Mathematical Noology

Intellectual Machines, Logic, Tongues and Algebra

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Ever-Growing WWW Publication
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*Led by the given name against Japanese tradition.
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Contents

0.1 Academic profile of the author

Kensaku Gomi, PhD,[*] engaged in researches in finite group theory (a branch of algebra) for the Department of Mathematics, College of General Education, University of Tokyo from Apr. 1973 through Mar. 1992 and in new mathematical psychology, especially noology for the Graduate School of Mathematical Sciences thereof from Apr. 1992 through Mar. 2010 and for himself ever after.

0.2 Acknowledgments of the author

His thanks are due to all known and unknown individuals who have contributed to the development of today's civilization, which makes it possible for him to publish this monograph. He is especially grateful to the founders of grammar, logic, algebra, evolutionism and set theory, as well as to the inventors of the electronic computer, the World Wide Web and the tools built upon them for editing and publishing, all listed in chronological order.

0.3 Acronyms inclusive of initialisms

ABCL asymmetric bipartible case logic, BCL bipartible case logic, CL case logic, DDU descriptive or deductive utterance, DT descriptive tongue, DU descriptive utterance, DW denotable world, FPL first-order predicate logic, GL general logic, ICL impartible case logic, IU intellectual unit, KML Kripkean modal logic, MN mathematical noology, MP mathematical psychology, PCL partible case logic, PL propositional logic, PLQ partially latticed quasialgebra, PU perceiving unit, RU ratiocinating unit, RDT rephrased descriptive tongue, RDU rephrased descriptive utterance, SW semasiological world, USA universal sorted algebra, UTA universal total algebra.

0.4 AMS Euler font for mathematics

$ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz$
$\mathcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$
$\mathbb{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$
$\mathfrak{ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz}$
$\mathrm{A}\mathrm{B}\Gamma\Delta\mathrm{E}\mathrm{Z}\mathrm{H}\Theta\mathrm{I}\mathrm{K}\Lambda\mathrm{M}\mathrm{N}\Xi\mathrm{O}\Pi\mathrm{P}\Sigma\mathrm{T}\Upsilon\Phi\mathrm{X}\Psi\Omega\alpha\beta\gamma\delta\epsilon\varepsilon\zeta\eta\theta\vartheta\iota\kappa\varkappa\lambda\mu\nu\xi o\pi\varpi\rho\sigma\tau\upsilon\phi\varphi\chi\psi\omega$
0123456789
Confusable symbols: b, $\flat$ (flat); Δ, $\triangle$ (triangle); E, $\sqsubseteq$ (basic relation); $\mathcal{E}$, ε, ϵ, $\in$ (set membership); φ, ϕ, $\emptyset$ (empty set); Λ, $\wedge$ (meet); ι, l, I, 1; 0, O, o, $\circ$ (circle); υ, ν, v, V, $\vee$ (join); K, κ, $\varkappa$, x, χ, X, $\times$ (direct product, multiplication).

[*] His doctoral thesis "Characterizations of linear groups of low rank" has been published in *J. Fac. Sci. Univ. Tokyo* 23 (1976) and archived in http://hdl.handle.net/2261/7339.

Chapter 1

From Noology to Case Logic via General Logic

Know thyself[1.1]

Noology is the science of intellectual phenomena. Its origin is probably logic, or the philosophic knowledge of intellection, in ancient Greece. The present monograph, however, is neither philosophic nor exegetic. Assuming that you are interested in your own intellection and appreciate creativity and intellectual independence, I aim this monograph at acquainting you with *mathematical noology* (MN) which I originated and have developed in collaboration with my pupils, notably Yasuaki Mizumura and Yôsuke Takaoka[1.2].

This monograph will also serve as an abridged and partly improved version of my other ever-growing WWW publication of wider scope[1.3] by which I launched new *mathematical psychology* (MP) in 1997 and educated some students at the Graduate School of Mathematical Sciences and the Department of Mathematics, University of Tokyo (s. §0.1)[1.4]. Since it is written in Japanese and has grown too long by providing every inspiration for the students, this monograph will help everyone get its gist, particularly about intellection.

The main purpose of this introductory chapter is to explain how noology leads to mathematics, especially algebra and logic. This chapter as such is transdisciplinary and not quite self-contained. The transcended disciplines and the expected range of knowledge will be implicit in the text and notes.

[1.1] The ancient aphorism I read as Remark 1.2.8 (s. [1.10][1.49]). See also Remarks 1.2.9 and 1.2.10 for its pre-Christian amplification *Know thyself, and thou shalt know the universe and God* and my own conclusion *Know God, and thou shalt know what must be done.*

[1.2] Takaoka romanizes his given name into *Yohsuke* (s. [1.72]).

[1.3] *Sûri Sinrigaku* (*Mathematical Psychology*), https://gomikensaku.github.io/homepage/ (s. [1.72]). Its scope includes emotion, volition, etc. as well as intellection.

[1.4] The abbreviation *s.* stands for the word *see.*

The other chapters 2–6 are intended for the mathematical steps of the basic flow chart (Fig. 1.1) below and so written in a style of mathematics for the most part (s. §1.4). The mathematical part is almost self-contained except that it requires a rudimentary knowledge of sets and orders including a knowledge of lattices (esp. Boolean ones and complete ones) and ordinal numbers.

Chapter 2 on logic spaces and deduction systems presents an abstract theory of semantics and deduction, amplifying a published paper[1.5] of mine. Chapter 3 on logic systems presents a general theory of syntax and semantics and links it to logic spaces by means of basic facts on (general) algebras[1.6]. Syntax, semantics and deduction are the pillars of logic. Thus Chapters 2 and 3 together present a theory of *general logic* (GL). You can read it regardless of MN, if you like, and reading it up, you will have an understanding of what logic is and find that GL should be dealt with by a branch of algebra.

Chapters 4–6 present a practical and specific theory of *case logic* (CL) designed for MN and organized in line with the theory of GL in Chapters 2 and 3. Of course, it is specific as contrasted with the theory of GL. The logic system CL in itself is rather general in that it will explain a variety of intellectual phenomena and others. In particular, it has a concept of partibility of entities, and Chapters 4 and 5 focus on *impartible CL* (ICL) and bipartible CL respectively. They are the simplest CL and the next simplest CL and so will help you understand the general CL in Chapter 6.

Notes including bibliographic ones are usually given at the foot of the primary page concerned, and the bracketed numbers refer you to them; for example, [1.3] refers you to Footnote 1.3. Bibliographic ones are few in number, because I enjoy a stand-alone theory on the basis of public knowledge (s. §0.2) and have put exegesis aside. Thus most notes are warnings against hasty reading.

No index is given, because you can locate every word on your monitor.

1.1 Noology as mathematical science

When our department of mathematics merged into the newborn graduate school of mathematical sciences in 1992, I understood that I had to engage in a kind of mathematical science distinguishing it from mathematics, which was one of my motives for converting from finite group theory to new MP (s. §0.1).

Mathematics in the modern sense is the totality of the study by deductive thinking based on the concept of sets and starting with definitions (s. §1.4),[1.7] although some mathematicians also resort to empirical thinking without sufficient set-theoretic bases or explicit definitions in the prior process of mathematizing the scientific problems they face, and so shall I in this chapter.

Mathematical science is still broader. It is the totality of the actions which follow the following four-step flow chart involving both mathematics and em-

[1.5] "Theory of completeness for logical spaces," *Logica Universalis* 3 (2009), 243–291.

[1.6] **(Algebra)** The noun *algebra* means both a branch of study and its object and is countable in the latter sense. The same remark applies to the noun *logic* (s. [2.3]) and others.

[1.7] The study of groups is one of the best examples of mathematics in this modern sense.

piricism and may also be called the science by means of mathematical models.

Figure 1.1: The flow chart for mathematical science

$\boxed{\text{Observation of a phenomenon}}$

$\downarrow$

$\boxed{\text{Construction of a mathematical model based on the observation}}$

$\downarrow$

$\boxed{\text{Mathematical analysis of the constructed model}}$

$\downarrow$

$\boxed{\text{Understanding of the phenomenon by the results of the analysis}}$

The understanding of the phenomenon will furthermore lead to predictions about the phenomenon by it or its practical applications or both.[1.8] Although we should not be endlessly addicted to the mathematical steps, we should make them as large as possible, because mathematics provides one of the most effective tools and the most expressive and rigorous languages of science.

As of today, I am more than halfway through the flow chart specialized for MN thanks to my pupils. The logic system CL together with a certain deduction system on it is intended to provide a principal mathematical model for MN, and our mathematical analysis of the model is well under way (s. §1.3.11). I will clarify in §1.2.6 what it models and what we observed in order to construct it. A possible utilitarian end of the analysis is to have an understanding of human intelligence in order to apply it to artificial intelligence. A spin-off of the model is a concept of God which leads us to the perspective that science, logic, ethics and religion can harmonize into what I call eusophy, i.e. pursuit of human omniscience for survival (s. [1.38]) of the human species.[1.9]

Remark 1.1.1 (The convention and principle of mathematical science) Regarding an intellectual object O as a mathematical concept M means abstracting M from O and sometimes means adopting M as a mathematical model of O. Adopting M as a good model of O means abstracting M from a characteristic of O, and this is why M and its mathematical analysis help understand O.

[1.8] The study of heredity by Gregor Mendel I will mention below is one of the best examples of mathematical sciences in this sense, although mathematics he used was elementary.

[1.9] *Foundations of Eusophy: A proscience general concept of God with a mathematical model*, https://gomikensaku.github.io/homepage/, ever-growing WWW publication, since 2014. See also Remarks 1.2.9 and 1.2.10.

1.2 Aims and methods of mathematical noology

Here I lift the curtain on MN, which is woven from the four nominals in the subtitle: *algebra*[1.10] is the weft and the others are the warp (s. Remark 1.2.4).

1.2.1 An intermediate aim and Mendel's method

MN is the science of intellection that is the process of perceiving and ratiocinating about intellectual objects, and the subtitle is led by the nominal phrase *intellectual machines*. Thus I aim to study human perception and ratiocination, comparing them to computers' under the following three assumptions.

1. There exists an *intellectual unit* (IU) in the human brain.
2. The IU consists of a *perceiving unit* (PU) and a *ratiocinating unit* (RU).
3. The three units IU, PU and RU are machines.

My intermediate aim under certain additional assumptions (s. the text near [1.16]) is to find out some universal truth about the possible[1.11] ability of the IU. You will, however, find in §1.2.6 that it can even be our ultimate aim.

I will soon clarify the meanings of the words and phrases around Assumptions 1–3. The three machines in Assumption 3 are different from ordinary ones especially in that they are believed to have been created by organic evolution. The Darwinian belief and the fact that we each have our native tongues will propel MN (s. §1.2.5) and separate it from the computer theory and others.

While direct observation of the IU still seems neither possible nor prudent, the indirect method urged below seems practicable and productive. It may be compared to Mendel's method (s. [1.8]). As his factors (or genes) were unknowns in *organisms*, so my IU is an unknown in the human brain. Still, as he was able to grasp the nature of the factors by observing phenotypes of pea plants which were supposed to be modified expressions of their genotypes, so I shall be able to grasp the nature of the IU by observing something which is supposed to be a modified expression of the structure of the IU. Metaphorically speaking, MN seeks a theory of the genotypes on observation of the phenotypes; in fact, we also observe the environment which modifies the phenotypes. I will clarify in §1.2.6 what the environment and phenotypes for MN should be.

1.2.2 Machines as algebras and their possible ability

I mean by the word *machine* above any device that transforms various tuples of materials into various products according to various processes, disregarding other components of the device such as its storage, searcher, transporter and

[1.10] **Quotations** and titles of publications will be printed in slant letters $a\,b\,c \dots z\,A\,B\,C \dots Z$ without quotation marks. Distinguish them from italics $a\,b\,c \dots z\,A\,B\,C \dots Z$ for emphasis.

[1.11] **(Possible, existing and potential)** The adjective *possible* means *capable of existing*, where *existing* is a gerund, and so the adjective *existing* implies *possible*, while the adjective *potential* means *possible and not existing*.

motivity.[1.12] To give an example, any kitchen together with a cook is a machine in this sense, because it transforms various tuples of ingredients into various dishes according to various cooking processes, or recipes, ranging from simple ones (e.g. *cut*, *mix*, *sauté*, etc.) to complex ones. To give another example, any calculator is naturally a machine in this sense. Ideal ones transform each couple (a, b) of numbers into the four numbers $a+b$, $a-b$, $a \times b$ and $a \div b$ according to the four basic calculation processes, or arithmetic operations ($b \neq 0$ for division), while actual ones deal with some limited numbers and approximations.

Each machine may be regarded as an algebra as defined in §3.1.2, i.e. a set S equipped with an algebraic structure $\mathcal{O}$ that is a family of (partial) operations on S. Here S is a suitable set containing all possible materials and products of the machine (s. [1.11]), and $\mathcal{O}$ is the family of all its possible processes regarded as operations on S, that is, if a process α in $\mathcal{O}$ transforms a tuple $(a_1, \dots, a_k)$ of materials into a product a, then a is the value $\alpha(a_1, \dots, a_k)$ of α regarded as an operation.[1.13] As for kitchens, S should be the set of (the names for) all possible ingredients and dishes (s. [1.13]), $\mathcal{O}$ is the family of all possible recipes,[1.14] and if a dish a is made from a tuple $(a_1, \dots, a_k)$ of ingredients by a recipe α, then $\alpha(a_1, \dots, a_k) = a$. As for ideal calculators, S should be the set of all real numbers, $\mathcal{O}$ is the family $(+, -, \times, \div)$ of the basic arithmetic operations and $*(a, b) = a * b$ for each $* \in \{+, -, \times, \div\}$ and each $(a, b) \in S^2$ ($b \neq 0$ for $\div$).

We may abstract an algebra $(S, \mathcal{O})$ from each machine in this way (s. Remark 1.1.1) and refer to S and $\mathcal{O}$ as its domain and processual algebraic structure. By virtue of the abstraction, understanding the structure of the machine is reduced to understanding its domain and processual algebraic structure.

Moreover, the possible ability of the machine may be defined as the mapping which associates each subset X of S with its closure $[X]_{\mathcal{O}}$ in S (here and elsewhere, I assume you acquainted with the basic concepts on algebras and quasialgebras to be given in Chapters 3 and 2). Thus the possible ability is determined by the algebra $(S, \mathcal{O})$ and may be analyzed by means of the descriptions of $[X]_{\mathcal{O}}$ given by Theorems 3.1.2–3.1.4 as well as its definition in §3.1.3 (s. [3.6]), while the existing ability depends on the above disregarded components and is difficult to mathematically analyze (s. [1.11][1.12] and Example 1.2.2).

By virtue of Assumptions 1–3 in §1.2.1 and the abstraction of algebras from machines, we may regard the IU, PU and RU as algebras (s. Remark 1.1.1) so that the IU is the algebraic union of the PU and RU, that is, the domain of the IU is the union of those of the PU and RU and the processual algebraic structure of the IU is the union of those of the PU and RU.[1.15]

[1.12] **(Memories)** Such components for some machines, particularly the IU, PU and RU, may be collectively referred to as memories.

[1.13] **(Symbolization)** In fact, in order for S to be a constant set, it is sometimes necessary to replace the materials and products with their symbols, e.g. names (s. [1.26]). It is necessary for kitchens and unnecessary for calculators because numbers are already symbols for quantities (s. §3.6.2). Similar remarks apply to $\mathcal{O}$ and others because of mathematical symbolism.

[1.14] In fact, recipes such as *mix* and *sauté* are applied to k-tuples of ingredients for various integers $k \geq 1$ and so each of them must be regarded as a family $(\alpha_1, \alpha_2, \dots)$ of k-ary recipes α_k ($k = 1, 2, \dots$), and likewise for other machines. See Theorem 3.1.1 for a generalization.

[1.15] Consequently, the PU and RU are subreducts of the IU (s. Remark 3.1.2).

What is possible to one person may also be possible to others after sufficient development or cure. Therefore, while we focus on the intellectual possibilities as was suggested right after Assumptions 1–3, we should furthermore assume that the algebraized IU, PU and RU do not essentially[1.16] depend on persons. I refer to the additional assumption as their algebraic homogeneity.

To tell the truth, we may abstract various algebras even from a machine, because we may choose which of its processes to treat as operations. More importantly, we may abstract various algebras other than from machines, particularly from tongues as in Remark 1.2.2 and from the *noocosmos*[1.17], i.e. the totality of the intellectual objects for all kinds of persons, as in §1.2.4.

Remark 1.2.1 (Deduction, induction and ratiocination) Regard the RU as an algebra $(\mathsf{S}, \mathcal{O})$ as above. Then deduction is the process of finding an element $\mathsf{y} \in [\mathsf{X}]_{\mathcal{O}}$ for a given subset X of S, while induction is that of finding a subset X of S which satisfies $\mathsf{y} \in [\mathsf{X}]_{\mathcal{O}}$ for all given elements $\mathsf{y} \in \mathsf{S}$, possibly excluding the trivial case $\mathsf{y} \in \mathsf{X}$ in both deduction and induction. Ratiocination is a finite sequence of deduction and induction in arbitrary combination and order, and so ratiocination of length $\mathsf{1}$ is deduction or induction.[1.18]

Both deduction and induction imply verifying $\mathsf{y} \in [\mathsf{X}]_{\mathcal{O}}$ for each found pair (X, y) of a subset X and an element y of S. Verifying $\mathsf{y} \in [\mathsf{X}]_{\mathcal{O}}$ implies finding an $\mathsf{X}/\mathcal{O}$-sequent $\mathcal{S}$ for y according to Theorem 3.1.3 (s. [3.7]). Finding (X, y) and $\mathcal{S}$ implies searching the storage of the RU for processes and their materials, transporting the materials and storing their products, all by its motivity. They are matters of the above disregarded components, which I call the memories of the RU here (s. [1.12]). Then deduction and induction are differentiated by the role of the memories in them: those in deduction first associate a given subset X of S with an element $\mathsf{y} \in \mathsf{S}$ and those in induction first associate all given elements $\mathsf{y} \in \mathsf{S}$ with a subset X of S (an $\mathsf{X}/\mathcal{O}$-sequent for y may happen to be simultaneously found out). Thus, while we disregard the memories in order to study the possible ability of the RU, we may regard ratiocination as a sequence only of deduction (s. Remark 1.1.1) contrary to the truth (s. [1.39]). This is why deduction is one of the pillars of logic as was noted in the preface.

Remark 1.2.2 (Algebras abstracted from tongues) Let T be a (natural, artificial or hybrid) tongue,[1.19] and let N and $\mathcal{R}$ be proposed or defined sets of its category names and syntactical rules, i.e. the rules under which the composites in T are composed of their constituents in T. Then $\mathcal{R}$ may usually be regarded as equipping N with an algebraic structure $\mathcal{O}$. Let L be a lexicon of T and ν be the mapping which associates each word in L with its category name in N. Then the quadruple $(\mathsf{N}, \mathcal{O}, \mathsf{L}, \nu)$ uniquely yields a certain universal sorted algebra, which may be regarded as a formal language (s. §3.1.7 and [3.22]). Unless T is an

[1.16] **(Essentially)** Such expressions as *do not essentially depend on* and *essentially the same* mean that the concept concerned is unique up to some parameters.
[1.17] I coined the word *noocosmos* from Greek words after the word *noosphere*.
[1.18] You may extend these three concepts to other machines, if you like.
[1.19] When you do mathematics, for example, you use a hybrid tongue (s. Remark 1.2.11).

artificial or dead tongue, the quadruple is hardly unique or complete for T, so that we abstract various formal languages from T and they only largely overlap with T up to punctuation marks. Their algebraic structures may, however, be called syntactical algebraic structures abstracted from T, or from $\mathcal{R}$.

1.2.3 Percepts as internal symbols for intellectual objects

We have regarded the IU as an algebra in §1.2.2 (s. Remark 1.1.1). Here I consider its domain. The consideration is preceded by a consideration of the noocosmos, or of the intellectual objects. I manage it by everyday words here and will improve it in a mathematical way in §1.2.4 (s. §1.4).

You will find on both introspection and inspection of tongues that the intellectual objects may be divided into entities and events.

An intellectual object is called an entity if we can ask whether it exists. We need not be able to answer the question. Therefore, each entity may or may not exist. Then you will find on inspection of tongues that the nominals are the names or descriptions of entities, or rather you should define the nominals as such,[1.20] and that they may be immaterial, imaginary, impossible or of whatever kind. For example, *idea* is immaterial, *Peter Rabbit* is imaginary and *king of a republic* is impossible; nevertheless, they are names or descriptions of entities, and so are *Peter*, *rabbit*, *king* and *republic*.

An intellectual object is called an event if it is not an entity and we can ask whether it occurs. We need not be able to answer the question. Therefore, each event may or may not occur. Then you will find on inspection of tongues that the declaratives are the descriptions of events, or rather you should define the declaratives as such (s. [1.20]), and that each event is either a relationship (between entities) or an attribute (of an entity). For example, the declarative sentence *An N exists* for each nominal N describes an attribute of the entity named (or described) N and the attribute is an event which occurs iff[1.21] an N exists.[1.22] The declarative sentence *A king rules a republic in Korea* describes a relationship between the three entities named *king*, *republic* and *Korea* and the relationship is an event which occurs iff a king rules a republic in Korea. Not only the sentences *An N exists* and *A king rules a republic in Korea* but also the four verbal phrases *exists*, *rules a republic in Korea*, *rules a republic* and *rules* in them are declaratives, i.e. descriptions of events (s. Remark 1.3.3). This is the reason why I have parenthesized the prepositional phrases *between entities* and *of an entity* in the above explanation of events.

Events may involve variable entities.[1.23] For example, the event described by the declarative *An X exists* for a variable nominal X involves a variable entity. While the event described by the declarative *A king rules a republic in Korea*

[1.20] **(Materials depend on aims)** You should have your own grammar, as well as your own logic, algebra and others, which is appropriate to MN (s. §0.2, Remark 1.2.4 and [1.48]).
[1.21] **(Iff)** The word *iff* is a conjunction meaning *if and only if*.
[1.22] **Existence and occurrence** are correlated, as I clarify in Chapters 4–6 on CL (s. §1.3.5).
[1.23] In contrast, a vicious circle prevents *definitions* of entities from involving variable entities, although subsequent *descriptions* of entities may involve variable entities and others (s. [1.35]).

involves no variable entity, the event described by the declarative *A Y rules a Z in Korea* for variable nominals Y and Z involves two variable entities.

Each event which involves k variable entities may be regarded as a mapping which associates each k-tuple of entities with a truth value, i.e. an element of the set $\{truth, falsity\}$[1.24], while each event which involves no variable entity may be regarded as a truth value (s. Remark 1.1.1). For example, the event described by the declarative *An X exists* may be regarded as a mapping which associates each entity named N with *truth* if an N exists (that is, if the event described by the declarative *An N exists* occurs) and *falsity* otherwise. The event described by the declarative *A Y rules a Z in Korea* may be regarded as a mapping which associates each couple of the entities named P and Q with *truth* if a P rules a Q in Korea (that is, if the event described by the declarative *A P rules a Q in Korea* occurs) and *falsity* otherwise, while the event described by the declarative *A king rules a republic in Korea* may be regarded as *truth* if the event occurs (that is, if a king rules a republic in Korea) and *falsity* otherwise.[1.25]

Now then, what the domain of the IU should be? It is the union of those of the PU and RU because the IU is the algebraic union of the PU and RU.

As for the PU (perceiving unit), I suppose that an intellectual object is perceived when an internal symbol[1.26] for it comes into existence. The words *internal* here and *external* below refer to the inside and outside of the nervous system of a person,[1.27] and so their range depends on the person; in particular, the external world depends on the person unlike the noocosmos. Although yet unknown and not required to be known for the present, the internal symbols must be some substantial existences in the nervous system, which I call percepts.[1.28] Then each percept is an internal symbol for the only intellectual object at a time. I call it the object of the percept (at the time).

Table 1.1: Dichotomies of percepts, their objects and their descriptions

percepts	objects of the percepts	descriptions of the objects
NOMINALS	entities	nominals
DECLARATIVES	events	declaratives

Percepts for entities are also called NOMINALs[1.29] because entities are described by nominals. Percepts for events are also called DECLARATIVEs (s. [1.29]) because events are described by declaratives. I suppose that a NOMINAL at some time is not a DECLARATIVE at other times and vice versa. Then

[1.24] **(Truth)** You may extend the set by intermediate truth values, if you like (s. [1.33]).

[1.25] **(Occurrence and truth)** The three related words *occur*, *occurrence* and *inevitable* for events here and below are respectively correlated with the three related words *hold true*, *truth* and *indubitable* for declaratives. I will clarify the correlation in Chapters 4–6 on CL.

[1.26] A **symbol** in its broadest sense is something that stands for something else (s. [1.13]).

[1.27] Thus the viscera of a person and their functions are external objects even for the person.

[1.28] They will be electrochemical existences due to the structure of the nervous system.

[1.29] **Small capital**s will be used for common words given uncommon meanings.

since the intellectual objects are divided into the entities and events, the percepts are divided into the NOMINALs and DECLARATIVEs. Thus we have Table 1.1 showing dichotomies of percepts, their objects and their descriptions.

In another dichotomy, I suppose that some percepts called the composite ones are produced from percepts by the processes of the PU and the other percepts called the prime ones are inputted from the external world (s. [1.27]): the prime ones include the percepts inputted as instinct before birth and those inputted by stimuli experienced through sense organs at any time.[1.30] Thus we may assume that the domain of the PU consists of all possible percepts (s. [1.11]). Then the possible materials of the machine PU are some possible percepts and its possible products are all of the possible composite percepts.

As for the domain of the RU (ratiocinating unit), distinguish ratiocination from perception and notice that not all utterances[1.31] precisely describe internal processes. Then you will find on both introspection and inspection of tongues that you ratiocinate only on DECLARATIVES (s. [1.39]). Indeed, even if we utter the nouns *Peter* and *rabbit*, when we must have perceived the entities named *Peter* and *rabbit* respectively and so possess NOMINALs for them, we must be ratiocinating on DECLARATIVEs for such events involving the entities as are described by the declaratives *Peter is a rabbit* and *A rabbit runs fast*. Thus we may assume that the domain of the RU consists of all possible DECLARATIVEs. Then the possible materials of the machine RU are some possible DECLARATIVEs and its possible products are some, perhaps other, possible DECLARATIVEs.

Thus we may assume that the domain of the IU, as well as that of the PU, consists of all possible percepts. Moreover, we may consider the mapping of the domain into the noocosmos which associates each possible percept with its possible object at a certain time. I refer to the mapping and its restriction to the prime percepts as a (full) PERCEPTION (s. [1.29]) and a prime PERCEPTION respectively. Now, intellection is limited to a class of intellectual objects which varies with persons, development of each person and the passage of time. Therefore, there is more than one such class, which I call the *nooworlds* (s. [1.17]), and the noocosmos is their union. Likewise, there is more than one PERCEPTION depending on persons, development of each person and the passage of time, and the image of each PERCEPTION is contained in a nooworld. Also, the nooworlds may overlap and the PERCEPTIONs may be neither injective nor surjective.

Leaving aside further consideration of the domain of the IU for Chapters 4–6 on CL, our next concern here in §1.2 is its processual algebraic structure, which will turn out in §1.2.5 to be closely related to syntactical algebraic structures abstracted from tongues as in Remark 1.2.2 and to the algebraic structures on the noocosmos, or on the nooworlds, to be found in §1.2.4.

[1.30] Thus *prime* and *composite* do not mean *prior* and *posterior*, although at least one prime percept priorly exists and the first composite one is posteriorly produced from prime ones.
[1.31] An **utterance** is something that a person *grammatically* says.

1.2.4 Nooworlds as algebras and causality in them

Here we consider algebraic structures on the nooworlds. We begin by improving §1.2.3 in a mathematical way: we start with definitions of basic sets (s. §1.1).

In §1.2.3, we defined entities and events using undefined concepts of existence and occurrence and then defined each nooworld as a class of entities and events. The quasi-definition still shows that we may regard each nooworld as the disjoint union $W = \mathsf{E} \cup \mathsf{F}$ of a nonempty set E and a set F of multiary relations on E (s. Remark 1.1.1) and refer to the elements of E and F as the entities and events of W respectively. The set F may or may not be empty (s. the definition of the SWs in §1.2.5 and the text near [1.34]). The word *multiary* means the phrase k*-ary for an integer* $\mathsf{k} \geq 0$. A k-ary relation on E is a mapping f of E^{k} into the set $\mathbb{T}$ of the truth values; if $\mathsf{k} = 0$, f may be identified with a truth value.[1.32]

By abstracting the set W from the nooworld and defining its entities and events as above, we can define occurrence for the events. Assume $\mathbb{T} = \{0, 1\}$, replacing *truth* and *falsity* with 1 and 0 respectively.[1.33] Then if a k-ary event f and a k-tuple $\vec{\alpha}$ of entities satisfy $\mathsf{f}\vec{\alpha} = 1$, we say that f occurs for $\vec{\alpha}$; if $\mathsf{k} = 0$ and $\mathsf{f} = 1$ under the above identification, we say that f occurs (s. [1.25]). Thus each event may or may not occur according to the entities concerned. We can also define existence for the entities after refining the definition of E in Chapters 4–6 on CL, and each entity may or may not exist (s. §1.3.5).

Furthermore, we may abstract various operations and so various algebraic structures on W from its compositionality, i.e. the totality of the rules under which elements of W compose elements of W. For example, if $\mathbb{T}$ is contained in W, then so is $\mathsf{E} \cup \mathbb{T}$, and so each k-ary relation f on E with $\mathsf{k} \geq 1$ is such a rule showing how elements of E compose elements of $\mathbb{T}$ and f itself is a k-ary operation on W irrespective of whether $\mathsf{f} \in \mathsf{F}$ or $\mathsf{f} \notin \mathsf{F}$. Moreover, modifying the definition of F in Chapters 4–6 on CL (s. Remark 1.2.3), we can abstract k binary operations $\alpha_1, \ldots, \alpha_{\mathsf{k}}$ on W from f and we have $\alpha_1, \ldots, \alpha_{\mathsf{k}}, \mathsf{f} \notin \mathsf{F}$.

Therefore, I will assume until a turning point of MN in §1.2.5 that every nooworld W contains $\mathbb{T}$, or rather assume that F contains $\mathbb{T}$ by identifying $\mathbb{T}$ with the set of the 0-ary relations on E (s. [1.32]),[1.34] and consequently F is nonempty. Then W is also called a nooworld with the truth values, and it seems reasonable to regard W as an algebra (s. Remark 1.1.1) by equipping it with one of the various algebraic structures abstracted from its compositionality, which I call the compositional one. Then the variety raises the question *What is the compositional algebraic structure of* W*?* It is a mathematical rendition of the question *What is the essence of compositionality on the intellectual objects?* I will show in §1.2.5 that it gets to the heart of MN.

By regarding W as an algebra in this way, we may also regard E and F as its subreducts and so as algebras. We should, however, regard F as another algebra

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[1.32]Set-theoretic definitions imply that $\mathsf{S}^0 = \{\emptyset\}$ for each set S (s. [3.23]), and each mapping f of $\{\emptyset\}$ into a set T may be identified with its value $\mathsf{f}\emptyset \in \mathsf{T}$ at $\emptyset$ (s. [3.24]).

[1.33]If you like, you may replace the set $\{0, 1\}$ with some sets such as the interval $[0, 1]$ of real numbers by allowing intermediate truth values (s. [1.24]).

[1.34]In CL, F contains $\mathbb{T}$ by definition (s. [1.82]).

in the following way. We first regard it as a quasialgebra as defined in §3.1.2, i.e. a set S equipped with a quasialgebraic structure R that is an association on S, i.e. a relation between S^* and S. Here S^* is the free monoid over S, or the set of all formal products $x_1 \cdots x_n$ of elements $x_1, \ldots, x_n$ of S of finite length $n \geq 0$ (s. Remark 3.1.14). Its identity element is the formal product of length 0 and denoted ε. The set $\{x \in S : \varepsilon \, R \, x\}$ is called the R-core of S and denoted S_R.

We equip F with the following association $\vDash$. Let $f_1, \ldots, f_n$ and g be arbitrary elements of F, $k_1, \ldots, k_n$ and l be their respective arities ($n \geq 0$) and k be the largest of the arities. Then by definition, the element $(f_1 \cdots f_n, g)$ of $F^* \times F$ satisfies $f_1 \cdots f_n \vDash g$ iff the following holds for all $(a_1, \ldots, a_k) \in E^k$:

$$\inf\{f_1(a_1, \ldots, a_{k_1}), \ldots, f_n(a_1, \ldots, a_{k_n})\} \leq g(a_1, \ldots, a_l).$$

Since $\mathbb{T} = \{0, 1\}$, this inequality means that if $f_i(a_1, \ldots, a_{k_i}) = 1$ for all $i \in \{1, \ldots, n\}$ then $g(a_1, \ldots, a_l) = 1$, that is, if the event f_i occurs for the tuple $(a_1, \ldots, a_{k_i})$ of entities for each $i \in \{1, \ldots, n\}$, then the event g occurs for the tuple $(a_1, \ldots, a_l)$ of entities. Consequently, the $\vDash$-core $F_{\vDash}$ of F consists of the events which occur for all entities concerned (s. [1.97]) and so may be called the inevitables (s. [1.25]).[1.35] Thus we call $\vDash$ cause-effect association or *causality*. As such, $\vDash$ is a partially latticed association as defined in Theorem 2.2.10.

Each algebraic structure $\mathcal{O}$ on F yields an association $R_{\mathcal{O}}$ on F as defined by (3.1.1) (s. [3.4]). Each pair (R, D) of an association R on F and a subset D of F yields an association R^D on F as in §2.5.1. Therefore, each pair $(\mathcal{O}, D)$ of an algebraic structure $\mathcal{O}$ on F and a subset D of F yields an association ${R_{\mathcal{O}}}^D$ on F. Moreover, being partially latticed, $\vDash$ is equal to ${R_{\mathcal{O}}}^D$ for various algebraic structures $\mathcal{O}$ on F and subsets D of $F_{\vDash}$ (s. Remark 2.5.4). The equality means that elements $f_1, \ldots, f_n$ and g of F satisfy $f_1 \cdots f_n \vDash g$ iff g has a $(\{f_1, \ldots, f_n\} \cup D)/\mathcal{O}$-sequent (s. §3.1.3), or in short $\vDash$ is ruled by $\mathcal{O}$ and D.[1.36] Thus it seems reasonable to regard F as an algebra (s. Remark 1.1.1) by equipping it with one of such algebraic structures $\mathcal{O}$, which I call the *causal* one. Then the variety of $\mathcal{O}$ raises the question *What is the causal algebraic structure of* F? It is a mathematical rendition of the question *What is the essence of causality among the events?* I will show in §1.2.5 that it also gets to the heart of MN.

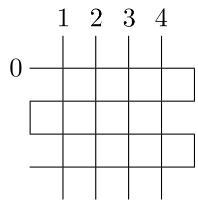
Remark 1.2.3 (Appropriate definitions of events and others) To tell the truth, the above definition that the events in F are multiary relations on the set E of entities does not seem quite appropriate in light of MN and was made for convenience of explanation. Therefore, I will modify it and the related definitions of occurrence and causality in Chapters 4–6 on CL (s. §1.3.6).

[1.35] **(Definition vs. description)** The inevitables *defined* in this way are obviously the events whose values are all equal to 1. In general, however, it is not obvious whether an event *described* in another way is an inevitable or not (s. [1.23]). Indeed, every scientific law (e.g. the evolutionary law of organisms) may be regarded, or rather should be defined as a description of an event which is supposed to be an inevitable.

[1.36] **(Indeterminism over determinism)** See Remark 1.2.9 for details of the rule, which shows that, although elements of $\mathcal{O} \cup D$ each are deterministic in a sense, $\vDash$ is in no sense so.

Remark 1.2.4 (Algebra! She rules them all) Whatever answers we give to the above two questions, algebra rules the noocosmos as well as machines, logic and tongues. Machines are so ruled by virtue of their processes (s. §1.2.2), logic is so ruled by virtue of the GL in Chapters 2 and 3 and tongues are so ruled by virtue of their syntactical rules (s. Remark 1.2.2), while the noocosmos is so ruled by virtue of its compositionality and causality. Thus algebra rules the main materials for MN all, and the rule is well illustrated by Fig. 1.2. This is why I began §1.2 by saying that the noun *algebra* is the weft and the other nominals in the subtitle are the warp.

Figure 1.2: A weave of MN



0	algebra
1	the intellectual machines
2	logic
3	tongues
4	the noocosmos

1.2.5 Intellect, the noocosmos and utterances

As for wisdom, what she is, and how she came up,
I will tell you, and will not hide mysteries from you:
but will seek her out from the beginning of her nativity,
and bring the knowledge of her into light,
and will not pass over the truth.[1.37]

We now proceed to consider the processual algebraic structure of the IU. We assume that it does not essentially depend on persons because of the algebraic homogeneity of the IU (s. [1.16] and the nearby text). It is the union of those of the PU and RU because the IU is the algebraic union of the PU and RU.

In light of organic evolutionism, it seems undoubted that the processual algebraic structure of the PU has produced composite percepts for intellectual objects in the nooworlds and that of the RU has processed DECLARATIVEs for events in them (s. Table 1.1) in order for the human species to survive[1.38] by perceiving and ratiocinating about *causal* problems of how to behave in

[1.37] King James version of *Wisdom*, or the *Book of the Wisdom of Solomon*, v. 6:22 (s. [1.47]).

[1.38] **(Survival)** While useful, the words *survival* and *survive* are misleading. The fact is, or a Darwinian belief is, environment allows those who possess some inheritable trait to leave more offspring than those who do not, and thus the trait spreads through the population in a long period provided that the environment is limited. The word *survival* means the spread, and sentences like *The human species has behaved A in order to survive* for adverbial phrases *A* mean *The human species has behaved A and therefore has spread.*

them for instinctual purposes.[1.39] The compositional algebraic structures of the nooworlds produce intellectual objects in them and the causal algebraic structures of their event sets together with inevitables rule causality among the events in them. Thus it seems reasonable to believe that the algebraic structure of the PU resembles[1.40] those of the nooworlds because it has adapted to them in its evolution so that human beings can more efficiently perceive intellectual objects in them. It also seems reasonable to believe that the algebraic structure of the RU similarly resembles those of the event sets of the nooworlds (s. [1.40]). Not only does algebra rule the intellectual machines and the noocosmos as was noted in Remark 1.2.4, but also intellect must have evolved as the algebraic structures of the machines adapted to those on the noocosmos. Thus the two questions raised in §1.2.4 get to the heart of MN as was noted there. They are now read as *What is the human perception of compositionality on the intellectual objects?* and *What is the human conception of causality among the events?*

This paragraph, however, also applies to the ape and other animals in varying degrees in contrast to the following ones involving tongues. As for intellect, tongues have differentiated the human species from other ones as is explained in the following paragraphs and in §1.2.6 around (1.2.1).

In light of organic evolutionism, it also seems undoubted that human beings have uttered their percepts, deduction and others and have longed to understand utterances (s. [1.31]) in order to survive (s. [1.38]) by communicating with their verbal communities and by self-communicating, or introspecting. Here a verbal community is an evolutional group of people with close verbal cultures, and a verbal culture is a verbal behavior pattern (s. §1.2.8). Obvious verbal communities are our native communities. See Remark 1.2.11 for less obvious ones.

For each verbal community, I refer to the possible utterances of percepts (s. [1.11]) as the *descriptive utterances* (DU) and refer to the totality of the DUs as the *descriptive tongue* (DT). Then since the percepts are the NOMINALs and DECLARATIVEs, i.e. the internal symbols for entities and events, we may assume that the DUs are the descriptions of entities and events, i.e. the nominals and declaratives (s. Table 1.1), as is illustrated by the following basic UPO-diagram.

$$\mathsf{U} \ \rightarrow \ \mathsf{P} \ \rightarrow \ \mathsf{O} \qquad \text{(UPO)}$$

Here U is a DU, P is a percept of which U is an utterance and O is the only object of P at the time, and so the right-hand arrow denotes a PERCEPTION,[1.41] O is an entity or an event and P is accordingly a NOMINAL or a DECLARATIVE.

[1.39] **Ratiocination** about the problem of how you behave in the nooworlds for your instinctual purpose implies finding the truth values of the truly causal declaratives *If I behave A, then my instinctual purpose is attained* for adverbial phrases A (this is probably the reason why human beings ratiocinate only on DECLARATIVEs as was noted in §1.2.3). It moreover implies inducing the declaratives *I behave A* from the declarative *My instinctual purpose is attained*. Thus human ratiocination is primarily a sequence of induction (s. Remark 1.2.1).

[1.40] **(Resemblance)** There are several kinds of algebraic resemblances and analogies (s. §3.1).

[1.41] Each DU is an utterance of a percept and each percept has its only object at a time, while not every percept is uttered and not every intellectual object is perceived, hence also the direction of the arrows here (s. [1.42][1.44]). The left-hand arrow never denotes a mapping, because a DU may be an utterance of more than one percept at any time.

The above assumption in fact means that U is a description of O and therefore is a nominal or a declarative. Then you will find on inspection of nominals and declaratives that the composite DUs are composed of their constituent DUs under some rules, from which we may abstract some syntactical algebraic structure (s. Remark 1.2.2), while the composite percepts are produced from percepts by the processual algebraic structure of the PU. Thus it seems reasonable to believe that, in each verbal community, the algebraic structure abstracted from the DT resembles that of the PU (s. [1.40]) and so also resembles the compositional algebraic structures of the nooworlds by the first belief about the PU because the DT has adapted to the PU in its evolution so that the DUs can more efficiently convey utterers' percepts. It also seems reasonable to believe that, in each verbal community, some algebraic structure abstracted from syntactical rules for the deductive utterances similarly resembles that of the RU (s. [1.40]) and so also resembles the causal algebraic structures of the event sets of the nooworlds.

The whole above belief about the PU may be illustrated by the following e^2-diagram, where each arrow means the evolutional relationship, i.e. the resemblance between the algebraic structures caused by the adaptation in evolution, and henceforth, the symbol NW stands for the word *nooworld*.

$$\boxed{\text{DT}} \xrightarrow{\text{evolutional}} \boxed{\text{PU}} \xrightarrow{\text{evolutional}} \boxed{\text{NWs}} \qquad (e^2)$$

You can draw an analogous diagram about the RU.[1.42] In contrast to the RU, however, the processual algebraic structure of the PU must also have adapted to nooworlds of another kind, i.e. SWs introduced soon. A similar remark applies to the DT. The DUs are tools of communication and introspection created by the verbal community rather than organismal phenomena out of the PU. Therefore, the syntactical algebraic structure abstracted from the DT must also have adapted to preference of the verbal community.[1.43]

The following p^2-diagram illustrates in advance the phraseological relationship and the perceptual relationship which are introduced shortly and supplement the above explanation of the evolutional relationship.[1.44]

$$\boxed{\text{DT}} \xrightarrow{\text{phraseological}} \boxed{\text{PU}} \xrightarrow{\text{perceptual}} \boxed{\text{NWs}} \qquad (p^2)$$

The perceptual relationship is as follows. Each operation α in the processual algebraic structure of the PU associates each tuple $(P_1, \ldots, P_k)$ of percepts in its domain with a composite percept P, and $P_1, \ldots, P_k, P$ respectively have their objects $O_1, \ldots, O_k, O$ under PERCEPTIONs. Therefore, questions arise as to how α relates $(O_1, \ldots, O_k)$ to O and what PERCEPTIONs are involved in the relation,

[1.42] The direction of the arrows in them two shows that of the adaptation (s. [1.41][1.44]).

[1.43] **(Selection)** As artificial selection causes adaptation to human preference, so communal selection causes adaptation to communal preference, while Darwinians metaphorically assert that natural selection causes adaptation to natural preference, or to environment (s. [1.38]).

[1.44] The direction of the arrows here derives from that in the UPO-diagram (s. [1.41][1.42]).

as is illustrated by the right half of the following composite of UPO-diagrams.

$$\begin{array}{ccccc} (\mathsf{U}_1, \ldots, \mathsf{U}_k) & \xrightarrow{\text{k times}} & (\mathsf{P}_1, \ldots, \mathsf{P}_k) & \xrightarrow{\text{k times}} & (\mathsf{O}_1, \ldots, \mathsf{O}_k) \\ \searrow & & \alpha\downarrow & & \swarrow \\ \mathsf{U} & \longrightarrow & \mathsf{P} & \longrightarrow & \mathsf{O} \end{array} \tag{cUPO}$$

In view of the concepts of the nooworlds and PERCEPTIONs as introduced in §1.2.3, we may assume that $\mathsf{O}_1, \ldots, \mathsf{O}_k, \mathsf{O}$ belong to the same nooworld and the images of the PERCEPTIONs involved are contained in it. Thus there is some perceptual relationship between the processual algebraic structure of the PU and each of the nooworlds. Being an organismal phenomenon and so a result of evolution, however, the relationship may be explained by the evolutional relationship between the PU and the nooworlds illustrated by the e^2-diagram and also by the adaptation of the algebraic structure of the PU to the SWs (s. §1.2.6 around (1.2.2) and (1.2.3)). Conversely, it supplements the above explanation of the evolutional relationship.

The phraseological relationship is similarly introduced and illustrated by the left half of the cUPO-diagram. Although not all percepts are uttered (s. [1.41]), if the percepts $\mathsf{P}_1, \ldots, \mathsf{P}_k, \mathsf{P}$ are uttered and phrased by DUs $\mathsf{U}_1, \ldots, \mathsf{U}_k, \mathsf{U}$ respectively, a question arises as to how α relates $(\mathsf{U}_1, \ldots, \mathsf{U}_k)$ to U. Thus there is some phraseological relationship between the processual algebraic structure of the PU and the DT. It must be explained not only by the evolutional relationship between the DT and PU illustrated by the e^2-diagram but also by the adaptation of the syntactical algebraic structure abstracted from the DT to the preference of the verbal community (s. [1.43]). Although difficult to explain so, it supplements the above explanation of the evolutional relationship (s. §4.3).

Now, we have reached the turning point of MN noted in §1.2.4. Here I alter my assumption that every nooworld contains the truth values (s. the text near [1.34]). We have seen that human beings have longed to understand utterances, especially DUs, in order to survive (s. [1.38]) in their verbal communities. Therefore, I should assume that each verbal community has nooworlds of another kind to which the PU has adapted in organic evolution. Each of them is associated with a person in the community and called a *semasiological world* (SW) or a nooworld without the truth values for convenience of explanation although it even lacks events. Each its entity is a perceivable symbol S (s. [1.26]) for a MEANING P (s. [1.29]) of a DU U for the person, that is, P is a percept in the PU of the person which the person utters by U, as is illustrated by the following SUPO-diagram.

$$\begin{array}{ccccc} & & \mathsf{S} & & \\ & & \downarrow & & \\ \mathsf{U} & \rightarrow & \mathsf{P} & \rightarrow & \mathsf{O} \end{array} \tag{SUPO}$$

This extends the UPO-diagram, although the person was implicit there, P was not called a MEANING of U there and the object O of P is subsidiary here. We may also call P an internal meaning of U for the person, while we may call O an (external) meaning of U for the person, as I have implicitly done so. Being an evolutional group, the verbal community consists of more than one person. Therefore, it has more than one SW depending on the persons. Since the persons have close verbal cultures, however, the personal SWs must be essentially the same (s. [1.16]). Thus I assume that each verbal community has its SW (or nooworld without the truth values)[1.45] to which the PU has adapted.

This turning brings the following paragraphs and remarks into view.

In each verbal community, DUs are usually unfaithful to their MEANINGS in the sense clarified in §1.2.7. Therefore, the DT itself cannot serve as the communal SW. However, it will turn out also in §1.2.7 that, for each DU U and each its MEANING P, we can rephrase U by a real or imaginary DU S which is faithful to P provided we belong to the verbal community,[1.46] as is illustrated by the following modification of the SUPO-diagram.

$$\mathsf{U} \quad \to \quad \mathsf{S} \quad \to \quad \mathsf{P} \quad \to \quad \mathrm{O} \tag{USPO}$$

Thus we may identify the SW with the collection of all such real or imaginary DUs, which I call the *rephrased DU*s (RDU). Then the SW is a hybrid tongue obtained by modifying the DT. They overlap up to punctuation marks because some DUs are faithful to their MEANINGs and so need not be rephrased. We may abstract some syntactical algebraic structure from the rules under which the composite RDUs are composed of their constituent RDUs (s. Remark 1.2.2).

Remark 1.2.5 (The DT and SW resemble each other only as a whole) Since the SW is a modification of the DT as above, their syntactical rules resemble each other. However, each DU U and the RDU S for each its MEANING P as shown in the USPO-diagram are not necessarily composed under similar syntactical rules, because U is usually unfaithful to P as was noted above.

Remark 1.2.6 (DUs are irregularly unfaithful to their MEANINGs) The syntactical rules for the SW are also semasiological ones for the DT in that the RDUs are symbols for the MEANINGs of the DUs and composed under the rules. The remaining semasiological rules for the DT, if any, associate each DU U with the RDUs S for its MEANINGs P. There are no such rules, however, because the association depends on CONTEXT (s. [1.29] and Remark 1.2.7) which is irregular.

Remark 1.2.7 (A CONTEXT as part of a MEANING) Let U be a DU and P be one of its MEANINGs. Then U is usually unfaithful to P, but there may exist another MEANING of U to which U is faithful. Let P' denote such a MEANING if it exists and let P' be void otherwise. Then P may or may not be contained in P' (and vice versa) and I refer to the part of P outside P' as a CONTEXT of U. The four words *void*, *contain*, *part* and *outside* here make sense because we may identify P and P' with the RDUs for them (s. Example 1.2.1).

[1.45] A nooworld without the truth values can be contained in one with the truth values.

[1.46] **(Dictionary)** Some monolingual dictionaries attempt to explain internal or external meanings of some DUs by rephrasing them only by real DUs.

Example 1.2.1 (Irony) In Remark 1.2.7, suppose U is the utterance *Peter is good.* Then the utterer may have meant *Peter is not good, but I say the opposite in fun.* Then we may identify P' and P with the RDUs *Peter is good* and *It is not that Peter is good, but I say the opposite in fun* respectively (since they are also DUs, the SW and DT overlap). Thus $\mathsf{P}' = \mathsf{U}$ and P is not contained in P', but P' is contained in P so that the CONTEXT of U is *It is not that, but I say the opposite in fun.* In view of other possible CONTEXTs of U, you will see that there are no rules for associating each DU with the RDUs for its MEANINGS, as was noted in Remark 1.2.6. As for Remark 1.2.5, notice that U and P are composed under dissimilar syntactical rules. See §1.2.7 for more.

As before, I now suppose that the PU has algebraically adapted to the SW, that is, the said adaptation of the PU to the SW has resulted in resemblance between the algebraic structures abstracted from them (s. [1.40]). We may conversely regard the SW as having algebraically adapted to the PU, because the DT has done so and the SW is a modification of the DT as above. In view of the remarks on the e^2-diagram, however, I rather suppose that the SW has algebraically adapted to the PU by natural selection and the DT is its modification by communal selection (s. [1.43]). Refining the e^2-diagram, the following e^3-diagram illustrates the evolutional relationship between the DT, SW, PU and nooworlds *with or without* the truth values.

$$\boxed{\quad \mathrm{DT} \quad} \xrightarrow[\text{communal}]{\text{evolutional}} \boxed{\mathrm{SW}} \xrightarrow[\text{natural}]{\text{evolutional}} \boxed{\mathrm{PU}} \xrightarrow[\text{natural}]{\text{evolutional}} \begin{cases} \boxed{\mathrm{NW}} \\ \cdots \\ \boxed{\mathrm{NW}} \end{cases} \qquad (\mathrm{e}^3)$$

Here, and only here, largeness of their boxes shows richness of the algebraic structures abstracted from them. Examples in §1.2.7 suggest that the box for the DT in general is larger than that for the SW. The boxes for the SW and PU are of the same size for the reason clarified shortly. The box for the PU is larger than or of the same size as those for the nooworlds with or without the truth values because the PU has algebraically adapted to all the nooworlds.

We have seen that the PU has algebraically adapted to the SW and vice versa. The mutual adaptation is illustrated by the e^3-diagram because the SW is a nooworld without the truth values. Still, since organic evolution is a change over successive generations, it is better illustrated by the following e^∞-diagram, where the symbols SW and PU stand for the SW and PU of the parent generation.

$$\cdots \longrightarrow \boxed{\mathrm{SW}} \xrightarrow[\text{natural}]{\text{evolutional}} \boxed{\mathrm{PU}} \xrightarrow[\text{natural}]{\text{evolutional}} \boxed{\textsc{sw}} \xrightarrow[\text{natural}]{\text{evolutional}} \boxed{\textsc{pu}} - \cdots \qquad (\mathrm{e}^\infty)$$

In view of the long period of evolution, I conjecture that the algebraic structures abstracted from the SW and PU have been essentially the same (s. [1.16]) for a long time now, to which I refer as the algebraic analogy between the SW and PU (s. [1.40]). The analogy is the evolutional relationship between the SW and PU illustrated by the e^3-diagram, and so their boxes are of the same size therein.

God hath granted me to speak as I would,

and to conceive as is meet for the things that are given me:
because it is he that leadeth unto wisdom, and directeth the wise.[1.47]

1.2.6 Logic for a model of the nootrinity

Enough has been said about the relationship between intellect, the noocosmos and utterances. In short, the nooworlds with the truth values, their event sets and the *descriptive or deductive utterance*s (DDU) in each verbal community constitute a modified expression of the structure of the IU under the assumption of algebraic homogeneity (s. the text near [1.16]). Metaphorically speaking as in §1.2.1, the IU is the genotype and the DDUs are the phenotypes, while the nooworlds with the truth values, their event sets and communal preference (s. [1.43]) constitute the environment which modifies the phenotypes.

Metaphor, however, has merits and demerits. In non-metaphoric terms of the flow chart (Fig. 1.1) for mathematical science, we should observe the nooworlds with the truth values, their event sets and the DDUs in some verbal communities and then carry out the following three empirical ologies[1.48] based on the observation in order to construct a principal mathematical model for MN.

Ontology: gaining an insight into the compositional algebraic structures of the nooworlds with the truth values and the causal algebraic structures of the event sets of the nooworlds, i.e. a mathematical insight into the questions *What is the essence of compositionality on the intellectual objects?* and *What is the essence of causality among the events?*

Semasiology: gaining an insight into the syntactical algebraic structures abstracted from the communal SW and the totality of the deductive utterances, i.e. a mathematical insight into the semasiological rules for the DT (s. Remark 1.2.6) and into the deduction rules reflected in the deductive utterances.

Phraseology: gaining an insight into the phraseological relationship between the processual algebraic structure of the PU and the DT illustrated by the p^2-diagram and the cUPO-diagram in §1.2.5, i.e. a mathematical insight into the way human beings phrase their percepts by DUs.

Here, by virtue of the algebraic analogy between the SW and PU, semasiology makes phraseology possible, and the resemblance between the syntactical algebraic structures abstracted from the DT and SW sheds light on the evolutional relationship between the DT and PU illustrated by the e^2-diagram in §1.2.5, while the difference between them sheds light on the evolutional and communal relationship between the DT and SW illustrated by the e^3-diagram therein.

[1.47] *Wisdom* v. 7:15 (s. [1.37]). Feeling obliged not to be incarnational here in a scientific monograph, I define God as a collection of laws which together rule causality in the universe (s. [1.9] and Remark 1.2.9). Then the evolutionary law of organisms will be one of the laws, and so the word *God* in this pre-Christian verse may now be read as *Evolution*. Thus the verse is perfect(!) for the epilogue of §1.2.5 as with v. 6:22 for its prologue.

[1.48] I borrow their names and neither their aims nor their methods from philosophy, linguistics and other disciplines (s. [1.20]). I distinguish between the terms *semasiology* and *semantics*, and save the latter and the term *syntax* for formal languages or logic systems (s. Chapter 3).

The principal mathematical model for MN, however, should not be a model of the IU alone but be a model of what I call *nootrinity* (s. [1.17]), i.e. the triple $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ for the collection $\mathcal{W}$ of the nooworlds with the truth values and the union $\mathcal{R}$ of the evolutional relationship $\mathcal{E}$ and the perceptual relationship $\mathcal{P}$ between the IU and $\mathcal{W}$ illustrated by the e^2-diagram, its analogue for the RU, the p^2-diagram and the cUPO-diagram. Therefore, the model of the IU means models of the PU and RU equipped with the processual algebraic structures, and the model of $\mathcal{W}$ means models of the nooworlds in $\mathcal{W}$ and their event sets equipped with the compositional algebraic structures and the causal algebraic structures respectively. Thus the mathematical model of the nootrinity and the above empirical ologies for constructing it together shed light on every detail of the diagrams in §1.2.5. In other words, MN aims at understanding not only the IU but also the nooworlds with or without the truth values and the relationship between the IU and each of the nooworlds, because intellect has evolved and exists by virtue of its relationship with the noocosmos.

Recall from §1.2.1–1.2.5 that I abstracted the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ from the triple $(\mathfrak{I}, \mathfrak{W}, \mathfrak{R})$ of intellect $\mathfrak{I}$, the noocosmos $\mathfrak{W}$ and the relationship $\mathfrak{R}$ between $\mathfrak{I}$ and $\mathfrak{W}$. Therefore, $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ is already a mathematical model of $(\mathfrak{I}, \mathfrak{W}, \mathfrak{R})$ (s. Remark 1.1.1). However, $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ is a rough and expedient model. Indeed, in §1.2.2, I only equipped the IU with the processual algebraic structure, which is unspecified yet. In §1.2.4, I regarded each nooworld with the truth values as the disjoint union $W = \mathsf{E} \cup \mathsf{F}$ of a nonempty set E and a nonempty set F of multiary relations on E for convenience of explanation (s. Remark 1.2.3), and I only equipped W and F with the compositional algebraic structure and the causal algebraic structure, which are unspecified yet. In §1.2.5, I only pointed out that there exists the relationship $\mathcal{R}$, which is unspecified yet. Thus the above model of $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ is in fact a model of $(\mathfrak{I}, \mathfrak{W}, \mathfrak{R})$ obtained by refining and modifying the rough and expedient model $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ of $(\mathfrak{I}, \mathfrak{W}, \mathfrak{R})$. I call it a model of $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ because I treat $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ as a characteristic of $(\mathfrak{I}, \mathfrak{W}, \mathfrak{R})$ rather than a model thereof (s. Remark 1.1.1).

Although I can carry out the above ologies only for a few verbal communities, I deem them enough for MN to have wide scope, because I conjecture that the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ does not essentially depend on verbal communities, that is, it is unique up to some parameters varying with them (s. [1.16]). The conjecture agrees with the algebraic homogeneity of the IU and seems reasonable because human beings constitute a single species, or rather MN should characterize the species by the unique nature of the nootrinity and the results of the above ologies for understanding it, regardless of characterizations based on other principles.

The following three remarks together show that my intermediate aim noted in §1.2.1 can even be our ultimate aim (s. [1.1]).

Remark 1.2.8 (Know thyself) The very nootrinity is what we all should try to understand sometime in our lifetime, because it is the essence of us *Homo sapiens* (wise man) associated with what we as such think about[1.49].

[1.49] The following is my prose translation of a passage from *Môsô* (Ôgai Mori, 1911) I read

Remark 1.2.9 (Know thyself, and thou shalt know the universe and God) Here I outline how the model of the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ spins off models of the universe and God (s. [1.9]). They suggest defining God as a collection of laws which together rule causality in the universe (s. [1.47]).

Being free of the dogmas in physical cosmology and others, we regard the universe as a nooworld W in $\mathcal{W}$, that is, W is a model of the universe (s. Remark 1.1.1). Let F and $\vDash$ be its event set and causality respectively (s. §1.2.4 including Remark 1.2.3). Then the $\vDash$-core $\mathsf{F}_{\vDash}$ of F consists of the inevitables of F and, being partially latticed, $\vDash$ is equal to ${\mathsf{R}_{\mathcal{O}}}^{\mathsf{D}}$ for an algebraic structure $\mathcal{O}$ on F, which need not be the causal one, and a subset D of $\mathsf{F}_{\vDash}$. The equality means that elements $\mathsf{f}_1, \ldots, \mathsf{f}_n$ and g of F satisfy $\mathsf{f}_1 \cdots \mathsf{f}_n \vDash \mathsf{g}$ iff g has a $(\{\mathsf{f}_1, \ldots, \mathsf{f}_n\} \cup \mathsf{D})/\mathcal{O}$-sequent, that is, $\mathcal{O}$ and D together have the following property (O).

(O) Let $\mathsf{f}_1, \ldots, \mathsf{f}_n$ and g be events. Then they satisfy $\mathsf{f}_1 \cdots \mathsf{f}_n \vDash \mathsf{g}$ iff there exists a sequence $\mathsf{h}_1, \ldots, \mathsf{h}_m$ $(m \geq 1)$ of events which satisfies $\mathsf{h}_m = \mathsf{g}$ and one of the following two conditions for each $i \in \{1, \ldots, m\}$:

  (i) There exist numbers $j_1, \ldots, j_k \in \{1, \ldots, i-1\}$ and an operation $\alpha \in \mathcal{O}$ such that $\mathsf{h}_i = \alpha(\mathsf{h}_{j_1}, \ldots, \mathsf{h}_{j_k})$.

  (ii) $\mathsf{h}_i \in \{\mathsf{f}_1, \ldots, \mathsf{f}_n\} \cup \mathsf{D}$.

Each operation in $\mathcal{O}$ associates each tuple of events in its domain with an event and so may be regarded as a law about events. Each k-ary inevitable in D occurs for every k-tuple of entities and so may be regarded as a law which holds for every k-tuple of entities $(k \geq 0)$. The condition (i) means that $i \geq 2$ and the event h_i is derived from some of the events $\mathsf{h}_1, \ldots, \mathsf{h}_{i-1}$ by one of the laws in $\mathcal{O}$. The condition (ii) means that h_i is one of the events $\mathsf{f}_1, \ldots, \mathsf{f}_n$ or one of the laws in D. The property (O) means that the laws in $\mathcal{O} \cup \mathsf{D}$ together rule and explain $\vDash$ in this way (s. [1.36]). Therefore, I call $\mathcal{O} \cup \mathsf{D}$ a GOD of W (s. [1.29]) and call (O) omnipotence or omniscience of the GOD. There may exist more than one GOD and they are models of God in various monotheistic religions.

Remark 1.2.10 (Know God, and thou shalt know what must be done) The concept of God suggested by Remark 1.2.9 leads us to eusophy, i.e. pursuit of human omniscience for survival (s. [1.38]) of the human species (s. [1.9]).

Henceforth, I assume that you have read up the GL in Chapters 2 and 3 to have an understanding of what logic is and to be acquainted with its basic concepts, particularly newly defined concepts of formal languages, syntax and semantics. Then you probably agree with me that the right mathematical model of the nootrinity should be provided by some logic system $\mathcal{L}$ and some deduction system on it both involving some parameters. Proceeding to Chapters 4–6 on CL (s. §1.3), you probably agree with me that $\mathcal{L}$ should be CL.

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also as Remark 1.2.8: *It is a pity and a matter for regret that we lose the so-called self without gaining a knowledge or making a study of its nature while it is alive.* Its romanized original is as follows: *sono ziga to iu mono ga aru aida ni, sore o donna mono da to hakkiri kangae te mo mizu ni, sirazu ni, sore o nakusi te simau no ga kuyasii. zannen de aru.*

To be precise, the formal language A of $\mathcal{L}$ is a model of the PU and a deduction system on it provides a model of the RU (and another),[1.50] and so they together provide a model of the algebraic union IU of the PU and RU, while the semantics of $\mathcal{L}$ provides a model of $\mathcal{W}$ and, together with the model of the IU, furthermore provides a model of $\mathcal{R}$. Thus $\mathcal{L}$ and a deduction system on it together provide a model of the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$.

Incidentally, I scorn the dogma that formal languages should model the DT and their semantics should explain the (external) meanings of the DUs. I suspect its propounders to ignore Remark 1.2.6 or be trapped in a vicious circle, that is, they have subjectively analyzed the meanings of the DUs in order to construct their logic systems for objectively analyzing the meanings of the DUs.[1.51]

From my perspective free of the dogma, the MEANINGs of the DUs may be faithfully phrased by the RDUs in the SW, whose syntactical rules will emerge from semasiology, which is reasonably subjective as is explained in §1.2.7, and will suggest the nature of our logic system $\mathcal{L}$ and deduction system on it for a model of the nootrinity by virtue of the relationship illustrated by the diagrams in §1.2.5; in particular, the syntax of the formal language A should be analogous to that of a formal language abstracted from the SW (s. Remark 1.2.2) because of the algebraic analogy between the SW and PU of which A is a model.

To be more precise as to the PU, the elements of A are models of the possible percepts (s. [1.11]), and the algebraic structure of A is a model of the processual algebraic structure of the PU. Being a formal language, A has the subset P of its primes (s. Remark 3.1.16) and $\mathsf{A} - \mathsf{P}$ consists of its composites. The elements of P and $\mathsf{A} - \mathsf{P}$ are models of the possible prime percepts and the possible composite ones respectively. Since P is a basis of A and so the operations of A inductively[1.52] produce the elements of $\mathsf{A} - \mathsf{P}$ from the elements of P (s. Theorems 3.1.6 and 3.1.7), these models imply that the processes of the PU inductively produce the composite percepts from the prime ones (this does not contradict [1.30]). The basis P is almost arbitrary except that it is the direct union of the sets C and X of the constants and variables respectively. The totality of the possible prime percepts varies with persons, development of each person and the passage of time, and the arbitrariness of P provides a model of the variety. A possible prime percept is said to be variable if its object may momently vary,[1.53] and the elements of C and X are models of the possible invariable prime percepts and the possible variable ones respectively.

[1.50] Distinguish between *to be a model of* and *to provide a model of.*

[1.51] **(Theory laundering)** The vicious circle may be compared to the following one in teaching calculus. We have $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ under the condition $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$ among others. However, if you use it and $(\sin x)' = \cos x$ to *prove* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, you probably commit circular reasoning because you have derived $(\sin x)' = \cos x$ from the *intuition* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ as is often the case.

[1.52] The induction here means mathematical induction rather than that in Remark 1.2.1.

[1.53] Possible prime percepts are divided into variable ones and invariable ones independently of the division into existing ones and potential ones (s. [1.11]).

To be more precise as to the RU, being models of the possible percepts, the elements of A must be divided into 'nominals' and 'declaratives' which are models of the possible NOMINALS and the possible DECLARATIVES respectively (s. Table 1.1). Let H the set of the 'declaratives' of A. Then a deduction system on H is a pair (R, D) of an association R on H and a subset D of H, and Theorem 3.1.1 shows that R may be derived from an algebraic structure $\mathcal{O}$ on H if (and only if) the R-core H_R of H is empty. Some such deduction system provides models $(H, \mathcal{O})$ and D of the RU equipped with the processual algebraic structure and a collection of stored DECLARATIVES.

To be more precise as to $\mathcal{W}$, the M-reduct A_M of A for the set M of the invariable indices of A is a sorted algebra (s. Remark 3.1.6), and the *denotable worlds* (DW) for A given by the semantics of $\mathcal{L}$ are certain sorted algebras of the same type as A_M. The algebras are models of the nooworlds in $\mathcal{W}$ equipped with the compositional algebraic structures. Therefore, the elements of the *given*[1.54] DWs are models of the intellectual objects in the nooworlds and must be divided into 'entities' and 'events' so that the latter provide models of the event sets of the nooworlds equipped with the causal algebraic structures. The following diagram illustrates the four paragraphs from here on.

$$
\begin{tikzcd}
\boxed{\text{SW}} & \boxed{\text{PU}} \arrow[l, "\text{evolutional}"'] \arrow[rr, "\text{evolutional}"] \arrow[d, "\text{model}"] & & \boxed{\text{NWs in } \mathcal{W}} \arrow[d, "\text{model}"'] \\
& \boxed{A} \arrow[r, "\text{reductive}"] & \boxed{A_M} \arrow[r, "\text{homotypic}"] & \boxed{\textit{given} \text{ DWs}}
\end{tikzcd}
\tag{1.2.1}
$$

To be more precise as to $\mathcal{R}$, being sorted algebras of the same type as A_M, the *given* DWs are similar to A_M (s. [3.13]). The algebraic relationship between A and the DWs via A_M is a model of half of the evolutional relationship $\mathcal{E}$, i.e. the evolutional relationship between the PU and the nooworlds in $\mathcal{W}$ illustrated by the e^2-diagram. A model of another half of $\mathcal{E}$ will be provided by the above model $\mathcal{O}$ of the processual algebraic structure of the RU and the above models of the causal algebraic structures of the event sets of the nooworlds in $\mathcal{W}$.

The above models of the PU, $\mathcal{W}$ and the former half of $\mathcal{E}$ together have the added bonus of providing the following extra models, which have an important interpretation. The PU has algebraically adapted to the SW as well as to the nooworlds in $\mathcal{W}$, and the operations of A_M are models of the processes of the PU evolved by its adaptation to the nooworlds in $\mathcal{W}$. Therefore, the algebraic structure of A may well be richer than that of A_M, that is, A may well have variable operations, and they are models of the processes of the PU evolved by its adaptation to the SW and not to the nooworlds in $\mathcal{W}$. Indeed, in CL, the variable operations of A are the nominalizers $\triangledown x$, which have no counterparts in any DWs and are models of the processes of the PU adapted to the nominalizers in the SW to be mentioned in §1.2.7. Thus the models provided by CL imply that the SW and none of the nooworlds in $\mathcal{W}$ evolved the processes of the PU for abstraction of NOMINALS from DECLARATIVES.

[1.54]The word *given* here and below means *given by the semantics of $\mathcal{L}$*.

The semantics of $\mathcal{L}$ also provides a model of the perceptual relationship $\mathcal{P}$ in the following way. Each pair (δ, υ) of a *given* denotation δ of C into a *given* DW W and any valuation υ of X into W yields a denotation $\varphi^{\delta}_{\upsilon}$ of A into W defined by (3.3.15) (s. (3.0.1)), which satisfies $\varphi^{\delta}_{\upsilon}|_{\mathsf{C}} = \delta$, $\varphi^{\delta}_{\upsilon}|_{\mathsf{X}} = \upsilon$ and $\varphi^{\delta}_{\upsilon}|_{\mathsf{P}} = \delta \cup \upsilon$ by (3.3.18) and (3.3.19). It is a model of a PERCEPTION, that is, the pair $(\mathsf{a}, \varphi^{\delta}_{\upsilon}\mathsf{a})$ for each element $\mathsf{a} \in \mathsf{A}$ is a model of the pair of a possible percept and its possible object. In particular, the pair $(\mathsf{c}, \delta\mathsf{c})$ for each constant c is a model of the pair of a possible invariable prime percept and its possible object, and the pair $(\mathsf{x}, \upsilon\mathsf{x})$ for each variable x is a model of the pair of a possible variable prime percept and its possible object. Although the object of each invariable prime percept does not momently vary, it may vary in the long term, and the set Δ_W of the *given* denotations of C into W provides a model of the variation. The objects of the variable prime percepts may momently vary, and the set Υ_W of *all* valuations of X into W provides a model of the variation. Thus the mapping $\delta \cup \upsilon$ of P into W is a model of a prime PERCEPTION which may vary with persons, development of each person and the passage of time. Since the pair (δ, υ) yields $\varphi^{\delta}_{\upsilon}$, these models imply that the prime PERCEPTION determines the PERCEPTION (this does not contradict [1.30]). Finally, the equations (3.3.20) and (3.3.21) on $\varphi^{\delta}_{\upsilon}$ together provide a model of the perceptual relationship $\mathcal{P}$. The equation (3.3.21) shows that $\varphi^{\delta}_{\upsilon}$ is a homomorphism of A_{M} into W, that is, the following diagram is commutative for each invariable operation α_λ $(\lambda \in \mathsf{M})$ of A, the associated operation ω_λ of W and each $(\mathsf{a}_1, \ldots, \mathsf{a}_k) \in \mathrm{Dm}\, \alpha_\lambda$.

$$\begin{array}{ccc} (\mathsf{a}_1, \ldots, \mathsf{a}_k) & \xrightarrow{\text{k times } \varphi^{\delta}_{\upsilon}} & (\varphi^{\delta}_{\upsilon}\mathsf{a}_1, \ldots, \varphi^{\delta}_{\upsilon}\mathsf{a}_k) \\ \alpha_\lambda \downarrow & & \downarrow \omega_\lambda \\ \alpha_\lambda(\mathsf{a}_1, \ldots, \mathsf{a}_k) & \xrightarrow[\varphi^{\delta}_{\upsilon}]{} & \varphi^{\delta}_{\upsilon}(\alpha_\lambda(\mathsf{a}_1, \ldots, \mathsf{a}_k)) \end{array} \tag{1.2.2}$$

Therefore, (3.3.21) gives an answer to the questions I raised about the right half of the cUPO-diagram with α modeled by α_λ. The equation (3.3.20) gives $\varphi^{\delta}_{\upsilon}(\alpha_\lambda(\mathsf{a}_1, \ldots, \mathsf{a}_k))$ for each variable index λ and each $(\mathsf{a}_1, \ldots, \mathsf{a}_k) \in \mathrm{Dm}\, \alpha_\lambda$ in terms of the *given* significance λ_W of λ on W and thereby gives an answer to the questions when λ_W is specified in CL. Particularly in ICL (s. Example 3.3.1), $\alpha_\lambda = \triangledown\mathsf{x}$ with $k = 1$,[1.55] and we may identify $\varphi^{\delta}_{\upsilon}(\triangledown\mathsf{x}\,\mathsf{a})$ for each $\mathsf{a} \in \mathrm{Dm}\, \triangledown\mathsf{x}$ with the subset $\{w \in W_\epsilon : \varphi^{\delta}_{\upsilon(\mathsf{x}/w)}\mathsf{a} = 1\}$ of the set W_ϵ of the basic entities in W, as is illustrated by the following diagram (s. (3.3.22)).

$$\begin{array}{ccc} \mathsf{a} & \xrightarrow{\varphi^{\delta}_{\upsilon(\mathsf{x}/w)}} & \varphi^{\delta}_{\upsilon(\mathsf{x}/w)}\mathsf{a} \quad (w \in W_\epsilon) \\ \triangledown\mathsf{x} \downarrow & & \downarrow \\ \triangledown\mathsf{x}\,\mathsf{a} & \xrightarrow[\varphi^{\delta}_{\upsilon}]{} & \varphi^{\delta}_{\upsilon}(\triangledown\mathsf{x}\,\mathsf{a}) = \{w \in W_\epsilon : \varphi^{\delta}_{\upsilon(\mathsf{x}/w)}\mathsf{a} = 1\} \end{array} \tag{1.2.3}$$

A generalization of this holds in the general CL. Thus a model of the perceptual relationship $\mathcal{P}$ may be derived from a model of the former half of the evolutional relationship $\mathcal{E}$ and models of the processes of the PU evolved by its adaptation to

[1.55] In fact, the operation symbol $\triangledown\mathsf{x}$ in CL is postpositive.

the SW, that is, the perceptual relationship may be explained by the evolutional relationship and the adaptation of the processual algebraic structure of the PU to the SW, as was noted right after the cUPO-diagram.

Enough has been said in terms of the GL in Chapters 2 and 3 as to how logic alone can provide a principal mathematical model for MN.[1.56]

Moreover, certain long-established methods of logic promise to be right for mathematical analysis of the model. Recall the flow chart (Fig. 1.1) for mathematical science and the intermediate or ultimate aim of MN noted in §1.2.1. They show that we should mathematically analyze the model of the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ provided by $\mathcal{L}$ and a deduction system (R, D) on H in order to obtain some results about the possible ability of the IU. Since the IU is the algebraic union of the PU and RU, the possible ability of the IU is also the 'algebraic union' of those of the PU and RU. Any expert logicians (you will be one of them provided you pursue the GL in Chapters 2 and 3) would agree with me that the questions about the possible ability of the PU are not challenging because the model A of the PU has a basis P (s. Remark 3.1.18). Any expert logicians would also agree with me that completeness theorems and incompleteness theorems in Gödel's sense for (R, D) and for subsets of H respectively imply certain results about the possible ability of the RU (s. §2.7, §2.8 and §3.5). The theorems hold for the logic space which is made out of P and the pairs $(\delta, \upsilon) \in \Delta_{W} \times \Upsilon_{W}$ for all *given* DWs W (s. §3.3.3), and the noocosmos is the union of the nooworlds in $\mathcal{W}$ of which the DWs are models. Therefore, the results about the RU apply to the ratiocination about the whole noocosmos at every instant by every person at every developmental stage and in every verbal community.

Thus I even think that modern logic was founded for the very purpose of MN, although the founders may have formulated neither the GL in Chapters 2 and 3 nor the above perspective of MN and we need to proceed to the above three empirical ologies in order to pick valuable specific logic for MN.

You will find a guide to semasiology in §1.2.7. It will turn out that phraseology is a converse of semasiology. Thus the guide also serves as one to phraseology and suggests how I and my pupils carried out the two ologies.

It is difficult to supply a guide to ontology or explain how we carried out it, because its essence is mathematics for a tool and a language of science (s. §1.1) and mathematics as such is a matter of intellectual inspiration and aesthetical appreciation which hardly admit of guides or explanation. Furthermore, it will turn out that semasiology requires ontology (and vice versa). Thus ontology is the key to MN and the most challenging of the three ologies.

In order to carry out semasiology and phraseology, one need to be an expert of the MEANINGS and the usage of the utterances in the verbal communities concerned. One is most expert at them in one's native community. For this reason and another,[1.57] we carried out the ologies for the present-day Japanese community rather than foreign ones.[1.58] That turned out to be rather fortunate for us, and we could derive from them the logic system CL and a certain

[1.56]This shows a raison d'être for GL (s. [1.69]).
[1.57]See §1.2.8 around [1.65] for another more important reason.
[1.58]Every ancient community should be regarded as a foreign one for everyone.

deduction system on it for a model of the nootrinity. I will explain the whys and wherefores of the fortune and the how of the derivation in due course.

The definition of CL and the deduction system implies the conclusions of the three ologies we carried out. Therefore, its value will imply value of the ologies which I do not detail in this monograph. It should be evaluated by how results of the ongoing mathematical analysis of CL and the deduction system are helpful in understanding intellectual phenomena in the semi-mathematical final step of the flow chart (Fig. 1.1) for mathematical science.

You may derive another logic system from the three ologies for your native community, but a carefully derived one would not essentially depend on the community (s. [1.16]), for I have conjectured that the nootrinity did not do so.

Remark 1.2.11 (Know the limit of mathematics) *What are the things of which we can construct mathematical models? To what extent can we mathematically analyze the models?* These questions fall within the scope of MN, because they concern both the mathematical steps of the flow chart (Fig. 1.1) for mathematical science and the possible ability of the IUs of mathematicians who constitute a verbal community with a hybrid DT (s. [1.19]). They are related to foundations of mathematics and an incompleteness theorem (s. §3.5).

Remark 1.2.12 (Know models from materials and tools) Mathematical scientists construct and analyze mathematical models by mathematical materials and tools (s. §1.1). Moreover, even a mathematical concept may provide various models, materials and tools. Particularly in MN, a logic system and a deduction system on it may provide not only a model of the nootrinity but also materials, tools and other models, and likewise for other mathematical concepts.

Example 1.2.2 (PLQs for models) A quasialgebra S is called a PLQ provided its quasialgebraic structure Q is a partially latticed association as defined in Theorem 2.2.10. The quasialgebra is a PLQ iff Q is equal to the association R^D derived from a deduction system (R, D) on S and iff $Q = {R_{\mathcal{O}}}^D$ for the association $R_{\mathcal{O}}$ derived from an algebraic structure $\mathcal{O}$ on S by (3.1.1) and a subset D of the Q-core $S_Q = \{x \in S : \varepsilon\, Q\, x\}$ of S (s. §2.5.1). Thus PLQs have provided models of the causal algebraic structures of the event sets of the nooworlds in $\mathcal{W}$ (s. §1.2.4) and a model of God in [1.9] and Remark 1.2.9.

Furthermore, a PLQ provides a model of the IU with certain of its memories taken into consideration. We have abstracted an algebra $(S, \mathcal{O})$ from the IU in §1.2.2, disregarding its memories (s. [1.12]). The set S consists of all possible percepts by §1.2.3, and $\mathcal{O}$ yields an association $R_{\mathcal{O}}$ on S as above. Let D the subset of S of all stored percepts and regard the association of percepts as an association M on S (s. [2.25]), that is, an element $(x_1 \cdots x_n, y) \in S^* \times S$ satisfies $x_1 \cdots x_n\, M\, y$ iff $n \geq 1$ and the tuple $(x_1, \ldots, x_n)$ of percepts recalls the percept y. Then Theorem 3.1.1 shows that $M = R_{\mathcal{M}}$ for some algebraic structure $\mathcal{M}$ on S, and so the union $R_{\mathcal{O}} \cup M$ is equal to $R_{\mathcal{O} \cup \mathcal{M}}$ (s. §1.5.2) and yields an association $(R_{\mathcal{O} \cup \mathcal{M}})^D$ on S. Thus we have abstracted a PLQ $(S, (R_{\mathcal{O} \cup \mathcal{M}})^D)$ from the IU taking certain of its memories into consideration (s. Remark 1.1.1). This seems reasonable because $(x_1 \cdots x_n, y) \in S^* \times S$ satisfies $x_1 \cdots x_n \, (R_{\mathcal{O} \cup \mathcal{M}})^D \, y$ iff y has a $(\{x_1, \ldots, x_n\} \cup D)/(\mathcal{O} \cup \mathcal{M})$-sequent.

1.2.7 Semasiology and phraseology

In semasiology for MN, you should deal with the DDUs in verbal communities to which you belong, especially your native community.[1.59] However, I focus on the DUs in the English community for convenience of explanation, disregarding the deductive utterances. Then semasiology is reduced to gaining an insight into the syntactical algebraic structure abstracted from the English SW, or into the syntactical rules for the MEANINGs of the English DUs. The insight will suggest the syntax of your logic system for a model of the nootrinity by virtue of the algebraic analogy between the SW and PU.

In explaining methods of semasiology, it is impossible to overemphasize the importance of distinguishing DUs from their MEANINGs and paying attention to their contexts, because DUs are usually unfaithful to their MEANINGs, while their MEANINGs may be faithfully phrased by real or imaginary DUs according to their contexts (s. the USPO-diagram). Here a DU is said to be unfaithful to its MEANINGs provided it is obscure or redundant for decrease of obscurity. This is best explained by the following examples.

I suppose that the utterer of

He ate many radishes at the garden

means the following or something like that or more according to its context:

Sometime in 1902 Peter ate four or five radishes at Mr. McGregor's garden.

This is because the utterer must know whom the pronoun *he* indicates, when he ate radishes, how many radishes he ate and how the definite article *the* restricts the noun *garden*, even if the knowledge is not precise; indeed, the adverbial phrase *sometime in 1902* does not show a precise time and the quantifier *four or five* does not show a precise number.[1.60] The utterer must also know and mean that the year 1902 is past, that *Peter* and *Mr. McGregor* are male names and that creatures like Peter usually eat at most one radish at a time.

The latter utterance is less obscure and so more faithful to its MEANING than the former. However, the latter is still unfaithful thereto because of redundancy caused by the decrease of obscurity. Although required by English grammar and useful, the past tense conjugation of *ate* and the plural declension of *radishes* therein are not necessary for expressing its MEANING, because *sometime in 1902* already shows preteritness and *four or five* already shows plurality. In other words, as the pronoun *he* and the definite article *the* in the former utterance are obscure substitutes for *Peter* and *Mr. McGregor's* respectively or something like them or more, so the past tense conjugation and the plural declension together

[1.59] I suppose that you belong to several verbal communities, because I belong to the Japanese community, worldwide mathematicians' community (s. Remark 1.2.11) and others.

[1.60] However, they may be regarded as indicating definite intervals of $\mathbb{R}$ and $\mathbb{N}$ respectively.

with the quantifier *many* therein are obscure substitutes for *sometime in 1902* and *four or five* respectively or something like them or more and so redundant in the latter utterance. I refer to this kind of redundancy as redundancy for decrease of obscurity (and so tautology is not redundant in this sense). Similar remarks apply to the conjugation of verbs by person and number such as *eats*, *gets* and *runs* in the utterances below. The conjugations are obscure substitutes for certain expressions about the persons concerned and so will be redundant in utterances which are less obscure, containing the expressions.

The quantifier *many* above is obscure because it is contextual, i.e. dependent on a context of the utterance such as that creatures like Peter usually eat at most one radish at a time, while *sometime in 1902* and *four or five* are not obscure in that they are context-free and intervallic definite, that is, they indicate definite numerical intervals (s. [1.60]) which do not depend on any context.

As for quantifiers, *almost all* is also obscure because it is contextual, while the quantifiers *all* and *some* are also not obscure in that they are context-free and intervallic definite (s. §1.3.7). Thus quantifiers are divided into contextual ones and context-free ones. The contextual ones are generally obscure, while the context-free ones are mostly intervallic definite and not obscure as such.

Also for example, I suppose that the utterer of

Peter must not go into Mr. McGregor's garden

means the following or something like that or more according to its context:

If Peter goes into Mr. McGregor's garden, then Mr. McGregor gets angry.

The utterer must also mean that Mr. McGregor uses violence to those who make him angry and that she wants no one to be hurt. In other words, the negation *must not* of the auxiliary verb *must* is an obscure substitute for the expression *If . . . , then Mr. McGregor gets angry* or something like that or more.

Similar remarks apply to the auxiliary verb *can* as in the utterance

Peter can go into Mr. McGregor's garden.

Also for example, I suppose that the utterer of

Peter runs fast

means the following or something like that or more according to its context:

Peter runs in a manner a cat runs in.

The utterer must also mean that humans cannot catch up with cats. In other words, the adverb *fast* is an obscure substitute for the adverbial phrase *in a manner a cat runs in* or something like that or more. Moreover, I regard the nominal *manner a cat runs in* as the composite

(*manner x*)(*a cat runs in x*).

Here x is a silent indefinite noun working as an appositive of the noun *manner*, and so *manner* x is a nominal and *a cat runs in* x is a sentence. While explained by the sentence *a cat runs in* x, the nominal *manner* x may be regarded as nominalizing the sentence. Thus the MEANING of *Peter runs fast* may be more faithfully phrased by the following imaginary utterance or something like that or more which involves the nominalizer *manner* x:

Peter runs in a (*manner* x)(*a cat runs in* x).

This also serves for a MEANING of the utterance *Peter runs as a cat*.

You can similarly obtain the nominalizer *degree* x.

Suppose also that the utterer of *Mr. McGregor lives there* means *Mr. McGregor lives in a place rabbits live in*. Then the latter utterance has the same structure as the above utterance *Peter runs in a manner a cat runs in*, and so the MEANING of the former utterance may be more faithfully phrased by the imaginary utterance *Mr. McGregor lives in a* (*place* x)(*rabbits live in* x) or something like that or more which involves the nominalizer *place* x.

You can similarly obtain the nominalizer *time* x.

Similarly, I deem that the MEANINGs of the two utterances

Mr. McGregor knows the radish that rabbits eat,
Mr. McGregor knows that rabbits eat the radish

may be more faithfully phrased respectively by the following imaginary utterances or something like them or more (s. §1.3.3):

Mr. McGregor knows the radish $\sqcap$ (*that* x)(*rabbits eat* x),
Mr. McGregor knows (*that* x)(*rabbits eat the radish*).

Here $\sqcap$ is a silent conjunction for appositive nominals, x is a silent indefinite noun and *that* qualifies x as an adjective, and so *that* x is a nominal and *rabbits eat* x is a sentence. A key idea here is to phrase both the relative pronoun *that* and the conjunction *that* in the original utterances by the nominal *that* x and regard it as nominalizing the sentences *rabbits eat* x and *rabbits eat the radish*.

I have thus extracted several kinds of nominalizers such as *manner* x, *degree* x, *place* x, *time* x and *that* x and a conjunction $\sqcap$ from the MEANINGs of the DUs. This suggests that the syntactical algebraic structure abstracted from the SW has such operations. Therefore, in view of the algebraic analogy between the SW and PU, I equip the formal language of CL with nominalizers $\triangledown\mathtt{x}$ for certain variables $\mathtt{x}$ and a conjunctive operation $\sqcap$, which together with the semantics of CL explain verbal phenomena concerning the nominalizers extracted from the MEANINGs of the DUs.[1.61] The nature of $\triangledown$ and $\mathtt{x}$ should vary with the nominals concerned, as was suggested by the above examples. In order to deal with the variety, I also equip CL with a concept of partibility of entities (s. Table 1.1); ICL has the smallest partibility $\mathtt{1}$, which is too small for MN (s. §1.3.1).

[1.61] By the semantics of ICL, the nominalizer $\triangledown\mathtt{x}$ signifies the set $\{\mathtt{s} \in \mathtt{S} : \mathtt{C}(\mathtt{s})\}$ of the elements $\mathtt{s}$ of a set $\mathtt{S}$ satisfying a condition $\mathtt{C}(\mathtt{s})$ (s. (1.2.3)), and the conjunctive operation $\sqcap$ signifies the intersections of subsets of $\mathtt{S}$ (s. §1.3.5). The semantics of CL is its generalization.

Exemplification continues. The word *of* connecting nominals is mostly an obscure substitute for expressions which involve the relative pronoun *that* and so may be more faithfully phrased by means of the nominalizer *that x* and the silent conjunction $\sqcap$, as is illustrated by the following examples.

$$
\begin{gathered}
\textit{dictator of Rome} \to \textit{dictator} \sqcap (\textit{that } x)(x \textit{ rules Rome}) \\
\textit{people of Rome} \to \textit{people} \sqcap (\textit{that } x)(x \textit{ live in Rome}) \\
\textit{man of Rome} \to \textit{man} \sqcap (\textit{that } x)(x \textit{ came from Rome}) \\
\textit{name of Rome} \to \textit{name} \sqcap (\textit{that } x)(x \textit{ is Rome}) \\
\textit{property of Rome} \to \textit{property} \sqcap (\textit{that } x)(x \textit{ is owned by Rome}) \\
\textit{progress of Rome} \to \textit{progress} \sqcap (\textit{that } x)(x \textit{ is made by Rome}) \\
\textit{history of Rome} \to \textit{history} \sqcap (\textit{that } x)(x \textit{ is concerned with Rome}) \\
\textit{north of Rome} \to \textit{north} \sqcap (\textit{that } x)(x \textit{ is viewed from Rome}) \\
\textit{love of Rome} \to \textit{love} \sqcap (\textit{that } x)(x \textit{ is directed to Rome}) \\
\textit{city of Rome} \to \textit{city} \sqcap (\textit{that } x)(x \textit{ is called Rome})
\end{gathered}
$$

Thus the word *of* here is an ellipsis indicating omission of regular expressions dependent on the nominals it connects.

Most adjectives have both attributive use and predicative use, and neither of them may appear obscure. However, the attributive use is judged obscure and may be more faithfully phrased by means of the predicative use, the nominalizer *that x* and the silent conjunction $\sqcap$, as is illustrated by the following example.

$$\textit{little bunny} \to (\textit{that } x)(x \textit{ is little}) \sqcap \textit{bunny}$$

Similar remarks apply to appositive nouns.

$$\textit{farmer McGregor} \to (\textit{that } x)(x \textit{ is a farmer}) \sqcap \textit{McGregor}$$

Enough has been said about methods of semasiology for MN. To empirically generalize from the above examples in the English community, DUs in each verbal community are usually unfaithful to their MEANINGs, but they may be rephrased according to their contexts by real or imaginary DUs which are faithful to their MEANINGs (s. the USPO-diagram) in the proper sense of the word *imaginary*, that is, *existing only in the mind*. Although context is irregular, the examples suggest the following rules for rephrasing the DUs among others.

- Replace obscure words by clear ones, phrases or clauses.
- Supply absent words, phrases, clauses and symbols[1.62].
- Remove redundancy for decrease of obscurity as to tense, plural, etc.
- Decompose nominals into pairs of a nominalizer and a sentence.

The totality of the rules for rephrasing the DUs, which I call rephraseology, will lead to removing the grammatical categories which cause obscurity such as the pronoun, definite article, contextual quantifier, auxiliary verb, adverb, ellipsis

[1.62] The symbols here include operation symbols, silent words and punctuation marks.

of and attributive adjective. In semasiology for MN, you must thus pursue rephraseology, from which the SW will emerge.

You need not worry that rephraseology is subjective. Indeed science in general should be objective, but rephraseology should be rather subjective, because the way you subjectively apply rephraseology to the DUs will reflect the processual algebraic structure of your PU, which you seek. To be precise, suppose that you rephrase a DU U by a real or imaginary DU U'. Then you have an UNDERSTANDING P' of U (s. [1.29]) and phrase it by U', as is illustrated by the following UPU-diagram. Your UNDERSTANDING P' of U, as well as a MEANING P of U for you (s. the SUPO-diagram in §1.2.5), is a percept in your PU.[1.63]

$$\mathsf{U} \;\leftarrow\; \mathsf{P}' \;\leftarrow\; \mathsf{U}' \tag{UPU}$$

Suppose moreover that you faithfully phrase P' by U' and that P' is produced from percepts $\mathsf{P}'_1, \dots, \mathsf{P}'_k$ by an operation α in the processual algebraic structure of your PU. Then you will faithfully phrase $\mathsf{P}'_1, \dots, \mathsf{P}'_k$ by real or imaginary DUs $\mathsf{U}'_1, \dots, \mathsf{U}'_k$ and form U' with $\mathsf{U}'_1, \dots, \mathsf{U}'_k$ under a syntactical rule R which reflects α, as is illustrated by the following diagram.

$$\begin{tikzcd}
(\mathsf{P}'_1, \dots, \mathsf{P}'_k) \arrow[dr, "\alpha"'] & & (\mathsf{U}'_1, \dots, \mathsf{U}'_k) \arrow[ll, "\text{k times}"'] \arrow[dl, "\mathsf{R}"] \\
& \mathsf{P}' \quad\longleftarrow\quad \mathsf{U}' &
\end{tikzcd}$$

In this sense, objectivity in semasiology for MN is equal to subjectivity.

On the other hand, the above examples of nominalizers and [1.61] suggest that semasiology for MN should not only be subjective but also be supported by the semantics of the logic system you construct, and semantics requires an insight into the compositional algebraic structures of the nooworlds with the truth values. Therefore, semasiology requires both ontology and an insight into semantics. The e^3-diagram in §1.2.5 shows that ontology conversely requires semasiology. Thus you need to synchronize ontology with semasiology. Furthermore, the ability to think logically is needed in considering the deduction rules reflected in the deductive utterances. Therefore, you cannot base semasiology for MN on any statistical survey of people's opinions about the DDUs.

Now then, what can you attain by rephraseology? Infinitely many DUs are possible (s. [1.11]) and you can hardly find their complete syntactical rules and lexicon. Furthermore, context is irregular and rephraseology is subjective. Therefore, you cannot apply rephraseology to all DUs. However, the empirical knowledge of the MEANINGs of the DUs obtained by pursuing rephraseology and the mathematical method of making formal languages noted in Remark 1.2.2 will together enable you to make a tongue out of the DT so that you can expect that it consists of faithful expressions of the MEANINGs of the DUs. I refer to the hybrid tongue and its elements as the *rephrased DT* (RDT) and *rephrased DUs* (RDU) respectively; I have used the term *RDU* also for the communal SW

[1.63]The duality of MEANING and UNDERSTANDING defined in §1.2.8 shows that P' is also a MEANING of U, and U is usually unfaithful to both P and P'.

in §1.2.5 because of (1) below. Since the persons in the verbal community have close verbal cultures, the RDT will not essentially depend on the person who makes it (s. [1.16]) provided the person is careful.

Thus I assume that each verbal community has its RDT. Although hybrid, the RDT possesses a definite syntactical algebraic structure abstracted from the rules under which the composite RDUs are composed of their constituent RDUs (s. Remark 1.2.2). To conclude, the RDT is the heart of semasiology for MN, as is explained in the following three steps.

(1) The RDT serves as a set of perceivable symbols for the MEANINGs of the DUs and so may be identified with the communal SW. However, the SW exists irrespective of whether we make the RDT, and so we can define CONTEXT as in Remark 1.2.7, and we find that it is what we have called context.

(2) The algebraic analogy between the SW and PU is that between the RDT and PU because of (1). Consequently, the algebraic structure of the RDT must not essentially depend on the verbal community because of the algebraic homogeneity of the PU (s. [1.16] and the nearby text).

(3) The duality of MEANING and UNDERSTANDING defined in §1.2.8 means that the way you phrase percepts in your PU by DUs is the converse of the way you have UNDERSTANDINGs of DUs. In rephraseology, you faithfully phrase the UNDERSTANDINGs by RDUs. Thus phraseology may be regarded as a converse of rephraseology, or of semasiology, as was noted in §1.2.6.

Thus semasiology by rephraseology will imply phraseology and suggest the nature of your logic system and deduction system on it for a model of the nootrinity as was noted in the paragraph of §1.2.6 on my perspective.

1.2.8 Meaning and understanding

Here I supplementally explain a correlation between MEANINGs and UNDERSTANDINGs of the DUs. I urge you to review §1.2.5–1.2.7 with this explanation in mind to find that the correlation has been taken for granted there.

Suppose a lady phrases her percept P by a DU U and I have an UNDERSTANDING P' of U (s. the UPU-diagram); for example, suppose Ms. Potter utters *He ate many radishes at the garden* and I understand it as *Sometime in 1902 Peter ate four or five radishes at Mr. McGregor's garden* (s. §1.2.7). By definition, the percept P is a MEANING of U for her (s. the SUPO-diagram). Thus we obtain the following PUP-diagram in view of the above two diagrams.

$$\text{her PU } \boxed{\quad \mathsf{P}} \longleftarrow \mathsf{U} \hookleftarrow \boxed{\mathsf{P}' \quad} \text{ my PU} \qquad \text{(PUP)}$$

This is underlain by the inner system $\mathcal{S}$ of each person which produces DUs from their MEANINGs for the person and produces UNDERSTANDINGs of DUs by the person. As such, $\mathcal{S}$ involves the vocal organ, auditory organ and other organs related to communication. More importantly, $\mathcal{S}$ also involves the processes of the IU and its presently disregarded memories. I refer to $\mathcal{S}$ as the verbal culture of the person because it rules verbal behavior of the person.[1.64]

The duality of MEANING and UNDERSTANDING noted in §1.2.7 may be defined as the invertibility of $\mathcal{S}$, that is, I suppose that the following holds for each pair $(\mathfrak{u}, \mathfrak{p})$ of a DU $\mathfrak{u}$ and a percept $\mathfrak{p}$ in the PU of the person:

$$\begin{aligned}&\mathfrak{p} \text{ is a MEANING of } \mathfrak{u} \text{ for the person,}\\ &\text{that is, } \mathfrak{u} \text{ is an utterance of } \mathfrak{p} \text{ by the person}\\ &\iff \mathfrak{p} \text{ is an UNDERSTANDING of } \mathfrak{u} \text{ by the person.}\end{aligned}$$

Now let $\mathcal{S}$ and $\mathcal{S}'$ be the lady's verbal culture and mine respectively. Unless she is I, her PU and my PU lie in different places, and so the percepts P and P' in the PUP-diagram are different existences. However, the PUs have essentially the same processual algebraic structure (s. [1.16]) because of the algebraic homogeneity of the PU. Recall from §1.2.5 that a verbal community is a group of people with close verbal cultures. Therefore, if she and I belong to the same verbal community, then $\mathcal{S}$ and $\mathcal{S}'$ are close, and so there is a good chance that P and P' have close constructions for each DU $\mathfrak{U}$ in the PUP-diagram, and thus I shall be able to study the common algebraic structure of the PUs by analyzing my UNDERSTANDINGs of her DUs by the method of semasiology as explained in §1.2.7. Especially if she is I, then $\mathcal{S}$ and $\mathcal{S}'$ are equal, and so there is a better chance that P and P' are equal. Thus I shall be better able to study the algebraic structure of my PU by analyzing my UNDERSTANDINGs of my DUs by the method of semasiology, and the results of the analysis will apply to other persons as well. For this very reason[1.65], I carried out ontology, semasiology and phraseology for my native community, especially myself.

1.2.9 A summary for going ahead

Mathematical noology is the science of intellectual phenomena based on the assumption that there exists an intellectual unit in the human brain and it is a machine, i.e. a transformational device. An intermediate aim of mathematical noology is to investigate the possible ability of the machine by means of some mathematical model of the triple consisting of the machine, the noocosmos and the relationship between them. It models the essence of us *Homo sapiens* (wise man) associated with what we as such think about and spins off a concept of God which leads us to eusophy for survival of us the human species in evolution. Therefore, the aim can even be our ultimate aim.

Algebraic abstraction of machines, logic, tongues and the noocosmos and a belief in evolutionism led me to the belief that the model should be provided by some logic system and some deduction system on it, which will emerge from

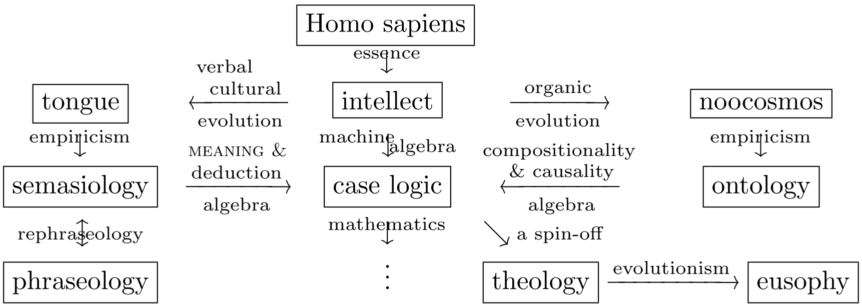
[1.64] I only described *part of* verbal phenomena as they are, and I intended no definition. Therefore, I do not assert that any ordinary machine has its verbal culture for uttering its percepts and understanding utterances in its verbal community.

[1.65] This is the reason I referred to as another more important reason in [1.57].

ontology, semasiology and phraseology for our native communities in an algebraic light. Thus I obtained case logic and a deduction system on it by faithfully dealing with the results of the ologies for the present-day Japanese community. The model should be evaluated by how results of their ongoing mathematical analysis help us understand intellectual phenomena.

This summary may be amplified by the schematic chart in Fig. 1.3.

Figure 1.3: A schematic chart for mathematical noology



1.3 Features of case logic besides nominalizers

The logic system CL (case logic) together with a certain deduction system on it is intended to provide a mathematical model of the nootrinity and is faithful to the three ologies urged in §1.2.6. On the other hand, CL can be more general than is necessary for MN. As such, CL has the following features in addition to those about the nominalizers ∇x mentioned in §1.2.6 and §1.2.7.

1.3.1 It is parameterized by nomina for entities

The logic system CL is parameterized by a nonempty set N whose elements are called the *nomina*. The singular form *nomen* of *nomina* has meant an ancient Roman male citizen's second name indicating the ancestral group of families to which his family belongs. Each nomen in MN is intended to be a model of the name for a basic class of entities such as *individual*, *place*, *time*, *matter*, *manner* and *degree* (s. §1.2.7 and §1.3.5). Therefore, as far as MN is concerned, N cannot be too small. As far as mathematics is concerned, however, N can be an arbitrary nonempty set, and CL with $\#\mathsf{N} = 1$ has been called ICL (impartible CL),[1.66] while CL with $\#\mathsf{N} > 1$ will be called *partible CL* (PCL); in particular,

[1.66] The symbol # denotes the cardinality of sets.

CL with $\#\mathsf{N} = 2$ will also be called *bipartible CL* (BCL).[1.67] Thus $\#\mathsf{N}$ may be called partibility of entities (s. Preface and §1.2.7).

The theory of CL is constructed in Chapters 4–6 in line with the theory of GL in Chapters 2 and 3. Phraseological illustrations apart, their mathematical organization is as follows. Chapters 4 and 5 are of introductory nature, focusing on ICL and BCL respectively. Chapter 4 begins with the definition of ICL and ends with a proof that *first-order predicate logic* (FPL) is embedded in it and so not quite valuable for MN as with ICL. There the set N of nomina is hidden because $\#\mathsf{N} = 1$. Chapter 5 begins with the definition of BCL and ends with a proof that *Kripkean modal logic* (KML) is embedded in *asymmetric BCL* (ABCL)[1.68] and so not quite valuable for MN as with BCL. In order for FPL and KML to be so embedded, they need reformulation in terms of GL (s. §3.4).[1.69] Focusing on PCL, Chapter 6 begins with the definition of CL and culminates in a completeness theorem and related theorems on it (s. §1.3.11).

Being faithful to the three ologies urged in §1.2.6, CL is naturally elaborate. Therefore, I advise you to master ICL and BCL first and proceed to CL. It is a reason why Chapters 4 and 5 on ICL and BCL precede Chapter 6 on CL, although they are special cases of CL and not quite valuable for MN.

1.3.2 It may replace some existing logic systems

The above completeness theorem is relevant to MN because it implies a certain result about the possible ability of the RU (s. §2.7). In contrast, embedding theorems in general concern researchers of the embedded logic systems. The above two of them are digressions which may possibly be helpful in understanding CL, and so I will not pursue embedding theorems any longer.

Since CL is faithful to the three ologies urged in §1.2.6, some more existing logic systems are rightly expected to be embedded in CL. Interested readers can extend the embedding of KML in ABCL to that of polymodal logic in PCL or to those of other existing logic systems (e.g. temporal logic and fuzzy logic) in PCL or some modifications of PCL (s. [1.69]).

The embedding theorems will imply that CL is more expressive than the embedded logic systems and that quantity, modality, tense, fuzziness, and so on are understood within CL all together and better than within the embedded logic systems. Then the embedded logic systems should be replaced by CL.

1.3.3 It supplies semantics for the "that" clauses

I defined an antecedent of CL before 1997 without any mathematical mistakes. As far as MN is concerned, however, it had certain defects which were the same

[1.67] I formerly referred to the elements of N, ICL and PCL as phases, *monophasic CL* (MCL or MPCL) and *polyphasic CL* (PCL or PPCL) respectively, translating the original Japanese name *sô* of the elements of N into *phase*. I am sorry that my pupils Mizumura and Takaoka have followed my mistranslation (s. [1.91]–[1.93]).

[1.68] The word *asymmetric* means that the two nomina are distinguishable by their nature.

[1.69] More generally in order to compare a specific branch of logic with another, we need to reformulate both in terms of GL. This shows another raison d'être for GL (s. [1.56]).

as those of ICL suggested in §1.3.1, that is, the antecedent of CL could not provide the right semantics for such a DECLARATIVE as is phrased (in view of phraseology) by the existential sentence *Peter exists in a wood*, such a DECLARATIVE as is phrased by the temporal copular sentence *Mrs. Rabbit*[1.70] *was a widow in 1902* and such a NOMINAL as is phrased by the *that* clause in the sentence *Mr. McGregor knows that rabbits eat the radish*.

In order to remove the existential defect and the temporal copular defect, Takaoka defined an antecedent of ABCL.[1.71] It turned out to be a breakthrough, and I made it up into CL in 2009 taking a certain observation of Mizumura into consideration. I will try to show that CL of large partibility is rid of all the defects. I have shown a key idea for the *that* clause defect in §1.2.7 by using the above sentence. The nominalizers $\triangledown \mathsf{x}$ of CL are relevant to the existential defect as well as to the *that* clause defect.

1.3.4 It has explicit postpositional case markers

Being faithful to the three ologies urged in §1.2.6 for the present-day Japanese community, CL furthermore has the following features.

The name of CL (case logic) derives from one of its parameters called the set of the case markers or cases, which in turn derives from a work of a linguist Charles Fillmore. I have not read his publications but only heard that he initiated the so-called case grammar of the English tongue, yet I know that the case in my native Japanese tongue is explicitly marked by postpositions such as *ga*, *o*, *ni* and *de*.[1.72] For example, *ga* is nominative; *o* is accusative; *ni* is dative and also indicative of occasions, locations, directions, and so on; *de* is indicative of scenes, means, and so on (s. Remark 1.3.1).[1.73] My knowledge of the Japanese case markers and the name of case grammar naturally inspired me to initiate logic featuring case. Thus both syntax and semantics of CL are parameterized by an arbitrary nonempty set K whose elements should be called the case markers in syntax and the cases in semantics.

Remark 1.3.1 (Genuine case markers) To tell the truth, *de* is a corrupt form of the combination *ni te* of the case marker *ni* and the conjunctive postposition *te*. Moreover, *ni te* is a clipped form of *ni ari te*, *ni oi te*, *ni yori te*, and so on, where *ari*, *oi* and *yori* are conjugations of the verbs *aru*, *oku* and *yoru* which mean *be situated*, *situate* and *depend* respectively. Therefore, *de* is not a genuine case marker but an abbreviation for expressions containing the genuine case marker *ni* (and so should be removed by semasiology as explained

[1.70] *Peter Rabbit* is not apposition but the full name of Peter, who is a rabbit.

[1.71] *Zyôkyôsô o motu kaku ronrigaku* (*Case logic with a class of situational entities*), Master's thesis, Graduate School Math. Sci., Univ. Tokyo, 2006.

[1.72] **(Romanization)** The Japanese writing system consists of kanji (Chinese ideograms) and two kinds of kana (phonograms born of kanji), and we can phoneticize kanji by kana. Thus we can write the Japanese tongue in Roman letters by romanizing kana. There are several romanization systems of kana, and this monograph follows the standardized system ISO 3602.

[1.73] The case markers are replaced or combined with the topic marker *wa* according to context.

in §1.2.7). Even *ni* is often a similar abbreviation. Thus *ni* and *de* can be indicative of occasions, locations, directions, scenes, means, and so on.

The Japanese tongue is characterized by the explicit case markers and postpositional constructions thereby, and so is the logic system CL. To give an example in the former, the upper row (1.3.1) of the following is a typical Japanese sentence meaning *Peter eats radishes in a garden*:

pêtâ	*ga*	*hatake*	*de*	*hatuka daikon*	*o*	*taberu*,	(1.3.1)
Peter		*garden*	*in*	*radish*		*eat*.	

Here the lower row shows English counterparts (or equivalents) of the Japanese words in (1.3.1), and likewise for the Japanese expressions shown below. The counterparts of nouns are shown in the singular forms without articles and those of verbs are shown in the basic forms, because Japanese nouns have neither plural forms nor articles, Japanese verbs conjugate neither by person nor by number and tense will be removed by semasiology as explained in §1.2.7.[1.74] The blanks as between the words *Peter* and *garden* in the lower row show that the Japanese words right over them have no English counterparts. As for the three postpositions *ga*, *de* and *o* in (1.3.1), only *de* has an English counterpart *in*, because the nominative case marker and the accusative case marker in the English tongue are silent and implicit in the word order.[1.75]

To put it differently, the English sentence *Peter eats radishes in a garden* has a Japanese counterpart of the same word order as

Peter ga garden in radish o eat, (1.3.2)

and (1.3.1) is a real counterpart. Therefore, in comparing Japanese sentences with English ones, it will be helpful to supply English ones with the nominative case marker *ga* and the accusative case marker *o*. For example, we obtain

ga Peter eats o radishes in a garden (1.3.3)

from the above sentence *Peter eats radishes in a garden*. Here *ga* and *o* should be prepositions in contrast to (1.3.2) because the English tongue is characterized by prepositional constructions in contrast to the Japanese tongue.[1.76]

By virtue of the explicit case markers, the Japanese tongue enjoys loose phrase order, and so does the logic system CL (s. §1.3.6). To give an example in the former, we can grammatically permute the three postpositional phrases *Peter ga*, *garden in* and *radish o* in (1.3.2) without affecting its meaning.[1.77]

[1.74] English plural forms and articles will also be removed by semasiology as explained in §1.2.7, and likewise for conjugations of English verbs by person and number.
[1.75] Some Japanese case markers are sometimes also silent and implicit in the word order.
[1.76] This suggests that the RDT for the English community (s. §1.2.7) should be equipped with a nominative preposition and an accusative preposition (s. (1.3.5)).
[1.77] In informal Japanese, even the verb *eat* can be permuted with the postpositional phrases.

1.3.5 Its entities have a basis and basic relations

In §1.2.4, I regarded each nooworld with the truth values as the disjoint union $W = \mathsf{E} \cup \mathsf{F}$ of a nonempty set E and a nonempty set F of multiary relations on E (s. Remark 1.1.1) and referred to the elements of E and F as the entities and events of W respectively. Furthermore, I equipped W with the compositional algebraic structure, which is unspecified yet. The algebra W is a rough and expedient model of the nooworld as was noted in §1.2.6. Therefore in Chapters 4–6 on CL, I will refine the definition of E, modify the definition of F and specify the compositional algebraic structure of W. The resultant algebra is called a DW and is the definitive model of the nooworld, provided partibility of CL is large enough. Here I outline the refinement of the definition of E and part of the compositional algebraic structure of W related to E.

As for ICL, E is defined as the direct union $\mathsf{S} \amalg \mathfrak{P}\mathsf{S}$ of a set S and its power set $\mathfrak{P}\mathsf{S}$.[1.78] The set S is called the basis of W and its elements are called the basic entities of W, while those of $\mathfrak{P}\mathsf{S}$ are called the derived entities of W.

If the basis S satisfies $\mathsf{S}_0 \amalg \mathfrak{P}\mathsf{S}_0 \subseteq \mathsf{S}$ for a set S_0 such as the basis of another nooworld, then the relation $x \in X$ between elements $x \in \mathsf{S}_0$ and $X \in \mathfrak{P}\mathsf{S}_0$ becomes a relation on S, and such recurs, for example, if $\mathsf{S} = \coprod_{n=0}^{\infty} \mathsf{S}_n$ with $\mathsf{S}_n = \mathsf{S}_{n-1} \amalg \mathfrak{P}\mathsf{S}_{n-1}$ $(n = 1, 2, \ldots)$. Thus I equip the basis S with a relation $\sqsubseteq$ from the outset, which I call the basic relation, and extend it to a relation between S and $\mathsf{E} = \mathsf{S} \amalg \mathfrak{P}\mathsf{S}$ by the following for each $s \in \mathsf{S}$ and each $X \in \mathfrak{P}\mathsf{S}$:

$$s \sqsubseteq X \iff s \in X.$$

Then as we can define the intersection $X \cap Y$, union $X \cup Y$ and complement X° for subsets X and Y of a set by the relation $\in$, so we can define the elements $a \sqcap b$, $a \sqcup b$ and $a^\square$ of $\mathfrak{P}\mathsf{S}$ for elements a and b of E by the extended relation $\sqsubseteq$ (s. [1.61]).[1.79] We can thus define operations $\sqcap$, $\sqcup$ and $\square$ on W, or on E.

Furthermore, we can associate each entity $a \in \mathsf{E}$ with the unary relation $a\triangle$ on S defined by the following for each $s \in \mathsf{S}$:

$$(a\triangle)s = 1 \iff s \sqsubseteq a.$$

Modifying the definition of the event set F so that $a\triangle \in \mathsf{F}$ for all $a \in \mathsf{E}$, we can thus define an operation $\triangle$ on W.

In this connection, an entity $a \in \mathsf{E}$ is said to exist if there is an element $s \in \mathsf{S}$ such that $s \sqsubseteq a$, that is, if the event $a\triangle$ occurs for some $s \in \mathsf{S}$ (s. [1.22]). The entitic existence is thus not equal yet related to the set membership.

Moreover, for each unary relation f on S, the subset $\{s \in \mathsf{S} : f(s) = 1\}$ of S belongs to E. This fact enables us to define the nominalizers $\triangledown x$ mentioned in §1.2.7 on the formal language of ICL (s. [1.61]).

Thus E has an almost sufficient structure despite its simple definition. Since it is not quite sufficient, however, ICL should be extended to CL, where the basis S is partitioned into subbases S_v indexed by the set N of nomina (s. §1.3.1) and each subbasis has a certain family of basic relations.

[1.78] In fact, $\mathfrak{P}\mathsf{S}$ is replaced with its equivalent $\mathbb{T}^{\mathsf{S}}$, which is denoted $\mathsf{S}{\to}\mathbb{T}$ (s. §1.5.2).

[1.79] See §1.5.2 for why we use the superscript symbols $\circ, \Diamond, \square$ for complements and negations.

1.3.6 Its events are loosely linearized by case markers

In §1.2.4, I defined the event set F of each nooworld W with the truth values to consist of multiary relations on the entity set E of W, and I will modify the definition in Chapters 4–6 on CL. This is because a remarkable relationship between a sentence like (1.3.1) and a multiary relation on E is suggested by the following well-known theorem (proof omitted), where the symbol $\mathsf{X}{\to}\mathsf{Y}$ for sets X and Y denotes the set of all functions, or mappings of X into Y (s. §1.5.2).[1.80]

Theorem 1.3.1 Let S and T be sets and n be a nonnegative integer. Define $\mathsf{M} = \mathsf{S}^n{\to}\mathsf{T}$ and $\overrightarrow{\mathsf{M}} = \overbrace{\mathsf{S}{\to}(\mathsf{S}{\to}(\cdots{\to}(\mathsf{S}{\to}}^{n}\mathsf{T})\cdots))$. Then for each $\mathsf{f} \in \mathsf{M}$, there exists a unique element $\overrightarrow{\mathsf{f}} \in \overrightarrow{\mathsf{M}}$ which satisfies $\mathsf{f}(s_1,\dots,s_n) = (\cdots((\overrightarrow{\mathsf{f}}\,s_1)s_2)\cdots)s_n$ for each $(s_1,\dots,s_n) \in \mathsf{S}^n$, and the mapping $\mathsf{f} \mapsto \overrightarrow{\mathsf{f}}$ is a bijection of M onto $\overrightarrow{\mathsf{M}}$. Here if $n = 0$, then $\mathsf{S}^n = \{\emptyset\}$ (s. [3.23]), $\overrightarrow{\mathsf{M}} = \mathsf{T}$ and $(\cdots((\overrightarrow{\mathsf{f}}\,s_1)s_2)\cdots)s_n = \overrightarrow{\mathsf{f}}$ by definition (s. [1.32]).

Remark 1.3.2 (Linearization) More generally, let $\mathsf{S}_1,\dots,\mathsf{S}_n$ and T be sets ($n \geq 0$). Define $\mathsf{M} = (\mathsf{S}_1\times\cdots\times\mathsf{S}_n){\to}\mathsf{T}$ and $\overrightarrow{\mathsf{M}} = \mathsf{S}_1{\to}(\mathsf{S}_2{\to}(\cdots{\to}(\mathsf{S}_n{\to}\mathsf{T})\cdots))$. Then for each $\mathsf{f} \in \mathsf{M}$, there exists a unique element $\overrightarrow{\mathsf{f}} \in \overrightarrow{\mathsf{M}}$ which satisfies $\mathsf{f}(s_1,\dots,s_n) = (\cdots((\overrightarrow{\mathsf{f}}\,s_1)s_2)\cdots)s_n$ for each $(s_1,\dots,s_n) \in \mathsf{S}_1\times\cdots\times\mathsf{S}_n$, and the mapping $\mathsf{f} \mapsto \overrightarrow{\mathsf{f}}$ is a bijection of M onto $\overrightarrow{\mathsf{M}}$. See Theorem 1.3.1 for the case $n = 0$. I refer to $\overrightarrow{\mathsf{f}}$ there and here as the linearization of f.

This theorem first suggests decomposing each function $\mathsf{f} \in \mathsf{M}$ by each n-tuple $(s_1,\dots,s_n) \in \mathsf{S}^n$ into the following series of $n+1$ functions:

$$\overrightarrow{\mathsf{f}},\ \overrightarrow{\mathsf{f}}\,s_1,\ (\overrightarrow{\mathsf{f}}\,s_1)s_2,\ \dots,\ (\cdots((\overrightarrow{\mathsf{f}}\,s_1)s_2)\cdots)s_n.$$

Here $(\cdots((\overrightarrow{\mathsf{f}}\,s_1)s_2)\cdots)s_n$ may be regarded as belonging to $\mathsf{S}^0{\to}\mathsf{T}$ (s. [1.32]).

Notice that mathematical notation follows the Indo-European word order. In the Japanese word order, $\mathsf{f}(s_1,\dots,s_n)$ is denoted $(s_1,\dots,s_n)\mathsf{f}$,[1.81] Theorem 1.3.1 is stated on the relationship between

$$\mathsf{M} = \mathsf{T}{\leftarrow}\mathsf{S}^n \qquad \text{and} \qquad \overleftarrow{\mathsf{M}} = ((\cdots(\mathsf{T}\overbrace{{\leftarrow}\mathsf{S})\cdots){\leftarrow}\mathsf{S}){\leftarrow}\mathsf{S}}^{n}$$

and f is decomposed into the series

$$\overleftarrow{\mathsf{f}},\ s_n\overleftarrow{\mathsf{f}},\ s_{n-1}(s_n\overleftarrow{\mathsf{f}}),\ \dots,\ s_1(\cdots(s_{n-1}(s_n\overleftarrow{\mathsf{f}}))\cdots).$$

Notice also that the numerical subscripts $1,\dots,n-1,n$ are not intrinsic to the functions in M. For example, each ternary function f is decomposed by each

[1.80] I will not italicize but romanize propositions in this monograph (s. §1.5.1).

[1.81] The Japanese counterpart of the English expression *a of b* is *B no A* for the postposition *no* and counterparts *A* and *B* of *a* and *b* respectively (s. §1.3.4).

triple $(\mathtt{p}, \mathtt{h}, \mathtt{d})$ in its domain into the series $\overleftarrow{\mathtt{f}}$, $\mathtt{d}\overleftarrow{\mathtt{f}}$, $\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}})$, $\mathtt{p}(\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}}))$ of four functions without explicit numerical subscripts. Therefore, it seems appropriate to accompany $\mathtt{p}$, $\mathtt{h}$ and $\mathtt{d}$ with the numbers $\mathtt{1}$, $\mathtt{2}$ and $\mathtt{3}$ respectively in order to indicate their positions in the triple $(\mathtt{p}, \mathtt{h}, \mathtt{d})$ (it will turn out to be actually appropriate after (1.3.4)). Then the functions $\mathtt{d}\overleftarrow{\mathtt{f}}$, $\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}})$ and $\mathtt{p}(\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}}))$ are denoted also by $\mathtt{d3}\overleftarrow{\mathtt{f}}$, $\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}})$ and $\mathtt{p1}(\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}}))$ respectively. This suggests regarding the numbers $\mathtt{1}$, $\mathtt{2}$ and $\mathtt{3}$ as binary operation symbols.

Notice also that we need not use the numbers $\mathtt{1}$, $\mathtt{2}$ and $\mathtt{3}$. We may use any symbols instead. Moreover, we need not use the Roman letters $\mathtt{p}$, $\mathtt{h}$, $\mathtt{d}$ and $\overleftarrow{\mathtt{f}}$. We may use any symbols instead (s. [1.26]). Therefore, let us replace $\mathtt{1}$, $\mathtt{2}$ and $\mathtt{3}$ with the Japanese case markers *ga*, *de* and *o*, and replace $\mathtt{p}$, $\mathtt{h}$, $\mathtt{d}$ and $\overleftarrow{\mathtt{f}}$ with the Japanese words *pêtâ*, *hatake*, *hatuka daikon* and *taberu*. Then the functions $\overleftarrow{\mathtt{f}}$, $\mathtt{d3}\overleftarrow{\mathtt{f}}$, $\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}})$ and $\mathtt{p1}(\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}}))$ are denoted also by the four Japanese declaratives on Table 1.2 (s. Remark 1.3.3), the fourth of which is equal to the sentence (1.3.1) meaning *Peter eats radishes in a garden*.

Table 1.2: The relationship between declaratives and functions

declaratives (with English counterparts at the bottom)							functions
						taberu	$\overleftarrow{\mathtt{f}}$
				hatuka daikon	*o*	*taberu*	$\mathtt{d3}\overleftarrow{\mathtt{f}}$
		hatake	*de*	*hatuka daikon*	*o*	*taberu*	$\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}})$
pêtâ	*ga*	*hatake*	*de*	*hatuka daikon*	*o*	*taberu*	$\mathtt{p1}(\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}}))$
Peter		*garden*	*in*	*radish*		*eat*	

Thus the declaratives may be regarded as alternative expressions of the functions $\overleftarrow{\mathtt{f}}$, $\mathtt{d}\overleftarrow{\mathtt{f}}$, $\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}})$ and $\mathtt{p}(\mathtt{h}(\mathtt{d}\overleftarrow{\mathtt{f}}))$ respectively. In particular, the sentence (1.3.1) may be regarded as a linearized expression of the $\mathtt{0}$-ary function, or the value $(\mathtt{p}, \mathtt{h}, \mathtt{d})\mathtt{f}$. This finally suggests that all Japanese declaratives are linearized expressions of events in the nooworlds and that the case markers indicate the positions of the arguments of the events and should be regarded as binary operations in the syntactical algebraic structure abstracted from the DT. The Japanese people must have invented linearization not by a knowledge of Theorem 1.3.1 but out of necessity because utterances are lines of words.

Still, we need to half abandon the linearization given by Theorem 1.3.1. This is because the Japanese tongue enjoys loose phrase order by virtue of the explicit case markers, as was noted in §1.3.4. For example, the sentence (1.3.1) has the same external meaning as the following sentence among others:

$$\textit{hatake} \quad \textit{de} \quad \textit{pêtâ} \quad \textit{ga} \quad \textit{hatuka daikon} \quad \textit{o} \quad \textit{taberu}. \tag{1.3.4}$$

Therefore, we need to be able to decompose $\mathtt{f}$ into the series $\overleftarrow{\mathtt{f}}$, $\mathtt{d3}\overleftarrow{\mathtt{f}}$, $\mathtt{p1}(\mathtt{d3}\overleftarrow{\mathtt{f}})$, $\mathtt{h2}(\mathtt{p1}(\mathtt{d3}\overleftarrow{\mathtt{f}}))$ as well as into the series $\overleftarrow{\mathtt{f}}$, $\mathtt{d3}\overleftarrow{\mathtt{f}}$, $\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}})$, $\mathtt{p1}(\mathtt{h2}(\mathtt{d3}\overleftarrow{\mathtt{f}}))$. The rigid linearization of Theorem 1.3.1 does not work for such looseness, and Theorem 1.3.2 below supplies the right loose linearization (Corollary 4.2.6.1 serves as its proof). It is based on the following simple observation.

Let S and T be sets and Q be a finite set. Suppose $\mathsf{Q} = \{\mathsf{k}_1, \ldots, \mathsf{k}_n\}$. Then there is a bijection $i \mapsto \mathsf{k}_i$ $(i = 1, \ldots, n)$ of $\{1, \ldots, n\}$ onto Q. Therefore, there is a bijection of $\mathsf{Q}{\to}\mathsf{S}$ onto S^n which associates each $\theta \in \mathsf{Q}{\to}\mathsf{S}$ with the element $(\theta\mathsf{k}_1, \ldots, \theta\mathsf{k}_n) \in \mathsf{S}^n$. Moreover, there is a bijection of $\mathsf{S}^n{\to}\mathsf{T}$ onto $(\mathsf{Q}{\to}\mathsf{S}){\to}\mathsf{T}$ which associates each $f \in \mathsf{S}^n{\to}\mathsf{T}$ with the element $\vec{f} \in (\mathsf{Q}{\to}\mathsf{S}){\to}\mathsf{T}$ defined by $\vec{f}\theta = f(\theta\mathsf{k}_1, \ldots, \theta\mathsf{k}_n)$ for each $\theta \in \mathsf{Q}{\to}\mathsf{S}$. Thus $\mathsf{S}^n{\to}\mathsf{T}$ is abstracted from $(\mathsf{Q}{\to}\mathsf{S}){\to}\mathsf{T}$ by introducing a linear order on Q. Therefore, we will obtain the right loose linearization by canceling the abstraction.

Theorem 1.3.2 Let S, T and K be sets. For each finite subset $\mathsf{Q} = \{\mathsf{k}_1, \ldots, \mathsf{k}_n\}$ of K and each $f \in \mathsf{S}^n{\to}\mathsf{T}$, define $\vec{f} \in (\mathsf{Q}{\to}\mathsf{S}){\to}\mathsf{T}$ as above. For each triple (s, k, g) of $s \in \mathsf{S}$, $k \in \mathsf{K}$ and $g \in (\mathsf{R}{\to}\mathsf{S}){\to}\mathsf{T}$ such that $k \in \mathsf{R} \in \mathfrak{P}\mathsf{K}$, define an element $s\,\mathfrak{o}k\,g \in ((\mathsf{R} - \{k\}){\to}\mathsf{S}){\to}\mathsf{T}$ by the following for each $\theta \in (\mathsf{R} - \{k\}){\to}\mathsf{S}$:

$$(s\,\mathfrak{o}k\,g)\theta = g((k/s)\theta),$$

where $(k/s)\theta$ is the element of $\mathsf{R}{\to}\mathsf{S}$ defined by

$$((k/s)\theta)l = \begin{cases} \theta l & \text{if } l \in \mathsf{R} - \{k\}, \\ s & \text{if } l = k. \end{cases}$$

Then $f(s_1, \ldots, s_n) = s_{\rho 1}\,\mathfrak{o}k_{\rho 1}\,(\cdots(s_{\rho(n-1)}\,\mathfrak{o}k_{\rho(n-1)}\,(s_{\rho n}\,\mathfrak{o}k_{\rho n}\vec{f}))\cdots)$ for each $(s_1, \ldots, s_n) \in \mathsf{S}^n$ and each permutation ρ on $\{1, \ldots, n\}$ (s. [1.80]).

Based on the above observations, I redefine the event set F of each nooworld W with the truth values in ICL as $\coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T})$ for the set K of the cases and the basis S of W,[1.82] and so $W = \mathsf{E} \amalg \mathsf{F} = \mathsf{S} \amalg \mathfrak{P}\mathsf{S} \amalg \coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T})$ (s. [1.78] and §4.1.1). Furthermore, I equip W with a family $(\mathfrak{o}k)_{k\in\mathsf{K}}$ of binary operations defined as in Theorem 1.3.2. The definition of W in CL is a generalization of this and is too elaborate to be outlined.

Remark 1.3.3 (Declaratives) As for the Japanese tongue, the declaratives defined as the descriptions of events (s. §1.2.3) are certain modifications of core declaratives including those on Table 1.2.

The simplest declaratives are commonly classified into the following four categories according to their core declaratives: (1) verbal declaratives such as the declaratives on Table 1.2 whose core declarative is the verb *taberu* and the declarative *Maguregâ san ga tomu* meaning *Mr. McGregor is rich* or *Mr. McGregor becomes rich* whose core declarative is the (literary but often used) verb *tomu* meaning *be rich* or *become rich*; (2) adjective declaratives such as

[1.82] Identified with $(\emptyset{\to}\mathsf{S}){\to}\mathbb{T}$ (s. [3.2][1.32]), $\mathbb{T}$ is contained in F (s. [1.34]).

Maguregâ san ga mazusii meaning *Mr. McGregor is poor* whose core declarative is the adjective *mazusii* meaning *be poor*; (3) nominal adjective declaratives such as *Maguregâ san ga yûhukuda* meaning *Mr. McGregor is rich* whose core declarative is the nominal adjective *yûhukuda* which means *be rich* and is compounded of the noun[1.83] *yûhuku* of Chinese origin and the ending *da* for making adjectives;[1.84] (4) nominal declaratives such as *Maguregâ san ga kanemoti da* meaning *Mr. McGregor is a rich person* whose core declarative is the combination *kanemoti da* of the nominal *kanemoti* meaning *rich person* and a postpositive counterpart *da* of the English copula *be*.

In view of the examples in (1)–(4) and others, however, I classify the declaratives into two categories by putting (1)–(3) together, or putting verbs, adjectives and nominal adjectives together, and design CL according to the classification.

1.3.7 Its basic quantifiers are subsets of quantitative sets

Being faithful to the three ologies urged in §1.2.6, CL furthermore has the following features about quantification.

Semasiology for MN will reduce quantification to context-free quantification, especially intervallic definite quantification (s. §1.2.7), which should vary according to the nature of the quantified entities and human ability to quantify. For example, some entities are quantified by nonnegative integers, others are quantified by nonnegative real numbers and infants at a certain developmental stage cannot distinguish quantities greater than 10.

Thus I introduce the concept of *quantitative sets*[1.85] in §3.6.1. They are linearly ordered commutative monoids and so there are innumerable examples. To give a few of them, the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers and the set $\mathbb{R}_{\geq 0}$ of nonnegative real numbers are quantitative sets and, for each positive integer n, there exists a quantitative set $\mathbb{Z}_n$ of n elements.

I equip CL with a family $(\mathbb{P}_\nu)_{\nu \in \mathsf{N}}$ of quantitative sets indexed by the set N of nomina and divide the quantifiers of CL into the absolute ones and the proportional ones and into the positive ones and the negative ones. The absolute positive ones are the subsets $\mathfrak{p}$ of the quantitative sets and the absolute negative ones are their copies $\neg\mathfrak{p}$. For example, for $0 = \min \mathbb{P}_\nu$ ($\nu \in \mathsf{N}$), the intervals $(\leftarrow 0] = \{x \in \mathbb{P}_\nu : x \leq 0\} = \{0\}$ and $(0 \rightarrow) = \{x \in \mathbb{P}_\nu : 0 < x\}$ of $\mathbb{P}_\nu$ provide the absolute negative quantifier $\neg(\leftarrow 0]$ and the absolute positive quantifier $(0 \rightarrow)$, which are designed to be the universal quantifier $\forall_\nu$ and the existential quantifier $\exists_\nu$ respectively by virtue of the semantics of CL illustrated by *The Tale of the CL Quantifiers* in §1.3.8. Therefore, the quantifiers *all* and *some* are intervallic definite in light of CL as was noted in §1.2.7. The proportional quantifiers are the subsets $\mathfrak{r}$ of the interval $[0, 1]$ of $\mathbb{R}$ and their copies $\neg\mathfrak{r}$ and related to the nomina $\nu \in \mathsf{N}$ such that $\mathbb{P}_\nu$ are nontrivial unital quantitative sets.

[1.83] Every Japanese word of foreign origin works as a noun.

[1.84] Japanese adjectives and nominal adjectives do not require copulas even in predicative use.

[1.85] They were called quantity systems in certain earlier manuscripts (s. Remark 1.4.1).

1.3.8 Its semantics employs measures for quantification

The semantics for the quantifiers of CL is defined by means of an arbitrary family of positive definite, increasing and subadditive $\mathbb{P}_\nu$-valued measure $|X|_\nu$ for all subsets X of the subbasis S_ν ($\nu \in \mathrm{N}$) (s. §3.6.4) and by means of the fact that if $\mathbb{P}_\nu$ is nontrivial unital then each pair (a, b) of elements of $\mathbb{P}_\nu$ such that $a \leq b \neq 0$ has its ratio a/b in the interval $[0, 1]$ of $\mathbb{R}$ (s. §3.6.3). The definition for the absolute quantifiers may be best illustrated by the following tale[1.86].

THE TALE OF THE CL QUANTIFIERS
BY KENSAKU GOMI

ONCE upon a time, mathematicians had to deal with the values $\xi_f, \eta_f \in \mathbb{T} = \{0, 1\}$ defined for each set S and each mapping $f \in S {\to} \mathbb{T}$ by the following:

$$\xi_f = 1 \iff f(s) = 1 \text{ for all } s \in S,$$
$$\eta_f = 1 \iff f(s) = 1 \text{ for some } s \in S.$$

The designer of FPL denoted ξ_f and η_f by $\forall f$ and $\exists f$ respectively and obtained the quantifiers $\forall$ and $\exists$. Meanwhile the designer of CL first tried to redefine ξ_f and η_f by using an arbitrary quantitative set $\mathbb{P}$ and an arbitrary $\mathbb{P}$-valued positive definite measure $|X|$ for all $X \in \mathfrak{P}S$. It was clear that the following held:

$$\xi_f = 1 \iff |\{s : s \in S, f(s) = 0\}| \in (\leftarrow 0],$$
$$\eta_f = 1 \iff |\{s : s \in S, f(s) = 1\}| \in (0 \rightarrow).$$

He next generalized this by replacing the set S and the relation $\in$ with an arbitrary element $a \in S \amalg \mathfrak{P}S$ and the extended relation $\sqsubseteq$ defined in §1.3.5:

$$\xi_{f,a} = 1 \iff |\{s : s \sqsubseteq a, f(s) = 0\}| \in (\leftarrow 0],$$
$$\eta_{f,a} = 1 \iff |\{s : s \sqsubseteq a, f(s) = 1\}| \in (0 \rightarrow).$$

He next denoted $\xi_{f,a}$ and $\eta_{f,a}$ by $a\neg(\leftarrow 0]f$ and $a(0\rightarrow)f$ respectively. Then

$$a\neg(\leftarrow 0]f = 1 \iff |\{s : s \sqsubseteq a, f(s) = 0\}| \in (\leftarrow 0],$$
$$a(0\rightarrow)f = 1 \iff |\{s : s \sqsubseteq a, f(s) = 1\}| \in (0 \rightarrow).$$

He finally replaced $(\leftarrow 0]$ and $(0 \rightarrow)$ with an arbitrary set $\mathfrak{p} \in \mathfrak{P}\mathbb{P}$:

$$a\neg\mathfrak{p}f = 1 \iff |\{s : s \sqsubseteq a, f(s) = 0\}| \in \mathfrak{p},$$
$$a\mathfrak{p}f = 1 \iff |\{s : s \sqsubseteq a, f(s) = 1\}| \in \mathfrak{p}.$$

Thus he obtained the negative quantifier $\neg\mathfrak{p}$ and the positive quantifier $\mathfrak{p}$ for each $\mathfrak{p} \in \mathfrak{P}\mathbb{P}$. Furthermore, for each $f \in S^2 {\to} \mathbb{T}$, he defined the four elements $a\neg\mathfrak{p}1f$, $a\mathfrak{p}1f$, $a\neg\mathfrak{p}2f$ and $a\mathfrak{p}2f$ of $S {\to} \mathbb{T}$ by the following for each $r \in S$:

$$(a\neg\mathfrak{p}1f)r = 1 \iff |\{s : s \sqsubseteq a, f(s, r) = 0\}| \in \mathfrak{p},$$

[1.86] It is a tale because it may depart from historical evidence for convenience of explanation.

$$(\mathfrak{a}\mathfrak{p}1f)r = 1 \iff |\{s : s \sqsubseteq \mathfrak{a}, f(s,r) = 1\}| \in \mathfrak{p},$$

$$(\mathfrak{a}\neg\mathfrak{p}2f)r = 1 \iff |\{s : s \sqsubseteq \mathfrak{a}, f(r,s) = 0\}| \in \mathfrak{p},$$

$$(\mathfrak{a}\mathfrak{p}2f)r = 1 \iff |\{s : s \sqsubseteq \mathfrak{a}, f(r,s) = 1\}| \in \mathfrak{p}.$$

Continuing this way, he defined binary operations $\neg\mathfrak{p}k$ and $\mathfrak{p}k$ with $\mathfrak{p} \in \mathfrak{P}\mathbb{P}$ and $k \in \mathbb{N}$ on $S \amalg \mathfrak{P}S \amalg \coprod_{n=0}^{\infty}(S^n \to \mathbb{T})$, and he lived happily ever after. THE END

As was noted around Theorem 1.3.2, we should replace $S^n \to \mathbb{T}$ $(n = 0, 1, \ldots)$ with $(Q \to S) \to \mathbb{T}$ for all subsets Q of $\mathbb{N}$ and so replace the above $S \amalg \mathfrak{P}S \amalg \coprod_{n=0}^{\infty}(S^n \to \mathbb{T})$ with $S \amalg \mathfrak{P}S \amalg \coprod_{Q \in \mathfrak{P}\mathbb{N}}((Q \to S) \to \mathbb{T})$, which is a nooworld with the truth values, or a DW in ICL with $K = \mathbb{N}$ (s. [1.78]).

1.3.9 Its quantifiers are accompanied by case markers

As *The Tale of the CL Quantifiers* has suggested, the absolute quantifiers $\mathfrak{p}$ and $\neg\mathfrak{p}$ of CL together with certain case markers (or cases) k form binary quantifying operations $\mathfrak{p}k$ and $\neg\mathfrak{p}k$ both on the formal language and on the DWs, and likewise for the proportional quantifiers. This is in contrast to FPL, where the quantifiers $\forall$ and $\exists$ are mere tokens always accompanied by variables x, and only the formal language has quantifying operations $\forall x$ and $\exists x$.[1.87]

What seems important in light of phraseology for MN, the above syntactical feature of quantification in CL is consistent with the usage of quantifiers in the Japanese utterances. I have, however, noticed it thanks to the following extract from *The Tale of Peter Rabbit* by Beatrix Potter (1902):

First he ate some lettuces
and some French beans;
and then he ate some radishes.

