark 1: Based on the circuit design of memristor-based neural networks [29] and value trends of the memristor [30], it is obvious that the values of the connection weights of memristor-based neural networks can be described as the effect of the difference between the network output voltage and the voltage on the capacitor. As stated in [28], two memory states are required in digital computer applications, and here memristor also has two sufficiently distinct

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equilibrium states. Therefore, a more realistic mathematical description of the connection weights of memristor-based neural networks is proposed in this brief.

Consider the MRNN model with time-varying delays described as follows:

$$\begin{aligned}\frac{du_k(t)}{dt} &= -d_k u_k(t) + \sum_{l=1}^{n} a_{kl}(t) f_l(u_l(t))\\ &\quad + \sum_{l=1}^{n} b_{kl}(t) f_l(u_l(t-\tau_{kl}(t))) + I_k \end{aligned}\tag{2}$$

where

$$a_{kl}(t) = \begin{cases} a'_{kl}, & \varrho_{kl}(s)\downarrow, s\in(t-\delta,t]\\ a''_{kl}, & \varrho_{kl}(s)\uparrow, s\in(t-\delta,t]\\ \lim\limits_{s\to t^-} a_{kl}(\varrho_{kl}(s)), & \varrho_{kl}(s)\rightarrow, s\in(t-\delta,t]\end{cases}\tag{3}$$

$$b_{kl}(t) = \begin{cases} b'_{kl}, & \rho_{kl}(s)\downarrow, s\in(t-\delta,t]\\ b''_{kl}, & \rho_{kl}(s)\uparrow, s\in(t-\delta,t]\\ \lim\limits_{s\to t^-} b_{kl}(\rho_{kl}(s)), & \rho_{kl}(s)\rightarrow, s\in(t-\delta,t]\end{cases}\tag{4}$$

where $\varrho_{kl}(t) = f_l(u_l(t)) - u_k(t)$, $\rho_{kl}(t) = f_l(u_l(t-\tau_{kl}(t))) - u_k(t)$, u_k is the state of the kth neuron, and $f_l(\cdot)$ denotes the activation function. I_k denotes the input of the kth neuron and $\tau_{kl}(t)$, k, and $l = 1,2,\ldots,n$ are the transmission delays, which are bounded.

For convenience, the dynamical differential equations of MRNN matrix format are given by

$$\begin{aligned}\frac{du(t)}{dt} &= -Du(t) + A(u(t))f(u(t))\\ &\quad + B(u(t))f(u(t-\tau_1(t))) + \mathbf{I}\end{aligned}\tag{5}$$

where $\tau_1(t) = [\tau_{kl}(t)]_{n\times n}$, $0 \le \tau_1(t) \le \tau_1$, $u(t) = (u_1(t), u_2(t),\ldots,u_n(t))^T$ is the state vector, and $D = \mathrm{diag}(d_1,d_2,\ldots,d_n)$ is a real diagonal matrix, where $d_i > 0$, $i = 1,2,\ldots,n$ are the neuron self-inhibitions; $A(u(t)) = [a_{kl}(t)]_{n\times n}$ and $B(u(t)) = [b_{kl}(t)]_{n\times n}$ are the feedback and delayed feedback connection weight matrices, respectively and, $f(u(t)) = (f_1(u(t)),\ldots,f_n(u(t)))^T$ represents the neuron activation function. $\mathbf{I} = [I_1,I_2,\ldots,I_n]^T \in \mathbb{R}^n$ is an input or a bias vector. For convenience, we denote $t_{\tau_1} = t - \tau_1(t)$ and $t_{\bar{k}\tau_2} = t_{\bar{k}} - \tau_2(t_{\bar{k}})$.

