A Completeness Theorem for Monophasic Case Logic

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Abstract Monophasic case logic (MPCL) is a prototype of case logic designed for mathematical psychology. A sentential deduction system on it will be proved to be complete. The semantics of MPCL are parameterized by measures with regard to quantification. Therefore, in proving any completeness theorem for MPCL, we have to construct a measure. The heart of the paper lies in capturing the measure by axioms which imply the pigeonhole principle.

Keywords completeness, deduction system, generalized quantification, logical system, logical space, mathematical psychology, measure, pigeonhole principle.

The purpose of this paper is to prove a sentential completeness theorem for the logical system MPCL (monophasic case logic), which is a prototype of the logical system CL (case logic) introduced by Gomi (2009c) from the viewpoint of MP (mathematical psychology).

Motivations, backgrounds and bibliographic remarks for MP and CL have been given in previous papers by Gomi (2009 a, b, c), and this paper is a direct continuation of them. So I will concentrate on the mathematical detail of the proof using their notation, terminology, definitions and results without reviewing or restating them.

This paper borrows many ideas from the thesis by Mizumura (2000) who presented a sequential completeness theorem for an antecedent of MPCL. Its denotable worlds, as well as those in the present MPCL, were parameterized by a measure with regard to quantification. Therefore in proving the completeness theorem, he had to construct a measure, which was expected to be difficult. Mizumura worked it out by introducing an axiom meaning the pigeonhole principle (cf. Lemma 2.8).

Meanwhile Takaoka (2009) has taken an alternative comprehensive (and elegant in my opinion) approach to the theorem and related problems. He managed with (Dedekind) cuts essentially due to Henkin (1949) instead of resolution trees

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in Mizumura's thesis and this paper (cf. Lemma 3.19). Takaoka's approach, however, also relies on the definitions, methods and intermediate results of this paper (cf. Remark 1.1). Takaoka's approach will be applied to CL as well in a subsequent paper.

1 A predicate deduction system

Let $(A, T, \sigma, \mathbb{S}, \mathbb{C}, \mathbb{X}, \Gamma)$ be a MPC language, $(A, \mathcal{W}, (I_W)_{W\in\mathcal{W}})$ be the logical system MPCL on it, and $(H, \mathcal{G})$ be its predicate logical space. The purpose of this paper is to prove that the deduction system $(\wp, \nabla)$ on H presented below is $\mathcal{G}$-complete under the following four conditions.

1. The quantity system $\mathbb{P}$ is linear.
2. The range K_f of each predicates $f \in H$ is a finite set.
3. Both the sets A_ε and $\mathbb{X}_\varepsilon$ of the simple nominals and the variable simple nominals are enumerable.
4. The set $\mathfrak{P}$ of the positive quantifiers is the smallest of the subsets of $\mathcal{P}\mathbb{P}$ which contain the intervals $(p\rightarrow) = \{x \in \mathbb{P} \mid p < x\}$ of $\mathbb{P}$ for all $p \in \mathbb{P}$ and are closed under the meet $\mathfrak{p} \cap \mathfrak{q}$, the join $\mathfrak{p} \cup \mathfrak{q}$ and the complement $\mathfrak{p}^\circ$.

Remark 1.1 The conditions 1 and 4 mean that $\mathfrak{P}$ consists of the finite unions of the intervals of $\mathbb{P}$ in the shape $(p\rightarrow)$ or $(\leftarrow q] = \{x \in \mathbb{P} \mid x \leq q\}$ or $(pq] = \{x \in \mathbb{P} \mid p < x \leq q\}$ for some $p, q \in \mathbb{P}$. Thus $\emptyset = (00]$ and $\mathbb{P} = (\leftarrow 0] \cup (0\rightarrow)$ belong to $\mathfrak{P}$.

As was shown in §2.2 of Gomi (09c), A_ε is equal to the closure of the set $\mathbb{S}_\varepsilon$ of the prime simple nominals under the set $\mathfrak{F}$ of the functionals:

$$A_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}. \tag{1.1}$$

Also $\mathbb{S}_\varepsilon = \mathbb{C}_\varepsilon \cup \mathbb{X}_\varepsilon$ for the set $\mathbb{C}_\varepsilon$ of the constant simple nominals. Therefore the condition 3 means that $\mathbb{X}_\varepsilon$ is enumerable and both $\mathbb{C}_\varepsilon$ and $\mathfrak{F}$ are at most enumerable. Takaoka (09) assumes a condition different from the condition 3.

The presentation of the logic $\wp$ on H is simple and familiar:

$$\wp = \frac{f \quad f \Rightarrow g}{g}, \qquad \text{(modus ponens)}$$

where $f, g \in H$. This means that elements $f_1, \ldots, f_n, g \in H$ satisfy $f_1 \cdots f_n \,\wp\, g$ iff $n = 2$ and $f_2 = f_1 \Rightarrow g$, and similarly for the fractional expressions below.

In contrast, the presentation of the subset ∇ of H is complex. First let x_0 be an arbitrary element of $\mathbb{X}_\varepsilon$ and define the complex nominal $\mathsf{one} \in A_\delta$ by

$$\mathsf{one} = (x_0 \,\forall\pi\, x_0 \triangle)\, \Omega x_0. \tag{1.2}$$

Next define the four logics $\&, \perp, \top$ and $\forall$ on H also by the fractional expressions, where and elsewhere $x \not\ll f$ means as in Gomi (09b) that x does not occur free in f:

$$\& = \frac{f \quad g}{f \wedge g} \qquad (f, g \in H), \qquad (\wedge+ \text{ logic})$$

$$\perp = \frac{f}{a \,\breve{o} k\, f} \qquad (f \in H,\ k \in K_f,\ a \in A_\varepsilon), \qquad (\text{case}+ \text{ logic})$$

$$\top = \frac{x \,\breve{o} k\, f}{f} \qquad (f \in H,\ k \in K_f,\ x \in \mathbb{X}_\varepsilon,\ x \not\ll f), \qquad (\text{case}- \text{ logic})$$

$$\forall = \frac{f}{\mathsf{one} \forall \pi (f\, \Omega x) \triangle} \qquad (f \in A_\emptyset,\ x \in \mathbb{X}_\varepsilon). \qquad (\forall+ \text{ logic})$$

Then ∇ is defined as the $(\wp \cup \& \cup \perp \cup \top \cup \forall)$-closure of the set ∂ of the **MPC predicates** presented below:

$$\nabla = [\partial]_{\wp \cup \& \cup \perp \cup \top \cup \forall}. \tag{1.3}$$

Remark 1.2 The equation (1.3) means that ∇ is the union of the inductively defined $(\wp \cup \& \cup \perp \cup \top \cup \forall)$-descendants ∂_n $(n = 0, 1, \dots)$ of ∂, where $\partial_0 = \partial$ and ∂_n with $n \geq 1$ consists of the following elements:

$$\begin{array}{rl}
f & (f \in H \text{ and } g \Rightarrow f \in \partial_{n_1} \text{ for some } g \in \partial_{n_2} \text{ with } n_1 + n_2 = n - 1), \\
f \wedge g & (f \in \partial_{n_1},\ g \in \partial_{n_2},\ n_1 + n_2 = n - 1), \\
a \,\breve{o} k\, f & (f \in \partial_{n-1},\ k \in K_f,\ a \in A_\varepsilon), \\
f & (f \in H \text{ and } x \,\breve{o} k\, f \in \partial_{n-1} \text{ for some } k \in K_f \text{ and } x \in \mathbb{X}_\varepsilon \text{ with } x \not\ll f), \\
\forall x\, f & (f \in \partial_{n-1} \cap A_\emptyset,\ x \in \mathbb{X}_\varepsilon),
\end{array}$$

where $\forall x f$ is an abbreviation of $\mathsf{one} \forall \pi (f\, \Omega x) \triangle$ (cf. Lemma 3.2 and §2.6 of Gomi (09c)). In other words, an element $h \in H$ belongs to ∇ if and only if there exists a sequence $h_1, \dots, h_n$ of elements of H such that $h_n = h$ and, for each $i \in \{1, \dots, n\}$, either $h_i \in \partial$ or h_i satisfies one of the following five conditions.

- $h_j = h_k \Rightarrow h_i$ for some $j, k \in \{1, \dots, i-1\}$.
- $h_i = h_j \wedge h_k$ for some $j, k \in \{1, \dots, i-1\}$.
- $h_i = a \,\breve{o} k\, h_j$ for some $j \in \{1, \dots, i-1\}$, $k \in K_{h_j}$, and $a \in A_\varepsilon$.
- $h_j = x \,\breve{o} k\, h_i$ for some $j \in \{1, \dots, i-1\}$, $k \in K_{h_i}$, and $x \in \mathbb{X}_\varepsilon$ with $x \not\ll h_i$.
- $h_i = \forall x\, h_j$ for some $j \in \{1, \dots, i-1\}$ with $h_j \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

The equation (1.3) means also that an element $h \in H$ belongs to ∇ if and only if there exists a $(\wp \cup \& \cup \perp \cup \top \cup \forall)$-proof figure with elements of ∂ at the top and h at the bottom.

The set ∂ of the MPC predicates consists of the following twenty three kinds of predicates, the first of which is the collection of familiar ones:

$$\left.\begin{array}{r} f^{\Diamond} \vee f, \\ (f \wedge g) \Rightarrow f, \\ (f \wedge g) \Rightarrow g, \\ f \Rightarrow (f \vee g), \\ g \Rightarrow (f \vee g), \\ (f^{\Diamond} \vee g) \Rightarrow (f \Rightarrow g), \\ \big((f \Rightarrow h) \wedge (g \Rightarrow h)\big) \Rightarrow \big((f \vee g) \Rightarrow h\big), \\ \big((h \Rightarrow f) \wedge (h \Rightarrow g)\big) \Rightarrow \big(h \Rightarrow (f \wedge g)\big), \\ \big((h \Rightarrow f) \wedge \big(h \Rightarrow (f \Rightarrow g)\big)\big) \Rightarrow (h \Rightarrow g), \end{array}\right\} \qquad \text{(Boolean pred.)}$$

where $f, g, h \in H$. The remaining twenty two kinds of MPC predicates are characteristic of MPCL and so are called the **proper** MPC predicates.

$$a\,\mathfrak{r}k\,(b\,\breve{o}l\,f) \Leftrightarrow b\,\breve{o}l\,(a\,\mathfrak{r}k\,f), \qquad (\mathfrak{Q}, \breve{o} \text{ pred.})$$

where $a \in G,\ b \in A_\varepsilon,\ f \in H,\ k, l \in K_f,\ k \neq l$, and $\mathfrak{r} \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $a \in A_\varepsilon$ in case $\mathfrak{r} = \breve{o}$. Both here and elsewhere, the two-way arrow $\Leftrightarrow$ is a device for showing a predicate $g \Rightarrow h$ and its reverse $h \Rightarrow g$ together.

$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\dots,m} f \wedge (a_i\,\breve{o}k_i)_{i=n+1,\dots,l} g\big), \quad (\wedge \text{ pred.})$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\dots,m} f \vee (a_i\,\breve{o}k_i)_{i=n+1,\dots,l} g\big), \quad (\vee \text{ pred.})$$
$$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\dots,m} f \Rightarrow (a_i\,\breve{o}k_i)_{i=n+1,\dots,l} g\big), \quad (\Rightarrow \text{ pred.})$$

where $a_1, \dots, a_l \in A_\varepsilon$, $f, g \in H$, and $k_1, \dots, k_l$ are distinct cases such that $k_1, \dots, k_n \in K_f - K_g$, $k_{n+1}, \dots, k_m \in K_f \cap K_g$, and $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Also, $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g)$ is an abbreviation for

$$a_1\,\breve{o}k_1\,(a_2\,\breve{o}k_2\,(\cdots(a_l\,\breve{o}k_l\,(f \wedge g))\cdots)),$$

and similarly for analogous expressions.

$$\big((a_i\,\breve{o}k_i)_{i=1,\dots,n}(f^{\Diamond})\big) \Leftrightarrow \big((a_i\,\breve{o}k_i)_{i=1,\dots,n} f\big)^{\Diamond}, \qquad (\Diamond \text{ pred.})$$

where $a_1, \dots, a_n \in A_\varepsilon$, $f \in H$, and $k_1, \dots, k_n$ are distinct cases in K_f.

$$a\,\neg\mathfrak{p}k\,f \Leftrightarrow a\,\mathfrak{p}k\,f^{\Diamond}, \qquad (\neg \text{ pred.})$$
$$a\,\mathfrak{p}^{\circ}k\,f \Leftrightarrow (a\,\mathfrak{p}k\,f)^{\Diamond}, \qquad (\circ \text{ pred.})$$

where $a \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$.

$$a\,(\mathfrak{p} \cap \mathfrak{q})k\,f \Leftrightarrow (a\,\mathfrak{p}k\,f \wedge a\,\mathfrak{q}k\,f), \qquad (\cap \text{ pred.})$$
$$a\,(\mathfrak{p} \cup \mathfrak{q})k\,f \Leftrightarrow (a\,\mathfrak{p}k\,f \vee a\,\mathfrak{q}k\,f), \qquad (\cup \text{ pred.})$$

where $\mathfrak{a} \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$.

