Case Logic for Mathematical Psychology I

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Abstract It is shown how the end and the method of mathematical psychology lead us to the definition of a logical system called CL (case logic) by illustrative translations into the Japanese language, and certain of its sequential tautologies are presented for the study in progress of the deduction systems on it. CL is parameterized by a set N of *phases*, and CL with $\#\mathsf{N} = 1$ is called MPCL (monophasic case logic), while CL with $\#\mathsf{N} \geq 2$ is called PPCL (polyphasic case logic). The syntax of MPCL features the operation $\triangle$ modeled after copulas, the intensional operations $\Omega\mathsf{x}$ called the *nominalizers*, and the operations $\neg\mathfrak{p}\mathsf{k}$ and $\mathfrak{p}\mathsf{k}$ indexed by *cases* k and *positive quantifiers* $\mathfrak{p}$, which are subsets of an ordered commutative monoid $\mathbb{P}$ called a *quantity system*, while its semantics features a *basic relation* $\ni$ between the *entities* and a $\mathbb{P}$-valued *measure* $|\mathsf{X}|$ for the sets X of entities. PPCL in addition has the *modal operations* $\Lambda\mathfrak{r}$ indexed by the *modal quantifiers* $\mathfrak{r}$. The first-order predicate logic and modal logic are embedded in MPCL and PPCL respectively.

Keywords case logic, generalized quantifier, intensional operation, logical system, logical space, mathematical psychology, modality, semantics, syntax, tautology

1 From MP to CL

The purpose of this paper is to introduce a logical system CL (case logic) designed for MP (mathematical psychology), which I launched by an electronic publication Gomi (97) capable of frequent updating and will abridge by this paper and others[1].

1.1 Outlines of MP and CL

I presuppose that the reader is acquainted with Gomi (09b), where I proposed a general definition of logical systems with comments from the viewpoint of MP and explained the end of MP by the following paragraph.

"The end of MP is to comprehend human mind by analyzing a *comprehensive* mathematical model of the triple of the human system of cognizing and

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[1]It goes without saying that every researcher is indebted to predecessors. As for this paper, I avow in advance that I have been inspired by but departed from Montague (70) and N. Chomsky.

thinking, the outer worlds which humans cognize and think about, and the relationship between a human and the outer worlds. Logic is a tool for that, just as probability is a tool for genetics. The formal language A in MP is a model of the human system of cognizing, and the semantics of A consists of models of pairs of an outer world and a relationship between the world and the human system, while deduction systems on the sentences of A are models of the human system of thinking. Those human systems are unknowns in the brain, just as Mendel's factors (genes) were unknowns in organisms. For the time being, adequacy of the models can be examined solely by observations of natural languages which are supposed to be deformed expressions of the human systems, just as Mendel's theory was once examined solely by observations of phenotypes of pea plants which were supposed to be deformed expressions of their genotypes. Metaphorically speaking, MP seeks for a theory of genotypes by observations of phenotypes."

I refer to the human brain system of cognizing outer worlds[2] as ML (mental language) for convenience. To tell the truth, the model of ML is not A but its syntax in the sense of Gomi (09b), and A is a model of the totality of the 'ideas' which the ML can *theoretically* generates. Although I can not define what ideas are, they must be electro-chemical beings in our brain, just as factors are molecular beings on the DNA although Mendel could not define what they were. Thus the idea is not the same that philosophers refer to as an idea in Latin and English.

The logical system CL together with certain deduction systems on it studied by Gomi (09c), Mizumura (09) and Takaoka (09) is intended to be a model of the triple mentioned in the above excerpt. Therefore the syntax of CL is a model of ML. CL is parameterized by a set N of **phases**, and CL with $\#\mathsf{N} = 1$ is called MPCL (monophasic case logic), while CL with $\#\mathsf{N} \geq 2$ is called PPCL (polyphasic case logic).

The syntax of MPCL features the unary operation $\triangle$ modeled after copulas, the unary intensional operations $\Omega\mathsf{x}$ called the **nominalizers**, and the binary operations $\neg\mathfrak{p}\mathsf{k}$ and $\mathfrak{p}\mathsf{k}$ indexed by the **cases** k and the **positive quantifiers** $\mathfrak{p}$, which are the subsets of an ordered commutative monoid $\mathbb{P}$ called a **quantity system**, while its semantics features a **basic relation** $\ni$ between the **entities**[3] and a $\mathbb{P}$-valued **measure** $|\mathsf{X}|$ for the sets X of entities. In PPCL, these concepts are modified by phases, and there also exist the unary **modal operations** $\Lambda\mathfrak{r}$ indexed by the **modal quantifiers** $\mathfrak{r}$, certain of which become the usual modal operations $\Box_\mu$ and $\Diamond_\mu$ indexed by the **modal phases** μ.

The logical system FPL (first-order predicate logic) may be regarded as providing a model of a person who cognizes and thinks about only mathematical worlds. We can behave ourselves like him/her when devoted to mathematics. Thus MPCL has been designed to include FPL as its fragment (cf. Gomi (09d)). For a similar reason, PPCL has been designed to include modal logic as its fragment (cf. Gomi (09e))[4].

[2]For humans, even their inner organs and other beings are part of an outer world.
[3]For the purpose of MP, it is appropriate to assume that the basic relation is reflexive.
[4]It seems possible to extend PPCL to include intuitionistic logic as its fragment.

1.2 Methods of MP

As is usual with the realm of mathematical sciences, the method of MP is based on several non-mathematical suppositions, the first of which states in the above excerpt that natural languages are deformed expressions of the human brain systems. To be precise, (declarative) utterances are deformed expressions of the utterer's ideas in the sense of §1.1. To be more precise, notions carried by utterances may be identified with the utterer's ideas. Thus the syntax of A will emerge out of the classification of notions carried by utterances.

It is impossible to overemphasize the importance of the distinction between utterances and notions. For example, I suppose that the utterance "He ate many radishes at the garden" carries the notion "Yesterday Peter ate four or five radishes at Mr. McGregor's garden" or something like that as the case may be, because the utterer must know whom "he" indicates, when he ate radishes, how many radishes he ate, and what "the" indicates. Furthermore, while the past tense form of "ate" and the plural form of "radishes" are grammatically required, they are logically redundant, because "yesterday" already shows that it is a past event and "four or five" already shows plurality. There are numbers of similar examples. In general, utterances which are ambiguous to listeners who do not know the context are supposed to be deformed expressions of the utterer's notions, because the utterer's notions must be unambiguous to the utterer himself/herself. Thus classifying notions carried by utterances is tantamount to rephrasing utterances by first replacing ambiguous words by unambiguous ones, secondly supplying absent words or symbols, and thirdly removing redundant grammatical categories such as the pronouns, definite article, tense and plurals, and A should be identified with the totality of the so rephrased utterances.

In fact, more drastic rephrasing is necessary. As for quantifiers, I suppose that the utterance "Almost all boys love Mary" is a deformed expression of the notion "There are at most ten boys in the class, and at least eight boys love Mary" or something like that. Therefore I do not think that the logical system in MP need to be furnished with quantifiers meaning "almost all" or other words which show ratios between quantities. We only need a logical system by which we can express and analyze notions like "At least eight boys love Mary" and "There are at most ten boys." The same remark applies to words "many," "a few," and so on, as was suggested in the above paragraph.

As for adverbs, I suppose that the utterance "Peter runs fast" is a deformed expression of the notion "Peter runs as a rabbit runs" or something like that. Therefore we only need a logical system by which we can express and analyze notions like "Peter runs in the manner $\cdots$" and "the manner in which a rabbit runs." I do not think that the logical system in MP needs to be furnished with adverbs of manner. For certain reasons which I will not show here, the same remark applies to all kinds of adverbs.

Also, I suppose that as we so rephrase utterances, the differences between natural languages and between utterers will fade away into the differences in the word order and the lexicon, that is, a minimal set of expressions from which all other expressions are generated. The difference in the word order is considered to have resulted from the difference in the notation for operations. For instance, the images of a binary operation f may be denoted by fxy, xfy, xyf, fyx, and so on. The difference in the lexicon may be regarded as that in the set of constants. Both differences can be disregarded in discussing the general syntactic structure of A. In this sense, I suppose that the syntax of A is universal. Therefore mathematical psychicists may observe any language. Nevertheless, they should first observe their native language so as to grasp notions carried by utterances, because how to grasp notions is not only a matter of logic but also a matter of culture and so can not be universal, just as a phenotype is not only a matter of genotypes but also a matter of environment. Thus I have mainly observed the Japanese language. That turned out to be rather fortunate for me, as I will show below. It should be noted that I have attempted in Gomi (97) a theory of the deformation of notions into Japanese utterances. However, I do not dare to launch on a *universal* theory of the deformation, just as no one dares to launch on a universal theory of culture or environment.

1.3 Cases in CL and the Japanese language

The name of CL is derived from one of its parameters which is called the set of the **cases** (kaku in Japanese). They are models of some electro-chemical units which are supposed to exist in the human brain, and some of which are supposed to be expressed by some of the Japanese postpositional particles generally termed teniwoha such as “de,” “ni,” “wo,” “ga” and “wa.”

The term “case” may remind the reader of the case grammar originated by Fillmore (68) and developed by Cook (89) and others. A glance at the name of “case grammar” certainly inspired me to study the case logic and eventually create CL. I did not, however, need to learn the case grammar from any foreign literature, because the cases in the Japanese language are clearly indicated by teniwoha, as is perceived by Japanese people since the Kamakura period (1185?–1333), and they have been called case markers (kaku joshi in Japanese) since the Meiji period (1867–1912) (cf. Hashimoto (69)). For instance, “ga” is nominative; “wo” is accusative; “ni” is dative, accusative and also indicates occasions, locations, directions, and so on; “de” indicates scenes, means, and so on[5]; “wa” is a topic marker but also indicates various cases[6].

The following is a typical Japanese sentence meaning “Peter eats radishes in a garden.” In fact, the Japanese language is mostly written in kanji (Chinese

[5]In fact, “de” is a clipped form of the combination “ni te” of the case marker “ni” and the conjunctive postposition “te.” Furthermore, “ni te” is a clipped form of “ni ari te,” “ni oi te,” “ni yori te,” and so on, where “ari,” “oi” and “yori” are conjugations of the verbs “aru,” “oku” and ‘yoru” which mean “exist,” “put” and “depend on” respectively. Therefore “de” is not a genuine case marker, but a clipped form of expressions containing the genuine case marker “ni.” Even the postposition “ni” itself is often a clipped form of expressions containing the case marker “ni,” and therefore “ni” and “de” can indicate occasions, locations, directions, scenes, means, and so on.

[6]The postposition “wa” stresses the preceding word and therefore can work as a topic marker. When it succeeds teniwoha, it stresses the case of the preceding teniwoha. However, “ga-wa,” “wo-wa,” “ni-wa,” “de-wa,” are always or often abbreviated to “wa,” and then “wa” looks as if it were a case marker.

ideograms) and kana (Japanese phonograms).

pētā	ga	hatake	de	daikon	wo	taberu,	(1.1)
Peter		*garden*	*in*	*radish*		*eat.*	

Here and henceforth, a counterpart written by English words but in Japanese word order will be placed underneath in italics, with verbs in the basic forms and nouns in the singular forms without articles, because Japanese verbs conjugate neither by the person nor by the number and Japanese nouns have neither plural forms nor articles. The blanks as between *Peter* and *garden* show that the Japanese words over them have no English equivalents. Thus as for the three postpositions in (1.1), only "de" has an English equivalent "in," because the nominative case marker and accusative case marker in English are deep[7].

As (1.1) suggests, the Japanese language is locally a mirror image of English. It is mathematically trivial because every permutation is a composite of transpositions, and so what is important is how one is a mirror image of another. It will be clarified by comparing the English original with the lower row of (1.1) both with the case markers "ga" and "wo" supplied in boldfaces:

ga Peter eats **wo** radishes in a garden,
Peter **ga** *garden in* *radish* **wo** *eat.*

Both "ga" and "wo" in the upper row should be prepositions, because their functions are similar to that of the preposition "in." To the contrary in the lower row, "in" as well as "ga" and "wo" is a postposition.

1.4 What is an outer world?

How to finish the syntax of A depends on how to define its semantics and in particular its denotable worlds W, because the algebraic structure of A should be almost similar to that of W, as was defined from the viewpoint of MP in §3.2 of Gomi (09b). Also, denotable worlds W are models of the outer worlds. Thus we have to answer the ultimate question "What is an outer world?" in advance.

I suppose that each outer world consists of entities and affairs, where an entity is something that exists, concrete or abstract, material or immaterial, living or nonliving, physical or mental, and so on, and an affair is a relationship between entities. For instance, the sentence (1.1) describes an affair which is a relationship between the three entities Peter, radishes and a garden.

In terms of mathematics, the entities and affairs in an outer world may be regarded as the elements of some specified subsets E and F of a denotable world W respectively, and the affairs in the outer world may also be regarded as some partial multary relations on the entity set E, where a partial n-ary relation on E is a mapping f of some subset of E^n into $\mathbb{T} = \{0, 1\}$, and elements $a_1, \ldots, a_n$ of E satisfy the relation f if and only if $\mathsf{f}(a_1, \ldots, a_n) = 1$.

[7] The nominative and accusative case markers in archaic or literary style Japanese are also deep.

As for MPCL, the entity set E of each denotable world W will be defined as the union $\mathcal{P}\mathsf{S} \cup \mathsf{S}$ of a set S and its power set $\mathcal{P}\mathsf{S}$ (cf. §2.3), and S is called the base of W. If $\mathcal{P}\mathsf{S}' \cup \mathsf{S}' \subseteq \mathsf{S}$ for a set S', then the relation $\mathsf{X} \ni x$ between elements $\mathsf{X} \in \mathcal{P}\mathsf{S}'$ and $x \in \mathsf{S}'$ becomes a relation on S. Therefore I will assume from the outset that the base S of each denotable world W is equipped with a **basic relation** $\exists$ on it, and will extend it to the relation between E and S by defining "$\mathsf{X} \exists x \iff \mathsf{X} \ni x$" for each $\mathsf{X} \in \mathcal{P}\mathsf{S}$ and each $x \in \mathsf{S}$ (cf. (2.3)). Then we can also define the intersection $a \sqcap b$, union $a \sqcup b$ and complement $a^{\square}$ on E by means of the relation $\exists$ (cf. (W6) and (W7) in §2.3). Also, for each partial n-ary relation f on E and each $i \in \{1, \ldots, n\}$, the set $\{s \in \mathsf{S} \mid f(a_1, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_n) = 1\}$ belongs to E. Therefore the intensional operations Ωx on A can be defined as usual (cf. (2.6)). Thus E has an almost sufficient structure despite its simple definition. Since it is not quite sufficient, however, MPCL should be extended to CL.

Although the affair set F consists of partial multary relations on E, it seems careless to define F as such without regard to how the sentence (1.1) describes a ternary relation. In fact, a remarkable relationship between a sentence like (1.1) and a partial multary relation on E is suggested by the following probably well-known theorem (proof omitted), where $\mathsf{X} \to \mathsf{Y}$ for sets X and Y denotes the set of all mappings or functions of X into Y.

Theorem 1.1 *Let* $\mathsf{S}_1, \ldots, \mathsf{S}_n$ $(n \geq 0)$ *and* T *be sets, and define* $\mathsf{M} = (\mathsf{S}_1 \times \cdots \times \mathsf{S}_n) \to \mathsf{T}$ *and* $\mathsf{M}^* = \mathsf{S}_1 \to (\mathsf{S}_2 \to (\cdots \to (\mathsf{S}_n \to \mathsf{T}) \cdots))$. *Then for each* $f \in \mathsf{M}$, *there exists a unique element* $f^* \in \mathsf{M}^*$ *which satisfies* $f(s_1, \ldots, s_n) = (\cdots((f^* s_1) s_2) \cdots) s_n$ *for each* $(s_1, \ldots, s_n) \in \mathsf{S}_1 \times \cdots \times \mathsf{S}_n$, *and the mapping* $f \mapsto f^*$ *is a bijection of* M *onto* M^* (f^* *will be called the* ***linearization*** *of* f).

This theorem first suggests decomposing each function $f \in \mathsf{M}$ by each n-tuple $(s_1, \ldots, s_n) \in \mathsf{S}_1 \times \cdots \times \mathsf{S}_n$ into the series

$$f^*,\ f^* s_1,\ (f^* s_1) s_2,\ \ldots,\ (\cdots((f^* s_1) s_2) \cdots) s_n$$

of $n+1$ functions, each of which is a linearization of a certain function.

Notice that mathematical notation follows the word order of Indo-European languages. In Japanese word order, $f(s_1, \ldots, s_n)$ is denoted by $(s_1, \ldots, s_n) f$, Theorem 1.1 is stated on the relationship between $\mathsf{T} \leftarrow (\mathsf{S}_1 \times \cdots \times \mathsf{S}_n)$ and $((\cdots(\mathsf{T} \leftarrow \mathsf{S}_1) \cdots) \leftarrow \mathsf{S}_{n-1}) \leftarrow \mathsf{S}_n$, and f is decomposed into the series f^*, $s_n f^*$, $s_{n-1}(s_n f^*)$, $\ldots$, $s_1(s_2(\cdots(s_n f^*) \cdots))$.

Notice also that the numerical subscripts are not intrinsic to functions. For example, each ternary function f is decomposed by each triple (p, h, d) into the series f^*, $d f^*$, $h(d f^*)$, $p(h(d f^*))$ of four functions. So it seems appropriate to accompany p, h and d with the numbers $1, 2$ and 3 respectively in order to indicate their position in the triple (p, h, d). Then the functions $d f^*$, $h(d f^*)$ and $p(h(d f^*))$ are denoted also by $d3f^*$, $h2(d3f^*)$ and $p1(h2(d3f^*))$ respectively. This suggests regarding the numbers $1, 2$ and 3 as binary operation symbols.

Notice also that we need not use the numbers $1, 2$ and 3. We may use any symbols instead. Also, we need not use Roman letters p, h, d and f^*. We may use any symbols instead. So let us replace 1, 2 and 3 by Japanese case markers "ga," "de" and "wo," and replace $\mathsf{p}, \mathsf{h}, \mathsf{d}$ and f^* by Japanese words "pētā," "hatake," "daikon" and "taberu." Then the functions f^*, $\mathsf{d}3\mathsf{f}^*$, $\mathsf{h}2(\mathsf{d}3\mathsf{f}^*)$ and $\mathsf{p}1(\mathsf{h}2(\mathsf{d}3\mathsf{f}^*))$ are denoted also by the four Japanese predicates (verbal phrases[8]) on Table 1.1, the fourth of which is equal to the sentence (1.1).

Table 1.1: Functions and Predicates

f^*							taberu
counterpart							*eat*
meaning							eats
$\mathsf{d}3\mathsf{f}^*$					daikon	wo	taberu
counterpart					*radish*		*eat*
meaning					eats radishes		
$\mathsf{h}2(\mathsf{d}3\mathsf{f}^*)$			hatake	de	daikon	wo	taberu
counterpart			*garden*	*in*	*radish*		*eat*
meaning			eats radishes in a garden				
$\mathsf{p}1(\mathsf{h}2(\mathsf{d}3\mathsf{f}^*))$	pētā	ga	hatake	de	daikon	wo	taberu
counterpart	*Peter*		*garden*	*in*	*radish*		*eat*
meaning	Peter eats radishes in a garden						

Thus these predicates may be regarded as alternative expressions of the functions f^*, $\mathsf{d}\mathsf{f}^*$, $\mathsf{h}(\mathsf{d}\mathsf{f}^*)$ and $\mathsf{p}(\mathsf{h}(\mathsf{d}\mathsf{f}^*))$ respectively. In particular, the sentence (1.1) may be regarded as a linearized expression of the value $(\mathsf{p}, \mathsf{h}, \mathsf{d})\mathsf{f}$. This finally suggests that every predicate is a linearized expression of a function on an outer world and that the case markers indicate the positions of the arguments of those functions and also are binary operation symbols.