According to [31] and [32], let $c_{ii} = -\sum_{j=1,j\neq i} c_{ij}$, an array of linearly coupled identical delayed MRNN with N identical networks can be described as

$$\begin{aligned}\frac{\mathrm{d}u_{ik}(t)}{dt} &= -d_k u_{ik}(t) + \sum_{l=1}^{n} a_{kl}(t) f_l(u_{il}(t)) + \sum_{l=1}^{n} b_{kl}(t)\\ &\quad \times f_l(u_{il}(t-\tau_{kl}(t))) + I_k + \alpha \sum_{j=1,j\neq i}^{N} c_{ij}\gamma_k u_{jk}(t).\end{aligned}\tag{6}$$

This can be written in matrix form as

$$\begin{aligned}\frac{\mathrm{d}u_i(t)}{dt} &= -Du_i(t) + A(u_i(t))f(u_i(t)) + B(u_i(t))\\ &\quad \times f(u_i(t_{\tau_1})) + \mathbf{I} + \alpha \sum_{j=1,j\neq i}^{N} c_{ij}\Gamma u_j(t)\end{aligned}\tag{7}$$

where $u_i(t) = (u_{i1}(t),u_{i2}(t),\ldots,u_{in}(t))^T$, $C = (c_{ij})_{N\times N} \in R^{N\times N}$ denotes the coupling configuration of the coupled network with $c_{ij} \ge 0$, $i\neq j$, $c_{ii} = -\sum_{j=1,j\neq i} c_{ij}$, $i,j = 1,2,\ldots,N$, and $\Gamma = \mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_n)$, and each differential equation has a unique solution with initial value

$$u_i(s) = \phi_i(s) \qquad \forall i = 1,2,\ldots,n,\ s\in[t_0-\tau,t_0]\tag{8}$$

where $\phi_i = (\phi_{i1},\phi_{i2},\ldots,\phi_{in})^T$.

For a constant vector $u_i^* = (u_{i1}^*,u_{i2}^*,\ldots,u_{in}^*)^T$, we have that $\tilde{f}_l(u_l^*) - u_k^*$ is a constant and its derivative is equal to 0. Then $a_{kl}(\tilde{f}_l(u_l^*) - u_k^*) = a_{kl}^0$ and $b_{kl}(\tilde{f}_l(u_l^*) - u_k^*) = b_{kl}^0$. Hence, we can present the following definition of equilibrium of (7).

Definition 1: A constant $u_i^* = (u_{i1}^*,u_{i2}^*,\ldots,u_{in}^*)^T$ is called an equilibrium of (6), if

$$-d_k u_{ik}^* + \sum_{l=1}^{n} a_{kl}^0 \tilde{f}(u_{il}^*) + \sum_{l=1}^{n} b_{kl}^0 \tilde{f}(u_{il}^*) + I_k = 0\tag{9}$$

for $k,l = 1,2,\ldots,n$.

By introducing the impulsive effects into system (7), one can obtain the following coupled MRNN with time-varying impulsive delay effects:

$$\begin{cases}\dfrac{du_i(t)}{dt} = -Du_i(t) + A(u_i(t))f(u_i(t)) + B(u_i(t))f(u_i(t_{\tau_1})) + \mathbf{I}\\ \qquad + \alpha \sum\limits_{j=1,j\neq i}^{N} c_{ij}\Gamma u_j(t), \quad t\neq t_{\bar{k}}\\ \Delta u_i(t_{\bar{k}}) = \varpi u_i(t_{\bar{k}}^-) + \mu u_i(t_{\bar{k}\tau_2}), \quad t = t_{\bar{k}},\ \bar{k}\in\mathbb{N}^+\end{cases}\tag{10}$$

where ϖ and μ are the impulsive strengths without and with delay, respectively. $\{t_1,t_2,\ldots\}$ is a sequence of strictly increasing impulsive instants satisfying $\lim_{k\to\infty} t_{\bar{k}} = +\infty$, $u(t_{\bar{k}}) = u(t_{\bar{k}}^+) = \lim_{t\to t_{\bar{k}}^+} u(t)$, $u(t_{\bar{k}}^-) = \lim_{t\to t_{\bar{k}}^-} u(t)$, $\Delta u_i(t_{\bar{k}}) = u_i(t_{\bar{k}}) - u_i(t_{\bar{k}}^-)$, and $0 \le \tau_2(t) \le \tau_2$.

For the activation function $f(\cdot)$, we have the following assumptions.

Assumption 1: The activation function $f(\cdot) = (f_1(\cdot),\ldots,f_n(\cdot))^T$ satisfies the Lipschitz condition, i.e., there exist positive constants l_i, and $\forall x,y \in \mathbb{R}$, such that $||f_i(x) - f_i(y)|| \le l_i||x-y||$, $i = 1,2,\ldots,n$.