$$\mathfrak{a}\,\overline{\mathfrak{p}}k\,f \Leftrightarrow \mathfrak{a}\,\overline{\mathfrak{p}}\pi\big((x\,\breve{o}k\,f)\,\Omega x\big)\triangle, \tag{$\mathfrak{P}$ pred.}$$

where $\mathfrak{a} \in G,\ f \in H,\ x \in \mathbb{X}_\varepsilon,\ K_f = \{k\},\ \mathfrak{p} \in \mathbb{P}$, and x does not occur free in f.

$$\mathfrak{a}\,\overline{\mathfrak{p}}\pi\,b\triangle \Leftrightarrow (\mathfrak{a} \sqcap b)\,\overline{\mathfrak{p}}\pi\,\mathrm{one}\triangle, \tag{$\triangle$ pred.}$$

where $\mathfrak{a}, b \in G$ and $\mathfrak{p} \in \mathbb{P}$.

$$\big(\mathrm{one}\,\forall\pi\,\big((f \Rightarrow g)\,\Omega x\big)\triangle\big) \Rightarrow \big(f \Rightarrow \mathrm{one}\,\forall\pi\,(g\,\Omega x)\triangle\big), \tag{$\forall, \Rightarrow$ pred.}$$

where $f, g \in A_\emptyset,\ x \in \mathbb{X}_\varepsilon$, and x does not occur free in f.

$$\big(\mathrm{one}\,\forall\pi\,\big(\big((x\,\breve{o}\pi\,\mathfrak{a}\triangle) \Rightarrow (x\,\breve{o}k\,f)\big)\,\Omega x\big)\triangle\big) \Rightarrow \mathfrak{a}\,\forall k\,f, \tag{$\forall$ pred.}$$

where $x \in \mathbb{X}_\varepsilon,\ \mathfrak{a} \in G,\ f \in H,\ K_f = \{k\}$, and x does not occur free in $\mathfrak{a}, f$.

$$(\mathfrak{a}\,\forall\pi\,b\triangle \wedge \mathfrak{a}\,\overline{\mathfrak{p}}k\,f) \Rightarrow b\,\overline{\mathfrak{p}}k\,f, \tag{$\forall, \mathfrak{P}$ pred.}$$

where $\mathfrak{a}, b \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p} \in \mathbb{P}$.

$$(\mathfrak{a} \sqcup b)\,\overline{\mathfrak{p} + \mathfrak{q}}k\,f \Rightarrow (\mathfrak{a}\,\overline{\mathfrak{p}}k\,f \vee b\,\overline{\mathfrak{q}}k\,f), \tag{$\sqcup, +$ pred.}$$

where $\mathfrak{a}, b \in G,\ f \in H,\ k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathbb{P}$.

$$(\mathrm{one}^{\square}\,\overline{\mathfrak{p}}k\,f)^{\Diamond}, \tag{$\mathrm{one}^{\square}$ pred.}$$

where $f \in H,\ k \in K_f$, and $\mathfrak{p} \in \mathbb{P}$.

$$b\,\breve{o}\pi\,\mathfrak{a}\triangle \Rightarrow \mathfrak{a}\,\exists\pi\,\mathrm{one}\triangle, \tag{$\exists$ pred.}$$

where $\mathfrak{a} \in G$ and $b \in A_\varepsilon$.

$$(\mathfrak{a} \sqcap b)\triangle \Leftrightarrow (\mathfrak{a}\triangle \wedge b\triangle), \tag{$\sqcap$ pred.}$$
$$(\mathfrak{a} \sqcup b)\triangle \Leftrightarrow (\mathfrak{a}\triangle \vee b\triangle), \tag{$\sqcup$ pred.}$$
$$(\mathfrak{a}^{\square})\triangle \Leftrightarrow (\mathfrak{a}\triangle)^{\Diamond}, \tag{$\square$ pred.}$$

where $\mathfrak{a}, b \in G$.

$$\mathfrak{a}\,\breve{o}\pi\,(f\,\Omega x)\triangle \Leftrightarrow f(x/\mathfrak{a}), \tag{Ω pred.}$$

where $\mathfrak{a} \in A_\varepsilon,\ f \in A_\emptyset,\ x \in \mathbb{X}_\varepsilon$, and x is free from $\mathfrak{a}$ in f. The $(x/\mathfrak{a})$ on the right-hand side of $\Leftrightarrow$ denotes the substitution of $\mathfrak{a}$ for x.

$$\mathrm{one}\,\forall\pi\,(f\,\Omega x)\triangle \Rightarrow f, \tag{$\forall-$ pred.}$$

where $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

This completes the presentation of $(\wp, \nabla)$. The main result of this paper can now be stated in due form.

Theorem 1 *The deduction system $(\wp, \nabla)$ on H is $\mathcal{G}$-complete under the conditions 1–4 stated at the beginning of this section.*

The proof of this theorem will begin at §2 and end at §4.

Corollary 1.1 *The subset ∇ of H is equal to the core C of $(\mathsf{H}, \mathcal{G})$, or the set of the $\mathcal{G}$-tautologies in H, under the conditions 1–4.*

Proof We have $\mathsf{C} = [\nabla]_\wp$ by Theorem 7.2 of Gomi (09a) and Theorem 1, while (1.3) implies that $[\nabla]_\wp = \nabla$, hence $\mathsf{C} = \nabla$.

Remark 1.3 For the proof of Theorem 1, it will be shown that the deduction system $(\wp \cup \&, \nabla)$ is $\mathcal{G}$-complete, and in particular the logics $\wp$ and $\&$ are $\mathcal{G}$-sound and close C. Also, Corollary 1.1 means $\mathsf{C} = [\partial]_{\wp \cup \& \cup \perp \cup \top \cup \forall}$ (cf. Remark 1.2), and in particular the logics $\perp, \top$ and $\forall$ also close C. However, $\perp, \top$ and $\forall$ are not $\mathcal{G}$-sound, and so, for instance, the deduction system $(\wp \cup \& \cup \forall, \nabla)$ is not $\mathcal{G}$-complete, although obviously $\mathcal{G}$-sufficient.

Also, there are proper subsets ∂' of ∂ which satisfy $\mathsf{C} = [\partial']_{\wp \cup \& \cup \perp \cup \top \cup \forall}$. Since a deduction system in MP is a model of the human system of thinking, I have not pursued mathematical minimalism in the presentation of ∂.

2 A generational law on the predicate sequences

The proof of Theorem 1 follows the two step general program given in Remark 7.3 of Gomi (09a) based on the fundamental theorem of completeness (Theorem 7.12 of Gomi (09a)).

The purpose of this section is to present a generational law called the **MPC law** on the relations $\preccurlyeq_*$ on H^* and show that it is satisfied by the validity relation $\preccurlyeq$ of the predicate logical space $(\mathsf{H}, \mathcal{G})$.

It will be shown in §3 that the MPC law is a characteristic law of $(\mathsf{H}, \mathcal{G})$ as to the first step of the program. Also, it will be shown in §4 that the deduction system $(\wp \cup \&, \nabla)$ on H is $\mathcal{G}$-sound and the deduction relation $\preccurlyeq_{\wp \cup \&, \nabla}$ satisfies the MPC law as to the second step of the program. Therefore $(\wp \cup \&, \nabla)$ is $\mathcal{G}$-complete by the fundamental theorem, and hence $\mathsf{C} = \nabla$ as in the proof of Corollary 1.1. Since $(\mathsf{H}, \mathcal{G})$ is a Boolean logical space with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ by Theorem 3.3 of Gomi (09c), Corollary 7.11.1 of Gomi (09a) finally shows that $(\wp, \nabla)$ is $\mathcal{G}$-complete. Thus Theorem 1 will be proved.

Since $\preccurlyeq$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ by Theorem 6.8 of Gomi (09a), the MPC law is defined as the extension of the Boolean law with respect to $\wedge, \vee, \Diamond, \Rightarrow$ by the **proper** MPC law consisting of the twenty five laws presented below, which are in one to one correspondence with the three logics $\perp, \top$ and $\forall$ on H together with the twenty two kinds of the proper MPC predicates used for defining ∇ by (1.3).

Both in the presentation and elsewhere, the element of H^* of length 0 will be denoted by a blank and $\asymp_*$ will denote the symmetric core of $\preccurlyeq_*$.

$$\preccurlyeq_* \mathsf{f} \implies \preccurlyeq_* \mathfrak{a}\,\breve{o}\mathsf{k}\,\mathsf{f}, \tag{case+ law}$$

where $\mathsf{f} \in \mathsf{H}$, $\mathfrak{a} \in \mathsf{A}_\varepsilon$, and $\mathsf{k} \in \mathsf{K}_\mathsf{f}$.

$$\preccurlyeq_* \mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f} \implies \preccurlyeq_* \mathsf{f}, \tag{case− law}$$

where $\mathsf{x} \in \mathbb{X}_\varepsilon$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and x does not occur free in f. Theorem 3.15 of Gomi (09c) shows that $\preccurlyeq$ satisfies the above two laws.

$$\preccurlyeq_* \mathsf{f} \implies \preccurlyeq_* \mathsf{one}\,\forall\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle, \tag{$\forall$+ law}$$

where $\mathsf{f} \in \mathsf{A}_\emptyset$ and $\mathsf{x} \in \mathbb{X}_\varepsilon$. Theorem 3.21 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

$$\mathfrak{a}\,\mathfrak{x}\mathsf{k}\,(\mathsf{b}\,\breve{o}\mathsf{l}\,\mathsf{f}) \asymp_* \mathsf{b}\,\breve{o}\mathsf{l}\,(\mathfrak{a}\,\mathfrak{x}\mathsf{k}\,\mathsf{f}), \tag{$\mathfrak{Q}, \breve{o}$ law}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{b} \in \mathsf{A}_\varepsilon$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k}, \mathsf{l} \in \mathsf{K}_\mathsf{f}$, $\mathsf{k} \neq \mathsf{l}$, and $\mathfrak{x} \in \{\breve{o}\} \cup \mathfrak{Q}$. Also $\mathfrak{a} \in \mathsf{A}_\varepsilon$ in case $\mathfrak{x} = \breve{o}$. Theorems 3.4 and 3.14 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$(\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,l}(\mathsf{f} \wedge \mathsf{g}) \asymp_* (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,m}\mathsf{f} \wedge (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=n+1,\dots,l}\mathsf{g}, \tag{$\wedge$ law}$$
$$(\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,l}(\mathsf{f} \vee \mathsf{g}) \asymp_* (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,m}\mathsf{f} \vee (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=n+1,\dots,l}\mathsf{g}, \tag{$\vee$ law}$$
$$(\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,l}(\mathsf{f} \Rightarrow \mathsf{g}) \asymp_* (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,m}\mathsf{f} \Rightarrow (\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=n+1,\dots,l}\mathsf{g}, \tag{$\Rightarrow$ law}$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_l \in \mathsf{A}_\varepsilon$, $\mathsf{f}, \mathsf{g} \in \mathsf{H}$, and $\mathsf{k}_1, \dots, \mathsf{k}_l$ are distinct cases such that $\mathsf{k}_1, \dots, \mathsf{k}_n \in \mathsf{K}_\mathsf{f} - \mathsf{K}_\mathsf{g}$, $\mathsf{k}_{n+1}, \dots, \mathsf{k}_m \in \mathsf{K}_\mathsf{f} \cap \mathsf{K}_\mathsf{g}$, and $\mathsf{k}_{m+1}, \dots, \mathsf{k}_l \in \mathsf{K}_\mathsf{g} - \mathsf{K}_\mathsf{f}$ $(0 \le n \le m \le l)$. Theorems 3.4 and 3.6 of Gomi (09c) show that $\preccurlyeq$ satisfies these three.

$$(\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,n}(\mathsf{f}^\Diamond) \asymp_* \big((\mathfrak{a}_i\,\breve{o}\mathsf{k}_i)_{i=1,\dots,n}\mathsf{f}\big)^\Diamond, \tag{$\Diamond$ law}$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathsf{A}_\varepsilon$, $\mathsf{f} \in \mathsf{H}$, and $\mathsf{k}_1, \dots, \mathsf{k}_n$ are distinct cases in K_f. Theorem 3.4 and Corollary 3.5.3 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$\mathfrak{a}\,\neg\mathfrak{p}\mathsf{k}\,\mathsf{f} \asymp_* \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f}^\Diamond, \tag{$\neg$ law}$$
$$\mathfrak{a}\,\mathfrak{p}^\circ\mathsf{k}\,\mathsf{f} \asymp_* (\mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f})^\Diamond, \tag{$\circ$ law}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p} \in \mathfrak{P}$. Theorems 3.4 and 3.10 of Gomi (09c) show that $\preccurlyeq$ satisfies these two.