The following is its Japanese translation by Momoko Isii (1971). As in §1.3.4, English counterparts are placed underneath each of the three lines.

sore kara	*mazu*	*retasu*	*o*	*nanmai ka*	*tabe,*
and then	*first*	*lettuce*		*some*	*ate,*

sore kara	*saya ingen*	*o*	*tabe,*
and then	*French bean*		*ate,*

sore kara	*hatuka daikon*	*o*	*nanbon ka*	*tabe masi ta.*
and then	*radish*		*some*	*ate.*

Isii was one of Japanese outstanding authors of children's books and was the 2002 nominee for the Hans Christian Andersen Awards from Japan, which shows among other things that her translations are polished orthodox Japanese.

[1.87] In CL also, a mere token $\triangledown$ is always accompanied by certain variables x, and only the formal language has nominalizing operations $\triangledown x$. In contrast to quantification, however, they are called the nominalizers and the token $\triangledown$ is given no name.

Notice that the quantifier *some* in the English original precedes the nouns *lettuces*, *French beans* and *radishes* and so seems to modify them, while its Japanese equivalents *nanmai ka* and *nanbon ka* do not adjoin any of the equivalents of *lettuces*, *French beans* and *radishes* but succeeds the case marker *o* and so do not seem to modify the equivalents. Moreover, *nanmai ka* and *nanbon ka* may be grammatically put also before *o*, although orthodox authors never do so, and they do not grammatically succeed but precede case markers other than *ga* and *o*. Therefore, neither *nanmai ka* nor *nanbon ka* seems to modify the conjugation *tabe* of the verb *taberu*.[1.88]

Thus the only reasonable conclusion seems to be that the Japanese quantifiers modify the case markers and not the nouns or any others.[1.89] In other words, the Japanese quantifiers are accompanied by the case markers. This conclusion agrees with *The Tale of the CL Quantifiers*, because the quantifiers there were accompanied by the numbers $1, 2, \dots$ which indicated the positions of the arguments of a function and the case markers *ga*, *de* and *o* on Table 1.2 indicated the positions of the arguments of functions. I suppose that the English quantifiers may also be considered to be accompanied by the (silent) case markers as is suggested by the following sentence (s. (1.3.3) and [1.76]):

$$(ga)\ \textit{one rabbit ate}\ (o)\ \textit{two radishes in three gardens.} \tag{1.3.5}$$

1.3.10 Its quantification can be precise or approximate

As *The Tale of the CL Quantifiers* has also suggested, CL can provide various kinds of quantification such as precise quantification and approximate quantification, because the absolute positive quantifiers $\mathfrak{p}$ are arbitrary subsets of a quantitative set $\mathbb{P}_\nu$ which is arbitrarily chosen and varies with the nomen $\nu \in \mathsf{N}$, and likewise for the proportional positive quantifiers.

The various kinds of quantification may be illustrated by the element $\mathsf{a}\mathfrak{p}\mathsf{f} \in \mathbb{T}$ defined in the tale. Suppose $\#\mathsf{S} < \infty$, $\mathbb{P} = \mathbb{Z}_{\geq 0}$ and $|\mathsf{Y}| = \#\mathsf{Y}$ for each $\mathsf{Y} \in \mathfrak{P}\mathsf{S}$. Pick an interval $[\mathsf{pq}] = \{\mathsf{x} \in \mathbb{P} : \mathsf{p} \leq \mathsf{x} \leq \mathsf{q}\}$ of $\mathbb{P}$ for $\mathfrak{p}$. Then $\mathsf{a}[\mathsf{pq}]\mathsf{f} = 1$ holds iff $\mathsf{p} \leq \#\{\mathsf{s} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{f}(\mathsf{s}) = 1\} \leq \mathsf{q}$. Therefore, $\mathsf{a}[\mathsf{pp}]\mathsf{f} = 1$ holds iff there exist precisely p elements s such that $\mathsf{s} \sqsubseteq \mathsf{a}$ and $\mathsf{f}(\mathsf{s}) = 1$. Thus the absolute positive quantifier $[\mathsf{pp}]$ provides precise quantification. In contrast if $\mathsf{p} < \mathsf{q}$, then the absolute positive quantifier $[\mathsf{pq}]$ provides approximate quantification, and $\mathsf{q} - \mathsf{p}$ may be called its approximation degree.

[1.88] The word *ikutu ka* is another equivalent of *some* and the expression *pêtâ ga hatake ikutu ka ni hairu* means *Peter goes into some gardens*, while the expression *pêtâ ga hatake ni ikutu ka hairu* sounds strange, and the word *ikutu ka* in the former expression does not seem to modify the verb *hairu*, because the case marker *ni* separates them.

[1.89] In the expressions like *nanbon ka no hatuka daikon*, the nominal *hatuka daikon* is considered to be qualified by the combination *nanbon ka no* of the quantifier *nanbon ka* and the postposition *no*. Such expressions are grammatical but foreign-sounding and are supposed to have come from mistranslation of European or Chinese utterances based on the misunderstanding that *some* modifies nouns, because orthodox Japanese novelists such as Naoya Siga seldom use such expressions, while Tatuo Hori who is said to be Europeanized often uses such expressions in his novel *Mugiwara Bôsi* (Straw Hats, 1933).

1.3.11 Its completeness theorem rests on the box principle

It is Mizumura who first attacked on proving completeness theorems on CL. The semantics of CL is parameterized by measures with regard to quantification (s. §1.3.8). Therefore, proof of any completeness theorem on CL requires construction of measures, which was challenging then because we knew no precedent about it. He worked it out by introducing an axiom meaning Dirichlet's box principle[1.90] and obtained a characteristic law, or presented a so-called sequential completeness theorem for an antecedent of ICL.[1.91] Later on, he generalized the theorem by one on an antecedent of CL.[1.92]

Meanwhile, following my advice, Takaoka took an alternative approach to Mizumura's earlier theorem and related new theorems.[1.93] He replaced resolution trees Mizumura used with what I call (Dedekind) cuts, although he also relied on Mizumura's construction of measure. His method is elegant in my opinion and better than Mizumura's in that it can link one of his theorems to an incompleteness theorem (Theorem 3.5.1). The sequential completeness theorem in Chapter 6 will be proved by his method, and it will be converted to a true completeness theorem by a principle given by Theorem 2.7.13.

It is not the sequential completeness but the true completeness that MN aims at, because sequents do not take part in any mathematical models for MN but are tools of analyzing the models (s. Remark 1.2.12).

1.4 A postscript for proceeding to mathematics

I have exhausted what I can tell without mathematics. As was noted in §1.1, mathematics in the modern sense is the totality of the study by deductive thinking based on the concept of sets and starting with definitions. The adjective *mathematical* means *having the character of the study*.

You may think that I have already used mathematics above, but it was quasi-mathematics especially in that some concepts were given no explicit definitions and so we cannot discuss them, much less obtain reliable results about them. Moreover, as was noted also in §1.1, mathematics provides one of the most effective tools and the most expressive and rigorous languages of science, which implies that some scientific truth can be found out only by mathematics and its non-mathematical expressions are more difficult and liable to be inaccurate than mathematical ones (compare §1.2.3 and §1.2.4). Thus I will proceed to genuine mathematics in the subsequent chapters.

Remark 1.4.1 (The right reunion in the right place) The subsequent chapters 2–6 reunite and amplify the following ealier manuscripts and others

[1.90] It is also called the drawer principle or pigeonhole principle (s. §3.6.4).

[1.91] *Ronri taikei MCL no kanzensei* (*The completeness of the logical system MCL*), Master's thesis, Graduate School Math. Sci., Univ. Tokyo, 2000 (s. [1.67]).

[1.92] *A completeness theorem for the logical system PCL*, manuscript, 2009 (s. [1.67]).

[1.93] *On existence of models for the logical system MPCL*, UTMS Preprint Series 2009, http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2009-24.pdf (s. [1.67]).

which were all extracted from or born of [1.3].

- *Theory of completeness for logical spaces.*
  https://gomikensaku.github.io/homepage/TCLS.pdf and [1.5]
- *From logical systems to logical spaces.*
  https://gomikensaku.github.io/homepage/FLSLS.pdf
- *Case logic for mathematical psychology I.*
  https://gomikensaku.github.io/homepage/CLMP.pdf
- *A completeness theorem for monophasic case logic.*
  https://gomikensaku.github.io/homepage/CTMPCL.pdf
- *Embedding first-order predicate logic in monophasic case logic.*
  https://gomikensaku.github.io/homepage/EFPLMPCL.pdf
- *Case logic for mathematical psychology II.*
  https://gomikensaku.github.io/homepage/S2CLMP.pdf (sample)
- *Completeness, models and classes in case logic.* This will be a generalization of Takaoka's paper [1.93] on ICL.

1.5 Mathematical convention

1.5.1 Organization

The subsequent chapters 2–6 are written in a style of mathematics for the most part. The mathematical part consists of definitions, propositions (i.e. theorems, corollaries and lemmas), proofs, remarks and examples, which are mostly sectioned off and numbered as such and sometimes nested. Some remarks and examples should be read as mixtures of propositions and their abridged proofs or as propositions whose proofs are left to you or given elsewhere. No exercise is given, but it will be a good exercise to try to figure out every detail.

Even a long mathematical paper may be devoted to a proof of a theorem. Chapters 2–6, however, are devoted to construction of a theory by means of numerous theorems and others. In general, the definitions, propositions, proofs, remarks and examples there are equally important for the theory. This is why I will not italicize but romanize all of them (mathematical symbols will be printed in AMS Euler shown in §0.4), particularly the propositions (s. [1.80]).

The mathematical part in Chapters 2–6 is almost self-contained except that it requires a rudimentary knowledge of sets and orders including a knowledge of lattices (esp. Boolean ones and complete ones) and ordinal numbers.

1.5.2 Notation and terminology on sets

Our set theory is neither naive nor axiomatic, but noological. A set S is a collection of perceived entities as explained in §1.2.3. If an entity a belongs to S, we say that a is an element of S and write $a \in S$.

Sets are entities. Entities other than sets will be called prime entities. For example, the symbol I is a prime entity, and so are all its sequences I ⋯ I. We call the sequences I, II, III, ... the natural numbers as the ancient romans probably did, and as usual denote them respectively by $1, 2, 3, \ldots$ in the decimal system. Using them, we can produce the integers, rational numbers and real numbers in a well-known way. They are all prime entities.

Every paradox is an incorrect thought, which we should correct rather than avoid. In particular, the Russell's paradox in naive set theory is an incorrect thought, which we should correct rather than avoid by the axiomatic set theory. From the noological point of view, sets are constructed by human beings who are not intellectually omnipotent, and not have been created by an omnipotent being, or God. Therefore, *every set is different from its elements, that is, if* A *is a set, then* $A \notin A$. This is a noological principle and may be compared to the fact that we cannot include any dish among its ingredients before producing it. Consequently, there does not exist the set of all sets, or in other words, we cannot perceive all of the sets. Thus, the Russell's paradox is an incorrect thought as stated above. Likewise, we cannot perceive all of the entities, which is one of the reasons why the noocosmos is divided into nooworlds.

We have denoted the cardinality of a set A by $\#A$ (s. [1.66]). The above noological principle, however, prevents us from defining the cardinality as in the naive set theory. Therefore, we regard $\#A$ as a duplicate of the letter A, regard the expression $\#A = \#B$ for sets A and B as showing that there is a bijection of A onto B, regard the expression $\#A \leq \#B$ as showing that there is an injection of A into B and regard the expression $\#A < \#B$ as showing that $\#A \leq \#B$ and $\#A \neq \#B$. Also, for the expressions

$$\#A = n, \qquad \#A \leq n, \qquad \#A < n, \qquad \#A \geq n, \qquad \#A > n$$

for a set A and a nonnegative integer n, we regard n as $\#\{1, \ldots, n\}$ or $\#\emptyset$ according as $n \neq 0$ or $n = 0$. Also, we regard the expression $\#A < \infty$ as "$\#A = n$ for some nonnegative integer n." With this convention, we will be able to push our noological set theory as the naive set theory.

My notation and terminology about sets will be standard except that I denote the set of the finite subsets of a set S by $\mathfrak{P}'S$ and that I denote the set of the mappings of a set X into a set Y by $X \to Y$ (I have already used it in §1.3.6) instead of Y^X. Therefore, $f \in X \to Y$ means $f : X \to Y$.[1.94] Some other notation and terminology may not be standard. Therefore, I will propose notation and terminology as well as a theory in this monograph.

If f is an injection of a set X into a set, I sometimes refer to its image $fX = \{fx : x \in X\}$ as a **copy**[1.95] of X by the symbol f.

Let X and Y be sets and Z be a subset of $X \to Y$. Then for each element $x \in X$, the mapping $z \mapsto zx$ of Z into Y is called the **projection** by x or x-projection and denoted x_p. Thus $x_p z = zx$ for each $(x, z) \in X \times Z$. In particular, identifying

[1.94] Thus I need not explain the set-theoretic meaning of the colon in the expression $f : X \to Y$.

[1.95] **(Boldface)** I use boldface for newly defined mathematical concepts.

Y^n with $\{1,\dots,n\}\to Y$ as usual ($n \geq 1$), we have that the i-projection i_p for each $i \in \{1,\dots,n\}$ satisfies $i_p(y_1,\dots,y_n) = y_i$ for each $(y_1,\dots,y_n) \in Y^n$.

In dealing with several lattices simultaneously, I wish to use different symbols for meets and joins in different lattices; for example, $\cap$ and $\cup$ for a lattice A, $\wedge$ and $\vee$ for a lattice B, $\sqcap$ and $\sqcup$ for a lattice C, and so on. Then, how about complements and **cojoin**s in Boolean lattices? Here *cojoin* is the term I coined as a set-theoretic and semantical counterpart of the logical and syntactical term *implication*.[1.96] The best way is to use symbols made of those for meets and joins. For example, if the above lattices A, B and C are Boolean, then use $\circ$ and $\Supset$ for A, $\lozenge$(lozenge) and $\Rrightarrow$ for B and $\square$(rectangle) and $\equiv$ for C. Therefore, I use the symbols $\circ$, $\lozenge$, $\square$, and so on for complements (and negations). Since they are superscript while modal operations $\Diamond$(diamond) and $\Box$(box) are prepositive and subsidiary in MN (s. §1.2.7 and §1.3.2), there is no fear of confusion. Fortunately, we only need $\Rrightarrow$ for cojoin (and implication). Thus $a \Rrightarrow b = a^{\lozenge} \vee b$ for each pair (a, b) of elements of a Boolean lattice of which $\wedge$, $\vee$ and $\lozenge$ are meet, join and complement respectively, hence the name of *cojoin*.

An ordered set is best described by the pair $(A, \leq)$ of a set A and an order $\leq$ on A. The ordered set $(A, \leq)$ is called a complete lattice if every subset X of A has both its infimum $\inf X$ and its supremum $\sup X$ in A with respect to $\leq$. Every complete lattice A has both $\max A = \inf \emptyset$ and $\min A = \sup \emptyset$.[1.97]

Orders are assumed partial, or not necessarily linear throughout this monograph. Thus I can treat subsets of the power set $\mathfrak{P}S$ of each set S as ordered sets with respect to the inclusion $\subseteq$. I denote the strict inclusion by $\subset$ because the strict order[1.98] for each order $\leq$ is usually denoted $<$. The ordered set $(\mathfrak{P}S, \subseteq)$ is a complete lattice and the infimum and supremum in it are also denoted $\bigcap$ and $\bigcup$ respectively, that is, if $\mathfrak{X}$ is a subset of $\mathfrak{P}S$, then $\inf \mathfrak{X} = \bigcap \mathfrak{X} = \bigcap_{X \in \mathfrak{X}} X$ and $\sup \mathfrak{X} = \bigcup \mathfrak{X} = \bigcup_{X \in \mathfrak{X}} X$. Notice $\bigcap \emptyset = S$ and $\bigcup \emptyset = \emptyset$ (s. [1.97]).

Let S and T be sets. Then the set of the relations between S and T may be identified with $\mathfrak{P}(S \times T)$. Thus we may consider the order $R \subseteq Q$ for relations R and Q between S and T, and if $R \subseteq Q$, we may say that R is contained in Q. Moreover, if $\mathfrak{X}$ is a set of relations between S and T, we may consider its infimum $\bigcap \mathfrak{X}$ and supremum $\bigcup \mathfrak{X}$. We may also consider whether there exists the largest or the smallest of the relations in $\mathfrak{X}$. In particular, there exists the largest of the relations between S and T, which may be identified with $S \times T$ and is called the **trivial relation**. This paragraph mainly concerns Chapter 2.[1.99]

[1.96] The terms *meet*, *join* and *complement* are set-theoretic and semantical counterparts of the logical and syntactical terms *conjunction*, *disjunction* and *negation* respectively.

[1.97] Let $(A, \leq)$ be an ordered set and X be a subset of A. Then by definition, $\inf X = \max X_*$ for $X_* = \{a \in A : a \leq x \text{ for all } x \in X\}$. If $X = \emptyset$, then $X_* = A$. Thus $\inf \emptyset = \max A$.

[1.98] A strict order is a transitive and asymmetric (and so irreflexive) relation $<$ and its reflexive closure $\leq$ (that is, $a \leq b$ iff $a < b$ or $a = b$) is an order. Conversely, the irreflexive core $<$ of an order $\leq$ (that is, $a < b$ iff $a \leq b$ and $a \neq b$) is a strict order. Furthermore, $\leq$ is linear (that is, every pair (a, b) of *distinct* elements satisfies $a \leq b$ or $b \leq a$) iff $<$ is similarly linear.

[1.99] See Theorems 2.2.11–2.2.13, 2.5.5, 2.5.8, 2.5.10, 2.6.1, 2.6.6–2.6.8, 2.7.9, 2.7.12–2.7.16 and 2.9.1, Corollaries 2.5.10.1 and 2.7.9.1, Lemmas 2.5.3, 2.5.4, 2.5.6 and 2.7.1, Remarks 2.2.10, 2.5.9 and 2.7.5 and Definition 2.5.2 for examples.

Chapter 2

Logic Spaces and Deduction Systems

A logic space[2.1] is a pair $(A, \mathfrak{B})$ of a nonempty set A and a subset $\mathfrak{B}$ of the power set $\mathfrak{P}A$ of A.[2.2] Since $\mathfrak{P}A$ is identified with $A \to \mathbb{T}$ for $\mathbb{T} = \{0, 1\}$, which is a lattice, a pair $(A, \mathcal{F})$ of a nonempty set A and a subset $\mathcal{F}$ of $A \to \mathbb{B}$ for a lattice $\mathbb{B}$ is also called a $\mathbb{B}$-valued functional logic space provided that $\mathbb{B}$ is bounded (that is, $\min \mathbb{B}$ and $\max \mathbb{B}$ exist) and is nontrivial, that is, $\#\mathbb{B} \geq 2$ or equivalently $\min \mathbb{B} \neq \max \mathbb{B}$. A deduction system on a set A is a pair (R, D) of an association R on A and a subset D of A, and an association on A is a relation between the free monoid A^* over A (s. Remark 3.1.14) and A.[2.3] In terms of these simplest concepts, this chapter presents an abstract theory of semantics and deduction. It is linked to the general theory of syntax and semantics in Chapter 3 and syntax, semantics and deduction are the pillars of logic (s. Remark 1.2.1). Thus Chapters 2 and 3 together present a theory of GL.

Logic spaces naturally occur in logic. To give an example, let A be the set of the formulas in *propositional logic* (PL). Then A is generated by the set X of the variables, and the mappings υ of X into the set $\mathbb{T}$ of the truth values are extended to mappings $f_\upsilon \in A \to \mathbb{T}$ in a certain process, and they form a $\mathbb{T}$-valued functional logic space $(A, \{f_\upsilon\})$. To give another example, let A be the set of the formulas in FPL. Then A is determined by the sets C and X of the constants and variables, and the pairs (δ, υ) of the mappings δ and υ of C and X into the sets E of entities yield mappings $f_{E,\delta,\upsilon} \in A \to \mathbb{T}$ in a certain process, and they form a $\mathbb{T}$-valued functional logic space $(A, \{f_{E,\delta,\upsilon}\})$. We are thus led to the concept of $\mathbb{T}$-valued functional logic spaces $(A, \mathcal{F})$, and then to that of logic spaces by identifying $A \to \mathbb{T}$ with $\mathfrak{P}A$. Furthermore, replacing $\mathbb{T}$ with nontrivial bounded lattices $\mathbb{B}$ in view of modal logic, intuitionistic logic, and so on, we are

[2.1] Logic spaces were called logical spaces in [1.5] which this chapter amplifies and in certain earlier manuscripts (s. Remark 1.4.1).

[2.2] Topological spaces may be regarded as degenerate logic spaces.

[2.3] Associations were called logics (s. [1.6]) in [1.5] and others (s. [2.1] and §2.6).

led to the concept of $\mathbb{B}$-valued functional logic spaces.[2.4] The specific processes in PL, FPL, and so on are unified and generalized in §3.3.3 so that any logic system with a truth type yields a logic space.

Deduction systems also naturally occur in logic. Let $(A, \mathcal{F})$ be one of the logic spaces in the above examples. Then people consider deduction rules on A such as $\dfrac{x \quad y}{x \wedge y}$ and $\dfrac{x \quad x \Rightarrow y}{y}$ together with formulas such as $(x \wedge y) \Rightarrow x$ and $x \Rightarrow (x \vee y)$. The union of the deduction rules may be regarded as an association R on A, and the formulas constitute a subset D of A. We are thus led to the concept of deduction systems (R, D).

2.1 Finitarities and closure operators

The theory of logic spaces and deduction systems in this chapter is based on finitary relations, and so certain kinds of finitarities necessarily play important roles in it. Moreover, a Galois connection underlies the basic concepts on logic spaces, and every Galois connection is accompanied by closure operators and vice versa.[2.5] Furthermore, the concept of deduction is related to closure operators, as is shown by Theorem 2.4.4 and Remark 2.4.3 (s. Theorem 3.1.4). Thus closure operators also necessarily play important roles in this chapter.

In particular, the quasi-finitary closure (Theorem 2.1.6) is a key concept and leads to Theorem 2.6.10, which shows how all the $\mathfrak{B}$-theories (§2.6) of a logic space $(A, \mathfrak{B})$ are generated by $\mathfrak{B}$ and in turn leads to a classification of the logic spaces into three classes (Definition 2.6.3). The classification is pertinent to the $\mathfrak{B}$-completeness (Definition 2.7.1) of deduction systems on A and also to the existence of nontrivial $\mathfrak{B}$-models (Definition 2.8.2) of $\mathfrak{B}$-consistent, esp. $\mathfrak{B}$-complete (Definition 2.8.1) subsets of A.

2.1.1 Covers and finitarities

Definition 2.1.1 Let $(A, \mathfrak{B})$ be a logic space and X be a subset of A. Then X is said to be **extra-covered** by $\mathfrak{B}$, if for each $Y \in \mathfrak{P}'X$ there exists a set $B \in \mathfrak{B}$ such that $Y \subseteq B$. Moreover, X is said to be **super-covered** by $\mathfrak{B}$, if for each $Y \in \mathfrak{P}'X$ there exists a set $B \in \mathfrak{B}$ such that $Y \subseteq B \subseteq X$. Furthermore, X is said to be **ultra-covered** by $\mathfrak{B}$, if $\mathfrak{P}'X \subseteq \mathfrak{B}$.

Remark 2.1.1 Definition 2.1.1 has the following obvious consequences. If X is ultra-covered by $\mathfrak{B}$, then so is every subset of X, and X is super-covered by $\mathfrak{B}$. If X is super-covered by $\mathfrak{B}$, then X is extra-covered by $\mathfrak{B}$ and also **exactly covered** by $\mathfrak{B}$ in the sense that $X = \bigcup\{B \in \mathfrak{B} : B \subseteq X\}$. If X is extra-covered by $\mathfrak{B}$, then so is every subset of X, and X is **covered** by $\mathfrak{B}$ in the usual sense

[2.4] Remark 2.2.2 gives another reason for replacing $\mathbb{T}$ with other lattices $\mathbb{B}$.

[2.5] Let $(A_i, \leq_i)$ be ordered sets $(i = 0, 1)$ and $f_i \in A_i \to A_{1-i}$ be a decreasing mapping such that $f_{1-i}f_i$ is increasing (s. [2.10]). Then the pair (f_0, f_1) is called a Galois connection, and $f_{1-i}f_i$ is a closure operator on A_i. Conversely, every closure operator is so constructed.

that $\mathsf{X} \subseteq \bigcup \mathfrak{B}$. If $\mathfrak{B}$ is **downward**[2.6] in the sense that $\mathfrak{P}\mathsf{B} \subseteq \mathfrak{B}$ for all $\mathsf{B} \in \mathfrak{B}$, then the concepts of extra-cover, super-cover and ultra-cover by $\mathfrak{B}$ are identical. Every set in $\mathfrak{B}$ is super-covered by $\mathfrak{B}$.

Definition 2.1.2 Let $(\mathsf{A}, \mathfrak{B})$ be a logic space. Then $\mathfrak{B}$ is said to be **finitary**, if every subset of A which is extra-covered by $\mathfrak{B}$ belongs to $\mathfrak{B}$. Moreover, $\mathfrak{B}$ is said to be **quasi-finitary**, if every subset of A which is super-covered by $\mathfrak{B}$ belongs to $\mathfrak{B}$. Furthermore, $\mathfrak{B}$ is said to be **semi-finitary**, if every subset of A which is ultra-covered by $\mathfrak{B}$ belongs to $\mathfrak{B}$.

Remark 2.1.2 It follows from Definition 2.1.2 and Remark 2.1.1 that if $\mathfrak{B}$ is finitary then $\mathfrak{B}$ is quasi-finitary and that if $\mathfrak{B}$ is quasi-finitary then $\mathfrak{B}$ is semi-finitary. Furthermore, the following theorem holds, which particularly shows that the above definition of the finitarity is equivalent to the usual one.

Theorem 2.1.1 Let $(\mathsf{A}, \mathfrak{B})$ be a logic space. Then the following four conditions are equivalent (the usual definition of the finitarity probably uses (4)).

(1) $\mathfrak{B}$ is finitary.

(2) $\mathfrak{B}$ is downward and quasi-finitary.

(3) $\mathfrak{B}$ is downward and semi-finitary.

(4) A subset X of A belongs to $\mathfrak{B}$ iff X is ultra-covered by $\mathfrak{B}$.

Proof This is derived from Remark 2.1.1 in the following way. Since every set in $\mathfrak{B}$ is super-covered and so extra-covered by $\mathfrak{B}$, (1) is equivalent to the following condition.

(5) A subset X of A belongs to $\mathfrak{B}$ iff X is extra-covered by $\mathfrak{B}$.

Furthermore, (4) and (5) imply that $\mathfrak{B}$ is downward, and so also does (1). If $\mathfrak{B}$ is downward, then the concepts of extra-cover, super-cover and ultra-cover by $\mathfrak{B}$ are identical. Thus (1)–(5) are equivalent.

Theorem 2.1.2 Let $(\mathsf{A}, \mathfrak{B})$ be a logic space. Assume that $\mathfrak{B}$ is quasi-finitary. Then $\mathfrak{B}$ is inductive[2.7] with respect to the inclusion $\subseteq$.

Proof Let $\mathfrak{X}$ be a nonempty linearly ordered subset of $\mathfrak{B}$. Define $\mathsf{X} = \bigcup \mathfrak{X}$. Then $\mathsf{X} = \sup_{\mathfrak{P}\mathsf{A}} \mathfrak{X}$. Let $\mathsf{Y} \in \mathfrak{P}'\mathsf{X}$. Then for each $\mathsf{y} \in \mathsf{Y}$, there exists a set $\mathsf{X}_\mathsf{y} \in \mathfrak{X}$ such that $\mathsf{y} \in \mathsf{X}_\mathsf{y}$. Since Y is finite and $\mathfrak{X}$ is linearly ordered, $\mathsf{B} = \max\{\mathsf{X}_\mathsf{y} : \mathsf{y} \in \mathsf{Y}\}$ exists and satisfies $\mathsf{B} \in \mathfrak{B}$ and $\mathsf{Y} \subseteq \mathsf{B} \subseteq \mathsf{X}$. Therefore X is super-covered by $\mathfrak{B}$, hence $\mathsf{X} \in \mathfrak{B}$, and hence $\mathsf{X} = \sup_{\mathfrak{B}} \mathfrak{X}$. Thus $\mathfrak{B}$ is inductive.

[2.6] The concept is generalized by that for the ordered sets in Theorem 2.1.8

[2.7] An ordered set A is said to be inductive or inductively ordered, if every nonempty linearly ordered subset of A has its supremum in A. The so-called Zorn's lemma asserts that every nonempty inductively ordered set has a maximal element.

Remark 2.1.3 Let $(A, \leq)$ be an ordered set and X be a set. Then the order $\leq$ on A yields the **power order** $\leq$ on $X{\to}A$ defined by

$$f \leq g \iff fx \leq gx \text{ for all } x \in X$$

for each $(f, g) \in (X{\to}A)^2$. Let $\mathcal{F}$ be a subset of $X{\to}A$. Then if the subset $\mathcal{F}x = \{fx : f \in \mathcal{F}\}$ of A has its infimum for each $x \in X$, then $\inf \mathcal{F}$ exists and is characterized by the equations $(\inf \mathcal{F})x = \inf(\mathcal{F}x)$ for all $x \in X$, and likewise for $\sup \mathcal{F}$. This holds even if $\mathcal{F} = \emptyset$, that is, if $\max A$ exists, then $\max(X{\to}A)$ exists and is characterized by the equations $(\max(X{\to}A))x = \max A$ for all $x \in X$, and likewise for $\min(X{\to}A)$. Thus, if $(A, \leq)$ is a lattice, then so is $(X{\to}A, \leq)$ and the projection $f \mapsto fx$ by each $x \in X$ (s. §1.5.2) is a homomorphism of $X{\to}A$ into A with respect to the meet and join on the lattices.[2.8] The same applies to Boolean lattices and the complement and cojoin (s. §1.5.2) on them and to complete lattices and the infimum and supremum on them.[2.9]

In particular, if S is a set, then $X{\to}\mathfrak{P}S$ is a complete lattice with respect to the power order $\subseteq$ defined by

$$f \subseteq g \iff fx \subseteq gx \text{ for all } x \in X$$

for each $(f, g) \in (X{\to}\mathfrak{P}S)^2$. Let $\mathcal{F}$ be a subset of $X{\to}\mathfrak{P}S$ and let $\bigcap \mathcal{F}$ and $\bigcup \mathcal{F}$ denote $\inf \mathcal{F}$ and $\sup \mathcal{F}$. Then the following hold for all $x \in X$:

$$(\bigcap \mathcal{F})x = \bigcap(\mathcal{F}x) = \bigcap_{f \in \mathcal{F}} fx, \qquad (\bigcup \mathcal{F})x = \bigcup(\mathcal{F}x) = \bigcup_{f \in \mathcal{F}} fx.$$

Definition 2.1.3 Let A and A' be sets and $\varphi \in \mathfrak{P}A{\to}\mathfrak{P}A'$. Then φ is said to be **finitary**, if $\varphi X = \bigcup_{Y \in \mathfrak{P}'X} \varphi Y$ for all $X \in \mathfrak{P}A$ (s. (2.11.1)).

This definition relates to Definition 2.1.2. Indeed, it follows from Theorem 2.1.1 that φ is finitary iff $\{X \in \mathfrak{P}A : y' \notin \varphi X\}$ is finitary for all $y' \in A'$.

Theorem 2.1.3 Let A and A' be sets. Then the following hold, where (2) and (4) deal with the complete power order $\subseteq$ on $\mathfrak{P}A{\to}\mathfrak{P}A'$ defined in Remark 2.1.3.

(1) Let $\varphi \in \mathfrak{P}A{\to}\mathfrak{P}A'$. Then φ is finitary iff it is increasing[2.10] and $\varphi X \subseteq \bigcup_{Y \in \mathfrak{P}'X} \varphi Y$ for all $X \in \mathfrak{P}A$.

[2.8] As for the meet $\wedge$, $(f \wedge g)x = (\inf\{f, g\})x = \inf\{fx, gx\} = fx \wedge gx$ for each $(f, g) \in (X{\to}A)^2$ and each $x \in X$, and likewise for the join $\vee$.

[2.9] Suppose A is a Boolean lattice and let $\lozenge$ and $\Rrightarrow$ be its complement (s. [1.79]) and cojoin. Let $f, g, h \in X{\to}A$. Then $((f \vee g) \wedge h)x = ((fx \vee gx) \wedge hx) = (fx \wedge hx) \vee (gx \wedge hx) = ((f \wedge h) \vee (g \wedge h))x$ for all $x \in X$ by [2.8]. Therefore, $(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h)$ (s. [2.17]). Let $f \in X{\to}A$ and define $f^{\lozenge} \in X{\to}A$ by $f^{\lozenge}x = (fx)^{\lozenge}$ for all $x \in X$. Then $(f \wedge f^{\lozenge})x = fx \wedge (fx)^{\lozenge} = \min A = (\min(X{\to}A))x$ for all $x \in X$, hence $f \wedge f^{\lozenge} = \min(X{\to}A)$, and likewise $f \vee f^{\lozenge} = \max(X{\to}A)$. Let $f, g \in X{\to}A$. Then $(f \Rrightarrow g)x = (f^{\lozenge} \vee g)x = (fx)^{\lozenge} \vee gx = fx \Rrightarrow gx$ for each $x \in X$. Thus $X{\to}A$ is also a Boolean lattice and the x-projection is also a homomorphism with respect to the complement and cojoin for each $x \in X$.

[2.10] Let $(A, \leq)$ and $(A', \leq)$ be ordered sets and $\varphi \in A{\to}A'$. Then φ is said to be **increasing** if $x \leq y$ implies $\varphi x \leq \varphi y$, while φ is said to be **decreasing** if $x \leq y$ implies $\varphi x \geq \varphi y$.

(2) Assume that $\varphi \in \mathfrak{P}A \to \mathfrak{P}A'$ is finitary and $\psi \in \mathfrak{P}A \to \mathfrak{P}A'$ is increasing. Then the subset $\{B \in \mathfrak{P}A : \varphi B \subseteq \psi B\}$ of $\mathfrak{P}A$ is quasi-finitary (in particular, if $\varphi \in \mathfrak{P}A \to \mathfrak{P}A$ is finitary, then $\{B \in \mathfrak{P}A : \varphi B \subseteq B\}$ is quasi-finitary). Moreover, $\varphi \subseteq \psi$ if (and only if) $\varphi Y \subseteq \psi Y$ for all $Y \in \mathfrak{P}'A$.

(3) If $\varphi \in \mathfrak{P}A \to \mathfrak{P}A'$ and $\psi \in \mathfrak{P}A' \to \mathfrak{P}A''$ are finitary for a set A'', then so is their composite $\psi \cdot \varphi \in \mathfrak{P}A \to \mathfrak{P}A''$.

(4) If Φ is a subset of $\mathfrak{P}A \to \mathfrak{P}A'$ and each mapping in Φ is finitary, then so is its supremum $\bigcup \Phi$ in $\mathfrak{P}A \to \mathfrak{P}A'$.

(5) If D is a subset of A and $\varphi \in \mathfrak{P}A \to \mathfrak{P}A'$ is finitary, then so is the mapping $X \mapsto \varphi(X \cup D)$ of $\mathfrak{P}A$ into $\mathfrak{P}A'$.

Proof (1) If subsets X and Y of A satisfy $X \subseteq Y$, then $\mathfrak{P}'X \subseteq \mathfrak{P}'Y$. Therefore, if φ is finitary, then it is increasing. Conversely if φ is increasing, then $\varphi X \supseteq \bigcup_{Y \in \mathfrak{P}'X} \varphi Y$ for all $X \in \mathfrak{P}A$. Thus (1) holds.

(2) Suppose $X \in \mathfrak{P}A$ is super-covered by $\mathfrak{B} = \{B \in \mathfrak{P}A : \varphi B \subseteq \psi B\}$. Then for each $Y \in \mathfrak{P}'X$, there exists a set $B \in \mathfrak{B}$ such that $Y \subseteq B \subseteq X$, and so $\varphi Y \subseteq \varphi B \subseteq \psi B \subseteq \psi X$ by (1) and our assumption. Therefore $\varphi X = \bigcup_{Y \in \mathfrak{P}'X} \varphi Y \subseteq \psi X$, and so $X \in \mathfrak{B}$. Thus $\mathfrak{B}$ is quasi-finitary, and in particular the statement in parentheses holds because $\mathrm{id}_{\mathfrak{P}A}$ is increasing.[2.11] Assume that $\varphi Y \subseteq \psi Y$ for all $Y \in \mathfrak{P}'A$. Then every subset of A is ultra-covered by $\mathfrak{B}$ and therefore super-covered by $\mathfrak{B}$, as was noted in Remark 2.1.1. Thus $\mathfrak{P}A \subseteq \mathfrak{B}$, which means $\varphi \subseteq \psi$.

(3) Let $X \in \mathfrak{P}A$. Furthermore let $Z \in \mathfrak{P}'(\varphi X)$. Then $Z \subseteq \varphi X = \bigcup_{Y \in \mathfrak{P}'X} \varphi Y$, and so for each $z \in Z$, there exists a set $Y_z \in \mathfrak{P}'X$ such that $z \in \varphi Y_z$. Define $Y_Z = \bigcup_{z \in Z} Y_z$. Then $Y_Z \in \mathfrak{P}'X$ and $Z \subseteq \bigcup_{z \in Z} \varphi Y_z \subseteq \varphi Y_Z$ by (1), and so $\psi Z \subseteq \psi(\varphi Y_Z)$ also by (1). Therefore $\psi(\varphi X) = \bigcup_{Z \in \mathfrak{P}'(\varphi X)} \psi Z \subseteq \bigcup_{Z \in \mathfrak{P}'(\varphi X)} \psi(\varphi Y_Z) \subseteq \bigcup_{Y \in \mathfrak{P}'X} \psi(\varphi Y)$. Moreover, since φ and ψ are increasing by (1), so is $\psi \cdot \varphi$. Therefore $\psi \cdot \varphi$ is finitary by (1).

(4) Let $X \in \mathfrak{P}A$. Then $(\bigcup \Phi)X = \bigcup_{\varphi \in \Phi} \varphi X = \bigcup_{\varphi \in \Phi} (\bigcup_{Y \in \mathfrak{P}'X} \varphi Y) = \bigcup_{Y \in \mathfrak{P}'X} (\bigcup_{\varphi \in \Phi} \varphi Y) = \bigcup_{Y \in \mathfrak{P}'X} ((\bigcup \Phi) Y)$ by Remark 2.1.3. Thus (4) holds.

(5) Let δ be the constant mapping $X \mapsto D$ on $\mathfrak{P}A$. Then the mapping $X \mapsto \varphi(X \cup D)$ is equal to $\varphi \cdot (\mathrm{id}_{\mathfrak{P}A} \cup \delta)$, which is finitary by (4) and (3) because $\mathrm{id}_{\mathfrak{P}A}$ and δ are finitary.

2.1.2 Closure operators

A **closure operator** on an ordered set $(A, \leq)$ is a mapping $\varphi \in A \to A$ which satisfies the following three conditions.

- $x \leq \varphi x$ for all $x \in A$.
- $\varphi(\varphi x) \leq \varphi x$ for all $x \in A$.

[2.11] For each set S, id_S denotes the identity transformation on S.

- φ is increasing (s. [2.10]).

Then $\varphi(\varphi x) = \varphi x$ for all $x \in A$, and the image $\varphi A = \{\varphi x : x \in A\}$ of φ satisfies $\varphi A = \{x \in A : \varphi x \leq x\} = \{x \in A : \varphi x = x\}$. Therefore, φA is also called the **closure domain** or **fixture domain** of φ (s. §2.11).

Remark 2.1.4 Let $(A, \leq)$ be an ordered set and $x \in A$. Then $[x\rightarrow)$ and $(\leftarrow x]$ denote the **upward closed interval** $\{y \in A : x \leq y\}$ and the **downward closed interval** $\{y \in A : y \leq x\}$ respectively. We need not assume the order $\leq$ here to be linear.

The following theorem is fundamental and probably well-known, and so I shall often use it without notice.

Theorem 2.1.4 Let $(A, \leq)$ be an ordered set. Then the following hold.

(1) If φ is a closure operator on A, then $\varphi x = \min(\varphi A \cap [x\rightarrow))$ for all $x \in A$.

(2) If B is a subset of A such that there exists $\min(B \cap [x\rightarrow))$ for each $x \in A$, then the mapping $x \mapsto \min(B \cap [x\rightarrow))$ is a closure operator on A of which B is the fixture domain.

(3) Two closure operators φ and ψ on A satisfy $\varphi \leq \psi$ (with respect to the power order $\leq$ on $A\rightarrow A$ defined in Remark 2.1.3) iff $\varphi A \supseteq \psi A$.

Proof (1) Let $x \in A$. Then $x \leq \varphi x \in \varphi A$. Moreover, if $x \leq y \in \varphi A$, then $\varphi x \leq \varphi y = y$. Thus $\varphi x = \min(\varphi A \cap [x\rightarrow))$.

(2) Define $B_x = B \cap [x\rightarrow)$ and $\varphi x = \min B_x$ for each $x \in A$. Then $\varphi x \in B_x$, and so $x \leq \varphi x$. Moreover, since $\varphi x \in B_x \subseteq B$, we have $\varphi x \in B_{\varphi x}$, and so $\varphi(\varphi x) = \min B_{\varphi x} \leq \varphi x$. Furthermore, if elements x and y of A satisfy $x \leq y$, then $B_x \supseteq B_y$, and so $\varphi x \leq \varphi y$. Thus φ is a closure operator. If $\varphi x = x$, then $x = \varphi x \in B$. Conversely if $x \in B$, then $x \in B_x$ and so $\varphi x \leq x$. Thus $B = \varphi A$.

(3) If $\varphi A \supseteq \psi A$, then $\varphi x = \min(\varphi A \cap [x\rightarrow)) \leq \min(\psi A \cap [x\rightarrow)) = \psi x$ for all $x \in A$ by (1), and so $\varphi \leq \psi$. Conversely if $\varphi \leq \psi$, then $\varphi x \leq \psi x$ for all $x \in A$, and so $\varphi A = \{x \in A : \varphi x \leq x\} \supseteq \{x \in A : \psi x \leq x\} = \psi A$.

Lemma 2.1.1 Let $(A, \leq)$ be a complete lattice, $(X_i)_{i\in I}$ be a family of subsets of A, and X be a subset of A. Then $\inf\{\inf X_i : i \in I\} = \inf(\bigcup_{i\in I} X_i)$ and $\inf(X - \{\max A\}) = \inf X$.

Proof Let $y = \inf\{\inf X_i : i \in I\}$ and $z = \inf(\bigcup_{i\in I} X_i)$. Then $y \leq x$ for all $i \in I$ and all $x \in X_i$, and so $y \leq x$ for all $x \in \bigcup_{i\in I} X_i$, hence $y \leq z$. Moreover, since $\bigcup_{i\in I} X_i \supseteq X_j$ for all $j \in I$, we have $z \leq \inf X_j$ for all $j \in I$, hence $z \leq y$. Thus $y = z$. We may derive the latter equation from the former by substituting $\inf(X - \{\max A\})$ for x in the obvious equation $x = \inf\{x, \inf\{\max A\}\}$.

Theorem 2.1.5 Let $(A, \leq)$ be a complete lattice and, for each subset B of A, let $B^{\cap}$ denote the subset $\{\inf X : X \subseteq B\}$ of A. Let us say that B is **$\cap$-closed** in A if $B^{\cap} \subseteq B$. Then the mapping $B \mapsto B^{\cap}$ is a closure operator on $\mathfrak{P}A$, and its fixture domain is equal to the set of the $\cap$-closed subsets of A. Therefore, $B^{\cap}$ is the smallest of the $\cap$-closed subsets of A which contain B (and so we call $B^{\cap}$ the **$\cap$-closure** of B in A). Furthermore the following hold.

(1) $\inf(B \cap [y\rightarrow)) = \inf(B^{\cap} \cap [y\rightarrow))$ for all $y \in A$.

(2) $\sup(B - \{\max A\}) = \sup(B^{\cap} - \{\max A\})$ (note that $\max A = \inf \emptyset \in B^{\cap}$).

In particular, if $(A, \mathfrak{B})$ is a logic space, then the $\cap$-closure $\mathfrak{B}^{\cap}$ of $\mathfrak{B}$ in $\mathfrak{P}A$ is defined by $\mathfrak{B}^{\cap} = \{\bigcap \mathfrak{X} : \mathfrak{X} \subseteq \mathfrak{B}\}$ and satisfies the following.

- $\bigcap_{Y \subseteq B \in \mathfrak{B}} B = \bigcap_{Y \subseteq X \in \mathfrak{B}^{\cap}} X$ for all $Y \in \mathfrak{P}A$.
- $\bigcup(\mathfrak{B} - \{A\}) = \bigcup(\mathfrak{B}^{\cap} - \{A\})$ (note that $A = \max \mathfrak{P}A = \bigcap \emptyset \in \mathfrak{B}^{\cap}$).

Proof Let B be a subset of A. Then since $x = \inf\{x\} \in B^{\cap}$ for all $x \in B$, we have $B \subseteq B^{\cap}$. Moreover, if $X \subseteq B^{\cap}$, then for each $x \in X$ there exists a subset Y_x of B such that $x = \inf Y_x$, and so $\inf X = \inf\{\inf Y_x : x \in X\} = \inf(\bigcup_{x \in X} Y_x) \in B^{\cap}$ by Lemma 2.1.1. Therefore $(B^{\cap})^{\cap} \subseteq B^{\cap}$. Furthermore, if $B \subseteq B' \subseteq A$, then $\mathfrak{P}B \subseteq \mathfrak{P}B'$, and so $B^{\cap} \subseteq B'^{\cap}$. Thus the former half of the theorem holds.

(1) If $y \leq x \in B^{\cap}$, then $x = \inf X$ for a subset X of B, and it satisfies $X \subseteq B \cap [y\rightarrow)$, and so $\inf(B \cap [y\rightarrow)) \leq x$. Therefore $\inf(B \cap [y\rightarrow)) \leq \inf(B^{\cap} \cap [y\rightarrow))$. Since $B \subseteq B^{\cap}$, we conversely have $\inf(B \cap [y\rightarrow)) \geq \inf(B^{\cap} \cap [y\rightarrow))$.

(2) Let $x \in B^{\cap} - \{\max A\}$. Then $x = \inf X$ for a subset X of $B - \{\max A\}$ by Lemma 2.1.1, and $X \neq \emptyset$ because $\inf X = x \neq \max A = \inf \emptyset$. Therefore $\sup(B - \{\max A\}) \geq \sup X \geq \inf X = x$ (notice here that $\sup X \geq \inf X$ holds because $X \neq \emptyset$). Therefore $\sup(B - \{\max A\}) \geq \sup(B^{\cap} - \{\max A\})$. Since $B \subseteq B^{\cap}$, we conversely have $\sup(B - \{\max A\}) \leq \sup(B^{\cap} - \{\max A\})$.

Theorem 2.1.6 Let A be a set and, for each subset $\mathfrak{B}$ of $\mathfrak{P}A$, let $\overline{\mathfrak{B}}$ denote the set of the subsets of A which are super-covered by $\mathfrak{B}$. Then the mappping $\mathfrak{B} \mapsto \overline{\mathfrak{B}}$ is a closure operator on $\mathfrak{P}(\mathfrak{P}A)$, and its fixture domain is equal to the set of the quasi-finitary subsets of $\mathfrak{P}A$. Therefore, $\overline{\mathfrak{B}}$ is the smallest of the quasi-finitary subsets of $\mathfrak{P}A$ which contain $\mathfrak{B}$ (and so we call $\overline{\mathfrak{B}}$ the **quasi-finitary closure** of $\mathfrak{B}$ in $\mathfrak{P}A$). Furthermore the following hold.

(1) $\bigcap_{Y \subseteq B \in \mathfrak{B}} B = \bigcap_{Y \subseteq X \in \overline{\mathfrak{B}}} X$ for all $Y \in \mathfrak{P}'A$.

(2) $\bigcup(\mathfrak{B} - \{A\}) = \bigcup(\overline{\mathfrak{B}} - \{A\})$.

(3) $\mathfrak{P}'A \cap \mathfrak{B} = \mathfrak{P}'A \cap \overline{\mathfrak{B}}$.

Proof Let $\mathfrak{B}$ be a subset of $\mathfrak{P}A$. Then $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$ by Remark 2.1.1. Moreover, if $X \in \overline{\overline{\mathfrak{B}}}$, then for each $Y \in \mathfrak{P}'X$, there exists a set $B' \in \overline{\mathfrak{B}}$ such that $Y \subseteq B' \subseteq X$, and so there exists a set $B \in \mathfrak{B}$ such that $Y \subseteq B \subseteq B' \subseteq X$, and so

$X \in \overline{\mathfrak{B}}$. Therefore $\overline{\overline{\mathfrak{B}}} \subseteq \overline{\mathfrak{B}}$. Moreover, it follows from Definition 2.1.1 that if $\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{P}A$ then $\overline{\mathfrak{B}} \subseteq \overline{\mathfrak{B}'}$. Furthermore, by Definition 2.1.2, $\overline{\mathfrak{B}} \subseteq \mathfrak{B}$ iff $\mathfrak{B}$ is quasi-finitary. Thus the former half of the theorem holds.

(1) If $Y \subseteq X \in \overline{\mathfrak{B}}$, then there exists a set $B \in \mathfrak{B}$ such that $Y \subseteq B \subseteq X$, and so $\bigcap_{Y \subseteq B \in \mathfrak{B}} B \subseteq X$. Therefore $\bigcap_{Y \subseteq B \in \mathfrak{B}} B \subseteq \bigcap_{Y \subseteq X \in \overline{\mathfrak{B}}} X$. Since $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$, we conversely have $\bigcap_{Y \subseteq B \in \mathfrak{B}} B \supseteq \bigcap_{Y \subseteq X \in \overline{\mathfrak{B}}} X$.

(2) If $X \in \overline{\mathfrak{B}} - \{A\}$, then for each $x \in X$, there exists a set $B \in \mathfrak{B} - \{A\}$ such that $\{x\} \subseteq B \subseteq X$, and so $\bigcup(\mathfrak{B} - \{A\}) \supseteq X$. Therefore $\bigcup(\mathfrak{B} - \{A\}) \supseteq \bigcup(\overline{\mathfrak{B}} - \{A\})$. Since $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$, we conversely have $\bigcup(\mathfrak{B} - \{A\}) \subseteq \bigcup(\overline{\mathfrak{B}} - \{A\})$.

(3) If X is a finite set in $\overline{\mathfrak{B}}$, then since $X \in \mathfrak{P}'X$, there exists a set $B \in \mathfrak{B}$ such that $X \subseteq B \subseteq X$, and so $X = B \in \mathfrak{B}$. Thus (3) holds.

Theorem 2.1.7 Let A be a set and, for each subset $\mathfrak{B}$ of $\mathfrak{P}A$, let $\overline{\mathfrak{B}^\cap}$ be the quasi-finitary closure of the $\cap$-closure $\mathfrak{B}^\cap$ of $\mathfrak{B}$ in $\mathfrak{P}A$. Then the mapping $\mathfrak{B} \mapsto \overline{\mathfrak{B}^\cap}$ is a closure operator on $\mathfrak{P}(\mathfrak{P}A)$, and its fixture domain is equal to the set of the subsets of $\mathfrak{P}A$ which are $\cap$-closed in $\mathfrak{P}A$ and quasi-finitary. Therefore, $\overline{\mathfrak{B}^\cap}$ is the smallest of the subsets of $\mathfrak{P}A$ which contain $\mathfrak{B}$ and are $\cap$-closed in $\mathfrak{P}A$ and quasi-finitary (and so we call $\overline{\mathfrak{B}^\cap}$ the **quasi-finitary $\cap$-closure** of $\mathfrak{B}$ in $\mathfrak{P}A$).

Proof By virtue of Theorems 2.1.5 and 2.1.6, it suffices to show that if $\mathfrak{B}$ is a $\cap$-closed subset of $\mathfrak{P}A$ then so is $\overline{\mathfrak{B}}$. Therefore let $\{X_i : i \in I\}$ be a subset of $\overline{\mathfrak{B}}$ and define $X = \bigcap_{i \in I} X_i$. Let $Y \in \mathfrak{P}'X$. Then for each $i \in I$, there exists a set $B_i \in \mathfrak{B}$ such that $Y \subseteq B_i \subseteq X_i$. Define $B = \bigcap_{i \in I} B_i$. Then $Y \subseteq B \subseteq X$, and $B \in \mathfrak{B}$ because $\mathfrak{B}$ is $\cap$-closed. Therefore $X \in \overline{\mathfrak{B}}$. Thus $\overline{\mathfrak{B}}$ is $\cap$-closed as desired.

Theorem 2.1.8 Let $(A, \leq)$ be an ordered set and, for each subset B of A, let $\overleftarrow{B}$ denote the subset $\bigcup_{x \in B} (\leftarrow x]$ of A. Let us say that B is **downward** if $\overleftarrow{B} \subseteq B$. Then $\overleftarrow{B}$ is the smallest of the downward subsets of A which contain B (and so we call $\overleftarrow{B}$ the **downward closure** of B in A). Therefore, the mapping $B \mapsto \overleftarrow{B}$ is a closure operator on $\mathfrak{P}A$, and its fixture domain is equal to the set of the downward subsets of A.

Proof Let B be a subset of A. Then since $x \in (\leftarrow x]$ for all $x \in B$, we have $B \subseteq \overleftarrow{B}$. Moreover, if $y \in \overleftarrow{B}$, then there exists an element $x \in B$ such that $y \in (\leftarrow x]$, and so $(\leftarrow y] \subseteq (\leftarrow x] \subseteq \overleftarrow{B}$. Therefore $\overleftarrow{B}$ is downward. Furthermore, if $B \subseteq C \subseteq A$ and C is downward, then $(\leftarrow x] \subseteq C$ for all $x \in B$, and so $\overleftarrow{B} \subseteq C$. Thus $\overleftarrow{B}$ is the smallest of the downward subsets of A which contain B.

Remark 2.1.5 Let $(A, \mathfrak{B})$ be a logic space. Then $\mathfrak{B}$ is downward in the sense defined in Theorem 2.1.8 iff it is downward in the sense defined in Remark 2.1.1. Moreover, the downward closure $\overleftarrow{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{P}A$ is equal to $\bigcup_{B \in \mathfrak{B}} \mathfrak{P}B$.

Theorem 2.1.9 Let A be a set. Then for each subset $\mathfrak{B}$ of $\mathfrak{P}\mathsf{A}$, the quasi-finitary closure $\overleftarrow{\overleftarrow{\mathfrak{B}}}$ of the downward closure $\overleftarrow{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{P}\mathsf{A}$ is equal to the set of the subsets of A which are extra-covered by $\mathfrak{B}$. Moreover, the mapping $\mathfrak{B} \mapsto \overleftarrow{\overleftarrow{\mathfrak{B}}}$ is a closure operator on $\mathfrak{P}(\mathfrak{P}\mathsf{A})$, and its fixture domain is equal to the set of the finitary subsets of $\mathfrak{P}\mathsf{A}$. Therefore, $\overleftarrow{\overleftarrow{\mathfrak{B}}}$ is the smallest of the finitary subsets of $\mathfrak{P}\mathsf{A}$ which contain $\mathfrak{B}$ (and so we call $\overleftarrow{\overleftarrow{\mathfrak{B}}}$ the **finitary closure** of $\mathfrak{B}$ in $\mathfrak{P}\mathsf{A}$).

Proof Let X be a subset of A. Then it follows from Remark 2.1.5 that X is extra-covered by $\mathfrak{B}$ iff it is extra-covered by $\overleftarrow{\mathfrak{B}}$. Since $\overleftarrow{\mathfrak{B}}$ is downward by Theorem 2.1.8, X is extra-covered by $\overleftarrow{\mathfrak{B}}$ iff it is super-covered by $\overleftarrow{\mathfrak{B}}$, as was noted in Remark 2.1.1. Thus the first statement holds. Hence it follows that $\overleftarrow{\overleftarrow{\mathfrak{B}}}$ is downward, which together with Theorems 2.1.8, 2.1.6 and Definition 2.1.2 is sufficient to prove the rest.

Theorem 2.1.10 Let φ be a closure operator on a complete lattice $(\mathsf{A}, \leq)$ and B be the fixture domain of φ. Then the following hold.

(1) $\varphi(\sup_{\mathsf{A}} \mathsf{X}) = \sup_{\mathsf{B}} \mathsf{X}$ and $\varphi(\inf_{\mathsf{A}} \mathsf{X}) = \inf_{\mathsf{B}} \mathsf{X} = \inf_{\mathsf{A}} \mathsf{X}$ for all $\mathsf{X} \in \mathfrak{P}\mathsf{B}$.

(2) $\varphi(\min \mathsf{A}) = \min \mathsf{B}$ and $\varphi(\max \mathsf{A}) = \max \mathsf{B} = \max \mathsf{A}$.

(3) $(\mathsf{B}, \leq)$ is also a complete lattice and B is $\cap$-closed in A.

(4) $\varphi(\sup_{\mathsf{A}} \varphi\mathsf{Y}) = \varphi(\sup_{\mathsf{A}} \mathsf{Y})$ for all $\mathsf{Y} \in \mathfrak{P}\mathsf{A}$, where $\varphi\mathsf{Y} = \{\varphi y : y \in \mathsf{Y}\}$.

Proof (1) Let $\mathsf{X} \in \mathfrak{P}\mathsf{B}$. Then every element $x \in \mathsf{X}$ satisfies $x \leq \sup_{\mathsf{A}} \mathsf{X}$, and so $x = \varphi x \leq \varphi(\sup_{\mathsf{A}} \mathsf{X}) \in \mathsf{B}$. Moreover, if an element $y \in \mathsf{B}$ satisfies $x \leq y$ for all $x \in \mathsf{X}$, then $\sup_{\mathsf{A}} \mathsf{X} \leq y$, and so $\varphi(\sup_{\mathsf{A}} \mathsf{X}) \leq \varphi y = y$. Thus $\varphi(\sup_{\mathsf{A}} \mathsf{X}) = \sup_{\mathsf{B}} \mathsf{X}$, and similarly $\varphi(\inf_{\mathsf{A}} \mathsf{X}) = \inf_{\mathsf{B}} \mathsf{X}$. Furthermore, since $\varphi(\inf_{\mathsf{A}} \mathsf{X}) \leq \varphi x = x$ for all $x \in \mathsf{X}$, we have $\varphi(\inf_{\mathsf{A}} \mathsf{X}) \leq \inf_{\mathsf{A}} \mathsf{X}$, and so $\inf_{\mathsf{A}} \mathsf{X} \in \mathsf{B}$. Thus $\inf_{\mathsf{B}} \mathsf{X} = \inf_{\mathsf{A}} \mathsf{X}$.

(2) We have these equations from those in (1) by substituting $\emptyset$ for X.

(3) This is a direct consequence of (1).

(4) Let $\mathsf{Y} \in \mathfrak{P}\mathsf{A}$. Then, since $\varphi y \geq y$ for all $y \in \mathsf{Y}$, we have $\sup_{\mathsf{A}} \varphi\mathsf{Y} \geq \sup_{\mathsf{A}} \mathsf{Y}$, and so $\varphi(\sup_{\mathsf{A}} \varphi\mathsf{Y}) \geq \varphi(\sup_{\mathsf{A}} \mathsf{Y})$. Moreover, since $\varphi y \leq \varphi(\sup_{\mathsf{A}} \mathsf{Y})$ for all $y \in \mathsf{Y}$, we have $\sup_{\mathsf{A}} \varphi\mathsf{Y} \leq \varphi(\sup_{\mathsf{A}} \mathsf{Y})$, and so $\varphi(\sup_{\mathsf{A}} \varphi\mathsf{Y}) \leq \varphi(\sup_{\mathsf{A}} \mathsf{Y})$. Thus $\varphi(\sup_{\mathsf{A}} \varphi\mathsf{Y}) = \varphi(\sup_{\mathsf{A}} \mathsf{Y})$.

Theorem 2.1.11 Let A be a set, φ be a closure operator on $\mathfrak{P}\mathsf{A}$, $\mathfrak{B}$ be the fixture domain of φ, and $\mathsf{D} \in \mathfrak{P}\mathsf{A}$. Define $\psi \in \mathfrak{P}\mathsf{A} \rightarrow \mathfrak{P}\mathsf{A}$ by $\psi\mathsf{X} = \varphi(\mathsf{X} \cup \mathsf{D})$ for each $\mathsf{X} \in \mathfrak{P}\mathsf{A}$. Then ψ is also a closure operator, and its fixture domain is equal to $\{\mathsf{B} \in \mathfrak{B} : \mathsf{D} \subseteq \mathsf{B}\}$. If furthermore φ is finitary, so is ψ.

Proof Let $\mathfrak{B}' = \{\mathsf{B} \in \mathfrak{B} : \mathsf{D} \subseteq \mathsf{B}\}$. Then $\psi\mathsf{X} = \min\{\mathsf{B} \in \mathfrak{B} : \mathsf{X} \cup \mathsf{D} \subseteq \mathsf{B}\} = \min\{\mathsf{B} \in \mathfrak{B}' : \mathsf{X} \subseteq \mathsf{B}\}$, and so the former statement holds, both by Theorem 2.1.4. The latter is a consequence of Theorem 2.1.3.

2.2 Latticed representations and relations

Throughout this section, we let A be a nonempty set. As always, A^* denotes the free monoid over A, or the set of all formal products $x_1 \cdots x_n$ of elements $x_1, \ldots, x_n$ of A of finite length $n \geq 0$ (s. Remark 3.1.14).

We use the following conventions about A^* throughout the remainder of this chapter. First by **alphabet convention**, the letters $\alpha, \beta, \gamma, \delta$ and ε denote elements of A^*, while the letters x, y and z denote those of A, both with or without numerical subscripts. In particular, ε denotes the identity element of A^*, i.e. the formal product of length 0. If $\alpha = x_1 \cdots x_n \in A^*$, then by **word convention**, the subset $\{x_1, \ldots, x_n\}$ of A is also denoted α, where if $n = 0$, then $\alpha = \varepsilon$ and $\{x_1, \ldots, x_n\} = \emptyset$. Lastly by **dot convention**, the multiplication of A^* is sometimes denoted by a dot; for example, $x * y \cdot \alpha = (x * y)\alpha$ for binary operations $*$ on A as in Definition 2.2.4.

2.2.1 Validity relations of latticed representations

Throughout this subsection, we let $\mathbb{B}$ be a bounded lattice, and denote its order, meet, join, minimum and maximum by $\leq$, $\wedge$, $\vee$, 0 and 1 as we do so henceforth. Then a **latticed representation** of A on $\mathbb{B}$ is simply a mapping $f \in A \to \mathbb{B}$. We also call the triple $(A, \mathbb{B}, f)$ a latticed representation.

For the latticed representation $(A, \mathbb{B}, f)$, we define a relation $\preccurlyeq_f$ on A^* by

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta \tag{2.2.1}$$

for each $(\alpha, \beta) \in A^* \times A^*$. Without the word convention, we may define it by

$$x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n \iff \inf\{fx_1, \ldots, fx_m\} \leq \sup\{fy_1, \ldots, fy_n\}$$

for each $(x_1 \cdots x_m, y_1 \cdots y_n) \in A^* \times A^*$, and the right-hand side is equivalent to $fx_1 \wedge \cdots \wedge fx_m \leq fy_1 \vee \cdots \vee fy_n$. We call $\preccurlyeq_f$ the **f-validity relation**.

Remark 2.2.1 Suppose f is a latticed representation of A on the binary lattice $\mathbb{T} = \{0, 1\}$. Then we may think that A is a set of propositions, $\mathbb{T}$ is the set of the truth values, and f is a truth function which assigns each $x \in A$ its truth value fx, that is, the proposition x is true iff $fx = 1$. Under this interpretation, $x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n$ means that if all of the propositions $x_1, \ldots, x_m$ are true then so are some of the propositions $y_1, \ldots, y_n$. In this sense, we may interpret $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ in $x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n$ as assumptions and consequences respectively. Therefore, the f-validity relation $\preccurlyeq_f$ may also be called the **f-assumption-consequence relation**.

Remark 2.2.2 If $(A, \mathbb{B}, f)$ is a latticed representation with $\mathbb{B}$ nontrivial, then $(A, \{f\})$ is a $\mathbb{B}$-valued functional logic space, and $\preccurlyeq_f$ is equal to the $\{f\}$-validity relation $\preccurlyeq_{\{f\}}$ for $(A, \{f\})$ defined by (2.6.6). Conversely, if $(A, \mathcal{F})$ is a $\mathbb{B}$-valued functional logic space and nontrivial in the sense $\mathcal{F} \neq \emptyset$, then it follows from

Remark 2.1.3 that $\mathbb{B}^{\mathcal{F}} = \mathcal{F} \to \mathbb{B}$ is a nontrivial bounded lattice with respect to the power order and that, for the latticed representation $\varphi \in A \to \mathbb{B}^{\mathcal{F}}$ defined by

$$(\varphi x) f = f x \qquad\qquad (x \in A,\ f \in \mathcal{F}),$$

the φ-validity relation $\preccurlyeq_\varphi$ is equal to the $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ for $(A, \mathcal{F})$, and so $(A, \mathcal{F})$ is equivalent to the $\mathbb{B}^{\mathcal{F}}$-valued functional logic space $(A, \{\varphi\})$ (s. Remarks 2.6.4–2.6.7). This is a reason why latticed representations are relevant to the theory of logic spaces. Moreover, the extension of the concept of $\mathbb{T}$-valued functional logic spaces to that of lattice-valued functional logic spaces means not only extending the truth values but also bundling the truth functions.

In this subsection, we will find out that the f-validity relation $\preccurlyeq_f$ satisfies various laws[2.12] according to various algebraic additional conditions on the latticed representation $(A, \mathbb{B}, f)$. Because of Remark 2.2.2, it means that we study the $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ for a functional logic space $(A, \mathcal{F})$ under various algebraic additional conditions on it.[2.13] Therefore from now on, I assume you familiar with the basic concepts on algebras to be given in Chapter 3 such as those of subalgebras, subreducts, homomorphisms, bases, ranks, ramifications[2.14], and so on. However, *all algebras and operations in this chapter are assumed to be total* in contrast to Chapter 3, and the knowledge of subreducts, bases, ranks and ramifications are needed only in §2.10.

Remark 2.2.3 Let $\mathbb{B}^*$ be the dual[2.15] of $\mathbb{B}$. Then $\mathbb{B}^*$ is a bounded lattice with respect to the order $\geq$, meet $\vee$, join $\wedge$, minimum 1 and maximum 0. Therefore $(A, \mathbb{B}^*, f)$ is also a latticed representation and its validity relation is equal to the dual $\succcurlyeq_f$ of $\preccurlyeq_f$. Thus $\succcurlyeq_f$ usually has the same properties as $\preccurlyeq_f$. This fact will be referred to as the **duality** between $\preccurlyeq_f$ and $\succcurlyeq_f$.

Theorem 2.2.1 The f-validity relation $\preccurlyeq_f$ satisfies the following four laws[2.16]:

$$x \preccurlyeq_f x, \qquad \text{(repetition)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ \alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(weakening)}$$

$$\left.\begin{array}{l} xx\alpha \preccurlyeq_f \beta \implies x\alpha \preccurlyeq_f \beta, \\ xx\alpha \succcurlyeq_f \beta \implies x\alpha \succcurlyeq_f \beta, \end{array}\right\} \qquad \text{(contraction)}$$

$$\left.\begin{array}{l} \alpha xy\beta \preccurlyeq_f \gamma \implies \alpha yx\beta \preccurlyeq_f \gamma, \\ \alpha xy\beta \succcurlyeq_f \gamma \implies \alpha yx\beta \succcurlyeq_f \gamma. \end{array}\right\} \qquad \text{(exchange)}$$

[2.12]The laws will usually be tagged and not numbered.
[2.13]It also implies that we study $\mathbb{B}$ itself because $(\mathbb{B}, \mathbb{B}, \mathrm{id}_{\mathbb{B}})$ is a latticed representation.
[2.14]The rank and ramification defined in Theorem 2.3.1 for trees are related concepts.
[2.15]Let A be a set and R be a relation on A. Then the **dual** R^* of R is the relation on A defined so that $x \, R^* \, y$ iff $y \, R \, x$. For example, if $\leq$ is an order on A, then its dual is $\geq$, which is also an order. If $(A, \leq)$ is an ordered set, then its **dual** is the ordered set $(A, \geq)$.
[2.16]The long right arrow $\implies$ stands for the word *imply* or *implies*.

Proof The repetition law holds because $\inf\{fx\} = fx = \sup\{fx\}$. The weakening law holds because $\alpha \subseteq \{x\} \cup \alpha$. The contraction law holds because $\{x, x\} \cup \alpha = \{x\} \cup \alpha$. The exchange law holds because $\alpha \cup \{x, y\} \cup \beta = \alpha \cup \{y, x\} \cup \beta$.

Remark 2.2.4 The repetition law is related to but different from the reflexion law $\alpha \preccurlyeq_f \alpha$ (s. Theorem 2.2.10). If $\alpha \neq \varepsilon$ then $\alpha \preccurlyeq_f \alpha$ holds, because $\inf \mathbb{X} \leq \sup \mathbb{X}$ for all $\mathbb{X} \in \mathfrak{P}'\mathbb{B} - \{\emptyset\}$ (s. the extension law in Remark 2.2.8). However, since $\inf \emptyset = 1 \geq 0 = \sup \emptyset$, $\varepsilon \preccurlyeq_f \varepsilon$ holds iff $1 = 0$, or iff $\mathbb{B}$ is trivial.

Theorem 2.2.2 The f-validity relation $\preccurlyeq_f$ satisfies the following law:

$$\left.\begin{array}{l} \alpha \preccurlyeq_f x,\ x\beta \preccurlyeq_f \delta \implies \alpha\beta \preccurlyeq_f \delta, \\ \alpha \succcurlyeq_f x,\ x\beta \succcurlyeq_f \delta \implies \alpha\beta \succcurlyeq_f \delta. \end{array}\right\} \qquad \text{(cut)}$$

If fA is contained in a distributive[2.17] sublattice of $\mathbb{B}$, then $\preccurlyeq_f$ satisfies the following law:

$$\alpha \preccurlyeq_f x\gamma,\ x\beta \preccurlyeq_f \delta \implies \alpha\beta \preccurlyeq_f \delta\gamma. \qquad \text{(strong cut)}$$

Remark 2.2.5 The cut law is derived from the strong cut law by assuming $\gamma = \varepsilon$ or $\beta = \varepsilon$, and so the strong cut law implies the cut law.

Proof As for the strong cut law, let $\mathbb{C}$ be the distributive sublattice of $\mathbb{B}$ containing fA, and define $a = \inf f\alpha$, $b = \inf f\beta$, $c = \sup f\gamma$, $d = \sup f\delta$ and $e = fx$. Then a, b, c, d, e are contained in $\mathbb{C} \cup \{0, 1\}$, which is also a distributive sublattice, and the premise of the law means $a \leq e \vee c$ and $e \wedge b \leq d$. Therefore

$$\inf f(\alpha\beta) = a \wedge b \leq (e \vee c) \wedge b \leq (e \wedge b) \vee c \leq d \vee c = \sup f(\delta\gamma),$$

and thus $\alpha\beta \preccurlyeq_f \delta\gamma$. If $\gamma = \varepsilon$ or $\beta = \varepsilon$, then $c = 0$ or $b = 1$, and the above holds without distributivity. Thus the cut law holds without additional conditions.

Definition 2.2.1 If a relation $\preccurlyeq$ on A^* satisfies the five laws of repetition, weakening, contraction, exchange and cut, we say that $\preccurlyeq$ is **latticed**. Likewise, if a relation $\preccurlyeq$ on A^* satisfies the five laws of repetition, weakening, contraction, exchange and strong cut, we say that $\preccurlyeq$ is **strongly latticed**.

Remark 2.2.6 Theorems 2.2.1 and 2.2.2 show that $\preccurlyeq_f$ is latticed and that if fA is contained in a distributive sublattice of $\mathbb{B}$ then $\preccurlyeq_f$ is strongly latticed. Moreover, strongly latticed relations are latticed by Remark 2.2.5.

Theorem 2.2.3 The image fA of f contains 0 and 1 iff $\preccurlyeq_f$ satisfies the following **end laws**:

$$\text{there exists an element } x \in A \text{ such that } x \preccurlyeq_f \varepsilon, \qquad \text{(lower end)}$$
$$\text{there exists an element } x \in A \text{ such that } x \succcurlyeq_f \varepsilon. \qquad \text{(upper end)}$$

[2.17] The distributivity law of lattices is equivalent to any of the following laws and their duals:

- $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.
- $(a \vee b) \wedge c \leq (a \wedge c) \vee b$.
- $a \leq e \vee c,\ e \wedge b \leq d \implies a \wedge b \leq d \vee c$.
- $a \wedge c \leq d,\ b \wedge c \leq d \implies (a \vee b) \wedge c \leq d$.

Proof This is because $x \preccurlyeq_f \varepsilon$ iff $fx = 0$, and $x \succcurlyeq_f \varepsilon$ iff $fx = 1$.

Theorem 2.2.4 Assume $0 = \inf fA$ and $1 = \sup fA$. Then $\preccurlyeq_f$ satisfies the following **quasi-end law**s (s. Lemma 2.2.5):

$$\alpha \preccurlyeq_f \varepsilon \iff \alpha \preccurlyeq_f y \text{ for all } y \in A, \qquad \text{(lower quasi-end)}$$
$$\alpha \succcurlyeq_f \varepsilon \iff \alpha \succcurlyeq_f y \text{ for all } y \in A. \qquad \text{(upper quasi-end)}$$

Proof This is because of the following and the duality between $\preccurlyeq_f$ and $\succcurlyeq_f$:

$$\begin{aligned} \alpha \preccurlyeq_f \varepsilon &\iff \inf f\alpha \leq \inf fA \\ &\iff \inf f\alpha \leq fy \text{ for all } y \in A \\ &\iff \alpha \preccurlyeq_f y \text{ for all } y \in A. \end{aligned}$$

Theorem 2.2.5 The image fA of f is a sublattice of $\mathbb{B}$, that is, it is closed by $\wedge$ and $\vee$, iff $\preccurlyeq_f$ satisfies the following **quasi-junction law**s:

for each $(x, y) \in A \times A$, there exists an element $z \in A$ such that
$z \preccurlyeq_f x,\ z \preccurlyeq_f y$ and $xy \preccurlyeq_f z$, (quasi-conjunction)
for each $(x, y) \in A \times A$, there exists an element $z \in A$ such that
$z \succcurlyeq_f x,\ z \succcurlyeq_f y$ and $xy \succcurlyeq_f z$. (quasi-disjunction)

Proof This is because of the following and the duality between $\preccurlyeq_f$ and $\succcurlyeq_f$:

$$\begin{aligned} fx \wedge fy = fz &\iff fz \leq fx,\ fz \leq fy,\ fx \wedge fy \leq fz \\ &\iff z \preccurlyeq_f x,\ z \preccurlyeq_f y,\ xy \preccurlyeq_f z. \end{aligned}$$

Theorem 2.2.6 Let $x \wedge y$ and $x \vee y$ be binary operations[2.18] on A. Then f is a $\{\wedge, \vee\}$-homomorphism iff $\preccurlyeq_f$ satisfies the following **junction law**s (s. [1.96]):

$$x \wedge y \preccurlyeq_f x, \qquad x \wedge y \preccurlyeq_f y, \qquad xy \preccurlyeq_f x \wedge y, \qquad \text{(conjunction)}$$
$$x \vee y \succcurlyeq_f x, \qquad x \vee y \succcurlyeq_f y, \qquad xy \succcurlyeq_f x \vee y. \qquad \text{(disjunction)}$$

Proof This is because of the following and the duality between $\preccurlyeq_f$ and $\succcurlyeq_f$:

$$\begin{aligned} f(x \wedge y) = fx \wedge fy &\iff f(x \wedge y) \leq fx,\ \ f(x \wedge y) \leq fy,\ \ fx \wedge fy \leq f(x \wedge y) \\ &\iff x \wedge y \preccurlyeq_f x,\ \ x \wedge y \preccurlyeq_f y,\ \ xy \preccurlyeq_f x \wedge y. \end{aligned}$$

Remark 2.2.7 A relation on A^* satisfies the quasi-conjunction law iff it satisfies the conjunction law for some binary operation on A, and likewise for the relationship between the laws of quasi-disjunction and disjunction.

[2.18] The expressions $x \wedge y$ and $x \vee y$ are abbreviations of the expressions $(x, y) \mapsto x \wedge y$ and $(x, y) \mapsto x \vee y$ respectively, and likewise for the expressions $a^{\Diamond}$, $x^{\Diamond}$, $a \Rrightarrow b$ and $x \Rrightarrow y$ below.

Theorem 2.2.7 Assume that $\mathbb{B}$ is a Boolean lattice and let $a^\lozenge$ be its complement (s. [1.79][2.18]). Let $x^\lozenge$ be a unary operation on A. Then f is a $\{\lozenge\}$-homomorphism iff $\preccurlyeq_f$ satisfies the following **negation laws** (s. [1.96]):

$$xx^\lozenge \preccurlyeq_f \varepsilon, \qquad \text{(lower negation)}$$
$$xx^\lozenge \succcurlyeq_f \varepsilon. \qquad \text{(upper negation)}$$

Proof This is because the following holds for all $x \in A$ by virtue of the uniqueness of the complement $a^\lozenge$ of each element $a \in \mathbb{B}$:

$$f(x^\lozenge) = (fx)^\lozenge \iff \left\{ \begin{array}{l} fx \wedge f(x^\lozenge) = 0, \\ fx \vee f(x^\lozenge) = 1 \end{array} \right\} \iff \left\{ \begin{array}{l} xx^\lozenge \preccurlyeq_f \varepsilon, \\ xx^\lozenge \succcurlyeq_f \varepsilon. \end{array} \right.$$

Theorem 2.2.8 Assume that $\mathbb{B}$ is a Boolean lattice and let $a^\lozenge$ and $a \Rightarrow b$ be its complement and cojoin (s. §1.5.2 and [2.18]). Let $x^\lozenge$ and $x \Rightarrow y$ be unary and binary operations on A and assume that f is a $\{\lozenge\}$-homomorphism (s. Theorem 2.2.7). Then f is a $\{\Rightarrow\}$-homomorphism iff $\preccurlyeq_f$ satisfies the following three **implication laws**[2.19] (s. [1.96]):

$$x^\lozenge \preccurlyeq_f x \Rightarrow y, \qquad \text{(contradictory implication)}$$
$$y \preccurlyeq_f x \Rightarrow y, \qquad \text{(reflexive implication)}$$
$$x \Rightarrow y \preccurlyeq_f x^\lozenge y. \qquad \text{(negative implication)}$$

Proof This is because the following holds for all $(x, y) \in A \times A$ by virtue of the definition $a \Rightarrow b = a^\lozenge \vee b$ of the cojoin $\Rightarrow$ on $\mathbb{B}$:

$$\begin{aligned}
& f(x \Rightarrow y) = fx \Rightarrow fy \\
& \iff f(x \Rightarrow y) = (fx)^\lozenge \vee fy \\
& \iff f(x \Rightarrow y) = f(x^\lozenge) \vee fy \\
& \iff f(x^\lozenge) \le f(x \Rightarrow y), \quad fy \le f(x \Rightarrow y), \quad f(x \Rightarrow y) \le f(x^\lozenge) \vee fy \\
& \iff x^\lozenge \preccurlyeq_f x \Rightarrow y, \quad y \preccurlyeq_f x \Rightarrow y, \quad x \Rightarrow y \preccurlyeq_f x^\lozenge y.
\end{aligned}$$

Definition 2.2.2 Assume that $\mathbb{B}$ is a Boolean lattice, and let $a^\lozenge$ and $a \Rightarrow b$ be its complement and cojoin as above and as we do so henceforth. Moreover, let $x \wedge y$, $x \vee y$, $x^\lozenge$ and $x \Rightarrow y$ be operations on A, and assume that $f \in A \to \mathbb{B}$ is a homomorphism with respect to the operations $\wedge, \vee, \lozenge$ and $\Rightarrow$ on A and the Boolean operations $\wedge, \vee, \lozenge$ and $\Rightarrow$ on $\mathbb{B}$. Then we say that f is a **Boolean representation** of A into $\mathbb{B}$ with respect to the operations $\wedge, \vee, \lozenge$ and $\Rightarrow$ on A; in case $\mathbb{B}$ is the binary lattice $\mathbb{T}$, we call it a **binary representation**.

Definition 2.2.3 Let $x \wedge y$, $x \vee y$, $x^\lozenge$ and $x \Rightarrow y$ be operations on A. Then a relation $\preccurlyeq$ on A^* is said to be **Boolean** with respect to the operations $\wedge, \vee, \lozenge$

[2.19] They were called the first, second and third implication laws in [1.5] and others (s. [2.1]).

and $\Rightarrow$, if it satisfies the laws of repetition, weakening, contraction, exchange, strong cut (that is, it is strongly latticed) and the laws of junction, negation and implication with respect to $\wedge, \vee, \diamond$ and $\Rightarrow$. The union of these eight kinds of laws will be called the **Boolean law** with respect to the operations.

Theorem 2.2.9 Assume that $\mathbb{B}$ is a Boolean lattice and f is a Boolean representation with respect to operations $x \wedge y$, $x \vee y$, $x^{\diamond}$ and $x \Rightarrow y$ on A. Then $\preccurlyeq_f$ is Boolean with respect to the operations.

Proof This is a consequence of Theorems 2.2.1, 2.2.2, 2.2.6, 2.2.7 and 2.2.8.

2.2.2 Restrictions of and extensions to latticed relations

We considered relations on A^* in §2.2.1, while we will treat **associations** on A, i.e. relations between A^* and A, in due course. Therefore, we consider interrelations between the two kinds of relations here. First of all, the following theorem obviously holds.

Theorem 2.2.10 Let $\preccurlyeq$ be a (strongly) latticed relation on A^* and $\vDash$ be the restriction of $\preccurlyeq$ to $A^* \times A$. Then $\vDash$ is a **partially latticed association** in the sense that it satisfies the following five laws:

$$
\begin{aligned}
& x \vDash x, && \text{(repetition)} \\
& \alpha \vDash y \implies x\alpha \vDash y, && \text{(partial weakening)} \\
& xx\alpha \vDash y \implies x\alpha \vDash y, && \text{(partial contraction)} \\
& \alpha x y \beta \vDash z \implies \alpha y x \beta \vDash z, && \text{(partial exchange)} \\
& \alpha \vDash x,\ x\beta \vDash y \implies \alpha\beta \vDash y. && \text{(partial cut)}
\end{aligned}
$$

If $\vDash$ is a partially latticed association on A, then its restriction $\sqsubseteq$ to $A \times A$ is a **preorder** in the sense that it satisfies the following two laws:

$$
\begin{aligned}
& x \sqsubseteq x, && \text{(reflexion)} \\
& x \sqsubseteq y,\ y \sqsubseteq z \implies x \sqsubseteq z. && \text{(transition)}
\end{aligned}
$$

In the remainder of this subsection, we conversely consider extensions of preorders and partially latticed associations to (partially) latticed relations.

Theorem 2.2.11 Let $\sqsubseteq$ be a preorder on A, and define an association $\vDash$ on A so that $x_1 \cdots x_m \vDash y$ iff the following holds for all $z \in A$:

$$z \sqsubseteq x_i \ (i = 1, \ldots, m) \implies z \sqsubseteq y.$$

In particular, $\varepsilon \vDash y$ iff $z \sqsubseteq y$ for all $z \in A$. Then $\vDash$ is the largest of the partially latticed associations on A which extend $\sqsubseteq$ (and so we call $\vDash$ the **largest partially latticed extension** of $\sqsubseteq$), and $\vDash$ satisfies the upper quasi-end law in the sense that $\varepsilon \vDash y$ iff $z \vDash y$ for all $z \in A$ (s. Theorem 2.2.4).

Proof Assume $x \sqsubseteq y$. Then if $z \sqsubseteq x$, we have $z \sqsubseteq y$ by the transition law. Therefore $x \vDash y$. Conversely if $x \vDash y$, then since $x \sqsubseteq x$ by the reflexion law, we have $x \sqsubseteq y$. Thus $\vDash$ extends $\sqsubseteq$, and so the definition of $\varepsilon \vDash y$ and the reflexion law for $\sqsubseteq$ imply the laws of upper quasi-end and repetition for $\vDash$.

If $x_1 \cdots x_m \vDash y$ and $x \in A$, then since $\{x, x_1, \ldots, x_m\} \supseteq \{x_1, \ldots, x_m\}$, we have $xx_1 \cdots x_m \vDash y$. Therefore the partial weakening law holds. If $xxx_1 \cdots x_m \vDash y$, then since $\{x, x_1, \ldots, x_m\} = \{x, x, x_1, \ldots, x_m\}$, we have $xx_1 \cdots x_m \vDash y$. Therefore the partial contraction law holds. If $x_1 \cdots x_m \vDash y$ and p is a permutation on $\{1, \ldots, m\}$, then since $\{x_{p1}, \ldots, x_{pm}\} = \{x_1, \ldots, x_m\}$, we have $x_{p1} \cdots x_{pm} \vDash y$. Therefore the partial exchange law holds. In order to verify the partial cut law, assume $x_1 \cdots x_m \vDash x$ and $xy_1 \cdots y_n \vDash y$. Then if $z \sqsubseteq x_i$ $(i = 1, \ldots, m)$, we have $z \sqsubseteq x$, and so if furthermore $z \sqsubseteq y_j$ $(j = 1, \ldots, n)$, then $z \sqsubseteq y$. Therefore $x_1 \cdots x_m y_1 \cdots y_n \vDash y$ as desired. Thus $\vDash$ is partially latticed.

Let $\vdash$ be a partially latticed extension of $\sqsubseteq$. In order to prove $\vdash\ \subseteq\ \vDash$, first assume $x_1 \cdots x_m \vdash y$ with $m \geq 1$. Then if $z \sqsubseteq x_i$ $(i = 1, \ldots, m)$, then $z \vdash x_i$ $(i = 1, \ldots, m)$, and by repeated application of the laws of partial cut, partial contraction and partial exchange to these and $x_1 \cdots x_m \vdash y$, we have $z \vdash y$, and so $z \sqsubseteq y$. Therefore $x_1 \cdots x_m \vDash y$. Next assume $\varepsilon \vdash y$. Then $z \vdash y$ for all $z \in A$ by the partial weakening law, and so $z \sqsubseteq y$ for all $z \in A$. Therefore $\varepsilon \vDash y$. Thus $\vdash\ \subseteq\ \vDash$.

Theorem 2.2.12 Let $\vDash$ be a partially latticed association on A, and define a relation $\preccurlyeq$ on A^* so that $\alpha \preccurlyeq y_1 \cdots y_n$ iff the condition

$$y_i \vDash z \ (i = 1, \ldots, n) \implies \alpha \vDash z$$

holds for all $z \in A$. In particular, $\alpha \preccurlyeq \varepsilon$ iff $\alpha \vDash z$ for all $z \in A$. Then $\preccurlyeq$ is the largest of the latticed relations on A^* which extend $\vDash$ (and so we call $\preccurlyeq$ the **largest latticed extension** of $\vDash$), and $\preccurlyeq$ satisfies the lower quasi-end law.

Proof Assume $\alpha \vDash y$. Then if $y \vDash z$, we have $\alpha \vDash z$ by the partial cut law. Therefore $\alpha \preccurlyeq y$. Conversely if $\alpha \preccurlyeq y$, then since $y \vDash y$ by the repetition law, we have $\alpha \vDash y$. Thus $\preccurlyeq$ extends $\vDash$, and so the definition of $\alpha \preccurlyeq \varepsilon$ and the repetition law for $\vDash$ imply the laws of lower quasi-end and repetition for $\preccurlyeq$.

In order to verify the weakening law, assume $\alpha \preccurlyeq y_1 \cdots y_n$ and let $x \in A$. Then if $y_i \vDash z$ $(i = 1, \ldots, n)$, then $\alpha \vDash z$, and so $x\alpha \vDash z$ by the partial weakening law. Therefore $x\alpha \preccurlyeq y_1 \cdots y_n$. Moreover, since $\{x, y_1, \ldots, y_n\} \supseteq \{y_1, \ldots, y_n\}$, we have $\alpha \preccurlyeq xy_1 \cdots y_n$.

In order to verify the contraction law, first assume $xx\alpha \preccurlyeq y_1 \cdots y_n$. Then if $y_i \vDash z$ $(i = 1, \ldots, n)$, then $xx\alpha \vDash z$, and so $x\alpha \vDash z$ by the partial contraction law. Therefore $x\alpha \preccurlyeq y_1 \cdots y_n$. Next assume $\alpha \preccurlyeq xxy_1 \cdots y_n$. Then since $\{x, y_1, \ldots, y_n\} = \{x, x, y_1, \ldots, y_n\}$, we have $\alpha \preccurlyeq xy_1 \cdots y_n$.

In order to verify the exchange law, assume $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$, and let p and q be permutations on $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively. Then if $y_i \vDash z$ $(i = 1, \ldots, n)$, then $x_1 \cdots x_m \vDash z$, and so $x_{p1} \cdots x_{pm} \vDash z$ by the partial exchange law. Therefore $x_{p1} \cdots x_{pm} \preccurlyeq y_1 \cdots y_n$. Moreover, since $\{y_{q1}, \ldots, y_{qn}\} = \{y_1, \ldots, y_n\}$, we have $x_1 \cdots x_m \preccurlyeq y_{q1} \cdots y_{qn}$.

In order to verify the cut law, first assume $\alpha \preccurlyeq x$ and $x\beta \preccurlyeq y_1 \cdots y_n$. Then as $\preccurlyeq$ extends $\vDash$, we have $\alpha \vDash x$. Moreover, if $y_i \vDash z$ $(i = 1, \ldots, n)$, then $x\beta \vDash z$. Applying the partial cut law to $\alpha \vDash x$ and $x\beta \vDash z$, we have $\alpha\beta \vDash z$. Thus $\alpha\beta \preccurlyeq y_1 \cdots y_n$. Next assume $x \preccurlyeq x_1 \cdots x_m$ and $\delta \preccurlyeq xy_1 \cdots y_n$. Then if $x_i \vDash z$ $(i = 1, \ldots, m)$, we have $x \vDash z$, and so if furthermore $y_j \vDash z$ $(j = 1, \ldots, n)$, then $\delta \vDash z$. Thus $\delta \preccurlyeq x_1 \cdots x_m y_1 \cdots y_n$.

Let $\prec$ be a latticed extension of $\vDash$. In order to prove $\prec \subseteq \preccurlyeq$, first assume $\alpha \prec y_1 \cdots y_n$ with $n \geq 1$. Then if $y_i \vDash z$ $(i = 1, \ldots, n)$, then $y_i \prec z$ $(i = 1, \ldots, n)$, and by repeated application of the laws of cut, exchange and contraction to these and $\alpha \prec y_1 \cdots y_n$, we have $\alpha \prec z$, and so $\alpha \vDash z$. Therefore $\alpha \preccurlyeq y_1 \cdots y_n$. Next assume $\alpha \prec \varepsilon$. Then $\alpha \prec z$ for all $z \in A$ by the weakening law, and so $\alpha \vDash z$ for all $z \in A$. Therefore $\alpha \preccurlyeq \varepsilon$. Thus $\prec \subseteq \preccurlyeq$.

Theorem 2.2.13 Let $\sqsubseteq$ be a preorder on A, and define a relation $\preccurlyeq$ on A^* so that $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$ iff the following holds for all $(x, y) \in A \times A$:

$$x \sqsubseteq x_i \ (i = 1, \ldots, m), \ y_j \sqsubseteq y \ (j = 1, \ldots, n) \implies x \sqsubseteq y.$$

Then $\preccurlyeq$ is the largest of the latticed relations on A^* which extend $\sqsubseteq$ (and so we call $\preccurlyeq$ the **largest latticed extension** of $\sqsubseteq$), and is the largest latticed extension of the largest partially latticed extension of $\sqsubseteq$, and satisfies the quasi-end laws.

Proof The above condition is equivalent to the following:

$$y_j \sqsubseteq y \ (j = 1, \ldots, n) \implies (x \sqsubseteq x_i \ (i = 1, \ldots, m) \implies x \sqsubseteq y).$$

This means the following for the largest partially latticed extension $\vDash$ of $\sqsubseteq$:

$$y_j \vDash y \ (j = 1, \ldots, n) \implies x_1 \cdots x_m \vDash y.$$

Thus $\preccurlyeq$ is the largest latticed extension of $\vDash$ and therefore is a latticed extension of $\sqsubseteq$ satisfying the lower quasi-end law. The dual $\succcurlyeq$ of $\preccurlyeq$ is derived from the dual $\sqsupseteq$ of $\sqsubseteq$ in the same way that $\preccurlyeq$ is derived from $\sqsubseteq$. Therefore, $\preccurlyeq$ also satisfies the upper quasi-end law.

Let $\prec$ be a latticed extension of $\sqsubseteq$. In order to prove $\prec \subseteq \preccurlyeq$, first assume $x_1 \cdots x_m \prec y_1 \cdots y_n$ with $m \geq 1$ and $n \geq 1$. Then if $x \sqsubseteq x_i$ $(i = 1, \ldots, m)$ and $y_j \sqsubseteq y$ $(j = 1, \ldots, n)$, then $x \prec x_i$ $(i = 1, \ldots, m)$ and $y_j \prec y$ $(j = 1, \ldots, n)$, and by repeated application of the laws of cut, exchange and contraction to these and $x_1 \cdots x_m \prec y_1 \cdots y_n$, we have $x \prec y$, and so $x \sqsubseteq y$. Therefore $x_1 \cdots x_m \preccurlyeq y_1 \cdots y_n$. Next assume $\alpha \prec \varepsilon$ with $\alpha \neq \varepsilon$. Then $\alpha \prec y$ for all $y \in A$ by the weakening law, and so $\alpha \preccurlyeq y$ for all $y \in A$ as above. Therefore $\alpha \preccurlyeq \varepsilon$ by the lower quasi-end law. Lastly assume $\varepsilon \prec \beta$. Then $x \prec \beta$ for all $x \in A$ by the weakening law, and so $x \preccurlyeq \beta$ for all $x \in A$ as above. Therefore $\varepsilon \preccurlyeq \beta$ by the upper quasi-end law. Thus $\prec \subseteq \preccurlyeq$.

We have defined three kinds of the largest (partially) latticed extensions in Theorems 2.2.11–2.2.13. We next consider the conditions under which a given (partially) latticed relation is the largest (partially) latticed extension. In particular, if a relation $\preccurlyeq$ on A^* is the largest latticed extension of a partially latticed association on A, we say that $\preccurlyeq$ is **extendedly latticed**.

Lemma 2.2.1 Let $\preccurlyeq$ be a latticed relation on A^*. Then the quasi-conjunction law and the quasi-disjunction law for $\preccurlyeq$ imply the following n-tuple quasi-conjunction law and the n-tuple quasi-disjunction law respectively for all $n \in \mathbb{N}$:

for each $(x_1, \dots, x_n) \in A^n$, there exists an element $y \in A$ such that
$y \preccurlyeq x_i$ $(i = 1, \dots, n)$ and $x_1 \cdots x_n \preccurlyeq y$, ($n$-tuple quasi-conjunction)
for each $(x_1, \dots, x_n) \in A^n$, there exists an element $y \in A$ such that
$y \succcurlyeq x_i$ $(i = 1, \dots, n)$ and $x_1 \cdots x_n \succcurlyeq y$. ($n$-tuple quasi-disjunction)

Proof In view of the duality between $\preccurlyeq$ and $\succcurlyeq$ to be noted in Remark 2.2.9, we only consider the case that $\preccurlyeq$ satisfies the quasi-conjunction law, and prove that $\preccurlyeq$ satisfies the n-tuple quasi-conjunction law for all $n \in \mathbb{N}$ by induction. The repetition law implies the 1-tuple quasi-conjunction law. Therefore assume $n \geq 2$ and $(x_1, \dots, x_n) \in A^n$. Then there exists an element $z \in A$ such that $z \preccurlyeq x_i$ $(i = 1, \dots, n-1)$ and $x_1 \cdots x_{n-1} \preccurlyeq z$ by the induction hypothesis. Moreover, there exists an element $y \in A$ such that $y \preccurlyeq z$, $y \preccurlyeq x_n$ and $zx_n \preccurlyeq y$ by the quasi-conjunction law. Applying the cut law to $y \preccurlyeq z$ and $z \preccurlyeq x_i$, we have $y \preccurlyeq x_i$ $(i = 1, \dots, n-1)$. Furthermore, applying the cut law to $x_1 \cdots x_{n-1} \preccurlyeq z$ and $zx_n \preccurlyeq y$, we have $x_1 \cdots x_n \preccurlyeq y$. This completes the proof by induction.

Lemma 2.2.2 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the laws of quasi-disjunction and lower quasi-end. Then $\preccurlyeq$ is extendedly latticed.

Proof Let $\vDash$ be the restriction of $\preccurlyeq$ to $A^* \times A$. Then $\vDash$ is partially latticed by Theorem 2.2.10. Let $\prec$ be the largest latticed extension of $\vDash$. Then as $\preccurlyeq$ is also a latticed extension of $\vDash$, we have $\preccurlyeq \subseteq \prec$. In order to prove the converse, first assume $\alpha \prec y_1 \cdots y_n$ with $n \geq 1$. Then, since $\preccurlyeq$ satisfies the quasi-disjunction law, there exists an element $z \in A$ such that $y_i \preccurlyeq z$ $(i = 1, \dots, n)$ and $z \preccurlyeq y_1 \cdots y_n$ by Lemma 2.2.1. Then $y_i \vDash z$ $(i = 1, \dots, n)$, and so the definition of $\alpha \prec y_1 \cdots y_n$ shows $\alpha \vDash z$. Therefore $\alpha \preccurlyeq z$, and applying the cut law to it and $z \preccurlyeq y_1 \cdots y_n$, we have $\alpha \preccurlyeq y_1 \cdots y_n$. Next assume $\alpha \prec \varepsilon$. Then $\alpha \prec z$ for all $z \in A$ by the weakening law, and so $\alpha \preccurlyeq z$ for all $z \in A$ as above. Therefore $\alpha \preccurlyeq \varepsilon$ by the lower quasi-end law. Thus $\prec \subseteq \preccurlyeq$.

Lemma 2.2.3 Let $\vDash$ be a partially latticed association on A, and assume that $\vDash$ satisfies the quasi-conjunction law in the same sense as in Theorem 2.2.5 and also satisfies the upper quasi-end law as defined in Theorem 2.2.11. Then $\vDash$ is the largest partially latticed extension of a preorder on A.

Proof Let $\sqsubseteq$ be the restriction of $\vDash$ to $A \times A$. Then $\sqsubseteq$ is a preorder by Theorem 2.2.10. Let $\vdash$ be the largest partially latticed extension of $\sqsubseteq$. Then as

$\vDash$ is also a partially latticed extension of $\sqsubseteq$, we have $\vDash \subseteq \vdash$. In order to prove the converse, first assume $x_1 \cdots x_m \vdash y$ with $m \geq 1$. Then, since $\vDash$ satisfies the quasi-conjunction law, so does its largest latticed extension, and therefore there exists an element $z \in A$ such that $z \vDash x_i$ $(i = 1, \dots, m)$ and $x_1 \cdots x_m \vDash z$ by Lemma 2.2.1. Then $z \sqsubseteq x_i$ $(i = 1, \dots, m)$, and so the definition of $x_1 \cdots x_m \vdash y$ shows $z \sqsubseteq y$. Therefore $z \vDash y$, and applying the partial cut law to it and $x_1 \cdots x_m \vDash z$, we have $x_1 \cdots x_m \vDash y$. Next assume $\varepsilon \vdash y$. Then $z \vdash y$ for all $z \in A$ by the partial weakening law, and so $z \vDash y$ for all $z \in A$ as above. Therefore $\varepsilon \vDash y$ by the upper quasi-end law. Thus $\vdash \subseteq \vDash$.

Theorem 2.2.14 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the laws of quasi-junction and quasi-end. Then $\preccurlyeq$ is the largest latticed extension both of a partially latticed association on A and of a preorder on A.

Proof Lemma 2.2.2 shows that $\preccurlyeq$ is the largest latticed extension of a partially latticed association $\vDash$ on A, and then Lemma 2.2.3 shows that $\vDash$ is the largest partially latticed extension of a preorder $\sqsubseteq$ on A. Thus $\preccurlyeq$ is the largest latticed extension of $\sqsubseteq$ by Theorem 2.2.13.

2.2.3 Boolean and weakly Boolean relations

The purpose of this subsection is to show that the laws obtained in §2.2.1 are equivalent or related to certain laws such as those in the following definition and that Boolean relations satisfy all the tagged or bold-faced laws in §2.2.1–2.2.3.

Definition 2.2.4 Let $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ be operations on A. Then a relation $\preccurlyeq$ on A^* is said to be **weakly Boolean** with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$, if it satisfies the laws of repetition, weakening, contraction, exchange and the following four laws (notice that the dot convention is used in the strong conjunction law, strong disjunction law and strong implication law, and distinguish between the operation symbol $\Rrightarrow$ and the long right arrow $\Longrightarrow$ (s. [2.16]) in the strong implication law):

$$\left.\begin{array}{l} xy\alpha \preccurlyeq \beta \implies x \wedge y \cdot \alpha \preccurlyeq \beta, \\ \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y \cdot \beta, \end{array}\right\} \qquad \text{(strong conjunction)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y \cdot \beta, \\ x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y \cdot \alpha \preccurlyeq \beta, \end{array}\right\} \qquad \text{(strong disjunction)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq x\beta \implies x^{\Diamond}\alpha \preccurlyeq \beta, \\ x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^{\Diamond}\beta, \end{array}\right\} \qquad \text{(strong negation)}$$

$$\left.\begin{array}{l} x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rrightarrow y \cdot \beta, \\ \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rrightarrow y \cdot \alpha \preccurlyeq \beta. \end{array}\right\} \qquad \text{(strong implication)}$$

The union of these eight kinds of laws will be called the **weakly Boolean law** with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$. Moreover, the union of the laws of strong conjunction and strong disjunction will be called the **strong junction laws** with respect to the operations $\wedge$ and $\vee$.

Remark 2.2.8 Under the laws of weakening and exchange, the repetition law is equivalent to the following law written by the word convention:

$$\alpha \cap \beta \neq \emptyset \implies \alpha \preccurlyeq \beta. \qquad \text{(intersection)}$$

Moreover, the union of the laws of weakening, contraction and exchange is equivalent to the following law written by the word convention:

$$\alpha \preccurlyeq \beta,\ \alpha \subseteq \alpha',\ \beta \subseteq \beta' \implies \alpha' \preccurlyeq \beta'. \qquad \text{(extension)}$$

Furthermore, under the exchange law, the laws of cut and strong cut are equivalent to the following laws of substitution and strong substitution respectively, where $\beta(x/\alpha)$ is the element of A^* obtained by substituting α for the occurrences[2.20] of x in β and likewise for $\gamma(x/\delta)$:

$$\left.\begin{array}{l} \alpha \preccurlyeq x,\ \beta \preccurlyeq \delta,\ x \text{ occurs in } \beta \implies \beta(x/\alpha) \preccurlyeq \delta, \\ \alpha \succcurlyeq x,\ \beta \succcurlyeq \delta,\ x \text{ occurs in } \beta \implies \beta(x/\alpha) \succcurlyeq \delta, \end{array}\right\} \qquad \text{(substitution)}$$

$$\left.\begin{array}{l} \alpha \preccurlyeq \gamma,\ \beta \preccurlyeq \delta, \\ x \text{ occurs in } \gamma \text{ and } \beta \end{array}\right\} \implies \beta(x/\alpha) \preccurlyeq \gamma(x/\delta). \qquad \text{(strong substitution)}$$

Similar remarks apply to the laws of partial weakening, partial contraction, partial exchange and partial cut on the associations on A provided that we introduce the laws of **partial extension** and **partial substitution**.

Remark 2.2.9 The conjunction law for $(\wedge, \preccurlyeq)$ and the disjunction law for $(\vee, \preccurlyeq)$ are the disjunction law for $(\wedge, \succcurlyeq)$ and the conjunction law for $(\vee, \succcurlyeq)$ respectively, and so the junction laws for $(\wedge, \vee, \preccurlyeq)$ are those for $(\vee, \wedge, \succcurlyeq)$, and likewise for the strong junction laws. The laws of lower and upper negation for $\preccurlyeq$ are those of upper and lower negation for $\succcurlyeq$ respectively, and so the negation laws for $\preccurlyeq$ are those for $\succcurlyeq$, and likewise for the strong negation law. The laws of (strong) junction and (strong) negation are **self-dual** in this sense, and likewise for the laws of repetition, weakening, contraction, exchange, (strong) cut, (quasi-)end, quasi-junction, intersection, extension and (strong) substitution. Consequently, if $\preccurlyeq$ is a (strongly) latticed relation on A^*, then so is $\succcurlyeq$. These facts will be referred to as the **duality** between $\preccurlyeq$ and $\succcurlyeq$ (s. [2.15]).

Theorem 2.2.15 Let $\preccurlyeq$ be a latticed relation on A^*. Furthermore let $x \wedge y$ be a binary operation on A. Then the following three laws are equivalent.

($\wedge$1) $x \wedge y \preccurlyeq x,\ x \wedge y \preccurlyeq y$ (first two parts of conjunction).

($\wedge$2) $xy\alpha \preccurlyeq \beta \implies x \wedge y \cdot \alpha \preccurlyeq \beta$ (first half of strong conjunction).

($\wedge$3) $\alpha \preccurlyeq x \wedge y \cdot \beta \implies \alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta$ (converse of ($\wedge$6) below).

Moreover, the following laws ($\wedge$4) and ($\wedge$5) are equivalent and ($\wedge$6) implies them. If $\preccurlyeq$ satisfies the strong cut law, ($\wedge$6) is equivalent to ($\wedge$4) and to ($\wedge$5).

[2.20] See Example 3.1.8 and Remark 3.1.21 for the concepts of occurrences and substitutions.

($\wedge$4) $xy \preccurlyeq x \wedge y$ (third part of conjunction).

($\wedge$5) $x \wedge y \cdot \alpha \preccurlyeq \beta \implies xy\alpha \preccurlyeq \beta$ (converse of ($\wedge$2)).

($\wedge$6) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \wedge y \cdot \beta$ (second half of strong conjunction).

Consequently, the strong conjunction law implies the conjunction law, and they are equivalent under the strong cut law.

Furthermore let $x \vee y$ be a binary operation on A. Then the following three laws are equivalent.

($\vee$1) $x \preccurlyeq x \vee y,\ y \preccurlyeq x \vee y$ (first two parts of disjunction).

($\vee$2) $\alpha \preccurlyeq xy\beta \implies \alpha \preccurlyeq x \vee y \cdot \beta$ (first half of strong disjunction).

($\vee$3) $x \vee y \cdot \alpha \preccurlyeq \beta \implies x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta$ (converse of ($\vee$6) below).

Moreover, the following laws ($\vee$4) and ($\vee$5) are equivalent and ($\vee$6) implies them. If $\preccurlyeq$ satisfies the strong cut law, ($\vee$6) is equivalent to ($\vee$4) and to ($\vee$5).

($\vee$4) $x \vee y \preccurlyeq xy$ (third part of disjunction).

($\vee$5) $\alpha \preccurlyeq x \vee y \cdot \beta \implies \alpha \preccurlyeq xy\beta$ (converse of ($\vee$2)).

($\vee$6) $x\alpha \preccurlyeq \beta,\ y\alpha \preccurlyeq \beta \implies x \vee y \cdot \alpha \preccurlyeq \beta$ (second half of strong disjunction).

Consequently, the strong disjunction law implies the disjunction law, and they are equivalent under the strong cut law.

Proof ($\wedge$2) is derived from ($\wedge$1) by the laws of substitution and contraction. We have $xy \preccurlyeq x$ and $xy \preccurlyeq y$ by the intersection law, and so ($\wedge$2) with $\alpha = \varepsilon$ and $\beta = x$ or y implies ($\wedge$1). ($\wedge$3) is derived from ($\wedge$1) by the cut law. We have $x \wedge y \preccurlyeq x \wedge y$ by the repetition law, and so ($\wedge$3) with $\alpha = x \wedge y$ and $\beta = \varepsilon$ implies ($\wedge$1). ($\wedge$5) is derived from ($\wedge$4) by the cut law. ($\wedge$5) with $\alpha = \varepsilon$ and $\beta = x \wedge y$ implies ($\wedge$4). ($\wedge$6) is derived from ($\wedge$4) by the laws of strong substitution and extension. ($\wedge$6) with $\alpha = xy$ and $\beta = \varepsilon$ implies ($\wedge$4). Thus the former half holds. The latter may be derived from the former by the duality between $\preccurlyeq$ and $\succcurlyeq$ noted in Remark 2.2.9.

Remark 2.2.10 Let $\sqsubseteq$ be a relation on a set S and $\equiv$ be the intersection of $\sqsubseteq$ and its dual $\sqsupseteq$, that is, $x \equiv y$ iff $x \sqsubseteq y$ and $x \sqsupseteq y$. Then $\equiv$ is the largest of the symmetric relations on S which are contained in $\sqsubseteq$, and therefore we call $\equiv$ the **symmetric core** of $\sqsubseteq$ (and of $\sqsupseteq$). If $\sqsubseteq$ is a preorder, then $\equiv$ is an equivalence relation. Therefore, if $\preccurlyeq$ is a latticed relation on A^* and $\asymp$ is its symmetric core as in Corollary 2.2.15.1, then since the restriction of $\preccurlyeq$ to $A \times A$ is a preorder by Theorem 2.2.10, that of $\asymp$ is an equivalence relation.

Corollary 2.2.15.1 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the strong junction laws with respect to binary operations $x \wedge y$ and $x \vee y$ on A. Then $\preccurlyeq$ and its symmetric core $\asymp$ satisfy the following laws.

(1) $\begin{cases} \alpha xy\beta \preccurlyeq \gamma \iff \alpha \cdot x \wedge y \cdot \beta \preccurlyeq \gamma, \\ \gamma \preccurlyeq \alpha x\beta,\ \gamma \preccurlyeq \alpha y\beta \iff \gamma \preccurlyeq \alpha \cdot x \wedge y \cdot \beta. \end{cases}$

(2) $\begin{cases} \gamma \preccurlyeq \alpha xy\beta \iff \gamma \preccurlyeq \alpha \cdot x \vee y \cdot \beta, \\ \alpha x\beta \preccurlyeq \gamma,\ \alpha y\beta \preccurlyeq \gamma \iff \alpha \cdot x \vee y \cdot \beta \preccurlyeq \gamma. \end{cases}$

(3) $\left.\begin{array}{l} x_1 \wedge \cdots \wedge x_n \asymp (\cdots(x_1 \wedge x_2) \wedge \cdots) \wedge x_n, \\ x_1 \vee \cdots \vee x_n \asymp (\cdots(x_1 \vee x_2) \vee \cdots) \vee x_n \end{array}\right\}$ irrespective of the order of applying the operations $\wedge$ and $\vee$ on the left-hand side of $\asymp$.

(4) $\left.\begin{array}{l} x_1 \preccurlyeq y_1, \\ x_2 \preccurlyeq y_2 \end{array}\right\} \implies \begin{cases} x_1 \wedge x_2 \preccurlyeq y_1 \wedge y_2, \\ x_1 \vee x_2 \preccurlyeq y_1 \vee y_2. \end{cases}$

Proof In view of the duality between $\preccurlyeq$ and $\succcurlyeq$, we only consider the laws for $\wedge$. All the laws in Theorem 2.2.15 hold, and (1) is a consequence of ($\wedge$2), ($\wedge$3), ($\wedge$5) and ($\wedge$6). As for (3), since $x_1 \cdots x_n \preccurlyeq x_i$ $(i = 1, \dots, n)$ by the intersection law, we have $x_1 \cdots x_n \preccurlyeq x_1 \wedge \cdots \wedge x_n$ and hence $(\cdots(x_1 \wedge x_2) \wedge \cdots) \wedge x_n \preccurlyeq x_1 \wedge \cdots \wedge x_n$ both by (1). Similarly $x_1 \wedge \cdots \wedge x_n \preccurlyeq (\cdots(x_1 \wedge x_2) \wedge \cdots) \wedge x_n$. Thus $x_1 \wedge \cdots \wedge x_n \asymp (\cdots(x_1 \wedge x_2) \wedge \cdots) \wedge x_n$. As for (4), since $x_1 \preccurlyeq y_1$ and $x_2 \preccurlyeq y_2$, we have $x_1 x_2 \preccurlyeq y_1$ and $x_1 x_2 \preccurlyeq y_2$ by the extension law, hence $x_1 x_2 \preccurlyeq y_1 \wedge y_2$ and hence $x_1 \wedge x_2 \preccurlyeq y_1 \wedge y_2$ both by (1).

Theorem 2.2.16 Let $\preccurlyeq$ be a latticed relation on A^* and $x^\Diamond$ be a unary operation on A. Then any of the following laws ($\Diamond$2), ($\Diamond$3) and ($\Diamond$4) implies the law ($\Diamond$1). If $\preccurlyeq$ satisfies the strong cut law, then ($\Diamond$1) implies ($\Diamond$2), ($\Diamond$3) and ($\Diamond$4).

($\Diamond$1) $xx^\Diamond \preccurlyeq \varepsilon$ (lower negation).

($\Diamond$2) $\alpha \preccurlyeq x\beta,\ \alpha \preccurlyeq x^\Diamond\beta \implies \alpha \preccurlyeq \beta$.

($\Diamond$3) $\alpha \preccurlyeq x\beta \implies x^\Diamond\alpha \preccurlyeq \beta$ (first half of strong negation).

($\Diamond$4) $\alpha \preccurlyeq x^\Diamond\beta \implies x\alpha \preccurlyeq \beta$ (converse of ($\Diamond$7) below).

Consequently, the first half of the strong negation law implies the lower negation law, and they are equivalent under the strong cut law.

Moreover, any of the following laws ($\Diamond$6), ($\Diamond$7) and ($\Diamond$8) implies the law ($\Diamond$5). If $\preccurlyeq$ satisfies the strong cut law, then ($\Diamond$5) implies ($\Diamond$6), ($\Diamond$7) and ($\Diamond$8).

($\Diamond$5) $\varepsilon \preccurlyeq xx^\Diamond$ (upper negation).

($\Diamond$6) $x\alpha \preccurlyeq \beta,\ x^\Diamond\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq \beta$.

($\Diamond$7) $x\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x^\Diamond\beta$ (second half of strong negation).

($\Diamond$8) $x^\Diamond\alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta$ (converse of ($\Diamond$3)).

Consequently, the second half of the strong negation law implies the upper negation law, and they are equivalent under the strong cut law.

Proof All of ($\Diamond$2), ($\Diamond$3) and ($\Diamond$4) are derived from ($\Diamond$1) by the laws of strong substitution and extension. We have $xx^\Diamond \preccurlyeq x$ and $xx^\Diamond \preccurlyeq x^\Diamond$ by the intersection law, and so ($\Diamond$2) with $\alpha = xx^\Diamond$ and $\beta = \varepsilon$ implies ($\Diamond$1). Similarly, ($\Diamond$3) with $\alpha = x$ and $\beta = \varepsilon$ together with the exchange law implies ($\Diamond$1), and ($\Diamond$4) with $\alpha = x^\Diamond$ and $\beta = \varepsilon$ implies ($\Diamond$1). Thus the former half holds. The latter may be derived from the former by the duality between $\preccurlyeq$ and $\succcurlyeq$.

Theorem 2.2.17 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the laws of strong cut and negation with respect to a unary operation $x^\Diamond$ on A. Let $x \Rightarrow y$ be a binary operation on A. Then the following three laws are equivalent.

($\Rightarrow$1) $x^\Diamond \preccurlyeq x \Rightarrow y,\ \ y \preccurlyeq x \Rightarrow y$ (contradictory and reflexive implication).

($\Rightarrow$2) $x\alpha \preccurlyeq y\beta \implies \alpha \preccurlyeq x \Rightarrow y \cdot \beta$ (first half of strong implication).

($\Rightarrow$3) $x \Rightarrow y \cdot \alpha \preccurlyeq \beta \implies \alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta$ (converse of ($\Rightarrow$6) below).

Moreover, the following four laws are equivalent.

($\Rightarrow$4) $x \Rightarrow y \preccurlyeq x^\Diamond y$ (negative implication).

($\Rightarrow$5) $\alpha \preccurlyeq x \Rightarrow y \cdot \beta \implies x\alpha \preccurlyeq y\beta$ (converse of ($\Rightarrow$2)).

($\Rightarrow$6) $\alpha \preccurlyeq x\beta,\ y\alpha \preccurlyeq \beta \implies x \Rightarrow y \cdot \alpha \preccurlyeq \beta$ (second half of strong implication).

($\Rightarrow$7) $x \cdot x \Rightarrow y \preccurlyeq y$ (cut implication).

Consequently, a latticed relation satisfying the laws of strong cut and negation satisfies the strong implication law iff it satisfies the three laws in Theorem 2.2.8.

Proof As was noted in Remark 2.2.5, $\preccurlyeq$ satisfies the cut law. Moreover, $\preccurlyeq$ satisfies the laws ($\Diamond$1)–($\Diamond$8) in Theorem 2.2.16.

Assume ($\Rightarrow$1) and $x\alpha \preccurlyeq y\beta$. Then applying the cut law to $x\alpha \preccurlyeq y\beta$ and $y \preccurlyeq x \Rightarrow y$, we have $x\alpha \preccurlyeq x \Rightarrow y \cdot \beta$. Moreover, applying the extension law to $x^\Diamond \preccurlyeq x \Rightarrow y$, we have $x^\Diamond\alpha \preccurlyeq x \Rightarrow y \cdot \beta$. Therefore $\alpha \preccurlyeq x \Rightarrow y \cdot \beta$ by ($\Diamond$6). Thus ($\Rightarrow$1) implies ($\Rightarrow$2). We have $xx^\Diamond \preccurlyeq y$ by the laws of lower negation and weakening, and so ($\Rightarrow$2) with $\alpha = x^\Diamond$ and $\beta = \varepsilon$ implies $x^\Diamond \preccurlyeq x \Rightarrow y$. Moreover, we have $xy \preccurlyeq y$ by the intersection law, and so ($\Rightarrow$2) with $\alpha = y$ and $\beta = \varepsilon$ implies $y \preccurlyeq x \Rightarrow y$. Thus ($\Rightarrow$2) implies ($\Rightarrow$1).

Assume ($\Rightarrow$1) and $x \Rightarrow y \cdot \alpha \preccurlyeq \beta$. Then applying the cut law to $x \Rightarrow y \cdot \alpha \preccurlyeq \beta$ and $x^\Diamond \preccurlyeq x \Rightarrow y$, we have $x^\Diamond\alpha \preccurlyeq \beta$, and so $\alpha \preccurlyeq x\beta$ by ($\Diamond$8). Moreover, applying the cut law to $x \Rightarrow y \cdot \alpha \preccurlyeq \beta$ and $y \preccurlyeq x \Rightarrow y$, we have $y\alpha \preccurlyeq \beta$. Thus ($\Rightarrow$1) implies ($\Rightarrow$3). We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so ($\Rightarrow$3) with $\alpha = \varepsilon$ and $\beta = x \Rightarrow y$ implies $\varepsilon \preccurlyeq x \cdot x \Rightarrow y$ and $y \preccurlyeq x \Rightarrow y$. Applying ($\Diamond$3) to $\varepsilon \preccurlyeq x \cdot x \Rightarrow y$, we have $x^\Diamond \preccurlyeq x \Rightarrow y$. Thus ($\Rightarrow$3) implies ($\Rightarrow$1).

($\Diamond$4) and ($\Diamond$7) show that ($\Rightarrow$4) and ($\Rightarrow$7) are equivalent.

Assume ($\Rightarrow$7) and $\alpha \preccurlyeq x \Rightarrow y \cdot \beta$. Then applying the strong substitution law, we have $x\alpha \preccurlyeq y\beta$. Thus ($\Rightarrow$7) implies ($\Rightarrow$5). We have $x \Rightarrow y \preccurlyeq x \Rightarrow y$ by the repetition law, and so ($\Rightarrow$5) with $\alpha = x \Rightarrow y$ and $\beta = \varepsilon$ implies ($\Rightarrow$7).

Assume ($\Rrightarrow$7), $\alpha \preccurlyeq x\beta$ and $y\alpha \preccurlyeq \beta$. Then applying the laws of strong substitution and extension, we have $x \Rrightarrow y \cdot \alpha \preccurlyeq \beta$. Thus ($\Rrightarrow$7) implies ($\Rrightarrow$6). We have $x \preccurlyeq xy$ and $yx \preccurlyeq y$ by the intersection law, and so ($\Rrightarrow$6) with $\alpha = x$ and $\beta = y$ implies $x \Rrightarrow y \cdot x \preccurlyeq y$, hence $x \cdot x \Rrightarrow y \preccurlyeq y$ by the exchange law. Thus ($\Rrightarrow$6) implies ($\Rrightarrow$7).

Theorem 2.2.18 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the strong implication law with respect to a binary operation $x \Rrightarrow y$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the extension law, and so $x \Rrightarrow x \cdot \alpha\beta \preccurlyeq \delta\gamma$ by the strong implication law. Moreover, $\varepsilon \preccurlyeq x \Rrightarrow x$ by the laws of repetition and strong implication. Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law.

Lemma 2.2.4 Let $\preccurlyeq$ be a latticed relation on A^* and $x^\Diamond$ be a unary operation on A. Then the following hold on the negation laws on $\preccurlyeq$ with respect to $\Diamond$.

- Assume that $\preccurlyeq$ satisfies the conjunction law with respect to a binary operation $x \wedge y$ on A. Then the lower negation law is equivalent to both of the laws $x \wedge x^\Diamond \preccurlyeq \varepsilon$ and $x^\Diamond \wedge x \preccurlyeq \varepsilon$, and implies the lower end law.
- Assume that $\preccurlyeq$ satisfies the disjunction law with respect to a binary operation $x \vee y$ on A. Then the upper negation law is equivalent to both of the laws $x \vee x^\Diamond \succcurlyeq \varepsilon$ and $x^\Diamond \vee x \succcurlyeq \varepsilon$, and implies the upper end law.

Proof This follows from the laws ($\wedge$2), ($\wedge$5), ($\vee$2), ($\vee$5) in Theorem 2.2.15 and the exchange law, and the assumption $A \neq \emptyset$ throughout this section.

Theorem 2.2.19 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the strong negation law with respect to a unary operation $x^\Diamond$ on A and either of the strong junction laws with respect to binary operations $x \wedge y$ and $x \vee y$ on A. Then $\preccurlyeq$ satisfies the strong cut law.

Proof In view of the duality between $\preccurlyeq$ and $\succcurlyeq$, we only consider the case that $\preccurlyeq$ satisfies the strong conjunction law. Assume $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$. Then $\alpha\beta \preccurlyeq x\delta\gamma$ and $x\alpha\beta \preccurlyeq \delta\gamma$ by the extension law, and applying the strong negation law to $x\alpha\beta \preccurlyeq \delta\gamma$, we have $\alpha\beta \preccurlyeq x^\Diamond\delta\gamma$, and so $\alpha\beta \preccurlyeq x \wedge x^\Diamond \cdot \delta\gamma$ by the strong conjunction law. Moreover, $x \wedge x^\Diamond \preccurlyeq \varepsilon$ by Theorems 2.2.15, 2.2.16 and Lemma 2.2.4 (or by the laws of repetition, strong negation, exchange and strong conjunction). Therefore $\alpha\beta \preccurlyeq \delta\gamma$ by the cut law. Thus $\preccurlyeq$ satisfies the strong cut law.

Lemma 2.2.5 Let $\preccurlyeq$ be a latticed relation on A^*. Then the following hold on the relationship between the end laws and the quasi-end laws.

- Assume that an element $x \in A$ satisfies $x \preccurlyeq \varepsilon$. Then an element $(\alpha, \beta) \in A^* \times A^*$ satisfies $\alpha \preccurlyeq \beta$ iff $\alpha \preccurlyeq x\beta$ and iff $\alpha \preccurlyeq y\beta$ for all $y \in A$. In particular, $\preccurlyeq$ satisfies the lower quasi-end law.
- Assume that an element $x \in A$ satisfies $x \succcurlyeq \varepsilon$. Then an element $(\alpha, \beta) \in A^* \times A^*$ satisfies $\alpha \succcurlyeq \beta$ iff $\alpha \succcurlyeq x\beta$ and iff $\alpha \succcurlyeq y\beta$ for all $y \in A$. In particular, $\preccurlyeq$ satisfies the upper quasi-end law.

Proof In view of the duality between $\preccurlyeq$ and $\succcurlyeq$, we only consider the former statement. If $\alpha \preccurlyeq \beta$, then $\alpha \preccurlyeq y\beta$ for all $y \in A$ by the weakening law. If in particular $\alpha \preccurlyeq x\beta$, then $\alpha \preccurlyeq \beta$ by the cut law applied to $\alpha \preccurlyeq x\beta$ and $x \preccurlyeq \varepsilon$.

Lemma 2.2.6 Let $(A, \mathbb{B}, f)$ be a latticed representation satisfying

$$\inf f\alpha = \sup(fA \cap (\leftarrow \inf f\alpha]), \qquad \sup f\alpha = \inf(fA \cap [\sup f\alpha \rightarrow)) \tag{2.2.2}$$

for all $\alpha \in A^*$. Then $\preccurlyeq_f$ is the largest latticed extension of a preorder on A.

Remark 2.2.11 (2.2.2) for $\alpha = \varepsilon$ means $1 = \sup fA$ and $0 = \inf fA$ (s. Theorem 2.2.4). (2.2.2) for $\alpha \neq \varepsilon$ holds if fA is a sublattice of $\mathbb{B}$ (s. Theorem 2.2.5). (2.2.2) is also related to the completion of ordered sets (s. Remark 2.2.12).

Proof Let $(x_1 \cdots x_m, y_1 \cdots y_n) \in A^* \times A^*$ and define $a = \inf\{fx_1, \ldots, fx_m\}$ and $b = \sup\{fy_1, \ldots, fy_n\}$. Then $a = \sup(fA \cap (\leftarrow a])$ and $b = \inf(fA \cap [b \rightarrow))$ by (2.2.2). Therefore

$$\begin{aligned}
&x_1 \cdots x_m \preccurlyeq_f y_1 \cdots y_n \\
&\iff \sup(fA \cap (\leftarrow a]) \leq \inf(fA \cap [b \rightarrow)) \\
&\iff \text{if elements } c, d \in fA \text{ satisfy } c \leq a \text{ and } b \leq d, \text{ then } c \leq d \\
&\iff \text{if elements } x, y \in A \text{ satisfy } fx \leq a \text{ and } b \leq fy, \text{ then } fx \leq fy.
\end{aligned}$$

This completes the proof because the following hold:

$$\begin{aligned}
&fx \leq a \iff fx \leq fx_i \ (i = 1, \ldots, m) \iff x \preccurlyeq_f x_i \ (i = 1, \ldots, m), \\
&b \leq fy \iff fy_j \leq fy \ (j = 1, \ldots, n) \iff y_j \preccurlyeq_f y \ (j = 1, \ldots, n), \\
&fx \leq fy \iff x \preccurlyeq_f y.
\end{aligned}$$

Remark 2.2.12 Let A be an ordered set. Then A can be extended to a complete lattice B such that $x = \sup_B(A \cap (\leftarrow x]) = \inf_B(A \cap [x \rightarrow))$ for all $x \in B$. Such a complete lattice is unique up to order isomorphism extending id_A and is called the **completion** of A. If A is a lattice, then it is a sublattice of B.

Lemma 2.2.7 Let $\preccurlyeq$ be a latticed relation on A^* satisfying the laws of quasi-junction and quasi-end. Then there exists a latticed representation $(A, \mathbb{B}, f)$ such that $\preccurlyeq_f$ is equal to $\preccurlyeq$.

Proof Let $\sqsubseteq$ be the restriction of $\preccurlyeq$ to $A \times A$. Then $\sqsubseteq$ is a preorder by Theorem 2.2.10, and its symmetric core $\equiv$ is an equivalence relation by Remark 2.2.10. Let $\mathbb{A}$ be the quotient set $A/\equiv$ and $f \in A \to \mathbb{A}$ be the canonical mapping. Then $fA = \mathbb{A}$, and we can define an order $\leq$ on $\mathbb{A}$ so that the elements $(x, y) \in A \times A$ satisfy $fx \leq fy$ iff $x \sqsubseteq y$, that is, iff $x \preccurlyeq y$.

Let $(x, y) \in A \times A$. Then by the quasi-conjunction law, there exists an element $\underline{z} \in A$ such that $\underline{z} \preccurlyeq x$, $\underline{z} \preccurlyeq y$ and $xy \preccurlyeq \underline{z}$, hence $f\underline{z} \leq fx$ and $f\underline{z} \leq fy$. Conversely, if an element $z' \in A$ satisfies $fz' \leq fx$ and $fz' \leq fy$, then $z' \preccurlyeq x$ and $z' \preccurlyeq y$, and applying the laws of substitution and extension to these and $xy \preccurlyeq \underline{z}$, we have $z' \preccurlyeq \underline{z}$, hence $fz' \leq f\underline{z}$. Therefore $f\underline{z} = \inf\{fx, fy\}$. Similarly, there exists an element $\overline{z} \in A$ such that $\overline{z} \succcurlyeq x$, $\overline{z} \succcurlyeq y$ and $xy \succcurlyeq \overline{z}$ by the quasi-disjunction law, and it satisfies $f\overline{z} = \sup\{fx, fy\}$. Thus $\mathbb{A}$ is a lattice.

Let $\mathbb{C}$ be the completion of $\mathbb{A}$ and define $\mathbb{B} = \mathbb{A} \cup \{\min \mathbb{C}, \max \mathbb{C}\}$. Then $\min \mathbb{C} = \min \mathbb{B}$ and $\max \mathbb{C} = \max \mathbb{B}$. Moreover, since $\mathbb{A}$ is a sublattice of $\mathbb{C}$ by Remark 2.2.12, so also is $\mathbb{B}$. Therefore $(A, \mathbb{B}, f)$ is a latticed representation. Remark 2.2.12 also shows that if $b \in \mathbb{B}$ then $b = \sup_{\mathbb{C}}(\mathbb{A} \cap (\leftarrow b]) = \sup_{\mathbb{B}}(\mathbb{A} \cap (\leftarrow b])$ and likewise $b = \inf_{\mathbb{B}}(\mathbb{A} \cap [b \to))$. Therefore $(A, \mathbb{B}, f)$ satisfies (2.2.2) for all $\alpha \in A^*$, and so Lemma 2.2.6 shows that $\preccurlyeq_f$ is equal to the largest latticed extension of the preorder $\leq_f$ obtained by restricting $\preccurlyeq_f$ to $A \times A$. Furthermore, Theorem 2.2.14 shows that $\preccurlyeq$ is the largest latticed extension of $\sqsubseteq$. Since $\leq_f$ is equal to $\sqsubseteq$ by the definition of $\leq$, we conclude that $\preccurlyeq$ is equal to $\preccurlyeq_f$.

Lemma 2.2.8 Let $(A, \mathbb{B}, f)$ be a latticed representation and assume that $\preccurlyeq_f$ either satisfies the strong conjunction law with respect to a binary operation $x \wedge y$ on A and the quasi-disjunction law or satisfies the strong disjunction law with respect to a binary operation $x \vee y$ on A and the quasi-conjunction law. Then (fA is a distributive sublattice of $\mathbb{B}$ and) $\preccurlyeq_f$ satisfies the strong cut law.

Proof In view of the duality between $\preccurlyeq_f$ and $\succcurlyeq_f$ noted in Remark 2.2.3, we only consider the case that $\preccurlyeq_f$ satisfies the strong conjunction law and the quasi-disjunction law. Then, since $\preccurlyeq_f$ is latticed by Theorems 2.2.1 and 2.2.2, Theorem 2.2.15 shows that $\preccurlyeq_f$ satisfies the conjunction law. Therefore the proof of Theorem 2.2.6 shows that $f(x \wedge y) = fx \wedge fy$ for all $(x, y) \in A \times A$. Moreover, Theorem 2.2.5 and Remark 2.2.7 show that fA is a sublattice of $\mathbb{B}$.

Let $a, b, c, d \in fA$, and assume $d \leq a \vee c$ and $d \leq b \vee c$. Then there exist elements $x, y, z, w \in A$ such that $a = fx$, $b = fy$, $c = fz$ and $d = fw$, and they satisfy $w \preccurlyeq_f xz$ and $w \preccurlyeq_f yz$. Therefore $w \preccurlyeq_f x \wedge y \cdot z$ by the strong conjunction law, and so $fw \leq f(x \wedge y) \vee fz = (fx \wedge fy) \vee fz$. Therefore $d \leq (a \wedge b) \vee c$. Thus fA is a distributive sublattice by [2.17], and so $\preccurlyeq_f$ satisfies the strong cut law by Theorem 2.2.2.

Theorem 2.2.20 Let $\preccurlyeq$ be a latticed relation on A^*. Assume that $\preccurlyeq$ satisfies the quasi-end laws and that $\preccurlyeq$ either satisfies the strong conjunction law with respect to a binary operation $x \wedge y$ on A and the quasi-disjunction law or satisfies the strong disjunction law with respect to a binary operation $x \vee y$ on A and the quasi-conjunction law. Then $\preccurlyeq$ satisfies the strong cut law.

Proof Theorem 2.2.15 and Remark 2.2.7 show that $\preccurlyeq$ satisfies the quasi-junction laws. Therefore Lemma 2.2.7 shows that there exists a latticed representation $(A, \mathbb{B}, f)$ such that $\preccurlyeq_f$ is equal to $\preccurlyeq$. Thus $\preccurlyeq$ satisfies the strong cut law by Lemma 2.2.8.

Theorem 2.2.21 Let $\preccurlyeq$ be a relation on A^* and $x \wedge y$, $x \vee y$, $x^\Diamond$ and $x \Rrightarrow y$ be operations on A. Then $\preccurlyeq$ is Boolean with respect to the operations iff it is weakly Boolean with respect to the operations and satisfies the cut law.

Remark 2.2.13 It follows from Remark 2.7.7 that, with respect to any operations $x \wedge y$, $x \vee y$, $x^\Diamond$ and $x \Rrightarrow y$ on A, there exists a weakly Boolean relation on A^* which does not satisfy the cut law.

Proof If $\preccurlyeq$ is Boolean, then it is (strongly) latticed and particularly satisfies the (strong) cut law by Remark 2.2.6, and therefore is weakly Boolean by Theorems 2.2.15–2.2.17. If $\preccurlyeq$ is weakly Boolean and satisfies the cut law, then it is latticed, and therefore is Boolean by the three theorems and any of Theorems 2.2.18–2.2.20[2.21].

Theorem 2.2.22 Let $\preccurlyeq$ be a Boolean relation on A^* with respect to operations $x \wedge y$, $x \vee y$, $x^\Diamond$ and $x \Rrightarrow y$ on A. Then $\preccurlyeq$ is the largest latticed extension both of a partially latticed association on A and of a preorder on A.

Proof As we have seen in the proof of Theorem 2.2.21, $\preccurlyeq$ is latticed, and satisfies the laws of junction and negation, hence the laws of quasi-junction and end by Remark 2.2.7 and Lemma 2.2.4, and hence the quasi-end laws by Lemma 2.2.5. Thus the statement holds by Theorem 2.2.14.

2.2.4 Hidden latticed representations

The purpose of this subsection is to show that certain associations on A are the restrictions of the validity relations of latticed representations of A and therefore partially latticed. Theorem 2.2.24 has a more elementary proof than is given, but it is irrelevant to latticed representations and not interesting.

Theorem 2.2.23 Let X be a subset of A, and define a relation $\preccurlyeq_X$ on A^* by

$$\alpha \preccurlyeq_X \beta \iff \alpha \nsubseteq X \text{ or } \beta \nsubseteq A - X$$

for each $(\alpha, \beta) \in A^* \times A^*$ by the word convention. Then $\preccurlyeq_X$ is strongly latticed. Moreover, if we define an association $\vDash_X$ on A by

$$\alpha \vDash_X y \iff \alpha \nsubseteq X \text{ or } y \in X$$

for each $(\alpha, y) \in A^* \times A$, then $\vDash_X$ is partially latticed.

[2.21] Theorem 2.2.20 is not recommendable here, though, because before using it, you need to derive the laws of quasi-end and quasi-junction as in the proof of Theorem 2.2.22.

Proof Let $1_X \in A \rightarrow \mathbb{T}$ be the characteristic function[2.22] of X. Then $\preccurlyeq_X$ is equal to the 1_X-validity relation $\preccurlyeq_{1_X}$, and therefore is strongly latticed by Theorems 2.2.1 and 2.2.2 because $\mathbb{T}$ is distributive. Being the restriction of $\preccurlyeq_X$, $\vDash_X$ is partially latticed by Theorem 2.2.10.

Remark 2.2.14 The relation $\preccurlyeq_X$ is equal to the relation $\preccurlyeq_{X,A-X}$ in Remark 2.7.7. Moreover, if $X \neq A$, $\preccurlyeq_X$ and $\vDash_X$ are equal to $\preccurlyeq_\varphi$ and $\vDash_\varphi$ in Theorem 2.2.24 for the closure operator φ on $\mathfrak{P}A$ with the fixture domain $\{X, A\}$.

Theorem 2.2.24 Let φ be a closure operator on $\mathfrak{P}A$, and define a relation $\preccurlyeq_\varphi$ on A^* by

$$\alpha \preccurlyeq_\varphi \beta \iff \varphi\alpha \supseteq \bigcap_{y\in\beta} \varphi\{y\}$$

for each $(\alpha, \beta) \in A^* \times A^*$ by the word convention. Then $\preccurlyeq_\varphi$ is latticed. Moreover, if we define an association $\vDash_\varphi$ on A by

$$\alpha \vDash_\varphi y \iff \varphi\alpha \ni y$$

for each $(\alpha, y) \in A^* \times A$ (s. (2.11.2)), then $\vDash_\varphi$ is partially latticed and $\preccurlyeq_\varphi$ is its largest latticed extension.

Proof Let $\mathfrak{B}$ be the fixture domain of φ. Then Theorem 2.1.10 and Remark 2.2.3 show that the dual ordered set $(\mathfrak{B}, \supseteq)$ is a lattice with $\min \mathfrak{B} = A$ and $\max \mathfrak{B} = \varphi\emptyset$. Define the latticed representation $f \in A \rightarrow \mathfrak{B}$ by $fx = \varphi\{x\}$ for each $x \in A$. Then $\preccurlyeq_f$ is latticed by Theorems 2.2.1 and 2.2.2.

Let $(\alpha, \beta) \in A^* \times A^*$ and define $\mathfrak{X} = \{\varphi\{x\} : x \in \alpha\}$, $\mathfrak{Y} = \{\{x\} : x \in \alpha\}$ and $\mathfrak{Z} = \{\varphi\{y\} : y \in \beta\}$. Then $\mathfrak{X} \subseteq \mathfrak{B}$, $\mathfrak{Z} \subseteq \mathfrak{B}$, $\mathfrak{Y} \subseteq \mathfrak{P}A$ and the following hold:

$$\begin{aligned} f\alpha &= \{fx : x \in \alpha\} = \{\varphi\{x\} : x \in \alpha\} = \mathfrak{X} = \{\varphi\eta : \eta \in \mathfrak{Y}\} = \varphi\mathfrak{Y}, \\ f\beta &= \{fy : y \in \beta\} = \{\varphi\{y\} : y \in \beta\} = \mathfrak{Z}. \end{aligned}$$

Therefore Theorem 2.1.10 shows that the following hold:

$$\begin{aligned} &\inf_{\mathfrak{B}} f\alpha = \inf_{\mathfrak{B}} \mathfrak{X} = \varphi\left(\sup_{\mathfrak{P}A} \mathfrak{X}\right) = \varphi\left(\sup_{\mathfrak{P}A} \varphi\mathfrak{Y}\right) = \varphi\left(\sup_{\mathfrak{P}A} \mathfrak{Y}\right) = \varphi\alpha, \\ &\sup_{\mathfrak{B}} f\beta = \sup_{\mathfrak{B}} \mathfrak{Z} = \inf_{\mathfrak{P}A} \mathfrak{Z} = \textstyle\bigcap_{y\in\beta} \varphi\{y\}. \end{aligned}$$

Thus $\preccurlyeq_\varphi$ is equal to $\preccurlyeq_f$ and therefore is latticed.