Remark 2: Various kinds of frequently used activation functions satisfy Assumption 2. For example, the sigmoid activation functions $f_i(x) = \tanh(x)$ used in the Hopfield neural networks, and the piecewise linear activation functions $f_i(x) = 0.5(|x+1| - |x|)$ in cellular neural networks.

Definition 2: The coupled MRNN with impulsive effects is said to be exponential synchronized, if there exist $\eta > 0$, $t_0 > 0$, and $M_0 > 0$ such that for any initial values $\phi_i(s)$

$$||x_i(t) - x_j(t)|| \le M_0 e^{-\eta(t-t_0)}\tag{11}$$

holds for all $t \ge t_0$, and for any $i,j = 1,2,\ldots,N$, where M_0 and η are called the decay coefficient and decay rate, respectively.

III. Main Results

In the following, we shall discuss the globally exponential synchronization problem for MRNNs with impulsive delays.

To present concisely, here we let $u(t) = [u_1(t), u_2(t), \ldots, u_N(t)]^T$, $A^N(u(t)) = (E_N \otimes A(u(t)))$, $f(u(t)) = [f^T(x_1(t)),\ldots,f^T(u_N(t))]^T$, $B^N(u(t)) = (E_N \otimes B(u(t)))$, $D^N = (E_N\otimes D)$, $\mathbf{C} = (C\otimes\Gamma)$, $A^+ = [a_{kl}^+]_{n\times n}$, $B^+ = [b_{kl}^+]_{n\times n}$, $a_{kl}^+ = \max\{a'_{kl}, a''_{kl}\}$, $b_{kl}^+ = \max\{b'_{kl}, b''_{kl}\}$, $A^{N+} = (I_N\otimes A^+)$, $B^{N+} = (E_N\otimes B^+)$, and $\bar{\mathbf{I}} = \mathbf{1}_N\otimes\mathbf{I}$.

Then coupled neural networks (10) can be rewritten as

$$\begin{cases}\dfrac{du(t)}{dt} = -D^N u(t) + A^N(u(t))f(u(t))\\ \qquad + B^N(u(t))f(u(t_{\tau_1})) + \bar{\mathbf{I}} + \alpha\mathbf{C}u(t), \quad t\neq t_{\bar{k}}\\ u(t_{\bar{k}}^+) - u(t_{\bar{k}}^-) = \varpi u(t_{\bar{k}}^-) + \mu u(t_{\bar{k}\tau_2}),\ t = t_{\bar{k}},\ \bar{k}\in\mathbb{N}^+.\end{cases}\tag{12}$$

Theorem 1: Assume that Assumption 1 holds, and there exist a small enough number $\lambda > 0$, a positive define matrix P and diagonal positive definite matrices Q and R such for all $\bar{k} \in \mathbb{N}^+$, the following conditions are satisfied:

$$\Omega = \begin{bmatrix} \Pi & PA^{(N-1)+} & PB^{(N-1)+} & 0 \\ * & -Q & 0 & 0 \\ * & * & -R & 0 \\ * & * & * & \Xi \end{bmatrix} < 0 \tag{13}$$

$$\eta \geq 0, \quad \lambda_1 + \lambda_2 e^{\lambda\tau} \leq 1 \tag{14}$$

$$\left(\xi + \frac{\eta}{\lambda_1 + \lambda_2 e^{\lambda\tau}}\right)\left(t_{\bar{k}+1} - t_{\bar{k}}\right) < -\ln\left(\lambda_1 + \lambda_2 e^{\lambda\tau}\right) \tag{15}$$

where $\Pi = -PD^{N-1} - (D^{N-1})^T P + 2\alpha P\tilde{\mathbf{C}} + L^T QL - \xi P$, $\Xi = L^T RL - \eta P$; $\lambda_1 = [(1+w)^2 + (1+w)\mu\beta\lambda_{\max}(P)]$ and $\lambda_2 = (\mu^2 + (1+w)\mu)\lambda_{\min}(P)/\beta$. Then the MRNN with delayed impulses (12) is exponentially synchronized.