$$\mathfrak{a}\,(\mathfrak{p} \cap \mathfrak{q})\mathsf{k}\,\mathsf{f} \asymp_* \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \wedge \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}, \tag{$\cap$ law}$$
$$\mathfrak{a}\,(\mathfrak{p} \cup \mathfrak{q})\mathsf{k}\,\mathsf{f} \asymp_* \mathfrak{a}\,\mathfrak{p}\mathsf{k}\,\mathsf{f} \vee \mathfrak{a}\,\mathfrak{q}\mathsf{k}\,\mathsf{f}, \tag{$\cup$ law}$$

where $\mathfrak{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. Theorems 3.4 and 3.11 of Gomi (09c) show that $\preccurlyeq$ satisfies these two.

$$\mathfrak{a}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f} \asymp_* \mathfrak{a}\,\overline{\mathfrak{p}}\pi\big((\mathsf{x}\,\breve{o}\mathsf{k}\,\mathsf{f})\,\Omega\mathsf{x}\big)\triangle, \tag{$\mathfrak{P}$ law}$$

where $a \in G,\ f \in H,\ x \in \mathbb{X}_\varepsilon,\ K_f = \{k\},\ p \in \mathbb{P}$, and x does not occur free in f. Theorem 3.16 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

$$a\,\overline{p}\pi\,b\triangle \asymp_* (a \sqcap b)\,\overline{p}\pi\,one\triangle, \tag{$\triangle$ law}$$

where $a, b \in G$ and $p \in \mathbb{P}$. Theorems 3.4, 3.9 and 3.19 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$f, one\,\forall\pi\big((f \Rightarrow g)\,\Omega x\big)\triangle \preccurlyeq_* one\,\forall\pi\,(g\,\Omega x)\triangle, \tag{$\forall, \Rightarrow$ law}$$

where $f, g \in A_\emptyset,\ x \in \mathbb{X}_\varepsilon$, and x does not occur free in f. Theorem 3.17 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

$$one\,\forall\pi\Big(\big((x\,\breve{o}\pi\,a\triangle) \Rightarrow (x\,\breve{o}k\,f)\big)\,\Omega x\Big)\triangle \preccurlyeq_* a\,\forall k\,f, \tag{$\forall$ law}$$

where $x \in \mathbb{X}_\varepsilon,\ a \in G,\ f \in H,\ K_f = \{k\}$, and x does not occur free in a, f. Theorem 3.22 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

$$a\,\forall\pi\,b\triangle, a\,\overline{p}k\,f \preccurlyeq_* b\,\overline{p}k\,f, \tag{$\forall, \mathfrak{P}$ law}$$

where $a, b \in G,\ f \in H,\ k \in K_f$, and $p \in \mathbb{P}$. Theorems 3.4 and 3.12 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$(a \sqcup b)\,\overline{p+q}k\,f \preccurlyeq_* a\,\overline{p}k\,f, b\,\overline{q}k\,f, \tag{$\sqcup, +$ law}$$

where $a, b \in G,\ f \in H,\ k \in K_f$, and $p, q \in \mathbb{P}$. Theorems 3.4 and 3.13 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$one^{\Box}\,\overline{p}k\,f \preccurlyeq_* , \tag{$one^{\Box}$ law}$$

where $f \in H,\ k \in K_f$, and $p \in \mathbb{P}$. Theorems 3.4 and 3.19 of Gomi (09c) and the definition of the operation $\overline{p}k$ on the MPC worlds show that $\preccurlyeq$ satisfies this.

$$b\,\breve{o}\pi\,a\triangle \preccurlyeq_* a\,\exists\pi\,one\triangle, \tag{$\exists$ law}$$

where $a \in G$ and $b \in A_\varepsilon$. Theorems 3.4, 3.9 and 3.19 of Gomi (09c) show that $\preccurlyeq$ satisfies this.

$$(a \sqcap b)\triangle \asymp_* a\triangle \wedge b\triangle, \tag{$\sqcap$ law}$$

$$(a \sqcup b)\triangle \asymp_* a\triangle \vee b\triangle, \tag{$\sqcup$ law}$$

$$(a^{\Box})\triangle \asymp_* (a\triangle)^{\Diamond}, \tag{$\Box$ law}$$

where $a, b \in G$. Theorems 3.1 and 3.4 of Gomi (09c) show that $\preccurlyeq$ satisfies these three.

$$a\,\breve{o}\pi\,(f\,\Omega x)\triangle \asymp_* f(x/a), \tag{Ω law}$$

where $a \in A_\varepsilon,\ f \in A_\emptyset,\ x \in \mathbb{X}_\varepsilon$, and x is free from a in f. Theorem 3.18 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

$$one\,\forall\pi\,(f\,\Omega x)\triangle \preccurlyeq_* f, \tag{$\forall-$ law}$$

where $\mathsf{f} \in A_\emptyset$ and $\mathsf{x} \in \mathbb{X}_\varepsilon$. Theorem 3.20 of Gomi (09c) shows that $\preccurlyeq$ satisfies this.

This completes the presentation of the MPC law. Henceforth, relations on H^* which satisfy the MPC law will be referred to **MPC relations**. Thus we have proved the following in the course of the presentation.

Theorem 2 *The validity relation* $\preccurlyeq$ *of the predicate logical space* $(\mathsf{H}, \mathcal{G})$ *is a MPC relation.*

We will proceed to derive more useful laws from the MPC law in the following ten lemmas. So we assume there without notice that $\preccurlyeq_*$ is a MPC relation on H^* and in particular is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, and so the restriction to $\mathsf{H} \times \mathsf{H}$ of the symmetric core $\asymp_*$ of $\preccurlyeq_*$ is an equivalence relation by virtue of the repetition law and cut law. Because of Theorem 2, the lemmas apply to $\preccurlyeq$.

Elements of H will be denoted by $\mathsf{f}, \mathsf{g}, \ldots$ with or without numerical subscripts, while those of H^* will be denoted by $\alpha, \beta, \ldots$.

Lemma 2.1 *The following holds as a consequence of the Boolean law.*

(1) $$\begin{cases} \alpha \mathsf{f} \mathsf{g} \beta \preccurlyeq_* \gamma \iff \alpha, \mathsf{f} \wedge \mathsf{g}, \beta \preccurlyeq_* \gamma, \\ \gamma \preccurlyeq_* \alpha \mathsf{f} \beta,\ \gamma \preccurlyeq_* \alpha \mathsf{g} \beta \iff \gamma \preccurlyeq_* \alpha, \mathsf{f} \wedge \mathsf{g}, \beta. \end{cases}$$

(2) $$\begin{cases} \alpha \mathsf{f} \beta \preccurlyeq_* \gamma,\ \alpha \mathsf{g} \beta \preccurlyeq_* \gamma \iff \alpha, \mathsf{f} \vee \mathsf{g}, \beta \preccurlyeq_* \gamma, \\ \gamma \preccurlyeq_* \alpha \mathsf{f} \mathsf{g} \beta \iff \gamma \preccurlyeq_* \alpha, \mathsf{f} \vee \mathsf{g}, \beta. \end{cases}$$

(3) $$\begin{cases} \alpha \preccurlyeq_* \mathsf{f} \beta \iff \mathsf{f}^\Diamond \alpha \preccurlyeq_* \beta, \\ \mathsf{f} \beta \preccurlyeq_* \alpha \iff \beta \preccurlyeq_* \mathsf{f}^\Diamond \alpha. \end{cases}$$

(4) $$\begin{cases} \alpha \preccurlyeq_* \mathsf{f} \beta,\ \mathsf{g} \alpha \preccurlyeq_* \beta \iff \mathsf{f} \Rightarrow \mathsf{g}, \alpha \preccurlyeq_* \beta, \\ \mathsf{f} \alpha \preccurlyeq_* \mathsf{g} \beta \iff \alpha \preccurlyeq_* \mathsf{f} \Rightarrow \mathsf{g}, \beta, \\ \mathsf{f}, \mathsf{f} \Rightarrow \mathsf{g} \preccurlyeq_* \mathsf{g}\ (\textit{cut-implication law}). \end{cases}$$

(5) $$\left.\begin{array}{l} \mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_n \asymp_* (\cdots(\mathsf{f}_1 \wedge \mathsf{f}_2) \wedge \cdots) \wedge \mathsf{f}_n, \\ \mathsf{f}_1 \vee \cdots \vee \mathsf{f}_n \asymp_* (\cdots(\mathsf{f}_1 \vee \mathsf{f}_2) \vee \cdots) \vee \mathsf{f}_n \end{array}\right\}$$ *irrespective of the order of applying the operations* $\wedge$ *and* $\vee$ *on the left-hand side of* $\asymp_*$.

(6) $$\left.\begin{array}{l} \mathsf{f}_1 \preccurlyeq_* \mathsf{g}_1, \\ \mathsf{f}_2 \preccurlyeq_* \mathsf{g}_2 \end{array}\right\} \Longrightarrow \begin{cases} \mathsf{f}_1 \wedge \mathsf{f}_2 \preccurlyeq_* \mathsf{g}_1 \wedge \mathsf{g}_2, \\ \mathsf{f}_1 \vee \mathsf{f}_2 \preccurlyeq_* \mathsf{g}_1 \vee \mathsf{g}_2. \end{cases}$$

(7) $\alpha \preccurlyeq_* \beta \iff \alpha \preccurlyeq_* \mathsf{f} \wedge \mathsf{f}^\Diamond, \beta \iff \mathsf{f} \vee \mathsf{f}^\Diamond, \alpha \preccurlyeq_* \beta.$

Proof As for (1)–(4), consult Theorems 3.12–3.14 of Gomi (09a).

(5) Since $\mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_n \preccurlyeq_* \mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_n$ by the repetition law, we first have $\mathsf{f}_1 \cdots \mathsf{f}_n \preccurlyeq_* \mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_n$, and next $(\cdots(\mathsf{f}_1 \wedge \mathsf{f}_2) \wedge \cdots) \wedge \mathsf{f}_n \preccurlyeq_* \mathsf{f}_1 \wedge \cdots \wedge \mathsf{f}_n$

both by (1). Similarly $f_1 \wedge \cdots \wedge f_n \preccurlyeq_* (\cdots(f_1 \wedge f_2) \wedge \cdots) \wedge f_n$. Hence the former $\asymp_*$ equation. The proof of the latter is similar.

(6) Since $f_1 \preccurlyeq_* g_1$ and $f_2 \preccurlyeq_* g_2$, we first have $f_1 f_2 \preccurlyeq_* g_1$ and $f_1 f_2 \preccurlyeq_* g_2$ by the weakening law, next $f_1 f_2 \preccurlyeq_* g_1 \wedge g_2$, and finally $f_1 \wedge f_2 \preccurlyeq_* g_1 \wedge g_2$ both by (1). The proof of $f_1 \vee f_2 \preccurlyeq_* g_1 \vee g_2$ is similar.

(7) If $\alpha \preccurlyeq_* \beta$, then $\alpha \preccurlyeq_* f \wedge f^{\Diamond}, \beta$ and $f \vee f^{\Diamond}, \alpha \preccurlyeq_* \beta$ by the weakening law. If $\alpha \preccurlyeq_* f \wedge f^{\Diamond}, \beta$ or $f \vee f^{\Diamond}, \alpha \preccurlyeq_* \beta$, then since $f \wedge f^{\Diamond} \preccurlyeq_* \varepsilon$ and $\varepsilon \preccurlyeq_* f \vee f^{\Diamond}$ by the negation laws and (1) (2), we have $\alpha \preccurlyeq_* \beta$ by the cut law.

Lemma 2.2 *Let* $a_1, \ldots, a_n \in A_\varepsilon$, $f_1, \ldots, f_m \in H$, *and* $k_1, \ldots, k_n$ *be distinct cases in* $K_{f_1} \cap \cdots \cap K_{f_m}$. *Then the following holds irrespective of the order of applying the operations* $\wedge$ *and* $\vee$ *:*

$$(a_i \check{o} k_i)_{i=1,\ldots,n}(f_1 \wedge \cdots \wedge f_m) \asymp_* (a_i \check{o} k_i)_{i=1,\ldots,n} f_1 \wedge \cdots \wedge (a_i \check{o} k_i)_{i=1,\ldots,n} f_m, \qquad \text{(gen.} \wedge \text{ law)}$$

$$(a_i \check{o} k_i)_{i=1,\ldots,n}(f_1 \vee \cdots \vee f_m) \asymp_* (a_i \check{o} k_i)_{i=1,\ldots,n} f_1 \vee \cdots \vee (a_i \check{o} k_i)_{i=1,\ldots,n} f_m. \qquad \text{(gen.} \vee \text{ law)}$$

Proof We may assume $m > 1$ and argue by induction on m. As for the generalized $\vee$ law, suppose $\vee$ has been applied in such an order that $f_1 \vee \cdots \vee f_m = (f_1 \vee \cdots \vee f_j) \vee (f_{j+1} \vee \cdots \vee f_m)$ holds. Then

$$(a_i \check{o} k_i)_i (f_1 \vee \cdots \vee f_m) \asymp_* (a_i \check{o} k_i)_i (f_1 \vee \cdots \vee f_j) \vee (a_i \check{o} k_i)_i (f_{j+1} \vee \cdots \vee f_m)$$

by the $\vee$ law, where $(a_i \check{o} k_i)_i$ is an abbreviation for $(a_i \check{o} k_i)_{i=1,\ldots,n}$. Also,

$$(a_i \check{o} k_i)_i (f_1 \vee \cdots \vee f_j) \asymp_* (a_i \check{o} k_i)_i f_1 \vee \cdots \vee (a_i \check{o} k_i)_i f_j,$$
$$(a_i \check{o} k_i)_i (f_{j+1} \vee \cdots \vee f_m) \asymp_* (a_i \check{o} k_i)_i f_{j+1} \vee \cdots \vee (a_i \check{o} k_i)_i f_m$$

by the induction hypothesis. Applying Lemma 2.1 to the above three displayed $\asymp_*$ relations, we see that the generalized $\vee$ law holds. The generalized $\wedge$ law may be proved similarly.