Nevertheless, we have to half abandon the linearization. This is because Japanese word order is loose except for the rigid constraint that the main verb in a predicate must come at its end. For instance, the sentence (1.1) carries the same notion as the sentence

hatake	de	pētā	ga	daikon	wo	taberu,
garden	*in*	*Peter*		*radish*		*eat,*

although their emphases are different. Therefore we wish to be able to decompose f into the series f^*, $\mathsf{d}3\mathsf{f}^*$, $\mathsf{p}1(\mathsf{d}3\mathsf{f}^*)$, $\mathsf{h}2(\mathsf{p}1(\mathsf{d}3\mathsf{f}^*))$ as well as into the series f^*, $\mathsf{d}3\mathsf{f}^*$, $\mathsf{h}2(\mathsf{d}3\mathsf{f}^*)$, $\mathsf{p}1(\mathsf{h}2(\mathsf{d}3\mathsf{f}^*))$. The rigid linearization of Theorem 1.1 does not work for such looseness, and the following theorem (cf. Theorem 3.5) supplies the right loose linearization.

Theorem 1.2 *Let* $\mathsf{S}, \mathsf{T}, \mathsf{K}$ *be sets. For each triple* $(\mathsf{s}, \mathsf{k}, \mathsf{f})$ *of* $\mathsf{s} \in \mathsf{S}$, $\mathsf{k} \in \mathsf{K}$ *and* $\mathsf{f} \in (\mathsf{P} \to \mathsf{S}) \to \mathsf{T}$ *with* $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, *define the element* $\mathsf{skf} \in \big((\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}\big) \to \mathsf{T}$

[8] In fact, there are three more types of predicates in the Japanese language; predicate adjective phrases, predicate nominal adjective phrases and nominals accompanied by copulas.

by $(skf)\theta = f\big((k/s)\theta\big)$ *for each* $\theta \in (P-\{k\})\to S$*, where* $(k/s)\theta$ *is the element of* $P\to S$ *defined by* $\big((k/s)\theta\big)l = s$ *if* $l = k$ *and by* $\big((k/s)\theta\big)l = \theta l$ *if* $l \in P-\{k\}$*. Let* $f \in S^n\to T$ *with* $n \le \#K$ *and let* $k_1,\dots,k_n$ *be distinct elements of* K*. Let* $P = \{k_1,\dots,k_n\}$ *and define* $f^* \in (P\to S)\to T$ *by* $f^*\theta = f(\theta k_1,\dots,\theta k_n)$ *for each* $\theta \in P\to S$*. Then* $f(s_1,\dots,s_n) = s_{\rho 1}k_{\rho 1}\,(s_{\rho 2}k_{\rho 2}\,(\cdots(s_{\rho n}k_{\rho n}f^*)\cdots))$ *for each* $(s_1,\dots,s_n) \in S^n$ *and each permutation* ρ *of the letters* $1,\dots,n$*.*

In view of this theorem, the affair set F of the denotable world W in MPCL will be defined as $\bigcup_{P\in\mathcal{P}K}\big((P\to S)\to\mathbb{T}\big)$ for the set K of the cases (cf. §2.3).

1.5 The tale of quantifiers

The quantifiers of CL are quite different in both syntax and semantics from those of FPL, as may be explained by the following tale.

Once upon a time, mathematicians had to deal with the values $a, b \in \mathbb{T}$ defined for a function $f \in S\to\mathbb{T}$ on a set S as follows:

$$\begin{aligned} a = 1 &\iff f(x) = 1 \text{ for all } x \in S,\\ b = 1 &\iff f(x) = 1 \text{ for some } x \in S. \end{aligned}$$

The designer of FPL simply denoted a and b by $\forall f$ and $\exists f$ respectively, and thus obtained the quantifiers $\forall$ and $\exists$. Meanwhile the designer of CL first redefined a and b by using an $\mathbb{R}_{\ge 0}$-valued measure $|X|$ for each $X \in \mathcal{P}S$ such that $|X| = 0$ if and only if $X = \emptyset$ (cf. §2.1). Then it was clear that the following held.

$$\begin{aligned} a = 1 &\iff \big|\{x \in S \mid f(x) = 0\}\big| \in \{0\},\\ b = 1 &\iff \big|\{x \in S \mid f(x) = 1\}\big| \in (0,\infty). \end{aligned}$$

He next generalized this by replacing S by an arbitrary set $X \in \mathcal{P}S$:

$$\begin{aligned} a_X = 1 &\iff \big|\{x \in X \mid f(x) = 0\}\big| \in \{0\},\\ b_X = 1 &\iff \big|\{x \in X \mid f(x) = 1\}\big| \in (0,\infty). \end{aligned}$$

He next denoted a_X and b_X by $X\neg\{0\}f$ and $X(0,\infty)f$ respectively. Then

$$\begin{aligned} X\neg\{0\}f = 1 &\iff \big|\{x \in X \mid f(x) = 0\}\big| \in \{0\},\\ X(0,\infty)f = 1 &\iff \big|\{x \in X \mid f(x) = 1\}\big| \in (0,\infty). \end{aligned}$$

He finally generalized this by replacing $\{0\}$ and $(0,\infty)$ by arbitrary subsets $\mathfrak{p}$ of $\mathbb{R}_{\ge 0}$ such as the intervals $[0,p]$ and (p,∞) for all $p \in \mathbb{R}_{\ge 0}$:

$$\begin{aligned} X\neg\mathfrak{p}f = 1 &\iff \big|\{x \in X \mid f(x) = 0\}\big| \in \mathfrak{p},\\ X\mathfrak{p}f = 1 &\iff \big|\{x \in X \mid f(x) = 1\}\big| \in \mathfrak{p}. \end{aligned}$$

Thus he obtained quantifiers $\neg\mathfrak{p}$ and $\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{P}(\mathbb{R}_{\ge 0})$. Since he sought for a model of the human brain system and not of natural languages, there was no reason for him to hesitate at adopting these quantifiers which appeared unusual from the viewpoint of linguistics. Furthermore, for all $f \in S^2 \to \mathbb{T}$, he defined elements $X\neg\mathfrak{p}1f$, $X\mathfrak{p}1f$, $X\neg\mathfrak{p}2f$ and $X\mathfrak{p}2f$ of $S \to \mathbb{T}$ by

$$\begin{aligned}
(X\neg\mathfrak{p}1f)y = 1 \iff& \big|\{x \in X \mid f(x,y) = 0\}\big| \in \mathfrak{p},\\
(X\mathfrak{p}1f)y = 1 \iff& \big|\{x \in X \mid f(x,y) = 1\}\big| \in \mathfrak{p},\\
(X\neg\mathfrak{p}2f)y = 1 \iff& \big|\{x \in X \mid f(y,x) = 0\}\big| \in \mathfrak{p},\\
(X\mathfrak{p}2f)y = 1 \iff& \big|\{x \in X \mid f(y,x) = 1\}\big| \in \mathfrak{p}
\end{aligned}$$

for each $y \in S$. Continuing this way, he finally defined binary operations $\neg\mathfrak{p}k$ and $\mathfrak{p}k$ with $\mathfrak{p} \in \mathcal{P}(\mathbb{R}_{\geq 0})$ and $k \in \mathbb{N}$ on the set $\mathcal{P}S \cup \bigcup_{n=1}^{\infty}(S^n \to \mathbb{T})$. The end.

1.6 Quantifiers modify cases!

To tell the truth, the quantifiers of MPCL are $\neg\mathfrak{p}$ and $\mathfrak{p}$ for all subsets $\mathfrak{p}$ of an ordered commutative monoid $\mathbb{P}$ called a **quantity system**, and a $\mathbb{P}$-valued **measure** $|X|$ is used. Furthermore, $\neg\mathfrak{p}$ and $\mathfrak{p}$ are always accompanied by cases k, and not only the formal language but also the denotable worlds of MPCL have binary operations $\neg\mathfrak{p}k$ and $\mathfrak{p}k$ (§2.3 (W2)). This is in contrast to FPL, where the quantifiers $\forall$ and $\exists$ are mere symbols, which are always accompanied by variables x, and only the formal language has operations $\forall x$ and $\exists x$.

More generally, CL is equipped with a family $(\mathbb{P}_\nu)_{\nu \in N}$ of quantity systems indexed by the set N of phases, because the denotable worlds contain models of various kinds of entities in the outer worlds such as individuals, matters, manners and times, and accordingly various kinds of measures are needed. For the same reason, the set K of the cases has a family $(K_\mu)_{\mu \in N'}$ of subsets, and some quantifiers have the form $\neg\mathfrak{p}$ or $\mathfrak{p}$ with $\mathfrak{p} \in \mathcal{P}(\mathbb{P}_\mu)$ for $\mu \in N'$ and are accompanied by the cases $k \in K_\mu$.

The above syntax of the quantifiers of CL is supported by an observation of the Japanese language. The following is an extract from *The Tale of Peter Rabbit* by Beatrix Potter.

First he ate some lettuces
and some French beans;
and then he ate some radishes.

The following is its Japanese translation by Momoko Ishii. As before, English counterparts in italics are placed underneath each of the three lines.

sore-kara mazu retasu wo nanmai-ka tabe,
and first lettuce some ate,
sore-kara saya-ingen wo tabe,
and French bean ate,
sore-kara hatsuka-daikon wo nanbon-ka tabe-masita.
and radish some ate.

Ishii was one of Japanese outstanding authors of children's books and was the 2002 nominee for the Hans Christian Andersen Awards from Japan, which shows among other things that her translations are well elaborated orthodox Japanese.

Notice that the quantifier "some" in the English original precedes "lettuces," "French beans" and "radishes" and so seems to modify them, while its Japanese equivalents "nanmai-ka" and "nanbon-ka" do not adjoin any of the equivalents of "lettuces," "French beans" and "radishes" but succeeds the case marker "wo" and so do not seem to modify the equivalents. Moreover, "nanmai-ka" and "nanbon-ka" may be grammatically put also before "wo" although orthodox authors never do so, and they do not succeed but precede case markers other than "ga" and "wo". Therefore neither "nanmai-ka" nor "nanbon-ka" modifies the verb "tabe"[9]. Thus the only reasonable conclusion seems to be that the quantifiers modify the case markers, not the nouns. This tallies with the tale in §1.5, where quantifiers $\neg\mathfrak{p}$ and $\mathfrak{p}$ were accompanied by the numbers $1, 2, \ldots$ which indicated the positions of the arguments of a function, as the case markers "ga," "de" and "wo" on Table 1.1 indicated the positions of the arguments of functions. I would like to ask native English speakers whether English quantifiers are also considered to modify (deep) case markers as in

(ga) *one* rabbit ate (wo) *two* radishes in *three* gardens[10].

1.7 Start for the study of deduction systems

I feel I have exhausted what I can tell without defining CL, so I will proceed to the definition of MPCL. It has an obvious defect that it can not deal with notions like "Peter was a student last year" involved with the copula "be" modified by the words "last year" which indicate an occasion. Certain other defects will be mentioned at the end of §2.6. They will be removed by generalizing MPCL to CL in Part II. The reason why I precede CL with MPCL is that familiarity with MPCL will facilitate understanding CL which is sophisticated.

The definition of a logical system in MP is not a goal but a start, for instance, for the study of deduction systems on it, and much has already been done on deduction systems on CL. Mizumura (00, 09) has presented sequential completeness theorems for antecedents of MPCL and PPCL. Using Mizumura's ideas, I have proved a sentential completeness theorem for MPCL in Gomi (09c). Takaoka (09) has taken an alternative comprehensive approach to the theorem and related problems. It will probably apply also to PPCL. Meanwhile, a general incompleteness theorem has been obtained in Gomi (97) for the logical systems in the sense of Gomi (09b). Hopefully, these results together will lead to an incompleteness theorem for CL like that of Gödel (31).

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[9] "ikutsu-ka" is another equivalent of "some" and the expression "pētā ga hatake ikutsu-ka ni yuku" means "Peter goes to some gardens," while "pētā ga hatake ni ikutsu-ka yuku" is peculiar, and "ikutsu-ka" in the former expression does not seem to modify "yuku."

[10] In the expressions like "nanbon-ka no hatsuka-daikon," the noun "hatsuka-daikon" is considered to be modified by the combination "nanbon-ka no" of the quantifier "nanmai-ka" and the particle "no." Such expressions are grammatical but foreign-sounding and are supposed to have come from a kind of mistranslations of the European languages based on the understanding that "some" modifies nouns, because orthodox Japanese novelists such as Naoya Shiga seldom use such expressions, while Tatsuo Hori who is said to be Europeanized often uses such expressions for instance in his novel "Mugiwara Bōshi" (Straw Hats).

2 Definition of MPCL

Following §3 of Gomi (09b), here we define the logical system MPCL. Since MPCL is monophasic, the concept of phases does not appear here.

2.1 Quantities and measures

A **quantity system** is a set $\mathbb{P}$ equipped with a total binary associative and commutative operation $(p, q) \mapsto p + q$ with the identity element 0 and an order $\leq$ which satisfy the following two conditions.

- If elements $p, p', q, q' \in \mathbb{P}$ satisfy $p \leq p'$ and $q \leq q'$, then $p + q \leq p' + q'$.
- $0 \leq p$ for every element $p \in \mathbb{P}$, that is, $0 = \min \mathbb{P}$.

The quantity system $\mathbb{P}$ is said to be **linear** if the order $\leq$ is linear[11], or total. The set $\mathbb{R}_{\geq 0}$ of non-negative real numbers is a linear quantity system. The power set $\mathcal{P}O$ of an arbitrary set O with $\#O \geq 2$ is a non-linear quantity system with respect to the operation $\cup$ with the identity element $\emptyset$ and the order $\subseteq$.

Among properties of quantity systems, the following found by Mizumura (00) has proved to be valuable also in Gomi (09c) and Takaoka (09).

Theorem 2.1 *Let $\mathbb{P}$ be a linear quantity system and $\mathbb{Q}$ be a finitely generated subalgebra of $\mathbb{P}$ with respect to the operation $+$ of $\mathbb{P}$. Then $\mathbb{Q}$ is well-ordered with respect to the order $\leq$ of $\mathbb{P}$.*

Proof We let $q_1, \ldots, q_k$ be generators of $\mathbb{Q}$ and argue by induction on k. We may assume $q_1 \geq q_j$ $(j = 2, \ldots, k)$. If $k = 1$, then there exists an increasing mapping $n \mapsto nq_1$ of $\mathbb{N}$ onto $\mathbb{Q}$, and so $\mathbb{Q}$ is well-ordered. Therefore assume $k > 1$, and let $\mathbb{Q}'$ be the subalgebra of $\mathbb{P}$ generated by $q_2, \ldots, q_k$. Then $\mathbb{Q}'$ is well-ordered by the induction hypothesis. We only need to show that every downward closed interval $(\leftarrow r] = \{q \in \mathbb{Q} \mid q \leq r\}$ of $\mathbb{Q}$ is well-ordered. There exist non-negative integers $n_1, \ldots, n_k$ such that $r = n_1 q_1 + \cdots + n_k q_k$. Define $n_0 = n_1 + \cdots + n_k$. Then $r \leq n_0 q_1$. Let $q \in (\leftarrow r]$. Then $q = nq_1 + q'$ for some non-negative integer n and $q' \in \mathbb{Q}'$. We may assume $n \leq n_0$, because if $n_0 < n$, then $r \leq n_0 q_1 \leq nq_1 \leq nq_1 + q' = q$, so $q = r = n_1 q_1 + \cdots + n_k q_k$ and $n_1 \leq n_0$. Therefore, defining $\mathbb{Q}'_n = \{nq_1 + q' \mid q' \in \mathbb{Q}'\}$ for $n = 0, \ldots, n_0$, we have $(\leftarrow r] \subseteq \bigcup_{n=0}^{n_0} \mathbb{Q}'_n$. Since the mapping $q' \mapsto nq_1 + q'$ of $\mathbb{Q}'$ onto $\mathbb{Q}'_n$ is increasing, $\mathbb{Q}'_n$ is well-ordered for $n = 0, \ldots, n_0$. Thus $(\leftarrow r]$ is well-ordered.

Let S be a set and $\mathbb{P}$ be a quantity system. Then a **$\mathbb{P}$-measure** on S is a mapping $X \mapsto |X|$ of $\mathcal{P}S$ into $\mathbb{P}$ which satisfies the following three conditions:

$$X \neq \emptyset \iff |X| > 0, \qquad \text{(positive definite)}$$

$$X \subseteq Y \implies |X| \leq |Y|, \qquad \text{(increasing)}$$

$$|X \cup Y| \leq |X| + |Y|. \qquad \text{(subadditive)}$$

[11] Linear quantity systems were referred to as quantity systems in Gomi (09a).

Among constructions of $\mathbb{P}$-measures, the following two have proved to be valuable in Gomi (09c) and Takaoka (09). The former is obvious and the latter is due to Mizumura (00).

Remark 2.1 Let S be a non-empty set and $\mathbb{P}$ be a quantity system. Assume $\mathsf{p} \in \mathbb{P} - \{0\}$. Define $|\emptyset| = 0$ and $|\mathsf{X}| = \mathsf{p}$ for all $\mathsf{X} \in \mathcal{P}\mathsf{S} - \{\emptyset\}$. Then the mapping $\mathsf{X} \mapsto |\mathsf{X}|$ is a $\mathbb{P}$-measure on S.

Theorem 2.2 *Let* S *be a non-empty set,* $\mathbb{P}$ *be a quantity system,* $0 < \acute{o} \in \mathbb{P}$*, and* R *be a relation between* $\mathcal{P}\mathsf{S}$ *and* $\mathbb{P}$ *which satisfies the following three conditions for all* $\mathsf{X}, \mathsf{Y} \in \mathcal{P}\mathsf{S}$ *and all* $\mathsf{p}, \mathsf{q} \in \mathbb{P}$:

- $\mathsf{X} = \emptyset \iff \mathsf{X}\,\mathsf{R}\,0$,
- $\mathsf{X} \subseteq \mathsf{Y}$ *and* $\mathsf{Y}\,\mathsf{R}\,\mathsf{p} \implies \mathsf{X}\,\mathsf{R}\,\mathsf{p}$,
- $\mathsf{X}\,\mathsf{R}\,\mathsf{p}$ *and* $\mathsf{Y}\,\mathsf{R}\,\mathsf{q} \implies (\mathsf{X} \cup \mathsf{Y})\,\mathsf{R}\,(\mathsf{p} + \mathsf{q})$.

Assume that, for each $\mathsf{X} \in \mathcal{P}\mathsf{S}$*, there exists* $\min\bigl(\{\mathsf{p} \in \mathbb{P} \mid \mathsf{X}\,\mathsf{R}\,\mathsf{p}\} \cup \{\acute{o}\}\bigr)$*, and let* $|\mathsf{X}|$ *denote the minimum. Then the mapping* $\mathsf{X} \mapsto |\mathsf{X}|$ *is a* $\mathbb{P}$*-measure on* S.

Proof (1) Since $\emptyset\,\mathsf{R}\,0$, we have $|\emptyset| = 0$. Conversely if $|\mathsf{X}| = 0$, then since $0 < \acute{o}$, we have $\mathsf{X}\,\mathsf{R}\,0$ and so $\mathsf{X} = \emptyset$. Thus the mapping is positive definite.