Proof: Let us construct a Lyapunov function of the form

$$V(t) = u^T(t)\mathbf{M}P\mathbf{M}u(t) \tag{16}$$

where $\mathbf{M} = M \otimes E$

$$M = \begin{bmatrix} 1 & -1 & & & \\ & & \ddots & & \\ & & & 1 & -1 \end{bmatrix}_{(N-1)\times N}. \tag{17}$$

For $t \neq t_{\bar{k}} (\bar{k} \in \mathbb{N}^+)$, by calculating the upper right-hand derivative of $V(t)$, we can obtain

$$\begin{aligned} D^+V(t) = \; & 2u^T(t)\mathbf{M}^T P\mathbf{M} \\ & \times\big[-D^N u(t) + A^N(u(t))f(x(t)) \\ & + B^N(u(t))f(u(t_{\tau_1})) + \alpha C^N u(t)\big]. \end{aligned} \tag{18}$$

By [32, Lemma 1], we have following:

$$\begin{aligned} \mathbf{M}D^N = D^{N-1}\mathbf{M}, \quad & \mathbf{M}A^N(u(t)) = A^{N-1}(u(t))\mathbf{M} \\ \mathbf{M}B^N(u(t)) = B^{N-1}(u(t))\mathbf{M}, \quad & \mathbf{M}\mathbf{C} = \tilde{\mathbf{C}}\mathbf{M} \end{aligned} \tag{19}$$

where $\tilde{\mathbf{C}} = \tilde{C} \otimes \Gamma$ and $\tilde{C} = MCJ$. Substituting (19) into (18), we have

$$\begin{aligned} D^+V(t) = \; & -2u^T(t)\mathbf{M}^T PD^{N-1}\mathbf{M}u(t) \\ & + 2u^T(t)\mathbf{M}^T PA^{N-1}(u(t))\mathbf{M}f(u(t)) \\ & + 2u^T(t)\mathbf{M}^T PB^{N-1}(u(t))\mathbf{M}f(u(t_{\tau_1})) \\ & + 2\alpha u^T(t)\mathbf{M}^T P\tilde{\mathbf{C}}\mathbf{M}u(t). \end{aligned} \tag{20}$$

In view of the Lipschitz condition, we have

$$\begin{aligned} & 2u^T(t)\mathbf{M}^T PA^{N-1}(u(t))\mathbf{M}f(u(t)) \\ & \leq 2u^T(t)\mathbf{M}^T PA^{(N-1)+}\mathbf{M}f(u(t)) \\ & \leq u^T(t)\mathbf{M}^T PA^{(N-1)+}Q^{-1}(A^{(N-1)+})^T P\mathbf{M}u(t) \\ & \quad + u^T(t)\mathbf{M}^T L^T QL\mathbf{M}u(t) \end{aligned} \tag{21}$$

and

$$\begin{aligned} & 2u^T(t)\mathbf{M}^T PB^{N-1}(u)\mathbf{M}f(u(t_{\tau_1})) \\ & \leq u^T(t)\mathbf{M}^T PB^{(N-1)+}R^{-1}(B^{(N-1)+})^T P\mathbf{M}u(t) \\ & \quad + u^T(t_{\tau_1})\mathbf{M}^T L^T RL\mathbf{M}u(t_{\tau_1}). \end{aligned} \tag{22}$$

By (18)–(22), we can obtain

$$\begin{aligned} & D^+V(t) \\ & \leq u^T(t)\mathbf{M}^T\big[-PD^{N-1} - (D^{N-1})^T P \\ & \qquad + 2\alpha P\tilde{\mathbf{C}} + PA^{(N-1)+}Q^{-1}(A^{(N-1)+})^T P \\ & \qquad + L^T QL + PB^{(N-1)+}R^{-1}(B^{(N-1)+})^T P - \xi P\big] \\ & \quad \times\mathbf{M}u(t) + u^T(t_{\tau_1})\mathbf{M}^T(L^T RL - \eta P)\mathbf{M}u(t_{\tau_1}) + \xi V(t) + \eta V(t_{\tau_1}) \\ & \leq \xi V(t) + \eta V(t_{\tau_1}). \end{aligned} \tag{23}$$

For any positive constant β, one obtains from the second equation (12) that

$$\begin{aligned} V(t_{\bar{k}}) = \; & (1+w)^2 u^T\big(t_{\bar{k}}^-\big)\mathbf{M}^T P\mathbf{M}u\big(t_{\bar{k}}^-\big) + \mu(1+w) \\ & \times\big[u^T\big(t_{\bar{k}}^-\big)\mathbf{M}^T P\mathbf{M}u(t_{\bar{k}\tau_2}) + u^T(t_{\bar{k}\tau_2})\mathbf{M}^T P\mathbf{M}u\big(t_{\bar{k}}^-\big)\big] \\ & + \mu^2 u^T(t_{\bar{k}\tau_2})\mathbf{M}^T P\mathbf{M}u(t_{\bar{k}\tau_2}) \\ \leq \; & \lambda_1 V\big(t_{\bar{k}}^-\big) + \lambda_2 V(t_{\bar{k}\tau_2}). \end{aligned} \tag{24}$$

Thus, in view of (23) and (24), all the conditions of [25, Lemma 1] are satisfied. This completes the proof of Theorem 1.