Since φ is a closure operator, an element $(\alpha, y) \in A^* \times A$ satisfies $\varphi\alpha \supseteq \varphi\{y\}$ iff $\varphi\alpha \ni y$. Therefore $\vDash_\varphi$ is the restriction of $\preccurlyeq_\varphi$, and therefore is partially latticed by Theorem 2.2.10. By definition, an element $(\alpha, y_1 \cdots y_n) \in A^* \times A^*$ satisfies $\varphi\alpha \supseteq \varphi\{y_1\} \cap \cdots \cap \varphi\{y_n\}$ iff the following holds for all $z \in A$:

$$\varphi\{y_i\} \ni z \ (i = 1, \ldots, n) \implies \varphi\alpha \ni z.$$

Thus $\preccurlyeq_\varphi$ is the largest latticed extension of $\vDash_\varphi$.

Remark 2.2.15 The lattice $(\mathfrak{B}, \supseteq)$ in the above proof is not necessarily a sublattice of $(\mathfrak{P}A, \supseteq)$, and so not necessarily distributive. Therefore $\preccurlyeq_\varphi$ is not necessarily strongly latticed.

[2.22] The **characteristic function** $1_X \in A \rightarrow \mathbb{T}$ of X is defined so that $1_X y = 1$ iff $y \in X$.

2.3 Trees as elements of a free monoid

Here we define the concept of trees. It is indispensable to the definitions of R-deductions in §2.4 and algebraic deductions in §3.1.3 and to the proof of the existence of USAs in Theorem 3.1.5.

Let G and N be sets and $M = (G \amalg N \amalg \{[,]\})^*$ be the free monoid over the direct union $G \amalg N \amalg \{[,]\}$ of G, N and the set $\{[,]\}$ of the left bracket [and the right bracket] (s. Remark 3.1.14). Then we inductively define subsets T_n $(n = 0, 1, \ldots)$ of M by $T_0 = G$ and the following for each $n \geq 1$:

$$T_n = \left\{ [\nu t_1 \cdots t_k] : \begin{array}{l} t_j \in T_{n_j} \ (j = 1, \ldots, k) \ \textit{for some nonnegative} \\ \textit{integers } n_1, \ldots, n_k \textit{ such that } n - 1 = \sum_{j=1}^{k} n_j, \\ \text{and } \nu \in N \end{array} \right\}. \quad (2.3.1)$$

Here the italicized condition on $(t_1, \ldots, t_k)$ is called the **ramification condition** and denoted $\mathcal{R}(T_0, \ldots, T_{n-1})$. It makes sense because of the induction, that is, when we define T_n by (2.3.1), we have already defined $T_0, \ldots, T_{n-1}$. We refer to the elements of G, N, $\bigcup_{n\geq 0} T_n$ and $\bigcup_{n \geq 1} T_n$ as **germ**s, **node**s, **tree**s and **ramified tree**s respectively. We furthermore define the **germ** $G(t)$ of each tree t as the set of the germs which occur in t (s. Example 3.1.8). Then $G(t) = \{t\}$ if $t \in T_0$, while $G(t) = \bigcup_{j=1}^{k} G(t_j)$ if $t = [\nu t_1 \cdots t_k] \in \bigcup_{n\geq 1} T_n$ as in (2.3.1). It will play a role in §2.4 and §3.1.3 by the name of **premise** of t.

Remark 2.3.1 If $k = 0$ in the definition of T_n $(n \geq 1)$, then $[\nu t_1 \cdots t_k] = [\nu]$ and $\sum_{j=1}^{k} n_j = 0$. The definition also implies that if $\nu \in N$ and $t_j \in T_{n_j}$ $(j = 1, \ldots, k)$ then $[\nu t_1 \cdots t_k] \in T_n$ for $n = 1 + \sum_{j=1}^{k} n_j$.

There are variations in the definition of trees. You may delete either or both of the brackets from the above definition. If you delete the right bracket, however, the latter half of Theorem 2.3.1 below does not hold.

Possibly you wish to define ramified trees as diagrams $\dfrac{t_1, \ldots, t_k}{\nu}$ so that they look like actual trees. The definition, however, is mathematically inadequate unless you define the set to which the diagrams belong and afterward define the operations $(t_1, \ldots, t_k) \mapsto \dfrac{t_1, \ldots, t_k}{\nu}$ in the set.[2.23]

The purpose of this section is to prove and illustrate the following.

Theorem 2.3.1 Each tree t has a unique nonnegative integer n such that $t \in T_n$ and if $n \geq 1$, that is, if t is ramified, it has a unique expression $t = [\nu t_1 \cdots t_k]$ by a node ν and trees $t_1, \ldots, t_k$ (we refer to n and the expression as the **rank** and the **ramification** of t respectively).

Example 2.3.1 Let $f, g \in G$ and $\lambda, \mu, \nu \in N$. Define the element $l \in M$ by

$$l = [\nu g f[\mu f g[\lambda f g]g]f[\mu[\nu[\nu g f[\mu f g f[\nu[\lambda f f]f g[\mu f[\lambda g f]f f][\lambda f f]]][\lambda g f]f]f f[\lambda g f]f]g f f]].$$

[2.23] As was noted in §1.1, mathematics in the modern sense is the totality of the study by deductive thinking based on the concept of sets and starting with definitions (s. §1.4).

The following proof of Theorem 2.3.1 also shows how to judge whether l is a tree or not and, in case it is a tree, find its rank and ramification.

If you delete the brackets from the definition of trees, then l becomes

$$m = \nu g f \mu f g \lambda f g g f \mu \nu \nu g f \mu f g f \nu \lambda f f f g \mu f \lambda g f f f \lambda f f \lambda g f f f f \lambda g f f g f f.$$

Obviously, m is a tree, but its ramification is not unique.

As Remark 2.3.1 and Example 2.3.1 suggest, the brackets play important roles below in the proofs of Lemma 2.3.1 and Theorem 2.3.1. A tree t is ramified iff the brackets occur in t. Every ramified tree $[\nu t_1 \cdots t_k]$ has the leftmost left bracket and the rightmost right bracket. Other occurrences of the brackets in $[\nu t_1 \cdots t_k]$ will be referred to as **inner** ones, which are therefore inner ones in t_j or the leftmost left bracket or the rightmost right bracket of t_j for some $j \in \{1, \dots, k\}$. Occurrences of germs in a tree t is inner iff $t \notin T_0$.

Lemma 2.3.1 (i) If $t \in T_n$ $(n \geq 0)$, then nodes occur in t exactly n times, and likewise for each of the brackets.

(ii) If trees $t, t_1, \dots, t_k$ and a node ν satisfy $t = [\nu t_1 \cdots t_k]$, then the following hold for $j = 1, \dots, k$ as to the occurrences of the brackets in t.

(1) The left bracket occurs on the left-hand side of t_j exactly one more times than the right bracket.

(2) The left bracket occurs on the left-hand side of any inner occurrence of the left bracket or germs in t_j at least two more times than the right bracket.

The words *left* and *right* interchanged, the above remains true.

Proof (i) We argue by induction on n. If $n = 0$, then $t \in G$ and so no nodes occur in t. Therefore assume $n \geq 1$. Then $t = [\nu t_1 \cdots t_k]$ with $\nu \in N$, $t_j \in T_{n_j}$ $(j = 1, \dots, k)$ and $n - 1 = \sum_{j=1}^{k} n_j$, and nodes occur in t_j exactly n_j times by the induction hypothesis $(j = 1, \dots, k)$. Exactly one node ν in addition occurs in t. Since $n - 1 = \sum_{j=1}^{k} n_j$, nodes occur in t exactly n times. It similarly follows that each of the brackets occur in t exactly n times.

(ii) (i) implies that the left bracket occurs in $t_1 \cdots t_{j-1}$ as many times as the right bracket. Therefore, the left bracket occurs in $[\nu t_1 \cdots t_{j-1}$ exactly one more times than the right bracket. Thus (1) holds.

Suppose $t \in T_n$. Then (i) implies that $n \geq 1$ and $t_j \in T_{n_j}$ for some $n_j < n$, and therefore we argue for (2) by induction on n. If $n = 1$, then $t_j \in T_0$, and so (2) obviously holds because there are no inner occurrences of the left bracket or germs in t_j. Therefore assume that $n \geq 2$ and that there is an inner occurrence of the left bracket or a germ in t_j. Then the induction hypothesis and (1) applied to t_j imply that, in t_j, the left bracket occurs on the left-hand side of the inner occurrence in question at least one more times than the right bracket. The left bracket occurs in $t_1 \cdots t_{j-1}$ as many times as the right bracket, and in addition t has the leftmost left bracket. Thus (2) holds.

The same proof works with the words *left* and *right* interchanged, $t_1 \cdots t_{j-1}$ replaced by $t_{j+1} \cdots t_k$, and $[\nu t_1 \cdots t_{j-1}$ replaced by $t_{j+1} \cdots t_k]$.

Proof of Theorem 2.3.1 The former half (i) of Lemma 2.3.1 implies that of Theorem 2.3.1. Moreover, the rank of a tree t is equal to the number of the occurrences of nodes and of each of the brackets in t.

Let t be a ramified tree. Then t has an expression $t = [\nu t_1 \cdots t_k]$ by a node ν and trees $t_1, \ldots, t_k$. Suppose $t_j \in T_{n_j}$ $(j = 1, \ldots, k)$. Then (ii) of Lemma 2.3.1 implies that the leftmost left bracket of any tree t_j with $n_j \geq 1$ is characterized as an occurrence of the left bracket in t on the left-hand side of which the left bracket occurs exactly one more times than the right bracket (we call it a **ramifying occurrence**). This also holds with the words *left* and *right* interchanged. Any pair of successive ramifying occurrences of the left bracket and the right bracket in t is the pair of the leftmost left bracket and the rightmost right bracket of a tree t_j with $n_j \geq 1$. Cut all those trees out of t. Then the occurrences of germs in the remainder are the trees t_j with $n_j = 0$. Alternatively, the trees t_j with $n_j = 0$ are characterized as the occurrences of germs on the left-hand side of which the left bracket occurs exactly one more times than the right bracket and also as the occurrences of germs on the right-hand side of which the right bracket occurs exactly one more times than the left bracket. Cutting them all out, we are left with $[\nu]$. Thus ν and $t_1, \ldots, t_k$ are uniquely determined by t.

Example 2.3.2 Let l be the element of M given in Example 2.3.1. Then the proof of Theorem 2.3.1 shows that the following **depth map** of l enables us to judge whether l is a tree and, in case it is so, find its ramification and rank:

$$l = \overset{0}{[}\nu gf\overset{1}{[}\mu fg\overset{2}{[}\lambda fg\underset{2}{]}g\underset{1}{]}f\overset{1}{[}\mu\overset{2}{[}\nu\overset{3}{[}\nu gf\overset{4}{[}\mu fgf\overset{5}{[}\nu\overset{6}{[}\lambda ff\underset{6}{]}fg\overset{6}{[}\mu f\overset{7}{[}\lambda gf\underset{7}{]}ff\underset{6}{]}\overset{6}{[}\lambda ff\underset{6}{]}\underset{5}{]}\underset{4}{]}\overset{4}{[}\lambda gf\underset{4}{]}f\underset{3}{]}ff\overset{3}{[}\lambda gf\underset{3}{]}f\underset{2}{]}gff\underset{1}{]}\underset{0}{]}.$$

Here the number placed over each left bracket is its **depth** which is defined as the number of the occurrences of the left bracket minus that of the right bracket on the left-hand side of the left bracket in question, and likewise for the number placed under each right bracket. The ramifying occurrences are nothing but the occurrences of depth 1. We find there two pairs of successive ramifying occurrences of the left bracket and the right bracket. They show that the possible ramification of l is $[\nu gfl_1fl_2]$ with $l_1 = \overset{0}{[}\mu fg\overset{1}{[}\lambda fg\underset{1}{]}g\underset{0}{]}$ and

$$l_2 = \overset{0}{[}\mu\overset{1}{[}\nu\overset{2}{[}\nu gf\overset{3}{[}\mu fgf\overset{4}{[}\nu\overset{5}{[}\lambda ff\underset{5}{]}fg\overset{5}{[}\mu f\overset{6}{[}\lambda gf\underset{6}{]}ff\underset{5}{]}\overset{5}{[}\lambda ff\underset{5}{]}\underset{4}{]}\underset{3}{]}\overset{3}{[}\lambda gf\underset{3}{]}f\underset{2}{]}ff\overset{2}{[}\lambda gf\underset{2}{]}f\underset{1}{]}gff\underset{0}{]}.$$

The depth map of l_1 shows that it is a tree of ramification $[\mu fgrg]$, where r is a tree of ramification $[\lambda fg]$. Therefore l is a tree iff l_2 is so. The depth map of l_2 shows that its possible ramification is $[\mu l_3gff]$ with

$$l_3 = \overset{0}{[}\nu\overset{1}{[}\nu gf\overset{2}{[}\mu fgf\overset{3}{[}\nu\overset{4}{[}\lambda ff\underset{4}{]}fg\overset{4}{[}\mu f\overset{5}{[}\lambda gf\underset{5}{]}ff\underset{4}{]}\overset{4}{[}\lambda ff\underset{4}{]}\underset{3}{]}\underset{2}{]}\overset{2}{[}\lambda gf\underset{2}{]}f\underset{1}{]}ff\overset{1}{[}\lambda gf\underset{1}{]}f\underset{0}{]},$$

and so l_2 is a tree iff l_3 is so. The possible ramification of l_3 is $[\nu l_4ffsf]$, where $l_4 = \overset{0}{[}\nu gf\overset{1}{[}\mu fgf\overset{2}{[}\nu\overset{3}{[}\lambda ff\underset{3}{]}fg\overset{3}{[}\mu f\overset{4}{[}\lambda gf\underset{4}{]}ff\underset{3}{]}\overset{3}{[}\lambda ff\underset{3}{]}\underset{2}{]}\underset{1}{]}\overset{1}{[}\lambda gf\underset{1}{]}f\underset{0}{]}$ and s is a tree of ramification $[\lambda gf]$, and so l_3 is a tree iff l_4 is so. The possible ramification of l_4 is $[\nu gf l_5 sf]$ with $l_5 = \overset{0}{[}\mu fgf\overset{1}{[}\nu\overset{2}{[}\lambda ff\underset{2}{]}fg\overset{2}{[}\mu f\overset{3}{[}\lambda gf\underset{3}{]}ff\underset{2}{]}\overset{2}{[}\lambda ff\underset{2}{]}\underset{1}{]}\underset{0}{]}$, and so l_4 is a tree iff l_5 is so. The possible ramification of l_5 is $[\mu fgf l_6]$ with $l_6 = \overset{0}{[}\nu\overset{1}{[}\lambda ff\underset{1}{]}fg\overset{1}{[}\mu f\overset{2}{[}\lambda gf\underset{2}{]}ff\underset{1}{]}\overset{1}{[}\lambda ff\underset{1}{]}\underset{0}{]}$, and so l_5 is a tree iff l_6 is so. The possible ramification of l_6 is $[\nu tfg l_7 t]$, where t is a tree of ramification $[\lambda ff]$ and $l_7 = \overset{0}{[}\mu f\overset{1}{[}\lambda gf\underset{1}{]}ff\underset{0}{]}$, and so l_6 is a tree iff l_7 is so. Finally l_7 is a tree of ramification $[\mu fsff]$. Thus l is a tree. Its rank is 14 because nodes and brackets each occur 14 times in it.

2.4 Closure by associations

Let A be a set and R be an association on A. Suppose R and a subset B of A satisfy the following condition for all $(\alpha, y) \in A^* \times A$:

$$\alpha \subseteq B,\ \alpha\, R\, y \implies y \in B. \tag{2.4.1}$$

Then we call B an **R-subset**[2.24] or say that B is **R-closed** or that R **close**s B. Without the word convention, we may phrase (2.4.1) as follows:

$$\alpha \in B^*,\ \alpha\, R\, y \implies y \in B.$$

Remark 2.4.1 Obviously, A itself is an R-subset of A. In contrast, $\emptyset$ is not necessarily an R-subset of A. Indeed, (2.4.1) with $B = \emptyset$ means that there exist no elements $y \in A$ such that $\varepsilon\, R\, y$. Therefore, defining the **R-core** A_R of A by

$$A_R = \{y \in A : \varepsilon\, R\, y\},$$

we have that $\emptyset$ is an R-subset of A iff $A_R = \emptyset$ (Theorem 3.1.1 gives another necessary and sufficient condition). More importantly, (2.4.1) with $\alpha = \varepsilon$ implies that every R-subset of A contains A_R.

The purpose of this section is to study closure by associations in terms of the closure operator to be given in Remark 2.4.3 by virtue of Theorem 2.4.1. The main results are Theorems 2.4.5 and 2.4.6. Their consequences Theorems 2.4.7 and 2.5.3 are crucial to the main result Theorem 2.5.6 of §2.5.1. Another consequence Theorem 2.6.3 plays an important role in §2.8. The latter conclusion of Theorem 2.4.5 is crucial to Theorem 2.6.10.

Remark 2.4.2 The deduction rules $\dfrac{x \quad x \Rightarrow y}{y}$ and $\dfrac{x \quad y}{x \wedge y}$ mentioned in the introduction of this chapter and their union may be regarded as associations on the set of the formulas of PL or FPL (in fact, Remark 2.5.5 conversely formulates the fractional expressions of associations). Furthermore, the f-validity relation

[2.24] R-Subsets were called R-theories in [1.5] and others (s. [2.1]).

(2.2.1) derived from a latticed representation $(A, \mathbb{B}, f)$ becomes an association on A when restricted to $A^* \times A$. Thus an association can be not only a model of a union of deduction rules but also a relation which is logically important (s. Theorem 2.6.6 for the importance of the restriction of (2.2.1)) and cannot appropriately be regarded as a union of deduction rules (s. Remark 2.4.7).[2.25] This is the key to the theory of logic spaces and deduction systems and also is a reason why we use the term *association* instead of the term *rule*.

Let us, however, regard an association R on a set A as a union of deduction rules here. Then if an element $(x_1 \cdots x_n, y) \in A^* \times A$ satisfies $x_1 \cdots x_n \, R \, y$, we may think y to be a **direct consequence** of or to **directly follow** from the elements $x_1, \ldots, x_n \in A$ by virtue of a rule in R (s. Remark 2.4.6). Under this interpretation, a subset B of A is an R-subset iff every direct consequence of elements of B by virtue of R belongs to B.

A set A equipped with an association R on it may be regarded as a generalized algebra, and therefore is called a **quasialgebra** and R is also called its (**quasialgebraic**) **structure** in §3.1.2. Moreover, R-subsets of A may be regarded as subalgebras of the quasialgebra. Thus, in view of our experience in proper algebra, the following theorem seems to be a natural point of departure.

Theorem 2.4.1 Let R be an association on a set A. Then the set of all R-subsets of A is $\cap$-closed in $\mathfrak{P}A$.

Proof Let $\mathfrak{X}$ be a set of R-subsets and define $B = \bigcap \mathfrak{X}$. Assume $\alpha \in B^*$ and $\alpha \, R \, y$. Then $\alpha \in X^*$ for each $X \in \mathfrak{X}$, and so since X is an R-subset, we have $y \in X$. Therefore $y \in B$. This holds even if $\mathfrak{X} = \emptyset$ because $\bigcap \emptyset = \inf_{\mathfrak{P}A} \emptyset = A$. Thus B is an R-subset.

Definition 2.4.1 Let R be an association on a set A and X be a subset of A. Then Theorem 2.4.1 implies that the intersection of the R-subsets of A which contain X is the smallest of the R-subsets of A which contain X. We denote it by $[X]_R$ and call it the **R-closure** of X in A (Remark 2.4.6 explains its meaning).

Remark 2.4.3 It follows from Theorem 2.1.4 that the mapping $X \mapsto [X]_R$ is a closure operator on $\mathfrak{P}A$ whose fixture domain is equal to the set of the R-subsets of A. Theorem 2.4.5 generalizes and refines this result.

The above definition of $[X]_R$ may be regarded as a downward one. Theorems 2.4.2–2.4.4 implicitly give equivalent upward ones, which show that $[X]_R$ consists of the elements of A that are generated by X by virtue of R (and so we also call $[X]_R$ the R-subset of A **generated** by X). The first one given in Theorem 2.4.2 is indispensable to the proofs of Theorems 2.4.3 and 2.4.4 and Lemmas 2.5.4 and 2.9.1, while the others should be recalled in order to understand the meaning of every concept related to $[X]_R$.

[2.25] Certain of memories of the IU may also be regarded as an association (s. Example 1.2.2).

Theorem 2.4.2 Let R be an association on a set A and X be a subset of A. Then $[X]_R$ is the union $\bigcup_{n \geq 0} X_n$ of the **R-descendants** X_n $(n = 0, 1, \ldots)$ of X in A which are inductively defined by $X_0 = X$ and the following for each $n \geq 1$:

$$X_n = \left\{ y \in A : \begin{array}{l} x_1 \cdots x_k \, R \, y \text{ for an element } x_1 \cdots x_k \in A^* \text{ satisfying} \\ x_j \in X_{n_j} \ (j = 1, \ldots, k) \textit{ for some nonnegative} \\ \textit{integers } n_1, \ldots, n_k \textit{ such that } n - 1 = \sum_{j=1}^{k} n_j. \end{array} \right\}.$$

In particular, $X_1 = \{y \in A : \alpha \, R \, y \text{ for an element } \alpha \in X^*\}$.

Remark 2.4.4 If $k = 0$ in the definition of X_n $(n \geq 1)$, then $x_1 \cdots x_k = \varepsilon$ and $\sum_{j=1}^{k} n_j = 0$ and so $n = 1$. The italicized condition on $(x_1, \ldots, x_k)$ in the definition is the ramification condition $\mathcal{R}(X_0, \ldots, X_{n-1})$ in (2.3.1).

Proof We first show that every element $y \in X_n$ $(n = 0, 1, \ldots)$ belongs to $[X]_R$ by induction on n. This holds for $n = 0$ because $X_0 = X \subseteq [X]_R$. Therefore assume $n \geq 1$. Then $x_1 \cdots x_k \, R \, y$ for some elements $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^{k} n_j$, and $x_1, \ldots, x_k \in [X]_R$ by the induction hypothesis. Therefore $y \in [X]_R$. This holds even if $k = 0$ because $A_R \subseteq [X]_R$ as was noted in Remark 2.4.1. Thus, defining $B = \bigcup_{n \geq 0} X_n$, we have $X \subseteq B \subseteq [X]_R$, and so it suffices to show that B is R-closed. Therefore assume $x_1 \cdots x_k \in B^*$ and $x_1 \cdots x_k \, R \, y$. Then $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ for some nonnegative integers $n_1, \ldots, n_k$, and so $y \in X_n \subseteq B$ for $n = 1 + \sum_{j=1}^{k} n_j$. This holds even if $k = 0$ because $A_R \subseteq X_1$ by the definition of X_1 or Remark 2.4.4. Thus B is R-closed.

The second upward definition of $[X]_R$ is implicitly given in the following.

Theorem 2.4.3 Let R be an association on a set A and X be a subset of A. Then an element $y \in A$ belongs to $[X]_R$ iff there exists an element $x_1 \cdots x_n \in A^*$ $(n \geq 1)$ which satisfies $x_n = y$ and one of the following conditions for each $i \in \{1, \ldots, n\}$ (we call $x_1 \cdots x_n$ an **X/R-sequent** or **X/R-sqnt** for y).

(1) $x_i \in X$.

(2) There exist numbers $j_1, \ldots, j_k \in \{1, \ldots, i-1\}$ such that $x_{j_1} \cdots x_{j_k} \, R \, x_i$.

Remark 2.4.5 The condition (2) with $k = 0$ means $x_i \in A_R$. Therefore the conditions for $i = 1$ mean $x_1 \in X \cup A_R$.

Proof In view of Theorem 2.4.2, we first show by induction on n that if y belongs to the n-th R-descendant X_n of X then y has an X/R-sqnt. If $n = 0$, then $y \in X$, and so y is an X/R-sqnt for y. Therefore assume $n \geq 1$. Then $x_1 \cdots x_k \, R \, y$ for some elements $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^{k} n_j$, and x_j has an X/R-sqnt α_j $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $\alpha_1 \cdots \alpha_k y$ is an X/R-sqnt for y, even if $k = 0$ because of Remark 2.4.5.

We next show by induction on n that if y has an X/R-sqnt $x_1 \cdots x_n$ then $y \in [X]_R$. If $n = 1$, then $y = x_1 \in X \cup A_R \subseteq [X]_R$. Therefore assume $n \geq 2$ and

$y \notin X \cup A_R$. Then for each $i \in \{1, \ldots, n-1\}$, $x_1 \cdots x_i$ is an X/R-sqnt for x_i, and so $x_i \in [X]_R$ by the induction hypothesis. Moreover, there exist numbers $j_1, \ldots, j_k \in \{1, \ldots, n-1\}$ $(k \geq 1)$ such that $x_{j_1} \cdots x_{j_k} \, R \, y$. Thus $y \in [X]_R$.

The third upward definition of $[X]_R$ uses trees as defined in §2.3. Let R be an association on a set A and $M = (A \amalg \bar{A} \amalg \{[,]\})^*$ be the free monoid over the direct union $A \amalg \bar{A} \amalg \{[,]\}$ of A, its copy $\bar{A} = \{\bar{x} : x \in A\}$ by the bar (s. §1.5.2) and the set $\{[,]\}$ of the left bracket [and the right bracket] (s. Remark 3.1.14). Then we inductively define subsets D_n $(n = 0, 1, \ldots)$ of M and an element $c(d) \in A$ for each $d \in D_n$ by $D_0 = A$ and $c(x) = x$ for each $x \in D_0$ and by the following for each $n \geq 1$, where the italicized condition on $(d_1, \ldots, d_k)$ is the ramification condition $\mathcal{R}(D_0, \ldots, D_{n-1})$ in (2.3.1):

$$D_n = \left\{ [\bar{x} d_1 \cdots d_k] : \begin{array}{l} d_j \in D_{n_j} \ (j = 1, \ldots, k) \textit{ for some nonnegative} \\ \textit{integers } n_1, \ldots, n_k \textit{ such that } n - 1 = \sum_{j=1}^{k} n_j, \\ \bar{x} \in \bar{A} \text{ and } c(d_1) \cdots c(d_k) \, R \, x. \end{array} \right\},$$

$$c([\bar{x} d_1 \cdots d_k]) = x \text{ for each } [\bar{x} d_1 \cdots d_k] \in D_n \text{ as above.}$$

It inductively follows that D_n $(n = 0, 1, \ldots)$ consists of trees of rank n whose germs and nodes are elements of A and $\bar{A}$ respectively. We have thus defined a set $D = \bigcup_{n \geq 0} D_n$ and a mapping $c \in D \to A$ so that $c|_A = \mathrm{id}_A$.

We refer to the elements of D, D_0 and $\bigcup_{n \geq 1} D_n$ as **R-deductions**, **prime R-deductions** and **composite R-deductions** on A respectively. We also refer to the element $c(d) \in A$ for each R-deduction d as its **conclusion**, while we refer to the mapping c as **concluding**. Furthermore, we define the **premise** $P(d)$ of d as the set of the elements of A which occur in d (s. Example 3.1.8). Then $P(x) = \{x\}$ for $x \in D_0$ and $P(d) = \bigcup_{j=1}^{k} P(d_j)$ for $d = [\bar{x} d_1 \cdots d_k] \in \bigcup_{n \geq 1} D_n$ as above. Moreover, $P(d)$ is equal to the germ $G(d)$ defined in §2.3.

In terms of the above concepts, the third upward definition of $[X]_R$ is implicitly given in the following.

Theorem 2.4.4 Let R be an association on a set A and X be a subset of A. Then an element $y \in A$ belongs to $[X]_R$ iff there exists an R-deduction d on A such that $P(d) \subseteq X$ and $c(d) = y$ (we call d an **X/R-proof** or **X/R-pf** of y). More precisely in terms of Theorem 2.4.2, $y \in X_n$ iff y has an X/R-proof in D_n $(n = 0, 1, \ldots)$.

Proof We first show by induction on n that if $y \in X_n$ then y has an X/R-pf in D_n. If $n = 0$, then $y \in X$, and so y is an X/R-pf of y in D_0. Therefore assume $n \geq 1$. Then $x_1 \cdots x_k \, R \, y$ for some elements $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^{k} n_j$, and x_j has an X/R-pf $d_j \in D_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $[\bar{y} d_1 \cdots d_k]$ is an X/R-pf of y in D_n. This holds even if $k = 0$ because of Remark 2.4.4 and the similar remark on D_n $(n \geq 1)$.

We next show by induction on n that if y has an X/R-pf $d \in D_n$ then $y \in X_n$. If $n = 0$, then $y = c(d) \in P(d) \subseteq X = X_0$. Therefore assume $n \geq 1$. Then $d = [\bar{y} d_1 \cdots d_k]$ with $d_j \in D_{n_j}$ $(j = 1, \ldots, k)$, $n - 1 = \sum_{j=1}^{k} n_j$ and

$c(d_1)\cdots c(d_k)\, R\, y$. Moreover, d_j is an X/R-pf of $c(d_j)$, and so $c(d_j) \in X_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $y \in X_n$. This holds even if $k = 0$ because $A_R \subseteq X_1$ by the definition of X_1 or Remark 2.4.4.

Remark 2.4.6 Theorems 2.4.2–2.4.4 suggest an interpretation of closures. If a subset X and an element y of a set A satisfy $y \in [X]_R$ for an association R on A, and if it is appropriate to regard R as a union of deduction rules (s. Remark 2.4.7), then we may think y to be an **indirect consequence** of or to **indirectly follow** from finite elements of $X \cup A_R$ by virtue of the rules in R (s. Remarks 2.4.2 and 2.5.1). The finiteness is verified by the following theorem.

Theorem 2.4.5 Let R be an association on a set A and D be a subset of A. Then the mapping $X \mapsto [X \cup D]_R$ is a finitary closure operator on $\mathfrak{P}A$ whose fixture domain is equal to the set of the R-subsets of A which contain D. Moreover, the fixture domain is $\cap$-closed in $\mathfrak{P}A$ and quasi-finitary.

Proof The latter conclusion is derived from the former by Theorems 2.1.10 and 2.1.3. In proving the former, we may assume $D = \emptyset$ by virtue of Theorem 2.1.11. As was noted in Remark 2.4.3, the mapping $X \mapsto [X]_R$ is a closure operator whose fixture domain is equal to the set of the R-subsets. Therefore, defining $Z = \bigcup_{Y \in \mathfrak{P}'X} [Y]_R$, we only need to show $[X]_R \subseteq Z$. Since $\{x\} \subseteq [\{x\}]_R \subseteq Z$ for all $x \in X$, we have $X \subseteq Z$, and so it suffices to show that Z is R-closed. Therefore assume $x_1 \cdots x_n \in Z^*$ and $x_1 \cdots x_n\, R\, y$. Then there exist sets $Y_1, \ldots, Y_n \in \mathfrak{P}'X$ such that $x_i \in [Y_i]_R$ $(i = 1, \ldots, n)$. Define $Y = \bigcup_{i=1}^{n} Y_i$. Then $Y \in \mathfrak{P}'X$ and $x_i \in [Y_i]_R \subseteq [Y]_R$ $(i = 1, \ldots, n)$. Therefore $y \in [Y]_R \subseteq Z$. This holds even if $n = 0$ because $A_R \subseteq [\emptyset]_R \subseteq Z$. Thus Z is R-closed.

Theorem 2.4.6 Let R be a partially latticed association on a set A. Then the following hold for all $X \in \mathfrak{P}A$ and for all $\alpha \in A^*$ (s. (2.11.3)):

$$[X]_R = \{y \in A : \alpha\, R\, y \text{ for an element } \alpha \in X^*\}, \qquad [\alpha]_R = \{y \in A : \alpha\, R\, y\}.$$

Consequently $A_R = [\emptyset]_R$, that is, A_R is the smallest R-subset of A.

Remark 2.4.7 It follows from Theorem 2.5.4 that an association R on a set A is partially latticed iff it satisfies the second equation for all $\alpha \in A^*$.

Because of Theorem 2.4.4, the first equation means that an element $y \in A$ has an X/R-proof iff y has one of rank 1, because its right-hand side is the first R-descendant X_1 of X in A by Theorem 2.4.2 (and so we use the symbol X_1 in the proof below). Therefore, it seems appropriate not to regard any partially latticed association as the union of a finite number of deduction rules (s. Remarks 2.4.2, 2.4.6 and 2.5.1).

Proof We have $X \subseteq X_1$ by the repetition law and $X_1 \subseteq [X]_R$, and so in order to prove the first equation, it suffices to show that X_1 is R-closed. Therefore assume $x_1, \ldots, x_n \in X_1$ and $x_1 \cdots x_n\, R\, y$. Then $\alpha_i\, R\, x_i$ for an element $\alpha_i \in$

X^* $(i = 1, \ldots, n)$. By repeated application of the partial substitution law to $\alpha_i \, R \, x_i$ $(i = 1, \ldots, n)$ and $x_1 \cdots x_n \, R \, y$, we have $\alpha_1 \cdots \alpha_n \, R \, y$, hence $y \in X_1$. This holds even if $n = 0$ because $A_R \subseteq X_1$. Thus X_1 is R-closed. The second equation is derived from the first by the partial extension law.

Theorem 2.4.7 Let Q and R be associations on a set A. Then the following conditions (2) and (3) are equivalent and the condition (1) implies them. If Q is partially latticed, then the conditions (1)–(3) are equivalent.

(1) $R \subseteq Q$.

(2) $[X]_R \subseteq [X]_Q$ for every subset X of A.

(3) Every Q-subset of A is an R-subset of A.

Remark 2.4.8 Since the mappings $X \mapsto [X]_R$ and $X \mapsto [X]_Q$ are finitary by Theorem 2.4.5, Theorem 2.1.3 shows that the condition (2) is equivalent to the one obtained from (2) by replacing the word *subset* with the phrase *finite subset*. Similar remarks apply to Theorems 2.5.6, 2.5.7 and 2.7.1.

Proof Assume (1) and let B be a Q-subset. If $\alpha \in B^*$ and $\alpha \, R \, y$, then $\alpha \, Q \, y$ by (1), and so $y \in B$ because B is a Q-subset. Therefore B is an R-subset. Thus (1) implies (3). Assume (3) and let X be a subset of A. Then $[X]_Q$ is an R-subset by (3) and contains X, and so $[X]_R \subseteq [X]_Q$ by Definition 2.4.1. Thus (3) implies (2). If (2) holds, then $[X]_R \subseteq [X]_Q = X$ for every Q-subset, and so (3) holds. Assume that Q is partially latticed and (2). If an element $(\alpha, y) \in A^* \times A$ satisfies $\alpha \, R \, y$, then $y \in [\alpha]_R$, hence $y \in [\alpha]_Q$ by (2), and so $\alpha \, Q \, y$ by Theorem 2.4.6. Thus (1) holds.

2.5 Deduction systems

A **deduction system** on a set A is a pair (R, D) of an association R on A and a subset D of A. Here we study it abstractly in §2.5.1, take a novel view of it in §2.5.2, and have its examples in §2.5.3.

2.5.1 Deduction relations of deduction systems

Here we study deduction systems in terms of the relations defined below. The main result is Theorem 2.5.6 on the preorder between deduction systems defined in Remark 2.5.3. Its consequence Theorem 2.5.7 is crucial to the fundamental Definition 2.7.1 of the completeness for deduction systems on logic spaces.

For a deduction system (R, D) on a set A, we define an association R^D on A by the following for each $(\alpha, y) \in A^* \times A$:

$$\alpha \, R^D \, y \iff [\alpha \cup D]_R \ni y. \tag{2.5.1}$$

We call R^D the **D-closure** of R because the mapping $R \mapsto R^D$ is a closure operator (s. Remark 2.5.2). We denote R^D also by $\vDash_{R,D}$ and call it the **deduction association** derived from (R, D), because Theorem 2.5.2 relates R^D to the **deduction relation** $\preccurlyeq_{R,D}$ on A^* defined by

$$\alpha \preccurlyeq_{R,D} \beta \iff [\alpha \cup D]_R \supseteq \bigcap_{y \in \beta} [\{y\} \cup D]_R$$

for each $(\alpha, \beta) \in A^* \times A^*$ and in particular R^D is the restriction of $\preccurlyeq_{R,D}$.

Remark 2.5.1 Remark 2.4.6 gives an interpretation of $\alpha\, R^D\, y$, or $\alpha \vDash_{R,D} y$. If it is appropriate to regard R as a union of deduction rules (s. Remark 2.4.7), then $\alpha\, R^D\, y$ iff y indirectly follows from finite elements of $\alpha \cup D$ by virtue of R. This kind of interpretation of $\alpha \preccurlyeq_{R,D} \beta$ is also possible, but $\preccurlyeq_{R,D}$ is important because of another interpretation given by Theorem 2.5.2 (s. Remark 2.6.2).

Theorem 2.5.1 Let (R, D) be a deduction system on a set A. Then the following hold for the D-closure R^D of R and the R^D-core A_{R^D} of A.

(1) $R \subseteq R^D$.

(2) $A_{R^D} = [D]_R$, hence in particular $D \subseteq A_{R^D}$.

(3) If a deduction system (Q, C) on A satisfies $R \subseteq Q$ and $D \subseteq C$, then $R^D \subseteq Q^C$ (s. Theorem 2.5.6 and Remark 2.5.3).

Proof (1) If $(\alpha, y) \in A^* \times A$ satisfies $\alpha\, R\, y$, then $y \in [\alpha]_R$, and since $[\alpha]_R \subseteq [\alpha \cup D]_R$ by Theorem 2.4.5, we have $\alpha\, R^D\, y$. Thus $R \subseteq R^D$.

(2) This is a direct consequence of (2.5.1) with $\alpha = \varepsilon$.

(3) If $(\alpha, y) \in A^* \times A$ satisfies $\alpha\, R^D\, y$, then $y \in [\alpha \cup D]_R$, and since $[\alpha \cup D]_R \subseteq [\alpha \cup C]_Q$ by Theorems 2.4.5 and 2.4.7, we have $\alpha\, Q^C\, y$. Thus $R^D \subseteq Q^C$.

Theorem 2.5.2 Let (R, D) be a deduction system on a set A. Then R^D is partially latticed, and $\preccurlyeq_{R,D}$ is the largest latticed extension of R^D.

Proof Define $\varphi \in \mathfrak{P}A \to \mathfrak{P}A$ by $\varphi X = [X \cup D]_R$ for each $X \in \mathfrak{P}A$. Then φ is a closure operator by Theorem 2.4.5, and R^D and $\preccurlyeq_{R,D}$ are equal to the relations $\vDash_\varphi$ and $\preccurlyeq_\varphi$ defined in Theorem 2.2.24. Thus Theorem 2.5.2 holds.

Theorem 2.5.3 Let (R, D) be a deduction system on a set A. Then the following hold.

(1) $[X]_{R^D} = [X \cup D]_R$ for every subset X of A.

(2) The set of the R^D-subsets of A is equal to that of the R-subsets of A which contain D.

Consequently, if R is an association on A, then $[X]_{R^\emptyset} = [X]_R$ for every subset X of A, and the set of the $R^\emptyset$-subsets of A is equal to that of the R-subsets of A.

Proof Since R^D is partially latticed by Theorem 2.5.2, Theorem 2.4.6 shows $[X]_{R^D} = \bigcup_{\alpha \in X^*} [\alpha \cup D]_R = \bigcup_{Y \in \mathfrak{P}'X} [Y \cup D]_R$. Since the mapping $X \mapsto [X \cup D]_R$ is finitary by Theorem 2.4.5, we have $\bigcup_{Y \in \mathfrak{P}'X} [Y \cup D]_R = [X \cup D]_R$. Thus (1) holds. Theorem 2.4.5 also shows that the fixture domains of the closure operators $X \mapsto [X]_{R^D}$ and $X \mapsto [X \cup D]_R$ are equal to the set of the R^D-subsets and that of the R-subsets which contain D respectively. Thus (2) is a consequence of (1).

Theorem 2.5.4 A deduction system (R, D) on a set A satisfies $R^D = R$ iff R is partially latticed and $D \subseteq A_R$. Consequently, an association R on A satisfies $R^\emptyset = R$ iff R is partially latticed.

Proof Since R^D is partially latticed by Theorem 2.5.2, $R^D = R$ only if R is partially latticed. Therefore assume that R is partially latticed. Then Theorems 2.4.7 and 2.5.3 show that $R^D = R$ iff every R-subset contains D. Moreover, Theorem 2.4.6 shows that every R-subset contains D iff $D \subseteq A_R$.

Theorem 2.5.5 Let (R, D) be a deduction system on a set A. Then R^D is the smallest of the associations Q on A which satisfy the following conditions.

(1) Q is partially latticed.

(2) $R \subseteq Q$ and $D \subseteq A_Q$.

Proof If (1) and (2) hold, then $R^D \subseteq Q^D = Q$ by Theorems 2.5.1 and 2.5.4. Moreover, Theorems 2.5.2 and 2.5.1 show that if $Q = R^D$ then (1) and (2) hold.

Remark 2.5.2 Let D be a subset of a set A. Then Theorem 2.5.5 together with Theorem 2.1.4 means that the mapping $R \mapsto R^D$ is a closure operator on the set of the associations on A and its fixture domain is equal to the set of the partially latticed associations R on A which satisfy $D \subseteq A_R$.

Theorem 2.5.6 Let (Q, C) and (R, D) be deduction systems on a set A. Then the following four conditions are equivalent (s. Theorem 2.5.1).

(1) $R^D \subseteq Q^C$.

(2) $[X \cup D]_R \subseteq [X \cup C]_Q$ for every subset X of A (s. Remark 2.4.8).

(3) Every Q-subset of A containing C is an R-subset of A containing D.

(4) $R \subseteq Q^C$ and $D \subseteq A_{Q^C}$.

Proof Since Q^C is partially latticed by Theorem 2.5.2, Theorems 2.4.7 and 2.5.3 show that (1)–(3) are equivalent, and Theorem 2.5.5 shows that (4) implies (1). Conversely under (1), we have $R \subseteq R^D \subseteq Q^C$ and $D \subseteq A_{R^D} \subseteq A_{Q^C}$ by Theorem 2.5.1, and thus (4) holds.

Remark 2.5.3 If deduction systems (Q, C) and (R, D) on a set A satisfy the four equivalent conditions of Theorem 2.5.6, we say that (Q, C) is **stronger** than (R, D) or that (R, D) is **weaker** than (Q, C). Moreover, we say that they are **equivalent** if $Q^C = R^D$. Then Theorem 2.5.6 particularly shows that (Q, C) and (R, D) are equivalent iff the set of the Q-subsets of A containing C is equal to that of the R-subsets of A containing D. The strength and weakness are preorders and the equivalence is their symmetric core (s. Remark 2.2.10).

Theorem 2.5.7 Let Q be a partially latticed association on a set A and (R, D) be a deduction system on A. Then the following four conditions are equivalent.

(1) $R^D \subseteq Q$.

(2) $[X \cup D]_R \subseteq [X]_Q$ for every subset X of A (s. Remark 2.4.8).

(3) Every Q-subset of A is an R-subset of A containing D.

(4) $R \subseteq Q$ and $D \subseteq A_Q$.

Moreover, the following three conditions are equivalent.

(5) $Q \subseteq R^D$.

(6) $[X]_Q \subseteq [X \cup D]_R$ for every subset X of A (s. Remark 2.4.8).

(7) Every R-subset of A containing D is a Q-subset of A.

Therefore, the following four conditions are equivalent.

(8) $Q = R^D$.

(9) $[X]_Q = [X \cup D]_R$ for every subset X of A (s. Remark 2.4.8).

(10) The set of the Q-subsets of A is equal to the set of the R-subsets of A which contain D.

(11) $R \subseteq Q,\ D \subseteq A_Q$, and $Q \subseteq R^D$.

Proof This is derived from Theorem 2.5.6. Indeed, since $Q^{\emptyset} = Q$ by Theorem 2.5.4, (1)–(4) here are derived from (1)–(4) of Theorem 2.5.6 by replacing C with $\emptyset$. Moreover, (5)–(7) are derived from (1)–(3) of Theorem 2.5.6 by first interchanging (Q, C) and (R, D) and next replacing C with $\emptyset$ ((5) is also so derived from (4) of Theorem 2.5.6). The condition (i) for $i = 8, 9, 10$ is the union of the conditions $(i–7)$ and $(i–3)$, and (11) is the union of (4) and (5).

Remark 2.5.4 An association R on a set A is said to be **singular** if $A_R = \emptyset$. For each association R on A, the association R' on A defined by

$$\alpha\, R'\, y \iff \alpha\, R\, y \text{ and } \alpha \neq \varepsilon$$

for each $(\alpha, y) \in A^* \times A$ is the largest of the singular associations on A contained in R. Therefore, it follows from Theorem 3.1.1 that $R' = R_\Lambda$ for some algebraic structure $(\alpha_\lambda)_{\lambda\in\Lambda}$ on A. Moreover, it follows from Theorem 2.4.3 that $[X]_R = [X \cup A_R]_{R'}$ for every subset X of A, and so $R^{\emptyset} = R'^{A_R}$ (s. Theorem 2.5.6). Thus if R is partially latticed, then $R = R'^{A_R} = R_\wedge{}^{A_R}$ by Theorem 2.5.4, and if furthermore $D \subseteq A_R \subseteq [D]_\wedge$, that is, if $D \subseteq A_R \subseteq [D]_{R'}$, then $[X \cup A_R]_{R'} = [X \cup D]_{R'}$ for every subset X of A, and so $R = R'^{D} = R_\wedge{}^{D}$.

2.5.2 Deductive laws for relations

The purpose of this subsection is to view deduction systems on the direct product $A \times B$ of sets A and B as laws on relations between A and B. This view is crucial to Theorem 2.7.13 mentioned in §1.3.11.

Definition 2.5.1 Let (R', D') be a deduction system on the direct product $A' = A \times B$ of sets A and B. Then we also call (R', D') a **deductive law**[2.26] on the relations between A and B, and if a relation R between A and B regarded as a subset of A' is an R'-subset containing D', we say that R **satisfies** (R', D') or call R an (R', D')**-relation** (therefore, the trivial relation $A' = A \times B$ between A and B satisfies (R', D')).

Let (R'', D'') be another deductive law on the relations between A and B. Then (R', D') is said to **imply** (R'', D'') or to be **stronger** than (R'', D'') if (R', D') is stronger than (R'', D'') as deduction systems on A' (Theorem 2.5.6 therefore shows that (R', D') implies (R'', D'') iff every (R', D')-relation is an (R'', D'')-relation). We similarly define **weakness** and **equivalence** between deductive laws on the relations between A and B.

Theorem 2.5.8 Let (R', D') be a deduction system on the direct product $A' = A \times B$ of sets A and B. Then the R'-closure $[D']_{R'}$ of D' in A' regarded as a relation between A and B is the smallest (R', D')-relation (therefore the smallest (R', D')-relation is generated by D' by virtue of R' as in Theorems 2.4.2–2.4.4).

Proof This is because $[D']_{R'}$ is the smallest of the R'-subsets of A' which contain D', as was noted in Definition 2.4.1.

Remark 2.5.5 The following notation and terminology are convenient in illustrating Definition 2.5.1 by the examples below.

Suppose we are to define an association R on a set A. Regard R as a subset of $A^* \times A$. Then in view of [2.23], we should define R by a set-theoretic notation

$$\{(x_1 \cdots x_n, y) \in A^* \times A : \text{a condition on } x_1, \ldots, x_n, y\}.$$

People, however, prefer simpler notation. Denoting each element $(x_1 \cdots x_n, y) \in A^* \times A$ by the fraction[2.27] $\dfrac{x_1 \cdots x_n}{y}$, and by abuse of the set-theoretic notation, they often define R by the list of the fractions which satisfy the above condition on $x_1, \ldots, x_n, y$. We will use the **fractional list**s also in this monograph.

[2.26] Deductive laws were called generational laws in [1.5] and others (s. [2.1]).

[2.27] The fraction is well-defined in contrast to those in Remark 2.3.1.

Example 2.5.1 The equivalence law may be regarded as a deductive law.

Let A be a set and define $\vec{A} = A \times A$. Denote the elements $(x, y) \in \vec{A}$ by $x \to y$. Define the subset $\vec{D}$ of $\vec{A}$ by $\vec{D} = \{x \to x : x \in A\}$. Define associations $\vec{S}$ and $\vec{T}$ on $\vec{A}$ by the fractional lists $\dfrac{x \to y}{y \to x}$ and $\dfrac{x \to y \quad y \to z}{x \to z}$ respectively. Their proper set-theoretic definitions are $\vec{S} = \{(x \to y, y \to x) : x, y \in A\}$ and $\vec{T} = \{((x \to y)(y \to z), x \to z) : x, y, z \in A\}$. Let $\vec{R}$ be the union of $\vec{S}$ and $\vec{T}$ in $\vec{A}^* \times \vec{A}$. Let R be a relation on A and regard it as a subset of $\vec{A}$. Then R is reflexive iff R contains $\vec{D}$, R is symmetric iff R is $\vec{S}$-closed, and R is transitive iff R is $\vec{T}$-closed. Thus R satisfies the equivalence law iff it is an $\vec{R}$-subset containing $\vec{D}$, or iff it satisfies the deductive law $(\vec{R}, \vec{D})$.

Example 2.5.2 All the laws which define the (partially, strongly) latticed relations and (weakly) Boolean relations may be regarded as deductive laws.

For example as for the whole Boolean law, let A be a set and define $\vec{A} = A^* \times A^*$. Denote the elements $(\alpha, \beta) \in \vec{A}$ by $\alpha \to \beta$ or $\beta \leftarrow \alpha$, and call them the **sequent**s on A. Define an association $\vec{R}$ on $\vec{A}$ by the fractional list $\dfrac{\alpha \to \beta}{x\alpha \to \beta}$, $\dfrac{\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}$, $\dfrac{xx\alpha \to \beta}{x\alpha \to \beta}$, $\dfrac{xx\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}$, $\dfrac{\alpha x y \beta \to \gamma}{\alpha y x \beta \to \gamma}$, $\dfrac{\alpha x y \beta \leftarrow \gamma}{\alpha y x \beta \leftarrow \gamma}$, $\dfrac{\alpha \to x\gamma \quad x\beta \to \delta}{\alpha\beta \to \delta\gamma}$. Let $\vec{D}$ be the set of the sequents on A in any of the following twelve forms, where $x \wedge y$, $x \vee y$, $x^\Diamond$ and $x \Rightarrow y$ are operations on A: $x \to x$, $x \wedge y \to x$, $x \wedge y \to y$, $xy \to x \wedge y$, $x \vee y \leftarrow x$, $x \vee y \leftarrow y$, $xy \leftarrow x \vee y$, $xx^\Diamond \to \varepsilon$, $xx^\Diamond \leftarrow \varepsilon$, $x^\Diamond \to x \Rightarrow y$, $y \to x \Rightarrow y$, $x \Rightarrow y \to x^\Diamond y$. Let R be a relation on A^* and regard R as a subset of $\vec{A}$. Then R is an $\vec{R}$-subset containing $\vec{D}$ iff R satisfies the laws of weakening, contraction, exchange, strong cut, repetition and the laws of junction, negation and implication with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. Thus R satisfies the deductive law $(\vec{R}, \vec{D})$ iff it satisfies the Boolean law with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$.

Definition 2.5.2 Let L be a law on the relations between sets A and B. Then L is said to be **$\cap$-closed** if the set of the relations between A and B which satisfy L is $\cap$-closed, when regarded as a subset of $\mathfrak{P}(A \times B)$.

Theorem 2.5.9 The deductive laws are $\cap$-closed.

Proof Let L be a deductive law on the relations between sets A and B. Then L is a deduction system (R', D') on $A' = A \times B$, and the set $\mathfrak{R}$ of the relations between A and B which satisfy L is equal to the set of the R'-subsets of A' which contain D', and so $\mathfrak{R}$ is $\cap$-closed in $\mathfrak{P}A'$ by Theorem 2.4.5.

Corollary 2.5.9.1 All the laws which define the (partially, strongly) latticed relations and (weakly) Boolean relations are $\cap$-closed.

Proof This is because the laws are deductive, as was shown by Example 2.5.2.

Remark 2.5.6 The concept of deductive laws on the relations on a set A as defined in Definition 2.5.1 depends on A. We can also define a concept of deductive laws which does not depend on any set.

Let (R', D') be a deduction system on $\mathbb{N}^2$, or a deductive law on the relations on $\mathbb{N}$. Then we also call (R', D') a **deductive law** on the relations on sets, and if a relation R on a set A satisfies the following two conditions for each $\varphi \in \mathbb{N} \to A$, we say that R **satisfies** (R', D') or call R an (R', D')**-relation**.

(1) $\varphi d \, R \, \varphi e$ for each $(d, e) \in D'$.

(2) If elements (d_i, e_i) $(i = 1, \ldots, n)$ and (d, e) of $\mathbb{N}^2$ satisfy $\varphi d_i \, R \, \varphi e_i$ $(i = 1, \ldots, n)$ and $(d_1, e_1) \cdots (d_n, e_n) \, R' \, (d, e)$, then $\varphi d \, R \, \varphi e$.

For example, if (R', D') is defined by $(1,2) \, R' \, (2,1)$, $(1,2)(2,3) \, R' \, (1,3)$ and $D' = \{(1,1)\}$, then the (R', D')-relations are the equivalence relations. The trivial relation $A \times A$ on each set A satisfies every deductive law.

The definition here is related to Definition 2.5.1 in the following way. Let A be a set. Then each $\varphi \in \mathbb{N} \to A$ naturally induces a mapping of $\mathbb{N}^2$ into A^2 and furthermore a mapping of $(\mathbb{N}^2)^* \times \mathbb{N}^2$ into $(A^2)^* \times A^2$. Denote them also by φ and regard R' as a subset of $(\mathbb{N}^2)^* \times \mathbb{N}^2$. Then $\varphi D' \subseteq A^2$ and $\varphi R' \subseteq (A^2)^* \times A^2$, and so $(\varphi R', \varphi D')$ may be regarded as a deduction system on A^2. The conditions (1) and (2) imply that the relation R on A regarded as a subset of A^2 is an $\varphi R'$-set containing $\varphi D'$. Thus R satisfies the deductive law (R', D') as defined here iff it satisfies the deductive law $(\bigcup_{\varphi \in \mathbb{N} \to A} \varphi R', \bigcup_{\varphi \in \mathbb{N} \to A} \varphi D')$ as defined in Definition 2.5.1.

2.5.3 Boolean deduction systems

Throughout this subsection, we let A be a nonempty set with operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ and (R, D) be a deduction system on A. In view of Remark 2.7.2 and Theorem 2.7.11, here we seek conditions on (R, D) under which the deduction relation $\preccurlyeq_{R,D}$ is Boolean with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. Although one of a variety of examples, Theorem 2.5.11 is our goal[2.28], where $\&$ and $\wp$ are associations on A defined by the fractional lists:

$$\& = \frac{x \quad y}{x \wedge y}, \qquad \wp = \frac{x \quad x \Rightarrow y}{y}. \tag{2.5.2}$$

Since $\preccurlyeq_{R,D}$ is (extendedly) latticed by Theorem 2.5.2, the results in §2.2.3 apply to $\preccurlyeq_{R,D}$, and we only need to seek conditions on (R, D) under which $\preccurlyeq_{R,D}$ satisfies the laws of strong cut, junction, negation and implication.

Remark 2.5.7 Let $\preccurlyeq$ be an extendedly latticed relation on A^*. Then $\preccurlyeq \; = \; \preccurlyeq_{R,D}$ for the restriction R of $\preccurlyeq$ to $A^* \times A$ and any subset D of A_R, because $R = R^D$ by Theorems 2.2.10 and 2.5.4 and $\preccurlyeq_{R,D}$ is the largest latticed extension of R^D by Theorem 2.5.2. Therefore, some of the results on $\preccurlyeq_{R,D}$ here may be regarded as those on extendedly latticed relations, and vice versa.

[2.28] Example 2.7.1 shows why it can be a goal and Theorem 2.5.12 shows another goal.

Lemma 2.5.1 Let $\preccurlyeq$ be an extendedly latticed relation on A^* satisfying the following three laws with respect to the operations $\vee$ and $\Rrightarrow$:[2.29]

$$x \preccurlyeq x \vee y, \;\; y \preccurlyeq x \vee y, \tag{2.5.3}$$

$$x \vee y \cdot x \Rrightarrow z \cdot y \Rrightarrow z \preccurlyeq z, \tag{2.5.4}$$

$$x\alpha \preccurlyeq y \implies \alpha \preccurlyeq x \Rrightarrow y. \qquad \text{(forward implication)}$$

Then $\preccurlyeq$ satisfies the strong disjunction law with respect to $\vee$.

Proof Assume $x\alpha \preccurlyeq z$ and $y\alpha \preccurlyeq z$. Then $\alpha \preccurlyeq x \Rrightarrow z$ and $\alpha \preccurlyeq y \Rrightarrow z$ by the forward implication law. Applying the laws of substitution and extension to these and (2.5.4), we have $x \vee y \cdot \alpha \preccurlyeq z$. Therefore $\preccurlyeq$ satisfies the following law:

$$x\alpha \preccurlyeq z, \; y\alpha \preccurlyeq z \implies x \vee y \cdot \alpha \preccurlyeq z. \tag{2.5.5}$$

Since $\preccurlyeq$ is extendedly latticed, this law with $\alpha = \varepsilon$ means that $\preccurlyeq$ satisfies the law $x \vee y \preccurlyeq xy$. This law and (2.5.3) constitute the disjunction law. Therefore $\preccurlyeq$ satisfies the following union of the laws ($\vee$2) and ($\vee$5) of Theorem 2.2.15:

$$\alpha \preccurlyeq xy\beta \iff \alpha \preccurlyeq x \vee y \cdot \beta. \tag{2.5.6}$$

Hence the first half of the strong disjunction law. In proving the second half

$$x\alpha \preccurlyeq \beta, \; y\alpha \preccurlyeq \beta \implies x \vee y \cdot \alpha \preccurlyeq \beta, \tag{2.5.7}$$

we may assume $\beta \neq \varepsilon$, because $\preccurlyeq$ satisfies the lower quasi-end law by Theorem 2.2.12. Moreover, (2.5.6) implies that (2.5.7) with $\beta \neq \varepsilon$ is equivalent to (2.5.5). Thus $\preccurlyeq$ satisfies (2.5.7).

Lemma 2.5.2 The following hold.

- If $\alpha \, R \, y$ then $\alpha \preccurlyeq_{R,D} y$.
- An element $x \in A$ belongs to $[D]_R$ iff $\varepsilon \preccurlyeq_{R,D} x$.

Proof This follows from Theorem 2.5.1 because $\preccurlyeq_{R,D}$ is an extension of R^D.

Lemma 2.5.3 If R contains $\wp$, then $\preccurlyeq_{R,D}$ satisfies the following two laws:[2.30]

$$x \cdot x \Rrightarrow y \preccurlyeq_{R,D} y, \qquad \text{(cut implication)}$$

$$\alpha \preccurlyeq_{R,D} x \Rrightarrow y \implies x\alpha \preccurlyeq_{R,D} y. \qquad \text{(backward implication)}$$

Proof Since $x \cdot x \Rrightarrow y \, \wp \, y$ and $\wp \subseteq R$, we have $x \cdot x \Rrightarrow y \, R \, y$, hence the cut implication law by Lemma 2.5.2. The backward implication law is derived from the cut implication law by the substitution law (s. Theorem 2.2.17).

[2.29] The third law was called the deduction law in [1.5] and others (s. [2.1]).
[2.30] The second law was called the reverse deduction law in [1.5] and others (s. [2.1]).

Lemma 2.5.4 The deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the forward implication law provided that it satisfies the following three laws:

(1) $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow x$,

(2) $y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ (reflexive implication),

(3) if $x_1 \cdots x_k \, \mathsf{R} \, y$ $(k \geq 1)$ then $z \Rrightarrow x_1 \cdots z \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} z \Rrightarrow y$.

The converse is true provided that R contains $\wp$.

Proof Under the laws (1)–(3), assume $x\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} y$. Then $y \in [X]_{\mathsf{R}}$ for $X = \{x\} \cup \alpha \cup D$, and so Theorem 2.4.2 shows that y belongs to the n-th R-descendant X_n of X for some nonnegative integer n. We will show $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ by induction on n. Assume $n = 0$, or $y \in X$. If $y = x$, then $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ by (1) and the weakening law. If $y \in \alpha \cup D$, then $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} y$, and applying the cut law to this and (2), we have $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$. Therefore assume $n \geq 1$. Then $x_1 \cdots x_k \, \mathsf{R} \, y$ for some elements $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^{k} n_j$. If $k = 0$, then $\alpha \, \mathsf{R} \, y$ by the weakening law, and so $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} y$ by Lemma 2.5.2, hence $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ as above. Therefore assume $k \geq 1$. Then $x \Rrightarrow x_1 \cdots x \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ by (3), and $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow x_j$ $(j = 1, \ldots, k)$ by the induction hypothesis. By repeated application of the laws of substitution and extension to these $k+1$ relations, we have $\alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ as desired.

Conversely, assume that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the forward implication law and R contains $\wp$. Then (1) and (2) are derived from the repetition law by the laws of forward implication and weakening. Assume $x_1 \cdots x_k \, \mathsf{R} \, y$ $(k \geq 1)$ in order to verify (3). Then $x_1 \cdots x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} y$ by Lemma 2.5.2, and $z \cdot z \Rrightarrow x_1 \cdots z \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} x_j$ $(j = 1, \ldots, k)$ by the cut implication law of Lemma 2.5.3 and the extension law. By repeated application of the laws of substitution and extension to the $k+1$ relations, we have $z \cdot z \Rrightarrow x_1 \cdots z \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} y$. Therefore $z \Rrightarrow x_1 \cdots z \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} z \Rrightarrow y$ by the forward implication law. Thus (3) holds.

Lemma 2.5.5 The deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws (1) and (2) of Lemma 2.5.4 provided that it satisfies the following three laws:

(1) $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee x$, (2) $y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \vee y$, (3) $x^{\Diamond} \vee y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$.

Proof The law $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow x$ is derived from (1) and (3) with $x = y$ by the cut law. The reflexive implication law $y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$ is derived from (2) with x replaced by $x^{\Diamond}$ and (3) by the cut law.

Lemma 2.5.6 If R contains &, then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the following laws:

(1) $xy \preccurlyeq_{\mathsf{R},\mathsf{D}} x \wedge y$, (2) $x \wedge y \cdot \beta \preccurlyeq_{\mathsf{R},\mathsf{D}} \alpha \implies xy\beta \preccurlyeq_{\mathsf{R},\mathsf{D}} \alpha$.

Proof Since $xy \, \& \, x \wedge y$ and $\& \subseteq \mathsf{R}$, we have $xy \, \mathsf{R} \, x \wedge y$, hence (1) by Lemma 2.5.2. (2) is derived from (1) by the cut law (s. Theorem 2.2.15).

Theorem 2.5.10 Assume that R contains $\& \cup \wp$. Then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$ iff it satisfies the following eight laws:

(1) $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee x$,
(2) $x \wedge y \preccurlyeq_{\mathsf{R},\mathsf{D}} x$,
(3) $x \wedge y \preccurlyeq_{\mathsf{R},\mathsf{D}} y$,
(4) $x \preccurlyeq_{\mathsf{R},\mathsf{D}} x \vee y$,
(5) $y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \vee y$,
(6) $x^{\Diamond} \vee y \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$,
(7) $x \vee y \cdot x \Rrightarrow z \cdot y \Rrightarrow z \preccurlyeq_{\mathsf{R},\mathsf{D}} z$,
(8) if $x_1 \cdots x_k \mathsf{R} y$ $(k \geq 1)$, then $z \Rrightarrow x_1 \cdots z \Rrightarrow x_k \preccurlyeq_{\mathsf{R},\mathsf{D}} z \Rrightarrow y$.

Proof Assume that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies (1)–(8). Then as was noted before, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is extendedly latticed, results in §2.2.3 apply to it, and we only need to show that it satisfies the laws of strong cut, junction, negation and implication.

Since R contains &, Lemma 2.5.6 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the law $xy \preccurlyeq_{\mathsf{R},\mathsf{D}} x \wedge y$. This law and (2) and (3) constitute the conjunction law.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies (1), (5) and (6), Lemma 2.5.5 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the law $\varepsilon \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow x$ and the reflexive implication law.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ also satisfies (8), Lemma 2.5.4 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the forward implication law. Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ furthermore satisfies (4), (5) and (7) and is extendedly latticed, Lemma 2.5.1 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong disjunction law. Therefore Theorem 2.2.15 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the disjunction law.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the disjunction law and (1), Lemma 2.2.4 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the upper negation law.

We have $x^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} x^{\Diamond} \vee y$ by (4). Applying the cut law to this and (6), we have the contradictory implication law $x^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} x \Rrightarrow y$.

Since R contains $\wp$, Lemma 2.5.3 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws of cut implication and backward implication.

Applying the backward implication law to the contradictory implication law, we have the law $xx^{\Diamond} \preccurlyeq_{\mathsf{R},\mathsf{D}} y$, hence the lower negation law, because $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is extendedly latticed and so satisfies the lower quasi-end law by Theorem 2.2.12.

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws of negation and junction, Lemmas 2.2.4, 2.2.5 and Remark 2.2.7 show that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws of quasi-end and quasi-conjunction, and so since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ also satisfies the strong disjunction law, Theorem 2.2.20 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the strong cut law (s. [2.21]).

Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws of cut implication, negation and strong cut, Theorem 2.2.17 shows that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the negative implication law.

Thus, if $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws (1)–(8), then it is Boolean. The converse is a consequence of the following lemma and Lemma 2.5.2.

Lemma 2.5.7 Let $\preccurlyeq$ be a Boolean relation on A^* with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$. Then $\preccurlyeq$ satisfies the laws (1*)–(8*) obtained from the laws (1)–(8) of Theorem 2.5.10 by replacing the relations $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ and R with $\preccurlyeq$.

Proof As was noted in Remark 2.2.6, $\preccurlyeq$ is latticed. Therefore Lemma 2.2.4 shows that $\preccurlyeq$ satisfies (1*). (2*)–(5*) are part of the junction laws. (6*) is derived from the laws of contradictory implication and reflexive implication by the law ($\vee$6) of Theorem 2.2.15. (7*) is derived from the law ($\Rrightarrow$7) of Theorem 2.2.17 by the laws of extension and ($\vee$6). (8*) is derived from ($\Rrightarrow$7) by the laws of extension, substitution and ($\Rrightarrow$2) of Theorem 2.2.17.

Remark 2.5.8 Theorem 2.5.10 implies Lemma 2.5.7 because of Remark 2.5.7 and Theorem 2.2.22, and likewise for Corollary 2.5.10.1 and its Lemma 2.5.8.

Corollary 2.5.10.1 Assume that R contains $\& \cup \wp$. Then $\preccurlyeq_{R,D}$ is Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$ iff it satisfies the following eight laws:

(1) $\varepsilon \preccurlyeq_{R,D} x^{\Diamond} \vee x$,
(2) $\varepsilon \preccurlyeq_{R,D} (x \wedge y) \Rrightarrow x$,
(3) $\varepsilon \preccurlyeq_{R,D} (x \wedge y) \Rrightarrow y$,
(4) $\varepsilon \preccurlyeq_{R,D} x \Rrightarrow (x \vee y)$,
(5) $\varepsilon \preccurlyeq_{R,D} y \Rrightarrow (x \vee y)$,
(6) $\varepsilon \preccurlyeq_{R,D} (x^{\Diamond} \vee y) \Rrightarrow (x \Rrightarrow y)$,
(7) $\varepsilon \preccurlyeq_{R,D} ((x \Rrightarrow z) \wedge (y \Rrightarrow z)) \Rrightarrow ((x \vee y) \Rrightarrow z)$,
(8) if $x_1 \cdots x_k \, R \, y$ $(k \geq 1)$, then $\varepsilon \preccurlyeq_{R,D} ((z \Rrightarrow x_1) \wedge \cdots \wedge (z \Rrightarrow x_k)) \Rrightarrow (z \Rrightarrow y)$, where the order of applying the operation $\wedge$ is arbitrary.

Proof This is because we can derive the laws (1)–(8) of Theorem 2.5.10 from the laws (1)–(8) here by the backward implication law of Lemma 2.5.3 and the law (2) of Lemma 2.5.6 and, assuming that $\preccurlyeq_{R,D}$ is Boolean, we can either conversely derive (1)–(8) here from (1)–(8) of Theorem 2.5.10 by the law ($\Rrightarrow$2) of Theorem 2.2.17 and the law ($\wedge$2) of Theorem 2.2.15, or prove (1)–(8) here by the following lemma and Lemma 2.5.2.

Lemma 2.5.8 Let $\preccurlyeq$ be a Boolean relation on A^* with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$. Then $\preccurlyeq$ satisfies the laws obtained from the laws (1)–(8) of Corollary 2.5.10.1 by replacing the relations $\preccurlyeq_{R,D}$ and R with $\preccurlyeq$.

Proof This is derived from Lemma 2.5.7 by the laws ($\Rrightarrow$2) and ($\wedge$2) as above.

Theorem 2.5.11 Assume $R = \& \cup \wp$. Then $\preccurlyeq_{R,D}$ is Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$ iff $[D]_R$ contains the elements of A in any of the following forms (we call them the **Boolean element**s of A with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$):

(1) $x^{\Diamond} \vee x$,
(2) $(x \wedge y) \Rrightarrow x$,
(3) $(x \wedge y) \Rrightarrow y$,
(4) $x \Rrightarrow (x \vee y)$,
(5) $y \Rrightarrow (x \vee y)$,
(6) $(x^{\Diamond} \vee y) \Rrightarrow (x \Rrightarrow y)$,
(7) $((x \Rrightarrow z) \wedge (y \Rrightarrow z)) \Rrightarrow ((x \vee y) \Rrightarrow z)$,
(8) $((z \Rrightarrow x) \wedge (z \Rrightarrow y)) \Rrightarrow (z \Rrightarrow (x \wedge y))$,
(9) $((z \Rrightarrow x) \wedge (z \Rrightarrow (x \Rrightarrow y))) \Rrightarrow (z \Rrightarrow y)$.

Proof This is a consequence of Corollary 2.5.10.1. Indeed, since $\mathsf{R} = \& \cup \wp$, the elements $((z \Rightarrow x_1) \wedge \cdots \wedge (z \Rightarrow x_k)) \Rightarrow (z \Rightarrow y)$ with $x_1 \cdots x_k \,\mathsf{R}\, y$ $(k \geq 1)$ are exactly the elements in the form (8) or (9). Therefore Lemma 2.5.2 shows that $[\mathsf{D}]_{\mathsf{R}}$ contains the elements in the forms (1)–(9) iff $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the laws (1)–(8) of Corollary 2.5.10.1.

Theorem 2.5.11 is one of a variety of examples. To give another example, the following was proved in [1.3] by the way[2.31].

Theorem 2.5.12 Assume $\mathsf{R} = \wp$. Then $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$ iff $[\mathsf{D}]_{\mathsf{R}}$ contains the elements of A in any of the following seven forms (we call them the **Lukasiewicz element**s of A with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$):

$y \Rightarrow (x \Rightarrow y)$,
$(z \Rightarrow (x \Rightarrow y)) \Rightarrow ((z \Rightarrow x) \Rightarrow (z \Rightarrow y))$,
$(y^{\Diamond} \Rightarrow x^{\Diamond}) \Rightarrow (x \Rightarrow y)$,
$(x \vee y) \Rightarrow (x^{\Diamond} \Rightarrow y)$, $(x^{\Diamond} \Rightarrow y) \Rightarrow (x \vee y)$,
$(x \wedge y) \Rightarrow (x \Rightarrow y^{\Diamond})^{\Diamond}$, $(x \Rightarrow y^{\Diamond})^{\Diamond} \Rightarrow (x \wedge y)$.

Remark 2.5.9 Let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be as in Example 2.5.2 and regard $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ as a relation $\preccurlyeq$ on A^*. Then Theorem 2.5.8 shows that $\preccurlyeq$ is the smallest Boolean relation on A^* with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$. Let R be its restriction to $\mathsf{A}^* \times \mathsf{A}$.

Let B be the set of the Boolean elements of A with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$. Then $\preccurlyeq_{\& \cup \wp, \mathsf{B}}$ is Boolean by Theorem 2.5.11, and so $\preccurlyeq \; \subseteq \; \preccurlyeq_{\& \cup \wp, \mathsf{B}}$, hence $\mathsf{R} \subseteq (\& \cup \wp)^{\mathsf{B}}$. Moreover, since $\preccurlyeq$ is Boolean, we have $\& \cup \wp \subseteq \mathsf{R}$ by Definition 2.2.3 and Theorem 2.2.17, and $\mathsf{B} \subseteq \mathsf{A}_{\mathsf{R}}$ by Lemma 2.5.8, and so $(\& \cup \wp)^{\mathsf{B}} \subseteq \mathsf{R}$ by Theorem 2.5.7. Therefore $\mathsf{R} = (\& \cup \wp)^{\mathsf{B}}$, and so $\preccurlyeq \; = \; \preccurlyeq_{\& \cup \wp, \mathsf{B}}$ by Theorems 2.2.22 and 2.5.2.

Similarly, $\preccurlyeq \; = \; \preccurlyeq_{\wp, \mathsf{L}}$ for the set L of the Lukasiewicz elements of A with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$. In light of MN explained in Chapter 1, however, $(\wp, \mathsf{L})$ does not seem better than $(\& \cup \wp, \mathsf{B})$ because $(\wp, \mathsf{L})$ is simpler than $(\& \cup \wp, \mathsf{B})$.

2.6 Logic spaces

A **logic space** is a pair $(\mathsf{A}, \mathfrak{B})$ of a nonempty set A and a subset $\mathfrak{B}$ of $\mathfrak{P}\mathsf{A}$ whose elements are called the **given theorie**s of the logic space. An association R on A is said to be **$\mathfrak{B}$-sound** or called a **$\mathfrak{B}$-logic**, if it closes every set in $\mathfrak{B}$. A subset X of A is called a **$\mathfrak{B}$-theory**, if X is closed by every $\mathfrak{B}$-logic on A. We call $\bigcap \mathfrak{B}$ the **$\mathfrak{B}$-core** of A. Elements and subsets of A are said to be **$\mathfrak{B}$-sound** if they are contained in the $\mathfrak{B}$-core. A $\mathfrak{B}$-sound element is also called a **$\mathfrak{B}$-tautology**.

As immediate consequences of the above definitions, we have that the given theories are $\mathfrak{B}$-theories and hence that $\mathfrak{B}$-logics are the only associations on A that close every $\mathfrak{B}$-theory. A Galois connection underlies these facts (s. [2.5]).

[2.31] Henceforth, the phrase *by the way* means *with interest outside MN*.

2.6.1 Control of logic spaces by their largest logics

The purpose of this subsection is to show that all the above concepts are controlled by the largest $\mathfrak{B}$-logic and clarify its nature.

Theorem 2.6.1 Let $(A, \mathfrak{B})$ be a logic space. Then there exists the largest $\mathfrak{B}$-logic on A. Let Q denote it. Then the following hold.

(1) An association R on A is a $\mathfrak{B}$-logic if (and only if) R is contained in Q.

(2) The set of the $\mathfrak{B}$-theories in A is equal to that of the Q-subsets of A.

(3) Let X be a subset of A. Then the Q-closure $[X]_Q$ of X in A is the smallest of the $\mathfrak{B}$-theories in A which contain X.

Proof Let Q be the union of all $\mathfrak{B}$-logics on A regarded as subsets of $A^* \times A$. If $\alpha \subseteq B \in \mathfrak{B}$ and $\alpha \, Q \, y$, then $\alpha \, R \, y$ for some $\mathfrak{B}$-logic R, and so $y \in B$. Thus Q is also a $\mathfrak{B}$-logic and therefore is the largest one.

(1) Let R be an association on A contained in Q. Then every set in $\mathfrak{B}$ is a Q-subset, and therefore is an R-subset by Theorem 2.4.7. Thus R is a $\mathfrak{B}$-logic.

(2) Every $\mathfrak{B}$-theory is a Q-subset because Q is a $\mathfrak{B}$-logic. Every Q-subset is closed by every $\mathfrak{B}$-logic again by Theorem 2.4.7 and therefore is a $\mathfrak{B}$-theory.

(3) This is a direct consequence of (2) and Definition 2.4.1.

Theorem 2.6.2 Let $(A, \mathfrak{B})$ be a logic space and Q be the largest $\mathfrak{B}$-logic on A. Then the following holds for all $(\alpha, y) \in A^* \times A$:

$$\alpha \, Q \, y \iff y \in \textstyle\bigcap_{\alpha \subseteq B \in \mathfrak{B}} B.$$

Proof Define an association P on A so that $\alpha \, P \, y$ iff $y \in \bigcap_{\alpha \subseteq B \in \mathfrak{B}} B$. If $\alpha \subseteq B \in \mathfrak{B}$ and $\alpha \, P \, y$, then $y \in B$ by the definition of P. Therefore P is a $\mathfrak{B}$-logic, and so $P \subseteq Q$. If $\alpha \, Q \, y$ and $\alpha \subseteq B \in \mathfrak{B}$, then $y \in B$ because Q is a $\mathfrak{B}$-logic. Therefore $Q \subseteq P$. Thus $Q = P$.

Theorem 2.6.3 Let $(A, \mathfrak{B})$ be a logic space and Q be the largest $\mathfrak{B}$-logic on A. Then Q is partially latticed, and the following hold for all $X \in \mathfrak{P}A$ and for all $\alpha \in A^*$:

$$[X]_Q = \{y \in A : \alpha \, Q \, y \text{ for an element } \alpha \in X^*\}, \qquad [\alpha]_Q = \{y \in A : \alpha \, Q \, y\}.$$

Proof For each $B \in \mathfrak{B}$, define an association $\models_B$ on A as in Theorem 2.2.23. Then $Q = \bigcap_{B \in \mathfrak{B}} \models_B$ by Theorem 2.6.2. Thus the former statement holds by Theorem 2.2.23 and Corollary 2.5.9.1. The latter statement is a direct consequence of the former and Theorem 2.4.6.

Alternative proof It follows from Theorems 2.5.3 and 2.5.1 that $Q^\emptyset$ is a $\mathfrak{B}$-logic and $Q \subseteq Q^\emptyset$. Therefore $Q = Q^\emptyset$, and so Q is partially latticed by Theorem 2.5.2, and the latter equation holds. Since the mapping $X \mapsto [X]_Q$ is finitary by Theorem 2.4.5, the former follows from the latter.

Theorem 2.6.4 Let $(A, \mathfrak{B})$ be a logic space and Q be the largest $\mathfrak{B}$-logic on A. Then the $\mathfrak{B}$-core C of A satisfies $C = A_Q = [\emptyset]_Q$.

Proof This is a consequence of Theorems 2.6.2 and 2.6.3 (s. Theorem 2.4.6).

Theorem 2.6.5 Let $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ be logic spaces. Then the following three conditions are equivalent.

(1) The set of the $\mathfrak{B}_0$-logics on A is equal to that of the $\mathfrak{B}_1$-logics on A.

(2) The set of the $\mathfrak{B}_0$-theories in A is equal to that of the $\mathfrak{B}_1$-theories in A.

(3) The largest $\mathfrak{B}_0$-logic on A is equal to the largest $\mathfrak{B}_1$-logic on A.

Under these conditions, the $\mathfrak{B}_0$-core of A is equal to the $\mathfrak{B}_1$-core of A.

Proof As for the former statement, obviously (1) implies (3), while Theorem 2.6.1 and the remark before it show that (3) implies (2) and (2) implies (1). The latter statement holds by virtue of Theorem 2.6.4.

2.6.2 Functional logic spaces and their validity relations

Let $\mathbb{B}$ be a nontrivial bounded lattice. Then a $\mathbb{B}$-valued **functional logic space** is a pair $(A, \mathcal{F})$ of a nonempty set A and a subset $\mathcal{F}$ of $A \to \mathbb{B}$. In order to associate it with a logic space, we define a subset $A_{f,a}$ of A for each $(f, a) \in \mathcal{F} \times \mathbb{B}$ by

$$A_{f,a} = \{x \in A : fx \geq a\}. \tag{2.6.1}$$

Then the following holds for $0 = \min \mathbb{B}$, $1 = \max \mathbb{B}$ and all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$A = A_{f,0} \supseteq A_{f,a} \supseteq A_{f,1} = \{x \in A : fx = 1\} = f^{-1}\{1\}. \tag{2.6.2}$$

Furthermore, we define a subset $\mathfrak{B}_{\mathcal{F}}$ of $\mathfrak{P}A$ by

$$\mathfrak{B}_{\mathcal{F}} = \begin{cases} \{A_{f,a} : (f, a) \in \mathcal{F} \times \mathbb{B}\} & \text{if } \mathcal{F} \neq \emptyset, \\ \{A\} & \text{if } \mathcal{F} = \emptyset. \end{cases} \tag{2.6.3}$$

Thus we associate $(A, \mathcal{F})$ with the logic space $(A, \mathfrak{B}_{\mathcal{F}})$, for which we have defined various concepts, i.e. the $\mathfrak{B}_{\mathcal{F}}$-logics, $\mathfrak{B}_{\mathcal{F}}$-theories, $\mathfrak{B}_{\mathcal{F}}$-core, and so on. We call them the **$\mathcal{F}$-logic**s, **$\mathcal{F}$-theorie**s, **$\mathcal{F}$-core**, and so on, and likewise for the concepts[2.32] to be defined for logic spaces except that of **extension**s in §2.9.

The purpose of this subsection is to relate the above concepts with the values of the functions in $\mathcal{F}$ as in Theorems 2.6.6 and 2.6.7, which together with Remark 2.4.2 will help you understand the meanings of the concepts especially when $\mathbb{B} = \mathbb{T}$ or more generally when $(A, \mathcal{F})$ is **extremal** in the sense that $fA \subseteq \{0, 1\}$ for all $f \in \mathcal{F}$.

[2.32] They will be found in §2.7.2, Remark 2.8.2, Theorems 2.8.3, 2.8.9, 2.8.11, and so on.

Remark 2.6.1 If $(A, \mathcal{F})$ is an extremal functional logic space, then $A_{f,a} = f^{-1}\{1\}$ for all $(f, a) \in \mathcal{F} \times (\mathbb{B} - \{0\})$, and so $\mathfrak{B}_{\mathcal{F}} = \{f^{-1}\{1\} : f \in \mathcal{F}\} \cup \{A\}$.

To start with, we define an association $\vDash_f$ on A for each $f \in \mathcal{F}$ by

$$\alpha \vDash_f y \iff \inf f\alpha \leq fy. \tag{2.6.4}$$

We call $\vDash_f$ the **f-validity association**,[2.33] because it is equal to the restriction to $A^* \times A$ of the f-validity relation $\preccurlyeq_f$ defined on A^* by (2.2.1):

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta.$$

Furthermore, we define an association $\vDash_{\mathcal{F}}$ on A by

$$\alpha \vDash_{\mathcal{F}} y \iff \alpha \vDash_f y \text{ for all } f \in \mathcal{F}, \tag{2.6.5}$$

which we call the **$\mathcal{F}$-validity association** (s. [2.33]). It is equal to the restriction to $A^* \times A$ of the **$\mathcal{F}$-validity relation** $\preccurlyeq_{\mathcal{F}}$ defined on A^* by

$$\alpha \preccurlyeq_{\mathcal{F}} \beta \iff \alpha \preccurlyeq_f \beta \text{ for all } f \in \mathcal{F}. \tag{2.6.6}$$

Remark 2.6.2 Theorem 2.6.6 shows why $\vDash_{\mathcal{F}}$ is important, while $\preccurlyeq_{\mathcal{F}}$ is as important as the deduction relation $\preccurlyeq_{R,D}$, for $\preccurlyeq_{R,D}$ is the largest latticed extension of the deduction association $\vDash_{R,D}$ by Theorem 2.5.2 and $\preccurlyeq_{\mathcal{F}}$ is that of $\vDash_{\mathcal{F}}$ in certain cases as in Theorem 2.6.9.

Theorem 2.6.6 Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space. Then the largest $\mathcal{F}$-logic on A is equal to $\vDash_{\mathcal{F}}$. Moreover, elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \vDash_{\mathcal{F}} y$ iff they satisfy the following condition for all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a \implies fy \geq a. \tag{2.6.7}$$

Proof The **largest $\mathcal{F}$-logic** is the largest $\mathfrak{B}_{\mathcal{F}}$-logic, as was defined above. Let Q denote it. If $\mathcal{F} = \emptyset$, then (2.6.3) and (2.6.5) show that both Q and $\vDash_{\mathcal{F}}$ are the trivial relation on A, and (2.6.7) is empty. Therefore assume $\mathcal{F} \neq \emptyset$. Then Theorem 2.6.2 shows that elements $x_1, \ldots, x_n, y \in A$ satisfy $x_1 \cdots x_n \, Q \, y$ iff they satisfy (2.6.7) for all $(f, a) \in \mathcal{F} \times \mathbb{B}$. Elements $x_1, \ldots, x_n, y \in A$ and $f \in \mathcal{F}$ satisfy (2.6.7) for all $a \in \mathbb{B}$ iff they satisfy $\inf\{fx_1, \ldots, fx_n\} \leq fy$, that is, iff $x_1 \cdots x_n \vDash_f y$. Thus Q is equal to $\vDash_{\mathcal{F}}$. This completes the proof.

Theorem 2.6.7 Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space. Then the following hold.

(1) An association R on A is an $\mathcal{F}$-logic iff it is contained in $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition for all $(f, a) \in \mathcal{F} \times \mathbb{B}$:

$$fx_1 \geq a, \ \ldots, \ fx_n \geq a, \ x_1 \cdots x_n \, R \, y \implies fy \geq a.$$

[2.33] A validity association was called a partial validity relation in [1.5] and others (s. [2.1]).

(2) A subset X of A is an $\mathcal{F}$-theory iff it is closed by $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition:

$$x_1, \ldots, x_n \in X,\ y \in A - X \\ \implies fx_1 \geq a,\ \ldots,\ fx_n \geq a \text{ and } fy \not\geq a \text{ for some } (f, a) \in \mathcal{F} \times \mathbb{B}.$$

(3) An element $x \in A$ belongs to the $\mathcal{F}$-core iff $\varepsilon \vDash_{\mathcal{F}} x$, that is, iff $fx = 1$ for all $f \in \mathcal{F}$.

Proof This is a restatement of part of Theorems 2.6.1 and 2.6.4 by means of Theorem 2.6.6, (2.6.4) and (2.6.5).

Theorem 2.6.8 Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space. Then $\preccurlyeq_{\mathcal{F}}$ is latticed. If fA is contained in a distributive sublattice of $\mathbb{B}$ for each $f \in \mathcal{F}$, then $\preccurlyeq_{\mathcal{F}}$ is strongly latticed. Moreover, $\vDash_{\mathcal{F}}$ is partially latticed.

Proof In view of (2.6.5), (2.6.6) and Corollary 2.5.9.1, we assume $\mathcal{F} = \{f\}$. Then $\preccurlyeq_{\mathcal{F}} = \preccurlyeq_f$ and $\vDash_{\mathcal{F}} = \vDash_f$. Theorems 2.2.1 and 2.2.2 show that $\preccurlyeq_f$ is latticed and that if fA is contained in a distributive sublattice of $\mathbb{B}$ then $\preccurlyeq_f$ is strongly latticed. Therefore $\vDash_f$ is partially latticed by Theorem 2.2.10.

Definition 2.6.1 If two logic spaces $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ satisfy the three equivalent conditions of Theorem 2.6.5, we say that they are **equivalent** or write $(A, \mathfrak{B}_0) \sim (A, \mathfrak{B}_1)$. Likewise, they are said to be **core equivalent** if their cores are equal. By extension, two functional logic spaces $(A, \mathcal{F})$ and $(A, \mathcal{F}')$ are said to be equivalent if $(A, \mathfrak{B}_{\mathcal{F}}) \sim (A, \mathfrak{B}_{\mathcal{F}'})$. Moreover, a functional logic space $(A, \mathcal{F})$ and a logic space $(A, \mathfrak{B})$ are said to be equivalent if $(A, \mathfrak{B}_{\mathcal{F}}) \sim (A, \mathfrak{B})$. The concept of core equivalence and the use of the symbol $\sim$ are similarly extended to functional logic spaces.

Remark 2.6.3 It follows from Definition 2.6.1 and Theorem 2.6.5 that the equivalence between (functional) logic spaces implies the core equivalence between them. Moreover, both equivalences satisfy the equivalence law. Obviously, every functional logic space is equivalent to its associated logic space.

Lemma 2.6.1 Let $(A, \mathfrak{B})$ be a logic space. Then the following hold.

(1) $(A, \mathfrak{B})$ is equivalent to the logic spaces $(A, \mathfrak{B} \cup \{A\})$ and $(A, \mathfrak{B} - \{A\})$.

(2) Let $\mathcal{F}_{\mathfrak{B}}$ be the set of the characteristic functions of the sets in $\mathfrak{B}$. Then the $\mathbb{T}$-valued functional logic space $(A, \mathcal{F}_{\mathfrak{B}})$ is equivalent to $(A, \mathfrak{B})$ (we call $(A, \mathcal{F}_{\mathfrak{B}})$ the **functionalization** of $(A, \mathfrak{B})$).

Proof (1) is because every association on A closes A. (2) follows from (1) because $\mathfrak{B}_{\mathcal{F}_{\mathfrak{B}}} = \mathfrak{B} \cup \{A\}$ by Remark 2.6.1.

Remark 2.6.4 Let $(A, \mathcal{F})$ be a nontrivial $\mathbb{B}$-valued functional logic space and $(A, \{\varphi\})$ be the $\mathbb{B}^{\mathcal{F}}$-valued functional logic space constructed in Remark 2.2.2. Then Remark 2.1.3 shows that the following holds for all $(x_1 \cdots x_m, y_1 \cdots y_n) \in A^* \times A^*$:

$$
\begin{aligned}
&\inf\{\varphi x_1, \ldots, \varphi x_m\} \leq \sup\{\varphi y_1, \ldots, \varphi y_n\} \\
&\iff (\inf\{\varphi x_1, \ldots, \varphi x_m\})\, f \leq (\sup\{\varphi y_1, \ldots, \varphi y_n\})\, f \text{ for all } f \in \mathcal{F} \\
&\iff \inf\{(\varphi x_1)f, \ldots, (\varphi x_m)f\} \leq \sup\{(\varphi y_1)f, \ldots, (\varphi y_n)f\} \text{ for all } f \in \mathcal{F} \\
&\iff \inf\{f x_1, \ldots, f x_m\} \leq \sup\{f y_1, \ldots, f y_n\} \text{ for all } f \in \mathcal{F}.
\end{aligned}
$$

Thus $\preccurlyeq_\varphi$ is equal to $\preccurlyeq_{\mathcal{F}}$, and so $(A, \mathcal{F}) \sim (A, \{\varphi\})$ by Theorem 2.6.6.

Remark 2.6.5 Remark 2.6.3 and Lemma 2.6.1 show that every (functional) logic space is equivalent to a $\mathbb{T}$-valued functional logic space. Moreover, Remark 2.6.4 shows that every nontrivial functional logic space is equivalent to the functional logic space derived from a latticed representation.

These facts, however, do not reduce the study of (functional) logic spaces to that of $\mathbb{T}$-valued functional logic spaces or to that of latticed representations, because certain concepts on logic spaces $(A, \mathfrak{B})$ such as those of classes (Definition 2.6.3), $\mathfrak{B}$-models (Definition 2.8.2) and $\mathfrak{B}$-negations (Definition 2.8.3) are characteristic of $\mathfrak{B}$ itself and not of the largest $\mathfrak{B}$-logic, and so equivalent logic spaces may make marked differences with respect to the concepts.

Moreover, $\mathfrak{B}$ is characteristic of the semantics of the logic system which yields $(A, \mathfrak{B})$, and semantics for MN should provide a mathematical model of the nooworlds and the relationship between the IU and the nooworlds as was explained in Chapter 1. Thus we are not free to transfer to equivalent logic spaces.

2.6.3 Boolean validity relations for Boolean logic spaces

Definition 2.6.2 Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space and $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ be operations on A. Assume that $\mathbb{B}$ is a Boolean lattice and every function in $\mathcal{F}$ is a Boolean representation of A with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$. Then we call $(A, \mathcal{F})$ a **Boolean logic space** with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$; in case $\mathbb{B} = \mathbb{T}$, we call it a **binary logic space**.

Example 2.6.1 As is shown in Chapters 3–6, the functional logic space on CL as well as PL and FPL is binary.

Theorem 2.6.9 Let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ on A. Then the following hold.

(1) The $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ is Boolean with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ and is the largest latticed extension of $\vDash_{\mathcal{F}}$ and of a preorder on A.

(2) Define the functions 1_A and 0_A in $A \to \mathbb{B}$ by $1_A x = 1$ and $0_A x = 0$ for each $x \in A$, where $1 = \max \mathbb{B}$ and $0 = \min \mathbb{B}$ as usual. Then $1_A, 0_A \notin \mathcal{F}$.

(3) $(A, \mathcal{F})$ is extremal iff the operation $\Diamond$ satisfies the following law for all $f \in \mathcal{F}$:

$$x^{\Diamond} \in f^{-1}\{1\} \iff x \notin f^{-1}\{1\}. \qquad (f^{-1}\{1\}\text{-negation})$$

Proof (1) Since $\preccurlyeq_f$ is Boolean for each $f \in \mathcal{F}$ by Theorem 2.2.9, so is $\preccurlyeq_{\mathcal{F}}$ by Corollary 2.5.9.1, and so $\preccurlyeq_{\mathcal{F}}$ is the largest latticed extension both of $\vDash_{\mathcal{F}}$ and of a preorder on A by Theorem 2.2.22.

(2) If $1_A \in \mathcal{F}$, then $1 = 1_A(x^{\Diamond}) = (1_A x)^{\Diamond} = 1^{\Diamond} = 0$ which is a contradiction, and likewise for 0_A.

(3) First assume that $(A, \mathcal{F})$ is extremal, and let $f \in \mathcal{F}$. If $x^{\Diamond} \in f^{-1}\{1\}$, then $1 = f(x^{\Diamond}) = (fx)^{\Diamond}$, and so $x \notin f^{-1}\{1\}$. Conversely if $x \notin f^{-1}\{1\}$, then since $(A, \mathcal{F})$ is extremal, we have $fx = 0$, and so $f(x^{\Diamond}) = (fx)^{\Diamond} = 1$. Thus the $f^{-1}\{1\}$-negation law holds. Next assume that the $f^{-1}\{1\}$-negation law holds for all $f \in \mathcal{F}$. If $fx \neq 1$ for some $f \in \mathcal{F}$ and $x \in A$, then $(fx)^{\Diamond} = f(x^{\Diamond}) = 1$ by the $f^{-1}\{1\}$-negation law, and so $fx = (fx)^{\Diamond\Diamond} = 1^{\Diamond} = 0$. Thus $(A, \mathcal{F})$ is extremal.

Remark 2.6.6 Every $\mathbb{B}$-valued Boolean logic space $(A, \mathcal{F})$ is equivalent to a binary logic space by Corollary 2.8.12.3. It also follows that if $\mathcal{F} \neq \emptyset$ then the latticed representation $\varphi \in A \to \mathbb{B}^{\mathcal{F}}$ constructed in Remark 2.2.2 is Boolean, and so $(A, \mathcal{F})$ is equivalent to the Boolean logic space $(A, \{\varphi\})$. For the same reason as in Remark 2.6.5, however, these facts do not reduce the study of Boolean logic spaces to that of binary ones or Boolean representations.

2.6.4 Generation of theories and classes of logic spaces

The purpose of this subsection it to show in Theorem 2.6.10 how the theories of a logic space are generated by the given theories and use it for classifying logic spaces in Definition 2.6.3. Theorem 2.6.10 is also crucial to §2.8.

Theorem 2.6.10 Let $(A, \mathfrak{B})$ be a logic space. Then the set of the $\mathfrak{B}$-theories in A is equal to the quasi-finitary $\cap$-closure $\overline{\mathfrak{B}^{\cap}}$ of $\mathfrak{B}$ in $\mathfrak{P}A$.

Proof Let $\mathfrak{T}$ be the set of the $\mathfrak{B}$-theories and Q be the largest $\mathfrak{B}$-logic. Then $\mathfrak{B} \subseteq \mathfrak{T}$. Moreover, $\mathfrak{T}$ is the set of the Q-subsets by Theorem 2.6.1, and therefore is $\cap$-closed in $\mathfrak{P}A$ and quasi-finitary by Theorem 2.4.5. Therefore $\overline{\mathfrak{B}^{\cap}} \subseteq \mathfrak{T}$ by Theorem 2.1.7. It remains to show that every set X in $\mathfrak{T}$ is super-covered by $\mathfrak{B}^{\cap}$. Therefore let $Y \in \mathfrak{P}'X$. Define $B' = \bigcap_{Y \subseteq B \in \mathfrak{B}} B$. Then $Y \subseteq B' \in \mathfrak{B}^{\cap}$, and so it suffices to show $B' \subseteq X$. Let $Y = \{y_1, \ldots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in A^*$. Then $B' = \bigcap_{\alpha \subseteq B \in \mathfrak{B}} B$, and so Theorem 2.6.2 shows that every element $y \in B'$ satisfies $\alpha \, Q \, y$. Since $\alpha \subseteq X$ and X is Q-closed, we have $B' \subseteq X$ as desired. This proof and the definition of the quasi-finitarity in Definition 2.1.2 are essentially due to Hitoki Matuda[2.34]. §2.11 suggests an alternative proof.

[2.34] *Kankei syûgô no sonzai teiri* (*Existence theorem for relational sets*), Master's thesis, Graduate School Math. Sci., Univ. Tokyo, 2004.

Corollary 2.6.10.1 Two logic spaces $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ are equivalent iff $\overline{\mathfrak{B}_0{}^{\cap}} = \overline{\mathfrak{B}_1{}^{\cap}}$. Therefore if $\mathfrak{B}_0{}^{\cap} = \mathfrak{B}_1{}^{\cap}$, then they are equivalent.

Proof This is because of Definition 2.6.1.

Remark 2.6.7 Let $(A, \mathcal{F})$ be a nontrivial $\mathbb{B}$-valued functional logic space and $(A, \{\varphi\})$ be the $\mathbb{B}^{\mathcal{F}}$-valued functional logic space constructed in Remark 2.2.2. Then $\mathfrak{B}_{\mathcal{F}} = \{A_{f,a} : (f, a) \in \mathcal{F} \times \mathbb{B}\}$ and $\mathfrak{B}_{\{\varphi\}} = \{A_{\varphi,a} : a \in \mathbb{B}^{\mathcal{F}}\}$ by (2.6.3). Moreover, (2.6.1) shows $A_{\varphi,a} = \{x \in A : \varphi x \geq a\} = \{x \in A : fx \geq af \text{ for all } f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} A_{f,af}$ and $A_{f,a} = A_{\varphi,a}$ for the element $a \in \mathbb{B}^{\mathcal{F}}$ such that $af = a$ and $ag = 0$ for all $g \in \mathcal{F} - \{f\}$. Therefore $\mathfrak{B}_{\mathcal{F}} \subseteq \mathfrak{B}_{\{\varphi\}} \subseteq \mathfrak{B}_{\mathcal{F}}{}^{\cap}$, hence $\mathfrak{B}_{\mathcal{F}}{}^{\cap} = \mathfrak{B}_{\{\varphi\}}{}^{\cap}$ by Theorems 2.1.5 (if $\mathbb{B}$ is complete with respect to supremum, then $\mathfrak{B}_{\{\varphi\}}{}^{\cap} = \mathfrak{B}_{\{\varphi\}}$). Thus $(A, \mathcal{F}) \sim (A, \{\varphi\})$ by Corollary 2.6.10.1.

Definition 2.6.3 Theorem 2.6.10 suggests that we should divide logic spaces into the following three **class**es.

Class 1 consists of the logic spaces $(A, \mathfrak{B})$ such that $\overline{\mathfrak{B}^{\cap}} = \mathfrak{B}$, that is, $\mathfrak{B}$ is $\cap$-closed in $\mathfrak{P}A$ and quasi-finitary.

Class 2 consists of the logic spaces $(A, \mathfrak{B})$ such that $\overline{\mathfrak{B}^{\cap}} = \mathfrak{B}^{\cap} \neq \mathfrak{B}$, that is, $\mathfrak{B}$ is not $\cap$-closed in $\mathfrak{P}A$ but the $\cap$-closure $\mathfrak{B}^{\cap}$ of $\mathfrak{B}$ in $\mathfrak{P}A$ is quasi-finitary.

Class 3 consists of the logic spaces $(A, \mathfrak{B})$ such that $\overline{\mathfrak{B}^{\cap}} \neq \mathfrak{B}^{\cap}$, that is, the $\cap$-closure $\mathfrak{B}^{\cap}$ of $\mathfrak{B}$ in $\mathfrak{P}A$ is not quasi-finitary.

Since $\overline{\mathfrak{B}^{\cap}} \supseteq \mathfrak{B}^{\cap} \supseteq \mathfrak{B}$, each logic space belongs to exactly one of the above classes. We denote the **class number** of the logic space $(A, \mathfrak{B})$ by $\mathrm{Cn}(A, \mathfrak{B})$. Then, $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$ iff $\overline{\mathfrak{B}^{\cap}} = \mathfrak{B}^{\cap}$, and $\mathrm{Cn}(A, \mathfrak{B}) = 1$ iff $\mathfrak{B}^{\cap} = \mathfrak{B} = \overline{\mathfrak{B}}$.

Example 2.6.2 As for Example 2.6.1, the binary logic spaces on PL and FPL belong to the class 2, while CL belongs to the class 2 or 3 according to a certain parameter related to quantification. Example 2.8.1 will give further details.

Remark 2.6.8 It follows from Theorems 2.4.5, 2.6.1 and 2.6.10 or from Theorem 2.7.5 and Corollary 2.7.1.1 that a logic space $(A, \mathfrak{B})$ belongs to the class 1 iff there exists a deduction system (R, D) on A such that $\mathfrak{B}$ is the set of the R-subsets of A which contain D.

Remark 2.6.9 It follows from Corollary 2.6.10.1 that every logic space $(A, \mathfrak{B})$ is equivalent to the logic space $(A, \overline{\mathfrak{B}^{\cap}})$ which belongs to the class 1. Moreover, every Boolean logic space is equivalent to a binary one in the class 1 or 2 by Corollary 2.8.12.3 (s. Theorem 2.8.11). These facts, however, do not reduce the study of logic spaces to that of the ones in the class 1 or 2 (s. Remark 2.6.5).

2.7 Completeness of deduction systems

Here we define the completeness of deduction systems on logic spaces and formulate the fundamental Theorem 2.7.13 on it to obtain a program for finding a complete deduction system.

2.7.1 Completeness for logic spaces

Throughout this subsection, we let $(\mathsf{A}, \mathfrak{B})$ be a logic space, Q be the largest $\mathfrak{B}$-logic on A, C be the $\mathfrak{B}$-core of A, and (R, D) be a deduction system on A.

Definition 2.7.1 In view of §2.5.1, we define the following eight concepts:[2.35]

- (R, D) is said to be **$\mathfrak{B}$-sound** if $\mathsf{R}^{\mathsf{D}} \subseteq \mathsf{Q}$.
- (R, D) is said to be **$\mathfrak{B}$-sufficient** if $\mathsf{Q} \subseteq \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be **$\mathfrak{B}$-complete** if $\mathsf{Q} = \mathsf{R}^{\mathsf{D}}$.
- (R, D) is said to be **core $\mathfrak{B}$-sound** if $[\mathsf{D}]_{\mathsf{R}} \subseteq \mathsf{C}$.
- (R, D) is said to be **core $\mathfrak{B}$-sufficient** if $\mathsf{C} \subseteq [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be **core $\mathfrak{B}$-complete** if $\mathsf{C} = [\mathsf{D}]_{\mathsf{R}}$.
- (R, D) is said to be **extra-$\mathfrak{B}$-complete**, if it is $\mathfrak{B}$-sound and all R-subsets of A which contain D belong to the $\cap$-closure $\mathfrak{B}^{\cap}$ of $\mathfrak{B}$ in $\mathfrak{P}\mathsf{A}$.
- (R, D) is said to be **super-$\mathfrak{B}$-complete**, if $\mathfrak{B}$ is the set of the R-subsets of A which contain D.

The purpose of the rest of this subsection is to explain meanings of the above concepts and interrelations between them and others. First of all, it is obvious that (R, D) is $\mathfrak{B}$-complete iff it is $\mathfrak{B}$-sound and $\mathfrak{B}$-sufficient. Similarly, (R, D) is core $\mathfrak{B}$-complete iff it is core $\mathfrak{B}$-sound and core $\mathfrak{B}$-sufficient. It should be mentioned in this regard that $[\mathsf{D}]_{\mathsf{R}}$ and C are the R^{D}-core and the Q-core of A by Theorems 2.5.1 and 2.6.4 respectively (s. Theorems 2.7.2 and 2.7.3).

Theorem 2.7.1 The following four conditions are equivalent.

(1) (R, D) is $\mathfrak{B}$-sound.

(2) $[\mathsf{X} \cup \mathsf{D}]_{\mathsf{R}} \subseteq [\mathsf{X}]_{\mathsf{Q}}$ for every subset X of A (s. Remark 2.4.8).

(3) Every $\mathfrak{B}$-theory in A is an R-subset of A containing D.

(4) $\mathsf{R} \subseteq \mathsf{Q}$ and $\mathsf{D} \subseteq \mathsf{C}$.

Moreover, the following three conditions are equivalent.

[2.35] Here we slightly alter the terminology of [1.5]. For example, the core $\mathfrak{B}$-completeness was called the $\mathfrak{B}$-core-completeness there.

(5) (R, D) is $\mathfrak{B}$-sufficient.

(6) $[X]_Q \subseteq [X \cup D]_R$ for every subset X of A (s. Remark 2.4.8).

(7) Every R-subset of A containing D is a $\mathfrak{B}$-theory in A.

Therefore, the following four conditions are equivalent.

(8) (R, D) is $\mathfrak{B}$-complete.

(9) $[X]_Q = [X \cup D]_R$ for every subset X of A (s. Remark 2.4.8).

(10) The set of the $\mathfrak{B}$-theories in A is equal to the set of the R-subsets of A which contain D.

(11) $R \subseteq Q$, $D \subseteq C$, and $Q \subseteq R^D$.

Proof This follows from Definition 2.7.1 and Theorem 2.5.7, because Q is partially latticed by Theorem 2.6.3, the set of the $\mathfrak{B}$-theories is equal to that of the Q-subsets by Theorem 2.6.1, and $C = A_Q$ by Theorem 2.6.4.

Corollary 2.7.1.1 Let D be a subset of A. Then the deduction system (Q, D) on A is $\mathfrak{B}$-sufficient. Moreover, it is $\mathfrak{B}$-sound iff $D \subseteq C$. In particular, the deduction system $(Q, \emptyset)$ on A is $\mathfrak{B}$-complete.

Proof Since $Q \subseteq Q^D$ by Theorem 2.5.1, (Q, D) is $\mathfrak{B}$-sufficient by Definition 2.7.1. Moreover, Theorem 2.7.1 shows that it is $\mathfrak{B}$-sound iff $D \subseteq C$.

Theorem 2.7.2 If (R, D) is $\mathfrak{B}$-sound, then it is core $\mathfrak{B}$-sound. If it is $\mathfrak{B}$-sufficient, then it is core $\mathfrak{B}$-sufficient. Therefore if it is $\mathfrak{B}$-complete, then it is core $\mathfrak{B}$-complete.

Proof If (R, D) is $\mathfrak{B}$-sound, then $A_{R^D} \subseteq A_Q$, and so (R, D) is core $\mathfrak{B}$-sound by Theorems 2.5.1 and 2.6.4. The rest may be proved similarly.

Theorem 2.7.3 Assume that (R, D) is core $\mathfrak{B}$-sufficient and there exists a mapping $\phi \in A^* \times A \to A$ which satisfies the following for all $(\alpha, y) \in A^* \times A$:

$$\alpha \, Q \, y \implies \varepsilon \, Q \, \phi(\alpha, y), \qquad \varepsilon \, R^D \, \phi(\alpha, y) \implies \alpha \, R^D \, y.$$

Then (R, D) is $\mathfrak{B}$-sufficient (s. Theorem 2.7.12).

Proof This is because $A_Q \subseteq A_{R^D}$ by Theorems 2.6.4 and 2.5.1, and so the following holds: $\alpha \, Q \, y \implies \varepsilon \, Q \, \phi(\alpha, y) \implies \varepsilon \, R^D \, \phi(\alpha, y) \implies \alpha \, R^D \, y$.

Theorem 2.7.4 The following conditions are equivalent.

(1) (R, D) is extra-$\mathfrak{B}$-complete.

(2) (R, D) is $\mathfrak{B}$-complete and $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$.

Proof Let $\mathfrak{X}$ be the set of the R-subsets which contain D, and assume (1). Then $\mathfrak{X} \subseteq \mathfrak{B}^\cap$, and $\mathfrak{B}^\cap \subseteq \overline{\mathfrak{B}^\cap}$ by Theorem 2.1.6. Consequently $\mathfrak{X} \subseteq \overline{\mathfrak{B}^\cap}$, and so (R, D) is $\mathfrak{B}$-sufficient by Theorems 2.6.10 and 2.7.1. Therefore (R, D) is $\mathfrak{B}$-complete, and so $\overline{\mathfrak{B}^\cap} = \mathfrak{X}$ by the two theorems. Therefore $\overline{\mathfrak{B}^\cap} = \mathfrak{B}^\cap$, that is, $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) \leq 2$. Thus (2) holds. Conversely if (2) holds, then (R, D) is $\mathfrak{B}$-sound and $\mathfrak{X} = \overline{\mathfrak{B}^\cap} = \mathfrak{B}^\cap$ by the two theorems, and so (1) holds.

Theorem 2.7.5 The following conditions are equivalent.

(1) (R, D) is super-$\mathfrak{B}$-complete.

(2) (R, D) is extra-$\mathfrak{B}$-complete and $\mathfrak{B}^\cap = \mathfrak{B}$.

(3) (R, D) is $\mathfrak{B}$-complete and $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = \mathsf{1}$.

Proof Theorem 2.7.4 shows that (2) is equivalent to (3), and so it suffices to show that (1) and (3) are equivalent. Let $\mathfrak{X}$ be the set of the R-subsets which contain D, and assume (1). Then $\mathfrak{X} = \mathfrak{B}$, and hence the following three consequences. First, $\mathsf{R} \subseteq \mathsf{Q}$ because R closes every set in $\mathfrak{X}$. Secondly, $\mathsf{D} \subseteq \bigcap \mathfrak{X} = \bigcap \mathfrak{B} = \mathsf{C}$. Thirdly, every set in $\mathfrak{X}$ is a $\mathfrak{B}$-theory. Therefore, (R, D) is $\mathfrak{B}$-complete by Theorem 2.7.1. Furthermore, $\overline{\mathfrak{B}^\cap} = \mathfrak{X} = \mathfrak{B}$ by Theorems 2.6.10 and 2.7.1. Thus (3) holds. Conversely if (3) holds, then $\mathfrak{X} = \overline{\mathfrak{B}^\cap} = \mathfrak{B}$ by the same theorems, and so (1) holds.

Theorem 2.7.6 The following hold on the $\mathfrak{B}$-completeness of the deduction systems on A and $\mathrm{Cn}(\mathsf{A}, \mathfrak{B})$.

(1) If a deduction system on A is super-$\mathfrak{B}$-complete, then $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = \mathsf{1}$. Conversely if $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = \mathsf{1}$, then every $\mathfrak{B}$-complete deduction system on A is super-$\mathfrak{B}$-complete.

(2) If a deduction system on A is extra-$\mathfrak{B}$-complete and not super-$\mathfrak{B}$-complete, then $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 2$. Conversely if $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 2$, then every $\mathfrak{B}$-complete deduction system on A is extra-$\mathfrak{B}$-complete and not super-$\mathfrak{B}$-complete.

(3) If a deduction system on A is $\mathfrak{B}$-complete and not extra-$\mathfrak{B}$-complete, then $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 3$. Conversely if $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 3$, then no deduction system on A is extra-$\mathfrak{B}$-complete.

Proof (1) and (3) are restatements of part of Theorems 2.7.5 and 2.7.4 respectively. If a deduction system on A is extra-$\mathfrak{B}$-complete and not super-$\mathfrak{B}$-complete, then it is $\mathfrak{B}$-complete and $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) \leq 2$ by Theorem 2.7.4, and so $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 2$ by Theorem 2.7.5. Conversely if $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) = 2$, then every $\mathfrak{B}$-complete deduction system on A is extra-$\mathfrak{B}$-complete by Theorem 2.7.4 and not super-$\mathfrak{B}$-complete by Theorem 2.7.5.

Theorem 2.7.7 Two logic spaces $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ are equivalent iff the set of the $\mathfrak{B}_0$-sound deduction systems on A is equal to that of the $\mathfrak{B}_1$-sound deduction systems on A, and likewise for the sufficiency and the completeness.

Moreover, $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ are core equivalent iff the set of the core $\mathfrak{B}_0$-sound deduction systems on A is equal to that of the core $\mathfrak{B}_1$-sound deduction systems on A, and likewise for the core sufficiency and the core completeness.

Proof Let Q_i be the largest $\mathfrak{B}_i$-logic on A $(i = 0, 1)$. If $(A, \mathfrak{B}_0) \sim (A, \mathfrak{B}_1)$, then $Q_0 = Q_1$ by Definition 2.6.1, and so Definition 2.7.1 shows that the set of the $\mathfrak{B}_0$-sound deduction systems on A is equal to that of the $\mathfrak{B}_1$-sound deduction systems on A, and likewise for the sufficiency and the completeness.

The deduction system $(Q_i, \emptyset)$ on A is $\mathfrak{B}_i$-complete by Corollary 2.7.1.1, and ${Q_i}^{\emptyset} = Q_i$ by Theorems 2.5.4 and 2.6.3 $(i = 0, 1)$. Therefore, if the set of the $\mathfrak{B}_0$-sound deduction systems on A is equal to that of the $\mathfrak{B}_1$-sound deduction systems on A, then $Q_0 = Q_1$ by Definition 2.7.1 and so $(A, \mathfrak{B}_0) \sim (A, \mathfrak{B}_1)$, and likewise for the sufficiency and the completeness.

The statement on the core equivalence may be proved similarly, because $(Q_i, \emptyset)$ is core $\mathfrak{B}_i$-complete by Theorem 2.7.2 and $[\emptyset]_{Q_i}$ is the $\mathfrak{B}_i$-core of A by Theorem 2.6.4 $(i = 0, 1)$.

Theorem 2.7.8 Let $(A, \mathfrak{B}_0)$ and $(A, \mathfrak{B}_1)$ be logic spaces. Then if ${\mathfrak{B}_0}^{\cap} = {\mathfrak{B}_1}^{\cap}$, the set of the extra-$\mathfrak{B}_0$-complete deduction systems on A is equal to that of the extra-$\mathfrak{B}_1$-complete deduction systems on A. The converse is true provided that the sets are nonempty.

Proof Assume ${\mathfrak{B}_0}^{\cap} = {\mathfrak{B}_1}^{\cap}$. Then $(A, \mathfrak{B}_0) \sim (A, \mathfrak{B}_1)$ by Corollary 2.6.10.1, and a deduction system on A is $\mathfrak{B}_0$-sound iff it is $\mathfrak{B}_1$-sound by Theorem 2.7.7. Thus it is extra-$\mathfrak{B}_0$-complete iff it is extra-$\mathfrak{B}_1$-complete.

Conversely assume that the set of the extra-$\mathfrak{B}_0$-complete deduction systems on A is equal to that of the extra-$\mathfrak{B}_1$-complete deduction systems on A and that they are nonempty. Then Theorem 2.7.4 shows that $\mathrm{Cn}(A, \mathfrak{B}_i) \leq 2$ $(i = 0, 1)$ and that the set of the $\mathfrak{B}_0$-complete deduction systems on A is equal to that of the $\mathfrak{B}_1$-complete deduction systems on A. Therefore $(A, \mathfrak{B}_0) \sim (A, \mathfrak{B}_1)$ by Theorem 2.7.7, and so ${\mathfrak{B}_0}^{\cap} = \overline{{\mathfrak{B}_0}^{\cap}} = \overline{{\mathfrak{B}_1}^{\cap}} = {\mathfrak{B}_1}^{\cap}$ by Corollary 2.6.10.1.

2.7.2 Functional aspects of completeness

Throughout this subsection, we let $(A, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space and (R, D) be a deduction system on A. We have associated $(A, \mathcal{F})$ with the logic space $(A, \mathfrak{B}_{\mathcal{F}})$ in §2.6.2, and defined the $\mathfrak{B}_{\mathcal{F}}$-soundness, $\mathfrak{B}_{\mathcal{F}}$-sufficiency, $\mathfrak{B}_{\mathcal{F}}$-completeness, and so on of (R, D) in §2.7.1. We call them the **$\mathcal{F}$-soundness, $\mathcal{F}$-sufficiency, $\mathcal{F}$-completeness,** and so on of (R, D), as [2.32] noticed.

The purpose of this subsection is to relate these concepts with the values of the functions in $\mathcal{F}$ as in Theorem 2.7.9, which together with Remark 2.5.1

will help you understand the meanings of the concepts especially when $(\mathsf{A}, \mathcal{F})$ is extremal, and compare them with their prototypes[2.36].

Theorem 2.7.9 The following hold (s. Theorem 2.6.7).

(1) (R, D) is $\mathcal{F}$-sound iff $\vDash_{\mathsf{R},\mathsf{D}}$ is contained in $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition for all $(\mathsf{f}, \mathsf{a}) \in \mathcal{F} \times \mathbb{B}$:

$$\mathsf{f}x_1 \geq \mathsf{a},\ \ldots,\ \mathsf{f}x_n \geq \mathsf{a},\ x_1 \cdots x_n \vDash_{\mathsf{R},\mathsf{D}} y \implies \mathsf{f}y \geq \mathsf{a}.$$

(2) (R, D) is $\mathcal{F}$-sufficient iff $\vDash_{\mathcal{F}}$ is contained in $\vDash_{\mathsf{R},\mathsf{D}}$, that is, iff it satisfies the following condition:

$$\begin{aligned} &x_1 \cdots x_n \nvDash_{\mathsf{R},\mathsf{D}} y \\ &\quad \implies \mathsf{f}x_1 \geq \mathsf{a},\ \ldots,\ \mathsf{f}x_n \geq \mathsf{a} \text{ and } \mathsf{f}y \ngeq \mathsf{a} \text{ for some } (\mathsf{f}, \mathsf{a}) \in \mathcal{F} \times \mathbb{B}. \end{aligned}$$

(3) (R, D) is $\mathcal{F}$-complete iff $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to $\vDash_{\mathcal{F}}$, that is, iff it satisfies the following condition:

$$\begin{aligned} &x_1 \cdots x_n \nvDash_{\mathsf{R},\mathsf{D}} y \\ &\quad \iff \mathsf{f}x_1 \geq \mathsf{a},\ \ldots,\ \mathsf{f}x_n \geq \mathsf{a} \text{ and } \mathsf{f}y \ngeq \mathsf{a} \text{ for some } (\mathsf{f}, \mathsf{a}) \in \mathcal{F} \times \mathbb{B}. \end{aligned}$$

Proof Since $\vDash_{\mathcal{F}}$ is the largest $\mathcal{F}$-logic on A by Theorem 2.6.6 and $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to R^{D}, this is a restatement of part of Definition 2.7.1.

Corollary 2.7.9.1 The following hold.

(1) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is contained in $\preccurlyeq_{\mathcal{F}}$, then (R, D) is $\mathcal{F}$-sound.

(2) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ contains $\preccurlyeq_{\mathcal{F}}$, then (R, D) is $\mathcal{F}$-sufficient.

(3) If $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$, then (R, D) is $\mathcal{F}$-complete.

Proof Since the restrictions of $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ and $\preccurlyeq_{\mathcal{F}}$ to $\mathsf{A}^* \times \mathsf{A}$ are equal to $\vDash_{\mathsf{R},\mathsf{D}}$ and $\vDash_{\mathcal{F}}$ respectively, (1)–(3) here are consequences of (1)–(3) of Theorem 2.7.9.

Theorem 2.7.10 Assume that $\preccurlyeq_{\mathcal{F}}$ is extendedly latticed. Then (R, D) is $\mathcal{F}$-complete iff $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$.

Proof Assume that (R, D) is $\mathcal{F}$-complete. Then $\vDash_{\mathsf{R},\mathsf{D}}$ is equal to $\vDash_{\mathcal{F}}$ by Theorem 2.7.9. Since $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ and $\preccurlyeq_{\mathcal{F}}$ are the largest latticed extensions of $\vDash_{\mathsf{R},\mathsf{D}}$ and $\vDash_{\mathcal{F}}$ by Theorem 2.5.2 and our assumption respectively, $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $\preccurlyeq_{\mathcal{F}}$. The converse has been proved in Corollary 2.7.9.1.

[2.36] Kurt Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls (The completeness of the axioms of the functional calculus of logic)," *Monatshefte für Mathematik und Physik* 37 (1930), 349–360.

2.7.3 Completeness for Boolean logic spaces

In view of Example 2.6.1, here we consider the completeness of deduction systems on Boolean logic spaces.

Theorem 2.7.11 Let $(A, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A and (R, D) be a deduction system on A. Then (R, D) is $\mathcal{F}$-complete iff $\preccurlyeq_{R,D}$ is equal to $\preccurlyeq_{\mathcal{F}}$ and only if $\preccurlyeq_{R,D}$ is Boolean with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$.

Proof Since $\preccurlyeq_{\mathcal{F}}$ is extendedly latticed by Theorem 2.6.9, the former statement holds by Theorem 2.7.10. Since $\preccurlyeq_{\mathcal{F}}$ is Boolean by Theorem 2.6.9, the latter follows from the former.

Example 2.7.1 Let $(A, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A and C be the $\mathcal{F}$-core of A. Let $\&$ and $\wp$ be associations on A defined by (2.5.2) and B be the set of the Boolean elements of A with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$. Then Remark 2.5.9 shows that $\preccurlyeq_{\& \cup \wp, B}$ is the smallest Boolean relation on A^* with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$. Therefore Theorem 2.7.11 shows that if (R, D) is an $\mathcal{F}$-complete deduction system on A then $\preccurlyeq_{\& \cup \wp, B} \subseteq \preccurlyeq_{R,D}$ and so $B^{\& \cup \wp} \subseteq R^D$ (s. Theorem 2.5.6). Moreover, $\preccurlyeq_{\& \cup \wp, B} \subseteq \preccurlyeq_{\mathcal{F}}$ by Theorem 2.6.9, and so the deduction system $(\& \cup \wp, B)$ is $\mathcal{F}$-sound by Corollary 2.7.9.1. Thus $\& \cup \wp \subseteq \models_{\mathcal{F}}$ and $B \subseteq C$ by Theorems 2.7.1 and 2.6.6.

As for Example 2.6.1, if $(A, \mathcal{F})$ is the binary logic space on PL, then $(\& \cup \wp, B)$ is furthermore $\mathcal{F}$-complete by Theorem 2.10.2. Suppose $(A, \mathcal{F})$ is the binary logic space on FPL or CL. Then for a specific deduction system (R, D) such that $\& \cup \wp \subseteq R$ and $B \subseteq D$, the deduction system $(\& \cup \wp, [D]_R)$ has been proved to be $\mathcal{F}$-complete. Therefore $C = [[D]_R]_{\& \cup \wp} = [D]_R$ by Theorems 2.7.2 and 2.4.7. Thus, although not $\mathcal{F}$-sound, (R, D) is core $\mathcal{F}$-complete and so $\mathcal{F}$-sufficient by Theorem 2.7.12 below. Moreover, the deduction system $(\wp, [D]_R)$ is $\mathcal{F}$-complete by Corollary 2.7.12.1 below. The details for FPL were given in [1.3] by the way (s. [2.31]), while those for CL will be given in Chapters 4–6.

Theorem 2.7.12 Let $(A, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A. Assume that (R, D) is a core $\mathcal{F}$-sufficient deduction system on A and R contains the association $\wp = \dfrac{x \quad x \Rightarrow y}{y}$. Then (R, D) is $\mathcal{F}$-sufficient.

Proof Define the mapping $\phi \in A^* \times A \to A$ by

$$\phi(x_1 \cdots x_n, y) = x_n \Rightarrow (\cdots \Rightarrow (x_2 \Rightarrow (x_1 \Rightarrow y)) \cdots).$$

Then since $\preccurlyeq_{\mathcal{F}}$ is Boolean by Theorem 2.6.9 and $\models_{\mathcal{F}}$ is the restriction of $\preccurlyeq_{\mathcal{F}}$ to $A^* \times A$, Theorem 2.2.17 shows that the following holds:

$$x_1 \cdots x_n \models_{\mathcal{F}} y \iff \varepsilon \models_{\mathcal{F}} \phi(x_1 \cdots x_n, y).$$

In particular, ϕ satisfies

$$\alpha \vDash_{\mathcal{F}} y \implies \varepsilon \vDash_{\mathcal{F}} \phi(\alpha, y)$$

for all $(\alpha, y) \in A^* \times A$. Assume $\varepsilon \vDash_{R,D} \phi(x_1 \cdots x_n, y)$. Then

$$x_n \Rrightarrow (\cdots \Rrightarrow (x_2 \Rrightarrow (x_1 \Rrightarrow y)) \cdots) \in [D]_R \subseteq [\{x_1, \ldots, x_n\} \cup D]_R$$

by (2.5.1) and Theorem 2.4.5, and since $\wp \subseteq R$, it inductively follows that

$$x_{n-i} \Rrightarrow (\cdots \Rrightarrow (x_2 \Rrightarrow (x_1 \Rrightarrow y)) \cdots) \in [\{x_1, \ldots, x_n\} \cup D]_R$$

for $i = 1, \ldots, n$ by Theorem 2.4.7. In particular $y \in [\{x_1, \ldots, x_n\} \cup D]_R$, and so $x_1 \cdots x_n \vDash_{R,D} y$ by (2.5.1). Therefore ϕ satisfies

$$\varepsilon \vDash_{R,D} \phi(\alpha, y) \implies \alpha \vDash_{R,D} y$$

for all $(\alpha, y) \in A^* \times A$. Thus (R, D) is $\mathcal{F}$-sufficient by Theorems 2.7.3 and 2.6.6.

Corollary 2.7.12.1 Let $(A, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ on A. Then the deduction system $(\wp, C)$ on A consisting of the association $\wp = \dfrac{x \quad x \Rrightarrow y}{y}$ and the $\mathcal{F}$-core C of A is $\mathcal{F}$-complete.

Proof This is because $(\wp, C)$ is $\mathcal{F}$-sufficient by Theorem 2.7.12 and $\mathcal{F}$-sound by the first paragraph of Example 2.7.1 and Theorems 2.7.1 and 2.6.6.

2.7.4 Completeness and characteristic laws

Throughout this subsection, we let $(A, \mathcal{F})$ be a functional logic space. Here we formulate the fundamental Theorem 2.7.13 to obtain a practically necessary and sufficient program for finding an $\mathcal{F}$-complete deduction system on A.

First as in Example 2.5.2, we define $\vec{A} = A^* \times A^*$, denote its elements (α, β) by $\alpha \to \beta$ or $\beta \leftarrow \alpha$ and call them the **sequent**s on A. Then $\preccurlyeq_{\mathcal{F}}$ and $\preccurlyeq_f$ $(f \in \mathcal{F})$ may be regarded as subsets of $\vec{A}$, but we define

$$\vec{C}_{\mathcal{F}} = \{\alpha \to \beta \in \vec{A} : \alpha \preccurlyeq_{\mathcal{F}} \beta\}, \qquad \vec{A}_f = \{\alpha \to \beta \in \vec{A} : \alpha \preccurlyeq_f \beta\} \ \ (f \in \mathcal{F}),$$

distinguishing relations on A^* and subsets of $\vec{A}$. Next we define

$$\vec{\mathcal{F}} = \{\vec{A}_f : f \in \mathcal{F}\}.$$

Then $(\vec{A}, \vec{\mathcal{F}})$ is a logic space, which we call the **sequential logic space** associated with $(A, \mathcal{F})$. The $\vec{\mathcal{F}}$-core of $\vec{A}$ is equal to $\vec{C}_{\mathcal{F}}$ because (2.6.6) means $\vec{C}_{\mathcal{F}} = \bigcap_{f \in \mathcal{F}} \vec{A}_f$. Finally in view of Definition 2.5.1 and Theorem 2.5.8, we call a deduction system $(\vec{R}, \vec{D})$ on $\vec{A}$ a **characteristic law** of $(A, \mathcal{F})$ if it is core $\vec{\mathcal{F}}$-complete, that is, if $\vec{C}_{\mathcal{F}} = [\vec{D}]_{\vec{R}}$. In other words, a characteristic law of $(A, \mathcal{F})$ is a deductive law $(\vec{R}, \vec{D})$ on the relations on A^* such that $\preccurlyeq_{\mathcal{F}}$ is equal to $[\vec{D}]_{\vec{R}}$ regarded as the smallest $(\vec{R}, \vec{D})$-relation on A^*, and so $\preccurlyeq_{\mathcal{F}}$ is generated by $\vec{D}$ by virtue of $\vec{R}$ as in Theorems 2.4.2–2.4.4.

Remark 2.7.1 Let $\vec{\mathsf{Q}}$ be the largest $\vec{\mathcal{F}}$-logic on $\vec{\mathsf{A}}$. Then the deduction system $(\vec{\mathsf{Q}}, \emptyset)$ on $\vec{\mathsf{A}}$ is $\vec{\mathcal{F}}$-complete by Corollary 2.7.1.1, and therefore is core $\vec{\mathcal{F}}$-complete by Theorem 2.7.2. Thus $(\vec{\mathsf{Q}}, \emptyset)$ is a characteristic law of $(\mathsf{A}, \mathcal{F})$. More generally, if $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is an $\vec{\mathcal{F}}$-complete deduction system on $\vec{\mathsf{A}}$, then it is a characteristic law of $(\mathsf{A}, \mathcal{F})$ and is equivalent to $(\vec{\mathsf{Q}}, \emptyset)$ because $\vec{\mathsf{R}}^{\vec{\mathsf{D}}} = \vec{\mathsf{Q}} = \vec{\mathsf{Q}}^{\emptyset}$. A functional logic space $(\mathsf{A}, \mathcal{F})$ may have characteristic laws inequivalent to $(\vec{\mathsf{Q}}, \emptyset)$. Indeed as for PL, both the Boolean law and the weakly Boolean law are characteristic laws by Theorem 2.10.1, and the former is equivalent to $(\vec{\mathsf{Q}}, \emptyset)$, while the latter is not for the reason shown in Remark 2.2.13. The details were given in [1.3] by the way (s. [2.31]).

Theorem 2.7.13 Let (R, D) be an $\mathcal{F}$-sound deduction system on A and assume that $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies a characteristic law of $(\mathsf{A}, \mathcal{F})$. Then (R, D) is $\mathcal{F}$-complete.

Proof It suffices to show that a deduction system (R, D) on A is $\mathcal{F}$-sufficient if a deduction system $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$ satisfies the following two conditions.

(1) $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is core $\vec{\mathcal{F}}$-sufficient, that is, $\vec{\mathsf{C}}_{\mathcal{F}} \subseteq [\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$.

(2) $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies the deductive law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

Define $\vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}} = \{\alpha \to \beta \in \vec{\mathsf{A}} : \alpha \preccurlyeq_{\mathsf{R},\mathsf{D}} \beta\}$. Then (2) means that $\vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ is $\vec{\mathsf{R}}$-closed and contains $\vec{\mathsf{D}}$, hence $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}} \subseteq \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$. Therefore $\vec{\mathsf{C}}_{\mathcal{F}} \subseteq \vec{\mathsf{A}}_{\mathsf{R},\mathsf{D}}$ by (1), which means that $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq_{\mathsf{R},\mathsf{D}}$. Thus (R, D) is $\mathcal{F}$-sufficient by Corollary 2.7.9.1.

The following theorem contains a partial converse of Theorem 2.7.13.

Theorem 2.7.14 Assume that $\preccurlyeq_{\mathcal{F}}$ is extendedly latticed. Let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a characteristic law of $(\mathsf{A}, \mathcal{F})$ and (R, D) be a deduction system on A. Then (R, D) is $\mathcal{F}$-complete, iff $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ is equal to $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ regarded as the smallest $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation on A^*, and iff R^{D} is equal to the restriction of $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ to $\mathsf{A}^* \times \mathsf{A}$.

Proof Since $\preccurlyeq_{\mathcal{F}}$ is equal to $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ regarded as a relation on A^* by the definition of characteristic laws, the statement holds by Theorems 2.7.10 and 2.7.9.

Remark 2.7.2 As for Example 2.6.1, Theorem 2.6.9 shows that the binary logic space in CL as well as PL and FPL satisfies the assumption of Theorem 2.7.14. Thus Theorem 2.7.13 gives the following practically necessary and sufficient two-step program for finding an $\mathcal{F}$-complete deduction system on A.

A program for finding an $\mathcal{F}$-complete deduction system (R, D) on A

Step 1. Find a characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$, that is, find a core $\vec{\mathcal{F}}$-complete deduction system $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on $\vec{\mathsf{A}}$.

Step 2. Find an $\mathcal{F}$-sound deduction system (R, D) on A whose deduction relation $\preccurlyeq_{\mathsf{R},\mathsf{D}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

Step 1 is examined in the succeeding subsections. Theorem 2.5.11 is relevant to Step 2 for CL as well as PL and FPL, because their $\mathcal{F}$-validity relations are Boolean by Theorem 2.6.9 and the (weakly) Boolean law has been extended to their characteristic laws. The details for FPL were given in [1.3] by the way (s. [2.31]), while those for PL and CL will be given in §2.10 and Chapters 4–6.

2.7.5 Characteristic laws and Dedekind cuts

Throughout this subsection, we let $(\mathsf{A}, \mathcal{F})$ be a $\mathbb{B}$-valued functional logic space. Here we formulate Theorems 2.7.15 and 2.7.16 to obtain a practically necessary and sufficient program for finding a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Definition 2.7.2 Let $(\mathsf{X}, \mathsf{Y}) \in \mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$. Then a function $\mathsf{f} \in \mathcal{F}$ is called an **$\mathcal{F}$-model** of (X, Y) if $\inf \mathsf{fX}$ and $\sup \mathsf{fY}$ exist and satisfy $\inf \mathsf{fX} \nleq \sup \mathsf{fY}$ in $\mathbb{B}$.

Remark 2.7.3 It follows from Definition 2.7.2 that if $(\mathsf{A}, \mathcal{F})$ is extremal then a function $\mathsf{f} \in \mathcal{F}$ is an $\mathcal{F}$-model of (X, Y) iff $\mathsf{X} \subseteq \mathsf{f}^{-1}\{1\}$ and $\mathsf{Y} \subseteq \mathsf{f}^{-1}\{0\}$ for $1 = \max \mathbb{B}$ and $0 = \min \mathbb{B}$. Moreover, (2.2.1) and (2.6.6) show that, if we regard an element $(\alpha, \beta) \in \mathsf{A}^* \times \mathsf{A}^*$ as belonging to $\mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$, then a function $\mathsf{f} \in \mathcal{F}$ is an $\mathcal{F}$-model of (α, β) iff $\alpha \not\preccurlyeq_{\mathsf{f}} \beta$, and so (α, β) has an $\mathcal{F}$-model iff $\alpha \not\preccurlyeq_{\mathcal{F}} \beta$.

Definition 2.7.3 Let $\preccurlyeq$ be a relation on A^*. Then an element $(\mathsf{X}, \mathsf{Y}) \in \mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$ is called a (**Dedekind**) **cut** of A by $\preccurlyeq$ if every element $(\alpha, \beta) \in \mathsf{X}^* \times \mathsf{Y}^*$ satisfies $\alpha \not\preccurlyeq \beta$. The cut (X, Y) is said to be **finite** if $(\mathsf{X}, \mathsf{Y}) \in \mathfrak{P}'\mathsf{A} \times \mathfrak{P}'\mathsf{A}$.

Remark 2.7.4 It follows from Definition 2.7.3 and Remark 2.7.3 that an element $(\mathsf{X}, \mathsf{Y}) \in \mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$ iff each element $(\alpha, \beta) \in \mathsf{X}^* \times \mathsf{Y}^*$ regarded as belonging to $\mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$ has an $\mathcal{F}$-model.

Lemma 2.7.1 Let $\preccurlyeq$ be a relation on A^*. Assume that $\preccurlyeq$ satisfies the extension law and each finite cut of A by $\preccurlyeq$ has an $\mathcal{F}$-model. Then $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq$.

Proof Assume that an element $(\alpha, \beta) \in \mathsf{A}^* \times \mathsf{A}^*$ satisfies $\alpha \not\preccurlyeq \beta$, and regard it as belonging to $\mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$. Then since $\preccurlyeq$ satisfies the extension law, (α, β) is a finite cut of A by $\preccurlyeq$, and so has an $\mathcal{F}$-model. Therefore $\alpha \not\preccurlyeq_{\mathcal{F}} \beta$, as was noted in Remark 2.7.3. Thus $\preccurlyeq_{\mathcal{F}}$ is contained in $\preccurlyeq$.

Theorem 2.7.15 Let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a deductive law on the relations on A^*. Assume the following conditions.

(1) The $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ satisfies $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$.

(2) $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ *implies* the extension law in the sense of the word *imply* as defined in Definition 2.5.1.

(3) Each finite cut of A by each $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation contained in $\preccurlyeq_{\mathcal{F}}$ has an $\mathcal{F}$-model.

Then $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Proof To be more precise in the condition (2), define an association $\vec{\mathsf{E}}$ on $\vec{\mathsf{A}} = \mathsf{A}^* \times \mathsf{A}^*$ by the following sequential and fractional list as in Example 2.5.2:

$$\frac{\alpha \to \beta}{\alpha' \to \beta'} \qquad\qquad (\alpha \subseteq \alpha',\ \beta \subseteq \beta').$$

Then the extension law in (2) means the deductive law $(\vec{\mathsf{E}}, \emptyset)$.

Regard $[\vec{\mathsf{D}}]_{\vec{\mathsf{R}}}$ as a relation on A^*, and denote it by $\preccurlyeq$. Then $\preccurlyeq$ is the smallest $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation by Theorem 2.5.8. Therefore $\preccurlyeq$ satisfies the extension law by (2) and is contained in $\preccurlyeq_{\mathcal{F}}$ by (1), and so each finite cut of A by $\preccurlyeq$ has an $\mathcal{F}$-model by (3). Therefore $\preccurlyeq_{\mathcal{F}}$ is equal to $\preccurlyeq$ by Lemma 2.7.1. Thus $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ is a characteristic law of $(\mathsf{A}, \mathcal{F})$.

Remark 2.7.5 Under the assumption of Theorem 2.7.15, let (X, Y) be a finite cut of A by a $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation $\preccurlyeq$. Then the intersection $\preccurlyeq'$ of $\preccurlyeq$ and $\preccurlyeq_{\mathcal{F}}$ is a $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation contained in $\preccurlyeq_{\mathcal{F}}$ by (1) and Theorem 2.5.9, and (X, Y) is a finite cut of A by $\preccurlyeq'$, and so has an $\mathcal{F}$-model by (3).

Theorem 2.7.15 has the following partial converse.

Theorem 2.7.16 Let $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ be a characteristic law of $(\mathsf{A}, \mathcal{F})$. Then each finite cut of A by each $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation on A^* has an $\mathcal{F}$-model.

Proof Let $\preccurlyeq$ be a $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation and (X, Y) be a finite cut of A by $\preccurlyeq$. Then, since $\preccurlyeq_{\mathcal{F}}$ is the smallest $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$-relation, $\not\preccurlyeq$ is contained in $\not\preccurlyeq_{\mathcal{F}}$, and so (X, Y) is a finite cut of A by $\preccurlyeq_{\mathcal{F}}$. Thus (X, Y) has an $\mathcal{F}$-model by Remark 2.7.4.

Remark 2.7.6 Theorems 2.7.15 and 2.7.16 give the following practically necessary and sufficient program for finding a characteristic law of $(\mathsf{A}, \mathcal{F})$.

A program for finding a characteristic law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ of $(\mathsf{A}, \mathcal{F})$
Pick a deductive law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on the relations on A^* and show that it satisfies the conditions (1)–(3) of Theorem 2.7.15.

Suppose $(\mathsf{A}, \mathcal{F})$ is extremal and $\preccurlyeq_{\mathcal{F}}$ satisfies the lower end law (s. Lemma 2.2.4). Let X be a subset of A. Then Theorem 2.8.3 shows that the element $(\mathsf{X}, \emptyset) \in \mathfrak{P}\mathsf{A} \times \mathfrak{P}\mathsf{A}$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$ iff X is $\mathcal{F}$-consistent. Moreover, it follows from Remarks 2.7.3 and 2.8.2 that $(\mathsf{X}, \emptyset)$ has an $\mathcal{F}$-model different from 1_{A} iff X has a nontrivial $\mathcal{F}$-model. Thus the above program unites with the problem of how to construct nontrivial $\mathcal{F}$-models of $\mathcal{F}$-consistent subsets of A.