Lemma 2.3 *Let* $a \in G$, $f \in H$, $k \in K_f$, *and* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathfrak{P}$. *Then the following holds irrespective of the order of applying the operations* $\wedge$ *and* $\vee$ *:*

$$a\,(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n) k f \asymp_* a\,\mathfrak{p}_1 k f \wedge \cdots \wedge a\,\mathfrak{p}_n k f, \qquad \text{(gen.} \cap \text{ law)}$$
$$a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n) k f \asymp_* a\,\mathfrak{p}_1 k f \vee \cdots \vee a\,\mathfrak{p}_n k f. \qquad \text{(gen.} \cup \text{ law)}$$

Proof We may assume $n > 1$ and argue by induction on n. As for the gen. $\cup$ law, $a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n) k f \asymp_* a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) k f \vee a\,\mathfrak{p}_n k f$ by the $\cup$ law, and $a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) k f \asymp_* a\,\mathfrak{p}_1 k f \vee \cdots \vee a\,\mathfrak{p}_{n-1} k f$ by the induction hypothesis. Applying Lemma 2.1 to these $\asymp_*$ relations, we see that the gen. $\cup$ law holds. The gen. $\cap$ law may be proved similarly.

Lemma 2.4 *Let* $a_1, \ldots, a_n \in G,\ f \in H,\ k \in K_f,$ *and* $p_1, \ldots, p_n \in \mathbb{P}$. *Then the following holds irrespective of the order of applying the operation* $\sqcup$:

$$(a_1 \sqcup \cdots \sqcup a_n)\,\overline{p_1 + \cdots + p_n}k\,f \preccurlyeq_* a_1\,\overline{p_1}k\,f, \cdots, a_n\,\overline{p_n}k\,f. \qquad \text{(gen.}\,\sqcup, + \text{ law)}$$

Proof We may assume $n > 1$ and argue by induction on n. Suppose $\sqcup$ has been applied in such an order that $a_1 \sqcup \cdots \sqcup a_n = (a_1 \sqcup \cdots \sqcup a_i) \sqcup (a_{i+1} \sqcup \cdots \sqcup a_n)$ holds. Then

$$\begin{aligned}&(a_1 \sqcup \cdots \sqcup a_n)\,\overline{p_1 + \cdots + p_n}k\,f\\ &\preccurlyeq_* (a_1 \sqcup \cdots \sqcup a_i)\,\overline{p_1 + \cdots + p_i}k\,f, (a_{i+1} \sqcup \cdots \sqcup a_n)\,\overline{p_{i+1} + \cdots + p_n}k\,f\end{aligned}$$

by the $\sqcup, +$ law, and

$$\begin{aligned}(a_1 \sqcup \cdots \sqcup a_i)\,\overline{p_1 + \cdots + p_i}k\,f &\preccurlyeq_* a_1\,\overline{p_1}k\,f, \cdots, a_i\,\overline{p_i}k\,f,\\ (a_{i+1} \sqcup \cdots \sqcup a_n)\,\overline{p_{i+1} + \cdots + p_n}k\,f &\preccurlyeq_* a_{i+1}\,\overline{p_{i+1}}k\,f, \cdots, a_n\,\overline{p_n}k\,f\end{aligned}$$

by the induction hypothesis. Applying the cut law twice to the above three displayed $\preccurlyeq_*$ relations, we see that the gen. $\sqcup, +$ law holds.

Lemma 2.5 *Let* $a_1, \ldots, a_n \in G$. *Then the following holds irrespective of the order of applying the operations* $\sqcap, \sqcup, \wedge$ *and* $\vee$:

$$\begin{aligned}(a_1 \sqcap \cdots \sqcap a_n)\triangle &\asymp_* a_1\triangle \wedge \cdots \wedge a_n\triangle, &\text{(gen.}\,\sqcap\text{ law)}\\ (a_1 \sqcup \cdots \sqcup a_n)\triangle &\asymp_* a_1\triangle \vee \cdots \vee a_n\triangle. &\text{(gen.}\,\sqcup\text{ law)}\end{aligned}$$

Proof We may assume $n > 1$ and argue by induction on n. As to the gen. $\sqcup$ law, suppose $\sqcup$ has been applied in such an order that $a_1 \sqcup \cdots \sqcup a_n = (a_1 \sqcup \cdots \sqcup a_i) \sqcup (a_{i+1} \sqcup \cdots \sqcup a_n)$ holds. Then

$$\begin{aligned}(a_1 \sqcup \cdots \sqcup a_n)\triangle &\asymp_* (a_1 \sqcup \cdots \sqcup a_i)\triangle \vee (a_{i+1} \sqcup \cdots \sqcup a_n)\triangle,\\ (a_1 \sqcup \cdots \sqcup a_i)\triangle &\asymp_* a_1\triangle \vee \cdots \vee a_i\triangle,\\ (a_{i+1} \sqcup \cdots \sqcup a_n)\triangle &\asymp_* a_{i+1}\triangle \vee \cdots \vee a_n\triangle\end{aligned}$$

by the $\sqcup$ law and the induction hypothesis. Applying Lemma 2.1 to the above displayed $\asymp_*$ relations, we see that the gen. $\sqcup$ law holds. The gen. $\sqcap$ law may be proved similarly.

From now on, When $\alpha = f_1 \cdots f_n \in H^*$ with $f_1, \ldots, f_n \in H$, the subset $\{f_1, \ldots, f_n\}$ of H will also be denoted by α.

Lemma 2.6 *Let* $f_1, \ldots, f_m, g_1, \ldots, g_n \in H,\ \alpha, \beta \in H^*,\ a \in A_\varepsilon,$ *and* $k \in K$. *Assume that* k *belongs to the ranges of* $f_1, \ldots, f_m, g_1, \ldots, g_n$ *but does not belong to those of the predicates in* $\alpha \cup \beta$. *Then the following holds:*

$$\begin{aligned}&f_1 \cdots f_m\alpha \preccurlyeq_* g_1 \cdots g_n\beta\\ &\implies a\,\breve{o}k\,f_1, \cdots, a\,\breve{o}k\,f_m, \alpha \preccurlyeq_* a\,\breve{o}k\,g_1, \cdots, a\,\breve{o}k\,g_n, \beta.\end{aligned} \qquad \text{(gen. case+ law)}$$

Proof We may assume that either $m \geq 1$ or $n \geq 1$. Let $\alpha = f'_1 \cdots f'_{m'}$ and $\beta = g'_1 \cdots g'_{n'}$ with $f'_1, \ldots, f'_{m'}, g'_1, \ldots, g'_{n'} \in H$. Let $x \in \mathbb{X}_\varepsilon$ and define $e = x \,\breve{o}\pi\, x \triangle$. Then k does not belong to the ranges of $e \wedge e^\lozenge$ or $e \vee e^\lozenge$. Therefore by Lemma 2.1, we may assume $m' \neq 0 \neq n'$. Define

$$h = (f_1 \wedge \cdots \wedge f_m) \wedge (g_1 \vee \cdots \vee g_n)^\lozenge,$$
$$h' = (f'_1 \wedge \cdots \wedge f'_{m'})^\lozenge \vee (g'_1 \vee \cdots \vee g'_{n'}),$$

where the orders of applying the operations $\wedge$ and $\vee$ within parentheses are arbitrary. Then Lemma 2.1 shows that the premise of the gen. case+ law implies $\preccurlyeq_* h \Rightarrow h'$, and so $\preccurlyeq_* a \,\breve{o}k\, (h \Rightarrow h')$ by the case+ law. Furthermore, since $k \in K_h - K_{h'}$, we have $a \,\breve{o}k\, (h \Rightarrow h') \asymp_* a \,\breve{o}k\, h \Rightarrow h'$ by the $\Rightarrow$ law. Thus

$$a \,\breve{o}k\, h \preccurlyeq_* h'$$

by the cut law and Lemma 2.1. Furthermore,

$$\begin{aligned}&\big(a \,\breve{o}k\, (f_1 \wedge \cdots \wedge f_m)\big) \wedge \big(a \,\breve{o}k\, (g_1 \vee \cdots \vee g_n)\big)^\lozenge \\ &\asymp_* \big(a \,\breve{o}k\, (f_1 \wedge \cdots \wedge f_m)\big) \wedge \big(a \,\breve{o}k\, (g_1 \vee \cdots \vee g_n)^\lozenge\big) \asymp_* a \,\breve{o}k\, h\end{aligned}$$

by the $\lozenge$ law, Lemma 2.1 and the $\wedge$ law. Thus

$$a \,\breve{o}k\, (f_1 \wedge \cdots \wedge f_m) \preccurlyeq_* a \,\breve{o}k\, (g_1 \vee \cdots \vee g_n), h'$$

by the cut law and Lemma 2.1. Furthermore,

$$a \,\breve{o}k\, f_1 \wedge \cdots \wedge a \,\breve{o}k\, f_m \asymp_* a \,\breve{o}k\, (f_1 \wedge \cdots \wedge f_m),$$
$$a \,\breve{o}k\, (g_1 \vee \cdots \vee g_n) \asymp_* a \,\breve{o}k\, g_1 \vee \cdots \vee a \,\breve{o}k\, g_n$$

by the gen. $\wedge$ law and gen. $\vee$ law. Applying the cut law to the above three displayed relations, we have $a \,\breve{o}k\, f_1 \wedge \cdots \wedge a \,\breve{o}k\, f_m \preccurlyeq_* a \,\breve{o}k\, g_1 \vee \cdots \vee a \,\breve{o}k\, g_n, h'$, hence the conclusion of the gen. case+ law by Lemma 2.1.

The following proposition is a continuation and a refinement of §4.1 and §4.2 of Gomi (09b) and collects basic and general facts about occurrences and substitutions.

Proposition 1 *Let $(A, (\alpha_\lambda)_{\lambda \in L})$ be an algebra. In addition to and on the basis of the notation and terminology introduced in previous papers, define*

$$L^a = \{\lambda \in L \mid (\mathrm{Im}\, \alpha_\lambda)^a \neq \emptyset\}$$

*for each element $a \in A$, and if $\lambda \in L^a$, then say that λ **occurs** in a. Assume that $(A, (\alpha_\lambda)_{\lambda \in L})$ has a basis S. Then the following holds.*

(1) If $a = \alpha_\lambda(a_1, \ldots, a_k) \in A$, then $L^a = \{\lambda\} \cup \bigcup_{j=1}^k L^{a_j}$. If $a \in S$, then $L^a = \emptyset$.

(2) For each element $a \in A$, L^a is a finite set.

Assume furthermore that L *is a subset of the free multiplicative semigroup over* $\Gamma \amalg \mathsf{S}$ *for a set* Γ*. Then the following holds.*

(3) If $a = \alpha_\lambda(a_1, \dots, a_k) \in A$*, then* $S^a_{\text{free}} = \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda$*. If* $a \in S$*, then* $S^a_{\text{free}} = \{a\}$*.*

(4) If $a, b \in A$ *and* $(S^\lambda)^b_{\text{free}} = \emptyset$ *for each* $\lambda \in L^a$*, then every element of* S *is free from* b *in* a*.*

Assume furthermore that (A, T, σ, S) *is a USA. Then the following holds.*

(5) Let $a, b, c \in A$*,* $s \in S$ *and assume* $b = a(s/c)$*, where* (s/c) *denotes the substitution of* c *for* s*. Then* $S^b_{\text{free}} \subseteq S^c_{\text{free}} \cup \left(S^a_{\text{free}} - \{s\}\right)$ *and* $L^b \subseteq L^a \cup L^c$*.*

Proof (0) If $a \in S$, then a has no word form, hence $A^a = \{a\}$. If $a \in A - S$, then a has the unique word form $\alpha_\lambda(a_1, \dots, a_k)$, hence $A^a = \{a\} \cup \bigcup_{j=1}^k A^{a_j}$. A proof of this equation is as follows. It is clear that $A^a \supseteq \{a\} \cup \bigcup_{j=1}^k A^{a_j}$. Suppose $b \in A^a - \{a\}$. Then there exists an occurrence $b_0, \dots, b_n$ of b in a such that $b_i \prec a$ for some $i \in \{1, \dots, n\}$, and so $b_i = a_j$ for some $j \in \{1, \dots, k\}$ by the uniqueness of the word form of a, hence $b \in A^{a_j}$. Therefore $A^a = \{a\} \cup \bigcup_{j=1}^k A^{a_j}$.