Because the resistors are used in the no-delay-feedback self-connection, here we assume that the connections between coupled identical networks are realized by resistors. Thus, $a_{kk}(t) = a_{kk}$ is a constant. We define the synchronization error system $e_i(t) = u_i(t) - y(t)$, where $y(t)$ is an isolated delayed neural network, as

$$\begin{aligned} \frac{dy(t)}{dt} = \; & -Dy(t) + A(y(t))f(y(t)) \\ & + B(y(t))f(y(t_{\tau_1})) + I + \Delta_i \end{aligned} \tag{25}$$

where $\Delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in})^T$ denotes the control input and maybe there exist an equilibrium point, a periodic orbit, or a chaotic attractor for $y(t)$.

Let $\Delta_i = R(e_i(t) - \mathrm{sign}(e_i(t)))$, the error dynamical system is governed as follows:

$$\begin{cases} \dfrac{de_i(t)}{dt} = -De_i(t) + A(u_i(t))g(e_i(t)) \\ \qquad + B(u_i(t))g(e_i(t_{\tau_1})) \\ \qquad + \alpha \displaystyle\sum_{j=1, j\neq i}^{N} c_{ij}\Gamma e_j(t) + \Theta + \Delta_i, \quad t \neq t_{\bar{k}} \\ e_i\big(t_{\bar{k}}^+\big) - e_i\big(t_{\bar{k}}^-\big) = we_i\big(t_{\bar{k}}^-\big) + \mu e_i\big(t_{\bar{k}\tau_2}\big), \quad t = t_{\bar{k}} \end{cases} \tag{26}$$

where $\bar{k} \in \mathbb{N}^+$, $g(e_i(t)) = f(e_i(t) + y(t)) - f(y(t))$, and $\Theta = (A(u_i(t)) - A(y(t)))f(y(t)) + (B(u_i(t)) - B(y(t)))f(y(t_{\tau_1}))$.

Assumption 2: The activation function $f(\cdot) = (f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot))^T$ satisfies the Lipschitz condition and is bounded, i.e., $\forall x, y \in \mathbb{R}$, such that $||f_i(x) - f_i(y)|| \leq l_i||x - y||$, $|f_i(x)| \leq h_i$, and $i = 1, 2, \ldots, n$.

Theorem 2: Suppose that Assumption 2 holds and there exist positive numbers r_k, $k = 1, \ldots, n$, such that the following inequalities hold:

$$d_{\bar{k}} = |w + 1| + |\mu| < 1 \tag{27}$$

$$p + \frac{1}{d_{\bar{k}}}q + \frac{\ln d_{\bar{k}}}{t_{\bar{k}+1} - t_{\bar{k}}} < 0 \tag{28}$$

$$r_k \geq \sum_{l=1}^{n} \Upsilon_{kl} \tag{29}$$

where $\bar{A}^{+} = (a_{kl}^{+})_{n\times n}$, $a_{kl}^{+} = a_{kk}$ only if $k = l$, otherwise $a_{kl}^{+} = \max\{|a_{kl}'|, |a_{kl}''|\}$, $\bar{B}^{+} = [b_{kl}^{+}]_{n\times n}$, $b_{kl}^{+} = \max\{|b_{kl}'|, |b_{kl}''|\}$, $p = \|R - D\| + \|\bar{A}^{+}L\|$, $q = \|\bar{B}^{+}L\|$, and $\Upsilon_{kl} = \sup_{s\in(t-\delta,t]}|a_{kl}'(s) - a_{kl}''(s)|h_k + \sup_{s\in(t-\delta,t]}|b_{kl}'(s) - b_{kl}''(s)|h_k$. Then the MRNN with time-varying impulsive delay (26) is globally exponentially stability.

