From Logical Systems to Logical Spaces

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Abstract A definition of *logical systems* as pairs of syntax and semantics is proposed, and it is shown how each logical system with a *truth* yields a *logical space*, which the author introduced as a framework for studying logic, in particular completeness of deduction systems [Theory of completeness for logical spaces, Logica Universalis 3 (2009)]. The definition is tested for adequacy by deriving two theorems from it on the relationship between syntax and semantics, and also is illustrated by a few examples. Although the whole theory is general and purely logical, it is oriented to and commented on from the special viewpoint of Mathematical Psychology (MP), and this paper is intended as both an introduction to MP and a theoretical basis for subsequent papers on completeness theorems for the logical systems designed for MP. Linguists may read this paper as providing a framework for categorial grammar.

Keywords abstract logic, categorial grammar, logical space, logical system, mathematical psychology, semantics, sorted algebra, syntax, universal logic

1 Introduction

The main purpose of this paper is to show how each logical system yields a logical space. As I introduced [5], a **logical space** is a pair $(\mathsf{A}, \mathcal{B})$ of a nonempty set A and a subset $\mathcal{B}$ of the power set $\mathcal{P}\mathsf{A}$ of A and is expected to be a framework for logic. On the other hand, there seems to be a debate on the question "What is a logical system?" around computer theorists [1]. Some people there seem to even mean "logic" by "logical system." Answers may well vary according to researchers' intent. My answer is as follows according to my intent mentioned later apart from computer theory.

First, a logical system is a pair of syntax and semantics. I mean by "syntax" and "semantics" the rules governing the composition and the interpretation respectively of expressions in a language. Secondly, logic is the study of deduction systems on logical systems, where a deduction system is a pair (R, S) of a subset S of a set A and a relation R between the free monoid A^* over A and A.

There also is a question "Is there a real difference between syntax and semantics from the viewpoint of computer applications?" in the AMS review of [1]. It sounded strange to me at first, but it makes sense, and my answer

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is "Syntax and semantics are different, but they may overlap if they are abstracted." Indeed, certain abstract definitions of syntax and semantics, and even of deductions, are quite similarly stated in terms of relations. It is rather natural because abstraction is the act of neglecting differences and focusing on similarities.

Semantics in the above sense for a natural language is typically provided by dictionaries. Let A be the set of the expressions of the language and D be the set of the words listed in a dictionary. Then the dictionary is regarded as defining a relation R between A and D. In consulting the dictionary, we pick an expression $a \in A$ and a subset E of D, and check whether $a\,R\,e$ holds for some word $e \in E$, that is, whether a belongs to the inverse image $R^{-1}E$ of E under R:

$$R^{-1}E = \{b \in A \mid b\,R\,e \text{ for some } e \in E\}.$$

The set E varies as the case may be and there are various dictionaries. Thus, I have been led to the following *tentative and informal* definition.

Definition 1.1 A *semantics* of a non-empty set A is a family of triples (D_i, R_i, E_i) of a set D_i, a relation R_i between A and D_i, and a subset E_i of D_i for all indices i in a set I. A *logical system* is a pair (A, Σ) of a non-empty set A defined by some syntax and a semantics Σ of A.

Logicians actually study semantics in this sense [6]. For instance, they pick an algebra A of sentences, a family $(D_i)_{i\in I}$ of algebras D_i similar to A, a homomorphism R_i of A into D_i and a subset E_i of D_i for each $i \in I$. For some reason or other, their study centers around the relation $\vDash$ between A^* and A defined so that an element $(\alpha, b) \in A^* \times A$ satisfies $\alpha \vDash b$ iff it satisfies

$$\alpha \subseteq {R_i}^{-1}E_i \implies b \in {R_i}^{-1}E_i \tag{1.1}$$

for all $i \in I$. Here and elsewhere, we regard elements $\alpha \in A^*$ as elements of $\mathcal{P}A$ for convenience sake. Logicians pick $D_i = \{0, 1\}$ and $E_i = \{1\}$ for all $i \in I$ in classical cases, where (1.1) reads "If $R_i a = 1$ for all $a \in \alpha$, then $R_i b = 1$" and $R_i a$ for $a \in A$ is intended as the truth value of a under the valuation R_i.

Under Definition 1.1, define the subset $\mathcal{B}$ of $\mathcal{P}A$ by

$$\mathcal{B} = \{{R_i}^{-1}E_i \mid i \in I\}. \tag{1.2}$$

Then we obtain a logical space $(A, \mathcal{B})$. Furthermore, according to [5], the study of a logical space $(A, \mathcal{B})$ necessarily centers around the relation Q between A^* and A, which is called the largest $\mathcal{B}$-logic and is defined to be the largest of the relations between A^* and A which close every member of $\mathcal{B}$. It also turned out to satisfy

$$\alpha\,Q\,b \iff b \in \textstyle\bigcap_{\alpha \subseteq B \in \mathcal{B}} B$$

for each $(\alpha, b) \in A^* \times A$. Therefore, as for the logical space $(A, \mathcal{B})$ defined by (1.2), $\alpha\,Q\,b$ holds iff (α, b) satisfies (1.1) for all $i \in I$, and so Q coincides with the relation $\vDash$ which logicians study.

We have seen how each logical system in the sense of Definition 1.1 yields a logical space. The concept of logical systems, however, is not very distant from that of logical spaces, because each logical space conversely and immediately yields a classical logical system. Moreover, it includes no syntax, and the definition of semantics will overlap with an equally abstract definition of syntax as mentioned before. Above all, it is too abstract to be immediately applied to practical logics. Thus, it does not yet answer the purpose of this paper.

The definition of logical systems which I will settle in §3 distinguishes between syntax and semantics and yields logical spaces by the same principle as above. While it is essentially a practical specialization of Definition 1.1, it is still highly general. Although it is not as simple as Definition 1.1, it relies only on the basic knowledge of sorted algebras collected in §2. The rest of the paper is to amplify it. In order to test it for adequacy and prepare for its application to the studies mentioned below, I will derive in §4 two theorems from it on the relationship between the syntax and semantics. Also, I will illustrate it by the propositional logics, classical first-order predicate logic, and typed λ-calculus in §5.

In fact, despite its generality, the whole theory is oriented to my special intent described below inspired by but departing from Montague's theory of the syntax and semantics of natural languages [8,11] which I learned from Shirai [9].

This paper is intended to give a theoretical basis for the papers [2,7,10] by my collaborators and myself who will prove completeness theorems for the logical systems MPCL (monophasic case logic) and PPCL (polyphasic case logic) introduced in [3] for MP (mathematical psychology), which I launched by an electronic publication [4] capable of frequent updating and will abridge by this paper and others.

The end of MP is to comprehend human mind by analyzing a *comprehensive* mathematical model of the triple of the human system of cognizing and thinking, the outer worlds which humans cognize and think about, and the relationship between a human and the outer worlds. Logic is a tool for that, just as probability is a tool for genetics. The formal language A in MP is a model of the human system of cognizing, and the semantics of A consists of models of pairs of an outer world and a relationship between the world and the human system, while deduction systems on the sentences of A are models of the human system of thinking. Those human systems are unknowns in the brain, just as Mendel's factors (genes) were unknowns in organisms. For the time being, adequacy of the models can be examined solely by observations of natural languages which are supposed to be deformed expressions of the human systems, just as Mendel's theory was once examined solely by observations of phenotypes of pea plants which were supposed to be deformed expressions of their genotypes. Metaphorically speaking, MP seeks for a theory of genotypes by observations of phenotypes.

MP should be comprehensive. Common logics such as the first-order predicate classical logic and modal logic may be regarded as providing theories of genotypes. However, they are not comprehensive and should be embedded together in some comprehensive logic. At present MP is in the process of trial and error in order to create such a logic, and PPCL is that which I could have created so far with MPCL being its prototype. As such, PPCL is sophisticated rather than complicated. In particular, its formal language A has operations Ωx which are analogous to and more sophisticated than the λ-abstractions λx in the typed λ-calculus. The semantics of A includes the interpretation of Ωx. The general definition of semantics to be proposed in §3 therefore includes the sophisticated interpretation of generalizations of Ωx. The sophistication is inevitable and essential, as neither human's cognition nor the outer world as a whole is simple.

2 Preliminaries on sorted algebras and universality

Here are collected notation, terminology, and basic facts about algebras. Some of them may be standard but others may not. Thus, this section may include proposals for notation and terminology, just as subsequent sections propose a theory. However, our notation and terminology about sets will be standard except that we denote the set of the mappings of a set X into a set Y by $X \to Y$ instead of Y^X. Thus $f \in X \to Y$ means $f : X \to Y$.

2.1 Operations

For each set A and each natural number n, an n-ary **operation** on A is a mapping α of a subset D of A^n into A. The set D is called the **domain** of α and denoted by $\mathrm{Dom}\,\alpha$, while the image αD is denoted by $\mathrm{Im}\,\alpha$. The number n is called an **arity** of α. If $D \neq \emptyset$, α has the unique arity. If $D = \emptyset$, every natural number is an arity of α, and $\alpha = \emptyset$ because $\emptyset \to A = \{\emptyset\}$. If $D = A^n$, we say that α is **total**. A subset B of A is said to be **closed** under α if $\alpha(B^n \cap D) \subseteq B$. If B is closed under α, the **restriction** $\beta = \alpha|_{B^n \cap D}$ of α to B becomes an operation on B.

2.2 Algebras

An **algebra** is a set A equipped with a family $(\alpha_\lambda)_{\lambda \in L}$ of operations α_λ on A, and is sometimes called an $(\alpha_\lambda)_{\lambda \in L}$**-algebra**. Strictly speaking, an algebra is a pair $(A, (\alpha_\lambda)_{\lambda \in L})$ of a set and a family of operations on it. Conventionally, we often identify $(\alpha_\lambda)_{\lambda \in L}$ with the family $(\lambda)_{\lambda \in L}$ and call A an L**-algebra**. The algebra A is said to be **total** if α_λ is total for every $\lambda \in L$. Two algebras A and B are said to be **similar**, if their operations $(\alpha_\lambda)_{\lambda \in L}$ and $(\beta_\lambda)_{\lambda \in L}$ are indexed by the same set L and α_λ and β_λ have a common arity for each $\lambda \in L$. The similarity is reflexive and symmetric, but not transitive when involved with empty operations.

2.3 Subalgebras, reducts and closures

Let $(A, (\alpha_\lambda)_{\lambda \in L})$ be an algebra. Then its **subalgebra** is an algebra $(B, (\beta_\lambda)_{\lambda \in L})$ consisting of a subset B of A closed under α_λ for every $\lambda \in L$ and the family $(\beta_\lambda)_{\lambda \in L}$ of restrictions β_λ of α_λ to B for all $\lambda \in L$. Also its M**-reduct** for a subset M of L is the algebra $(A, (\alpha_\mu)_{\mu \in M})$, which will be abbreviated as A_M.

For each subset S of A, there exists the smallest of the subalgebras of A which contain S. We call it the **closure** of S and denote it by $[S], [S]_L$, and so on.

Define the **descendants** S_n $(n = 0, 1, \dots)$ of S inductively as follows. First $S_0 = S$. Next for each $n \geq 1$, S_n is the set of all elements $\alpha_\lambda(a_1, \dots, a_k)$ with $\lambda \in L$, $(a_1, \dots, a_k) \in \mathrm{Dom}\, \alpha_\lambda$, and $a_j \in S_{n_j}$ $(j = 1, \dots, k)$ for some non-negative integers $n_1, \dots, n_k$ such that $n = 1 + \sum_{j=1}^{k} n_j$. Then $[S] = \bigcup_{n \geq 0} S_n$ holds.

2.4 Holomorphisms and homomorphisms

Let $(A, (\alpha_\lambda)_{\lambda \in L})$ and $(B, (\beta_\lambda)_{\lambda \in L})$ be similar algebras and $f \in A \to B$. Then f is called an **homomorphism**, if it satisfies the following condition for each $\lambda \in L$.

- If $(a_1, \dots, a_n) \in \mathrm{Dom}\, \alpha_\lambda$, then $(fa_1, \dots, fa_n) \in \mathrm{Dom}\, \beta_\lambda$ and $f(\alpha_\lambda(a_1, \dots, a_n)) = \beta_\lambda(fa_1, \dots, fa_n)$.