Example 2.7.2 Suppose $(\mathsf{A}, \mathcal{F})$ is the binary logic space on FPL. Then, if it has infinitely many variables, a specific deductive law $(\vec{\mathsf{R}}, \vec{\mathsf{D}})$ on the relations on A^* has been found to satisfy the conditions (1)–(3) of Theorem 2.7.15 and therefore is a characteristic law of $(A, \mathcal{F})$. Moreover, the deduction relation $\preccurlyeq_{R,D}$ of a specific $\mathcal{F}$-sound deduction system (R, D) on A has been found to satisfy $(\vec{R}, \vec{D})$, and so (R, D) is $\mathcal{F}$-complete by Theorem 2.7.13. Furthermore, $1_A \notin \mathcal{F}$ by Theorem 2.6.9, and irrespective of the cardinality of the variables, each (not necessarily finite) cut of A by $\preccurlyeq_{\mathcal{F}}$ has been found to have an $\mathcal{F}$-model, and so the latter half of Remark 2.7.6 shows that each $\mathcal{F}$-consistent subset of A has a nontrivial $\mathcal{F}$-model. The details were given in [1.3] by the way (s. [2.31]). Takaoka [1.93] obtained similar results for ICL, which will be generalized in Chapter 6 for CL.

2.7.6 Dedekind cuts by weakly Boolean relations

In this subsection, in view of Remarks 2.7.2 and 2.7.6, we consider (Dedekind) cuts by (weakly) Boolean relations. Throughout, we let A be a nonempty set, $\preccurlyeq$ be a relation on A^* and $\mathfrak{C}_\preccurlyeq$ be the set of the cuts of A by $\preccurlyeq$. Furthermore, we define the **product order** $\subseteq$ on $\mathfrak{P}A \times \mathfrak{P}A$ by

$$(X, Y) \subseteq (X', Y') \iff X \subseteq X' \text{ and } Y \subseteq Y',$$

and in view of the following lemma, we denote by $\mathfrak{D}_\preccurlyeq$ the set of the maximal elements of $\mathfrak{C}_\preccurlyeq$ with respect to the product order.

Lemma 2.7.2 The cut set $\mathfrak{C}_\preccurlyeq$ is downward in $\mathfrak{P}A \times \mathfrak{P}A$ and inductive with respect to the product order.

Proof Suppose $(X, Y) \in \mathfrak{C}_\preccurlyeq$, $(X', Y') \in \mathfrak{P}A \times \mathfrak{P}A$ and $(X', Y') \subseteq (X, Y)$. Then if $(\alpha, \beta) \in X'^* \times Y'^*$, then $(\alpha, \beta) \in X^* \times Y^*$, and so $\alpha \not\preccurlyeq \beta$. Therefore $(X', Y') \in \mathfrak{C}_\preccurlyeq$. Thus $\mathfrak{C}_\preccurlyeq$ is downward in $\mathfrak{P}A \times \mathfrak{P}A$.

Let $\{(X_i, Y_i) : i \in I\}$ be a nonempty linearly ordered subset of $\mathfrak{C}_\preccurlyeq$. Define $X = \bigcup_{i \in I} X_i$ and $Y = \bigcup_{i \in I} Y_i$. Then $(X, Y) = \sup_{\mathfrak{P}A \times \mathfrak{P}A}\{(X_i, Y_i) : i \in I\}$. Let $(\alpha, \beta) \in X^* \times Y^*$. Then since $(\alpha, \beta) \in \mathfrak{P}'X \times \mathfrak{P}'Y$ by the word convention and $\{(X_i, Y_i) : i \in I\}$ is linearly ordered, there exists an index $i \in I$ such that $(\alpha, \beta) \in {X_i}^* \times {Y_i}^*$, hence $\alpha \not\preccurlyeq \beta$ because $(X_i, Y_i) \in \mathfrak{C}_\preccurlyeq$. Therefore $(X, Y) \in \mathfrak{C}_\preccurlyeq$, and so $(X, Y) = \sup_{\mathfrak{C}_\preccurlyeq}\{(X_i, Y_i) : i \in I\}$. Thus $\mathfrak{C}_\preccurlyeq$ is inductive.

Lemma 2.7.3 If $\preccurlyeq$ satisfies the repetition law, then every element $(X, Y) \in \mathfrak{C}_\preccurlyeq$ satisfies $X \cap Y = \emptyset$.

Proof If $x \in X \cap Y$ for an element $(X, Y) \in \mathfrak{C}_\preccurlyeq$, then $x \not\preccurlyeq x$, and so $\preccurlyeq$ does not satisfy the repetition law. Remark 2.7.7 gives an alternative proof.

Lemma 2.7.4 If $\preccurlyeq$ satisfies the laws of extension and strong cut, then every element $(X, Y) \in \mathfrak{D}_\preccurlyeq$ satisfies $X \cup Y = A$.

Proof Suppose $X \cup Y \neq A$ for an element $(X, Y) \in \mathfrak{D}_\preccurlyeq$. Let $x \in A - X \cup Y$. Then neither $(X, \{x\} \cup Y)$ nor $(\{x\} \cup X, Y)$ belongs to $\mathfrak{C}_\preccurlyeq$ by the maximality of (X, Y), and so the extension law shows that there exist elements $(\alpha, \gamma), (\beta, \delta) \in X^* \times Y^*$ satisfying $\alpha \preccurlyeq x\gamma$ and $x\beta \preccurlyeq \delta$, and also $\alpha\beta \not\preccurlyeq \delta\gamma$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus $\preccurlyeq$ does not satisfy the strong cut law, which is a contradiction.

Lemma 2.7.5 If $\preccurlyeq$ satisfies the laws of extension, strong junction, strong negation and strong implication with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ on A, then every element $(X, Y) \in \mathfrak{D}_{\preccurlyeq}$ satisfies the following eight conditions.

(1) $x \wedge y \in X \implies x, y \in X$.

(2) $x \wedge y \in Y \implies x \in Y$ or $y \in Y$.

(3) $x \vee y \in X \implies x \in X$ or $y \in X$.

(4) $x \vee y \in Y \implies x, y \in Y$.

(5) $x^{\Diamond} \in X \implies x \in Y$.

(6) $x^{\Diamond} \in Y \implies x \in X$.

(7) $x \Rrightarrow y \in X \implies x \in Y$ or $y \in X$.

(8) $x \Rrightarrow y \in Y \implies x \in X,\ y \in Y$.

Proof Assume $x \notin X$ or $y \notin X$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $xy\alpha \preccurlyeq \beta$. Therefore $x \wedge y \cdot \alpha \preccurlyeq \beta$ by the strong conjunction law, and so $x \wedge y \notin X$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (1) holds.

Assume $x \notin Y$ and $y \notin Y$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $\alpha \preccurlyeq x\beta$ and $\alpha \preccurlyeq y\beta$. Therefore $\alpha \preccurlyeq x \wedge y \cdot \beta$ by the strong conjunction law, and so $x \wedge y \notin Y$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (2) holds.

Assume $x \notin X$ and $y \notin X$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $x\alpha \preccurlyeq \beta$ and $y\alpha \preccurlyeq \beta$. Therefore $x \vee y \cdot \alpha \preccurlyeq \beta$ by the strong disjunction law, and so $x \vee y \notin X$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (3) holds.

Assume $x \notin Y$ or $y \notin Y$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $\alpha \preccurlyeq xy\beta$. Therefore $\alpha \preccurlyeq x \vee y \cdot \beta$ by the strong disjunction law, and so $x \vee y \notin Y$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (4) holds.

Assume $x \notin Y$. Then the maximality of (X, Y) shows that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $\alpha \preccurlyeq x\beta$. Therefore $x^{\Diamond}\alpha \preccurlyeq \beta$ by the strong negation law, and so $x^{\Diamond} \notin X$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (5) holds.

Assume $x \notin X$. Then the maximality of (X, Y) shows that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $x\alpha \preccurlyeq \beta$. Therefore $\alpha \preccurlyeq x^{\Diamond}\beta$ by the strong negation law, and so $x^{\Diamond} \notin Y$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (6) holds.

Assume $x \notin Y$ and $y \notin X$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $\alpha \preccurlyeq x\beta$ and

$y\alpha \preccurlyeq \beta$. Therefore $x \Rrightarrow y \cdot \alpha \preccurlyeq \beta$ by the strong implication law, and so $x \Rrightarrow y \notin X$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (7) holds.

Assume $x \notin X$ or $y \notin Y$. Then the maximality of (X, Y) and the extension law show that there exists an element $(\alpha, \beta) \in X^* \times Y^*$ satisfying $x\alpha \preccurlyeq y\beta$. Therefore $\alpha \preccurlyeq x \Rrightarrow y \cdot \beta$ by the strong implication law, and so $x \Rrightarrow y \notin Y$ because $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$. Thus (8) holds.

Lemma 2.7.6 If $\preccurlyeq$ is Boolean with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ on A and $(X, Y) \in \mathfrak{D}_{\preccurlyeq}$, then $Y = A - X$, and the characteristic function 1_X of X is a binary representation of A with respect to the operations.

Proof Since $\preccurlyeq$ satisfies the extension law by Remark 2.2.8, $Y = A - X$ by Lemmas 2.7.3 and 2.7.4, and so X satisfies the following four conditions by Theorem 2.2.21 and Lemma 2.7.5.

- $x \wedge y \in X \iff x, y \in X$.
- $x \vee y \in X \iff x \in X$ or $y \in X$.
- $x^{\Diamond} \in X \iff x \notin X$.
- $x \Rrightarrow y \in X \iff x \notin X$ or $y \in X$.

This means that 1_X is a binary representation with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$.

Remark 2.7.7 For each $(X, Y) \in \mathfrak{P}A \times \mathfrak{P}A$, define a relation $\preccurlyeq_{X,Y}$ on A^* by

$$\alpha \preccurlyeq_{X,Y} \beta \iff (\alpha, \beta) \notin X^* \times Y^*$$

for each $(\alpha, \beta) \in A^* \times A^*$. Then the following hold for all $(X, Y) \in \mathfrak{P}A \times \mathfrak{P}A$.

- $(X, Y) \in \mathfrak{D}_{\preccurlyeq_{X,Y}}$, and $\preccurlyeq_{X,Y}$ is the largest of the relations $\preccurlyeq$ on A^* which satisfy $(X, Y) \in \mathfrak{C}_{\preccurlyeq}$.
- The relation $\preccurlyeq_{X,Y}$ satisfies the extension law and $\varepsilon \not\preccurlyeq_{X,Y} \varepsilon$.
- The relation $\preccurlyeq_{X,Y}$ satisfies the repetition law iff $X \cap Y = \emptyset$.
- The relation $\preccurlyeq_{X,Y}$ satisfies the (strong) cut law iff $X \cup Y = A$.
- If $A = X \amalg Y$, then $\preccurlyeq_{X,Y}$ is equal to the relation $\preccurlyeq_X$ defined in Theorem 2.2.23 and to the 1_X-validity relation $\preccurlyeq_{1_X}$.
- Let $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ be operations on A. Then $\preccurlyeq_{X,Y}$ satisfies the laws of strong junction, strong negation and strong implication with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ iff (X, Y) satisfies the conditions (1)–(8) of Lemma 2.7.5. Therefore, if $X \cap Y = \emptyset$ and $X \cup Y \subseteq A - \mathrm{Im} \wedge \cup \mathrm{Im} \vee \cup \mathrm{Im} \Diamond \cup \mathrm{Im} \Rrightarrow$, then $\preccurlyeq_{X,Y}$ is weakly Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rrightarrow$ but does not satisfy the (strong) cut law (s. Remark 2.2.13).

2.8 Consistency, models and classes

In Theorems 2.7.4–2.7.6, we observed interrelations between the completeness of deduction systems on logic spaces and the classification of the logic spaces given in Definition 2.6.3. In this section, we observe interrelations between the classification and the existence of models of consistent sets. The main result is Theorem 2.8.12. Throughout this section, we let $(\mathsf{A}, \mathfrak{B})$ be a logic space and Q be the largest $\mathfrak{B}$-logic on A.

2.8.1 Consistency and models

Definition 2.8.1 A subset X and an element x of A are said to be **$\mathfrak{B}$-consistent** if $[\mathsf{X}]_\mathsf{Q} \neq \mathsf{A}$ and $[\{\mathsf{x}\}]_\mathsf{Q} \neq \mathsf{A}$ respectively (thus x is $\mathfrak{B}$-consistent iff so is $\{\mathsf{x}\}$). Let $\mathfrak{C}$ be the set of the $\mathfrak{B}$-consistent subsets of A, $\mathfrak{D}$ be the set of the maximal $\mathfrak{B}$-consistent subsets of A, i.e. the maximal elements of $\mathfrak{C}$, and $\mathfrak{E}$ be the set $\{\mathsf{X} \in \mathfrak{P}\mathsf{A} : [\mathsf{X}]_\mathsf{Q} \in \mathfrak{D}\}$. We call the elements of $\mathfrak{E}$ the **$\mathfrak{B}$-complete** subsets of A.

Remark 2.8.1 Let $(\mathsf{A}, \mathfrak{B}')$ be a logic space equivalent to $(\mathsf{A}, \mathfrak{B})$. Then Q is also the largest $\mathfrak{B}'$-logic. Therefore, $\mathfrak{C}$ is equal to the $\mathfrak{B}'$-consistent subsets of A, and likewise for $\mathfrak{D}$ and $\mathfrak{E}$.

Definition 2.8.2 A **$\mathfrak{B}$-model** of a subset X of A is a set $\mathsf{B} \in \mathfrak{B}$ containing X. If furthermore $\mathsf{B} \neq \mathsf{A}$, we say that B is **nontrivial**[2.37].

Remark 2.8.2 Let $(\mathsf{A}, \mathcal{F})$ be a functional logic space and $(\mathsf{A}, \mathfrak{B}_\mathcal{F})$ be the associated logic space. Then by definition, an **$\mathcal{F}$-model** of a subset X of A is its $\mathfrak{B}_\mathcal{F}$-model, as [2.32] noticed. Therefore if $(\mathsf{A}, \mathcal{F})$ is extremal, then Remark 2.6.1 shows that a nontrivial $\mathcal{F}$-model of X is the inverse image $\mathsf{f}^{-1}\{1\}$ of a function $\mathsf{f} \in \mathcal{F} - \{1_\mathsf{A}\}$ containing X, where 1_A was defined in Theorem 2.6.9.

Theorem 2.8.1 The following hold on $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$.

(1) $\mathfrak{C}$ is downward (Theorem 2.8.5 (1) refines this).

(2) A subset X of A belongs to $\mathfrak{C}$ iff $[\mathsf{X}]_\mathsf{Q} \in \mathfrak{C}$. If $\mathsf{X} \in \mathfrak{D}$, then $\mathsf{X} = [\mathsf{X}]_\mathsf{Q}$.

(3) $\mathfrak{D} \subseteq \mathfrak{E} \subseteq \mathfrak{C}$ and $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{C}$ (Theorem 2.8.5 (1) refines this).

(4) If $\mathsf{X} \in \mathfrak{E}$, then $[\mathsf{X}]_\mathsf{Q}$ is its only nontrivial $\mathfrak{B}$-model, if any.

Proof The mapping $\mathsf{X} \mapsto [\mathsf{X}]_\mathsf{Q}$ is a closure operator, as was noted in Remark 2.4.3, and (1) is because it is increasing and (2) is because $[\mathsf{X}]_\mathsf{Q} = [[\mathsf{X}]_\mathsf{Q}]_\mathsf{Q}$ and $\mathsf{X} \subseteq [\mathsf{X}]_\mathsf{Q}$. If $\mathsf{X} \in \mathfrak{D}$, then $[\mathsf{X}]_\mathsf{Q} = \mathsf{X} \in \mathfrak{D}$ by (2) and so $\mathsf{X} \in \mathfrak{E}$. If $\mathsf{X} \in \mathfrak{E}$, then $\mathsf{X} \subseteq [\mathsf{X}]_\mathsf{Q} \in \mathfrak{D} \subseteq \mathfrak{C}$ and so $\mathsf{X} \in \mathfrak{C}$ by (1). If $\mathsf{B} \in \mathfrak{B} - \{\mathsf{A}\}$, then $[\mathsf{B}]_\mathsf{Q} = \mathsf{B} \neq \mathsf{A}$ and so $\mathsf{B} \in \mathfrak{C}$. Thus (3) holds. If $\mathsf{X} \in \mathfrak{E}$ and $\mathsf{X} \subseteq \mathsf{B} \in \mathfrak{B} - \{\mathsf{A}\}$, then $\mathfrak{D} \ni [\mathsf{X}]_\mathsf{Q} \subseteq \mathsf{B} \in \mathfrak{C}$ by (3) and so $\mathsf{B} = [\mathsf{X}]_\mathsf{Q}$. Thus (4) holds.

[2.37] Nontrivial $\mathfrak{B}$-models were simply called $\mathfrak{B}$-models in [1.5] and others (s. [2.1]).

Lemma 2.8.1 A finite subset $\{x_1, \ldots, x_n\}$ of A is $\mathfrak{B}$-inconsistent iff every element $y \in A$ satisfies $x_1 \cdots x_n \, Q \, y$.

Proof This is a direct consequence of Theorem 2.6.3.

Theorem 2.8.2 Assume that A has a $\mathfrak{B}$-inconsistent finite subset $\{x_1, \ldots, x_n\}$ ($n \geq 1$). Then the following five conditions on a subset X of A are equivalent.

(1) X is $\mathfrak{B}$-inconsistent.

(2) $\{x_1, \ldots, x_n\} \subseteq [X]_Q$.

(3) There exists an element $\alpha \in X^*$ such that $\alpha \, Q \, x_i$ $(i = 1, \ldots, n)$.

(4) There exists an element $\alpha \in X^*$ such that $\alpha \, Q \, y$ for all $y \in A$.

(5) There exists a $\mathfrak{B}$-inconsistent finite subset of X.

Therefore $\mathfrak{C}$ is a finitary subset of $\mathfrak{P}A$ (Theorem 2.8.5 (4) refines this).

Proof If (1) holds, then $[X]_Q = A$, and so (2) holds.

Assume (2). Then Theorem 2.6.3 shows that there exists an element $\alpha_i \in X^*$ such that $\alpha_i \, Q \, x_i$ for each $i \in \{1, \ldots, n\}$. Thus (3) holds with $\alpha = \alpha_1 \cdots \alpha_n$, because Q satisfies the partial extension law (and the partial substitution law) by Theorem 2.6.3 and Remark 2.2.8.

Assume (3) and let $y \in A$. Then $\alpha \, Q \, x_i$ $(i = 1, \ldots, n)$, and $x_1 \cdots x_n \, Q \, y$ by Lemma 2.8.1. By repeated application of the laws of partial substitution and partial extension to these $n + 1$ relations, we have $\alpha \, Q \, y$. Thus (4) holds.

If (4) holds, then α regarded as a finite subset of X is $\mathfrak{B}$-inconsistent by Lemma 2.8.1, and so (5) holds.

Since $\mathfrak{C}$ is downward by Theorem 2.8.1, (5) implies (1).

Thus (1)–(5) are equivalent. Theorem 2.1.1 shows that (1) and (5) are equivalent iff $\mathfrak{C}$ is finitary. Thus the final statement holds.

Theorem 2.8.3 Let $(A, \mathcal{F})$ be a functional logic space. Then a finite subset $\{x_1, \ldots, x_n\}$ of A is $\mathcal{F}$-inconsistent provided that $x_1 \cdots x_n \preccurlyeq_{\mathcal{F}} \varepsilon$. Furthermore, if $\preccurlyeq_{\mathcal{F}}$ satisfies the lower end law (s. Lemma 2.2.4), then the following hold.

(1) A has an $\mathcal{F}$-inconsistent element.

(2) A finite subset $\{x_1, \ldots, x_n\}$ of A is $\mathcal{F}$-inconsistent iff $x_1 \cdots x_n \preccurlyeq_{\mathcal{F}} \varepsilon$.

(3) A subset X of A is $\mathcal{F}$-consistent iff $(X, \emptyset) \in \mathfrak{P}A \times \mathfrak{P}A$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$.

(4) If X is a maximal $\mathcal{F}$-consistent subset of A, then there exists a subset Y of A such that (X, Y) is a maximal cut of A by $\preccurlyeq_{\mathcal{F}}$.

Proof Let $(A, \mathfrak{B}_{\mathcal{F}})$ be the logic space associated with $(A, \mathcal{F})$. Then by definition, a subset X of A is **$\mathcal{F}$-consistent** iff it is $\mathfrak{B}_{\mathcal{F}}$-consistent, as [2.32] noticed. The largest $\mathfrak{B}_{\mathcal{F}}$-logic on A is equal to $\vDash_{\mathcal{F}}$ by Theorem 2.6.6. Its extension $\preccurlyeq_{\mathcal{F}}$ is latticed by Theorem 2.6.8, and the lower end law for $\preccurlyeq_{\mathcal{F}}$ implies the lower quasi-end law by Lemma 2.2.5. Therefore, Lemma 2.8.1 shows that the former statement holds and that (1) and (2) hold under the lower end law.

Under (1) and (2), Theorem 2.8.2 shows that a subset X of A is $\mathcal{F}$-consistent iff every finite subset $\{x_1, \ldots, x_n\}$ of X satisfies $x_1 \cdots x_n \not\preccurlyeq_{\mathcal{F}} \varepsilon$. Thus (3) holds.

Under (3), let X be a maximal $\mathcal{F}$-consistent subset of A. Then $(X, \emptyset)$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$, and so Lemma 2.7.2 and Zorn's lemma show that $X \subseteq X'$ for some maximal cut (X', Y) of A by $\preccurlyeq_{\mathcal{F}}$. Lemma 2.7.2 also shows that $(X', \emptyset)$ is a cut of A by $\preccurlyeq_{\mathcal{F}}$, and so X' is $\mathcal{F}$-consistent. Therefore $X = X'$ by the maximality of X, and so (X, Y) is a maximal cut of A by $\preccurlyeq_{\mathcal{F}}$. Thus (4) holds.

Remark 2.8.3 A subset X of A has a nontrivial $\mathfrak{B}$-model iff $\bigcap_{X \subseteq B \in \mathfrak{B}} B \neq A$ and iff X belongs to the downward closure $\overleftarrow{\mathfrak{B} - \{A\}}$ of $\mathfrak{B} - \{A\}$ in $\mathfrak{P}A$ defined in Theorem 2.1.8 (s. Remark 2.1.5). Theorem 2.1.9 shows that X belongs to the finitary closure $\overleftarrow{\overline{\mathfrak{B} - \{A\}}}$ of $\mathfrak{B} - \{A\}$ in $\mathfrak{P}A$ iff each finite subset of X has a nontrivial $\mathfrak{B}$-model. Theorems 2.8.4 and 2.8.5 are relevant to these facts.

Theorem 2.8.4 The following hold on the Q-closures of subsets of A.

(1) $[Y]_Q = \bigcap_{Y \subseteq B \in \mathfrak{B}} B$ for all $Y \in \mathfrak{P}'A$.

(2) $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$ iff $[X]_Q = \bigcap_{X \subseteq B \in \mathfrak{B}} B$ for all $X \in \mathfrak{P}A$.

Proof (1) Let $Y = \{y_1, \ldots, y_n\}$ and define $\alpha = y_1 \cdots y_n \in A^*$. Then $[Y]_Q = [\alpha]_Q = \{y \in A : \alpha \, Q \, y\} = \bigcap_{\alpha \subseteq B \in \mathfrak{B}} B = \bigcap_{Y \subseteq B \in \mathfrak{B}} B$ by Theorems 2.6.3 and 2.6.2.

Alternatively, $[Y]_Q = \bigcap_{Y \subseteq X \in \overline{\mathfrak{B}^\cap}} X = \bigcap_{Y \subseteq X \in \mathfrak{B}^\cap} X = \bigcap_{Y \subseteq B \in \mathfrak{B}} B$ by Theorems 2.6.1, 2.6.10, 2.1.6 and 2.1.5.

(2) If $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$, that is, if $\overline{\mathfrak{B}^\cap} = \mathfrak{B}^\cap$, then $[X]_Q = \bigcap_{X \subseteq Y \in \overline{\mathfrak{B}^\cap}} Y = \bigcap_{X \subseteq Y \in \mathfrak{B}^\cap} Y = \bigcap_{X \subseteq B \in \mathfrak{B}} B$ for all $X \in \mathfrak{P}A$ by Theorems 2.6.1, 2.6.10 and 2.1.5. Conversely, if $[X]_Q = \bigcap_{X \subseteq B \in \mathfrak{B}} B$ for all $X \in \mathfrak{P}A$, then every set X in $\overline{\mathfrak{B}^\cap}$ satisfies $X = [X]_Q = \bigcap_{X \subseteq B \in \mathfrak{B}} B \in \mathfrak{B}^\cap$ by Theorem 2.6.10, and therefore $\overline{\mathfrak{B}^\cap} = \mathfrak{B}^\cap$.

Theorem 2.8.5 The following hold on $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ (s. Remark 2.8.3).

(1) $\mathfrak{B} - \{A\} \subseteq \overleftarrow{\mathfrak{B} - \{A\}} \subseteq \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}} = \mathfrak{C} \subseteq \overleftarrow{\overline{\mathfrak{B} - \{A\}}}$.

(2) $\mathfrak{P}'A \cap \mathfrak{C} = \mathfrak{P}'A \cap \overleftarrow{\mathfrak{B} - \{A\}}$.

(3) If $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$, then $\mathfrak{C} = \overleftarrow{\mathfrak{B} - \{A\}}$ (s. Remark 2.8.6).

(4) If A has a $\mathfrak{B}$-inconsistent finite subset, then $\mathfrak{C} = \overleftarrow{\overline{\mathfrak{B} - \{A\}}} = \overleftarrow{\mathfrak{D}}$.

(5) $\mathfrak{D}$ is equal to the set of the maximal elements of $\overline{\mathfrak{B}^\cap} - \{A\}$.

Proof (1) Theorems 2.1.7 and 2.1.8 show $\mathfrak{B} - \{A\} \subseteq \overleftarrow{\mathfrak{B} - \{A\}} \subseteq \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$. If $X \in \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$, then there exists a set $Y \in \overline{\mathfrak{B}^\cap} - \{A\}$ such that $X \subseteq Y$, and since $[X]_Q \subseteq Y \neq A$ by Theorem 2.6.10, we have $X \in \mathfrak{C}$. Conversely if $X \in \mathfrak{C}$, then $X \subseteq [X]_Q \in \overline{\mathfrak{B}^\cap} - \{A\}$ by Theorems 2.6.1 and 2.6.10, and so $X \in \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$. Thus $\overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}} = \mathfrak{C}$. If $X \in \mathfrak{C}$, then each set $Y \in \mathfrak{P}'X$ satisfies $\bigcap_{Y \subseteq B \in \mathfrak{B}} B = [Y]_Q \neq A$ by Theorems 2.8.4 and 2.8.1 (1). Therefore $\mathfrak{C} \subseteq \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$ by Remark 2.8.3.

(2) is because $\overleftarrow{\mathfrak{B} - \{A\}} \subseteq \mathfrak{C} \subseteq \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$ by (1) and $\mathfrak{P}'A \cap \overleftarrow{\mathfrak{B} - \{A\}} = \mathfrak{P}'A \cap \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$ by Theorem 2.1.6, or is because Theorem 2.8.4 and Remark 2.8.3 show that an element $Y \in \mathfrak{P}'A$ satisfies $[Y]_Q \neq A$ iff $Y \in \overleftarrow{\mathfrak{B} - \{A\}}$.

(3) follows from Theorem 2.8.4 and Remark 2.8.3 as above, or is because $\mathfrak{C} = \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$ by (1), $\overline{\mathfrak{B}^\cap} = \mathfrak{B}^\cap$ and $\overleftarrow{\mathfrak{B}^\cap - \{A\}} = \overleftarrow{\mathfrak{B} - \{A\}}$.

Under the premise of (4), $\mathfrak{C}$ is finitary by Theorem 2.8.2. Therefore, applying the closure operator in Theorem 2.1.9 to the inclusion $\mathfrak{B} - \{A\} \subseteq \mathfrak{C} \subseteq \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$ in (1), we have $\mathfrak{C} = \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$. Being finitary, $\mathfrak{C}$ is also inductive by Theorem 2.1.2 and Remark 2.1.2. Moreover, $\mathfrak{C}$ is downward by Theorem 2.8.1. Therefore $\mathfrak{C} = \overleftarrow{\mathfrak{D}}$ by Zorn's lemma and Definition 2.8.1.

(5) is a consequence of the equation $\mathfrak{C} = \overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$ in (1) and Definition 2.8.1.

Remark 2.8.4 It follows from Theorems 2.8.5 and 2.8.1 that $\bigcup(\mathfrak{B} - \{A\})$ is the set of the $\mathfrak{B}$-consistent elements in A and $\bigcup \mathfrak{X} = \bigcup(\mathfrak{B} - \{A\})$ for $\mathfrak{X} = \overleftarrow{\mathfrak{B} - \{A\}}$, $\overline{\mathfrak{B}^\cap} - \{A\}$, $\overleftarrow{\overline{\mathfrak{B}^\cap} - \{A\}}$ and $\mathfrak{C}$. Furthermore if A has a $\mathfrak{B}$-inconsistent finite subset, then $\bigcup \mathfrak{X} = \bigcup(\mathfrak{B} - \{A\})$ for $\mathfrak{X} = \overline{\overleftarrow{\mathfrak{B} - \{A\}}}$, $\mathfrak{D}$ and $\mathfrak{C}$ as well.

2.8.2 Consistency and complements

Definition 2.8.3 Let $x^\diamond$ be a unary operation on A. Then it is called a **$\mathfrak{B}$-negation**, if it satisfies the following law for all $B \in \mathfrak{B} - \{A\}$:

$$x^\diamond \in B \iff x \notin B. \qquad \text{(B-negation)}$$

Moreover, it is called a **$\mathfrak{B}$-complement**, if it satisfies the following two laws:

$$x x^\diamond \, Q \, y, \qquad \text{(contradiction)}$$

$$x\alpha \, Q \, y, \ x^\diamond \alpha \, Q \, y \implies \alpha \, Q \, y. \qquad \text{(excluded middle)}$$

Theorem 2.8.6 Let $x^\diamond$ be a unary operation on A. Then the following hold.

(1) It satisfies the contradiction law iff $\{x, x^\diamond\}$ is $\mathfrak{B}$-inconsistent for all $x \in A$.

(2) It satisfies the excluded middle law iff it satisfies the law

$$[\{x\} \cup X]_Q \cap [\{x^\Diamond\} \cup X]_Q \subseteq [X]_Q \tag{2.8.1}$$

for all $X \in \mathfrak{P}A$ and iff it satisfies this law for all $X \in \mathfrak{P}'A$.

Proof (1) is a direct consequence of Lemma 2.8.1.

(2) First, under the excluded middle law, assume $y \in [\{x\} \cup X]_Q \cap [\{x^\Diamond\} \cup X]_Q$ for a set $X \in \mathfrak{P}A$. Then it follows from Theorem 2.6.3 that there exists an element $\alpha \in X^*$ such that $x\alpha \, Q \, y$ and $x^\Diamond \alpha \, Q \, y$, and so $\alpha \, Q \, y$ by the excluded middle law, hence $y \in [X]_Q$. Thus the law (2.8.1) is satisfied for all $X \in \mathfrak{P}A$.

Next, under the law (2.8.1) for all $X \in \mathfrak{P}'A$, assume $x\alpha \, Q \, y$ and $x^\Diamond \alpha \, Q \, y$. Then $y \in [\{x\} \cup \alpha]_Q \cap [\{x^\Diamond\} \cup \alpha]_Q$, and so $y \in [\alpha]_Q$ by (2.8.1) for α regarded as belonging to $\mathfrak{P}'A$, hence $\alpha \, Q \, y$ by Theorem 2.6.3. Thus the excluded middle law is satisfied.

Corollary 2.8.6.1 Assume that A has a $\mathfrak{B}$-complement $x^\Diamond$. Let X and x be a subset and an element of A. Then $\{x, x^\Diamond\}$ is $\mathfrak{B}$-inconsistent, and the following three conditions are equivalent (Theorem 2.8.2 has given further equivalent conditions (4) and (5) under a weaker assumption).

- X is $\mathfrak{B}$-inconsistent.
- $\{x, x^\Diamond\} \subseteq [X]_Q$.
- There exists an element $\alpha \in X^*$ such that $\alpha \, Q \, x$ and $\alpha \, Q \, x^\Diamond$.

Proof This is a direct consequence of Theorems 2.8.6 and 2.8.2.

Theorem 2.8.7 Assume that A has a $\mathfrak{B}$-complement $x^\Diamond$. Let X be a subset of A. Then the following three conditions are equivalent.

(1) X is a maximal $\mathfrak{B}$-consistent subset, that is, $X \in \mathfrak{D}$.

(2) $[X]_Q = X$ and $\Diamond$ satisfies the X-negation law: $x^\Diamond \in X \iff x \notin X$.

(3) X is $\mathfrak{B}$-consistent and each element $x \in A$ satisfies $x \in X$ or $x^\Diamond \in X$.

Proof Assume (1). Then $[X]_Q = X$ by Theorem 2.8.1. Let $x \in A$. Then $\{x, x^\Diamond\} \nsubseteq X$ by the $\mathfrak{B}$-consistency of X and Corollary 2.8.6.1, and so if $x^\Diamond \in X$ then $x \notin X$. Conversely if $x \notin X$, then $[\{x\} \cup X]_Q = A$ by the maximality of X, and so $x^\Diamond \in [X]_Q = X$ by (2.8.1). Thus (2) holds.

Since $A \neq \emptyset$, the X-negation law implies $X \neq A$. Thus (2) implies (3).

Assume (3) and $X \subset Y \subseteq A$. Then any element $y \in Y - X$ satisfies $y^\Diamond \in X$, and so $\{y, y^\Diamond\} \subseteq Y$. Therefore $Y \notin \mathfrak{C}$ by Corollary 2.8.6.1. Thus (1) holds.

Corollary 2.8.7.1 Assume that A has a $\mathfrak{B}$-complement $x^\Diamond$. Let X be a subset of A. Then the following three conditions are equivalent (Theorem 2.8.13 and Corollary 2.8.13.1 will give further equivalent conditions under stronger assumptions and under the same one respectively).

(1) X is $\mathfrak{B}$-complete, that is, $\mathsf{X} \in \mathfrak{E}$.

(2) $\Diamond$ satisfies the $[\mathsf{X}]_\mathsf{Q}$-negation law: $\mathsf{x}^\Diamond \in [\mathsf{X}]_\mathsf{Q} \iff \mathsf{x} \notin [\mathsf{X}]_\mathsf{Q}$.

(3) X is $\mathfrak{B}$-consistent and each element $\mathsf{x} \in \mathsf{A}$ satisfies $\mathsf{x} \in [\mathsf{X}]_\mathsf{Q}$ or $\mathsf{x}^\Diamond \in [\mathsf{X}]_\mathsf{Q}$.

Proof This is Theorem 2.8.7 applied to $[\mathsf{X}]_\mathsf{Q}$, because $[[\mathsf{X}]_\mathsf{Q}]_\mathsf{Q} = [\mathsf{X}]_\mathsf{Q}$ and Theorem 2.8.1 shows that X is $\mathfrak{B}$-consistent iff $[\mathsf{X}]_\mathsf{Q}$ is $\mathfrak{B}$-consistent. This proof and its requisites are essentially due to Kazuhiro Takahasi.

Corollary 2.8.7.2 Every $\mathfrak{B}$-complement on A is a $\mathfrak{D}$-negation.

Proof This is because the condition (1) in Theorem 2.8.7 implies the condition (2) there for every $\mathfrak{B}$-complement $\mathsf{x}^\Diamond$ on A.

Theorem 2.8.8 The following relate $\mathfrak{B}$-negations and $\mathfrak{B}$-complements.

(1) Every $\mathfrak{B}$-negation on A is a $\mathfrak{B}$-complement.

(2) If A has a $\mathfrak{B}$-negation, then $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$. Conversely if $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$, then every $\mathfrak{B}$-complement on A is a $\mathfrak{B}$-negation.

Proof (1) Let $\mathsf{x}^\Diamond$ be a $\mathfrak{B}$-negation. Then every element $\mathsf{x} \in \mathsf{A}$ satisfies $\{\mathsf{x}, \mathsf{x}^\Diamond\} \notin \overleftarrow{\mathfrak{B} - \{\mathsf{A}\}}$, and so $\{\mathsf{x}, \mathsf{x}^\Diamond\} \notin \mathfrak{C}$ by Theorem 2.8.5 (2). Therefore $\Diamond$ satisfies the contradiction law by Theorem 2.8.6. Let $\mathsf{x} \in \mathsf{A}$ and $\mathsf{X} \in \mathfrak{P}'\mathsf{A}$. Then since $\Diamond$ is a $\mathfrak{B}$-negation, we also have

$$\{\mathsf{B} \in \mathfrak{B} : \{\mathsf{x}\} \cup \mathsf{X} \subseteq \mathsf{B}\} \cup \{\mathsf{B} \in \mathfrak{B} : \{\mathsf{x}^\Diamond\} \cup \mathsf{X} \subseteq \mathsf{B}\} = \{\mathsf{B} \in \mathfrak{B} : \mathsf{X} \subseteq \mathsf{B}\},$$

and so $\left(\bigcap_{\{\mathsf{x}\}\cup\mathsf{X}\subseteq\mathsf{B}\in\mathfrak{B}} \mathsf{B}\right) \cap \left(\bigcap_{\{\mathsf{x}^\Diamond\}\cup\mathsf{X}\subseteq\mathsf{B}\in\mathfrak{B}} \mathsf{B}\right) = \bigcap_{\mathsf{X}\subseteq\mathsf{B}\in\mathfrak{B}} \mathsf{B}$, hence (2.8.1) by Theorem 2.8.4. Therefore $\Diamond$ satisfies the excluded middle law. Thus (1) holds.

(2) Assume that A has a $\mathfrak{B}$-negation $\Diamond$. Then it is a $\mathfrak{B}$-complement by (1), and if $\mathsf{B} \in \mathfrak{B} - \{\mathsf{A}\}$, then $[\mathsf{B}]_\mathsf{Q} = \mathsf{B}$ and $\Diamond$ satisfies the B-negation law, and so $\mathsf{B} \in \mathfrak{D}$ by Theorem 2.8.7. Thus $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$. Conversely if $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$, then every $\mathfrak{B}$-complement is a $\mathfrak{B}$-negation by Corollary 2.8.7.2.

Alternative proof of (1). We may assume $\mathsf{A} \notin \mathfrak{B}$ by virtue of Lemma 2.6.1. Let $(\mathsf{A}, \mathcal{F}_\mathfrak{B})$ be the functionalization of $(\mathsf{A}, \mathfrak{B})$. Then since $\mathbb{T}$ is a Boolean lattice, $\preccurlyeq_{\mathcal{F}_\mathfrak{B}}$ is strongly latticed by Theorem 2.6.8.

Let $\mathsf{x}^\Diamond$ be a $\mathfrak{B}$-negation. Then since $\mathcal{F}_\mathfrak{B} = \{1_\mathsf{B} : \mathsf{B} \in \mathfrak{B}\}$ and $\mathsf{A} \notin \mathfrak{B}$, every function $\mathsf{f} \in \mathcal{F}_\mathfrak{B}$ satisfies $\mathsf{f}(\mathsf{x}^\Diamond) = (\mathsf{f}\mathsf{x})^\Diamond$ for all $\mathsf{x} \in \mathsf{A}$, where $\Diamond$ on the right-hand side is the complement on $\mathbb{T}$. Therefore $\preccurlyeq_\mathsf{f}$ satisfies the negation laws by Theorem 2.2.7, and so also does $\preccurlyeq_{\mathcal{F}_\mathfrak{B}}$ by Corollary 2.5.9.1. Therefore $\preccurlyeq_{\mathcal{F}_\mathfrak{B}}$ satisfies the laws $\mathsf{x}\mathsf{x}^\Diamond \preccurlyeq_{\mathcal{F}_\mathfrak{B}} \mathsf{y}$ and $(\Diamond 6)$ in Theorem 2.2.16. Since $\mathsf{Q} = \vDash_{\mathcal{F}_\mathfrak{B}}$ by Lemma 2.6.1 and Theorem 2.6.6, it follows that $\Diamond$ is a $\mathfrak{B}$-complement.

Theorem 2.8.9 Let $(\mathsf{A}, \mathcal{F})$ be a functional logic space and $\mathsf{x}^\Diamond$ be a unary operation on A. Then $\Diamond$ is an $\mathcal{F}$-negation on A iff it is an $\mathcal{F}$-complement on A and satisfies the $\mathsf{f}^{-1}\{1\}$-negation law in Theorem 2.6.9 for all $\mathsf{f} \in \mathcal{F} - \{1_\mathsf{A}\}$.

Proof Let $(A, \mathfrak{B}_{\mathcal{F}})$ be the associated logic space. Then by definition, an **$\mathcal{F}$-negation** is a $\mathfrak{B}_{\mathcal{F}}$-negation and an **$\mathcal{F}$-complement** is a $\mathfrak{B}_{\mathcal{F}}$-complement, as [2.32] noticed. Therefore if $\Diamond$ is an $\mathcal{F}$-negation, then it is an $\mathcal{F}$-complement by Theorem 2.8.8, and since $\{f^{-1}\{1\} : f \in \mathcal{F} - \{1_A\}\} \subseteq \mathfrak{B}_{\mathcal{F}} - \{A\}$ by (2.6.2) and (2.6.3), it satisfies the $f^{-1}\{1\}$-negation law for all $f \in \mathcal{F} - \{1_A\}$.

Conversely assume that $\Diamond$ is an $\mathcal{F}$-complement and satisfies the $f^{-1}\{1\}$-negation law for all $f \in \mathcal{F} - \{1_A\}$. Then Theorem 2.8.7 shows that $f^{-1}\{1\}$ is a maximal $\mathcal{F}$-consistent subset for all $f \in \mathcal{F} - \{1_A\}$, because $f^{-1}\{1\} \in \mathfrak{B}_{\mathcal{F}}$. Moreover, if $B \in \mathfrak{B}_{\mathcal{F}} - \{A\}$, then B is $\mathcal{F}$-consistent and (2.6.2) shows that $B \supseteq f^{-1}\{1\}$ for some $f \in \mathcal{F} - \{1_A\}$. Therefore $\mathfrak{B}_{\mathcal{F}} - \{A\} \subseteq \{f^{-1}\{1\} : f \in \mathcal{F} - \{1_A\}\}$. Thus $\Diamond$ is an $\mathcal{F}$-negation.

Theorem 2.8.10 Let $(A, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A. Then $\Diamond$ is an $\mathcal{F}$-complement. Moreover, A has an $\mathcal{F}$-negation iff $\Diamond$ is an $\mathcal{F}$-negation and iff $(A, \mathcal{F})$ is extremal.

Proof The largest $\mathfrak{B}_{\mathcal{F}}$-logic on A is equal to $\models_{\mathcal{F}}$ by Theorem 2.6.6, and its extension $\preccurlyeq_{\mathcal{F}}$ is Boolean by Theorem 2.6.9. Thus it follows from Theorem 2.2.16 that $\Diamond$ is an $\mathcal{F}$-complement. Therefore Theorem 2.8.8 shows that A has an $\mathcal{F}$-negation iff $\Diamond$ is an $\mathcal{F}$-negation, and Theorems 2.8.9 and 2.6.9 show that $\Diamond$ is an $\mathcal{F}$-negation iff $(A, \mathcal{F})$ is extremal.

Lemma 2.8.2 The following two conditions are equivalent.

(1) $\mathrm{Cn}(A, \mathfrak{B}) = 1$ and A has a $\mathfrak{B}$-negation.

(2) $\mathfrak{B} = \{A, B\}$ for some nonempty subset B of A.

Proof Assume (1). Then $\mathfrak{B}^{\cap} - \{A\} = \mathfrak{B} - \{A\} \subseteq \mathfrak{D}$ by Theorem 2.8.8. Therefore, if B_0 and B_1 are distinct sets in $\mathfrak{B} - \{A\}$, then they together with $B_0 \cap B_1$ belong to $\mathfrak{D}$ and either B_0 or B_1 is larger than $B_0 \cap B_1$, which is a contradiction. Therefore $\mathfrak{B} = \{A, B\}$ for some subset B of A. If $B \neq A$, then the B-negation law implies $B \neq \emptyset$. Thus (2) holds.

Assume (2). Then obviously $\mathfrak{B}^{\cap} = \mathfrak{B}$. Remark 2.1.1 shows that $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$ and that every set in $\overline{\mathfrak{B}}$ is exactly covered by $\mathfrak{B}$. Hence $\overline{\mathfrak{B}} = \mathfrak{B}$. Thus $\mathrm{Cn}(A, \mathfrak{B}) = 1$. If $\mathfrak{B} = \{A\}$, then any unary operation on A is obviously a $\mathfrak{B}$-negation. If $B \neq A$, then $A - B \neq \emptyset \neq B$, and so A has a $\mathfrak{B}$-negation. Thus (1) holds.

Theorem 2.8.11 Assume that $(A, \mathcal{F})$ is an extremal Boolean logic space. Then $\mathrm{Cn}(A, \mathcal{F}) = 1$ iff $\#\mathcal{F} \leq 1$.

Proof Let $(A, \mathfrak{B}_{\mathcal{F}})$ be the associated logic space. Then $\mathrm{Cn}(A, \mathcal{F}) = \mathrm{Cn}(A, \mathfrak{B}_{\mathcal{F}})$ by definition, as [2.32] noticed. Remark 2.6.1 and Theorem 2.6.9 show that $\emptyset \notin \mathfrak{B}_{\mathcal{F}} = \{f^{-1}\{1\} : f \in \mathcal{F}\} \amalg \{A\}$, and the mapping $f \mapsto f^{-1}\{1\}$ of $\mathcal{F}$ into $\mathfrak{B}_{\mathcal{F}}$ is injective. Moreover, A has an $\mathcal{F}$-negation by Theorem 2.8.10. Thus Lemma 2.8.2 shows that $\mathrm{Cn}(A, \mathcal{F}) = 1$ iff $\#\mathcal{F} \leq 1$.

2.8.3 Classes and the existence of models

Theorem 2.8.12 Assume that A has a $\mathfrak{B}$-complement. Then the following three conditions on a logic space $(\mathsf{A}, \mathfrak{B}')$ are equivalent.

(1) $(\mathsf{A}, \mathfrak{B}')$ is equivalent to $(\mathsf{A}, \mathfrak{B})$ and $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}') \le 2$.

(2) $\mathfrak{B}' \subseteq \overline{\mathfrak{B}^\cap}$ and $\mathfrak{C} = \overleftarrow{\mathfrak{B}' - \{\mathsf{A}\}}$.

(3) $\mathfrak{D} \subseteq \mathfrak{B}' - \{\mathsf{A}\} \subseteq \overline{\mathfrak{B}^\cap} - \{\mathsf{A}\}$.

Remark 2.8.5 The condition (3) is equivalent to the condition $\mathfrak{D} \subseteq \mathfrak{B}' \subseteq \overline{\mathfrak{B}^\cap}$ because $\mathsf{A} \in \overline{\mathfrak{B}^\cap} - \mathfrak{D}$. Both make sense because $\mathfrak{D} \subseteq \overline{\mathfrak{B}^\cap} - \{\mathsf{A}\}$ by Theorem 2.8.5 (5). Moreover, Corollary 2.6.10.1 shows that the logic space $(\mathsf{A}, \mathfrak{B}')$ is equivalent to $(\mathsf{A}, \mathfrak{B})$ and satisfies $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}') = 1$ iff $\mathfrak{B}' = \overline{\mathfrak{B}^\cap}$. Therefore, the logic space $(\mathsf{A}, \mathfrak{B}')$ is equivalent to $(\mathsf{A}, \mathfrak{B})$ and satisfies $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}') = 2$ iff $\mathfrak{D} \subseteq \mathfrak{B}' \subset \overline{\mathfrak{B}^\cap}$.

Proof We show that the above three conditions are equivalent to the following.

(4) $[\mathsf{X}]_\mathsf{Q} = \bigcap_{\mathsf{X} \subseteq \mathsf{B}' \in \mathfrak{B}'} \mathsf{B}'$ for all $\mathsf{X} \in \mathfrak{P}\mathsf{A}$.

Assume (4) and let Q' be the largest $\mathfrak{B}'$-logic on A. Then $[\mathsf{Y}]_\mathsf{Q} = [\mathsf{Y}]_{\mathsf{Q}'}$ for all $\mathsf{Y} \in \mathfrak{P}'\mathsf{A}$ by Theorem 2.8.4, and so $\mathsf{Q}' = \mathsf{Q}$ by Theorem 2.6.3. Therefore $(\mathsf{A}, \mathfrak{B}') \sim (\mathsf{A}, \mathfrak{B})$ by Definition 2.6.1, and $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}') \le 2$ by Theorem 2.8.4. Thus (1) holds. If (1) holds, then $\mathfrak{B}' \subseteq \overline{\mathfrak{B}'^\cap} = \overline{\mathfrak{B}^\cap}$ by Corollary 2.6.10.1, and $\mathfrak{C} = \overleftarrow{\mathfrak{B}' - \{\mathsf{A}\}}$ by Remark 2.8.1 and Theorem 2.8.5 (3), and thus (2) holds. If $\mathfrak{C} = \overleftarrow{\mathfrak{B}' - \{\mathsf{A}\}}$, then $\mathfrak{D} \subseteq \mathfrak{B}' - \{\mathsf{A}\}$ by Definition 2.8.1. Thus (2) implies (3).

Assume (3) in order to prove (4). Then since $\mathfrak{B}' \subseteq \overline{\mathfrak{B}^\cap}$, we have $[\mathsf{X}]_\mathsf{Q} \subseteq \bigcap_{\mathsf{X} \subseteq \mathsf{B}' \in \mathfrak{B}'} \mathsf{B}'$ by Theorem 2.6.10. Therefore, it remains to show that every element $\mathsf{x} \in \bigcap_{\mathsf{X} \subseteq \mathsf{B}' \in \mathfrak{B}'} \mathsf{B}'$ belongs to $[\mathsf{X}]_\mathsf{Q}$. If $\mathsf{X} \subseteq \mathsf{B}' \in \mathfrak{B}' - \{\mathsf{A}\}$, then $\mathsf{B}' \in \overline{\mathfrak{B}^\cap} - \{\mathsf{A}\} \subseteq \mathfrak{C}$ by Theorem 2.8.5 (1) and $\mathsf{x} \in \mathsf{B}'$, and so the $\mathfrak{B}$-complement $\Diamond$ satisfies $\mathsf{x}^\Diamond \notin \mathsf{B}'$ by Corollary 2.8.6.1. Therefore, there does not exist a set B' such that $\{\mathsf{x}^\Diamond\} \cup \mathsf{X} \subseteq \mathsf{B}' \in \mathfrak{B}' - \{\mathsf{A}\}$, while $\mathfrak{C} = \overleftarrow{\mathfrak{D}} \subseteq \overleftarrow{\mathfrak{B}' - \{\mathsf{A}\}}$ by Theorems 2.8.6 and 2.8.5 (4). Therefore $[\{\mathsf{x}^\Diamond\} \cup \mathsf{X}]_\mathsf{Q} = \mathsf{A}$. Thus $\mathsf{x} \in [\mathsf{X}]_\mathsf{Q}$ by (2.8.1). This latter part of the proof is essentially due to Takaoka.

Corollary 2.8.12.1 Assume that A has a $\mathfrak{B}$-complement. Then the following three conditions are equivalent (s. Remark 2.8.3 for the meaning of (2)).

(1) $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) \le 2$. (2) $\mathfrak{C} = \overleftarrow{\mathfrak{B} - \{\mathsf{A}\}}$. (3) $\mathfrak{D} \subseteq \mathfrak{B} - \{\mathsf{A}\}$.

Proof This is Theorem 2.8.12 with $\mathfrak{B}' = \mathfrak{B}$.

Remark 2.8.6 We already have $\overleftarrow{\mathfrak{B} - \{\mathsf{A}\}} \subseteq \mathfrak{C} = \overleftarrow{\overleftarrow{\mathfrak{B} - \{\mathsf{A}\}}}$ by Theorems 2.8.5. Therefore, the three equivalent conditions in Corollary 2.8.12.1 are furthermore equivalent to both of the conditions $\mathfrak{C} \subseteq \overleftarrow{\mathfrak{B} - \{\mathsf{A}\}}$ and $\overleftarrow{\mathfrak{B} - \{\mathsf{A}\}} = \overleftarrow{\overleftarrow{\mathfrak{B} - \{\mathsf{A}\}}}$ (s.

Remark 2.8.3 for their meanings). Corollary 2.8.12.1 also established the converse of Theorem 2.8.5 (3) under the assumption that A has a $\mathfrak{B}$-complement. If we assume instead that A has a $\mathfrak{B}$-negation, then it is a $\mathfrak{B}$-complement and $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$ by Theorem 2.8.8, and so the above conditions are furthermore equivalent to the condition $\mathfrak{D} = \mathfrak{B} - \{\mathsf{A}\}$.

Corollary 2.8.12.2 Assume that A has a $\mathfrak{B}$-complement. Then the logic space $(\mathsf{A}, \mathfrak{D})$ is equivalent to $(\mathsf{A}, \mathfrak{B})$ and $\mathrm{Cn}(\mathsf{A}, \mathfrak{D}) \leq 2$.

Proof This is because $\mathfrak{D} \subseteq \overline{\mathfrak{B}^{\cap}} - \{\mathsf{A}\}$ by Theorem 2.8.5 (5).

Corollary 2.8.12.3 Let $(\mathsf{A}, \mathcal{F})$ be a Boolean logic space with respect to operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A, $\mathfrak{D}$ be the set of the maximal $\mathcal{F}$-consistent subsets of A and $(\mathsf{A}, \mathcal{F}_{\mathfrak{D}})$ be the functionalization of the logic space $(\mathsf{A}, \mathfrak{D})$. Then $(\mathsf{A}, \mathcal{F}_{\mathfrak{D}})$ is a binary logic space with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, $(\mathsf{A}, \mathcal{F}) \sim (\mathsf{A}, \mathcal{F}_{\mathfrak{D}})$ and $\mathrm{Cn}(\mathsf{A}, \mathcal{F}_{\mathfrak{D}}) \leq 2$ (s. Remarks 2.6.6 and 2.6.9).

Proof The validity relation $\preccurlyeq_{\mathcal{F}}$ is Boolean by Theorem 2.6.9, and so satisfies the lower end law by Lemma 2.2.4. Therefore if $\mathsf{X} \in \mathfrak{D}$, then Theorem 2.8.3 shows that there exists a subset Y of A such that (X, Y) is a maximal cut of A by $\preccurlyeq_{\mathcal{F}}$, and so Lemma 2.7.6 shows that the characteristic function 1_{X} of X is a binary representation of A. Thus $(\mathsf{A}, \mathcal{F}_{\mathfrak{D}})$ is a binary logic space. Let $(\mathsf{A}, \mathfrak{B}_{\mathcal{F}})$ be the logic space associated with $(\mathsf{A}, \mathcal{F})$. Then since A has an $\mathcal{F}$-complement by Theorem 2.8.10, $(\mathsf{A}, \mathfrak{B}_{\mathcal{F}}) \sim (\mathsf{A}, \mathfrak{D})$ and $\mathrm{Cn}(\mathsf{A}, \mathfrak{D}) \leq 2$ by Corollary 2.8.12.2. Therefore $(\mathsf{A}, \mathcal{F}) \sim (\mathsf{A}, \mathcal{F}_{\mathfrak{D}})$ by Remark 2.6.3 and Lemma 2.6.1, and $\mathrm{Cn}(\mathsf{A}, \mathcal{F}_{\mathfrak{D}}) = \mathrm{Cn}(\mathsf{A}, \mathfrak{B}_{\mathcal{F}_{\mathfrak{D}}}) \leq 2$ because $\mathfrak{B}_{\mathcal{F}_{\mathfrak{D}}} = \mathfrak{D} \cup \{\mathsf{A}\}$ and $(\mathfrak{D} \cup \{\mathsf{A}\})^{\cap} = \mathfrak{D}^{\cap} = \overline{\mathfrak{D}^{\cap}}$.

Theorem 2.8.13 Assume that A has a $\mathfrak{B}$-negation. Let X be a subset of A, and assume that either X is finite or $\mathrm{Cn}(\mathsf{A}, \mathfrak{B}) \leq 2$. Then the following four conditions are equivalent (Corollary 2.8.7.1 has given further equivalent conditions under a weaker assumption. s. Theorem 2.8.8).

(1) X is $\mathfrak{B}$-complete, that is, $\mathsf{X} \in \mathfrak{C}$.

(2) $[\mathsf{X}]_{\mathsf{Q}} \in \mathfrak{B} - \{\mathsf{A}\}$.

(3) X has a unique nontrivial $\mathfrak{B}$-model.

(4) $[\mathsf{X}]_{\mathsf{Q}}$ is the unique nontrivial $\mathfrak{B}$-model of X.

Proof Assume (1). Then Theorem 2.8.1 shows that $\mathsf{X} \in \mathfrak{C}$ and $[\mathsf{X}]_{\mathsf{Q}}$ is its only nontrivial $\mathfrak{B}$-model, if any. Moreover, (2) and (3) of Theorem 2.8.5 show that X has a nontrivial $\mathfrak{B}$-model. Thus (4) holds. Obviously (4) implies (3). We have $[\mathsf{X}]_{\mathsf{Q}} = \bigcap_{\mathsf{X} \subseteq \mathsf{B} \in \mathfrak{B}} \mathsf{B}$ by Theorem 2.8.4. Therefore, if B is the unique nontrivial $\mathfrak{B}$-model of X, then $[\mathsf{X}]_{\mathsf{Q}} = \mathsf{B} \in \mathfrak{B} - \{\mathsf{A}\}$. Thus (3) implies (2). If (2) holds, then $\mathsf{X} \in \mathfrak{C}$ because $\mathfrak{B} - \{\mathsf{A}\} \subseteq \mathfrak{D}$ by Theorem 2.8.8.

Corollary 2.8.13.1 Assume that A has a $\mathfrak{B}$-complement. Let X be a subset of A. Then the following three conditions are equivalent (Corollary 2.8.7.1 has given further equivalent conditions under the same assumption).

(1) X is $\mathfrak{B}$-complete, that is, $\mathsf{X} \in \mathfrak{C}$.

(2) X has a unique $\mathfrak{D}$-model.

(3) $[\mathsf{X}]_{\mathsf{Q}}$ is the unique $\mathfrak{D}$-model of X.

Proof Corollaries 2.8.12.2 and 2.8.7.2 show that $(\mathsf{A}, \mathfrak{D}) \sim (\mathsf{A}, \mathfrak{B})$, $\mathrm{Cn}(\mathsf{A}, \mathfrak{D}) \leq 2$ and A has a $\mathfrak{D}$-negation. Moreover $\mathsf{A} \notin \mathfrak{D}$. Therefore Remark 2.8.1 and Theorem 2.8.13 show that (1)–(3) are equivalent.

Example 2.8.1 Let $(\mathsf{A}, \mathcal{F})$ be a Boolean logic space. Then Theorem 2.8.10 shows that A has an $\mathcal{F}$-complement and, if $(\mathsf{A}, \mathcal{F})$ is binary, then A has an $\mathcal{F}$-negation. Therefore we can apply the results in this subsection to $(\mathsf{A}, \mathcal{F})$.

Indeed, it follows from Corollary 2.8.12.1 that the binary logic space on PL belongs to the class 2 (Theorem 2.10.3).

Let $(\mathsf{A}, \mathcal{F})$ be the binary logic space on FPL. Then, as was mentioned in Example 2.7.2, each $\mathcal{F}$-consistent subset of A has a nontrivial $\mathcal{F}$-model. Therefore $\mathrm{Cn}(\mathsf{A}, \mathcal{F}) = 2$ by Corollary 2.8.12.1 (s. Example 2.9.1) and Theorem 2.8.11.

Takaoka [1.93] has shown that ICL belongs to the class 2 or 3 and that it belongs to the class 2 iff its quantitative set is well-ordered and bounded. Takaoka's result will be generalized in Chapter 6 for CL.

2.9 Embedding logic spaces

As was mentioned in §1.3.2, some existing logic systems will be embedded in CL, and the embedding of logic systems implies that of logic spaces.

Suppose logic spaces $(\mathsf{A}, \mathfrak{B})$ and $(\mathsf{A}', \mathfrak{B}')$ satisfy $\mathsf{A}' \subseteq \mathsf{A}$ and $\mathfrak{B}' = \mathfrak{B} \cap \mathsf{A}'$, where $\mathfrak{B} \cap \mathsf{A}' = \{\mathsf{B} \cap \mathsf{A}' : \mathsf{B} \in \mathfrak{B}\}$ by definition. Then we say that $(\mathsf{A}', \mathfrak{B}')$ is **embedded** in $(\mathsf{A}, \mathfrak{B})$, or call $(\mathsf{A}', \mathfrak{B}')$ a **restriction** of $(\mathsf{A}, \mathfrak{B})$, or call $(\mathsf{A}, \mathfrak{B})$ an **extension** of $(\mathsf{A}', \mathfrak{B}')$.

Suppose $\mathbb{B}$-valued functional logic spaces $(\mathsf{A}, \mathcal{F})$ and $(\mathsf{A}', \mathcal{F}')$ satisfy $\mathsf{A}' \subseteq \mathsf{A}$ and $\mathcal{F}' = \{\mathsf{f}|_{\mathsf{A}'} : \mathsf{f} \in \mathcal{F}\}$. It then follows from (2.6.1) and (2.6.3) that the associated logic space of $(\mathsf{A}', \mathcal{F}')$ is embedded in that of $(\mathsf{A}, \mathcal{F})$, and therefore we say that $(\mathsf{A}', \mathcal{F}')$ is **embedded** in $(\mathsf{A}, \mathcal{F})$, or call $(\mathsf{A}', \mathcal{F}')$ a **restriction** of $(\mathsf{A}, \mathcal{F})$, or call $(\mathsf{A}, \mathcal{F})$ an **extension** of $(\mathsf{A}', \mathcal{F}')$.

Refining our notation and terminology for logic spaces $(\mathsf{A}, \mathfrak{B})$, we refer to the $\mathfrak{B}$-logics, $\mathfrak{B}$-theories, $\mathfrak{B}$-core, $\mathfrak{B}$-completeness, core $\mathfrak{B}$-completeness, etc. as the logics, theories, core, completeness, core completeness, etc. of/in $(\mathsf{A}, \mathfrak{B})$, and likewise for functional logic spaces $(\mathsf{A}, \mathcal{F})$. Moreover, for an association R on a set A and a subset X of A, we denote the R-core $[\mathsf{X}]_{\mathsf{R}}$ of X in A by $[\mathsf{X}]_{(\mathsf{A},\mathsf{R})}$.

Lemma 2.9.1 Let A be a set and A' be its subset. Let R be an association on A and R' be its restriction to $\mathsf{A}'^{*} \times \mathsf{A}'$. Then the following hold.

(1) If B is an R-subset of A, then $B \cap A'$ is an R'-subset of A'.

(2) Assume that R is partially latticed. Then every subset X' of A' satisfies $[X']_{(A',R')} = [X']_{(A,R)} \cap A'$, and the set of the R'-subsets of A' is equal to the set of the intersections $B \cap A'$ for the R-subsets B of A.

(3) Assume that R satisfies the following condition written by the word convention (then we say that R is **tame** on A'):

$$\alpha \subseteq A,\ y \in A',\ \alpha\, R\, y \implies \alpha \subseteq A'.$$

Then every subset X of A satisfies $[X \cap A']_{(A',R')} = [X]_{(A,R)} \cap A'$.

Proof (1) If elements $x_1, \ldots, x_n \in B \cap A'$ and $y \in A'$ satisfy $x_1 \cdots x_n\, R'\, y$, then since $x_1, \ldots, x_n \in B$ and $x_1 \cdots x_n\, R\, y$, we have $y \in B$, and so $y \in B \cap A'$. Thus $B \cap A'$ is an R'-subset of A'.

(2) Let $B' = [X']_{(A',R')}$ and $B = [X']_{(A,R)}$. Then since $X' \subseteq B \cap A'$, we have $B' \subseteq B \cap A'$ by (1). Assume $y \in B \cap A'$. Then Theorem 2.4.6 shows that $x_1 \cdots x_n\, R\, y$ for an element $x_1 \cdots x_n \in X'^*$, and so $x_1 \cdots x_n\, R'\, y$. Therefore $y \in B'$. Thus the former statement holds. In particular if X' is an R'-subset of A', then $X' = B \cap A'$. This together with (1) proves the latter.

(3) Let $B' = [X \cap A']_{(A',R')}$ and $B = [X]_{(A,R)}$. Then since $X \cap A' \subseteq B \cap A'$, we have $B' \subseteq B \cap A'$ by (1). In order to prove $B \cap A' \subseteq B'$, it suffices to show, by virtue of Theorem 2.4.2, that the n-th descendant X_n of X in (A, R) and the n-th descendant $(X \cap A')_n$ of $X \cap A'$ in (A', R') satisfy $X_n \cap A' \subseteq (X \cap A')_n$ for $n = 0, 1, \ldots$. We argue by induction on n. First, $X_0 \cap A' = X \cap A' = (X \cap A')_0$. Therefore assume $n \geq 1$, and let $y \in X_n \cap A'$. Then $x_1 \cdots x_k\, R\, y$ with $x_j \in X_{n_j}$ $(j = 1, \ldots, k)$ and $n - 1 = \sum_{j=1}^{k} n_j$, and so $x_j \in X_{n_j} \cap A'$ because R is tame on A'. Therefore $x_1 \cdots x_k\, R'\, y$, and $x_j \in (X \cap A')_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $y \in (X \cap A')_n$ as desired.

Theorem 2.9.1 Let $(A, \mathfrak{B})$ be a logic space and $(A', \mathfrak{B}')$ be its restriction. Then the following hold.

(1) The cores C and C' of $(A, \mathfrak{B})$ and $(A', \mathfrak{B}')$ satisfy $C' = C \cap A'$.

(2) The largest logic Q' of $(A', \mathfrak{B}')$ is the restriction of the largest logic Q of $(A, \mathfrak{B})$ to $A'^* \times A'$.

(3) The set of the logics of $(A', \mathfrak{B}')$ is equal to the set of the restrictions of the logics of $(A, \mathfrak{B})$ to $A'^* \times A'$.

(4) Every subset X' of A' satisfies $[X']_{(A',Q')} = [X']_{(A,Q)} \cap A'$, and the set of the theories of $(A', \mathfrak{B}')$ is equal to the set of the intersections $B \cap A'$ for the theories B of $(A, \mathfrak{B})$.

(5) The logic spaces $(A', \mathfrak{B}'^{\cap})$ and $(A', \overline{\mathfrak{B}'^{\cap}})$ are restrictions of the logic spaces $(A, \mathfrak{B}^{\cap})$ and $(A, \overline{\mathfrak{B}^{\cap}})$ respectively.

(6) $\mathrm{Cn}(A', \mathfrak{B}') \leq \mathrm{Cn}(A, \mathfrak{B})$.

(7) Let $x^{\Diamond}$ be a unary operation on A which closes A'. Then if $\Diamond$ is a $\mathfrak{B}$-complement on A, its restriction to A' is a $\mathfrak{B}'$-complement on A', and A' is inconsistent in $(A, \mathfrak{B})$, and likewise for $\mathfrak{B}$-negations.

(8) Let X' be a subset of A'. Then if X' is consistent in $(A', \mathfrak{B}')$, it is consistent in $(A, \mathfrak{B})$. The converse is true provided that A' is inconsistent in $(A, \mathfrak{B})$.

(9) Let X' be a subset of A'. Then if X' has a nontrivial $\mathfrak{B}'$-model, it has a nontrivial $\mathfrak{B}$-model. The converse is true provided that A' is inconsistent in $(A, \mathfrak{B})$.

(10) Let (R, D) be a deduction system on A and (R', D') be its restriction to $(A'^* \times A', A')$. Assume that (R, D) is core complete in $(A, \mathfrak{B})$ and R is tame on A'. Then (R', D') is core complete in $(A', \mathfrak{B}')$.

Proof (1) is because $C' = \bigcap \mathfrak{B}' = \bigcap(\mathfrak{B} \cap A') = (\bigcap \mathfrak{B}) \cap A' = C \cap A'$. This also follows from Theorem 2.6.4 and either (2) or (4).

(2) is because the following holds for all $(\alpha, y) \in A'^* \times A'$:

$$
\begin{aligned}
\alpha\, Q\, y &\iff y \in \textstyle\bigcap_{\alpha \subseteq B \in \mathfrak{B}} B && \text{(by Theorem 2.6.2)} \\
&\iff y \in \textstyle\bigcap_{\alpha \subseteq B \in \mathfrak{B}} (B \cap A') \\
&\iff y \in \textstyle\bigcap_{\alpha \subseteq B \cap A',\ B \in \mathfrak{B}} (B \cap A') \\
&\iff y \in \textstyle\bigcap_{\alpha \subseteq B' \in \mathfrak{B}'} B' \\
&\iff \alpha\, Q'\, y && \text{(by Theorem 2.6.2).}
\end{aligned}
$$

(3) follows from (2) and Theorem 2.6.1. Indeed, if R' is a logic of $(A', \mathfrak{B}')$, then $R' \subseteq Q'$, hence $R' \subseteq Q$ and therefore R' is a logic of $(A, \mathfrak{B})$, and its restriction to $A'^* \times A'$ is R' itself. Moreover, if R is a logic of $(A, \mathfrak{B})$, then $R \subseteq Q$, and so its restriction to $A'^* \times A'$ is contained in Q' and therefore is a logic of $(A', \mathfrak{B}')$.

(4) Q is partially latticed by Theorem 2.6.3. Theorem 2.6.1 shows that the theories of $(A', \mathfrak{B}')$ are the Q'-subsets of A' and the theories of $(A, \mathfrak{B})$ are the Q-subsets of A. Therefore (4) follows from (2) and Lemma 2.9.1.

(5) Since $\mathfrak{B}' = \mathfrak{B} \cap A'$, we immediately have $\mathfrak{B}'^{\cap} = \mathfrak{B}^{\cap} \cap A'$. Since $\overline{\mathfrak{B}'^{\cap}}$ and $\overline{\mathfrak{B}^{\cap}}$ are the sets of the theories of $(A', \mathfrak{B}')$ and $(A, \mathfrak{B})$ respectively by Theorem 2.6.10, we have $\overline{\mathfrak{B}'^{\cap}} = \overline{\mathfrak{B}^{\cap}} \cap A'$ by (4).

(6) If $\mathrm{Cn}(A, \mathfrak{B}) = 1$, then $\overline{\mathfrak{B}'^{\cap}} = \overline{\mathfrak{B}^{\cap}} \cap A' = \mathfrak{B} \cap A' = \mathfrak{B}'$ by (5), and so $\mathrm{Cn}(A', \mathfrak{B}') = 1$. If $\mathrm{Cn}(A, \mathfrak{B}) = 2$, then $\overline{\mathfrak{B}'^{\cap}} = \overline{\mathfrak{B}^{\cap}} \cap A' = \mathfrak{B}^{\cap} \cap A' = \mathfrak{B}'^{\cap}$ by (5), and so $\mathrm{Cn}(A', \mathfrak{B}') \leq 2$.

(7) Assume that $\Diamond$ is a $\mathfrak{B}$-complement. Then $\Diamond$ satisfies the laws of contradiction and excluded middle with respect to Q. Since Q' is the restriction of Q by (2), the restriction $\Diamond'$ of $\Diamond$ to A' satisfies the laws with respect to Q'. Thus $\Diamond'$ is a $\mathfrak{B}'$-complement. Moreover, since $\{x, x^{\Diamond}\} \subseteq A'$ for all $x \in A'$, A' is inconsistent in $(A, \mathfrak{B})$ by Corollary 2.8.6.1.

Assume that $\Diamond$ is a $\mathfrak{B}$-negation. Let $B' \in \mathfrak{B}' - \{A'\}$. Then $B' = B \cap A'$ for some $B \in \mathfrak{B} - \{A\}$, and so $\Diamond$ satisfies the B-negation law. Moreover, each element $x \in A'$ satisfies $x \in B'$ iff $x \in B$. Therefore $\Diamond'$ satisfies the B'-negation law. Thus $\Diamond'$ is a $\mathfrak{B}'$-negation. Furthermore, since $\Diamond$ is a $\mathfrak{B}$-complement by Theorem 2.8.8, A' is inconsistent in $(A, \mathfrak{B})$ by the above.

(8) If $[X']_{(A',Q')} \neq A'$, then $[X']_{(A,Q)} \neq A$ by the equation $[X']_{(A',Q')} = [X']_{(A,Q)} \cap A'$ in (4). Conversely if $[X']_{(A,Q)} \neq A$ and A' is inconsistent in $(A, \mathfrak{B})$, then $A' \nsubseteq [X']_{(A,Q)}$ and so $[X']_{(A',Q')} \neq A'$ by this equation.

(9) If $X' \subseteq B' \in \mathfrak{B}' - \{A'\}$, then $B' = B \cap A'$ for some $B \in \mathfrak{B} - \{A\}$ and $X' \subseteq B$. Conversely if $X' \subseteq B \in \mathfrak{B} - \{A\}$ and A' is inconsistent in $(A, \mathfrak{B})$, then $A' \nsubseteq B$ and so $X' \subseteq B \cap A' \in \mathfrak{B}' - \{A'\}$.

(10) This is because $C' = C \cap A' = [D]_{(A,R)} \cap A' = [D \cap A']_{(A',R')} = [D']_{(A',R')}$ by (1) and Lemma 2.9.1.

Example 2.9.1 Takaoka [1.93] has shown that the binary logic space $(A, \mathcal{F})$ on ICL belongs to the class 2 under certain conditions. As is shown in Chapter 4, the binary logic space $(A', \mathcal{F}')$ on FPL may be embedded in such $(A, \mathcal{F})$. Thus it follows from Theorem 2.9.1 that $\mathrm{Cn}(A', \mathcal{F}') \leq 2$ (s. Example 2.8.1).

Theorem 2.9.2 Let $(A, \mathcal{F})$ be a functional logic space and $(A', \mathcal{F}')$ be its restriction. Let $(\vec{A}, \vec{\mathcal{F}})$ and $(\vec{A'}, \vec{\mathcal{F}'})$ be the sequential logic spaces associated with $(A, \mathcal{F})$ and $(A', \mathcal{F}')$ respectively. Then $(\vec{A'}, \vec{\mathcal{F}'})$ is embedded in $(\vec{A}, \vec{\mathcal{F}})$.

Furthermore, let $(\vec{R}, \vec{D})$ be a characteristic law of $(A, \mathcal{F})$ and $(\vec{S}, \vec{E})$ be its restriction to $(\vec{A'}^* \times \vec{A'}, \vec{A'})$. Assume that $\vec{R}$ is tame on $\vec{A'}$. Then $(\vec{S}, \vec{E})$ is a characteristic law of $(A', \mathcal{F}')$.

Proof By definition, $\vec{A} = A^* \times A^*$, $\vec{A}_f = \{\alpha \to \beta \in \vec{A} : \alpha \preccurlyeq_f \beta\}$ $(f \in \mathcal{F})$ and $\vec{\mathcal{F}} = \{\vec{A}_f : f \in \mathcal{F}\}$, where

$$\alpha \preccurlyeq_f \beta \iff \inf f\alpha \leq \sup f\beta$$

for all $\alpha \to \beta \in \vec{A}$. Likewise $\vec{A'} = A'^* \times A'^*$, $\vec{A'}_g = \{\gamma \to \delta \in \vec{A'} : \gamma \preccurlyeq_g \delta\}$ $(g \in \mathcal{F}')$ and $\vec{\mathcal{F}'} = \{\vec{A'}_g : g \in \mathcal{F}'\}$, where

$$\gamma \preccurlyeq_g \delta \iff \inf g\gamma \leq \sup g\delta$$

for all $\gamma \to \delta \in \vec{A'}$. If $f \in \mathcal{F}$ and $g = f|_{A'} \in \mathcal{F}'$, then $\vec{A}_f \cap \vec{A'} = \vec{A'}_g$. Therefore $\vec{\mathcal{F}} \cap \vec{A'} = \vec{\mathcal{F}'}$ and thus the former statement holds. The latter is a consequence of the former and Theorem 2.9.1.

2.10 Propositional logic as a prototype

Throughout this section, we let A be a UTA as defined in §3.1.7 with respect to a nonempty basis P and operations $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rightarrow y$ on A, and let $\mathcal{F}$ be the set of the binary representations of A with respect to the operations $\wedge, \vee, \Diamond$

and $\Rrightarrow$. The purpose of this section is to illustrate the theory of logic spaces and deduction systems developed in the preceding sections with the binary logic space $(A, \mathcal{F})$, which in fact is what I have called the binary logic space on PL.

Lemma 2.10.1 Let $\preccurlyeq$ be a weakly Boolean relation on A^* with respect to $\wedge, \vee, \diamond$ and $\Rrightarrow$. Then each cut of A by $\preccurlyeq$ has an $\mathcal{F}$-model.

Proof Let (X, Y) be a cut of A by $\preccurlyeq$. Then Lemma 2.7.2 and Zorn's lemma show that there exists a maximal cut (X', Y') of A by $\preccurlyeq$ satisfying $(X, Y) \subseteq (X', Y')$, hence $\inf fX' \leq \inf fX$ and $\sup fY \leq \sup fY'$ for all $f \in \mathcal{F}$. Therefore we may assume that (X, Y) itself is maximal. Then Lemma 2.7.5 shows that (X, Y) satisfies the following eight conditions, because $\preccurlyeq$ satisfies the extension law by Remark 2.2.8.

(1) $x \wedge y \in X \implies x, y \in X$.

(2) $x \wedge y \in Y \implies x \in Y$ or $y \in Y$.

(3) $x \vee y \in X \implies x \in X$ or $y \in X$.

(4) $x \vee y \in Y \implies x, y \in Y$.

(5) $x^\diamond \in X \implies x \in Y$.

(6) $x^\diamond \in Y \implies x \in X$.

(7) $x \Rrightarrow y \in X \implies x \in Y$ or $y \in X$.

(8) $x \Rrightarrow y \in Y \implies x \in X,\ y \in Y$.

Since $X \cap Y = \emptyset$ by Lemmas 2.7.3, there exists a mapping $\varphi \in P \to \mathbb{T}$ such that $X \cap P \subseteq \varphi^{-1}\{1\}$ and $Y \cap P \subseteq \varphi^{-1}\{0\}$. Since (A, P) is a UTA, φ is extended to a binary representation f of A, and $f \in \mathcal{F}$ by the definition of $\mathcal{F}$. Therefore, because of Remark 2.7.3, it suffices to show that $fz = 1$ for all $z \in X$ and $fz = 0$ for all $z \in Y$. In view of Remark 3.1.17 and Theorem 3.1.7, we argue by induction on the rank n of z. If $n = 0$, then $z \in P$, and so $fz = \varphi z = 1$ if $z \in X$, and $fz = \varphi z = 0$ if $z \in Y$. Therefore assume $n \geq 1$. Then Theorem 3.1.7 shows that the ramification of z is $x \wedge y$, $x \vee y$, $x^\diamond$ or $x \Rrightarrow y$ with $\operatorname{rk} x,\ \operatorname{rk} y < n$.

Assume $z = x \wedge y \in X$. Then $x, y \in X$ by (1), and so $fx = fy = 1$ by the induction hypothesis, hence $fz = f(x \wedge y) = fx \wedge fy = 1 \wedge 1 = 1$.

Assume $z = x \wedge y \in Y$. Then $x \in Y$ or $y \in Y$ by (2), and so $fx = 0$ or $fy = 0$ by the induction hypothesis, hence $fz = f(x \wedge y) = fx \wedge fy = 0$.

Assume $z = x \vee y \in X$. Then $x \in X$ or $y \in X$ by (3), and so $fx = 1$ or $fy = 1$ by the induction hypothesis, hence $fz = f(x \vee y) = fx \vee fy = 1$.

Assume $z = x \vee y \in Y$. Then $x, y \in Y$ by (4), and so $fx = fy = 0$ by the induction hypothesis, hence $fz = f(x \vee y) = fx \vee fy = 0 \vee 0 = 0$.

Assume $z = x^\diamond \in X$. Then $x \in Y$ by (5), and so $fx = 0$ by the induction hypothesis, hence $fz = f(x^\diamond) = (fx)^\diamond = 0^\diamond = 1$.

Assume $z = x^{\Diamond} \in Y$. Then $x \in X$ by (6), and so $fx = 1$ by the induction hypothesis, hence $fz = f(x^{\Diamond}) = (fx)^{\Diamond} = 1^{\Diamond} = 0$.

Assume $z = x \Rightarrow y \in X$. Then $x \in Y$ or $y \in X$ by (7), and so $fx = 0$ or $fy = 1$ by the induction hypothesis, hence $fz = f(x \Rightarrow y) = fx \Rightarrow fy = 1$.

Assume $z = x \Rightarrow y \in Y$. Then $x \in X$, $y \in Y$ by (8), and so $fx = 1$, $fy = 0$ by the induction hypothesis, hence $fz = f(x \Rightarrow y) = fx \Rightarrow fy = 1 \Rightarrow 0 = 0$.

Theorem 2.10.1 The (weakly) Boolean law with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$ is a characteristic law of $(A, \mathcal{F})$.

Proof The (weakly) Boolean law is a deductive law by Example 2.5.2 and the $\mathcal{F}$-validity relation $\preccurlyeq_{\mathcal{F}}$ satisfies it by Theorems 2.6.9 and 2.2.21. Moreover, the (weakly) Boolean law implies the extension law by Remark 2.2.8. Furthermore, each cut of A by each (weakly) Boolean relation on A^* has an $\mathcal{F}$-model by Lemma 2.10.1 and Theorem 2.2.21. Thus the (weakly) Boolean law is a characteristic law of $(A, \mathcal{F})$ by Theorems 2.7.15.

Theorem 2.10.2 Let & and $\wp$ be associations on A defined by (2.5.2) and B be the set of the Boolean elements of A with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$. Then the deduction system $(\& \cup \wp, B)$ on A is $\mathcal{F}$-complete.

Proof Example 2.7.1 and Theorem 2.2.21 show that $(\& \cup \wp, B)$ is $\mathcal{F}$-sound and $\preccurlyeq_{\& \cup \wp, B}$ satisfies the (weakly) Boolean law which is a characteristic law of $(A, \mathcal{F})$ by Theorem 2.10.1. Thus $(\& \cup \wp, B)$ is $\mathcal{F}$-complete by Theorems 2.7.13.

Theorem 2.10.3 $\mathrm{Cn}(A, \mathcal{F}) = 2$.

Proof Theorems 2.6.9 and Theorem 2.2.21 show that $1_A \notin \mathcal{F}$ and $\preccurlyeq_{\mathcal{F}}$ is weakly Boolean, and so $\preccurlyeq_{\mathcal{F}}$ satisfies the lower end law by Lemma 2.2.4. Therefore, as was noted in Remark 2.7.6, Lemma 2.10.1 shows that each $\mathcal{F}$-consistent subset of A has a nontrivial $\mathcal{F}$-model. Moreover, A has an $\mathcal{F}$-complement by Theorem 2.8.10. Therefore $\mathrm{Cn}(A, \mathcal{F}) \leq 2$ by Corollary 2.8.12.1 together with Remark 2.8.6.

Since (A, P) is a UTA, each member of $P \to \mathbb{T}$ is extended to a member of $\mathcal{F}$, and so $\#\mathcal{F} > 1$. Therefore $\mathrm{Cn}(A, \mathcal{F}) \neq 1$ by Theorem 2.8.11.

Alternative proof of $\mathrm{Cn}(A, \mathcal{F}) \leq 2$**.** Let $\mathfrak{D}$ be the set of the maximal $\mathcal{F}$-consistent subsets of A and $(A, \mathcal{F}_{\mathfrak{D}})$ be the functionalization of the logic space $(A, \mathfrak{D})$. Then $(A, \mathcal{F}_{\mathfrak{D}})$ is a binary logic space by Corollary 2.8.12.3. Therefore $\mathcal{F}_{\mathfrak{D}} \subseteq \mathcal{F}$, and so $\mathfrak{D} \subseteq \mathfrak{B}_{\mathcal{F}} - \{A\}$ by Remark 2.6.1. Moreover, A has an $\mathcal{F}$-complement by Theorem 2.8.10. Thus $\mathrm{Cn}(A, \mathcal{F}) \leq 2$ by Corollary 2.8.12.1.

We can study variations $(A', \mathcal{F}')$ of $(A, \mathcal{F})$ by embedding them in $(A, \mathcal{F})$. To give an example, let A' be a UTA with respect to a nonempty basis P' and operations $x^{\Diamond}$ and $x \Rightarrow y$ on A', and let $\mathcal{F}'$ be the set of the homomorphisms of A' into $\mathbb{T}$ with respect to the operations $\Diamond$ and $\Rightarrow$ on A' and the complement $\Diamond$ and the cojoin $\Rightarrow$ on $\mathbb{T}$ (s. §1.5.2).

Since the basis P of A is an arbitrary nonempty set, we assume $P = P'$. Then A' is identified with the subreduct $[P]_{\{\Diamond,\Rrightarrow\}}$ of A by Remark 3.1.12, Corollary 3.1.7.3 and the uniqueness of USAs in Theorem 3.1.5.

Lemma 2.10.2 The functional logic space $(A', \mathcal{F}')$ is a restriction of $(A, \mathcal{F})$.

Proof If $f \in \mathcal{F}$, then $f|_{A'} \in \mathcal{F}'$ because $\{\Diamond, \Rrightarrow\} \subseteq \{\wedge, \vee, \Diamond, \Rrightarrow\}$. Moreover if $f' \in \mathcal{F}'$, then since (A, P) is a UTA, $f'|_P$ is extended to a function $f \in \mathcal{F}$, and since $f|_{A'} \in \mathcal{F}'$ and $f'|_P = (f|_{A'})|_P$, Lemma 3.1.4 shows $f' = f|_{A'}$.

Theorem 2.10.4 The union of the laws of repetition, weakening, contraction, exchange, strong negation and strong implication with respect to the operations $\Diamond$ and $\Rrightarrow$ is a characteristic law of $(A', \mathcal{F}')$.

Proof Lemma 2.10.2 allows us to use Theorem 2.9.2. Define an association $\vec{R}$ on $\vec{A} = A^* \times A^*$ by the fractional list $\dfrac{\alpha \to \beta}{x\alpha \to \beta}, \dfrac{\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \dfrac{xx\alpha \to \beta}{x\alpha \to \beta}, \dfrac{xx\alpha \leftarrow \beta}{x\alpha \leftarrow \beta},$ $\dfrac{\alpha xy\beta \to \gamma}{\alpha yx\beta \to \gamma}, \dfrac{\alpha xy\beta \leftarrow \gamma}{\alpha yx\beta \leftarrow \gamma}, \dfrac{xy\alpha \to \beta}{x \wedge y \cdot \alpha \to \beta}, \dfrac{\alpha \to x\beta \quad \alpha \to y\beta}{\alpha \to x \wedge y \cdot \beta}, \dfrac{\alpha \to xy\beta}{\alpha \to x \vee y \cdot \beta},$ $\dfrac{x\alpha \to \beta \quad y\alpha \to \beta}{x \vee y \cdot \alpha \to \beta}, \dfrac{\alpha \to x\beta}{x^{\Diamond}\alpha \to \beta}, \dfrac{x\alpha \to \beta}{\alpha \to x^{\Diamond}\beta}, \dfrac{x\alpha \to y\beta}{\alpha \to x \Rrightarrow y \cdot \beta}, \dfrac{\alpha \to x\beta \quad y\alpha \to \beta}{x \Rrightarrow y \cdot \alpha \to \beta}$ (s. Example 2.5.2). Define the subset $\vec{D}$ of $\vec{A}$ by $\vec{D} = \{x \to x : x \in A\}$. Then the deduction system $(\vec{R}, \vec{D})$ on $\vec{A}$ is a characteristic law of $(A, \mathcal{F})$ by Theorem 2.10.1. Let $(\vec{S}, \vec{E})$ be its restriction to $(\vec{A'}^* \times \vec{A'}, \vec{A'})$, where $\vec{A'} = A'^* \times A'^*$. Remark 3.1.17 and Theorem 3.1.7 show that each element of $A - P$ has a unique ramification in one of the forms $x \wedge y$, $x \vee y$, $x^{\Diamond}$ and $x \Rrightarrow y$ and each element of $A' - P$ has a ramification in one of the forms $x^{\Diamond}$ and $x \Rrightarrow y$. Therefore $x \wedge y \notin A'$ and $x \vee y \notin A'$ for all $(x, y) \in A \times A$. Moreover, $x^{\Diamond} \in A'$ iff $x \in A'$. Moreover, $x \Rrightarrow y \in A'$ iff $(x, y) \in A' \times A'$. Therefore $\vec{S}$ is the union of the associations $\dfrac{\alpha \to \beta}{x\alpha \to \beta}, \dfrac{\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \dfrac{xx\alpha \to \beta}{x\alpha \to \beta}, \dfrac{xx\alpha \leftarrow \beta}{x\alpha \leftarrow \beta}, \dfrac{\alpha xy\beta \to \gamma}{\alpha yx\beta \to \gamma}, \dfrac{\alpha xy\beta \leftarrow \gamma}{\alpha yx\beta \leftarrow \gamma}, \dfrac{\alpha \to x\beta}{x^{\Diamond}\alpha \to \beta},$ $\dfrac{x\alpha \to \beta}{\alpha \to x^{\Diamond}\beta}, \dfrac{x\alpha \to y\beta}{\alpha \to x \Rrightarrow y \cdot \beta}, \dfrac{\alpha \to x\beta \quad y\alpha \to \beta}{x \Rrightarrow y \cdot \alpha \to \beta}$ with $x, y \in A'$ and $\alpha, \beta, \gamma \in A'^*$, and $\vec{E} = \{x \to x : x \in A'\}$. Furthermore $\vec{R}$ is tame on $\vec{A'}$. Thus $(\vec{S}, \vec{E})$ is a characteristic law of $(A', \mathcal{F}')$.

Theorem 2.10.5 Let $\wp'$ be the restriction of $\wp$ to $A'^* \times A'$ and L' be the set of the elements of A' in any of the following four forms: $x \Rrightarrow x$, $y \Rrightarrow (x \Rrightarrow y)$, $(z \Rrightarrow (x \Rrightarrow y)) \Rrightarrow ((z \Rrightarrow x) \Rrightarrow (z \Rrightarrow y))$, $(y^{\Diamond} \Rrightarrow x^{\Diamond}) \Rrightarrow (x \Rrightarrow y)$. Then the deduction system $(\wp', L')$ on A' is $\mathcal{F}'$-complete.

Proof Define the binary operations $\wedge$ and $\vee$ on A' by $x \wedge y = (x \Rrightarrow y^{\Diamond})^{\Diamond}$ and $x \vee y = x^{\Diamond} \Rrightarrow y$ for all $(x, y) \in A' \times A'$ (A' is neither a UTA with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ on it nor a subalgebra of A with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ on A). Then $\mathcal{F}'$ is equal to the set of the binary representations of A' with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$, because

$a \wedge b = (a \Rightarrow b^\Diamond)^\Diamond$ and $a \vee b = a^\Diamond \Rightarrow b$ for all $(a, b) \in \mathbb{T} \times \mathbb{T}$. Therefore, the $\mathcal{F}'$-validity relation $\preccurlyeq_{\mathcal{F}'}$ on A'^* is Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$ by Theorem 2.6.9, and hence it follows that $(\wp', L')$ is $\mathcal{F}'$-sound. Moreover, L' contains all Lukasiewicz elements of A' with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$, because the elements $(x \vee y) \Rightarrow (x^\Diamond \Rightarrow y)$, $(x^\Diamond \Rightarrow y) \Rightarrow (x \vee y)$, $(x \wedge y) \Rightarrow (x \Rightarrow y^\Diamond)^\Diamond$ and $(x \Rightarrow y^\Diamond)^\Diamond \Rightarrow (x \wedge y)$ all have the form $z \Rightarrow z$. Therefore, the deduction relation $\preccurlyeq_{\wp', L'}$ on A'^* is weakly Boolean with respect to $\wedge, \vee, \Diamond$ and $\Rightarrow$ by Theorems 2.5.12 and 2.2.21. In particular, it satisfies the characteristic law of $(A', \mathcal{F}')$ obtained in Theorem 2.10.4. Thus $(\wp', L')$ is $\mathcal{F}'$-complete by Theorem 2.7.13.

Theorem 2.10.6 $\mathrm{Cn}(A', \mathcal{F}') = 2$.

Proof Since $(A', \mathcal{F}')$ is a restriction of $(A, \mathcal{F})$ by Lemma 2.10.2, Theorems 2.10.3 and 2.9.1 show that $\mathrm{Cn}(A', \mathcal{F}') \leq 2$. The definition of $(A', \mathcal{F}')$ shows $\#\mathcal{F}' > 1$ as in Theorem 2.10.3. As was shown in the proof of Theorem 2.10.5, $(A', \mathcal{F}')$ is a binary logic space with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ on A'. Therefore $\mathrm{Cn}(A', \mathcal{F}') \neq 1$ by Theorem 2.8.11.

2.11 Associations vs operators

The purpose of this section is to suggest and outline an alternative approach to logic spaces and deduction systems. Full details were given in [1.3].

Let A be a set. Then an **operator** on A is a unary operation φ on $\mathfrak{P}A$. If $X \in \mathfrak{P}A$ satisfy $\varphi X \subseteq X$, then we call X a **φ-subset** of A or say that X is φ-closed or that φ **close**s X (s. §2.1.2).

For each increasing operator φ on A, the **finitary core** φ' is the operator on A defined by the following for each $X \in \mathfrak{P}A$ (s. Definition 2.1.3):

$$\varphi' X = \bigcup\nolimits_{Y \in \mathfrak{P}' X} \varphi Y. \tag{2.11.1}$$

Then, with respect to the power order $\subseteq$ on $\mathfrak{P}A \to \mathfrak{P}A$ defined in Remark 2.1.3, φ' is the largest of the finitary operators ψ on A which satisfy $\psi \subseteq \varphi$. Moreover, if φ is a closure operator, then so is φ' and its fixture domain is equal to the quasi-finitary closure $\overline{\mathfrak{B}}$ of the fixture domain $\mathfrak{B}$ of φ in $\mathfrak{P}A$, and so it follows from Theorem 2.1.4 that $\varphi = \varphi'$ iff $\mathfrak{B} = \overline{\mathfrak{B}}$.

For each operator φ on A, we define an association R_φ on A by

$$\alpha \, R_\varphi \, y \iff \varphi\alpha \ni y \tag{2.11.2}$$

for each $(\alpha, y) \in A^* \times A$ (s. Theorem 2.2.24). Conversely for each association R on A, we define an operator φ_R on A by

$$\varphi_R X = \{y \in A : \alpha \, R \, y \text{ for an element } \alpha \in X^*\} \tag{2.11.3}$$

for each $X \in \mathfrak{P}A$ (s. Theorem 2.4.6). Then φ_R is finitary, and the set of the φ_R-subsets of A is equal to that of the R-subsets of A. Ken Sasaki[2.38] showed

[2.38] *Syazô ronri no hakken to sono eikyô* (*Discovery of logic operators and its influence*), Master's thesis, Graduate School Math. Sci., Univ. Tokyo, 2008.

among other things that the mappings $\varphi \mapsto R_\varphi$ and $R \mapsto \varphi_R$ yield mutually inverting order isomorphisms between the set of the finitary operators on A and that of the associations on A which satisfy the partial extension law.

Let $(A, \mathfrak{B})$ be a logic space. Then an operator φ on A is said to be **$\mathfrak{B}$-sound**, if every set in $\mathfrak{B}$ is φ-closed. There exist the largest increasing $\mathfrak{B}$-sound operator and the largest finitary $\mathfrak{B}$-sound operator on A. Let μ and ν denote them. Then $\mu X = \bigcap_{X \subseteq B \in \mathfrak{B}} B$ for all $X \in \mathfrak{P}A$, and so it follows from Theorem 2.1.4 that μ is a closure operator and its fixture domain is equal to the $\cap$-closure $\mathfrak{B}^\cap$ of $\mathfrak{B}$ in $\mathfrak{P}A$. Moreover, ν is equal to the finitary core μ' of μ, and so ν is a closure operator and its fixture domain is equal to the quasi-finitary $\cap$-closure $\overline{\mathfrak{B}^\cap}$ of $\mathfrak{B}$ in $\mathfrak{P}A$. Thus $\mu = \nu$ iff $\mathfrak{B}^\cap = \overline{\mathfrak{B}^\cap}$, that is, iff $\mathrm{Cn}(A, \mathfrak{B}) \leq 2$.

Let Q be the largest $\mathfrak{B}$-logic on A. Then $Q = R_\nu$ and $\varphi_Q = \nu$. The former equation gives an alternative proof of Theorem 2.6.2. The latter shows that the set of the Q-subsets of A is equal to the set $\overline{\mathfrak{B}^\cap}$ of the ν-subsets of A. This together with Theorem 2.6.1 gives an alternative proof of Theorem 2.6.10.

Operators also give alternative proofs of certain other theorems in this chapter. The concept of completeness and Theorem 2.7.1 can be restated in terms of operators. However, I have not been able to even formulate Theorem 2.7.13 in terms of operators. This is a reason why operators have not been able to play a principal role in the theory of logic spaces and deduction systems.

Chapter 3

Logic Systems

The purpose of this chapter is to present a general theory of syntax and semantics and link it to the abstract theory of semantics and deduction presented in Chapter 2. Syntax, semantics and deduction are the pillars of logic (s. Remark 1.2.1). Thus Chapters 2 and 3 together present a theory of GL.

As the concepts of logic spaces and deduction systems defined and analyzed in Chapter 2 were abstracted from semantics and deduction in various specific branches of logic, so the concept of logic systems is generalized from the combination of syntax and semantics therein. Its definition in §3.2 and analysis in §3.3 are based on the fundamentals of sorted algebras, especially *universal sorted algebras* (USA), given in §3.1. The GL in §3.1–3.3 and Chapter 2 is illustrated in §3.4, §3.5 and Chapters 4–6 by means of its earliest prototype FPL, an incompleteness theorem for it and its ultimate application to CL designed for MN, respectively. What follows is an introduction to §3.1–3.3, particularly newly defined concepts of formal languages, syntax and semantics.

Syntax in logic may have been understood as the method of defining formal languages. A formal language is commonly defined by specifying the rules for forming all its elements by induction starting from certain prime elements. The formation rules are commonly described in terms of the prime elements, certain tokens, certain punctuation marks and the names of the categories in which elements of the formal language are placed. Thus a formal language is commonly defined as a subset of the free monoid over the set consisting of the prime elements, tokens and punctuation marks.[3.1]

According to common textbooks of FPL, for example, its formal language consists of terms and formulas, which are formed by induction starting from the prime elements, i.e. constants and variables. The formation rules are described in terms of the prime elements, certain tokens (i.e. function symbols, predicate symbols, logical symbols and quantifiers), certain punctuation marks and the category names *term* and *formula*.

[3.1] Some such definitions are mathematically inadequate because they lack enough punctuation marks, the concept of ranks which are necessary for correct induction and/or the free monoid as a set-theoretic basis (s. Remark 3.1.14, the proof of Theorem 3.1.5 and [2.23]).

Such a definition of a formal language L is underlain by the triple $(\mathsf{T}, \tau, \mathsf{P})$ of the set T of the category names, the set P of the prime elements and the naming $\tau \in \mathsf{P} \to \mathsf{T}$ which associates each element of P with its category name, and also underlain by algebraic structures which the formation rules give to L and T so that the extended naming $\tau' \in \mathsf{L} \to \mathsf{T}$ is a kind of homomorphism.

In FPL, for example, the formation rule for the terms of L may be regarded as assigning each k-ary function symbol f the k-ary operation λ_f on L whose domain $\mathrm{Dm}\,\lambda_f$ and values are defined by the following equations:

$$\begin{aligned} \mathrm{Dm}\,\lambda_f &= \{(a_1, \ldots, a_k) \in \mathsf{L}^k : a_1, \ldots, a_k \text{ are terms}\}, \\ \lambda_f(a_1, \ldots, a_k) &= f(a_1, \ldots, a_k). \end{aligned}$$

Therefore, its image $\mathrm{Im}\,\lambda_f$ consists of terms. The formation rule may also be regarded as assigning f the k-ary operation τ_f on the set $\mathsf{T} = \{\mathit{term}, \mathit{formula}\}$ of the category names of L defined by the following equations:

$$\mathrm{Dm}\,\tau_f = \{(\overbrace{\mathit{term}, \ldots, \mathit{term}}^{\text{k-tuple}})\}, \qquad \tau_f(\overbrace{\mathit{term}, \ldots, \mathit{term}}^{\text{k-tuple}}) = \mathit{term}.$$

The definitions of λ_f and τ_f imply that

$$\mathrm{Dm}\,\lambda_f = \{(a_1, \ldots, a_k) \in \mathsf{L}^k : (\tau' a_1, \ldots, \tau' a_k) \in \mathrm{Dm}\,\tau_f\}$$

and that the following holds for each $(a_1, \ldots, a_k) \in \mathrm{Dm}\,\lambda_f$:

$$\tau'(\lambda_f(a_1, \ldots, a_k)) = \tau_f(\tau' a_1, \ldots, \tau' a_k).$$

By definition, these two equations mean that $\tau' \in \mathsf{L} \to \mathsf{T}$ is a holomorphism (or exact homomorphism) with respect to the operations λ_f and τ_f on L and T respectively. Similar remarks apply to the formation rule for the formulas of L (s. §3.4). Thus the formation rules give algebraic structures to L and T so that the extended naming $\tau' \in \mathsf{L} \to \mathsf{T}$ is a holomorphism.

The concept of USAs may be derived from this example among others. An algebra A is said to be sorted if it is equipped with an algebra T and a holomorphism $\sigma \in \mathsf{A} \to \mathsf{T}$. The sorted algebra $(\mathsf{A}, \mathsf{T}, \sigma)$ is said to be universal if A has a subset P which satisfies a certain universality condition similar to that on the bases of a vector space. Theorem 3.1.5 shows that each triple $(\mathsf{T}, \tau, \mathsf{P})$ of an algebra T, a set P and a mapping $\tau \in \mathsf{P} \to \mathsf{T}$ yields a USA $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P})$ such that $\sigma|_{\mathsf{P}} = \tau$ and that it is unique up to homotypic isomorphism extending id_{P}. Theorems 3.1.6 and 3.1.7 show that the elements of A are uniquely formed by induction on their ranks starting from the elements of P.

Thus the triple $(\mathsf{T}, \tau, \mathsf{P})$ underlying the definition of a formal language L yields a USA $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P})$. Moreover, if the definition is adequate in view of [3.1], we may identify L with A by the uniqueness of A or because their elements are formed by induction starting from the elements of P under the same rules. Furthermore, we may identify τ' with σ by virtue of Lemma 3.1.4 and so identify the categories of L with the subsets $\sigma^{-1}\{t\}$ $(t \in \mathsf{T})$ of A.

In another context, each operation in the algebraic structure of L is commonly assigned to a token or the formal product of a token and a variable. In FPL, for example, it is assigned to a function symbol, a predicate symbol, a logical symbol or the product of a quantifier and a variable.

Based on the above observations, I define a **formal language** as a USA (A, T, σ, P) equipped with three sets C, X and Γ which satisfy a certain condition $(\star)$ whose core is the following (s. §3.2.1).

$(*)$ P is the direct union $C \amalg X$, and the index set Λ of the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ of T is a subset of the free monoid $(\Gamma \amalg X)^*$ over the direct union $\Gamma \amalg X$, that is, elements of Λ are formal products of elements of $\Gamma \amalg X$.

Then A also has an algebraic structure $(\alpha_\lambda)_{\lambda\in\Lambda}$ indexed by Λ, and the elements of A are placed in the categories $\sigma^{-1}\{t\}$ $(t \in T)$.

The elements of the sets P, C, X and Γ are called the primes, constants, variables and tokens respectively. The indices in $M = \Lambda \cap \Gamma^*$ and $\Lambda - M$ are said to be invariable and variable respectively, that is, invariable indices are products of tokens and variable ones are products of tokens and at least one variable. Therefore, the latter half of $(*)$ gives us room, for example, to extend the formal language of FPL by furthermore assigning operations to products $f + g$, $f \circ g$ and $\forall xy$ of function symbols f and g, the arithmetic symbol $+$, the composition symbol $\circ$, the quantifier $\forall$ and variables x and y.

Finally, Theorem 3.1.5 shows that each sextuple $(T, \tau, P, C, X, \Gamma)$ of an algebra T, sets P, C, X and Γ satisfying $(\star)$ and a mapping $\tau \in P \to T$ yields a formal language $(A, T, \sigma, P, C, X, \Gamma)$ such that $\sigma|_P = \tau$ up to homotypic isomorphism extending id_P. Thus I call the sextuple the **syntax** of the formal language.

Now, semantics in logic may have been understood as the method of defining the worlds which a given formal language is capable of denoting and the ways the formal language denotes each *denotable world* (DW).

Since the formal language may well be defined as a USA (A, T, σ, P) equipped with certain sets, we should define the concept of DWs in terms of concepts related to sorted algebras. Indeed, people implicitly do so.

In FPL, for example, they define each DW to be the disjoint union $W = E \cup \mathbb{T}$ of a set E and the binary lattice $\mathbb{T} = \{0, 1\}$. Therefore, the elements of W are placed in two categories E and $\mathbb{T}$, whose elements are called the entities and truth values respectively. Furthermore, they equip W with a certain algebraic structure $(\omega_\mu)_{\mu\in M}$ indexed by the set M of the invariable indices, i.e. function symbols, predicate symbols and logical symbols. For example, the operation ω_f assigned to each k-ary function symbol f is defined so that the following hold:

$$\mathrm{Dm}\,\omega_f = E^k, \qquad\qquad \mathrm{Im}\,\omega_f \subseteq E.$$

We may also assign f the k-ary operation τ'_f on the set $T' = \{\mathit{entity}, \mathit{truth\ value}\}$ of the category names of W defined by the following equations:

$$\mathrm{Dm}\,\tau'_f = \{(\overbrace{\mathit{entity}, \ldots, \mathit{entity}}^{\text{k-tuple}})\}, \qquad \tau'_f(\overbrace{\mathit{entity}, \ldots, \mathit{entity}}^{\text{k-tuple}}) = \mathit{entity}.$$

Let π be the mapping of W into T' which associates each element of W with its category name, that is, $\pi w = \textit{entity}$ for each $w \in \mathsf{E}$ and $\pi w = \textit{truth value}$ for each $w \in \mathbb{T}$. Then the above four conditions mean that

$$\mathrm{Dm}\,\omega_f = \{(w_1, \dots, w_k) \in W^k : (\pi w_1, \dots, \pi w_k) \in \mathrm{Dm}\,\tau'_f\}$$

and that the following holds for each $(w_1, \dots, w_k) \in \mathrm{Dm}\,\omega_f$:

$$\pi(\omega_f(w_1, \dots, w_k)) = \tau'_f(\pi w_1, \dots, \pi w_k).$$

As before, these two equations mean that $\pi \in W \to \mathsf{T}'$ is a holomorphism with respect to the operations ω_f and τ'_f on W and T' respectively. Similar remarks apply to the predicate symbols and logical symbols (s. §3.4). Therefore, we may give M-algebraic structures to W and T' so that π is a holomorphism. Furthermore, we may identify the M-algebra T' with the M-reduct T_M of the algebra $\mathsf{T} = \{\textit{term}, \textit{formula}\}$ of the category names of A by the M-isomorphism

$$\textit{entity} \mapsto \textit{term}, \qquad \textit{truth value} \mapsto \textit{formula}$$

of T' onto T_M. Thus we obtain a sorted algebra $(W, \mathsf{T}_\mathsf{M}, \pi)$.

Based on the above observations, I define each DW for the formal language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ as a sorted algebra $(W, \mathsf{T}_\mathsf{M}, \pi)$ for the set M of the invariable indices of the algebraic structure of T (s. §3.2.2). Then W has an algebraic structure $(\omega_\mu)_{\mu \in \mathsf{M}}$ similar to the algebraic structure $(\alpha_\mu)_{\mu \in \mathsf{M}}$ of the M-reduct A_M of A, and elements of W are placed in the categories $\pi^{-1}\{t\}$ $(t \in \mathsf{T})$.

Each way A denotes W should be described by a mapping which associates each element $a \in \mathsf{A}$ with the element of W which a denotes. Therefore, in order to define the ways A denotes W, we only need to assign W a subset Φ_W of $\mathsf{A} \to W$. What conditions should each element $\varphi \in \Phi_W$ satisfy? Since A_M and W are similar algebras categorized by the same algebra T_M, it seems reasonable to assume the following conditions on each way A denotes W:

(1) If an element $a \in \mathsf{A}$ denotes an element $w \in W$, then they belong to the categories of the same name, that is, $\sigma a = \pi w$.

(2) If elements $a_1, \dots, a_k \in \mathsf{A}$ denote elements $w_1, \dots, w_k \in W$ respectively and $(a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\mu$ for an index $\mu \in \mathsf{M}$, then $(w_1, \dots, w_k) \in \mathrm{Dm}\,\omega_\mu$ and $\alpha_\mu(a_1, \dots, a_k)$ denotes $\omega_\mu(w_1, \dots, w_k)$.

The condition (1) means that $\sigma = \pi\varphi$, or $\varphi(\sigma^{-1}\{t\}) \subseteq \pi^{-1}\{t\}$ for each $t \in \mathsf{T}$, that is, φ is homotypic. The condition (2) means that φ is an M-homomorphism.

Since A is generated by $\mathsf{P} = \mathsf{C} \amalg \mathsf{X}$, it seems reasonable to assume in addition that (3) each $\varphi \in \Phi_W$ is determined by $\varphi|_\mathsf{C}$ and $\varphi|_\mathsf{X}$. Therefore, I focus on the mappings $\delta \in \mathsf{C} \to W$ and $\upsilon \in \mathsf{X} \to W$ satisfying $\delta(\mathsf{C} \cap \sigma^{-1}\{t\}) \subseteq \pi^{-1}\{t\}$ and $\upsilon(\mathsf{X} \cap \sigma^{-1}\{t\}) \subseteq \pi^{-1}\{t\}$ for each $t \in \mathsf{T}$, and call them the denotations of C into W and the valuations of X into W respectively (s. §3.2.3). Let Δ_W and Υ_W be the sets of the denotations of C into W and the valuations of X into W respectively. Then the condition (1) implies that $\varphi|_\mathsf{C} \in \Delta_W$ and $\varphi|_\mathsf{X} \in \Upsilon_W$ for each $\varphi \in \Phi_W$.

Thus, in order to define the ways A denotes W under (1)–(3), it is sufficient, if not necessary, to assign W a family $(\varphi_{\delta,\upsilon})_{(\delta,\upsilon)\in\Delta_W\times\Upsilon_W}$ of homotypic M-homomorphisms $\varphi_{\delta,\upsilon} \in A{\to}W$ such that $\varphi_{\delta,\upsilon}|_C = \delta$ and $\varphi_{\delta,\upsilon}|_X = \upsilon$. In fact, we may replace Δ_W with some nonempty subset according to our purposes.

Let $(W^{\Upsilon_W}, T_M, \rho)$ be the Υ_W-power algebra over the DW (W, T_M, π) defined in §3.1.6. Then Remark 3.1.8 shows that each family $(\varphi_{\delta,\upsilon})_{(\delta,\upsilon)\in\Delta_W\times\Upsilon_W}$ as above yields a family $(\delta^\sharp)_{\delta\in\Delta_W}$ of homotypic M-homomorphisms $\delta^\sharp \in A{\to}W^{\Upsilon_W}$ defined by the following for each $a \in A$ and each $\upsilon \in \Upsilon_W$:

$$(\delta^\sharp a)\upsilon = \varphi_{\delta,\upsilon} a. \tag{3.0.1}$$

Furthermore, their restrictions $\varphi_\delta = \delta^\sharp|_P$ to $P = C \amalg X$ satisfy

$$(\varphi_\delta a)\upsilon = \begin{cases} \delta a & \text{if } a \in C, \\ \upsilon a & \text{if } a \in X \end{cases} \tag{3.0.2}$$

for each $a \in P$ and each $\upsilon \in \Upsilon_W$, because $\varphi_{\delta,\upsilon}|_C = \delta$ and $\varphi_{\delta,\upsilon}|_X = \upsilon$.

Conversely, if $(\delta^\sharp)_{\delta\in\Delta_W}$ is a family of homotypic M-homomorphisms $\delta^\sharp \in A{\to}W^{\Upsilon_W}$ whose restrictions $\varphi_\delta = \delta^\sharp|_P$ satisfy (3.0.2) for each $a \in P$ and each $\upsilon \in \Upsilon_W$, then for each $(\delta,\upsilon) \in \Delta_W\times\Upsilon_W$, the mapping $\varphi_{\delta,\upsilon} \in A{\to}W$ defined by the reverse of (3.0.1) for each $a \in A$ is a homotypic M-homomorphism because $\varphi_{\delta,\upsilon}$ is the composite $\upsilon_{\mathrm{p}}\delta^\sharp$ of $\delta^\sharp$ and the υ-projection $\upsilon_{\mathrm{p}} \in W^{\Upsilon_W}{\to}W$ (s. §1.5.2), and satisfies $\varphi_{\delta,\upsilon}|_C = \delta$ and $\varphi_{\delta,\upsilon}|_X = \upsilon$ because of (3.0.2). Thus, in order to define the ways A denotes W, it suffices to assign W such a family $(\delta^\sharp)_{\delta\in\Delta_W}$.

Assume that W^{Υ_W} is the M-reduct of a Λ-algebra $W^\sharp$ and $(W^\sharp, T, \rho)$ is a sorted algebra for the type T of A and the sorting ρ of W^{Υ_W}. For each $\delta \in \Delta_W$, define a mapping $\varphi_\delta \in P{\to}W^{\Upsilon_W}$ by (3.0.2) for each $a \in P$ and each $\upsilon \in \Upsilon_W$. Then $\varphi_\delta \in P{\to}W^\sharp$, and the universality of (A, T, σ, P) enables us to extend φ_δ to a homotypic Λ-homomorphism $\delta^\sharp \in A{\to}W^\sharp$, which consequently is a homotypic M-homomorphism of A into W^{Υ_W} (s. §3.3.2). Thus, in order to define the ways A denotes W, it is sufficient, if not necessary, to extend $(W^{\Upsilon_W}, T_M, \rho)$ to a sorted algebra $(W^\sharp, T, \rho)$ so that W^{Υ_W} is the M-reduct of $W^\sharp$, i.e. to assign an operation β_λ on W^{Υ_W} to each variable index $\lambda \in \Lambda - M$ so that ρ is a $(\Lambda - M)$-holomorphism as well as an M-holomorphism. Such a family $(\beta_\lambda)_{\lambda\in\Lambda-M}$ is not unique in general, and what to pick depends on our purposes.

In order to deal with the purposes, I introduce the concept of interpretations of $\Lambda - M$ on W (s. §3.2.4). Each of them is a certain family $(\lambda_W)_{\lambda\in\Lambda-M}$ of mappings $\lambda_W \in V_\lambda{\to}W$ for a certain set V_λ. Moreover, I show in §3.3.1 how to extend $(W^{\Upsilon_W}, T_M, \rho)$ to $(W^\sharp, T, \rho)$ by means of each interpretation of $\Lambda - M$ on W. The concept of interpretations was abstracted from treatment of variable operations in certain specific branches of logic such as the quantifying ones $\forall x$ and $\exists x$ in FPL and the nominalizers $\triangledown x$ in CL mentioned in §1.2.7. There is a room for extending it, but I hope that it will widely serve our purposes.

Thus finally in §3.2.5, I define the **semantics** of the formal language $(A, T, \sigma, P, C, X, \Gamma)$ as a triple $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ of a nonempty collection $\mathfrak{W}$ of DWs for A, a family $(\mathrm{I}_W)_{W\in\mathfrak{W}}$ of interpretations I_W of $\Lambda - M$ on $W \in \mathfrak{W}$ and a family $(\Delta_W)_{W\in\mathfrak{W}}$ of nonempty sets Δ_W of denotations of C into $W \in \mathfrak{W}$.

3.1 Sorted algebras

Here I collect notation, terminology and basic concepts on sorted algebras.

3.1.1 Operations

For each set A, a (partial) **operation** on A is a mapping α of a subset D of A^k into A for a positive integer k. If $D = A^k$, α is said to be **total**, and so the word *partial* above in fact means *not necessarily total*. The set D is called the **domain** of α and denoted $\mathrm{Dm}\,\alpha$. The set $\alpha D = \{\alpha(x_1, \ldots, x_k) : (x_1, \ldots, x_k) \in D\}$ is called the **image** of α and denoted $\mathrm{Im}\,\alpha$, and each its element $\alpha(x_1, \ldots, x_k)$ is called the **image** of $(x_1, \ldots, x_k)$ by α or the **value** of α at $(x_1, \ldots, x_k)$. The positive integer k is called an **arity** of α. If $D \neq \emptyset$, α has a unique arity. If $D = \emptyset$, every positive integer is an arity of α, and $\alpha = \emptyset$ because $\emptyset \to A = \{\emptyset\}$.[3.2] If α is **unary**, that is, if $k = 1$, its value αx is also denoted $x\alpha$, x^α, x_α, and so on, that is, unary operation symbols may be postpositive, superscript, subscript, and so on, as well as prepositive. If α is **binary**, that is, if $k = 2$, its value $\alpha(x, y)$ is often denoted $x\alpha y$, that is, binary operation symbols are often interpositions. If a subset B of A satisfies $\alpha(B^k \cap D) \subseteq B$, we say that B is **α-closed** or that α **close**s B, and we may regard the **restriction** $\beta = \alpha|_{B^k \cap D}$ as an operation on B with $\mathrm{Dm}\,\beta = B^k \cap D$. The restriction β is sometimes also denoted α.

3.1.2 Algebras and quasialgebras

An **algebra** (or **algebraic system**) is a set A equipped with an (**algebraic**) **structure** $(\alpha_\lambda)_{\lambda\in\Lambda}$ that is a family of operations α_λ on A indexed by a set Λ. Thus an algebra is best described by the pair $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ of a set and an algebraic structure on it. We sometimes call it a **Λ-algebra** and abbreviate α_λ to λ for some $\lambda \in \Lambda$. The algebra is said to be **total** if α_λ is total for each $\lambda \in \Lambda$. The elements of the set $\bigcup_{\lambda\in\Lambda} \mathrm{Im}\,\alpha_\lambda$ are called the **composite**s, while those of the set $A - \bigcup_{\lambda\in\Lambda} \mathrm{Im}\,\alpha_\lambda$ are called the **prime**s, that is, an element $a \in A$ is a prime iff it has no expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$. Two algebras A and B are said to be **similar**, if their structures $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ are indexed by the same set Λ and α_λ and β_λ have a common arity for each $\lambda \in \Lambda$. The similarity is intransitive.[3.3]

A **quasialgebra** is a set A equipped with a (**quasialgebraic**) **structure** R that is an association on A, i.e. a relation between the free monoid A^* over A (s. Remark 3.1.14) and A. Thus a quasialgebra is best described by the pair

[3.2] The set $A \to B$ of the mappings of a set A into a set B is defined as the set of the elements f of $\mathfrak{P}(A \times B)$ such that each element $a \in A$ has exactly one element $b \in B$ such that $(a, b) \in f$ (the element b is denoted fa). If $A = \emptyset$, then $A \times B = \emptyset$ and the unique element $\emptyset$ of $\mathfrak{P}\emptyset$ obviously satisfies the condition. Thus $\emptyset \to B = \{\emptyset\}$ (s. [3.23]). In contrast, $A \to \emptyset = \emptyset$ if $A \neq \emptyset$.

[3.3] The similarity is reflexive and symmetric, that is, each algebra is similar to itself, and if an algebra A is similar to an algebra B, then B is similar to A. However, it is intransitive. If A is similar to B and B is similar to an algebra C, then their structures $(\alpha_\lambda)_{\lambda\in\Lambda}$, $(\beta_\lambda)_{\lambda\in\Lambda}$ and $(\gamma_\lambda)_{\lambda\in\Lambda}$ are indexed by the same set Λ, but A is not necessarily similar to C, because it is possible that α_λ and γ_λ have no common arity iff $\mathrm{Dm}\,\alpha_\lambda \neq \emptyset \neq \mathrm{Dm}\,\gamma_\lambda$ and $\mathrm{Dm}\,\beta_\lambda = \emptyset$.

(A, R) of a set and a quasialgebraic structure on it. Since the quasialgebraic structure R is an association on A, some concepts in Chapter 2 are relevant to the quasialgebra (A, R). For example, (A, R) and R are said to be **singular**, if the R-core $A_R = \{x \in A : \varepsilon \, R \, x\}$ of A is empty (s. Remark 2.5.4). Here and below in (3.1.1) and Theorem 3.1.1, we use part of the alphabet convention introduced in §2.2, that is, ε denotes the identity element of A^*, and x and y with or without numerical subscripts denote elements of A.

Let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ be an algebra. Then we define an association R_Λ on A by

$$x_1 \cdots x_k \, R_\Lambda \, y \iff \alpha_\lambda(x_1, \ldots, x_k) = y \text{ for some } \lambda \in \Lambda \tag{3.1.1}$$

as was mentioned in §1.2.4 [3.4]. Then the quasialgebra (A, R_Λ) is singular. Its converse is also true, that is, the following theorem holds, and has been used particularly in §1.2.6 and Remark 2.5.4 (s. [1.14] and Remark 2.4.1).

Theorem 3.1.1 Let (A, R) be a singular quasialgebra. Then there exists an algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ such that $R = R_\Lambda$.

Proof There exists a family $(R_i)_{i \in I}$ of associations on A such that $R = \bigcup_{i \in I} R_i$ and R_i is univalent for each $i \in I$, that is, for each $x_1 \cdots x_k \in A^*$ with $k \geq 1$ there exists at most one element $y \in A$ such that $x_1 \cdots x_k \, R_i \, y$. For example, let $I = A$ and define R_i for each $i \in I$ by the following for each $(x_1 \cdots x_k, y) \in A^* \times A$:

$$x_1 \cdots x_k \, R_i \, y \iff x_1 \cdots x_k \, R \, y \text{ and } y = i.$$

Define $\Lambda = I \times \mathbb{N}$. For each $(i, k) \in \Lambda$, let $A_{(i,k)}$ be the set of the elements $(x_1, \ldots, x_k) \in A^k$ for which there exists exactly one element $y \in A$ such that $x_1 \cdots x_k \, R_i \, y$, and define an operation $\alpha_{(i,k)}$ on A by $\mathrm{Dm}\, \alpha_{(i,k)} = A_{(i,k)}$ and

$$\alpha_{(i,k)}(x_1, \ldots, x_k) = y \iff x_1 \cdots x_k \, R_i \, y.$$

Assume $x_1 \cdots x_k \, R \, y$. Then $k \geq 1$ because (A, R) is singular, and there exists an element $i \in I$ such that $x_1 \cdots x_k \, R_i \, y$ because $R = \bigcup_{i \in I} R_i$. Furthermore $(i, k) \in \Lambda$, and $(x_1, \ldots, x_k) \in A_{(i,k)}$ by the univalence of R_i. Therefore $\alpha_{(i,k)}(x_1, \ldots, x_k) = y$. Conversely if $\alpha_{(i,k)}(x_1, \ldots, x_k) = y$ for some $(i, k) \in \Lambda$, then $x_1 \cdots x_k \, R_i \, y$ and so $x_1 \cdots x_k \, R \, y$ because $R_i \subseteq R$. Thus $R = R_\Lambda$.

3.1.3 Subalgebras, reducts, subreducts and closures

Let A be an algebra and $(\alpha_\lambda)_{\lambda \in \Lambda}$ be its structure. Then its **subalgebra** is a subset B of A which is α_λ-closed for all $\lambda \in \Lambda$ and conventionally regarded as an algebra equipped with the structure $(\beta_\lambda)_{\lambda \in \Lambda}$ consisting of the restrictions β_λ of α_λ to B for all $\lambda \in \Lambda$. Also, its M**-reduct** for a subset M of Λ is the algebra $(A, (\alpha_\mu)_{\mu \in M})$, which is often abbreviated to A_M. Its M**-subreduct** (or M-subalgebra) is a subalgebra of A_M and so regarded as an M-algebra as

[3.4] There R_Λ was denoted $R_{\mathcal{O}}$ for $\mathcal{O} = (\alpha_\lambda)_{\lambda \in \Lambda}$.

above. Its **reduct** is an M-reduct for a subset M of Λ, and its **subreduct** is an M-subreduct for a subset M of Λ, i.e. a subalgebra of a reduct.[3.5]

Let R_Λ be the association on A derived from the structure $(\alpha_\lambda)_{\lambda\in\Lambda}$ of A by (3.1.1). Then a subset B of A is a subalgebra of A iff it is an R_Λ-subset of A as defined in §2.4. Therefore, for each subset S of A, the R_Λ-closure $[S]_{R_\Lambda}$ of S in A is the smallest of the subalgebras of A which contain S. We call it the **closure** of S in A and denote it by $[S]_\Lambda$, $[S]_A$, $[S]$, and so on[3.6]. Then we may derive the following descriptions of $[S]_\Lambda$ mentioned in §1.2.2 from Theorems 2.4.2 and 2.4.3, and in view of them, we also call $[S]_\Lambda$ the subalgebra **generated** by S.

Theorem 3.1.2 Let $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra and S be a subset of A. Then $[S]_\Lambda$ is the union $\bigcup_{n\geq 0} S_n$ of the **descendant**s S_n $(n = 0, 1, \dots)$ of S in A which are inductively defined by $S_0 = S$ and the following for each $n \geq 1$:

$$S_n = \left\{ \alpha_\lambda(a_1,\dots,a_k) : \begin{array}{l} \lambda\in\Lambda,\ (a_1,\dots,a_k)\in \operatorname{Dm}\alpha_\lambda \text{ and} \\ a_j \in S_{n_j}\ (j=1,\dots,k) \textit{ for some nonnegative} \\ \textit{integers } n_1,\dots,n_k \textit{ such that } n-1=\sum_{j=1}^k n_j. \end{array} \right\}.$$

Remark 3.1.1 The definition of S_n $(n \geq 1)$ implies $\bigcup_{n\geq 1} S_n \subseteq \bigcup_{\lambda\in\Lambda} \operatorname{Im}\alpha_\lambda$, and so Theorem 3.1.2 implies $[S]_\Lambda \subseteq S \cup \bigcup_{\lambda\in\Lambda} \operatorname{Im}\alpha_\lambda$, although we had better derive it from the fact that $S \cup \bigcup_{\lambda\in\Lambda} \operatorname{Im}\alpha_\lambda$ is a subalgebra of A and regard Theorem 3.1.2 as one of its refinements. The italicized condition on $(a_1,\dots,a_k)$ in the definition is the ramification condition $\mathcal{R}(S_0,\dots,S_{n-1})$ in (2.3.1).

Example 3.1.1 Suppose the algebra A in Theorem 3.1.2 is a semigroup as explained in §3.6.1. Then its algebraic structure consists of an associative multiplication, hence it follows that the descendant S_n of a subset S of A is equal to the set S^{n+1} of all products of $n+1$ elements of S and so $[S]_A = \bigcup_{n\geq 1} S^n$.

Theorem 3.1.3 Let $(A,(\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra and S be a subset of A. Then an element $a \in A$ belongs to $[S]_\Lambda$ iff there exists a sequence $a_1,\dots,a_n$ $(n \geq 1)$ of elements of A which satisfies $a_n = a$ and one of the following conditions for each $i \in \{1,\dots,n\}$ (we call it an S/Λ**-sequent** or S/Λ**-sqnt** for a) [3.7].

(1) $a_i \in S$.

(2) There exist numbers $j_1,\dots,j_k \in \{1,\dots,i-1\}$ and an index $\lambda\in\Lambda$ such that $(a_{j_1},\dots,a_{j_k}) \in \operatorname{Dm}\alpha_\lambda$ and $\alpha_\lambda(a_{j_1},\dots,a_{j_k}) = a_i$.

[3.5]Each algebra is both its subalgebra and its reduct, and its subalgebras and reducts are its subreducts. Moreover, subalgebras of its subalgebras are its subalgebras, reducts of its reducts are its reducts, and reducts of its subalgebras are its subreducts. Therefore, subreducts of its subreducts are its subreducts. Thus the concept of subreducts is the most comprehensive.

Its subreduct is not necessarily a reduct of its subalgebra. For example, regard $\mathbb{Z}$ as an algebra with respect to addition $+$ and subtraction $-$. Then $\mathbb{N}$ is a subalgebra of the reduct $\mathbb{Z}_{\{+\}}$. However, it is not closed by $-$ and therefore is not a reduct of a subalgebra of $\mathbb{Z}$.

[3.6]It was denoted $[S]_{\mathcal{O}}$ for $\mathcal{O} = (\alpha_\lambda)_{\lambda\in\Lambda}$ in §1.2.2.

[3.7]It was called an $S/\mathcal{O}$-sequent for $\mathcal{O} = (\alpha_\lambda)_{\lambda\in\Lambda}$ in Remark 1.2.1 and elsewhere in §1.2.

We may similarly derive another description of $[S]_\Lambda$ from Theorem 2.4.4 on R-deductions, but we should redefine **Λ-deduction**s on the algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ in the following way. Let $M = (A \amalg \Lambda \amalg \{[,]\})^*$ be the free monoid over the direct union $A \amalg \Lambda \amalg \{[,]\}$ of A, the index set Λ of the structure $(\alpha_\lambda)_{\lambda\in\Lambda}$ and the set $\{[,]\}$ of the left bracket [and the right bracket] (s. Remark 3.1.14). Then we inductively define subsets D_n $(n = 0, 1, \ldots)$ of M and an element $c(d) \in A$ for each $d \in D_n$ by $D_0 = A$ and $c(a) = a$ for each $a \in D_0$ and by the following for each $n \geq 1$, where the italicized condition on $(d_1, \ldots, d_k)$ is the ramification condition $\mathcal{R}(D_0, \ldots, D_{n-1})$ in (2.3.1):

$$D_n = \left\{ [\lambda d_1 \cdots d_k] : \begin{array}{l} d_j \in D_{n_j} \ (j = 1, \ldots, k) \textit{ for some nonnegative} \\ \textit{integers } n_1, \ldots, n_k \textit{ such that } n - 1 = \sum_{j=1}^k n_j, \\ \lambda \in \Lambda \text{ and } (c(d_1), \ldots, c(d_k)) \in \mathrm{Dm}\, \alpha_\lambda. \end{array} \right\},$$

$$c([\lambda d_1 \cdots d_k]) = \alpha_\lambda(c(d_1), \ldots, c(d_k)) \text{ for each } [\lambda d_1 \cdots d_k] \in D_n \text{ as above.}$$

Here $c([\lambda d_1 \cdots d_k])$ as well as D_n is well-defined, because it inductively follows from the definition of D_i $(i = 0, \ldots, n)$ that D_i consists of trees of rank i whose germs and nodes are elements of A and Λ respectively, and so Theorem 2.3.1 shows that $D_n \cap (D_0 \cup \cdots \cup D_{n-1}) = \emptyset$ and that each element $d \in D_n$ has a unique expression $d = [\lambda d_1 \cdots d_k]$ by an index $\lambda \in \Lambda$ and elements $d_j \in D_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^k n_j$ (and so the expression necessarily satisfies $(c(d_1), \ldots, c(d_k)) \in \mathrm{Dm}\, \alpha_\lambda$). We have thus defined a set $D = \bigcup_{n\geq 0} D_n$ and a mapping $c \in D \to A$ so that $c|_A = \mathrm{id}_A$.

We refer to the elements of D, D_0 and $\bigcup_{n\geq 1} D_n$ as **Λ-deduction**s, **prime Λ-deduction**s and **composite Λ-deduction**s on A respectively. We also refer to the element $c(d) \in A$ for each Λ-deduction d as its **conclusion**, while we refer to the mapping c as **concluding**. Furthermore, we define the **premise** $P(d)$ of d as the set of the elements of A which occur in d (s. Example 3.1.8). Then $P(a) = \{a\}$ for $a \in D_0$ and $P(d) = \bigcup_{j=1}^k P(d_j)$ for $d = [\lambda d_1 \cdots d_k] \in \bigcup_{n\geq 1} D_n$ as above. Moreover, $P(d)$ is equal to the germ $G(d)$ defined in §2.3.

Under the above definitions, we have the following description of $[S]_\Lambda$.

Theorem 3.1.4 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra and S be a subset of A. Then an element $a \in A$ belongs to $[S]_\Lambda$ iff there exists a Λ-deduction d on A such that $P(d) \subseteq S$ and $c(d) = a$ (we call d an **S/Λ-proof** or **S/Λ-pf** of a). More precisely in terms of Theorem 3.1.2, $a \in S_n$ iff a has an S/Λ-pf in D_n $(n = 0, 1, \ldots)$.

Proof We first show by induction on n that if $a \in S_n$ then a has an S/Λ-pf in D_n. If $n = 0$, then $a \in S$, and so a is an S/Λ-pf of a in D_0. Therefore assume $n \geq 1$. Then $a = \alpha_\lambda(a_1, \ldots, a_k)$ with $\lambda \in \Lambda$, $a_j \in S_{n_j}$ $(j = 1, \ldots, k)$ and $n - 1 = \sum_{j=1}^k n_j$, and so a_j has an S/Λ-pf $d_j \in D_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $[\lambda d_1 \cdots d_k]$ is an S/Λ-pf of a in D_n.

We next show by induction on n that if a has an S/Λ-pf $d \in D_n$ then $a \in S_n$. If $n = 0$, then $a = c(d) = d \in P(d) \subseteq S = S_0$. Therefore assume $n \geq 1$. Then $d = [\lambda d_1 \cdots d_k]$ with $\lambda \in \Lambda$, $d_j \in D_{n_j}$ $(j = 1, \ldots, k)$, $n - 1 = \sum_{j=1}^k n_j$

and $a = \alpha_\lambda(c(d_1), \dots, c(d_k))$. Furthermore, d_j is an S/Λ-pf of $c(d_j)$, and so $c(d_j) \in S_{n_j}$ $(j = 1, \dots, k)$ by the induction hypothesis. Thus $a \in S_n$.

Example 3.1.2 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra and $D = \bigcup_{n\geq 0} D_n$ be the set of the Λ-deductions on A. Then we may regard D as an algebra by equipping it with the structure $(\delta_\lambda)_{\lambda\in\Lambda}$ defined by the following for each $\lambda \in \Lambda$:

$$\mathrm{Dm}\,\delta_\lambda = \{(d_1, \dots, d_k) \in \textstyle\bigcup_{n=1}^{\infty} D^n : (c(d_1), \dots, c(d_k)) \in \mathrm{Dm}\,\alpha_\lambda\},$$
$$\delta_\lambda(d_1, \dots, d_k) = [\lambda d_1 \cdots d_k] \text{ for each } (d_1, \dots, d_k) \in \mathrm{Dm}\,\delta_\lambda.$$

Notice that $\mathrm{Dm}\,\delta_\lambda \subseteq D^k$ for an arity k of α_λ.[3.8] If $(d_1, \dots, d_k) \in \mathrm{Dm}\,\delta_\lambda$, then $d_j \in D_{n_j}$ for some n_j $(j = 1, \dots, k)$ and $[\lambda d_1 \cdots d_k] \in D_n$ for $n = 1 + \sum_{j=1}^{k} n_j$. Therefore, the operation δ_λ on D is well-defined. We call $(D, (\delta_\lambda)_{\lambda\in\Lambda})$ the **deduction algebra** on/of A.

We can show by induction on n that D_n $(n = 0, 1, \dots)$ is equal to the n-th descendant A_n of A in D, and so $D = \bigcup_{n\geq 0} A_n = [A]$ by Theorem 3.1.2. First, $D_0 = A = A_0$ by definition. Therefore assume $n \geq 1$. In order to show $D_n = A_n$, first assume $d \in D_n$. Then $d = [\lambda d_1 \cdots d_k]$ with $\lambda \in \Lambda$, $d_j \in D_{n_j}$ $(j = 1, \dots, k)$, $n - 1 = \sum_{j=1}^{k} n_j$ and $(c(d_1), \dots, c(d_k)) \in \mathrm{Dm}\,\alpha_\lambda$. Therefore $d = \delta_\lambda(d_1, \dots, d_k)$, and $d_j \in A_{n_j}$ $(j = 1, \dots, k)$ by the induction hypothesis. Thus $d \in A_n$. Conversely assume $d \in A_n$. Then $d = \delta_\lambda(d_1, \dots, d_k)$ with $\lambda \in \Lambda$, $d_j \in A_{n_j}$ $(j = 1, \dots, k)$ and $n - 1 = \sum_{j=1}^{k} n_j$. Therefore $d = [\lambda d_1 \cdots d_k]$ with $(c(d_1), \dots, c(d_k)) \in \mathrm{Dm}\,\alpha_\lambda$, and $d_j \in D_{n_j}$ $(j = 1, \dots, k)$ by the induction hypothesis. Thus $d \in D_n$.

Remark 3.1.2 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ and $(B, (\beta_\mu)_{\mu\in M})$ be algebras. Then their **algebraic union** is the algebra $(C, (\gamma_\nu)_{\nu\in N})$ defined by $C = A \cup B$ and the union $(\gamma_\nu)_{\nu\in N}$ of $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\mu)_{\mu\in M}$ regarded as algebraic structures on C, that is, $N = \Lambda \amalg M$, $\gamma_\nu = \alpha_\nu$ for $\nu \in \Lambda$ and $\gamma_\nu = \beta_\nu$ for $\nu \in M$. Then the two parent algebras are subreducts of the algebraic union.

3.1.4 Holomorphisms and homomorphisms

Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ and $(B, (\beta_\lambda)_{\lambda\in\Lambda})$ be algebras with the structures indexed by the same set Λ. Then a mapping $f \in A \to B$ is called a **homomorphism**, if it satisfies the following condition for each $\lambda \in \Lambda$.

- (**Homomorphy**) If $(a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$ for a positive integer k, then $(fa_1, \dots, fa_k) \in \mathrm{Dm}\,\beta_\lambda$ and $f(\alpha_\lambda(a_1, \dots, a_k)) = \beta_\lambda(fa_1, \dots, fa_k)$.

The existence of a homomorphism implies the similarity.[3.9] The mapping f is said to be **exact**, if it satisfies the following condition for each $\lambda \in \Lambda$.

[3.8] If $\mathrm{Dm}\,\alpha_\lambda \neq \emptyset$ then $\mathrm{Dm}\,\delta_\lambda \subseteq D^k$ for a unique arity k of α_λ, while if $\mathrm{Dm}\,\alpha_\lambda = \emptyset$ then $\mathrm{Dm}\,\delta_\lambda = \emptyset \subseteq D^k$ for any arity k of α_λ.

[3.9] Homomorphy implies that if $\mathrm{Dm}\,\alpha_\lambda \neq \emptyset$ then $\mathrm{Dm}\,\beta_\lambda \neq \emptyset$ and α_λ and β_λ have the same unique arity. Therefore, if $f \in A \to B$ is a homomorphism, then A and B are similar (s. [3.3]).

- (**Exactness**) If $(a_1, \dots, a_k) \in A^k$ and $(fa_1, \dots, fa_k) \in \mathrm{Dm}\,\beta_\lambda$ for a positive integer k, then $(a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$.

A **holomorphism** is an exact homomorphism.

Remark 3.1.3 It follows from the above definition that the mapping $f \in A \to B$ is a holomorphism iff the following hold for each $\lambda \in \Lambda$:

$$\mathrm{Dm}\,\alpha_\lambda = \{(a_1, \dots, a_k) \in \textstyle\bigcup_{n=1}^{\infty} A^n : (fa_1, \dots, fa_k) \in \mathrm{Dm}\,\beta_\lambda\},$$
$$f(\alpha_\lambda(a_1, \dots, a_k)) = \beta_\lambda(fa_1, \dots, fa_k) \text{ for each } (a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\lambda.$$

These equations may be rewritten in the following way:

$$\mathrm{Dm}\,\alpha_\lambda = \textstyle\bigcup_{(b_1,\dots,b_k)\in\mathrm{Dm}\,\beta_\lambda} \prod_{j=1}^{k} f^{-1}\{b_j\},$$
$$\alpha_\lambda(\textstyle\prod_{j=1}^{k} f^{-1}\{b_j\}) \subseteq f^{-1}\{\beta_\lambda(b_1, \dots, b_k)\} \text{ for each } (b_1, \dots, b_k) \in \mathrm{Dm}\,\beta_\lambda.$$

Thus, when identified with a partition $(A_b)_{b\in B}$ of A,[3.10] the mapping f is a holomorphism iff the following hold for each $\lambda \in \Lambda$:

$$\mathrm{Dm}\,\alpha_\lambda = \textstyle\bigcup_{(b_1,\dots,b_k)\in\mathrm{Dm}\,\beta_\lambda} \prod_{j=1}^{k} A_{b_j},$$
$$\alpha_\lambda(\textstyle\prod_{j=1}^{k} A_{b_j}) \subseteq A_{\beta_\lambda(b_1,\dots,b_k)} \text{ for each } (b_1, \dots, b_k) \in \mathrm{Dm}\,\beta_\lambda.$$

The three equations for $\mathrm{Dm}\,\alpha_\lambda$ imply that $\mathrm{Dm}\,\alpha_\lambda \subseteq A^k$ for an arity k of β_λ (s. [3.8]). The latter two in effect mean that $\big(\prod_{j=1}^{k} f^{-1}\{b_j\}\big)_{(b_1,\dots,b_k)\in\mathrm{Dm}\,\beta_\lambda}$ and $\big(\prod_{j=1}^{k} A_{b_j}\big)_{(b_1,\dots,b_k)\in\mathrm{Dm}\,\beta_\lambda}$ are partitions of $\mathrm{Dm}\,\alpha_\lambda$.