Proof: Construct a Lyapunov function

$$V(t) = \sum_{i=1}^{N}\|e_i(t)\| \tag{30}$$

where $\|e_i(t)\| = \sum_{k=1}^{n}|e_{ik}(t)|$.

Differentiating $V(t)$ along the solution of (26) $t \le t_{\bar{k}}$, $\bar{k} \in \mathbb{N}^{+}$, one obtain that

$$\begin{aligned}
D^{+}V(t) \le{}& \sum_{i=1}^{N}\sum_{k=1}^{n}\mathrm{sign}(e_{ik}(t))\\
&\Bigg[-d_k e_{ik}(t) + r_k e_{ik}(t) + \sum_{l=1}^{n} b_{kl}(t)g_l(e_{il}(t_{\tau_1}))\\
&+\alpha\sum_{j=1,j\neq i}^{N} c_{ij}\gamma_k e_{jk}(t)\\
&-r_k\mathrm{sign}(e_{ik}(t)) + \sum_{l=1}^{n} a_{kl}(t)g_l(e_{il}(t))\Bigg]\\
&+\sum_{i=1}^{N}[\|(A(u_i(t)) - A(y(t)))f(y(t))\|\\
&\quad+\|(B(u_i(t)) - B(y(t)))f(y(t_{\tau_1}))\|]\\
\le{}& \sum_{i=1}^{N}\sum_{k=1}^{n}\Bigg[-d_k|e_{ik}(t)| + l_k\sum_{l=1}^{n}|a_{kl}(t)||e_{il}(t)|\\
&+l_k\sum_{l=1}^{n}|b_{kl}(t)||e_{il}(t_{\tau_1})| - r_k + r_k|e_{ik}(t)|\\
&+\alpha\sum_{j=1,j\neq i}^{N} c_{ij}\gamma_k|e_{jk}(t)|\Bigg]\\
&+\sum_{i=1}^{N}[\|(A(u_i(t)) - A(y(t)))f(y(t))\|\\
&\quad+\|(B(u_i(t)) - B(y(t)))f(y(t_{\tau_1}))\|].
\end{aligned} \tag{31}$$

From $a_{kl}(t)$ and $b_{kl}(t)$ are definition, we get

$$\begin{aligned}
\sum_{l=1}^{n}|a_{kl}(t)||e_{il}(t)| &= \left(\sum_{l=1,k\neq l}^{n} a_{kl}(t)|e_{il}(t)| + \sum_{l=k}^{n} a_{kk}|e_{il}(t)|\right)\\
&\le \sum_{l=1}^{n} a_{kl}^{+}|e_{il}(t)|
\end{aligned} \tag{32}$$

and

$$\sum_{l=1}^{n}|b_{kl}(t)||e_{il}(t_{\tau_1})| \le \sum_{l=1}^{n} b_{kl}^{+}|e_{il}(t_{\tau_1})|. \tag{33}$$

From the diffusive property of symmetric matrix C, one observes that

$$\begin{aligned}
\alpha&\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\Gamma\|e_j(t)\|\\
&= \alpha\sum_{k=1}^{n}\gamma_k\left[\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}|e_{jk}(t)|\right]\\
&= -\alpha\sum_{k=1}^{n}\gamma_k\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}\left|e_{ik}^{T}(t) - e_{jk}^{T}(t)\right| \le 0.
\end{aligned} \tag{34}$$

Because the activation function is bounded, we can obtain

$$\begin{aligned}
&\|(A(u_i(t)) - A(y(t)))f(y(t))\|\\
&\quad\le \sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{s\in(t-\delta,t]}\left|a_{kl}'(s) - a_{kl}''(s)\right||f_k(y_k(t))|\\
&\quad\le \sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{s\in(t-\delta,t]}\left|a_{kl}'(s) - a_{kl}''(s)\right|h_k
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\|(B(u_i(t)) - B(y(t)))f(y(t_{\tau_1}))\|\\
&\quad\le \sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{s\in(t-\delta,t]}\left|b_{kl}'(s) - b_{kl}''(s)\right|h_k.
\end{aligned} \tag{36}$$