(1) If $a \in S$, then $(\operatorname{Im} \alpha_\lambda)^a \subseteq \operatorname{Im} \alpha_\lambda \cap \{a\} = \emptyset$ by (0) for all $\lambda \in L$, and so $L^a = \emptyset$. Therefore assume $a = \alpha_\lambda(a_1, \dots, a_k)$. Then since $a \in (\operatorname{Im} \alpha_\lambda)^a$ and $(\operatorname{Im} \alpha_\mu)^{a_j} \subseteq (\operatorname{Im} \alpha_\mu)^a$ for all $\mu \in L$ and $j \in \{1, \dots, k\}$ by (0), we have $\{\lambda\} \cup \bigcup_{j=1}^k L^{a_j} \subseteq L^a$. Suppose $\mu \in L^a - \{\lambda\}$. Then there exists an element $b \in (\operatorname{Im} \alpha_\mu)^a$, and $b \neq a$ because $\mu \neq \lambda$, and so $b \in (\operatorname{Im} \alpha_\mu)^{a_j}$ for some $j \in \{1, \dots, k\}$ by (0), hence $\mu \in L^{a_j}$. Therefore $L^a = \{\lambda\} \cup \bigcup_{j=1}^k L^{a_j}$.

(2) We argue by induction on $r = \operatorname{Rank} a$. If $r = 0$, then $a \in S$, and so $L^a = \emptyset$ by (1). Therefore assume $r \geq 1$, and let $\alpha_\lambda(a_1, \dots, a_k)$ be the word form of a. Then $r = 1 + \sum_{j=1}^k \operatorname{Rank} a_j$, and so L^{a_j} is a finite set for all $j \in \{1, \dots, k\}$ by the induction hypothesis. Therefore L^a are finite set by (1).

(3) If $a \in S$, then $S^a_{\text{free}} \subseteq \{a\}$ and a is a free occurrence of a in itself by (0), hence $S^a_{\text{free}} = \{a\}$. Therefore assume $a = \alpha_\lambda(a_1, \dots, a_k)$. Let $s \in S^a_{\text{free}}$. Then there exists a free occurrence $s_0, \dots, s_n$ of s in a. Since $s_0 = a \in \operatorname{Im} \alpha_\lambda$, we have $s \notin S^\lambda$. Also, since $a \neq s$ by (0), $s_i \prec a$ for some $i \in \{1, \dots, n\}$, and so $s_i = a_j$ for some $j \in \{1, \dots, k\}$ by (0), hence $s \in S^{a_j}_{\text{free}}$. Thus $S^a_{\text{free}} \subseteq \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda$. The converse has been proved in Lemma 4.2 of Gomi (09b).

(4) Let $s_0, \dots, s_n$ be a free occurrence of an element $s \in S$ in a. If $s_i \in \operatorname{Im} \alpha_\lambda$ for some $i \in \{0, \dots, n\}$ and $\lambda \in L$, then $\lambda \in L^a$, and so $(S^\lambda)^b_{\text{free}} = \emptyset$. Therefore $s_0, \dots, s_n$ is free from b. Thus, every element of S is free from b in a.

(5) We argue by induction on $r = \operatorname{Rank} a$. If $a = s$ then $b = c$, while if $S \ni a \neq s$ then $b = a$ and $s \notin S^a_{\text{free}}$, both by (4.5) of Gomi (09b) and (3). Thus the assertions hold if $r = 0$. Therefore assume $r \geq 1$, and let $\alpha_\lambda(a_1, \dots, a_k)$ be the word form of a. Then $r = 1 + \sum_{j=1}^k \operatorname{Rank} a_j$. If $s \in S^\lambda$, then $b = a$ by (4.6) of Gomi (09b) and $s \notin S^a_{\text{free}}$ by (3), and the assertions hold. Therefore assume

$s \notin S^\lambda$, and define $b_j = a_j(s/c)$ for $j = 1, \dots, k$. Then $b = \alpha_\lambda(b_1, \dots, b_k)$ by (4.6) of Gomi (09b). Therefore

$$S^a_{\text{free}} = \bigcup_{j=1}^k S^{a_j}_{\text{free}} - S^\lambda, \qquad S^b_{\text{free}} = \bigcup_{j=1}^k S^{b_j}_{\text{free}} - S^\lambda,$$
$$L^a = \{\lambda\} \cup \bigcup_{j=1}^k L^{a_j}, \qquad L^b = \{\lambda\} \cup \bigcup_{j=1}^k L^{b_j}$$

by (3) and (1). Also, $S^{b_j}_{\text{free}} \subseteq S^c_{\text{free}} \cup \big(S^{a_j}_{\text{free}} - \{s\}\big)$ and $L^{b_j} \subseteq L^{a_j} \cup L^c$ for $j = 1, \dots, k$ by the induction hypothesis. Therefore the assertions hold.

Lemma 2.7 *Let* $f_1, \dots, f_m, g_1, \dots, g_n \in H$, $\alpha, \beta \in H^*$, $x \in \mathbb{X}_\varepsilon$, *and* $k \in K$. *Assume that* k *belongs to the ranges of* $f_1, \dots, f_m$, $g_1, \dots, g_n$ *but does not belong to those of the predicates in* $\alpha \cup \beta$ *and that* x *does not occur free in the predicates in* $\{f_1, \dots, f_m, g_1, \dots, g_n\} \cup \alpha \cup \beta$. *Then the following holds:*

$$\begin{aligned} &x\,\breve{o}k\, f_1, \cdots, x\,\breve{o}k\, f_m, \alpha \preccurlyeq_* x\,\breve{o}k\, g_1, \cdots, x\,\breve{o}k\, g_n, \beta \\ &\Longrightarrow\ f_1 \cdots f_m \alpha \preccurlyeq_* g_1 \cdots g_n \beta. \end{aligned} \qquad \text{(gen. case$-$ law)}$$

Proof We may assume that either $m \geq 1$ or $n \geq 1$. Let $\alpha = f'_1 \cdots f'_{m'}$ and $\beta = g'_1 \cdots g'_{n'}$ with $f'_1, \dots, f'_{m'}, g'_1, \dots, g'_{n'} \in H$. Define $e = \mathrm{one}\,\forall\pi\,\mathrm{one}\triangle$. Then k does not belong to the range of $e \wedge e^\Diamond$ or of $e \vee e^\Diamond$, and x does not occur free in $e \wedge e^\Diamond$ or in $e \vee e^\Diamond$ by Proposition 1 (3). Therefore by Lemma 2.1, we may assume $m' \neq 0 \neq n'$. Define $h' = (f'_1 \wedge \cdots \wedge f'_{m'})^\Diamond \vee (g'_1 \vee \cdots \vee g'_{n'})$, where the orders of applying the operations $\wedge$ and $\vee$ within parentheses are arbitrary. Then by Lemma 2.1 the premise of the gen. case$-$ law implies

$$x\,\breve{o}k\, f_1 \wedge \cdots \wedge x\,\breve{o}k\, f_m \preccurlyeq_* x\,\breve{o}k\, g_1 \vee \cdots \vee x\,\breve{o}k\, g_n, h'$$

irrespective of the orders of applying the operations $\wedge$ and $\vee$. Furthermore,

$$x\,\breve{o}k\,(f_1 \wedge \cdots \wedge f_m) \asymp_* x\,\breve{o}k\, f_1 \wedge \cdots \wedge x\,\breve{o}k\, f_m,$$
$$x\,\breve{o}k\, g_1 \vee \cdots \vee x\,\breve{o}k\, g_n \asymp_* x\,\breve{o}k\,(g_1 \vee \cdots \vee g_n)$$

by the gen. $\wedge$ law and gen. $\vee$ law. Applying the cut law to the above three displayed relations and using Lemma 2.1, we have

$$\big(x\,\breve{o}k\,(f_1 \wedge \cdots \wedge f_m)\big) \wedge \big(x\,\breve{o}k\,(g_1 \vee \cdots \vee g_n)\big)^\Diamond \preccurlyeq_* h'.$$

Define $h = (f_1 \wedge \cdots \wedge f_m) \wedge (g_1 \vee \cdots \vee g_n)^\Diamond$. Then

$$\begin{aligned} x\,\breve{o}k\, h &\asymp_* \big(x\,\breve{o}k\,(f_1 \wedge \cdots \wedge f_m)\big) \wedge \big(x\,\breve{o}k\,(g_1 \vee \cdots \vee g_n)^\Diamond\big) \\ &\asymp_* \big(x\,\breve{o}k\,(f_1 \wedge \cdots \wedge f_m)\big) \wedge \big(x\,\breve{o}k\,(g_1 \vee \cdots \vee g_n)\big)^\Diamond \end{aligned}$$

by the $\wedge$ law, $\Diamond$ law, and Lemma 2.1. Therefore $x\,\breve{o}k\, h \preccurlyeq_* h'$ by the cut law, hence $\preccurlyeq_* x\,\breve{o}k\, h \Rightarrow h'$. Furthermore, $x\,\breve{o}k\, h \Rightarrow h' \asymp_* x\,\breve{o}k\,(h \Rightarrow h')$ by the $\Rightarrow$ law. Therefore $\preccurlyeq_* x\,\breve{o}k\,(h \Rightarrow h')$ by the cut law, and since x does not occur free in $h \Rightarrow h'$ by Proposition 1 (3), we have $\preccurlyeq_* h \Rightarrow h'$ by the case$-$ law, hence the conclusion of the gen. case$-$ law.

Lemma 2.8 *Let* $x \in \mathbb{X}_\varepsilon$, $a, b_1, \dots, b_n \in G$, $\alpha, \beta \in (A_\emptyset)^*$, $f \in H$, $k \in K_f$, $p, q_1, \dots, q_n \in \mathbb{P}$, *and assume that* x *does not occur free in the elements of* $\{a, b_1, \dots, b_n\} \cup \alpha \cup \beta$ *and that* $p \geq \sum_{i=1}^n q_i$ *holds, where if* $n = 0$ *then* $\sum_{i=1}^n q_i = 0$ *by definition. Then the following holds:*

$$\begin{aligned} & x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* x\,\breve{o}\pi\, b_1\triangle, \cdots, x\,\breve{o}\pi\, b_n\triangle, \beta \\ & \implies a\,\overline{p}k\,f, \alpha \preccurlyeq_* b_1\,\overline{q_1}k\,f, \cdots, b_n\,\overline{q_n}k\,f, \beta. \end{aligned} \qquad \text{(pigeonhole principle)}$$

Proof Assume $n = 0$. Let $b = \mathsf{one}^\square$ and $q = 0$. Then x does not occur free in b, $p \geq q$, and the premise $x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* \beta$ of the pigeonhole principle and the weakening law imply $x\,\breve{o}\pi\, a\triangle, \alpha \preccurlyeq_* x\,\breve{o}\pi\, b\triangle, \beta$. If we show $a\,\overline{p}k\,f, \alpha \preccurlyeq_* b\,\overline{q}k\,f, \beta$, then since $b\,\overline{q}k\,f \preccurlyeq_*$ by the $\mathsf{one}^\square$ law, we have $a\,\overline{p}k\,f, \alpha \preccurlyeq_* \beta$ by the cut law, which is the conclusion of the pigeonhole principle.

