From Graphs to Algebras

From What We See to What We Penetrate

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Preface

Graphs with labels, orientations and weights are generally regarded as visual representations of relatively simple algebraic systems. In contrast, algebra provides a means of handling all such "graphs," including those that lie beyond the reach of human visual intuition. This note explores the correspondence between graphs and algebraic systems, and suggests a way to penetrate invisible structures through algebraic methods. A precise and thorough account of a generalized correspondence, i.e. one between sorted graphs and sorted algebras, will be given in Section 3.1 (Sorted algebras) of [1], where machines are also discussed algebraically.

From algebras to graphs and the way home

For each set A, a (partial) **operation** on A is a mapping α of a subset D of the direct product A^k into A for a positive integer k. The set D is called the **domain** of α and denoted by $\mathrm{Dm}\,\alpha$. The positive integer k is called an **arity** of α. If D is nonempty, α has a unique arity.

An **algebra** (or **algebraic system**) is a set A equipped with an **algebraic structure**, here defined to be a family $(\alpha_\lambda)_{\lambda\in\Lambda}$ of operations α_λ on A indexed by a set Λ. Thus an algebra is best described by the pair $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ of a set and an algebraic structure on it. If $\mathrm{Dm}\,\alpha_\lambda$ is nonempty, the unique arity of α_λ will often be denoted by k_λ. Another algebra $(B, (\beta_\lambda)_{\lambda\in\Lambda})$ indexed by the same set Λ is said to be **equivalent** to $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ if $\mathrm{Dm}\,\beta_\lambda = \mathrm{Dm}\,\alpha_\lambda$ for each $\lambda \in \Lambda$ and $\beta_\lambda(b_1, \dots, b_{k_\lambda}) = \alpha_\lambda(b_1, \dots, b_{k_\lambda})$ for each $(b_1, \dots, b_{k_\lambda}) \in \mathrm{Dm}\,\beta_\lambda$.

*Led by the given name against Japanese tradition.

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The algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ yields a **set-theoretic graph**, denoted by $\mathcal{G}_A$, in the following way. For each $\lambda \in \Lambda$ and each $(a_1, \ldots, a_{k_\lambda}) \in \mathrm{Dm}\,\alpha_\lambda$, the triplet

$$\lambda \quad a_1 - \cdots \rightarrow a_{k_\lambda} \quad \alpha_\lambda(a_1, \ldots, a_{k_\lambda})$$

of λ, the symbol $a_1 - \cdots \rightarrow a_{k_\lambda}$ and $\alpha_\lambda(a_1, \ldots, a_{k_\lambda})$ is called an **lwa** (labeled weighted arrow) with **label** λ, **arrow** $a_1 - \cdots \rightarrow a_{k_\lambda}$ and **weight** $\alpha_\lambda(a_1, \ldots, a_{k_\lambda})$. Notice that the arrow satisfies the following:

$$a_1 - \cdots \rightarrow a_{k_\lambda} = \begin{cases} a_1 & \text{if } k_\lambda = 1, \\ a_1 \rightarrow a_2 & \text{if } k_\lambda = 2, \\ a_1 - \cdots - a_{k_\lambda - 1} \rightarrow a_{k_\lambda} & \text{if } k_\lambda > 2. \end{cases}$$

The set-theoretic graph $\mathcal{G}_A$ is defined to be the totality of the arrows for all $\lambda \in \Lambda$ and all $(a_1, \ldots, a_{k_\lambda}) \in \mathrm{Dm}\,\alpha_\lambda$.

Although the following two remarks may be familiar to practitioners, they are included here for completeness. First, the above symbol $a_1 - \cdots \rightarrow a_{k_\lambda}$ is an alternative expression of $(a_1, \ldots, a_{k_\lambda})$, and so the above triplet is an alternative expression of the triplet

$$\lambda \quad (a_1, \ldots, a_{k_\lambda}) \quad \alpha_\lambda(a_1, \ldots, a_{k_\lambda}).$$

Therefore, when printed on a sheet of paper as above, the set-theoretic graph $\mathcal{G}_A$ may be regarded as an explicit tabular description of the algebraic structure $(\alpha_\lambda)_{\lambda\in\Lambda}$ of A. Second, in the context of [1], where machines are discussed algebraically as stated in the preface, the terms *label*, *arrow* and *weight* should be replaced by *process name*, *input* and *output*, respectively.