(2) Assume $\mathsf{X} \subseteq \mathsf{Y}$ and let $\mathsf{p} = |\mathsf{Y}|$. If $\mathsf{Y}\,\mathsf{R}\,\mathsf{p}$, then $\mathsf{X}\,\mathsf{R}\,\mathsf{p}$ and so $|\mathsf{X}| \le \mathsf{p}$. If $\mathsf{p} = \acute{o}$, then $|\mathsf{X}| \le \acute{o} = \mathsf{p}$. Thus the mapping is increasing.

(3) Let $\mathsf{p} = |\mathsf{X}|$ and $\mathsf{q} = |\mathsf{Y}|$. If $\mathsf{X}\,\mathsf{R}\,\mathsf{p}$ and $\mathsf{Y}\,\mathsf{R}\,\mathsf{q}$, then $(\mathsf{X} \cup \mathsf{Y})\,\mathsf{R}\,(\mathsf{p} + \mathsf{q})$ and so $|\mathsf{X} \cup \mathsf{Y}| \le \mathsf{p} + \mathsf{q}$. If $\mathsf{p} = \acute{o}$ or $\mathsf{q} = \acute{o}$, then $|\mathsf{X} \cup \mathsf{Y}| \le \acute{o} \le \mathsf{p} + \mathsf{q}$. Thus the mapping is subadditive.

2.2 MPC language

Following §3.1 of Gomi (09b), here we define the formal language $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ called the **MPC language**. Its syntax $(\mathsf{T}, \tau, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ is as follows.

As for the primes, constants and variables, we let $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{X}$ be sets with $\mathbb{S} = \mathbb{C} \amalg \mathbb{X}$ and $\mathbb{X} \ne \emptyset$. As for the carrier of the sorter, we let K be a set equipped with a specific element π and define $\mathsf{T} = \{\delta, \varepsilon\} \amalg \mathcal{P}\mathsf{K}$ by distinct symbols δ and ε. We call K the set of the **cases** and call π the **nominative case**. As for the presorting $\tau \in \mathbb{S} \to \mathsf{T}$, we assume that the set $\mathbb{X}_\varepsilon = \{\mathsf{x} \in \mathbb{X} \mid \tau\mathsf{x} = \varepsilon\}$ is non-empty. As for the operations of T, we first let $\mathbb{P}$ be a quantity system with $\#\mathbb{P} > 1$, $\mathfrak{P}$ be a subset of $\mathcal{P}\mathbb{P}$, $\neg\mathfrak{P} = \{\neg\mathfrak{p} \mid \mathfrak{p} \in \mathfrak{P}\}$ be a copy of $\mathfrak{P}$, and define $\mathfrak{Q} = \neg\mathfrak{P} \amalg \mathfrak{P}$. We call $\mathfrak{Q}, \neg\mathfrak{P}$ and $\mathfrak{P}$ the sets of the **quantifiers**, **negative quantifiers** and **positive quantifiers** respectively. Also, we let $\breve{o}$ be a symbol which does not belong to $\mathfrak{Q}$. Next we let $(\mathsf{n}_\mathfrak{f})_{\mathfrak{f} \in \mathfrak{F}}$ be a family of positive integers indexed by a set $\mathfrak{F}$ called the set of the **functionals**. Finally we let T be the algebra equipped with the following nine families (T1)–(T9) of operations indexed by the set

$$\mathsf{L} = \{\breve{o}\mathsf{k}, \mathfrak{x}\mathsf{k}, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f}, \Omega\mathsf{x} \mid \mathsf{k} \in \mathsf{K},\ \mathfrak{x} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F},\ \mathsf{x} \in \mathbb{X}_\varepsilon\}. \quad (2.1)$$

(T1) The family of binary operations $\breve{o}k$ $(k \in K)$ such that $\mathrm{Dom}\,\breve{o}k = \{\varepsilon\} \times \{P \in \mathcal{P}K \mid k \in P\}$ and $\varepsilon\,\breve{o}k\,P = P - \{k\}$ for each $P \in \mathcal{P}K$ with $k \in P$.

(T2) The family of binary operations $\mathfrak{x}k$ $\big((\mathfrak{x}, k) \in \mathfrak{Q} \times K\big)$ such that $\mathrm{Dom}\,\mathfrak{x}k = \{\delta, \varepsilon\} \times \{P \in \mathcal{P}K \mid k \in P\}$ and $\delta\,\mathfrak{x}k\,P = \varepsilon\,\mathfrak{x}k\,P = P - \{k\}$ for each $P \in \mathcal{P}K$ with $k \in P$.

(T3) The three binary operations $\wedge, \vee$ and $\Rightarrow$ such that $\mathrm{Dom} \wedge = \mathrm{Dom} \vee = \mathrm{Dom} \Rightarrow = (\mathcal{P}K)^2$ and $P \wedge Q = P \vee Q = P \Rightarrow Q = P \cup Q$ for each $(P, Q) \in (\mathcal{P}K)^2$.

(T4) The unary operation $\Diamond$ such that $\mathrm{Dom}\,\Diamond = \mathcal{P}K$ and $P^{\Diamond} = P$ for each $P \in \mathcal{P}K$.

(T5) The unary operation $\triangle$ such that $\mathrm{Dom}\,\triangle = \{\delta, \varepsilon\}$ and $\delta\triangle = \varepsilon\triangle = \{\pi\}$.

(T6) The binary operations $\sqcap$ and $\sqcup$ such that $\mathrm{Dom}\,\sqcap = \mathrm{Dom}\,\sqcup = \{\delta, \varepsilon\}^2$ and $\xi \sqcap \eta = \xi \sqcup \eta = \delta$ for each $(\xi, \eta) \in \{\delta, \varepsilon\}^2$.

(T7) The unary operation $\square$ such that $\mathrm{Dom}\,\square = \{\delta, \varepsilon\}$ and $\delta^{\square} = \varepsilon^{\square} = \delta$.

(T8) The family of $n_{\mathfrak{f}}$-ary operations $\mathfrak{f} \in \mathfrak{F}$ such that $\mathrm{Dom}\,\mathfrak{f} = \{\varepsilon\}^{n_{\mathfrak{f}}}$ and $\mathfrak{f}(\varepsilon, \ldots, \varepsilon) = \varepsilon$ for the unique element $(\varepsilon, \ldots, \varepsilon) \in \{\varepsilon\}^{n_{\mathfrak{f}}}$.

(T9) The family of unary operations Ωx $(x \in \mathbb{X}_\varepsilon)$ such that $\mathrm{Dom}\,\Omega x = \{\emptyset\}$ and $\emptyset\,\Omega x = \delta$. Notice that $\emptyset \in \mathcal{P}K \subseteq T$.

Therefore defining

$$\Gamma = \{\breve{o}k, \mathfrak{x}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f}, \Omega \mid k \in K,\ \mathfrak{x} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F}\},$$

we have $L \subseteq \Gamma \cup \Gamma\mathbb{X}$ with $L \cap \Gamma\mathbb{X} = \{\Omega x \mid x \in \mathbb{X}_\varepsilon\}$ consisting of unary operations. Thus the MPC language $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ has been defined. The variable operations Ωx $(x \in \mathbb{X}_\varepsilon)$ will be called the **nominalizers**.

Since (A, T, σ) is a sorted algebra, A is divided into its t-parts A_t $(t \in T)$, and since $T = \{\delta, \varepsilon\} \cup \mathcal{P}K$, we have $A = A_\delta \cup A_\varepsilon \cup \bigcup_{P \in \mathcal{P}K} A_P$. We define $G = A_\delta \cup A_\varepsilon$, and call G, A_δ and A_ε the sets of the **nominals**, **complex nominals** and **simple nominals** respectively. We also define $H = \bigcup_{P \in \mathcal{P}K} A_P$, which we call the set of the **predicates**. The elements of $A_\emptyset$ are also called the **sentences**. For each $f \in H$, we denote by K_f the element $P \in \mathcal{P}K$ satisfying $f \in A_P$ and call it the **range** of f.

Also since (A, T, σ) is a sorted algebra, the following holds on the domains and images of the operations of A.

(A1) $\mathrm{Dom}\,\breve{o}k = A_\varepsilon \times \bigcup_{k \in P \in \mathcal{P}K} A_P$ and $a\,\breve{o}k\,f \in A_{P - \{k\}}$ for all $k \in K$ and all $(a, f) \in A_\varepsilon \times A_P$ with $k \in P \in \mathcal{P}K$.

(A2) $\mathrm{Dom}\,\mathfrak{x}k = G \times \bigcup_{k \in P \in \mathcal{P}K} A_P$ and $a\,\mathfrak{x}k\,f \in A_{P - \{k\}}$ for all $(\mathfrak{x}, k) \in \mathfrak{Q} \times K$ and all $(a, f) \in G \times A_P$ with $k \in P \in \mathcal{P}K$.

(A3) $\mathrm{Dom}\wedge = \mathrm{Dom}\vee = \mathrm{Dom}\Rightarrow = H^2$, and $f \wedge g, f \vee g, f \Rightarrow g \in A_{P\cup Q}$ for all $(f, g) \in A_P \times A_Q$ with $P, Q \in \mathcal{P}K$.

(A4) $\mathrm{Dom}\Diamond = H$, and $f^{\Diamond} \in A_P$ for all $f \in A_P$ with $P \in \mathcal{P}K$.

(A5) $\mathrm{Dom}\triangle = G$, and $a\triangle \in A_{\{\pi\}}$ for all $a \in G$.

(A6) $\mathrm{Dom}\sqcap = \mathrm{Dom}\sqcup = G^2$, and $a \sqcap b, a \sqcup b \in A_\delta$ for all $(a, b) \in G^2$.

(A7) $\mathrm{Dom}\,\Box = G$, and $a^{\Box} \in A_\delta$ for all $a \in G$.

(A8) $\mathrm{Dom}\,\mathfrak{f} = (A_\varepsilon)^{n_\mathfrak{f}}$ and $\mathfrak{f}(a_1, \ldots, a_{n_\mathfrak{f}}) \in A_\varepsilon$ for all $\mathfrak{f} \in \mathfrak{F}$ and all $(a_1, \ldots, a_{n_\mathfrak{f}}) \in (A_\varepsilon)^{n_\mathfrak{f}}$.

(A9) $\mathrm{Dom}\,\Omega x = A_\emptyset$ and $f\,\Omega x \in A_\delta$ for all $x \in \mathbb{X}_\varepsilon$ and all $f \in A_\emptyset$.

Consequently, the following also holds.

(1) Let $a_1, \ldots, a_n \in G$, $f \in H$, $\mathfrak{r}_1, \ldots, \mathfrak{r}_n \in \{\breve{o}\} \cup \mathfrak{Q}$, and $k_1, \ldots, k_n$ be distinct cases in K_f. Assume that $a_i \in A_\varepsilon$ for all $i \in \{1, \ldots, n\}$ with $\mathfrak{r}_i = \breve{o}$. Then $a_1\, \mathfrak{r}_1 k_1\, (\cdots\, (a_n\, \mathfrak{r}_n k_n\, f) \cdots) \in A_{K_f - \{k_1, \ldots, k_n\}}$.

(2) H and A_P for each $P \in \mathcal{P}K$ are closed under the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, whose restrictions to H and A_P are total.

(3) G and A_δ are closed under the operations $\sqcap, \sqcup$ and $\Box$, whose restrictions to G and A_δ are total.

(4) $A_\delta - \mathbb{S}$ is non-empty and consists of the nominals in one of the word forms $b \sqcap c$, $b \sqcup c$, $b^{\Box}$ and $f\,\Omega x$.

(5) A_ε is closed under the operations $\mathfrak{f} \in \mathfrak{F}$, whose restrictions to A_ε are total. Furthermore $A_\varepsilon = [\mathbb{S}_\varepsilon]_\mathfrak{F} \neq \emptyset$, where $[\mathbb{S}_\varepsilon]_\mathfrak{F}$ is the closure of $\mathbb{S}_\varepsilon$ in the $\mathfrak{F}$-reduct $A_\mathfrak{F}$ of A.

(6) $H - \mathbb{S}$ is non-empty and consists of the predicates in one of the word forms $a\,\breve{o}k\,f$, $a\,\mathfrak{r}k\,f$, $f \wedge g$, $f \vee g$, $f \Rightarrow g$, $f^{\Diamond}$ and $a\triangle$.

(7) $A_\emptyset - \mathbb{S}$ is non-empty and consists of the sentences in one of the word forms $a\,\breve{o}k\,f$, $a\,\mathfrak{r}k\,f$, $f \wedge g$, $f \vee g$, $f \Rightarrow g$ and $f^{\Diamond}$.

For instance as for (4) and (7), we have $x\,\breve{o}\pi\,x\triangle \in A_\emptyset - \mathbb{S}$ and $(x\,\breve{o}\pi\,x\triangle)\,\Omega x \in A_\delta - \mathbb{S}$ for all $x \in \mathbb{X}_\varepsilon$ by (A1), (A5) and (A9). A proof of the equation $A_\varepsilon = [\mathbb{S}_\varepsilon]_\mathfrak{F}$ in (5) is as follows. First, since A_ε is closed under the operations in $\mathfrak{F}$, we have $A_\varepsilon \supseteq [\mathbb{S}_\varepsilon]_\mathfrak{F}$. We will show $a \in [\mathbb{S}_\varepsilon]_\mathfrak{F}$ for each $a \in A_\varepsilon$ by induction on $r = \mathrm{Rank}\, a$. Since $\mathbb{S} \cap A_\varepsilon = \mathbb{S}_\varepsilon$, we may assume $r \geq 1$. Then since $a \in A_\varepsilon$, the facts (A1)–(A9) show that $a = \mathfrak{f}(a_1, \ldots, a_n)$ for some $\mathfrak{f} \in \mathfrak{F}$ and $a_1, \ldots, a_n \in A_\varepsilon$. Since $a_1, \ldots, a_n \in [\mathbb{S}_\varepsilon]_\mathfrak{F}$ by the induction hypothesis, we have $a \in [\mathbb{S}_\varepsilon]_\mathfrak{F}$ as desired.

2.3 MPC worlds

Following §3.2 of Gomi (09b), here we define a non-empty collection $\mathcal{W}$ of denotable worlds for the MPC language $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ defined in §2.2.

First, following (3.1) of Gomi (09b), we define

$$\mathsf{M} = \mathsf{L} \cap \Gamma = \{\breve{o}\mathsf{k}, \mathfrak{x}\mathsf{k}, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f} \mid \mathsf{k} \in \mathsf{K},\ \mathfrak{x} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F}\}, \tag{2.2}$$

and let T_M be the M-reduct of T. As a set, T_M is equal to T.

Next, we pick a non-empty set S, define

$$W = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S} \cup \bigcup_{\mathsf{P} \in \mathcal{P}\mathsf{K}} \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big),$$

and call S the **base** of W. Notice that $\emptyset \to \mathsf{S} = \{\emptyset\}$ and so $(\emptyset \to \mathsf{S}) \to \mathbb{T} = \{\emptyset\} \to \mathbb{T}$, which we regard as $\mathbb{T}$, identifying $\mathsf{f} \in \{\emptyset\} \to \mathbb{T}$ with $\mathsf{f}\emptyset \in \mathbb{T}$.

Next, we define the sorting ρ of W into $\mathsf{T}_\mathsf{M} = \{\delta, \varepsilon\} \cup \mathcal{P}\mathsf{K}$ so that the t-parts W_t $(\mathsf{t} \in \mathsf{T}_\mathsf{M})$ satisfy $W_\delta = \mathsf{S} \to \mathbb{T}$, $W_\varepsilon = \mathsf{S}$ and $W_\mathsf{P} = (\mathsf{P} \to \mathsf{S}) \to \mathbb{T}$ for each $\mathsf{P} \in \mathcal{P}\mathsf{K}$. In particular $W_\emptyset = \mathbb{T}$ by the above remark.

We define $\mathsf{E} = W_\delta \cup W_\varepsilon$, and call E, W_δ and W_ε the sets of the **entities**, **complex entities** and **simple entities** respectively. We also define $\mathsf{F} = \bigcup_{\mathsf{P} \in \mathcal{P}\mathsf{K}} W_\mathsf{P}$, which we call the set of the **affairs**. For each $\mathsf{f} \in \mathsf{F}$, we denote by K_f the element $\mathsf{P} \in \mathcal{P}\mathsf{K}$ satisfying $\mathsf{f} \in W_\mathsf{P}$ and call it the **frame** of f.

The definition of the operations of W depends on two parameters. The one is a $\mathbb{P}$-measure $\mathsf{X} \mapsto |\mathsf{X}|$ on S (cf. Remark 2.1). The other is a relation $\exists$ on S, which we call the **basic relation** of W. Preparatory to defining the operations, we first extend $\exists$ to the relation between $\mathsf{E} = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$ and S by defining

$$\mathsf{a} \mathrel{\exists} \mathsf{s} \iff \mathsf{as} = 1 \tag{2.3}$$

for each $\mathsf{a} \in \mathsf{S} \to \mathbb{T}$ and each $\mathsf{s} \in \mathsf{S}$. Next when $\mathsf{s} \in \mathsf{S}$ and $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, we define, for each $\theta \in (\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}$, the element $(\mathsf{k}/\mathsf{s})\theta \in \mathsf{P} \to \mathsf{S}$ by

$$\big((\mathsf{k}/\mathsf{s})\theta\big)\mathsf{l} = \begin{cases} \mathsf{s} & \text{if } \mathsf{l} = \mathsf{k}, \\ \theta\mathsf{l} & \text{if } \mathsf{l} \in \mathsf{P} - \{\mathsf{k}\}. \end{cases} \tag{2.4}$$

If $\mathsf{P} = \{\mathsf{k}\}$, then $(\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S} = \emptyset \to \mathsf{S} = \{\emptyset\}$, and so we denote $(\mathsf{k}/\mathsf{s})\emptyset$ by (k/s). Thus $(\mathsf{k}/\mathsf{s}) \in \{\mathsf{k}\} \to \mathbb{T}$ with $(\mathsf{k}/\mathsf{s})\mathsf{k} = \mathsf{s}$. Next we define $\neg(\neg\mathfrak{p}) = \mathfrak{p}$ for each $\mathfrak{p} \in \mathfrak{P}$, and so if $\mathfrak{x} \in \neg\mathfrak{P}$ then $\neg\mathfrak{x} \in \mathfrak{P}$. Finally, we let W be the algebra equipped with the following eight families (W1)–(W8) of operations indexed by M.

(W1) The family of binary operations $\breve{o}\mathsf{k}$ $(\mathsf{k} \in \mathsf{K})$ such that

$$\mathrm{Dom}\, \breve{o}\mathsf{k} = \mathsf{S} \times \bigcup_{\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}} W_\mathsf{P} = \mathsf{S} \times \bigcup_{\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}} \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big)$$

and, for each $(\mathsf{s}, \mathsf{f}) \in \mathsf{S} \times \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big)$ with $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, $\mathsf{s}\,\breve{o}\mathsf{k}\,\mathsf{f}$ is the element of $\big((\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}\big) \to \mathbb{T}$ which satisfies $(\mathsf{s}\,\breve{o}\mathsf{k}\,\mathsf{f})\theta = \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)$ for all $\theta \in (\mathsf{P} - \{\mathsf{k}\}) \to \mathsf{S}$. Notice that if $\mathsf{P} = \{\mathsf{k}\}$ then $\mathsf{s}\,\breve{o}\mathsf{k}\,\mathsf{f} = \mathsf{f}(\mathsf{k}/\mathsf{s}) \in \mathbb{T}$.