Substituting (32)–(36) into (31), we have

$$\begin{aligned}
D^{+}V(t) \le{}& \sum_{i=1}^{N}\sum_{k=1}^{n}\Bigg[-d_k|e_{ik}(t)| + l_k\sum_{l=1}^{n} a_{kl}^{+}|e_{il}(t)|\\
&+l_k\sum_{l=1,k\neq l}^{n} b_{kl}^{+}|e_{il}(t_{\tau_1})| + r_k|e_{ik}(t)| - r_k\\
&+\sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{s\in(t-\delta,t]}\left|a_{kl}'(s) - a_{kl}''(s)\right|h_k\\
&+\sum_{k=1}^{n}\sum_{l=1}^{n}\sup_{s\in(t-\delta,t]}\left|b_{kl}'(s) - b_{kl}''(s)\right|h_k\Bigg]\\
\le{}& \sum_{i=1}^{N}\sum_{k=1}^{n}[-d_k + r_k]|e_{ik}(t)|\\
&+l_k\sum_{l=1}^{n} a_{kl}^{+}|e_{il}(t)| + l_k\sum_{l=1}^{n} b_{kl}^{+}|e_{il}(t_{\tau_1})|]\\
\le{}& pV(t) + qV(t_{\tau_1}).
\end{aligned} \tag{37}$$

On the other hand, from the construction of $V(t)$, we have

$$\begin{aligned}
V(t) &\le \sum_{i=1}^{N}|w+1|\left\|e_i\left(t_{\bar{k}}^{-}\right)\right\| + |\mu|\left\|e_i\left(t_{\bar{k}\tau_2}\right)\right\|\\
&= |w+1|V\left(t_{\bar{k}}^{-}\right) + |\mu|V\left(t_{\bar{k}\tau_2}\right).
\end{aligned} \tag{38}$$

By (37) and (38), it follows immediately that all the conditions of [26, Lemma 5] are satisfied. This completes the proof of Theorem 2.

IV. Numerical Examples

In order to verify the effectiveness of the theoretical results, we give two numerical examples. In order to facilitate, we set $\mathbf{I} = 0$.

TABLE I
EXAMPLE 1 PARAMETER SETUP

a_{11}	a_{12}	a_{21}	a_{22}	b_{11}	b_{12}	b_{21}	b_{22}
$-\frac{1}{6}$	-1	-1.5	-1.2	0.8	0.5	0.5	1.1
$\frac{1}{6}$	1	2	1	1	1.2	0.8	0.5

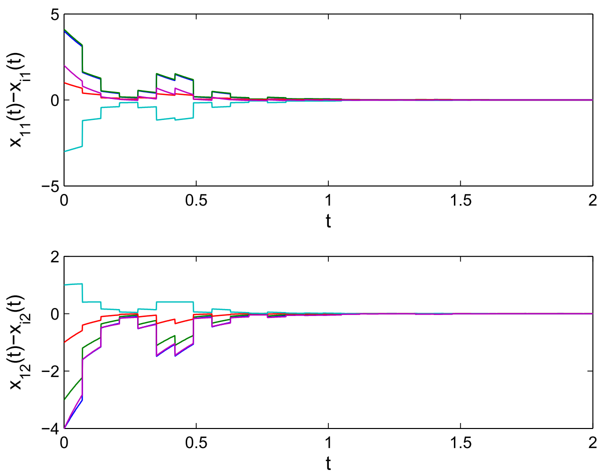

Fig. 1. Synchronization errors of $x_{11} - x_{i1}$ and $i = 2, \ldots, 6$, in Example 1.

We will consider the following coupled neural network models:

$$\frac{du_i(t)}{dt} = -Du_i(t) + A(u_i(t))f(u_i(t)) + B(u_i(t))f(u_i(t_{\tau_1})) + \alpha \sum_{j=1, j\neq i}^{N} c_{ij}\Gamma u_j(t). \tag{39}$$

Example 1: Consider a two-neuron MRNN model with six subsystems. The parameters are given in Table I.

Here, δ is a sufficiently small positive constant, $D = \text{diag}(3, 3)$, $\Gamma = \text{diag}(1, 1)$, $f(t) = (\tanh(u_{i1}(t)), \tanh(u_{i2}(t)))^T$, $\tau_1(t) = 0.25\sin(t) + 0.25$, and $\tau_2(t) = 0.2\cos(t) + 0.2$. The coupling matrix is given $c_{ij} = 0.5, i \neq j$.