The homomorphism is called a **holomorphism**, if it is furthermore **exact** in the sense that it satisfies the following condition for each $\lambda \in L$.

- If $(a_1, \dots, a_n) \in A^n$ and $(fa_1, \dots, fa_n) \in \mathrm{Dom}\, \beta_\lambda$, then $(a_1, \dots, a_n) \in \mathrm{Dom}\, \alpha_\lambda$.

Homomorphisms of total algebras are necessarily holomorphisms. Bijective holomorphisms are called **isomorphisms**. Holomorphisms of L-algebras are sometimes called L**-holomorphisms**, and similarly for homomorphisms and isomorphisms.

2.5 Sorted algebras and sort-consistent mapping

A **sorted algebra** is an algebra A equipped with an algebra T similar to A and a holomorphism $\sigma \in A \to T$. Strictly speaking, a sorted algebra is a triple (A, T, σ) of similar algebras and a holomorphism between them. We call T and σ the **sorter** and **sorting** of the sorted algebra. For each element $a \in A$, we call σa the **type of** a. Conversely for each element $t \in T$ and each subset B of A, the inverse image $\sigma^{-1} t \cap B$ of t in B is called the t**-part** of B and denoted by B_t, and its elements are said to be **of type** t.

Let (A, T, σ) and (B, T, τ) be sorted algebras with the same sorter T. Then A, B are similar. Also, a mapping $f \in A \to B$ is said to be **sort-consistent**, if it satisfies $\tau f = \sigma$, or equivalently $f(A_t) \subseteq B_t$ for each $t \in T$. Sort-consistent homomorphisms are necessarily holomorphisms.

Every total algebra A may be regarded as a sorted algebra (A, T, σ), where T is an arbitrary singleton made into a total algebra similar to A in the obvious unique manner and σ is the unique element of $A \to T$. Conversely if (A, T, σ) is a sorted algebra with T total (and a singleton), then A is a total algebra. For similar total algebras A and B, a mapping $f \in A \to B$ is a homomorphism iff it is a sort-consistent holomorphism when A, B are regarded as sorted algebras as above.

2.6 Universal sorted algebras and free algebras

As for this paper, "universal" is not a synonym of "general." A sorted algebra (A, T, σ) is said to be **universal** or called a USA (**universal sorted algebra**) if A has a subset S which satisfies the following two conditions:

Generation $A = [S]$.

Universality If (A', T, σ') is a sorted algebra and a mapping $\varphi \in S \to A'$ satisfies $\sigma' \varphi = \sigma|_S$, then φ is extended to a sort-consistent holomorphism $f \in A \to A'$.

Also, a total algebra A is said to be **universal** or called a UTA (**universal total algebra**) if A has a subset S which satisfies the following two conditions:

Generation $A = [S]$.

Universality If A' is a total algebra similar to A and $\varphi \in S \to A'$, then φ is extended to a homomorphism $f \in A \to A'$.

The elements of the set S in the above conditions are called the **primes**.

The mapping f in each universality condition is uniquely determined by φ. If (A, T, σ, S) is a USA and T is a total singleton, then (A, S) is a UTA. Conversely if (A, S) is a UTA, then the sorted algebra (A, T, σ) made of a singleton T and the unique mapping $\sigma \in A \to T$ as in §2.5 together with S becomes a USA.

The theory in §3 is based solely on the following theorem on USA's.

Theorem 2.1 *Let T be an algebra, S be a set, and $\tau \in S \to T$. Then there exists a USA (A, T, σ, S) with $\sigma|_S = \tau$. If (A', T, σ', S) is also a USA with $\sigma'|_S = \tau$, then there exists a sort-consistent isomorphism of A onto A' extending* id_S.

Therefore, in order to construct a USA, we only need to pick a triple (T, τ, S) of an algebra T, a set S, and a mapping $\tau \in S \to T$, so we call it the **syntax** of the USA. Picking τ is equivalent to partitioning S into disjoint subsets S_t $(t \in T)$. We call τ and $S = \bigcup_{t \in T} S_t$ a **pre-sorting** and a **pre-partition**.

Theorem 2.1 is so pertinent also to linguists and logicians that I guess it as known to them, but I could not yet find it in any literature. Therefore, I will supply its proof in Appendix.

Example 2.1 The concept of USA's was probably known to Montague and his predecessors, because I extracted it from his linguistics-oriented quasimathematical definition [8] of "disambiguated languages." The advantage of our genuinely mathematical definition of USA's will be evident shortly.

Everyone will soon be convinced that Theorem 2.1 is true, if he/she compares (T, τ, S) to an elementary grammar of his/her mother tongue, that is, compares T to a set of the names of words and expressions; for instance $T = \{$sentence, noun, noun phrase, verb, verbal phrase, $\dots$ $\}$, S to the set of the words listed in a dictionary, and τ to the mapping which assigns each word in S its name, e.g. τ(rabbit) = noun, τ(eat) = verb, and so on for English. Equip T with some operations, e.g. the product $\bullet$ defined by "noun $\bullet$ (intransitive verb) = sentence," "(transitive verb) $\bullet$ noun = (verbal phrase)," and so on neglecting articles and paradigms for convenience. Also, regard the copula, conjunctions, some of the prepositions, postpositions, and so on as operations on T. For instance, consider that "be noun = (verbal phrase)" is part of the definition of the English copula "be" as a unary operation on T. It is our empirical knowledge that any such grammar uniquely determines the set A of the correct expressions and their names.

Surprisingly enough, we have seen that USA's and the phenomena described by Theorem 2.1 are typically observed in natural languages, which are the most noteworthy expressions of human mind. Another surprisingly relevant example is the USA (F, A, σ, A) determined by the syntax (A, id_A, A) for an algebra A. The algebra F is identified with the algebra of proof figures on A. It goes without saying that proof figures are processes of deduction. These facts are the reasons why USA's are the right algebras for MP.

The following theorem is indispensable in the inductive treatment of the elements of the USA's (cf. §4).

Theorem 2.2 *Let (A, T, σ, S) be a USA on an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$. Then the algebra is **free** over S, or S is its **basis**, in the sense that the following holds.*

(1) $A = [S]$.

(2) $S \cap \bigcup_{\lambda \in L} \mathrm{Im}\, \alpha_\lambda = \emptyset$*, that is, no element* $a \in S$ *has an expression* $a = \alpha_\lambda(a_1, \dots, a_k)$ *with* $\lambda \in L$ *and* $(a_1, \dots, a_k) \in \mathrm{Dom}\, \alpha_\lambda$*.*

(3) Each element $a \in A - S$ *has a unique expression* $a = \alpha_\lambda(a_1, \dots, a_k)$ *with* $\lambda \in L$ *and* $(a_1, \dots, a_k) \in \mathrm{Dom}\, \alpha_\lambda$*, which we call the **word form** of* a*.*

*If an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$ has a basis S, then A is the direct union $\coprod_{n=0}^{\infty} S_n$ of the descendants S_n $(n = 0, 1, \dots)$ of S, and so for each element $a \in A$, there exists a unique non-negative integer n satisfying $a \in S_n$ which we call the **rank** of a and denote by $\mathrm{Rank}\, a$, and if $\mathrm{Rank}\, a \geq 1$, then the unique word form $\alpha_\lambda(a_1, \dots, a_k)$ of a satisfies $\mathrm{Rank}\, a = 1 + \sum_{j=1}^{k} \mathrm{Rank}\, a_j$.*

Proof The former half is obtained in the course of the proof of Theorem 2.1. As for the latter half, we only need to show $S_n \cap S_{n'} = \emptyset$ for each pair n, n' of distinct non-negative integers. We argue by induction on $\min\{n, n'\}$. If $\min\{n, n'\} = 0$, then $S_n \cap S_{n'} \subseteq S \cap \bigcup_{\lambda \in L} \mathrm{Im}\, \alpha_\lambda = \emptyset$. Assume $a \in S_n \cap S_{n'}$ for some pair n, n' with $\min\{n, n'\} \geq 1$. Then a has an expression $a = \alpha_\lambda(a_1, \dots, a_k)$ with $a_j \in S_{n_j} \cap S_{n'_j}$ $(j = 1, \dots, k)$, $n = 1 + \sum_{j=1}^{k} n_j$ and $n' =$

$1 + \sum_{j=1}^{k} n'_j$. Since $n_j = n'_j$ for $j = 1, \ldots, k$ by the induction hypothesis, we have $n = n'$ as desired.

2.7 Power algebras

Let (A, T, σ) be a sorted algebra and V be a non-empty set. We will construct a sorted algebra (A^V, T, ρ) with $A^V = \bigcup_{t \in T}(V \to A_t)$ and call it the **V-power** of A.

First define the sorting $\rho \in A^V \to T$ so that $\rho b = t$ for each $t \in T$ and each $b \in V \to A_t$. Then the following holds for each $b \in A^V$ and each $v \in V$:

$$\rho b = \sigma(bv). \tag{2.1}$$

Next, let $(\alpha_\lambda)_{\lambda \in L}$ and $(\tau_\lambda)_{\lambda \in L}$ be the operations of A and T respectively, and let n_λ be an arity of α_λ and τ_λ. For each $\lambda \in L$, define the operation β_λ on A^V as follows. First define the domain of β_λ to be

$$D_\lambda = \big\{(b_1, \ldots, b_{n_\lambda}) \in (A^V)^{n_\lambda} \mid (\rho b_1, \ldots, \rho b_{n_\lambda}) \in \mathrm{Dom}\, \tau_\lambda\big\}. \tag{2.2}$$

If $(b_1, \ldots, b_{n_\lambda}) \in D_\lambda$, then $\big(\sigma(b_1 v), \ldots, \sigma(b_{n_\lambda} v)\big) = (\rho b_1, \ldots, \rho b_{n_\lambda}) \in \mathrm{Dom}\, \tau_\lambda$ by (2.1), so $(b_1 v, \ldots, b_{n_\lambda} v) \in \mathrm{Dom}\, \alpha_\lambda$ for each $v \in V$ because σ is exact, and we can define the mapping $\beta_\lambda(b_1, \ldots, b_{n_\lambda}) \in V \to A$ by

$$\big(\beta_\lambda(b_1, \ldots, b_{n_\lambda})\big)v = \alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v) \tag{2.3}$$

for each $v \in V$. Furthermore, since σ is a homomorphism and (2.1) holds,

$$\sigma\big(\alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda}), \tag{2.4}$$

and $t = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda})$ does not depend on $v \in V$, hence $\beta_\lambda(b_1, \ldots, b_{n_\lambda}) \in V \to A_t$. Thus β_λ is an operation on A^V for each $\lambda \in L$, and so $(A^V, (\beta_\lambda)_{\lambda \in L})$ becomes an algebra. Furthermore, combining (2.1), (2.3), and (2.4), we have

$$\begin{aligned}\rho(\beta_\lambda(b_1, \ldots, b_{n_\lambda})) &= \sigma\big(\big(\beta_\lambda(b_1, \ldots, b_{n_\lambda})\big)v\big)\\ &= \sigma\big(\alpha_\lambda(b_1 v, \ldots, b_{n_\lambda} v)\big) = \tau_\lambda(\rho b_1, \ldots, \rho b_{n_\lambda})\end{aligned}$$

with $v \in V$ arbitrary. This together with (2.2) shows that ρ is a holomorphism of A^V into T. Thus we have constructed the sorted algebra (A^V, T, ρ) as noticed.

Furthermore, it follows from (2.1) and (2.3) that, for each $v \in V$, the mapping $b \mapsto bv$ of A^V into A is a sort-consistent holomorphism, which we call the **projection** by v and denote by pr_v.

3 Logical systems as pairs of syntax and semantics

Here I will settle the definition of logical systems and indicate that each of them appropriately yields a logical space under a certain reasonable condition. The theory is very general, but it is oriented to and commented on from the viewpoint of MP so that this section will serve as an introduction to MP.

3.1 Languages

A **formal language** is a USA (A, T, σ, S) equipped with sets C, X, Γ which satisfy the following conditions, where $(\tau_\lambda)_{\lambda \in L}$ are the operations of the sorter T.