Bijective holomorphisms are also called **isomorphism**s, while bijective homomorphisms have no other names.

Let A be an M-algebra, B be an N-algebra and Λ be a subset of $M \cap N$. Then homomorphisms of A_Λ into B_Λ are also called **Λ-homomorphism**s of A into B (particularly, homomorphisms of Λ-algebras are also called Λ-homomorphisms), and likewise for holomorphisms and isomorphisms.

Remark 3.1.4 Holomorphisms are homomorphisms, and homomorphisms of total algebras are holomorphisms. Composites of homomorphisms are homomorphisms, and likewise for holomorphisms. If $f \in C \to D$ is a homomorphism and A and B are Λ-subreducts of C and D respectively such that $fA \subseteq B$, then the restriction $f|_A$ regarded as belonging to $A \to B$ is a Λ-homomorphism, and likewise for holomorphisms. If $f \in A \to E$ is a Λ-homomorphism and E is a subreduct of an algebra B, then f regarded as belonging to $A \to B$ is a Λ-homomorphism, and likewise for holomorphisms. Identity transformations id_A

[3.10] Let A and B be sets. Then if $f \in A \to B$, the family $(f^{-1}\{b\})_{b\in B}$ partitions A. Conversely if $(A_b)_{b\in B}$ is a partition of A, then there is a unique mapping $f \in A \to B$ such that $f^{-1}\{b\} = A_b$ for each $b \in B$. Mappings are thus identified with partitions.

on algebras A are holomorphisms. More generally, inclusion mappings of Λ-subreducts into parent algebras are Λ-holomorphisms.[3.11]

Example 3.1.3 Let A be an algebra and D be its deduction algebra defined in Example 3.1.2. Then the concluding $c \in D \to A$ is a holomorphism. To see this, let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\delta_\lambda)_{\lambda\in\Lambda}$ be the structures of A and D. Then the definitions of δ_λ and c show that if $(d_1, \dots, d_k) \in \mathrm{Dm}\,\delta_\lambda$ then $(c(d_1), \dots, c(d_k)) \in \mathrm{Dm}\,\alpha_\lambda$ and $c(\delta_\lambda(d_1, \dots, d_k)) = c([\lambda d_1 \cdots d_k]) = \alpha_\lambda(c(d_1), \dots, c(d_k))$. Furthermore, the definition of $\mathrm{Dm}\,\delta_\lambda$ shows that c is exact.

Lemma 3.1.1 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$, $(A', (\alpha'_\lambda)_{\lambda\in\Lambda})$ and B be algebras, $f \in A \to B$ be a holomorphism and $f' \in A' \to B$ be a homomorphism. Assume $(a_1, \dots, a_k) \in A^k$, $\lambda \in \Lambda$, $(a'_1, \dots, a'_k) \in \mathrm{Dm}\,\alpha'_\lambda$, and $(fa_1, \dots, fa_k) = (f'a'_1, \dots, f'a'_k)$. Then $(a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$ and $f(\alpha_\lambda(a_1, \dots, a_k)) = f'(\alpha'_\lambda(a'_1, \dots, a'_k))$.

Proof Let β_λ be the operation of B assigned to λ. Then $(fa_1, \dots, fa_k) = (f'a'_1, \dots, f'a'_k) \in \mathrm{Dm}\,\beta_\lambda$ because f' is a homomorphism and $(a'_1, \dots, a'_k) \in \mathrm{Dm}\,\alpha'_\lambda$. Therefore, $(a_1, \dots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$ by the exactness of f. Since f is also a homomorphism, we furthermore have $f(\alpha_\lambda(a_1, \dots, a_k)) = \beta_\lambda(fa_1, \dots, fa_k) = \beta_\lambda(f'a'_1, \dots, f'a'_k) = f'(\alpha'_\lambda(a'_1, \dots, a'_k))$.

3.1.5 Sorted algebras and homotypisms

A **sorted algebra** is an algebra A equipped with an algebra T and a holomorphism $\sigma \in A \to T$. Thus a sorted algebra is best described by the triple (A, T, σ) of algebras A and T and a holomorphism $\sigma \in A \to T$. We call T and σ the **type** and **sorting** of A. For each element $a \in A$, we call σa the **type** of a. For each subset S of A and each element $t \in T$, we call $\sigma|_S{}^{-1}\{t\} = S \cap \sigma^{-1}\{t\}$ the **t-part** of S and often abbreviate it to S_t. In particular, A_t for each $t \in T$ consists of the elements of A of type t. Each subset S of A is partitioned by $(S_t)_{t\in T}$.

Remark 3.1.5 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ and $(T, (\tau_\lambda)_{\lambda\in\Lambda})$ be algebras. Let $(A_t)_{t\in T}$ be a partition of A and identify it with a mapping σ of A into T (s. [3.10]). Then it follows from Remark 3.1.3 that (A, T, σ) is a sorted algebra iff the following hold for each $\lambda \in \Lambda$, where the former equation in effect means that $\left(\prod_{j=1}^k A_{t_j}\right)_{(t_1,\dots,t_k)\in\mathrm{Dm}\,\tau_\lambda}$ is a partition of $\mathrm{Dm}\,\alpha_\lambda$:

$$\mathrm{Dm}\,\alpha_\lambda = \bigcup_{(t_1,\dots,t_k)\in\mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^k A_{t_j},$$
$$\alpha_\lambda\Big(\prod_{j=1}^k A_{t_j}\Big) \subseteq A_{\tau_\lambda(t_1,\dots,t_k)} \text{ for each } (t_1, \dots, t_k) \in \mathrm{Dm}\,\tau_\lambda.$$

[3.11] Conversely, if A is a Λ-algebra and B is an M-algebra such that $\Lambda \subseteq M$, $A \subseteq B$ and the inclusion mapping of A into B is a Λ-holomorphism, then A is a Λ-subreduct of B.

This fact is related to extension of the concept of numbers. The inclusion mapping of $\mathbb{N}$ into $\mathbb{Z}$ is an inexact $\{-\}$-homomorphism. That of $\mathbb{Z}$ into $\mathbb{Q}$ is an inexact $\{\div\}$-homomorphism. That of $\mathbb{Q}$ into $\mathbb{R}$ is an inexact homomorphism with respect to the infinitary operation limit. Indeed, $\mathbb{N}$ is not a $\{-\}$-subreduct of $\mathbb{Z}$, and likewise for the other inclusion mappings.

Remark 3.1.6 Let (A, T, σ) be a sorted Λ-algebra, and M be a subset of Λ. Then if B and U are M-subreducts of A and T respectively such that $\sigma B \subseteq U$, then $(B, U, \sigma|_B)$ is a sorted M-algebra by Remark 3.1.4. In particular, (A_M, T_M, σ) is a sorted M-algebra.

Example 3.1.4 Example 3.1.3 shows that the triple (D, A, c) of an algebra A, its deduction algebra D and the concluding $c \in D \to A$ is a sorted algebra.

Let (A, T, σ) and (B, T, τ) be sorted algebras of the same type T. Then a mapping $f \in A \to B$ is called a **homotypism** or said to be **homotypic**, if $f(A_t) \subseteq B_t$ for all $t \in T$. If (C, T, υ) is also a sorted algebra of the same type T and both f and $g \in B \to C$ are homotypic, so also is their composite $gf \in A \to C$.

The following lemma gives equivalent definitions of homotypisms.

Lemma 3.1.2 Let (A, T, σ) and (B, T, τ) be sorted algebras of the same type T and let $f \in A \to B$. Then f is homotypic iff $fa \in B_{\sigma a}$ for all $a \in A$ and iff $\tau f = \sigma$. More generally, if S is a subset of A and $\varphi \in S \to B$, then $\varphi(S_t) \subseteq B_t$ for all $t \in T$ iff $\varphi a \in B_{\sigma a}$ for all $a \in S$ and iff $\tau\varphi = \sigma|_S$.

Proof First assume that $\varphi(S_t) \subseteq B_t$ for all $t \in T$, and let $a \in S$. Then $a \in S_{\sigma a}$, and so $\varphi a \in B_{\sigma a}$. Next if $\varphi a \in B_{\sigma a}$ for all $a \in S$, then $\tau(\varphi a) = \sigma a$ for all $a \in S$, and so $\tau\varphi = \sigma|_S$. Lastly assume $\tau\varphi = \sigma|_S$, and let $t \in T$. If $a \in S_t$, then $\tau(\varphi a) = \sigma a = t$, and so $\varphi a \in B_t$. Thus $\varphi(S_t) \subseteq B_t$.

Lemma 3.1.3 A homotypic homomorphism is necessarily a holomorphism. More generally, if $\sigma \in A \to T$ is a holomorphism, $\tau \in B \to T$ is a homomorphism and $f \in A \to B$ satisfies $\tau f = \sigma$, then f is exact.

Proof The former half follows from the latter and Lemma 3.1.2. As for the latter, let $(\alpha_\lambda)_{\lambda \in \Lambda}$ and $(\beta_\lambda)_{\lambda \in \Lambda}$ be the structures of A and B. If $(a_1, \ldots, a_k) \in A^k$ and $(fa_1, \ldots, fa_k) \in \mathrm{Dm}\, \beta_\lambda$ for an index $\lambda \in \Lambda$, then since $(\sigma a_1, \ldots, \sigma a_k) = (\tau(fa_1), \ldots, \tau(fa_k))$, $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ by Lemma 3.1.1. Thus f is exact.

Remark 3.1.7 Every total algebra A may be regarded as a sorted algebra whose type is an arbitrary singleton $\{t\}$ made into a total algebra similar to A and whose sorting is the unique element of $A \to \{t\}$. Conversely if $(A, \{t\}, \sigma)$ is a sorted algebra and the type $\{t\}$ is total, then A is total because the sorting σ is exact. Furthermore, if A and B are similar total algebras, they may be regarded as sorted algebras of the same total type $\{t\}$, and a mapping $f \in A \to B$ is a homomorphism iff it is a homotypic holomorphism.

3.1.6 Power algebras and projections

Let (A, T, σ) be a sorted algebra and V be a nonempty set. Then we construct a sorted algebra (A^V, T, ρ) for the subset $A^V = \bigcup_{t \in T}(V \to A_t)$ of $V \to A$ and call it the V-**power algebra** over (A, T, σ). Since $A^V \subseteq V \to A$, each $\upsilon \in V$ yields the mapping $\varphi \mapsto \varphi\upsilon$ of A^V into A, which is called the **projection by υ** or **υ-projection** and denoted υ_p (s. §1.5.2). The following construction of (A^V, T, ρ) is that which makes υ_p a homotypic holomorphism for each $\upsilon \in V$.

Since $A = \coprod_{t\in T} A_t$ and $V \neq \emptyset$, we have $A^V = \coprod_{t\in T}(V\to A_t)$ (s. [3.2]), and so we define the mapping $\rho \in A^V\to T$ by the following for each $t \in T$ (s. [3.10]):

$$\rho^{-1}\{t\} = V\to A_t. \tag{3.1.2}$$

Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ be the algebraic structures of A and T respectively. Then we equip A^V with the following algebraic structure $(\beta_\lambda)_{\lambda\in\Lambda}$. Let λ be an arbitrary element of Λ. Then, in view of (3.1.2) and Remark 3.1.5, we define

$$\mathrm{Dm}\,\beta_\lambda = \bigcup_{(t_1,\ldots,t_k)\in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k}(V\to A_{t_j}). \tag{3.1.3}$$

Let $(\varphi_1, \ldots, \varphi_k)$ be an arbitrary element of the right-hand side. Then there is a unique element $(t_1, \ldots, t_k) \in \mathrm{Dm}\,\tau_\lambda$ such that $(\varphi_1, \ldots, \varphi_k) \in \prod_{j=1}^{k}(V\to A_{t_j})$. Let υ be an arbitrary element of V. Then $(\sigma(\varphi_1\upsilon), \ldots, \sigma(\varphi_k\upsilon)) = (t_1, \ldots, t_k) \in \mathrm{Dm}\,\tau_\lambda$, and so Lemma 3.1.1 applied to σ and id_T (s. Remark 3.1.4) shows that $(\varphi_1\upsilon, \ldots, \varphi_k\upsilon) \in \mathrm{Dm}\,\alpha_\lambda$ and $\sigma(\alpha_\lambda(\varphi_1\upsilon, \ldots, \varphi_k\upsilon)) = \tau_\lambda(t_1, \ldots, t_k)$. Therefore, we define $\beta_\lambda(\varphi_1, \ldots, \varphi_k) \in V\to A_{\tau_\lambda(t_1,\ldots,t_k)}$ by

$$(\beta_\lambda(\varphi_1, \ldots, \varphi_k))\upsilon = \alpha_\lambda(\varphi_1\upsilon, \ldots, \varphi_k\upsilon) \text{ for each } \upsilon \in V. \tag{3.1.4}$$

We have thus equipped A^V with an algebraic structure $(\beta_\lambda)_{\lambda\in\Lambda}$ which satisfies (3.1.3) and the following for each $\lambda \in \Lambda$:

$$\beta_\lambda\left(\prod_{j=1}^{k}(V\to A_{t_j})\right) \subseteq V\to A_{\tau_\lambda(t_1,\ldots,t_k)} \text{ for each } (t_1, \ldots, t_k) \in \mathrm{Dm}\,\tau_\lambda.$$

Thus (A^V, T, ρ) is a sorted algebra by Remark 3.1.5. Moreover, υ_p for each $\upsilon \in V$ is a homomorphism by (3.1.4), and since $\upsilon_p(V\to A_t) \subseteq A_t$ for each $t \in T$, it is a homotypic holomorphism by Lemma 3.1.3.

Remark 3.1.8 If elements $\varphi, \psi \in A^V$ satisfy $\upsilon_p\varphi = \upsilon_p\psi$ for all $\upsilon \in V$, then $\varphi, \psi \in V\to A$ and $\varphi\upsilon = \psi\upsilon$ for all $\upsilon \in V$, and so $\varphi = \psi$.

Let C be a sorted algebra of type T and $(f_\upsilon)_{\upsilon\in V}$ be a family of homotypic homomorphisms of C into A (s. Lemma 3.1.3). Then we can define a homotypism $f \in C\to A^V$ by $(fc)\upsilon = f_\upsilon c$ for each $c \in C$ and each $\upsilon \in V$, that is, $\upsilon_p f = f_\upsilon$ for each $\upsilon \in V$. Let $(\gamma_\lambda)_{\lambda\in\Lambda}$ be the algebraic structure of C, and assume $(c_1, \ldots, c_k) \in \mathrm{Dm}\,\gamma_\lambda$ for an index $\lambda \in \Lambda$. Then $(f_\upsilon c_1, \ldots, f_\upsilon c_k) = (\upsilon_p(fc_1), \ldots, \upsilon_p(fc_k))$ for each $\upsilon \in V$. Therefore, $(fc_1, \ldots, fc_k) \in \mathrm{Dm}\,\beta_\lambda$ and $\upsilon_p(f(\gamma_\lambda(c_1, \ldots, c_k))) = f_\upsilon(\gamma_\lambda(c_1, \ldots, c_k)) = \upsilon_p(\beta_\lambda(fc_1, \ldots, fc_k))$ for each $\upsilon \in V$ by Lemma 3.1.1. Therefore, $f(\gamma_\lambda(c_1, \ldots, c_k)) = \beta_\lambda(fc_1, \ldots, fc_k)$ by the preceding paragraph. Thus f is a homotypic homomorphism.

Remark 3.1.9 Let V' be a nonempty set and $f \in V\to V'$. Then the V'-power algebra $A^{V'}$ is also constructed, and if $\varphi \in V'\to A_t$ for an element $t \in T$, then $\varphi f \in V\to A_t$. Therefore, we have a homotypism $\varphi \mapsto \varphi f$ of $A^{V'}$ into A^V, which we denote by f^A. Then $(f^A\varphi)\upsilon = \varphi(f\upsilon)$ for each $\varphi \in A^{V'}$ and each $\upsilon \in V$, that is, $\upsilon_p f^A = (f\upsilon)_p$ for each $\upsilon \in V$. Let β'_λ be the operation of $A^{V'}$ assigned to $\lambda \in \Lambda$ and let $(\varphi_1, \dots, \varphi_k) \in \mathrm{Dm}\,\beta'_\lambda$. Then $((f\upsilon)_p\varphi_1, \dots, (f\upsilon)_p\varphi_k) = (\upsilon_p(f^A\varphi_1), \dots, \upsilon_p(f^A\varphi_k))$ for each $\upsilon \in V$, so $(f^A\varphi_1, \dots, f^A\varphi_k) \in \mathrm{Dm}\,\beta_\lambda$ and $\upsilon_p(f^A(\beta'_\lambda(\varphi_1, \dots, \varphi_k))) = (f\upsilon)_p(\beta'_\lambda(\varphi_1, \dots, \varphi_k)) = \upsilon_p(\beta_\lambda(f^A\varphi_1, \dots, f^A\varphi_k))$ for each $\upsilon \in V$ by Lemma 3.1.1, i.e. $f^A(\beta'_\lambda(\varphi_1, \dots, \varphi_k)) = \beta_\lambda(f^A\varphi_1, \dots, f^A\varphi_k)$. Thus f^A is a homotypic holomorphism by Lemma 3.1.3.

Remark 3.1.10 Let M be a subset of Λ. Then (A_M, T_M, σ) is a sorted algebra by Remark 3.1.6. Therefore, the V-power algebra $(A_M)^V$ is also constructed as above, and comparing its algebraic structure with that of A^V, we find that it is equal to the M-reduct of A^V, that is, $(A_M)^V = (A^V)_M$. Its sorting is equal to ρ. Suppose B is a subalgebra of A. Then $(B, T, \sigma|_B)$ is a sorted algebra by Remark 3.1.6. Therefore, the V-power algebra B^V is also constructed as above, and comparing its algebraic structure with that of A^V, we find that it is a subalgebra of A^V. Its sorting is equal to $\rho|_{B^V}$. Thus if B is a subreduct of A, then B^V is a subreduct of A^V and its sorting is equal to $\rho|_{B^V}$.

3.1.7 Universal sorted algebras

A sorted algebra (A, T, σ) is said to be **universal** or called a USA (**universal sorted algebra**) if it satisfies the following two conditions for a subset P of A.

- (**Generativity**) $A = [P]$.
- (**Universality**) If (A', T, σ') is a sorted algebra and a mapping $\varphi \in P \to A'$ satisfies $\sigma'\varphi = \sigma|_P$, then φ is extended to a homotypic holomorphism $f \in A \to A'$.

We refer to P and $\sigma|_P$ as the (**universality**) **basis** (s. Theorem 3.1.6) and the **basic sorting** of (A, T, σ) respectively. Then a USA is best described by the quadruple (A, T, σ, P) made of a sorted algebra (A, T, σ) and its basis P.

Remark 3.1.11 Lemma 3.1.4 below shows that the generativity and the universality together imply the following.

- (Strong universality) If (A', T, σ') is a sorted algebra and a mapping $\varphi \in P \to A'$ satisfies $\sigma'\varphi = \sigma|_P$, then φ is *uniquely* extended to a homotypic holomorphism $f \in A \to A'$.[3.12]

The same remark applies to UTAs below.

Lemma 3.1.4 Let A and B be algebras, and $f \in A \to B$ and $g \in A \to B$ be homomorphisms. Assume $f|_S = g|_S$ for a subset S of A. Then $f|_{[S]} = g|_{[S]}$.

[3.12] The strong universality conversely implies the generativity (and the universality).

Proof Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ be the structures of A and B. Define $C = \{a \in A : fa = ga\}$. If $\lambda \in \Lambda$ and $(a_1, \ldots, a_k) \in C^k \cap \mathrm{Dm}\,\alpha_\lambda$, then since f and g are homomorphisms, we have $f(\alpha_\lambda(a_1, \ldots, a_k)) = \beta_\lambda(fa_1, \ldots, fa_k) = \beta_\lambda(ga_1, \ldots, ga_k) = g(\alpha_\lambda(a_1, \ldots, a_k))$, and so $\alpha_\lambda(a_1, \ldots, a_k) \in C$. Therefore C is a subalgebra of A. Thus if S is contained in C, so is $[S]$.

The two kinds of universalities of the USA (A, T, σ, P) are illustrated by the following diagrams, where i is the inclusion mapping.

$$\begin{array}{ccc} P & \xrightarrow{i} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A' & \xrightarrow[\sigma']{} & T \end{array} \qquad\qquad \begin{array}{ccc} P & \xrightarrow{i} & A \\ {\scriptstyle\varphi}\downarrow & \swarrow_{f} & \downarrow{\scriptstyle\sigma} \\ A' & \xrightarrow[\sigma']{} & T \end{array} \tag{3.1.5}$$

Lemmas 3.1.2 and 3.1.3 show that a mapping $f \in A \to A'$ is a homotypic holomorphism iff it is a homomorphism and satisfies $\sigma = \sigma' f$. Moreover, $\sigma|_P = \sigma i$ and $f|_P = fi$. Therefore, the (strong) universality means that if the left rectangular diagram is commutative for a sorted algebra (A', T, σ') and a mapping $\varphi \in P \to A'$, then there exists a (unique) homomorphism $f \in A \to A'$ which makes the two triangles in the right diagram commutative. For this reason, we call f the **triangulating homomorphism** of the left diagram.

A total algebra A is said to be **universal** or called a UTA (**universal total algebra**) if it satisfies the following two conditions for a subset P of A.

- (**Generativity**) $A = [P]$.
- (**Universality**) If A' is a total algebra similar to A and $\varphi \in P \to A'$, then φ is extended to a homomorphism $f \in A \to A'$.

We call P the (**universality**) **basis** of A (s. Remark 3.1.17). Thus a UTA is best described by the pair (A, P) of a total algebra and its basis.

Remark 3.1.12 The concept of USAs generalizes that of UTAs.

Let (A, P) be a UTA. Then the sorted algebra $(A, \{t\}, \sigma)$ made of a total singleton $\{t\}$ and the unique mapping $\sigma \in A \to \{t\}$ as in Remark 3.1.7 together with P is a USA. As for its key universality, if $(A', \{t\}, \sigma')$ is a sorted algebra and a mapping $\varphi \in P \to A'$ satisfies $\sigma'\varphi = \sigma|_P$, then A' is a total algebra similar to A,[3.13] and so φ is extended to a homomorphism $f \in A \to A'$ by the universality of (A, P), and f is furthermore a homotypic holomorphism.

Conversely if $(A, \{t\}, \sigma, P)$ is a USA and $\{t\}$ is total, then (A, P) is a UTA. As for its key universality, if A' is a total algebra similar to A and $\varphi \in P \to A'$, then $(A', \{t\}, \sigma')$ is a sorted algebra for the unique mapping $\sigma' \in A' \to \{t\}$ and φ satisfies $\sigma'\varphi = \sigma|_P$, and so φ is extended to a homotypic holomorphism $f \in A \to A'$ by the universality of $(A, \{t\}, \sigma, P)$, and f is a homomorphism.

The following theorem is of fundamental importance in this chapter.

[3.13] More generally, any two sorted algebras of the same type are similar by [3.9].

Theorem 3.1.5 (Unique existence of USAs) Let T be an algebra, P be a set and $\tau \in \mathsf{P} \rightarrow \mathsf{T}$. Then there exists a USA (A, T, σ, P) such that $\sigma|_P = \tau$. If (A', T, σ', P) is also a USA such that $\sigma'|_P = \tau$, then there exists a homotypic isomorphism of A onto A' extending id_P.

We refer to the triple $(T, \sigma|_P, P)$ for a USA (A, T, σ, P) as its **syntax**. Then Theorem 3.1.5 means that each triple (T, τ, P) of an algebra T, a set P and a mapping $\tau \in P \rightarrow T$ yields a USA of syntax (T, τ, P) and that it is unique up to homotypic isomorphism extending id_P. Thus, in order to construct a USA, we only need to pick such a triple.[3.14] Picking a mapping $\tau \in P \rightarrow T$ is equivalent to picking a partition $(P_t)_{t \in T}$ of P (s. [3.10]), and the above key condition $\sigma|_P = \tau$ means that $\sigma|_P{}^{-1}\{t\} = P_t$ for each $t \in T$.

Remark 3.1.13 You will soon be convinced that Theorem 3.1.5 is true once you compare the triple (T, τ, P) therein to that underlying a common definition of a formal language discussed in the introduction of this chapter, that is, T is the set of the category names equipped with an algebraic structure derived from the formation rules, P is the set of the prime elements and τ associates each element of P with its category name in T. You must empirically know that such a triple uniquely determines a formal language and the category names of its elements, especially as to a formal language abstracted from your native tongue (s. Remark 1.2.2). Theorem 3.1.5 formulates and establishes the empirical knowledge.

Remark 3.1.14 (Free monoid) The concept of USAs is one of the mathematical concepts characterized by some universality conditions such as those of free monoids, free groups, polynomial rings and tensor products of modules (s. §3.6.1 for the concept of monoids). For example, a monoid A is said to be **free** over its subset P if it satisfies the following generativity and universality .

- A is generated by P and its identity element e, that is, $A = [P \cup \{e\}]_A$.
- If A' is a monoid and $\varphi \in P \rightarrow A'$, then φ is extended to a homomorphism $f \in A \rightarrow A'$ which associates e with the identity element of A'.

For each set S, we let S^* denote the set of all formal products $x_1 \cdots x_n$ of elements $x_1, \ldots, x_n$ of S of finite length $n \geq 0$. Then S^* is a monoid with respect to the multiplication defined by $(x_1 \cdots x_m)(y_1 \cdots y_n) = x_1 \cdots x_m y_1 \cdots y_n$ for each $(x_1 \cdots x_m, y_1 \cdots y_n) \in S^* \times S^*$ with $x_1, \ldots, x_m, y_1, \ldots, y_n \in S$. Its identity element is the formal product of length 0 and denoted ε. Furthermore, S^* is free over S. Therefore, as in the first paragraph of the proof of Theorem 3.1.5 below, it follows from the above two conditions that id_P is extended to an isomorphism of A onto P^* which associates e with ε, that is, we may identify A and e with P^* and ε respectively, hence $A - \{e\} = P \amalg (A - \{e\})^2 = [P]_A = \coprod_{n \geq 1} P^n$ (s. Example 3.1.1). Consequently, a monoid is free over at most one subset.

[3.14] The proof of Theorem 3.1.5 below naturally uses another method of constructing USAs, but it should never be used elsewhere, because USAs can be analyzed by means of Theorems 3.1.6 and 3.1.7 more neatly than by means of the construction in the proof.

Proof of Theorem 3.1.5 The latter half is a routine consequence of the definitions. Let i and i' be the inclusion mappings of P into A and A' respectively. Then $\sigma i = \sigma|_P = \tau = \sigma'|_P = \sigma' i'$, and so the universality of A and A' shows that i' and i are extended to homotypic holomorphisms $f \in A \to A'$ and $f' \in A' \to A$ respectively, hence $f'f|_P = f'i' = f'|_P = i = \mathrm{id}_A|_P$ and similarly $ff'|_P = \mathrm{id}_{A'}|_P$. Moreover, $A = [P]_A$, $A' = [P]_{A'}$ and all of $f'f$, id_A, ff' and $\mathrm{id}_{A'}$ are homomorphisms by Remark 3.1.4. Therefore $f'f = \mathrm{id}_A$ and $ff' = \mathrm{id}_{A'}$ by Lemma 3.1.4, and thus f is a homotypic isomorphism extending id_P.

As for the former half, we construct a USA (A, T, σ, P) such that $\sigma|_P = \tau$, generalizing the construction of deduction algebras in Examples 3.1.2–3.1.4.

Let $(\tau_\lambda)_{\lambda \in \Lambda}$ be the algebraic structure of T and $M = (P \amalg \Lambda \amalg \{[,]\})^*$ be the free monoid over the direct union $P \amalg \Lambda \amalg \{[,]\}$ of the set P, the index set Λ of $(\tau_\lambda)_{\lambda \in \Lambda}$ and the set $\{[,]\}$ of the left bracket [and the right bracket] (s. Remark 3.1.14). Then we inductively define subsets A_n $(n = 0, 1, \ldots)$ of M and an element $\sigma a \in T$ for each $a \in A_n$ by $A_0 = P$ and $\sigma a = \tau a$ for each $a \in A_0$ and by the following for each $n \geq 1$, where the italicized condition on $(a_1, \ldots, a_k)$ is the ramification condition $\mathcal{R}(A_0, \ldots, A_{n-1})$ in (2.3.1):

$$A_n = \left\{ [\lambda a_1 \cdots a_k] : \begin{array}{c} a_j \in A_{n_j} \ (j = 1, \ldots, k) \textit{ for some nonnegative} \\ \textit{integers } n_1, \ldots, n_k \textit{ such that } n - 1 = \sum_{j=1}^k n_j, \\ \lambda \in \Lambda \text{ and } (\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda. \end{array} \right\},$$

$$\sigma[\lambda a_1 \cdots a_k] = \tau_\lambda(\sigma a_1, \ldots, \sigma a_k) \text{ for each } [\lambda a_1 \cdots a_k] \in A_n \text{ as above.}$$

Here $\sigma[\lambda a_1 \cdots a_k]$ as well as A_n is well-defined, because it inductively follows from the definition of A_i $(i = 0, \ldots, n)$ that A_i consists of trees of rank i whose germs and nodes are elements of P and Λ respectively, and so Theorem 2.3.1 shows that $A_n \cap (A_0 \cup \cdots \cup A_{n-1}) = \emptyset$ and that each element $a \in A_n$ has a unique expression $a = [\lambda a_1 \cdots a_k]$ by an index $\lambda \in \Lambda$ and elements $a_j \in A_{n_j}$ $(j = 1, \ldots, k)$ such that $n - 1 = \sum_{j=1}^k n_j$ (and so the expression necessarily satisfies $(\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda$). Thus we have defined a set $A = \bigcup_{n \geq 0} A_n$ and a mapping $\sigma \in A \to T$ so that $\sigma|_P = \tau$.

We regard A as an algebra by equipping it with the algebraic structure $(\alpha_\lambda)_{\lambda \in \Lambda}$ defined by the following for each $\lambda \in \Lambda$:

$$\mathrm{Dm}\, \alpha_\lambda = \{(a_1, \ldots, a_k) \in \textstyle\bigcup_{n=1}^\infty A^n : (\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda\},$$
$$\alpha_\lambda(a_1, \ldots, a_k) = [\lambda a_1 \cdots a_k] \text{ for each } (a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda.$$

Notice that $\mathrm{Dm}\, \alpha_\lambda \subseteq A^k$ for an arity k of τ_λ (s. [3.8]). If $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$, then $a_j \in A_{n_j}$ for some n_j $(j = 1, \ldots, k)$ and $[\lambda a_1 \cdots a_k] \in A_n$ for $n = 1 + \sum_{j=1}^k n_j$. Therefore, the operation α_λ on A is well-defined. Furthermore, the definitions of α_λ and σ show that if $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ then $(\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda$ and $\sigma(\alpha_\lambda(a_1, \ldots, a_k)) = \sigma[\lambda a_1 \cdots a_k] = \tau_\lambda(\sigma a_1, \ldots, \sigma a_k)$. Therefore σ is a homomorphism. Moreover, the definition of $\mathrm{Dm}\, \alpha_\lambda$ shows that σ is exact. Thus σ is a holomorphism and so (A, T, σ) is a sorted algebra.

We argue by induction on n that A_n $(n = 0, 1, \ldots)$ is equal to the n-th descendant P_n of P in A. First, $A_0 = P = P_0$ by definition. Therefore assume

$n \geq 1$. In order to show $A_n = P_n$, first assume $a \in A_n$. Then $a = [\lambda a_1 \cdots a_k]$ with $\lambda \in \Lambda$, $a_j \in A_{n_j}$ $(j = 1, \ldots, k)$, $n - 1 = \sum_{j=1}^{k} n_j$ and $(\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda$. Therefore $a = \alpha_\lambda(a_1, \ldots, a_k)$, and $a_j \in P_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $a \in P_n$. Conversely assume $a \in P_n$. Then $a = \alpha_\lambda(a_1, \ldots, a_k)$ with $\lambda \in \Lambda$, $a_j \in P_{n_j}$ $(j = 1, \ldots, k)$ and $n - 1 = \sum_{j=1}^{k} n_j$. Therefore $a = [\lambda a_1 \cdots a_k]$ with $(\sigma a_1, \ldots, \sigma a_k) \in \mathrm{Dm}\, \tau_\lambda$, and $a_j \in A_{n_j}$ $(j = 1, \ldots, k)$ by the induction hypothesis. Thus $a \in A_n$.

Here we interrupt the proof by the following.

Remark 3.1.15 The descendants P_n $(n = 0, 1, \ldots)$ of P in A satisfy the following two conditions.

(a1) Each element $a \in A$ has a unique nonnegative integer n such that $a \in P_n$.

(a2) Each element $a \in P_n$ $(n \geq 1)$ has a unique expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ which satisfies the ramification condition $\mathcal{R}(P_0, \ldots, P_{n-1})$ in (2.3.1), that is,

$$\begin{aligned} &a_j \in P_{n_j} \ (j = 1, \ldots, k) \text{ for some nonnegative} \\ &\text{integers } n_1, \ldots, n_k \text{ such that } n - 1 = \textstyle\sum_{j=1}^{k} n_j. \end{aligned} \tag{3.1.6}$$

Indeed $P_n = A_n$ $(n = 0, 1, \ldots)$, hence $A = \bigcup_{n \geq 0} P_n$, and as was noted right after the definitions of A_n and σ, P_n consists of trees of rank n whose germs and nodes are elements of P and Λ respectively $(n = 0, 1, \ldots)$. Thus Theorem 2.3.1 shows that (a1) holds. The theorem together with the definitions of $(\alpha_\lambda)_{\lambda \in \Lambda}$ and the descendants in Theorem 3.1.2 shows that each element $a \in P_n$ $(n \geq 1)$ has a unique expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ and it necessarily satisfies (3.1.6). Thus (a2) holds.

Therefore, the proof will be complete once we prove the following.

Lemma 3.1.5 Let (A, T, σ) be a sorted algebra and P be a subset of A. Assume that the descendants P_n $(n = 0, 1, \ldots)$ of P in A satisfy the conditions (a1) and (a2) given in Remark 3.1.15 for the algebraic structure $(\alpha_\lambda)_{\lambda \in \Lambda}$ of A. Then (A, T, σ, P) is a USA.

Proof The condition (a1) and Theorem 3.1.2 imply $A = [P]$, that is, (A, T, σ, P) satisfies the generativity. In order to show that it also satisfies the universality, let (A', T, σ') be a sorted algebra, and assume that $\varphi \in P \rightarrow A'$ satisfies $\sigma' \varphi = \sigma|_P$. Let $(\alpha'_\lambda)_{\lambda \in \Lambda}$ be the structure of A'. Then we inductively define an element $fa \in A'$ for each $a \in P_n$ $(n = 0, 1, \ldots)$ so that $\sigma'(fa) = \sigma a$. First if $n = 0$, then $a \in P$ and therefore we define $fa = \varphi a$, hence $\sigma'(fa) = \sigma'(\varphi a) = \sigma a$ as desired. Therefore assume $n \geq 1$. Then a has a unique expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ with $a_j \in P_{n_j}$ $(j = 1, \ldots, k)$ and $n - 1 = \sum_{j=1}^{k} n_j$, and $(\sigma'(fa_1), \ldots, \sigma'(fa_k)) = (\sigma a_1, \ldots, \sigma a_k)$ by induction. Therefore $(fa_1, \ldots, fa_k) \in \mathrm{Dm}\, \alpha'_\lambda$ and $\sigma'(\alpha'_\lambda(fa_1, \ldots, fa_k)) = \sigma(\alpha_\lambda(a_1, \ldots, a_k)) = \sigma a$ by

Lemma 3.1.1. Moreover, $P_n \cap (P_0 \cup \cdots \cup P_{n-1}) = \emptyset$ by (a1). Therefore we define $fa = \alpha'_\lambda(fa_1, \ldots, fa_k)$, hence $\sigma'(fa) = \sigma a$.

We have thus defined a mapping $f \in A \rightarrow A'$ so that $f|_P = \varphi$ and $\sigma' f = \sigma$. If $\lambda \in \Lambda$ and $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$, then $a_j \in P_{n_j}$ for some n_j $(j = 1, \ldots, k)$ and $\alpha_\lambda(a_1, \ldots, a_k) \in P_n$ for $n = 1 + \sum_{j=1}^{k} n_j$, and so $(fa_1, \ldots, fa_k) \in \mathrm{Dm}\, \alpha'_\lambda$ and $f(\alpha_\lambda(a_1, \ldots, a_k)) = \alpha'_\lambda(fa_1, \ldots, fa_k)$ by the definition of f. Therefore f is a homomorphism. Thus (A, T, σ, P) also satisfies the universality.

Example 3.1.5 Let A be an algebra, and replace (T, τ, P) in the proof of Theorem 3.1.5 with (A, id_A, A). Then the resultant USA is equal to (D, A, c, A) for the deduction algebra D on A and the concluding $c \in D \rightarrow A$. Thus D and c may also be defined as the USA of syntax (A, id_A, A) and its sorting.

3.1.8 Bases, rank and universality

Here we consider algebras closely related to USAs.

Definition 3.1.1 Let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ be an algebra and P be a subset of A. Assume that the descendants P_n $(n = 0, 1, \ldots)$ of P in A satisfy the following two conditions given in Remark 3.1.15.

(a1) Each element $a \in A$ has a unique nonnegative integer n such that $a \in P_n$ (we call n the **rank** of a and denote it by $\mathrm{rk}\, a$).

(a2) Each element $a \in P_n$ $(n \geq 1)$ has a unique expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ which satisfyies (3.1.6), i.e. the ramification condition $\mathcal{R}(P_0, \ldots, P_{n-1})$ in (2.3.1).

Then we call P a (**generativity**) **basis** of A or say that A is **based on** P or that (A, P) is a **based algebra**.

The following two theorems are indispensable in the analysis of USAs, and together with Remark 3.1.16, give other definitions of the concept of USAs.

Theorem 3.1.6 Let (A, T, σ) be a sorted algebra and P be a subset of A. Then (A, T, σ, P) is a USA iff (A, P) is a based algebra, that is, P is a universality basis of (A, T, σ) iff it is a generativity basis of A.

Theorem 3.1.7 Let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ be an algebra and P be a subset of A. Then P is a basis of A iff it satisfies the following three conditions.

(b1) $A = [P]$.

(b2) P consists of primes of A, that is, no element $a \in P$ has an expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$.

(b3) Each element $a \in A - P$ has a unique expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$ (we call it the **ramification** of a for the reason to be clarified in [3.15]).

If P is a basis of A, then the ramification $a = \alpha_\lambda(a_1, \dots, a_k)$ of each element $a \in A - P$ satisfies $\mathrm{rk}\, a - 1 = \sum_{j=1}^{k} \mathrm{rk}\, a_j$.

Some remarks and corollaries are in order before proving the theorems.

Remark 3.1.16 The conditions (b2) and (b3) together imply the following condition where P does not occur.

(b4) Each element $a \in A$ has at most one expression $a = \alpha_\lambda(a_1, \dots, a_k)$ by an index $\lambda \in \Lambda$ and an element $(a_1, \dots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$.

Remark 3.1.1 shows that (b1) implies $A - P \subseteq \bigcup_{\lambda\in\Lambda} \mathrm{Im}\, \alpha_\lambda$. Therefore (b1) and (b4) together imply (b3). Thus, under (b1) and (b2), (b3) is equivalent to (b4). The condition (b1) also implies $A - \bigcup_{\lambda\in\Lambda} \mathrm{Im}\, \alpha_\lambda \subseteq P$, while (b2) means $P \subseteq A - \bigcup_{\lambda\in\Lambda} \mathrm{Im}\, \alpha_\lambda$. Therefore (b1) and (b2) together imply $P = A - \bigcup_{\lambda\in\Lambda} \mathrm{Im}\, \alpha_\lambda$. The condition (b4) is violated iff A satisfies a relation $\alpha_\lambda(a_1, \dots, a_k) = \alpha_\mu(b_1, \dots, b_l)$ for some distinct tuples $(\lambda, a_1, \dots, a_k)$ and $(\mu, b_1, \dots, b_l)$, and so an algebra satisfying (b4) is said to be **relation-free**. Thus Theorems 3.1.6 and 3.1.7 show that a (sorted) algebra A has a (universality) basis iff A is generated by primes and relation-free and also show that if A has a basis, then it is equal to the prime set of A.

Remark 3.1.17 Let A be a total algebra and P be a subset of A. Then A may be regarded as a sorted algebra $(A, \{t\}, \sigma)$ for a total singleton $\{t\}$ by Remark 3.1.7, and (A, P) is a UTA iff $(A, \{t\}, \sigma, P)$ is a USA by Remark 3.1.12. Therefore, Theorem 3.1.6 implies that (A, P) is a UTA iff (A, P) is a based algebra, that is, P is a universality basis of A iff it is a generativity basis of A.

Example 3.1.6 The set $\mathbb{N}$ of all positive integers together with its element 1 and the successor function f is characterized by the following three conditions, the first of which is the principle of (mathematical) induction.

(c1) If a subset P of $\mathbb{N}$ satisfies $1 \in P$ and $fa \in P$ for all $a \in P$, then $P = \mathbb{N}$.

(c2) $fa \neq 1$ for all $a \in \mathbb{N}$.

(c3) If a and b are distinct elements of $\mathbb{N}$, then $fa \neq fb$.

The conditions (c1), (c2) and (c3) mean that the algebra $(\mathbb{N}, \{f\})$ and the subset $\{1\}$ of $\mathbb{N}$ satisfy the conditions (b1), (b2) and (b4). Therefore $(\mathbb{N}, \{f\}, \{1\})$ is a based algebra by Remark 3.1.16 and Theorem 3.1.7, and therefore is a UTA by Remark 3.1.17. The set $\mathbb{Z}$ of all integers equipped with the successor function f is a relation-free algebra, and not a based algebra because $f\mathbb{Z} = \mathbb{Z}$.

Corollary 3.1.7.1 Let A be a based algebra and B be its subreduct. Then B is based on its prime set.

Proof Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ be the structure of A. Then B is an M-subreduct for a subset M of Λ, and so its structure consists of the restrictions β_μ of α_μ to B for all $\mu \in M$. Remark 3.1.16 shows that A is relation-free, and therefore so is B, and thus it suffices to prove $B = [R]_B$ for the prime set R of B. By way of contradiction, assume $B \supset [R]_B$, and pick an element $b \in B - [R]_B$ so that its rank $\operatorname{rk} b$ in A is minimal. Then since $b \in B - R$, we have $b = \beta_\mu(b_1, \ldots, b_k) = \alpha_\mu(b_1, \ldots, b_k)$ for some $\mu \in M$ and $(b_1, \ldots, b_k) \in \operatorname{Dm}\beta_\mu = B^k \cap \operatorname{Dm}\alpha_\mu$. Therefore $\operatorname{rk} b - 1 = \sum_{j=1}^{k} \operatorname{rk} b_j$ by Theorem 3.1.7, and so $b_j \in [R]_B$ for all $j \in \{1, \ldots, k\}$ by the minimality of $\operatorname{rk} b$. But then $b = \beta_\mu(b_1, \ldots, b_k) \in [R]_B$, which is a contradiction.

Remark 3.1.18 Continuing the proof of Corollary 3.1.7.1, we have that an element $(a_1, \ldots, a_k) \in \operatorname{Dm}\alpha_\lambda$ $(\lambda \in \Lambda)$ satisfies $\alpha_\lambda(a_1, \ldots, a_k) \in B - R$ iff $\lambda \in M$ and $a_1, \ldots, a_k \in B$. Indeed, if $\lambda \in M$ and $a_1, \ldots, a_k \in B$, then $\alpha_\lambda(a_1, \ldots, a_k) = \beta_\lambda(a_1, \ldots, a_k) \in B - R$ by (b2) for B. Conversely if $\alpha_\lambda(a_1, \ldots, a_k) \in B - R$, then (b3) for B and (b4) for A show that $\lambda \in M$ and $a_1, \ldots, a_k \in B$. Moreover, a prime of A belongs to B iff it belongs to R.

Suppose we have an algorithm for determining whether a given element $a \in A$ belongs to R or not. Then the above fact together with Theorem 3.1.7 shows that we also have an algorithm for determining whether a given element $a \in A$ belongs to B or not. Thus the questions about the possible ability of the PU are not challenging, as was mentioned in §1.2.6.

Corollary 3.1.7.2 Let $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ be an algebra and M be a subset of Λ. Then the following hold on the M-reduct A_M of A.

(1) If (A, P) is a based algebra, then so is $(A_M, P \cup \bigcup_{\lambda\in\Lambda-M} \operatorname{Im}\alpha_\lambda)$.

(2) If (A, T, σ, P) is a USA, then so is $(A_M, T_M, \sigma, P \cup \bigcup_{\lambda\in\Lambda-M} \operatorname{Im}\alpha_\lambda)$.

Proof (1) is a consequence of Corollary 3.1.7.1 because $A = P \amalg \coprod_{\lambda\in\Lambda} \operatorname{Im}\alpha_\lambda$ by (b2) and (b3). (2) is a consequence of (1) and Theorem 3.1.6 because (A_M, T_M, σ) is a sorted algebra by Remark 3.1.6.

Corollary 3.1.7.3 Let (A, P) be a based Λ-algebra, M be a subset of Λ, and Q be a subset of P. Then $([Q]_M, Q)$ is a based M-algebra.

Let (A, T, σ, P) be a universal sorted Λ-algebra, M be a subset of Λ, Q be a subset of P, and U be an M-subreduct of T such that $\sigma Q \subseteq U$. Then $([Q]_M, U, \sigma|_{[Q]_M}, Q)$ is a universal sorted M-algebra.

Proof The former half is also a consequence of Corollary 3.1.7.1 because Q is the prime set of $[Q]_M$ by (b2) and Remark 3.1.1. The latter half is a consequence of the former and Theorem 3.1.6 because $([Q]_M, U, \sigma|_{[Q]_M})$ is a sorted M-algebra by Lemma 3.1.9 below and Remark 3.1.6.

Example 3.1.7 Let D be the deduction algebra on an algebra A and c be its concluding. Then (D, A, c, A) is a USA by Example 3.1.5, and so (D, A) is a based algebra by Theorem 3.1.6. Furthermore, Example 3.1.2 shows that the rank and ramification of each element of the based algebra D are equal to its rank and ramification as a tree.[3.15]

Proof of Theorem 3.1.7 Assume that P is a basis of A. Then (b1) holds by (a1) and Theorem 3.1.2. Assume that an element $a \in A$ has an expression $a = \alpha_\lambda(a_1, \ldots, a_k)$ with $\lambda \in \Lambda$ and $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$. Then $a_j \in P_{n_j}$ $(j = 1, \ldots, k)$ for some n_j by (a1), and $a \in P_n$ for $n = 1 + \sum_{j=1}^{k} n_j \geq 1$, and so $a \in A - P$ by (a1). Thus (b2) and $\mathrm{rk}\, a - 1 = \sum_{j=1}^{k} \mathrm{rk}\, a_j$ hold. Assume that a also has an expression $a = \alpha_{\lambda'}(a'_1, \ldots, a'_{k'})$ with $\lambda' \in \Lambda$ and $(a'_1, \ldots, a'_{k'}) \in \mathrm{Dm}\, \alpha_{\lambda'}$. Then $a'_j \in P_{n'_j}$ $(j = 1, \ldots, k')$ for some n'_j, and $a \in P_{n'}$ for $n' = 1 + \sum_{j=1}^{k'} n'_j$. Therefore $n = n'$ by (a1), and so $\lambda = \lambda'$ (hence $k = k'$) and $a_j = a'_j$ $(j = 1, \ldots, k)$ by (a2). Thus (b4) holds, and so does (b3) by Remark 3.1.16.

Conversely assume (b1)–(b3). Then $A = \bigcup_{n \geq 0} P_n$ by (b1) and Theorem 3.1.2. Therefore, in order to prove (a1), we only need to show $P_n \cap P_{n'} = \emptyset$ for each pair n, n' of distinct nonnegative integers. If $\min\{n, n'\} = 0$, then $P_n \cap P_{n'} \subseteq P \cap \bigcup_{\lambda \in \Lambda} \mathrm{Im}\, \alpha_\lambda = \emptyset$ by the definition of descendants and (b2). Therefore we may argue by induction on $\min\{n, n'\}$. Assume $a \in P_n \cap P_{n'}$ for integers n, n' such that $\min\{n, n'\} \geq 1$. Then there are expressions

$$\begin{aligned} a &= \alpha_\lambda(a_1, \ldots, a_k) && (a_j \in P_{n_j}\ (j = 1, \ldots, k),\ n - 1 = \textstyle\sum_{j=1}^{k} n_j), \\ a &= \alpha_{\lambda'}(a'_1, \ldots, a'_{k'}) && (a'_j \in P_{n'_j}\ (j = 1, \ldots, k'),\ n' - 1 = \textstyle\sum_{j=1}^{k'} n'_j), \end{aligned}$$

and $a \in A - P$. Therefore $\lambda = \lambda'$ (hence $k = k'$) and $a_j = a'_j \in P_{n_j} \cap P_{n'_j}$ $(j = 1, \ldots, k)$ by (b3), and so $n_j = n'_j$ $(j = 1, \ldots, k)$ by the induction hypothesis, hence $n = n'$. Thus we have proved (a1). In order to prove (a2), assume $n = n'$ for the above two expressions. Then $\lambda = \lambda'$ (hence $k = k'$) and $a_j = a'_j$ $(j = 1, \ldots, k)$ by (b3). Thus (a2) holds.

Theorem 3.1.6 will be proved in a series of basic lemmas.

Lemma 3.1.6 The inverses of isomorphisms of algebras are isomorphisms.[3.16]

Proof Let f be an isomorphism of an algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ onto an algebra $(B, (\beta_\lambda)_{\lambda \in \Lambda})$. Let $\lambda \in \Lambda$ and $(b_1, \ldots, b_k) \in \bigcup_{n=1}^{\infty} B^n$. Then $(b_1, \ldots, b_k) = (f(f^{-1}b_1), \ldots, f(f^{-1}b_k))$, and so Lemma 3.1.1 applied to holomorphisms id_B and f shows that $(b_1, \ldots, b_k) \in \mathrm{Dm}\, \beta_\lambda$ iff $(f^{-1}b_1, \ldots, f^{-1}b_k) \in \mathrm{Dm}\, \alpha_\lambda$ and that if $(b_1, \ldots, b_k) \in \mathrm{Dm}\, \beta_\lambda$ then $\beta_\lambda(b_1, \ldots, b_k) = f(\alpha_\lambda(f^{-1}b_1, \ldots, f^{-1}b_k))$ and so $f^{-1}(\beta_\lambda(b_1, \ldots, b_k)) = \alpha_\lambda(f^{-1}b_1, \ldots, f^{-1}b_k)$. Thus f^{-1} is an isomorphism.

[3.15] If A is a based algebra, then it can be embedded in D by a mapping which preserves the ranks and ramifications, hence the term *ramification* for based algebras (s. Theorem 3.1.7).
[3.16] The inverses of bijective homomorphisms are not necessarily homomorphisms.

Lemma 3.1.7 Let $f \in A \to B$ be a holomorphism of algebras. Then the images by f of subalgebras of A are subalgebras of B.[3.17]

Proof Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ be the structures of A and B, and C be a subalgebra of A. Assume $(b_1,\ldots,b_k) \in (fC)^k \cap \mathrm{Dm}\,\beta_\lambda$. Then $(b_1,\ldots,b_k) = (fc_1,\ldots,fc_k)$ for some $(c_1,\ldots,c_k) \in C^k$, and since f is a holomorphism, we have $(c_1,\ldots,c_k) \in \mathrm{Dm}\,\alpha_\lambda$ and $\beta_\lambda(b_1,\ldots,b_k) = f(\alpha_\lambda(c_1,\ldots,c_k))$. Moreover, since C is a subalgebra of A, we have $\alpha_\lambda(c_1,\ldots,c_k) \in C$. Therefore $\beta_\lambda(b_1,\ldots,b_k) \in fC$. Thus fC is a subalgebra of B.

Lemma 3.1.8 Let $f \in A \to B$ be a homomorphism of algebras. Then the inverse images by f of subalgebras of B are subalgebras of A.

Proof Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ be the structures of A and B, and D be a subalgebra of B. Assume $(a_1,\ldots,a_k) \in (f^{-1}D)^k \cap \mathrm{Dm}\,\alpha_\lambda$. Then since f is a homomorphism, we have $(fa_1,\ldots,fa_k) \in D^k \cap \mathrm{Dm}\,\beta_\lambda$ and $f(\alpha_\lambda(a_1,\ldots,a_k)) = \beta_\lambda(fa_1,\ldots,fa_k)$. Moreover $\beta_\lambda(fa_1,\ldots,fa_k) \in D$ because D is a subalgebra of B. Therefore $\alpha_\lambda(a_1,\ldots,a_k) \in f^{-1}D$. Thus $f^{-1}D$ is a subalgebra of A.

Lemma 3.1.9 Let $f \in A \to B$ be a homomorphism of algebras and S be a subset of A. Then $f([S]_A) \subseteq [fS]_B$. If f is a holomorphism, then $f([S]_A) = [fS]_B$.[3.18]

Proof Since $f^{-1}([fS]_B)$ is a subalgebra of A by Lemma 3.1.8 and $S \subseteq f^{-1}(fS) \subseteq f^{-1}([fS]_B)$, we have $[S]_A \subseteq f^{-1}([fS]_B)$, and so $f([S]_A) \subseteq [fS]_B$. If f is a holomorphism, then $f([S]_A)$ is a subalgebra of B by Lemma 3.1.7, and so since $f([S]_A) \supseteq fS$, we have $f([S]_A) \supseteq [fS]_B$.

Lemma 3.1.10 Let $f \in A \to A'$ be an isomorphism of algebras and assume that A' is based on a subset P'. Then A is based on $f^{-1}P'$.

Proof Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\alpha'_\lambda)_{\lambda\in\Lambda}$ be the structures of A and A'. Theorem 3.1.7 shows that (A', P') satisfies the conditions (b1)–(b4). Therefore $A = f^{-1}A' = f^{-1}[P'] = [f^{-1}P']$ by Lemma 3.1.6 and Lemma 3.1.9, that is, $(A, f^{-1}P')$ satisfies (b1). If an element $s \in f^{-1}P'$ has an expression $s = \alpha_\lambda(a_1,\ldots,a_k)$, then $P' \ni fs = \alpha'_\lambda(fa_1,\ldots,fa_k)$, which contradicts (b2) for (A', P'). Therefore $(A, f^{-1}P')$ satisfies (b2). If $\alpha_\lambda(a_1,\ldots,a_k) = \alpha_\mu(b_1,\ldots,b_l)$, then $\alpha'_\lambda(fa_1,\ldots,fa_k) = \alpha'_\mu(fb_1,\ldots,fb_l)$, and so $\lambda = \mu$ (hence $k = l$) and $fa_j = fb_j$ $(j = 1,\ldots,k)$ by (b4) for (A', P'), hence $a_j = b_j$ $(j = 1,\ldots,k)$. Therefore $(A, f^{-1}P')$ satisfies (b4). Thus A is based on $f^{-1}P'$ by Theorem 3.1.7 and Remark 3.1.16.

[3.17] Images by homomorphisms are not necessarily subalgebras (s. [3.11] for examples).
[3.18] This does not necessarily hold for homomorphisms (s. [3.11] for examples).

Proof of Theorem 3.1.6 If (A, P) is a based algebra, then (A, T, σ, P) is a USA by Lemma 3.1.5. Conversely assume that (A, T, σ, P) is a USA. Construct a USA $(\overline{A}, T, \overline{\sigma}, P)$ such that $\overline{\sigma}|_P = \sigma|_P$ by the method of the proof of Theorem 3.1.5. Then $(\overline{A}, P)$ is a based algebra by Remark 3.1.15. Moreover, there exists an isomorphism $f \in A \to \overline{A}$ such that $f|_P = \mathrm{id}_P$ by the latter half of Theorem 3.1.5. Thus (A, P) is a based algebra by Lemma 3.1.10.

3.1.9 Occurrences for based algebras

Throughout this subsection, we let $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ be an algebra. We will soon make two assumptions on it italicized below.

It goes without saying that every mathematical concept should be explicitly defined and dealt with by rigorous deduction, and so should be those of occurrences here and substitutions in §3.1.10, however intuitively obvious they may appear. Indeed, I have seen that people intuitively deal with them without explicit definitions and some draw erroneous conclusions about them.

If elements a and b of A satisfy $a = \alpha_\lambda(\dots, b, \dots)$ for an index $\lambda \in \Lambda$, we say that b **immediately occur**s in a or write $b \prec a$. Moreover, if there exists a sequence $(b_i)_{i=0,\dots,n}$ $(n \geq 0)$ of elements of A such that $b_0 = a$, $b_n = b$ and either $b_i \prec b_{i-1}$ or $b_i = b_{i-1}$ for each $i \in \{1, \dots, n\}$, then we say that b **occur**s in a or write $b \preceq a$, and call the sequence an **occurrence** of b in a. Furthermore, for each element $a \in A$ and each subset B of A, we define

$$B^a = \{b \in B : b \preceq a\}. \tag{3.1.7}$$

Notice that $a \in A^a$ for all $a \in A$, that is, the relation $\preceq$ is reflexive.

Example 3.1.8 Let A be a free monoid P^* over a set P (s. Remark 3.1.14). Then its algebraic structure consists of an associative multiplication. Therefore, an element $b \in A$ immediately occurs in an element $a \in A$ iff $a = bc$ or $a = cb$ for an element $c \in A$, and so the identity element of A occurs in every element of A. Moreover, each element $a \in A$ is the product $x_1 \cdots x_n$ of a unique sequence $x_1, \dots, x_n$ $(n \geq 0)$ of elements of P. Therefore, a non-identity element $b \in A$ immediately occurs in a iff $b = x_1 \cdots x_i$ or $b = x_i \cdots x_n$ for some $i \in \{1, \dots, n\}$, and so b occurs in a iff $b = x_i \cdots x_j$ for some $i, j \in \{1, \dots, n\}$ such that $i \leq j$. Thus, an element $b \in P$ occurs in a iff $b = x_i$ for some $i \in \{1, \dots, n\}$, and so we also call such x_i an **occurrence** of b in a. We have used the concept of occurrences in this sense in Remark 2.2.8, §2.3, §2.4 and §3.1.3.

Henceforth in this subsection, we assume that A *has a basis* P*.* Then

$$A^a = \{a\} \qquad (a \in P) \tag{3.1.8}$$

by Theorem 3.1.7. See the proof of the following lemma for A^a for $a \in A - P$.

Lemma 3.1.11 For each element $a \in A$, A^a is a finite set.

Proof We argue by induction on $r = \mathrm{rk}\, a$. If $r = 0$, that is, if $a \in P$, then $A^a = \{a\}$ by (3.1.8). Therefore assume $r \geq 1$, i.e. $a \in A - P$. Then Theorem 3.1.7 shows that its ramification $a = \alpha_\lambda(a_1, \dots, a_k)$ satisfies $r - 1 = \sum_{j=1}^{k} \mathrm{rk}\, a_j$, and so A^{a_j} is a finite set for all $j \in \{1, \dots, k\}$ by the induction hypothesis. Therefore it suffices to show $A^a - \{a\} \subseteq \bigcup_{j=1}^{k} A^{a_j}$ (in fact, $A^a = \{a\} \amalg \bigcup_{j=1}^{k} A^{a_j}$). Assume $b \in A^a - \{a\}$. Then there exists an occurrence $(b_i)_{i=0,\dots,n}$ of b in a. Since $b_n = b \neq a = b_0$, there exists an integer $i \in \{1, \dots, n\}$ such that $b_i \prec a$, and so $b_i = a_j$ for some $j \in \{1, \dots, k\}$ by Theorem 3.1.7. Therefore $(b_i, \dots, b_n)$ is an occurrence of b in a_j, and so $b \in A^{a_j}$. Thus $A^a - \{a\} \subseteq \bigcup_{j=1}^{k} A^{a_j}$ as desired.

Henceforth in this subsection, we furthermore assume that the index set Λ *of* $(\alpha_\lambda)_{\lambda \in \Lambda}$ *satisfies the following condition for a set* Γ *and a subset* X *of* P:

$$\Lambda \subseteq (\Gamma \amalg X)^*. \tag{3.1.9}$$

Then if $\lambda \in \Lambda$, since $\lambda \in (\Gamma \amalg X)^*$ and $X \subseteq (\Gamma \amalg X)^*$, we may consider the set

$$X^\lambda = \{b \in X : b \preccurlyeq \lambda\} \tag{3.1.10}$$

defined by (3.1.7) for $(\Gamma \amalg X)^*$, that is, we may consider the occurrences of elements of X in λ discussed in Example 3.1.8.

Let a and b be elements of A. Then an occurrence $(b_i)_{i=0,\dots,n}$ of b in a is said to be **free**, if $\{b_0, \dots, b_n\} \cap \mathrm{Im}\, \alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $b \in X^\lambda$. If there exists a free occurrence of b in a, we say that b **occurs free** in a or write $b \preccurlyeq_{\mathrm{fr}} a$. For each element $a \in A$ and each subset B of A, we define

$$B^a_{\mathrm{fr}} = \{b \in B : b \preccurlyeq_{\mathrm{fr}} a\}. \tag{3.1.11}$$

Then $A^a_{\mathrm{fr}} \subseteq A^a$ for all $a \in A$. Furthermore, Theorem 3.1.7 shows that $a \in A^a_{\mathrm{fr}}$ for all $a \in A$, because the occurrences of a in a are the sequences $(a, \dots, a)$ of a of arbitrary length, and if $a \in X^\lambda$ for some $\lambda \in \Lambda$, then $a \in X \subseteq P$ and so $\{a\} \cap \mathrm{Im}\, \alpha_\lambda = \emptyset$. Therefore

$$A^a_{\mathrm{fr}} = \{a\} \qquad (a \in P) \tag{3.1.12}$$

by (3.1.8). See the following lemma for A^a_{fr} for $a \in A - P$.

Lemma 3.1.12 If $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, then $A^a_{\mathrm{fr}} - \{a\} = \bigcup_{j=1}^{k} A^{a_j}_{\mathrm{fr}} - X^\lambda$ (therefore, $A^a_{\mathrm{fr}} = \{a\} \amalg (\bigcup_{j=1}^{k} A^{a_j}_{\mathrm{fr}} - X^\lambda)$) and $P^a_{\mathrm{fr}} = \bigcup_{j=1}^{k} P^{a_j}_{\mathrm{fr}} - X^\lambda$.

Proof The latter conclusion follows from the former because $a \in A - P$ by Theorem 3.1.7. As for the former, first assume $b \in A^{a_j}_{\mathrm{fr}} - X^\lambda$ for some $j \in \{1, \dots, k\}$. Then since $b \in A^{a_j}_{\mathrm{fr}}$, there exists a free occurrence $(b_i)_{i=1,\dots,n}$ ($n \geq 1$) of b in a_j. Define $b_0 = a$. Then since $b_1 = a_j$, we have $b_1 \prec b_0$, and so $(b_i)_{i=0,\dots,n}$ is an occurrence of b in a. Assume $b \in X^\mu$ for some $\mu \in \Lambda$. Then since $(b_i)_{i=1,\dots,n}$ is a free occurrence of b in a_j, we have $\{b_1, \dots, b_n\} \cap \mathrm{Im}\, \alpha_\mu = \emptyset$. Furthermore, since $b \in X^\mu - X^\lambda$, we have $\mu \neq \lambda$, and so since $b_0 = a \in \mathrm{Im}\, \alpha_\lambda$, we have $b_0 \notin \operatorname{Im} \alpha_\mu$ by Theorem 3.1.7. Therefore, $(b_i)_{i=0,\dots,n}$ is a free occurrence of b in a, and so $b \in A_{\mathrm{fr}}^a$. Moreover, since $\operatorname{rk} b = \operatorname{rk} b_n \leq \cdots \leq \operatorname{rk} b_1 = \operatorname{rk} a_j < \operatorname{rk} a$ by Theorem 3.1.7, we have $b \neq a$. Thus $\bigcup_{j=1}^k A_{\mathrm{fr}}^{a_j} - X^\lambda \subseteq A_{\mathrm{fr}}^a - \{a\}$.

Next assume $b \in A_{\mathrm{fr}}^a - \{a\}$. Then there exists a free occurrence $(b_i)_{i=0,\dots,n}$ of b in a, and so since $b_0 = a \in \operatorname{Im} \alpha_\lambda$, we have $b \notin X^\lambda$. Furthermore, since $b_n = b \neq a = b_0$, there exists an integer $i \in \{1, \dots, n\}$ such that $b_i \prec a$, and so $b_i = a_j$ for some $j \in \{1, \dots, k\}$ by Theorem 3.1.7. Therefore, $(b_i, \dots, b_n)$ is an occurrence of b in a_j. If $b \in X^\mu$ for some $\mu \in \Lambda$, then since $(b_i)_{i=0,\dots,n}$ is a free occurrence of b in a, we have $\{b_0, \dots, b_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$, and so $\{b_i, \dots, b_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$. Therefore, $(b_i, \dots, b_n)$ is a free occurrence of b in a_j, and so $b \in A_{\mathrm{fr}}^{a_j} - X^\lambda$. Thus $A_{\mathrm{fr}}^a - \{a\} \subseteq \bigcup_{j=1}^k A_{\mathrm{fr}}^{a_j} - X^\lambda$.

Definition 3.1.2 Let a, b and c be elements of A. Then an occurrence $(b_0, \dots, b_n)$ of b in a is said to be **free from** c, if $\{b_0, \dots, b_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in \Lambda$ such that $(X^\lambda)_{\mathrm{fr}}^c \neq \emptyset$. Furthermore, we say that b is **free from** c **in** a, if every free occurrence of b in a is free from c.

Lemma 3.1.13 Let $a = \alpha_\lambda(a_1, \dots, a_k)$, b and c be elements of A. Assume that b is free from c in a and $b \precneqq_{\mathrm{fr}} a$. Then $(X^\lambda)_{\mathrm{fr}}^c = \emptyset$ and b is free from c in a_j for all $j \in \{1, \dots, k\}$.

Proof Since $b \precneqq_{\mathrm{fr}} a$, there exists a free occurrence $(b_i)_{i=0,\dots,n}$ of b in a, and so since it is free from c and $b_0 = a \in \operatorname{Im} \alpha_\lambda$, we have $(X^\lambda)_{\mathrm{fr}}^c = \emptyset$. Moreover, since $a \notin X^\lambda$ by Theorem 3.1.7, we have $b \notin X^\lambda$ by Lemma 3.1.12.

Now let $(b_i)_{i=1,\dots,n}$ $(n \geq 1)$ be a free occurrence of b in a_j for some $j \in \{1, \dots, k\}$. Then defining $b_0 = a$ and arguing as in the proof of Lemma 3.1.12, we have that $(b_i)_{i=0,\dots,n}$ is a free occurrence of b in a, and so it is free from c. Therefore if $(X^\mu)_{\mathrm{fr}}^c \neq \emptyset$ for some $\mu \in \Lambda$, then $\{b_0, \dots, b_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$, and so $\{b_1, \dots, b_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$. Thus b is free from c in a_j for all $j \in \{1, \dots, k\}$.

Lemma 3.1.14 Let $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, $b \in A - X^\lambda$ and $c \in A$. Assume that b is free from c in a. Then b is free from c in a_j for all $j \in \{1, \dots, k\}$.

Proof Let $j \in \{1, \dots, k\}$. If $b \not\precneqq_{\mathrm{fr}} a_j$, then obviously b is free from c in a_j. If $b \precneqq_{\mathrm{fr}} a_j$, then $b \precneqq_{\mathrm{fr}} a$ by Lemma 3.1.12, and so b is free from c in a_j by Lemma 3.1.13.

3.1.10 Substitutions for universal sorted algebras

Throughout this subsection, we let (A, T, σ, P) be a USA and $(\alpha_\lambda)_{\lambda \in \Lambda}$ be the structure of A, and assume that the index set Λ of $(\alpha_\lambda)_{\lambda \in \Lambda}$ satisfies (3.1.9) for a set Γ and a subset X of P. Then (A, P) is a based algebra by Theorem 3.1.6, and so we can apply Theorem 3.1.7 and the concepts in §3.1.9 to it.

Let $b_1, \dots, b_n$ $(n \geq 0)$ be *distinct* elements of P and let $(c_1, \dots, c_n) \in A_{\sigma b_1} \times \cdots \times A_{\sigma b_n}$. Then for each element $a \in A$, we will define an element

$a\left(\dfrac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right) \in A$ satisfying

$$\sigma\left(a\left(\frac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right)\right) = \sigma a \tag{3.1.13}$$

by induction on n and $r = \operatorname{rk} a$ arranged in lexicographical order.

First if $n = 0$, then we define $a\left(\dfrac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right) = a$, hence (3.1.13).

Assume $n \geq 1$. First if $r = 0$, that is, if $a \in P$, then we define

$$a\left(\frac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right) = \begin{cases} c_i & \text{if } a = b_i \text{ for some } i \in \{1,\ldots,n\}, \\ a & \text{if } a \notin \{b_1,\ldots,b_n\}. \end{cases} \tag{3.1.14}$$

This definition is possible and meets (3.1.13) because $b_1,\ldots,b_n$ are distinct and $\sigma c_i = \sigma b_i$ for all $i \in \{1,\ldots,n\}$.

Next assume $r \geq 1$, i.e. $a \in A - P$. Then a has a *unique* ramification $a = \alpha_\lambda(a_1,\ldots,a_k)$ by Theorem 3.1.7. Let $\{b_{i_1},\ldots,b_{i_\nu}\} = \{b_1,\ldots,b_n\} - X^\lambda$ with $i_1 < \cdots < i_\nu$ and $0 \leq \nu \leq n$. Then we define

$$a\left(\frac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right) = \alpha_\lambda\left(a_1\left(\frac{b_{i_1},\ldots,b_{i_\nu}}{c_{i_1},\ldots,c_{i_\nu}}\right),\ldots,a_k\left(\frac{b_{i_1},\ldots,b_{i_\nu}}{c_{i_1},\ldots,c_{i_\nu}}\right)\right). \tag{3.1.15}$$

Since $r - 1 = \sum_{j=1}^{k} \operatorname{rk} a_j$ by Theorem 3.1.7, $a'_j = a_j\left(\dfrac{b_{i_1},\ldots,b_{i_\nu}}{c_{i_1},\ldots,c_{i_\nu}}\right)$ has already been defined and satisfies $\sigma a'_j = \sigma a_j$ for all $j \in \{1,\ldots,k\}$ by induction. Since $(a_1,\ldots,a_k) \in \operatorname{Dm}\alpha_\lambda$, $(a'_1,\ldots,a'_k) \in \operatorname{Dm}\alpha_\lambda$ and $\sigma(\alpha_\lambda(a'_1,\ldots,a'_k)) = \sigma(\alpha_\lambda(a_1,\ldots,a_k)) = \sigma a$ by Lemma 3.1.1. Therefore, the definition (3.1.15) is possible and meets (3.1.13).

Thus we have defined a homotypic transformation

$$a \mapsto a\left(\frac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right)$$

on A, which we call the (simultaneous) **substitution** of $c_1,\ldots,c_n$ for $b_1,\ldots,b_n$ and denote by $\left(\dfrac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right)$, or by $(b_1,\ldots,b_n/c_1,\ldots,c_n)$ especially when $n = 1$. The lemmas below deal with it.

Remark 3.1.19 The simultaneous substitution $\left(\dfrac{b_1,\ldots,b_n}{c_1,\ldots,c_n}\right)$ is not always equal to the one by one substitution $\left(\dfrac{b_1}{c_1}\right)\cdots\left(\dfrac{b_n}{c_n}\right)$. If any of $b_i,\ldots,b_n$ does not occur free in c_{i-1} for each $i \in \{2,\ldots,n\}$, then they are equal.

Remark 3.1.20 Suppose $X^\lambda = \emptyset$ for all $\lambda \in \Lambda$, that is, Λ satisfies the condition $\Lambda \subseteq \Gamma^*$ stronger than (3.1.9). Define $\varphi \in P \to A$ by $\varphi b_i = c_i$ $(i = 1,\ldots,n)$ and

$\varphi a = a$ for each $a \in P - \{b_1, \dots, b_n\}$. Then $\sigma\varphi = \sigma|_P$, and so φ is uniquely extended to a homotypic homomorphism $f \in A \to A$ by the universality of A. The uniqueness together with (3.1.14) and (3.1.15) shows $f = \left(\dfrac{b_1, \dots, b_n}{c_1, \dots, c_n}\right)$.

Remark 3.1.21 We may similarly define substitution for the free monoid $A = P^*$ over a set P (s. Remark 3.1.14). Let $b_1, \dots, b_n$ ($n \geq 0$) be *distinct* elements of P and $c_1, \dots, c_n$ be elements of A. Define $\varphi \in P \to A$ by $\varphi b_i = c_i$ ($i = 1, \dots, n$) and $\varphi a = a$ for each $a \in P - \{b_1, \dots, b_n\}$. Then φ is uniquely extended to a homomorphism $f \in A \to A$ such that $f\varepsilon = \varepsilon$. We define f to be the (simultaneous) **substitution** of $c_1, \dots, c_n$ for $b_1, \dots, b_n$ in A. Each element $a \in A$ is a product $x_1 \cdots x_m$ of elements $x_1, \dots, x_m$ of P, and since f is a homomorphism extending φ, we have $fa = (\varphi x_1) \cdots (\varphi x_m)$. Thus fa is obtained by replacing all occurrences of $b_1, \dots, b_n$ in a with $c_1, \dots, c_n$ respectively (s. Example 3.1.8).

Lemma 3.1.15 If $a \in \operatorname{Im} \alpha_\lambda$ and $b_i \in X^\lambda$ for some $i \in \{1, \dots, n\}$, then

$$a\left(\frac{b_1, \dots, b_n}{c_1, \dots, c_n}\right) = a\left(\frac{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right).$$

Proof Since $\{b_1, \dots, b_n\} - X^\lambda = \{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\} - X^\lambda$, this holds by (3.1.15).

Lemma 3.1.16 Assume $a \in A$ and $b_i \not\preceq_{\text{fr}} a$ for some $i \in \{1, \dots, n\}$. Then

$$a\left(\frac{b_1, \dots, b_n}{c_1, \dots, c_n}\right) = a\left(\frac{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right).$$

Proof We argue by induction on n and $r = \operatorname{rk} a$ as in the definition of the substitutions. Assume $r = 0$, i.e. $a \in P$. Then since $b_i \preceq_{\text{fr}} b_i$ by (3.1.12) and $b_i \not\preceq_{\text{fr}} a$ by our assumption, we have $a \neq b_i$, and so (3.1.14) gives the desired result for $n = 1, 2, \dots$. Therefore assume $r \geq 1$, i.e. $a \in A - P$. Then Theorem 3.1.7 shows that its ramification $a = \alpha_\lambda(a_1, \dots, a_k)$ satisfies $r - 1 = \sum_{j=1}^{k} \operatorname{rk} a_j$. Since $a \in \operatorname{Im} \alpha_\lambda$, we assume $b_i \notin X^\lambda$ in view of Lemma 3.1.15. Then $b_i \not\preceq_{\text{fr}} a_j$ for all $j \in \{1, \dots, k\}$ by Lemma 3.1.12, and so

$$a_j\left(\frac{b_1, \dots, b_n}{c_1, \dots, c_n}\right) = a_j\left(\frac{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right)$$

by the induction hypothesis, and (3.1.15) gives the desired result provided that $\{b_1, \dots, b_n\} \cap X^\lambda = \emptyset$. Therefore assume $b_h \in X^\lambda$ for some $h \in \{1, \dots, n\}$. Then $h \neq i$, hence $n \geq 2$, and so the proof is complete if $n = 1$. If $h < i$, then

$$\begin{aligned} a\left(\frac{b_1, \dots, b_n}{c_1, \dots, c_n}\right) &= a\left(\frac{b_1, \dots, b_{h-1}, b_{h+1}, \dots, b_n}{c_1, \dots, c_{h-1}, c_{h+1}, \dots, c_n}\right) \\ &= a\left(\frac{b_1, \dots, b_{h-1}, b_{h+1}, \dots, b_{i-1}, b_{i+1}, \dots, b_n}{c_1, \dots, c_{h-1}, c_{h+1}, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right) \end{aligned}$$

$$= a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$$

by Lemma 3.1.15 and the induction hypothesis, and likewise for the case $i < h$.

Lemma 3.1.17 If $b_i = c_i$ for some $i \in \{1,\dots,n\}$, then

$$\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = \left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right).$$

Proof We will prove that $a\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$ for all $a \in A$ by induction on $r = \operatorname{rk} a$. Assume $r = 0$, i.e. $a \in P$. If $a = b_j$ for some $j \in \{1,\dots,i-1,i+1,\dots,n\}$, then $a\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = c_j = a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$ by (3.1.14). If $a = b_i$, then $a\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = c_i = b_i = a = a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$ by (3.1.14). If $a \notin \{b_1,\dots,b_n\}$, then $a\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = a = a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$ by (3.1.14). Therefore assume $r \geq 1$, i.e. $a \in A - P$. Then Theorem 3.1.7 shows that its ramification $a = \alpha_\lambda(a_1,\dots,a_k)$ satisfies $r - 1 = \sum_{j=1}^{k} \operatorname{rk} a_j$. Let $\{b_{i_1},\dots,b_{i_\nu}\} = \{b_1,\dots,b_n\} - X^\lambda$ with $i_1 < \cdots < i_\nu$ and $0 \leq \nu \leq n$. Since $a \in \operatorname{Im} \alpha_\lambda$, we assume $b_i \notin X^\lambda$ in view of Lemma 3.1.15. Then $b_i = b_{i_h}$ for some $h \in \{1,\dots,\nu\}$, and since $b_1,\dots,b_n$ are distinct, we have $i = i_h$ and hence $b_{i_h} = c_{i_h}$. Therefore $a_j\left(\frac{b_{i_1},\dots,b_{i_\nu}}{c_{i_1},\dots,c_{i_\nu}}\right) = a_j\left(\frac{b_{i_1},\dots,b_{i_{h-1}},b_{i_{h+1}},\dots,b_{i_\nu}}{c_{i_1},\dots,c_{i_{h-1}},c_{i_{h+1}},\dots,c_{i_\nu}}\right)$ for all $j \in \{1,\dots,k\}$ by the induction hypothesis. Furthermore, $\{b_{i_1},\dots,b_{i_{h-1}},b_{i_{h+1}},\dots,b_{i_\nu}\} = \{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n\} - X^\lambda$ because $i_h = i$. Therefore $a\left(\frac{b_1,\dots,b_n}{c_1,\dots,c_n}\right) = a\left(\frac{b_1,\dots,b_{i-1},b_{i+1},\dots,b_n}{c_1,\dots,c_{i-1},c_{i+1},\dots,c_n}\right)$ by (3.1.15).

3.2 The concept of logic systems

The purpose of this section is to define the concept of logic systems in a series of short subsections 3.2.1–3.2.5 as was outlined in the introduction of this chapter.

3.2.1 Formal languages as universal sorted algebras

A **formal language** is a USA (A, T, σ, P) equipped with three sets C, X and Γ which satisfy the following two conditions on the type T and the basis P.

- P is the direct union $C \amalg X$ of C and X, and $X \neq \emptyset$.[3.19]

[3.19] You may remove the condition $X \neq \emptyset$, if you like.

- The index set Λ of the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ of T is a subset of the free monoid $(\Gamma \amalg X)^*$ over the direct union $\Gamma \amalg X$ (s. Remark 3.1.14).

We refer to the elements of C, X and Γ as the **constant**s, **variable**s and **token**s respectively. Moreover, in view of the inclusion $\Lambda \cup \Gamma^* \subseteq (\Gamma \amalg X)^*$,[3.20] we define

$$M = \Lambda \cap \Gamma^* \tag{3.2.1}$$

and refer to the elements of M and $\Lambda - M$ as the **invariable indice**s and **variable indice**s respectively. Furthermore, we say that the operation τ_λ of T is **invariable** or **variable** according as $\lambda \in M$ or $\lambda \in \Lambda - M$, and likewise for A and other algebras whose structures are indexed by Λ.

In order to classify the indices in Λ, especially the variable indices, we let $X^{\circledast}$ denote the subset of X^* which consists of all formal products $x_1 \cdots x_n$ of *distinct* elements $x_1, \ldots, x_n$ of X of finite length $n \geq 0$; for example, X and the formal product ε of length 0 are obviously contained in $X^{\circledast}$. Then since $\Lambda \subseteq (\Gamma \amalg X)^*$, we may assign each index $\lambda \in \Lambda$ the formal product $\xi_\lambda = x_1 \cdots x_n \in X^{\circledast}$ of all distinct elements $x_1, \ldots, x_n$ of X that occur in λ (s. Example 3.1.8) and are picked from the right to the left, that is, x_i occurs in λ on the right-hand side of all occurrences of $x_1, \ldots, x_{i-1}$ in λ $(i = 2, \ldots, n)$;[3.21] for example, if $\lambda = azyxyzb$ for elements a and b of Γ and distinct elements x, y and z of X, then $\xi_\lambda = xyz$, while if $\lambda = ab$, then $\xi_\lambda = \varepsilon$. We call ξ_λ the **variable qualifier** in λ. Conversely, we assign each element $\xi \in X^{\circledast}$ the subset

$$\Lambda_\xi = \{\lambda \in \Lambda : \xi_\lambda = \xi\}$$

of Λ. Then we have

$$\Lambda = \textstyle\coprod_{\xi\in X^{\circledast}} \Lambda_\xi, \qquad M = \Lambda_\varepsilon, \qquad \Lambda - M = \textstyle\coprod_{\xi\in X^{\circledast}-\{\varepsilon\}} \Lambda_\xi. \tag{3.2.2}$$

Thus we classify the indices according to variable qualifiers in them.

Extending the syntax $(T, \sigma|_P, P)$ of the USA (A, T, σ, P), we refer to the sextuple $(T, \sigma|_P, P, C, X, \Gamma)$ for the formal language $(A, T, \sigma, P, C, X, \Gamma)$ as its **syntax**.[3.22] The above two conditions and both (3.2.1) and (3.2.2) have been described in terms of the syntax. Theorem 3.1.5 shows that each sextuple $(T, \tau, P, C, X, \Gamma)$ of an algebra T, four sets P, C, X and Γ satisfying the above two conditions and a mapping $\tau \in P \to T$ yields a formal language of syntax $(T, \tau, P, C, X, \Gamma)$ up to homotypic isomorphism extending id_P. Thus, in order to construct a formal language, we only need to pick such a sextuple (s. [3.14]).

Example 3.2.1 In the most usual case that Λ is contained in the subset $\Gamma^* \cup \Gamma X$ of $(\Gamma \amalg X)^*$ as in FPL and CL, we have $\Lambda - M = \Lambda \cap \Gamma X = \coprod_{x\in X}(\Lambda \cap \Gamma x)$ and $\Lambda \cap \Gamma x = \Lambda_x$ for each $x \in X$ (notice here that $X \subseteq X^{\circledast} - \{\varepsilon\}$). In FPL (s. §3.4), $\Lambda \cap \Gamma x = \{\forall x, \exists x\}$ for quantifiers $\forall$ and $\exists$ in Γ $(x \in X)$. In CL (s. Chapters 4–6), $\Lambda \cap \Gamma x \subseteq \{\nabla x\}$ for a token $\nabla \in \Gamma$ $(x \in X)$.

[3.20] If Y is a set and Z is its subset, Z^* is naturally embedded in Y^*, so that $\Gamma^* \subseteq (\Gamma \amalg X)^*$.

[3.21] This definition of ξ_λ presupposes that every variable operation symbol is prepositive.

[3.22] Any USA of syntax (T, τ, P) with $P \neq \emptyset$ (s. [3.19]) may be regarded as a formal language of syntax $(T, \tau, P, \emptyset, P, \Lambda)$ for the index set Λ of the algebraic structure of T.

3.2.2 Denotable worlds as sorted algebras

Let $(A, T, \sigma, P, C, X, \Gamma)$ be a formal language as defined in §3.2.1. Then a **denotable world** (DW) for it is a sorted algebra W whose type and sorting satisfy the following two conditions described in terms of its syntax $(T, \sigma|_P, P, C, X, \Gamma)$.

- The type of W is the M-reduct T_M of T for the set M of the invariable indices (3.2.1) of the algebraic structure of T.

Under this first condition, since $T_M = T$ as sets, the implicit sorting of W partitions it into the t-parts W_t $(t \in T)$, which concern the second condition.

- $W_t \neq \emptyset$ for each $t \in \sigma P$, that is, for each $t \in T$ such that $P_t \neq \emptyset$.

We call it the P-**denotability**. It implies $W \neq \emptyset$ because $P = \bigcup_{t\in T} P_t \neq \emptyset$.

Remark 3.2.1 Let $(\tau_\lambda)_{\lambda\in M}$ be the algebraic structure of T_M. Then Remark 3.1.5 shows that a DW for A may also be defined as an algebra $(W, (\omega_\lambda)_{\lambda\in M})$ equipped with a partition $W = \coprod_{t\in T} W_t$ which satisfies the P-denotability and the following for each $\lambda \in M$, where the former equation in effect means that $\left(\prod_{j=1}^{k} W_{t_j}\right)_{(t_1,\dots,t_k)\in \mathrm{Dm}\,\tau_\lambda}$ is a partition of $\mathrm{Dm}\,\omega_\lambda$:

$$\begin{aligned} &\mathrm{Dm}\,\omega_\lambda = \textstyle\bigcup_{(t_1,\dots,t_k)\in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k} W_{t_j}, \\ &\omega_\lambda(\textstyle\prod_{j=1}^{k} W_{t_j}) \subseteq W_{\tau_\lambda(t_1,\dots,t_k)} \text{ for each } (t_1, \dots, t_k) \in \mathrm{Dm}\,\tau_\lambda. \end{aligned} \tag{3.2.3}$$

The implicit sorting $\pi \in W \to T_M$ is defined by $\pi^{-1}\{t\} = W_t$ for each $t \in T$.

Example 3.2.2 Let (W, T, π) be a sorted algebra of the same type T as A. Then (W_M, T_M, π) is a sorted algebra by Remark 3.1.6. If W_M moreover satisfies the P-denotability, that is, if W_M is a DW for A, we call W a **signifiable world** for A. For example if $P \subseteq W$ and $\sigma|_P = \pi|_P$, then $P_t \subseteq W_t$ for each $t \in T$, and so W_M satisfies the P-denotability. In this particular case, we call W a **transferable world** for A. For example, A itself is a transferable world for A, and therefore we particularly call it the **self-world** for A. Henceforth, we will delete the subscript M of the DWs W_M for the signifiable worlds W for A.

3.2.3 Prime denotations and valuations

Let W be a DW for the formal language $(A, T, \sigma, P, C, X, \Gamma)$ as defined in §3.2.1 and §3.2.2. Then a **denotation** of C into W is a mapping $\delta \in C \to W$ which satisfies $\delta(C_t) \subseteq W_t$ for each $t \in T$ (s. Lemma 3.1.2). There exists at least one denotation because the P-denotability of W shows that $W_t \neq \emptyset$ whenever $C_t \neq \emptyset$. If $C = \emptyset$, then since $\emptyset \to W = \{\emptyset\}$ by [3.2], $\emptyset$ is the unique and formal denotation. Likewise, a **valuation** of X into W is a mapping $\upsilon \in X \to W$ which satisfies $\upsilon(X_t) \subseteq W_t$ for each $t \in T$. We denote the set of the valuations of X into W by Υ_W or $\Upsilon_{X,W}$:

$$\Upsilon_W = \Upsilon_{X,W} = \{\upsilon \in X \to W : \upsilon(X_t) \subseteq W_t \text{ for each } t \in T\}.$$

Then $\Upsilon_W \neq \emptyset$ because the P-denotability of W also shows that $W_t \neq \emptyset$ whenever $X_t \neq \emptyset$. Moreover $\Upsilon_W \neq \{\emptyset\}$, because $X \neq \emptyset$ by definition.

Example 3.2.3 Suppose W is a transferable world for A as defined in Example 3.2.2. Then since $P_t \subseteq W_t$ for each $t \in T$, the inclusion mappings of C and X into W are a denotation and a valuation respectively.

3.2.4 Interpretations of the variable indices

Let W be a DW for the formal language $(A, T, \sigma, P, C, X, \Gamma)$ as defined in §3.2.1 and §3.2.2. Then its algebraic structure $(\omega_\lambda)_{\lambda \in M}$ is indexed by the set M of the invariable indices (3.2.1) of the algebraic structure $(\tau_\lambda)_{\lambda \in \Lambda}$ of T. In other words, each invariable index $\lambda \in M$ is assigned an operation ω_λ on W.

Here we consider assigning each variable index $\lambda \in \Lambda - M$ a mapping $\lambda_W \in V_\lambda \to W$ for a certain set V_λ by means of the operation τ_λ, the partition $W = \coprod_{t \in T} W_t$ given by the implicit sorting of W and the variable qualifier $\xi_\lambda = x_1 \cdots x_n$ in λ (*henceforth, the expressions like this imply that* $x_1, \ldots, x_n$ *are distinct elements of* X). Since $\lambda \in \Lambda - M$, we have $\xi_\lambda \neq \varepsilon$, that is, $n \geq 1$ by (3.2.2). However, we also consider the case $\lambda \in M$, where $\xi_\lambda = \varepsilon$, that is, $n = 0$.

For each element $\xi = x_1 \cdots x_n \in X^{\circledast}$ $(n \geq 0)$, we define

$$W_{\sigma\xi} = \prod_{i=1}^{n} W_{\sigma x_i}. \tag{3.2.4}$$

Then $W_{\sigma\xi} \neq \emptyset$, because if $\xi \neq \varepsilon$ then $n \geq 1$ and $W_{\sigma x_i} \neq \emptyset$ for each $i \in \{1, \ldots, n\}$ by the P-denotability of W, while if $\xi = \varepsilon$ then $n = 0$ and $W_{\sigma\xi} = \{\emptyset\}$.[3.23]

A **significance** of $\lambda \in \Lambda$ on W is a mapping

$$\lambda_W \in \left(\bigcup_{(t_1,\ldots,t_k) \in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k} (W_{\sigma\xi_\lambda} \to W_{t_j})\right) \to W \tag{3.2.5}$$

which satisfies the following condition for each $(t_1, \ldots, t_k) \in \mathrm{Dm}\,\tau_\lambda$:

$$\lambda_W\left(\prod_{j=1}^{k} (W_{\sigma\xi_\lambda} \to W_{t_j})\right) \subseteq W_{\tau_\lambda(t_1,\ldots,t_k)}. \tag{3.2.6}$$

Therefore, $V_\lambda = \bigcup_{(t_1,\ldots,t_k) \in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k} (W_{\sigma\xi_\lambda} \to W_{t_j})$.

An **interpretation** of $\Lambda - M$ on W is a family $(\lambda_W)_{\lambda \in \Lambda - M}$ of significances λ_W of the variable indices $\lambda \in \Lambda - M$ on W. Thus, interpreting $\Lambda - M$ on W means assigning each variable index λ a significance on W.

Remark 3.2.2 If $\xi = \varepsilon$ in (3.2.4), then $W_{\sigma\xi} = \{\emptyset\}$. Therefore if $\lambda \in M$, then $W_{\sigma\xi_\lambda} \to W_{t_j}$ in (3.2.5) and (3.2.6) may be identified with W_{t_j} for each $j \in \{1, \ldots, k\}$,[3.24] and so V_λ may be identified with $\bigcup_{(t_1,\ldots,t_k) \in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k} W_{t_j}$.

[3.23] The direct product $\prod_{j=1}^{k} S_k$ of k sets $S_1, \ldots, S_k$ may be defined as the set of the mappings f of $K = \{1, \ldots, k\} = \{j \in \mathbb{N} : 1 \leq j \leq k\}$ into $\bigcup_{j=1}^{k} S_j = \bigcup_{j \in K} S_j$ such that $fj \in S_j$ for each $j \in K$. If $k = 0$, then $K = \emptyset$ (also $\bigcup_{j \in K} S_j = \emptyset$) and so $\prod_{j=1}^{k} S_k = \{\emptyset\}$ by [3.2]. Thus we denote f by the tuple $(f1, \ldots, fk)$ understanding that if $k = 0$ then it is $\emptyset$.

[3.24] Each mapping f of a singleton $\{s\}$ into a set T may be identified with its value $fs \in T$ at s (s. [1.32]). The identification will be referred to as $\{s\}$ **convention**.

Therefore, we may regard the significances of each $\lambda \in M$ on W as the operations λ_W on W which satisfy the following:

$$\operatorname{Dm}\lambda_W = \bigcup_{(t_1,\ldots,t_k)\in\operatorname{Dm}\tau_\lambda} \prod_{j=1}^k W_{t_j},$$
$$\lambda_W\left(\prod_{j=1}^k W_{t_j}\right) \subseteq W_{\tau_\lambda(t_1,\ldots,t_k)} \text{ for each } (t_1,\ldots,t_k) \in \operatorname{Dm}\tau_\lambda.$$

Therefore, Remark 3.2.1 shows that the algebraic structure $(\omega_\lambda)_{\lambda\in M}$ of W is nothing but a family of significances of the invariable indices $\lambda \in M$ on W. Thus, defining $\lambda_W = \omega_\lambda$ for each $\lambda \in M$, we may extend $(\lambda_W)_{\lambda\in\Lambda-M}$ to the family $(\lambda_W)_{\lambda\in\Lambda}$ and call it an **(extended) interpretation** of Λ on W.

Example 3.2.4 Let $(W, (\omega_\lambda)_{\lambda\in\Lambda})$ be a signifiable world for A as defined in Example 3.2.2. Then we may define a significance λ_W of each index $\lambda \in \Lambda$ on W by the following equation for each $(t_1,\ldots,t_k) \in \operatorname{Dm}\tau_\lambda$ and each $(f_1,\ldots,f_k) \in \prod_{j=1}^k (W_{\sigma\xi_\lambda} \to W_{t_j})$:

$$\lambda_W(f_1,\ldots,f_k) = \omega_\lambda(f_1(\upsilon_{11}x_1,\ldots,\upsilon_{1n}x_n),\ldots,f_k(\upsilon_{k1}x_1,\ldots,\upsilon_{kn}x_n)),$$

where $\xi_\lambda = x_1\cdots x_n$ and υ_{ji} $(i = 1,\ldots,n,\ j = 1,\ldots,k)$ are arbitrary elements of Υ_W. Thus $\lambda_W = \omega_\lambda$ for each $\lambda \in M$ as in Remark 3.2.2.

This definition is possible and meets (3.2.6), for

$$(\upsilon_{j1}x_1,\ldots,\upsilon_{jn}x_n) \in \prod_{i=1}^n \upsilon_{ji}(X_{\sigma x_i}) \subseteq \prod_{i=1}^n W_{\sigma x_i} = W_{\sigma\xi_\lambda} \quad (j = 1,\ldots,k),$$
$$(f_1(\upsilon_{11}x_1,\ldots,\upsilon_{1n}x_n),\ldots,f_k(\upsilon_{k1}x_1,\ldots,\upsilon_{kn}x_n)) \in \prod_{j=1}^k W_{t_j},$$
$$\prod_{j=1}^k W_{t_j} \subseteq \operatorname{Dm}\omega_\lambda \text{ and } \omega_\lambda\left(\prod_{j=1}^k W_{t_j}\right) \subseteq W_{\tau_\lambda(t_1,\ldots,t_k)}.$$

If W is a transferable world for A, then Example 3.2.3 shows that the inclusion mapping of X into W belongs to Υ_W, and so we may replace the above with

$$\lambda_W(f_1,\ldots,f_k) = \omega_\lambda(f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n)). \tag{3.2.7}$$

Example 3.2.5 In the usual case as in FPL and ICL that $\Lambda \subseteq \Gamma^* \cup \Gamma X$ and every variable index λ is unary and satisfies $\operatorname{Dm}\tau_\lambda = \{\phi\}$ for an element $\phi \in T$, we have $\Lambda - M = \coprod_{x\in X}(\Lambda \cap \Gamma x)$ by Example 3.2.1, and the significance of $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ on W can be an arbitrary mapping

$$\lambda_W \in (W_{\sigma x} \to W_\phi) \to W_{\tau_\lambda\phi}. \tag{3.2.8}$$

In FPL, $\lambda = \forall x$ or $\exists x$ for quantifiers $\forall$ and $\exists$ in Γ, $W_{\sigma x}$ is the subset W_ϵ of the entities, W_ϕ is the set $\mathbb{T} = \{0, 1\}$ of the truth values, and $\tau_\lambda\phi = \phi$. Therefore, the significance λ_W belongs to $(W_\epsilon \to \mathbb{T}) \to \mathbb{T}$, and is defined by

$$\lambda_W f = \begin{cases} \inf\{fw : w \in W_\epsilon\} & \text{if } \lambda = \forall x, \\ \sup\{fw : w \in W_\epsilon\} & \text{if } \lambda = \exists x \end{cases} \tag{3.2.9}$$

for each $f \in W_\epsilon \to \mathbb{T}$, so that the variable operations $\forall x$ and $\exists x$ of A signify *for all* and *for some* respectively. The details will be given in §3.4.

In ICL, $\lambda = \triangledown x$ for a token $\triangledown \in \Gamma$, $W_{\sigma x}$ is the subset W_ϵ of the basic entities, $W_\phi = \mathbb{T}$ and $W_{\tau_\lambda \phi} = W_\epsilon \to \mathbb{T}$ (s. [1.55]).[3.25] Therefore, the significance λ_W belongs to $(W_\epsilon \to \mathbb{T}) \to (W_\epsilon \to \mathbb{T})$, and is defined as the identity transformation on $W_\epsilon \to \mathbb{T}$, so that the variable operation $\triangledown x$ of A works as the nominalizer mentioned in §1.2.7. The details will be given in §4.1.3.

3.2.5 Logic systems as formal languages with semantics

A **logic system** is a pair of a formal language $(A, T, \sigma, P, C, X, \Gamma)$ as defined in §3.2.1 and a triple $(\mathfrak{W}, (I_W)_{W \in \mathfrak{W}}, (\Delta_W)_{W \in \mathfrak{W}})$ of

- a nonempty collection $\mathfrak{W}$ of DWs for A as defined in §3.2.2,
- a family $(I_W)_{W \in \mathfrak{W}}$ of interpretations I_W on $W \in \mathfrak{W}$ of the set $\Lambda - M$ of the variable indices of the algebraic structure of T as defined in §3.2.4 and
- a family $(\Delta_W)_{W \in \mathfrak{W}}$ of nonempty sets Δ_W of denotations of C into $W \in \mathfrak{W}$ as defined in §3.2.3.

The triple is called the **semantics** of the logic system and of the formal language. Since the formal language is determined by its syntax $(T, \sigma|_P, P, C, X, \Gamma)$ up to homotypic isomorphism extending id_P and since the semantics can be described in terms of the syntax, the logic system may be regarded as the pair of the syntax and the semantics.

3.3 Basic analysis of logic systems

Throughout this section, we let $\mathfrak{L} = (A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W \in \mathfrak{W}}, (\Delta_W)_{W \in \mathfrak{W}})$ be a logic system as defined in §3.2.5. Here we relate its seemingly unrelated components and derive further concepts from them in a series of subsections 3.3.1–3.3.6 part of which was mentioned in the introduction of this chapter.

3.3.1 Metaworlds derived by the interpretations

Throughout this subsection, we let $W \in \mathfrak{W}$. Then since W is a DW for the formal language $(A, T, \sigma, P, C, X, \Gamma)$ as defined in §3.2.1 and §3.2.2, W is a sorted algebra whose type is the M-reduct T_M of T for the set M of the invariable indices (3.2.1) of the algebraic structure $(\tau_\lambda)_{\lambda \in \Lambda}$ of T. Moreover, the set Υ_W of the valuations of X into W is nonempty as was noted in §3.2.3. Therefore, the sorted algebra $(W^{\Upsilon_W}, T_M, \rho)$ called the Υ_W-power algebra of W is constructed as in §3.1.6, that is,

$$W^{\Upsilon_W} = \bigcup_{t \in T} (\Upsilon_W \to W_t)$$

and the algebraic structure $(\beta_\lambda)_{\lambda \in M}$ and the sorting ρ of W^{Υ_W} satisfy the following four equations for each $\lambda \in M$, each $\upsilon \in \Upsilon_W$ and each $t \in T$, where

[3.25] In fact, ϕ in ICL is replaced with $\emptyset$.

ω_λ is the operation of W assigned to λ:

$$\begin{gathered}\operatorname{Dm}\beta_\lambda = \{(\varphi_1,\ldots,\varphi_k) \in \textstyle\bigcup_{n=1}^\infty (W^{\Upsilon_W})^n : (\rho\varphi_1,\ldots,\rho\varphi_k) \in \operatorname{Dm}\tau_\lambda\},\\ \rho(\beta_\lambda(\varphi_1,\ldots,\varphi_k)) = \tau_\lambda(\rho\varphi_1,\ldots,\rho\varphi_k) \text{ for each } (\varphi_1,\ldots,\varphi_k) \in \operatorname{Dm}\beta_\lambda,\\ (\beta_\lambda(\varphi_1,\ldots,\varphi_k))\upsilon = \omega_\lambda(\varphi_1\upsilon,\ldots,\varphi_k\upsilon) \text{ for each } (\varphi_1,\ldots,\varphi_k) \in \operatorname{Dm}\beta_\lambda,\\ \rho^{-1}\{t\} = \Upsilon_W \to W_t. \end{gathered} \tag{3.3.1}$$

The first implies that $\operatorname{Dm}\beta_\lambda \subseteq (W^{\Upsilon_W})^k$ for an arity k of τ_λ (s. [3.8]), and the first two for all $\lambda \in M$ together mean that ρ is an M-holomorphism (s. Remark 3.1.3). The third and (3.3.1) for all $\lambda \in M$ and all $t \in T$ together with Lemma 3.1.3 imply that the υ-projection $\upsilon_p \in W^{\Upsilon_W} \to W$ is a homotypic M-holomorphism. Moreover, $\rho^{-1}\{t\} \neq \emptyset$ for each $t \in \sigma P$ by (3.3.1) and the P-denotability of W. Thus W^{Υ_W} is a DW for A. We call it the **power world** derived from W.[3.26] The index set M of the algebraic structure of W^{Υ_W} is in general smaller than the index set Λ of the algebraic structure of T.

Here we assign each variable index $\lambda \in \Lambda - M$ an operation β_λ on W^{Υ_W} by means of the interpretation I_W of $\Lambda - M$ on W given by the logic system $\mathfrak{L}$, and extending the structure and type of W^{Υ_W} from $(\beta_\lambda)_{\lambda\in M}$ and T_M to $(\beta_\lambda)_{\lambda\in\Lambda}$ and T, we construct a sorted algebra $(W^{\Upsilon_W}, T, \rho)$. It plays the title role here.

The construction takes three steps. First, let $\xi = x_1 \cdots x_n \in X^{\circledast}$ and $(w_1,\ldots,w_n) \in W_{\sigma\xi}$, where $W_{\sigma\xi} = \prod_{i=1}^n W_{\sigma x_i}$ as in (3.2.4). Then for each valuation $\upsilon \in \Upsilon_W$, we define a valuation $\upsilon\left(\dfrac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right) \in \Upsilon_W$ by

$$\left(\upsilon\left(\frac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right)\right)x = \begin{cases} w_i & \text{if } x = x_i \text{ for some } i \in \{1,\ldots,n\},\\ \upsilon x & \text{if } x \in X - \{x_1,\ldots,x_n\}\end{cases} \tag{3.3.2}$$

for each $x \in X$. It is a well-defined element of $X \to W$ because $x_1,\ldots,x_n$ are distinct elements of X. It belongs to Υ_W because $w_i \in W_{\sigma x_i}$ for all $i \in \{1,\ldots,n\}$ and $\upsilon \in \Upsilon_W$ (s. Lemma 3.1.2). Thus we have defined a transformation

$$\upsilon \mapsto \upsilon\left(\frac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right)$$

on Υ_W. We call it the **transvaluation** of $x_1,\ldots,x_n$ by $w_1,\ldots,w_n$ and denote it by $\left(\dfrac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right)$ or by $(x_1,\ldots,x_n/w_1,\ldots,w_n)$ particularly when $n = 1$.

Next, for each triple (φ,υ,ξ) of $\varphi \in W^{\Upsilon_W}$, $\upsilon \in \Upsilon_W$ and $\xi = x_1 \cdots x_n \in X^{\circledast}$, we define a mapping

$$\varphi(\upsilon(\xi/\Box)) \in W_{\sigma\xi} \to W_{\rho\varphi} \tag{3.3.3}$$

by the following for each $(w_1,\ldots,w_n) \in W_{\sigma\xi}$:

$$(\varphi(\upsilon(\xi/\Box)))(w_1,\ldots,w_n) = \varphi\left(\upsilon\left(\frac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right)\right). \tag{3.3.4}$$

[3.26] The power world W^{Υ_W} may be defined even if the DW W does not belong to $\mathfrak{W}$. Moreover, W^{Υ_W} may not belong to $\mathfrak{W}$ even if $W \in \mathfrak{W}$, and likewise for the metaworld $W^\sharp$ below.

Here $\upsilon\left(\dfrac{x_1,\dots,x_n}{w_1,\dots,w_n}\right) \in \Upsilon_W$ has been defined above and $\varphi\left(\upsilon\left(\dfrac{x_1,\dots,x_n}{w_1,\dots,w_n}\right)\right)$ as well as $\varphi\upsilon$ belongs to $W_{\rho\varphi}$ because $\varphi \in \rho^{-1}\{\rho\varphi\} = \Upsilon_W \to W_{\rho\varphi}$ by (3.3.1).

Lastly, we assign each variable index $\lambda \in \Lambda - M$ an operation β_λ on W^{Υ_W}. Similarly to the case $\lambda \in M$, we define

$$\operatorname{Dm}\beta_\lambda = \{(\varphi_1,\dots,\varphi_k) \in \textstyle\bigcup_{n=1}^{\infty}(W^{\Upsilon_W})^n : (\rho\varphi_1,\dots,\rho\varphi_k) \in \operatorname{Dm}\tau_\lambda\}.$$

Then $\operatorname{Dm}\beta_\lambda \subseteq (W^{\Upsilon_W})^k$ for an arity k of τ_λ (s. [3.8]). Let $(\varphi_1,\dots,\varphi_k) \in \operatorname{Dm}\beta_\lambda$. Then $(\rho\varphi_1,\dots,\rho\varphi_k) \in \operatorname{Dm}\tau_\lambda$ and

$$(\varphi_1(\upsilon(\xi_\lambda/\square)),\dots,\varphi_k(\upsilon(\xi_\lambda/\square))) \in \textstyle\prod_{j=1}^{k}(W_{\sigma\xi_\lambda} \to W_{\rho\varphi_j})$$

for each $\upsilon \in \Upsilon_W$ by (3.3.3), and so (3.2.5) and (3.2.6) show that

$$\lambda_W(\varphi_1(\upsilon(\xi_\lambda/\square)),\dots,\varphi_k(\upsilon(\xi_\lambda/\square))) \in W_{\tau_\lambda(\rho\varphi_1,\dots,\rho\varphi_k)}$$

for the significance λ_W of λ on W in the given interpretation I_W of $\Lambda - M$ on W. Therefore, we define $\beta_\lambda(\varphi_1,\dots,\varphi_k) \in \Upsilon_W \to W_{\tau_\lambda(\rho\varphi_1,\dots,\rho\varphi_k)}$ by

$$(\beta_\lambda(\varphi_1,\dots,\varphi_k))\upsilon = \lambda_W(\varphi_1(\upsilon(\xi_\lambda/\square)),\dots,\varphi_k(\upsilon(\xi_\lambda/\square)))$$

for each $\upsilon \in \Upsilon_W$. Then $\beta_\lambda(\varphi_1,\dots,\varphi_k) \in \rho^{-1}\{\tau_\lambda(\rho\varphi_1,\dots,\rho\varphi_k)\}$ by (3.3.1). Thus we have assigned each variable index $\lambda \in \Lambda - M$ the operation β_λ on W^{Υ_W}, which satisfies the following for each $(\varphi_1,\dots,\varphi_k) \in \operatorname{Dm}\beta_\lambda$:

$$\rho(\beta_\lambda(\varphi_1,\dots,\varphi_k)) = \tau_\lambda(\rho\varphi_1,\dots,\rho\varphi_k).$$

This equation and the definition of $\operatorname{Dm}\beta_\lambda$ together mean that ρ is a $(\Lambda - M)$-holomorphism as well as an M-holomorphism.

Thus we have constructed the sorted algebra $(W^{\Upsilon_W}, T, \rho)$. We denote it by $(W^\sharp, T, \rho)$ in order to distinguish it from the power algebra $(W^{\Upsilon_W}, T_M, \rho)$. Then

$$W^\sharp = \textstyle\bigcup_{t\in T}(\Upsilon_W \to W_t) \tag{3.3.5}$$

and its sorting ρ satisfies (3.3.1) for each $t \in T$. Moreover, $W^\sharp$ is a signifiable world for A because its M-reduct is the power world W^{Υ_W}. We call $W^\sharp$ the **metaworld** derived from the DW $W \in \mathfrak{W}$. The algebraic structure $(\beta_\lambda)_{\lambda\in\Lambda}$ of $W^\sharp$ satisfies the following three equations for each $\lambda \in \Lambda$ and each $\upsilon \in \Upsilon_W$:

$$\operatorname{Dm}\beta_\lambda = \{(\varphi_1,\dots,\varphi_k) \in \textstyle\bigcup_{n=1}^{\infty}(W^\sharp)^n : (\rho\varphi_1,\dots,\rho\varphi_k) \in \operatorname{Dm}\tau_\lambda\}, \tag{3.3.6}$$

$$\rho(\beta_\lambda(\varphi_1,\dots,\varphi_k)) = \tau_\lambda(\rho\varphi_1,\dots,\rho\varphi_k), \tag{3.3.7}$$

$$\begin{aligned}&(\beta_\lambda(\varphi_1,\dots,\varphi_k))\upsilon\\ &= \begin{cases}\omega_\lambda(\varphi_1\upsilon,\dots,\varphi_k\upsilon) & \text{if } \lambda \in M,\\ \lambda_W(\varphi_1(\upsilon(\xi_\lambda/\square)),\dots,\varphi_k(\upsilon(\xi_\lambda/\square))) & \text{if } \lambda \in \Lambda - M.\end{cases}\end{aligned} \tag{3.3.8}$$

(3.3.6) implies that $\operatorname{Dm}\beta_\lambda \subseteq (W^\sharp)^k$ for an arity k of τ_λ (s. [3.8]). (3.3.6) and (3.3.7) for all $\lambda \in \Lambda$ together mean that ρ is a Λ-holomorphism. Moreover, they together with (3.3.1) show that the following hold for each $\lambda \in \Lambda$, where the former equation in effect means that $\left(\prod_{j=1}^{k}(\Upsilon_W \to W_{t_j})\right)_{(t_1,\ldots,t_k)\in \operatorname{Dm}\tau_\lambda}$ is a partition of $\operatorname{Dm}\beta_\lambda$ (s. Remark 3.1.5):

$$\begin{aligned}
&\operatorname{Dm}\beta_\lambda = \bigcup_{(t_1,\ldots,t_k)\in \operatorname{Dm}\tau_\lambda} \prod_{j=1}^{k}(\Upsilon_W \to W_{t_j}),\\
&\beta_\lambda(\prod_{j=1}^{k}(\Upsilon_W \to W_{t_j})) \subseteq \Upsilon_W \to W_{\tau_\lambda(t_1,\ldots,t_k)}\\
&\qquad\qquad\qquad\qquad \text{for each } (t_1,\ldots,t_k) \in \operatorname{Dm}\tau_\lambda.
\end{aligned} \tag{3.3.9}$$

Remark 3.3.1 If $\xi = \varepsilon$, that is, if $n = 0$, in (3.3.4), then $\upsilon\left(\dfrac{x_1,\ldots,x_n}{w_1,\ldots,w_n}\right) = \upsilon$ and $W_{\sigma\xi} = \{\emptyset\}$ by [3.23] and so $W_{\sigma\xi} \to W_{\rho\varphi}$ in (3.3.3) may be identified with $W_{\rho\varphi}$ by the $\{\emptyset\}$ convention (s. [3.24]). Therefore, we may regard (3.3.4) as defining $\varphi(\upsilon(\varepsilon/\Box))$ to be the element $\varphi\upsilon$ of $W_{\rho\varphi}$. Thus, using the extended interpretation $(\lambda_W)_{\lambda\in\Lambda}$ of Λ on W defined in Remark 3.2.2, we may unite the two equations in (3.3.8) for $\lambda \in M$ and for $\lambda \in \Lambda - M$ into one for all $\lambda \in \Lambda$:

$$(\beta_\lambda(\varphi_1,\ldots,\varphi_k))\upsilon = \lambda_W(\varphi_1(\upsilon(\xi_\lambda/\Box)),\ldots,\varphi_k(\upsilon(\xi_\lambda/\Box))).$$

Example 3.3.1 In the usual case as in Example 3.2.5 that $\Lambda \subseteq \Gamma^* \cup \Gamma X$ and every variable index λ is unary and satisfies $\operatorname{Dm}\tau_\lambda = \{\phi\}$ for an element $\phi \in T$, we have $\Lambda - M = \coprod_{x\in X}(\Lambda \cap \Gamma x)$, and it follows from (3.3.8) and (3.3.9) that the operation β_λ of the metaworld $W^\sharp$ assigned to $\lambda \in \Lambda \cap \Gamma x$ $(x \in X)$ satisfies

$$\operatorname{Dm}\beta_\lambda = \Upsilon_W \to W_\phi, \qquad \operatorname{Im}\beta_\lambda \subseteq \Upsilon_W \to W_{\tau_\lambda\phi}$$

and the following for each $\varphi \in \Upsilon_W \to W_\phi$ and each $\upsilon \in \Upsilon_W$:

$$(\beta_\lambda\varphi)\upsilon = \lambda_W(\varphi(\upsilon(x/\Box))).$$

Here $\lambda_W \in (W_{\sigma x} \to W_\phi) \to W_{\tau_\lambda\phi}$ by (3.2.8), $\varphi(\upsilon(x/\Box)) \in W_{\sigma x} \to W_\phi$ by (3.3.1) and (3.3.3), and $(\varphi(\upsilon(x/\Box)))w = \varphi(\upsilon(x/w))$ for each $w \in W_{\sigma x}$ by (3.3.4).

In FPL, W_ϕ is the set $\mathbb{T}$ of the truth values and $\tau_\lambda\phi = \phi$. Therefore

$$\operatorname{Dm}\beta_\lambda = \Upsilon_W \to \mathbb{T}, \qquad \operatorname{Im}\beta_\lambda \subseteq \Upsilon_W \to \mathbb{T}.$$

Moreover, $W_{\sigma x}$ is the subset W_ϵ of the entities. Therefore, $\varphi(\upsilon(x/\Box)) \in W_\epsilon \to \mathbb{T}$ for each $\varphi \in \Upsilon_W \to \mathbb{T}$ and each $\upsilon \in \Upsilon_W$. Moreover, $\lambda = \forall x$ or $\exists x$ for quantifiers $\forall$ or $\exists$ in Γ, and $\lambda_W \in (W_\epsilon \to \mathbb{T}) \to \mathbb{T}$ is defined by (3.2.9). Thus, abbreviating β_λ to λ, we have the following for each $\varphi \in \Upsilon_W \to \mathbb{T}$ and each $\upsilon \in \Upsilon_W$:

$$\begin{aligned}
(\forall x\, \varphi)\upsilon &= \inf\{\varphi(\upsilon(x/w)) : w \in W_\epsilon\},\\
(\exists x\, \varphi)\upsilon &= \sup\{\varphi(\upsilon(x/w)) : w \in W_\epsilon\}.
\end{aligned} \tag{3.3.10}$$

In ICL, $\lambda = \triangledown x$ for a token $\triangledown \in \Gamma$, $W_\phi = \mathbb{T}$ and $W_{\tau_\lambda\phi} = W_\epsilon \to \mathbb{T}$ for the subset W_ϵ of the basic entities. Therefore, abbreviating $\beta_{\triangledown x}$ to $\triangledown x$, we have

$$\operatorname{Dm}\triangledown x = \Upsilon_W \to \mathbb{T}, \qquad \operatorname{Im}\triangledown x \subseteq \Upsilon_W \to (W_\epsilon \to \mathbb{T}).$$

Moreover $W_{\sigma x} = W_\epsilon$. Therefore, $\varphi(\upsilon(x/\square)) \in W_\epsilon \to \mathbb{T}$ for each $\varphi \in \Upsilon_W \to \mathbb{T}$ and each $\upsilon \in \Upsilon_W$. Moreover, $\lambda_W \in (W_\epsilon \to \mathbb{T}) \to (W_\epsilon \to \mathbb{T})$ is defined as the identity transformation on $W_\epsilon \to \mathbb{T}$. Thus

$$((\nabla x\, \varphi)\upsilon)w = \varphi(\upsilon(x/w)) \tag{3.3.11}$$

for each $\varphi \in \Upsilon_W \to \mathbb{T}$, each $\upsilon \in \Upsilon_W$ and each $w \in W_\epsilon$. Identifying $W_\epsilon \to \mathbb{T}$ with $\mathfrak{P}(W_\epsilon)$, we have $(\nabla x\, \varphi)\upsilon = \{w \in W_\epsilon : \varphi(\upsilon(x/w)) = 1\}$ (s. [1.55][3.25]).

3.3.2 Full denotations derived by the universality

Throughout this subsection, we let $W \in \mathfrak{W}$ and $\delta \in \Delta_W$, that is, W is a DW for the formal language $(A, T, \sigma, P, C, X, \Gamma)$ and δ is a denotation of C into W both given by the semantics $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$.

In §3.3.1, we have derived the metaworld $(W^\sharp, T, \rho)$ from W by the interpretation I_W. Here we derive a homotypic holomorphism $\delta^\sharp \in A \to W^\sharp$ from δ by the universality of the USA (A, T, σ, P). Then since $W^\sharp = W^{\Upsilon_W}$ as sets by (3.3.5), we may consider the composite $\upsilon_\mathrm{p}\delta^\sharp \in A \to W$ of $\delta^\sharp$ and the projection $\upsilon_\mathrm{p} \in W^{\Upsilon_W} \to W$ by each valuation $\upsilon \in \Upsilon_W$. It plays the title role here.

In view of the partition $P = C \amalg X$, we define a mapping φ_δ of P into $\Upsilon_W \to W$ by the following for each $a \in P$ and each $\upsilon \in \Upsilon_W$ (s. (3.0.2)):

$$(\varphi_\delta a)\upsilon = \begin{cases} \delta a & \text{if } a \in C, \\ \upsilon a & \text{if } a \in X. \end{cases}$$

Then since $\delta(C_t) \subseteq W_t$ and $\upsilon(X_t) \subseteq W_t$ for each $t \in T$, we have $\varphi_\delta(P_t) \subseteq \Upsilon_W \to W_t \subseteq W^\sharp$ by (3.3.5) for each $t \in T$. Therefore, we may regard φ_δ as belonging to $P \to W^\sharp$, and it satisfies $\rho\varphi_\delta = \sigma|_P$ by Lemma 3.1.2 and (3.3.1). Thus the universality of (A, T, σ, P) and Remark 3.1.11 show that φ_δ is *uniquely* extended to a homotypic holomorphism of A into $W^\sharp$. It is denoted $\delta^\sharp$ and called the **metadenotation** of A derived from δ.

Since $\delta^\sharp$ is homotypic, Lemma 3.1.2 and (3.3.1) show that

$$\delta^\sharp a \in \Upsilon_W \to W_{\sigma a} \tag{3.3.12}$$

for each $a \in A$. Since $\delta^\sharp$ is a holomorphism, the algebraic structures $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ of A and $W^\sharp$ satisfy the following equations for each $\lambda \in \Lambda$:

$$\begin{aligned} &\mathrm{Dm}\,\alpha_\lambda = \{(a_1, \ldots, a_k) \in \textstyle\bigcup_{n=1}^\infty A^n : (\delta^\sharp a_1, \ldots, \delta^\sharp a_k) \in \mathrm{Dm}\,\beta_\lambda\}, \\ &\delta^\sharp(\alpha_\lambda(a_1, \ldots, a_k)) = \beta_\lambda(\delta^\sharp a_1, \ldots, \delta^\sharp a_k) \\ &\qquad\qquad\qquad\qquad \text{for each } (a_1, \ldots, a_k) \in \mathrm{Dm}\,\alpha_\lambda. \end{aligned} \tag{3.3.13}$$

The former implies that $\mathrm{Dm}\,\alpha_\lambda \subseteq A^k$ for an arity k of β_λ (s. [3.8]).

Now let $\upsilon \in \Upsilon_W$. Then, since $W^\sharp = W^{\Upsilon_W}$ as sets, we may consider the composite $\varphi_\upsilon^\delta \in A \to W$ of $\delta^\sharp \in A \to W^\sharp$ and the υ-projection $\upsilon_\mathrm{p} \in W^{\Upsilon_W} \to W$:

$$\varphi_\upsilon^\delta = \upsilon_\mathrm{p}\delta^\sharp. \tag{3.3.14}$$

It is called the (full) **denotation** of A derived from (δ, υ). It plays the principal role in GL and MN in that it is intended for the mapping $\varphi_{\delta,\upsilon}$ considered around (3.0.1) and for a model of a PERCEPTION as explained in §1.2.6 and moreover it yields a logic space on part of A as in §3.3.3 and functional expressions of elements of A as in §3.3.6. The remainder of this subsection deals with its values

$$\varphi^{\delta}_{\upsilon} a = (\delta^{\sharp} a)\upsilon \qquad\qquad (a \in A). \tag{3.3.15}$$

First, (3.3.12) shows that the following holds for each $a \in A$:

$$\varphi^{\delta}_{\upsilon} a \in W_{\sigma a}. \tag{3.3.16}$$

Thus $\varphi^{\delta}_{\upsilon}$ is homotypic by Lemma 3.1.2. Since $\varphi^{\delta}_{\upsilon} = \upsilon_{\mathrm{p}}\delta^{\sharp}$, this may also be derived from the homotypicity of $\delta^{\sharp}$ and υ_{p}.

Next, since $\delta^{\sharp}$ is an extension of φ_{δ}, the following holds for each $a \in P$:

$$(\delta^{\sharp} a)\upsilon = \begin{cases} \delta a & \text{if } a \in C, \\ \upsilon a & \text{if } a \in X. \end{cases} \tag{3.3.17}$$

Because of (3.3.15), this means

$$\varphi^{\delta}_{\upsilon}|_{C} = \delta, \qquad\qquad \varphi^{\delta}_{\upsilon}|_{X} = \upsilon. \tag{3.3.18}$$

Since $P = C \amalg X$, the united mapping $\delta \cup \upsilon \in P \to W$ is defined to be equal to δ and υ on C and X respectively. Thus

$$\varphi^{\delta}_{\upsilon}|_{P} = \delta \cup \upsilon. \tag{3.3.19}$$

Therefore assume $a \in A - P$. Then a has a ramification $a = \alpha_{\lambda}(a_1, \ldots, a_k)$ for some $\lambda \in \Lambda$ and $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_{\lambda}$ by Theorems 3.1.6 and 3.1.7, and so

$$\delta^{\sharp}(\alpha_{\lambda}(a_1, \ldots, a_k)) = \beta_{\lambda}(\delta^{\sharp} a_1, \ldots, \delta^{\sharp} a_k)$$

by (3.3.13). Remark 3.3.1 shows that the following holds for the significance λ_W of λ on W under the extended interpretation $(\lambda_W)_{\lambda\in\Lambda}$ of Λ on W:

$$(\delta^{\sharp}(\alpha_{\lambda}(a_1, \ldots, a_k)))\upsilon = \lambda_W((\delta^{\sharp} a_1)(\upsilon(\xi_{\lambda}/\square)), \ldots, (\delta^{\sharp} a_k)(\upsilon(\xi_{\lambda}/\square))). \tag{3.3.20}$$

Therefore, if λ belongs to the set M of the invariable indices (3.2.1) of the algebraic structure of T, then

$$(\delta^{\sharp}(\alpha_{\lambda}(a_1, \ldots, a_k)))\upsilon = \omega_{\lambda}((\delta^{\sharp} a_1)\upsilon, \ldots, (\delta^{\sharp} a_k)\upsilon)$$

for the operation ω_{λ} of W assigned to λ, that is,

$$\varphi^{\delta}_{\upsilon}(\alpha_{\lambda}(a_1, \ldots, a_k)) = \omega_{\lambda}(\varphi^{\delta}_{\upsilon} a_1, \ldots, \varphi^{\delta}_{\upsilon} a_k). \tag{3.3.21}$$

Thus $\varphi^{\delta}_{\upsilon}$ is an M-homomorphism. Since $\varphi^{\delta}_{\upsilon} = \upsilon_{\mathrm{p}}\delta^{\sharp}$ and $M \subseteq \Lambda$, this may also be derived from the homomorphy of $\delta^{\sharp}$ and υ_{p} by Remark 3.1.4.

We cannot relate the right-hand side of (3.3.20) in general to φ^δ_υ, and so $\delta^\sharp$ plays the principal role instead of φ^δ_υ in part of GL. However, we can do so in FPL and ICL, for example. The counterparts of (3.3.20) in them may be derived from (3.3.10) and (3.3.11) in Example 3.3.1 and (3.3.13), that is,

$$\begin{aligned}
&(\delta^\sharp(\forall x\, a))\upsilon = (\forall x\,(\delta^\sharp a))\upsilon = \inf\{(\delta^\sharp a)(\upsilon(x/w)) : w \in W_\epsilon\},\\
&(\delta^\sharp(\exists x\, a))\upsilon = (\exists x\,(\delta^\sharp a))\upsilon = \sup\{(\delta^\sharp a)(\upsilon(x/w)) : w \in W_\epsilon\},\\
&((\delta^\sharp(\triangledown x\, a))\upsilon)w = ((\triangledown x\,(\delta^\sharp a))\upsilon)w = (\delta^\sharp a)(\upsilon(x/w))
\end{aligned}$$

for each $a \in A_\phi$, each $\upsilon \in \Upsilon_W$ and each $w \in W_\epsilon$ (s. [1.55][3.25]). Thus

$$\begin{aligned}
&\varphi^\delta_\upsilon(\forall x\, a) = \inf\{\varphi^\delta_{\upsilon(x/w)} a : w \in W_\epsilon\},\\
&\varphi^\delta_\upsilon(\exists x\, a) = \sup\{\varphi^\delta_{\upsilon(x/w)} a : w \in W_\epsilon\},\\
&(\varphi^\delta_\upsilon(\triangledown x\, a))w = \varphi^\delta_{\upsilon(x/w)} a.
\end{aligned}$$

Identifying $W_\epsilon \to \mathbb{T}$ with $\mathfrak{P}(W_\epsilon)$ for the last equation, we have

$$\varphi^\delta_\upsilon(\triangledown x\, a) = \{w \in W_\epsilon : \varphi^\delta_{\upsilon(x/w)} a = 1\} \tag{3.3.22}$$

as was illustrated by (1.2.3) (s. §4.1.3).

Remark 3.3.2 The following diagram illustrates the process of constructing φ^δ_υ, although the 3D diagram with the sorting π of W added is desirable.

$$\begin{array}{ccccc}
C, X & \xrightarrow{i_C,\, i_X} & P & \xrightarrow{j} & A\\
{}^{\delta}\!\downarrow\ \downarrow^{\upsilon} & & {}^{\varphi_\delta}\!\downarrow & \swarrow_{\delta^\sharp} & \downarrow^{\sigma}\\
W & \xleftarrow[\upsilon_p]{\quad} & W^\sharp & \xrightarrow[\rho]{} & T
\end{array}$$

Here i_C, i_X and j denote the inclusion mappings of C, X and P respectively. We defined φ_δ so that $\upsilon_p\varphi_\delta|_C = \delta$ and $\upsilon_p\varphi_\delta|_X = \upsilon$. Therefore, $\upsilon_p\varphi_\delta i_C = \delta$ and $\upsilon_p\varphi_\delta i_X = \upsilon$, that is, the left square is commutative for δ and υ. Moreover, $\rho = \pi\upsilon_p$, $\pi\delta = \sigma|_C$ and $\pi\upsilon = \sigma|_X$ by Lemma 3.1.2. Therefore, $\rho\varphi_\delta|_C = \rho\varphi_\delta i_C = \pi\upsilon_p\varphi_\delta i_C = \pi\delta = \sigma|_C$ and $\rho\varphi_\delta|_X = \rho\varphi_\delta i_X = \pi\upsilon_p\varphi_\delta i_X = \pi\upsilon = \sigma|_X$. Thus $\rho\varphi_\delta = \sigma|_P$, that is, the right square is commutative, and $\delta^\sharp$ is its triangulating homomorphism (s. (3.1.5)). Uniting υ_p to $\delta^\sharp$, we obtain φ^δ_υ.

3.3.3 Logic spaces for logic systems with a truth type

Here we link the logic system $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ to a logic space under the following condition.

- There exists an element $\phi \in T$ for which the ϕ-part A_ϕ of A is nonempty and the ϕ-part W_ϕ of each DW $W \in \mathfrak{W}$ is a nontrivial bounded lattice.[3.27]

[3.27]The lattice operations $\wedge$ and $\vee$ of W_ϕ are not assumed here to be the restrictions of operations in the algebraic structure of W as a DW for A, but so are they in practice.

We call ϕ a **truth type**[3.28] and refer to the elements of A_ϕ and W_ϕ as the **ϕ-declarative**s and the **ϕ-truth value**s respectively.

Let $\mathcal{D}$ be the set of the triples (W, δ, υ) of $W \in \mathfrak{W}$, $\delta \in \Delta_W$ and $\upsilon \in \Upsilon_W$. Then for each such triple, the denotation φ^δ_υ defined by (3.3.14) is a homotypism (and an M-holomorphism) of A into W. In particular, it satisfies $\varphi^\delta_\upsilon A_\phi \subseteq W_\phi$, and so we may regard its restriction

$$\varphi^\delta_{\upsilon,\phi} = \varphi^\delta_\upsilon|_{A_\phi} \tag{3.3.23}$$

to A_ϕ as belonging to $A_\phi \to W_\phi$. Therefore, defining

$$\mathcal{F}_W = \{\varphi^\delta_{\upsilon,\phi} : (\delta, \upsilon) \in \Delta_W \times \Upsilon_W\} \tag{3.3.24}$$

for each $W \in \mathfrak{W}$, we obtain a W_ϕ-valued functional logic space $(A_\phi, \mathcal{F}_W)$. Let $(A_\phi, \mathfrak{B}_W)$ be its associated logic space, and define $\mathfrak{B} = \bigcup_{W\in\mathfrak{W}} \mathfrak{B}_W$, that is, $\mathfrak{B}$ consists of the following subsets $A_{\phi,W,\delta,\upsilon,\mathfrak{b}}$ of A_ϕ for the quadruples $(W, \delta, \upsilon, \mathfrak{b})$ consisting of $(W, \delta, \upsilon) \in \mathcal{D}$ and $\mathfrak{b} \in W_\phi$:

$$A_{\phi,W,\delta,\upsilon,\mathfrak{b}} = \{a \in A_\phi : \varphi^\delta_{\upsilon,\phi} a \geq \mathfrak{b}\}. \tag{3.3.25}$$

Then $(A_\phi, \mathfrak{B})$ is a logic space. We call it the **ϕ-logic space** derived from the logic system $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$.

In particular, if W_ϕ is equal to a lattice $\mathbb{B}$ for all $W \in \mathfrak{W}$, then defining

$$\mathcal{F} = \bigcup_{W\in\mathfrak{W}} \mathcal{F}_W = \{\varphi^\delta_{\upsilon,\phi} : (W, \delta, \upsilon) \in \mathcal{D}\}, \tag{3.3.26}$$

we obtain a $\mathbb{B}$-valued functional logic space $(A_\phi, \mathcal{F})$, whose associated logic space is equal to $(A_\phi, \mathfrak{B})$. We call $(A_\phi, \mathcal{F})$ the **ϕ-functional logic space** derived from the logic system.

Remark 3.3.3 (The functional logic space in CL) In CL as well as FPL (s. §3.4), there exists a truth type ϕ such that $W_\phi = \mathbb{T}$ for all $W \in \mathfrak{W}$, and so we obtain the ϕ-functional logic space $(A_\phi, \mathcal{F})$. In CL, however, there also exist truth types ψ such that W_ψ is a complete lattice $S_\psi \to \mathbb{T}$ for some set S_ψ varying with ψ, and they together yield a $\mathbb{T}$-valued functional logic space $(H, \mathcal{G})$ which is an extension of, and more worth studying than $(A_\phi, \mathcal{F})$.

Example 3.3.2 (Generalized PL as a degenerate logic system) Here we consider the logic system $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ whose syntax $(T, \sigma|_P, P, C, X, \Gamma)$ satisfies the following three degeneracy conditions.

(1) The type T is a singleton, say $\{\phi\}$, and a total algebra.

(2) The index set Λ of the structure of T is contained in Γ^*.

(3) $C = \emptyset$, that is, $P = X$.

[3.28] A truth type was simply called a *truth* in the introduction of [1.5].

The condition (1) means that (A, P) is a UTA as was noted in Remark 3.1.12. The condition (2) means $M = \Lambda$ as to (3.2.1). Therefore, the DWs for A are the nonempty total algebras similar to A as was noted in Remark 3.1.7 and, since $\Lambda - M = \emptyset$, we have nothing to do with the interpretations I_W of $\Lambda - M$ on $W \in \mathfrak{W}$. The condition (3) means that $\emptyset$ is the unique denotation of C into W and so $\Delta_W = \{\emptyset\}$ for each $W \in \mathfrak{W}$.

Thus the logic system satisfying (1)–(3) may be identified with the triple $(A, P, \mathfrak{W})$ for a UTA (A, P) with $P \neq \emptyset$ (s. [3.22]) and a nonempty collection $\mathfrak{W}$ of nonempty total algebras similar to A. It may look general in some lights, but it is degenerate in light of GL (general logic).

Let $W \in \mathfrak{W}$. Then $W = W_\phi$ and $P = X = X_\phi$ by (1) and (3), and so $\Upsilon_W = P \rightarrow W$. If $\upsilon \in P \rightarrow W$, then the denotation $\varphi^\emptyset_\upsilon \in A \rightarrow W$ is a homomorphism, and $\varphi^\emptyset_\upsilon|_P = \upsilon$ by (3.3.18) and (3). Conversely, if $f \in A \rightarrow W$ is a homomorphism, then its restriction $\upsilon = f|_P$ belongs to $P \rightarrow W$, and since (A, P) is a UTA, υ is uniquely extended to a homomorphism of A into W, hence $f = \varphi^\emptyset_\upsilon$. Thus $\{\varphi^\emptyset_\upsilon : \upsilon \in P \rightarrow W\}$ is equal to the set of the homomorphisms of A into W.

Assume that the logic system $(A, P, \mathfrak{W})$ has a truth type. Since $T = \{\phi\}$ and $A \neq \emptyset$, it means that each member $W \in \mathfrak{W}$ is a nontrivial bounded lattice (s. [3.27]). Since $T = \{\phi\}$ and $\Delta_W = \{\emptyset\}$, the set $\mathcal{F}_W$ defined by (3.3.24) is equal to the set $\{\varphi^\emptyset_\upsilon : \upsilon \in P \rightarrow W\}$ of the homomorphisms of A into W.

In particular, if $\mathfrak{W}$ consists of a single nontrivial bounded lattice $\mathbb{B}$, then the set $\mathcal{F}$ defined by (3.3.26) is equal to the set of the homomorphisms of A into $\mathbb{B}$, and since $T = \{\phi\}$, $(A, \mathcal{F})$ is the ϕ-functional logic space derived from the degenerate logic system $(A, P, \mathfrak{W})$.

A typical example is the logic system on PL (hence the title of this example), where (A, P) is a UTA with respect to operations $x \wedge y$, $x \vee y$, $x^\diamondsuit$ and $x \Rightarrow y$ on A and $\mathfrak{W}$ consists only of the binary lattice $\mathbb{T}$ whose algebraic structure as a DW for A consists of the Boolean operations $\wedge, \vee, \diamondsuit$ and $\Rightarrow$ (s. [1.79][2.18]).

3.3.4 Regular extensions of logic systems and logic spaces

Let $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W \in \mathfrak{W}}, (\Delta_W)_{W \in \mathfrak{W}})$ be a logic system with a truth type ϕ. Then, in studying the ϕ-logic space derived from it in §3.3.3, we sometimes need its extensions defined below (s. Remark 3.5.2).

Let $(A', T', \sigma', P', C', X', \Gamma', \mathfrak{W}', (I'_{W'})_{W' \in \mathfrak{W}'}, (\Delta'_{W'})_{W' \in \mathfrak{W}'})$ be another logic system with a truth type ϕ'. Then the latter is called a **regular extension** of the former provided that they satisfy the following eight conditions (1)–(8), where Λ and Λ' are the index sets of the algebraic structures of T and T', and $M = \Lambda \cap \Gamma^*$ and $M' = \Lambda' \cap \Gamma'^*$ are the sets of the invariable indices of the structures of T and T'. The first four conditions concern their syntaxes $(T, \sigma|_P, P, C, X, \Gamma)$ and $(T', \sigma'|_{P'}, P', C', X', \Gamma')$.

(1) $C \subseteq C'$, $X \subseteq X'$ and $\Gamma \subseteq \Gamma'$.

(2) $\Lambda \subseteq \Lambda'$.

(3) T is equal to the Λ-reduct T'_Λ of T'.

(4) A is a Λ-subreduct of A' and $\sigma|_P = \sigma'|_P$.

Some remarks are in order before stating the remaining conditions (5)–(8).

Remark 3.3.4 The condition (3) and the former half of (4) make sense because of (2). The latter half of (4) makes sense because $P = C \cup X \subseteq C' \cup X' = P'$ by (1) and $T = T'$ as sets by (3) and so both $\sigma|_P$ and $\sigma'|_P$ belong to $P{\to}T$. Moreover, (1)–(4) have the following five consequences among others.

(a) $\sigma P \subseteq \sigma' P'$ and $\sigma = \sigma'|_A$.

(b) $M \subseteq M'$ (and so (c) and (d) make sense).

(c) $T_M = (T'_{M'})_M$.

(d) If W' is a DW for A', then its M-reduct W'_M is a DW for A with respect to the sorting of W', and $\Upsilon_{W'_M} = \{\upsilon'|_X : \upsilon' \in \Upsilon_{W'}\}$ as to the valuations of X and X' into W'_M and W' respectively, and likewise for the denotations of C and C' into W'_M and W' respectively.

(e) $\Lambda - M \subseteq \Lambda' - M'$.

The former half of (a) holds because $\sigma P = \sigma' P$ by (4) and $P \subseteq P'$ by (1). The latter half of (a) follows from Lemma 3.1.4, because σ and $\sigma'|_A$ are homomorphisms of $A = [P]$ into T and $\sigma|_P = \sigma'|_P = (\sigma'|_A)|_P$ by (3), (4) and Remark 3.1.4. The condition (b) holds because of (1) and (2). The condition (c) holds because $T_M = (T'_\Lambda)_M = T'_M = (T'_{M'})_M$ by (3) and (b). Let W' be a DW for A'. Then since W' is a sorted algebra of type $T'_{M'}$, W'_M is a sorted algebra of type T_M with respect to the sorting of W' by Remark 3.1.6 and (c). Since W' satisfies the P'-denotability, W'_M satisfies the P-denotability by (a). Thus the former conclusion of (d) holds. As to the latter, $T = T'$ as sets by (3), and if $t \in T$, then $X_t \subseteq X'_t$ by (1) and (4), and $W'_t = (W'_M)_t$ because W' and W'_M have the same sorting. Therefore if $\upsilon' \in \Upsilon_{W'}$, then $\upsilon'(X_t) \subseteq \upsilon'(X'_t) \subseteq W'_t = (W'_M)_t$ for each $t \in T$, and so $\upsilon'|_X \in \Upsilon_{W'_M}$. Conversely if $\upsilon \in \Upsilon_{W'_M}$, then $\upsilon(X_t) \subseteq (W'_M)_t = W'_t$ for each $t \in T'$ and the P'-denotability of W' shows that $W'_t \neq \emptyset$ whenever $X_t \subset X'_t$, and so we can extend υ to an element of $\Upsilon_{W'}$. The same argument applies to the denotations of C and C' into W'_M and W' respectively. Since $\Lambda \subseteq (\Gamma \amalg X)^*$, $\Lambda - M$ consists of the elements of Λ which are products of elements of Γ and at least one element of X. Likewise, $\Lambda' - M'$ consists of the elements of Λ' which are products of elements of Γ' and at least one element of X'. Thus (e) holds by (1) and (2).