Therefore assume $n \geq 1$. Let $\alpha = g_1 \cdots g_l$ and $\beta = h_1 \cdots h_m$ with $g_1, \dots, g_l, h_1, \dots, h_m \in A_\emptyset$. We may assume $l \neq 0 \neq m$ as in the proof of Lemma 2.7. Define $e = (g_1 \wedge \cdots \wedge g_l) \wedge (h_1 \vee \cdots \vee h_m)^\diamond$, where and elsewhere the orders of applying the operations are arbitrary. Then $e \in A_\emptyset$, and the premise of the pigeonhole principle implies

$$x\,\breve{o}\pi\, a\triangle, e \preccurlyeq_* x\,\breve{o}\pi\, b_1\triangle \vee \cdots \vee x\,\breve{o}\pi\, b_n\triangle,$$

and $x\,\breve{o}\pi\, b_1\triangle \vee \cdots \vee x\,\breve{o}\pi\, b_n\triangle \asymp_* x\,\breve{o}\pi\,(b_1\triangle \vee \cdots \vee b_n\triangle)$ by the gen. $\vee$ law. Furthermore, since $b_1\triangle \vee \cdots \vee b_n\triangle \asymp_* (b_1 \sqcup \cdots \sqcup b_n)\triangle$ by the gen. $\sqcup$ law, we have $x\,\breve{o}\pi\,(b_1\triangle \vee \cdots \vee b_n\triangle) \asymp_* x\,\breve{o}\pi\,(b_1 \sqcup \cdots \sqcup b_n)\triangle$ by the gen. case+ law. Therefore for $b = b_1 \sqcup \cdots \sqcup b_n$, the premise of the pigeonhole principle implies $\preccurlyeq_* e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)$, and applying the $\forall$+ law, we have

$$\preccurlyeq_* \mathsf{one}\,\forall\pi\,\Big(\big(e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\big)\,\Omega x\Big)\triangle.$$

Furthermore, since x does not occur free in e by Proposition 1 (3), we have

$$\begin{gathered} e, \mathsf{one}\,\forall\pi\,\Big(\big(e \Rightarrow \big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\big)\,\Omega x\Big)\triangle \\ \preccurlyeq_* \mathsf{one}\,\forall\pi\,\Big(\big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\,\Omega x\Big)\triangle \end{gathered}$$

by the $\forall, \Rightarrow$ law, and since x does not occur free in a or in $b\triangle$ by Proposition 1 (3), we have

$$\mathsf{one}\,\forall\pi\,\Big(\big((x\,\breve{o}\pi\, a\triangle) \Rightarrow (x\,\breve{o}\pi\, b\triangle)\big)\,\Omega x\Big)\triangle \preccurlyeq_* a\,\forall\pi\, b\triangle$$

by the $\forall$ law, and we have

$$a\,\forall\pi\, b\triangle, a\,\overline{p}k\,f \preccurlyeq_* b\,\overline{p}k\,f$$

by the $\forall, \mathfrak{P}$ law, and since $\overline{p} \subseteq \overline{q_1 + \cdots + q_n}$ by the assumption $p \geq q_1 + \cdots + q_n$, we have

$$b\,\overline{p}k\,f \asymp_* (b\,\overline{q_1 + \cdots + q_n}k\,f) \wedge (b\,\overline{p}k\,f)$$

by the $\cap$ law, and we have

$$(\mathsf{b}\,\overline{\mathsf{q}_1 + \cdots + \mathsf{q}_n}\mathsf{k}\,\mathsf{f}) \wedge (\mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \preccurlyeq_* \mathsf{b}\,\overline{\mathsf{q}_1 + \cdots + \mathsf{q}_n}\mathsf{k}\,\mathsf{f}$$

by the conjunction law, and we have

$$\mathsf{b}\,\overline{\mathsf{q}_1 + \cdots + \mathsf{q}_n}\mathsf{k}\,\mathsf{f} \preccurlyeq_* \mathsf{b}_1\,\overline{\mathsf{q}_1}\mathsf{k}\,\mathsf{f}, \cdots, \mathsf{b}_n\,\overline{\mathsf{q}_n}\mathsf{k}\,\mathsf{f}$$

by the gen. $\sqcup, +$ law. Applying the cut law to the above seven displayed $\preccurlyeq_*$ relations, we finally obtain $\mathsf{e}, \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq_* \mathsf{b}_1\,\overline{\mathsf{q}_1}\mathsf{k}\,\mathsf{f}, \cdots, \mathsf{b}_n\,\overline{\mathsf{q}_n}\mathsf{k}\,\mathsf{f}$, hence the conclusion of the pigeonhole principle.

Lemma 2.9 *Let* $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \mathsf{A}_\varepsilon$, $\mathsf{f} \in \mathsf{H}$, *and* $\mathsf{k}_1, \ldots, \mathsf{k}_n$ *be distinct cases in* K_f. *Then the following holds for every* $\rho \in \mathfrak{S}_n$, *where and elsewhere* $\mathfrak{S}_n$ *denotes the symmetric group on the letters* $1, \ldots, n$:

$$(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}\mathsf{f} \asymp_* (\mathsf{a}_{\rho i}\,\breve{\mathsf{o}}\mathsf{k}_{\rho i})_{i=1,\ldots,n}\mathsf{f}. \qquad \text{(permutation law)}$$

Proof We only need to consider the case where ρ is a transposition, in which case, the result follows from the $\mathfrak{Q}, \breve{\mathsf{o}}$ law and gen. case+ law.

Lemma 2.10 *Let* $\mathsf{a} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{x} \in \mathbb{X}_\varepsilon$, $\mathsf{k} \in \mathsf{K}_\mathsf{f}$, *and* $\mathsf{p} \in \mathbb{P}$. *Also, let* $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \mathsf{A}_\varepsilon$ *and* $\mathsf{k}_1, \ldots, \mathsf{k}_n$ *be the set of distinct cases in* $\mathsf{K}_\mathsf{f} - \{\mathsf{k}\}$. *Assume that* x *does not occur free in* $(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}\mathsf{f}$. *Then the following holds:*

$$(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \asymp_* \big(\mathsf{a} \sqcap \big(\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}\mathsf{f}\big)\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle.$$

Proof Define $\mathsf{g} = (\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}\mathsf{f}$. Then, as in the proof of the permutation law, it follows from the $\mathfrak{Q}, \breve{\mathsf{o}}$ law and gen. case+ law that

$$(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \asymp_* \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{g}$$

holds. Also, since $\mathsf{K}_\mathsf{g} = \{\mathsf{k}\}$ and x does not occur free in g, we have

$$\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{g} \asymp_* \mathsf{a}\,\overline{\mathsf{p}}\pi\big((\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\triangle$$

by the $\mathfrak{P}$ law. Also we have

$$\mathsf{a}\,\overline{\mathsf{p}}\pi\big((\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\triangle \asymp_* \big(\mathsf{a} \sqcap (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$$

by the $\triangle$ law. Applying the cut law to the above three $\asymp_*$ equations, we have $(\mathsf{a}_i\,\breve{\mathsf{o}}\mathsf{k}_i)_{i=1,\ldots,n}(\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \asymp_* \big(\mathsf{a} \sqcap (\mathsf{x}\,\breve{\mathsf{o}}\mathsf{k}\,\mathsf{g})\,\Omega\mathsf{x}\big)\,\overline{\mathsf{p}}\pi\,\mathsf{one}\triangle$.

3 The law is characteristic of the predicate logical space

The purpose of this section it to show that the MPC law is a characteristic law of the predicate logical space $(\mathsf{H}, \mathcal{G})$ under the conditions 1–4 stated at the beginning of §1. Because of Theorem 2, we only need to prove the following.

Theorem 3 *The validity relation* $\preccurlyeq$ *of the predicate logical space* $(\mathsf{H}, \mathcal{G})$ *is contained in every MPC relation* $\preccurlyeq_*$ *on* H^* *under the conditions 1–4.*

Elements (α, β) of $\mathsf{H}^* \times \mathsf{H}^*$ will be denoted by $\alpha \to \beta$ and called the **sequents**. A sequent is said to be **sentential** if it consists of sentences.

In order to argue by contradiction, we assume that there exists a sequent $\alpha \to \beta$ such that $\alpha \preccurlyeq \beta$ and $\alpha \not\preccurlyeq_* \beta$, which we call a **counter sequent**. A contradiction will be obtained in a series of twenty four lemmas.

Lemma 3.1 *There exists a sentential counter sequent.*

Proof Let $f_1 \cdots f_m \alpha \to g_1 \cdots g_n \beta$ be a counter sequent and assume that a case k belongs to the ranges of $f_1, \ldots, f_m, g_1, \ldots, g_n$ but does not belong to those of the predicates in $\alpha \cup \beta$. Since we are assuming that $\mathbb{X}_\varepsilon$ is enumerable, Lemma 4.1 of Gomi (09b) shows that there exists a variable $x \in \mathbb{X}_\varepsilon$ which does not occur free in the predicates in $\{f_1, \ldots, f_m\} \cup \{g_1, \ldots, g_n\} \cup \alpha \cup \beta$. Since $\preccurlyeq$ satisfies the gen. case+ law by Lemma 2.6 and $\preccurlyeq_*$ satisfies the gen. case− law by Lemma 2.7, it follows that $x\,\breve{o}k\, f_1, \cdots, x\,\breve{o}k\, f_m, \alpha \to x\,\breve{o}k\, g_1, \cdots, x\,\breve{o}k\, g_n, \beta$ is a counter sequent. Since we are assuming that there exists a counter sequent and that the ranges of the predicates are finite sets, we conclude that there exists a sentential counter sequent.

Lemma 3.2 *There exists a sentential counter sequent* $\alpha_0 \to \beta_0$ *which satisfies the following condition G.*

G *If the nominalizer* Ωx *occurs in a sentence in* $\alpha_0 \cup \beta_0$ *for some* $x \in \mathbb{X}_\varepsilon$*, then the variable* x *does not occur free in the sentences in* $\alpha_0 \cup \beta_0$*. Also the nominalizer* Ωx_0 *in* (1.2) *occurs in a sentence in* $\alpha_0 \cup \beta_0$*.*

Proof There exists a sentential counter sequent $\alpha \to \beta$ by Lemma 3.1. Since the φ-validity relation for each $\varphi \in \mathcal{G}$ is non-trivial, $\preccurlyeq$ is also non-trivial, hence $\alpha \cup \beta \neq \emptyset$. Let $\alpha = f_1 \cdots f_m$, $\beta = g_1 \cdots g_n$, and define

$$h = (f_1 \wedge \cdots \wedge f_m)^\Diamond \vee (g_1 \vee \cdots \vee g_n),$$

where the orders of applying the operations $\wedge$ and $\vee$ within parentheses are arbitrary. Then h is a sentence. Since both $\preccurlyeq$ and $\preccurlyeq_*$ are Boolean relations, Lemma 2.1 shows that $\to h$ is a counter sequent.

For each $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$, let $\forall x\, f$ denote the sentence $\mathsf{one}\,\forall\pi\,(f\,\Omega x)\triangle$ for the time being. If $f \in A_\emptyset$ satisfies $\preccurlyeq f$, then we have $\preccurlyeq \forall x\, f$ for all $x \in \mathbb{X}_\varepsilon$ by the $\forall$+ law. Conversely, if $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$ satisfy $\preccurlyeq_* \forall x\, f$, then we have $\preccurlyeq_* f$ by the $\forall$− law and cut law. Therefore if $\to f$ is a sentential counter sequent, so is $\to \forall x\, f$ for all $x \in \mathbb{X}_\varepsilon$. Furthermore, Proposition 1 (3) (1) show that $\mathbb{X}_{\text{free}}^{\forall x\, f} = \mathbb{X}_{\text{free}}^{f} - \{x\}$ and Ωx_0 occurs in $\forall x\, f$ for each $f \in A_\emptyset$ and $x \in \mathbb{X}_\varepsilon$.

There exists a non-empty subset $\{x_1, \ldots, x_k\}$ of $\mathbb{X}_\varepsilon$ which contains every element of $\mathbb{X}_\varepsilon$ occurring free in h. The above shows that $\to \forall x_k \cdots \forall x_1\, h$ is a sentential counter sequent and no element of $\mathbb{X}_\varepsilon$ occurs free in $\forall x_k \cdots \forall x_1\, h$

but Ωx_0 occurs in $\forall x_k \cdots \forall x_1 \, h$. Thus defining $\alpha_0 \to \beta_0$ as $\to \forall x_k \cdots \forall x_1 \, h$, we have that it satisfies G.

Our aim is to show $\alpha_0 \not\preccurlyeq \beta_0$, which is a desired contradiction.

Lemma 3.3 *$\mathbb{X}_\varepsilon$ is partitioned into subsets $\mathbb{X}'_\varepsilon$ and $\mathbb{X}''_\varepsilon$ which satisfy the following three conditions.*

- *If $x \in \mathbb{X}'_\varepsilon$, then Ωx does not occur in the sentences in $\alpha_0 \cup \beta_0$.*
- *Elements of $\mathbb{X}''_\varepsilon$ do not occur free in the sentences in $\alpha_0 \cup \beta_0$.*
- *$\mathbb{X}'_\varepsilon$ is an enumerable set, while $\mathbb{X}''_\varepsilon$ is a non-empty finite set.*

Proof Let $\mathbb{X}''_\varepsilon$ be the set of the elements $x \in \mathbb{X}_\varepsilon$ such that Ωx occurs in a sentence in $\alpha_0 \cup \beta_0$, and define $\mathbb{X}'_\varepsilon = \mathbb{X}_\varepsilon - \mathbb{X}''_\varepsilon$. Then $\mathbb{X}''_\varepsilon$ is a finite set by Proposition 1 (2). Since $\alpha_0 \to \beta_0$ satisfies the condition G and we are assuming that $\mathbb{X}_\varepsilon$ is enumerable, $\mathbb{X}'_\varepsilon$ and $\mathbb{X}''_\varepsilon$ partition $\mathbb{X}_\varepsilon$ and satisfy the conditions.

We say that an element $a \in A$ is **good** if it satisfies the following conditions.

- If $x \in \mathbb{X}'_\varepsilon$, then Ωx does not occur in a.
- Elements of $\mathbb{X}''_\varepsilon$ do not occur free in a.

We denote the $\mathfrak{F}$-closure of $\mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon$ by A'_ε:

$$A'_\varepsilon = [\mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon]_{\mathfrak{F}}. \tag{3.1}$$

Lemma 3.4 *The following holds as to the good elements.*

(1) Let λ be an operation of A other than the nominalizers and $(a_1, \ldots, a_n) \in \operatorname{Dom} \lambda$, hence $n \leq 2$. Then $\lambda(a_1, \ldots, a_n)$ is good iff $a_1, \ldots, a_n$ are good.

