Case Logic for Mathematical Psychology II

Kensaku Gomi*

Abstract The purpose of this Part II is to generalize MPCL (monophasic case logic) defined in Part I to CL (case logic) of arbitrary number of phases. The generalization was necessitated because MPCL could deal with neither the notion involved with the conjunction "that" nor the copular or existential notion modified by the notion of location in space or time. The defects are removed in PPCL (polyphasic case logic) by introducing a degenerate phase and by the parameterized basic relations between entities. The parameterization also enables us to equip PPCL with modal operations indexed by modal quantifiers. Certain extremal modal operations play the roles of the modal operations in (poly)modal logic, and consequently (poly)modal logic is embedded in PPCL as a fragment.

Keywords case logic, generalized quantifier, intensional operation, logical system, logical space, mathematical psychology, modality, semantics, syntax, tautology

1 Introduction

The purpose of this Part II is to introduce the logical system CL (case logic) designed for MP (mathematical psychology). It has a parameter called the set of the phases, and MPCL (monophasic case logic) defined in Part I without the notion of phases will turn out to be equivalent to the CL with only one phase. In this sense, CL is a generalization of MPCL, and so familiarity with MPCL will facilitate understanding CL which is sophisticated.

The generalization of MPCL to CL was necessitated because MPCL had certain defects. First, it could not deal with notions like "Mr. McGregor knows that Peter eats the beans" involved with the conjunction "that" used to introduce a noun clause giving what has been perceived, cognized, said, etc (cf. §2.6 of Part I). Secondly, it could not deal with notions like "Peter was a student last year" involved with the copula "be" modified by the words "last year" which indicate an occasion (cf. §1.7 of Part I) or notions like "John and Mary are present at the hall" involved with the existential predicate "be present" modified by the words "at the hall" which indicate a location (cf. §2.6 of Part I).

As for the first defect, we should take notice that "that" also works as a relative pronoun as in "Mr. McGregor knows the beans that Peter eats." In §2.6

*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914 Japan. email: mathpsy@mb.infoweb.ne.jp

of Part I, we saw that the nominalizers Ωx ($x \in \mathbb{X}_\varepsilon$) translated into equivalents of the relative pronoun "that." Therefore it seemed nice that some peculiar nominalizers could translate into the conjunction "that." But it was impossible in MPCL, because MPCL had only one phase and so $\mathbb{X}_\varepsilon$ could have no peculiar elements. Thus the first defect will be removed in PPCL (polyphasic case logic) of at least two phases by making one of them peculiar. The notion of *degeneracy* to be defined in §**??** is intended as the peculiarity.

The second defect was caused by the definition of the algebraic structure of the MPC worlds W. It was determined by two parameters, one of which was a basic relation $\ni$ on the base S of W, and the relation $a \ni b$ between elements $a, b \in S$ was intended to mean the copular notion "b is a," or "b belongs to a" between the entities a and b. Since $\ni$ was a constant relation, however, $a \ni b$ could mean neither "b is a on c" for any occasion c nor "b is a at c" for any location c. Therefore the second defect will be removed by replacing the single basic relation $\ni$ by a family $(\ni_\theta)_{\theta\in\Theta}$ of basic relations indexed by the direct product $\Theta = S_o \times S_\lambda$ of the sets S_o and S_λ consisting of occasions and locations respectively. This idea is essentially due to Takaoka (06).

Takaoka's idea has brought about unexpected byproducts. It enables us to equip PPCL with modal operations $\Lambda\mathfrak{x}$ indexed by the modal quantifiers $\mathfrak{x}$, among which are the extremal modal quantifiers $\forall_\mu$ and $\exists^\mu$ indexed by the modal phases μ. It turned out that the extremal modal operations $\Lambda\forall_\mu$ and $\Lambda\exists^\mu$ can play the roles of the operations $\Box_n$ and $\Diamond_n$ in the polymodal logic. Indeed, Gomi (10a) has shown how the usual modal logic under Kripke semantics is embedded in ADPCL (asymmetric diphasic case logic), with the pair $(S_\mu, \ni_\mu)$ of the μ-base S_μ and the basic μ-relation $\ni_\mu$ for a unique modal phase μ playing the role of the Kripke frame. Interested readers might extend the embedding to that of polymodal logic in PPCL.

Even though modal logic seems to have been regarded as important and the modal operations $\Lambda\mathfrak{x}$ of PPCL are necessary for embedding (poly)modal logic in PPCL, they do not make their presence felt within PPCL. The equation

$$s\,\breve{o}\varpi_\mu\,(f\Lambda\mathfrak{x}) = s\,\mathfrak{x}\varpi_\mu\,f \tag{1.1}$$

in Theorem **??** even suggests that $\Lambda\mathfrak{x}$ can be disregarded. Indeed, because of (1.1), they neither play roles nor causes difficulties in the proof of a completeness theorem for CL, while the operations $\breve{o}\varpi_\mu$ and $\mathfrak{x}\varpi_\mu$ play roles and raise challenging questions there.

(1.1) also shows how the modal operations $\Lambda\mathfrak{x}$ can be interpreted by means of the operations $\breve{o}\varpi_\mu$ and $\mathfrak{x}\varpi_\mu$. The semantics of CL is designed so that $s\,\exists^\mu\varpi_\mu\,f$ may be interpreted as "The event f happens in some of the situations which belong to the situation s." Also $s\,\breve{o}\varpi_\mu\,g$ may be interpreted as "The event g happens in the situation s." Therefore, (1.1) with $\mathfrak{x} = \exists^\mu$ means that the proposition "The event $f\Lambda\exists^\mu$ happens in the situation s" is equivalent to the proposition "The event f happens in *some* of the situations which belong to s." The meaning of (1.1) with $\mathfrak{x} = \forall_\mu$ is similar with "some" replaced by "any." Thus, $\Lambda\exists^\mu$ and $\Lambda\forall_\mu$ may be regarded as showing, among other things, the so-called logical possibility and logical necessity respectively.

Just as there are various systems in Kripkean modal logic according to the laws on accessibility relations between possible worlds, CL may be of various *relational types* according to the laws on basic relations between entities. Relational types are, however, connected with tautologies on the operations $\breve{o}\pi$ and $\triangle$ not on the modal operations. For instance, the reflexivity, symmetricity and transitivity of the basic relations are identical to the validity of the sequents

$$
\begin{aligned}
&\to \mathfrak{a}\,\breve{o}\pi\,\mathfrak{a}\triangle,\\
\mathfrak{a}\,\breve{o}\pi\,\mathfrak{b}\triangle &\to \mathfrak{b}\,\breve{o}\pi\,\mathfrak{a}\triangle,\\
\mathfrak{a}\,\breve{o}\pi\,\mathfrak{b}\triangle, \mathfrak{b}\,\breve{o}\pi\,\mathfrak{c}\triangle &\to \mathfrak{a}\,\breve{o}\pi\,\mathfrak{c}\triangle,
\end{aligned}
$$

respectively, where $\mathfrak{d}\,\breve{o}\pi\,\mathfrak{e}\triangle$ may be interpreted as "$\mathfrak{d}$ is $\mathfrak{e}$" for all $\mathfrak{d}, \mathfrak{e}$.

The organization of this Part II is similar to that of Part I. §**??** contains the definition of CL and its illustrative translations into the Japanese language. §**??** collects those tautologies in CL which have proved to be valuable for the study of deduction systems on CL and for the embedding of (poly)modal logic in PPCL.