Remark 3.3.5 We can extend any formal language $(A, T, \sigma, P, C, X, \Gamma)$ to plenty of formal languages $(A', T', \sigma', P', C', X', \Gamma')$ satisfying (1)–(4) by the following five-step process.

- Pick a set Q and a mapping $\rho \in Q{\to}T$ such that $P \cap Q = \emptyset$. Decompose Q into a disjoint union $D \cup Y$. Define $P' = P \cup Q$, $C' = C \cup D$ and $X' = X \cup Y$. Define $\tau' \in P'{\to}T$ by $\tau'|_P = \sigma|_P$ and $\tau'|_Q = \rho$.
  Then $P \subseteq P'$, $C \subseteq C'$, $X \subseteq X'$, $P' = C' \amalg X'$ and $X' \neq \emptyset$.

- Pick a set Γ' such that $\Gamma \subseteq \Gamma'$.
  Then $M \subseteq \Gamma'^*$ and $\Lambda - M \subseteq (\Gamma' \amalg X')^* - \Gamma'^*$ similarly to (e).

- Pick sets M' and N' such that $M \subseteq M' \subseteq \Gamma'^*$ and $\Lambda - M \subseteq N' \subseteq (\Gamma' \amalg X')^* - \Gamma'^*$. Define $\Lambda' = M' \cup N'$.
  Then $\Lambda \subseteq \Lambda' \subseteq (\Gamma' \amalg X')^*$, $\Lambda' \cap \Gamma'^* = M'$ and $\Lambda' - M' = N'$.

- Assign each element $\lambda \in \Lambda' - \Lambda$ an operation τ_λ on T. Let T' be the algebra obtained by extending the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ of T to $(\tau_\lambda)_{\lambda\in\Lambda'}$. Then T is equal to the Λ-reduct T'_Λ of T'.

- Let (A', T', σ', P') be the USA of syntax (T', τ', P'). Define $B = [P]_{A'_\Lambda}$. Then $(B, T, \sigma'|_B, P)$ is a USA by Corollary 3.1.7.3 and may be identified with (A, T, σ, P) by Theorem 3.1.5 because $(\sigma'|_B)|_P = \sigma'|_P = (\sigma'|_{P'})|_P = \tau'|_P = \sigma|_P$ and so they have the same syntax.

Thus $(A', T', \sigma', P', C', X', \Gamma')$ is a formal language satisfying (1)–(4).

The conditions (5)–(8) on the logic systems can now be stated by virtue of the conditions (1)–(4) and their consequences (d)–(e). They concern the semantics $(\mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ and $(\mathfrak{W}', (I'_{W'})_{W'\in\mathfrak{W}'}, (\Delta'_{W'})_{W'\in\mathfrak{W}'})$ and the truth types ϕ and ϕ' of the logic systems.

(5) $\mathfrak{W} = \{W'_M : W' \in \mathfrak{W}'\}$.

(6) $\lambda_{W'_M} = \lambda_{W'}$ for each $\lambda \in \Lambda - M$ and each $W' \in \mathfrak{W}'$.

(7) $\Delta_{W'_M} = \{\delta'|_C : \delta' \in \Delta'_{W'}\}$ for each $W' \in \mathfrak{W}'$.

(8) $\phi = \phi'$.

Remark 3.3.6 The condition (5) is based on (d), and so for each $W' \in \mathfrak{W}'$, $W'_M \in \mathfrak{W}$ is implicitly supposed to have the same sorting as W'.

Denote W'_M in (6) by W. Then $W \in \mathfrak{W}$ by (5), and λ_W is the significance of λ on W under the given interpretation I_W of $\Lambda - M$. Likewise, $\lambda \in \Lambda' - M'$ by (e), and $\lambda_{W'}$ is the significance of λ on W' under the given interpretation $I'_{W'}$ of $\Lambda' - M'$. Assume $\lambda \in \Lambda_\xi$ ($\xi = x_1 \cdots x_n \in X^{\circledast} - \{\varepsilon\}$) in view of (3.2.2) for A. Then λ_W belongs to the set

$$(\textstyle\bigcup_{(t_1,\ldots,t_k)\in \mathrm{Dm}\,\tau_\lambda} \prod_{j=1}^{k}(W_{\sigma\xi} \to W_{t_j})) \to W$$

by (3.2.5). Here $W_{\sigma\xi} = \prod_{i=1}^{n} W_{\sigma x_i}$ by (3.2.4) and τ_λ is the variable operation of T assigned to λ. Likewise, $\lambda_{W'}$ belongs to the above set by (3.2.5), and so (6) makes sense. Indeed, $\lambda \in \Lambda' - M'$ by (e), $\lambda \in \Lambda'_\xi$ ($\xi = x_1 \cdots x_n \in X'^{\circledast} - \{\varepsilon\}$) as to (3.2.2) for A' by (1) and (2), $T = T'$ as sets and τ_λ is also the variable operation of T' assigned to λ by (3). Moreover, $\sigma x_i = \sigma' x_i$ for each $i \in \{1, \ldots, n\}$ by (4), $W = W'$ as sets and W has the same sorting as W' by (5). Thus $W_{\sigma\xi} = \prod_{i=1}^{n} W'_{\sigma' x_i} = W'_{\sigma'\xi}$ and $W_{t_j} = W'_{t_j}$ for each $j \in \{1, \ldots, k\}$.

The condition (7) is based on (d). In particular if Δ_W and $\Delta'_{W'}$ consist of all denotations of C and C' into W and W' respectively, then (7) holds by (d).

Suppose $C = C'$ and $M = M'$. Then (5) means $\mathfrak{W} = \mathfrak{W}'$, and (6) and (7) mean $I_W = (I'_W)|_{\Lambda - M}$ and $\Delta_W = \Delta'_W$ respectively for each $W \in \mathfrak{W}$.

Finally, the conditions (5)–(7) give us a process of extending the semantics, and together with the process of extending the syntaxes given in Remark 3.3.5, it shows that any logic system has plenty of regular extensions.

Remark 3.3.6 completes the definition of the concept of regular extensions of logic systems with truth types. Our purpose here is to prove the following.

Theorem 3.3.1 Assume that $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W \in \mathfrak{W}}, (\Delta_W)_{W \in \mathfrak{W}})$ and $(A', T', \sigma', P', C', X', \Gamma', \mathfrak{W}', (I'_{W'})_{W' \in \mathfrak{W}'}, (\Delta'_{W'})_{W' \in \mathfrak{W}'})$ are logic systems with truth types ϕ and ϕ' respectively and that the latter is a regular extension of the former. Then the ϕ'-logic space $(A'_{\phi'}, \mathfrak{B}')$ is an extension of the ϕ-logic space $(A_\phi, \mathfrak{B})$.

Proof Since $A_\phi \subseteq A'_{\phi'}$ by (4), (a) and (8), it suffices to prove the following.

(f) Assume that triples (W, δ, υ) and (W', δ', υ') for $W \in \mathfrak{W}$, $\delta \in \Delta_W$, $\upsilon \in \Upsilon_W$, $W' \in \mathfrak{W}'$, $\delta' \in \Delta'_{W'}$ and $\upsilon' \in \Upsilon_{W'}$ satisfy $W = W'_M$, $\delta = \delta'|_C$ and $\upsilon = \upsilon'|_X$. Then $W_\phi = W'_{\phi'}$ and $\varphi^{\delta}_{\upsilon,\phi} = \varphi^{\delta'}_{\upsilon',\phi'}|_{A_\phi}$ as to (3.3.23).

Indeed, its conclusion implies that $A_{\phi,W,\delta,\upsilon,b} = A'_{\phi',W',\delta',\upsilon',b} \cap A_\phi$ for each $b \in W_\phi = W'_{\phi'}$ as to (3.3.25). Thus $\mathfrak{B} = \mathfrak{B}' \cap A_\phi$ by (5), (7) and (d).

The proof of (f) is as follows. Since W and W' have the same sorting as was noted in Remark 3.3.6, $W_\phi = W'_{\phi'}$ by (8).

Associating each $\nu' \in \Upsilon_{W'}$ with $\nu'|_X$ is a mapping of $\Upsilon_{W'}$ onto Υ_W by (d), and so Remark 3.1.9 shows that there exists a homotypic holomorphism f of the power algebra W^{Υ_W} into the power algebra $W^{\Upsilon_{W'}}$ which satisfies the following for each $\varphi \in W^{\Upsilon_W}$ and each $\nu' \in \Upsilon_{W'}$:

$$\varphi(\nu'|_X) = (f\varphi)\nu'. \tag{$*$}$$

The definition of the metaworld $W^\sharp$ in §3.3.1 shows that W^{Υ_W} is the M-reduct of $W^\sharp$. Since $W = W'_M$, Remark 3.1.10 shows that $W^{\Upsilon_{W'}}$ is the M-reduct of the power algebra $W'^{\Upsilon_{W'}}$ which is the M'-reduct of the metaworld $W'^\sharp$. Therefore, $W^{\Upsilon_{W'}}$ is the M-reduct of $W'^\sharp$. Thus f is a homotypic M-holomorphism of $W^\sharp$ into $W'^\sharp$ by Remark 3.1.4.

Let $a \in P$. Then the following holds for each $\nu' \in \Upsilon_{W'}$ by ($*$) and (3.3.17):

$$(f(\delta^\sharp a))\nu' = (\delta^\sharp a)(\nu'|_X) = \begin{cases} \delta a = \delta' a = (\delta'^\sharp a)\nu' & \text{if } a \in C, \\ (\nu'|_X)a = \nu' a = (\delta'^\sharp a)\nu' & \text{if } a \in X. \end{cases}$$

This means $(f\delta^\sharp)|_P = \delta'^\sharp|_P$. Therefore, if f is a $(\Lambda - M)$-homomorphism of $W^\sharp$ into $W'^\sharp$, then $f\delta^\sharp = \delta'^\sharp|_A$ by Lemma 3.1.4, because both are Λ-homomorphisms of $A = [P]_\Lambda$ into $W'^\sharp$ by (3.3.13) and Remark 3.1.4. Moreover, $(\delta^\sharp a)(\nu'|_X) = (f(\delta^\sharp a))\nu' = (\delta'^\sharp a)\nu'$ for each $a \in A$ and $\nu' \in \Upsilon_{W'}$. In particular $\varphi^\delta_{\upsilon,\phi} a =$

$\varphi^{\delta'}_{\upsilon',\phi'}a$ for each $a \in A_\phi$, and therefore $\varphi^{\delta}_{\upsilon,\phi} = \varphi^{\delta'}_{\upsilon',\phi'}|_{A_\phi}$. Thus it suffices to prove that f is a $(\Lambda - M)$-homomorphism of $W^\sharp$ into $W'^\sharp$.

Let $\lambda \in \Lambda - M$ and assume $\lambda \in \Lambda_\xi$ $(\xi = x_1 \cdots x_n \in X^\circledast - \{\varepsilon\})$ in view of (3.2.2) for A. Then $\lambda \in \Lambda'_\xi$ $(\xi = x_1 \cdots x_n \in X'^\circledast - \{\varepsilon\})$ and $W_{\sigma\xi} = W'_{\sigma'\xi}$ as to (3.2.2) for A' as was shown in Remark 3.3.6. Furthermore, we find that the following holds for each $\nu' \in \Upsilon_{W'}$ and each $(w_1, \ldots, w_n) \in W_{\sigma\xi}$:

$$(\nu'|_X)\left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right) = \left(\nu'\left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)\right)\Big|_X,$$

where $\left(\dfrac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)$ on the left-hand side denotes the transvaluation on Υ_W, while it on the right-hand side denotes the transvaluation on $\Upsilon_{W'}$. Therefore,

$$\varphi\left((\nu'|_X)\left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)\right) = (f\varphi)\left(\nu'\left(\frac{x_1, \ldots, x_n}{w_1, \ldots, w_n}\right)\right)$$

for each $\varphi \in W^\sharp$ by $(*)$. Moreover, $W_{\rho\varphi} = W'_{\rho'(f\varphi)}$ for the sortings ρ and ρ' of $W^\sharp$ and $W'^\sharp$, because W and W' have the same sorting and $\rho' f = \rho$ by Lemmas 3.1.2. Therefore, (3.3.3) and (3.3.4) show that

$$\varphi((\nu'|_X)(\xi/\Box)) = (f\varphi)(\nu'(\xi/\Box)). \qquad (\star)$$

for each $\varphi \in W^\sharp$ and each $\nu' \in \Upsilon_{W'}$. Let β_λ and β'_λ be the operations of $W^\sharp$ and $W'^\sharp$ assigned to λ, and let $(\varphi_1, \ldots, \varphi_k) \in \mathrm{Dm}\,\beta_\lambda$. Then $(\rho\varphi_1, \ldots, \rho\varphi_k) = (\rho'(f\varphi_1), \ldots, \rho'(f\varphi_k))$, and so $(f\varphi_1, \ldots, f\varphi_k) \in \mathrm{Dm}\,\beta'_\lambda$ by Lemma 3.1.1, and the following holds for each $\nu' \in \Upsilon_{W'}$:

$$\begin{aligned}
&(f(\beta_\lambda(\varphi_1, \ldots, \varphi_k)))\nu' \\
&= (\beta_\lambda(\varphi_1, \ldots, \varphi_k))(\nu'|_X) && \text{(by } (*)) \\
&= \lambda_W(\varphi_1((\nu'|_X)(\xi/\Box)), \ldots, \varphi_k((\nu'|_X)(\xi/\Box))) && \text{(by (3.3.8))} \\
&= \lambda_W((f\varphi_1)(\nu'(\xi/\Box)), \ldots, (f\varphi_k)(\nu'(\xi/\Box))) && \text{(by } (\star)) \\
&= \lambda_{W'}((f\varphi_1)(\nu'(\xi/\Box)), \ldots, (f\varphi_k)(\nu'(\xi/\Box))) && \text{(by (6))} \\
&= (\beta'_\lambda(f\varphi_1, \ldots, f\varphi_k))\nu'. && \text{(by (3.3.8))}
\end{aligned}$$

Therefore, $f(\beta_\lambda(\varphi_1, \ldots, \varphi_k)) = \beta'_\lambda(f\varphi_1, \ldots, f\varphi_k)$. Thus f is a $(\Lambda - M)$-homomorphism of $W^\sharp$ into $W'^\sharp$.

3.3.5 Denotations vs transvaluations and substitutions

Throughout this subsection, we let $W \in \mathfrak{W}$ and $\delta \in \Delta_W$. Here we prove two theorems on the denotations $\varphi^\delta_\upsilon \in A \to W$ for the valuations $\upsilon \in \Upsilon_W$. The former theorem is necessary for the proof of the latter.

In the proofs, we let $(\alpha_\lambda)_{\lambda\in\Lambda}$, $(\beta_\lambda)_{\lambda\in\Lambda}$ and $(\omega_\lambda)_{\lambda\in M}$ be the algebraic structures of A, $W^\sharp$ and W and let $(\lambda_W)_{\lambda\in\Lambda}$ be the interpretation of Λ on W given by I_W and then extended as in Remark 3.2.2.

We also apply the concepts and results in §3.1.9 and §3.1.10 to A. We are allowed to do so, because (A, T, σ, P) is a USA by definition, (A, P) is a based algebra by Theorem 3.1.6 and $(\alpha_\lambda)_{\lambda\in\Lambda}$ satisfies $\Lambda \subseteq (\Gamma \amalg X)^*$ which is the basic assumption (3.1.9) of the subsections. In particular, the following holds as to (3.1.10) for $\lambda \in \Lambda$:

$$X^\lambda = \{x_1, \ldots, x_m\} \text{ for the variable qualifier } x_1 \cdots x_m \text{ in } \lambda. \tag{3.3.27}$$

Here if $m = 0$, then $x_1 \cdots x_m = \varepsilon$ and $\{x_1, \ldots, x_m\} = \emptyset$.

The former theorem concerns (3.1.11) and transvaluations. In particular, it shows that if $x \in X$ does not occur free in $a \in A$ then $\varphi^\delta_\upsilon a = \varphi^\delta_{\upsilon(x/w)} a$ for all $w \in W_{\sigma x}$. Obviously, it implies that if a satisfies $X^a_{\text{fr}} = \emptyset$ then $\varphi^\delta_\upsilon a$ does not depend on $\upsilon \in \Upsilon_W$.

Theorem 3.3.2 Let $a \in A$ and assume that valuations $\upsilon, \upsilon' \in \Upsilon_W$ satisfy $\upsilon x = \upsilon' x$ for all $x \in X^a_{\text{fr}}$. Then $\varphi^\delta_\upsilon a = \varphi^\delta_{\upsilon'} a$, that is, $(\delta^\sharp a)\upsilon = (\delta^\sharp a)\upsilon'$.

Proof We argue by induction on $r = \text{rk}\, a$. Assume $r = 0$, i.e. $a \in P = C \cup X$. If $a \in C$, then $(\delta^\sharp a)\upsilon = \delta a = (\delta^\sharp a)\upsilon'$ by (3.3.17). If $a \in X$, then since $a \in X^a_{\text{fr}}$ by (3.1.12), we have $(\delta^\sharp a)\upsilon = \upsilon a = \upsilon' a = (\delta^\sharp a)\upsilon'$ by (3.3.17) and our assumption.

Therefore, assume $r \geq 1$, i.e. $a \in A - P$. Then Theorem 3.1.7 shows that its ramification $a = \alpha_\lambda(a_1, \ldots, a_k)$ satisfies $r - 1 = \sum_{j=1}^k \text{rk}\, a_j$. Let $\xi = x_1 \cdots x_m$ be the variable qualifier in λ. Then

$$(\delta^\sharp a)\upsilon = \lambda_W((\delta^\sharp a_1)(\upsilon(\xi/\Box)), \ldots, (\delta^\sharp a_k)(\upsilon(\xi/\Box)))$$

by (3.3.20), and likewise for υ'. Therefore, it suffices to prove

$$(\delta^\sharp a_j)(\upsilon(\xi/\Box)) = (\delta^\sharp a_j)(\upsilon'(\xi/\Box))$$

for all $j \in \{1, \ldots, k\}$. By definition, (3.3.3) and (3.3.4), this equation means that the following holds for each $(w_1, \ldots, w_m) \in W_{\sigma\xi} = \prod_{i=1}^m W_{\sigma x_i}$:

$$(\delta^\sharp a_j)\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right) = (\delta^\sharp a_j)\left(\upsilon'\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right).$$

This follows from the induction hypothesis, because

$$\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)x = \left(\upsilon'\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)x$$

for all $x \in X^{a_j}_{\text{fr}}$. Indeed, if $x \in \{x_1, \ldots, x_m\}$, this holds by (3.3.2). If $x \notin \{x_1, \ldots, x_m\}$, then $x \notin X^\lambda$ by (3.3.27), and so $x \in X^a_{\text{fr}}$ by Lemma 3.1.12, and so $\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)x = \upsilon x = \upsilon' x = \left(\upsilon'\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)x$ by (3.3.2) and our assumption.

The latter theorem in addition concerns Definition 3.1.2 and substitutions.

Theorem 3.3.3 Let $y_1, \ldots, y_n$ be distinct variables of A, and $c_1, \ldots, c_n$ and a be elements of A ($n \geq 0$). Assume that $\sigma y_i = \sigma c_i$ and y_i is free from c_i in a for each $i \in \{1, \ldots, n\}$. Then

$$\varphi^\delta_\upsilon(a(y_1, \ldots, y_n/c_1, \ldots, c_n)) = \varphi^\delta_{\upsilon(y_1,\ldots,y_n/\varphi^\delta_\upsilon c_1,\ldots,\varphi^\delta_\upsilon c_n)} a$$

for each $\upsilon \in \Upsilon_W$, that is,

$$\left(\delta^\sharp \left(a \left(\frac{y_1, \ldots, y_n}{c_1, \ldots, c_n}\right)\right)\right)\upsilon = (\delta^\sharp a)\left(\upsilon\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon, \ldots, (\delta^\sharp c_n)\upsilon}\right)\right).$$

Proof Define $b = a\left(\frac{y_1, \ldots, y_n}{c_1, \ldots, c_n}\right)$ and $u = \upsilon\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon, \ldots, (\delta^\sharp c_n)\upsilon}\right)$. Our goal is to prove $(\delta^\sharp b)\upsilon = (\delta^\sharp a)u$. We argue by induction on n and $r = \operatorname{rk} a$ as in the definition of the substitution $\left(\frac{y_1, \ldots, y_n}{c_1, \ldots, c_n}\right)$.

If $n = 0$, then $b = a$ and $u = \upsilon$, and $(\delta^\sharp b)\upsilon = (\delta^\sharp a)u$ obviously holds. Therefore assume $n \geq 1$.

Furthermore assume $r = 0$, i.e. $a \in P = C \cup X$. If $a \in C$, then $a \notin \{y_1, \ldots, y_n\}$, and so $(\delta^\sharp b)\upsilon = (\delta^\sharp a)\upsilon = \delta a = (\delta^\sharp a)u$ by (3.1.14) and (3.3.17). If $a = y_i$ for some $i \in \{1, \ldots, n\}$, then $(\delta^\sharp b)\upsilon = (\delta^\sharp c_i)\upsilon = uy_i = ua = (\delta^\sharp a)u$ by (3.1.14), (3.3.2) and (3.3.17). If $a \in X - \{y_1, \ldots, y_n\}$, then similarly $(\delta^\sharp b)\upsilon = (\delta^\sharp a)\upsilon = \upsilon a = ua = (\delta^\sharp a)u$.

Therefore, assume $r \geq 1$, i.e. $a \in A - P$. Then its ramification

$$a = \alpha_\lambda(a_1, \ldots, a_k)$$

satisfies $r - 1 = \sum_{j=1}^k \operatorname{rk} a_j$ by Theorem 3.1.7.

Suppose $y_i \not\preccurlyeq_{\mathrm{fr}} a$, i.e. $y_i \notin X^a_{\mathrm{fr}}$ for some $i \in \{1, \ldots, n\}$. Then

$$(\delta^\sharp b)\upsilon = \left(\delta^\sharp \left(a\left(\frac{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n}{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n}\right)\right)\right)\upsilon$$

by Lemma 3.1.16, and

$$(\delta^\sharp a)u = (\delta^\sharp a)\left(\upsilon\left(\frac{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n}{(\delta^\sharp c_1)\upsilon, \ldots, (\delta^\sharp c_{i-1})\upsilon, (\delta^\sharp c_{i+1})\upsilon, \ldots, (\delta^\sharp c_n)\upsilon}\right)\right).$$

by Theorem 3.3.2. Therefore, $(\delta^\sharp b)\upsilon = (\delta^\sharp a)u$ by the induction hypothesis.

Therefore, assume that $y_i \in X^a_{\mathrm{fr}}$ for all $i \in \{1, \ldots, n\}$. Then

$$X^\lambda \cap \{y_1, \ldots, y_n\} = \emptyset$$

by Lemma 3.1.12. Therefore, defining

$$b_j = a_j\left(\frac{y_1, \ldots, y_n}{c_1, \ldots, c_n}\right)$$

for each $j \in \{1, \ldots, k\}$, we have

$$b = \alpha_\lambda(b_1, \ldots, b_k)$$

by (3.1.15). Furthermore, since y_i is free from c_i in $a = \alpha_\lambda(a_1, \ldots, a_k)$ by assumption and $y_i \notin X^\lambda$ for each $i \in \{1, \ldots, n\}$, y_i is free from c_i in a_j for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, k\}$ by Lemma 3.1.14.

Let $\xi = x_1 \cdots x_m$ be the variable qualifier in λ. Then since $b = \alpha_\lambda(b_1, \ldots, b_k)$ and $a = \alpha_\lambda(a_1, \ldots, a_k)$, we have

$$(\delta^\sharp b)\upsilon = \lambda_W((\delta^\sharp b_1)(\upsilon(\xi/\Box)), \ldots, (\delta^\sharp b_k)(\upsilon(\xi/\Box))),$$
$$(\delta^\sharp a)u = \lambda_W((\delta^\sharp a_1)(u(\xi/\Box)), \ldots, (\delta^\sharp a_k)(u(\xi/\Box)))$$

by (3.3.20). Therefore, it suffice to prove

$$(\delta^\sharp b_j)(\upsilon(\xi/\Box)) = (\delta^\sharp a_j)(u(\xi/\Box))$$

for each $j \in \{1, \ldots, k\}$. By definition, (3.3.3) and (3.3.4), this means that

$$(\delta^\sharp b_j)\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right) = (\delta^\sharp a_j)\left(u\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)$$

for each $(w_1, \ldots, w_m) \in W_{\sigma\xi} = \prod_{i=1}^m W_{\sigma x_i}$. Its proof is as follows.

We are assuming that $y_i \in X^a_{\mathrm{fr}}$ for all $i \in \{1, \ldots, n\}$ and hence it has followed that y_i is free from c_i in a_j for each $i \in \{1, \ldots, n\}$. Furthermore, $b_j = a_j\left(\frac{y_1, \ldots, y_n}{c_1, \ldots, c_n}\right)$. Therefore, defining $\upsilon' = \upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)$, we have

$$\begin{aligned}(\delta^\sharp b_j)\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right) &= (\delta^\sharp b_j)\upsilon' \\ &= (\delta^\sharp a_j)\left(\upsilon'\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon', \ldots, (\delta^\sharp c_n)\upsilon'}\right)\right)\end{aligned}$$

by the induction hypothesis. Furthermore, since y_i is free from c_i in $a = \alpha_\lambda(a_1, \ldots, a_k)$ and $y_i \in X^a_{\mathrm{fr}}$ for each $i \in \{1, \ldots, n\}$ by assumption, $(X^\lambda)^{c_i}_{\mathrm{fr}} = \emptyset$ for each $i \in \{1, \ldots, n\}$ by Lemma 3.1.13 and so, since $X^\lambda = \{x_1, \ldots, x_m\}$ by (3.3.27), $x_h \notin X^{c_i}_{\mathrm{fr}}$ for each $h \in \{1, \ldots, m\}$ and each $i \in \{1, \ldots, n\}$. Therefore, $(\delta^\sharp c_i)\upsilon' = (\delta^\sharp c_i)\upsilon$ for each $i \in \{1, \ldots, n\}$ by Theorem 3.3.2. Since $y_i \in X^a_{\mathrm{fr}}$ for each $i \in \{1, \ldots, n\}$, it has also followed that $X^\lambda \cap \{y_1, \ldots, y_n\} = \emptyset$, that is, $\{x_1, \ldots, x_m\} \cap \{y_1, \ldots, y_n\} = \emptyset$. Therefore,

$$\begin{aligned}\upsilon'\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon', \ldots, (\delta^\sharp c_n)\upsilon'}\right) &= \left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon, \ldots, (\delta^\sharp c_n)\upsilon}\right) \\ &= \left(\upsilon\left(\frac{y_1, \ldots, y_n}{(\delta^\sharp c_1)\upsilon, \ldots, (\delta^\sharp c_n)\upsilon}\right)\right)\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right) = u\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right).\end{aligned}$$

Thus $(\delta^\sharp b_j)\left(\upsilon\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right) = (\delta^\sharp a_j)\left(u\left(\frac{x_1, \ldots, x_m}{w_1, \ldots, w_m}\right)\right)$ as desired.

3.3.6 Functional expressions and denotable functions

Throughout this subsection, we also let $W \in \mathfrak{W}$ and $\delta \in \Delta_W$. Our purpose here is to show that δ assigns each element of A a function on W. Then a function on W is said to be **δ-denotable** if δ assigns it to an element of A.

Let $a \in A$. Then since X^a is a finite set by Lemma 3.1.11 and $X^a_{\text{fr}} \subseteq X^a$, there exist a tuple $(x_1, \dots, x_n)$ of *distinct* variables of A which satisfy

$$X^a_{\text{fr}} \subseteq \{x_1, \dots, x_n\}. \tag{3.3.28}$$

We call it a (free variable) **basis** of a. Each element of A has at least two bases unless X is a singleton. Any finite elements of A have a common basis.

We assign a a function

$$a^\delta(x_1, \dots, x_n) \in (\textstyle\prod_{i=1}^n W_{\sigma x_i}) \to W_{\sigma a},$$

and call it the **functional expression** of a under δ with respect to the basis $(x_1, \dots, x_n)$ of a. Its definition is as follows.

Let $(w_1, \dots, w_n) \in \prod_{i=1}^n W_{\sigma x_i}$. Then since $x_1, \dots, x_n$ are distinct, there exists a valuation $\upsilon \in \Upsilon_W$ which satisfies $\upsilon x_i = w_i$ for each $i \in \{1, \dots, n\}$. Although such $\upsilon \in \Upsilon_W$ may not be unique, Theorem 3.3.2 shows that $\varphi^\delta_\upsilon a$ does not depend on the choice of such $\upsilon \in \Upsilon_W$. Moreover, $\varphi^\delta_\upsilon a \in W_{\sigma a}$ by (3.3.16). Therefore, we define the value of $a^\delta(x_1, \dots, x_n)$ at $(w_1, \dots, w_n)$ by

$$(a^\delta(x_1, \dots, x_n))(w_1, \dots, w_n) = \varphi^\delta_\upsilon a$$

with such $\upsilon \in \Upsilon_W$, and abbreviate the left-hand side to $a^\delta(w_1, \dots, w_n)$.

Suppose $n = 0$ in the above definition. Then (3.3.28) means $X^a_{\text{fr}} = \emptyset$. Moreover, $\prod_{i=1}^n W_{\sigma x_i} = \{\emptyset\}$ by [3.23], and therefore we identify $(\prod_{i=1}^n W_{\sigma x_i}) \to W_{\sigma a}$ with $W_{\sigma a}$. Then $a^\delta(x_1, \dots, x_n)$ is an element of $W_{\sigma a}$.

Now that $a^\delta(x_1, \dots, x_n)$ is defined, every valuation $\upsilon \in \Upsilon_W$ gives its values $a^\delta(w_1, \dots, w_n)$ for all $(w_1, \dots, w_n) \in \prod_{i=1}^n W_{\sigma x_i}$ by the equation

$$\begin{aligned} a^\delta(w_1, \dots, w_n) &= \varphi^\delta_{\upsilon(x_1, \dots, x_n / w_1, \dots, w_n)} a \\ &= (\delta^\sharp a)\left(\upsilon\left(\frac{x_1, \dots, x_n}{w_1, \dots, w_n}\right)\right). \end{aligned} \tag{3.3.29}$$

Therefore, since $\upsilon\left(\dfrac{x_1, \dots, x_n}{\upsilon x_1, \dots, \upsilon x_n}\right) = \upsilon$,

$$a^\delta(\upsilon x_1, \dots, \upsilon x_n) = \varphi^\delta_\upsilon a = (\delta^\sharp a)\upsilon.$$

These two equations together show that two elements a and b of A satisfy $\delta^\sharp a = \delta^\sharp b$ iff $a^\delta(x_1, \dots, x_n) = b^\delta(x_1, \dots, x_n)$ for any/some of their common basis $(x_1, \dots, x_n)$. Thus the metadenotation $\delta^\sharp$ may be regarded as categorizing the elements of A according to their functional expressions.

Example 3.3.3 Suppose $a \in C$. Then since $X^a_{\text{fr}} = \emptyset$ by (3.1.12), an arbitrary tuple $(x_1, \dots, x_n)$ of distinct variables is a basis of a, and so a has a functional expression $a^\delta(x_1, \dots, x_n) \in (\prod_{i=1}^n W_{\sigma x_i}) \to W_{\sigma a}$. It is of constant value δa by (3.3.18) and (3.3.29).

Suppose $x \in X$. Then since $X^x_{\text{fr}} = \{x\}$ by (3.1.12), an arbitrary tuple $(x_1, \dots, x_n)$ of distinct variables such that $x \in \{x_1, \dots, x_n\}$ is a basis of x, and so if $x = x_i$, x has a functional expression $x^\delta(x_1, \dots, x_n) \in (\prod_{j=1}^n W_{\sigma x_j}) \to W_{\sigma x_i}$. It is the i-projection $(w_1, \dots, w_n) \mapsto w_i$ (s. §1.5.2) by (3.3.18) and (3.3.29).

Functional expressions help us understand or explain the intention of the definition of the semantics $(\mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ in specific branches of logic (s. §3.4 on FPL and Chapters 4–6 on CL).

Since δ-denotable functions are pertinent to the incompleteness theorem 3.5.1, it seems important to know about δ-denotable functions. Example 3.3.3 has given the basic δ-denotable functions. The following theorem shows how other δ-denotable functions are generated by the basic ones.

Theorem 3.3.4 Let $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\omega_\lambda)_{\lambda\in M}$ be the algebraic structures of A and $W \in \mathfrak{W}$, and extend I_W to an interpretation $(\lambda_W)_{\lambda\in\Lambda}$ of Λ on W as in Remark 3.2.2. Assume that an element $a \in A$ has a ramification $a = \alpha_\lambda(a_1, \dots, a_k)$ and let $(y_1, \dots, y_n)$ be a basis of a. Let $\xi = x_1 \cdots x_m$ be the variable qualifier in λ and define

$$\{y_1, \dots, y_n\} - \{x_1, \dots, x_m\} = \{y_{i_1}, \dots, y_{i_\nu}\}, \quad (i_1 < \cdots < i_\nu, 0 \leq \nu \leq n).$$

Then $(y_{i_1}, \dots, y_{i_\nu}, x_1, \dots, x_m)$ is a basis of a_j for all $j \in \{1, \dots, k\}$, and the functional expressions $a^\delta(y_1, \dots, y_n)$ and $a_j^\delta(y_{i_1}, \dots, y_{i_\nu}, x_1, \dots, x_m)$ of a and a_j $(j = 1, \dots, k)$ satisfy the following for each $(w_1, \dots, w_n) \in \prod_{i=1}^n W_{\sigma y_i}$:

$$\begin{aligned} &a^\delta(w_1, \dots, w_n) \\ &= \lambda_W(a_1^\delta(w_{i_1}, \dots, w_{i_\nu}, \square_1, \dots, \square_m), \dots, a_k^\delta(w_{i_1}, \dots, w_{i_\nu}, \square_1, \dots, \square_m)), \end{aligned}$$

where $a_j^\delta(w_{i_1}, \dots, w_{i_\nu}, \square_1, \dots, \square_m)$ is the mapping of $W_{\sigma\xi} = \prod_{i=1}^m W_{\sigma x_i}$ into $W_{\sigma a_j}$ defined by the following for each $(v_1, \dots, v_m) \in \prod_{i=1}^m W_{\sigma x_i}$:

$$(a_j^\delta(w_{i_1}, \dots, w_{i_\nu}, \square_1, \dots, \square_m))(v_1, \dots, v_m) = a_j^\delta(w_{i_1}, \dots, w_{i_\nu}, v_1, \dots, v_m).$$

Remark 3.3.7 This implies that if $\lambda \in M$ then $(y_1, \dots, y_n)$ is also a basis of a_j for all $j \in \{1, \dots, k\}$, and the functional expressions $a^\delta(y_1, \dots, y_n)$ and $a_j^\delta(y_1, \dots, y_n)$ of a and a_j $(j = 1, \dots, k)$ satisfy the following for each $(w_1, \dots, w_n) \in \prod_{i=1}^n W_{\sigma y_i}$:

$$a^\delta(w_1, \dots, w_n) = \omega_\lambda(a_1^\delta(w_1, \dots, w_n), \dots, a_k^\delta(w_1, \dots, w_n)).$$

Proof Since $X^\lambda = \{x_1, \dots, x_m\}$ by (3.3.27), $X^{a_j}_{\text{fr}} \subseteq \{y_1, \dots, y_n, x_1, \dots, x_m\}$ by Lemma 3.1.12, and so $(y_{i_1}, \dots, y_{i_\nu}, x_1, \dots, x_m)$ is a basis of a_j for all $j \in$

$\{1, \ldots, k\}$. Let $\upsilon \in \Upsilon_W$ and define $\upsilon' = \upsilon \left(\dfrac{y_1, \ldots, y_n}{w_1, \ldots, w_n} \right)$. Then (3.3.20) and (3.3.29) show that the following holds:

$$a^\delta(w_1, \ldots, w_n) = (\delta^\sharp a)\upsilon' = \lambda_W((\delta^\sharp a_1)(\upsilon'(\xi/\Box)), \ldots, (\delta^\sharp a_k)(\upsilon'(\xi/\Box))).$$

Therefore, it suffices to show that

$$(\delta^\sharp a_j)(\upsilon'(\xi/\Box)) = a_j^\delta(w_{i_1}, \ldots, w_{i_\nu}, \Box_1, \ldots, \Box_m)$$

for each $j \in \{1, \ldots, k\}$. Its proof is as follows.

First, $(\delta^\sharp a_j)(\upsilon'(\xi/\Box)) \in W_{\sigma\xi} \to W_{\sigma a_j}$ by (3.3.3), (3.3.1) and (3.3.12). Next, (3.3.4) shows that the following holds for each $(v_1, \ldots, v_m) \in W_{\sigma\xi}$:

$$((\delta^\sharp a_j)(\upsilon'(\xi/\Box)))(v_1, \ldots, v_m) = (\delta^\sharp a_j) \left(\upsilon' \left(\frac{x_1, \ldots, x_m}{v_1, \ldots, v_m} \right) \right).$$

This is equal to $a_j^\delta(w_{i_1}, \ldots, w_{i_\nu}, v_1, \ldots, v_m)$ by (3.3.29), because

$$\begin{aligned} \left(\upsilon' \left(\frac{x_1, \ldots, x_m}{v_1, \ldots, v_m} \right) \right) &= \left(\upsilon \left(\frac{y_1, \ldots, y_n}{w_1, \ldots, w_n} \right) \right) \left(\frac{x_1, \ldots, x_m}{v_1, \ldots, v_m} \right) \\ &= \upsilon \left(\frac{y_{i_1}, \ldots, y_{i_\nu}, x_1, \ldots, x_m}{w_{i_1}, \ldots, w_{i_\nu}, v_1, \ldots, v_m} \right). \end{aligned}$$

Thus $(\delta^\sharp a_j)(\upsilon'(\xi/\Box)) = a_j^\delta(w_{i_1}, \ldots, w_{i_\nu}, \Box_1, \ldots, \Box_m)$ as desired.

Example 3.3.4 Suppose W is the self-world for A as defined in Example 3.2.2 and the significance λ_W of each $\lambda \in \Lambda - M$ is defined by (3.2.7). Assume that δ is the inclusion mapping of C into W. Let $a \in A$ and let $(y_1, \ldots, y_n)$ be a basis of a. Then the functional expression $a^\delta(y_1, \ldots, y_n)$ of a satisfies

$$a^\delta(w_1, \ldots, w_n) = a \left(\frac{y_1, \ldots, y_n}{w_1, \ldots, w_n} \right)$$

for each $(w_1, \ldots, w_n) \in \prod_{i=1}^n W_{\sigma y_i}$. The following proof is due to Takaoka.

We argue by induction on $r = \operatorname{rk} a$. Assume $r = 0$, i.e. $a \in P = C \cup X$. If $a \in C$, then $a^\delta(w_1, \ldots, w_n) = \delta a = a = a \left(\dfrac{y_1, \ldots, y_n}{w_1, \ldots, w_n} \right)$ by Example 3.3.3, our assumption on δ and (3.1.14). If $a \in X$, then $a = y_i$ for some $i \in \{1, \ldots, n\}$, and $a^\delta(w_1, \ldots, w_n) = w_i = a \left(\dfrac{y_1, \ldots, y_n}{w_1, \ldots, w_n} \right)$ by Example 3.3.3 and (3.1.14).

Therefore assume $r \geq 1$, i.e. $a \in A - P$. Then Theorem 3.1.7 shows that its ramification $a = \alpha_\lambda(a_1, \ldots, a_k)$ satisfies $r - 1 = \sum_{j=1}^k \operatorname{rk} a_j$. Let $x_1 \cdots x_m$ be the variable qualifier in λ and define $\{y_1, \ldots, y_n\} - \{x_1, \ldots, x_m\} = \{y_{i_1}, \ldots, y_{i_\nu}\}$ $(i_1 < \cdots < i_\nu, 0 \leq \nu \leq n)$. Then

$$\begin{aligned} &a^\delta(w_1, \ldots, w_n) \\ &= \lambda_W(a_1^\delta(w_{i_1}, \ldots, w_{i_\nu}, \Box_1, \ldots, \Box_m), \ldots, a_k^\delta(w_{i_1}, \ldots, w_{i_\nu}, \Box_1, \ldots, \Box_m)) \end{aligned}$$

$$= \alpha_\lambda(a_1^\delta(w_{i_1},\ldots,w_{i_\nu},x_1,\ldots,x_m),\ldots,a_k^\delta(w_{i_1},\ldots,w_{i_\nu},x_1,\ldots,x_m))$$
$$= \alpha_\lambda\left(a_1\left(\frac{y_{i_1},\ldots,y_{i_\nu},x_1,\ldots,x_m}{w_{i_1},\ldots,w_{i_\nu},x_1,\ldots,x_m}\right),\ldots,a_k\left(\frac{y_{i_1},\ldots,y_{i_\nu},x_1,\ldots,x_m}{w_{i_1},\ldots,w_{i_\nu},x_1,\ldots,x_m}\right)\right)$$
$$= \alpha_\lambda\left(a_1\left(\frac{y_{i_1},\ldots,y_{i_\nu}}{w_{i_1},\ldots,w_{i_\nu}}\right),\ldots,a_k\left(\frac{y_{i_1},\ldots,y_{i_\nu}}{w_{i_1},\ldots,w_{i_\nu}}\right)\right)$$
$$= a\left(\frac{y_1,\ldots,y_n}{w_1,\ldots,w_n}\right)$$

by Theorem 3.3.4, (3.2.7), the induction hypothesis, Lemma 3.1.17 and (3.1.15), because $X^\lambda = \{x_1,\ldots,x_m\}$ by (3.3.27).

3.4 First-order predicate logic as a prototype

The purpose of this section is to illustrate the GL in §3.1–3.3 by means of its earliest prototype FPL. In particular, FPL is reformulated in terms of the GL, and the formulation will be used for embedding FPL in ICL (s. §1.3.2).

FPL is defined as a logic system $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ whose syntax $(T, \sigma|_P, P, C, X, \Gamma)$ and semantics $(\mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ satisfy the following conditions (1)–(3) and (4)–(6) respectively. I refer to them as the FPL syntax and FPL semantics.

The definition of the FPL syntax The FPL syntax $(T, \sigma|_P, P, C, X, \Gamma)$ is characterized by the following conditions (1)–(3) on Γ, T and $\sigma|_P$.

(1) The set Γ of tokens is the direct union $\Phi \amalg \Pi \amalg \{\wedge, \vee, \Diamond, \Rightarrow\} \amalg \{\forall, \exists\}$ of a set Φ of function symbols, a nonempty set Π of predicate symbols, the set $\{\wedge, \vee, \Diamond, \Rightarrow\}$ of the Boolean symbols (s. [1.79]) and the set $\{\forall, \exists\}$ of quantifiers.

(2) The type T is a set $\{\epsilon, \phi\}$ of distinct elements ϵ and ϕ equipped with an algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$, which is indexed by the subset $\Lambda = \Phi \cup \Pi \cup \{\wedge, \vee, \Diamond, \Rightarrow\} \cup \{\forall x, \exists x : x \in X\}$ of $(\Gamma \amalg X)^*$ and consists of the operations defined by the following, where τ_λ is abbreviated to λ, f and p are arbitrary elements of Φ and Π respectively and k denotes the arities of τ_f and τ_p:

$$\begin{aligned}
&\mathrm{Dm}\, f = \{(\overbrace{\epsilon,\ldots,\epsilon}^{\text{k-tuple}})\}, && f(\overbrace{\epsilon,\ldots,\epsilon}^{\text{k-tuple}}) = \epsilon,\\
&\mathrm{Dm}\, p = \{(\overbrace{\epsilon,\ldots,\epsilon}^{\text{k-tuple}})\}, && p(\overbrace{\epsilon,\ldots,\epsilon}^{\text{k-tuple}}) = \phi,\\
&\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow = \{(\phi,\phi)\}, && \phi\wedge\phi = \phi\vee\phi = \phi\Rightarrow\phi = \phi,\\
&\mathrm{Dm}\Diamond = \mathrm{Dm}\,\forall x = \mathrm{Dm}\,\exists x = \{\phi\}, && \phi^\Diamond = \forall x\phi = \exists x\phi = \phi.
\end{aligned}$$

Thus the binary Boolean operations $\wedge$, $\vee$ and $\Rightarrow$ are interpositions, the unary Boolean operation $\Diamond$ is superscript and the others are prepositive.

(3) The basic sorting $\sigma|_P$ satisfies $\sigma a = \epsilon$ for all $a \in P$.

Theorem 3.1.5 shows that the FPL syntax $(\mathsf{T}, \sigma|_{\mathsf{P}}, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ yields a formal language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ and it is unique up to homotypic isomorphism extending id_{P}. I call it the FPL language.

The definition of the FPL semantics Before stating the conditions (4)–(6) on the FPL semantics $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$, remarks are in order on implications of the conditions (1)–(3) on the FPL syntax.

As to the collection $\mathfrak{W}$ of DWs for A, the conditions (1) and (2) show that $\Lambda \subseteq \Gamma \cup \Gamma\mathsf{X}$, and so the set (3.2.1) of the invariable indices satisfies $\mathsf{M} = \Lambda \cap \Gamma = \Phi \cup \Pi \cup \{\wedge, \vee, \lozenge, \Rrightarrow\}$. Thus the conditions (2) and (3) together with Remark 3.2.1 show that an algebra W is a DW for A iff it is the disjoint union $W_\epsilon \cup W_\phi$ of its subsets $W_\epsilon \neq \emptyset$ and W_ϕ and its structure consists of the operations ω_λ $(\lambda \in \mathsf{M})$ which satisfy the following, where ω_λ is abbreviated to λ, and f and p are arbitrary elements of Φ and Π respectively:

$$\begin{array}{ll}
\mathrm{Dm}\,\mathsf{f} = {W_\epsilon}^{\mathsf{k}} \text{ for the arity } \mathsf{k} \text{ of } \tau_{\mathsf{f}}, & \mathrm{Im}\,\mathsf{f} \subseteq W_\epsilon, \\
\mathrm{Dm}\,\mathsf{p} = {W_\epsilon}^{\mathsf{k}} \text{ for the arity } \mathsf{k} \text{ of } \tau_{\mathsf{p}}, & \mathrm{Im}\,\mathsf{p} \subseteq W_\phi, \\
\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rrightarrow = W_\phi \times W_\phi, & \mathrm{Im}\wedge \cup \mathrm{Im}\vee \cup \mathrm{Im}\Rrightarrow \subseteq W_\phi, \\
\mathrm{Dm}\,\lozenge = W_\phi, & \mathrm{Im}\,\lozenge \subseteq W_\phi.
\end{array}$$

The first line shows that W_ϵ is a Φ-subreduct of W. The second line implies $W_\phi \neq \emptyset$ because $\Pi \neq \emptyset$ and $W_\epsilon \neq \emptyset$. By virtue of the last two lines, we may regard $\wedge, \vee, \lozenge$ and $\Rrightarrow$ as total operations on W_ϕ.

As to the family $(\mathrm{I}_W)_{W\in\mathfrak{W}}$ of interpretations I_W on $W \in \mathfrak{W}$ of the set $\Lambda - \mathsf{M}$ of the variable indices, we have $\Lambda - \mathsf{M} = \Lambda \cap \Gamma\mathsf{X} = \coprod_{\mathsf{x}\in\mathsf{X}}\{\forall\mathsf{x}, \exists\mathsf{x}\}$, as was noted in Example 3.2.1. Therefore, each $\lambda \in \Lambda - \mathsf{M}$ satisfies $\mathrm{Dm}\,\tau_\lambda = \{\phi\}$ and $\tau_\lambda\phi = \phi$ by the condition (2). Moreover, $\sigma\mathsf{x} = \epsilon$ for each $\mathsf{x} \in \mathsf{X}$ by the condition (3). Thus (3.2.8) shows that the significance of λ on $W \in \mathfrak{W}$ under the interpretation I_W can be an arbitrary mapping $\lambda_W \in (W_\epsilon \to W_\phi) \to W_\phi$.

The set Δ_W of denotations of C into $W \in \mathfrak{W}$ can be an arbitrary nonempty subset of $\mathsf{C} \to W_\epsilon$, because $\mathsf{C} = \mathsf{C}_\epsilon$ by the condition (3) (likewise, the set Υ_W of the valuations of X into W is equal to $\mathsf{X} \to W_\epsilon$).

The conditions on $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ can now be stated.

(4) $\mathfrak{W}$ consists of the DWs W for A such that W_ϕ is the binary lattice $\mathbb{T}$ and the operations $\wedge, \vee, \lozenge$ and $\Rrightarrow$ of W regarded as total operations on W_ϕ are the meet, join, complement and cojoin respectively on $\mathbb{T}$ (s. §1.5.2).

(5) The significance $\lambda_W \in (W_\epsilon \to W_\phi) \to W_\phi$ of each variable index $\lambda \in \{\forall\mathsf{x}, \exists\mathsf{x}\}$ $(\mathsf{x} \in \mathsf{X})$ on each $W \in \mathfrak{W}$ under the interpretation I_W is defined by the following for each $\mathsf{f} \in W_\epsilon \to W_\phi$, as was noted in Example 3.2.5:

$$\lambda_W \mathsf{f} = \begin{cases} \inf\{\mathsf{f}w : w \in W_\epsilon\} & \text{if } \lambda = \forall\mathsf{x}, \\ \sup\{\mathsf{f}w : w \in W_\epsilon\} & \text{if } \lambda = \exists\mathsf{x}, \end{cases}$$

where inf and sup are taken with respect to the usual order on $W_\phi = \mathbb{T}$.

(6) $\Delta_W = \mathsf{C} \to W_\epsilon$ for each $W \in \mathfrak{W}$.

The FPL language The structure of the FPL language A may be clarified by means of the concepts in §3.1, especially Theorems 3.1.6 and 3.1.7.

First of all, since (A, T, σ) is a sorted algebra, the condition (2) together with Remark 3.1.5 means that A is the disjoint union $A_\epsilon \cup A_\phi$ of its ϵ-part A_ϵ and ϕ-part A_ϕ and that its algebraic structure consists of the operations α_λ $(\lambda \in \Lambda)$ which satisfy the following, where α_λ is abbreviated to λ, and f and p are arbitrary elements of Φ and Π respectively:

$$\begin{array}{ll} \mathrm{Dm}\, f = {A_\epsilon}^k \text{ for the arity } k \text{ of } \tau_f, & \mathrm{Im}\, f \subseteq A_\epsilon, \\ \mathrm{Dm}\, p = {A_\epsilon}^k \text{ for the arity } k \text{ of } \tau_p, & \mathrm{Im}\, p \subseteq A_\phi, \\ \mathrm{Dm} \wedge = \mathrm{Dm} \vee = \mathrm{Dm} \Rrightarrow = A_\phi \times A_\phi, & \mathrm{Im} \wedge \cup \mathrm{Im} \vee \cup \mathrm{Im} \Rrightarrow \subseteq A_\phi, \\ \mathrm{Dm} \Diamond = \mathrm{Dm}\, \forall x = \mathrm{Dm}\, \exists x = A_\phi, & \mathrm{Im} \Diamond \cup \mathrm{Im}\, \forall x \cup \mathrm{Im}\, \exists x \subseteq A_\phi. \end{array} \tag{3.4.1}$$

By virtue of the last two lines, we may regard $\wedge, \vee, \Diamond, \Rrightarrow, \forall x$ and $\exists x$ as total operations on A_ϕ (we may also regard Φ as a set of total operations on A_ϵ).

The conditions (3) and (3.4.1) show that A_ϵ contains P and is a Φ-subreduct of A. Therefore, defining B to be the closure $[P]_\Phi$ of P in the Φ-reduct A_Φ, we have $B \subseteq A_\epsilon$. Define $R = \bigcup_{p \in \Pi} \mathrm{Im}\, p$ and $K = \{\wedge, \vee, \Diamond, \Rrightarrow, \forall x, \exists x : x \in X\}$. Then (3.4.1) shows that A_ϕ contains R and is a K-subreduct of A. Therefore, defining D to be the closure $[R]_K$ of R in the K-reduct A_K, we have $D \subseteq A_\phi$. Furthermore since $A_\epsilon \cap A_\phi = \emptyset$, we have $(B \cup D) \cap A_\epsilon = B$ and $(B \cup D) \cap A_\phi = D$.

It also follows from (3.4.1) that $B \cup D$ is a subalgebra of A. In order to prove it, assume $(a_1, \dots, a_k) \in (B \cup D)^k \cap \mathrm{Dm}\, \alpha_\lambda$ for $\lambda \in \Lambda$ and let $a = \alpha_\lambda(a_1, \dots, a_k)$. First if $\lambda \in \Phi$, then $a_1, \dots, a_k \in (B \cup D) \cap A_\epsilon = B$, and since B is a Φ-subreduct of A, we have $a \in B \subseteq B \cup D$. Next if $\lambda \in K$, then $a_1, \dots, a_k \in (B \cup D) \cap A_\phi = D$, and since D is a K-subreduct of A, we have $a \in D \subseteq B \cup D$. Lastly if $\lambda \in \Pi$, then $a \in R \subseteq D \subseteq B \cup D$. Since $\Lambda = \Phi \cup \Pi \cup K$, we have shown that $B \cup D$ is a subalgebra of A. Moreover, since $A = [P]$ and $P \subseteq B \subseteq B \cup D$, we have $A = B \cup D$. Thus $A_\epsilon = (B \cup D) \cap A_\epsilon = B = [P]_\Phi$ and $A_\phi = (B \cup D) \cap A_\phi = D = [R]_K$.

Since (A, T, σ, P) is a USA, (A, P) is a based algebra by Theorem 3.1.6. Therefore, since $A_\epsilon = [P]_\Phi$, (A_ϵ, P) is a based Φ-algebra by Corollary 3.1.7.3. Moreover, $(A_K, P \cup (\bigcup_{f \in \Phi} \mathrm{Im}\, f) \cup R)$ is a based algebra by Corollary 3.1.7.2, and so since $A_\phi = [R]_K$, (A_ϕ, R) is a based K-algebra also by Corollary 3.1.7.3.

Theorem 3.1.7 applied to the based Φ-algebra (A_ϵ, P) shows that each element $a \in A_\epsilon - P$ has a ramification $a = f(a_1, \dots, a_k)$ for some $f \in \Phi$ and $a_1, \dots, a_k \in A_\epsilon$. The theorem applied to the based K-algebra (A_ϕ, R) shows that each element $a \in A_\phi - R$ has a ramification $a = a_1 \wedge a_2$, $a = a_1 \vee a_2$, $a = a_1 \Rrightarrow a_2$, $a = b^\Diamond$, $a = \forall x b$ or $a = \exists x b$ for some $a_1, a_2, b \in A_\phi$ and $x \in X$. Each of the ramifications is one in the based algebra (A, P). Each element $a \in R = \bigcup_{p \in \Pi} \mathrm{Im}\, p$ obviously has a ramification $a = p(a_1, \dots, a_k)$ in (A, P) for some $p \in \Pi$ and $a_1, \dots, a_k \in A_\epsilon$. Theorem 3.1.7 also shows that every ramification $a = \alpha_\lambda(a_1, \dots, a_k)$ satisfies $\mathrm{rk}\, a - 1 = \sum_{j=1}^k \mathrm{rk}\, a_j$. Thus the elements of A can be enumerated by induction on their ranks starting from the elements of P, as was noted in the introduction of this chapter.

Functional expressions in FPL The intention of the condition (4) on each DW $W \in \mathfrak{W}$ may be understood by means of functional expressions of elements of A under denotations $\delta \in \Delta_W$ of C into W. For example, assume that an element $a \in A_\phi$ has a ramification $a = a_1 \wedge a_2$ with $a_j \in A_\phi$ $(j = 1, 2)$. Let $(y_1, \ldots, y_n)$ be a basis of a. Then Theorem 3.3.4 together with the conditions (3) and (4) shows that $(y_1, \ldots, y_n)$ is also a basis of a_1 and a_2 and that the functional expressions $a^\delta(y_1, \ldots, y_n)$, $a_1{}^\delta(y_1, \ldots, y_n)$ and $a_2{}^\delta(y_1, \ldots, y_n)$ belong to $W_\epsilon{}^n \!\rightarrow\! \mathbb{T}$ and satisfy the following for each $(w_1, \ldots, w_n) \in W_\epsilon{}^n$:

$$a^\delta(w_1, \ldots, w_n) = a_1{}^\delta(w_1, \ldots, w_n) \wedge a_2{}^\delta(w_1, \ldots, w_n),$$

that is, $a^\delta(w_1, \ldots, w_n) = 1$ iff $a_1{}^\delta(w_1, \ldots, w_n) = 1$ *and* $a_2{}^\delta(w_1, \ldots, w_n) = 1$. Similar remarks apply to ramifications $a = a_1 \vee a_2$, $a = b^\Diamond$ and $a = a_1 \Rrightarrow a_2$. Thus the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ of A signify *and*, *or*, *not* and *imply*.

The intention of the condition (5) on each I_W $(W \in \mathfrak{W})$ may also be understood by means of functional expressions of elements of A under $\delta \in \Delta_W$. For example, assume that an element $a \in A_\phi$ has a ramification $a = \forall x b$ with $x \in X$ and $b \in A_\phi$. Let $(y_1, \ldots, y_n)$ be a basis of a and assume $x \notin \{y_1, \ldots, y_n\}$. Then Theorem 3.3.4 together with the conditions (3) and (4) shows that $(y_1, \ldots, y_n, x)$ is a basis of b and that the functional expressions $a^\delta(y_1, \ldots, y_n)$ and $b^\delta(y_1, \ldots, y_n, x)$ belong to $W_\epsilon{}^n \!\rightarrow\! \mathbb{T}$ and $W_\epsilon{}^{n+1} \!\rightarrow\! \mathbb{T}$ respectively and satisfy the following for each $(w_1, \ldots, w_n) \in W_\epsilon{}^n$:

$$a^\delta(w_1, \ldots, w_n) = \lambda_W(b^\delta(w_1, \ldots, w_n, \Box)),$$

where $\lambda = \forall x$ and $b^\delta(w_1, \ldots, w_n, \Box)$ is the mapping of W_ϵ into $\mathbb{T}$ defined by

$$(b^\delta(w_1, \ldots, w_n, \Box))w = b^\delta(w_1, \ldots, w_n, w)$$

for each $w \in W_\epsilon$. Therefore,

$$a^\delta(w_1, \ldots, w_n) = \inf\{b^\delta(w_1, \ldots, w_n, w) : w \in W_\epsilon\}$$

by the condition (5). This means

$$a^\delta(w_1, \ldots, w_n) = 1 \iff b^\delta(w_1, \ldots, w_n, w) = 1 \textit{ for all } w \in W_\epsilon.$$

Likewise, if an element $a \in A_\phi$ has a ramification $a = \exists x b$ with $x \in X$ and $b \in A_\phi$, and if $(y_1, \ldots, y_n)$ is a basis of a with $x \notin \{y_1, \ldots, y_n\}$, then $(y_1, \ldots, y_n, x)$ is a basis of b and the functional expression $b^\delta(y_1, \ldots, y_n, x)$ of b satisfies

$$a^\delta(w_1, \ldots, w_n) = 1 \iff b^\delta(w_1, \ldots, w_n, w) = 1 \textit{ for some } w \in W_\epsilon.$$

Thus the operations $\forall x$ and $\exists x$ of A signify *for all* and *for some* respectively, as was mentioned in Example 3.2.5.

The FPL functional logic space The condition (4) also shows that the logic system FPL has a truth type ϕ and the set W_ϕ of the ϕ-truth values of each DW $W \in \mathfrak{W}$ is the binary lattice $\mathbb{T}$. Therefore, it yields the ϕ-functional logic space $(A_\phi, \mathcal{F})$ defined by (3.3.23)–(3.3.26). Moreover, the conditions (4) and (3.3.21) show that $(A_\phi, \mathcal{F})$ is a binary logic space with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ regarded as total operations on A_ϕ.

3.5 An incompleteness theorem

The purpose of this section is to prove a so-called incompleteness theorem for a certain logic system $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma, \mathfrak{W}, (I_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ and thereby illustrate the GL in §3.1–3.3 and Chapter 2.

We assume that the logic system satisfies the following conditions (1)–(4).

(1) There exists a truth type $\phi \in \mathsf{T}$, and the set W_ϕ of the ϕ-truth values of each DW $W \in \mathfrak{W}$ is the binary lattice $\mathbb{T}$.

Under this first condition, the ϕ-functional logic space $(\mathsf{A}_\phi, \mathcal{F})$ is defined on the ϕ-part A_ϕ of A by (3.3.26) and (3.3.23), which concerns the second condition.

(2) There exists a unary operation $\Diamond$ in the algebraic structure of A such that A_ϕ is a $\Diamond$-closed subset of $\mathrm{Dm}\,\Diamond$ and the restriction of $\Diamond$ to A_ϕ, which is a total unary operation on A_ϕ, is an $\mathcal{F}$-complement (s. Theorem 2.8.9).

(3) There exists an element $\epsilon \in \mathsf{T}-\{\phi\}$ such that the ϵ-part X_ϵ of X is nonempty and the subset $\mathsf{A}_{\phi,1} = \{\mathsf{a} \in \mathsf{A}_\phi : \mathsf{X}^{\mathsf{a}}_{\mathrm{fr}} \subseteq \mathsf{X}_\epsilon,\ \#\mathsf{X}^{\mathsf{a}}_{\mathrm{fr}} \leq 1\}$ of A_ϕ and the subset $\mathsf{A}_{\epsilon,0} = \{\mathsf{b} \in \mathsf{A}_\epsilon : \mathsf{X}^{\mathsf{b}}_{\mathrm{fr}} = \emptyset\}$ of the ϵ-part A_ϵ of A satisfy $\#\mathsf{A}_{\phi,1} \leq \#\mathsf{A}_{\epsilon,0}$.

(4) There exists a subset Φ of the set M of the invariable indices (3.2.1) of the algebraic structure of T such that A_ϵ is equal to the closure $[\mathsf{P}_\epsilon]_\Phi$ of the ϵ-part P_ϵ of the basis P in the Φ-reduct A_Φ of A and such that the ϵ-part W_ϵ of each DW $W \in \mathfrak{W}$ is a Φ-subreduct of W.

Before stating the incompleteness theorem, a few definitions and remarks are in order on the conditions (a) and (b) in the theorem.

The ϕ-functional logic space $(\mathsf{A}_\phi, \mathcal{F})$ is defined by

$$\mathcal{F} = \{\varphi^\delta_{\upsilon,\phi} : (W, \delta, \upsilon) \in \mathcal{D}\}, \qquad \varphi^\delta_{\upsilon,\phi} = \varphi^\delta_\upsilon|_{\mathsf{A}_\phi}.$$

Let $(\mathsf{A}_\phi, \mathfrak{B})$ be the associated logic space and Q be the largest $\mathfrak{B}$-logic on A_ϕ. Then $\mathfrak{B} = \{\varphi^{-1}\{1\} : \varphi \in \mathcal{F}\} \cup \{\mathsf{A}_\phi\}$ because $W_\phi = \mathbb{T}$ for each $W \in \mathfrak{W}$ by the condition (1), and each subset U of A_ϕ has its Q-closure $[\mathsf{U}]_\mathsf{Q}$ in A_ϕ.

The condition (3) implies that there exists an injection $\mathsf{a} \mapsto \overline{\mathsf{a}}$ of $\mathsf{A}_{\phi,1}$ into $\mathsf{A}_{\epsilon,0}$, which we call an $\mathsf{A}_{\epsilon,0}$**-coding** of $\mathsf{A}_{\phi,1}$. Let $\mathsf{a} \in \mathsf{A}_{\phi,1}$. Then since $\mathsf{X}_\epsilon \neq \emptyset$, the definition of $\mathsf{A}_{\phi,1}$ implies that there exists a basis $\mathsf{x}_\mathsf{a} \in \mathsf{X}_\epsilon$ of a. Moreover, for each element $\mathsf{b} \in \mathsf{A}_\epsilon$, we can apply the substitution $(\mathsf{x}_\mathsf{a}/\mathsf{b})$ to obtain an element $\mathsf{a}(\mathsf{x}_\mathsf{a}/\mathsf{b}) \in \mathsf{A}_\phi$. In particular $\mathsf{a}(\mathsf{x}_\mathsf{a}/\overline{\mathsf{a}}) \in \mathsf{A}_\phi$, and so the condition (2) allows us to apply the operation $\Diamond$ to obtain an element $\mathsf{a}(\mathsf{x}_\mathsf{a}/\overline{\mathsf{a}})^\Diamond \in \mathsf{A}_\phi$ (as usual, the operation symbol $\Diamond$ is superscript henceforth).

The condition (4) implies that A_ϵ and W_ϵ for each $W \in \mathfrak{W}$ are Φ-algebras.

Example 3.5.1 The logic system CL as well as FPL satisfies the conditions (1), (2) and (4) by its definition. As for FPL, see §3.4 and Theorem 2.8.10. They satisfy the condition (3) if every set involved in the syntax of A is countable and $\mathsf{A}_{\epsilon,0}$ is countably infinite,[3.29] for example.

[3.29] A set S is said to be **countable** if there exists an injection of S into the set $\mathbb{N}$ of positive integers, and it is said to be **countably infinite** if there exists a bijection of S onto $\mathbb{N}$.

Now we can state the incompleteness theorem.

Theorem 3.5.1 Let U be a subset of A_ϕ, and assume that it has a nontrivial $\mathfrak{B}$-model ${\varphi^{\delta}_{\upsilon,\phi}}^{-1}\{1\}$ $((\delta,\upsilon) \in \Delta_W \times \Upsilon_W,\ W \in \mathfrak{W})$ which satisfies the following two conditions. Then U is $\mathcal{F}$-incomplete.

(a) The ϵ-part A_ϵ of A is a Φ-subreduct of W_ϵ, and δ and υ satisfy $\delta c = c$ and $\upsilon x = x$ for all $c \in \mathsf{C}_\epsilon$ and all $x \in \mathsf{X}_\epsilon$ respectively.

(b) There exists an $\mathsf{A}_{\epsilon,0}$-coding $a \mapsto \overline{a}$ of $\mathsf{A}_{\phi,1}$ such that the function $f_{\mathsf{U}} \in W_\epsilon \to W_\phi$ defined by the following for each $w \in W_\epsilon$ is δ-denotable:

$$f_{\mathsf{U}} w = 1 \iff w = \overline{a} \text{ for an element } a \in \mathsf{A}_{\phi,1} \text{ such that } a(x_a/\overline{a})^{\Diamond} \in [\mathsf{U}]_{\mathsf{Q}}.$$

Remark 3.5.1 The latter half of the condition (a) is reasonable because $\mathsf{C}_\epsilon \cup \mathsf{X}_\epsilon \subseteq W_\epsilon$ by the former half. The function f_{U} is well-defined because $W_\phi = \mathbb{T}$ by the condition (1) and $\{\overline{a} : a \in \mathsf{A}_{\phi,1}\} \subseteq \mathsf{A}_{\epsilon,0} \subseteq W_\epsilon$ by the former half of the condition (a).

Remark 3.5.2 Assume moreover that $(\mathsf{A}_\phi, \mathcal{F})$ belongs to the class 2. Then if U is an $\mathcal{F}$-consistent subset of A_ϕ, it has a nontrivial $\mathfrak{B}$-model by Corollary 2.8.12.1. Moreover, we can expect to construct a nontrivial $\mathfrak{B}$-model satisfying the condition (a) by means of a regular extension of the logic system (s. §3.3.4). Therefore, this theorem for the class 2 logic systems in practice shows that an $\mathcal{F}$-consistent subset U of A_ϕ is $\mathcal{F}$-incomplete provided that f_{U} is δ-denotable for some $\mathsf{A}_{\epsilon,0}$-coding of $\mathsf{A}_{\phi,1}$.

Remark 3.5.3 If there exists a specific $\mathcal{F}$-complete deduction system (R, D) on A_ϕ, as is the case with CL as well as FPL, then the closure $[\mathsf{U}]_{\mathsf{Q}}$ in the definition of f_{U} may be replaced by $[\mathsf{U} \cup \mathsf{D}]_{\mathsf{R}}$ by virtue of Theorem 2.7.1, which will help us verify that the condition (b) is satisfied.

Remark 3.5.4 The $\mathsf{A}_{\epsilon,0}$-coding of $\mathsf{A}_{\phi,1}$ for MN is a model of self-perception (s. Remark 1.2.12) in contrast to its prototype[3.30] for metamathematics. It is assumed to be injective, which means that the self-perception is clear.

Lemma 3.5.1 The denotation $\varphi^{\delta}_{\upsilon} \in \mathsf{A} \to W$ satisfies $\varphi^{\delta}_{\upsilon} b = b$ for all $b \in \mathsf{A}_\epsilon$.

Proof The condition (4) shows that both A_ϵ and W_ϵ are Φ-algebras and moreover $\mathsf{A}_\epsilon = [\mathsf{P}_\epsilon]_\Phi$. Since $\Phi \subseteq \mathsf{M}$ and $\varphi^{\delta}_{\upsilon}$ is a homotypic M-homomorphism of A into W by (3.3.16) and (3.3.21), $\varphi^{\delta}_{\upsilon}|_{\mathsf{A}_\epsilon}$ is an Φ-homomorphism of A_ϵ into W_ϵ. The former half of the condition (a) shows that the inclusion mapping f of A_ϵ into W_ϵ is also an Φ-homomorphism, while the latter half and (3.3.18) show $\varphi^{\delta}_{\upsilon}|_{\mathsf{P}_\epsilon} = f|_{\mathsf{P}_\epsilon}$. Therefore, $\varphi^{\delta}_{\upsilon}|_{\mathsf{A}_\epsilon} = f$ by Lemma 3.1.4.

[3.30] Kurt Gödel, "Über formal unentscheidbare Sätze der *Principia mathematica* und verwandter Systeme I (On formally undecidable propositions of *Principia mathematica* and related systems I)," *Monatshefte für Mathematik und Physik* 38 (1931), 173–198.

Proof of Theorem 3.5.1 By way of contradiction, assume that U is $\mathcal{F}$-complete, that is, $\mathfrak{B}$-complete. Then since ${\varphi^{\delta}_{\upsilon,\Phi}}^{-1}\{1\}$ is a nontrivial $\mathfrak{B}$-model of U by our assumption, we have $[U]_Q = {\varphi^{\delta}_{\upsilon,\Phi}}^{-1}\{1\}$ by Theorem 2.8.1.

The condition (b) shows that there exists an element $c \in A_{\Phi,1}$ such that $f_U = c^{\delta}(x_c)$, and it satisfies the following for all $a \in A_{\Phi,1}$:

$$\begin{aligned}
\varphi^{\delta}_{\upsilon}(c(x_c/\overline{a})) &= \varphi^{\delta}_{\upsilon(x_c/\varphi^{\delta}_{\upsilon}\overline{a})}c && \text{(for the reason below)}\\
&= \varphi^{\delta}_{\upsilon(x_c/\overline{a})}c && \text{(by Lemma 3.5.1)}\\
&= c^{\delta}(\overline{a}) && \text{(by (3.3.29))}\\
&= f_U\overline{a} && \text{(since } f_U = c^{\delta}(x_c)\text{)}.
\end{aligned}$$

Here notice that $\overline{a} \in A_{\epsilon,0} \subseteq W_{\epsilon}$ by the condition (a). The first equality follows from Theorem 3.3.3 because $X^{\overline{a}}_{\mathrm{fr}} = \emptyset$ by the condition (3), hence $(X^{\lambda})^{\overline{a}}_{\mathrm{fr}} = \emptyset$ for each $\lambda \in \Lambda$, and so x_c is free from $\overline{a}$ in c.

Let a be an arbitrary element of $A_{\Phi,1}$. Then it follows from the above two paragraphs that a satisfies $c(x_c/\overline{a}) \in [U]_Q$ iff $f_U\overline{a} = 1$.

Moreover, a satisfies $f_U\overline{a} = 1$ iff $a(x_a/\overline{a})^{\Diamond} \in [U]_Q$. Indeed, if $a(x_a/\overline{a})^{\Diamond} \in [U]_Q$, then since $\overline{a} \in A_{\epsilon,0} \subseteq W_{\epsilon}$ and obviously $\overline{a} = \overline{a}$, we have $f_U\overline{a} = 1$ by the definition of f_U. Conversely if $f_U\overline{a} = 1$, then the definition of f_U shows that $\overline{a} = \overline{b}$ for some element $b \in A_{\Phi,1}$ such that $b(x_b/\overline{b})^{\Diamond} \in [U]_Q$, and so since the $A_{\epsilon,0}$-coding is injective, we have $a = b$, and so $a(x_a/\overline{a})^{\Diamond} \in [U]_Q$.

Furthermore, Corollary 2.8.7.1 shows that a satisfies $a(x_a/\overline{a})^{\Diamond} \in [U]_Q$ iff $a(x_a/\overline{a}) \notin [U]_Q$, because $\Diamond$ is a $\mathfrak{B}$-complement by the condition (2). Thus a satisfies $c(x_c/\overline{a}) \in [U]_Q$ iff $a(x_a/\overline{a}) \notin [U]_Q$. But then, since a is an arbitrary element of $A_{\Phi,1}$ and $c \in A_{\Phi,1}$, it follows that c satisfies $c(x_c/\overline{c}) \in [U]_Q$ iff $c(x_c/\overline{c}) \notin [U]_Q$, which is a contradiction completing the proof.

3.6 Quantity, ratio and measure

Here we abstract algebraic concepts from our notion of quantity such as length, volume, weight and degree. They will be used throughout Chapters 4–6 on CL.

3.6.1 Quantitative sets and qualitative sets

Recall that a semigroup is a set P equipped with a binary operation $*$ which is associative, that is, $a * (b * c) = (a * b) * c$ for all $(a, b, c) \in P^3$. The operation $*$ is called a multiplication and $a * b$ is called the product of a and b especially when the symbol $*$ is omitted as usual. Recall also that the semigroup is called a monoid if it has an identity element, i.e. a unique element e such that $a * e = e * a = a$ for all $a \in P$. Thus a semigroup and a monoid are best described by the pair $(P, *)$ and the triple $(P, *, e)$ which satisfy these conditions respectively. If a subset Q of P satisfies $a * b \in Q$ for all $(a, b) \in Q^2$, then Q is a subalgebra of the algebra $(P, *)$ and $(Q, *)$ is also a semigroup, and so Q is called a subsemigroup of the semigroup $(P, *)$. If the subsemigroup Q

furthermore contains the identity element e of P, then $(Q, *, e)$ is a monoid, and so Q is called a submonoid of the monoid $(P, *, e)$. The symbols $*$ and e are conventionally replaced by $+$ and 0 if the operation $*$ is commutative, that is, if $a * b = b * a$ for all $(a, b) \in P^2$. The operation $+$ is called an addition and $a + b$ is called the sum of a and b. The commutativity, as well as the associativity, is inherited by subsemigroups and submonoids.

A **qualitative set** is a commutative monoid $(P, +, 0)$ equipped with an order $\leq$ on P which satisfies the following two conditions.

- $0 \leq a$ for all $a \in P$, that is, $0 = \min P$.
- If elements $a, b \in P$ satisfy $a \leq b$, then $a + c \leq b + c$ for all $c \in P$.

Thus a qualitative set is best described by the quadruple $(P, +, 0, \leq)$ made of a commutative monoid $(P, +, 0)$ and an order $\leq$ on P which satisfies the two conditions. The latter condition is called the **monotonicity** (of the transformation $x \mapsto x + c$ on P for all $c \in P$) and is equivalent to the following.

- If elements $a, b, c, d \in P$ satisfy $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

The qualitative set is said to be **linear** or called a **quantitative set** if the order $\leq$ is linear, and is said to be **bounded** (from above) if $\max P$ exists.

Remark 3.6.1 Every qualitative set $(P, +, 0, \leq)$ satisfies the following conditions, which will be used below without notice.

- $a \leq a + b$ and $b \leq a + b$ for all $(a, b) \in P^2$.
- $0a \leq 1a \leq 2a \leq \cdots$ for all $a \in P$.
- If elements $a, b \in P$ satisfy $a \leq b$, then $na \leq nb$ $(n = 1, 2, \ldots)$.
- If Q is a submonoid of the monoid $(P, +, 0)$, then $(Q, +, 0, \leq)$ is a qualitative set, and if $(P, +, 0, \leq)$ is linear, then so is $(Q, +, 0, \leq)$.

Remark 3.6.2 Let $(P, +, 0, \leq)$ be a qualitative set and $m \in P$. Define addition $+_m$ on P by the following for all $(a, b) \in P^2$:

$$a +_m b = \begin{cases} a + b & \text{if } a + b < m, \\ m & \text{otherwise.} \end{cases}$$

Then $(P, +_m)$ is a commutative semigroup. In proving its key associativity $(a +_m b) +_m c = a +_m (b +_m c)$, we may assume that either $(a +_m b) + c < m$ or $a + (b +_m c) < m$. In the former case, $(a +_m b) +_m c = (a +_m b) + c = (a + b) + c = a + (b + c) = a + (b +_m c) = a +_m (b +_m c)$ because every element $x \in P$ involved here satisfies $x < m$, and likewise for the latter. Moreover, the downward closed interval $(\leftarrow m]$ of P (s. Remark 2.1.4) is a subsemigroup of $(P, +_m)$, and $((\leftarrow m], +_m, 0, \leq)$ is a bounded qualitative set, which is called the **m-cut** of $(P, +, 0, \leq)$. As for its key monotonicity, if elements

$a, b \in (\leftarrow m]$ satisfy $a \le b$, then $a + c \le b + c$ for all $c \in (\leftarrow m]$, and either $a +_m c = a + c \le b + c = b +_m c$ or $a +_m c \le m = b +_m c$. If $(P, +, 0, \le)$ is linear, then so is the m-cut and $a +_m b = \min\{a + b, m\}$ for all $(a, b) \in P^2$.

Example 3.6.1 The set $\mathbb{R}_{\ge 0}$ of all nonnegative real numbers is a quantitative set with respect to its usual addition $+$, zero 0 and order $\le$, and likewise for the set $\mathbb{Z}_{\ge 0}$ of all nonnegative integers. More generally, if $\mathbb{P}$ is a submonoid of $(\mathbb{R}_{\ge 0}, +, 0)$, then $(\mathbb{P}, +, 0, \le)$ is a quantitative set, which is called a **real** quantitative set. For each positive integer n, the $(n-1)$-cut $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ of $\mathbb{Z}_{\ge 0}$ is a bounded quantitative set of n elements with respect to the addition $a +_{n-1} b = \min\{a + b, n - 1\}$, 0 and $\le$ (s. §1.3.7). Any singleton $\{e\}$ is a trivial bounded quantitative set with respect to a unique triple $(+, e, \le)$.

Example 3.6.2 Suppose $(P, \le)$ is an ordered set and each subset X of P such that $\#X \le 2$ has its supremum in P. Then P has $\min P = \sup \emptyset$ (s. [1.97]), and P is a qualitative set with respect to the addition $a + b = \sup\{a, b\}$, $\min P$ and $\le$. In particular, the power set $\mathfrak{P}S$ of an arbitrary set S is a qualitative set with respect to the set-theoretic addition $\cup$, empty set $\emptyset$ and inclusion $\subseteq$, and likewise for its equivalent $S \rightarrow \mathbb{T}$ for $\mathbb{T} = \{0, 1\}$,[3.31] and more generally, the set $S^n \rightarrow \mathbb{T}$ of all n-ary relations on S $(n = 1, 2, \ldots)$. They are linear iff $\#S \le 1$.

Remark 3.6.3 Let $(P_i, +_i, 0_i, \le_i)$ be qualitative sets $(i = 1, 2)$. Then we can extend the additions $+_i$ and orders $\le_i$ $(i = 1, 2)$ to the addition $+$ and order $\le$ on the direct union $P_1 \amalg P_2$ so that $a_1 + a_2 = a_2 + a_1 = a_2$ and $a_1 < a_2$ for each $(a_1, a_2) \in P_1 \times P_2$. Moreover $(P_1 \amalg P_2, +, 0_1, \le)$ is a qualitative set, which is called the **extension** of $(P_1, +_1, 0_1, \le_1)$ by $(P_2, +_2, 0_2, \le_2)$. It is linear iff $(P_i, +_i, 0_i, \le_i)$ $(i = 1, 2)$ are linear, and it is bounded iff $(P_2, +_2, 0_2, \le_2)$ is bounded. In particular, every qualitative set can be extended to a bounded one by a singleton regarded as a bounded quantitative set, and the unique element of the singleton is the largest element of the extension and therefore is usually denoted ∞ in advance (s. §1.3.10).

Mizumura [1.91] has shown that the following theorem is of value for the study of semantics and deduction on CL. Recall that a **well-ordered set** is an ordered set $(A, \le)$ such that every nonempty subset of A has the smallest element. Well-ordered sets are necessarily linear. Recall also that a subalgebra of an algebra A is said to be **finitely generated** if it is generated by a finite subset of A, that is, if it is the closure $[S]_A$ of a finite subset S of A.

Theorem 3.6.1 Let $(P, +, 0, \le)$ be a quantitative set and Q a finitely generated subsemigroup of the semigroup $(P, +)$. Then $(Q, \le)$ is a well-ordered set.

Proof Let $\{q_1, \ldots, q_k\}$ be the finite subset of P generating Q and argue by induction on k. Since $\le$ is linear, we may assume $q_j \le q_1$ $(j = 2, \ldots, k)$. If

[3.31] A quality is an attribute, which is a unary event, and mathematically defined as an element of $S \rightarrow \mathbb{T}$ for a set S (s. §1.2.3 and §1.2.4). This explains the name of qualitative set.

$k = 1$, then there exists an increasing mapping $n \mapsto nq_1$ of $\mathbb{N}$ onto Q, and since $\mathbb{N}$ is well-ordered, so is Q. Therefore assume $k > 1$, and let Q' be the subsemigroup of P generated by $\{q_2, \ldots, q_k\}$. Then Q' is well-ordered by the induction hypothesis. Thus $Q' \cup \{0\}$ is well-ordered.

Since Q is linearly ordered, we only need to show that every downward closed interval $(\leftarrow r]$ of Q is well-ordered. There exist nonnegative integers $n_1, \ldots, n_k$ such that $r = n_1 q_1 + \cdots + n_k q_k$ (and at least one of $n_1, \ldots, n_k$ is positive). Define $n_0 = n_1 + \cdots + n_k$. Then $r \leq n_0 q_1$. Suppose $q \in (\leftarrow r]$. Then $q = nq_1 + q'$ for a nonnegative integer n and an element $q' \in Q' \cup \{0\}$. We may assume $n \leq n_0$, because if $n_0 < n$ then $q \leq r \leq n_0 q_1 \leq nq_1 \leq nq_1 + q' = q$ and so $q = n_0 q_1$. Therefore, defining $Q'_n = \{nq_1 + q' : q' \in Q' \cup \{0\}\}$ for each $n \in \{0, \ldots, n_0\}$, we have $(\leftarrow r] \subseteq \bigcup_{n=0}^{n_0} Q'_n$. Since the mapping $q' \mapsto nq_1 + q'$ of $Q' \cup \{0\}$ onto Q'_n is increasing, Q'_n is well-ordered for each $n \in \{0, \ldots, n_0\}$, and so is $\bigcup_{n=0}^{n_0} Q'_n$ because $\leq$ is linear. Thus $(\leftarrow r]$ is well-ordered.

Remark 3.6.4 If $(P, +, 0, \leq)$ in Theorem 3.6.1 is a qualitative set, Q is not necessarily well-ordered, even if its generators are linearly ordered. To give an example, the set $\mathbb{N}$ of all positive integers is a qualitative set with respect to its usual multiplication $\times$, one 1 and the divisibility $|$, that is, elements $a, b \in \mathbb{N}$ satisfy $a \mid b$ iff a divides b. Let Q be the subsemigroup of $\mathbb{N}$ generated by $\{2, 6\}$. Then $\{2, 6\}$ is linear with respect to the order $|$, whereas $\{4, 6\}$ is a subset of Q and does not have the smallest element with respect to $|$.

3.6.2 Unital quantitative sets and real quantitative sets

A quantitative set $(P, +, 0, \leq)$ is said to be **strict** if it satisfies the following two conditions which are equivalent and stronger than the monotonicity condition on qualitative sets. The former condition is called the **strict monotonicity**.

- If elements $a, b \in P$ satisfy $a < b$, then $a + c < b + c$ for all $c \in P$.
- If elements $a, b, c, d \in P$ satisfy $a < b$ and $c \leq d$, then $a + c < b + d$.

Remark 3.6.5 Every strict quantitative set $(P, +, 0, \leq)$ satisfies the following conditions, which will be used below without notice.

- If an element $a \in P$ satisfies $0 < a$, then $b < a + b$ for all $b \in P$.
- If an element $a \in P$ satisfies $0 < a$, then $0a < 1a < 2a < \cdots$.
- If elements $a, b \in P$ satisfy $a < b$, then $na < nb$ $(n = 1, 2, \ldots)$.
- If Q is a submonoid of the monoid $(P, +, 0)$, then $(Q, +, 0, \leq)$ is a strict quantitative set.

A (**quantitative**) **unit** of a strict quantitative set $(P, +, 0, \leq)$ is an element $e \in P$ which satisfies the following **unitarity** condition.

- If elements $a, b \in P$ satisfy $a < b$, then there exist positive integers m, n such that $na < me \leq nb$.

The condition $na < me \leq nb$ is satisfiable because if $a < b$ then $na < nb$ $(n = 1, 2, \ldots)$ by the strictness. The strict quantitative set is said to be **unital** if it has a unit. Thus a unital quantitative set is best described by the quintuple $(P, +, 0, \leq, e)$ made of a strict quantitative set $(P, +, 0, \leq)$ and its unit e.

Example 3.6.3 Any singleton $\{e\}$ is a trivial unital quantitative set with respect to a unique quadruple $(+, e, \leq, e)$, because it has no elements a, b such that $a < b$. Any real quantitative set $(\mathbb{P}, +, 0, \leq)$ is unital, and if it is nontrivial, that is, if $\mathbb{P} \neq \{0\}$, every element of $\mathbb{P} - \{0\}$ is its unit, because the set $\mathbb{Q}$ of all rationals is dense in the set $\mathbb{R}$ of all real numbers. Therefore if $1 \in \mathbb{P}$, then $(\mathbb{P}, +, 0, \leq, 1)$ is a unital quantitative set.

The $(n-1)$-cut $\mathbb{Z}_n$ of $\mathbb{Z}_{\geq 0}$ is strict only if $n = 1$. More generally, a bounded quantitative set is strict only if it is trivial. There exists a quantitative set which is strict and not unital (s. Remark 3.6.8).

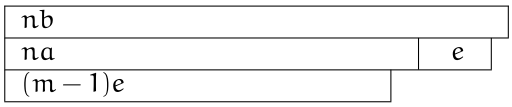
Remark 3.6.6 Here I explain background of the concept of unital quantitative sets. Notice that I use the auxiliary verb *will* indicating likelihood because I talk about empirical knowledge and not about mathematically proved facts.

Let P be the totality of the (spatial) line segments. Regard the points as line segments and identify any two line segments that can be superposed on each other. Then all points will be identified and so may be denoted by the single symbol 0. When connected straight, any two line segments a and b will make a line segment c, which we denote by $a + b$. The definition of the sum $a + b$ will be consistent with the identification of any two line segments that can be superposed on each other, that is, if a and b are superposed on line segments a' and b' respectively and a' and b' make a line segment c', then c will be superposed on c'. Moreover, the addition $+$ for line segments will be commutative and associative, that is, each combination $\{a_1, \ldots, a_k\}$ of line segments will make a unique line segment irrespective of how they all are connected straight. Thus $(P, +, 0)$ will be a commutative monoid.

Suppose we try to superpose two line segments and fail. Then we will find that one of them, say b, protrudes from the other, say a, when we write $a < b$. The definition of the relation $a < b$ will be consistent with the identification of any two line segments that can be superposed on each other, that is, if a and b are superposed on line segments a' and b' respectively, then b' will protrude from a'. Moreover, the relation $<$ will be transitive, that is, if a line segment b protrudes from a line segment a and a line segment c protrudes from b, then c will protrude from a. Therefore $<$ will be extended to a linear order $\leq$ on P, that is, $a \leq b$ iff b protrudes from or is superposed on a (s. [1.98]). Then 0 will be the smallest element of P and $+$ will be strictly monotonic with respect to $\leq$, that is, if a line segment b protrudes from a line segment a, then $b + c$ will protrude from $a + c$ for any line segment c. Thus $(P, +, 0, \leq)$ will be a strict quantitative set.

Let e be an arbitrary line segment which is not a point. For each line segment a and each positive integer n, any n copies of a make a line segment na when they are connected straight. Therefore, if a line segment b protrudes from a line segment a, then the line segment nb will protrude from na for any positive integer n, and moreover as n increases, the protrusion of nb from na will increase and eventually protrude from or is superposed on e. Then nb will protrude from or is superposed on $na + e$, that is, $na + e \leq nb$. Moreover, there will exist the smallest positive integer m such that the line segment me protrudes from na, that is, $na < me$, and so na protrude from or is superposed on $(m-1)e$, that is, $(m-1)e \leq na$. Therefore $na < me = (m-1)e + e \leq na + e \leq nb$. Thus $(P, +, 0, \leq, e)$ will be a unital quantitative set (s. Remark 3.6.9). This paragraph is illustrated by Fig. 3.1, where every line segment is given visible width.

Figure 3.1: Line segments e, a, b such that $na < me \leq nb$ for $m, n \in \mathbb{N}$



To cut the above explanation short, a unital quantitative set $(P, +, 0, \leq, e)$ will underlie the process of connecting line segments straight and comparing the connected line segments by trying to superpose them.

Generally speaking, a unital quantitative set $(P, +, 0, \leq, e)$ will underlie the real or imaginary process of connecting some entities in some manner and comparing some attribute of the connectives by some means. Moreover, although P consists of some geometrical or physical entities such as line segments, areas, bodies, particles and their connectives, and the process seemingly involves no numbers other than the positive integers, yet the following theorem shows that $(P, +, 0, \leq, e)$ is realizable, that is, it can be identified with a unital quantitative set $(\mathbb{P}, +, 0, \leq, 1)$ made of a real quantitative set $(\mathbb{P}, +, 0, \leq)$ such that $1 \in \mathbb{P}$. The realizability means that the attribute in question is representable by numbers.

Theorem 3.6.2 Let $(P, +, 0, \leq, e)$ be a nontrivial unital quantitative set. Then there exists a unique mapping f of P into $\mathbb{R}_{\geq 0}$ which satisfies the following three conditions (f is called the **e-realization** of $(P, +, 0, \leq, e)$ or of P).

- Each element $(a, b) \in P^2$ satisfies $f(a + b) = fa + fb$.
- If elements $a, b \in P$ satisfy $a < b$, then $fa < fb$.
- $fe = 1$.

Remark 3.6.7 Since the e-realization f is a homomorphism of the semigroup $(P, +)$ into the semigroup $(\mathbb{R}_{\geq 0}, +)$, it furthermore satisfies $f0 = 0$ and $f(na) = n(fa)$ for each $a \in P$ and each $n \in \mathbb{N}$, and so $e \neq 0$ because $fe = 1 \neq 0 = f0$.

For the same reason, the image $\mathbb{P} = fP$ of f is a submonoid of the monoid $(\mathbb{R}_{\geq 0}, +, 0)$. Therefore, $(\mathbb{P}, +, 0, \leq)$ is a real quantitative set, and since $1 = fe \in \mathbb{P}$, $(\mathbb{P}, +, 0, \leq, 1)$ is a unital quantitative set.

Moreover, since f is a strictly increasing mapping of a linearly ordered set $(P, \leq)$ into $(\mathbb{R}_{\geq 0}, \leq)$, elements $a, b \in P$ satisfy $fa \leq fb$ iff $a \leq b$, and so they satisfy $fa < fb$ iff $a < b$. Consequently, f is injective.

Thus f is an isomorphism between the monoids $(P, +, 0)$ and $(\mathbb{P}, +, 0)$ and between the ordered sets $(P, \leq)$ and $(\mathbb{P}, \leq)$, and associates the unit e of $(P, +, 0, \leq)$ with the unit 1 of $(\mathbb{P}, +, 0, \leq)$. In this sense, f is an isomorphism between the nontrivial unital quantitative sets $(P, +, 0, \leq, e)$ and $(\mathbb{P}, +, 0, \leq, 1)$. The uniqueness of f is strengthened by Lemma 3.6.7.

The proof of Theorem 3.6.2 is divided into the following seven lemmas.

Lemma 3.6.1 Each element $a \in P$ is associated with a unique sequence $\{a_n\}$ of nonnegative integers such that $a_n e \leq na < (a_n + 1)e$ for each $n \in \mathbb{N}$ (call it the **e-asymptote** of a).

Proof It suffices to show that a is associated with a unique nonnegative integer m such that $me \leq a < (m+1)e$. Since $(P, +, 0, \leq)$ is strict and nontrivial, there exists an element $b \in P$ such that $a < b$. Therefore, there exist positive integers k, l such that $la < ke \leq lb$ by the unitarity of e. Consequently $0e = 0 \leq a < ke \leq (k+1)e \leq \cdots$, and so there exists the largest nonnegative integer m such that $me \leq a$. Since $\leq$ is linear, it also satisfies $a < (m+1)e$. Suppose another nonnegative integer m' satisfies $m'e \leq a < (m'+1)e$. We may assume $m < m'$. Then $m + 1 \leq m'$, and so $a < (m+1)e \leq m'e \leq a$, which is a contradiction.

Assign each $a \in P$ the subset $\mathbb{Q}_a = \{m/n : m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}, me \leq na\}$ of the set $\mathbb{Q}_{\geq 0}$ of all nonnegative rationals. The following lemma is underlain by the construction of $\mathbb{R}$ by means of the Dedekind cuts of $\mathbb{Q}$.

Lemma 3.6.2 The following hold for each $a \in P$ and its e-asymptote $\{a_n\}$.

(1) If $k/l \leq s$ for elements $k \in \mathbb{Z}_{\geq 0}$, $l \in \mathbb{N}$ and $s \in \mathbb{Q}_a$, then $ke \leq la$.

(2) If $k/l \in \mathbb{Q}_a$ for elements $k \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{N}$, then $ke \leq la$.

(3) Every element of $\mathbb{Q}_{\geq 0} - \mathbb{Q}_a$ is an upper bound of $\mathbb{Q}_a$ with respect to $\leq$.

(4) $\dfrac{a_n}{n} \in \mathbb{Q}_a$ and $\dfrac{a_n + 1}{n} \in \mathbb{Q}_{\geq 0} - \mathbb{Q}_a$ for each $n \in \mathbb{N}$.

(5) There exists the supremum of $\mathbb{Q}_a$ in the ordered set $(\mathbb{R}, \leq)$.

Proof (2) and (3) are direct consequences of (1), because its conclusion implies $k/l \in \mathbb{Q}_a$. (4) is a consequence of (2) and the definitions of $\mathbb{Q}_a$ and $\{a_n\}$. (5) is a consequence of (3), (4) and the completeness of $\mathbb{R}$. Therefore it suffices to prove (1). Assume $k/l \le s \in \mathbb{Q}_a$ for elements $k \in \mathbb{Z}_{\ge 0}$, $l \in \mathbb{N}$ and $s \in \mathbb{Q}_a$. Then since $s \in \mathbb{Q}_a$, there exist elements $m \in \mathbb{Z}_{\ge 0}$ and $n \in \mathbb{N}$ such that $s = m/n$ and $me \le na$. Since $k/l \le s$, we have $nk \le lm$, and so $n(ke) = (nk)e \le (lm)e = l(me)$. Since $me \le na$, we have $l(me) \le l(na) = (ln)a = (nl)a = n(la)$. Therefore $n(ke) \le n(la)$. Thus $ke \le la$ by the linearity and strictness of $(P, +, 0, \le)$.

Lemma 3.6.2 enables us to define a mapping $F \in P \to \mathbb{R}_{\ge 0}$ by $Fa = \sup \mathbb{Q}_a$ for each $a \in P$.

Lemma 3.6.3 $Fa = \lim\limits_{n\to\infty} \dfrac{a_n}{n}$ for each $a \in P$ and its e-asymptote $\{a_n\}$.

Proof This is because $\dfrac{a_n}{n} \le Fa \le \dfrac{a_n + 1}{n}$ for each $n \in \mathbb{N}$ by Lemma 3.6.2.

Lemma 3.6.4 $Fe = 1$ and $F(a + b) = Fa + Fb$ for each $(a, b) \in P^2$.

Proof Since $0 < e$ by Lemma 3.6.1, we have $ne \le ne < (n + 1)e$ for each $n \in \mathbb{N}$ by the strictness of $(P, +, 0, \le)$. Therefore, the e-asymptote $\{e_n\}$ of e satisfies $e_n = n$ for each $n \in \mathbb{N}$. Thus $Fe = 1$ by Lemma 3.6.3.

The e-asymptotes $\{a_n\}$ and $\{b_n\}$ of a and b satisfy $a_n e \le na < (a_n+1)e$ and $b_n e \le nb < (b_n + 1)e$ and so $(a_n + b_n)e \le n(a + b) < (a_n + b_n + 2)e$ for each $n \in \mathbb{N}$ by the strictness of $(P, +, 0, \le)$. Moreover, either $n(a+b) < (a_n+b_n+1)e$ or $(a_n + b_n + 1)e \le n(a + b)$ for each $n \in \mathbb{N}$ by the linearity of $\le$. Therefore, the e-asymptote $\{(a + b)_n\}$ of $a + b$ satisfies either $(a + b)_n = a_n + b_n$ or $(a + b)_n = a_n + b_n + 1$ and so $\dfrac{a_n}{n} + \dfrac{b_n}{n} \le \dfrac{(a + b)_n}{n} \le \dfrac{a_n}{n} + \dfrac{b_n}{n} + \dfrac{1}{n}$ for each $n \in \mathbb{N}$. Thus $F(a + b) = Fa + Fb$ by Lemma 3.6.3.

Lemma 3.6.5 If elements $a, b \in P$ satisfy $a < b$ and $Fa = Fb$ (this will be proved impossible by Lemma 3.6.6), then Fa is a positive rational, and positive integers m, n satisfy $Fa = m/n$ (if and) only if $me = nb$.

Proof Since $a < b$, there exist positive integers k, l such that $la < ke \le lb$ by the unitarity of e. They satisfy $k/l \in \mathbb{Q}_b - \mathbb{Q}_a$ and $Fa \le k/l \le Fb$ by Lemma 3.6.2. Thus, since $Fa = Fb$, we have $Fa = Fb = k/l$ and so Fa is a positive rational. Moreover, $Fa = Fb \in \mathbb{Q}_b - \mathbb{Q}_a$.

Suppose positive integers m, n satisfy $Fa = m/n$. Then $m/n = Fb \in \mathbb{Q}_b$ by the above paragraph, and so $me \le nb$ by Lemma 3.6.2. Moreover, $F(me) = m(Fe) = m = n(Fb) = F(nb)$ by Lemma 3.6.4. Therefore if $me < nb$, then $F(me) \notin \mathbb{Q}_{me}$ by the above paragraph, whereas $F(me) = m = m/1 \in \mathbb{Q}_{me}$ because $me \le 1(me)$. Thus $me = nb$.

Lemma 3.6.6 If elements $a, b \in P$ satisfy $a < b$, then $Fa < Fb$.

Proof Since $a < b$, we have $na \leq nb$ for all $n \in \mathbb{N}$, and so $\mathbb{Q}_a \subseteq \mathbb{Q}_b$. Therefore $Fa \leq Fb$, and so we may assume $Fa = Fb$ by way of contradiction. Then there exist positive integers m, n such that $Fa = Fb = m/n$ and $me = nb$ by Lemma 3.6.5. Since $Fa = Fb = m/n$, we have $F(2a) = 2(Fa) = Fa + Fb = F(a + b) = 2m/n$ by Lemma 3.6.4. Since $a < b$, we have $2a < a + b$ by the strictness of $(P, +, 0, \leq)$. Therefore $(2m)e = n(a+b) = na+nb$ by Lemma 3.6.5. On the other hand, since $me = nb$, we have $(2m)e = 2(me) = nb + nb$. This is a contradiction because $na + nb < nb + nb$ by the strictness of $(P, +, 0, \leq)$.

The following lemma completes the proof of Theorem 3.6.2.

Lemma 3.6.7 The mapping F is the only e-realization of P. More generally, if f is a homomorphism of $(P, +)$ into $(\mathbb{R}_{\geq 0}, +)$ and an increasing mapping of $(P, \leq)$ into $(\mathbb{R}_{\geq 0}, \leq)$ satisfying $fe = 1$, then $f = F$ (s. Remark 3.6.7).

Proof Lemmas 3.6.4 and 3.6.6 show that F is an e-realization of P. Let $a \in P$. Then since its e-asymptote $\{a_n\}$ satisfies $a_n e \leq na < (a_n + 1)e$, we have $a_n = a_n(fe) = f(a_n e) \leq f(na) = n(fa)$ and $n(fa) = f(na) \leq f((a_n + 1)e) = (a_n + 1)(fe) = a_n + 1$ for all $n \in \mathbb{N}$. Therefore $\dfrac{a_n}{n} \leq fa \leq \dfrac{a_n}{n} + \dfrac{1}{n}$ for all $n \in \mathbb{N}$, and so $fa = Fa$ by Lemma 3.6.3. Thus $f = F$.

The following theorem implies a converse of Theorem 3.6.2.

Theorem 3.6.3 Suppose a quantitative set $(P, +, 0, \leq)$, an element $e \in P$ and a mapping $g \in P \to \mathbb{R}_{\geq 0}$ satisfy the following three conditions.

- Each element $(a, b) \in P^2$ satisfies $g(a + b) = ga + gb$.
- If elements $a, b \in P$ satisfy $a < b$, then $ga < gb$.
- $ge > 0$.

Then $(P, +, 0, \leq, e)$ is a unital quantitative set, $e \in P - \{0\}$ and the e-realization f of P satisfies $fx = (gx)(ge)^{-1}$ for each $x \in P$.

Proof Being a homomorphism of the semigroup $(P, +)$ into the semigroup $(\mathbb{R}_{\geq 0}, +)$, g satisfies $g0 = 0$ and $g(na) = n(ga)$ for each $a \in P$ and each $n \in \mathbb{N}$. Since $ge > g0$, we have $e \in P - \{0\}$. Since g is a strictly increasing mapping of a linearly ordered set $(P, \leq)$ into $(\mathbb{R}_{\geq 0}, \leq)$, elements $a, b \in P$ satisfy $ga \leq gb$ iff $a \leq b$, and so they satisfy $ga < gb$ iff $a < b$.

Suppose elements $a, b \in P$ satisfy $a < b$. Then $0 \leq ga < gb$. Therefore if $c \in P$, then $g(a+c) = ga+gc < gb+gc = g(b+c)$, and so $a+c < b+c$. Thus $(P, +, 0, \leq)$ is strict. Moreover, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist positive integers m, n such that $n(ga) < m(ge) \leq n(gb)$. Therefore $g(na) < g(me) \leq g(nb)$, and so $na < me \leq nb$. Thus $(P, +, 0, \leq, e)$ is a unital quantitative set. The e-realization of P is equal to the mapping $x \mapsto (gx)(ge)^{-1}$, because the latter satisfies the three conditions characterizing the former.

Corollary 3.6.3.1 Let $(P, +, 0, \leq)$ be a unital quantitative set and $e \in P - \{0\}$. Then $(P, +, 0, \leq, e)$ is a unital quantitative set.

Proof Let d be a unit of $(P, +, 0, \leq)$ and g be the d-realization of P. Then since $0 < e$, we have $0 = g0 < ge$. Thus the result follows from Theorem 3.6.3.

Remark 3.6.8 Every unital quantitative set $(P, +, 0, \leq)$ is **Archimedean**, that is, if $a, b \in P - \{0\}$, then there exists a positive integer n such that $a \leq nb$. Indeed, since a is a unit of $(P, +, 0, \leq)$ by Corollary 3.6.3.1 and $0 < b$, there exist positive integers m, n such that $n0 < ma \leq nb$, and so $a \leq nb$.

There exists, however, an Archimedean strict quantitative set which is not unital (s. Example 3.6.3). Define $P = \{(a_1, a_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : a_1 \geq a_2\}$. Then P is a submonoid of the twofold direct product of the commutative monoid $(\mathbb{Z}_{\geq 0}, +, 0)$, and so $(P, +, (0, 0))$ is a commutative monoid with $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ for each $((a_1, a_2), (b_1, b_2)) \in P^2$. Let $<_d$ be the dictionary order on P, that is, $(a_1, a_2) <_d (b_1, b_2)$ holds for $((a_1, a_2), (b_1, b_2)) \in P^2$ iff either $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$. Then $<_d$ is a linear strict order on P (s. [1.98]) and therefore is extended to a linear order $\leq_d$ on P. Moreover, $(P, +, (0, 0), \leq_d)$ is a strict quantitative set. If $(0, 0) <_d (a_1, a_2) \in P$, then $a_1 > 0$ because $a_1 \geq a_2$. Therefore $(P, +, (0, 0), \leq_d)$ is Archimedean.

Assume that $(P, +, (0, 0), \leq_d)$ is unital by way of contradiction. Then it has a $(1, 0)$-realization f by Corollary 3.6.3.1 and Theorem 3.6.2. Since $n(1, 0) <_d n(1, 1) <_d (n + 1)(1, 0)$ for all $n \in \mathbb{N}$, the $(1, 0)$-asymptote $\{(1, 1)_n\}$ of $(1, 1)$ satisfies $(1, 1)_n = n$ for all $n \in \mathbb{N}$, and so $f(1, 1) = 1$ by Lemmas 3.6.7 and 3.6.3. This is a contradiction because $f(1, 0) = 1$ and $(1, 0) <_d (1, 1)$.

Remark 3.6.9 Let $(G, +, 0)$ be a commutative group equipped with a linear order $\leq$ on G satisfying the monotonicity condition. Define $P = \{a \in G : 0 \leq a\}$. Then $(P, +, 0, \leq)$ is a quantitative set. It is necessarily strict, because if $a + c = b + c$ for elements $a, b, c \in P$ then $a = (a + c) - c = (b + c) - c = b$. Moreover, if it is Archimedean, then it is unital in contrast to the example in Remark 3.6.8. The following proof has been suggested by Remark 3.6.6.

Assume that $(P, +, 0, \leq)$ is Archimedean. Let $e \in P - \{0\}$. Suppose $a, b \in P$ and $a < b$. Then $b - a \in P - \{0\}$, and so there exists a positive integer n such that $e \leq n(b - a)$, which satisfies $na + e \leq nb$. There similarly exists the smallest positive integer m such that $na < me$, which satisfies $(m - 1)e \leq na$. Therefore $na < me = (m - 1)e + e \leq na + e \leq nb$. Thus $(P, +, 0, \leq, e)$ is a unital quantitative set. Moreover, the e-realization of $(P, +, 0, \leq)$ is uniquely extended to a mapping $f \in G \to \mathbb{R}$ which is a group homomorphism of $(G, +)$ into $(\mathbb{R}, +)$ and a strictly increasing mapping of $(G, \leq)$ into $(\mathbb{R}, \leq)$. Thus the Archimedean linearly ordered commutative groups are realizable.

3.6.3 Real valued ratio for unital quantitative sets

Let $(P, +, 0, \leq)$ be a unital quantitative set and $a \in P - \{0\}$. Then $(P, +, 0, \leq, a)$ is a unital quantitative set by Corollary 3.6.3.1. Therefore, there exists a unique

a-realization of P by Theorem 3.6.2. We refer to it also as the **ratio to a**. Moreover, we refer to its value at each $b \in P$ as the **ratio of b to a** and denote it by b/a. Then, being the a-realization, the ratio to a is a mapping $x \mapsto x/a$ of P into $\mathbb{R}_{\geq 0}$ and satisfies the following three conditions.

- Each element $(b, c) \in P^2$ satisfies $(b + c)/a = b/a + c/a$.
- If elements $b, c \in P$ satisfy $b < c$, then $b/a < c/a$.
- $a/a = 1$.

We call them the **fundamental properties of ratio**.

Example 3.6.4 Let $(\mathbb{P}, +, 0, \leq)$ be a real quantitative set and $a \in \mathbb{P} - \{0\}$. Then $b/a = ba^{-1}$ for each $b \in \mathbb{P}$, because the mapping $x \mapsto xa^{-1}$ of $\mathbb{P}$ into $\mathbb{R}_{\geq 0}$ satisfies the three conditions characterizing the a-realization of $\mathbb{P}$.

Remark 3.6.10 Let $(P, +, 0, \leq)$ be a unital quantitative set and $a \in P - \{0\}$. Then, being the a-realization, the ratio to a furthermore satisfies the following.

- $0/a = 0$.
- $(nb)/a = n(b/a)$ for each $b \in P$ and each $n \in \mathbb{N}$.
- Elements $b, c \in P$ satisfy $b/a \leq c/a$ iff $b \leq c$.

Consequently, the ratio to a is injective, and so in particular, an element $b \in P$ satisfies $b/a = 1$ iff $b = a$, and it satisfies $b/a = 0$ iff $b = 0$.

Moreover, $(nb)/(na) = b/a$ for each $b \in P$ and each $n \in \mathbb{N}$. Indeed, since $(na)/a = n(a/a) = n > 0$, Theorem 3.6.3 shows that $na \in P - \{0\}$ and that the na-realization of P is equal to the mapping $x \mapsto (x/a)n^{-1}$. Therefore $(nb)/(na) = (nb/a)n^{-1} = n(b/a)n^{-1} = b/a$.

Moreover, $(c/b)(b/a) = c/a$ for each $b \in P - \{0\}$ and each $c \in P$. Indeed, since $b/a > 0$, the b-realization of P is equal to the mapping $x \mapsto (x/a)(b/a)^{-1}$ by Theorem 3.6.3. Therefore $(c/b)(b/a) = (c/a)(b/a)^{-1}(b/a) = c/a$.

Remark 3.6.11 The fundamental properties of ratio are worthy of the name because they characterize ratio in the following sense.

Let $(P, +, 0, \leq)$ be a quantitative set and $a \in P$. Suppose each element $b \in P$ is associated with a nonnegative real number $b/\!/a$ and the following hold.

- Each element $(b, c) \in P^2$ satisfies $(b + c)/\!/a = b/\!/a + c/\!/a$.
- If elements $b, c \in P$ satisfy $b < c$, then $b/\!/a < c/\!/a$.
- $a/\!/a = 1$.

It then follows from Theorem 3.6.3 that $(P, +, 0, \leq)$ is a unital quantitative set, $a \in P - \{0\}$ and $b/a = b/\!/a$ for each $b \in P$.

3.6.4 Measures with values in quantitative sets

Let S be a nonempty set and $(P, +, 0, \leq)$ be a quantitative set. Then a mapping $X \mapsto |X|$ of $\mathfrak{P}S$ into P is called a **P-measure** on S if it is positive definite, increasing and subadditive, that is, if it satisfies the following three conditions.

- $|\emptyset| = 0$, while $|X| > 0$ for each nonempty set $X \in \mathfrak{P}S$.
- If sets $X, Y \in \mathfrak{P}S$ satisfy $X \subseteq Y$, then $|X| \leq |Y|$.
- Each element $(X, Y) \in (\mathfrak{P}S)^2$ satisfies $|X \cup Y| \leq |X| + |Y|$.

If an element $a \in P$ satisfies $|X| \leq a$ for all $X \in \mathfrak{P}S$, we say that the measure is **bounded** (from above) by a and call a its (upper) **bound**.

Remark 3.6.12 Being an increasing mapping, every P-measure $X \mapsto |X|$ is bounded by $|S|$. Moreover, $|S| \neq 0$ by the positive definiteness because $S \neq \emptyset$ from the outset. Therefore if $(P, +, 0, \leq)$ is trivial then there exists no P-measure on S, while Remark 3.6.15 below shows that if $a \in P - \{0\}$ then there exists a P-measure on S bounded by a.

Remark 3.6.13 Any P-measure $X \mapsto |X|$ on S satisfies the following condition.

- If sets $X, Y_1, \ldots, Y_n \in \mathfrak{P}S$ and elements $a_1, \ldots, a_n \in P$ satisfy $X \subseteq \bigcup_{i=1}^n Y_i$ and $|Y_i| \leq a_i$ $(i = 1, \ldots, n)$, then $|X| \leq \sum_{i=1}^n a_i$.

Here n is a nonnegative integer and if $n = 0$, then $\bigcup_{i=1}^n Y_i = \emptyset$ and $\sum_{i=1}^n a_i = 0$. Conversely, if a mapping $X \mapsto |X|$ of $\mathfrak{P}S$ into P satisfies this condition, then it is increasing, subadditive and satisfies $|\emptyset| = 0$. The following equivalent of this condition should be called the box principle (s. §1.3.11).

- If sets $X, Y_1, \ldots, Y_n \in \mathfrak{P}S$ and elements $a_1, \ldots, a_n \in P$ satisfy $X \subseteq \bigcup_{i=1}^n Y_i$ and $|X| > \sum_{i=1}^n a_i$, then $|Y_i| > a_i$ for some $i \in \{1, \ldots, n\}$.

Remark 3.6.14 The measures here are different from those in integral theory especially in that they are positive definite. Indeed, there exists no $\mathbb{R}_{\geq 0} \amalg \{\infty\}$-measure $X \mapsto |X|$ on $\mathbb{R}$ such that $|(a, b)| = b - a$ for each open interval (a, b) of $\mathbb{R}$. If it existed, then $0 < |\{0\}| \leq |(-1/n, 1/n)| = 2/n$ for each positive integer n, which is a contradiction. However, a $\mathbb{Z}_{\geq 0} \amalg \mathbb{R}_{\geq 0} \amalg \{\infty\}$-measure $X \mapsto |X|$ on $\mathbb{R}$ was constructed in [1.3] so that $|X| = \#X \in \mathbb{Z}_{\geq 0}$ for each finite subset X of $\mathbb{R}$, $|X| = \min \mathbb{R}_{\geq 0} = 0$ for each countably infinite subset X of $\mathbb{R}$ (s. [3.29]), and $|(a, b)| = b - a \in \mathbb{R}_{>0}$ for each open interval (a, b) of $\mathbb{R}$ such that $a < b$.

Among general methods of constructing P-measures, the following two have proved to be valuable for the study of semantics and deduction on CL. The former given by Remark 3.6.15 is obvious. The latter given by Lemma 3.6.8 is natural in view of Remark 3.6.16 and essentially due to Mizumura [1.91].

Remark 3.6.15 Suppose S is a nonempty set and $(P, +, 0, \leq)$ is a nontrivial quantitative set. Then there exists at least one P-measure on S. For example, pick $a \in P - \{0\}$ and define $|\emptyset| = 0$ and $|X| = a$ for all $X \in \mathfrak{P}S - \{\emptyset\}$. Then the mapping $X \mapsto |X|$ is a P-measure on S bounded by a.

Lemma 3.6.8 Let S be a nonempty set and $(P, +, 0, \leq)$ be a quantitative set. Assume that each set $X \in \mathfrak{P}S$ is assigned a subset P_X of P which satisfies the following conditions.

(1) $0 \in P_\emptyset$, while $0 \notin P_X$ for each nonempty set $X \in \mathfrak{P}S$.

(2) If sets $X, Y \in \mathfrak{P}S$ satisfy $X \subseteq Y$, then $P_Y \subseteq P_X$.

(3) Each element $(X, Y) \in (\mathfrak{P}S)^2$ satisfies $\{a + b : a \in P_X, b \in P_Y\} \subseteq P_{X \cup Y}$.

Suppose $m \in P - \{0\}$ and there exists $\min(P_X \cup \{m\})$ for each set $X \in \mathfrak{P}S$. Then the mapping $X \mapsto \min(P_X \cup \{m\})$ is a P-measure on S bounded by m.

Proof Define $|X| = \min(P_X \cup \{m\})$ for each $X \in \mathfrak{P}S$. Then since $0 \in P_\emptyset \subseteq P_\emptyset \cup \{m\}$ by (1), we have $|\emptyset| = 0$. If $|X| = 0$, then since $0 \notin \{m\}$, we have $0 \in P_X$, and so $X = \emptyset$ by (1). Therefore the mapping $X \mapsto |X|$ is positive definite.

If sets $X, Y \in \mathfrak{P}S$ satisfy $X \subseteq Y$, then $P_Y \cup \{m\} \subseteq P_X \cup \{m\}$ by (2), hence $|X| = \min(P_X \cup \{m\}) \leq \min(P_Y \cup \{m\}) = |Y|$. Therefore the mapping is increasing.

Let $X, Y \in \mathfrak{P}S$. Define $a = |X|$ and $b = |Y|$. If $a \in P_X$ and $b \in P_Y$, then $a + b \in P_{X \cup Y} \subseteq P_{X \cup Y} \cup \{m\}$ by (3), hence $|X \cup Y| \leq a + b$. Otherwise, $a = m$ or $b = m$, hence $|X \cup Y| \leq m \leq a + b$. Therefore the mapping is subadditive.

Thus the mapping is a P-measure on S. It is bounded by m by definition.

Remark 3.6.16 Let S be a nonempty set, $(P, +, 0, \leq)$ be a quantitative set and $X \mapsto |X|$ be a P-measure on S. Associate each set $X \in \mathfrak{P}S$ with the subset $P_X = \{a \in P : |X| \leq a\}$ of P. Then every bound of the P-measure is contained in P_X, $|X| = \min P_X$ and the conditions (1)–(3) of Lemma 3.6.8 is satisfied.

Remark 3.6.17 Let S be a nonempty set, $(P, +, 0, \leq)$ be a quantitative set and $X \mapsto |X|$ be a P-measure on S bounded by an element $m \in P$. Then the m-cut $((\leftarrow m], +_m, 0, \leq)$ of $(P, +, 0, \leq)$ is a quantitative set and the mapping $X \mapsto |X|$ is a $(\leftarrow m]$-measure on S (bounded by m). It is subadditive with respect to the addition $+_m$ because $|X \cup Y| \leq \min\{|X| + |Y|, m\} = |X| +_m |Y|$ for all $(X, Y) \in (\mathfrak{P}S)^2$ by its subadditivity with respect to the addition $+$.

Chapter 4

Impartible Case Logic

The purpose of the remaining chapters 4–6 is to construct a theory of CL. It has been outlined in Chapter 1 in light of the principle, aim and method of MN. It is organized in line with the theory of GL presented in Chapters 2 and 3. Chapters 4 and 5 are of introductory nature, focusing on the simplest and the next simplest CL. Chapter 6 is devoted to a thorough study of general CL.

The following is an alphabetical list of letters used for specific mathematical concepts throughout Chapters 4–6 (s. §0.4):

A	the C language, i.e. the formal language of CL.
C	the set of the constants of A.
Γ	the set of the tokens of CL.
$\mathfrak{D}$	the set of the triples of $W \in \mathfrak{W}$, $\delta \in \Delta_W$ and $\upsilon \in \Upsilon_W$.
Δ_W	the set of the given denotations of C into W.
δ	an element of Δ_W.
E	the set of the entities of W.
$\mathfrak{e}$	the type of the derived entities of W.
ϵ	the type of the basic entities of W.
ε	the formal product of length 0 on various sets.
η	the nomina of various elements.
F	the set of the events of W.
G	the set of the nominals of A.
H	the set of the declaratives of A.
$\mathfrak{I}$	the index set of the operations $\mathfrak{q}\mathsf{k}$ ($\mathfrak{q} \in \mathfrak{Q}$, $\mathsf{k} \in \mathsf{K}$) of CL.
K	the set of the case markers or cases of CL.
κ	the modality of CL.
Λ	the set of the indices of the algebraic structures of A and T.
M	the set of the invariable indices in Λ.
N	the set of the nomina of CL.
$\mathfrak{o}$	the token for the operations $\mathfrak{o}\mathsf{k}$ ($\mathsf{k} \in \mathsf{K}$) of CL.
P	the set of the primes of A.
$\mathbb{P}$	the quantitative set of CL.

$\mathfrak{P}$	the set of the positive quantifiers of CL.
π	the nominative or principal case in K.
$\mathfrak{Q}$	the set of the quantifiers of CL.
S	the basis of W.
σ	the sorting of A.
T	the type of A.
$\mathbb{T}$	the binary lattice $\{0, 1\}$.
Υ_W	the set of all valuations of X into W.
υ	an element of Υ_W.
W	a C world, i.e. a given DW of CL.
$\mathfrak{W}$	the set of the C worlds.
Φ	the set of the functionals in Γ.
X	the set of the variables of A.

Some of these will be parameterized by the set N of nomina[4.1] in Chapters 5–6. Some of them will also be hidden in Chapter 4 or Chapter 5; particularly in this chapter, the set N and related concepts are hidden because $\#\mathsf{N} = 1$.

4.1 Instructive construction of ICL

Following §3.2, here we construct the logic system ICL. A logic system in GL as defined in §3.2.5 is a pair of a formal language and its semantics. Their counterparts in ICL are called the **IC language** and **IC semantics**.

In contrast to §3.2, however, our slogan here is *DWs first*, that is, we construct certain DWs of ICL first and then adapt other components of ICL to the DWs. This is because CL of some partibility together with a deduction system on it is intended to provide a mathematical model of the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$ and its DWs provide a model of $\mathcal{W}$. The slogan agrees with the belief that the IU has adapted to $\mathcal{W}$ (and to the SW) in organic evolution. Thus this section is instructive and slightly informal, leaving aside formalities for Chapter 6.

4.1.1 The IC worlds as the geneses

Following Remark 3.2.1, here we construct certain DWs for the IC language $(A, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ and call them the **IC worlds**, that is, an IC world is an algebra $(W, (\omega_\lambda)_{\lambda\in\mathsf{M}})$ equipped with a partition $W = \coprod_{t\in\mathsf{T}} W_t$ which satisfies the P-denotability and (3.2.3) for the algebraic structure $(\tau_\lambda)_{\lambda\in\mathsf{M}}$ of T_M, where M is the set of the invariable indices (3.2.1) of the algebraic structure of T.

As for the set W, we pick nonempty sets S and K and define

$$W = \mathsf{S} \amalg (\mathsf{S}{\to}\mathbb{T}) \amalg \coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T})$$

by the binary lattice $\mathbb{T} = \{0, 1\}$ and the power set $\mathfrak{P}\mathsf{K}$ of K, where we assume

$$(\emptyset{\to}\mathsf{S}){\to}\mathbb{T} = \mathbb{T}$$

[4.1] The word *nomina* is the plural form of the noun *nomen* (s. §1.3.1).

by the $\{\emptyset\}$ convention (s. [3.24]) because $\emptyset \to \mathsf{S} = \{\emptyset\}$ (s. [3.2]). We call S the **basis** of W and refer to the elements of $\mathbb{T}$ and K as the **truth values** and **cases** (or **case markers**) respectively.

As for the type T and the partition $W = \coprod_{t \in \mathsf{T}} W_t$, we define

$$\mathsf{T} = \{\epsilon, \mathfrak{e}\} \amalg \mathfrak{P}\mathsf{K},$$

picking the symbols ϵ and $\mathfrak{e}$, and then define

$$W_\epsilon = \mathsf{S}, \qquad W_\mathfrak{e} = \mathsf{S} \to \mathbb{T}, \qquad W_\mathsf{Q} = (\mathsf{Q} \to \mathsf{S}) \to \mathbb{T} \quad (\mathsf{Q} \in \mathfrak{P}\mathsf{K}).$$

Then

$$W_\emptyset = \mathbb{T} \tag{4.1.1}$$

by the above assumption and so

$$W_\mathfrak{e} = W_\epsilon \to W_\emptyset. \tag{4.1.2}$$

Since $\mathsf{S} \neq \emptyset$, we have $W_t \neq \emptyset$ for all $t \in \mathsf{T}$, and so the P-denotability is satisfied.

Furthermore, we define

$$\mathsf{E} = W_\epsilon \amalg W_\mathfrak{e}, \qquad \mathsf{F} = \coprod_{\mathsf{Q} \in \mathfrak{P}\mathsf{K}} W_\mathsf{Q},$$

so that

$$W = \mathsf{E} \amalg \mathsf{F}, \qquad \mathbb{T} = W_\emptyset \subseteq \mathsf{F}.$$

We refer to the elements of E, W_ϵ and $W_\mathfrak{e}$ as the **entities**, **basic entities** and **derived entities** respectively, because W_ϵ is the basis S of W and $W_\mathfrak{e} = \mathsf{S} \to \mathbb{T}$ may be identified with $\mathfrak{P}\mathsf{S}$ which is derived from S. Each element of $\mathsf{Q} \to \mathsf{S}$ ($\mathsf{Q} \in \mathfrak{P}\mathsf{K}$) may be identified with a family of elements of S indexed by Q, and so if $\#\mathsf{Q} = n < \infty$, $W_\mathsf{Q} = (\mathsf{Q} \to \mathsf{S}) \to \mathbb{T}$ may be identified with the set $\mathsf{S}^n \to \mathbb{T}$ of the $\mathbb{T}$-valued n-ary functions on S. Therefore, we refer to the elements of F as the **events** and furthermore, if $f \in W_\mathsf{Q}$ ($\mathsf{Q} \in \mathfrak{P}\mathsf{K}$), we call f a **Q-event**, call Q the **arity**[4.2] of f and denote it by K^f. Therefore, the truth values are the $\emptyset$-events, i.e. the events of arity $\emptyset$.

The algebraic structure $(\omega_\lambda)_{\lambda \in \mathsf{M}}$ of W is divided into the families defined in (W1)–(W8) below, which also clarify the set M.

For (W1)–(W7), we pick the tokens $\sqcap$, $\sqcup$, $\square$, $\triangle$, $\wedge$, $\vee$, $\Rightarrow$, $\lozenge$ and $\mathfrak{o}$, and also pick an arbitrary set Φ of *other* tokens. We refer to the elements of Φ as the **functionals** and refer to $\mathfrak{o}$ as the **case operationalizer** because it only associates each case k with an operation $\mathfrak{o}k$ in (W7), (T7) and (A7) below.

For (W1), we equip Φ with a family $(n_\phi)_{\phi \in \Phi}$ of positive integers and call n_ϕ ($\phi \in \Phi$) the **arity** of the functional ϕ.

For (W2)–(W4) and (W8), we pick a relation $\sqsubseteq$ on the basis S, which of course exists, and call it the **basic relation**. Then we extend it to a relation between S and $\mathsf{E} = \mathsf{S} \amalg (\mathsf{S} \to \mathbb{T})$ by the following for each $(s, a) \in \mathsf{S} \times (\mathsf{S} \to \mathbb{T})$:

$$s \sqsubseteq a \iff as = 1. \tag{4.1.3}$$

[4.2] The arity was called the *frame* in certain earlier manuscripts (s. Remark 1.4.1).

Remark 4.1.1 In Chapter 6 on general CL, we assume that the basic relation $\sqsubseteq$ satisfies some deductive law as defined in Remark 2.5.6. However, we disregard it in this introductory chapter; in particular, we do not assume here that $\sqsubseteq$ is reflexive contrary to certain earlier manuscripts (s. Remark 1.4.1).

For (W4), we pick an element π of K and call it the **nominative case**.

For each $(s, k) \in S \times K$ in (W7) and (W8), we define $(k/s) \in \{k\} \rightarrow S$ by

$$(k/s)k = s.$$

For each quadruple (s, k, Q, θ) also in (W7) and (W8) satisfying $s \in S$, $k \in Q \in \mathfrak{P}K$ and $\theta \in (Q - \{k\}) \rightarrow S$, we define $(k/s)\theta \in Q \rightarrow S$ by $((k/s)\theta)|_{Q-\{k\}} = \theta$ and $((k/s)\theta)|_{\{k\}} = (k/s)$, that is,

$$((k/s)\theta)l = \begin{cases} \theta l & \text{if } l \in Q - \{k\}, \\ s & \text{if } l = k. \end{cases} \tag{4.1.4}$$

If $Q = \{k\}$, then $Q - \{k\} = \emptyset$, $\theta = \emptyset$ (s. [3.2]) and $(k/s)\emptyset = (k/s)$.

For (W8), we pick a nontrivial quantitative set $\mathbb{P}$ and a subset $\mathfrak{P}$ of the power set $\mathfrak{P}\mathbb{P}$ of $\mathbb{P}$. Then we define

$$\mathfrak{Q} = \mathfrak{P} \amalg \neg\mathfrak{P}$$

by a copy

$$\neg\mathfrak{P} = \{\neg\mathfrak{p} : \mathfrak{p} \in \mathfrak{P}\}$$

of $\mathfrak{P}$ by the symbol $\neg$ (s. §1.5.2), where if $\mathfrak{P} = \emptyset$ then $\neg\mathfrak{P} = \mathfrak{Q} = \emptyset$ by definition. We refer to the elements of $\mathfrak{Q}, \mathfrak{P}$ and $\neg\mathfrak{P}$ as the **quantifier**s, **positive quantifier**s and **negative quantifier**s respectively. We also pick a $\mathbb{P}$-measure $Y \mapsto |Y|$ on S, which exists by Remark 3.6.15, and abbreviate the expression

$$|\{\text{a specification of the members of } Y\}|$$

to the expression |a specification of the members of Y| without braces.

Remark 4.1.2 In Chapter 6 on general CL, we also deal with proportional quantifiers in case $\mathbb{P}$ is unital, but I disregard them in this introductory chapter.

The operation symbols ω_λ ($\lambda \in M$) in (W1)–(W8) are abbreviated to λ; for example, the operation symbols ω_ϕ ($\phi \in \Phi$) in (W1) are abbreviated to ϕ. Except for the postpositive operation symbols of various arities in (W1) and (W4), the binary ones in (W2), (W5), (W7) and (W8) are interpositions and the unary ones in (W3) and (W6) are superscript.

(W1) An arbitrary family of n_ϕ-ary operations $\phi \in \Phi$ such that $\mathrm{Dm}\,\phi = {W_\epsilon}^{n_\phi}$ and $(a_1, \ldots, a_{n_\phi})\phi \in W_\epsilon$ for all $(a_1, \ldots, a_{n_\phi}) \in {W_\epsilon}^{n_\phi}$.

(W2) The binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dm}\sqcap = \mathrm{Dm}\sqcup = \mathsf{E}^2 = \coprod_{(t,u)\in\{\epsilon,\mathfrak{e}\}^2}(W_t \times W_u)$$

and if $(a, b) \in \mathsf{E}^2$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $W_\mathfrak{e} = \mathsf{S}{\to}\mathbb{T}$ defined by the following for each $s \in \mathsf{S}$:

$$s \sqsubseteq a \sqcap b \iff s \sqsubseteq a \text{ and } s \sqsubseteq b,$$

$$s \sqsubseteq a \sqcup b \iff s \sqsubseteq a \text{ or } s \sqsubseteq b.$$

Because of (4.1.3), this certainly defines $a{\sqcap}b$ and $a{\sqcup}b$ as elements of $\mathsf{S}{\to}\mathbb{T}$ and if $(a, b) \in (\mathsf{S}{\to}\mathbb{T})^2$, then $(a \sqcap b)s = as \wedge bs$ and $(a \sqcup b)s = as \vee bs$ for each $s \in \mathsf{S}$, where $\wedge$ and $\vee$ are the meet and join on the binary lattice $\mathbb{T}$, that is, $u \wedge v = \inf\{u, v\}$ and $u \vee v = \sup\{u, v\}$ for each $(u, v) \in \mathbb{T}^2$.

(W3) The unary operation $\square$ such that

$$\mathrm{Dm}\,\square = \mathsf{E} = W_\epsilon \amalg W_\mathfrak{e}$$

and if $a \in \mathsf{E}$, then its image $a^\square$ is the element of $W_\mathfrak{e} = \mathsf{S}{\to}\mathbb{T}$ defined by the following for each $s \in \mathsf{S}$:

$$s \sqsubseteq a^\square \iff s \not\sqsubseteq a.$$

Because of (4.1.3), this certainly defines $a^\square$ as an element of $\mathsf{S}{\to}\mathbb{T}$ and if $a \in \mathsf{S}{\to}\mathbb{T}$, then $a^\square s = (as)^\Diamond$ for each $s \in \mathsf{S}$, where $\Diamond$ is the complement on the binary lattice $\mathbb{T}$, that is, $v^\Diamond = 1 - v$ for each $v \in \mathbb{T}$ (s. [1.79]).

(W4) The unary operation $\triangle$ such that

$$\mathrm{Dm}\,\triangle = \mathsf{E} = W_\epsilon \amalg W_\mathfrak{e}$$

and if $a \in \mathsf{E}$, then its image $a\triangle$ is the element of $W_{\{\pi\}} = (\{\pi\}{\to}\mathsf{S}){\to}\mathbb{T}$ defined by the following for each $\theta \in \{\pi\}{\to}\mathsf{S}$:

$$(a\triangle)\theta = 1 \iff \theta\pi \sqsubseteq a.$$

Thus if $a \in \mathsf{S}{\to}\mathbb{T}$, then $(a\triangle)\theta = a(\theta\pi)$ for each $\theta \in \{\pi\}{\to}\mathsf{S}$ by (4.1.3).

(W5) The three binary operations $\wedge$, $\vee$ and $\Rightarrow$ such that

$$\begin{aligned}\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow &= \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(((Q{\to}\mathsf{S}){\to}\mathbb{T}) \times ((R{\to}\mathsf{S}){\to}\mathbb{T})) \\ &= \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(W_Q \times W_R) = \mathsf{F}^2\end{aligned}$$

and if $(Q, R) \in (\mathfrak{P}K)^2$ and $(f, g) \in ((Q{\to}\mathsf{S}){\to}\mathbb{T}) \times ((R{\to}\mathsf{S}){\to}\mathbb{T})$, then its images $f \wedge g$, $f \vee g$ and $f \Rightarrow g$ are the elements of $W_{Q\cup R} = ((Q{\cup}R){\to}\mathsf{S}){\to}\mathbb{T}$ defined by the following for each $\theta \in (Q \cup R){\to}\mathsf{S}$:

$$(f \wedge g)\theta = f(\theta|_Q) \wedge g(\theta|_R),$$

$$(f \vee g)\theta = f(\theta|_Q) \vee g(\theta|_R),$$
$$(f \Rrightarrow g)\theta = f(\theta|_Q) \Rrightarrow g(\theta|_R).$$

Here $\wedge$, $\vee$ and $\Rrightarrow$ on the right-hand sides are the meet, join and cojoin on the binary lattice $\mathbb{T}$, and so $u \Rrightarrow v = u^{\Diamond} \vee v$ for each $(u, v) \in \mathbb{T}^2$ and for the complement $\Diamond$ on $\mathbb{T}$ (s. §1.5.2).

(W6) The unary operation $\Diamond$ such that

$$\mathrm{Dm}\,\Diamond = \coprod_{Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T}) = \coprod_{Q \in \mathfrak{P}K} W_Q = F$$

and if $Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then its image $f^{\Diamond}$ is the element of $W_Q = (Q \to S) \to \mathbb{T}$ defined by the following for each $\theta \in Q \to S$:

$$(f^{\Diamond})\theta = (f\theta)^{\Diamond}.$$

Here $\Diamond$ on the right-hand side is the complement on the binary lattice $\mathbb{T}$.

(W7) The family of binary operations $\mathfrak{o}k$ $(k \in K)$ such that

$$\mathrm{Dm}\,\mathfrak{o}k = S \times \coprod_{k \in Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T}) = \coprod_{k \in Q \in \mathfrak{P}K}(W_{\epsilon} \times W_Q)$$

and if $s \in S$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then the image $s\,\mathfrak{o}k\,f$ of (s, f) is the element of $W_{Q-\{k\}} = ((Q - \{k\}) \to S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q - \{k\}) \to S$:

$$(s\,\mathfrak{o}k\,f)\theta = f((k/s)\theta).$$

Here if $Q = \{k\}$, then $s\,\mathfrak{o}k\,f = f(k/s) \in \mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]).

(W8) The family of binary operations $\mathfrak{q}k$ $((\mathfrak{q}, k) \in \mathfrak{Q} \times K)$ such that

$$\mathrm{Dm}\,\mathfrak{q}k = E \times \coprod_{k \in Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T}) = \coprod_{t \in \{\epsilon, \mathfrak{e}\}, k \in Q \in \mathfrak{P}K}(W_t \times W_Q)$$

and if $a \in E$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then the image $a\,\mathfrak{q}k\,f$ of (a, f) is the element of $W_{Q-\{k\}} = ((Q - \{k\}) \to S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q - \{k\}) \to S$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}$:

$$(a\,\mathfrak{q}k\,f)\theta = 1 \iff |s \in S : s \sqsubseteq a, f((k/s)\theta) = \nu| \in \mathfrak{p}.$$

This definition makes sense because $\mathfrak{q} \in \mathfrak{Q} = \mathfrak{P} \amalg \neg\mathfrak{P}$. Notice that $f((k/s)\theta) = (s\,\mathfrak{o}k\,f)\theta$ and if $Q = \{k\}$ then $(a\,\mathfrak{q}k\,f)\theta = a\,\mathfrak{q}k\,f \in \mathbb{T}$ and $f((k/s)\theta) = f(k/s) = s\,\mathfrak{o}k\,f \in \mathbb{T}$ by (W7) and the $\{\emptyset\}$ convention.

This completes the definition of $(\omega_\lambda)_{\lambda \in M}$. Thus

$$M = \Phi \amalg \{\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond\} \amalg \{\mathfrak{o}k : k \in K\} \amalg \{\mathfrak{q}k : (\mathfrak{q}, k) \in \mathfrak{Q} \times K\},$$

and the partition $W = \coprod_{t \in T} W_t$ satisfies (3.2.3) for each $\lambda \in M$ provided that we define the algebraic structure $(\tau_\lambda)_{\lambda \in M}$ of T_M as the union of the following families (T1)–(T8). As with ω_λ ($\lambda \in \mathsf{M}$) in (W1)–(W8), the operation symbols τ_λ ($\lambda \in \mathsf{M}$) here are abbreviated to λ and, except for the postpositive operation symbols of various arities in (T1) and (T4), the binary ones in (T2), (T5), (T7) and (T8) are interpositions and the unary ones in (T3) and (T6) are superscript.

(T1) The family of n_ϕ-ary operations $\phi \in \Phi$ defined by $\mathrm{Dm}\,\phi = \{\epsilon\}^{n_\phi}$ and $(\overbrace{\epsilon, \ldots, \epsilon}^{n_\phi})\phi = \epsilon$ for the unique element $(\overbrace{\epsilon, \ldots, \epsilon}^{n_\phi})$ of $\{\epsilon\}^{n_\phi}$.

(T2) The binary operations $\sqcap$ and $\sqcup$ defined by $\mathrm{Dm}\sqcap = \mathrm{Dm}\sqcup = \{\epsilon, \mathfrak{e}\}^2$ and $t \sqcap u = t \sqcup u = \mathfrak{e}$ for each $(t, u) \in \{\epsilon, \mathfrak{e}\}^2$.

(T3) The unary operation $\Box$ defined by $\mathrm{Dm}\,\Box = \{\epsilon, \mathfrak{e}\}$ and $\epsilon^\Box = \mathfrak{e}^\Box = \mathfrak{e}$.

(T4) The unary operation $\triangle$ defined by $\mathrm{Dm}\,\triangle = \{\epsilon, \mathfrak{e}\}$ and $\epsilon\triangle = \mathfrak{e}\triangle = \{\pi\}$.

(T5) The binary operations $\wedge$, $\vee$ and $\Rrightarrow$ defined by $\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rrightarrow = (\mathfrak{P}\mathsf{K})^2$ and $\mathsf{Q} \wedge \mathsf{R} = \mathsf{Q} \vee \mathsf{R} = \mathsf{Q} \Rrightarrow \mathsf{R} = \mathsf{Q} \cup \mathsf{R}$ for each $(\mathsf{Q}, \mathsf{R}) \in (\mathfrak{P}\mathsf{K})^2$.