(W2) The family of binary operations $\mathfrak{x}\mathsf{k}$ $\big((\mathfrak{x},\mathsf{k}) \in \mathfrak{Q} \times \mathsf{K}\big)$ such that

$$\mathrm{Dom}\,\mathfrak{x}\mathsf{k} = \mathsf{E} \times \textstyle\bigcup_{\mathsf{k}\in\mathsf{P}\in\mathcal{P}\mathsf{K}} W_{\mathsf{P}} = \mathsf{E} \times \textstyle\bigcup_{\mathsf{k}\in\mathsf{P}\in\mathcal{P}\mathsf{K}} \big((\mathsf{P}\to\mathsf{S})\to\mathbb{T}\big)$$

and, for each $(\mathsf{a},\mathsf{f}) \in \mathsf{E} \times \big((\mathsf{P}\to\mathsf{S})\to\mathbb{T}\big)$ with $\mathsf{k} \in \mathsf{P} \in \mathcal{P}\mathsf{K}$, $\mathsf{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f}$ is the element of $\big((\mathsf{P}-\{\mathsf{k}\})\to\mathsf{S}\big)\to\mathbb{T}$ which satisfies

$$(\mathsf{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f})\theta = 1 \iff \begin{cases} \big|\{\mathsf{s}\in\mathsf{S} \mid \mathsf{a} \ni \mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big) = 0\}\big| \in \neg\mathfrak{x} & \text{if } \mathfrak{x} \in \neg\mathfrak{P}, \\ \big|\{\mathsf{s}\in\mathsf{S} \mid \mathsf{a} \ni \mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big) = 1\}\big| \in \mathfrak{x} & \text{if } \mathfrak{x} \in \mathfrak{P} \end{cases}$$

for all $\theta \in (\mathsf{P}-\{\mathsf{k}\})\to\mathsf{S}$. Notice that $\mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big) = (\mathsf{s}\,\breve{o}\mathsf{k}\,\mathsf{f})\theta$ and if $\mathsf{P} = \{\mathsf{k}\}$ then $(\mathsf{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f})\theta = \mathsf{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f} \in \mathbb{T}$ and $\mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big) = \mathsf{f}(\mathsf{k}/\mathsf{s}) = \mathsf{s}\,\breve{o}\mathsf{k}\,\mathsf{f} \in \mathbb{T}$.

(W3) The three binary operations $\wedge, \vee$ and $\Rightarrow$ such that

$$\mathrm{Dom}\wedge = \mathrm{Dom}\vee = \mathrm{Dom}\Rightarrow\ = \mathsf{F}^2 = \Big(\textstyle\bigcup_{\mathsf{P}\in\mathcal{P}\mathsf{K}}\big((\mathsf{P}\to\mathsf{S})\to\mathbb{T}\big)\Big)^2$$

and, for each $(\mathsf{f},\mathsf{g}) \in \big((\mathsf{P}\to\mathsf{S})\to\mathbb{T}\big) \times \big((\mathsf{Q}\to\mathsf{S})\to\mathbb{T}\big)$ with $\mathsf{P},\mathsf{Q} \in \mathcal{P}\mathsf{K}$, $\mathsf{f}\wedge\mathsf{g}$ and $\mathsf{f}\vee\mathsf{g}$ and $\mathsf{f}\Rightarrow\mathsf{g}$ are the elements of $\big((\mathsf{P}\cup\mathsf{Q})\to\mathsf{S}\big)\to\mathbb{T}$ which satisfy

$$\begin{aligned}(\mathsf{f}\wedge\mathsf{g})\theta &= \mathsf{f}(\theta|_{\mathsf{P}}) \wedge \mathsf{g}(\theta|_{\mathsf{Q}}),\\ (\mathsf{f}\vee\mathsf{g})\theta &= \mathsf{f}(\theta|_{\mathsf{P}}) \vee \mathsf{g}(\theta|_{\mathsf{Q}}),\\ (\mathsf{f}\Rightarrow\mathsf{g})\theta &= \mathsf{f}(\theta|_{\mathsf{P}}) \Rightarrow \mathsf{g}(\theta|_{\mathsf{Q}})\end{aligned}$$

for all $\theta \in (\mathsf{P}\cup\mathsf{Q})\to\mathsf{S}$, where $\wedge, \vee$ and $\Rightarrow$ on the right-hand sides are the meet, join and implication on the Boolean lattice $\mathbb{T}$ defined by $\mathsf{a}\wedge\mathsf{b} = \inf\{\mathsf{a},\mathsf{b}\}$, $\mathsf{a}\vee\mathsf{b} = \sup\{\mathsf{a},\mathsf{b}\}$ and $\mathsf{a}\Rightarrow\mathsf{b} = \sup\{1-\mathsf{a},\mathsf{b}\}$ for all $\mathsf{a},\mathsf{b} \in \mathbb{T}$.

(W4) The unary operation $\Diamond$ such that $\mathrm{Dom}\,\Diamond = \mathsf{F} = \bigcup_{\mathsf{P}\in\mathcal{P}\mathsf{K}}\big((\mathsf{P}\to\mathsf{S})\to\mathbb{T}\big)$ and, for each $\mathsf{f} \in (\mathsf{P}\to\mathsf{S})\to\mathbb{T}$ with $\mathsf{P} \in \mathcal{P}\mathsf{K}$, $\mathsf{f}^{\Diamond}$ is the element of $(\mathsf{P}\to\mathsf{S})\to\mathbb{T}$ which satisfies $(\mathsf{f}^{\Diamond})\theta = (\mathsf{f}\theta)^{\Diamond}$ for all $\theta \in \mathsf{P}\to\mathsf{S}$, where $\Diamond$ on the right-hand side is the complement on the Boolean lattice $\mathbb{T}$ defined by $\mathsf{a}^{\Diamond} = 1-\mathsf{a}$ for all $\mathsf{a} \in \mathbb{T}$.

(W5) The unary operation $\triangle$ such that $\mathrm{Dom}\,\triangle = \mathsf{E}$ and, for each $\mathsf{a} \in \mathsf{E}$, $\mathsf{a}\triangle$ is the element of $(\{\pi\}\to\mathsf{S})\to\mathbb{T}$ which satisfies

$$(\mathsf{a}\triangle)\theta = 1 \iff \mathsf{a} \ni \theta\pi$$

for all $\theta \in \{\pi\}\to\mathsf{S}$.

(W6) The binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dom}\,\sqcap = \mathrm{Dom}\,\sqcup = \mathsf{E}^2 = \big((\mathsf{S}\to\mathbb{T})\cup\mathsf{S}\big)^2$$

and, for each $(\mathsf{a},\mathsf{b}) \in \mathsf{E}^2$, $\mathsf{a}\sqcap\mathsf{b}$ and $\mathsf{a}\sqcup\mathsf{b}$ are the elements of $\mathsf{S}\to\mathbb{T}$ which satisfy

$$\begin{aligned}\mathsf{a}\sqcap\mathsf{b} \ni \mathsf{s} &\iff \mathsf{a} \ni \mathsf{s} \text{ and } \mathsf{b} \ni \mathsf{s},\\ \mathsf{a}\sqcup\mathsf{b} \ni \mathsf{s} &\iff \mathsf{a} \ni \mathsf{s} \text{ or } \mathsf{b} \ni \mathsf{s}\end{aligned}$$

for all $\mathsf{s} \in \mathsf{S}$.

(W7) The unary operation $\square$ such that $\mathrm{Dom}\,\square = \mathsf{E} = (\mathsf{S}\to\mathbb{T}) \cup \mathsf{S}$ and, for each $\mathsf{a} \in \mathsf{E}$, $\mathsf{a}^{\square}$ is the element of $\mathsf{S}\to\mathbb{T}$ which satisfies

$$\mathsf{a}^{\square} \ni \mathsf{s} \iff \mathsf{a} \not\ni \mathsf{s}$$

for all $\mathsf{s} \in \mathsf{S}$.

(W8) A family of $n_{\mathfrak{f}}$-ary operations $\mathfrak{f} \in \mathfrak{F}$ such that $\mathrm{Dom}\,\mathfrak{f} = \mathsf{S}^{n_{\mathfrak{f}}}$ and $\mathfrak{f}(\mathsf{S}^{n_{\mathfrak{f}}}) \subseteq \mathsf{S}$.

Thus a denotable world W for the MPC language $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ has been constructed.

We call the denotable worlds constructed as above the **MPC worlds** cognizable by $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ and define $\mathcal{W}$ to be the collection of all such worlds.

2.4 Interpretations of the nominalizers

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be the MPC language defined in §2.2 and $\mathcal{W}$ be the collection of the MPC worlds cognizable by it defined in §2.3. Following §3.3 of Gomi (09b), here we complete the definition of the logical system MPCL by defining the interpretation I_W of the set $\mathsf{L} \cap \Gamma\mathbb{X} = \{\Omega\mathsf{x} \mid \mathsf{x} \in \mathbb{X}_\varepsilon\}$ of the variable operations on each MPC world $W \in \mathcal{W}$.

Let $\lambda = \Omega\mathsf{x}$ $(\mathsf{x} \in \mathbb{X}_\varepsilon)$. Then the domain T_λ of λ on T is equal to $\{\emptyset\}$ and $\lambda\emptyset = \emptyset\,\Omega\mathsf{x} = \delta$, hence $W_{\lambda\emptyset} = W_\delta = \mathsf{S}\to\mathbb{T} = W_\varepsilon\to\mathbb{T} = W_{\sigma\mathsf{x}}\to W_\emptyset$. Thus the meaning λ_W of λ on W under I_W should be a mapping of $W_{\sigma\mathsf{x}}\to W_\emptyset$ into itself, and therefore we define λ_W to be the identity mapping of $W_{\sigma\mathsf{x}}\to W_\emptyset$. This completes the definition of the logical system MPCL.

The domain of the resultant operation $\Omega\mathsf{x}$ on the metaworld $W^{\mathsf{V}_{\mathbb{X},W}}$ is equal to $\mathsf{V}_{\mathbb{X},W}\to\mathbb{T}$ by (3.8) of Gomi (09b), and the image $\varphi\,\Omega\mathsf{x}$ of $\varphi \in \mathsf{V}_{\mathbb{X},W}\to\mathbb{T}$ under $\Omega\mathsf{x}$ belongs to $\mathsf{V}_{\mathbb{X},W}\to(\mathsf{S}\to\mathbb{T})$ by (3.10) of Gomi (09b), hence $(\varphi\,\Omega\mathsf{x})\nu \in \mathsf{S}\to\mathbb{T}$ for each $\nu \in \mathsf{V}_{\mathbb{X},W}$, and finally (3.9) and (3.6) of Gomi (09b) show that

$$\big((\varphi\,\Omega\mathsf{x})\nu\big)\mathsf{s} = \varphi\big((\mathsf{x}/\mathsf{s})\nu\big) \tag{2.5}$$

for each $\mathsf{s} \in \mathsf{S}$. This may be paraphrased in terms of the extended relation $\ni$:

$$\big((\varphi\,\Omega\mathsf{x})\nu\big) \ni \mathsf{s} \iff \varphi\big((\mathsf{x}/\mathsf{s})\nu\big) = 1. \tag{2.6}$$

2.5 Predicate logical space

Let $(\mathsf{A}, \mathcal{W}, (\mathrm{I}_W)_{W\in\mathcal{W}})$ be the logical system MPCL defined above. Then, $\emptyset \in \mathcal{P}\mathsf{K} \subseteq \mathsf{T}$, $\mathsf{A}_\emptyset \neq \emptyset$ by §2.2 (7), and $W_\emptyset = \mathbb{T}$ for each $W \in \mathcal{W}$. Therefore $(\mathsf{A}, \mathcal{W}, (\mathrm{I}_W)_{W\in\mathcal{W}})$ has a truth $\emptyset$ and yields the $\mathbb{T}$-valued $\emptyset$-sentential functional logical space $(\mathsf{A}_\emptyset, \mathcal{F})$, as was shown in §3.8 of Gomi (09b).

However, another larger logical space on the set $\mathsf{H} = \bigcup_{\mathsf{P}\in\mathcal{P}\mathsf{K}} \mathsf{A}_\mathsf{P}$ of the predicates of A is more worth studying. Let W be a MPC world in $\mathcal{W}$, Φ be a $\mathbb{C}$-denotation into W, and ν be an $\mathbb{X}$-denotation into W. Then the metadenotation $\Phi^* \in \mathsf{A}\to W^{\mathsf{V}_{\mathbb{X},W}}$ is sort-consistent, and so also is the projection $\mathrm{pr}_\nu \in W^{\mathsf{V}_{\mathbb{X},W}}\to W$. Therefore if $\mathsf{f} \in \mathsf{H}$, then since $\mathsf{f} \in \mathsf{A}_{\mathsf{K}_\mathsf{f}}$, $(\Phi^*\mathsf{f})\nu$

belongs to $W_{\mathsf{K}_f} = (\mathsf{K}_f {\to} W_\varepsilon){\to}\mathbb{T}$, and so if furthermore $\theta \in \mathsf{K}{\to}W_\varepsilon$, then $\big((\Phi^* f)\nu\big)(\theta|_{\mathsf{K}_f})$ belongs to $\mathbb{T}$. Thus the quadruple W, Φ, ν, θ yields the mapping $f \mapsto \big((\Phi^* f)\nu\big)(\theta|_{\mathsf{K}_f})$ of H into $\mathbb{T}$. Let $\mathcal{G}$ denote the set of those mappings obtained from all possible such quadruples. Then $(\mathsf{H}, \mathcal{G})$ is a $\mathbb{T}$-valued functional logical space, which we call the **predicate logical space**. It is an extension of $(\mathsf{A}_\emptyset, \mathcal{F})$ in the sense that $\mathsf{A}_\emptyset \subseteq \mathsf{H}$ and $\mathcal{F} = \big\{\varphi|_{\mathsf{A}_\emptyset} \bigm| \varphi \in \mathcal{G}\big\}$.

2.6 Translations of the MPC language into Japanese

As was explained in §1, the formal language in MP should be identified with the totality of the *rephrased* utterances, and the definition of MPCL is based on observations of the Japanese language. For this reason, plenty of elements of the MPC language A translate verbatim into *grammatical but not necessarily good* Japanese expressions, as I will show below.

As we so translate elements of A, we will come across numbers of sequents and predicates which are expected to be tautologies. They are certainly tautologies, as can be proved by the methods in §3. Thus those translations will serve as both tests of MPCL for adequacy and illustrations of the definition of MPCL.

As was noted in §3.4 of Gomi (09b), the constants of A are models of the neurons which have been connected with their percepts, while the variables of A are models of the neurons which are capable of connecting with some percepts and are not connected with anything. Also, the primes of A have no word forms. For these reasons, constant nominals in $\mathbb{C} \cap \mathsf{G}$ translate into nouns, and constant predicates in $\mathbb{C} \cap \mathsf{H}$ translate into verbs, predicate adjectives and predicate nominal adjectives, while variables translate into nothing.

The operations $\breve{o}k, \mathfrak{r}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f}, \Omega x$ ($k \in \mathsf{K}$, $\mathfrak{r} \in \mathfrak{Q}$, $\mathfrak{f} \in \mathfrak{F}$, $x \in \mathbb{X}_\varepsilon$) of A also have their translations unless variables occur in their arguments.

1. The cases k in $\breve{o}k$ translate into postpositions such as "ga," "wa" and "wo" (no English equivalents). In particular, the nominative case π translates into "ga" and "wa." The symbol $\breve{o}$ is to distinguish the operation $\breve{o}k$ from the case k, and so may be neglected in considering translations.

2. The operations $\mathfrak{r}k$ with $(\mathfrak{r}, k) \in \mathfrak{Q} \times \mathsf{K}$ translate into combinations of quantifiers and postpositions. Assume that the intervals $(p{\to}) = \{x \in \mathbb{P} \mid p < x\}$ and $({\leftarrow}p] = \{x \in \mathbb{P} \mid x \leq p\}$ of $\mathbb{P}$ belong to $\mathfrak{P}$ for all $p \in \mathbb{P}$. Define $\underline{p} = \neg({\leftarrow}p]$ and $\overline{p} = (p{\to})$ for all $p \in \mathbb{P}$, and $\forall = \underline{0}$ and $\exists = \overline{0}$ for $0 = \min \mathbb{P}$ (thus the symbol $\exists$ denotes both the basic relation and the quantifier $\overline{0}$). Then for instance, $\overline{p}$ translates into words meaning "more than p," and $\forall$ translates into equivalents of "all."

3. The operations $\wedge, \vee$ and $\Rightarrow$ translate into conjunctions meaning "and," "or" and "then" respectively

4. The operation $\Diamond$ translates into equivalents of "not."

5. The operation $\triangle$ translates into the words "dearu," "da," "desu," and so on meaning "be."
6. The operation $\sqcap$ translates into nothing or into words which joins appositional nominals such as the particle "no" as in "gaka no Gogh" meaning "painter Gogh" and the auxiliary verb "naru" as in "Gogh naru gaka" also meaning "painter Gogh." The operation $\sqcup$ translates into words meaning "and" or words meaning "or" according to its 'neighborhood.'
7. The operation $\square$ translates into words meaning "other than."
8. The operations in $\mathfrak{F}$ translate into function symbols in mathematical non-everyday expressions and therefore have been named functionals.
9. The nominalizer $\Omega\mathsf{x}$ ($\mathsf{x} \in \mathbb{X}_\varepsilon$) translates into equivalents of the relative pronouns "what," "that" and so on if it is applied to a sentence $\mathsf{f} \in \mathsf{A}_\emptyset$ in which x occurs free.

Translations of the elements $\mathsf{a} \in \mathsf{A} - \mathbb{S}$ are verbatim with a few exceptions in the sense that they are usually obtained by replacing the primes and operations occurring in a by their above-mentioned counterparts in Japanese.

By way of illustration, if we translate $\mathsf{a}, \mathsf{b} \in \mathbb{C}_\varepsilon$ into "pētā" (Peter) and "usagi" (rabbit), then since π translates into "wa" and $\triangle$ translates into "desu" (be), the predicate $\mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle$ translates verbatim into the sentence "pētā wa usagi desu."

predicate	a	$\breve{o}\pi$	b	$\triangle$
Japanese translation	pētā	wa	usagi	desu
English counterpart	*Peter*		*rabbit*	*be*
meaning	Peter is a rabbit			

Here and henceforth as in the introduction, a counterpart written by English words but in Japanese word order will be shown in italics, with verbs in the basic form and nouns in the singular form without articles. Theorem 3.9 shows that $\mathsf{s}\,\breve{o}\pi\,\mathsf{s}\triangle$ is a sentential tautology for each $\mathsf{s} \in \mathsf{A}_\varepsilon$ if we assume that the basic relations of the MPC worlds are reflexive.

Also, a verbatim translation of the predicate $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle$ is as follows[12].

a	$\forall$	π	b	$\triangle$
pētā	zenbu	ga	usagi	desu
Peter	*all*		*rabbit*	*be*
All Peters are rabbits				

Theorem 3.9 shows that $\mathsf{a}\,\forall\pi\,\mathsf{a}\triangle$ is a sentential tautology for each $\mathsf{a} \in \mathsf{G}$.

Also, a verbatim translation of the predicate $(\mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle)^\lozenge$ is as follows.

a	$\breve{o}\pi$	b	$\triangle$	$\lozenge$
pētā	wa	usagi	deari	masen
Peter		*rabbit*	*be*	*not*
Peter is not a rabbit				

[12] A much better Japanese is "pētā wa minna usagi desu."

The three predicates $(\mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle)^{\Diamond}$, $\mathsf{a}\,\breve{o}\pi\,(\mathsf{b}\triangle)^{\Diamond}$ and $\mathsf{a}\,\breve{o}\pi\,\mathsf{b}^{\square}\triangle$ are equivalent, that is, their truth values are equal in every denotable world.