We use the MATLAB LMI Control Toolbox to solve the LMIs in (13), and obtain the following feasible matrices:

$$P = \begin{pmatrix} 3.9536 & -0.1046 \\ -0.1046 & 3.8902 \end{pmatrix}, \quad Q = \begin{pmatrix} 13.5955 & 0 \\ 0 & 13.5955 \end{pmatrix}$$
$$R = \begin{pmatrix} 5.2354 & 0 \\ 0 & 5.2354 \end{pmatrix}.$$

Thus, condition (H1) of Theorem 1 is satisfied.

Let $w = -1$, $\mu = 0.4$, $\xi = 3$, $\eta = 2$, $\beta = 1$, $\alpha = 3$, and $\lambda = 0.001$. By simple computation, we obtain $\lambda_1 + \lambda_2 e^{\lambda\tau} = 0.1601 < 1$ and $\exp((\xi + \eta/(\lambda_1 + \lambda_2 e^{\lambda\tau}))(t_{\bar{k}+1} - t_{\bar{k}})) - (1/(\lambda_1 + \lambda_2 e^{\lambda\tau})) = -3.2887 < 0$. According to the condition of Theorem 1, it can be concluded that system (39) is exponential stability. Thus, MRNN with time-varying impulsive delay is globally exponential synchronization. Fig. 1 shows the synchronization errors of $x_{11} - x_{i1}$ and $x_{12} - x_{i2}$, respectively.

Example 2: In this example, we consider three-neuron MRNN model with six subsystems. The parameters are given in Table II.

Here, δ is a sufficiently small positive constant, $D = \text{diag}(10.5, 10.5, 10.5)$, $\Gamma = \text{diag}(1, 1, 1)$, $f(u) = (|u + 1| - |u - 1|)/2$,

TABLE II
EXAMPLE 2 PARAMETER SETUP

a_{12}	a_{13}	a_{21}	a_{23}	a_{31}	a_{32}
-1	-0.2	-1.5	-1	0.5	0.2
1	1.4	2	0.5	-1.8	-1.3
b_{12}	b_{13}	b_{21}	b_{23}	b_{31}	b_{32}
-0.5	-0.8	0.5	0.8	0.4	-0.9
1.2	0.6	0.8	0.1	-0.9	0.3
a_{11}	a_{22}	a_{33}	b_{11}	b_{22}	b_{33}
-1	-2	-2.5	-1.5	-1	-2

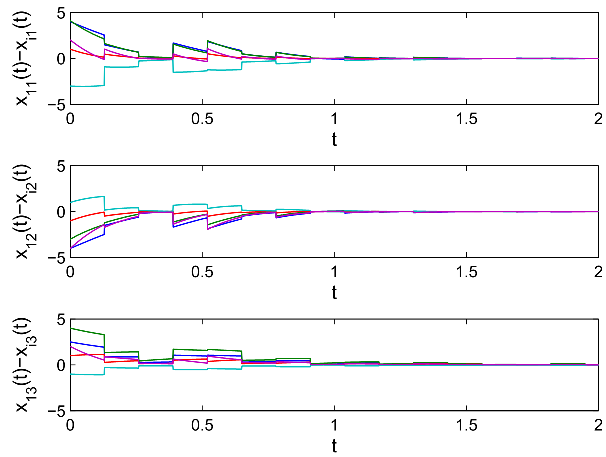

Fig. 2. Trajectories of system (39) with impulse.

$\tau_1(t) = 0.25\sin(t) + 0.25$, and $\tau_2(t) = 0.3\cos(t) + 0.3$. The coupling matrix is given as $c_{ij} = 1, i \neq j$.

Let $w = -1.2$ and $\mu = 0.5$. By solving (36) and (37), we can obtain $d_{\bar{k}} = 0.7 < 1$ and $p + (1/d_{\bar{k}})q + (\ln d_{\bar{k}}/t_{\bar{k}+1} - t_{\bar{k}}) = -0.2106 < 0$. According to Theorem 2, it can be concluded that system (39) is exponential synchronization. We select coupled neural networks consisting of six linearly coupled identical nodes. Fig. 2 shows the synchronization errors of $x_{11} - x_{i1}$, $x_{12} - x_{i2}$, and $x_{13} - x_{i3}$, respectively. Figs. 1 and 2 indicate that synchronization can be achieved.

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