(T6) The unary operation $\Diamond$ defined by $\mathrm{Dm}\,\Diamond = \mathfrak{P}\mathsf{K}$ and $\mathsf{Q}^\Diamond = \mathsf{Q}$ for each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$.

(T7) The family of binary operations $\mathfrak{o}\mathsf{k}$ ($\mathsf{k} \in \mathsf{K}$) defined by $\mathrm{Dm}\,\mathfrak{o}\mathsf{k} = \{\epsilon\} \times \{\mathsf{Q} \in \mathfrak{P}\mathsf{K} : \mathsf{k} \in \mathsf{Q}\}$ and $\epsilon\, \mathfrak{o}\mathsf{k}\, \mathsf{Q} = \mathsf{Q} - \{\mathsf{k}\}$ for each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$ satisfying $\mathsf{k} \in \mathsf{Q}$.

(T8) The family of binary operations $\mathfrak{q}\mathsf{k}$ ($(\mathfrak{q}, \mathsf{k}) \in \mathfrak{Q} \times \mathsf{K}$) defined by $\mathrm{Dm}\,\mathfrak{q}\mathsf{k} = \{\epsilon, \mathfrak{e}\} \times \{\mathsf{Q} \in \mathfrak{P}\mathsf{K} : \mathsf{k} \in \mathsf{Q}\}$ and $\epsilon\, \mathfrak{q}\mathsf{k}\, \mathsf{Q} = \mathfrak{e}\, \mathfrak{q}\mathsf{k}\, \mathsf{Q} = \mathsf{Q} - \{\mathsf{k}\}$ for each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$ satisfying $\mathsf{k} \in \mathsf{Q}$.

This completes the construction of the IC worlds.

4.1.2 The IC language adapted to the IC worlds

Following §3.2.1, here we construct the IC language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ by defining its syntax $(\mathsf{T}, \sigma|_\mathsf{P}, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ called the **IC syntax**. It is adapted to the IC worlds constructed in §4.1.1 (and to the SW). In particular $\mathsf{T} = \{\epsilon, \mathfrak{e}\} \amalg \mathfrak{P}\mathsf{K}$, whose algebraic structure is an extensioin of that of T_M in (T1)–(T8).

As for the sets P, C and X of its primes, constants and variables, we assume

$$\mathsf{P} = \mathsf{C} \amalg \mathsf{X}, \qquad\qquad \mathsf{X} \neq \emptyset,$$

that is, we only assume the conditions stated in §3.2.1.

As for the basic sorting $\sigma|_\mathsf{P}$, we assume that the subset X_ϵ of X defined by

$$\mathsf{X}_\epsilon = \sigma|_\mathsf{X}{}^{-1}\{\epsilon\}$$

is nonempty, strengthening the assumption $\mathsf{X} \neq \emptyset$. This is a condition on $\sigma|_\mathsf{P}$ because $\mathsf{X} \subseteq \mathsf{P}$ and $\sigma|_\mathsf{X} = (\sigma|_\mathsf{P})|_\mathsf{X}$.

As for the type T, we define the algebraic structure $(\tau_\lambda)_{\lambda \in \Lambda}$ of T as the union of the family $(\tau_\lambda)_{\lambda \in \mathsf{M}}$ in (T1)–(T8) and a family $(\tau_{\triangledown x})_{x \in \mathsf{X}_\epsilon}$ in the following

(T9), where the operation symbols $\tau_{\triangledown x}$ $(x \in X_\epsilon)$ are abbreviated to $\triangledown x$ as with τ_λ $(\lambda \in M)$ and are postpositive. We call $\triangledown x$ $(x \in X_\epsilon)$ the **nominalizer**s, while we give no name to $\triangledown$. Notice that $\{\emptyset\} \subseteq \mathfrak{P}K \subseteq T$ as to their domain.

(T9) The family of unary operations $\triangledown x$ $(x \in X_\epsilon)$ defined by $\mathrm{Dm}\, \triangledown x = \{\emptyset\}$ and $\emptyset\, \triangledown x = \mathfrak{e}$.

Therefore, the algebraic structure of T is indexed by the set

$$\begin{aligned}\Lambda &= M \amalg \{\triangledown x : x \in X_\epsilon\} \\ &= \Phi \amalg \{\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond\} \amalg \{\mathfrak{o}k : k \in K\} \amalg \{\mathfrak{q}k : (\mathfrak{q}, k) \in \mathfrak{Q} \times K\} \\ &\quad \amalg \{\triangledown x : x \in X_\epsilon\},\end{aligned}$$

and defining

$$\Gamma = \Phi \amalg \{\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond, \mathfrak{o}, \triangledown\} \amalg \mathfrak{Q} \amalg K,$$

we have

$$\Lambda \subseteq (\Gamma \amalg X)^*.$$

This completes the construction of the IC language $(A, T, \sigma, P, C, X, \Gamma)$ by means of the definition of the IC syntax $(T, \sigma|_P, P, C, X, \Gamma)$ which is adapted to the IC worlds constructed in §4.1.1.

Still, some definitions and remarks are in order.

First, M is equal to the set $\Lambda \cap \Gamma^*$ of the invariable indices of the algebraic structure $(\tau_\lambda)_{\lambda \in \Lambda}$ of T and so the IC worlds are DWs for the IC language.

Secondly, since (A, T, σ) is a sorted algebra, A is divided into its t-parts $A_t = \sigma^{-1}\{t\}$ $(t \in T)$, and since $T = \{\epsilon, \mathfrak{e}\} \amalg \mathfrak{P}K$, we have the partition

$$A = A_\epsilon \amalg A_\mathfrak{e} \amalg \textstyle\coprod_{Q \in \mathfrak{P}K} A_Q.$$

We define

$$G = A_\epsilon \amalg A_\mathfrak{e}, \qquad H = \textstyle\coprod_{Q \in \mathfrak{P}K} A_Q,$$

so that

$$A = G \amalg H, \qquad A_\emptyset \subseteq H.$$

We refer to the elements of G, A_ϵ and $A_\mathfrak{e}$ as the **nominal**s, **ϵ-nominal**s and **$\mathfrak{e}$-nominal**s respectively. We also refer to the elements of H and $A_\emptyset$ as the **declarative**s and **sentence**s respectively. Furthermore, if $f \in A_Q$ $(Q \in \mathfrak{P}K)$, we call f a **Q-declarative**, call Q the **arity**[4.3] of f and denote it by K^f. Therefore, the sentences are the $\emptyset$-declaratives, i.e. the declaratives of arity $\emptyset$.

Thirdly, also since (A, T, σ) is a sorted algebra, Remark 3.1.5 and (T1)–(T9) show that the algebraic structure $(\alpha_\lambda)_{\lambda \in \Lambda}$ of A satisfies the following (A1)–(A9); in particular, the definitions in (T9) are designed to imply (A9). As with τ_λ $(\lambda \in \Lambda)$ in (T1)–(T9), the operation symbols α_λ $(\lambda \in \Lambda)$ here are abbreviated to λ and, except for the postpositive operation symbols of various arities in (A1), (A4) and (A9), the binary ones in (A2), (A5), (A7) and (A8) are interpositions and the unary ones in (A3) and (A6) are superscript.

[4.3] The arity was called the *range* in certain earlier manuscripts (s. Remark 1.4.1).

(A1) $\mathrm{Dm}\,\phi = A_\epsilon{}^{n_\phi}$ for each $\phi \in \Phi$
and $(a_1, \dots, a_{n_\phi})\phi \in A_\epsilon$ for all $(a_1, \dots, a_{n_\phi}) \in A_\epsilon{}^{n_\phi}$.

(A2) $\mathrm{Dm}\sqcap = \mathrm{Dm}\sqcup = \coprod_{(t,u)\in\{\epsilon,\mathfrak{e}\}^2}(A_t \times A_u) = G^2$ and $a \sqcap b, a \sqcup b \in A_\mathfrak{e}$ for all $(a, b) \in G^2$.

(A3) $\mathrm{Dm}\,\Box = A_\epsilon \amalg A_\mathfrak{e} = G$ and $a^\Box \in A_\mathfrak{e}$ for all $a \in G$.

(A4) $\mathrm{Dm}\,\triangle = A_\epsilon \amalg A_\mathfrak{e} = G$ and $a\triangle \in A_{\{\pi\}}$ for all $a \in G$.

(A5) $\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow = \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(A_Q \times A_R) = H^2$ and if $(Q, R) \in (\mathfrak{P}K)^2$ and $(f, g) \in A_Q \times A_R$, then $f \wedge g, f \vee g, f \Rightarrow g \in A_{Q\cup R}$, that is, $f \wedge g, f \vee g, f \Rightarrow g \in A_{K^f \cup K^g}$ for each $(f, g) \in H^2$.

(A6) $\mathrm{Dm}\,\Diamond = \coprod_{Q\in\mathfrak{P}K} A_Q = H$ and if $Q \in \mathfrak{P}K$ and $f \in A_Q$, then $f^\Diamond \in A_Q$, that is, $f^\Diamond \in A_{K^f}$ for each $f \in H$. We refer to the operation $\Diamond$ on A and its values $f^\Diamond$ at $f \in H$ as the **negation**(s) (s. [1.96]).

(A7) $\mathrm{Dm}\,\mathfrak{o}k = \coprod_{k\in Q\in\mathfrak{P}K}(A_\epsilon \times A_Q) = A_\epsilon \times \coprod_{k\in Q\in\mathfrak{P}K} A_Q = A_\epsilon \times \{f \in H : k \in K^f\}$ for each $k \in K$ and if $a \in A_\epsilon$, $k \in Q \in \mathfrak{P}K$ and $f \in A_Q$, then $a\,\mathfrak{o}k\,f \in A_{Q-\{k\}}$, that is, $a\,\mathfrak{o}k\,f \in A_{K^f-\{k\}}$ for each $a \in A_\epsilon$ and each $f \in H$ such that $k \in K^f$.

(A8) $\mathrm{Dm}\,\mathfrak{q}k = \coprod_{t\in\{\epsilon,\mathfrak{e}\},k\in Q\in\mathfrak{P}K}(A_t \times A_Q) = G \times \coprod_{k\in Q\in\mathfrak{P}K} A_Q = G \times \{f \in H : k \in K^f\}$ for each $(\mathfrak{q}, k) \in \mathfrak{Q} \times K$ and if $a \in G$, $k \in Q \in \mathfrak{P}K$ and $f \in A_Q$, then $a\,\mathfrak{q}k\,f \in A_{Q-\{k\}}$, that is, $a\,\mathfrak{q}k\,f \in A_{K^f-\{k\}}$ for each $a \in G$ and each $f \in H$ such that $k \in K^f$.

(A9) $\mathrm{Dm}\,\triangledown x = A_\emptyset$ for all $x \in X_\epsilon$ and $f \triangledown x \in A_\mathfrak{e}$ for all $f \in A_\emptyset$.

Lastly, the following hold as consequences of (A1)–(A9).

(B1) Let $\mathfrak{q}_1, \dots, \mathfrak{q}_n \in \{\mathfrak{o}\} \cup \mathfrak{Q}$, $a_1, \dots, a_n \in G$, $f \in H$, and let $k_1, \dots, k_n$ be distinct cases in K^f. Assume that $a_i \in A_\epsilon$ for all $i \in \{1, \dots, n\}$ such that $\mathfrak{q}_i = \mathfrak{o}$. Then $a_1\,\mathfrak{q}_1 k_1\,(\cdots(a_n\,\mathfrak{q}_n k_n\,f)\cdots) \in A_{K^f-\{k_1,\dots,k_n\}}$.

(B2) The set A_ϵ is nonempty and closed by the operations in Φ, whose restrictions to A_ϵ are total. Moreover, A_ϵ is the closure $[P_\epsilon]_\Phi$ of the ϵ-part P_ϵ of P in the Φ-reduct A_Φ of A. Thus $A_\epsilon - P$ is nonempty iff $\Phi \neq \emptyset$, and each its element has a ramification $(a_1, \dots, a_{n_\phi})\phi$ for a functional $\phi \in \Phi$ and an element $(a_1, \dots, a_{n_\phi}) \in A_\epsilon{}^{n_\phi}$.

(B3) The sets G and $A_\mathfrak{e}$ are closed by the operations $\sqcap, \sqcup$ and $\Box$, whose restrictions to G and $A_\mathfrak{e}$ are total.

(B4) The set $A_\mathfrak{e} - P$ is nonempty, and each its element has a ramification $a \sqcap b$, $a \sqcup b$, $a^\Box$ or $f \triangledown x$ for elements $a, b, f, x \in A$ which satisfy the conditions shown in (A2), (A3) and (A9).

(B5) The sets H and A_{Q} ($\mathsf{Q} \in \mathfrak{P}\mathsf{K}$) are closed by the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, whose restrictions to H and A_{Q} are total.

(B6) The set $\mathsf{H} - \mathsf{P}$ is nonempty, and each its element has a ramification $\mathsf{a}\triangle$, $\mathsf{f} \wedge \mathsf{g}$, $\mathsf{f} \vee \mathsf{g}$, $\mathsf{f} \Rightarrow \mathsf{g}$, $\mathsf{f}^{\Diamond}$, $\mathsf{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f}$ or $\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$ for tokens $\mathsf{k}, \mathsf{q} \in \Gamma$ and elements $\mathsf{a}, \mathsf{f}, \mathsf{g} \in \mathsf{A}$ which satisfy the conditions shown in (A4)–(A8).

(B7) The set $\mathsf{A}_{\emptyset} - \mathsf{P}$ is nonempty, and each its element has a ramification $\mathsf{f} \wedge \mathsf{g}$, $\mathsf{f} \vee \mathsf{g}$, $\mathsf{f} \Rightarrow \mathsf{g}$, $\mathsf{f}^{\Diamond}$, $\mathsf{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f}$ or $\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$ for tokens $\mathsf{k}, \mathsf{q} \in \Gamma$ and elements $\mathsf{a}, \mathsf{f}, \mathsf{g} \in \mathsf{A}$ which satisfy the conditions shown in (A5)–(A8) including the conditions necessary for the ramifications to belong to $\mathsf{A}_{\emptyset}$.

As for the main part of (B2), A_{ϵ} is closed by the operations in Φ by (A1), and so $\mathsf{A}_{\epsilon} \supseteq [\mathsf{P}_{\epsilon}]_{\Phi}$. We can conversely show that every element $\mathsf{a} \in \mathsf{A}_{\epsilon}$ belongs to $[\mathsf{P}_{\epsilon}]_{\Phi}$ by induction on $\mathsf{r} = \mathrm{rk}\,\mathsf{a}$. Since $\mathsf{P} \cap \mathsf{A}_{\epsilon} = \mathsf{P}_{\epsilon}$, we may assume $\mathsf{r} \geq 1$. Then since $\mathsf{a} \in \mathsf{A}_{\epsilon}$, (A1)–(A9) show that $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{n}_\phi})\phi$ for some $\phi \in \Phi$ and some $\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{n}_\phi} \in \mathsf{A}_{\epsilon}$ such that $\mathrm{rk}\,\mathsf{a} - 1 = \sum_{\mathsf{j}=1}^{\mathsf{n}_\phi} \mathrm{rk}\,\mathsf{a}_{\mathsf{j}}$. Since $\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{n}_\phi} \in [\mathsf{P}_{\epsilon}]_{\Phi}$ by the induction hypothesis, we have $\mathsf{a} \in [\mathsf{P}_{\epsilon}]_{\Phi}$ as desired. We can derive $\mathsf{A}_{\mathfrak{e}} - \mathsf{P} \neq \emptyset$ in (B4) and $\mathsf{A}_{\emptyset} - \mathsf{P} \neq \emptyset$ in (B7) as well as $\mathsf{A}_{\epsilon} \neq \emptyset$ in (B2) from our assumption $\mathsf{X}_{\epsilon} \neq \emptyset$. Indeed, if $\mathsf{x} \in \mathsf{X}_{\epsilon}$, then $\mathsf{x}\,\mathfrak{o}\pi\,\mathsf{x}\triangle \in \mathsf{A}_{\emptyset} - \mathsf{P}$ and $(\mathsf{x}\,\mathfrak{o}\pi\,\mathsf{x}\triangle)\nabla\mathsf{x} \in \mathsf{A}_{\mathfrak{e}} - \mathsf{P}$ by (A4), (A7) and (A9).

4.1.3 The IC semantics and its aim at the nominalizers

As the final step of the instructive construction of ICL, here we define the IC semantics $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ for the IC language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ constructed in §4.1.2. Then its component $(\mathrm{I}_W)_{W\in\mathfrak{W}}$ will be illustrated by means of the metaworld $W^\sharp$ defined in §3.3.1 for $W \in \mathfrak{W}$, the denotation $\varphi^\delta_\upsilon \in \mathsf{A}{\to}W$ defined in §3.3.2 for $(\delta, \upsilon) \in \Delta_W \times \Upsilon_W$ and functional expressions a^δ of elements a of A on W defined in §3.3.6 for $\delta \in \Delta_W$.

As was noted above, M is equal to the set $\Lambda \cap \Gamma^*$ of the invariable indices of the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ of T and so the IC worlds constructed in §4.1.1 are DWs for the IC language. Thus we define $\mathfrak{W}$ to be the collection of all the IC worlds and then define Δ_W ($W \in \mathfrak{W}$) to be the set of all denotations of C into W. The set $\Lambda - \mathsf{M}$ of the variable indices of $(\tau_\lambda)_{\lambda\in\Lambda}$ is equal to $\{\nabla\mathsf{x} : \mathsf{x} \in \mathsf{X}_\epsilon\}$. We define its interpretation $\mathrm{I}_W = (\nabla\mathsf{x}_W)_{\mathsf{x}\in\mathsf{X}_\epsilon}$ on each $W \in \mathfrak{W}$ following §3.2.4. Let $\mathsf{x} \in \mathsf{X}_\epsilon$. Then $\nabla\mathsf{x} \in \Lambda_\mathsf{x}$ and $\sigma\mathsf{x} = \epsilon$ by definition, and $\mathrm{Dm}\,\tau_{\nabla\mathsf{x}} = \{\emptyset\}$ and $\tau_{\nabla\mathsf{x}}\,\emptyset = \mathfrak{e}$ by (T9). Therefore, (3.2.5), (3.2.6) and (4.1.2) show that the significance $\nabla\mathsf{x}_W$ of $\nabla\mathsf{x}$ on W can be an arbitrary element of $(W_\epsilon{\to}W_\emptyset){\to}(W_\epsilon{\to}W_\emptyset)$. Thus we define $\nabla\mathsf{x}_W$ to be the identity transformation on $W_\epsilon{\to}W_\emptyset$.

This completes the definition of the IC semantics and thereby completes the construction of ICL $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma, \mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$.

Now then, why do we so define $\mathrm{I}_W = (\nabla\mathsf{x}_W)_{\mathsf{x}\in\mathsf{X}_\epsilon}$ for each $W \in \mathfrak{W}$? Let $\beta_{\nabla\mathsf{x}}$ ($\mathsf{x} \in \mathsf{X}_\epsilon$) be the operation on the metaworld $W^\sharp$ defined in §3.3.1. Then (3.3.9), (T9), (4.1.2), and (3.3.8) show that

$$\mathrm{Dm}\,\beta_{\nabla\mathsf{x}} = \Upsilon_W{\to}W_\emptyset, \qquad \mathrm{Im}\,\beta_{\nabla\mathsf{x}} \subseteq \Upsilon_W{\to}(W_\epsilon{\to}W_\emptyset)$$

and the following holds for each $\varphi \in \Upsilon_W \to W_\emptyset$ and each $\upsilon \in \Upsilon_W$:

$$(\beta_{\triangledown x}\varphi)\upsilon = \triangledown x_W(\varphi(\upsilon(x/\Box))).$$

Here (3.3.1), (3.3.3) and (3.3.4) show that $\varphi(\upsilon(x/\Box)) \in W_\epsilon \to W_\emptyset$ and

$$(\varphi(\upsilon(x/\Box)))w = \varphi(\upsilon(x/w))$$

for each $w \in W_\epsilon$. Therefore, the definition of I_W is designed to imply that

$$((\beta_{\triangledown x}\varphi)\upsilon)w = \varphi(\upsilon(x/w))$$

for each $(\varphi, \upsilon, w) \in (\Upsilon_W \to W_\emptyset) \times \Upsilon_W \times W_\epsilon$. Thus, abbreviating $\beta_{\triangledown x}$ to $\triangledown x$ and assuming it to be postpositive as with the abbreviation $\triangledown x$ of $\tau_{\triangledown x}$, we have the following for each $(\varphi, \upsilon, w) \in (\Upsilon_W \to W_\emptyset) \times \Upsilon_W \times W_\epsilon$:

$$((\varphi \triangledown x)\upsilon)w = \varphi(\upsilon(x/w)).$$

Now, let $f \in A_\emptyset$, $x \in X_\epsilon$ and $\delta \in \Delta_W$. Then $f \in \mathrm{Dm}\,\triangledown x$ by (A9) and $\delta^\sharp(f \triangledown x) = (\delta^\sharp f)\triangledown x$ and $\delta^\sharp f \in \Upsilon_W \to W_\emptyset$ because $\delta^\sharp$ is a homotypic homomorphism of A into $W^\sharp$. Therefore,

$$((\delta^\sharp(f \triangledown x))\upsilon)w = (\delta^\sharp f)(\upsilon(x/w))$$

for each $(\upsilon, w) \in \Upsilon_W \times W_\epsilon$, that is, the denotations φ^δ_υ and $\varphi^\delta_{\upsilon(x/w)}$ of A satisfy

$$(\varphi^\delta_\upsilon(f \triangledown x))w = \varphi^\delta_{\upsilon(x/w)} f. \tag{4.1.5}$$

Identifying $W_\epsilon \to W_\emptyset$ with $\mathfrak{P}W_\epsilon$ by virtue of (4.1.1), we have

$$\varphi^\delta_\upsilon(f \triangledown x) = \{w \in W_\epsilon : \varphi^\delta_{\upsilon(x/w)} f = 1\}$$

as was illustrated by (1.2.3) (s. §3.3.2).

Now, let $f \in A_\emptyset$, $x \in X_\epsilon$ and $\delta \in \Delta_W$ again. Then $f \in \mathrm{Dm}\,\triangledown x$ and $f \triangledown x \in A_\epsilon$ by (A9). Let $(y_1, \ldots, y_n)$ be a basis of $f \triangledown x$, so that we may assume $x \notin \{y_1, \ldots, y_n\}$ by Lemma 3.1.12. Then Theorem 3.3.4 and (4.1.2) show that $(y_1, \ldots, y_n, x)$ is a basis of f and the functional expressions $(f \triangledown x)^\delta(y_1, \ldots, y_n)$ and $f^\delta(y_1, \ldots, y_n, x)$ of $f \triangledown x$ and f under δ satisfy

$$(f \triangledown x)^\delta(y_1, \ldots, y_n) \in (W_{\sigma y_1} \times \cdots \times W_{\sigma y_n}) \to (W_\epsilon \to W_\emptyset),$$
$$f^\delta(y_1, \ldots, y_n, x) \in (W_{\sigma y_1} \times \cdots \times W_{\sigma y_n} \times W_\epsilon) \to W_\emptyset$$

and the following for each $(w_1, \ldots, w_n) \in W_{\sigma y_1} \times \cdots \times W_{\sigma y_n}$:

$$(f \triangledown x)^\delta(w_1, \ldots, w_n) = \triangledown x_W(f^\delta(w_1, \ldots, w_n, \Box)).$$

Here $f^\delta(w_1, \ldots, w_n, \Box) \in W_\epsilon \to W_\emptyset$ and

$$(f^\delta(w_1, \ldots, w_n, \Box))w = f^\delta(w_1, \ldots, w_n, w)$$

for each $w \in W_\epsilon$. Therefore, the definition of I_W is also designed to imply

$$((f \triangledown x)^\delta(w_1, \ldots, w_n))w = f^\delta(w_1, \ldots, w_n, w)$$

for each $(w_1, \ldots, w_n, w) \in W_{\sigma y_1} \times \cdots \times W_{\sigma y_n} \times W_\epsilon$, that is, $(f \triangledown x)^\delta(y_1, \ldots, y_n)$ is a linearization of $f^\delta(y_1, \ldots, y_n, x)$ (s. Remark 1.3.2). Identifying $W_\epsilon \to W_\emptyset$ with $\mathfrak{P}W_\epsilon$ by virtue of (4.1.1), we have

$$(f \triangledown x)^\delta(w_1, \ldots, w_n) = \{w \in W_\epsilon : f^\delta(w_1, \ldots, w_n, w) = 1\}$$

for each $(w_1, \ldots, w_n) \in W_{\sigma y_1} \times \cdots \times W_{\sigma y_n}$.

4.2 Valuable sequential IC tautologies

Let $(A, T, \sigma, P, C, X, \Gamma, \mathfrak{W}, (\mathrm{I}_W)_{W \in \mathfrak{W}}, (\Delta_W)_{W \in \mathfrak{W}})$ be the logic system ICL defined in §4.1. This section collects its tautologies which have proved to be valuable for the study of deduction systems on it, its phraseology as in §4.3 and the embedding of FPL in it.

4.2.1 The IC logic space on the declaratives

ICL has a truth type $\emptyset$, because $\emptyset \in \mathfrak{P}K \subseteq T$, $A_\emptyset \neq \emptyset$ by (B7) and $W_\emptyset$ is the binary lattice $\mathbb{T}$ for all $W \in \mathfrak{W}$ by (4.1.1). Therefore, ICL yields the $\mathbb{T}$-valued $\emptyset$-functional logic space $(A_\emptyset, \mathcal{F})$ as was shown in §3.3.3, that is,

$$\mathcal{F} = \{\varphi^\delta_{\upsilon,\emptyset} : (W, \delta, \upsilon) \in \mathcal{D}\}$$

for the set $\mathcal{D}$ of the triples (W, δ, υ) of $W \in \mathfrak{W}$, $\delta \in \Delta_W$ and $\upsilon \in \Upsilon_W$ and the restriction $\varphi^\delta_{\upsilon,\emptyset} = \varphi^\delta_\upsilon|_{A_\emptyset}$ of the denotation φ^δ_υ. We call it the **sentential IC logic space** because $A_\emptyset$ is the set of the sentences of A.

More worth studying, however, is another larger logic space on the set $H = \bigcup_{Q \in \mathfrak{P}K} A_Q$ of the declaratives of A. For each $(W, \delta, \upsilon) \in \mathcal{D}$, φ^δ_υ is a homotypism of A into W. Therefore if $f \in H$, then since $f \in A_{K^f}$, $\varphi^\delta_\upsilon f$ belongs to $W_{K^f} = (K^f \to S) \to \mathbb{T}$ and so if $\theta \in K \to S$, then $(\varphi^\delta_\upsilon f)(\theta|_{K^f}) \in \mathbb{T}$. Thus the quadruple $(W, \delta, \upsilon, \theta)$ yields the mapping

$$f \mapsto (\varphi^\delta_\upsilon f)(\theta|_{K^f})$$

of H into $\mathbb{T}$. Let $\mathcal{G}$ denote the set of the mappings so obtained from such quadruples. Then $(H, \mathcal{G})$ is a $\mathbb{T}$-valued functional logic space, which we call the **IC logic space**. It is an extension of $(A_\emptyset, \mathcal{F})$ in the sense that $A_\emptyset \subseteq H$ and $\mathcal{F} = \{\varphi|_{A_\emptyset} : \varphi \in \mathcal{G}\}$, because if $f \in A_\emptyset$ then $K^f = \emptyset$ and so $(\varphi^\delta_\upsilon f)(\theta|_{K^f}) = \varphi^\delta_\upsilon f = \varphi^\delta_{\upsilon,\emptyset} f$ by the $\{\emptyset\}$ convention (s. [3.24]). We denote the validity relation of $(H, \mathcal{G})$ and its symmetric core (s. Remark 2.2.10) by $\preccurlyeq$ and $\asymp$ respectively.

We use the conventions introduced in §2.2 for the free monoid H^* over H (s. Remark 3.1.14). First by the alphabet convention, the letters α, β, γ and ε denote elements of H^*, while the letters f, g and h denote those of H, both with or without numerical subscripts. In particular, ε denotes the identity element of H^*. If $\alpha = f_1 \cdots f_n \in H^*$, then by the word convention, the subset $\{f_1, \ldots, f_n\}$ of H is also denoted α, where if $n = 0$, then $\alpha = \varepsilon$ and $\{f_1, \ldots, f_n\} = \emptyset$. In this notation, we have the following:

$$\begin{aligned} \alpha \preccurlyeq \beta &\iff \inf \varphi\alpha \le \sup \varphi\beta \text{ for all } \varphi \in \mathcal{G}, \\ f \asymp g &\iff f \preccurlyeq g \text{ and } f \succcurlyeq g \iff \varphi f = \varphi g \text{ for all } \varphi \in \mathcal{G}. \end{aligned}$$

A sequent $\alpha \to \beta$ on H is a (**sequential**) **IC tautology** iff $\alpha \preccurlyeq \beta$. An element $f \in H$ is an **IC tautology** iff $\varepsilon \preccurlyeq f$. Lastly by the dot convention, the multiplication of H^* is sometimes denoted by a dot. We use the conventions also for the set F of the events of each $W \in \mathfrak{W}$ (s. Lemma 4.2.5).

4.2.2 Boolean features and causal relation for events

IC tautologies emerge from the analysis in §4.2.2–4.2.4 of an arbitrary IC world

$$W = S \amalg (S \to \mathbb{T}) \amalg \coprod_{Q \in \mathfrak{P}K} ((Q \to S) \to \mathbb{T}).$$

Here we focus on its Boolean features.

Remark 2.1.3 shows that the set $W_{\mathfrak{e}} = S \to \mathbb{T}$ of the derived entities of W is a Boolean lattice with respect to the order $\sqsubseteq$ defined by

$$a \sqsubseteq b \iff as \le bs \text{ for all } s \in S \tag{4.2.1}$$

for each $(a, b) \in {W_{\mathfrak{e}}}^2$ and that the smallest element $0_{\mathfrak{e}}$ and the largest element $1_{\mathfrak{e}}$ of $W_{\mathfrak{e}}$ are characterized by the equations $0_{\mathfrak{e}}s = 0$ and $1_{\mathfrak{e}}s = 1$ for all $s \in S$. (W2), (W3) and Remark 2.1.3 show that the meet, join and complement of $W_{\mathfrak{e}}$ are the restrictions of the operations $\sqcap, \sqcup$ and $\square$ to it. They also show that the projection $a \mapsto as$ by each $s \in S$ (s. §1.5.2) is a homomorphism of $W_{\mathfrak{e}}$ into $\mathbb{T}$ with respect to the operations on $W_{\mathfrak{e}}$ and the operations $\wedge$, $\vee$ and $\Diamond$ on $\mathbb{T}$.

(W2) and (W3) also show that we may regard the operations $\sqcap, \sqcup$ and $\square$ as total operations on the set $E = W_{\mathfrak{e}} \cup W_{\mathfrak{e}} = S \cup (S \to \mathbb{T})$ of the entities of W. Since $\mathbb{T} = \{0, 1\}$, the order $\sqsubseteq$ on $W_{\mathfrak{e}}$ satisfies the following for each $(a, b) \in {W_{\mathfrak{e}}}^2$:

$$a \sqsubseteq b \iff \text{If } s \in S \text{ and } as = 1 \text{ then } bs = 1.$$

Therefore, by virtue of (4.1.3) for $(s, a) \in S \times (S \to \mathbb{T})$, we may extend $\sqsubseteq$ to the relation $\sqsubseteq$ on E defined by the following for each $(a, b) \in E^2$:

$$a \sqsubseteq b \iff \text{If } s \in S \text{ and } s \sqsubseteq a \text{ then } s \sqsubseteq b. \tag{4.2.2}$$

Then it is a preorder. Let $\equiv$ be its symmetric core, that is,

$$a \equiv b \iff \text{An element } s \in S \text{ satisfies } s \sqsubseteq a \text{ iff } s \sqsubseteq b \tag{4.2.3}$$

for each $(a, b) \in E^2$. Then $\equiv$ is an equivalence relation and its restriction to $W_{\mathfrak{e}}$ is the equality $=$. Every element $a \in E$ satisfies $0_{\mathfrak{e}} \sqsubseteq a$ and $a \sqsubseteq 1_{\mathfrak{e}}$, that is, $0_{\mathfrak{e}} \sqcup a \equiv a$ and $a \sqcap 1_{\mathfrak{e}} \equiv a$.

Remark 4.2.1 For each element $\mathsf{a} \in W_\epsilon$, there exists precisely one element $\mathsf{b} \in W_\mathfrak{e}$ such that $\mathsf{a} \equiv \mathsf{b}$. Therefore, since $\#W_\mathfrak{e} = 2^{\#W_\epsilon}$, there are plenty of elements $\mathsf{b} \in W_\mathfrak{e}$ such that $\mathsf{a} \not\equiv \mathsf{b}$ for any $\mathsf{a} \in W_\epsilon$. A notable example is the element $\mathsf{b} \in W_\mathfrak{e}$ such that an element $\mathsf{s} \in \mathsf{S}$ satisfies $\mathsf{bs} = 1$ iff $\mathsf{s} \not\equiv \mathsf{s}$.

Let $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$. Then as with $W_\mathfrak{e} = \mathsf{S}{\to}\mathbb{T}$, the set $W_\mathsf{Q} = (\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T}$ of the Q-events of W is a Boolean lattice with respect to the order $\leq$ defined by

$$\mathsf{f} \leq \mathsf{g} \iff \mathsf{f}\theta \leq \mathsf{g}\theta \text{ for all } \theta \in \mathsf{Q}{\to}\mathsf{S}$$

for each $(\mathsf{f}, \mathsf{g}) \in {W_\mathsf{Q}}^2$. The smallest element 0_Q and the largest element 1_Q of W_Q are characterized by the equations $0_\mathsf{Q}\theta = 0$ and $1_\mathsf{Q}\theta = 1$ for all $\theta \in \mathsf{Q}{\to}\mathsf{S}$. (W5), (W6) and Remark 2.1.3 show that the meet, join, complement and cojoin of W_Q are the restrictions of the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ to it (s. §1.5.2). They also show that the projection $\mathsf{f} \mapsto \mathsf{f}\theta$ by each $\theta \in \mathsf{Q}{\to}\mathsf{S}$ is a homomorphism of W_Q into $\mathbb{T}$ with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ on W_Q and $\mathbb{T}$. We refer to it also as the **θ-projection** (s. §1.5.2).

The Boolean feature of $W_{\{\pi\}}$ is particularly connected with that of E by the following consequence (proof left to you) of (W2)–(W6).

Theorem 4.2.1 The mapping $\mathsf{a} \mapsto \mathsf{a}\triangle$ of E into $W_{\{\pi\}}$ is a homomorphism with respect to the operations $\sqcap$, $\sqcup$ and $\Box$ on E and the operations $\wedge$, $\vee$ and $\Diamond$ on $W_{\{\pi\}}$, that is, the following hold for all $(\mathsf{a}, \mathsf{b}) \in \mathsf{E}^2$ and all $\mathsf{a} \in \mathsf{E}$:

$$(\mathsf{a} \sqcap \mathsf{b})\triangle = \mathsf{a}\triangle \wedge \mathsf{b}\triangle, \quad (\mathsf{a} \sqcup \mathsf{b})\triangle = \mathsf{a}\triangle \vee \mathsf{b}\triangle, \quad (\mathsf{a}^\Box)\triangle = (\mathsf{a}\triangle)^\Diamond.$$

(W5) and (W6) also show that we may regard the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ as total operations on the set $\mathsf{F} = \bigcup_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}} W_\mathsf{Q}$ of the events of W and that $\mathsf{f} \Rrightarrow \mathsf{g} = \mathsf{f}^\Diamond \vee \mathsf{g}$ for all $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$. Furthermore, we may extend the orders $\leq$ on W_Q $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$ all together to the relation $\lesssim$ on F defined by

$$\mathsf{f} \lesssim \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}^\mathsf{f}}) \leq \mathsf{g}(\theta|_{\mathsf{K}^\mathsf{g}}) \text{ for all } \theta \in (\mathsf{K}^\mathsf{f} \cup \mathsf{K}^\mathsf{g}){\to}\mathsf{S}$$

for each $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$. Since the mapping $\theta \mapsto \theta|_\mathsf{Q}$ of $\mathsf{K}{\to}\mathsf{S}$ into $\mathsf{Q}{\to}\mathsf{S}$ is surjective for each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$, we have the following for each $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$:

$$\mathsf{f} \lesssim \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}^\mathsf{f}}) \leq \mathsf{g}(\theta|_{\mathsf{K}^\mathsf{g}}) \text{ for all } \theta \in \mathsf{K}{\to}\mathsf{S}.$$

Therefore, $\lesssim$ is a preorder. Let $\doteq$ denote its symmetric core, that is,

$$\mathsf{f} \doteq \mathsf{g} \iff \mathsf{f}(\theta|_{\mathsf{K}^\mathsf{f}}) = \mathsf{g}(\theta|_{\mathsf{K}^\mathsf{g}}) \text{ for all } \theta \in \mathsf{K}{\to}\mathsf{S} \tag{4.2.4}$$

for each $(\mathsf{f}, \mathsf{g}) \in \mathsf{F}^2$. Then $\doteq$ is an equivalence relation and its restriction to W_Q $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$ is the equality $=$. Every element $\mathsf{f} \in \mathsf{F}$ satisfies $0_\mathsf{Q} \lesssim \mathsf{f}$ and $\mathsf{f} \lesssim 1_\mathsf{Q}$ for all $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$. Therefore, $0_\mathsf{Q} \doteq 0_\mathsf{R}$ and $1_\mathsf{Q} \doteq 1_\mathsf{R}$ for all $(\mathsf{Q}, \mathsf{R}) \in (\mathfrak{P}\mathsf{K})^2$.

For each element $\mathsf{f} \in \mathsf{F}$, we let $\mathsf{f}^\sharp$ be the element of W_K defined by

$$\mathsf{f}^\sharp\theta = \mathsf{f}(\theta|_{\mathsf{K}^\mathsf{f}}) \tag{4.2.5}$$

for each $\theta \in K \to S$, and call the mapping $f \mapsto f^\sharp$ the **inflation**. Then

$$f \leq g \iff f^\sharp \leq g^\sharp, \qquad f \doteq g \iff f^\sharp = g^\sharp,$$

for each $(f, g) \in F^2$, and (W5) and (W6) show that the inflation is a homomorphism of F into $\mathcal{W}_K$ with respect to the operations $\wedge, \vee, \lozenge$ and $\Rrightarrow$. Therefore, $\doteq$ is consistent with the operations, that is, if elements (f, g) and (f', g') of F^2 satisfy $f \doteq f'$ and $g \doteq g'$ then $f \wedge g \doteq f' \wedge g'$, and likewise for $\vee, \lozenge$ and $\Rrightarrow$.

Moreover, we may extended the preorder $\leq$ on F to the relation $\leq$ on F^* defined by the following for each $(f_1 \cdots f_m, g_1 \cdots g_n) \in F^* \times F^*$:

$$f_1 \cdots f_m \leq g_1 \cdots g_n \iff \inf\{f_1^\sharp, \ldots, f_m^\sharp\} \leq \sup\{g_1^\sharp, \ldots, g_n^\sharp\}.$$

Theorem 4.2.2 Let $(f_1 \cdots f_m, g_1 \cdots g_n) \in F^* \times F^*$. Then the following hold:

$$f_1 \cdots f_m \leq g_1 \cdots g_n \iff \inf\{f_1^\sharp\theta, \ldots, f_m^\sharp\theta\} \leq \sup\{g_1^\sharp\theta, \ldots, g_n^\sharp\theta\} \quad \text{for all } \theta \in K \to S.$$

More generally, if $\bigcup_{i=1}^m K^{f_i} \cup \bigcup_{j=1}^n K^{g_j} \subseteq L \subseteq K$, then $f_1 \cdots f_m \leq g_1 \cdots g_n$ iff $\inf\{f_1(\theta|_{K^{f_1}}), \ldots, f_m(\theta|_{K^{f_m}})\} \leq \sup\{g_1(\theta|_{K^{g_1}}), \ldots, g_n(\theta|_{K^{g_n}})\}$ for all $\theta \in L \to S$. In particular, an event $f \in F$ satisfies $\varepsilon \leq f$ iff $f = 1_{K^f}$.

Proof Since the projection by each $\theta \in K \to S$ is a lattice homomorphism of $\mathcal{W}_K$ into $\mathbb{T}$, we have $(\inf\{f_1^\sharp, \ldots, f_m^\sharp\})\theta = (f_1^\sharp \wedge \ldots \wedge f_m^\sharp)\theta = (f_1^\sharp\theta) \wedge \ldots \wedge (f_m^\sharp\theta) = \inf\{f_1^\sharp\theta, \ldots, f_m^\sharp\theta\}$ even if $m = 0$, and likewise for $(\sup\{g_1^\sharp, \ldots, g_n^\sharp\})\theta$, hence the former statement. The latter follows from the former and (4.2.5), because the mapping $\theta \mapsto \theta|_L$ of $K \to S$ into $L \to S$ is surjective.

Remark 4.2.2 (Existence, occurrence and causality) An entity $a \in E$ is said to **exist** if $s \sqsubseteq a$ for some $s \in S$. Therefore, all derived entities other than 0_ε exist, while a basic entity may or may not exist. An event $f \in F$ is said to **occur for** $\theta \in K^f \to S$ if $f\theta = 1$. Furthermore, it is said to **occur** if it occurs for some $\theta \in K^f \to S$. Then the entity a exists iff the event $a\triangle$ occurs (s. Remarks 4.2.3 and 4.2.4). Moreover, due to the latter statement of Theorem 4.2.2, $f_1 \cdots f_m \leq g_1 \cdots g_n$ means that if $\theta \in L \to S$ and the event f_i occurs for $\theta|_{K^{f_i}}$ for all $i \in \{1, \ldots, m\}$ then the event g_j occurs for $\theta|_{K^{g_j}}$ for some $j \in \{1, \ldots, n\}$. In particular, an event f satisfies $\varepsilon \leq f$ iff f occurs for all $\theta \in K^f \to S$, that is, iff $f = 1_{K^f}$. Thus we refer to the relation $\leq$ on F^* as the **causal relation** of F and refer to 1_Q ($Q \in \mathfrak{P}K$) as the **inevitable**s of F.

Theorem 4.2.3 The causal relation $\leq$ of F is a Boolean relation with respect to the operations $\wedge, \vee, \lozenge$ and $\Rrightarrow$ on F. Moreover, the set F equipped with the mappings $f \mapsto f^\sharp\theta$ for all $\theta \in K \to S$ is a binary logic space with respect to the operations, and its validity relation and core are equal to the causal relation and the set of the inevitables of F respectively.

Proof As was noted above, the inflation $f \mapsto f^\sharp$ of F into W_K is a homomorphism with respect to the operations, and so also is the θ-projection $f \mapsto f\theta$ of W_K into $\mathbb{T}$ for each $\theta \in K \to S$. This fact and Theorem 4.2.2 show that the latter statement holds. The former follows from the latter and Theorem 2.6.9.

When generalized in Chapter 6, the following corollary of Theorem 4.2.3 will give an answer to the question *What is the causal algebraic structure of the event sets of the nooworlds?* raised in §1.2.4.

Corollary 4.2.3.1 The deduction system $(\wp, I)$ on F consisting of the association $\wp = \dfrac{f \quad f \Rightarrow g}{g}$ and the set I of the inevitables of F is complete with respect to the logic space in Theorem 4.2.3.

Proof This follows from Theorem 4.2.3 and Corollary 2.7.12.1.

The following theorem clarifies the relationship between the validity relation $\preccurlyeq$ of the IC logic space $(H, \mathcal{G})$ and the causal relations $\trianglelefteq$ of the IC worlds and shows how IC tautologies are obtained from the results in §4.2.2–4.2.4.

Theorem 4.2.4 An element $(f_1 \cdots f_m, g_1 \cdots g_n)$ of $H^* \times H^*$ satisfies $f_1 \cdots f_m \preccurlyeq g_1 \cdots g_n$ iff $\varphi^\delta_\upsilon f_1 \cdots \varphi^\delta_\upsilon f_m \trianglelefteq \varphi^\delta_\upsilon g_1 \cdots \varphi^\delta_\upsilon g_n$ for all $(W, \delta, \upsilon) \in \mathcal{D}$. An element (f, g) of H^2 satisfies $f \asymp g$ iff $\varphi^\delta_\upsilon f \doteq \varphi^\delta_\upsilon g$ for all $(W, \delta, \upsilon) \in \mathcal{D}$. An element (f, g) of ${A_Q}^2$ $(Q \in \mathfrak{P}K)$ satisfies $f \asymp g$ iff $\varphi^\delta_\upsilon f = \varphi^\delta_\upsilon g$ for all $(W, \delta, \upsilon) \in \mathcal{D}$.

Proof This follows from the definition of $(H, \mathcal{G})$ and Theorem 4.2.2.

Now, (A5) and (A6) show that we may regard the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ on A as total operations on H, which concerns the following.

Theorem 4.2.5 The validity relation $\preccurlyeq$ of the IC logic space $(H, \mathcal{G})$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ on H. Moreover, $(H, \mathcal{G})$ is a binary logic space with respect to the operations. Consequently, when restricted to H, the symmetric core $\asymp$ of $\preccurlyeq$ is an equivalence relation (for this reason if $f \asymp g$, then we say that f and g are **equivalent**).

Proof Because of (4.2.5), $\mathcal{G}$ consists of the mappings $f \mapsto (\varphi^\delta_\upsilon f)^\sharp\theta$ of H into $\mathbb{T}$ for all $(W, \delta, \upsilon) \in \mathcal{D}$ and all $\theta \in K \to S$. Since φ^δ_υ, inflation $\sharp$ and θ-projection are all homomorphisms with respect to the operations, so are the members of $\mathcal{G}$. Thus the second statement holds. The first follows from the second and Theorem 2.6.9. The third follows from the first and Remark 2.2.10.

4.2.3 Operations $\mathfrak{o}k$ for case markers k

Let $k_1, \ldots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then, for each $(s_1, \ldots, s_n) \in S^n$ and each $f \in W_Q$, $s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f)\cdots)$ belongs to $W_{Q-\{k_1,\ldots,k_n\}}$ by (W7). We sometimes abbreviate it to $(s_i\,\mathfrak{o}k_i)_{i=1,\ldots,n} f$ or $(s_i\,\mathfrak{o}k_i)_i f$.

Let $k_1, \ldots, k_n$ and $(s_1, \ldots, s_n)$ be as above and let $\theta \in (Q - \{k_1, \ldots, k_n\}) \to S$. Then, generalizing (4.1.4), we define an element $\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta$ of $Q \to S$ by

$$\left(\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta \right) k = \begin{cases} \theta k & \text{if } k \in Q - \{k_1, \ldots, k_n\}, \\ s_i & \text{if } k = k_i \ (i = 1, \ldots, n). \end{cases} \tag{4.2.6}$$

We sometimes abbreviate $\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix}$ to $(k_i/s_i)_{i=1,\ldots,n}$ or $(k_i/s_i)_i$.

The proof of the following lemma is left to you.

Lemma 4.2.1 Assume $Q \subseteq R \in \mathfrak{P}K$. Let $k_1, \ldots, k_n$ be distinct cases in Q and $k_{n+1}, \ldots, k_m$ be distinct cases in $R - Q$ $(n \leq m)$. Then $Q - \{k_1, \ldots, k_n\} \subseteq R - \{k_1, \ldots, k_m\}$ and the following holds for each $(s_1, \ldots, s_m) \in S^m$ and each $\theta \in (R - \{k_1, \ldots, k_m\}) \to S$:

$$\left. \left(\begin{pmatrix} k_1, \ldots, k_m \\ \hline s_1, \ldots, s_m \end{pmatrix} \theta \right) \right|_Q = \begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta|_{Q - \{k_1, \ldots, k_n\}}.$$

Theorem 4.2.6 Let $k_1, \ldots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then

$$(s_1 \, \mathfrak{o} k_1 \, (\cdots (s_n \, \mathfrak{o} k_n \, f) \cdots))\theta = f \left(\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta \right)$$

for each $(s_1, \ldots, s_n) \in S^n$, each $f \in W_Q$ and each $\theta \in (Q - \{k_1, \ldots, k_n\}) \to S$.

Proof By virtue of (W7), we may assume $n > 1$ and argue by induction on n. Define $R = Q - \{k_n\}$ and $g = s_n \, \mathfrak{o} k_n \, f$. Then $g \in W_R$ and $\theta \in (R - \{k_1, \ldots, k_{n-1}\}) \to S$, and so $\theta' = \begin{pmatrix} k_1, \ldots, k_{n-1} \\ \hline s_1, \ldots, s_{n-1} \end{pmatrix} \theta$ belongs to $R \to S$ and $g\theta' = f((k_n/s_n)\theta')$. Obviously $(k_n/s_n)\theta' = \begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta$. Thus

$$\begin{aligned} &(s_1 \, \mathfrak{o} k_1 \, (\cdots (s_n \, \mathfrak{o} k_n \, f) \cdots))\theta = (s_1 \, \mathfrak{o} k_1 \, (\cdots (s_{n-1} \, \mathfrak{o} k_{n-1} \, g) \cdots))\theta \\ &= g \left(\begin{pmatrix} k_1, \ldots, k_{n-1} \\ \hline s_1, \ldots, s_{n-1} \end{pmatrix} \theta \right) = g\theta' = f((k_n/s_n)\theta') = f \left(\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta \right), \end{aligned}$$

where the second equality holds by the induction hypothesis.

Corollary 4.2.6.1 Let $k_1, \ldots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then

$$s_1 \, \mathfrak{o} k_1 \, (\cdots (s_n \, \mathfrak{o} k_n \, f) \cdots) = s_{\rho 1} \, \mathfrak{o} k_{\rho 1} \, (\cdots (s_{\rho n} \, \mathfrak{o} k_{\rho n} \, f) \cdots)$$

for each $(s_1, \ldots, s_n) \in S^n$, each $f \in W_Q$ and each permutation ρ on $\{1, \ldots, n\}$.

Proof This follows from Theorem 4.2.6 because $\{1, \ldots, n\} = \{\rho 1, \ldots, \rho n\}$ and $\begin{pmatrix} k_1, \ldots, k_n \\ \hline s_1, \ldots, s_n \end{pmatrix} \theta = \begin{pmatrix} k_{\rho 1}, \ldots, k_{\rho n} \\ \hline s_{\rho 1}, \ldots, s_{\rho n} \end{pmatrix} \theta$ for all $\theta \in (Q - \{k_1, \ldots, k_n\}) \to S$.

Corollary 4.2.6.2 Let $k_1, \dots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then the following holds for each $f \in W_Q$ and each $\theta \in Q \to S$:

$$f\theta = ((\theta k_1)\,\mathfrak{o}k_1\,(\cdots((\theta k_n)\,\mathfrak{o}k_n\,f)\cdots))\theta|_{Q-\{k_1,\dots,k_n\}}.$$

Proof This follows from Theorem 4.2.6 because

$$\theta = \left(\frac{k_1, \dots, k_n}{\theta k_1, \dots, \theta k_n}\right)\theta|_{Q-\{k_1,\dots,k_n\}}.$$

Remark 4.2.3 Corollary 4.2.6.2 concerns occurrence defined in Remark 4.2.2. It shows that $f \in W_Q$ occurs for $\theta \in Q \to S$ iff $(\theta k_1)\,\mathfrak{o}k_1\,(\cdots((\theta k_n)\,\mathfrak{o}k_n\,f)\cdots)$ occurs for $\theta|_{Q-\{k_1,\dots,k_n\}}$. Therefore in the case $Q = \{k_1, \dots, k_n\}$, f occurs iff $s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f)\cdots) = 1$ for some $(s_1, \dots, s_n) \in S^n$ (s. Remark 4.2.4).

Corollary 4.2.6.3 Let $k_1, \dots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then the following holds for each $(s_1, \dots, s_n) \in S^n$ and each $f \in W_Q$:

$$s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f^\Diamond)\cdots) = (s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f)\cdots))^\Diamond.$$

Proof This is because $((s_i\,\mathfrak{o}k_i)_i f^\Diamond)\theta = f^\Diamond((k_i/s_i)_i\theta) = (f((k_i/s_i)_i\theta))^\Diamond = (((s_i\,\mathfrak{o}k_i)_i f)\theta)^\Diamond = ((s_i\,\mathfrak{o}k_i)_i f)^\Diamond\theta$ for all $\theta \in (Q - \{k_1, \dots, k_n\}) \to S$ by Theorem 4.2.6 and (W6).

We have defined a mapping $f \mapsto f^\sharp$ of F into W_K called the inflation and a relation $\doteq$ on F in §4.2.2, which concern the following.

Lemma 4.2.2 Let $k_1, \dots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$ and $k_{n+1}, \dots, k_m$ be distinct cases in $K - Q$ ($n \le m$). Then $k_1, \dots, k_m$ are distinct cases and the following holds for each $(s_1, \dots, s_m) \in S^m$ and each $f \in W_Q$:

$$s_1\,\mathfrak{o}k_1\,(\cdots(s_m\,\mathfrak{o}k_m\,f^\sharp)\cdots) \doteq s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f)\cdots).$$

Proof This follows from (4.2.4) because the following holds for all $\theta \in (K - \{k_1, \dots, k_m\}) \to S$ by Theorem 4.2.6, (4.2.5) and Lemma 4.2.1:

$$\begin{aligned}
&(s_1\,\mathfrak{o}k_1\,(\cdots(s_m\,\mathfrak{o}k_m\,f^\sharp)\cdots))\theta = f^\sharp\left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m}\right)\theta\right)\\
&= f\left(\left(\left(\frac{k_1, \dots, k_m}{s_1, \dots, s_m}\right)\theta\right)\Big|_Q\right) = f\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta|_{Q-\{k_1,\dots,k_n\}}\right)\\
&= (s_1\,\mathfrak{o}k_1\,(\cdots(s_n\,\mathfrak{o}k_n\,f)\cdots))\theta|_{Q-\{k_1,\dots,k_n\}}.
\end{aligned}$$

Lemma 4.2.3 Let $k_1, \dots, k_n$ be distinct cases in $Q \in \mathfrak{P}K$. Then the following hold for each $(s_1, \dots, s_n) \in S^n$ and each $(f, g) \in {W_Q}^2$:

$$\begin{aligned}
(s_i\,\mathfrak{o}k_i)_i(f \wedge g) &= (s_i\,\mathfrak{o}k_i)_i f \wedge (s_i\,\mathfrak{o}k_i)_i g,\\
(s_i\,\mathfrak{o}k_i)_i(f \vee g) &= (s_i\,\mathfrak{o}k_i)_i f \vee (s_i\,\mathfrak{o}k_i)_i g,\\
(s_i\,\mathfrak{o}k_i)_i(f \Rightarrow g) &= (s_i\,\mathfrak{o}k_i)_i f \Rightarrow (s_i\,\mathfrak{o}k_i)_i g.
\end{aligned}$$

Proof This is because the following holds for any one $*$ of the operations $\wedge, \vee$ and $\Rrightarrow$ and all $\theta \in (\mathsf{Q} - \{k_1, \dots, k_n\}) \to \mathsf{S}$ by Theorem 4.2.6 and (W5):

$$\begin{aligned}&((s_i \,\mathfrak{o} k_i)_i (f * g))\theta = (f * g)((k_i/s_i)_i\theta) = f((k_i/s_i)_i\theta) * g((k_i/s_i)_i\theta)\\ &= ((s_i \,\mathfrak{o} k_i)_i f)\theta * ((s_i \,\mathfrak{o} k_i)_i g)\theta = ((s_i \,\mathfrak{o} k_i)_i f * (s_i \,\mathfrak{o} k_i)_i g)\theta.\end{aligned}$$

Theorem 4.2.7 Let $k_1, \dots, k_l$ be distinct case. Then the following hold for each $(s_1, \dots, s_l) \in \mathsf{S}^l$ and each $(f, g) \in \mathsf{F}^2$ such that $k_1, \dots, k_n \in \mathsf{K}^f - \mathsf{K}^g$, $k_{n+1}, \dots, k_m \in \mathsf{K}^f \cap \mathsf{K}^g$ and $k_{m+1}, \dots, k_l \in \mathsf{K}^g - \mathsf{K}^f$ $(0 \le n \le m \le l)$:

$$\begin{aligned}(s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f \wedge g) &= (s_i \,\mathfrak{o} k_i)_{i=1,\dots,m} f \wedge (s_i \,\mathfrak{o} k_i)_{i=n+1,\dots,l}\, g,\\ (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f \vee g) &= (s_i \,\mathfrak{o} k_i)_{i=1,\dots,m} f \vee (s_i \,\mathfrak{o} k_i)_{i=n+1,\dots,l}\, g,\\ (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f \Rrightarrow g) &= (s_i \,\mathfrak{o} k_i)_{i=1,\dots,m} f \Rrightarrow (s_i \,\mathfrak{o} k_i)_{i=n+1,\dots,l}\, g.\end{aligned}$$

Proof Let $*$ be any one of the operations $\wedge, \vee$ and $\Rrightarrow$. Then the inflation $\sharp$ is a homomorphism with respect to $*$ and the relation $\doteq$ is consistent with $*$. Therefore, we have the following by Lemmas 4.2.2, 4.2.3 and Corollary 4.2.6.1:

$$\begin{aligned}&(s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f * g) \doteq (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f * g)^\sharp = (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f^\sharp * g^\sharp)\\ &= (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l} f^\sharp * (s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}\, g^\sharp\\ &\doteq (s_i \,\mathfrak{o} k_i)_{i=1,\dots,m} f * (s_i \,\mathfrak{o} k_i)_{i=n+1,\dots,l}\, g.\end{aligned}$$

Since $\doteq$ is an equivalence relation and its restriction to $W_{\mathsf{K}^f \cup \mathsf{K}^g - \{k_1,\dots,k_l\}}$ is the equality, we have $(s_i \,\mathfrak{o} k_i)_{i=1,\dots,l}(f * g) = (s_i \,\mathfrak{o} k_i)_{i=1,\dots,m} f * (s_i \,\mathfrak{o} k_i)_{i=n+1,\dots,l}\, g$.

We have defined an order $\le$ on W_Q for each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$ in §4.2.2, which concerns the following.

Lemma 4.2.4 Let $k_1, \dots, k_n$ be distinct cases in $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $(f, g) \in {W_\mathsf{Q}}^2$. Then $f \le g$ iff $s_1 \,\mathfrak{o} k_1 \,(\cdots (s_n \,\mathfrak{o} k_n\, f) \cdots) \le s_1 \,\mathfrak{o} k_1 \,(\cdots (s_n \,\mathfrak{o} k_n\, g) \cdots)$ for all $(s_1, \dots, s_n) \in \mathsf{S}^n$.

Proof If $f \le g$ and $(s_1, \dots, s_n) \in \mathsf{S}^n$, then Theorem 4.2.6 shows that

$$\begin{aligned}&(s_1 \,\mathfrak{o} k_1 \,(\cdots (s_n \,\mathfrak{o} k_n\, f) \cdots))\theta = f\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta\right)\\ &\le g\left(\left(\frac{k_1, \dots, k_n}{s_1, \dots, s_n}\right)\theta\right) = (s_1 \,\mathfrak{o} k_1 \,(\cdots (s_n \,\mathfrak{o} k_n\, g) \cdots))\theta\end{aligned}$$

for all $\theta \in (\mathsf{Q} - \{k_1, \dots, k_n\}) \to \mathsf{S}$, hence $(s_i \,\mathfrak{o} k_i)_i f \le (s_i \,\mathfrak{o} k_i)_i g$. If $(s_i \,\mathfrak{o} k_i)_i f \le (s_i \,\mathfrak{o} k_i)_i g$ for all $(s_1, \dots, s_n) \in \mathsf{S}^n$, then Corollary 4.2.6.2 shows that

$$\begin{aligned}f\theta &= ((\theta k_1) \,\mathfrak{o} k_1 \,(\cdots ((\theta k_n) \,\mathfrak{o} k_n\, f) \cdots))\theta|_{\mathsf{Q} - \{k_1,\dots,k_n\}}\\ &\le ((\theta k_1) \,\mathfrak{o} k_1 \,(\cdots ((\theta k_n) \,\mathfrak{o} k_n\, g) \cdots))\theta|_{\mathsf{Q} - \{k_1,\dots,k_n\}} = g\theta\end{aligned}$$

for all $\theta \in \mathsf{Q} \to \mathsf{S}$, hence $f \le g$.

We have extended the orders $\leq$ on W_Q $(Q \in \mathfrak{P}K)$ all together to a relation $\lesssim$ on F in §4.2.2, which concerns the following.

Theorem 4.2.8 Let $k_1, \ldots, k_l$ be distinct cases and let $(f, g) \in F^2$ satisfy $k_1, \ldots, k_n \in K^f - K^g$, $k_{n+1}, \ldots, k_m \in K^f \cap K^g$ and $k_{m+1}, \ldots, k_l \in K^g - K^f$ $(0 \leq n \leq m \leq l)$. Then $f \lesssim g$ iff $(s_i \mathsf{o} k_i)_{i=1,\ldots,m} f \lesssim (s_i \mathsf{o} k_i)_{i=n+1,\ldots,l} g$ for all $(s_1, \ldots, s_l) \in S^l$.

Proof The relation $\lesssim$ is a preorder on F, and $f \lesssim g$ iff $f^\sharp \leq g^\sharp$. Furthermore, it extends the order $\leq$ on $W_{K^f \cup K^g - \{k_1,\ldots,k_l\}}$ and its symmetric core is $\doteq$. Thus we may argue in the following way by Lemmas 4.2.4, 4.2.2 and Corollary 4.2.6.1:

$$\begin{aligned} f \lesssim g &\iff (s_i \mathsf{o} k_i)_{i=1,\ldots,l} f^\sharp \leq (s_i \mathsf{o} k_i)_{i=1,\ldots,l} g^\sharp \text{ for all } (s_1, \ldots, s_l) \in S^l \\ &\iff (s_i \mathsf{o} k_i)_{i=1,\ldots,l} f^\sharp \lesssim (s_i \mathsf{o} k_i)_{i=1,\ldots,l} g^\sharp \text{ for all } (s_1, \ldots, s_l) \in S^l \\ &\iff (s_i \mathsf{o} k_i)_{i=1,\ldots,m} f \lesssim (s_i \mathsf{o} k_i)_{i=n+1,\ldots,l} g \text{ for all } (s_1, \ldots, s_l) \in S^l. \end{aligned}$$

We have extended the relation $\lesssim$ to the causal relation of F in §4.2.2, which concerns the following.

Lemma 4.2.5 The following hold for the causal relation $\lesssim$ of F.

(1) $\begin{cases} \alpha f g \beta \lesssim \gamma \implies \alpha g f \beta \lesssim \gamma, \\ \gamma \lesssim \alpha f g \beta \implies \gamma \lesssim \alpha g f \beta. \end{cases}$

(2) $\begin{cases} \alpha f g \beta \lesssim \gamma \iff \alpha \cdot f \wedge g \cdot \beta \lesssim \gamma, \\ \gamma \lesssim \alpha f g \beta \iff \gamma \lesssim \alpha \cdot f \vee g \cdot \beta. \end{cases}$

(3) $\begin{cases} \alpha \lesssim f \beta \iff f^\Diamond \alpha \lesssim \beta, \\ f \beta \lesssim \alpha \iff \beta \lesssim f^\Diamond \alpha. \end{cases}$

(4) $\alpha \lesssim \beta \iff \alpha \lesssim f \wedge f^\Diamond \cdot \beta \iff f \vee f^\Diamond \cdot \alpha \lesssim \beta$.

Proof Since $\lesssim$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond$ and $\Rrightarrow$ on F by Theorem 4.2.3, (1), (2) and (3) follow from Definition 2.2.3, Corollary 2.2.15.1 and Theorem 2.2.16 respectively. As for (4), if $\alpha \lesssim \beta$, then $\alpha \lesssim f \wedge f^\Diamond \cdot \beta$ and $f \vee f^\Diamond \cdot \alpha \lesssim \beta$ by the weakening law. If $\alpha \lesssim f \wedge f^\Diamond \cdot \beta$ or $f \vee f^\Diamond \cdot \alpha \lesssim \beta$, then since $f \wedge f^\Diamond \lesssim \varepsilon$ and $\varepsilon \lesssim f \vee f^\Diamond$ by the negation laws and (2), we have $\alpha \lesssim \beta$ by the cut law.

Theorem 4.2.9 Assume that a case marker $k \in K$ belongs to the arities of events $f_1, \ldots, f_m, g_1, \ldots, g_n \in F$ but does not belong to those of the events which occur in $\alpha, \beta \in F^*$ (s. Example 3.1.8). Then $f_1 \cdots f_m \alpha \lesssim g_1 \cdots g_n \beta$ iff $s \mathsf{o} k f_1 \cdots s \mathsf{o} k f_m \cdot \alpha \lesssim s \mathsf{o} k g_1 \cdots s \mathsf{o} k g_n \cdot \beta$ for all $s \in S$.

Proof We may assume $m \neq 0$ or $n \neq 0$. Suppose $\alpha = f'_1 \cdots f'_{m'}$ and $\beta = g'_1 \cdots g'_{n'}$. Let $s \in S$ and define $h = s\,\mathfrak{o}\pi\, s\triangle$. Then $h \in W_\emptyset$ by (W4) and (W7), and so k does not belong to the arities of $h \wedge h^\Diamond$ or $h \vee h^\Diamond$. Therefore, we may assume $m' \neq 0 \neq n'$ by virtue of Lemma 4.2.5. Define

$$\begin{aligned} f' &= f'_1 \wedge \cdots \wedge f'_{m'}, & g' &= g'_1 \vee \cdots \vee g'_{n'}, \\ f &= (f_1 \wedge \cdots \wedge f_m) \wedge f', & g &= (g_1 \vee \cdots \vee g_n) \vee g', \end{aligned}$$

where the orders of applying the operations $\wedge$ and $\vee$ within parentheses are arbitrary. Then if $m \neq 0 \neq n$, we have

$$f_1 \cdots f_m \alpha \leq g_1 \cdots g_n \beta \iff f \leq g \iff s\,\mathfrak{o}k\, f \leq s\,\mathfrak{o}k\, g \text{ for all } s \in S$$

by Lemma 4.2.5 and Theorem 4.2.8, and we have

$$\begin{aligned} & s\,\mathfrak{o}k\, f \leq s\,\mathfrak{o}k\, g \\ & \iff (s\,\mathfrak{o}k\, f_1 \wedge \cdots \wedge s\,\mathfrak{o}k\, f_m) \wedge f' \leq (s\,\mathfrak{o}k\, g_1 \vee \cdots \vee s\,\mathfrak{o}k\, g_n) \vee g' \\ & \iff s\,\mathfrak{o}k\, f_1 \ \cdots \ s\,\mathfrak{o}k\, f_m \cdot \alpha \leq s\,\mathfrak{o}k\, g_1 \ \cdots \ s\,\mathfrak{o}k\, g_n \cdot \beta \end{aligned}$$

for each $s \in S$ by Theorems 4.2.7 and Lemma 4.2.5. This completes the proof in case $m \neq 0 \neq n$. If $m = 0 \neq n$, argue similarly by replacing $s\,\mathfrak{o}k\, f$ with f. If $m \neq 0 = n$, replace $s\,\mathfrak{o}k\, g$ with g.

4.2.4 Operations $\mathfrak{q}k$ for quantifiers $\mathfrak{q}$ and case markers k

Henceforth we assume that the set $\mathfrak{P}$ of the positive quantifiers is closed by the meet $\mathfrak{p} \cap \mathfrak{q}$, the join $\mathfrak{p} \cup \mathfrak{q}$ and the complement $\mathfrak{p}^\circ$ on $\mathfrak{P}\mathbb{P}$ (s. [1.79]), and contains the intervals $(p{\rightarrow}) = \{x \in \mathbb{P} : p < x\}$ and $({\leftarrow}p] = \{x \in \mathbb{P} : x \leq p\}$ of $\mathbb{P}$ for all $p \in \mathbb{P}$. Then we define quantifiers $\underline{p} \in \neg\mathfrak{P}$ and $\overline{p} \in \mathfrak{P}$ by $\underline{p} = \neg({\leftarrow}p]$ and $\overline{p} = (p{\rightarrow})$ for all $p \in \mathbb{P}$. Furthermore, we define $\forall = \underline{0}$ and $\exists = \overline{0}$ for $0 = \min \mathbb{P}$.

We have defined a preorder $\sqsubseteq$ on E and its symmetric core $\equiv$ by (4.2.2) and (4.2.3) in §4.2.2, which concern (2)–(4) in the following theorem.

Theorem 4.2.10 The following hold on the operation $\triangle$.

(1) Let $a \in S$ and $b \in E$. Then $a\,\mathfrak{o}\pi\, b\triangle = 1$ iff $a \in\!\!\!\sqsubseteq b$. Therefore if $b \in W_{\mathfrak{e}} = S{\rightarrow}\mathbb{T}$, then $a\,\mathfrak{o}\pi\, b\triangle = ba$.

(2) Let $a, b \in E$. Then $a\,\forall\pi\, b\triangle = 1$ iff $a \sqsubseteq b$. Therefore, $a\,\forall\pi\, a\triangle = 1$ for all $a \in E$.

(3) If $a, b, a', b' \in E$ satisfy $a \sqcap b \equiv a' \sqcap b'$, then $a\,\mathfrak{p}\pi\, b\triangle = a'\,\mathfrak{p}\pi\, b'\triangle$ for all $\mathfrak{p} \in \mathfrak{P}$. Therefore if $a, b \in E$ and $\mathfrak{p} \in \mathfrak{P}$, then $a\,\mathfrak{p}\pi\, b\triangle = (a \sqcap b)\,\mathfrak{p}\pi\, 1_{\mathfrak{e}}\triangle$ and $a\,\mathfrak{p}\pi\, b\triangle = b\,\mathfrak{p}\pi\, a\triangle$.

(4) Let $a \in E$. Then $a\,\exists\pi\, 1_{\mathfrak{e}}\triangle = 1$ iff there exists an element $s \in S$ such that $s \in\!\!\!\sqsubseteq a$, that is, $s\,\mathfrak{o}\pi\, a\triangle = 1$.

Proof The main part of (1) holds because

$$\mathsf{a}\,\mathfrak{o}\pi\,\mathsf{b}\triangle = 1 \iff (\mathsf{b}\triangle)(\pi/\mathsf{a}) = 1 \iff \mathsf{a} \sqsubseteq \mathsf{b}$$

by (W7) and (W4). The other part of (1) holds because of (4.1.3). The main part of (2) holds because

$$\begin{aligned}\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle = 1 &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{s}\,\mathfrak{o}\pi\,\mathsf{b}\triangle = 0| \leq 0 \\ &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{s} \not\sqsubseteq \mathsf{b}| \leq 0 \\ &\iff \{\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{s} \not\sqsubseteq \mathsf{b}\} = \emptyset \\ &\iff \text{If } \mathsf{s} \in \mathsf{S} \text{ and } \mathsf{s} \sqsubseteq \mathsf{a} \text{ then } \mathsf{s} \sqsubseteq \mathsf{b} \\ &\iff \mathsf{a} \sqsubseteq \mathsf{b}\end{aligned}$$

by (W8), (1), the positive definiteness of the $\mathbb{P}$-measure and (4.2.2). The main part of (3) holds because of (4.2.3) and the following:

$$\begin{aligned}\mathsf{a}\,\mathfrak{p}\pi\,\mathsf{b}\triangle = 1 &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{s}\,\mathfrak{o}\pi\,\mathsf{b}\triangle = 1| \in \mathfrak{p} && \text{(by (W4) and (W8))} \\ &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{s} \sqsubseteq \mathsf{b}| \in \mathfrak{p} && \text{(by (1))} \\ &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a} \sqcap \mathsf{b}| \in \mathfrak{p} && \text{(by (W2)).}\end{aligned}$$

Since $\mathsf{a} \sqcap 1_{\mathfrak{e}} \equiv \mathsf{a}$ and $\exists = \overline{0} = (0\to)$, we particularly have

$$\mathsf{a}\,\exists\pi\,1_{\mathfrak{e}}\triangle = 1 \iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}| > 0 \iff \{\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}\} \neq \emptyset$$

by the positive definiteness of the $\mathbb{P}$-measure. This proves (4).

Remark 4.2.4 Theorem 4.2.10 and Remark 4.2.3 together correlate existence and occurrence defined in Remark 4.2.2 (s. [1.22]). They show that an entity $\mathsf{a} \in \mathsf{E}$ exists iff $\mathsf{s}\,\mathfrak{o}\pi\,\mathsf{a}\triangle = 1$ for some $\mathsf{s} \in \mathsf{S}$ and iff the event $\mathsf{a}\triangle$ occurs.

Theorem 4.2.11 Let $\mathsf{a} \in \mathsf{E}$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$, $\mathsf{f} \in W_{\mathsf{Q}}$ and $\mathfrak{p} \in \mathfrak{P}$. Then

$$\mathsf{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f} = \mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}^{\Diamond}, \qquad \mathsf{a}\,\mathfrak{p}^{\circ}\mathsf{k}\,\mathsf{f} = (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})^{\Diamond}.$$

Consequently, $(\mathsf{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f})^{\Diamond} = \mathsf{a}\,\mathfrak{p}^{\circ}\mathsf{k}\,\mathsf{f}^{\Diamond}$ and $(\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})^{\Diamond} = \mathsf{a}\,\neg\mathfrak{p}^{\circ}\mathsf{k}\,\mathsf{f}^{\Diamond}$. In particular, $(\mathsf{a}\,\forall\mathsf{k}\,\mathsf{f})^{\Diamond} = \mathsf{a}\,\exists\mathsf{k}\,\mathsf{f}^{\Diamond}$ and $(\mathsf{a}\,\exists\mathsf{k}\,\mathsf{f})^{\Diamond} = \mathsf{a}\,\forall\mathsf{k}\,\mathsf{f}^{\Diamond}$.

Proof The former part is because the following hold for all $\theta \in (\mathsf{Q} - \{\mathsf{k}\})\to\mathsf{S}$:

$$\begin{aligned}(\mathsf{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta = 1 &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 0| \in \mathfrak{p} && \text{(by (W8))} \\ &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathsf{a}, \mathsf{f}^{\Diamond}((\mathsf{k}/\mathsf{s})\theta) = 1| \in \mathfrak{p} && \text{(by (W6))} \\ &\iff (\mathsf{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}^{\Diamond})\theta = 1,\end{aligned}$$

$$\begin{aligned}(\mathfrak{a}\,\mathfrak{p}^{\circ}\mathsf{k}\,\mathsf{f})\theta = 1 &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1| \in \mathfrak{p}^{\circ} && \text{(by (W8))}\\ &\iff (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta = 0 && \text{(by (W8))}\\ &\iff (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})^{\Diamond}\theta = 1 && \text{(by (W6)).}\end{aligned}$$

As for the latter part, notice that $\mathsf{f} = (\mathsf{f}^{\Diamond})^{\Diamond}$.

Theorem 4.2.12 Let $\mathfrak{a} \in \mathsf{E}$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$, $\mathsf{f} \in W_{\mathsf{Q}}$ and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. Then

$$\begin{aligned}\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f} &= \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}, \quad & \mathfrak{a}\,(\mathfrak{p} \cup \mathfrak{q})\mathsf{k}\,\mathsf{f} &= \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \vee \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f},\\ \mathfrak{a}\,\neg(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f} &= \mathfrak{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\neg\mathfrak{q}\mathsf{k}\,\mathsf{f}, \quad & \mathfrak{a}\,\neg(\mathfrak{p} \cup \mathfrak{q})\mathsf{k}\,\mathsf{f} &= \mathfrak{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f} \vee \mathfrak{a}\,\neg\mathfrak{q}\mathsf{k}\,\mathsf{f}.\end{aligned}$$

Proof Let $\theta \in (\mathsf{Q} - \{\mathsf{k}\}) \to \mathsf{S}$. Then

$$\begin{aligned}&(\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f})\theta = 1\\ &\iff |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1| \in \mathfrak{p} \cap \mathfrak{q} && \text{(by (W8))}\\ &\iff \begin{cases}|\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1| \in \mathfrak{p},\\ |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1| \in \mathfrak{q}\end{cases}\\ &\iff (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta = (\mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1\\ &\iff (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})\theta \wedge (\mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1\\ &\iff (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1 && \text{(by (W5)).}\end{aligned}$$

Therefore, $\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f} = \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$. The rest may be derived from this by Theorem 4.2.11 or similarly proved.

Theorem 4.2.13 Let $\mathfrak{a}, \mathfrak{b} \in \mathsf{E}$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$, $\mathsf{f} \in W_{\mathsf{Q}}$ and $\mathsf{p} \in \mathbb{P}$. Then

$$\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle \cdot \mathfrak{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}, \qquad \mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle \cdot \mathfrak{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}.$$

Proof Assume $\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle = 1$ and let $\theta \in (\mathsf{Q} - \{\mathsf{k}\}) \to \mathsf{S}$. Then $\mathfrak{a} \sqsubseteq \mathfrak{b}$ by Theorem 4.2.10, and so since the $\mathbb{P}$-measure is increasing, we have

$$|\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{a}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1| \leq |\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{b}, \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = 1|.$$

Therefore, if $(\mathfrak{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f})\theta = 1$, then $(\mathfrak{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f})\theta = 1$ by (W8). Also, if $(\mathfrak{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f})\theta = 1$, then $(\mathfrak{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f})\theta = 1$. Thus $\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle \cdot \mathfrak{b}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{a}\,(\leftarrow\mathsf{p}]\mathsf{k}\,\mathsf{f}$ and $\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle \cdot \mathfrak{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$ by Theorem 4.2.2.

Theorem 4.2.14 Let $\mathfrak{a} \in \mathsf{S}$, $\mathfrak{b} \in \mathsf{E}$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in W_{\mathsf{Q}}$. Then

$$\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle \cdot \mathfrak{b}\,\forall\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f}, \qquad \mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle \cdot \mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{b}\,\exists\mathsf{k}\,\mathsf{f}.$$

Proof Assume $\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle = 1$ and let $\theta \in (\mathsf{Q} - \{\mathsf{k}\}) \to \mathsf{S}$. Then $\mathfrak{a} \sqsubseteq \mathfrak{b}$ by Theorem 4.2.10. Therefore if $(\mathfrak{b}\,\forall\mathsf{k}\,\mathsf{f})\theta = 1$, then $\{\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{b}, (\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta = 0\} = \emptyset$ by (W8) and the positive definiteness of the $\mathbb{P}$-measure, and so $(\mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta = 1$. Likewise if $(\mathfrak{b}\,\exists\mathsf{k}\,\mathsf{f})\theta = 0$, then $\{\mathsf{s} \in \mathsf{S} : \mathsf{s} \sqsubseteq \mathfrak{b}, (\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta = 1\} = \emptyset$, and so $(\mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta = 0$. Thus $\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle \cdot \mathfrak{b}\,\forall\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f}$ and $\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle \cdot \mathfrak{a}\,\mathfrak{o}\mathsf{k}\,\mathsf{f} \lessdot \mathfrak{b}\,\exists\mathsf{k}\,\mathsf{f}$ by Theorem 4.2.2.

Theorem 4.2.15 Let $a, b \in E$, $k \in Q \in \mathfrak{P}K$, $f \in W_Q$ and $p, q \in \mathbb{P}$. Then

$$a\,(\leftarrow p]k\,f \cdot b\,(\leftarrow q]k\,f \lessdot (a \sqcup b)\,(\leftarrow(p+q)]k\,f.$$

If $\mathbb{P}$ is linear, then $(a \sqcup b)\,\overline{p+q}k\,f \lessdot a\,\overline{p}k\,f \cdot b\,\overline{q}k\,f$.

Proof Assume $\theta \in (Q - \{k\}) \to S$ and $(a\,(\leftarrow p]k\,f)\theta = (b\,(\leftarrow q]k\,f)\theta = 1$. Then

$$|s \in S : s \sqsubseteq a, f((k/s)\theta) = 1| \leq p, \qquad |s \in S : s \sqsubseteq b, f((k/s)\theta) = 1| \leq q$$

by (W8) and so, by (W2) and the subadditivity of the $\mathbb{P}$-measure, we have

$$\begin{aligned}
&|s \in S : s \sqsubseteq a \sqcup b, f((k/s)\theta) = 1| \\
&= |s \in S : s \sqsubseteq a \text{ or } s \sqsubseteq b, f((k/s)\theta) = 1| \\
&\leq |s \in S : s \sqsubseteq a, f((k/s)\theta) = 1| + |s \in S : s \sqsubseteq b, f((k/s)\theta) = 1| \\
&\leq p + q,
\end{aligned}$$

hence $((a \sqcup b)\,(\leftarrow(p+q)]k\,f)\theta = 1$ by (W8). Thus Theorem 4.2.2 shows that the former statement holds. The latter may be similarly proved or derived from the former by Lemma 4.2.5 and Theorem 4.2.11.

Theorem 4.2.16 Let $a \in E$, $b \in S$, $\{k, l\} \subseteq Q \in \mathfrak{P}K$, $k \neq l$, $f \in W_Q$ and $\mathfrak{q} \in \{\mathfrak{o}\} \cup \mathfrak{Q}$. Let $a \in S$ in case $\mathfrak{q} = \mathfrak{o}$. Then $a\,\mathfrak{q}k\,(b\,\mathfrak{o}l\,f) = b\,\mathfrak{o}l\,(a\,\mathfrak{q}k\,f)$.

Proof When $\mathfrak{q} = \mathfrak{o}$, this holds by Corollary 4.2.6.1. Let ν be 1 or 0 according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}$. Let $\theta \in (Q - \{k, l\}) \to S$. Then

$$\begin{aligned}
(a\,\mathfrak{q}k\,(b\,\mathfrak{o}l\,f))\theta = 1 &\iff |s \in S : s \sqsubseteq a, (b\,\mathfrak{o}l\,f)((k/s)\theta) = \nu| \in \mathfrak{p} \\
&\iff |s \in S : s \sqsubseteq a, f((l/b)(k/s)\theta) = \nu| \in \mathfrak{p}, \\
(b\,\mathfrak{o}l\,(a\,\mathfrak{q}k\,f))\theta = 1 &\iff (a\,\mathfrak{q}k\,f)((l/b)\theta) = 1 \\
&\iff |s \in S : s \sqsubseteq a, f((k/s)(l/b)\theta) = \nu| \in \mathfrak{p}
\end{aligned}$$

by (W7) and (W8) and $(l/b)(k/s)\theta = (k/s)(l/b)\theta$. Therefore, $(a\,\mathfrak{q}k\,(b\,\mathfrak{o}l\,f))\theta = (b\,\mathfrak{o}l\,(a\,\mathfrak{q}k\,f))\theta$. Thus $a\,\mathfrak{q}k\,(b\,\mathfrak{o}l\,f) = b\,\mathfrak{o}l\,(a\,\mathfrak{q}k\,f)$.

4.2.5 Variables and nominalizers

Here we consider IC tautologies which involve variables, especially nominalizers ∇x $(x \in X_\epsilon)$. Recall that if $(W, \delta, \upsilon) \in \mathcal{D}$ then the denotation $\varphi^\delta_\upsilon \in A \to W$ is a homotypic M-homomorphism. However, φ^δ_υ is not a homomorphism with respect to the nominalizer ∇x but satisfies (4.1.5) for each $(f, w) \in A_\emptyset \times W_\epsilon$. Recall that (x/w) in (4.1.5) is the transvaluation of x by w on Υ_W.

Theorem 4.2.17 Assume that a case marker $k \in K$ belongs to the arities of declaratives $f_1, \dots, f_m, g_1, \dots, g_n \in H$ but does not belong to those of the declaratives which occur in $\alpha, \beta \in H^*$. Then the following holds for all $a \in A_\epsilon$:

$$\begin{aligned} & f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta \\ & \implies a \,\mathsf{ok}\, f_1 \cdots a \,\mathsf{ok}\, f_m \cdot \alpha \preccurlyeq a \,\mathsf{ok}\, g_1 \cdots a \,\mathsf{ok}\, g_n \cdot \beta. \end{aligned} \qquad \text{(gen. case } \oplus \text{ law)}$$

Assume furthermore that a variable $x \in X_\epsilon$ does not occur free in the declaratives in $\{f_1, \dots, f_m, g_1, \dots, g_n\} \cup \alpha \cup \beta$. Then the following holds:

$$\begin{aligned} & x \,\mathsf{ok}\, f_1 \cdots x \,\mathsf{ok}\, f_m \cdot \alpha \preccurlyeq x \,\mathsf{ok}\, g_1 \cdots x \,\mathsf{ok}\, g_n \cdot \beta \\ & \implies f_1 \cdots f_m \alpha \preccurlyeq g_1 \cdots g_n \beta. \end{aligned} \qquad \text{(gen. case } \ominus \text{ law)}$$

Proof Let $\alpha = f'_1 \cdots f'_{m'}$ and $\beta = g'_1 \cdots g'_{n'}$. Assume the premise of the gen. case $\oplus$ law and let $(W, \delta, \upsilon) \in \mathcal{D}$. Then

$$\varphi^\delta_\upsilon f_1 \cdots \varphi^\delta_\upsilon f_m \cdot \varphi^\delta_\upsilon f'_1 \cdots \varphi^\delta_\upsilon f'_{m'} \leq \varphi^\delta_\upsilon g_1 \cdots \varphi^\delta_\upsilon g_n \cdot \varphi^\delta_\upsilon g'_1 \cdots \varphi^\delta_\upsilon g'_{n'}$$

by Theorem 4.2.4 and, since φ^δ_υ is homotypic, k belongs to the arities of the events $\varphi^\delta_\upsilon f_1, \dots, \varphi^\delta_\upsilon f_m, \varphi^\delta_\upsilon g_1, \dots, \varphi^\delta_\upsilon g_n$ but does not belong to those of the events $\varphi^\delta_\upsilon f'_1, \dots, \varphi^\delta_\upsilon f'_{m'}, \varphi^\delta_\upsilon g'_1, \dots, \varphi^\delta_\upsilon g'_{n'}$, and $\varphi^\delta_\upsilon a \in W_\epsilon$. Therefore,

$$\begin{aligned} & \varphi^\delta_\upsilon a \,\mathsf{ok}\, \varphi^\delta_\upsilon f_1 \cdots \varphi^\delta_\upsilon a \,\mathsf{ok}\, \varphi^\delta_\upsilon f_m \cdot \varphi^\delta_\upsilon f'_1 \cdots \varphi^\delta_\upsilon f'_{m'} \\ & \leq \varphi^\delta_\upsilon a \,\mathsf{ok}\, \varphi^\delta_\upsilon g_1 \cdots \varphi^\delta_\upsilon a \,\mathsf{ok}\, \varphi^\delta_\upsilon g_n \cdot \varphi^\delta_\upsilon g'_1 \cdots \varphi^\delta_\upsilon g'_{n'} \end{aligned}$$

by Theorem 4.2.9, and so

$$\begin{aligned} & \varphi^\delta_\upsilon (a \,\mathsf{ok}\, f_1) \cdots \varphi^\delta_\upsilon (a \,\mathsf{ok}\, f_m) \cdot \varphi^\delta_\upsilon f'_1 \cdots \varphi^\delta_\upsilon f'_{m'} \\ & \leq \varphi^\delta_\upsilon (a \,\mathsf{ok}\, g_1) \cdots \varphi^\delta_\upsilon (a \,\mathsf{ok}\, g_n) \cdot \varphi^\delta_\upsilon g'_1 \cdots \varphi^\delta_\upsilon g'_{n'} \end{aligned}$$

by the M-homomorphy of φ^δ_υ. Thus Theorem 4.2.4 shows that the conclusion of the gen. case $\oplus$ law holds.

Assume the premise of the gen. case $\ominus$ law and let $(W, \delta, \upsilon) \in \mathcal{D}$. Furthermore, let $s \in W_\epsilon$ and define $\upsilon' = \upsilon(x/s)$. Then $(W, \delta, \upsilon') \in \mathcal{D}$ and

$$\begin{aligned} & \varphi^\delta_{\upsilon'} (x \,\mathsf{ok}\, f_1) \cdots \varphi^\delta_{\upsilon'} (x \,\mathsf{ok}\, f_m) \cdot \varphi^\delta_{\upsilon'} f'_1 \cdots \varphi^\delta_{\upsilon'} f'_{m'} \\ & \leq \varphi^\delta_{\upsilon'} (x \,\mathsf{ok}\, g_1) \cdots \varphi^\delta_{\upsilon'} (x \,\mathsf{ok}\, g_n) \cdot \varphi^\delta_{\upsilon'} g'_1 \cdots \varphi^\delta_{\upsilon'} g'_{n'} \end{aligned}$$

by Theorem 4.2.4, and so

$$\begin{aligned} & \varphi^\delta_{\upsilon'} x \,\mathsf{ok}\, \varphi^\delta_{\upsilon'} f_1 \cdots \varphi^\delta_{\upsilon'} x \,\mathsf{ok}\, \varphi^\delta_{\upsilon'} f_m \cdot \varphi^\delta_{\upsilon'} f'_1 \cdots \varphi^\delta_{\upsilon'} f'_{m'} \\ & \leq \varphi^\delta_{\upsilon'} x \,\mathsf{ok}\, \varphi^\delta_{\upsilon'} g_1 \cdots \varphi^\delta_{\upsilon'} x \,\mathsf{ok}\, \varphi^\delta_{\upsilon'} g_n \cdot \varphi^\delta_{\upsilon'} g'_1 \cdots \varphi^\delta_{\upsilon'} g'_{n'} \end{aligned}$$

by the M-homomorphy of $\varphi^\delta_{\upsilon'}$, and so

$$s \,\mathsf{ok}\, \varphi^\delta_\upsilon f_1 \cdots s \,\mathsf{ok}\, \varphi^\delta_\upsilon f_m \cdot \varphi^\delta_\upsilon f'_1 \cdots \varphi^\delta_\upsilon f'_{m'}$$

$$\leq s\,\mathfrak{ok}\,\varphi_\upsilon^\delta g_1 \cdots s\,\mathfrak{ok}\,\varphi_\upsilon^\delta g_n \cdot \varphi_\upsilon^\delta g_1' \cdots \varphi_\upsilon^\delta g_{n'}'$$

by (3.3.18), (3.3.2) and Theorem 3.3.2. Therefore,

$$\varphi_\upsilon^\delta f_1 \cdots \varphi_\upsilon^\delta f_m \cdot \varphi_\upsilon^\delta f_1' \cdots \varphi_\upsilon^\delta f_{m'}' \leq \varphi_\upsilon^\delta g_1 \cdots \varphi_\upsilon^\delta g_n \cdot \varphi_\upsilon^\delta g_1' \cdots \varphi_\upsilon^\delta g_{n'}'$$

by Theorem 4.2.9. Thus Theorem 4.2.4 shows that the conclusion of the gen. case $\ominus$ law holds.

Theorem 4.2.18 Let $a \in G$, $f \in H$, $K^f = \{k\}$, $\mathfrak{q} \in \mathfrak{Q}$ and $x \in X_\epsilon$. Assume that x does not occur free in f. Then $a\,\mathfrak{q}k\,f \asymp a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle$.

Proof Let $(W, \delta, \upsilon) \in \mathcal{D}$. Then since $a\,\mathfrak{q}k\,f$ and $a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle$ belong to $A_\emptyset$, their images by φ_υ^δ belong to $W_\emptyset = \mathbb{T}$, and

$$\begin{aligned}
&\varphi_\upsilon^\delta(a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle) = 1\\
&\iff \varphi_\upsilon^\delta a\,\mathfrak{q}\pi\,\varphi_\upsilon^\delta((x\,\mathfrak{o}k\,f)\triangledown x)\triangle = 1\\
&\iff |s \in W_\epsilon : s \sqsubseteq \varphi_\upsilon^\delta a, s\,\mathfrak{o}\pi\,\varphi_\upsilon^\delta((x\,\mathfrak{o}k\,f)\triangledown x)\triangle = \nu| \in \mathfrak{p}
\end{aligned}$$

by the M-homomorphy of φ_υ^δ and (W8).[4.4] Here $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}$, and

$$\begin{aligned}
&s\,\mathfrak{o}\pi\,\varphi_\upsilon^\delta((x\,\mathfrak{o}k\,f)\triangledown x)\triangle\\
&= (\varphi_\upsilon^\delta((x\,\mathfrak{o}k\,f)\triangledown x))s && \text{(by Theorem 4.2.10)}\\
&= \varphi_{\upsilon(x/s)}^\delta(x\,\mathfrak{o}k\,f) && \text{(by (4.1.5))}\\
&= \varphi_{\upsilon(x/s)}^\delta x\,\mathfrak{o}k\,\varphi_{\upsilon(x/s)}^\delta f && \text{(by the M-homomorphy of } \varphi_{\upsilon(x/s)}^\delta)\\
&= s\,\mathfrak{o}k\,\varphi_\upsilon^\delta f
\end{aligned}$$

by (3.3.18), (3.3.2) and Theorem 3.3.2. Therefore,

$$\begin{aligned}
&\varphi_\upsilon^\delta(a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle) = 1\\
&\iff |s \in W_\epsilon : s \sqsubseteq \varphi_\upsilon^\delta a, s\,\mathfrak{o}k\,\varphi_\upsilon^\delta f = \nu| \in \mathfrak{p}\\
&\iff \varphi_\upsilon^\delta a\,\mathfrak{q}k\,\varphi_\upsilon^\delta f = 1 && \text{(by (W8))}\\
&\iff \varphi_\upsilon^\delta(a\,\mathfrak{q}k\,f) = 1
\end{aligned}$$

by the M-homomorphy of φ_υ^δ, hence $\varphi_\upsilon^\delta(a\,\mathfrak{q}k\,f) = \varphi_\upsilon^\delta(a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle)$. Thus $a\,\mathfrak{q}k\,f \asymp a\,\mathfrak{q}\pi\,((x\,\mathfrak{o}k\,f)\triangledown x)\triangle$ by Theorem 4.2.4.

Theorem 4.2.19 Let $a \in G$, $f, g \in A_\emptyset$ and $x \in X_\epsilon$. Assume that x does not occur free in f. Then $f \cdot a\,\forall\pi\,((f \Rrightarrow g)\triangledown x)\triangle \preccurlyeq a\,\forall\pi\,(g\,\triangledown x)\triangle$.

[4.4] Since φ_υ^δ is an M-homomorphism, $\varphi_\upsilon^\delta(a\triangle) = (\varphi_\upsilon^\delta a)\triangle$ for each $a \in G$. Therefore, both sides of $=$ here may be denoted $\varphi_\upsilon^\delta a\triangle$, and likewise for the operation $\square$ as in Theorem 4.2.21.

Proof Let $(W, \delta, \upsilon) \in \mathcal{D}$. Then since f, $\mathsf{a}\,\forall\pi\,((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})\triangle$ and $\mathsf{a}\,\forall\pi\,(\mathsf{g}\,\triangledown\mathsf{x})\triangle$ belong to $A_\emptyset$, their images by φ^δ_υ belong to $W_\emptyset = \mathbb{T}$, and

$$\varphi^\delta_\upsilon(\mathsf{a}\,\forall\pi\,((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})\triangle) = \varphi^\delta_\upsilon\mathsf{a}\,\forall\pi\,\varphi^\delta_\upsilon((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})\triangle,$$
$$\varphi^\delta_\upsilon(\mathsf{a}\,\forall\pi\,(\mathsf{g}\,\triangledown\mathsf{x})\triangle) = \varphi^\delta_\upsilon\mathsf{a}\,\forall\pi\,\varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x})\triangle$$

by the M-homomorphy of φ^δ_υ (s. [4.4]), and so

$$\varphi^\delta_\upsilon(\mathsf{a}\,\forall\pi\,((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})\triangle) = 1 \iff \varphi^\delta_\upsilon\mathsf{a} \sqsubseteq \varphi^\delta_\upsilon((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x}),$$
$$\varphi^\delta_\upsilon(\mathsf{a}\,\forall\pi\,(\mathsf{g}\,\triangledown\mathsf{x})\triangle) = 1 \iff \varphi^\delta_\upsilon\mathsf{a} \sqsubseteq \varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x})$$

by Theorem 4.2.10. Therefore, assume $\varphi^\delta_\upsilon\mathsf{f} = 1$ and $\varphi^\delta_\upsilon\mathsf{a} \sqsubseteq \varphi^\delta_\upsilon((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})$. Assume furthermore that an element $\mathsf{s} \in W_\epsilon$ satisfies $\mathsf{s} \sqsubseteq \varphi^\delta_\upsilon\mathsf{a}$. Then $\varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})}\mathsf{f} = 1$ by Theorem 3.3.2, and

$$\begin{aligned}
1 &= (\varphi^\delta_\upsilon((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x}))\mathsf{s} && \text{(by (4.2.2) and (4.1.3))}\\
&= \varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})}(\mathsf{f} \Rightarrow \mathsf{g}) && \text{(by (4.1.5))}\\
&= \varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})}\mathsf{f} \Rightarrow \varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})}\mathsf{g} && \text{(by the M-homomorphy of } \varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})})\\
&= \varphi^\delta_{\upsilon(\mathsf{x}/\mathsf{s})}\mathsf{f} \Rightarrow (\varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x}))\mathsf{s} && \text{(by (4.1.5))},
\end{aligned}$$

hence $(\varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x}))\mathsf{s} = 1$, that is, $\mathsf{s} \sqsubseteq \varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x})$. Therefore, $\varphi^\delta_\upsilon\mathsf{a} \sqsubseteq \varphi^\delta_\upsilon(\mathsf{g}\,\triangledown\mathsf{x})$. Thus $\mathsf{f} \cdot \mathsf{a}\,\forall\pi\,((\mathsf{f} \Rightarrow \mathsf{g})\triangledown\mathsf{x})\triangle \preccurlyeq \mathsf{a}\,\forall\pi\,(\mathsf{g}\,\triangledown\mathsf{x})\triangle$ by Theorem 4.2.4.

Theorem 4.2.20 Let $\mathsf{a} \in A_\epsilon$, $\mathsf{f} \in A_\emptyset$ and $\mathsf{x} \in X_\epsilon$. Assume that x is free from a in f. Then $\mathsf{a}\,\mathfrak{o}\pi\,(\mathsf{f}\,\triangledown\mathsf{x})\triangle \asymp \mathsf{f}(\mathsf{x}/\mathsf{a})$ for the substitution (x/a) of a for x.

Proof Let $(W, \delta, \upsilon) \in \mathcal{D}$. Then $\mathsf{a}\,\mathfrak{o}\pi\,(\mathsf{f}\,\triangledown\mathsf{x})\triangle$ and $\mathsf{f}(\mathsf{x}/\mathsf{a})$ belong to $A_\emptyset$, and

$$\begin{aligned}
\varphi^\delta_\upsilon(\mathsf{a}\,\mathfrak{o}\pi\,(\mathsf{f}\,\triangledown\mathsf{x})\triangle) &= \varphi^\delta_\upsilon\mathsf{a}\,\mathfrak{o}\pi\,\varphi^\delta_\upsilon(\mathsf{f}\,\triangledown\mathsf{x})\triangle && \text{(by the M-homomorphy of } \varphi^\delta_\upsilon)\\
&= (\varphi^\delta_\upsilon(\mathsf{f}\,\triangledown\mathsf{x}))(\varphi^\delta_\upsilon\mathsf{a}) && \text{(by Theorem 4.2.10)}\\
&= \varphi^\delta_{\upsilon(\mathsf{x}/\varphi^\delta_\upsilon\mathsf{a})}\mathsf{f} && \text{(by (4.1.5))}\\
&= \varphi^\delta_\upsilon(\mathsf{f}(\mathsf{x}/\mathsf{a})) && \text{(by Theorem 3.3.3)}.
\end{aligned}$$

Thus $\mathsf{a}\,\mathfrak{o}\pi\,(\mathsf{f}\,\triangledown\mathsf{x})\triangle \asymp \mathsf{f}(\mathsf{x}/\mathsf{a})$ by Theorem 4.2.4.

4.2.6 The generic one and the existence

We pick a variable $\mathsf{x}_0 \in X_\epsilon$ and define

$$\breve{1} = (\mathsf{x}_0\,\forall\pi\,\mathsf{x}_0\triangle)\triangledown\mathsf{x}_0, \qquad\qquad \breve{e} = (\mathsf{x}_0\,\exists\pi\,\breve{1}\triangle)\triangledown\mathsf{x}_0.$$

Then $\breve{1}$ and $\breve{e}$ belong to $A_\mathfrak{e}$ and are called the **generic one** and **existence** respectively.[4.5]

[4.5] The existence here means *all that exist* and is defined as a nominal, while the existence in Remark 4.2.2 means *the state of existing* and was defined for entities.

Theorem 4.2.21 If $(W, \delta, \upsilon) \in \mathcal{D}$, then $\varphi_\upsilon^\delta \breve{1}$ and $\varphi_\upsilon^\delta \breve{1}^\square$ are the largest element 1_ϵ and the smallest element 0_ϵ of W_ϵ respectively (s. [4.4]).

More generally, $\varepsilon \preccurlyeq \mathrm{a} \forall\pi\, \mathrm{a} \triangle$ for all $\mathrm{a} \in \mathsf{G}$ and, if $\mathrm{f} \in A_\emptyset$ and $\varepsilon \preccurlyeq \mathrm{f}$, then $\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) = 1_\epsilon$ and $\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x})^\square = 0_\epsilon$ for all $\mathrm{x} \in X_\epsilon$.

Proof We have $\varphi_\upsilon^\delta(\mathrm{a} \forall\pi\, \mathrm{a} \triangle) = \varphi_\upsilon^\delta \mathrm{a} \forall\pi\, \varphi_\upsilon^\delta \mathrm{a} \triangle = 1$ for all $(W, \delta, \upsilon) \in \mathcal{D}$ by the M-homomorphy of φ_υ^δ and Theorem 4.2.10. Thus $\varepsilon \preccurlyeq \mathrm{a} \forall\pi\, \mathrm{a} \triangle$ by Theorems 4.2.2 and 4.2.4.

Since $\mathrm{f} \triangledown \mathrm{x} \in A_\epsilon$, $\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) \in W_\epsilon = W_\epsilon \to \mathbb{T}$. If $\mathrm{s} \in W_\epsilon$, then $(\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}))\mathrm{s} = \varphi_{\upsilon(\mathrm{x}/\mathrm{s})}^\delta \mathrm{f}$ by (4.1.5), and so $(\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}))\mathrm{s} = 1$ by Theorems 4.2.4 and 4.2.2. Thus $\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) = 1_\epsilon$, and so $\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x})^\square = 1_\epsilon{}^\square = 0_\epsilon$.

Theorem 4.2.22 If $\mathrm{a} \in A_\epsilon$, then $\mathrm{a}\, \mathsf{o}\pi\, \breve{\mathrm{e}} \triangle \asymp \mathrm{a} \exists\pi\, \breve{1} \triangle$.

Proof If $\mathrm{a} \in A_\epsilon$, then $\mathrm{a}\, \mathsf{o}\pi\, \breve{\mathrm{e}} \triangle = \mathrm{a}\, \mathsf{o}\pi\, ((\mathrm{x}_0 \exists\pi\, \breve{1} \triangle) \triangledown \mathrm{x}_0) \asymp \mathrm{a} \exists\pi\, \breve{1} \triangle$ by Theorem 4.2.20, because x_0 is free from a in $\mathrm{x}_0 \exists\pi\, \breve{1} \triangle = \mathrm{x}_0 \exists\pi\, ((\mathrm{x}_0 \forall\pi\, \mathrm{x}_0 \triangle) \triangledown \mathrm{x}_0) \triangle$ and $(\mathrm{x}_0 \exists\pi\, \breve{1} \triangle)(\mathrm{x}_0/\mathrm{a}) = \mathrm{a} \exists\pi\, \breve{1} \triangle$ by (3.1.14) and (3.1.15).

Lemma 4.2.6 Let $\mathrm{f} \in A_\emptyset$, $\mathrm{x} \in X_\epsilon$ and $(W, \delta, \upsilon) \in \mathcal{D}$. Then

$$\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle) = 1 \iff \varphi_{\upsilon(\mathrm{x}/\mathrm{s})}^\delta \mathrm{f} = 1 \text{ for all } \mathrm{s} \in W_\epsilon.$$

Proof The element $\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle)$ belongs to $W_\emptyset = \mathbb{T}$, and

$$\begin{aligned}
&\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle) = 1 && \\
&\iff \varphi_\upsilon^\delta \breve{1} \forall\pi\, \varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) \triangle = 1 && \text{(by the M-homomorphy of φ_υ^δ)} \\
&\iff 1_\epsilon \forall\pi\, \varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) \triangle = 1 && \text{(by Theorem 4.2.21)} \\
&\iff \varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) = 1_\epsilon && \text{(by Theorem 4.2.10)} \\
&\iff (\varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}))\mathrm{s} = 1 \text{ for all } \mathrm{s} \in W_\epsilon && \\
&\iff \varphi_{\upsilon(\mathrm{x}/\mathrm{s})}^\delta \mathrm{f} = 1 \text{ for all } \mathrm{s} \in W_\epsilon && \text{(by (4.1.5)).}
\end{aligned}$$

Theorem 4.2.23 Let $\mathrm{f} \in A_\emptyset$ and $\mathrm{x} \in X_\epsilon$. Then $\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle \preccurlyeq \mathrm{f}$.

Proof Let $(W, \delta, \upsilon) \in \mathcal{D}$ and assume $\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle) = 1$. Then $\varphi_\upsilon^\delta \mathrm{f} = \varphi_{\upsilon(\mathrm{x}/\upsilon\mathrm{x})}^\delta \mathrm{f} = 1$ by Lemma 4.2.6. Thus $\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle \preccurlyeq \mathrm{f}$ by Theorem 4.2.4.

Theorem 4.2.24 Let $\mathrm{f} \in A_\emptyset$ and $\mathrm{x} \in X_\epsilon$. Then $\varepsilon \preccurlyeq \breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle$ iff $\varepsilon \preccurlyeq \mathrm{f}$.

Proof Assume $\varepsilon \preccurlyeq \mathrm{f}$ and let $(W, \delta, \upsilon) \in \mathcal{D}$. Then $\varphi_{\upsilon(\mathrm{x}/\mathrm{s})}^\delta \mathrm{f} = 1$ for all $\mathrm{s} \in W_\epsilon$ by Theorems 4.2.4 and 4.2.2, and so $\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle) = 1$ by Lemma 4.2.6 (alternatively, $\varphi_\upsilon^\delta(\breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle) = \varphi_\upsilon^\delta \breve{1} \forall\pi\, \varphi_\upsilon^\delta(\mathrm{f} \triangledown \mathrm{x}) \triangle = 1_\epsilon \forall\pi\, 1_\epsilon \triangle = 1$ by Theorem 4.2.21). Thus $\varepsilon \preccurlyeq \breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle$ by Theorems 4.2.2 and 4.2.4.

The validity relation $\preccurlyeq$ is a Boolean relation by Theorem 4.2.5. Therefore, if $\varepsilon \preccurlyeq \breve{1} \forall\pi\, (\mathrm{f} \triangledown \mathrm{x}) \triangle$, then $\varepsilon \preccurlyeq \mathrm{f}$ by Theorem 4.2.23 and the cut law.

Theorem 4.2.25 Let $x \in X_\epsilon$, $a \in G$, $f \in H$ and $K^f = \{k\}$. Assume that x occurs free in neither a nor f. Then $\breve{1}\,\forall\pi\,(((x\,o\pi\,a\triangle) \Rrightarrow (x\,ok\,f))\nabla x)\triangle \asymp a\,\forall k\,f$.

Proof Let $(W, \delta, \upsilon) \in \mathcal{D}$. Then

$$\begin{aligned}
&\varphi^\delta_\upsilon(\breve{1}\,\forall\pi\,(((x\,o\pi\,a\triangle) \Rrightarrow (x\,ok\,f))\nabla x)\triangle) = 1\\
&\iff \varphi^\delta_{\upsilon(x/s)}((x\,o\pi\,a\triangle) \Rrightarrow (x\,ok\,f)) = 1 \text{ for all } s \in W_\epsilon\\
&\iff (\varphi^\delta_{\upsilon(x/s)}x\,o\pi\,\varphi^\delta_{\upsilon(x/s)}a\triangle) \Rrightarrow (\varphi^\delta_{\upsilon(x/s)}x\,ok\,\varphi^\delta_{\upsilon(x/s)}f) = 1 \text{ for all } s \in W_\epsilon
\end{aligned}$$

by Lemma 4.2.6 and the M-homomorphy of $\varphi^\delta_{\upsilon(x/s)}$. Here $\varphi^\delta_{\upsilon(x/s)}x = s$ by (3.3.18) and (3.3.2), and $\varphi^\delta_{\upsilon(x/s)}a = \varphi^\delta_\upsilon a$ and $\varphi^\delta_{\upsilon(x/s)}f = \varphi^\delta_\upsilon f$ by Theorem 3.3.2. Therefore,

$$\begin{aligned}
&\varphi^\delta_\upsilon(\breve{1}\,\forall\pi\,(((x\,o\pi\,a\triangle) \Rrightarrow (x\,ok\,f))\nabla x)\triangle) = 1\\
&\iff (s\,o\pi\,\varphi^\delta_\upsilon a\triangle) \Rrightarrow (s\,ok\,\varphi^\delta_\upsilon f) = 1 \text{ for all } s \in W_\epsilon\\
&\iff \text{if } s \in W_\epsilon \text{ and } s\,o\pi\,\varphi^\delta_\upsilon a\triangle = 1 \text{ then } s\,ok\,\varphi^\delta_\upsilon f = 1\\
&\iff \text{if } s \in W_\epsilon \text{ and } s \sqsubseteq \varphi^\delta_\upsilon a \text{ then } s\,ok\,\varphi^\delta_\upsilon f = 1 \qquad \text{(by Theorem 4.2.10)}\\
&\iff \{s \in W_\epsilon : s \sqsubseteq \varphi^\delta_\upsilon a, s\,ok\,\varphi^\delta_\upsilon f = 0\} = \emptyset\\
&\iff |s \in W_\epsilon : s \sqsubseteq \varphi^\delta_\upsilon a, s\,ok\,\varphi^\delta_\upsilon f = 0| \le 0\\
&\iff \varphi^\delta_\upsilon a\,\forall k\,\varphi^\delta_\upsilon f = 1 \qquad \text{(by (W8))}\\
&\iff \varphi^\delta_\upsilon(a\,\forall k\,f) = 1.
\end{aligned}$$

Thus $\breve{1}\,\forall\pi\,(((x\,o\pi\,a\triangle) \Rrightarrow (x\,ok\,f))\nabla x)\triangle \asymp a\,\forall k\,f$ by Theorem 4.2.4.

4.3 Sample phraseology for the English people

The logic system CL of enough partibility together with a certain deduction system on it is intended to provide a mathematical model of the nootrinity $(\mathrm{IU}, \mathcal{W}, \mathcal{R})$. In particular, the formal language of the CL provides a model of the PU. The IC language A is not quite adequate for that because ICL has the smallest partibility 1. Nevertheless, the best way for understanding ICL is to regard A as the PU (s. Remark 1.1.1) and consider the phraseological relationship between the DT and PU illustrated by the p^2-diagram and the cUPO-diagram in §1.2.5. The following is a generalization of the cUPO-diagram.

$$\begin{array}{ccccc}
(U_1, \ldots, U_k) & \xrightarrow{\text{k times}} & (P_1, \ldots, P_k) & \xrightarrow{\text{k times}} & (O_1, \ldots, O_k)\\
\searrow & & \{\alpha_i\}\downarrow & & \swarrow\\
U & \longrightarrow & P & \longrightarrow & O
\end{array} \qquad \text{(gcUPO)}$$

Phraseology mainly concerns its left half:

$$\begin{array}{ccc}
(U_1, \ldots, U_k) & \xrightarrow{\text{k times}} & (P_1, \ldots, P_k)\\
\searrow & & \swarrow\{\alpha_i\}\\
U & \longrightarrow & P
\end{array} \qquad \text{(gcUP)}$$

Here operations α_i $(i = 1, \ldots, n)$ in the processual algebraic structure of the PU together transform a tuple $(P_1, \ldots, P_k)$ of percepts into a percept P, and we assume that $P_1, \ldots, P_k$ and P are phrased by DUs $U_1, \ldots U_k$ and U respectively. Then in phraseology, we ask how $\alpha_1, \ldots, \alpha_n$ relate $(U_1, \ldots U_k)$ to U. As was noted after the cUPO-diagram, the question must be answered not only by the evolutional relationship between the DT and PU illustrated by the e^2-diagram but also by the adaptation of the syntactical algebraic structure abstracted from the DT to the preference of the verbal community (s. [1.43]).

The verbal community must be our native community. For convenience of explanation, however, I focus on the present-day English community. Furthermore, phraseology I present here is a sample. Therefore, I urge you to carry out phraseology for your native community after the sample. As you carry on phraseology, you will find sequents which are expected to be tautologies. You can prove that they are so by the methods in §4.2.

4.3.1 Adapting ICL to the English community

Since the definition of ICL is based on semasiology and phraseology as explained in §1.2.7 for the Japanese community, we must begin by adapting A to the English community, that is, we must alter the positions of the tokens, symbols and arguments for certain operations of A by the following rules of translation, which show that the English community is basically a mirror image of the Japanese community except that it places the nominative case marker π at the head.

Rule Table: The Rules of Translation

Japanese	English	key conditions
$a\,q\pi\,f$	$\pi q(a, f)$	$q \in \{\mathfrak{o}\} \cup \mathfrak{Q}$, $\pi \in K^f$
$a\,qk\,f$	$f\,kq\,a$	$q \in \{\mathfrak{o}\} \cup \mathfrak{Q}$, $k \in K^f - \{\pi\}$
$f^{\Diamond}$	${}^{\Diamond}f$	
$a\triangle$	$\triangle a$	
$a^{\square}$	${}^{\square}a$	
$\vec{a}\phi$	$\phi\vec{a}$	$\phi \in \Phi$
$f\,\triangledown x$	$\triangledown x\,f$	$f \in A_{\emptyset}$, $x \in X_{\epsilon}$

Combining these rules, we have the following examples of translation, where q and $\mathfrak{r}$ are elements of $\{\mathfrak{o}\} \cup \mathfrak{Q}$.

Translation Table: Examples of Translation

Japanese	English	key conditions
$a\,q\pi\,(b\,\mathfrak{r}k\,f)$	$\pi q(a, f\,k\mathfrak{r}\,b)$	$\pi \in K^f$, $k \in K^f - \{\pi\}$
$a\,qk\,(b\,\mathfrak{r}\pi\,f)$	$\pi\mathfrak{r}(b, f)\,kq\,a$	$\pi \in K^f$, $k \in K^f - \{\pi\}$
$a\,qk\,(b\,\mathfrak{r}l\,f)$	$(f\,l\mathfrak{r}\,b)\,kq\,a$	$k \in K^f - \{\pi\}$, $l \in K^f - \{\pi, k\}$
$a\,q\pi\,f^{\Diamond}$	$\pi q(a, {}^{\Diamond}f)$	$\pi \in K^f$
$a\,qk\,f^{\Diamond}$	${}^{\Diamond}f\,kq\,a$	$k \in K^f - \{\pi\}$
$a\,q\pi\,b\triangle$	$\pi q(a, \triangle b)$	

$$\begin{array}{ll} \mathsf{a}\,\mathfrak{q}\pi\,(\mathsf{b}^{\Box})\triangle & \pi\mathfrak{q}(\mathsf{a},\triangle(^{\Box}\mathsf{b})) \\ \mathsf{a}\,\mathfrak{q}\pi\,(\mathsf{b}\triangle)^{\Diamond} & \pi\mathfrak{q}(\mathsf{a},{}^{\Diamond}(\triangle\mathsf{b})) \\ \mathsf{a}\,\mathfrak{q}\pi\,(\mathsf{f}\,\triangledown\mathsf{x})\triangle & \pi\mathfrak{q}(\mathsf{a},\triangle(\triangledown\mathsf{x}\,\mathsf{f})) \end{array}$$

4.3.2 Peter eats some radishes. All rabbits eat some.

The first declarative $\pi\mathfrak{q}(\mathsf{a},\mathsf{f}\,\mathsf{k}\mathfrak{r}\,\mathsf{b})$ on the translation table with $\mathfrak{q}=\mathfrak{o}$ and $\mathfrak{r}=\exists$ can have the following gcUP-diagram among many others.

$$\begin{tikzcd} \text{(Peter, eat, radish)} \arrow[r, "\text{3 times}"] \arrow[d, swap] & (\mathsf{a},\mathsf{f},\mathsf{b}) \arrow[d, "{\pi\mathfrak{o},\,\mathsf{k}\exists}"] \\ \text{Peter eats some radishes} \arrow[r] & \pi\mathfrak{o}(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b}) \end{tikzcd}$$

This may be abbreviated yet partly amplified to the following ‘agcUP-table.’

π	$\mathfrak{o}$	(	a,	f	k	$\exists$	b	)
			Peter	eat(s)		some	radish(es)	

In addition to what the gcUP-diagram implies, the agcUP-table implies that the quantifier $\exists$ is phrased by the word *some*. It also implies that the nominative case marker π and the case marker k, which may be called accusative, are silent due to the communal preference, and so are the case operationalizer $\mathfrak{o}$, the parentheses and the comma. In other words, the RDT for the English community should be equipped with a nominative preposition and an accusative one (s. [1.76]) as well as the case operationalizer, parentheses and commas. It furthermore implies that semasiology for the English community should remove the plural declension of *radishes* and the conjugation of *eats* by person and number because they are redundant for decrease of obscurity (s. [1.74]). It moreover implies that, except for the silence, declension and conjugation due to communal preference, the lower left corner of the gcUP-diagram is obtained by replacing a, f, $\exists$ and b in $\pi\mathfrak{o}(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})$ with *Peter*, *eat*, *some* and *radish* because of the evolutional relationship between the DT and the PU. We have thus answered the phraseological question about $\pi\mathfrak{o}(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})$.

The declarative $\pi\mathfrak{q}(\mathsf{a},\mathsf{f}\,\mathsf{k}\mathfrak{r}\,\mathsf{b})$ with $\mathfrak{q}=\forall$ or $\exists$ instead of $\mathfrak{o}$ and $\mathfrak{r}=\exists$ can likewise have the following agcUP-tables.

π	$\forall$	(	a,	f	k	$\exists$	b	)
	All		rabbit(s)	eat		some	radish(es)	

π	$\exists$	(	a,	f	k	$\exists$	b	)
	Some		rabbit(s)	eat		some	radish(es)	

Here the quantifier $\forall$ is phrased by the word *all*, and semasiology should remove the plural declension of *rabbits* as well as *radishes*.

Now, Theorem 4.2.4, Corollary 4.2.6.3 and Theorem 4.2.11 together show that the following hold for the negations of the above declaratives:

$${}^{\Diamond}(\pi\mathfrak{o}(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\mathfrak{o}(\mathsf{a},{}^{\Diamond}(\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\mathfrak{o}(\mathsf{a},{}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b}),$$

$$
\begin{aligned}
&{}^{\Diamond}(\pi\forall(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\exists(\mathsf{a}, {}^{\Diamond}(\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\exists(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b}),\\
&{}^{\Diamond}(\pi\exists(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\forall(\mathsf{a}, {}^{\Diamond}(\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\forall(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b}).
\end{aligned}
$$

The rightmost declaratives of these lines can have the following agcUP-tables.

π	$\mathfrak{o}$	(	a,	$\Diamond$	f	k	$\forall$	b	)
			Peter	(does) not	eat		any	radish(es)	

π	$\exists$	(	a,	$\Diamond$	f	k	$\forall$	b	)
	Some		rabbit(s)	(do) not	eat		any	radish(es)	

π	$\forall$	(	a,	$\Diamond$	f	k	$\forall$	b	)
	Any		rabbit(s)	(do) not	eat		any	radish(es)	

Here the quantifier $\forall$ is phrased by *any* and the token $\Diamond$ is phrased by *does not* and *do not*, but semasiology should remove *does* and *do* because they are redundant for decrease of obscurity. Furthermore, the leftmost declarative of the second line can have the following agcUP-table.

$\Diamond$	(	π	$\forall$	(	a,	f	k	$\exists$	b	))
Not			all		rabbit(s)	eat		some	radish(es)	

Now that ${}^{\Diamond}(\pi\forall(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})) \asymp \pi\exists(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b})$, we have $\varphi^{\delta}_{\upsilon}({}^{\Diamond}(\pi\forall(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b}))) = \varphi^{\delta}_{\upsilon}(\pi\exists(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b}))$ for each $(\mathcal{W}, \delta, \upsilon) \in \mathcal{D}$ by Theorem 4.2.4. Since $\mathcal{W}$ is a model of a nooworld and $\varphi^{\delta}_{\upsilon}$ is a model of a PERCEPTION into the nooworld, this equality means that, when regarded as DECLARATIVEs in the PU, ${}^{\Diamond}(\pi\forall(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b}))$ and $\pi\exists(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b})$ have the same object O as shown in the gcUPO-diagram. Thus the possibility of the above agcUP-tables for them implies that the two English declaratives *Some rabbits do not eat any radishes* and *Not all rabbits eat some radishes* can have the same external meaning O, although their MEANINGS ${}^{\Diamond}(\pi\forall(\mathsf{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathsf{b}))$ and $\pi\exists(\mathsf{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathsf{b})$ are different (s. the remark on the SUPO-diagram) and any two declaratives (and any two nominals) can also have different external meanings because of difference in CONTEXT (s. Remark 1.2.7). The same remark applies to certain agcUP-tables below.

Moreover, the leftmost declaratives of the above three lines can have the following agcUP-tables, where *Iint* is an abbreviation of *It is not that*:

$\Diamond$	(π	$\mathfrak{o}$	(	a,	f	k	$\exists$	b	))
Iint				Peter	eat(s)		some	radish(es)	

$\Diamond$	(π	$\forall$	(	a,	f	k	$\exists$	b	))
Iint		all		rabbit(s)	eat		some	radish(es)	

$\Diamond$	(π	$\exists$	(	a,	f	k	$\exists$	b	))
Iint		some		rabbit(s)	eat		some	radish(es)	

To my knowledge of English, however, the remaining middle declaratives of the above three lines have no appropriate agcUP-tables, that is, when regarded as DECLARATIVEs in the PU, they are not appropriately phrased.

Incidentally, the Japanese community enjoys looseness about that. First, the Japanese counterparts $\mathsf{a}\,\mathsf{o}\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})$, $\mathsf{a}\,\forall\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})$ and $\mathsf{a}\,\exists\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})$ of the above declaratives $\pi\mathsf{o}(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})$, $\pi\forall(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})$ and $\pi\exists(\mathsf{a},\mathsf{f}\,\mathsf{k}\exists\,\mathsf{b})$ can have the following agcUP-tables.

a	o	π	(	b	$\exists$	k	f	)
pêtâ		ga		daikon	(ikuhon ka)	o	taberu	
Peter				radish	(some)		eat	

a	$\forall$	π	(	b	$\exists$	k	f	)
usagi	subete	ga		daikon	(ikuhon ka)	o	taberu	
rabbit	all			radish	(some)		eat	

a	$\exists$	π	(	b	$\exists$	k	f	)
usagi	ikuhiki ka	ga		daikon	(ikuhon ka)	o	taberu	
rabbit	some			radish	(some)		eat	

Here English counterparts, if any, of the Japanese words are shown underneath as always. The words *ga* and *o* are explicit case markers, which are nominative and accusative respectively and replaced with the topic marker *wa* according to CONTEXT (s. [1.73]). The word *daikon* means *hatuka daikon*. The parentheses for *ikuhon ka* mean that the word is usually disregarded and silent.

Secondly, the same propositions as above show that the following hold for the negations of the above declaratives:

$$
\begin{aligned}
(\mathsf{a}\,\mathsf{o}\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f}))^{\Diamond} &\asymp \mathsf{a}\,\mathsf{o}\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})^{\Diamond} \asymp \mathsf{a}\,\mathsf{o}\pi\,(\mathsf{b}\,\forall\mathsf{k}\,\mathsf{f}^{\Diamond}),\\
(\mathsf{a}\,\forall\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f}))^{\Diamond} &\asymp \mathsf{a}\,\exists\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})^{\Diamond} \asymp \mathsf{a}\,\exists\pi\,(\mathsf{b}\,\forall\mathsf{k}\,\mathsf{f}^{\Diamond}),\\
(\mathsf{a}\,\exists\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f}))^{\Diamond} &\asymp \mathsf{a}\,\forall\pi\,(\mathsf{b}\,\exists\mathsf{k}\,\mathsf{f})^{\Diamond} \asymp \mathsf{a}\,\forall\pi\,(\mathsf{b}\,\forall\mathsf{k}\,\mathsf{f}^{\Diamond}).
\end{aligned}
$$

Lastly, all of these nine declaratives can have some agcUP-tables by virtue of the disregard of counterparts of *some* and *any*. For example,

(	a	o	π	(	b	$\exists$	k	f	))	$\Diamond$	
	a	o	π	(	b	$\exists$	k	f	)	$\Diamond$	
	a	o	π	(	b	$\forall$	k	f		$\Diamond$	)
	pêtâ		ga		daikon		o	tabe		nai	
	Peter				radish			eat		not	

Here *tabe* under f is the conjugation of *taberu* which *nai* under $\Diamond$ requires, but semasiology should replace it with *taberu*. Furthermore, we have

a	$\exists$	π	(	b	$\exists$	k	f	)	$\Diamond$	
a	$\exists$	π	(	b	$\forall$	k	f		$\Diamond$	)
usagi	ikuhiki ka	ga		daikon		o	tabe		nai	
rabbit	some			radish			eat		not	

a	$\forall$	π	(	b	$\exists$	k	f	)	$\Diamond$	
a	$\forall$	π	(	b	$\forall$	k	f		$\Diamond$	)
usagi	subete	ga		daikon		o	tabe		nai	
rabbit	all			radish			eat		not	

Furthermore, we have the following, where *ndn* is an abbreviation of the expression *no de* (*wa*) *nai* which is a counterpart of *iint* (*it is not that*):

(	$\mathfrak{a}$	$\mathfrak{o}$	π	(	$\mathfrak{b}$	$\exists$	k	f	))	$\Diamond$
	pêtâ		ga		daikon		o	taberu		ndn
	Peter				radish			eat		iint

(	$\mathfrak{a}$	$\forall$	π	(	$\mathfrak{b}$	$\exists$	k	f	))	$\Diamond$
	usagi	subete	ga		daikon		o	taberu		ndn
	rabbit	all			radish			eat		iint

(	$\mathfrak{a}$	$\exists$	π	(	$\mathfrak{b}$	$\exists$	k	f	))	$\Diamond$
	usagi	ikuhiki ka	ga		daikon		o	taberu		ndn
	rabbit	some			radish			eat		iint

We may obtain other agcUP-tables for the declarative $\pi\mathfrak{q}(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\mathfrak{r}\,\mathfrak{b})$ by replacing the words *Peter*, *rabbit*, *radish* and *eat* with other appropriate words and picking $\mathfrak{q}$ and $\mathfrak{r}$ appropriately from $\{\mathfrak{o}\} \cup \mathfrak{Q}$, as we have already done so and as the following examples show, and likewise for other DUs.

The declarative $\pi\forall(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})$ can also have the following agcUP-table.

π	$\forall$	(	$\mathfrak{a}$,	f	k	$\exists$	$\mathfrak{b}$	)
	All		rabbit(s)	go	into	some	garden(s)	

Here the case marker k is not silent but phrased by the preposition *into* and so is different from the silent accusative case marker also denoted k above.

Suppose the quantitative set $\mathbb{P}$ is equal to $\mathbb{Z}_{\geq 0}$. Then its interval $(2\rightarrow) = \{\mathfrak{n} \in \mathbb{Z}_{\geq 0} : 2 < \mathfrak{n}\}$ is a positive quantifier, and the declarative $\pi(2\rightarrow)(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})$ can have the following agcUP-table.

π	$(2\rightarrow)$	(	$\mathfrak{a}$,	f	k	$\exists$	$\mathfrak{b}$	)
	More than 2		rabbit(s)	go	into	some	garden(s)	

To my knowledge of English, the declarative $\pi\neg(2\rightarrow)(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})$ for the negative quantifier $\neg(2\rightarrow)$ has no appropriate agcUP-table. However,

$$\pi\neg(2\rightarrow)(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b}) \asymp \pi(2\rightarrow)(\mathfrak{a}, {}^{\Diamond}(\mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})) \asymp \pi(2\rightarrow)(\mathfrak{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathfrak{b})$$

by Theorems 4.2.4 and 4.2.11, and the rightmost declarative can have the following agcUP-table.

π	$(2\rightarrow)$	(	$\mathfrak{a}$,	$\Diamond$	f	k	$\forall$	$\mathfrak{b}$	)
	More than 2		rabbit(s)	(do) not	go	into	any	garden	

Furthermore, the interval $(\leftarrow 2] = \{\mathfrak{a} \in \mathbb{Z}_{\geq 0} : \mathfrak{a} \leq 2\}$ is a positive quantifier, and the same theorems show that the negation of $\pi(2\rightarrow)(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})$ satisfies

$${}^{\Diamond}(\pi(2\rightarrow)(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b})) \asymp \pi(\leftarrow 2](\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b}).$$

These declaratives can have the following agcUP-tables.

$\Diamond$	$(\pi$	$(2{\rightarrow})$	$($	$\mathfrak{a},$	f	k	$\exists$	$\mathfrak{b}$	$))$
Iint		more than 2		rabbit(s)	go	into	some	garden(s)	

π	$({\leftarrow}2]$	$($	$\mathfrak{a},$	f	k	$\exists$	$\mathfrak{b}$	$)$
	At most 2		rabbit(s)	go	into	some	garden(s)	

The possibility of these agcUP-tables implies that the two English declaratives *It is not that more than two rabbits go into some gardens* and *At most two rabbits go into some gardens* can have the same external meaning.

4.3.3 Peter is fond of some radishes. Peter is wild.

As was noted in Remark 1.3.3, the definition of CL is based on the classification of the Japanese declaratives which put verbs and (nominal) adjectives together. Therefore, we put the English verbs and predicative adjectives (i.e. adjectives accompanied by copulas) together. Consequently, the declarative $\pi\mathfrak{q}(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\mathfrak{r}\,\mathfrak{b})$ considered in §4.3.2 and the simpler declarative $\pi\mathfrak{q}(\mathfrak{a}, \mathsf{f})$ can have the following agcUP-tables with $\mathfrak{q} = \mathfrak{o}$ and $\mathfrak{r} = \exists$.

π	$\mathfrak{o}$	$($	$\mathfrak{a},$	f	k	$\exists$	$\mathfrak{b}$	$)$
			Peter	(is) fond	of	some	radish(es)	

π	$\mathfrak{o}$	$($	$\mathfrak{a},$	f	$)$
			Peter	(is) wild	

Here f is phrased by *is fond* and *is wild*, but semasiology should remove *is*. The former agcUP-table is the same as the first one in §4.3.2 except that f thereof was phrased by the verb *eat* and k thereof was silent. This is a reason why verbs and predicative adjectives should be put together.

Furthermore, the negations ${}^{\Diamond}(\pi\mathfrak{o}(\mathfrak{a}, \mathsf{f}\,\mathsf{k}\exists\,\mathfrak{b}))$ and ${}^{\Diamond}(\pi\mathfrak{o}(\mathfrak{a}, \mathsf{f}))$ are equivalent to $\pi\mathfrak{o}(\mathfrak{a}, {}^{\Diamond}\mathsf{f}\,\mathsf{k}\forall\,\mathfrak{b})$ and $\pi\mathfrak{o}(\mathfrak{a}, {}^{\Diamond}\mathsf{f})$, which can have the following agcUP-tables.

π	$\mathfrak{o}$	$($	$\mathfrak{a},$	$\Diamond$	f	k	$\forall$	$\mathfrak{b}$	$)$
			Peter	(is) not	fond	of	any	radish(es)	

π	$\mathfrak{o}$	$($	$\mathfrak{a},$	$\Diamond$	f	$)$
			Peter	(is) not	wild	

Here the negation ${}^{\Diamond}\mathsf{f}$ of f is phrased by *is not fond* and *is not wild*, but semasiology should remove *is*.

4.3.4 Peter is a rabbit. All rabbits are wildlife.

The sixth declarative $\pi\mathfrak{q}(\mathfrak{a}, \triangle\mathfrak{b})$ on the translation table in §4.3.1 can have the following agcUP-tables for $\mathfrak{q} = \mathfrak{o}$, $\forall$ and $\exists$.

π	$\mathfrak{o}$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	)
			Peter	be[is]	(a) rabbit	

π	$\forall$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	)
	All		rabbit(s)	be[are]	wildlife	

π	$\exists$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	)
	Some		rabbit(s)	be[are]	wildlife	

Here the token $\triangle$ is phrased by the words *is* and *are*, but semasiology should replace them with the word *be* and remove the indefinite article *a* because of redundancy for decrease of obscurity.

Theorem 4.2.4, Corollary 4.2.6.3, Theorem 4.2.11 and Theorem 4.2.1 together show that the following hold for the negations of the above declaratives:

$$\begin{aligned} &{}^{\Diamond}(\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})) \asymp \pi\mathfrak{o}(\mathfrak{a},{}^{\Diamond}(\triangle\mathfrak{b})) \asymp \pi\mathfrak{o}(\mathfrak{a},\triangle({}^{\square}\mathfrak{b})),\\ &{}^{\Diamond}(\pi\forall(\mathfrak{a},\triangle\mathfrak{b})) \asymp \pi\exists(\mathfrak{a},{}^{\Diamond}(\triangle\mathfrak{b})) \asymp \pi\exists(\mathfrak{a},\triangle({}^{\square}\mathfrak{b})),\\ &{}^{\Diamond}(\pi\exists(\mathfrak{a},\triangle\mathfrak{b})) \asymp \pi\forall(\mathfrak{a},{}^{\Diamond}(\triangle\mathfrak{b})) \asymp \pi\forall(\mathfrak{a},\triangle({}^{\square}\mathfrak{b})). \end{aligned}$$

The rightmost declaratives of these three lines can have the following agcUP-tables, where the token $\square$ is phrased by the word *not*.

π	$\mathfrak{o}$	(	$\mathfrak{a}$,	$\triangle$	(	$\square$	$\mathfrak{b}$	))
			Peter	be[is]		not	(a) rabbit	

π	$\exists$	(	$\mathfrak{a}$,	$\triangle$	(	$\square$	$\mathfrak{b}$	))
	Some		rabbit(s)	be[are]		not	wildlife	

π	$\forall$	(	$\mathfrak{a}$,	$\triangle$	(	$\square$	$\mathfrak{b}$	))
	Any		rabbit	be[is]		not	wildlife	

The leftmost declaratives of the above three lines can have the following agcUP-tables.

$\Diamond$	(π	$\mathfrak{o}$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	))
Iint				Peter	be[is]	(a) rabbit	

$\Diamond$	(π	$\forall$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	))
Iint		all		rabbit(s)	be[are]	wildlife	

$\Diamond$	(π	$\exists$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	))
Iint		some		rabbit(s)	be[are]	wildlife	

The second declarative can also have the following agcUP-table.

$\Diamond$	(	π	$\forall$	(	$\mathfrak{a}$,	$\triangle$	$\mathfrak{b}$	))
Not			all		rabbit(s)	be[are]	wildlife	

The possibility of these agcUP-tables implies that the three English declaratives *Some rabbits are not wildlife*, *It is not that all rabbits are wildlife* and *Not all rabbits are wildlife* can have the same external meaning.

To my knowledge of English, however, the remaining middle declaratives in the above three lines have no appropriate agcUP-tables, that is, they are not appropriately phrased in the English community.

Incidentally, the Japanese community is different from the English community also about that. First, the Japanese counterparts $\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle$, $\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle$ and $\mathfrak{a}\,\exists\pi\,\mathfrak{b}\triangle$ of the above declaratives $\pi\mathfrak{o}(\mathfrak{a}, \triangle\mathfrak{b})$, $\pi\forall(\mathfrak{a}, \triangle\mathfrak{b})$ and $\pi\exists(\mathfrak{a}, \triangle\mathfrak{b})$ can have the following agcUP-tables (s. [1.73]).

$\mathfrak{a}$	$\mathfrak{o}$	π	$\mathfrak{b}$	$\triangle$
pêtâ		ga	usagi	da
Peter			rabbit	be

$\mathfrak{a}$	$\forall$	π	$\mathfrak{b}$	$\triangle$
usagi	subete	ga	yasei	da
Rabbit	all		wildlife	be

$\mathfrak{a}$	$\exists$	π	$\mathfrak{b}$	$\triangle$
usagi	ikuhiki ka	ga	yasei	da
Rabbit	some		wildlife	be

Secondly, the same propositions as above show that the following hold for the negations of the above declaratives:

$$(\mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\mathfrak{o}\pi\,(\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\mathfrak{o}\pi\,\mathfrak{b}^\Box\triangle,$$
$$(\mathfrak{a}\,\forall\pi\,\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\exists\pi\,(\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\exists\pi\,\mathfrak{b}^\Box\triangle,$$
$$(\mathfrak{a}\,\exists\pi\,\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\forall\pi\,(\mathfrak{b}\triangle)^\Diamond \asymp \mathfrak{a}\,\forall\pi\,\mathfrak{b}^\Box\triangle.$$

Lastly, in contrast to the English community, the rightmost declaratives of the three lines have no appropriate agcUP-tables, but the other declaratives of the lines can have the following agcUP-tables, where *de* and *na* under $\triangle$ are the conjugations of *da* which *nai* and *ndn* under $\Diamond$ require, but semasiology should replace them with *da*.

$\mathfrak{a}$	$\mathfrak{o}$	π	(	$\mathfrak{b}$	$\triangle$	)	$\Diamond$
pêtâ		ga		usagi	de		nai
Peter				rabbit	be		not

$\mathfrak{a}$	$\exists$	π	(	$\mathfrak{b}$	$\triangle$	)	$\Diamond$
usagi	ikuhiki ka	ga		yasei	de		nai
rabbit	some			wildlife	be		not

$\mathfrak{a}$	$\forall$	π	(	$\mathfrak{b}$	$\triangle$	)	$\Diamond$
usagi	subete	ga		yasei	de		nai
rabbit	all			wildlife	be		not

(	a	$\mathfrak{o}$	π	b	$\triangle$	)	$\Diamond$
	pêtâ		ga	usagi	na		ndn
	Peter			rabbit	be		iint

(	a	$\forall$	π	b	$\triangle$	)	$\Diamond$
	usagi	subete	ga	yasei	na		ndn
	Rabbit	all		wildlife	be		iint

(	a	$\exists$	π	b	$\triangle$	)	$\Diamond$
	usagi	ikuhiki ka	ga	yasei	na		ndn
	Rabbit	some		wildlife	be		iint

We may also obtain other agcUP-tables for the declarative $\pi\mathfrak{q}(\mathsf{a}, \triangle\mathsf{b})$ by replacing the words *Peter*, *rabbit* and *wildlife* with other appropriate words and picking $\mathfrak{q}$ appropriately from $\{\mathfrak{o}\} \cup \mathfrak{Q}$. For example,

π	$\mathfrak{o}$	(	a,	$\triangle$	b	)
			Mrs. Rabbit	be[is]	(a) widow	

However, there exists no agcUP-table whose lower line can read *Mrs. Rabbit is a widow in 1902* because of the smallest partibility 1 of ICL, as was suggested in §1.3.3. This is a reason why ICL should be generalized to CL.

4.3.5 Peter is a rabbit. All rabbits are wild. Then?

In relation to the declarative $\pi\mathfrak{q}(\mathsf{a}, \mathsf{f})$ and $\pi\mathfrak{q}(\mathsf{a}, \triangle\mathsf{b})$ considered in §4.3.3 and §4.3.4 respectively, we have

$$\pi\mathfrak{o}(\mathsf{a}, \triangle\mathsf{b}) \cdot \pi\forall(\mathsf{b}, \mathsf{f}) \preccurlyeq \pi\mathfrak{o}(\mathsf{a}, \mathsf{f}),$$
$$\pi\mathfrak{o}(\mathsf{a}, \triangle\mathsf{b}) \cdot \pi\mathfrak{o}(\mathsf{a}, \mathsf{f}) \preccurlyeq \pi\exists(\mathsf{b}, \mathsf{f})$$

by Theorems 4.2.4 and 4.2.14 and the M-homomorphy of $\varphi^{\delta}_{\upsilon}$ for all $(W, \delta, \upsilon) \in \mathcal{D}$, that is, the following sequents are IC tautologies:

$$\pi\mathfrak{o}(\mathsf{a}, \triangle\mathsf{b}) \cdot \pi\forall(\mathsf{b}, \mathsf{f}) \to \pi\mathfrak{o}(\mathsf{a}, \mathsf{f}),$$
$$\pi\mathfrak{o}(\mathsf{a}, \triangle\mathsf{b}) \cdot \pi\mathfrak{o}(\mathsf{a}, \mathsf{f}) \to \pi\exists(\mathsf{b}, \mathsf{f}).$$

Let a', b' and f' be the images of a, b and f by $\varphi^{\delta}_{\upsilon}$ for arbitrary $(W, \delta, \upsilon) \in \mathcal{D}$. Then the above means that the following hold for all $\theta \in (\mathsf{K}^{\mathsf{f}} - \{\pi\}) \to W_{\epsilon}$.

$$\inf\{\pi\mathfrak{o}(\mathsf{a}', \triangle\mathsf{b}'), (\pi\forall(\mathsf{b}', \mathsf{f}'))\theta\} \le (\pi\mathfrak{o}(\mathsf{a}', \mathsf{f}'))\theta,$$
$$\inf\{\pi\mathfrak{o}(\mathsf{a}', \triangle\mathsf{b}'), (\pi\mathfrak{o}(\mathsf{a}', \mathsf{f}'))\theta\} \le (\pi\exists(\mathsf{b}', \mathsf{f}'))\theta.$$

The former inequality means that if the event $\pi\mathfrak{o}(\mathsf{a}', \triangle\mathsf{b}')$ occurs and the event $\pi\forall(\mathsf{b}', \mathsf{f}')$ occurs for θ, then the event $\pi\mathfrak{o}(\mathsf{a}', \mathsf{f}')$ occurs for θ, and likewise for the latter. This may be illustrated by the following agcUP-tables for sequents similar to ones for declaratives.