If furthermore we translate $\mathsf{c} \in \mathbb{C}_{\varepsilon}$ into "wanpaku-bōzu" (naughty boy), then a verbatim translation of the predicate $\mathsf{a}\,\breve{o}\pi\,(\mathsf{b} \sqcap \mathsf{c})\triangle$ is as follows.

a	$\breve{o}\pi$	b	$\sqcap$	c	$\triangle$
pētā	wa	usagi	no	wanpaku-bōzu	desu
Peter		*rabbit*	*and*	*naughty boy*	*be*

Peter is a rabbit and a naughty boy

Moreover, $\mathsf{a}\,\breve{o}\pi\,(\mathsf{b}\sqcap\mathsf{c})\triangle$ is equivalent to $(\mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle) \wedge (\mathsf{a}\,\breve{o}\pi\,\mathsf{c}\triangle)$, which translates verbatim into the sentence "pētā wa usagi de pētā wa wanpaku-bōzu desu" meaning "Peter is a rabbit and Peter is a naughty boy."

Similarly, a verbatim translation of the predicate $\mathsf{a}\,\breve{o}\pi\,(\mathsf{b}\sqcup\mathsf{c})\triangle$ is as follows, and it is equivalent to $(\mathsf{a}\,\breve{o}\pi\,\mathsf{b}\triangle) \vee (\mathsf{a}\,\breve{o}\pi\,\mathsf{c}\triangle)$.

a	$\breve{o}\pi$	b	$\sqcup$	c	$\triangle$
pētā	wa	usagi	ka	wanpaku-bōzu	desu
Peter		*rabbit*	*or*	*naughty boy*	*be*

Peter is a rabbit or a naughty boy

If furthermore we translate $\mathsf{d} \in \mathbb{C}_{\varepsilon}$ into "mame" (bean), $\omega \in \mathsf{K} - \{\pi\}$ into "wo" and $\mathsf{f} \in \mathbb{C}_{\{\pi,\omega\}}$ into "taberu" (eat), then a verbatim translation of the predicate $\mathsf{a}\,\breve{o}\pi\,(\mathsf{d}\,\breve{o}\omega\,\mathsf{f})$ is as follows.

a	$\breve{o}\pi$	d	$\breve{o}\omega$	f
pētā	ga	mame	wo	taberu
Peter		*bean*		*eat*

Peter eats beans

Corollary 3.5.1 shows that $\mathsf{a}\,\breve{o}\pi\,(\mathsf{d}\,\breve{o}\omega\,\mathsf{f})$ is equivalent to $\mathsf{d}\,\breve{o}\omega\,(\mathsf{a}\,\breve{o}\pi\,\mathsf{f})$, which translates verbatim into the sentence "mame wo pētā ga taberu" of the same meaning but of a different emphasis.

If furthermore we translate $\mathsf{e} \in \mathbb{C}_{\varepsilon}$ into "yasai" (vegetable), then a verbatim translation of the predicate $\mathsf{a}\,\breve{o}\pi\,(\mathsf{d} \sqcup \mathsf{e})\,\forall\omega\,\mathsf{f}$ is as follows[13].

a	$\breve{o}\pi$	d	$\sqcup$	e	$\forall$	ω	f
pētā	ga	mame	**to**	yasai	zenbu	wo	taberu
Peter		*bean*	***and***	*vegetable*	*all*		*eat*

Peter eats all beans **and** vegetables

Moreover, $\mathsf{a}\,\breve{o}\pi\,(\mathsf{d} \sqcup \mathsf{e})\,\forall\omega\,\mathsf{f}$ is equivalent to $(\mathsf{a}\,\breve{o}\pi\,\mathsf{d}\,\forall\omega\,\mathsf{f}) \wedge (\mathsf{a}\,\breve{o}\pi\,\mathsf{e}\,\forall\omega\,\mathsf{f})$, which translates verbatim into the sentence "pētā ga mame zenbu wo tabe te pētā ga yasai zenbu wo taberu" meaning "Peter eats all beans **and** Peter eats all vegetables." On the other hand, a verbatim translation of the predicate $\mathsf{a}\,\breve{o}\pi\,(\mathsf{d}\sqcup$

[13]The expression with "zenbu" and "wo" interchanged is a better Japanese.

$\mathsf{e})\,\exists\omega\,\mathsf{f}$ is as follows[14].

a	$\breve{\mathsf{o}}\pi$	d	$\sqcup$	e	$\exists$	ω	f
pētā	ga	mame	**ka**	yasai	ikuraka	wo	taberu
Peter		*bean*	***or***	*vegetable*	*some*		*eat*

Peter eats some beans **or** vegetables

Moreover, $\mathsf{a}\,\breve{\mathsf{o}}\pi\,(\mathsf{d}\sqcup\mathsf{e})\,\exists\omega\,\mathsf{f}$ is equivalent to $(\mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{d}\,\exists\omega\,\mathsf{f})\vee(\mathsf{a}\,\breve{\mathsf{o}}\pi\,\mathsf{e}\,\exists\omega\,\mathsf{f})$, which translates verbatim into the sentence "pētā ga mame ikuraka wo taberu ka pētā ga yasai ikuraka wo taberu" meaning "Peter eats some beans **or** Peter eats some vegetables." Thus the translation of $\sqcup$ depends on its neighborhood. In this regard, recall from §1.2 that notions and utterances should be distinguished. While $\mathsf{d}\sqcup\mathsf{e}$ is a notion or more precisely an idea in the sense of §1.1, both "bean and vegetable" and "bean or vegetable" are its deformed expressions in utterances.

In this connection, let me introduce a sophism. "Let $\mathsf{s},\mathsf{t}\in\mathbb{C}_\varepsilon$ and $\mathsf{m}\in\mathbb{C}_{\{\pi\}}$. Then the predicates $(\mathsf{s}\sqcup\mathsf{t})\,\forall\pi\,\mathsf{m}$ and $(\mathsf{s}\,\forall\pi\,\mathsf{m})\wedge(\mathsf{t}\,\forall\pi\,\mathsf{m})$ are equivalent. Suppose s,t and m translate into 'student,' 'teacher' and the intransitive verb 'meet' respectively. Then $(\mathsf{s}\sqcup\mathsf{t})\,\forall\pi\,\mathsf{m}$ translates verbatim into a sentence meaning 'All students and teachers meet,' while $(\mathsf{s}\,\forall\pi\,\mathsf{m})\wedge(\mathsf{t}\,\forall\pi\,\mathsf{m})$ translates verbatim into a sentence meaning 'All students meet and all teachers meet.' These sentences do not have the same meaning, and therefore this theory is inadequate." This argument is sophistic because it does not distinguish between notions and utterances. The utterance "All students meet" is supposed to be a deformed expression of the notion "All students meet this evening at the hall" or something like that, and so is not an expression of the simple notion $\mathsf{s}\,\forall\pi\,\mathsf{m}$. The same remark applies to "All students and teachers meet" and "All teachers meet." Moreover as for "meet," for instance "John and Mary meet" is supposed to be a deformed expression of the notion "John and Mary are present at the hall and recognize each other" or something like that, and so is not an expression of the notion $(\mathsf{J}\sqcup\mathsf{M})\,\forall\pi\,\mathsf{m}$ (cf. Link (98), p.197).

Meanwhile, a verbatim translation of the nominal $\big(\mathsf{a}\,\breve{\mathsf{o}}\pi\,(\mathsf{x}\,\breve{\mathsf{o}}\omega\,\mathsf{f})\big)\,\Omega\mathsf{x}$ is as follows, where the operation $\breve{\mathsf{o}}\omega$ translates into nothing because the variable x is one of its arguments.

a	$\breve{\mathsf{o}}\pi$	x	$\breve{\mathsf{o}}\omega$	f	$\Omega\mathsf{x}$
pētā	ga			taberu	mono
Peter				*eat*	*what*

what Peter eats

Also, a verbatim translation of the nominal $\big(\mathsf{a}\,\breve{\mathsf{o}}\pi\,(\mathsf{x}\,\breve{\mathsf{o}}\omega\,\mathsf{f})\big)\,\Omega\mathsf{x}\sqcap\mathsf{d}$ is as follows.

a	$\breve{\mathsf{o}}\pi$	x	$\breve{\mathsf{o}}\omega$	f	$\Omega\mathsf{x}$	$\sqcap$	d
pētā	ga			taberu	(mono)	(no)	mame
Peter				*eat*	*that*		*bean*

beans that Peter eats

[14] The expression with "ikuraka" and "wo" interchanged is a better Japanese.

Here parentheses show that "mono" and "no" are not actually uttered.

The complex nominals $(\mathfrak{x}\,\forall\pi\,\mathfrak{x}\triangle)\,\Omega\mathfrak{x}$ have proved to be valuable in particular in the study of deduction systems by Gomi (09c), Mizumura (00, 09) and Takaoka (09). It translates verbatim into the noun "mono" which means "any indefinite person or thing," and so we denote it by $\mathfrak{one}$ (cf. Theorems 3.19–3.22). Then a verbatim translation of $\mathfrak{one}\,\forall\pi\,\mathfrak{b}\triangle$ is as follows.

$\mathfrak{one}$	$\forall$	π	$\mathfrak{b}$	$\triangle$
(mono)	zenbu	ga	usagi	desu
One	*every*		*rabbit*	*be*

Everyone is a rabbit

In this connection, $\mathfrak{one}\,\forall\pi\,(\mathfrak{f}\,\Omega\mathfrak{x})\triangle$ and $\mathfrak{one}\,\exists\pi\,(\mathfrak{f}\,\Omega\mathfrak{x})\triangle$ play roles of $\forall\mathfrak{x}\,\mathfrak{f}$ and $\exists\mathfrak{x}\,\mathfrak{f}$ respectively in embedding FPL in MPCL (cf. Gomi (09d), Theorems 3.20 and 3.21).

The sentences like $\mathfrak{b}\,\overline{\mathfrak{p}}\pi\,\mathfrak{one}\triangle$ also have proved to be valuable. Their verbatim translations, however, do not make sense. The predicate $\mathfrak{one}\triangle$ which succeeds the operation $\mathfrak{p}\pi$ for positive quantifiers $\mathfrak{p} \in \mathfrak{P}$ should be regarded as a prime and translated into words meaning "exist" such as "imasu" and "arimasu." For instance, if the quantity system $\mathbb{P}$ is the set $\mathbb{Z}_{\geq 0}$ of non-negative integers, then a quasi-verbatim translation of $\mathfrak{b}\,\overline{3}\pi\,\mathfrak{one}\triangle$ is as follows[15].

$\mathfrak{b}$	$\overline{3}$	π	$\mathfrak{one}\triangle$
usagi	3-hiki yori ōku	ga	imasu
rabbit	*more than 3*		*exist*

More than three rabbits exist

As mentioned in §1, MPCL has a defect that it can not deal with notions like "Peter was a student last year" involved with the copula "be" modified by the words "last year" which indicate an occasion, because the range of $\mathfrak{s}\triangle$ is equal to $\{\pi\}$. For the same reason, it can not deal with notions like "John and Mary are present at the hall" involved with the existential predicate "be present" modified by the words "at the hall" which indicate a location. For another reason, it can not deal with notions like "Mr. McGregor knows that Peter eats the beans" involved with the conjunction "that" used to introduces a noun clause giving what has been perceived, cognized, said, etc. The defects and others will be removed by generalizing MPCL to CL in Part II.

3 MPC sequential tautologies

This section collects those tautologies in MPCL which have proved to be valuable for the study of deduction systems on it by Gomi (09c) and Takaoka (09) and for the embedding of FPL in MPCL by Gomi (09d). I presuppose that

[15] The expression with "3-hiki yori ōku" and "ga" interchanged is a much better Japanese, and I prefer "takusan" to "ōku" contrary to NHK (Japan Broadcasting Corporation).

the reader is acquainted with the general theory of logical spaces developed by Gomi (09a).

Let $(\mathsf{H}, \mathcal{G})$ be the predicate logical space defined in §2.5. We will denote elements of H by $\mathsf{f}, \mathsf{g}, \ldots$ with or without numerical subscripts, while elements of H^* by $\alpha, \beta, \ldots$. When $\alpha = \mathsf{f}_1 \cdots \mathsf{f}_n \in \mathsf{H}^*$, we will denote the subset $\{\mathsf{f}_1, \ldots, \mathsf{f}_n\}$ of H also by α. The element of H^* of length 0 will be denoted by a blank. We will extend this notation to arbitrary sets.

Let $\preccurlyeq$ be the validity relation of $(\mathsf{H}, \mathcal{G})$ and $\asymp$ be its symmetric core. Then

$$\alpha \preccurlyeq \beta \iff \inf \varphi\alpha \leq \sup \varphi\beta \text{ for all } \varphi \in \mathcal{G},$$
$$\mathsf{f} \asymp \mathsf{g} \iff \mathsf{f} \preccurlyeq \mathsf{g} \text{ and } \mathsf{f} \succcurlyeq \mathsf{g} \iff \varphi\mathsf{f} = \varphi\mathsf{g} \text{ for all } \varphi \in \mathcal{G}.$$

A sequent $\alpha \to \beta$ on H is a MPC tautology iff $\alpha \preccurlyeq \beta$.

3.1 Boolean features

MPC tautologies will emerge from the analysis until §3.3 of the MPC world

$$W = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S} \cup \textstyle\bigcup_{\mathsf{P} \in \mathcal{P}\mathsf{K}} \big((\mathsf{P} \to \mathsf{S}) \to \mathbb{T}\big)$$

defined in §2.3. The set $W_\delta = \mathsf{S} \to \mathbb{T}$ of the complex entities is identified with $\mathcal{P}\mathsf{S}$, and so is a Boolean lattice with respect to the order $\sqsubseteq$ defined by

$$\begin{aligned} \mathsf{a} \sqsubseteq \mathsf{b} &\iff \text{If } \mathsf{s} \in \mathsf{S} \text{ and } \mathsf{as} = 1 \text{ then } \mathsf{bs} = 1 \\ &\iff \mathsf{as} \leq \mathsf{bs} \text{ for each } \mathsf{s} \in \mathsf{S}. \end{aligned}$$

The smallest element 0 and the largest element 1 of W_δ are characterized by the properties that $0\mathsf{s} = 0$ and $1\mathsf{s} = 1$ for all $\mathsf{s} \in \mathsf{S}$. Also (W6), (W7) and (2.3) show that the entity set $\mathsf{E} = W_\delta \cup W_\varepsilon = (\mathsf{S} \to \mathbb{T}) \cup \mathsf{S}$ and W_δ are closed under the operations $\sqcap, \sqcup$ and $\square$ and that their restrictions to W_δ are equal to the meet, join and complement on the Boolean lattice W_δ. Also because of (2.3), the order $\sqsubseteq$ on W_δ is extended to the preorder $\sqsubseteq$ on E defined by

$$\mathsf{a} \sqsubseteq \mathsf{b} \iff \text{If } \mathsf{s} \in \mathsf{S} \text{ and } \mathsf{a} \ni \mathsf{s} \text{ then } \mathsf{b} \ni \mathsf{s}. \tag{3.1}$$

Consequently $0 \sqsubseteq \mathsf{a}$ and $\mathsf{a} \sqsubseteq 1$ for all $\mathsf{a} \in \mathsf{E}$.

Similarly, the set $W_\mathsf{P} = (\mathsf{P} \to \mathsf{S}) \to \mathbb{T}$ of the affairs of the frame $\mathsf{P} \in \mathcal{P}\mathsf{K}$ is a Boolean lattice with respect to the order $\leq$ defined by

$$\mathsf{f} \leq \mathsf{g} \iff \mathsf{f}\theta \leq \mathsf{g}\theta \text{ for each } \theta \in \mathsf{P} \to \mathsf{S}.$$

The smallest element 0 and the largest element 1 of W_P are characterized by the properties that $0\theta = 0$ and $1\theta = 1$ for all $\theta \in \mathsf{P} \to \mathsf{S}$. Also (W3) and (W4) show that the set $\mathsf{F} = \bigcup_{\mathsf{P} \in \mathcal{P}\mathsf{K}} W_\mathsf{P}$ of the affairs and W_P ($\mathsf{P} \in \mathcal{P}\mathsf{K}$) are closed under the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, that their restrictions to W_P are equal to the meet, join, complement and implication on the Boolean lattice W_P, and that $\mathsf{f} \Rightarrow \mathsf{g} = \mathsf{f}^\Diamond \vee \mathsf{g}$ for all $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$. Moreover, for each $\theta \in \mathsf{P} \to \mathsf{S}$, the projection $\mathsf{f} \mapsto \mathsf{f}\theta$ of W_P onto $\mathbb{T}$ by θ is a homomorphism with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. The Boolean features of $W_{\{\pi\}}$ and E are connected by the following consequences (proof omitted) of (W3)–(W5).

Theorem 3.1 *The following holds for all* $a, b \in E$:

$$(a \sqcap b)\triangle = a\triangle \wedge b\triangle, \quad (a \sqcup b)\triangle = a\triangle \vee b\triangle, \quad (a^{\square})\triangle = (a\triangle)^{\Diamond}.$$

The orders $\leq$ on W_P $(P \in \mathcal{P}K)$ are extended to the relation $\lessdot$ on F defined by

$$f \lessdot g \iff f(\theta|_{K_f}) \leq g(\theta|_{K_g}) \text{ for each } \theta \in (K_f \cup K_g) \to S.$$

Since there exists the mapping $\theta \mapsto \theta|_P$ of $K \to S$ onto $P \to S$ for each $P \in \mathcal{P}K$,

$$f \lessdot g \iff f(\theta|_{K_f}) \leq g(\theta|_{K_g}) \text{ for each } \theta \in K \to S,$$

and therefore $\lessdot$ is a preorder. Let $\doteq$ denote its symmetric core:

$$\begin{aligned} f \doteq g &\iff f \lessdot g \text{ and } f \gtrdot g \\ &\iff f(\theta|_{K_f}) = g(\theta|_{K_g}) \text{ for each } \theta \in K \to S. \end{aligned} \tag{3.2}$$

Then $\doteq$ is an equivalence relation, and its restriction to W_P is the equality $=$. For each element $f \in F$, we define the element $f^\sharp \in W_K$ by

$$f^\sharp \theta = f(\theta|_{K_f}) \tag{3.3}$$

for each $\theta \in K \to S$, and call the mapping $f \mapsto f^\sharp$ the **inflation**. Then

$$f \lessdot g \iff f^\sharp \leq g^\sharp, \qquad f \doteq g \iff f^\sharp = g^\sharp,$$

and (W3) and (W4) show that the inflation is a homomorphism of F into W_K with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. Therefore we may extended the preorder $\lessdot$ on F to the relation $\lessdot$ on F^* by defining

$$f_1 \cdots f_m \lessdot g_1 \cdots g_n \iff \inf\{f_1^\sharp, \ldots, f_m^\sharp\} \leq \sup\{g_1^\sharp, \ldots, g_n^\sharp\}$$

for all $f_1, \ldots, f_m, g_1, \ldots, g_n \in F$, and the relation $\doteq$ is consistent with the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. The results so far will often be used without notice.