(2) Let $a \in A_\varepsilon$, $f \in A_\emptyset$, $x \in \mathbb{X}_\varepsilon$, and assume that $a \,\breve{o}\pi (f\, \Omega x) \triangle$ is good. Then $f(x/a)$ is also good and x is free from a in f.

(3) A'_ε is an enumerable subset of A_ε, and an element $a \in A_\varepsilon$ is good iff $a \in A'_\varepsilon$.

Proof (1) Proposition 1 (1) implies that Ωx occurs in $\lambda(a_1, \ldots, a_n)$ iff it occurs in some of $a_1, \ldots, a_n$. Proposition 1 (3) implies that $x \in \mathbb{X}_\varepsilon$ occurs free in $\lambda(a_1, \ldots, a_n)$ iff it occurs free in some of $a_1, \ldots, a_n$. Therefore (1) holds.

(2) It follow from (1) that a and $f\, \Omega x$ are good. Therefore if $y \in \mathbb{X}'_\varepsilon$, then Ωy does not occur in either a or $f\, \Omega x$, and so it does not occur in f either by Proposition 1 (1). Therefore Ωy does not occur in $f(x/a)$ by Proposition 1 (5). Also, elements of $\mathbb{X}''_\varepsilon$ do not occur free in a, and Proposition 1 (3) shows that elements of $\mathbb{X}''_\varepsilon$ other than x do not occur free in f. Therefore elements of $\mathbb{X}''_\varepsilon$ do not occur free in $f(x/a)$ by Proposition 1 (5). Thus, $f(x/a)$ is good. If a variable $y \in \mathbb{X}_\varepsilon$ occurs free in a, then $y \in \mathbb{X}'_\varepsilon$, and so Ωy does not occur in f as was shown above. Therefore x is free from a in f by Proposition 1 (4).

(3) (3.1) implies $\mathbb{X}'_\varepsilon \subseteq \mathsf{A}'_\varepsilon \subseteq [\mathbb{S}_\varepsilon]_{\mathfrak{F}} = \mathsf{A}_\varepsilon$, and since both $\mathbb{X}'_\varepsilon$ and A_ε are enumerable by Lemma 3.3 and our assumption, so is A'_ε.

In order to prove the latter half, let B be the set of the good elements of A. Since $\mathsf{A}_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}$ by (1.1), A_ε is the union of $\mathfrak{F}$-descendants D_n ($n = 0, 1, \dots$) of $\mathbb{S}_\varepsilon$. Similarly, A'_ε is the union of $\mathfrak{F}$-descendants D'_n ($n = 0, 1, \dots$) of $\mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon$. So we need only to show that $\mathsf{D}_n \cap \mathsf{B} = \mathsf{D}'_n$ for all n. Obviously $\mathbb{S}_\varepsilon \cap \mathsf{B} = (\mathbb{C}_\varepsilon \cup \mathbb{X}_\varepsilon) \cap \mathsf{B} = \mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon$, so we argue by induction on n and assume $n \geq 1$. Assume $a \in \mathsf{D}_n \cap \mathsf{B}$. Then $a = \mathfrak{f}(a_1, \dots, a_m)$ for some $\mathfrak{f} \in \mathfrak{F}$ and $a_1, \dots, a_m \in \mathsf{A}_\varepsilon$ such that $a_i \in \mathsf{D}_{n_i}$ ($i = 1, \dots, m$) with $n = 1 + \sum_{i=1}^m n_i$. Since $a_i \in \mathsf{B}$ by (1), we have $a_i \in \mathsf{D}'_{n_i}$ ($i = 1, \dots, m$) by the induction hypothesis. Therefore $a \in \mathsf{D}'_n$. Conversely assume $a \in \mathsf{D}'_n$. Then $a = \mathfrak{f}(a_1, \dots, a_m)$ for some $\mathfrak{f} \in \mathfrak{F}$ and $a_1, \dots, a_m \in \mathsf{A}'_\varepsilon$ such that $a_i \in \mathsf{D}'_{n_i}$ ($i = 1, \dots, m$) with $n = 1 + \sum_{i=1}^m n_i$. Since $a_i \in \mathsf{D}_{n_i} \cap \mathsf{B}$ ($i = 1, \dots, m$) by the induction hypothesis, we have $a \in \mathsf{D}_n \cap \mathsf{B}$ by (1). This completes the proof of the latter half by induction.

A sequent $\alpha \to \beta$ is said to be **good** if it consists of good predicates, and called a SS (**singular** sequent) if $\alpha \not\preccurlyeq_* \beta$. GSS and GSSS are abbreviations for Good Singular Sequent and Good Sentential Singular Sequent respectively. Thus $\alpha_0 \to \beta_0$ is a GSSS.

Lemma 3.5 *Let $\alpha \to \beta$ be a SS. Then $\alpha \cap \beta = \emptyset$ and the following holds.*

(1) If $f \in \alpha$, then $f\alpha \to \beta$ is a SS. If $g \in \beta$, then $\alpha \to g\beta$ is a SS.

(2) If $\alpha = f_1 \cdots f_l$, $\beta = g_1 \cdots g_m$ and elements $f'_1, \dots f'_l, g'_1, \dots, g'_m \in H$ satisfy $f_i \preccurlyeq_* f'_i$ ($i = 1, \dots, l$) and $g'_j \preccurlyeq_* g_j$ ($j = 1, \dots, m$), then $f'_1 \cdots f'_l \to g'_1 \cdots g'_m$ is a SS.

Also the following holds for $n = 1, 2, \dots,$ irrespective of the order of applying the operations $\wedge$ and $\vee$.

(3) If $f_1 \wedge \cdots \wedge f_n \in \alpha$, then $f_1 \cdots f_n\alpha \to \beta$ is a SS.

(4) If $f_1 \wedge \cdots \wedge f_n \in \beta$, then $\alpha \to f_i\beta$ is a SS for some $i \in \{1, \dots, n\}$.

(5) If $f_1 \vee \cdots \vee f_n \in \alpha$, then $f_i\alpha \to \beta$ is a SS for some $i \in \{1, \dots, n\}$.

(6) If $f_1 \vee \cdots \vee f_n \in \beta$, then $\alpha \to f_1 \cdots f_n\beta$ is a SS.

Also the following holds.

(7) If $f \Rightarrow g \in \alpha$, then either $\alpha \to f\beta$ or $g\alpha \to \beta$ is a SS.

(8) If $f \Rightarrow g \in \beta$, then $f\alpha \to g\beta$ is a SS.

(9) If $f^\Diamond \in \alpha$, then $\alpha \to f\beta$ is a SS.

(10) If $f^\Diamond \in \beta$, then $f\alpha \to \beta$ is a SS.

Proof If $\alpha \cap \beta \neq \emptyset$, then $\alpha \preccurlyeq_* \beta$ by the repetition law, weakening law and exchange law. Thus $\alpha \cap \beta = \emptyset$.

(1) If $f \in \alpha$ and $f\alpha \preccurlyeq_* \beta$, then $\alpha \preccurlyeq_* \beta$ by the exchange law and contraction law. Thus if $f \in \alpha$, then $f\alpha \to \beta$ is a SS. The latter assertion is proved similarly.

(2) If $f'_1 \cdots f'_l \preccurlyeq_* g'_1 \cdots g'_m$, then $\alpha \preccurlyeq_* \beta$ by the cut law and exchange law. Thus $f'_1 \cdots f'_l \to g'_1 \cdots g'_m$ is a SS.

(3) Suppose $f_1 \wedge \cdots \wedge f_n \in \alpha$. Then $f_1 \wedge \cdots \wedge f_n, \alpha \to \beta$ is a SS by (1). If $f_1 \cdots f_n \alpha \preccurlyeq_* \beta$, then $f_1 \wedge \cdots \wedge f_n, \alpha \preccurlyeq_* \beta$ by Lemma 2.1. Thus $f_1 \cdots f_n \alpha \to \beta$ is a SS.

(4) Suppose $f_1 \wedge \cdots \wedge f_n \in \beta$. Then $\alpha \to f_1 \wedge \cdots \wedge f_n, \beta$ is a SS by (1). If $\alpha \preccurlyeq_* f_i\beta$ for all $i \in \{1, \dots, n\}$, then $\alpha \preccurlyeq_* f_1 \wedge \cdots \wedge f_n, \beta$ by Lemma 2.1. Thus $\alpha \to f_i\beta$ is a SS for some $i \in \{1, \dots, n\}$.

The rest of the proof is similar and omitted.

Lemma 3.6 *Let $\alpha \to \beta$ be a GSS and $f \in H$. Also, let $a_1, \dots, a_n \in A_\varepsilon$, $k_1, \dots, k_n$ be distinct cases in K_f, and $\rho \in \mathfrak{S}_n$. Then the following holds.*

- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,n} f \in \alpha$, then $(a_{\rho i} \,\breve{o} k_{\rho i})_{i=1,\dots,n} f, \alpha \to \beta$ is a GSS.*
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,n} f \in \beta$, then $\alpha \to (a_{\rho i} \,\breve{o} k_{\rho i})_{i=1,\dots,n} f, \beta$ is a GSS.*

Proof The sequents in question are good by Lemma 3.4. Since $(a_i \,\breve{o} k_i)_i f \asymp^* (a_{\rho i} \,\breve{o} k_{\rho i})_i f$ by the permutation law, the sequents in question are singular by Lemma 3.5.

Lemma 3.7 *Let $\alpha \to \beta$ be a GSS and $f, g \in H$. Also, let $a_1, \dots, a_l \in A_\varepsilon$ and $k_1, \dots, k_l$ be distinct cases with $k_1, \dots, k_n \in K_f - K_g$, $k_{n+1}, \dots, k_m \in K_f \cap K_g$, and $k_{m+1}, \dots, k_l \in K_g - K_f$ $(0 \leq n \leq m \leq l)$. Then the following holds.*

- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \wedge g) \in \alpha$, then*
  $(a_i \,\breve{o} k_i)_{i=1,\dots,m} f, (a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \alpha \to \beta$ is a GSS.
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \wedge g) \in \beta$, then either*
  $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\dots,m} f, \beta$ or $\alpha \to (a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \beta$ is a GSS.
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \vee g) \in \alpha$, then either*
  $(a_i \,\breve{o} k_i)_{i=1,\dots,m} f, \alpha \to \beta$ or $(a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \alpha \to \beta$ is a GSS.
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \vee g) \in \beta$, then*
  $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\dots,m} f, (a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \beta$ is a GSS.
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \alpha$, then either*
  $\alpha \to (a_i \,\breve{o} k_i)_{i=1,\dots,m} f, \beta$ or $(a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \alpha \to \beta$ is a GSS.
- *If $(a_i \,\breve{o} k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \beta$, then*
  $(a_i \,\breve{o} k_i)_{i=1,\dots,m} f, \alpha \to (a_i \,\breve{o} k_i)_{i=n+1,\dots,l} g, \beta$ is a GSS.

Proof The sequents in question are good by Lemma 3.4, and singular by the $\wedge$ law, $\vee$ law, $\Rightarrow$ law, and Lemma 3.5.

Lemma 3.8 *Let $\alpha \to \beta$ a GSS and $f \in H$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in K_f. Then the following holds.*

- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(f^\Diamond) \in \alpha$, then $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,n} f, \beta$ is a GSS.*
- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(f^\Diamond) \in \beta$, then $(a_i \breve{o} k_i)_{i=1,\ldots,n} f, \alpha \to \beta$ is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the $\Diamond$ law and Lemma 3.5.

Lemma 3.9 *Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p} \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in $K_f - \{k\}$. Then the following holds.*

- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a \neg\mathfrak{p} k\, f) \in \alpha$, then $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f^\Diamond), \alpha \to \beta$ is a GSS.*
- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a \neg\mathfrak{p} k\, f) \in \beta$, then $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f^\Diamond), \beta$ is a GSS.*

Proof The sequents in question are good by Lemma 3.4, and singular by Lemma 3.5, because $(a_i \breve{o} k_i)_i(a \neg\mathfrak{p} k\, f) \asymp_* (a_i \breve{o} k_i)_i(a\, \mathfrak{p} k\, f^\Diamond)$ by the $\neg$ law and gen. case+ law.

Lemma 3.10 *Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in $K_f - \{k\}$. Then the following holds.*

- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m) k\, f) \in \alpha$, then $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_j k\, f), \alpha \to \beta$ is a GSS for some $j \in \{1, \ldots, m\}$.*
- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m) k\, f) \in \beta$, then $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_1 k\, f), \cdots, (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p}_m k\, f), \beta$ is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the gen. $\cup$ law, gen. case+ law, gen. $\vee$ law, and Lemma 3.5.