- The prime set S is the direct union $C \amalg X$ of C and $X \neq \emptyset$.
- Each element of L either belongs to Γ or is a formal product γx of an element $\gamma \in \Gamma$ and an element $x \in X$. Strictly speaking, L is contained in the subset $\Gamma \cup \Gamma X$ of the free multiplicative semigroup over the direct union $\Gamma \amalg S$.
- The arity of each operation τ_λ with $\lambda \in L \cap \Gamma X$ is equal to 1.

Theorem 2.1 shows that each quintuple (T, S, C, X, Γ) of an algebra T and sets S, C, X, Γ which satisfy the above conditions together with a pre-sorting $\tau \in S \rightarrow T$ or a pre-partition $S = \bigcup_{t \in T} S_t$ determines a formal language $(A, T, \sigma, S, C, X, \Gamma)$ with $\sigma|_S = \tau$ up to sort-consistent isomorphism. Therefore, in order to construct a formal language, we only need to pick such a sextuple $(T, \tau, S, C, X, \Gamma)$. Moreover, we can visibly reconstruct A and σ by $(T, \tau, S, C, X, \Gamma)$ as is illustrated in §5.2. This principle enables us to construct and analyze such sophisticated languages as in MPCL and PPCL. For these theoretical and practical reasons, we call $(T, \tau, S, C, X, \Gamma)$ the **syntax** of the formal language.

We call C and X the sets of the **constants** and **variables**, define

$$M = L \cap \Gamma, \tag{3.1}$$

and call M the set of the **invariable indices**, while $L \cap \Gamma X$ is called the set of the **variable indices**. Henceforth, however, we identify each variable index $\lambda \in L \cap \Gamma X$ with the operation τ_λ, call it a **variable operation**, and denote its domain $\operatorname{Dom} \tau_\lambda$ by T_λ, hence $T_\lambda \subseteq T$ because τ_λ is unary. Note the following:

$$L = M \amalg (L \cap \Gamma X), \qquad L \cap \Gamma X = \textstyle\coprod_{x \in X} (L \cap \Gamma x). \tag{3.2}$$

The above definition is based on such an insight as in Example 2.1, and the formal language A in MP or more precisely its syntax $(T, \tau, S, C, X, \Gamma)$ should be a comprehensive model of the human brain system for cognizing the outer worlds. I call the system the **mental language**, while the human brain system for thinking is called the **mental deduction system**.

3.2 Worlds

A **denotable world** for the formal language $(A, T, \sigma, S, C, X, \Gamma)$ is a sorted algebra W which satisfies the following two conditions.

(1) The sorter of W is the M-reduct T_M of T for (3.1) (the sorting of W is not shown here, but it exists and partitions W into the t-parts W_t for $t \in T$).

(2) $W_t \neq \emptyset$ for each $t \in \sigma S$, that is, for each $t \in T$ with $S_t \neq \emptyset$.

The condition (1) implies that the M-reduct A_{M} of A is similar to W. When $\mathsf{M} = \mathsf{L}$ and T is a total singleton, (1) means that A is similar to W (cf. §5.1).

Denotable worlds are called **cognizable worlds** from the viewpoint of MP, because they are intended, under suitable specialization, to be comprehensive models of the outer worlds which humans actually cognize. Each outer world consists of entities and relations among them, which naturally form an algebra. The formal language A in MP is a model of the mental language, which is supposed to have adapted to the algebras of outer worlds in its evolution. This is the reason why mathematical psychicists assume that denotable worlds satisfy (1).

3.3 Interpretations of variable operations

An **interpretation** of the set $\mathsf{L}\cap\Gamma\mathsf{X}$ of the variable operations on the denotable world W for the formal language $(\mathsf{A},\mathsf{T},\sigma,\ \mathsf{S},\mathsf{C},\mathsf{X},\Gamma)$ is a mapping I_W which assigns each $\lambda \in \mathsf{L}\cap\Gamma\mathsf{x}$ $(\mathsf{x}\in\mathsf{X})$ a mapping

$$\lambda_W \in \left(\bigcup_{\mathsf{t}\in\mathsf{T}_\lambda}(W_{\sigma\mathsf{x}}\to W_{\mathsf{t}})\right)\to W \tag{3.3}$$

satisfying the following condition for each $\mathsf{t}\in\mathsf{T}_\lambda$:

$$\lambda_W(W_{\sigma\mathsf{x}}\to W_{\mathsf{t}}) \subseteq W_{\lambda\mathsf{t}}. \tag{3.4}$$

We call $\lambda_W = \mathrm{I}_W(\lambda)$ the **meaning** of λ on W under the interpretation I_W.

The terms “interpretation” and “meaning” are conventional ones, but they are misleading from the viewpoint of MP for the following reasons. As for linguists, the formal language A is a model of a natural language, and they seem to be interested in the question of how to theoretically define or interpret or analyze the meaning of expressions of the natural language through A. On the other hand as for mathematical psychicists, the formal language A is a model of the *personal* mental language, and so it does not make sense to theoretically define or interpret or analyze the meaning of elements of A. They are instead interested in questions of how entities and affairs in the outer worlds are expressed by elements of A, how elements of A are expressed by natural languages, whether the mental deduction system is complete or not [2,7,10], and so on, that is, how humans cognize, speak about and can think about the outer worlds. For humans, even a natural language is part of an outer world. Therefore, these questions contain that of how each human understands the *meaning for himself/herself* of the natural language.

3.4 Denotations

A C**-denotation** into the denotable world W for the formal language $(\mathsf{A},\mathsf{T},\sigma,$ $\mathsf{S},\mathsf{C},\mathsf{X},\Gamma)$ is a mapping $\Phi\in\mathsf{C}\to W$ which satisfies $\Phi(\mathsf{C}_\mathsf{t})\subseteq W_\mathsf{t}$ for each $\mathsf{t}\in\mathsf{T}$. There exists at least one C-denotation because $W_\mathsf{t}\neq\emptyset$ whenever $\mathsf{C}_\mathsf{t}\neq\emptyset$. If $\mathsf{C}=\emptyset$, then since $\emptyset\to W=\{\emptyset\}$, $\emptyset$ is the unique C-denotation. Similarly, an X**-denotation** into W is a mapping $\nu\in\mathsf{X}\to W$ which satisfies $\nu(\mathsf{X}_\mathsf{t})\subseteq W_\mathsf{t}$ for each $t \in T$. We denote the set of all X-denotations into W by $V_{X,W}$. Then $V_{X,W} \neq \emptyset$ because $W_t \neq \emptyset$ whenever $X_t \neq \emptyset$. Recall $X \neq \emptyset$ from §3.1.

From the viewpoint of MP, denotations should be called **perceptions** for the following reason. First, both constants and variables are models of some physical units which are supposed to exist in the human brain. It is said that if we perceive for instance a dog in the outer world, then a specific dog neuron in our brain gets excited. The set C of the constants is a model of the collection of such specific neurons. Each such neuron is connected with and excited by a specific entity or affair in the world, which I call the **percept** for the neuron. The C-denotation Φ into W is a model of the set $\{(c, w)\}$ of the connections (c, w) of a neuron c and its percept w. On the other hand, each variable is a model of a neuron which is not connected with anything and is capable of connecting with something. Each X-denotation ν is a model of a set $\{(x, w)\}$ of possible connections (x, w). Thus, while C and X are physical, denotations and $V_{X,W}$ are metaphysical, and so also are the concepts to be introduced in §3.5–§3.8.

3.5 Metaworlds

Since the denotable world W for the formal language $(A, T, \sigma, S, C, X, \Gamma)$ is a sorted algebra whose sorter is the M-reduct T_M of T for (3.1) and $V_{X,W} \neq \emptyset$, the $V_{X,W}$-power $(W^{V_{X,W}}, T_M, \rho)$ of W is defined as in §2.7. Let $(\beta_\lambda)_{\lambda \in M}$ be the operations of $W^{V_{X,W}}$. Recall $L = M \amalg (L \cap \Gamma X)$ from (3.2).

In this subsection, we assume that there exists an interpretation I_W of $L \cap \Gamma X$ on W in the sense of §3.3. We will define the unary operation β_λ on $W^{V_{X,W}}$ for each $\lambda \in L \cap \Gamma X$, and extending the operations of $W^{V_{X,W}}$ from $(\beta_\lambda)_{\lambda \in M}$ to $(\beta_\lambda)_{\lambda \in L}$, we will construct the sorted algebra $(W^{V_{X,W}}, T, \rho)$. We call it the **metaworld** associated with the world W.

First we define, for each pair (x, w) of elements $x \in X$ and $w \in W_{\sigma x}$, the transformation $\nu \mapsto (x/w)\nu$ on $V_{X,W}$ by

$$\big((x/w)\nu\big)y = \begin{cases} w & \text{if } y = x, \\ \nu y & \text{if } y \in X - \{x\}. \end{cases} \tag{3.5}$$

Next we define, for each quadruple (t, φ, x, ν) consisting of elements $t \in T$, $\varphi \in V_{X,W} \to W_t$, $x \in X$ and $\nu \in V_{X,W}$, the mapping $\varphi\big((x/\Box)\nu\big) \in W_{\sigma x} \to W_t$ by

$$\Big(\varphi\big((x/\Box)\nu\big)\Big)\, w = \varphi\big((x/w)\nu\big) \tag{3.6}$$

for each $w \in W_{\sigma x}$. Recalling $L \cap \Gamma X = \coprod_{x \in X}(L \cap \Gamma x)$ from (3.2), we finally define, for each $\lambda \in L \cap \Gamma x$ $(x \in X)$, the unary operation β_λ on $W^{V_{X,W}}$. First we define

$$\operatorname{Dom} \beta_\lambda = \{\varphi \in W^{V_{X,W}} \mid \rho\varphi \in \operatorname{Dom} \tau_\lambda\}. \tag{3.7}$$

Then since $\operatorname{Dom} \tau_\lambda = T_\lambda$ and $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$,

$$\operatorname{Dom} \beta_\lambda = \textstyle\bigcup_{t \in T_\lambda} (V_{X,W} \to W_t). \tag{3.8}$$

Next for each $t \in T_\lambda$ and each $\varphi \in V_{X,W} \to W_t$, we define $\beta_\lambda \varphi$ to be the element of $V_{X,W} \to W_{\lambda t}$ such that

$$(\beta_\lambda \varphi)\nu = \lambda_W\big(\varphi\big((x/\square)\nu\big)\big) \tag{3.9}$$

for each $\nu \in V_{X,W}$. Certainly $(\beta_\lambda \varphi)\nu \in W_{\lambda t}$, because $\varphi\big((x/\square)\nu\big) \in W_{\sigma x} \to W_t$ and (3.4) holds. Thus, we have defined the operation β_λ with (3.8) so that

$$\beta_\lambda(V_{X,W} \to W_t) \subseteq V_{X,W} \to W_{\lambda t} \tag{3.10}$$

for each $t \in T_\lambda$. Since $V_{X,W} \to W_t$ is the t-part of $W^{V_{X,W}}$ for each $t \in T$, this shows that $\rho(\beta_\lambda \varphi) = \tau_\lambda(\rho\varphi)$ for each $\lambda \in L \cap \Gamma X$ and each $\varphi \in \operatorname{Dom}\beta_\lambda$. This together with (3.7) shows that ρ is an $L \cap \Gamma X$-holomorphism as well as an M-holomorphism. We have thus constructed the sorted algebra $(W^{V_{X,W}}, T, \rho)$ as noticed.

3.6 Metadenotations

Let Φ be a C-denotation into the denotable world W for the formal language $(A, T, \sigma, S, C, X, \Gamma)$. We will construct a sort-consistent holomorphism of A into the metaworld $(W^{V_{X,W}}, T, \rho)$ associated with W constructed in §3.5 under the assumption that there exists an interpretation I_W of $L \cap \Gamma X$ on W.