Lemma 3.1 *Let* $f_1, \ldots f_m, g_1, \ldots, g_n \in F$. *Then* $f_1 \cdots f_m \lessdot g_1 \cdots g_n$ *iff*

$$\inf\{f_1^\sharp\theta, \ldots, f_m^\sharp\theta\} \leq \sup\{g_1^\sharp\theta, \ldots, g_n^\sharp\theta\}$$

for all $\theta \in K \to S$. *More generally, if* $\bigcup_{i=1}^m P_i \cup \bigcup_{j=1}^n Q_j \subseteq R \subseteq K$ *for the frames* $P_1, \ldots, P_m$, $Q_1, \ldots, Q_n$ *of* $f_1, \ldots f_m, g_1, \ldots, g_n$, *then* $f_1 \cdots f_m \lessdot g_1 \cdots g_n$ *iff*

$$\inf\{f_1(\theta|_{P_1}), \ldots, f_m(\theta|_{P_m})\} \leq \sup\{g_1(\theta|_{Q_1}), \ldots, g_n(\theta|_{Q_n})\}$$

for all $\theta \in R \to S$.

Proof We have $(\inf\{f_1^\sharp, \ldots, f_m^\sharp\})\theta = (f_1^\sharp \wedge \ldots \wedge f_m^\sharp)\theta = (f_1^\sharp\theta) \wedge \ldots \wedge (f_m^\sharp\theta) = \inf\{f_1^\sharp\theta, \ldots, f_m^\sharp\theta\}$ even if $m = 0$, and similarly for the supremum, hence the former half. The latter half follows from the former, because there exists the mapping $\theta \mapsto \theta|_R$ of $K \to S$ onto $R \to S$.

Theorem 3.2 *The relation* $\lessdot$ *on* F^* *is a Boolean relation with respect to the operations* $\wedge, \vee, \Diamond$ *and* $\Rightarrow$.

Proof For each $\theta \in \mathsf{K} \to \mathsf{S}$, the mapping $\mathsf{f} \mapsto \mathsf{f}^\sharp\theta$ is a homomorphism of F into $\mathbb{T}$ with respect to the operations. Therefore the set F equipped with the mappings $\mathsf{f} \mapsto \mathsf{f}^\sharp\theta$ for all $\theta \in \mathsf{K} \to \mathsf{S}$ is a binary logical space. Since $\lessdot$ is its validity relation by Lemma 3.1, the above result follows from Theorem 6.8 of Gomi (09a).

Recall for the next theorem from §2.2 that H is closed under the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ and that their restrictions to H is total.

Theorem 3.3 *The predicate logical space* $(\mathsf{H}, \mathcal{G})$ *is a binary logical space with respect to the operations* $\wedge, \vee, \Diamond$ *and* $\Rightarrow$ *on* H.

Proof Because of (3.3), $\mathcal{G}$ consists of the mappings $\mathsf{f} \mapsto \big((\Phi^*\mathsf{f})\nu\big)^\sharp\theta$ of H into $\mathbb{T}$ determined by the quadruples W, Φ, ν, θ of a MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W, an $\mathbb{X}$-denotation ν into W and $\theta \in \mathsf{K} \to W_\varepsilon$. Since the metadenotation Φ^*, the projection by ν, the inflation $\sharp$ and the projection by θ are all homomorphisms with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$, so is the members of $\mathcal{G}$.

The following theorem shows how the tautologies of $(\mathsf{H}, \mathcal{G})$ are obtained from the results in §3.1–§3.3.

Theorem 3.4 *Let* $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n \in \mathsf{H}$. *Then* $\mathsf{f}_1 \cdots \mathsf{f}_m \preccurlyeq \mathsf{g}_1 \cdots \mathsf{g}_n$ *iff*

$$(\Phi^*\mathsf{f}_1)\nu \cdots (\Phi^*\mathsf{f}_m)\nu \lessdot (\Phi^*\mathsf{g}_1)\nu \cdots (\Phi^*\mathsf{g}_n)\nu$$

for each triple (W, Φ, ν) *of a MPC world* $W \in \mathcal{W}$, *a* $\mathbb{C}$*-denotation* Φ *into* W, *and an* $\mathbb{X}$*-denotation* ν *into* W.

Proof This follows from Lemma 3.1 and the definition of $(\mathsf{H}, \mathcal{G})$.

3.2 Operations $\breve{o}\mathsf{k}$

Let $\mathsf{s}_1, \ldots, \mathsf{s}_n \in \mathsf{S}$ and let $\mathsf{k}_1, \ldots, \mathsf{k}_n$ be distinct cases in $\mathsf{P} \in \mathcal{P}\mathsf{K}$. Then (W1) shows that each affair $\mathsf{f} \in W_\mathsf{P}$ yields the affair $\mathsf{s}_1\,\breve{o}\mathsf{k}_1\,(\cdots(\mathsf{s}_n\,\breve{o}\mathsf{k}_n\,\mathsf{f})\cdots) \in W_{\mathsf{P}-\{\mathsf{k}_1,\ldots,\mathsf{k}_n\}}$. We sometimes abbreviate it to $(\mathsf{s}_i\,\breve{o}\mathsf{k}_i)_{i=1,\ldots,n}\mathsf{f}$ or $(\mathsf{s}_i\,\breve{o}\mathsf{k}_i)_i\mathsf{f}$.

Let $\theta \in (\mathsf{P}-\{\mathsf{k}_1, \ldots, \mathsf{k}_n\}) \to \mathsf{S}$. Then, generalizing (2.4), we define the element $\left(\dfrac{\mathsf{k}_1, \ldots, \mathsf{k}_n}{\mathsf{s}_1, \ldots, \mathsf{s}_n}\right)\theta$ of $\mathsf{P} \to \mathsf{S}$ as follows:

$$\left(\left(\frac{\mathsf{k}_1, \ldots, \mathsf{k}_n}{\mathsf{s}_1, \ldots, \mathsf{s}_n}\right)\theta\right)\mathsf{k} = \begin{cases} \mathsf{s}_i & \text{if } \mathsf{k} = \mathsf{k}_i \ (i = 1, \ldots, n), \\ \theta\mathsf{k} & \text{if } \mathsf{k} \in \mathsf{P} - \{\mathsf{k}_1, \ldots, \mathsf{k}_n\}. \end{cases} \tag{3.4}$$

We sometimes abbreviate $\left(\dfrac{\mathsf{k}_1, \ldots, \mathsf{k}_n}{\mathsf{s}_1, \ldots, \mathsf{s}_n}\right)$ to $(\mathsf{k}_i/\mathsf{s}_i)_{i=1,\ldots,n}$ or $(\mathsf{k}_i/\mathsf{s}_i)_i$.

The proof of the following lemma is left to the reader.

Lemma 3.2 *Let* $k_1, \ldots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, $P \subseteq Q \in \mathcal{P}K$, $k_{n+1}, \ldots,$ k_m *be distinct cases in* $Q - P$, $\theta \in (Q - \{k_1, \ldots, k_m\}) \to S$, *and* $s_1, \ldots, s_m \in S$. *Then* $P - \{k_1, \ldots, k_n\} \subseteq Q - \{k_1, \ldots, k_m\}$ *and the following holds:*

$$\left(\left(\frac{k_1, \ldots, k_m}{s_1, \ldots, s_m}\right)\theta\right)\Big|_P = \left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta|_{P-\{k_1,\ldots,k_n\}}.$$

Theorem 3.5 *Let* $s_1, \ldots, s_n \in S$, $k_1, \ldots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, *and* $f \in W_P$. *Then the following holds for each* $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$:

$$(s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots))\theta = f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta\right).$$

Proof By virtue of (W1), we may assume $n > 1$ and argue by induction on n. Define $Q = P - \{k_n\}$ and $g = s_n \,\breve{o} k_n\, f$. Then $g \in W_Q$ and $\theta \in (Q - \{k_1, \ldots, k_{n-1}\}) \to S$, so $\theta' = \left(\frac{k_1, \ldots, k_{n-1}}{s_1, \ldots, s_{n-1}}\right)\theta$ belongs to $Q \to S$ and $g\theta' = f\big((k_n/s_n)\theta'\big)$. Obviously $(k_n/s_n)\theta' = \left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta$. Thus

$$\begin{aligned}
&(s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots))\theta = (s_1 \,\breve{o} k_1\, (\cdots (s_{n-1} \,\breve{o} k_{n-1}\, g) \cdots))\theta \\
&\quad = g\left(\left(\frac{k_1, \ldots, k_{n-1}}{s_1, \ldots, s_{n-1}}\right)\theta\right) = g\theta' = f\big((k_n/s_n)\theta'\big) = f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta\right),
\end{aligned}$$

where the second equality holds by the induction hypothesis.

Corollary 3.5.1 *Let* $s_1, \ldots, s_n \in S$, $k_1, \ldots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, *and* $f \in W_P$. *Then the following holds for every permutation* ρ *of the letters* $1, \ldots, n$:

$$s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots) = s_{\rho 1} \,\breve{o} k_{\rho 1}\, (\cdots (s_{\rho n} \,\breve{o} k_{\rho n}\, f) \cdots).$$

Proof Using Theorem 3.5, we have

$$\begin{aligned}
&(s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots))\theta = f\left(\left(\frac{k_1, \ldots, k_n}{s_1, \ldots, s_n}\right)\theta\right) \\
&\quad = f\left(\left(\frac{k_{\rho 1}, \ldots, k_{\rho n}}{s_{\rho 1}, \ldots, s_{\rho n}}\right)\theta\right) = (s_{\rho 1} \,\breve{o} k_{\rho 1}\, (\cdots (s_{\rho n} \,\breve{o} k_{\rho n}\, f) \cdots))\theta
\end{aligned}$$

for each $\theta \in (P - \{k_1, \ldots, k_n\}) \to S$, hence the above result.

Corollary 3.5.2 *Let* $k_1, \ldots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$ *and* $f \in W_P$. *Then the following holds for each* $\theta \in P \to S$:

$$f\theta = \big((\theta k_1) \,\breve{o} k_1\, \big(\cdots \big((\theta k_n) \,\breve{o} k_n\, f\big) \cdots \big)\big)\theta|_{P-\{k_1,\ldots,k_n\}}.$$

Proof This is because $\theta = \left(\frac{k_1, \ldots, k_n}{\theta k_1, \ldots, \theta k_n}\right)\theta|_{P-\{k_1,\ldots,k_n\}}$.

Corollary 3.5.3 *Let* $s_1, \dots, s_n \in S$, $k_1, \dots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, *and* $f \in W_P$. *Then* $(s_i \,\breve{o} k_i)_{i=1,\dots,n} f^\Diamond = \big((s_i \,\breve{o} k_i)_{i=1,\dots,n} f\big)^\Diamond$.

Proof Using Theorem 3.5 and (W4), we have

$$\begin{aligned}\big((s_i \,\breve{o} k_i)_i f^\Diamond\big)\theta &= f^\Diamond\big((k_i/s_i)_i\theta\big) = \big(f\big((k_i/s_i)_i\theta\big)\big)^\Diamond \\ &= \big(\big((s_i \,\breve{o} k_i)_i f\big)\theta\big)^\Diamond = \big((s_i \,\breve{o} k_i)_i f\big)^\Diamond\theta\end{aligned}$$

for each $\theta \in (P - \{k_1, \dots, k_n\}) \to S$, hence the above result.

Lemma 3.3 *Let* $k_1, \dots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, $k_{n+1}, \dots, k_m$ *be distinct cases in* $K - P$, $s_1, \dots, s_m \in S$, *and* $f \in W_P$. *Then*

$$s_1 \,\breve{o} k_1\, (\cdots (s_m \,\breve{o} k_m\, f^\sharp) \cdots) \doteq s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots).$$

Proof Using Theorem 3.5, (3.3) and Lemma 3.2, we have

$$\begin{aligned}&(s_1 \,\breve{o} k_1\, (\cdots (s_m \,\breve{o} k_m\, f^\sharp) \cdots))\theta = f^\sharp\left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m}\right)\theta\right) \\ &= f\left(\left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m}\right)\theta\right)\Big|_P\right) = f\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta|_{P-\{k_1,\dots,k_n\}}\right) \\ &= (s_1 \,\breve{o} k_1\, (\cdots (s_n \,\breve{o} k_n\, f) \cdots))\theta|_{P-\{k_1,\dots,k_n\}}\end{aligned}$$

for each $\theta \in (K - \{k_1, \dots, k_m\}) \to S$, and so (3.2) yields the above result.

Lemma 3.4 *Let* $s_1, \dots, s_n \in S$, $k_1, \dots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$, *and* $f, g \in W_P$. *Then the following holds:*

$$\begin{aligned}(s_i \,\breve{o} k_i)_i(f \wedge g) &= (s_i \,\breve{o} k_i)_i f \wedge (s_i \,\breve{o} k_i)_i g, \\ (s_i \,\breve{o} k_i)_i(f \vee g) &= (s_i \,\breve{o} k_i)_i f \vee (s_i \,\breve{o} k_i)_i g, \\ (s_i \,\breve{o} k_i)_i(f \Rightarrow g) &= (s_i \,\breve{o} k_i)_i f \Rightarrow (s_i \,\breve{o} k_i)_i g.\end{aligned}$$

Proof Let $*$ be any one of the operations $\wedge, \vee$ and $\Rightarrow$. Then, using Theorem 3.5 and (W3), we have

$$\begin{aligned}&\big((s_i \,\breve{o} k_i)_i(f * g)\big)\theta = (f * g)\big((k_i/s_i)_i\theta\big) = f\big((k_i/s_i)_i\theta\big) * g\big((k_i/s_i)_i\theta\big) \\ &= \big((s_i \,\breve{o} k_i)_i f\big)\theta * \big((s_i \,\breve{o} k_i)_i g\big)\theta = \big((s_i \,\breve{o} k_i)_i f * (s_i \,\breve{o} k_i)_i g\big)\theta\end{aligned}$$

for each $\theta \in (P - \{k_1, \dots, k_n\}) \to S$, hence the above results.

Theorem 3.6 *Let* $s_1, \dots, s_l \in S$, $f, g \in F$, *and* $k_1, \dots, k_l$ *be distinct cases with* $k_1, \dots, k_n \in K_f - K_g$, $k_{n+1}, \dots, k_m \in K_f \cap K_g$, *and* $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \le n \le m \le l)$. *Then the following holds:*

$$\begin{aligned}(s_i \,\breve{o} k_i)_{i=1,\dots,l}(f \wedge g) &= (s_i \,\breve{o} k_i)_{i=1,\dots,m} f \wedge (s_i \,\breve{o} k_i)_{i=n+1,\dots,l}\, g, \\ (s_i \,\breve{o} k_i)_{i=1,\dots,l}(f \vee g) &= (s_i \,\breve{o} k_i)_{i=1,\dots,m} f \vee (s_i \,\breve{o} k_i)_{i=n+1,\dots,l}\, g, \\ (s_i \,\breve{o} k_i)_{i=1,\dots,l}(f \Rightarrow g) &= (s_i \,\breve{o} k_i)_{i=1,\dots,m} f \Rightarrow (s_i \,\breve{o} k_i)_{i=n+1,\dots,l}\, g.\end{aligned}$$

Proof Let $*$ be any one of the operations $\wedge, \vee$ and $\Rightarrow$. Then using Lemmas 3.3, 3.4, Corollary 3.5.1, and remarks in §3.1, we have

$$\begin{aligned}
&(s_i \,\breve{o} k_i)_{i=1,\dots,l}(f * g) \doteq (s_i \,\breve{o} k_i)_{i=1,\dots,l}(f * g)^\sharp = (s_i \,\breve{o} k_i)_{i=1,\dots,l}(f^\sharp * g^\sharp) \\
&= (s_i \,\breve{o} k_i)_{i=1,\dots,l} f^\sharp * (s_i \,\breve{o} k_i)_{i=1,\dots,l} g^\sharp \\
&\doteq (s_i \,\breve{o} k_i)_{i=1,\dots,m} f * (s_i \,\breve{o} k_i)_{i=n+1,\dots,l} g,
\end{aligned}$$

hence the above result.

Lemma 3.5 *Let* $k_1, \dots, k_n$ *be distinct cases in* $P \in \mathcal{P}K$ *and* $f, g \in W_P$. *Then* $f \leq g$ *iff* $s_1 \,\breve{o} k_1 \,(\cdots(s_n \,\breve{o} k_n\, f)\cdots) \leq s_1 \,\breve{o} k_1 \,(\cdots(s_n \,\breve{o} k_n\, g)\cdots)$ *for any elements* $s_1, \dots, s_n \in S$.

Proof If $f \leq g$ and $s_1, \dots, s_n \in S$, then using Theorem 3.5, we have

$$\begin{aligned}
(s_1 \,\breve{o} k_1 \,(\cdots(s_n \,\breve{o} k_n\, f)\cdots))\theta &= f\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta\right) \\
\leq g\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta\right) &= (s_1 \,\breve{o} k_1 \,(\cdots(s_n \,\breve{o} k_n\, g)\cdots))\theta
\end{aligned}$$

for each $\theta \in (P - \{k_1, \dots, k_n\}) \to S$, hence $(s_i \,\breve{o} k_i)_i f \leq (s_i \,\breve{o} k_i)_i g$. Conversely if $(s_i \,\breve{o} k_i)_i f \leq (s_i \,\breve{o} k_i)_i g$ for any $s_1, \dots, s_n \in S$, then using Corollary 3.5.2, we have

$$\begin{aligned}
f\theta &= \big((\theta k_1) \,\breve{o} k_1 \,\big(\cdots\big((\theta k_n) \,\breve{o} k_n\, f\big)\cdots\big)\big)\theta|_{P-\{k_1,\dots,k_n\}} \\
&\leq \big((\theta k_1) \,\breve{o} k_1 \,\big(\cdots\big((\theta k_n) \,\breve{o} k_n\, g\big)\cdots\big)\big)\theta|_{P-\{k_1,\dots,k_n\}} = g\theta
\end{aligned}$$

for each $\theta \in P \to S$, hence $f \leq g$.

Theorem 3.7 *Let* $f, g \in F$ *and* $k_1, \dots, k_l$ *be distinct cases with* $k_1, \dots, k_n \in K_f - K_g$, $k_{n+1}, \dots, k_m \in K_f \cap K_g$, *and* $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. *Then* $f \lessdot g$ *iff* $(s_i \,\breve{o} k_i)_{i=1,\dots,m} f \lessdot (s_i \,\breve{o} k_i)_{i=n+1,\dots,l} g$ *for any elements* $s_1, \dots, s_l \in S$.

Proof This is derived from Lemmas 3.5, 3.3, Corollary 3.5.1, and remarks in §3.1. First, $f \lessdot g$ iff $f^\sharp \leq g^\sharp$. Secondly, $f^\sharp \leq g^\sharp$ iff $(s_i \,\breve{o} k_i)_i f^\sharp \leq (s_i \,\breve{o} k_i)_i g^\sharp$ for any $s_1, \dots, s_l \in S$, where $(s_i \,\breve{o} k_i)_i$ is an abbreviation for $(s_i \,\breve{o} k_i)_{i=1,\dots,l}$. Thirdly, $(s_i \,\breve{o} k_i)_i f^\sharp \leq (s_i \,\breve{o} k_i)_i g^\sharp$ iff $(s_i \,\breve{o} k_i)_i f^\sharp \lessdot (s_i \,\breve{o} k_i)_i g^\sharp$. Finally, $(s_i \,\breve{o} k_i)_i f^\sharp \doteq (s_i \,\breve{o} k_i)_{i=1,\dots,m} f$ and $(s_i \,\breve{o} k_i)_i g^\sharp \doteq (s_i \,\breve{o} k_i)_{i=n+1,\dots,l} g$. Hence the above result.