$\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})$	$\cdot$	$\pi\forall(\mathfrak{b},\mathfrak{f})$	$\to$	$\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})$
Peter is a rabbit	and	all rabbits are wild	then	Peter is wild

$\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})$	$\cdot$	$\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})$	$\to$	$\pi\exists(\mathfrak{b},\mathfrak{f})$
Peter is a rabbit	and	Peter is wild	then	some rabbits are wild

Here the arrow $\to$ is phrased by the conjunction *then* and the dot $\cdot$ on the left-hand side of $\to$ is phrased by the conjunction *and*, while that on the right-hand side is phrased by the conjunction *or*.

The first two inequalities on the relation $\preccurlyeq$ together with Theorems 4.2.5, 2.2.15 and 2.2.17 means that the declaratives $\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})\wedge\pi\forall(\mathfrak{b},\mathfrak{f})\Rrightarrow\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})$ and $\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})\wedge\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})\Rrightarrow\pi\exists(\mathfrak{b},\mathfrak{f})$ are tautologies, and they can have the following agcUP-tables.

$\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})$	$\wedge$	$\pi\forall(\mathfrak{b},\mathfrak{f})$	$\Rrightarrow$	$\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})$
Peter is a rabbit	and	all rabbits are wild	then	Peter is wild

$\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})$	$\wedge$	$\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f})$	$\Rrightarrow$	$\pi\exists(\mathfrak{b},\mathfrak{f})$
Peter is a rabbit	and	Peter is wild	then	some rabbits are wild

Here the tokens $\wedge$ and $\Rrightarrow$ are phrased by the conjunctions *and* and *then*.

4.3.6 Mrs. Rabbit is a mother and a widow.

In relation to the declarative $\pi\mathfrak{q}(\mathfrak{a},\triangle\mathfrak{b})$ considered in §4.3.4, the declarative $\pi\mathfrak{o}(\mathfrak{a},\triangle(\mathfrak{b}\sqcap\mathfrak{c}))$ can have the following agcUP-table.

$\pi\mathfrak{o}($	$\mathfrak{a},$	$\triangle$	$($	$\mathfrak{b}$	$\sqcap$	$\mathfrak{c}$	$))$
	Mrs. Rabbit	be[is]		(a) mother	and	(a) widow	

Here the token $\sqcap$ is phrased by the word *and*. This is because

$$\pi\mathfrak{o}(\mathfrak{a},\triangle(\mathfrak{b}\sqcap\mathfrak{c}))\asymp\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b}\wedge\triangle\mathfrak{c})\asymp\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{b})\wedge\pi\mathfrak{o}(\mathfrak{a},\triangle\mathfrak{c})$$

by Theorems 4.2.4, 4.2.1 and 4.2.7, and the token $\wedge$ is phrased by the conjunction *and* as was noted in §4.3.5.

The declarative $\pi\mathfrak{o}(\mathfrak{a},\triangle(\mathfrak{b}\sqcup\mathfrak{c}))$ can likewise have the following agcUP-table because the token $\vee$ is naturally phrased by the conjunction *or*.

$\pi\mathfrak{o}($	$\mathfrak{a},$	$\triangle$	$($	$\mathfrak{b}$	$\sqcup$	$\mathfrak{c}$	$))$
	Mr. McGregor	be[is]		(a) farmer	or	(a) peasant	

4.3.7 Radishes and lettuces. Radishes or lettuces.

In relation to the declarative $\pi\mathfrak{q}(\mathfrak{a},\mathfrak{f}\,\mathfrak{k}\mathfrak{r}\,\mathfrak{b})$ considered in §4.3.2, the declaratives $\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f}\,\mathfrak{k}\forall\,(\mathfrak{b}\sqcup\mathfrak{c}))$ and $\pi\mathfrak{o}(\mathfrak{a},\mathfrak{f}\,\mathfrak{k}\exists\,(\mathfrak{b}\sqcup\mathfrak{c}))$ can have the following agcUP-tables.

$\pi o($	$a,$	f	k	$\forall$	$($	b	$\sqcup$	c	$))$
	Peter	eat(s)		all		radish(es)	and	lettuce(s)	

$\pi o($	$a,$	f	k	$\exists$	$($	b	$\sqcup$	c	$))$
	Peter	eat(s)		some		radish(es)	or	lettuce(s)	

Here the token $\sqcup$ is phrased by both *and* and *or*. This is because

$$\pi o(a, f\,k\forall\,(b \sqcup c)) \asymp \pi o(a, f\,k\forall\, b) \wedge \pi o(a, f\,k\forall\, c),$$
$$\pi o(a, f\,k\exists\,(b \sqcup c)) \asymp \pi o(a, f\,k\exists\, b) \vee \pi o(a, f\,k\exists\, c)$$

as you can now easily prove by the methods in §4.2, and the tokens $\wedge$ and $\vee$ are phrased by the conjunctions *and* and *or* respectively. Thus how the token $\sqcup$ is phrased sometimes depends on its neighborhood.

4.3.8 What rabbits eat. Radishes that rabbits eat.

In relation to the declarative $\pi q(a, f\,k r\, b)$ considered in §4.3.2, the three nominals $\nabla x\,(\pi o(a, f\,k o\, x))$, $b \sqcap \nabla x\,(\pi o(a, f\,k o\, x))$ and $a \sqcap \nabla x\,(\pi o(x, f\,k\exists\, b))$ can have the following agcUP-tables (s. §1.2.7 for the second).

		∇x	$(\pi o($	$a,$	f	k	o	x	$))$
		what		rabbit(s)	eat				
b	$\sqcap$	∇x	$(\pi o($	$a,$	f	k	o	x	$))$
radish(es)		that		rabbit(s)	eat				
a	$\sqcap$	∇x	$(\pi o($	$x,$	f	k	$\exists$	b	$))$
rabbit(s)		that			eat		some	radish(es)	

Here the nominalizer ∇x is phrased by the relative pronouns *what* and *that*, while the token $\sqcap$ is silent and appositively connects nominals. The variable x is silent because it is (a model of) a variable prime percept whose object may momently vary.

Incidentally, the Japanese community manages without relative pronouns. For example, $(a\, o\pi\,(x\, o k\, f))\nabla x$, $(a\, o\pi\,(x\, o k\, f))\nabla x \sqcap b$ and $(x\, o\pi\,(b\, \exists k\, f))\nabla x \sqcap a$ are the Japanese counterparts of the above nominals and can have the following agcUP-tables.

$($	a	o	π	$($	x	o	k	f	$))$	∇x		
	usagi		ga					taberu		mono		
	rabbit							eat		thing		
$($	a	o	π	$($	x	o	k	f	$))$	∇x	$\sqcap$	b
	usagi		ga					taberu				daikon
	rabbit							eat				radish
$($	x	o	π	$($	b	$\exists$	k	f	$))$	∇x	$\sqcap$	a
					daikon		o	taberu				usagi
					radish			eat				rabbit

Here the quantifier $\exists$ is silent as usual. Since the variable x is silent, so are the operations ok and $o\pi$ applied to x.

4.3.9 Rabbits fond of some radishes. Wild rabbits.

In §4.3.3, we considered the declaratives $\pi q(a, f\, k r\, b)$ and $\pi q(a, f)$ in relation to the predicative adjectives. As for the attributive ones, the nominals $a \sqcap \triangledown x\,(\pi o\,(x, f\, k \exists\, b))$ and $\triangledown x\,(\pi o\,(x, f)) \sqcap a$ can have the following agcUP-tables.

a	$\sqcap$	$\triangledown x$	$(\pi o($	$x,$	f	k	$\exists$	b	$))$
rabbit(s)					fond	of	some	radish(es)	

$\triangledown x$	$(\pi o($	$x,$	f	$))$	$\sqcap$	a
			wild			rabbit(s)

However, the former nominal also has the following agcUP-table.

a	$\sqcap$	$\triangledown x$	$(\pi o($	$x,$	f	k	$\exists$	b	$))$
rabbit(s)		that			(are) fond	of	some	radish(es)	

This agcUP-table is the same as the third one in §4.3.8 except that f thereof was phrased by the verb *eat* and k thereof was silent. This is another reason why verbs and predicative adjectives should be put together.

Incidentally, all Japanese attributive (nominal) adjectives are prepositive in contrast to the English attributive adjectives. For example, $(x\, o\pi\,(b\, \exists k\, f))\triangledown x \sqcap a$ is a Japanese counterpart of the above nominal $a \sqcap \triangledown x\,(\pi o\,(x, f\, k\exists\, b))$ and can have the following agcUP-table.

$($	x	o	π	$($	b	$\exists$	k	f	$))$	$\triangledown x$	$\sqcap$	a
					daikon		ga/o	sukina				usagi
					radish		of	be fond				rabbit

Here the quantifier $\exists$ is silent as usual. The word *sukina* under f is the conjugation of the nominal adjective *sukida* used in §4.3.3. The silent nominalizer $\triangledown x$ requires the conjugation, but semasiology should replace it with *sukida*. This agcUP-table is the same as the last one of §4.3.8 except that f and k thereof were phrased by the verb *taberu* and the case marker *o*. This is another reason why verbs and (nominal) adjectives should be put together.

4.3.10 Some one is Peter. Peter exists.

We have defined the generic one $\breve{1} = \triangledown x_0\,(\pi\forall(x_0, \triangle x_0))$ and the existence $\breve{e} = \triangledown x_0\,(\pi\exists(x_0, \triangle\breve{1}))$ in §4.2.6. Although silent according to the above phraseology, $\breve{1}$ is phrased by the generic words, such as *one* and *thing*, which mean the totality of the basic entities, and $\breve{e}$ is phrased by the word *existence* (s. [4.5]).

If $a \in A_\epsilon$, then $\pi\exists(\breve{1}, \triangle a) \asymp \pi\exists(a, \triangle\breve{1}) \asymp \pi o(a, \triangle\breve{e})$ by Theorems 4.2.4, 4.2.10 and 4.2.22. Although $\pi\exists(a, \triangle\breve{1})$ has no appropriate agcUP-table, the other declaratives can have the following agcUP-tables.

$$\begin{array}{ccccccc} \pi & \exists & (& \check{1}, & \triangle & \mathfrak{a} &) \\ & \text{Some} & & \text{one} & \text{be[is]} & \text{Peter} & \end{array}$$

$$\begin{array}{ccccc} \pi & \mathfrak{o} & (& \mathfrak{a}, & \triangle\breve{e} &) \\ & & & \text{Peter} & \text{exist(s)} & \end{array}$$

Here $\triangle\breve{e}$ is phrased by the verb *exist(s)*, although it can be phrased by *is (an) existence* according to the above phraseology. The possibility of these agcUP-tables implies that the two English declaratives *Some one is Peter* and *Peter exists* can have the same external meaning.

However, there exists no agcUP-table whose lower line reads *Peter exists in a wood* because of the smallest partibility 1 of ICL, as was suggested in §1.3.3. This is another reason why ICL should be generalized to CL.

Chapter 4 to be continued (s. Remark 1.4.1).

Chapter 5

Bipartible Case Logic

To be written (s. Remark 1.4.1).

Chapter 6

Case Logic

Reading up Chapters 1–5 of introductory or preparatory nature, you are now ready to read this main and final chapter of thorough study on the general CL.

6.1 Construction of CL

Following §3.2, here we construct the logic system CL. A logic system in GL as defined in §3.2.5 is a pair of a formal language and its semantics. Their counterparts in CL are called the **C language** and **C semantics**, which have certain common parameters called the **C parameter**s.

6.1.1 The C parameters

Called case logic, CL is parameterized by a nonempty set K whose elements are called the **case**s or **case marker**s. Being general, CL is also parameterized by a nonempty set N equipped with *finite* subsets N_ν $(\nu \in \mathsf{N})$[6.1] which satisfy

$$\mathsf{N}_\nu \subseteq \mathsf{N} - \{\nu\} \qquad\qquad (\nu \in \mathsf{N}).$$

Its elements are called the **nomina** (s. [4.1]) and its cardinality $\#\mathsf{N}$ is called the **partibility** of entities. ICL and BCL in Chapters 4 and 5 are essentially CL of partibility 1 and 2 respectively (s. §6.2).

The two parameters are related by an injective and nonsurjective mapping

$$\kappa \in \mathsf{N}' {\to} \mathsf{K}$$

of the subset N' of N defined by

$$\mathsf{N}' = \textstyle\bigcup_{\nu \in \mathsf{N}} \mathsf{N}_\nu.$$

[6.1] They are together identified with the relation R on N such that $\mu \,\mathsf{R}\, \nu$ iff $\mu \in \mathsf{N}_\nu$. The condition $\mathsf{N}_\nu \subseteq \mathsf{N} - \{\nu\}$ means that R is irreflexive.

The mapping κ together with the underlying family $(\mathsf{N}_\nu)_{\nu\in\mathsf{N}}$ is called the **modality** and its value at $\mu \in \mathsf{N}'$ is denoted κ_μ. The nomina in the sets N' and N_ν $(\nu \in \mathsf{N})$ are called the **modal nomina** and **ν-modal nomina** respectively.[6.2] Accordingly, the subsets K' and K_ν $(\nu \in \mathsf{N})$ of K defined by

$$\mathsf{K}' = \kappa\mathsf{N}' = \{\kappa_\mu : \mu \in \mathsf{N}'\}, \qquad \mathsf{K}_\nu = \kappa\mathsf{N}_\nu = \{\kappa_\mu : \mu \in \mathsf{N}_\nu\} \quad (\nu \in \mathsf{N})$$

satisfy

$$\mathsf{K}' = \textstyle\bigcup_{\nu\in\mathsf{N}} \mathsf{K}_\nu,$$

and the cases in K' and K_ν $(\nu \in \mathsf{N})$ are called the **modal case**s and **ν-modal case**s respectively, while those in the set $\mathsf{K}-\mathsf{K}'$ are called the **nonmodal case**s. Since κ is not a surjection, i.e. $\mathsf{K}' \neq \mathsf{K}$, we may pick a nonmodal case

$$\pi \in \mathsf{K} - \mathsf{K}'.$$

We call it the **nominative** or **principal case**. Since κ is an injection, we have

$$\#\mathsf{K}' = \#\mathsf{N}', \qquad\qquad \#\mathsf{K}_\nu = \#\mathsf{N}_\nu < \infty \quad (\nu \in \mathsf{N}).$$

For the same reason, each modal case $\mathsf{k} \in \mathsf{K}'$ has a unique nomen $\mu \in \mathsf{N}'$ such that $\mathsf{k} = \kappa_\mu$, which we call the **nomen of** k and denote by $\eta\mathsf{k}$. Therefore, the nomen $\eta\mathsf{k}$ of each $\mathsf{k} \in \mathsf{K}'$ belongs to N' in contrast to the nomina $\eta\mathfrak{q}$, $\eta\phi$, ηx and ηa of $\mathfrak{q} \in \mathfrak{Q} \cup \breve{\mathfrak{Q}}$, $\phi \in \Phi$, $x \in \mathsf{X}_\epsilon$ and $a \in \mathsf{G} \cup \mathsf{E}$ defined below which may belong to $\mathsf{N} - \mathsf{N}'$ as well as to N'. The mapping $\mathsf{k} \mapsto \eta\mathsf{k}$ is the inverse of κ regarded as a bijection of N' onto K', that is, $\eta\kappa_\mu = \mu$ for each $\mu \in \mathsf{N}'$ and $\kappa_{\eta\mathsf{k}} = \mathsf{k}$ for each $\mathsf{k} \in \mathsf{K}'$. Nonmodal cases do not have their nomina, but we associate each case $\mathsf{k} \in \mathsf{K}$ with the subset N_k of N defined by

$$\mathsf{N}_\mathsf{k} = \begin{cases} \mathsf{N} & \text{if } \mathsf{k} \in \mathsf{K} - \mathsf{K}', \\ \{\eta\mathsf{k}\} & \text{if } \mathsf{k} \in \mathsf{K}'. \end{cases}$$

In particular, $\mathsf{N}_\pi = \mathsf{N}$ for the nominative case π.

Being general, CL is also parameterized by a family $(\mathbb{P}_\nu)_{\nu\in\mathsf{N}}$ of nontrivial quantitative sets for quantification, which is divided into absolute quantification and proportional quantification.

As for the absolute quantification, we pick a subset $\mathfrak{P}_\nu$ of the power set $\mathfrak{P}\mathbb{P}_\nu$ of $\mathbb{P}_\nu$ for each $\nu \in \mathsf{N}$. Then we pick a copy

$$\neg\mathfrak{P}_\nu = \{\neg\mathfrak{p} : \mathfrak{p} \in \mathfrak{P}_\nu\}$$

of $\mathfrak{P}_\nu$ by the symbol $\neg$ (s. §1.5.2) for each $\nu \in \mathsf{N}$ and define

$$\mathfrak{Q}_\nu = \mathfrak{P}_\nu \amalg \neg\mathfrak{P}_\nu \quad (\nu \in \mathsf{N}), \qquad\qquad \mathfrak{Q} = \textstyle\coprod_{\nu\in\mathsf{N}} \mathfrak{Q}_\nu.$$

[6.2]Since N may be equal to N', it may happen that every nomen is a modal nomen. Since $(\mathsf{N}_\nu)_{\nu\in\mathsf{N}}$ may not be disjoint, a ν-modal nomen for some $\nu \in \mathsf{N}$ may also be a ν'-modal nomen for some other $\nu' \in \mathsf{N}$, and likewise for the modal cases below.

Here if $\mathfrak{P}_\nu = \emptyset$ for some $\nu \in \mathsf{N}$, then $\neg\mathfrak{P}_\nu = \mathfrak{Q}_\nu = \emptyset$ by definition. We refer to the elements of the sets $\mathfrak{Q}$ and $\mathfrak{Q}_\nu$ ($\nu \in \mathsf{N}$) as the **quantifier**s and **ν-quantifier**s respectively. Furthermore, if $\mathfrak{q} \in \mathfrak{Q}_\nu$ ($\nu \in \mathsf{N}$), we call ν the **nomen of** $\mathfrak{q}$. Therefore, each quantifier $\mathfrak{q} \in \mathfrak{Q}$ has its nomen $\nu \in \mathsf{N}$, which we denote by $\eta\mathfrak{q}$. Then we let $\mathfrak{Q}'$ be the subset of $\mathfrak{Q}$ defined by

$$\mathfrak{Q}' = \{\mathfrak{q} \in \mathfrak{Q} : \eta\mathfrak{q} \in \mathsf{N}'\},$$

i.e. $\mathfrak{Q}' = \coprod_{\mu\in\mathsf{N}'} \mathfrak{Q}_\mu = \coprod_{k\in\mathsf{K}'} \mathfrak{Q}_{\eta k}$, and refer to its elements as the **modal quantifier**s. We also let $\mathfrak{I}$ be the subset of $\mathfrak{Q} \times \mathsf{K}$ defined by

$$\mathfrak{I} = \{(\mathfrak{q}, k) \in \mathfrak{Q} \times \mathsf{K} : \eta\mathfrak{q} \in \mathsf{N}_k\},$$

i.e. $\mathfrak{I} = (\mathfrak{Q}\times(\mathsf{K}-\mathsf{K}'))\amalg\coprod_{k\in\mathsf{K}'}(\mathfrak{Q}_{\eta k}\times\{k\}) = (\mathfrak{Q}\times(\mathsf{K}-\mathsf{K}'))\amalg\coprod_{\mu\in\mathsf{N}'}(\mathfrak{Q}_\mu\times\{\kappa_\mu\})$.

Incidentally, we define subsets $\mathfrak{P}$ and $\neg\mathfrak{P}$ of $\mathfrak{Q}$ by

$$\mathfrak{P} = \coprod_{\nu\in\mathsf{N}} \mathfrak{P}_\nu, \qquad\qquad \neg\mathfrak{P} = \coprod_{\nu\in\mathsf{N}} \neg\mathfrak{P}_\nu,$$

so that $\mathfrak{Q} = \mathfrak{P} \amalg \neg\mathfrak{P}$, and refer to their elements as the **positive quantifier**s and **negative quantifier**s respectively (s. [6.3]).

As for the proportional quantification, we pick a subset $\breve{\mathfrak{P}}_\nu$ of the power set $\mathfrak{P}[0,1]$ of the interval $[0,1]$ of real numbers for each $\nu \in \mathsf{N}$ such that $\mathbb{P}_\nu$ is unital, while we define $\breve{\mathfrak{P}}_\nu = \emptyset$ for each $\nu \in \mathsf{N}$ such that $\mathbb{P}_\nu$ is not unital. Then, parallelly to the absolute quantification, we pick a copy

$$\neg\breve{\mathfrak{P}}_\nu = \{\neg\mathfrak{p} : \mathfrak{p} \in \breve{\mathfrak{P}}_\nu\}$$

of $\breve{\mathfrak{P}}_\nu$ by the symbol $\neg$ (s. §1.5.2) for each $\nu \in \mathsf{N}$ and define

$$\breve{\mathfrak{Q}}_\nu = \breve{\mathfrak{P}}_\nu \amalg \neg\breve{\mathfrak{P}}_\nu \quad (\nu \in \mathsf{N}), \qquad\qquad \breve{\mathfrak{Q}} = \coprod_{\nu\in\mathsf{N}} \breve{\mathfrak{Q}}_\nu.$$

Here if $\breve{\mathfrak{P}}_\nu = \emptyset$ for some $\nu \in \mathsf{N}$, then $\neg\breve{\mathfrak{P}}_\nu = \breve{\mathfrak{Q}}_\nu = \emptyset$ by definition. We refer to the elements of the sets $\breve{\mathfrak{Q}}$ and $\breve{\mathfrak{Q}}_\nu$ ($\nu \in \mathsf{N}$) as the **proportional quantifier**s and **proportional ν-quantifier**s respectively. Furthermore, if $\mathfrak{q} \in \breve{\mathfrak{Q}}_\nu$ ($\nu \in \mathsf{N}$), we call ν the **nomen of** $\mathfrak{q}$. Therefore, each proportional quantifier $\mathfrak{q} \in \breve{\mathfrak{Q}}$ has its nomen $\nu \in \mathsf{N}$, which we denote by $\eta\mathfrak{q}$. Then we let $\breve{\mathfrak{Q}}'$ be the subset of $\breve{\mathfrak{Q}}$ defined by

$$\breve{\mathfrak{Q}}' = \{\mathfrak{q} \in \breve{\mathfrak{Q}} : \eta\mathfrak{q} \in \mathsf{N}'\},$$

i.e. $\breve{\mathfrak{Q}}' = \coprod_{\mu\in\mathsf{N}'} \breve{\mathfrak{Q}}_\mu = \coprod_{k\in\mathsf{K}'} \breve{\mathfrak{Q}}_{\eta k}$, and refer to its elements as the **modal proportional quantifier**s. We also let $\breve{\mathfrak{I}}$ be the subset of $\breve{\mathfrak{Q}} \times \mathsf{K}$ defined by

$$\breve{\mathfrak{I}} = \{(\mathfrak{q}, k) \in \breve{\mathfrak{Q}} \times \mathsf{K} : \eta\mathfrak{q} \in \mathsf{N}_k\},$$

i.e. $\breve{\mathfrak{I}} = (\breve{\mathfrak{Q}}\times(\mathsf{K}-\mathsf{K}'))\amalg\coprod_{k\in\mathsf{K}'}(\breve{\mathfrak{Q}}_{\eta k}\times\{k\}) = (\breve{\mathfrak{Q}}\times(\mathsf{K}-\mathsf{K}'))\amalg\coprod_{\mu\in\mathsf{N}'}(\breve{\mathfrak{Q}}_\mu\times\{\kappa_\mu\})$.

Incidentally, we define subsets $\breve{\mathfrak{P}}$ and $\neg\breve{\mathfrak{P}}$ of $\breve{\mathfrak{Q}}$ by

$$\breve{\mathfrak{P}} = \coprod_{\nu\in\mathsf{N}} \breve{\mathfrak{P}}_\nu, \qquad\qquad \neg\breve{\mathfrak{P}} = \coprod_{\nu\in\mathsf{N}} \neg\breve{\mathfrak{P}}_\nu,$$

so that $\breve{\mathfrak{Q}} = \breve{\mathfrak{P}} \amalg \neg\breve{\mathfrak{P}}$, and refer to their elements as the **positive proportional quantifier**s and **negative proportional quantifier**s respectively (s. [6.3]).

Being general, CL is also parameterized by a partitioned set

$$\Phi = \coprod_{\nu \in \mathsf{N}} \Phi_\nu.$$

We refer to the elements of Φ and Φ_ν ($\nu \in \mathsf{N}$) as the **functional**s and **ν-functional**s respectively. Furthermore, if $\phi \in \Phi_\nu$ ($\nu \in \mathsf{N}$), we call ν the **nomen of ϕ**. Therefore, each functional $\phi \in \Phi$ has it nomen $\nu \in \mathsf{N}$, which we denote by $\eta\phi$. We also equip Φ with a family $(n_\phi)_{\phi\in\Phi}$ of positive integers and call n_ϕ ($\phi \in \Phi$) the **arity** of the functional ϕ.

This completes the introduction of the C parameters, whose essence consists of the set K of the cases, the set N of the nomina, the modality $(\kappa, (\mathsf{N}_\nu)_{\nu\in\mathsf{N}})$, the nominative case π, the family $(\mathbb{P}_\nu)_{\nu\in\mathsf{N}}$ of nontrivial quantitative sets, the set $\mathfrak{P} = \coprod_{\nu\in\mathsf{N}} \mathfrak{P}_\nu$ of the positive quantifiers, the set $\breve{\mathfrak{P}} = \coprod_{\nu\in\mathsf{N}} \breve{\mathfrak{P}}_\nu$ of the positive proportional quantifiers, the set $\Phi = \coprod_{\nu\in\mathsf{N}} \Phi_\nu$ of the functionals and the family $(n_\phi)_{\phi\in\Phi}$ of their arities.

6.1.2 The C language

Following §3.2.1, here we construct the C language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ by defining its syntax $(\mathsf{T}, \sigma|_{\mathsf{P}}, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ called the **C syntax** on the basis of the C parameters introduced in §6.1.1.

As for the sets P, C and X of its primes, constants and variables, we assume

$$\mathsf{P} = \mathsf{C} \amalg \mathsf{X}, \qquad\qquad \mathsf{X} \neq \emptyset,$$

that is, we only assume the conditions stated in §3.2.1.

We define the set Γ of its tokens by

$$\Gamma = \Phi \amalg \{\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond, \mathfrak{o}, \natural, \triangledown\} \amalg \mathfrak{Q} \amalg \breve{\mathfrak{Q}} \amalg \mathsf{K},$$

picking the tokens $\sqcap$, $\sqcup$, $\square$, $\triangle$, $\wedge$, $\vee$, $\Rrightarrow$, $\Diamond$, $\mathfrak{o}$, $\natural$ and $\triangledown$.

As for its type, we first let T be the set defined by

$$\mathsf{T} = (\textstyle\coprod_{\nu\in\mathsf{N}}\{\epsilon_\nu, \mathfrak{e}_\nu\}) \amalg \mathfrak{P}\mathsf{K},$$

picking the symbols ϵ_ν and $\mathfrak{e}_\nu$ ($\nu \in \mathsf{N}$).

As for the basic sorting $\sigma|_{\mathsf{P}}$, we first let $\mathsf{C}_{\epsilon_\nu}$ and $\mathsf{X}_{\epsilon_\nu}$ ($\nu \in \mathsf{N}$) be the subsets of C and X respectively defined by

$$\mathsf{C}_{\epsilon_\nu} = \sigma|_{\mathsf{C}}{}^{-1}\{\epsilon_\nu\}, \qquad\qquad \mathsf{X}_{\epsilon_\nu} = \sigma|_{\mathsf{X}}{}^{-1}\{\epsilon_\nu\}$$

and let N_{d} be the subset of N defined by

$$\mathsf{N}_{\mathsf{d}} = \{\nu \in \mathsf{N} : \mathsf{C}_{\epsilon_\nu} = \emptyset, \#\mathsf{X}_{\epsilon_\nu} = 1, \mathsf{N}_\nu = \emptyset, \Phi_\nu = \emptyset\}.$$

The elements of N_{d} are called the **degenerate nomina**, and the unique element of $\mathsf{X}_{\epsilon_\nu}$ ($\nu \in \mathsf{N}_{\mathsf{d}}$) is called the **$\nu$-variable** and denoted x_ν. Then we assume

$$\mathsf{X}_{\epsilon_\nu} \neq \emptyset \quad (\nu \in \mathsf{N}), \qquad\qquad \mathsf{C}_{\epsilon_\mu} \neq \emptyset \quad (\mu \in \mathsf{N}' - \mathsf{N}_{\mathsf{d}}).$$

These are assumptions on $\sigma|_P$ because $\sigma|_C = (\sigma|_P)|_C$ and $\sigma|_X = (\sigma|_P)|_X$. The former assumption strengthens the assumption $X \neq \emptyset$ and is relevant to the definition of X_ϵ below, while the latter plays a role in §6.1.5.

We equip the set T with the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ defined in [T1]–[T12] below, which also clarify the index set Λ.

For [T7], we define subsets T_k $(k \in K)$ of T by

$$T_k = \{\epsilon_\nu : \nu \in N_k\}, \text{ i.e. } T_k = \begin{cases} \{\epsilon_\nu : \nu \in N\} & \text{if } k \in K - K', \\ \{\epsilon_{\eta k}\} & \text{if } k \in K'. \end{cases}$$

Therefore, $T_k \subseteq \{\epsilon_\nu : \nu \in N\}$ for all $k \in K$ and $T_\pi = \{\epsilon_\nu : \nu \in N\}$ for the nominative case π.

For [T12], we let X_ϵ be the subset of X defined by $X_\epsilon = \bigcup_{\nu\in N} X_{\epsilon_\nu}$. Then since $X_{\epsilon_\nu} = \sigma|_X{}^{-1}\{\epsilon_\nu\}$ $(\nu \in N)$, we have

$$X_\epsilon = \coprod_{\nu\in N} X_{\epsilon_\nu},$$

that is, each element $x \in X_\epsilon$ satisfies $x \in X_{\epsilon_\nu}$ for a unique nomen $\nu \in N$, which we call the **nomen of** x and denote by ηx.

The operation symbols τ_λ $(\lambda \in \Lambda)$ in [T1]–[T12] are abbreviated to λ; for example, the operation symbols τ_ϕ $(\phi \in \Phi)$ in [T1] are abbreviated to ϕ; Except for the postpositive operation symbols of various arities in [T1], [T4] and [T10]–[T12], the binary ones in [T2], [T5] and [T7]–[T9] are interpositions and the unary ones in [T3] and [T6] are superscript.

[T1] The family of n_ϕ-ary operations $\phi \in \Phi$ defined by $\mathrm{Dm}\,\phi = \{\epsilon_{\eta\phi}\}^{n_\phi}$ and $(\overbrace{\epsilon_{\eta\phi}, \dots, \epsilon_{\eta\phi}}^{n_\phi})\phi = \epsilon_{\eta\phi}$ for the unique element $(\overbrace{\epsilon_{\eta\phi}, \dots, \epsilon_{\eta\phi}}^{n_\phi})$ of $\{\epsilon_{\eta\phi}\}^{n_\phi}$.

[T2] The binary operations $\sqcap$ and $\sqcup$ defined by $\mathrm{Dm}\sqcap = \mathrm{Dm}\sqcup = \coprod_{\nu\in N}\{\epsilon_\nu, \mathfrak{e}_\nu\}^2$ and $t \sqcap u = t \sqcup u = \mathfrak{e}_\nu$ for each $\nu \in N$ and each $(t, u) \in \{\epsilon_\nu, \mathfrak{e}_\nu\}^2$.

[T3] The unary operation $\square$ defined by $\mathrm{Dm}\,\square = \coprod_{\nu\in N}\{\epsilon_\nu, \mathfrak{e}_\nu\}$ and $\epsilon_\nu{}^\square = \mathfrak{e}_\nu{}^\square = \mathfrak{e}_\nu$ for each $\nu \in N$.

[T4] The unary operation $\triangle$ defined by $\mathrm{Dm}\,\triangle = \coprod_{\nu\in N}\{\epsilon_\nu, \mathfrak{e}_\nu\}$ and $\epsilon_\nu\triangle = \mathfrak{e}_\nu\triangle = \{\pi\} \cup K_\nu$ for each $\nu \in N$.

[T5] The binary operations $\wedge$, $\vee$ and $\Rightarrow$ defined by $\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow = (\mathfrak{P}K)^2$ and $Q \wedge R = Q \vee R = Q \Rightarrow R = Q \cup R$ for each $(Q, R) \in (\mathfrak{P}K)^2$.

[T6] The unary operation $\Diamond$ defined by $\mathrm{Dm}\,\Diamond = \mathfrak{P}K$ and $Q^\Diamond = Q$ for each $Q \in \mathfrak{P}K$.

[T7] The family of binary operations $\mathfrak{o}k$ $(k \in K)$ defined by $\mathrm{Dm}\,\mathfrak{o}k = T_k \times \{Q \in \mathfrak{P}K : k \in Q\}$ and $t\,\mathfrak{o}k\,Q = Q - \{k\}$ for each $t \in T_k$ and each $Q \in \mathfrak{P}K$ satisfying $k \in Q$. We call $\mathfrak{o}$ the **case operationalizer** for this reason.[6.3]

[6.3] It may also be called the **null quantifier** because it is not quantifying yet related to [T8].

[T8] The family of binary operations $\mathfrak{q}k$ ($(\mathfrak{q},k) \in \mathfrak{I}$) defined by $\mathrm{Dm}\,\mathfrak{q}k = \{\epsilon_{\eta\mathfrak{q}}, \mathfrak{e}_{\eta\mathfrak{q}}\} \times \{Q \in \mathfrak{P}K : k \in Q\}$ and $\epsilon_{\eta\mathfrak{q}}\,\mathfrak{q}k\,Q = \mathfrak{e}_{\eta\mathfrak{q}}\,\mathfrak{q}k\,Q = Q - \{k\}$ for each $Q \in \mathfrak{P}K$ satisfying $k \in Q$.

[T9] The family of binary operations $\mathfrak{q}k$ ($(\mathfrak{q},k) \in \breve{\mathfrak{I}}$) defined by $\mathrm{Dm}\,\mathfrak{q}k = \{\epsilon_{\eta\mathfrak{q}}, \mathfrak{e}_{\eta\mathfrak{q}}\} \times \{Q \in \mathfrak{P}K : k \in Q\}$ and $\epsilon_{\eta\mathfrak{q}}\,\mathfrak{q}k\,Q = \mathfrak{e}_{\eta\mathfrak{q}}\,\mathfrak{q}k\,Q = Q - \{k\}$ for each $Q \in \mathfrak{P}K$ satisfying $k \in Q$.

[T10] The family of unary operations $\natural\mathfrak{q}$ ($\mathfrak{q} \in \mathfrak{Q}'$, that is, $\mathfrak{q} \in \mathfrak{Q}$ and $\eta\mathfrak{q} \in N'$) defined by $\mathrm{Dm}\,\natural\mathfrak{q} = \{Q \in \mathfrak{P}K : \kappa_{\eta\mathfrak{q}} \in Q\}$ and $Q\,\natural\mathfrak{q} = Q$ for each $Q \in \mathfrak{P}K$ satisfying $\kappa_{\eta\mathfrak{q}} \in Q$.

[T11] The family of unary operations $\natural\mathfrak{q}$ ($\mathfrak{q} \in \breve{\mathfrak{Q}}'$, that is, $\mathfrak{q} \in \breve{\mathfrak{Q}}$ and $\eta\mathfrak{q} \in N'$) defined by $\mathrm{Dm}\,\natural\mathfrak{q} = \{Q \in \mathfrak{P}K : \kappa_{\eta\mathfrak{q}} \in Q\}$ and $Q\,\natural\mathfrak{q} = Q$ for each $Q \in \mathfrak{P}K$ satisfying $\kappa_{\eta\mathfrak{q}} \in Q$.

[T12] The family of unary operations $\triangledown x$ ($x \in X_\epsilon$) defined by $\mathrm{Dm}\,\triangledown x = \mathfrak{P}K_{\eta x}$ (notice $\mathfrak{P}K_{\eta x} = \{Q \in \mathfrak{P}K : Q \subseteq K_{\eta x}\}$) and $Q\,\triangledown x = \mathfrak{e}_{\eta x}$ for each $Q \in \mathfrak{P}K_{\eta x}$.

The operation symbols τ_λ ($\lambda \in \Lambda$) were abbreviated to λ here. Thus the index set Λ is given by the following and so satisfies $\Lambda \subseteq (\Gamma \amalg X)^*$.

$$\begin{aligned}\Lambda = \Phi \amalg \{&\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond\}\\ &\amalg \{\mathfrak{o}k : k \in K\} \amalg \{\mathfrak{q}k : (\mathfrak{q},k) \in \mathfrak{I}\} \amalg \{\mathfrak{q}k : (\mathfrak{q},k) \in \breve{\mathfrak{I}}\}\\ &\amalg \{\natural\mathfrak{q} : \mathfrak{q} \in \mathfrak{Q}'\} \amalg \{\natural\mathfrak{q} : \mathfrak{q} \in \breve{\mathfrak{Q}}'\}\\ &\amalg \{\triangledown x : x \in X_\epsilon\}.\end{aligned}$$

This completes the construction of the C language $(A, T, \sigma, P, C, X, \Gamma)$ by means of the definition of the C syntax $(T, \sigma|_P, P, C, X, \Gamma)$.

Still, some definitions and remarks are in order.

First, the C language is parameterized only by the C parameters.

Secondly, the sets $M = \Lambda \cap \Gamma^*$ and $\Lambda - M$ of the invariable indices and variable indices are given by the following:

$$\begin{aligned}M = \Phi \amalg \{&\sqcap, \sqcup, \square, \triangle, \wedge, \vee, \Rrightarrow, \Diamond\}\\ &\amalg \{\mathfrak{o}k : k \in K\} \amalg \{\mathfrak{q}k : (\mathfrak{q},k) \in \mathfrak{I}\} \amalg \{\mathfrak{q}k : (\mathfrak{q},k) \in \breve{\mathfrak{I}}\}\\ &\amalg \{\natural\mathfrak{q} : \mathfrak{q} \in \mathfrak{Q}'\} \amalg \{\natural\mathfrak{q} : \mathfrak{q} \in \breve{\mathfrak{Q}}'\},\\ \Lambda - M = \{&\triangledown x : x \in X_\epsilon\}.\end{aligned}$$

Thirdly, since (A, T, σ) is a sorted algebra, A is divided into its t-parts $A_t = \sigma^{-1}\{t\}$ ($t \in T$), and since $T = (\coprod_{\nu \in N}\{\epsilon_\nu, \mathfrak{e}_\nu\}) \amalg \mathfrak{P}K$, we have the partition

$$A = \textstyle\coprod_{\nu \in N}(A_{\epsilon_\nu} \amalg A_{\mathfrak{e}_\nu}) \amalg \coprod_{Q \in \mathfrak{P}K} A_Q.$$

We define

$$G = \textstyle\coprod_{\nu \in N}(A_{\epsilon_\nu} \amalg A_{\mathfrak{e}_\nu}), \qquad H = \coprod_{Q \in \mathfrak{P}K} A_Q,$$

so that

$$A = G \amalg H, \qquad\qquad A_\emptyset \subseteq H.$$

We refer to the elements of G, A_{ϵ_ν} and $A_{\mathfrak{e}_\nu}$ $(\nu \in N)$ as the **nominal**s, **ϵ_ν-nominal**s and **$\mathfrak{e}_\nu$-nominal**s respectively. We also refer to the elements of H and $A_\emptyset$ as the **declarative**s and **sentence**s respectively. Furthermore, if $f \in A_Q$ $(Q \in \mathfrak{P}K)$, we call f a **Q-declarative**, call Q the **arity** of f (s. [4.3]) and denote it by K^f. Therefore, the sentences are the $\emptyset$-declaratives, i.e. the declaratives of arity $\emptyset$. Furthermore, we define

$$A_\epsilon = \textstyle\coprod_{\nu \in N} A_{\epsilon_\nu}, \qquad A_{\mathfrak{e}} = \textstyle\coprod_{\nu \in N} A_{\mathfrak{e}_\nu}, \qquad G_\nu = A_{\epsilon_\nu} \amalg A_{\mathfrak{e}_\nu} \quad (\nu \in N).$$

Then we have

$$G = A_\epsilon \amalg A_{\mathfrak{e}} = \textstyle\coprod_{\nu \in N} G_\nu,$$
$$A_{\epsilon_\nu} = A_\epsilon \cap G_\nu, \quad A_{\mathfrak{e}_\nu} = A_{\mathfrak{e}} \cap G_\nu \quad (\nu \in N).$$

We refer to the elements of A_ϵ, $A_{\mathfrak{e}}$ and G_ν $(\nu \in N)$ as the **ϵ-nominal**s, **$\mathfrak{e}$-nominal**s and **ν-nominal**s respectively. Furthermore, if $a \in G_\nu$ $(\nu \in N)$, we call ν the **nomen of a**. Therefore, each nominal $a \in G$ has its nomen $\nu \in N$, which we denote by ηa, ϵ_ν-nominals are the ϵ-nominals of nomen ν and the $\mathfrak{e}_\nu$-nominals are the $\mathfrak{e}$-nominals of nomen ν. This definition of ηa $(a \in G)$ extends the above definition of ηx $(x \in X_\epsilon)$ because

$$X_\epsilon = X \cap A_\epsilon, \qquad\qquad X_{\epsilon_\nu} = X \cap A_{\epsilon_\nu} = X_\epsilon \cap G_\nu \quad (\nu \in N).$$

Indeed, $X_\epsilon = \bigcup_{\nu \in N} X_{\epsilon_\nu}$ and $X_{\epsilon_\nu} = X \cap \sigma^{-1}\{\epsilon_\nu\} = X \cap A_{\epsilon_\nu}$ $(\nu \in N)$, and so $X_\epsilon = \bigcup_{\nu \in N}(X \cap A_{\epsilon_\nu}) = X \cap \bigcup_{\nu \in N} A_{\epsilon_\nu} = X \cap A_\epsilon$ and $X \cap A_{\epsilon_\nu} = X \cap A_\epsilon \cap G_\nu = X_\epsilon \cap G_\nu$ $(\nu \in N)$. We similarly have $C_{\epsilon_\nu} = C \cap A_{\epsilon_\nu}$ $(\nu \in N)$.

Lastly, also since (A, T, σ) is a sorted algebra, Remark 3.1.5 and [T1]–[T12] show that the algebraic structure $(\alpha_\lambda)_{\lambda \in \Lambda}$ of A satisfies [A1]–[A12] below.

For [A7], we define subsets A_k $(k \in K)$ of A by

$$A_k = \textstyle\coprod_{t \in T_k} A_t, \quad \text{i.e.}$$
$$A_k = \textstyle\coprod_{\nu \in N_k} A_{\epsilon_\nu} = \{a \in A_\epsilon : \eta a \in N_k\} = \begin{cases} A_\epsilon & \text{if } k \in K - K', \\ A_{\epsilon_{\eta k}} & \text{if } k \in K'. \end{cases}$$

In particular, $A_\pi = A_\epsilon$ for the nominative case π.

As with τ_λ $(\lambda \in \Lambda)$ in [T1]–[T12], the operation symbols α_λ $(\lambda \in \Lambda)$ in [A1]–[A12] are abbreviated to λ and, except for the postpositive operation symbols of various arities in [A1], [A4] and [A10]–[A12], the binary ones in [A2], [A5] and [A7]–[A9] are interpositions and the unary ones in [A3] and [A6] are superscript.

[A1] Let $\phi \in \Phi$ and $\nu = \eta\phi$. Then $\mathrm{Dm}\,\phi = {A_{\epsilon_\nu}}^{n_\phi}$ and $(a_1, \ldots, a_{n_\phi})\phi \in A_{\epsilon_\nu}$ for all $(a_1, \ldots, a_{n_\phi}) \in {A_{\epsilon_\nu}}^{n_\phi}$.

[A2] $\mathrm{Dm}\sqcap = \mathrm{Dm}\sqcup = \coprod_{\nu\in N}\coprod_{(t,u)\in\{\epsilon_\nu,\mathfrak{e}_\nu\}^2}(A_t\times A_u) = \coprod_{\nu\in N}{G_\nu}^2 = \{(a,b)\in G^2 : \eta a = \eta b\}$ and $a\sqcap b, a\sqcup b\in A_{\mathfrak{e}_\nu}$ for each $\nu\in N$ and each $(a,b)\in {G_\nu}^2$, that is, $a\sqcap b, a\sqcup b\in A_{\mathfrak{e}_{\eta a}} = A_{\mathfrak{e}_{\eta b}}$ for each $(a,b)\in G^2$ such that $\eta a = \eta b$.

[A3] $\mathrm{Dm}\,\Box = \coprod_{\nu\in N}(A_{\epsilon_\nu}\amalg A_{\mathfrak{e}_\nu}) = \coprod_{\nu\in N} G_\nu = G$ and $a^\Box\in A_{\mathfrak{e}_\nu}$ for each $\nu\in N$ and each $a\in G_\nu$, that is, $a^\Box\in A_{\mathfrak{e}_{\eta a}}$ for each $a\in G$.

[A4] $\mathrm{Dm}\,\triangle = \coprod_{\nu\in N}(A_{\epsilon_\nu}\amalg A_{\mathfrak{e}_\nu}) = \coprod_{\nu\in N} G_\nu = G$ and $a\triangle\in A_{\{\pi\}\cup K_\nu}$ for each $\nu\in N$ and each $a\in G_\nu$, that is, $a\triangle\in A_{\{\pi\}\cup K_{\eta a}}$ for each $a\in G$.

[A5] $\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow = \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(A_Q\times A_R) = H^2$ and if $(Q,R)\in(\mathfrak{P}K)^2$ and $(f,g)\in A_Q\times A_R$, then $f\wedge g, f\vee g, f\Rightarrow g\in A_{Q\cup R}$, that is, $f\wedge g, f\vee g, f\Rightarrow g\in A_{K^f\cup K^g}$ for each $(f,g)\in H^2$.

[A6] $\mathrm{Dm}\,\Diamond = \coprod_{Q\in\mathfrak{P}K} A_Q = H$ and if $Q\in\mathfrak{P}K$ and $f\in A_Q$, then $f^\Diamond\in A_Q$, that is, $f^\Diamond\in A_{K^f}$ for each $f\in H$.

[A7] Let $k\in K$. Then

$$\begin{aligned}\mathrm{Dm}\,\mathfrak{o}k &= \textstyle\coprod_{t\in T_k, k\in Q\in\mathfrak{P}K}(A_t\times A_Q) = A_k\times\coprod_{k\in Q\in\mathfrak{P}K}A_Q\\ &= \{(a,f)\in A_\epsilon\times H : \eta a\in N_k, k\in K^f\}\end{aligned}$$

and if $a\in A_k$, $k\in Q\in\mathfrak{P}K$ and $f\in A_Q$, then $a\,\mathfrak{o}k\,f\in A_{Q-\{k\}}$, that is, $a\,\mathfrak{o}k\,f\in A_{K^f-\{k\}}$ for each $(a,f)\in A_\epsilon\times H$ such that $\eta a\in N_k$ and $k\in K^f$.

[A8] Let $(q,k)\in\mathfrak{I}$ and $\nu = \eta q$. Then

$$\begin{aligned}\mathrm{Dm}\,qk &= \textstyle\coprod_{t\in\{\epsilon_\nu,\mathfrak{e}_\nu\}, k\in Q\in\mathfrak{P}K}(A_t\times A_Q) = G_\nu\times\coprod_{k\in Q\in\mathfrak{P}K}A_Q\\ &= \{(a,f)\in G\times H : \eta a = \nu, k\in K^f\}\end{aligned}$$

and if $a\in G_\nu$, $k\in Q\in\mathfrak{P}K$ and $f\in A_Q$, then $a\,qk\,f\in A_{Q-\{k\}}$, that is, $a\,qk\,f\in A_{K^f-\{k\}}$ for each $(a,f)\in G\times H$ such that $\eta a = \nu$ and $k\in K^f$.

[A9] Let $(q,k)\in\breve{\mathfrak{I}}$ and $\nu = \eta q$. Then

$$\begin{aligned}\mathrm{Dm}\,qk &= \textstyle\coprod_{t\in\{\epsilon_\nu,\mathfrak{e}_\nu\}, k\in Q\in\mathfrak{P}K}(A_t\times A_Q) = G_\nu\times\coprod_{k\in Q\in\mathfrak{P}K}A_Q\\ &= \{(a,f)\in G\times H : \eta a = \nu, k\in K^f\}\end{aligned}$$

and if $a\in G_\nu$, $k\in Q\in\mathfrak{P}K$ and $f\in A_Q$, then $a\,qk\,f\in A_{Q-\{k\}}$, that is, $a\,qk\,f\in A_{K^f-\{k\}}$ for each $(a,f)\in G\times H$ such that $\eta a = \nu$ and $k\in K^f$.

[A10] Let $q\in\mathfrak{Q}'$ (that is, $q\in\mathfrak{Q}$ and $\eta q\in N'$) and $k = \kappa_{\eta q}$. Then $\mathrm{Dm}\,\natural q = \coprod_{k\in Q\in\mathfrak{P}K}A_Q = \{f\in H : k\in K^f\}$ and if $k\in Q\in\mathfrak{P}K$ and $f\in A_Q$, then $f\natural q\in A_Q$, that is, $f\natural q\in A_{K^f}$ for each $f\in H$ such that $k\in K^f$.

[A11] Let $q\in\breve{\mathfrak{Q}}'$ (that is, $q\in\breve{\mathfrak{Q}}$ and $\eta q\in N'$) and $k = \kappa_{\eta q}$. Then $\mathrm{Dm}\,\natural q = \coprod_{k\in Q\in\mathfrak{P}K}A_Q = \{f\in H : k\in K^f\}$ and if $k\in Q\in\mathfrak{P}K$ and $f\in A_Q$, then $f\natural q\in A_Q$, that is, $f\natural q\in A_{K^f}$ for each $f\in H$ such that $k\in K^f$.

[A12] Let $x \in X_\epsilon$ and $\nu = \eta x$. Then $\mathrm{Dm}\,\triangledown x = \coprod_{Q \in \mathfrak{P}K_\nu} A_Q = \{f \in H : K^f \subseteq K_\nu\}$ and $f \triangledown x \in A_{\epsilon_\nu}$ for all $f \in \mathrm{Dm}\,\triangledown x$. We call $\triangledown x$ $(x \in X_\epsilon)$ the **nominalizer**s for this reason, while we give no name to $\triangledown$.

Consequently, the following also hold.

[B1] Let $q_1, \ldots, q_n \in \{\mathfrak{o}\} \cup \mathfrak{Q} \cup \breve{\mathfrak{Q}}$, $a_1, \ldots, a_n \in G$, $f \in H$, and let $k_1, \ldots, k_n$ be distinct cases in K^f. Assume $a_i \in A_\epsilon$ and $\eta a_i \in N_{k_i}$ for each $i \in \{1, \ldots, n\}$ such that $q_i = \mathfrak{o}$. Assume $\eta a_i = \eta q_i \in N_{k_i}$ for other $i \in \{1, \ldots, n\}$. Then $a_1\, q_1 k_1\, (\cdots (a_n\, q_n k_n\, f) \cdots) \in A_{K^f - \{k_1, \ldots, k_n\}}$.

[B2] The set A_{ϵ_ν} $(\nu \in N)$ is nonempty and closed by the operations in Φ_ν, whose restrictions to A_{ϵ_ν} are total. Moreover, A_{ϵ_ν} is the closure $[P_{\epsilon_\nu}]_{\Phi_\nu}$ of the ϵ_ν-part P_{ϵ_ν} of P in the Φ_ν-reduct A_{Φ_ν} of A. Thus $A_{\epsilon_\nu} - P$ is nonempty iff $\Phi_\nu \neq \emptyset$, and each its element has a ramification $(a_1, \ldots, a_{n_\phi})\phi$ for a ν-functional $\phi \in \Phi_\nu$ and an element $(a_1, \ldots, a_{n_\phi}) \in {A_{\epsilon_\nu}}^{n_\phi}$. Also, if $\nu \in N_d$, then $A_{\epsilon_\nu} = \{x_\nu\}$ for the ν-variable x_ν.

[B3] The sets G_ν and $A_{\mathfrak{e}_\nu}$ $(\nu \in N)$ are closed by the operations $\sqcap, \sqcup$ and $\Box$, whose restrictions to G_ν and $A_{\mathfrak{e}_\nu}$ are total.

[B4] The set $A_{\mathfrak{e}_\nu} - P$ $(\nu \in N)$ is nonempty, and each its element has a ramification $a \sqcap b$, $a \sqcup b$, $a^\Box$ or $f \triangledown x$ for elements $a, b, f, x \in A$ satisfying the conditions shown in [A2], [A3] and [A12].

[B5] The sets H and A_Q $(Q \in \mathfrak{P}K)$ are closed by the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, whose restrictions to H and A_Q are total.

[B6] The set $H - P$ is nonempty, and each its element has a ramification $a\triangle$, $f \wedge g$, $f \vee g$, $f \Rightarrow g$, $f^\Diamond$, $a\, \mathfrak{o}k\, f$, $a\, \mathfrak{q}k\, f$ or $f \natural \mathfrak{q}$ for tokens $k, \mathfrak{q} \in \Gamma$ and elements $a, f, g \in A$ satisfying the conditions shown in [A4]–[A11].

[B7] For each $\nu \in N$ and each $Q \in \mathfrak{P}K_\nu$, the set $A_Q - P$ is nonempty. In particular, $A_\emptyset - P$ is nonempty, and each its element has a ramification $f \wedge g$, $f \vee g$, $f \Rightarrow g$, $f^\Diamond$, $a\, \mathfrak{o}k\, f$ or $a\, \mathfrak{q}k\, f$ for tokens $k, \mathfrak{q} \in \Gamma$ and elements $f, g, a \in A$ satisfying the conditions shown in [A5]–[A9] including the conditions necessary for the ramifications to belong to $A_\emptyset$.

As for the main part of [B2], A_{ϵ_ν} is closed by the operations in Φ_ν by [A1], and so $A_{\epsilon_\nu} \supseteq [P_{\epsilon_\nu}]_{\Phi_\nu}$. We can conversely show that every element $a \in A_{\epsilon_\nu}$ belongs to $[P_{\epsilon_\nu}]_{\Phi_\nu}$ by induction on $r = \mathrm{rk}\, a$. Since $P \cap A_{\epsilon_\nu} = P_{\epsilon_\nu}$, we may assume $r \geq 1$. Then since $a \in A_{\epsilon_\nu}$, [A1]–[A12] show that $a = (a_1, \ldots, a_{n_\phi})\phi$ for some $\phi \in \Phi_\nu$ and some $a_1, \ldots, a_{n_\phi} \in A_{\epsilon_\nu}$ such that $\mathrm{rk}\, a - 1 = \sum_{j=1}^{n_\phi} \mathrm{rk}\, a_j$. Since $a_1, \ldots, a_{n_\phi} \in [P_{\epsilon_\nu}]_{\Phi_\nu}$ by the induction hypothesis, we have $a \in [P_{\epsilon_\nu}]_{\Phi_\nu}$ as desired. We can derive $A_{\mathfrak{e}_\nu} - P \neq \emptyset$ in [B4] and $A_Q - P \neq \emptyset$ in [B7] as well as $A_{\epsilon_\nu} \neq \emptyset$ in [B2] from our assumption $X_{\epsilon_\mu} \neq \emptyset$ $(\mu \in N)$. Indeed, if $x \in X_{\epsilon_\nu}$, then $x \in A_{\epsilon_\nu}$ by definition, and $x\, \mathfrak{o}\pi\, x\triangle \in A_{K_\nu}$ and $(x\, \mathfrak{o}\pi\, x\triangle) \triangledown x \in A_{\mathfrak{e}_\nu} - P$ by [A4], [A7] and [A12]. Since $Q \subseteq K_\nu$ and $\# K_\nu < \infty$, there are distinct elements $\mu_1, \ldots, \mu_n \in N_\nu$ such that $Q = K_\nu - \{\kappa_{\mu_1}, \ldots, \kappa_{\mu_n}\}$, and if $x_i \in X_{\epsilon_{\mu_i}}$ $(i = 1, \ldots, n)$, then $x_1\, \mathfrak{o}\kappa_{\mu_1}\, (\cdots (x_n\, \mathfrak{o}\kappa_{\mu_n}\, (x\, \mathfrak{o}\pi\, x\triangle)) \cdots) \in A_Q - P$ by [B1].

6.1.3 The C worlds

Let $(A, T, \sigma, P, C, X, \Gamma)$ be the C language constructed in §6.1.2. Using its C parameters introduced in §6.1.1 and following Remark 3.2.1, here we construct certain DWs for it, that is, each of them is an algebra $(W, (\omega_\lambda)_{\lambda \in M})$ equipped with a partition $W = \coprod_{t \in T} W_t$ which satisfies the P-denotability and (3.2.3) for the algebraic structure $(\tau_\lambda)_{\lambda \in M}$ of T_M defined in [T1]–[T11]. We call them the **C world**s *for the C language*. The phrase *for the C language* here is necessary because other C languages may have other C worlds.

Picking a family $(S_\nu)_{\nu \in N}$ of nonempty sets, we first define

$$W = \textstyle\coprod_{\nu \in N}(S_\nu \amalg (\Theta_\nu \to (S_\nu \to \mathbb{T}))) \amalg \coprod_{Q \in \mathfrak{P}K}((Q \rightsquigarrow S) \to \mathbb{T}),$$

where $\mathbb{T}$ is the binary lattice $\{0, 1\}$, whose elements are called the **truth value**s,

$$\begin{aligned} &S = \textstyle\coprod_{\nu \in N} S_\nu, \\ &Q \rightsquigarrow S = \{\theta \in Q \to S : \theta k \in S_{\eta k} \text{ for each } k \in K' \cap Q\} \quad (Q \in \mathfrak{P}K), \\ &\Theta_\nu = K_\nu \rightsquigarrow S \quad (\nu \in N) \end{aligned}$$

and we identify $(\emptyset \rightsquigarrow S) \to \mathbb{T}$ with $\mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]) because $\emptyset \rightsquigarrow S = \emptyset \to S = \{\emptyset\}$ (s. Remark 6.1.1 and [3.2]). We refer to S and S_ν $(\nu \in N)$ as the **basis** and **ν-basis** respectively.

Remark 6.1.1 The above definitions imply that if $Q \in \mathfrak{P}K$ and $K' \cap Q = \emptyset$ then $Q \rightsquigarrow S = Q \to S$ and that

$$\Theta_\nu = \{\theta \in K_\nu \to S : \theta k \in S_{\eta k} \text{ for each } k \in K_\nu\}$$

for each $\nu \in N$ because $K_\nu \subseteq K'$. Moreover, defining subsets S_k $(k \in K)$ of S by

$$S_k = \textstyle\coprod_{\nu \in N_k} S_\nu, \text{ i.e. } S_k = \begin{cases} S & \text{if } k \in K - K', \\ S_{\eta k} & \text{if } k \in K', \end{cases}$$

we have the following for each $Q \in \mathfrak{P}K$:

$$Q \rightsquigarrow S = \{\theta \in Q \to S : \theta k \in S_k \text{ for each } k \in Q\}.$$

Let $(Q, R) \in (\mathfrak{P}K)^2$ and $\theta \in (Q \cup R) \to S$. Then $\theta|_Q \in Q \to S$, $\theta|_R \in R \to S$ and we have that $\theta \in (Q \cup R) \rightsquigarrow S$ iff $\theta|_Q \in Q \rightsquigarrow S$ and $\theta|_R \in R \rightsquigarrow S$. In particular, if $Q \in \mathfrak{P}K$ and $\theta \in Q \rightsquigarrow S$, then $\theta|_R \in R \rightsquigarrow S$ for each subset R of Q.

Since $T = (\coprod_{\nu \in N}\{\epsilon_\nu, \mathfrak{e}_\nu\}) \amalg \mathfrak{P}K$, the partition $W = \coprod_{t \in T} W_t$ is as follows:

$$W = \textstyle\coprod_{\nu \in N}(W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}) \amalg \coprod_{Q \in \mathfrak{P}K} W_Q.$$

Comparing this with the above definition of W, we define

$$W_{\epsilon_\nu} = S_\nu, \qquad W_{\mathfrak{e}_\nu} = \Theta_\nu \to (S_\nu \to \mathbb{T}), \qquad W_Q = (Q \rightsquigarrow S) \to \mathbb{T}$$

for each $\nu \in \mathsf{N}$ and each $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$, where

$$W_\emptyset = \mathbb{T}$$

because we have identified $(\emptyset \rightsquigarrow \mathsf{S}) \to \mathbb{T}$ with $\mathbb{T}$. Then since $\mathsf{S}_\nu \neq \emptyset$ $(\nu \in \mathsf{N})$, we have $W_\mathsf{t} \neq \emptyset$ for all $\mathsf{t} \in \mathsf{T}$, and so the P-denotability is satisfied.

Furthermore, we define

$$\mathsf{E} = \coprod_{\nu \in \mathsf{N}} (W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}), \qquad \mathsf{F} = \coprod_{\mathsf{Q} \in \mathfrak{P}\mathsf{K}} W_\mathsf{Q},$$

so that

$$W = \mathsf{E} \amalg \mathsf{F}, \qquad \mathbb{T} = W_\emptyset \subseteq \mathsf{F}.$$

We refer to the elements of E and F as **entitie**s and **event**s respectively. Furthermore, if $\mathsf{f} \in W_\mathsf{Q}$ $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$, we call f a **Q-event**, call Q the **arity** of f (s. [4.2]) and denote it by K^f. Therefore, the truth values are the $\emptyset$-events, i.e. the events of arity $\emptyset$. Furthermore, we define

$$W_\epsilon = \coprod_{\nu \in \mathsf{N}} W_{\epsilon_\nu}, \quad W_\mathfrak{e} = \coprod_{\nu \in \mathsf{N}} W_{\mathfrak{e}_\nu}, \quad \mathsf{E}_\nu = W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu} \quad (\nu \in \mathsf{N}).$$

Then we have

$$\begin{gathered} W_\epsilon = \mathsf{S}, \\ \mathsf{E} = W_\epsilon \amalg W_\mathfrak{e} = \coprod_{\nu \in \mathsf{N}} \mathsf{E}_\nu, \\ W_{\epsilon_\nu} = W_\epsilon \cap \mathsf{E}_\nu, \quad W_{\mathfrak{e}_\nu} = W_\mathfrak{e} \cap \mathsf{E}_\nu \quad (\nu \in \mathsf{N}). \end{gathered}$$

We refer to the elements of W_ϵ, $W_\mathfrak{e}$ and E_ν $(\nu \in \mathsf{N})$ as the **basic entitie**s, **derived entitie**s and **ν-entitie**s respectively, and so the elements of W_{ϵ_ν} and $W_{\mathfrak{e}_\nu}$ are the **basic ν-entitie**s and **derived ν-entitie**s respectively. Furthermore, if $\mathsf{a} \in \mathsf{E}_\nu$ $(\nu \in \mathsf{N})$, we call ν the **nomen of a**. Therefore, each entity $\mathsf{a} \in \mathsf{E}$ has its nomen $\nu \in \mathsf{N}$, which we denote by $\eta\mathsf{a}$.

The algebraic structure $(\omega_\lambda)_{\lambda \in \mathsf{M}}$ of W is divided into the families defined in [W1]–[W11] below according to the division of the algebraic structure $(\tau_\lambda)_{\lambda \in \mathsf{M}}$ of T_M into the families in [T1]–[T11].

For [W2]–[W4] and [W8]–[W11], we pick a family $(\sqsubseteq_\theta)_{\theta \in \Theta_\nu}$ of relations on S_ν for each $\nu \in \mathsf{N}$, which we call the **basic ν-relations**. Then we extend $\sqsubseteq_\theta$ for each $\theta \in \Theta_\nu$ to a relation between S_ν and $\mathsf{E}_\nu = \mathsf{S}_\nu \amalg (\Theta_\nu \to (\mathsf{S}_\nu \to \mathbb{T}))$ by the following for each $(\mathsf{s}, \mathsf{a}) \in \mathsf{S}_\nu \times (\Theta_\nu \to (\mathsf{S}_\nu \to \mathbb{T}))$:

$$\mathsf{s} \sqsubseteq_\theta \mathsf{a} \iff (\mathsf{a}\theta)\mathsf{s} = 1. \tag{6.1.1}$$

For [W4] and [W8]–[W11], we also pick an element $\mathsf{s}_\mu \in \mathsf{S}_\mu$ for each $\mu \in \mathsf{N}'$ and call it the **μ-default**. Then, for each $\theta \in \mathsf{Q} \rightsquigarrow \mathsf{S}$ $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$ and each $\nu \in \mathsf{N}$, we let θ_ν be the element of $\mathsf{K}_\nu \to \mathsf{S}$ defined by the following for each $\mathsf{k} \in \mathsf{K}_\nu$:

$$\theta_\nu \mathsf{k} = \begin{cases} \theta\mathsf{k} & \text{if } \mathsf{k} \in \mathsf{K}_\nu \cap \mathsf{Q}, \\ \mathsf{s}_{\eta\mathsf{k}} & \text{if } \mathsf{k} \in \mathsf{K}_\nu - \mathsf{Q}. \end{cases} \tag{6.1.2}$$

Then $\theta_\nu|_{K_\nu\cap Q} = \theta|_{K_\nu\cap Q}$ and, since $\theta_\nu k \in S_{\eta k}$ for each $k \in K_\nu$, we have

$$\theta_\nu \in \Theta_\nu$$

by Remark 6.1.1. We refer to the extended ν-relation $\sqsubseteq_{\theta_\nu}$ between S_ν and E_ν as the (θ, ν)**-relation**.

For each $(s, k) \in S \times K$ in [W7]–[W9], we define $(k/s) \in \{k\}{\to}S$ by

$$(k/s)k = s.$$

Then Remark 6.1.1 shows that $(k/s) \in \{k\}{\rightsquigarrow}S$ iff $s \in S_k$.

For each quadruple (s, k, Q, θ) also in [W7]–[W9] satisfying $s \in S$, $k \in Q \in \mathfrak{P}K$ and $\theta \in (Q - \{k\}){\to}S$, we define $(k/s)\theta \in Q{\to}S$ by $((k/s)\theta)|_{Q-\{k\}} = \theta$ and $((k/s)\theta)|_{\{k\}} = (k/s)$, that is,

$$((k/s)\theta)l = \begin{cases} \theta l & \text{if } l \in Q - \{k\}, \\ s & \text{if } l = k. \end{cases}$$

In particular, if $Q = \{k\}$, then $Q - \{k\} = \emptyset$, $\theta = \emptyset$ (s. [3.2]) and $(k/s)\emptyset = (k/s)$. Remark 6.1.1 shows that $(k/s)\theta \in Q{\rightsquigarrow}S$ iff $\theta \in (Q - \{k\}){\rightsquigarrow}S$ and $s \in S_k$.

As for [W7], Remark 6.1.1 shows that the following holds for each $k \in K$:

$$\textstyle\coprod_{t\in T_k} W_t = \coprod_{\nu\in N_k} W_{\epsilon_\nu} = \coprod_{\nu\in N_k} S_\nu = S_k.$$

For [W8] and [W9], we pick a $\mathbb{P}_\nu$-measure $Y \mapsto |Y|_\nu$ on S_ν for each $\nu \in N$, which exists by Remark 3.6.15, and abbreviate the expression

$$|\{\text{a specification of the members of } Y\}|_\nu$$

to the expression $|$a specification of the members of $Y|_\nu$ without braces.

As with τ_λ $(\lambda \in M)$ in [T1]–[T11], the operation symbols ω_λ $(\lambda \in M)$ in [W1]–[W11] are abbreviated to λ and, except for the postpositive operation symbols of various arities in [W1], [W4], [W10] and [W11], the binary ones in [W2], [W5] and [W7]–[W9] are interpositions and the unary ones in [W3] and [W6] are superscript. You will see there that (3.2.3) is satisfied.

[W1] An arbitrary family of n_ϕ-ary operations $\phi \in \Phi$ such that if $\nu = \eta\phi$ then $\mathrm{Dm}\,\phi = {W_{\epsilon_\nu}}^{n_\phi}$ and $(a_1, \ldots, a_{n_\phi})\phi \in W_{\epsilon_\nu}$ for all $(a_1, \ldots, a_{n_\phi}) \in {W_{\epsilon_\nu}}^{n_\phi}$.

[W2] The binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dm}\,\sqcap = \mathrm{Dm}\,\sqcup = \textstyle\coprod_{\nu\in N} \coprod_{(t,u)\in\{\epsilon_\nu,\mathfrak{e}_\nu\}^2} (W_t \times W_u) = \coprod_{\nu\in N} {E_\nu}^2$$

and if $(a, b) \in {E_\nu}^2$ $(\nu \in N)$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $W_{\mathfrak{e}_\nu} = \Theta_\nu{\to}(S_\nu{\to}\mathbb{T})$ defined by the following for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$:

$$s \sqsubseteq_\theta a \sqcap b \iff s \sqsubseteq_\theta a \text{ and } s \sqsubseteq_\theta b,$$

$$s \sqsubseteq_\theta a \sqcup b \iff s \sqsubseteq_\theta a \text{ or } s \sqsubseteq_\theta b.$$

Because of (6.1.1), this certainly defines $a \sqcap b$ and $a \sqcup b$ as elements of $\Theta_\nu \to (S_\nu \to \mathbb{T})$ and if $(a, b) \in (\Theta_\nu \to (S_\nu \to \mathbb{T}))^2$, then $((a \sqcap b)\theta)s = (a\theta)s \wedge (b\theta)s$ and $((a \sqcup b)\theta)s = (a\theta)s \vee (b\theta)s$ for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$, where $\wedge$ and $\vee$ are the meet and join on the binary lattice $\mathbb{T}$.

[W3] The unary operation $\square$ such that

$$\mathrm{Dm}\,\square = \coprod_{\nu \in N}(W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}) = \coprod_{\nu \in N} E_\nu = E$$

and if $a \in E_\nu$ ($\nu \in N$), then its image $a^\square$ is the element of $W_{\mathfrak{e}_\nu} = \Theta_\nu \to (S_\nu \to \mathbb{T})$ defined by the following for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$:

$$s \sqsubseteq_\theta a^\square \iff s \not\sqsubseteq_\theta a.$$

Because of (6.1.1), this certainly defines $a^\square$ as an element of $\Theta_\nu \to (S_\nu \to \mathbb{T})$ and if $a \in \Theta_\nu \to (S_\nu \to \mathbb{T})$ then $(a^\square\theta)s = ((a\theta)s)^\lozenge$ for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$, where $\lozenge$ is the complement on the binary lattice $\mathbb{T}$ (s. [1.79]).

[W4] The unary operation $\triangle$ such that

$$\mathrm{Dm}\,\triangle = \coprod_{\nu \in N}(W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}) = \coprod_{\nu \in N} E_\nu = E$$

and if $a \in E_\nu$ ($\nu \in N$), then its image $a\triangle$ is the element of $W_{\{\pi\} \cup K_\nu} = ((\{\pi\} \cup K_\nu) \rightsquigarrow S) \to \mathbb{T}$ defined by the following for each $\theta \in (\{\pi\} \cup K_\nu) \rightsquigarrow S$:

$$(a\triangle)\theta = 1 \iff \theta\pi \in S_\nu \text{ and } \theta\pi \sqsubseteq_{\theta_\nu} a.$$

Notice that $\theta_\nu = \theta|_{K_\nu}$ here.

[W5] The three binary operations $\wedge$, $\vee$ and $\Rightarrow$ such that

$$\begin{aligned}\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rightarrow = F^2 &= \textstyle\coprod_{(Q,R) \in (\mathfrak{P}K)^2}(W_Q \times W_R) \\ &= \textstyle\coprod_{(Q,R) \in (\mathfrak{P}K)^2}(((Q \rightsquigarrow S) \to \mathbb{T}) \times ((R \rightsquigarrow S) \to \mathbb{T}))\end{aligned}$$

and if $(Q, R) \in (\mathfrak{P}K)^2$ and $(f, g) \in ((Q \rightsquigarrow S) \to \mathbb{T}) \times ((R \rightsquigarrow S) \to \mathbb{T})$, then its images $f \wedge g$, $f \vee g$ and $f \Rightarrow g$ are the elements of $W_{Q \cup R} = ((Q \cup R) \rightsquigarrow S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q \cup R) \rightsquigarrow S$:

$$\begin{aligned}(f \wedge g)\theta &= f(\theta|_Q) \wedge g(\theta|_R), \\ (f \vee g)\theta &= f(\theta|_Q) \vee g(\theta|_R), \\ (f \Rightarrow g)\theta &= f(\theta|_Q) \Rightarrow g(\theta|_R).\end{aligned}$$

Here $\wedge$, $\vee$ and $\Rightarrow$ on the right-hand sides are the meet, join and cojoin (s. §1.5.2) on the binary lattice $\mathbb{T}$. This definition makes sense because $\theta|_Q \in Q \rightsquigarrow S$ and $\theta|_R \in R \rightsquigarrow S$ for all $\theta \in (Q \cup R) \rightsquigarrow S$ by Remark 6.1.1.

[W6] The unary operation $\lozenge$ such that

$$\mathrm{Dm}\,\lozenge = \mathsf{F} = \textstyle\coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}} W_{\mathsf{Q}} = \coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T})$$

and if $\mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$, then its image $\mathsf{f}^{\lozenge}$ is the element of $W_{\mathsf{Q}} = (\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$ defined by the following for each $\theta \in \mathsf{Q}\rightsquigarrow\mathsf{S}$:

$$(\mathsf{f}^{\lozenge})\theta = (\mathsf{f}\theta)^{\lozenge}.$$

Here $\lozenge$ on the right-hand side is the complement on the binary lattice $\mathbb{T}$.

[W7] The family of binary operations $\mathfrak{o}\mathsf{k}$ ($\mathsf{k} \in \mathsf{K}$) such that

$$\mathrm{Dm}\,\mathfrak{o}\mathsf{k} = \textstyle\coprod_{\mathsf{t}\in\mathsf{T}_{\mathsf{k}},\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}(W_{\mathsf{t}} \times W_{\mathsf{Q}}) = \mathsf{S}_{\mathsf{k}} \times \coprod_{\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T})$$

and if $\mathsf{s} \in \mathsf{S}_{\mathsf{k}}$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$, then the image $\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f}$ of (s,f) is the element of $W_{\mathsf{Q}-\{\mathsf{k}\}} = ((\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S}$:

$$(\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta = \mathsf{f}((\mathsf{k}/\mathsf{s})\theta).$$

This definition makes sense because $(\mathsf{k}/\mathsf{s})\theta \in \mathsf{Q}\rightsquigarrow\mathsf{S}$ by Remark 6.1.1. Notice that if $\mathsf{Q} = \{\mathsf{k}\}$ then $\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f} = \mathsf{f}(\mathsf{k}/\mathsf{s}) \in \mathbb{T}$ by the $\{\emptyset\}$ convention.

[W8] The family of binary operations $\mathfrak{q}\mathsf{k}$ ($(\mathfrak{q},\mathsf{k}) \in \mathfrak{I}$) such that if $\nu = \eta\mathfrak{q}$, then

$$\mathrm{Dm}\,\mathfrak{q}\mathsf{k} = \textstyle\coprod_{\mathsf{t}\in\{\epsilon_\nu,\mathfrak{e}_\nu\},\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}(W_{\mathsf{t}} \times W_{\mathsf{Q}}) = \mathsf{E}_\nu \times \coprod_{\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T})$$

and if $\mathsf{a} \in \mathsf{E}_\nu$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$, then the image $\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$ of (a,f) is the element of $W_{\mathsf{Q}-\{\mathsf{k}\}} = ((\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S}$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}_\nu$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}_\nu$ (distinguish between ν and v (s. §0.4)):

$$(\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1 \iff |\mathsf{s} \in \mathsf{S}_\nu : \mathsf{s} \sqsubseteq_{\theta_\nu} \mathsf{a},\ \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = \mathsf{v}|_\nu \in \mathfrak{p}.$$

This definition makes sense because $\mathfrak{q} \in \mathfrak{Q}_\nu = \mathfrak{P}_\nu \amalg \neg\mathfrak{P}_\nu$ and $\nu \in \mathsf{N}_{\mathsf{k}}$ by the definition of $\mathfrak{I}$, and so $(\mathsf{k}/\mathsf{s})\theta \in \mathsf{Q}\rightsquigarrow\mathsf{S}$ for all $\mathsf{s} \in \mathsf{S}_\nu$ by Remark 6.1.1. Notice that $\mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = (\mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f})\theta$ and if $\mathsf{Q} = \{\mathsf{k}\}$ then $(\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = \mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f} \in \mathbb{T}$ and $\mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = \mathsf{f}(\mathsf{k}/\mathsf{s}) = \mathsf{s}\,\mathfrak{o}\mathsf{k}\,\mathsf{f} \in \mathbb{T}$ by [W7] and the $\{\emptyset\}$ convention.

[W9] The family of binary operations $\mathfrak{q}\mathsf{k}$ ($(\mathfrak{q},\mathsf{k}) \in \breve{\mathfrak{I}}$) such that if $\nu = \eta\mathfrak{q}$ then

$$\mathrm{Dm}\,\mathfrak{q}\mathsf{k} = \textstyle\coprod_{\mathsf{t}\in\{\epsilon_\nu,\mathfrak{e}_\nu\},\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}(W_{\mathsf{t}} \times W_{\mathsf{Q}}) = \mathsf{E}_\nu \times \coprod_{\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T})$$

and if $\mathsf{a} \in \mathsf{E}_\nu$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$, then the image $\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$ of (a,f) is the element of $W_{\mathsf{Q}-\{\mathsf{k}\}} = ((\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S})\rightarrow\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q}-\{\mathsf{k}\})\rightsquigarrow\mathsf{S}$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \breve{\mathfrak{P}}_\nu$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\breve{\mathfrak{P}}_\nu$ (distinguish between ν and v (s. §0.4)):

$$(\mathsf{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1 \iff \text{if } |\mathsf{s} \in \mathsf{S}_\nu : \mathsf{s} \sqsubseteq_{\theta_\nu} \mathsf{a}|_\nu \neq 0, \text{ then } \frac{|\mathsf{s} \in \mathsf{S}_\nu : \mathsf{s} \sqsubseteq_{\theta_\nu} \mathsf{a},\ \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = \mathsf{v}|_\nu}{|\mathsf{s} \in \mathsf{S}_\nu : \mathsf{s} \sqsubseteq_{\theta_\nu} \mathsf{a}|_\nu} \in \mathfrak{p}.$$

This definition makes sense for a similar reason as in [W8].

[W10] The family of unary operations $\natural\mathfrak{q}$ ($\mathfrak{q} \in \mathfrak{Q}'$, that is, $\mathfrak{q} \in \mathfrak{Q}$ and $\eta\mathfrak{q} \in N'$) such that if $k = \kappa_{\eta\mathfrak{q}}$ then

$$\mathrm{Dm}\,\natural\mathfrak{q} = \coprod_{k\in Q\in\mathfrak{P}K} W_Q = \coprod_{k\in Q\in\mathfrak{P}K}((Q \rightsquigarrow S) \to \mathbb{T})$$

and if $k \in Q \in \mathfrak{P}K$ and $f \in (Q \rightsquigarrow S) \to \mathbb{T}$, then the image $f\natural\mathfrak{q}$ of f is the element of $W_Q = (Q \rightsquigarrow S) \to \mathbb{T}$ defined by the following for each $\theta \in Q \rightsquigarrow S$, where $\mathfrak{q}k$ is the operation defined in [W8]:

$$(f\natural\mathfrak{q})\theta = ((\theta k)\,\mathfrak{q}k\,f)\theta|_{Q-\{k\}}.$$

This definition makes sense because $\eta\mathfrak{q} = \eta k \in N_k$ and so $(\mathfrak{q}, k) \in \mathfrak{I}$, $\theta k \in S_{\eta\mathfrak{q}} \subseteq E_{\eta\mathfrak{q}}$, $(\theta k)\,\mathfrak{q}k\,f \in ((Q-\{k\}) \rightsquigarrow S) \to \mathbb{T}$, and $\theta|_{Q-\{k\}} \in (Q-\{k\}) \rightsquigarrow S$ by Remark 6.1.1.

[W11] The family of unary operations $\natural\mathfrak{q}$ ($\mathfrak{q} \in \breve{\mathfrak{Q}}'$, that is, $\mathfrak{q} \in \breve{\mathfrak{Q}}$ and $\eta\mathfrak{q} \in N'$) such that if $k = \kappa_{\eta\mathfrak{q}}$ then

$$\mathrm{Dm}\,\natural\mathfrak{q} = \coprod_{k\in Q\in\mathfrak{P}K} W_Q = \coprod_{k\in Q\in\mathfrak{P}K}((Q \rightsquigarrow S) \to \mathbb{T})$$

and if $k \in Q \in \mathfrak{P}K$ and $f \in (Q \rightsquigarrow S) \to \mathbb{T}$, then the image $f\natural\mathfrak{q}$ of f is the element of $W_Q = (Q \rightsquigarrow S) \to \mathbb{T}$ defined by the following for each $\theta \in Q \rightsquigarrow S$, where $\mathfrak{q}k$ is the operation defined in [W9]:

$$(f\natural\mathfrak{q})\theta = ((\theta k)\,\mathfrak{q}k\,f)\theta|_{Q-\{k\}}.$$

This definition makes sense for a similar reason as in [W10].

This completes the construction of the C worlds W on the C parameters of the C language A. Thus they are also parameterized by the basis $S = \coprod_{\nu\in N} S_\nu$, the basic ν-relations on S_ν ($\nu \in N$), the μ-default in S_μ ($\mu \in N'$), the $\mathbb{P}_\nu$-measure $Y \mapsto |Y|_\nu$ on S_ν ($\nu \in N$) and the operations $(\omega_\phi)_{\phi\in\Phi}$ in [W1].

6.1.4 The C interpretation of the nominalizers

Let $(A, T, \sigma, P, C, X, \Gamma)$ be the C language constructed in §6.1.2 and W be any one of the C worlds for it constructed in §6.1.3. Following §3.2.4, here we introduce an interpretation $(\lambda_W)_{\lambda\in\Lambda-M}$, called the **C interpretation**, of the set $\Lambda - M$ of the variable indices of the algebraic structure $(\tau_\lambda)_{\lambda\in\Lambda}$ of T on W. The significance λ_W, called the **C significance**, of each $\lambda \in \Lambda - M$ on W by definition depends on τ_λ and the partitions $W = \coprod_{t\in T} W_t$ and $\Lambda - M = \coprod_{\xi\in X^{\circledast}-\{\varepsilon\}} \Lambda_\xi$.

Let $\lambda \in \Lambda - M$. Then since $\Lambda - M = \{\triangledown x : x \in X_\epsilon\}$, we have $\lambda = \triangledown x \in \Lambda_x$ for some $x \in X_\epsilon$. Let $\nu = \eta x$, that is, $x \in X_{\epsilon_\nu} = \sigma|_X{}^{-1}\{\epsilon_\nu\}$. Then $\mathrm{Dm}\,\tau_\lambda = \mathfrak{P}K_\nu$ and $\tau_\lambda Q = \mathfrak{e}_\nu$ for each $Q \in \mathfrak{P}K_\nu$ by [T12]. Therefore, (3.2.5) and (3.2.6) show

$$\lambda_W \in (\textstyle\bigcup_{Q\in\mathfrak{P}K_\nu}(W_{\sigma x} \to W_Q)) \to W_{\mathfrak{e}_\nu}$$

and since $W_{\sigma x} \to W_Q = W_{\epsilon_\nu} \to W_Q = S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T})$ for each $Q \in \mathfrak{P}K_\nu$ and $W_{\mathfrak{e}_\nu} = \Theta_\nu \to (S_\nu \to \mathbb{T})$, we have

$$\lambda_W \in (\textstyle\bigcup_{Q\in\mathfrak{P}K_\nu}(S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T}))) \to (\Theta_\nu \to (S_\nu \to \mathbb{T})).$$

Thus we define its value $\lambda_W f \in \Theta_\nu \to (S_\nu \to \mathbb{T})$ at each $f \in S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T})$ $(Q \in \mathfrak{P}K_\nu)$ by the following for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$:

$$((\lambda_W f)\theta)s = (fs)\theta|_Q.$$

This makes sense because $\Theta_\nu = K_\nu \rightsquigarrow S$ and so $\theta|_Q \in Q \rightsquigarrow S$ for each $\theta \in \Theta_\nu$ and each $Q \in \mathfrak{P}K_\nu$ by Remark 6.1.1; in particular, if $Q = K_\nu$ then $((\lambda_W f)\theta)s = (fs)\theta$, while if $Q = \emptyset$ then $f \in S_\nu \to \mathbb{T}$ by the $\{\emptyset\}$ convention and $((\lambda_W f)\theta)s = fs$.

The definition of λ_W is natural because we may identify $S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T})$ $(Q \in \mathfrak{P}K_\nu)$ and $\Theta_\nu \to (S_\nu \to \mathbb{T})$ with $(S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T}$ and $(\Theta_\nu \times S_\nu) \to \mathbb{T}$ respectively by linearization (s. Remark 1.3.2). Under the identification, we see that $\lambda_W \in (\bigcup_{Q \in \mathfrak{P}K_\nu} ((S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T})) \to ((\Theta_\nu \times S_\nu) \to \mathbb{T})$ and that its value $\lambda_W f \in (\Theta_\nu \times S_\nu) \to \mathbb{T}$ at each $f \in (S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T}$ $(Q \in \mathfrak{P}K_\nu)$ satisfies $(\lambda_W f)(\theta, s) = f(s, \theta|_Q)$ for each $(\theta, s) \in \Theta_\nu \times S_\nu$; in particular, if $Q = K_\nu$ then $(\lambda_W f)(\theta, s) = f(s, \theta)$, while if $Q = \emptyset$ then $f \in S_\nu \to \mathbb{T}$ and $(\lambda_W f)(\theta, s) = fs$, because $S_\nu \to ((\emptyset \rightsquigarrow S) \to \mathbb{T})$ has been identified with $S_\nu \to \mathbb{T}$ by the $\{\emptyset\}$ convention and so $(S_\nu \times (\emptyset \rightsquigarrow S)) \to \mathbb{T}$ must also be identified with $S_\nu \to \mathbb{T}$, or rather $S_\nu \times \{\emptyset\}$ must be identified with S_ν.

The resultant operation β_λ on the metaworld $W^\sharp$ satisfies

$$\operatorname{Dm} \beta_\lambda = \textstyle\bigcup_{Q \in \mathfrak{P}K_\nu} (\Upsilon_W \to ((Q \rightsquigarrow S) \to \mathbb{T})),$$
$$\beta_\lambda(\Upsilon_W \to ((Q \rightsquigarrow S) \to \mathbb{T})) \subseteq \Upsilon_W \to (\Theta_\nu \to (S_\nu \to \mathbb{T})) \text{ for each } Q \in \mathfrak{P}K_\nu$$

by (3.3.9). Let $\varphi \in \Upsilon_W \to ((Q \rightsquigarrow S) \to \mathbb{T})$ $(Q \in \mathfrak{P}K_\nu)$. Then its image $\beta_\lambda \varphi \in \Upsilon_W \to (\Theta_\nu \to (S_\nu \to \mathbb{T}))$ satisfies

$$(\beta_\lambda \varphi)\upsilon = \lambda_W(\varphi(\upsilon(x/\Box)))$$

for each $\upsilon \in \Upsilon_W$ by (3.3.8), where $\varphi(\upsilon(x/\Box)) \in S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T})$ and

$$(\varphi(\upsilon(x/\Box)))s = \varphi(\upsilon(x/s))$$

for each $s \in S_\nu$ by (3.3.1), (3.3.3) and (3.3.4). Therefore, the above definition of λ_W shows that the following holds for each $\theta \in \Theta_\nu$ and each $s \in S_\nu$:

$$(((\beta_\lambda \varphi)\upsilon)\theta)s = (\varphi(\upsilon(x/s)))\theta|_Q.$$

This may be paraphrased in terms of the ν-relation $\sqsubseteq_\theta$ extended by (6.1.1):

$$s \sqsubseteq_\theta (\beta_\lambda \varphi)\upsilon \iff (\varphi(\upsilon(x/s)))\theta|_Q = 1.$$

Now $\lambda = \triangledown x$, $x \in X_\epsilon$ and $\nu = \eta x$. Thus, abbreviating $\beta_{\triangledown x}$ to $\triangledown x$ and assuming it to be postpositive as with the abbreviation $\triangledown x$ of $\tau_{\triangledown x}$, we have

$$\operatorname{Dm} \triangledown x = \textstyle\bigcup_{Q \in \mathfrak{P}K_\nu} (\Upsilon_W \to ((Q \rightsquigarrow S) \to \mathbb{T}))$$

and for each $Q \in \mathfrak{P}K_\nu$ and each $\varphi \in \Upsilon_W \to ((Q \rightsquigarrow S) \to \mathbb{T})$, we have

$$\varphi \triangledown x \in \Upsilon_W \to (\Theta_\nu \to (S_\nu \to \mathbb{T}))$$

and the following for each $\upsilon \in \Upsilon_W$, each $\theta \in \Theta_\nu$ and each $s \in S_\nu$:

$$(((\varphi \triangledown x)\upsilon)\theta)s = (\varphi(\upsilon(x/s)))\theta|_Q,$$
$$s \sqsubseteq_\theta (\varphi \triangledown x)\upsilon \iff (\varphi(\upsilon(x/s)))\theta|_Q = 1.$$

6.1.5 The C semantics

Here we define the C semantics $(\mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ for the C language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$ constructed in §6.1.2 and thereby complete the construction of the logic system CL begun in §6.1.1.

As for $\mathfrak{W}$, we first pick a deductive law $\mathcal{L}_\nu$ on the relations on sets as defined in Remark 2.5.6 for each $\nu \in \mathsf{N}$, which we call the **basic ν-law**. Then we define $\mathfrak{W}$ to be the set of the C-worlds for A constructed in §6.1.3 whose basic ν-relations satisfy $\mathcal{L}_\nu$ for each $\nu \in \mathsf{N}$ (s. Remark 4.1.1). Then $\mathfrak{W} \neq \emptyset$, and we let I_W $(W \in \mathfrak{W})$ be the C interpretation of $\Lambda - \mathsf{M}$ on W introduced in §6.1.4. As for $(\Delta_W)_{W\in\mathfrak{W}}$, we first pick an element $\mathsf{c}_\mu \in \mathsf{C}_{\epsilon_\mu}$ for each $\mu \in \mathsf{N}' - \mathsf{N}_\mathsf{d}$ (this is possible because we have assumed $\mathsf{C}_{\epsilon_\mu} \neq \emptyset$), which we call the **μ-constant**. Then we define Δ_W $(W \in \mathfrak{W})$ to be the set of the denotations δ of C into W such that $\delta\mathsf{c}_\mu$ is the μ-default of W_{ϵ_μ} for each $\mu \in \mathsf{N}' - \mathsf{N}_\mathsf{d}$. Notice $\Delta_W \neq \emptyset$.

This completes the definition of the C semantics. Thus it is parameterized by the basic ν-laws $\mathcal{L}_\nu$ $(\nu \in \mathsf{N})$ and the μ-constant $\mathsf{c}_\mu \in \mathsf{C}_{\epsilon_\mu}$ $(\mu \in \mathsf{N}' - \mathsf{N}_\mathsf{d})$.

6.2 Extremal modality and partibility

Let $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma, \mathfrak{W}, (\mathrm{I}_W)_{W\in\mathfrak{W}}, (\Delta_W)_{W\in\mathfrak{W}})$ be the logic system CL constructed in §6.1. Here we consider how extremality of its modality $(\kappa, (\mathsf{N}_\nu)_{\nu\in\mathsf{N}})$ and partibility $\#\mathsf{N}$ defined in §6.1.1 affects on the construction and introduce relevant convention. We place particular emphasis on the construction of $\mathfrak{W}$ and $(\mathrm{I}_W)_{W\in\mathfrak{W}}$ in §6.1.3 and §6.1.4 in view of the slogan *DWs first* of §4.1 and importance of the interpretation of the nominalizers.

6.2.1 Isolated nomina and semisolated nomina

A nomen $\nu \in \mathsf{N}$ is said to be **isolated** if $\mathsf{N}_\nu = \emptyset$. The isolation simplifies the construction of each $W \in \mathfrak{W}$ and I_W in the following way.

First, we have $\mathsf{K}_\nu = \kappa\mathsf{N}_\nu = \emptyset$. Consequently, $\Theta_\nu = \emptyset \rightsquigarrow \mathsf{S} = \emptyset \to \mathsf{S} = \{\emptyset\}$ for the basis $\mathsf{S} = W_\epsilon$ of W (s. Remark 6.1.1 and [3.2]). Therefore, $\sqsubseteq_\emptyset$ is the only basic ν-relation on the ν-basis $\mathsf{S}_\nu = W_{\epsilon_\nu}$ of W, which we denote by $\sqsubseteq_\nu$. We also have $W_{\mathfrak{e}_\nu} = \{\emptyset\} \to (\mathsf{S}_\nu \to \mathbb{T})$, which we identify with $\mathsf{S}_\nu \to \mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]). Then $\mathsf{E}_\nu = W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu} = \mathsf{S}_\nu \amalg (\mathsf{S}_\nu \to \mathbb{T})$, and the extended ν-relation $\sqsubseteq_\nu$ between S_ν and E_ν satisfies the following for each $(\mathsf{s}, \mathsf{a}) \in \mathsf{S}_\nu \times (\mathsf{S}_\nu \to \mathbb{T})$:

$$\mathsf{s} \sqsubseteq_\nu \mathsf{a} \iff \mathsf{a}\mathsf{s} = 1.$$

Furthermore, for each $\theta \in \mathsf{Q} \rightsquigarrow \mathsf{S}$ $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$, the (θ, ν)-relation between S_ν and E_ν is equal to the extended ν-relation $\sqsubseteq_\nu$ irrespective of the μ-defaults $(\mu \in \mathsf{N}')$.

These remarks on $\sqsubseteq_\nu$ are concerned with the operations on W in [W2]–[W4] and [W8]–[W11]. For example, since $\mathsf{K}_\nu = \emptyset$ and $\{\pi\} \rightsquigarrow \mathsf{S} = \{\pi\} \to \mathsf{S}$ by Remark 6.1.1, the image $\mathsf{a}\triangle$ of $\mathsf{a} \in \mathsf{E}_\nu$ by the operation $\triangle$ in [W4] is the element of $W_{\{\pi\}} = (\{\pi\} \to \mathsf{S}) \to \mathbb{T}$ defined by the following for each $\theta \in \{\pi\} \to \mathsf{S}$:

$$(\mathsf{a}\triangle)\theta = 1 \iff \theta\pi \in \mathsf{S}_\nu \text{ and } \theta\pi \sqsubseteq_\nu \mathsf{a}.$$

Now let λ be the token $\triangledown x$ ($x \in X_{\epsilon_\nu}$). Then $\mathfrak{P}K_\nu = \mathfrak{P}\emptyset = \{\emptyset\}$, $W_{\sigma x}{\to}W_\emptyset = W_{\epsilon_\nu}{\to}W_\emptyset = S_\nu{\to}\mathbb{T}$ and $W_{\mathfrak{e}_\nu} = S_\nu{\to}\mathbb{T}$ by the $\{\emptyset\}$ convention. Therefore, the C significance λ_W defined in §6.1.4 is a transformation on $S_\nu{\to}\mathbb{T}$ and satisfies $(\lambda_W f)s = fs$ for each $f \in S_\nu{\to}\mathbb{T}$ and each $s \in S_\nu$, that is, $\lambda_W = \mathrm{id}_{S_\nu\to\mathbb{T}}$.

Semisolated nomina

A nomen $\nu \in N$ is said to be **semisolated** if $\#N_\nu = 1$. The semisolation simplifies the construction of each $W \in \mathfrak{W}$ and I_W in the following way.

Suppose $N_\nu = \{\mu\}$. Then $\mu \in N'$. Let $\varkappa$ denote κ_μ. Then $K_\nu = \{\varkappa\}$ and $\eta\varkappa = \mu$, and so by Remark 6.1.1, $\Theta_\nu = \{\varkappa\}{\rightsquigarrow}S = \{\varkappa\}{\to}S_\mu$ for the basis $S = W_\epsilon$ and the μ-basis $S_\mu = W_{\epsilon_\mu}$ of W. Therefore, as for the set $E_\nu = W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}$ of the ν-entities of W, we identify Θ_ν with S_μ by the $\{\varkappa\}$ convention (s. [3.24] and Remark 6.2.1). Then $W_{\mathfrak{e}_\nu} = S_\mu{\to}(S_\nu{\to}\mathbb{T})$ for the ν-basis $S_\nu = W_{\epsilon_\nu}$ of W, and the basic ν-relations on S_ν constitutes a family $(\sqsubseteq_s)_{s\in S_\mu}$ and are extended to relations between S_ν and $E_\nu = S_\nu \amalg (S_\mu{\to}(S_\nu{\to}\mathbb{T}))$ by the following for each $(s', a') \in S_\nu \times (S_\mu{\to}(S_\nu{\to}\mathbb{T}))$:

$$s' \sqsubseteq_s a' \iff (a's)s' = 1.$$

Furthermore, for each $\theta \in Q{\rightsquigarrow}S$ ($Q \in \mathfrak{P}K$), the (θ,ν)-relation between S_ν and E_ν is equal to the extended ν-relation $\sqsubseteq_{s_\theta}$ for the element $s_\theta \in S_\mu$ defined by the following, where s_μ is the μ-default:

$$s_\theta = \begin{cases} \theta\varkappa & \text{if } \varkappa \in Q, \\ s_\mu & \text{if } \varkappa \notin Q. \end{cases}$$

To be precise, we have identified $\Theta_\nu = \{\varkappa\}{\to}S_\mu$ with S_μ by the bijection c which associates each $\theta \in \Theta_\nu$ with $\theta\varkappa \in S_\mu$ (s. [3.24]). Therefore, we have identified each $s \in S_\mu$ with $\theta = c^{-1}s \in \Theta_\nu$ and denoted the basic ν-relation $\sqsubseteq_\theta$ by $\sqsubseteq_s$. We have also identified each $a' \in S_\mu{\to}(S_\nu{\to}\mathbb{T})$ with the composite $a = a'c \in \Theta_\nu{\to}(S_\nu{\to}\mathbb{T})$ of $c \in \Theta_\nu{\to}S_\mu$ and a'. Since $a\theta = (a'c)(c^{-1}s) = a's$, (6.1.1) shows that $\sqsubseteq_s$ is extended as above. For each $\theta \in Q{\rightsquigarrow}S$ ($Q \in \mathfrak{P}K$), since

$$\theta_\nu\varkappa = \begin{cases} \theta\varkappa & \text{if } \varkappa \in Q, \\ s_\mu & \text{if } \varkappa \notin Q \end{cases}$$

by (6.1.2), we have $\theta_\nu = c^{-1}s_\theta$. Therefore, the (θ,ν)-relation $\sqsubseteq_{\theta_\nu}$ between S_ν and E_ν is equal to the extended relation $\sqsubseteq_{s_\theta}$.

These remarks on $(\sqsubseteq_s)_{s\in S_\mu}$ are concerned with the operations on W in [W2]–[W4] and [W8]–[W11] in §6.1.3. For example, the image $a\triangle$ of $a \in E_\nu$ by the operation $\triangle$ in [W4] is the element of $W_{\{\pi,\varkappa\}} = (\{\pi,\varkappa\}{\rightsquigarrow}S){\to}\mathbb{T}$ defined by the following for each $\theta \in \{\pi,\varkappa\}{\rightsquigarrow}S$:

$$(a\triangle)\theta = 1 \iff \theta\pi \in S_\nu \text{ and } \theta\pi \sqsubseteq_{\theta\varkappa} a.$$

Remark 6.2.1 As for the set $W_{\{\varkappa\}} = (\{\varkappa\} \rightsquigarrow S) \to \mathbb{T}$ of the $\{\varkappa\}$-events of W, we will not identify $\{\varkappa\} \rightsquigarrow S$ with S_μ by the $\{\varkappa\}$ convention.

Now let λ be the token $\triangledown x$ ($x \in X_{\epsilon_\nu}$). Then identifying $S_\nu \to ((Q \rightsquigarrow S) \to \mathbb{T})$ ($Q \in \mathfrak{P}K_\nu$) and $\Theta_\nu \to (S_\nu \to \mathbb{T})$ with $(S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T}$ and $(\Theta_\nu \times S_\nu) \to \mathbb{T}$ respectively by linearization, we have seen in §6.1.4 that the C significance λ_W belongs to $(\bigcup_{Q \in \mathfrak{P}K_\nu} ((S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T})) \to ((\Theta_\nu \times S_\nu) \to \mathbb{T})$ and that its value $\lambda_W f \in (\Theta_\nu \times S_\nu) \to \mathbb{T}$ at each $f \in (S_\nu \times (Q \rightsquigarrow S)) \to \mathbb{T}$ ($Q \in \mathfrak{P}K_\nu$) satisfies $(\lambda_W f)(\theta, s') = f(s', \theta|_Q)$ for each $(\theta, s') \in \Theta_\nu \times S_\nu$; in particular, if $Q = K_\nu$ then $(\lambda_W f)(\theta, s') = f(s', \theta)$, while if $Q = \emptyset$ then $f \in S_\nu \to \mathbb{T}$ and $(\lambda_W f)(\theta, s') = fs'$.

Here $\mathfrak{P}K_\nu = \mathfrak{P}\{\varkappa\} = \{\emptyset, \{\varkappa\}\}$, $\Theta_\nu = \{\varkappa\} \rightsquigarrow S = \{\varkappa\} \to S_\mu$ and we have identified Θ_ν in the definition $W_{\epsilon_\nu} = \Theta_\nu \to (S_\nu \to \mathbb{T})$ with S_μ by the $\{\varkappa\}$ convention but will not identify $\{\varkappa\} \rightsquigarrow S$ in the definition $W_{\{\varkappa\}} = (\{\varkappa\} \rightsquigarrow S) \to \mathbb{T}$ so (s. Remark 6.2.1). Therefore, λ_W belongs to $(((S_\nu \times (\{\varkappa\} \to S_\mu)) \to \mathbb{T}) \cup (S_\nu \to \mathbb{T})) \to ((S_\mu \times S_\nu) \to \mathbb{T})$ and its value $\lambda_W f \in (S_\mu \times S_\nu) \to \mathbb{T}$ at each $f \in ((S_\nu \times (\{\varkappa\} \to S_\mu)) \to \mathbb{T}) \cup (S_\nu \to \mathbb{T})$ satisfies the following for each $(s, s') \in S_\mu \times S_\nu$:

$$(\lambda_W f)(s, s') = \begin{cases} f(s', (\varkappa/s)) & \text{if } f \in (S_\nu \times (\{\varkappa\} \to S_\mu)) \to \mathbb{T}, \\ fs' & \text{if } f \in S_\nu \to \mathbb{T}. \end{cases}$$

Canceling the identification by linearization, we conclude that

$$\lambda_W \in ((S_\nu \to ((\{\varkappa\} \to S_\mu) \to \mathbb{T})) \cup (S_\nu \to \mathbb{T})) \to (S_\mu \to (S_\nu \to \mathbb{T}))$$

and that its value $\lambda_W f \in S_\mu \to (S_\nu \to \mathbb{T})$ at each $f \in (S_\nu \to ((\{\varkappa\} \to S_\mu) \to \mathbb{T})) \cup (S_\nu \to \mathbb{T})$ satisfies the following for each $s \in S_\mu$ and each $s' \in S_\nu$:

$$((\lambda_W f)s)s' = \begin{cases} (fs')(\varkappa/s) & \text{if } f \in S_\nu \to ((\{\varkappa\} \to S_\mu) \to \mathbb{T}), \\ fs' & \text{if } f \in S_\nu \to \mathbb{T}. \end{cases}$$

6.2.2 Discrete case logic and impartible case logic

The logic system CL is said to be **discrete** if all its nomina are isolated. The discreteness has the following consequences in addition to those in §6.2.1.

We may forget about the modality $(\kappa, (N_\nu)_{\nu \in N})$. Indeed, $N' = \bigcup_{\nu \in N} N_\nu = \emptyset$, hence $K' = \bigcup_{\nu \in N} K_\nu = \emptyset$ and $N_k = N$ for all $k \in K$. Therefore, $\mathfrak{Q}' = \breve{\mathfrak{Q}}' = \emptyset$, $\mathfrak{I} = \mathfrak{Q} \times K$ and $\breve{\mathfrak{I}} = \breve{\mathfrak{Q}} \times K$. Consequently, the operations in [T10], [T11], [A10], [A11], [W10] and [W11] do not exist. Furthermore, $T_k = \{\epsilon_\nu : \nu \in N\}$ for all $k \in K$ as to the operations $\mathfrak{o}k$ ($k \in K$) in [T7], [A7] and [W7].

These remarks and those in §6.2.1 show that the construction of the C worlds W in discrete CL is simplified in the following way.

Picking a family $(S_\nu)_{\nu \in N}$ of nonempty sets, we first define

$$W = \coprod_{\nu \in N} (S_\nu \amalg (S_\nu \to \mathbb{T})) \amalg \coprod_{Q \in \mathfrak{P}K} ((Q \to S) \to \mathbb{T}),$$

where $S = \coprod_{\nu \in N} S_\nu$ and $(\emptyset \to S) \to \mathbb{T} = \mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]). Notice that $Q \to S = Q \rightsquigarrow S$ for each $Q \in \mathfrak{P}K$ because $K' = \emptyset$.

As for the partition $W = \coprod_{t \in T} W_t$, we define

$$W_{\epsilon_\nu} = S_\nu, \qquad W_{\mathfrak{e}_\nu} = S_\nu {\to} \mathbb{T}, \qquad W_Q = (Q {\to} S) {\to} \mathbb{T}$$

for each $\nu \in N$ and each $Q \in \mathfrak{P}K$, where $W_\emptyset = \mathbb{T}$. Furthermore, we define

$$E_\nu = W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu} \quad (\nu \in N), \qquad E = \coprod_{\nu \in N} E_\nu, \qquad F = \coprod_{Q \in \mathfrak{P}K} W_Q.$$

The algebraic structure of W is divided into the families defined in [W1)–[W9) below. [W1) is the same as [W1]. For [W2)–[W4), [W8) and [W9), we pick a basic ν-relation $\sqsubseteq_\nu$ on S_ν for each $\nu \in N$ and extend it to a relation between S_ν and $E_\nu = S_\nu \amalg (S_\nu {\to} \mathbb{T})$ by the following for each $(s, a) \in S_\nu \times (S_\nu {\to} \mathbb{T})$:

$$s \sqsubseteq_\nu a \iff as = 1.$$

[W1) An arbitrary family of n_ϕ-ary operations $\phi \in \Phi$ such that if $\nu = \eta\phi$ then $\mathrm{Dm}\,\phi = {W_{\epsilon_\nu}}^{n_\phi}$ and $(a_1, \ldots, a_{n_\phi})\phi \in W_{\epsilon_\nu}$ for all $(a_1, \ldots, a_{n_\phi}) \in {W_{\epsilon_\nu}}^{n_\phi}$.

[W2) The binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dm} \sqcap = \mathrm{Dm} \sqcup = \coprod_{\nu \in N} \coprod_{(t,u) \in \{\epsilon_\nu, \mathfrak{e}_\nu\}^2} (W_t \times W_u) = \coprod_{\nu \in N} {E_\nu}^2$$

and if $(a, b) \in {E_\nu}^2$ $(\nu \in N)$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $W_{\mathfrak{e}_\nu} = S_\nu {\to} \mathbb{T}$ defined by the following for each $s \in S_\nu$:

$$s \sqsubseteq_\nu a \sqcap b \iff s \sqsubseteq_\nu a \text{ and } s \sqsubseteq_\nu b,$$
$$s \sqsubseteq_\nu a \sqcup b \iff s \sqsubseteq_\nu a \text{ or } s \sqsubseteq_\nu b.$$

[W3) The unary operation $\square$ such that

$$\mathrm{Dm}\,\square = \coprod_{\nu \in N} (W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}) = \coprod_{\nu \in N} E_\nu = E$$

and if $a \in E_\nu$ $(\nu \in N)$, then its image $a^\square$ is the element of $W_{\mathfrak{e}_\nu} = S_\nu {\to} \mathbb{T}$ defined by the following for each $s \in S_\nu$:

$$s \sqsubseteq_\nu a^\square \iff s \not\sqsubseteq_\nu a.$$

[W4) The unary operation $\triangle$ such that

$$\mathrm{Dm}\,\triangle = \coprod_{\nu \in N} (W_{\epsilon_\nu} \amalg W_{\mathfrak{e}_\nu}) = \coprod_{\nu \in N} E_\nu = E$$

and if $a \in E_\nu$ $(\nu \in N)$, then its image $a\triangle$ is the element of $W_{\{\pi\}} = (\{\pi\} {\to} S) {\to} \mathbb{T}$ defined by the following for each $\theta \in \{\pi\} {\to} S$:

$$(a\triangle)\theta = 1 \iff \theta\pi \in S_\nu \text{ and } \theta\pi \sqsubseteq_\nu a.$$

[W5) The three binary operations $\wedge$, $\vee$ and $\Rrightarrow$ such that

$$\mathrm{Dm}\wedge = \mathrm{Dm}\vee = \mathrm{Dm}\Rrightarrow = \mathsf{F}^2 = \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(W_Q \times W_R)$$
$$= \coprod_{(Q,R)\in(\mathfrak{P}K)^2}(((Q\to S)\to\mathbb{T}) \times ((R\to S)\to\mathbb{T}))$$

and if $(Q,R) \in (\mathfrak{P}K)^2$ and $(f,g) \in ((Q\to S)\to\mathbb{T}) \times ((R\to S)\to\mathbb{T})$, then its images $f\wedge g$, $f\vee g$ and $f \Rrightarrow g$ are the elements of $W_{Q\cup R} = ((Q\cup R)\to S)\to\mathbb{T}$ defined by the following for each $\theta \in (Q\cup R)\to S$:

$$\begin{aligned}(f\wedge g)\theta &= f(\theta|_Q)\wedge g(\theta|_R),\\ (f\vee g)\theta &= f(\theta|_Q)\vee g(\theta|_R),\\ (f\Rrightarrow g)\theta &= f(\theta|_Q)\Rrightarrow g(\theta|_R).\end{aligned}$$

Here $\wedge$, $\vee$ and $\Rrightarrow$ on the right-hand sides are the meet, join and cojoin (s. §1.5.2) on the binary lattice $\mathbb{T}$.

[W6) The unary operation $\Diamond$ such that

$$\mathrm{Dm}\,\Diamond = \mathsf{F} = \coprod_{Q\in\mathfrak{P}K} W_Q = \coprod_{Q\in\mathfrak{P}K}((Q\to S)\to\mathbb{T})$$

and if $Q \in \mathfrak{P}K$ and $f \in (Q\to S)\to\mathbb{T}$, then its image $f^{\Diamond}$ is the element of $W_Q = (Q\to S)\to\mathbb{T}$ defined by the following for each $\theta \in Q\to S$:

$$(f^{\Diamond})\theta = (f\theta)^{\Diamond}.$$

Here $\Diamond$ on the right-hand side is the complement on the binary lattice $\mathbb{T}$.

[W7) The family of binary operations $\mathfrak{o}k$ $(k \in K)$ such that

$$\mathrm{Dm}\,\mathfrak{o}k = \coprod_{\nu\in N, k\in Q\in\mathfrak{P}K}(W_{\epsilon_\nu} \times W_Q) = S \times \coprod_{k\in Q\in\mathfrak{P}K}((Q\to S)\to\mathbb{T})$$

and if $s \in S$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q\to S)\to\mathbb{T}$, then the image $s\,\mathfrak{o}k\,f$ of (s,f) is the element of $W_{Q-\{k\}} = ((Q-\{k\})\to S)\to\mathbb{T}$ defined by the following for each $\theta \in (Q-\{k\})\to S$:

$$(s\,\mathfrak{o}k\,f)\theta = f((k/s)\theta).$$

[W8) The family of binary operations $\mathfrak{q}k$ $((\mathfrak{q},k) \in \mathfrak{Q}\times K)$ such that if $\nu = \eta\mathfrak{q}$, then

$$\mathrm{Dm}\,\mathfrak{q}k = \coprod_{t\in\{\epsilon_\nu,\mathfrak{e}_\nu\}, k\in Q\in\mathfrak{P}K}(W_t \times W_Q) = E_\nu \times \coprod_{k\in Q\in\mathfrak{P}K}((Q\to S)\to\mathbb{T})$$

and if $a \in E_\nu$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q\to S)\to\mathbb{T}$, then the image $a\,\mathfrak{q}k\,f$ of (a,f) is the element of $W_{Q-\{k\}} = ((Q-\{k\})\to S)\to\mathbb{T}$ defined by the following for each $\theta \in (Q-\{k\})\to S$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}_\nu$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}_\nu$:

$$(a\,\mathfrak{q}k\,f)\theta = 1 \iff |s \in S_\nu : s \sqsubseteq_\nu a,\ f((k/s)\theta) = \nu|_\nu \in \mathfrak{p}.$$

This definition makes sense because $\mathfrak{q} \in \mathfrak{Q}_\nu = \mathfrak{P}_\nu \amalg \neg\mathfrak{P}_\nu$.

[W9] The family of binary operations $\mathfrak{q}\mathsf{k}$ $((\mathfrak{q},\mathsf{k}) \in \breve{\mathfrak{Q}} \times \mathsf{K})$ such that if $\nu = \eta\mathfrak{q}$ then

$$\mathrm{Dm}\,\mathfrak{q}\mathsf{k} = \coprod_{t\in\{\epsilon_\nu,\mathfrak{e}_\nu\},\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}(W_t \times W_\mathsf{Q}) = \mathsf{E}_\nu \times \coprod_{\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\to\mathsf{S})\to\mathbb{T})$$

and if $a \in \mathsf{E}_\nu$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $f \in (\mathsf{Q}\to\mathsf{S})\to\mathbb{T}$, then the image $a\,\mathfrak{q}\mathsf{k}\,f$ of (a, f) is the element of $W_{\mathsf{Q}-\{\mathsf{k}\}} = ((\mathsf{Q}-\{\mathsf{k}\})\to\mathsf{S})\to\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q}-\{\mathsf{k}\})\to\mathsf{S}$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \breve{\mathfrak{P}}_\nu$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\breve{\mathfrak{P}}_\nu$:

$$(a\,\mathfrak{q}\mathsf{k}\,f)\theta = 1 \iff \text{if } |s \in \mathsf{S}_\nu : s \sqsubseteq_\nu a|_\nu \neq 0, \text{ then } \frac{|s \in \mathsf{S}_\nu : s \sqsubseteq_\nu a,\ f((\mathsf{k}/s)\theta) = \nu|_\nu}{|s \in \mathsf{S}_\nu : s \sqsubseteq_\nu a|_\nu} \in \mathfrak{p}.$$

This definition makes sense for a similar reason as in [W8].

Impartible case logic

In case the partibility $\#\mathsf{N}$ is equal to 1, CL is said to be **impartible** and has been called ICL. Obviously, ICL is discrete and so may be obtained from discrete CL by deleting subscripts in N and others.

To be more precise, in addition to the set K of the cases and the nominative case $\pi \in \mathsf{K}$, we pick a nontrivial quantitative set $\mathbb{P}$, a subset $\mathfrak{P}$ of the power set $\mathfrak{P}\mathbb{P}$ of $\mathbb{P}$ and a subset $\breve{\mathfrak{P}}$ of the power set $\mathfrak{P}[0,1]$ of the interval $[0,1]$ of real numbers in case $\mathbb{P}$ is unital, while we assume $\breve{\mathfrak{P}} = \emptyset$ in case $\mathbb{P}$ is not unital. Then we define $\mathfrak{Q} = \mathfrak{P} \amalg \neg\mathfrak{P}$ and $\breve{\mathfrak{Q}} = \breve{\mathfrak{P}} \amalg \neg\breve{\mathfrak{P}}$ by the copies $\neg\mathfrak{P}$ and $\neg\breve{\mathfrak{P}}$ of $\mathfrak{P}$ and $\breve{\mathfrak{P}}$ respectively by the symbol $\neg$. As for its C language $(\mathsf{A}, \mathsf{T}, \sigma, \mathsf{P}, \mathsf{C}, \mathsf{X}, \Gamma)$, we let $\mathsf{T} = \{\epsilon, \mathfrak{e}\} \amalg \mathfrak{P}\mathsf{K}$, picking the symbols ϵ and $\mathfrak{e}$. Finally, the construction of the C worlds W in ICL is as follows.

Picking a nonempty set S, we first define

$$W = \mathsf{S} \amalg (\mathsf{S}\to\mathbb{T}) \amalg \coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}\to\mathsf{S})\to\mathbb{T}),$$

where $(\emptyset\to\mathsf{S})\to\mathbb{T} = \mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]).

As for the partition $W = \coprod_{t\in\mathsf{T}} W_t$ for $\mathsf{T} = \{\epsilon, \mathfrak{e}\} \amalg \mathfrak{P}\mathsf{K}$, we define

$$W_\epsilon = \mathsf{S}, \qquad W_\mathfrak{e} = \mathsf{S}\to\mathbb{T}, \qquad W_\mathsf{Q} = (\mathsf{Q}\to\mathsf{S})\to\mathbb{T} \quad (\mathsf{Q} \in \mathfrak{P}\mathsf{K}),$$

where $W_\emptyset = \mathbb{T}$. Furthermore, we define

$$\mathsf{E} = W_\epsilon \amalg W_\mathfrak{e}, \qquad \mathsf{F} = \coprod_{\mathsf{Q}\in\mathfrak{P}\mathsf{K}} W_\mathsf{Q}.$$

The algebraic structure of W is divided into the families defined in (W1)–(W9) below. They are obtained from [W1]–[W9] by deleting subscript ν, expressions about it such as $\coprod_{\nu\in\mathsf{N}}$ and "$\nu \in \mathsf{N}$" and the redundant condition $\theta\pi \in \mathsf{S}$ in (W4). Therefore, (W5) and (W6) are the same as [W5] and [W6] respectively. Furthermore, for (W2)–(W4), (W8) and (W9), we pick a basic relation $\sqsubseteq$ on S and extend it to a relation between S and $\mathsf{E} = \mathsf{S} \amalg (\mathsf{S}{\to}\mathbb{T})$ by the following for each $(s, a) \in \mathsf{S} \times (\mathsf{S}{\to}\mathbb{T})$:

$$s \sqsubseteq a \iff as = 1.$$

Moreover, for (W8) and (W9), we pick a $\mathbb{P}$-measure $\mathsf{Y} \mapsto |\mathsf{Y}|$ on S. You will see that (W1)–(W8) here are the same as (W1)–(W8) in §4.1.1 (s. Remark 4.1.2) and so we have implicitly assumed there that $\breve{\mathfrak{P}} = \emptyset$ even if $\mathbb{P}$ is unital.

(W1) An arbitrary family of n_ϕ-ary operations $\phi \in \Phi$ such that $\mathrm{Dm}\,\phi = {W_\epsilon}^{n_\phi}$ and $(a_1, \ldots, a_{n_\phi})\phi \in W_\epsilon$ for all $(a_1, \ldots, a_{n_\phi}) \in {W_\epsilon}^{n_\phi}$.

(W2) The binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dm} \sqcap = \mathrm{Dm} \sqcup = \coprod_{(t,u)\in\{\epsilon,\mathfrak{e}\}^2} (W_t \times W_u) = \mathsf{E}^2$$

and if $(a, b) \in \mathsf{E}^2$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $W_\mathfrak{e} = \mathsf{S}{\to}\mathbb{T}$ defined by the following for each $s \in \mathsf{S}$:

$$s \sqsubseteq a \sqcap b \iff s \sqsubseteq a \text{ and } s \sqsubseteq b,$$
$$s \sqsubseteq a \sqcup b \iff s \sqsubseteq a \text{ or } s \sqsubseteq b.$$

(W3) The unary operation $\Box$ such that

$$\mathrm{Dm}\,\Box = W_\epsilon \amalg W_\mathfrak{e} = \mathsf{E}$$

and if $a \in \mathsf{E}$, then its image $a^\Box$ is the element of $W_\mathfrak{e} = \mathsf{S}{\to}\mathbb{T}$ defined by the following for each $s \in \mathsf{S}$:

$$s \sqsubseteq a^\Box \iff s \not\sqsubseteq a.$$

(W4) The unary operation $\triangle$ such that

$$\mathrm{Dm}\,\triangle = W_\epsilon \amalg W_\mathfrak{e} = \mathsf{E}$$

and if $a \in \mathsf{E}$, then its image $a\triangle$ is the element of $W_{\{\pi\}} = (\{\pi\}{\to}\mathsf{S}){\to}\mathbb{T}$ defined by the following for each $\theta \in \{\pi\}{\to}\mathsf{S}$:

$$(a\triangle)\theta = 1 \iff \theta\pi \sqsubseteq a.$$

(W5) The three binary operations $\wedge$, $\vee$ and $\Rightarrow$ such that

$$\begin{aligned}\mathrm{Dm} \wedge = \mathrm{Dm} \vee = \mathrm{Dm} \Rightarrow = \mathsf{F}^2 &= \textstyle\coprod_{(\mathsf{Q},\mathsf{R})\in(\mathfrak{P}\mathsf{K})^2} (W_\mathsf{Q} \times W_\mathsf{R}) \\ &= \textstyle\coprod_{(\mathsf{Q},\mathsf{R})\in(\mathfrak{P}\mathsf{K})^2} (((\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T}) \times ((\mathsf{R}{\to}\mathsf{S}){\to}\mathbb{T}))\end{aligned}$$

and if $(\mathsf{Q}, \mathsf{R}) \in (\mathfrak{P}\mathsf{K})^2$ and $(f, g) \in ((\mathsf{Q}{\to}\mathsf{S}){\to}\mathbb{T}) \times ((\mathsf{R}{\to}\mathsf{S}){\to}\mathbb{T})$, then its images $f \wedge g$, $f \vee g$ and $f \Rightarrow g$ are the elements of $W_{\mathsf{Q}\cup\mathsf{R}} = ((\mathsf{Q}\cup\mathsf{R}){\to}\mathsf{S}){\to}\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q} \cup \mathsf{R}){\to}\mathsf{S}$:

$$(f \wedge g)\theta = f(\theta|_\mathsf{Q}) \wedge g(\theta|_\mathsf{R}),$$

$$\begin{aligned}(f \vee g)\theta &= f(\theta|_Q) \vee g(\theta|_R),\\ (f \Rrightarrow g)\theta &= f(\theta|_Q) \Rrightarrow g(\theta|_R).\end{aligned}$$

Here $\wedge$, $\vee$ and $\Rrightarrow$ on the right-hand sides are the meet, join and cojoin on the binary lattice $\mathbb{T}$.

(W6) The unary operation $\Diamond$ such that

$$\mathrm{Dm}\,\Diamond = \mathsf{F} = \coprod_{Q \in \mathfrak{P}K} W_Q = \coprod_{Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T})$$

and if $Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then its image $f^{\Diamond}$ is the element of $W_Q = (Q \to S) \to \mathbb{T}$ defined by the following for each $\theta \in Q \to S$:

$$(f^{\Diamond})\theta = (f\theta)^{\Diamond}.$$

Here $\Diamond$ on the right-hand side is the complement on the binary lattice $\mathbb{T}$.

(W7) The family of binary operations $\mathfrak{o}k$ $(k \in K)$ such that

$$\mathrm{Dm}\,\mathfrak{o}k = \coprod_{k \in Q \in \mathfrak{P}K}(W_\epsilon \times W_Q) = S \times \coprod_{k \in Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T})$$

and if $s \in S$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then the image $s\,\mathfrak{o}k\,f$ of (s, f) is the element of $W_{Q-\{k\}} = ((Q - \{k\}) \to S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q - \{k\}) \to S$:

$$(s\,\mathfrak{o}k\,f)\theta = f((k/s)\theta).$$

(W8) The family of binary operations $\mathfrak{q}k$ $((\mathfrak{q}, k) \in \mathfrak{Q} \times K)$ such that

$$\mathrm{Dm}\,\mathfrak{q}k = \coprod_{t \in \{\epsilon, \mathfrak{e}\}, k \in Q \in \mathfrak{P}K}(W_t \times W_Q) = E \times \coprod_{k \in Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T})$$

and if $a \in E$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then the image $a\,\mathfrak{q}k\,f$ of (a, f) is the element of $W_{Q-\{k\}} = ((Q - \{k\}) \to S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q - \{k\}) \to S$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}$:

$$(a\,\mathfrak{q}k\,f)\theta = 1 \iff |s \in S : s \sqsubseteq a, f((k/s)\theta) = \nu| \in \mathfrak{p}.$$

This definition makes sense because $\mathfrak{q} \in \mathfrak{Q} = \mathfrak{P} \amalg \neg\mathfrak{P}$.

(W9) The family of binary operations $\mathfrak{q}k$ $((\mathfrak{q}, k) \in \breve{\mathfrak{Q}} \times K)$ such that

$$\mathrm{Dm}\,\mathfrak{q}k = \coprod_{t \in \{\epsilon, \mathfrak{e}\}, k \in Q \in \mathfrak{P}K}(W_t \times W_Q) = E \times \coprod_{k \in Q \in \mathfrak{P}K}((Q \to S) \to \mathbb{T})$$

and if $a \in E$, $k \in Q \in \mathfrak{P}K$ and $f \in (Q \to S) \to \mathbb{T}$, then the image $a\,\mathfrak{q}k\,f$ of (a, f) is the element of $W_{Q-\{k\}} = ((Q - \{k\}) \to S) \to \mathbb{T}$ defined by the following for each $\theta \in (Q - \{k\}) \to S$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \breve{\mathfrak{P}}$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\breve{\mathfrak{P}}$:

$$(a\,\mathfrak{q}k\,f)\theta = 1 \iff \text{if } |s \in S : s \sqsubseteq a| \neq 0, \text{ then } \frac{|s \in S : s \sqsubseteq a, f((k/s)\theta) = \nu|}{|s \in S : s \sqsubseteq a|} \in \mathfrak{p}.$$

This definition makes sense for a similar reason as in (W8).

6.2.3 Bipartible case logic

In case the partibility $\#\mathsf{N}$ is equal to 2, CL is said to be **bipartible** and has been called BCL. For BCL, we assume

$$\mathsf{N} = \{\mathsf{o}, \iota\}$$

without loss of generality. Then if $\#\mathsf{N}_\mathsf{o} = \#\mathsf{N}_\iota$, it is said to be **symmetric**. Discrete BCL is symmetric because $\mathsf{N}_\mathsf{o} = \mathsf{N}_\iota = \emptyset$. Indiscrete symmetric BCL satisfies $\mathsf{N}_\mathsf{o} = \{\iota\}$ and $\mathsf{N}_\iota = \{\mathsf{o}\}$, that is, o and ι are bound by the underlying relation on N (s. [6.1]), and so is also called **bound BCL**.

Here we concentrate on asymmetric BCL (ABCL), because it is a mixture of bound BCL and discrete BCL and so also indicative of the nature of the mixed two. Without loss of generality, we assume $\#\mathsf{N}_\mathsf{o} > \#\mathsf{N}_\iota$. Then

$$\mathsf{N}_\mathsf{o} = \{\iota\}, \qquad\qquad \mathsf{N}_\iota = \emptyset,$$

that is, o is semisolated and ι is isolated. This extremality particularly has the following consequences in addition to those shown in §6.2.1.

We first have $\mathsf{N}' = \{\iota\}$. Let $\varkappa$ denote κ_ι. Then $\mathsf{K}' = \{\varkappa\}$ and $\eta\varkappa = \iota$. Moreover, $\mathfrak{Q}' = \mathfrak{Q}_\iota$ and $\mathcal{J} = (\mathfrak{Q} \times (\mathsf{K} - \{\varkappa\})) \amalg (\mathfrak{Q}_\iota \times \{\varkappa\})$, and since $\mathfrak{Q} = \mathfrak{Q}_\mathsf{o} \amalg \mathfrak{Q}_\iota$,

$$\mathcal{J} = (\mathfrak{Q}_\mathsf{o} \times (\mathsf{K} - \{\varkappa\})) \amalg (\mathfrak{Q}_\iota \times \mathsf{K}).$$

These remarks and those in §6.2.1 show that the construction of the C worlds W in ABCL has the following features.

Picking a pair $(\mathsf{S}_\mathsf{o}, \mathsf{S}_\iota)$ of nonempty sets, we first define

$$W = \mathsf{S}_\mathsf{o} \amalg (\mathsf{S}_\iota \to (\mathsf{S}_\mathsf{o} \to \mathbb{T})) \amalg \mathsf{S}_\iota \amalg (\mathsf{S}_\iota \to \mathbb{T}) \amalg \textstyle\coprod_{\mathsf{Q} \in \mathfrak{P}\mathsf{K}} ((\mathsf{Q} \rightsquigarrow \mathsf{S}) \to \mathbb{T}),$$

where

$$\mathsf{S} = \mathsf{S}_\mathsf{o} \amalg \mathsf{S}_\iota, \qquad \mathsf{Q} \rightsquigarrow \mathsf{S} = \{\theta \in \mathsf{Q} \to \mathsf{S} : \theta\varkappa \in \mathsf{S}_\iota \text{ if } \varkappa \in \mathsf{Q}\} \quad (\mathsf{Q} \in \mathfrak{P}\mathsf{K})$$

and we identify $(\emptyset \rightsquigarrow \mathsf{S}) \to \mathbb{T}$ with $\mathbb{T}$ by the $\{\emptyset\}$ convention (s. [3.24]).

As for the partition $W = \coprod_{\mathsf{t} \in \mathsf{T}} W_\mathsf{t}$ for $\mathsf{T} = \{\epsilon_\mathsf{o}, \mathfrak{e}_\mathsf{o}, \epsilon_\iota, \mathfrak{e}_\iota\} \amalg \mathfrak{P}\mathsf{K}$, we define

$$\begin{aligned} &W_{\epsilon_\mathsf{o}} = \mathsf{S}_\mathsf{o}, && W_{\mathfrak{e}_\mathsf{o}} = \mathsf{S}_\iota \to (\mathsf{S}_\mathsf{o} \to \mathbb{T}), && \\ &W_{\epsilon_\iota} = \mathsf{S}_\iota, && W_{\mathfrak{e}_\iota} = \mathsf{S}_\iota \to \mathbb{T}, && W_\mathsf{Q} = (\mathsf{Q} \rightsquigarrow \mathsf{S}) \to \mathbb{T} \quad (\mathsf{Q} \in \mathfrak{P}\mathsf{K}), \end{aligned}$$

and so $W_\emptyset = \mathbb{T}$ by the above identification.

Furthermore, we define

$$\begin{aligned} &\mathsf{E} = W_{\epsilon_\mathsf{o}} \amalg W_{\mathfrak{e}_\mathsf{o}} \amalg W_{\epsilon_\iota} \amalg W_{\mathfrak{e}_\iota}, \\ &\mathsf{E}_\mathsf{o} = W_{\epsilon_\mathsf{o}} \amalg W_{\mathfrak{e}_\mathsf{o}}, \quad \mathsf{E}_\iota = W_{\epsilon_\iota} \amalg W_{\mathfrak{e}_\iota}, \\ &W_\epsilon = W_{\epsilon_\mathsf{o}} \amalg W_{\epsilon_\iota}, \quad W_\mathfrak{e} = W_{\mathfrak{e}_\mathsf{o}} \amalg W_{\mathfrak{e}_\iota}, \qquad \mathsf{F} = \textstyle\coprod_{\mathsf{Q} \in \mathfrak{P}\mathsf{K}} W_\mathsf{Q}, \end{aligned}$$

so that $W = \mathsf{E} \amalg \mathsf{F}$, $\mathsf{E} = \mathsf{E}_\mathsf{o} \amalg \mathsf{E}_\iota = W_\epsilon \amalg W_\mathfrak{e}$ and $W_\epsilon = \mathsf{S}$.

The algebraic structure of W is divided into the families ⌈W1⌋–⌈W11⌋ derived from the families [W1]–[W11] in §6.1.3.

For ⌈W2⌋–⌈W4⌋ and ⌈W8⌋–⌈W11⌋, we pick a family $(\sqsubseteq_s)_{s\in S_\iota}$ of relations on S_o and a relation $\sqsubseteq$ on S_ι. Then we extend $\sqsubseteq_s$ for each $s \in S_\iota$ to a relation between S_o and $E_o = S_o \amalg (S_\iota{\to}(S_o{\to}\mathbb{T}))$ by

$$s' \sqsubseteq_s a \iff (as)s' = 1$$

for each $(s', a) \in S_o \times (S_\iota{\to}(S_o{\to}\mathbb{T}))$ and extend $\sqsubseteq$ to a relation between S_ι and $E_\iota = S_\iota \amalg (S_\iota{\to}\mathbb{T})$ by the following for each $(s, a) \in S_\iota \times (S_\iota{\to}\mathbb{T})$:

$$s \sqsubseteq a \iff as = 1.$$

⌈W2⌋ consists of the binary operations $\sqcap$ and $\sqcup$ such that

$$\mathrm{Dm}\,\sqcap = \mathrm{Dm}\,\sqcup = {E_o}^2 \amalg {E_\iota}^2$$

and if $(a, b) \in {E_o}^2$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $S_\iota{\to}(S_o{\to}\mathbb{T})$ defined by

$$s' \sqsubseteq_s a \sqcap b \iff s' \sqsubseteq_s a \text{ and } s' \sqsubseteq_s b,$$
$$s' \sqsubseteq_s a \sqcup b \iff s' \sqsubseteq_s a \text{ or } s' \sqsubseteq_s b$$

for each $s \in S_\iota$ and each $s' \in S_o$, while if $(a, b) \in {E_\iota}^2$, then its images $a \sqcap b$ and $a \sqcup b$ are the elements of $S_\iota{\to}\mathbb{T}$ defined by the following for each $s \in S_\iota$:

$$s \sqsubseteq a \sqcap b \iff s \sqsubseteq a \text{ and } s \sqsubseteq b,$$
$$s \sqsubseteq a \sqcup b \iff s \sqsubseteq a \text{ or } s \sqsubseteq b.$$

⌈W3⌋ consists of the single unary operation $\square$ such that

$$\mathrm{Dm}\,\square = E = E_o \amalg E_\iota$$

and if $a \in E_o$, then its image $a^\square$ is the element of $S_\iota{\to}(S_o{\to}\mathbb{T})$ defined by

$$s' \sqsubseteq_s a^\square \iff s' \not\sqsubseteq_s a$$

for each $s \in S_\iota$ and each $s' \in S_o$, while if $a \in E_\iota$, then its image $a^\square$ is the element of $S_\iota{\to}\mathbb{T}$ defined by the following for each $s \in S_\iota$:

$$s \sqsubseteq a^\square \iff s \not\sqsubseteq a.$$

⌈W4⌋ consists of the single unary operation $\triangle$ such that

$$\mathrm{Dm}\,\triangle = E = E_o \amalg E_\iota$$

and if $a \in E_o$, then its image $a\triangle$ is the element of $(\{\pi, \varkappa\}{\rightsquigarrow}S){\to}\mathbb{T}$ defined by

$$(a\triangle)\theta = 1 \iff \theta\pi \in S_o \text{ and } \theta\pi \sqsubseteq_{\theta\varkappa} a$$

for each $\theta \in \{\pi, \varkappa\}{\rightsquigarrow}\mathsf{S}$, while if $\mathfrak{a} \in \mathsf{E}_\iota$, then its image $\mathfrak{a}\triangle$ is the element of $(\{\pi\}{\rightarrow}\mathsf{S}){\rightarrow}\mathbb{T}$ defined by the following for each $\theta \in \{\pi\}{\rightarrow}\mathsf{S}$:

$$(\mathfrak{a}\triangle)\theta = 1 \iff \theta\pi \in \mathsf{S}_\iota \text{ and } \theta\pi \sqsubseteq \mathfrak{a}.$$

For $\lceil$W8$\rfloor$–$\lceil$W11$\rfloor$, we furthermore pick an element $\mathsf{s}_\iota \in \mathsf{S}_\iota$ as the ι-default. Then for each $\theta \in \mathsf{Q}{\rightsquigarrow}\mathsf{S}$ $(\mathsf{Q} \in \mathfrak{P}\mathsf{K})$, we let s_θ be the element of S_ι defined by

$$\mathsf{s}_\theta = \begin{cases} \theta\varkappa & \text{if } \varkappa \in \mathsf{Q}, \\ \mathsf{s}_\iota & \text{if } \varkappa \notin \mathsf{Q}. \end{cases}$$

$\lceil$W8$\rfloor$ consists of the binary operations $\mathfrak{q}\mathsf{k}$ $((\mathfrak{q}, \mathsf{k}) \in (\mathfrak{Q}_o \times (\mathsf{K}-\{\varkappa\}))\amalg(\mathfrak{Q}_\iota \times \mathsf{K}))$ such that if $\mathfrak{q} \in \mathfrak{Q}_\nu$ $(\nu \in \mathsf{N})$, then

$$\mathrm{Dm}\,\mathfrak{q}\mathsf{k} = \mathsf{E}_\nu \times \coprod_{\mathsf{k}\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}{\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T})$$

and if $\mathfrak{a} \in \mathsf{E}_\nu$, $\mathsf{k} \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}{\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T}$, then the image $\mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}$ of $(\mathfrak{a}, \mathsf{f})$ is the element of $((\mathsf{Q}-\{\mathrm{k}\}){\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T}$ defined by the following for each $\theta \in (\mathsf{Q}-\{\mathrm{k}\}){\rightsquigarrow}\mathsf{S}$, where $\nu = 1$ or $\nu = 0$ according as $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}_\nu$ or $\mathfrak{q} = \neg\mathfrak{p} \in \neg\mathfrak{P}_\nu$:

$$(\mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f})\theta = 1 \iff \begin{cases} |\mathsf{s} \in \mathsf{S}_o : \mathsf{s} \sqsubseteq_{\mathsf{s}_\theta} \mathfrak{a},\ \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = \nu|_o \in \mathfrak{p} & \text{if } \nu = o, \\ |\mathsf{s} \in \mathsf{S}_\iota : \mathsf{s} \sqsubseteq \mathfrak{a},\ \mathsf{f}((\mathsf{k}/\mathsf{s})\theta) = \nu|_\iota \in \mathfrak{p} & \text{if } \nu = \iota. \end{cases}$$

$\lceil$W10$\rfloor$ consists of the unary operations $\natural\mathfrak{q}$ $(\mathfrak{q} \in \mathfrak{Q}_\iota)$ such that

$$\mathrm{Dm}\,\natural\mathfrak{q} = \coprod_{\varkappa\in\mathsf{Q}\in\mathfrak{P}\mathsf{K}}((\mathsf{Q}{\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T})$$

and if $\varkappa \in \mathsf{Q} \in \mathfrak{P}\mathsf{K}$ and $\mathsf{f} \in (\mathsf{Q}{\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T}$, then the image $\mathsf{f}\,\natural\mathfrak{q}$ of f is the element of $(\mathsf{Q}{\rightsquigarrow}\mathsf{S}){\rightarrow}\mathbb{T}$ defined by

$$(\mathsf{f}\,\natural\mathfrak{q})\theta = ((\theta\varkappa)\,\mathfrak{q}\varkappa\,\mathsf{f})\theta|_{\mathsf{Q}-\{\varkappa\}},$$

for each $\theta \in \mathsf{Q}{\rightsquigarrow}\mathsf{S}$, where $\mathfrak{q}\varkappa$ is the operation defined in $\lceil$W8$\rfloor$.

Chapter 6 to be continued (s. Remark 1.4.1).