First define the mapping φ of $S = C \amalg X$ into $V_{X,W} \to W$ so that, for each $\nu \in V_{X,W}$, $(\varphi a)\nu = \Phi a$ for all $a \in C$ and $(\varphi a)\nu = \nu a$ for all $a \in X$. Then $\varphi(S_t) \subseteq V_{X,W} \to W_t$ for each $t \in T$, because $\Phi(C_t) \subseteq W_t$ and $\nu(X_t) \subseteq W_t$. Therefore, φ belongs to $S \to W^{V_{X,W}}$ and satisfies $\rho\varphi = \sigma|_S$, and so the universality of A shows that φ is uniquely extended to a sort-consistent holomorphism of A into $W^{V_{X,W}}$. We call it the **metadenotation** associated with Φ and denote it by Φ^*. Being an extension of φ, Φ^* satisfies the following for each $\nu \in V_{X,W}$:

$$(\Phi^* a)\nu = \begin{cases} \Phi a & \text{if } a \in C, \\ \nu a & \text{if } a \in X. \end{cases} \tag{3.11}$$

Remark 4.1 will show that Φ^* may be regarded as sorting elements $a \in A$ by their functional expressions $a^\Phi(x_1, \dots, x_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n} \to W_{\sigma a}$, where $x_1, \dots, x_n$ are free variables of a. From the viewpoint of MP, this suggests that humans in fact cognize functions on the outer worlds.

3.7 Logical systems

A **logical system** is a formal language $(A, T, \sigma, S, C, X, \Gamma)$ equipped with a non-empty collection $\mathcal{W}$ of denotable worlds for A and a family $(I_W)_{W \in \mathcal{W}}$ of interpretations I_W of $L \cap \Gamma X$ on each $W \in \mathcal{W}$. We call the pair $\big(\mathcal{W}, (I_W)_{W \in \mathcal{W}}\big)$ the **semantics** of the formal language. As shown in §3.6, it assigns each C-denotation Φ the metadenotation Φ^*.

3.8 Sentential logical spaces

Once the concept of a logical system $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ on the formal language $(A, T, \sigma, S, C, X, \Gamma)$ is defined in §3.7, there are various ways to associate it with logical spaces. Here we concentrate on the most usual case where the following condition is satisfied.

- For an element $\phi \in T$, the ϕ-part A_ϕ of A is non-empty, and the ϕ-part W_ϕ of each $W \in \mathcal{W}$ is a lattice which has the smallest element and the largest element and is non-trivial in the sense that $\#W_\phi \geq 2$.

Then we call ϕ and A_ϕ a **truth** and the set of the **ϕ-sentences**, respectively.

Let $W \in \mathcal{W}$. Also, let Φ, ν be a C-denotation and an X-denotation into W, respectively. Then the interpretation I_W of $L \cap \Gamma X$ on W yields the metaworld $W^{V_{X,W}}$ as shown in §3.5, and Φ determines the metadenotation $\Phi^* \in A \to W^{V_{X,W}}$ as shown in §3.6, hence the composite $\mathrm{pr}_\nu \Phi^* \in A \to W$. Since Φ^* is a sort-consistent holomorphism and pr_ν is a sort-consistent M-holomorphism,

$$\mathrm{pr}_\nu \Phi^* \text{ is a sort-consistent } M\text{-holomorphism,} \tag{3.12}$$

and consequently $\mathrm{pr}_\nu \Phi^* A_t \subseteq W_t$ for each $t \in T$. Therefore, we can define the mappping $\Phi^\nu \in A_\phi \to W_\phi$ by

$$\Phi^\nu = \mathrm{pr}_\nu \Phi^*|_{A_\phi}, \tag{3.13}$$

that is, $\Phi^\nu a = (\Phi^* a)\nu$ for each $a \in A_\phi$.

Now, for each $W \in \mathcal{W}$, we define the subset $\mathcal{F}_W$ of $A_\phi \to W_\phi$ by

$$\mathcal{F}_W = \{\Phi^\nu \mid \Phi \text{ is a } C\text{-denotation into } W,\ \nu \in V_{X,W}\}. \tag{3.14}$$

Then $(A_\phi, \mathcal{F}_W)$ is a W_ϕ-valued functional logical space, and so a certain logical space $(A_\phi, \mathcal{B}_W)$ is associated with it as in §6.2 of [5]. We finally define $\mathcal{B} = \bigcup_{W\in\mathcal{W}} \mathcal{B}_W$. Then $(A_\phi, \mathcal{B})$ is a logical space.

Thus, we have seen that each logical system $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ with a truth ϕ yields the logical space $(A_\phi, \mathcal{B})$. We call it the **ϕ-sentential logical space** associated with the logical system.

In particular, if W_ϕ is equal to a lattice $\mathbb{L}$ for all $W \in \mathcal{W}$, then defining $\mathcal{F} = \bigcup_{W\in\mathcal{W}} \mathcal{F}_W$, we have an $\mathbb{L}$-valued functional logical space $(A_\phi, \mathcal{F})$, which is equivalent to $(A_\phi, \mathcal{B})$ in the sense that the largest $\mathcal{F}$-logic on A_ϕ is equal to the largest $\mathcal{B}$-logic on A_ϕ. We call $(A_\phi, \mathcal{F})$ the **ϕ-sentential functional logical space** associated with the logical system. According to [5] and (3.13), the largest $\mathcal{F}$-logic on A_ϕ, that is, the relation $\models$ mentioned in the introduction satisfies $a_1 \cdots a_m \models b$ for elements $a_1, \ldots, a_m, b \in A_\phi$ iff the following holds for each pair Φ, ν of a C-denotation and an X-denotation into each $W \in \mathcal{W}$ and each element $l \in \mathbb{L}$:

$$(\Phi^* a_i)\nu \geq l \ (i = 1, \ldots, m) \implies (\Phi^* b)\nu \geq l.$$

In terms of functional expressions defined in Remark 4.1, $a_1 \cdots a_m \models b$ holds iff

$$a_i{}^\Phi(w_1, \ldots, w_n) \geq l \ (i = 1, \ldots, m) \implies b^\Phi(w_1, \ldots, w_n) \geq l$$

holds for each C-denotation Φ into each $W \in \mathcal{W}$ and each $(w_1, \dots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$, where $x_1, \dots, x_n$ are common free variables of $a_1, \dots, a_m, b$. Also, this fact together with Theorem 4.2 yields the following result for instance.

Theorem 3.1 *Let $a_1, \dots, a_m, b$ be elements of A_ϕ which satisfy $a_1 \cdots a_m \vDash b$. Let $x_1, \dots, x_n$ be distinct variables, and let $c_i \in A_{\sigma x_i}$ for $i = 1, \dots, n$. Let $a'_1, \dots, a'_m, b'$ be the elements obtained from $a_1, \dots, a_m, b$ by the substitution of $c_1, \dots, c_n$ for $x_1, \dots, x_n$. Assume that x_i is free from c_i in each of $a_1, \dots, a_m, b$ for $i = 1, \dots, n$. Then $a'_1 \cdots a'_m \vDash b'$ holds.*

4 Denotations, occurrences and substitutions

In order to test the theory in §3 for adequacy and prepare for its application [2,7,10], I will prove in §4.3 two general theorems on the relationship between denotations, occurrences and substitutions. §4.1 and §4.2 collect notation, terminology, and basic facts on occurrences and substitutions.

4.1 Occurrences

Here we let $(A, (\alpha_\lambda)_{\lambda \in L})$ be an algebra. If, for two elements $a, b \in A$, there exists an index $\lambda \in L$ such that $a = \alpha_\lambda(\dots, b, \dots)$, then we write $b \prec a$. If there exists a sequence $(b_i)_{i=0,\dots,n}$ $(n \geq 0)$ of elements of A such that $b_0 = a$, $b_n = b$ and either $b_i \prec b_{i-1}$ or $b_i = b_{i-1}$ for $i = 1, \dots, n$, then we write $b \underline{\prec\prec} a$ or say that b **occurs** in a, and call the sequence an **occurrence** of b in a. For each subset X of A and each element $a \in A$, we define

$$X^a = \{x \in X \mid x \underline{\prec\prec} a\}. \tag{4.1}$$

Henceforth in this subsection, we assume that $(A, (\alpha_\lambda)_{\lambda \in L})$ has a basis S.

Lemma 4.1 *For each element $a \in A$, S^a is a finite set.*

Proof We argue by induction on $r = \operatorname{Rank} a$. If $r = 0$ or $a \in S$, then $S^a = \{a\}$ by Theorem 2.2. Assume $r \geq 1$. Then the word form $\alpha_\lambda(a_1, \dots, a_k)$ of a satisfies $\operatorname{Rank} a = 1 + \sum_{j=1}^k \operatorname{Rank} a_j$ and $S^a \subseteq \bigcup_{j=1}^k S^{a_j}$ by Theorem 2.2. Since S^{a_j} is finite for $j = 1, \dots, k$ by the induction hypothesis, so is S^a.

We furthermore assume that L is contained in the set of the formal products of the elements of $\Gamma \amalg S$ for a set Γ, or more precisely

$$L \text{ is a subset of the free multiplicative semigroup over } \Gamma \amalg S. \tag{4.2}$$

Then, as a special case of (4.1), we may argue about the set S^λ of the elements of S which occur in $\lambda \in L$ in the semigroup:

$$S^\lambda = \{s \in S \mid s \underline{\prec\prec} \lambda\}. \tag{4.3}$$

Let $a \in A$ and $s \in S$. Then an occurrence $(s_i)_{i=0,\dots,n}$ of s in a is said to be **free**, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in L$ such that $s \in S^\lambda$. If there exists a free occurrence of s in a, we say that s **occurs free** in a or write $s \ll a$. For each subset X of S and each element $a \in A$, we define

$$X^a_{\text{free}} = \{x \in X \mid x \ll a\}.$$

Let $a \in A$, $s \in S$, and $b \in A$. Then an occurrence $(s_i)_{i=0,\dots,n}$ of s in a is said to be **free from** b, if $\{s_0, \dots, s_n\} \cap \operatorname{Im} \alpha_\lambda = \emptyset$ for each $\lambda \in L$ such that $(S^\lambda)^b_{\text{free}} \neq \emptyset$. Furthermore we say that s is **free from** b **in** a, if every free occurrence of s in a is free from b. Therefore if $s \not\ll a$, then obviously s is free from b in a.

Lemma 4.2 *If* $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, *then* $S^a_{\text{free}} = \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda$.

Proof Let $s \in S^{a_j}_{\text{free}} - S^\lambda$ for some $j \in \{1, \dots, k\}$. Then there exists a free occurrence $(s_i)_{i=1,\dots,n}$ of s in a_j, and if $s \in S^\mu$ for some $\mu \in L$, then $\{s_1, \dots, s_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$ and $\mu \neq \lambda$, hence $a \notin \operatorname{Im} \alpha_\mu$ by Theorem 2.2. Thus for $s_0 = a$, $(s_i)_{i=0,\dots,n}$ is a free occurrence of s in a. We have shown $S^a_{\text{free}} \supseteq \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda$. The rest of the proof is left to the reader because it is needed only in examples in §5.

Lemma 4.3 *Let* $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, $s \in S$, *and* $b \in A$. *Assume that* s *is free from* b *in* a *and* $s \ll a$. *Then* $(S^\lambda)^b_{\text{free}} = \emptyset$ *and* s *is free from* b *in* a_j *for* $j = 1, \dots, k$.

Proof Since $a \in \operatorname{Im} \alpha_\lambda$, $(S^\lambda)^b_{\text{free}} = \emptyset$ and $s \notin S^\lambda$. Let $(s_i)_{i=1,\dots,n}$ be a free occurrence of s in a_j for some $j \in \{1, \dots, k\}$. Then for $s_0 = a$, $(s_i)_{i=0,\dots,n}$ is a free occurrence of s in a as in the proof of Lemma 4.2. Therefore if $(S^\mu)^b_{\text{free}} \neq \emptyset$ for some $\mu \in L$, then $\{s_0, s_1, \dots, s_n\} \cap \operatorname{Im} \alpha_\mu = \emptyset$. Thus s is free from b in a_j.

Lemma 4.4 *Let* $a = \alpha_\lambda(a_1, \dots, a_k) \in A$, $s \in S - S^\lambda$, *and* $b \in A$. *Assume that* s *is free from* b *in* a. *Then* s *is free from* b *in* a_j *for* $j = 1, \dots, k$.

Proof This is a consequence of Lemmas 4.2 and 4.3.

4.2 Substitutions

Here we let (A, T, σ, S) be a USA on an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$ and assume that (4.2) is satisfied for a set Γ. Then S is a basis of $(A, (\alpha_\lambda)_{\lambda \in L})$ by Theorem 2.2, and so we can apply the definitions and results in §4.1.