You may have seen what are called graphs with arrows carrying labels and weights. Let G denote one of them, and Λ be the collection of the labels for the arrows of G. Assume that each label λ is a tuple $(k_\lambda, \ldots)$ of a positive integer k_λ possibly with other entries and that each λ-arrow (an arrow of label λ) joins exactly k_λ nodes. Then we can construct an algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ in the following two steps. First, we define A to be the collection of all nodes and all weights of G. Next, we define the operation α_λ for each $\lambda \in \Lambda$ in the following way. Namely, $\mathrm{Dm}\,\alpha_\lambda$ consists of all elements $(a_1, \ldots, a_{k_\lambda})$ of A^{k_λ} such that there exists exactly one λ-arrow which joins $a_1, \ldots, a_{k_\lambda}$ in this order, that is, a_1 and a_{k_λ} are respectively the tail and head of the λ-arrow, and for each element $(a_1, \ldots, a_{k_\lambda})$ of the domain, $\alpha_\lambda(a_1, \ldots, a_{k_\lambda})$ is the weight of such unique λ-arrow. This algebra, denoted by $\mathcal{A}_G$, is relatively simple, because G is visible. Moreover, when drawn on a sheet of paper, G and the set-theoretic graph $\mathcal{G}_{\mathcal{A}_G}$ associated with $\mathcal{A}_G$ are similar, that is,

$$\mathcal{G}_{\mathcal{A}_G} \sim G,$$

where the symbol $\sim$ denotes the vague notion of similarity. Furthermore, if G is the set-theoretic graph $\mathcal{G}_B$ associated with an algebra B, then $\mathcal{A}_G$ is equivalent to B, that is,

$$\mathcal{A}_{(\mathcal{G}_B)} \sim B,$$

where the symbol $\sim$ denotes the above-mentioned equivalence among algebras. Thus, there is a one-to-one correspondence between the set of the equivalence classes of relatively simple algebras and the set of the similarity classes of the graphs with arrows carrying labels and weights.

Although simple, the following example illustrates both the above construction of $\mathcal{A}_{\mathsf{G}}$ and the above correspondence well. Assume that the graph G has three nodes a, b, c, two weights v, w and three labels λ, μ, ν with $k_\lambda = 3$, $k_\mu = 2$ and $k_\nu = 1$, and assume that its arrows are as follow:

labels	arrows	weights
λ	$a - b \to c$	v
λ	$b - c \to a$	w
μ	$a \to b$	v
μ	$b \to c$	w
ν	a	v
ν	b	w

Then

$$A = \{a, b, c, v, w\} \qquad\qquad \Lambda = \{\lambda, \mu, \nu\}$$

and the algebraic structure on A we obtain as above consists of the operations $\alpha_\lambda, \alpha_\mu, \alpha_\nu$ which satisfy the following:

$$\begin{array}{lrr}
\mathrm{Dm}\,\alpha_\lambda = \{(a,b,c),(b,c,a)\} & \alpha_\lambda(a,b,c) = v & \alpha_\lambda(b,c,a) = w \\
\mathrm{Dm}\,\alpha_\mu = \{(a,b),(b,c)\} & \alpha_\mu(a,b) = v & \alpha_\mu(b,c) = w \\
\mathrm{Dm}\,\alpha_\nu = \{a,b\} & \alpha_\nu a = v & \alpha_\nu b = w
\end{array}$$

The set-theoretic graphs $\mathcal{G}_A$ associated with more complicated algebras A, or rather the algebras A themselves, though barely visible, still await investigation and application not only through graph-theoretic methods but also directly through algebraic methods, as exemplified in [1]. Algebra, along with other mathematical disciplines, enables us to penetrate what lies beyond the reach of the naked eye, even when aided by microscopes, telescopes, or other instruments.

Referred author's works and profile

[1] *Mathematical Noology: Intellectual machines, logic, tongues and algebra*, https://gomikensaku.github.io/homepage/, ever-growing WWW publication, since 2010.

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