From Graphs to Algebras

From What We See to What We Penetrate

© **Kensaku Gomi***
mathpsy@nifty.com

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Abstract

Graphs with labels, orientations and weights are regarded in principle as visual representations of relatively simple algebraic systems. In contrast, algebra provides a means of handling all such "graphs," including those that lie beyond the reach of human visual intuition. This note explores the correspondence between graphs and algebras, and suggests a way to penetrate invisible structures through algebraic methods.

From algebras to graphs and the way home

For each set A, a (partial) **operation** on A is a mapping α of a nonempty subset D of the direct product A^k into A for a positive integer k. The set D is called the **domain** of α and denoted $\mathrm{Dm}\,\alpha$. The positive integer k is uniquely determined by α because D is nonempty, and is called the **arity** of α.

An **algebra** (or **algebraic system**) is a set A equipped with an **algebraic structure** $(\alpha_\lambda)_{\lambda\in\Lambda}$ which is a family of operations α_λ on A indexed by a set Λ. Thus an algebra is best described by the pair $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ of a set and an algebraic structure on it.

The algebra $(A, (\alpha_\lambda)_{\lambda\in\Lambda})$ gives rise to a set-theoretic graph in the following way: Suppose $\lambda \in \Lambda$, $(a_1, \ldots, a_k) \in \mathrm{Dm}\,\alpha_\lambda$, $b \in A$ and

$$\alpha_\lambda(a_1, \ldots, a_k) = b.$$

Then the triplet

$$(\lambda, a_1 - \cdots \to a_k, b)$$

*Led by the given name against Japanese tradition.

†Uploaded on February 19, 2026. To be updated irregularly to grow into a mature file.

of λ, b and the symbol

$$a_1 - \cdots \to a_k$$

is called an **arrow** with **label** λ and **weight** b, provided that $a_1 \neq a_k$. When $a_1 = a_k$, for instance when $k = 1$, it is called a **loop**. Notice that

$$a_1 - \cdots \to a_k = \begin{cases} a_1 & \text{if } k = 1, \\ a_1 \to a_2 & \text{if } k = 2, \\ a_1 - \cdots - a_{k-1} \to a_k & \text{if } k > 2. \end{cases}$$

The **set-theoretic graph** associated with the algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ is the totality of the arrows and loops

$$(\lambda, a_1 - \cdots \to a_k, \alpha_\lambda(a_1, \ldots, a_k))$$

for all $\lambda \in \Lambda$ and all $(a_1, \ldots, a_k) \in \mathrm{Dm}\, \alpha_\lambda$.

You may have seen what are called graphs with arrows carrying labels, orientations and weights. You are now able to identify them with one of the set-theoretic graphs associated with relatively simple algebras. Namely, letting Λ be the collection of the labels for the arrows, we can construct an algebra $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$ such that our graph may be identified with the graph associated with $(A, (\alpha_\lambda)_{\lambda \in \Lambda})$. The construction is as follows. The arrows of label λ will be called the λ-arrows. We first define the set A to be the collection of all nodes and all weights. Assuming that each label λ is a tuple $(k_\lambda, \ldots)$ of a positive integer k_λ possibly with other entries and that each λ-arrow joins exactly k_λ nodes, we next define the operation α_λ as follows: $\mathrm{Dm}\, \alpha_\lambda$ consists of all elements $(a_1, \ldots, a_{k_\lambda})$ of A^{k_λ} such that there exists exactly one λ-arrow which joins $a_1, \ldots, a_{k_\lambda}$ in this order, that is, a_1 and a_{k_λ} are respectively the tail and head of the λ-arrow, and for each element $(a_1, \ldots, a_{k_\lambda})$ of the domain, $\alpha_\lambda(a_1, \ldots, a_{k_\lambda})$ is the weight of such unique λ-arrow.

Conclusion

The graphs associated with more complicated algebras—or rather, the algebras themselves—though barely visible, still await investigation and application not only through graph-theoretic methods but also directly through algebraic methods, as exemplified in [1]. Algebra, along with other mathematical disciplines, enables us to penetrate what lies beyond the reach of the naked eye, even when aided by microscopes, telescopes, or other instruments.

Referred author's works and profile

[1] *Mathematical Noology: Intellectual machines, logic, tongues and algebra*, https://gomikensaku.github.io/homepage/, ever-growing WWW publication, since 2010.

[2] Kensaku Gomi, PhD,* engaged in researches in finite group theory (a branch of algebra) for the Department of Mathematics, College of General Education, University of Tokyo from Apr. 1973 through Mar. 1992 and in new mathematical psychology, especially noology for the Graduate School of Mathematical Sciences thereof from Apr. 1992 through Mar. 2010 and for himself ever after.

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*His doctoral thesis "Characterizations of linear groups of low rank" has been published in *J. Fac. Sci. Univ. Tokyo* 23 (1976) and archived in http://hdl.handle.net/2261/7339.