Let $s_1, \dots, s_n$ ($n \geq 0$) be distinct elements of S and $c_1, \dots, c_n \in A$ with $\sigma s_i = \sigma c_i$ ($i = 1, \dots, n$). Then we will define, for each element $a \in A$, the element $a\left(\dfrac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) \in A$ satisfying

$$\sigma\left(a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)\right) = \sigma a \tag{4.4}$$

by induction on n and $r = \operatorname{Rank} a$ arranged in lexicographical order.

First if $n = 0$, then we define $a\begin{pmatrix} s_1, \dots, s_n \\ \hline c_1, \dots, c_n \end{pmatrix} = a$, hence (4.4). Assume $n \geq 1$. First if $r = 0$ or $a \in S$, then we define

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = \begin{cases} c_i & \text{if } a = s_i \text{ for some } i \in \{1, \dots, n\}, \\ a & \text{if } a \notin \{s_1, \dots, s_n\}, \end{cases} \tag{4.5}$$

hence (4.4). If $r \geq 1$, then let $\alpha_\lambda(a_1, \dots, a_k)$ be the word form of a and $\{s_1, \dots, s_n\} - S^\lambda = \{s_{i_1}, \dots, s_{i_m}\}$ $(0 \leq m \leq n)$ with $i_1 < \cdots < i_m$, and define

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = \alpha_\lambda\left(a_1\left(\frac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right), \dots, a_k\left(\frac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right)\right). \tag{4.6}$$

This is possible, because $r = 1 + \sum_{j=1}^k \operatorname{Rank} a_j$ by Theorem 2.2, and so $a'_j = a_j\left(\frac{s_{i_1}, \dots, s_{i_m}}{c_{i_1}, \dots, c_{i_m}}\right)$ has already been defined and satisfies $\sigma a'_j = \sigma a_j$ for $j = 1, \dots, k$ by induction, hence $(a'_1, \dots, a'_k) \in \operatorname{Dom} \alpha_\lambda$ because σ is a holomorphism. Since $\sigma a'_j = \sigma a_j$ for $j = 1, \dots, k$ and σ is a homomorphism, (4.4) also holds.

We denote the transformation $a \mapsto a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)$ on A by $\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right)$ and call it the (simultaneous) **substitution** of $c_1, \dots, c_n$ for $s_1, \dots, s_n$.

Lemma 4.5 *If* $a = \alpha_\lambda(a_1, \dots, a_k) \in A$ *and* $s_i \in S^\lambda$ *for some* $i \in \{1, \dots, n\}$, *then*

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = a\left(\frac{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right).$$

Proof This is because $\{s_1, \dots, s_n\} - S^\lambda = \{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n\} - S^\lambda$.

Lemma 4.6 *If* $a \in A$ *and* $s_i \not\ll a$ *for some* $i \in \{1, \dots, n\}$, *then*

$$a\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = a\left(\frac{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right).$$

Proof We argue by induction on n and $r = \operatorname{Rank} a$ as in the definition of substitutions. If $r = 0$ or $a \in S$, then $a \neq s_i$ by our assumption, and (4.5) yields the desired result for $n = 1, 2, \dots$. Therefore assume $r \geq 1$, and let $\alpha_\lambda(a_1, \dots, a_k)$ be the word form of a. Then $r = 1 + \sum_{j=1}^k \operatorname{Rank} a_j$ by Theorem 2.2. We assume $s_i \notin S^\lambda$ in view of Lemma 4.5. Then $s_i \not\ll a_j$ by Lemma 4.2, and so

$$a_j\left(\frac{s_1, \dots, s_n}{c_1, \dots, c_n}\right) = a_j\left(\frac{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n}{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n}\right)$$

by the induction hypothesis for $j = 1, \ldots, k$, and (4.6) yields the desired result, if $\{s_1, \ldots, s_n\} \cap S^\lambda = \emptyset$. Therefore assume $s_h \in S^\lambda$ for some $h \in \{1, \ldots, n\}$. Then $h \neq i$, hence $n \geq 2$, and so the proof is complete if $n = 1$. If $h < i$, then

$$\begin{aligned} & a\left(\frac{s_1, \ldots, s_{h-1}, s_{h+1}, \ldots, s_n}{c_1, \ldots, c_{h-1}, c_{h+1}, \ldots, c_n}\right) \\ &= a\left(\frac{s_1, \ldots, s_{h-1}, s_{h+1}, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n}{c_1, \ldots, c_{h-1}, c_{h+1}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n}\right) \end{aligned}$$

by the induction hypothesis, and Lemma 4.5 yields the desired result, and similarly for the case $i < h$.

4.3 Two denotation theorems

Throughout this subsection, we let $(A, T, \sigma, S, C, X, \Gamma)$ be a formal language with index set L, W be its denotable world, and Φ be a C-denotation into W. Also we assume that the metaworld $(W^{V_{X,W}}, T, \rho)$ has been constructed by some interpretation I_W of $L \cap \Gamma X$ on W. Then the metadenotation $\Phi^* \in A \rightarrow W^{V_{X,W}}$ is defined. Let $(\alpha_\lambda)_{\lambda \in L}$, $(\beta_\lambda)_{\lambda \in L}$, and $(\omega_\lambda)_{\lambda \in M}$ be the operations of A, $W^{V_{X,W}}$, and W respectively, where $M = L \cap \Gamma$ by (3.1). Note that (4.2) is satisfied and so we can apply the definitions and results in §4.1 and §4.2. Furthermore, since $L \subseteq \Gamma \cup \Gamma X$, the following holds as to (4.3):

$$S^\lambda = \begin{cases} \emptyset & \text{if } \lambda \in M, \\ \{x\} & \text{if } \lambda \in L \cap \Gamma x \ (x \in X). \end{cases} \tag{4.7}$$

Theorem 4.1 *Let $a \in A$ and $v, v' \in V_{X,W}$. Assume $vx = v'x$ for all $x \in X^a_{\text{free}}$. Then $(\Phi^* a)v = (\Phi^* a)v'$.*

Proof We argue by induction on $r = \text{Rank}\, a$. Assume $r = 0$ or $a \in S = C \cup X$. If $a \in C$, then $(\Phi^* a)v = \Phi a = (\Phi^* a)v'$ by (3.11). If $a \in X$, then since $a \ll a$ by Theorem 2.2, $(\Phi^* a)v = va = v'a = (\Phi^* a)v'$ by (3.11) and our assumption.

Assume $r \geq 1$, and let $\alpha_\lambda(a_1, \ldots, a_k)$ be the word form of a. Then $r = 1 + \sum_{j=1}^k \text{Rank}\, a_j$ by Theorem 2.2. Also, since Φ^* is an L-homomorphism, we have $(\Phi^* a)v = \big(\beta_\lambda(\Phi^* a_1, \ldots, \Phi^* a_k)\big)v$.

Assume $\lambda \in M$. Then, since the projection by v is an M-homomorphism, the above equation may be rewritten $(\Phi^* a)v = \omega_\lambda\big((\Phi^* a_1)v, \ldots, (\Phi^* a_k)v\big)$, and a similar equation holds with v replaced by v'. Also since $\lambda \in M$, $S^\lambda = \emptyset$ by (4.7), and so Lemma 4.2 shows that $X^{a_j}_{\text{free}} \subseteq X^a_{\text{free}}$, hence $(\Phi^* a_j)v = (\Phi^* a_j)v'$ by the induction hypothesis for $j = 1, \ldots, k$. Therefore $(\Phi^* a)v = (\Phi^* a)v'$.

Assume $\lambda \notin M$. Then $\lambda \in \Gamma x$ for some $x \in X$ and $k = 1$. Therefore $(\Phi^* a)v = \big(\beta_\lambda(\Phi^* a_1)\big)v = \lambda_W\big((\Phi^* a_1)\big((x/\Box)v\big)\big)$ by (3.9), and a similar equation holds with v replaced by v'. Thus, we only need to show $(\Phi^* a_1)\big((x/\Box)v\big) = (\Phi^* a_1)\big((x/\Box)v'\big)$, which means that $(\Phi^* a_1)\big((x/w)v\big) = (\Phi^* a_1)\big((x/w)v'\big)$ for each $w \in W_{\sigma x}$ because of (3.6), This will follow from the induction hypothesis, once we show $\big((x/w)v\big)y = \big((x/w)v'\big)y$ for each $y \in X^{a_1}_{\text{free}}$. If $y = x$, this holds by (3.5). If $y \neq x$, then $y \notin S^\lambda$ by (4.7), and so $y \in X^{a}_{\text{free}}$ by Lemma 4.2, hence $\big((x/w)v\big)y = vy = v'y = \big((x/w)v'\big)y$ by (3.5) and our assumption.

Remark 4.1 Let $a \in A$. Then Lemma 4.1 shows that there exists a sequence $x_1, \dots, x_n$ of distinct variables with $n \geq 0$ which satisfy $X^{a}_{\text{free}} \subseteq \{x_1, \dots, x_n\}$. We call them **free variables** of a, and define the function $F \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n} \to W_{\sigma a}$ as follows. If $(w_1, \dots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$, then there exists an element $v \in V_{X,W}$ which satisfies $vx_i = w_i$ for $i = 1, \dots, n$. Using any one of such $v \in V_{X,W}$, we define $F(w_1, \dots, w_n) = (\Phi^* a)v$. Since Φ^* is sort-consistent, $(\Phi^* a)v$ belongs to $W_{\sigma a}$. Moreover, Theorem 4.1 shows that $(\Phi^* a)v$ does not depend on the choice of such v. Thus, the function F is determined by Φ, a, and $x_1, \dots, x_n$. We call it the **functional expression** of a under Φ with respect to $x_1, \dots, x_n$, and denote it by $a^\Phi(x_1, \dots, x_n)$, while we abbreviate its value at $(w_1, \dots, w_n) \in W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ to $a^\Phi(w_1, \dots, w_n)$.

For instance if $a \in C$, then arbitrary distinct variables $x_1, \dots, x_n$ are free variables of a, and (3.11) shows that $a^\Phi(x_1, \dots, x_n)$ is of constant value Φa. If $x \in X$, then arbitrary distinct variables $x_1, \dots, x_n$ with $x \in \{x_1, \dots, x_n\}$ are free variables of x, and if $x = x_i$, then (3.11) shows that $x^\Phi(x_1, \dots, x_n)$ is the projection of $W_{\sigma x_1} \times \cdots \times W_{\sigma x_n}$ onto $W_{\sigma x_i}$.

Functional expressions are useful for understanding the intention of the definitions of $(\omega_\lambda)_{\lambda \in M}$ and I_W (cf. §5.2 and §5.3).

Elements $a, b \in A$ satisfy $\Phi^* a = \Phi^* b$ iff $a^\Phi(x_1, \dots, x_n) = b^\Phi(x_1, \dots, x_n)$ for their common free variables $x_1, \dots, x_n$. Thus, the metadenotation Φ^* may be regarded as sorting elements of A according to their functional expressions.

In order to state the next theorem, we generalize the transformation (3.5). Let $x_1, \dots, x_n$ ($n \geq 0$) be distinct variables of A and let $w_i \in W_{\sigma x_i}$ ($i = 1, \dots, n$). Then we define the transformation $v \mapsto \left(\dfrac{x_1, \dots, x_n}{w_1, \dots, w_n}\right) v$ on $V_{X,W}$ by

$$\left(\left(\frac{x_1, \dots, x_n}{w_1, \dots, w_n}\right) v\right) y = \begin{cases} w_i & \text{if } y = x_i \text{ for some } i \in \{1, \dots, n\}, \\ vy & \text{if } y \in X - \{x_1, \dots, x_n\} \end{cases} \tag{4.8}$$

and denote it by $\left(\dfrac{x_1, \dots, x_n}{w_1, \dots, w_n}\right)$. Then the following holds for $i = 2, \dots, n$:

$$\left(\frac{x_1, \dots, x_{i-1}}{w_1, \dots, w_{i-1}}\right)\left(\frac{x_i, \dots, x_n}{w_i, \dots, w_n}\right) = \left(\frac{x_i, \dots, x_n}{w_i, \dots, w_n}\right)\left(\frac{x_1, \dots, x_{i-1}}{w_1, \dots, w_{i-1}}\right). \tag{4.9}$$

Theorem 4.2 *Let* $x_1, \dots, x_n$ *be distinct variables of* A *and* $a, c_1, \dots, c_n \in A$ ($n \geq 0$). *Assume that* $\sigma x_i = \sigma c_i$ *and* x_i *is free from* c_i *in* a *for* $i = 1, \dots, n$. *Then the following holds for each* $v \in V_{X,W}$:

$$\left(\Phi^*\left(a\left(\frac{x_1, \dots, x_n}{c_1, \dots, c_n}\right)\right)\right) v = (\Phi^* a)\left(\left(\frac{x_1, \dots, x_n}{(\Phi^* c_1)v, \dots, (\Phi^* c_n)v}\right) v\right). \tag{4.10}$$

Proof We argue by induction on n and $r = \mathrm{Rank}\, a$ as in the definition of the substitutions. If $n = 0$, then both $\begin{pmatrix} x_1, \ldots, x_n \\ \hline c_1, \ldots, c_n \end{pmatrix}$ and $\left(\dfrac{x_1, \ldots, x_n}{(\Phi^* c_1)v, \ldots, (\Phi^* c_n)v}\right)$ are identity transformations, and so (4.10) holds. Therefore assume $n \geq 1$.