Lemma 3.6 *The following holds.*

(1) $\begin{cases} \alpha f g \beta \lessdot \gamma \iff \alpha, f \wedge g, \beta \lessdot \gamma, \\ \gamma \lessdot \alpha f g \beta \iff \gamma \lessdot \alpha, f \vee g, \beta. \end{cases}$

(2) $\begin{cases} \alpha \lessdot f\beta \iff f^\diamond \alpha \lessdot \beta, \\ f\beta \lessdot \alpha \iff \beta \lessdot f^\diamond \alpha. \end{cases}$

(3) $\alpha \lessdot \beta \iff \alpha \lessdot f \wedge f^{\Diamond}, \beta \iff f \vee f^{\Diamond}, \alpha \lessdot \beta$.

Proof This follows solely from Theorem 3.2 and so also holds for every Boolean relation. Consult Theorems 3.12 and 3.13 of Gomi (09a) for (1) and (2) respectively. If $\alpha \lessdot \beta$, then $\alpha \lessdot f \wedge f^{\Diamond}, \beta$ and $f \vee f^{\Diamond}, \alpha \lessdot \beta$ by the weakening law. If $\alpha \lessdot f \wedge f^{\Diamond}, \beta$ or $f \vee f^{\Diamond}, \alpha \lessdot \beta$, then since $f \wedge f^{\Diamond} \lessdot \varepsilon$ and $\varepsilon \lessdot f \vee f^{\Diamond}$ by the negation laws and (1), we have $\alpha \lessdot \beta$ by the cut law.

Theorem 3.8 *Let $f_1, \dots, f_m, g_1, \dots, g_n \in F$, $\alpha, \beta \in F^*$, and $k \in K$. Assume that k belongs to the frames of $f_1, \dots, f_m, g_1, \dots, g_n$ but does not belong to those of the affairs in $\alpha \cup \beta$. Then $f_1 \cdots f_m \alpha \lessdot g_1 \cdots g_n \beta$ iff $s\,\breve{o}k\,f_1, \cdots, s\,\breve{o}k\,f_m, \alpha \lessdot s\,\breve{o}k\,g_1, \cdots, s\,\breve{o}k\,g_n, \beta$ for all $s \in S$.*

Proof We may assume $m \neq 0$ or $n \neq 0$. Suppose $\alpha = f'_1 \cdots f'_{m'}$, $\beta = g'_1 \cdots g'_{n'}$ with $f'_1, \dots, f'_{m'}, g'_1, \dots, g'_{n'} \in F$. Let $s \in S$ and define $h = s\,\breve{o}\pi\,s\triangle$. Then $h \in W_\emptyset$ by (W5) and (W1), and so k does not belong to the frames of $h \wedge h^{\Diamond}$ or $h \vee h^{\Diamond}$. Therefore we may assume $m' \neq 0 \neq n'$ by Lemma 3.6. Define

$$\begin{aligned} f' &= (f'_1 \wedge \cdots \wedge f'_{m'}), & g' &= (g'_1 \vee \cdots \vee g'_{n'}), \\ f &= (f_1 \wedge \cdots \wedge f_m) \wedge f', & g &= (g_1 \vee \cdots \vee g_n) \vee g', \end{aligned}$$

where the orders of applying the operations $\wedge$ and $\vee$ within parentheses are arbitrary. Then if $m \neq 0 \neq n$, using Theorems 3.6, 3.7 and Lemma 3.6, we have

$$f_1 \cdots f_m \alpha \lessdot g_1 \cdots g_n \beta \iff f \lessdot g \iff s\,\breve{o}k\,f \lessdot s\,\breve{o}k\,g \text{ for any } s \in S,$$

and for each $s \in S$, we have

$$\begin{aligned} & s\,\breve{o}k\,f \lessdot s\,\breve{o}k\,g \\ \iff & (s\,\breve{o}k\,f_1 \wedge \cdots \wedge s\,\breve{o}k\,f_m) \wedge f' \lessdot (s\,\breve{o}k\,g_1 \vee \cdots \vee s\,\breve{o}k\,g_n) \vee g' \\ \iff & s\,\breve{o}k\,f_1, \cdots, s\,\breve{o}k\,f_m, \alpha \lessdot s\,\breve{o}k\,g_1, \cdots, s\,\breve{o}k\,g_n, \beta. \end{aligned}$$

This completes the proof in case $m \neq 0 \neq n$. If $m = 0 \neq n$, argue similarly by replacing $s\,\breve{o}k\,f$ by f. If $m \neq 0 = n$, replace $s\,\breve{o}k\,g$ by g.

3.3 Operations $\mathfrak{r}k$

Henceforth we assume that $\mathfrak{P}$ is a subset of $\mathcal{P}\mathbb{P}$ which are closed under the meet $\mathfrak{p} \cap \mathfrak{q}$, the join $\mathfrak{p} \cup \mathfrak{q}$ and the complement $\mathfrak{p}^\circ$ and contain the intervals $(p{\rightarrow}) = \{x \in \mathbb{P} \mid p < x\}$ and $({\leftarrow}p] = \{x \in \mathbb{P} \mid x \leq p\}$ of $\mathbb{P}$ for all $p \in \mathbb{P}$, and define $\underline{p} = \neg({\leftarrow}p]$ and $\overline{p} = (p{\rightarrow})$ for all $p \in \mathbb{P}$, and $\forall = \underline{0}$ and $\exists = \overline{0}$ for $0 = \min \mathbb{P}$. Note that the symbol $\exists$ denotes both the basic relation and the quantifier $\overline{0}$.

Theorem 3.9 *Let $a, b \in E$. Then the following holds.*

(1) Let $s \in S$. Then $s\,\breve{o}\pi\,a\triangle = 1$ iff $a \exists s$. Therefore if $a \in W_\delta$, then $s\,\breve{o}\pi\,a\triangle = as$.

(2) $\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle = 1$ *iff* $\mathfrak{a} \sqsubseteq \mathfrak{b}$.

(3) $\mathfrak{a}\,\mathfrak{p}\pi\,\mathfrak{b}\triangle = (\mathfrak{a} \sqcap \mathfrak{b})\,\mathfrak{p}\pi\,1\triangle$ *for all* $\mathfrak{p} \in \mathfrak{P}$ *and the largest element* 1 *of* W_δ.

(4) $\mathfrak{a}\,\exists\pi\,1\triangle = 1$ *iff there exists an element* $s \in S$ *such that* $s\,\breve{o}\pi\,\mathfrak{a}\triangle = 1$.

Proof We have $s\,\breve{o}\pi\,\mathfrak{a}\triangle = (\mathfrak{a}\triangle)(\pi/s)$ by (W1), and so (W5) shows that the first assertion of (1) holds. (2) is proved by the following reasoning using (W2), (1), (3.1) and the positive definiteness of the $\mathbb{P}$-measure:

$$\begin{aligned}
\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle = 1 &\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ s\,\breve{o}\pi\,\mathfrak{b}\triangle = 0\right\}\right| \leq 0 \\
&\iff \left|\{s \in S \mid \mathfrak{a} \ni s,\ \mathfrak{b} \not\ni s\}\right| \leq 0 \\
&\iff \{s \in S \mid \mathfrak{a} \ni s,\ \mathfrak{b} \not\ni s\} = \emptyset \\
&\iff \text{If } s \in S \text{ and } \mathfrak{a} \ni s \text{ then } \mathfrak{b} \ni s \\
&\iff \mathfrak{a} \sqsubseteq \mathfrak{b}.
\end{aligned}$$

(3) is proved by the following reasoning using (W2), (1) and (W6):

$$\begin{aligned}
\mathfrak{a}\,\mathfrak{p}\pi\,\mathfrak{b}\triangle = 1 &\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ s\,\breve{o}\pi\,\mathfrak{b}\triangle = 1\right\}\right| \in \mathfrak{p} \\
&\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ \mathfrak{b} \ni s\right\}\right| \in \mathfrak{p} \\
&\iff \left|\left\{s \in S \mid \mathfrak{a} \sqcap \mathfrak{b} \ni s\right\}\right| \in \mathfrak{p} \\
&\iff \left|\left\{s \in S \mid \mathfrak{a} \sqcap \mathfrak{b} \ni s,\ 1 \ni s\right\}\right| \in \mathfrak{p} \\
&\iff (\mathfrak{a} \sqcap \mathfrak{b})\,\mathfrak{p}\pi\,1\triangle = 1.
\end{aligned}$$

(4) is proved by part of this reasoning with $\mathfrak{p} = (0\rightarrow)$ and $\mathfrak{b} = 1$ together with the positive definiteness of the $\mathbb{P}$-measure and (1).

Theorem 3.10 *Let* $\mathfrak{a} \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$ *and* $\mathfrak{p} \in \mathfrak{P}$. *Then*

$$\mathfrak{a}\,\neg\mathfrak{p}k\,f = \mathfrak{a}\,\mathfrak{p}k\,f^\Diamond, \qquad \mathfrak{a}\,\mathfrak{p}^\circ k\,f = (\mathfrak{a}\,\mathfrak{p}k\,f)^\Diamond.$$

Proof Let $\theta \in (P - \{k\})\rightarrow S$. Then using (W2) and (W4), we have

$$\begin{aligned}
(\mathfrak{a}\,\neg\mathfrak{p}k\,f)\theta = 1 &\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ f\big((k/s)\theta\big) = 0\right\}\right| \in \mathfrak{p} \\
&\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ f^\Diamond\big((k/s)\theta\big) = 1\right\}\right| \in \mathfrak{p} \\
&\iff (\mathfrak{a}\,\mathfrak{p}k\,f^\Diamond)\theta = 1, \\
(\mathfrak{a}\,\mathfrak{p}^\circ k\,f)\theta = 1 &\iff \left|\left\{s \in S \mid \mathfrak{a} \ni s,\ f\big((k/s)\theta\big) = 1\right\}\right| \in \mathfrak{p}^\circ \\
&\iff (\mathfrak{a}\,\mathfrak{p}k\,f)\theta = 0 \\
&\iff (\mathfrak{a}\,\mathfrak{p}k\,f)^\Diamond\theta = 1.
\end{aligned}$$

Hence the above result.

Theorem 3.11 *Let* $\mathfrak{a} \in E,\ k \in P \in \mathcal{P}K,\ f \in W_P$ *and* $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. *Then*

$$\begin{aligned}
\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})k\,f = \mathfrak{a}\,\mathfrak{p}k\,f \wedge \mathfrak{a}\,\mathfrak{q}k\,f, &\quad \mathfrak{a}\,(\mathfrak{p} \cup \mathfrak{q})k\,f = \mathfrak{a}\,\mathfrak{p}k\,f \vee \mathfrak{a}\,\mathfrak{q}k\,f, \\
\mathfrak{a}\,\neg(\mathfrak{p} \cap \mathfrak{q})k\,f = \mathfrak{a}\,\neg\mathfrak{p}k\,f \wedge \mathfrak{a}\,\neg\mathfrak{q}k\,f, &\quad \mathfrak{a}\,\neg(\mathfrak{p} \cup \mathfrak{q})k\,f = \mathfrak{a}\,\neg\mathfrak{p}k\,f \vee \mathfrak{a}\,\neg\mathfrak{q}k\,f.
\end{aligned}$$

Proof Let $\theta \in (\mathsf{P}-\{\mathsf{k}\})\to\mathsf{S}$. Then using (W2) and (W3), we have

$$\begin{aligned}\big(\mathsf{a}\,(\mathfrak{p}\cap\mathfrak{q})\mathsf{k}\,\mathsf{f}\big)\theta = 1 &\iff \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \in \mathfrak{p}\cap\mathfrak{q}\\ &\iff \begin{cases}\big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \in \mathfrak{p},\\ \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \in \mathfrak{q}\end{cases}\\ &\iff (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta = (\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1\\ &\iff (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta \wedge (\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1\\ &\iff (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}\wedge \mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1.\end{aligned}$$

Hence the first result. The rest of the proof is left to the reader.

Theorem 3.12 *Let* $\mathsf{a},\mathsf{b}\in\mathsf{E},\ \mathsf{k}\in\mathsf{P}\in\mathcal{P}\mathsf{K},\ \mathsf{f}\in W_{\mathsf{P}}$ *and* $\mathsf{p}\in\mathbb{P}$. *Then*

$$\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle,\ \mathsf{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f} \lessdot \mathsf{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}, \qquad \mathsf{a}\,\forall\pi\,\mathsf{b}\triangle,\ \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \lessdot \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}.$$

Proof Let $\theta\in(\mathsf{P}-\{\mathsf{k}\})\to\mathsf{S}$ and assume $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle = \big(\mathsf{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}\big)\theta = 1$. Then $\mathsf{a}\sqsubseteq\mathsf{b}$ by Theorem 3.9 and, since the $\mathbb{P}$-measure is increasing,

$$\big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \le \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \le \mathsf{p},$$

hence $\big(\mathsf{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}\big)\theta = 1$. Therefore $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle,\ \mathsf{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f} \lessdot \mathsf{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}$ by Lemma 3.1. The rest of the proof is left to the reader.

Theorem 3.13 *Let* $\mathsf{a},\mathsf{b}\in\mathsf{E},\ \mathsf{k}\in\mathsf{P}\in\mathcal{P}\mathsf{K},\ \mathsf{f}\in W_{\mathsf{P}}$ *and* $\mathsf{p},\mathsf{q}\in\mathbb{P}$. *Then*

$$\mathsf{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f},\ \mathsf{b}\,(\leftarrow\mathsf{q}]\mathsf{k}\,\mathsf{f} \lessdot (\mathsf{a}\sqcup\mathsf{b})\,\big(\leftarrow(\mathsf{p}+\mathsf{q})\big]\mathsf{k}\,\mathsf{f}.$$

If $\mathbb{P}$ *is linear, then* $(\mathsf{a}\sqcup\mathsf{b})\,\overline{\mathsf{p}+\mathsf{q}}\mathsf{k}\,\mathsf{f} \lessdot \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f},\ \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}$.

Proof Let $\theta\in(\mathsf{P}-\{\mathsf{k}\})\to\mathsf{S}$. Then

$$\begin{aligned}&\big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\sqcup\mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big|\\ &= \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\cup\{\mathsf{s}\in\mathsf{S}\mid \mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big|\\ &\le \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| + \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big|\end{aligned}$$

by (W6) and the subadditivity of the $\mathbb{P}$-measure. Suppose $\big(\mathsf{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}\big)\theta = \big(\mathsf{b}\,(\leftarrow\mathsf{q}]\mathsf{k}\,\mathsf{f}\big)\theta = 1$. Then

$$\big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \le \mathsf{p}, \quad \big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \le \mathsf{q},$$

so $\big|\{\mathsf{s}\in\mathsf{S}\mid \mathsf{a}\sqcup\mathsf{b}\,\exists\,\mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})\theta\big)=1\}\big| \le \mathsf{p}+\mathsf{q}$, hence $\big((\mathsf{a}\sqcup\mathsf{b})\,\big(\leftarrow(\mathsf{p}+\mathsf{q})\big]\mathsf{k}\,\mathsf{f}\big)\theta = 1$. Thus Lemma 3.1 shows that the former assertion holds. Lemma 3.6 and Theorem 3.10 show that the latter assertion follows from the former.

Theorem 3.14 *Let* $\mathsf{a}\in\mathsf{E},\ \mathsf{b}\in\mathsf{S},\ \mathsf{k},\mathsf{l}\in\mathsf{P}\in\mathcal{P}\mathsf{K},\ \mathsf{k}\neq\mathsf{l},\ \mathsf{f}\in W_{\mathsf{P}}$ *and* $\mathfrak{x}\in\{\breve{\mathsf{o}}\}\cup\mathfrak{Q}$. *Let* $\mathsf{a}\in\mathsf{S}$ *in case* $\mathfrak{x}=\breve{\mathsf{o}}$. *Then* $\mathsf{a}\,\mathfrak{x}\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f}) = \mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f})$.

Proof When $\mathfrak{x} = \breve{o}$, this holds by Corollary 3.5.1. Suppose $\mathfrak{x} = \mathfrak{p} \in \mathfrak{P}$, as the proof in the case $\mathfrak{x} = \neg\mathfrak{p} \in \neg\mathfrak{P}$ is similar. Let $\theta \in (\mathsf{P} - \{\mathsf{k}, \mathsf{l}\}) \to \mathsf{S}$. Then

$$\begin{aligned}
\big(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,(\mathsf{b}\,\breve{o}\mathsf{l}\,\mathsf{f})\big)\theta = 1 &\iff \big|\big\{\mathsf{s} \in \mathsf{S} \mid \mathsf{a} \ni \mathsf{s},\ (\mathsf{b}\,\breve{o}\mathsf{l}\,\mathsf{f})\big((\mathsf{k}/\mathsf{s})\theta\big) = 1\big\}\big| \in \mathfrak{p} \\
&\iff \big|\big\{\mathsf{s} \in \mathsf{S} \mid \mathsf{a} \ni \mathsf{s},\ \mathsf{f}\big((\mathsf{l}/\mathsf{b})(\mathsf{k}/\mathsf{s})\theta\big) = 1\big\}\big| \in \mathfrak{p}, \\
\big(\mathsf{b}\,\breve{o}\mathsf{l}\,(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\big)\theta = 1 &\iff (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\big((\mathsf{l}/\mathsf{b})\theta\big) = 1 \\
&\iff \big|\big\{\mathsf{s} \in \mathsf{S} \mid \mathsf{a} \ni \mathsf{s},\ \mathsf{f}\big((\mathsf{k}/\mathsf{s})(\mathsf{l}/\mathsf{b})\theta\big) = 1\big\}\big| \in \mathfrak{p},
\end{aligned}$$

and $(\mathsf{l}/\mathsf{b})(\mathsf{k}/\mathsf{s})\theta = (\mathsf{k}/\mathsf{s})(\mathsf{l}/\mathsf{b})\theta$, hence $\big(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,(\mathsf{b}\,\breve{o}\mathsf{l}\,\mathsf{f})\big)\theta = \big(\mathsf{b}\,\breve{o}\mathsf{l}\,(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\big)\theta$. Thus $\mathsf{a}\,\mathfrak{p}\mathsf{k}\,(\mathsf{b}\,\breve{o}\mathsf{l}\,\mathsf{f}) = \mathsf{b}\,\breve{o}\mathsf{l}\,(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})$.

3.4 Nominalizers and variables

Let $\mathcal{V}$ be the set of all triples (W, Φ, ν) of a MPC world $W \in \mathcal{W}$, a $\mathbb{C}$-denotation Φ into W and an $\mathbb{X}$-denotation ν into W. We will frequently use the fact that the metadenotation $\Phi^* \in \mathsf{A} \to W^{V_{\mathbb{X},W}}$ is an L-homomorphism for (2.1) and the projection $\mathrm{pr}_\nu \in W^{V_{\mathbb{X},W}} \to W$ is an M-homomorphism for (2.2).