Assume $r = 0$ or $a \in S = C \cup X$. If $a \in C$, then $a \notin \{x_1, \ldots, x_n\}$, and so the left-hand side of (4.10) is equal to $(\Phi^* a)v$ by (4.5), and therefore, both sides are equal to Φa by (3.11). If $a = x_i$ for some $i \in \{1, \ldots, n\}$, then the left-hand side is equal to $(\Phi^* c_i)v$ by (4.5), while the right-hand side is also equal to $(\Phi^* c_i)v$ by (3.11) and (4.8). If $a \in X - \{x_1, \ldots, x_n\}$, then the left-hand side is equal to va as in the case $a \in C$, while the right-hand side is also equal to va by (3.11) and (4.8).

Assume $r \geq 1$, and let $\alpha_\lambda(a_1, \ldots, a_k)$ be the word form of a. Define $u = \left(\dfrac{x_1, \ldots, x_n}{(\Phi^* c_1)v, \ldots, (\Phi^* c_n)v}\right) v$, $b = a \left(\dfrac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right)$, and $b_j = a_j \left(\dfrac{x_1, \ldots, x_n}{c_1, \ldots, c_n}\right)$ for $j = 1, \ldots, k$. We have to show $(\Phi^* b)v = (\Phi^* a)u$. Notice $r = 1 + \sum_{j=1}^{k} \mathrm{Rank}\, a_j$.

Suppose $x_i \not\ll a$ for some $i \in \{1, \ldots, n\}$. Then Lemma 4.6 shows that the left-hand side of (4.10) remains unchanged by the deletion of $\dfrac{x_i}{c_i}$, while Theorem 4.1 shows that the right-hand side remains unchanged by the deletion of $\dfrac{x_i}{(\Phi^* c_i)v}$. Therefore, (4.10) holds by the induction hypothesis.

Therefore assume $x_i \ll a$ for $i = 1, \ldots, n$. Then $\{x_1, \ldots, x_n\} \cap S^\lambda = \emptyset$. Hence $b = \alpha_\lambda(b_1, \ldots, b_k)$ by (4.6), and x_i is free from c_i in a_j by Lemma 4.4 for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Therefore $(\Phi^* b_j)v = (\Phi^* a_j)u$ by the induction hypothesis for $j = 1, \ldots, k$. Also, since Φ^* is an L-homomorphism, we have $(\Phi^* b)v = \big(\beta_\lambda(\Phi^* b_1, \ldots, \Phi^* b_k)\big)v$ and $(\Phi^* a)u = \big(\beta_\lambda(\Phi^* a_1, \ldots, \Phi^* a_k)\big)u$.

If $\lambda \in M$, then since the projections by v and u are M-homomorphisms, the above equations may be rewritten $(\Phi^* b)v = \omega_\lambda\big((\Phi^* b_1)v, \ldots, (\Phi^* b_k)v\big)$ and $(\Phi^* a)u = \omega_\lambda\big((\Phi^* a_1)u, \ldots, (\Phi^* a_k)u\big)$. Since $(\Phi^* b_j)v = (\Phi^* a_j)u$ for $j = 1, \ldots, k$, we conclude that $(\Phi^* b)v = (\Phi^* a)u$ holds.

Assume $\lambda \notin M$. Then $\lambda \in \Gamma x$ for some $x \in X$ and $k = 1$. Therefore we have $(\Phi^* b)v = \lambda_W\big((\Phi^* b_1)\big((x/\Box)v\big)\big)$ and $(\Phi^* a)u = \lambda_W\big((\Phi^* a_1)\big((x/\Box)u\big)\big)$ by (3.9), and so in view of (3.6), we only need to show $(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\big((x/w)u\big)$ for all $w \in W_{\sigma x}$. Recall that x_i is free from c_i in a_1 for $i = 1, \ldots, n$. Therefore

$$(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\left(\left(\frac{x_1, \ldots, x_n}{(\Phi^* c_1)v', \ldots, (\Phi^* c_n)v'}\right)\big((x/w)v\big)\right)$$

by the induction hypothesis, where $v' = (x/w)v$. We are assuming that x_i is free from c_i in $a = \alpha_\lambda a_1$ and $x_i \ll a$ for $i = 1, \ldots, n$, and $x \in S^\lambda$. Therefore, $x \not\ll c_i$ by Lemma 4.3, and so $(\Phi^* c_i)v' = (\Phi^* c_i)v$ by Theorem 4.1 for $i = 1, \ldots, n$. Recall $\{x_1, \ldots, x_n\} \cap S^\lambda = \emptyset$ while $x \in S^\lambda$. Therefore $x_1, \ldots, x_n, x$ are distinct, and so $\left(\dfrac{x_1, \ldots, x_n}{(\Phi^* c_1)v, \ldots, (\Phi^* c_n)v}\right)\big((x/w)v\big) = (x/w)u$ by (4.9). Thus $(\Phi^* b_1)\big((x/w)v\big) = (\Phi^* a_1)\big((x/w)u\big)$ as desired.

5 Examples

Here are given common examples of logical systems $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ on formal languages $(A, T, \sigma, S, C, X, \Gamma)$. Uncommon examples will be given in [3].

5.1 Propositional logics as degenerate logical systems

Here we assume the conditions (1) $S = X \neq \emptyset$ and $C = \emptyset$, (2) T is a total singleton, say $\{\phi\}$, and (3) $\Gamma = L$, hence $M = L$ as to (3.1) and $L \cap \Gamma X = \emptyset$.

Propositional logics, either classical or non-classical, fall within this case when given semantics by truth functions or Kripke frames, because the following holds as to §3.1–§3.8 under the conditions (1)–(3).

First, (2) means that (A, S) is a UTA and $A = A_\phi$. Secondly, (2) and (3) mean that the denotable worlds for A are the non-empty total algebras W similar to A and $W = W_\phi$. Thirdly, (3) shows that we are not involved with the interpretations of $L \cap \Gamma X$ on W. The results so far show that a logical system under (1)–(3) is a UTA (A, S) with $S \neq \emptyset$ equipped with a non-empty collection $\mathcal{W}$ of non-empty total algebras similar to A. Fourthly, (1) shows that $\emptyset$ is the unique C-denotation into W. Also, since $S = X = X_\phi$ and $W = W_\phi$, $V_{X,W}$ is equal to $S{\rightarrow}W$. Fifthly, since $M = L$ and $V_{X,W} = S{\rightarrow}W$, the metaworld associated with W is the $(S{\rightarrow}W)$-power $W^{S\rightarrow W}$ of W. Sixthly, the metadenotation $\emptyset^*$ associated with the C-denotation $\emptyset$ into W is a homomorphism of A into $W^{S\rightarrow W}$ which satisfies $(\emptyset^* s)\nu = \nu s$ for each $s \in S$ and each $\nu \in S{\rightarrow}W$. For each $\nu \in S{\rightarrow}W$ in view of (3.13), define the mapping $\emptyset^\nu \in A{\rightarrow}W$ by $\emptyset^\nu = \mathrm{pr}_\nu \emptyset^*$, that is, $\emptyset^\nu a = (\emptyset^* a)\nu$ for each $a \in A$. Also, define $\mathcal{F}_W = \{\emptyset^\nu \mid \nu \in S{\rightarrow}W\}$ in view of (3.14). Then $\emptyset^\nu$ is a homomorphism by (3.12) and satisfies $\emptyset^\nu|_S = \nu$. Conversely, if $f \in A{\rightarrow}W$ is a homomorphism, then $\nu = f|_S$ belongs to $S{\rightarrow}W$, and since (A, S) is a UTA, ν is uniquely extended to a homomorphism of A into W, hence $f = \emptyset^\nu$. Thus, $\mathcal{F}_W$ is the set of all homomorphisms of A into W. Now assume that the logical system $(A, \mathcal{W})$ has a truth, that is, $A \neq \emptyset$ and every member $W \in \mathcal{W}$ is a non-trivial lattice with the smallest element and the largest element. Then a logical space $(A, \mathcal{B})$ is associated with $(A, \mathcal{W})$ as in §3.8. In particular when $\mathcal{W}$ is a singleton $\{W\}$, $(A, \mathcal{B})$ is equivalent to the functional logical space $(A, \mathcal{F}_W)$.

5.2 Classical first-order predicate logic

The first-order predicate language may be defined to be the formal language $(A, T, \sigma, S, C, X, \Gamma)$ with the following syntax $(T, \tau, S, C, X, \Gamma)$. First, let S, C, and X be sets with $S = C \amalg X$ and $X \neq \emptyset$. Next, let $\Gamma = \{\wedge, \vee, \Rightarrow, \neg, \forall, \exists\} \amalg F \amalg P$ with distinct symbols $\wedge, \vee, \Rightarrow, \neg, \forall, \exists$ and sets F, P with $P \neq \emptyset$. Next, let T be a set $\{\epsilon, \phi\}$ of distinct elements ϵ, ϕ equipped with the operations denoted by the symbols in the subset $L = \{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\} \cup F \cup P$ of $(\Gamma \amalg S)^*$ and satisfying

$$
\begin{array}{ll}
\mathrm{Dom}\wedge = \mathrm{Dom}\vee = \mathrm{Dom}\Rightarrow = \{\phi\}\times\{\phi\}, & \phi\wedge\phi = \phi\vee\phi = \phi\Rightarrow\phi = \phi, \\
\mathrm{Dom}\neg = \mathrm{Dom}\,\forall x = \mathrm{Dom}\,\exists x = \{\phi\}, & \neg\phi = \forall x\phi = \exists x\phi = \phi, \\
\mathrm{Dom}\, f = \{\epsilon\}\times\cdots\times\{\epsilon\}, & f(\epsilon,\dots,\epsilon) = \epsilon, \\
\mathrm{Dom}\, p = \{\epsilon\}\times\cdots\times\{\epsilon\}, & p(\epsilon,\dots,\epsilon) = \phi
\end{array}
\tag{5.1}
$$

with arities of $f \in F$ and $p \in P$ arbitrary. Finally, define the pre-sorting $\tau \in S \to T$ by $\tau s = \epsilon$ for all $s \in S$. Then $L \subseteq \Gamma \cup \Gamma X$ and $L \cap \Gamma X$ consists of the unary operations $\forall x, \exists x$ ($x \in X$) as required in §3.1. Also, $M = \{\wedge, \vee, \Rightarrow, \neg\} \cup F \cup P$ as to (3.1).

The syntax implies that the denotable worlds W for A are the direct unions $W_\epsilon \amalg W_\phi$ equipped with operations $\wedge, \vee, \Rightarrow, \neg$, $f \in F$, and $p \in P$ satisfying

$$
\begin{array}{ll}
\mathrm{Dom}\wedge = \mathrm{Dom}\vee = \mathrm{Dom}\Rightarrow = W_\phi \times W_\phi, & \mathrm{Im}\wedge,\ \mathrm{Im}\vee,\ \mathrm{Im}\Rightarrow \subseteq W_\phi, \\
\mathrm{Dom}\neg = W_\phi, & \mathrm{Im}\neg \subseteq W_\phi, \\
\mathrm{Dom}\, f = W_\epsilon \times\cdots\times W_\epsilon, & \mathrm{Im}\, f \subseteq W_\epsilon, \\
\mathrm{Dom}\, p = W_\epsilon \times\cdots\times W_\epsilon, & \mathrm{Im}\, p \subseteq W_\phi.
\end{array}
$$

In order to consider the classical first-order predicate logic, we define $\mathcal{W}$ to be the collection of the denotable worlds W such that W_ϕ is the binary lattice $\mathbb{T} = \{0, 1\}$ on which $\wedge, \vee, \Rightarrow, \neg$ are the meet, join, implication and complement.

We finally define the interpretation I_W of the set $L \cap \Gamma X$ of the variable operations on each $W \in \mathcal{W}$. Let $\lambda \in L \cap \Gamma X$. Then $\lambda = \forall x, \exists x$ ($x \in X$), hence $T_\lambda = \{\phi\}$ and $\lambda\phi = \phi$ by (5.1). Therefore, (3.3) and (3.4) show that the meaning λ_W of λ on W is a mapping $\lambda_W \in (W_{\sigma x} \to W_\phi) \to W_\phi$. We define

$$
\lambda_W(f) = \begin{cases} \inf\{fw \mid w \in W_{\sigma x}\} & \text{if } \lambda = \forall x, \\ \sup\{fw \mid w \in W_{\sigma x}\} & \text{if } \lambda = \exists x \end{cases}
\tag{5.2}
$$

for each $f \in W_{\sigma x} \to W_\phi$, where the infimum and supremum are taken with respect to the usual order on $W_\phi = \mathbb{T}$. This completes the definition of the logical system for the classical first-order predicate logic.

The intention of (5.2) for $\lambda = \forall x$ is as follows. First, (3.8) and (3.10) show that the unary operation β_λ on the metaworld $W^{V_{X,W}}$ satisfies $\mathrm{Dom}\,\beta_\lambda = V_{X,W} \to W_\phi$ and $\mathrm{Im}\,\beta_\lambda \subseteq V_{X,W} \to W_\phi$. Next, (3.9), (5.2) and (3.6) show that

$$
(\beta_\lambda \varphi)v = \inf\bigl\{\varphi\bigl((x/w)v\bigr) \mid w \in W_{\sigma x}\bigr\}
\tag{5.3}
$$

for each $\varphi \in V_{X,W} \to W_\phi$ and $v \in V_{X,W}$.

Let $a \in A_\phi$ and Φ be a C-denotation into W. Then $\lambda a \in A_\phi$ by (5.4) below, hence $\Phi^* a$, $\Phi^*(\lambda a) \in V_{X,W} \to W_\phi$, and

$$
\bigl(\Phi^*(\lambda a)\bigr)v = \bigl(\beta_\lambda(\Phi^* a)\bigr)v = \inf\bigl\{(\Phi^* a)\bigl((x/w)v\bigr) \mid w \in W_{\sigma x}\bigr\}
$$

by (5.3) for each $v \in V_{X,W}$.

Assume that $x, x_2, \dots, x_n$ are free variables of a. Then $x_2, \dots, x_n$ are those of λa by Lemma 4.2, and the above equation shows that the functional expressions $a^\Phi(x, x_2, \dots, x_n)$ and $(\lambda a)^\Phi(x_2, \dots, x_n)$, which are both $\mathbb{T}$-valued, satisfy

$$(\lambda a)^\Phi(w_2, \dots, w_n) = \inf\bigl\{a^\Phi(w, w_2, \dots, w_n) \mid w \in W_{\sigma x}\bigr\}$$

for each $(w_2, \dots, w_n) \in W_{\sigma x_2} \times \cdots \times W_{\sigma x_n}$. Thus

$$(\forall x a)^\Phi(w_2, \dots, w_n) = 1 \iff a^\Phi(w, w_2, \dots, w_n) = 1 \text{ for } \textit{all } w \in W_{\sigma x}.$$

The intention of (5.2) for $\lambda = \exists x$ is similar with "all" replaced by "some."

We can visibly reconstruct A and σ as mentioned in §3.1. First, since σ is a holomorphism, (5.1) shows that the operations of A satisfy

$$\begin{array}{ll} \mathrm{Dom}\wedge = \mathrm{Dom}\vee = \mathrm{Dom}\Rightarrow = A_\phi \times A_\phi, & \mathrm{Im}\wedge, \mathrm{Im}\vee, \mathrm{Im}\Rightarrow \subseteq A_\phi, \\ \mathrm{Dom}\neg = \mathrm{Dom}\forall x = \mathrm{Dom}\exists x = A_\phi, & \mathrm{Im}\neg, \mathrm{Im}\forall x, \mathrm{Im}\exists x \subseteq A_\phi, \\ \mathrm{Dom}\, f = A_\epsilon \times \cdots \times A_\epsilon, & \mathrm{Im}\, f \subseteq A_\epsilon, \\ \mathrm{Dom}\, p = A_\epsilon \times \cdots \times A_\epsilon, & \mathrm{Im}\, p \subseteq A_\phi. \end{array} \tag{5.4}$$

Next, let B be the closure $[S]_F$ of S in the F-reduct A_F of A. Then, $B \neq \emptyset$ and B is the union $\bigcup_{n=0}^\infty S_n$ of the descendants S_n $(n = 0, 1, \dots)$ of S, and since S is contained in the ϵ-part A_ϵ of A, (5.4) shows that $B \subseteq A_\epsilon$ and S_n is inductively described as follows. First, $S_0 = S$. Next for $n \geq 1$, S_n consists of the elements $f(b_1, \dots, b_k)$, where $f \in F$ and $b_i \in S_{n_i}$ $(i = 1, \dots, k)$ for some non-negative integers $n_1, \dots, n_k$ satisfying $n = 1 + \sum_{i=1}^k n_i$.

Next, let B_1 be the first descendant of B in the P-reduct A_P of A. Then B_1 consists of the elements $p(b_1, \dots, b_k)$ with $p \in P$ and $b_1, \dots, b_k \in B$. Therefore $B_1 \neq \emptyset$ and (5.4) shows that B_1 is contained in the ϕ-part A_ϕ of A.

Next, let D be the closure of B_1 in the $\{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$-reduct of A. Then (5.4) shows that $D = \bigcup_{n=0}^\infty D_n \subseteq A_\phi$ and D_n $(n = 0, 1, \dots)$ are inductively described as follows. First, $D_0 = B_1$. Next for $n \geq 1$, D_n consists of the elements $d_1 \wedge d_2$, $d_1 \vee d_2$, $d_1 \Rightarrow d_2$, $\neg d$, $\forall x d$, and $\exists x d$, where $d_i \in D_{n_i}$ $(i = 1, 2)$, $n = 1 + n_1 + n_2$, $d \in D_{n-1}$, and $x \in X$.

Let $(a_1, \dots, a_k) \in (B \cup D)^k \cap \mathrm{Dom}\,\lambda$ for $\lambda \in L$ and $a = \lambda(a_1, \dots, a_k)$. Recall $B \subseteq A_\epsilon$ and $D \subseteq A_\phi$. Hence if $\lambda \in F$, then $a_1, \dots, a_k \in B$, and since B is closed under every operation in F, we have $a \in B$. Next if $\lambda \in P$, then also $a_1, \dots, a_k \in B$, hence $a \in B_1 \subseteq D$. Finally if $\lambda \in \{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$, then $a_1, \dots, a_k \in D$, and since D is closed under $\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x$ $(x \in X)$, we have $a \in D$. Therefore $B \cup D$ is closed under every operation of A, hence $A = B \cup D$ because $A = [S]$ and $S \subseteq B \cup D$. Since $B \subseteq A_\epsilon$, $D \subseteq A_\phi$, and $A_\epsilon \cap A_\phi = \emptyset$, we conclude that $B = A_\epsilon$ and $D = A_\phi$. Furthermore, it follows from Theorem 2.2 that A_ϵ as an F-algebra is a UTA over S and that A_ϕ as a $\{\wedge, \vee, \Rightarrow, \neg, \forall x, \exists x \mid x \in X\}$-algebra is a UTA over the set $D_0 = B_1 = \{p(a_1, \dots, a_k) \mid p \in P,\ a_1, \dots, a_k \in A_\epsilon\}$. Consequently, we have $A_\epsilon = \coprod_{n=0}^\infty S_n$ and $A_\phi = \coprod_{n=0}^\infty D_n$.

Thus, the formal language A defined above is the set of the terms and formulas in the usually defined first-order predicate logic with F and P serving as the sets of the function symbols and the predicate symbols respectively.

5.3 Typed λ-calculus

The set of the λ-terms in the typed λ-calculus may be defined to be the formal language $(A, T, \sigma, S, C, X, \Gamma)$ with the following syntax $(T, \tau, S, C, X, \Gamma)$. First, let S, C, and X be sets with $S = C \amalg X$ and $X \neq \emptyset$. Next, let Γ be a set $\{\bullet, \lambda\}$ of distinct elements $\bullet$ and λ. Next, let T be the UTA over a set T_0 with one binary operation $\to$. Then $T = \coprod_{n=0}^{\infty} T_n$ and T_n $(n = 1, 2, \dots)$ are the sets of the elements $t_1 \to t_2$ with $t_i \in T_{n_i}$ $(i = 1, 2)$ and $n = 1 + n_1 + n_2$. Equip the set T with the operations denoted by the symbols in the subset $L = \{\bullet, \lambda x \mid x \in X\}$ of $(\Gamma \amalg S)^*$ and satisfying

$$\operatorname{Dom} \bullet = \{(t \to u, t) \mid (t, u) \in T^2\}, \qquad (t \to u) \bullet t = u,$$
$$\operatorname{Dom} \lambda x = T, \qquad \lambda x\, t = \tau x \to t,$$

where the pre-sorting $\tau \in S \to T$ is arbitrary. Then $L \subseteq \Gamma \cup \Gamma X$ and $L \cap \Gamma X$ consists of the unary operations λx $(x \in X)$. Also, $M = \{\bullet\}$ as to (3.1).

The formal language A thus defined is the set of λ-terms in the typed λ-calculus, where T_0 is the set of the basic types, $\bigcup_{n=1}^{\infty} T_n$ is the set of the functional types, the operation $\bullet$ is the application, and the operation λx $(x \in X)$ is a λ-abstraction.

We will construct a specific denotable world W for A. Since the set W is the direct union $W = \coprod_{t \in T} W_t$ of its t-parts W_t $(t \in T)$, we first define a family of sets W_t $(t \in T)$. Since (T, T_0) is a UTA with respect to the operation $\to$, we define W_t by induction on the rank n of t with respect to $\to$. If $n = 0$ or $t \in T_0$, we define W_t to be an arbitrary non-empty set. Suppose $n \geq 1$. Then $t = t_1 \to t_2$ with $t_i \in T_{n_i}$ $(i = 1, 2)$ and $n = 1 + n_1 + n_2$, and W_{t_i} $(i = 1, 2)$ have already been defined. Therefore we define $W_t = W_{t_1} \to W_{t_2}$. It remains to define a binary operation $\bullet$ on W. First, we define

$$\operatorname{Dom} \bullet = \textstyle\bigcup_{(t,u) \in T^2} (W_{t \to u} \times W_t) = \coprod_{(t,u) \in T^2} (W_{t \to u} \times W_t).$$

Next for $a \in W_{t \to u}$ and $b \in W_t$, since $W_{t \to u} = W_t \to W_u$, we may define $a \bullet b$ to be the image ab in W_u of $b \in W_t$ by the mapping $a \in W_t \to W_u$.

We finally define the interpretation I_W of $L \cap \Gamma X$ on W. Let $\mu = \lambda x$. Then $T_\mu = T$, and $W_{\mu t} = W_{\sigma x \to t} = W_{\sigma x} \to W_t$ for each $t \in T$. Therefore, (3.3) and (3.4) show that the meaning μ_W of μ on W is a transformation on $\bigcup_{t \in T} (W_{\sigma x} \to W_t)$ satisfying $\mu_W(W_{\sigma x} \to W_t) \subseteq W_{\sigma x} \to W_t$ for each $t \in T$. Therefore we define

$$\mu_W(f) = f \tag{5.5}$$

for each $f \in \bigcup_{t \in T} (W_{\sigma x} \to W_t)$.