Theorem 3.15 *Let* $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n \in \mathsf{H}$, $\alpha, \beta \in \mathsf{H}^*$, *and* $\mathsf{k} \in \mathsf{K}$. *Assume that* k *belongs to the ranges of* $\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n$ *but does not belong to those of the predicates in* $\alpha \cup \beta$. *Then the following holds for all* $\mathsf{a} \in \mathsf{A}_\varepsilon$ *:*

$$\begin{aligned}
&\mathsf{f}_1 \cdots \mathsf{f}_m \alpha \preccurlyeq \mathsf{g}_1 \cdots \mathsf{g}_n \beta \\
&\implies \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_1, \cdots, \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{f}_m, \alpha \preccurlyeq \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_1, \cdots, \mathsf{a}\,\breve{o}\mathsf{k}\,\mathsf{g}_n, \beta.
\end{aligned} \qquad \text{(gen. case+ law)}$$

Assume furthermore that a variable $\mathsf{x} \in \mathbb{X}_\varepsilon$ *does not occur free in the predicates in* $\{\mathsf{f}_1, \ldots, \mathsf{f}_m, \mathsf{g}_1, \ldots, \mathsf{g}_n\} \cup \alpha \cup \beta$. *Then the following holds:*

$$\begin{aligned}
&\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_1, \cdots, \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f}_m, \alpha \preccurlyeq \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_1, \cdots, \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{g}_n, \beta \\
&\implies \mathsf{f}_1 \cdots \mathsf{f}_m \alpha \preccurlyeq \mathsf{g}_1 \cdots \mathsf{g}_n \beta.
\end{aligned} \qquad \text{(gen. case− law)}$$

Proof In order to prove the gen. case− law, let $\alpha = \mathsf{f}'_1 \cdots \mathsf{f}'_{m'}$ and $\beta = \mathsf{g}'_1 \cdots \mathsf{g}'_{n'}$ with $\mathsf{f}'_1, \ldots, \mathsf{f}'_{m'}, \mathsf{g}'_1, \ldots, \mathsf{g}'_{n'} \in \mathsf{H}$. Let $(W, \Phi, \nu) \in \mathcal{V}$. Then Theorem 3.4 shows that the premise of the gen. case− law implies the following:

$$\begin{aligned}
&(\Phi^*\mathsf{x})\nu\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{f}_1)\nu, \cdots, (\Phi^*\mathsf{x})\nu\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{f}_m)\nu, (\Phi^*\mathsf{f}'_1)\nu, \cdots, (\Phi^*\mathsf{f}'_{m'})\nu \\
&\lessdot (\Phi^*\mathsf{x})\nu\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{g}_1)\nu, \cdots, (\Phi^*\mathsf{x})\nu\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{g}_n)\nu, (\Phi^*\mathsf{g}'_1)\nu, \cdots, (\Phi^*\mathsf{g}'_{n'})\nu.
\end{aligned}$$

Let $\mathsf{s} \in W_\varepsilon$ and define $\nu' = (\mathsf{x}/\mathsf{s})\nu$. Then the above holds with ν replaced by ν', and so (3.11), (3.5) and Theorem 4.1 of Gomi (09b) show that

$$\begin{aligned}
&\mathsf{s}\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{f}_1)\nu, \cdots, \mathsf{s}\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{f}_m)\nu, (\Phi^*\mathsf{f}'_1)\nu, \cdots, (\Phi^*\mathsf{f}'_{m'})\nu \\
&\lessdot \mathsf{s}\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{g}_1)\nu, \cdots, \mathsf{s}\,\breve{o}\mathsf{k}\,(\Phi^*\mathsf{g}_n)\nu, (\Phi^*\mathsf{g}'_1)\nu, \cdots, (\Phi^*\mathsf{g}'_{n'})\nu
\end{aligned}$$

holds. Since $\mathsf{s} \in W_\varepsilon$ is arbitrary, Theorem 3.8 shows that

$$(\Phi^*\mathsf{f}_1)\nu, \cdots, (\Phi^*\mathsf{f}_m)\nu, (\Phi^*\mathsf{f}'_1)\nu, \cdots, (\Phi^*\mathsf{f}'_{m'})\nu$$

$$\lessdot (\Phi^* g_1)\nu, \cdots, (\Phi^* g_n)\nu, (\Phi^* g'_1)\nu, \cdots, (\Phi^* g'_{n'})\nu$$

holds. Since $(W, \Phi, \nu) \in \mathcal{V}$ is arbitrary, Theorem 3.4 shows that the conclusion of the gen. case− law holds. The proof of the gen. case+ law is left to the reader.

Theorem 3.16 *Let* $a \in G,\ f \in H,\ x \in \mathbb{X}_\varepsilon,\ K_f = \{k\}$ *and* $\mathfrak{x} \in \mathfrak{Q}$*. Assume that* x *does not occur free in* f*. Then* $a\,\mathfrak{x} k\, f \asymp a\,\mathfrak{x}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle$*.*

Proof Suppose $\mathfrak{x} = \mathfrak{p} \in \mathfrak{P}$, as the proof in the case $\mathfrak{x} = \neg\mathfrak{p} \in \neg\mathfrak{P}$ is similar. Both $a\,\mathfrak{p}k\,f$ and $a\,\mathfrak{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle$ belong to $A_\emptyset$. Let $(W, \Phi, \nu) \in \mathcal{V}$. Then

$$\begin{aligned}
&\Big(\Phi^*\big(a\,\mathfrak{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle\big)\Big)\nu = 1\\
&\iff (\Phi^* a)\nu\,\mathfrak{p}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle = 1\\
&\iff \Big|\Big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ s\,\breve{o}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle = 1\Big\}\Big| \in \mathfrak{p},
\end{aligned}$$

and as for the equation $s\,\breve{o}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle = 1$, we have

$$\begin{aligned}
&s\,\breve{o}\pi\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\triangle\\
&= \Big(\Big(\big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\,\Omega x\Big)\nu\Big)s && \text{(by Theorem 3.9)}\\
&= \big((\Phi^* x)\,\breve{o}k\,(\Phi^* f)\big)\big((x/s)\nu\big) && \text{(by (2.5))}\\
&= (\Phi^* x)\big((x/s)\nu\big)\,\breve{o}k\,(\Phi^* f)\big((x/s)\nu\big)\\
&= s\,\breve{o}k\,(\Phi^* f)\nu,
\end{aligned}$$

where the last equality follows from (3.11), (3.5) and Theorem 4.1 of Gomi (09b), hence

$$\begin{aligned}
&\Big(\Phi^*\big(a\,\mathfrak{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle\big)\Big)\nu = 1\\
&\iff \Big|\Big\{s \in W_\varepsilon \mid (\Phi^* a)\nu \ni s,\ s\,\breve{o}k\,(\Phi^* f)\nu = 1\Big\}\Big| \in \mathfrak{p}\\
&\iff (\Phi^* a)\nu\,\mathfrak{p}k\,(\Phi^* f)\nu = 1\\
&\iff \big(\Phi^*(a\,\mathfrak{p}k\,f)\big)\nu = 1.
\end{aligned}$$

Thus $a\,\mathfrak{p}k\,f \asymp a\,\mathfrak{p}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle$.

Theorem 3.17 *Let* $a \in G,\ f, g \in A_\emptyset$ *and* $x \in \mathbb{X}_\varepsilon$*. Assume that* x *does not occur free in* f*. Then* $f, a\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle \preccurlyeq a\,\forall\pi\,(g\,\Omega x)\triangle$*.*

Proof Since $f,\ a\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle,\ a\,\forall\pi\,(g\,\Omega x)\triangle \in A_\emptyset$, we need to show

$$(\Phi^* f)\nu = \Big(\Phi^*\big(a\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\Big)\nu = 1 \implies \Big(\Phi^*\big(a\,\forall\pi\,(g\,\Omega x)\triangle\big)\Big)\nu = 1$$

for each $(W, \Phi, \nu) \in \mathcal{V}$. We have

$$\begin{aligned}
\Big(\Phi^*\big(a\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle\big)\Big)\nu &= (\Phi^* a)\nu\,\forall\pi\big((\Phi^* f \Rightarrow \Phi^* g)\,\Omega x\big)\nu\triangle,\\
\Big(\Phi^*\big(a\,\forall\pi\,(g\,\Omega x)\triangle\big)\Big)\nu &= (\Phi^* a)\nu\,\forall\pi\big((\Phi^* g)\,\Omega x\big)\nu\triangle,
\end{aligned}$$

and so Theorem 3.9 shows that the following holds:

$$\big(\Phi^*\big(\mathsf{a}\,\forall\pi\,\big((\mathsf{f}\Rightarrow\mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big)\big)v=1 \iff (\Phi^*\mathsf{a})v\sqsubseteq\big((\Phi^*\mathsf{f}\Rightarrow\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v,$$
$$\big(\Phi^*\big(\mathsf{a}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big)\big)v=1 \iff (\Phi^*\mathsf{a})v\sqsubseteq\big((\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v.$$

Therefore assume $(\Phi^*\mathsf{f})v=1$, $(\Phi^*\mathsf{a})v\sqsubseteq\big((\Phi^*\mathsf{f}\Rightarrow\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v$, $\mathsf{s}\in W_\varepsilon$ and $(\Phi^*\mathsf{a})v\ni\mathsf{s}$. Then $(\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})v\big)=1$ by Theorem 4.1 of Gomi (09b), and

$$\begin{aligned}
1 &= \big(\big((\Phi^*\mathsf{f}\Rightarrow\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v\big)\mathsf{s} && \text{(by (3.1))}\\
&= (\Phi^*\mathsf{f}\Rightarrow\Phi^*\mathsf{g})\big((\mathsf{x}/\mathsf{s})v\big) && \text{(by (2.5))}\\
&= (\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})v\big)\Rightarrow(\Phi^*\mathsf{g})\big((\mathsf{x}/\mathsf{s})v\big)\\
&= (\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})v\big)\Rightarrow\big(\big((\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v\big)\mathsf{s} && \text{(by (2.5))}.
\end{aligned}$$

Thus $\big(\big((\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v\big)\mathsf{s}=1$, that is, $\big((\Phi^*\mathsf{g})\,\Omega\mathsf{x}\big)v\ni\mathsf{s}$. This completes the proof.

Theorem 3.18 *Let* $\mathsf{a}\in\mathsf{A}_\varepsilon$, $\mathsf{f}\in\mathsf{A}_\emptyset$ *and* $\mathsf{x}\in\mathbb{X}_\varepsilon$. *Assume that* x *is free from* a *in* f. *Then* $\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\asymp\mathsf{f}(\mathsf{x}/\mathsf{a})$, *where* (x/a) *denotes the substitution of* a *for* x.

Proof Both $\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle$ and $\mathsf{f}(\mathsf{x}/\mathsf{a})$ belong to $\mathsf{A}_\emptyset$ by (4.4) of Gomi (09b). Let $(W,\Phi,v)\in\mathcal{V}$. Then

$$\begin{aligned}
&\big(\Phi^*\big(\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\big)\big)v\\
&= (\Phi^*\mathsf{a})v\,\breve{o}\pi\,\big(\big((\Phi^*\mathsf{f})\,\Omega\mathsf{x}\big)v\big)\triangle\\
&= \big(\big((\Phi^*\mathsf{f})\,\Omega\mathsf{x}\big)v\big)\big((\Phi^*\mathsf{a})v\big) && \text{(by Theorem 3.9)}\\
&= (\Phi^*\mathsf{f})\big(\big(\mathsf{x}/(\Phi^*\mathsf{a})v\big)v\big) && \text{(by (2.5))}\\
&= \big(\Phi^*\big(\mathsf{f}(\mathsf{x}/\mathsf{a})\big)\big)v && \text{(by Theorem 4.2 of Gomi (09b))}.
\end{aligned}$$

Hence $\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\asymp\mathsf{f}(\mathsf{x}/\mathsf{a})$.

Let x_0 be an arbitrary element of $\mathbb{X}_\varepsilon$ and define $\mathsf{one}=(\mathsf{x}_0\,\forall\pi\,\mathsf{x}_0\triangle)\,\Omega\mathsf{x}_0$. Then $\mathsf{one}\in\mathsf{A}_\delta$. The following theorems center around this element.

Theorem 3.19 *If* $(W,\Phi,v)\in\mathcal{V}$, *then* $(\Phi^*\mathsf{one})v$ *is the largest element* 1 *of* W_δ, *while* $\big(\Phi^*(\mathsf{one}^\square)\big)v$ *is the smallest element* 0 *of* W_δ.

Proof Since $\mathsf{one}\in\mathsf{A}_\delta$, $(\Phi^*\mathsf{one})v\in W_\delta=W_\varepsilon\to\mathbb{T}$. Let $\mathsf{s}\in W_\varepsilon$. Then

$$\begin{aligned}
\big((\Phi^*\mathsf{one})v\big)\mathsf{s} &= \big(\big(\Phi^*\big((\mathsf{x}_0\,\forall\pi\,\mathsf{x}_0\triangle)\,\Omega\mathsf{x}_0\big)\big)v\big)\mathsf{s}\\
&= \big(\big(\big((\Phi^*\mathsf{x}_0)\,\forall\pi\,(\Phi^*\mathsf{x}_0)\triangle\big)\,\Omega\mathsf{x}_0\big)v\big)\mathsf{s}\\
&= \big((\Phi^*\mathsf{x}_0)\,\forall\pi\,(\Phi^*\mathsf{x}_0)\triangle\big)\big((\mathsf{x}_0/\mathsf{s})v\big) && \text{(by (2.5))}\\
&= (\Phi^*\mathsf{x}_0)\big((\mathsf{x}_0/\mathsf{s})v\big)\,\forall\pi\,\big((\Phi^*\mathsf{x}_0)\big((\mathsf{x}_0/\mathsf{s})v\big)\big)\triangle\\
&= \mathsf{s}\,\forall\pi\,\mathsf{s}\triangle.
\end{aligned}$$

The final equality holds by (3.11) and (3.5) of Gomi (09b). Thus $(\Phi^*\mathsf{one})v=1$ by Theorem 3.9, and so $\big(\Phi^*(\mathsf{one}^\square)\big)v=\big((\Phi^*\mathsf{one})v\big)^\square=1^\square=0$.

Theorem 3.20 *Let* $\mathsf{f} \in \mathsf{A}_\emptyset$ *and* $\mathsf{x} \in \mathbb{X}_\varepsilon$. *Then* $\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \preccurlyeq \mathsf{f}$.

Proof Both $\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle$ and f belong to $\mathsf{A}_\emptyset$. Let $(W, \Phi, \nu) \in \mathcal{V}$. Then

$$\begin{aligned}
&\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\big)\big)\nu = 1 && \\
&\iff (\Phi^*\mathsf{one})\nu\,\forall\pi\,\big((\Phi^*\mathsf{f})\,\Omega\mathsf{x}\big)\nu\triangle = 1 && \\
&\iff 1\,\forall\pi\,\big((\Phi^*\mathsf{f})\,\Omega\mathsf{x}\big)\nu\triangle = 1 && \text{(by Theorem 3.19)}\\
&\iff \big(\big((\Phi^*\mathsf{f})\,\Omega\mathsf{x}\big)\nu\big)\mathsf{s} = 1 \text{ for all } \mathsf{s} \in W_\varepsilon && \text{(by Theorem 3.9)}\\
&\iff (\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})\nu\big) = 1 \text{ for all } \mathsf{s} \in W_\varepsilon && \text{(by (2.5))}\\
&\implies (\Phi^*\mathsf{f})\nu = 1 && \text{(since } (\mathsf{x}/\nu\mathsf{x})\nu = \nu).
\end{aligned}$$

Therefore $\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \preccurlyeq \mathsf{f}$.

Theorem 3.21 *Let* $\mathsf{f} \in \mathsf{A}_\emptyset$ *and* $\mathsf{x} \in \mathbb{X}_\varepsilon$. *Then* $\preccurlyeq \mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle$ *iff* $\preccurlyeq \mathsf{f}$.

Proof Assume $\preccurlyeq \mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle$. Then $\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\big)\big)\nu = 1$ for all $(W, \Phi, \nu) \in \mathcal{V}$, and so $(\Phi^*\mathsf{f})\nu = 1$ for all $(W, \Phi, \nu) \in \mathcal{V}$ by Theorem 3.20, hence $\preccurlyeq \mathsf{f}$. Conversely assume $\preccurlyeq \mathsf{f}$. Then $(\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})\nu\big) = 1$ for all $(W, \Phi, \nu) \in \mathcal{V}$ and all $\mathsf{s} \in W_\varepsilon$, and so $\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle\big)\big)\nu = 1$ for all $(W, \Phi, \nu) \in \mathcal{V}$ by the reasoning in the proof of Theorem 3.20, hence $\preccurlyeq \mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle$.

Theorem 3.22 *Let* $\mathsf{x} \in \mathbb{X}_\varepsilon$, $\mathsf{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$ *and* $\mathsf{K}_\mathsf{f} = \{\mathsf{k}\}$. *Assume that* x *does not occur free in either* a *or* f. *Then* $\mathsf{one}\,\forall\pi\,\big(\big((\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f})\big)\,\Omega\mathsf{x}\big)\triangle \asymp \mathsf{a}\,\forall\mathsf{k}\,\mathsf{f}$.

Proof Let $(W, \Phi, \nu) \in \mathcal{V}$. Then the reasoning in the proof of Theorem 3.20 shows

$$\begin{aligned}
&\big(\Phi^*\big(\mathsf{one}\,\forall\pi\,\big(\big((\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f})\big)\,\Omega\mathsf{x}\big)\triangle\big)\big)\nu = 1\\
&\iff \big(\Phi^*\big((\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f})\big)\big)\big((\mathsf{x}/\mathsf{s})\nu\big) = 1 \text{ for all } \mathsf{s} \in W_\varepsilon.
\end{aligned}$$

Also $(\Phi^*\mathsf{x})\big((\mathsf{x}/\mathsf{s})\nu\big) = \mathsf{s}$, $(\Phi^*\mathsf{a})\big((\mathsf{x}/\mathsf{s})\nu\big) = (\Phi^*\mathsf{a})\nu$ and $(\Phi^*\mathsf{f})\big((\mathsf{x}/\mathsf{s})\nu\big) = (\Phi^*\mathsf{f})\nu$ as before. Also $\mathsf{s}\,\breve{\mathsf{o}}\pi\,(\Phi^*\mathsf{a})\nu\triangle = 1$ iff $(\Phi^*\mathsf{a})\nu \ni \mathsf{s}$ by Theorem 3.9. Therefore

$$\begin{aligned}
&\big(\Phi^*\big((\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f})\big)\big)\big((\mathsf{x}/\mathsf{s})\nu\big) = 1 \text{ for all } \mathsf{s} \in W_\varepsilon\\
&\iff \big(\mathsf{s}\,\breve{\mathsf{o}}\pi\,(\Phi^*\mathsf{a})\nu\triangle\big) \Rightarrow \big(\mathsf{s}\,\breve{\mathsf{o}}\mathsf{k}\,(\Phi^*\mathsf{f})\nu\big) = 1 \text{ for all } \mathsf{s} \in W_\varepsilon\\
&\iff \text{if } \mathsf{s} \in W_\varepsilon \text{ and } \mathsf{s}\,\breve{\mathsf{o}}\pi\,(\Phi^*\mathsf{a})\nu\triangle = 1 \text{ then } \mathsf{s}\,\breve{\mathsf{o}}\mathsf{k}\,(\Phi^*\mathsf{f})\nu = 1\\
&\iff \text{if } \mathsf{s} \in W_\varepsilon \text{ and } (\Phi^*\mathsf{a})\nu \ni \mathsf{s} \text{ then } \big((\Phi^*\mathsf{f})\nu\big)(\mathsf{k}/\mathsf{s}) = 1\\
&\iff \big|\big\{\mathsf{s} \in W_\varepsilon \mid (\Phi^*\mathsf{a})\nu \ni \mathsf{s},\ \big((\Phi^*\mathsf{f})\nu\big)(\mathsf{k}/\mathsf{s}) = 0\big\}\big| \le 0\\
&\iff (\Phi^*\mathsf{a})\nu\,\forall\mathsf{k}\,(\Phi^*\mathsf{f})\nu = 1\\
&\iff \big(\Phi^*(\mathsf{a}\,\forall\mathsf{k}\,\mathsf{f})\big)\nu = 1.
\end{aligned}$$

Thus $\mathsf{one}\,\forall\pi\,\big(\big((\mathsf{x}\,\breve{\mathsf{o}}\pi\,\mathsf{a}\triangle) \Rightarrow (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{f})\big)\,\Omega\mathsf{x}\big)\triangle \asymp \mathsf{a}\,\forall\mathsf{k}\,\mathsf{f}$.

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