The intention of this is as follows. First, (3.8) and (3.10) show that the unary operation β_μ on the metaworld $W^{V_{X,W}}$ satisfies $\operatorname{Dom} \beta_\mu = \bigcup_{t \in T} (V_{X,W} \to W_t)$ and $\beta_\mu(V_{X,W} \to W_t) \subseteq V_{X,W} \to (W_{\sigma x} \to W_t)$ for each $t \in T$. Next, (3.9), (5.5) and (3.6) show that

$$\big((\beta_\mu \varphi) v\big) w = \varphi\big((x/w) v\big) \tag{5.6}$$

for each $\varphi \in \bigcup_{t \in T}(V_{X,W} \to W_t)$, $v \in V_{X,W}$, and $w \in W_{\sigma x}$.

Let $a \in A_t$ and Φ be a C-denotation into W. Then $\mu a \in A_{\sigma x \to t}$, hence $\Phi^* a \in V_{X,W} \to W_t$, $\Phi^*(\mu a) \in V_{X,W} \to (W_{\sigma x} \to W_t)$, and

$$\big((\Phi^*(\mu a))v\big)w = \big((\beta_\mu(\Phi^* a))v\big)w = (\Phi^* a)\big((x/w)v\big)$$

by (5.6) for each $v \in V_{X,W}$ and $w \in W_{\sigma x}$.

Assume that $x, x_2, \dots, x_n$ are free variables of a. Then $x_2, \dots, x_n$ are those of μa by Lemma 4.2, and the above equation shows that the functional expressions $a^\Phi(x, x_2, \dots, x_n)$ and $(\mu a)^\Phi(x_2, \dots, x_n)$, which are W_t-valued and $(W_{\sigma x} \to W_t)$-valued respectively, satisfy

$$\big((\lambda x\, a)^\Phi(w_2, \dots, w_n)\big)w = a^\Phi(w, w_2, \dots, w_n)$$

for each $(w, w_2, \dots, w_n) \in W_{\sigma x} \times W_{\sigma x_2} \times \cdots \times W_{\sigma x_n}$. In particular if $W_t = \mathbb{T}$ and we identify $W_{\sigma x} \to W_t$ with $\mathcal{P}(W_{\sigma x})$, the above equation may be written

$$(\lambda x\, a)^\Phi(w_2, \dots, w_n) = \{w \in W_{\sigma x} \mid a^\Phi(w, w_2, \dots, w_n) = 1\}.$$

6 Appendix: Universal sorted algebras and free algebras

The latter half of Theorem 2.1 is a routine consequence of the universalities of A and A'. The proof of the former half is as follows.

Let $(t_\lambda)_{\lambda \in L}$ be the operations of T and m_λ be an arity of t_λ for each $\lambda \in L$. Let $K = S \amalg L \amalg \{\, [,] \,\}$ be the direct union of the prime set S, the index set L, and the bracket set $\{\, [,] \,\}$ consisting of the left bracket and the right bracket. Let K^+ be the set of words on the alphabet K.

For each non-negative integer n, we will construct the subset S_n of K^+ and $\sigma_n \in S_n \to T$ as follows by induction on n. First, we define $S_0 = S$ and $\sigma_0 = \tau$. Next for $n \geq 1$, we define S_n to be the set of all words $[\lambda a_1 \dots a_k]$, where $\lambda \in L$, $a_j \in S_{n_j}$ $(j = 1, \dots, k)$, $n = 1 + \sum_{j=1}^{k} n_j$, $(\sigma_{n_1} a_1, \dots, \sigma_{n_k} a_k) \in \operatorname{Dom} t_\lambda$, and $k = m_\lambda$ for a typographic reason, and define

$$\sigma_n[\lambda a_1 \dots a_k] = t_\lambda(\sigma_{n_1} a_1, \dots, \sigma_{n_k} a_k). \tag{6.1}$$

The latter half of the following lemma shows that σ_n is well-defined.

Lemma 6.1 *The words in S_n contain precisely n indices. Also, for each element $a \in S_n$ $(n \geq 1)$, there exists a unique tuple λ, a_j, n_j $(j = 1, \dots, m_\lambda)$ which satisfies $a = [\lambda a_1 \dots a_{m_\lambda}]$, $\lambda \in L$, $a_j \in S_{n_j}$ $(j = 1, \dots, m_\lambda)$, and $n = 1 + \sum_{j=1}^{m_\lambda} n_j$.*

The former half of this lemma is immediately proved by induction on n. The latter half is a consequence of the following lemma.

Lemma 6.2 *Each word in S_n $(n \geq 1)$ contains as many left brackets as right brackets. Let $a = [\lambda a_1 \dots a_{m_\lambda}] \in S_n$ $(n \geq 1)$ with $\lambda \in L$, $a_j \in S_{n_j}$ $(j = 1, \dots, m_\lambda)$, and $n = 1 + \sum_{j=1}^{m_\lambda} n_j$. Then the following holds for $j = 1, \dots, m_\lambda$.*

(1) a contains precisely one more left brackets than right brackets on the left-hand side of a_j.

(2) a contains at least two more left brackets than right brackets on the left-hand side of each left bracket of a_j other than the leftmost left bracket of a_j.

The above holds with "left" and "right" interchanged.

The former half of this lemma is immediately proved by induction on n, from which (1) immediately follows. We will derive (2) form (1) by induction on n. If $n = 1$, then $n_j = 0$, so a_j has no left brackets and (2) obviously holds. If $n \geq 2$ and a_j has a left bracket other than the leftmost left bracket, then it is a left bracket of some of $b_1, \dots, b_{m_\mu}$ in the expression $a_j = [\mu b_1 \dots b_{m_\mu}]$ with $\mu \in L$, $b_i \in S_{n_{ij}}$ $(i = 1, \dots, m_\mu)$, and $n_j = 1 + \sum_{i=1}^{m_\mu} n_{ij}$. The induction hypothesis and (1) show that a_j has at least one more left brackets than right brackets on the left-hand side of the left bracket in question. Each of $a_1, \dots, a_{j-1}$ has as many left brackets as right brackets, and a has the leftmost left bracket. Therefore (2) holds. The same proof works with "left" and "right" interchanged.

We will derive the latter half of Lemma 6.1 from Lemma 6.2. Let $a = [\lambda a_1 \dots a_{m_\lambda}] \in S_n$ $(n \geq 1)$ with $\lambda \in L$, $a_j \in S_{n_j}$ $(j = 1, \dots, m_\lambda)$, and $n = 1 + \sum_{j=1}^{m_\lambda} n_j$. Then the leftmost left bracket of each a_j with $n_j \geq 1$ is characterized as a left bracket of a on the left-hand side of which there are precisely one more left brackets than right brackets. This also holds with "left" and "right" interchanged. The pairs of adjacent such left and right brackets cut the elements a_j with $n_j \geq 1$ out of a. The remaining primes contained in a are the elements a_j with $n_j = 0$. The former half of Lemma 6.1 shows that n_j for $j = 1, \dots, m_\lambda$ is characterized as the number of the indices contained in a_j. Finally, λ is characterized as the index next to the leftmost left bracket of a.

The proof of Lemma 6.1 is complete, and so is the construction of S_n and σ_n $(n = 0, 1, \dots)$. We define $A = \bigcup_{n \geq 0} S_n$. Then the former half of Lemma 6.1 shows that A is the direct union of S_n $(n = 0, 1, \dots)$. Therefore, we define $\sigma \in A \to T$ by $\sigma|_{S_n} = \sigma_n$ $(n = 0, 1, \dots)$. Then $\sigma|_S = \tau$, and the definition (6.1) of σ_n may be rewritten $\sigma[\lambda a_1 \dots a_{m_\lambda}] = t_\lambda(\sigma a_1, \dots, \sigma a_{m_\lambda})$.

For each $\lambda \in L$, we define the operation α_λ on A by

$$\alpha_\lambda(a_1, \dots, a_{m_\lambda}) = [\lambda a_1 \dots a_{m_\lambda}], \tag{6.2}$$

where $\mathrm{Dom}\,\alpha_\lambda = \{(a_1, \dots, a_{m_\lambda}) \in A^{m_\lambda} \mid (\sigma a_1, \dots, \sigma a_{m_\lambda}) \in \mathrm{Dom}\,t_\lambda\}$. Certainly, α_λ is an operation on A, because there exists a non-negative integer n_j such that $a_j \in S_{n_j}$ for $j = 1, \dots, m_\lambda$, and $n = 1 + \sum_{j=1}^{m_\lambda} n_j$ satisfies $[\lambda a_1 \dots a_{m_\lambda}] \in S_n$. Thus, we have constructed an algebra $(A, (\alpha_\lambda)_{\lambda \in L})$ similar to T. Moreover, σ is necessarily a holomorphism of A into T. Therefore, (A, T, σ) is a sorted algebra. Also, S_n is equal to the n-th descendant of S. Therefore $A = [S]$.

Finally, in order to show that the universality is satisfied, we let (A', T, σ') be a sorted algebra, and assume that $\varphi \in S \to A'$ satisfies $\sigma'\varphi = \sigma|_S$. Let $(\alpha'_\lambda)_{\lambda \in L}$ be the operations of A'. We will construct $f_n \in S_n \to A'$ such that $\sigma' f_n = \sigma_n$ by induction on $n = 0, 1, \ldots$. First, we define $f_0 = \varphi$. Then $\sigma' f_0 = \sigma'\varphi = \sigma|_S = \sigma_0$ as desired. Next for $n \geq 1$, we wish to define f_n by

$$f_n a = \alpha'_\lambda(f_{n_1} a_1, \ldots, f_{n_k} a_k) \tag{6.3}$$

for each element $a \in S_n$, using its unique expression $a = [\lambda a_1 \ldots a_k]$ with $\lambda \in L$, $a_j \in S_{n_j}$ $(j = 1, \ldots, k)$, $n = 1 + \sum_{j=1}^{k} n_j$, and $(\sigma_{n_1} a_1, \ldots, \sigma_{n_k} a_k) \in \mathrm{Dom}\, t_\lambda$, where $k = m_\lambda$. This definition is possible, because $\sigma_{n_j} a_j = \sigma' f_{n_j} a_j$ for $j = 1, \ldots, k$ by the induction hypothesis and σ' is exact. Furthermore $\sigma' f_n = \sigma_n$ holds, because

$$\begin{aligned} \sigma' f_n a &= \sigma'(\alpha'_\lambda(f_{n_1} a_1, \ldots, f_{n_k} a_k)) = t_\lambda(\sigma' f_{n_1} a_1, \ldots, \sigma' f_{n_k} a_k) \\ &= t_\lambda(\sigma_{n_1} a_1, \ldots, \sigma_{n_k} a_k) = \sigma_n[\lambda a_1 \ldots a_k] = \sigma_n a \end{aligned}$$

by (6.1). We have defined f_n for each non-negative integer n so that $\sigma' f_n = \sigma_n$ holds. Finally, we define $f \in A \to A'$ by $f|_{S_n} = f_n$ $(n = 0, 1, \ldots)$. Then $f|_S = \varphi$ and $\sigma' f = \sigma$. Also, (6.2) and (6.3) together with the comments on them show that f is a holomorphism.

Proof of the former half of Theorem 2.2 In view of the latter half of Theorem 2.1, it only remains to show that the USA (A, T, σ, S) constructed above satisfies (3) of Theorem 2.2. Suppose an element $a \in A - S$ has expressions $a = \alpha_\lambda(a_1, \ldots, a_k)$ and $a = \alpha_{\lambda'}(a'_1, \ldots, a'_{k'})$ with $\lambda, \lambda' \in L$, $(a_1, \ldots, a_k) \in \mathrm{Dom}\, \alpha_\lambda$ and $(a'_1, \ldots, a'_{k'}) \in \mathrm{Dom}\, \alpha_{\lambda'}$. Suppose $a_j \in S_{n_j}$ $(j = 1, \ldots, k)$ and $a'_{j'} \in S_{n'_{j'}}$ $(j' = 1, \ldots, k')$. Define $n = 1 + \sum_{j=1}^{k} n_j$ and $n' = 1 + \sum_{j'=1}^{k'} n'_{j'}$. Then $a \in S_n \cap S_{n'}$. Thus, $\lambda = \lambda'$, $k = k'$ and $a_j = a'_j$ $(j = 1, \ldots, k)$ by Lemma 6.1.

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