Lemma 3.11 *Let $\alpha \to \beta$ be a GSS, $a \in G$, $f \in H$, $k \in K_f$, and $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$. Also, let $a_1, \ldots, a_n \in A_\varepsilon$ and $k_1, \ldots, k_n$ be distinct cases in $K_f - \{k\}$. Then the following holds.*

- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p} \cap \mathfrak{q}) k\, f) \in \alpha$, then $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f), (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{q} k\, f), \alpha \to \beta$ is a GSS.*
- *If $(a_i \breve{o} k_i)_{i=1,\ldots,n}(a\,(\mathfrak{p} \cap \mathfrak{q}) k\, f) \in \beta$, then either $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{p} k\, f), \beta$ or $\alpha \to (a_i \breve{o} k_i)_{i=1,\ldots,n}(a\, \mathfrak{q} k\, f), \beta$ is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the $\cap$ law, gen. case+ law, $\wedge$ law, and Lemma 3.5.

Lemma 3.12 *Let* $\alpha \to \beta$ *be a GSS,* $a \in G,\ f \in H,\ k \in K_f$, *and* $\mathfrak{p} \in \mathfrak{P}$. *Also, let* $a_1, \dots, a_n \in A_\varepsilon$ *and* $k_1, \dots, k_n$ *be distinct cases in* $K_f - \{k\}$. *Then the following holds.*

- *If* $(a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \mathfrak{p}^\circ k\, f) \in \alpha$, *then* $\alpha \to (a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \mathfrak{p} k\, f), \beta$ *is a GSS.*
- *If* $(a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \mathfrak{p}^\circ k\, f) \in \beta$, *then* $(a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \mathfrak{p} k\, f), \alpha \to \beta$ *is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the $\circ$ law, gen. case+ law, $\lozenge$ law, and Lemma 3.5.

Lemma 3.13 *Let* $\alpha \to \beta$ *be a GSS,* $a \in G,\ f \in H,\ k \in K_f$, *and* $p \in \mathbb{P}$. *Also, let* $a_1, \dots, a_n \in A_\varepsilon$ *and* $k_1, \dots, k_n$ *be all of the distinct cases in* $K_f - \{k\}$. *Then the following holds.*

- *If* $(a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \overline{p} k\, f) \in \alpha$, *then* $\big(a \sqcap \big(x\, \breve{o} k\, (a_i\, \breve{o} k_i)_{i=1,\dots,n} f\big)\, \Omega x\big)\, \overline{p} \pi\, \mathrm{one} \triangle, \alpha \to \beta$ *is a GSS for all* $x \in \mathbb{X}''_\varepsilon$.
- *If* $(a_i\, \breve{o} k_i)_{i=1,\dots,n}(a\, \overline{p} k\, f) \in \beta$, *then* $\alpha \to \big(a \sqcap \big(x\, \breve{o} k\, (a_i\, \breve{o} k_i)_{i=1,\dots,n} f\big)\, \Omega x\big)\, \overline{p} \pi\, \mathrm{one} \triangle, \beta$ *is a GSS for all* $x \in \mathbb{X}''_\varepsilon$.

Proof Define $g = (a_i\, \breve{o} k_i)_{i=1,\dots,n} f$. Lemma 3.4 shows that $a_1, \dots, a_n, a, f$ are good and therefore so is g. Therefore $\big(x\, \breve{o} k\, g\big)\, \Omega x$ is good by Proposition 1 (1) and (3). Similarly $\mathrm{one} = (x_0\, \forall \pi\, x_0 \triangle)\, \Omega x_0$ is good. Thus the sequents in question are good by Lemma 3.4 and singular by Lemmas 2.10 and 3.5.

Lemma 3.14 *Let* $\alpha \to \beta$ *be a GSSS,* $a, b_1, \dots, b_n \in G$, *and* $p, q_1, \dots, q_n \in \mathbb{P}$. *Assume* $a\, \overline{p} \pi\, \mathrm{one} \triangle \in \alpha$, $b_1\, \overline{q_1} \pi\, \mathrm{one} \triangle, \dots, b_n\, \overline{q_n} \pi\, \mathrm{one} \triangle \in \beta$, *and* $p \geq \sum_{i=1}^n q_i$, *where if* $n = 0$ *then* $\sum_{i=1}^n q_i = 0$ *by definition. Then*

$$x\, \breve{o} \pi\, a \triangle, \alpha \to x\, \breve{o} \pi\, b_1 \triangle, \cdots, x\, \breve{o} \pi\, b_n \triangle, \beta$$

is a GSSS for all $x \in \mathbb{X}'_\varepsilon$ *which do not occur free in the sentences in* $\alpha \cup \beta$.

Proof The sequent in question is good by Lemma 3.4 and singular by the pigeonhole principle and Lemma 3.5 because x does not occur free in $a, b_1, \dots, b_n$ by Proposition 1 (3).

Lemma 3.15 *Let* $\alpha \to \beta$ *be a GSS and* $a \in G$. *Assume* $a\, \exists \pi\, \mathrm{one} \triangle \in \beta$. *Then* $\alpha \to b\, \breve{o} \pi\, a \triangle, \beta$ *is a GSS for all* $b \in A'_\varepsilon$.

Proof The sequent in question is good by Lemma 3.4 and singular by the $\exists$ law and Lemma 3.5.

Lemma 3.16 *Let* $\alpha \to \beta$ *be a GSS,* $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, *and* $\mathsf{c} \in \mathsf{A}_\varepsilon$. *Then the following holds.*

- *If* $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a} \sqcap \mathsf{b})\triangle \in \alpha$, *then* $\mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \alpha \to \beta$ *is a GSS.*
- *If* $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a} \sqcap \mathsf{b})\triangle \in \beta$, *then either* $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \beta$ *or* $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \beta$ *is a GSS.*
- *If* $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a} \sqcup \mathsf{b})\triangle \in \alpha$, *then either* $\mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \alpha \to \beta$ *or* $\mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \alpha \to \beta$ *is a GSS.*
- *If* $\mathsf{c}\,\breve{o}\pi\,(\mathsf{a} \sqcup \mathsf{b})\triangle \in \beta$, *then* $\alpha \to \mathsf{c}\,\breve{o}\pi\,\mathsf{a}\triangle, \mathsf{c}\,\breve{o}\pi\,\mathsf{b}\triangle, \beta$ *is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the $\sqcap$ law, $\sqcup$ law, gen. case+ law, $\wedge$ law, $\vee$ law, and Lemma 3.5.

Lemma 3.17 *Let* $\alpha \to \beta$ *be a GSS,* $\mathsf{a} \in \mathsf{G}$, *and* $\mathsf{b} \in \mathsf{A}_\varepsilon$. *Then the following holds.*

- *If* $\mathsf{b}\,\breve{o}\pi\,(\mathsf{a}^\square)\triangle \in \alpha$, *then* $\alpha \to \mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle, \beta$ *is a GSS.*
- *If* $\mathsf{b}\,\breve{o}\pi\,(\mathsf{a}^\square)\triangle \in \beta$, *then* $\mathsf{b}\,\breve{o}\pi\,\mathsf{a}\triangle, \alpha \to \beta$ *is a GSS.*

Proof The sequents in question are good by Lemma 3.4 and singular by the $\square$ law, gen. case+ law, $\Diamond$ law, and Lemma 3.5.

Lemma 3.18 *Let* $\alpha \to \beta$ *be a GSS,* $\mathsf{a} \in \mathsf{A}_\varepsilon$, $\mathsf{f} \in \mathsf{A}_\emptyset$, *and* $\mathsf{x} \in \mathbb{X}_\varepsilon$. *Then the following holds.*

- *If* $\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \in \alpha$, *then* $\mathsf{f}(\mathsf{x}/\mathsf{a}), \alpha \to \beta$ *is a GSS.*
- *If* $\mathsf{a}\,\breve{o}\pi\,(\mathsf{f}\,\Omega\mathsf{x})\triangle \in \beta$, *then* $\alpha \to \mathsf{f}(\mathsf{x}/\mathsf{a}), \beta$ *is a GSS.*

Proof The sequents in question are good and x is free from a in f by Lemma 3.4. Therefore they are singular by the Ω law and Lemma 3.5.

Remark 3.1 Recall from Remark 1.1 that $\mathfrak{P}$ is the set of the finite unions of the intervals in the shape $(\mathsf{p}\!\to)$, $(\leftarrow\!\mathsf{q}]$ or $(\mathsf{p}\mathsf{q}]$. Here and elsewhere, let ∞ be the largest element of $\mathbb{P}$, if it exists. Then $(\infty\!\to) = \emptyset$, $(\leftarrow\!\infty] = \mathbb{P}$, $(\mathsf{p}\infty] = (\mathsf{p}\!\to)$, and $(\mathsf{p}\mathsf{q}] = \emptyset$ if $\mathsf{p} \geq \mathsf{q}$. So henceforth, unless otherwise stated, we assume $\mathsf{p} \neq \infty$ as to the intervals $(\mathsf{p}\!\to)$, $\mathsf{q} \neq \infty$ as to the intervals $(\leftarrow\!\mathsf{q}]$, and either $\mathsf{p} < \mathsf{q} \neq \infty$ or $\mathsf{p} = \mathsf{q} = 0$ as to the intervals $(\mathsf{p}\mathsf{q}]$. Then since $\mathbb{P}$ is linear, the following holds.

- The intervals $(\mathsf{p}\!\to)$ and $(\leftarrow\!\mathsf{q}]$ are different from $\mathbb{P}$ and $\emptyset$. The intervals $(\mathsf{p}\mathsf{q}]$ are different from $\mathbb{P}$ and equal to $\emptyset$ iff $\mathsf{p} = \mathsf{q} = 0$.
- $(\mathsf{p}\!\to) = (\mathsf{p}'\!\to) \iff \mathsf{p} = \mathsf{p}'$.
- $(\leftarrow\!\mathsf{q}] = (\leftarrow\!\mathsf{q}'] \iff \mathsf{q} = \mathsf{q}'$.

- $(\mathsf{pq}] = (\mathsf{p'q'}] \iff \mathsf{p} = \mathsf{p'},\ \mathsf{q} = \mathsf{q'}$.
- $(\mathsf{p}\rightarrow) \neq (\leftarrow\mathsf{q}] \neq (\mathsf{pq'}] \neq (\mathsf{p'}\rightarrow)$ for all $\mathsf{p}, \mathsf{p'}, \mathsf{q}, \mathsf{q'} \in \mathbb{P}$.

Therefore each of these intervals has uniquely determined one or two **ends**, which are different from ∞.

Also since $\mathbb{P}$ is linear, every element $\mathfrak{p} \in \mathfrak{P}$ is the direct union of its uniquely determined *distinct* **connected components** $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, each of which is equal to $\mathbb{P}$ or one of the above intervals. We call $\neg\mathfrak{p}_1, \dots, \neg\mathfrak{p}_n$ the connected components of $\neg\mathfrak{p} \in \neg\mathfrak{P}$, and call the ends of $\mathfrak{p}_i$ the **ends** of $\neg\mathfrak{p}_i$ also. We say that an element $\mathsf{p} \in \mathbb{P}$ **occurs** in an element $\mathsf{f} \in \mathsf{H}$, if p is equal to an end of a connected component of $\mathfrak{x} \in \mathfrak{Q}$ such that the operation $\mathfrak{x}\mathsf{k}$ occurs in f for some $\mathsf{k} \in \mathsf{K}$. Furthermore, we say that p **occurs** in a subset X of H, if p occurs in a predicate $\mathsf{f} \in \mathsf{X}$.

Lemma 3.19 *There exists a series $(\alpha_n \to \beta_n)_{n=1,2,\dots}$ of GSSS's which satisfies the following thirty three conditions, where "$n \equiv i$" is an abbreviation for "$n \equiv i \mod 32$" for $i \in \{1, \dots, 32\}$. For the condition 24, we let $A'_\varepsilon = \{a_1, a_2, \dots\}$, because A'_ε is an enumerable set by Lemma 3.4.*

(0) $\alpha_{n-1} \subseteq \alpha_n$ *and* $\beta_{n-1} \subseteq \beta_n$ *hold, and if an element of* $\mathbb{P} - \{0\}$ *occurs in* $\alpha_n \cup \beta_n$*, then it also occurs in* $\alpha_{n-1} \cup \beta_{n-1}$ $(n = 1, 2, \dots)$.

(1) If $n \equiv 1$ *and* $(a_i\, \breve{o}k_i)_{i=1,\dots,l}f \in \alpha_{n-1}$*, then* $(a_{\rho i}\, \breve{o}k_{\rho i})_{i=1,\dots,l}f \in \alpha_n$ *for all* $\rho \in \mathfrak{S}_l$.

(2) If $n \equiv 2$ *and* $(a_i\, \breve{o}k_i)_{i=1,\dots,l}f \in \beta_{n-1}$*, then* $(a_{\rho i}\, \breve{o}k_{\rho i})_{i=1,\dots,l}f \in \beta_n$ *for all* $\rho \in \mathfrak{S}_l$.

(3) If $n \equiv 3$, $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \in \alpha_{n-1}$, *and the range condition*

$$\begin{aligned} K_f - K_g &= \{k_1, \dots, k_\nu\}, \\ K_f \cap K_g &= \{k_{\nu+1}, \dots, k_m\}, \\ K_g - K_f &= \{k_{m+1}, \dots, k_l\} \end{aligned} \tag{3.2}$$

is satisfied, then $(a_i\, \breve{o}k_i)_{i=1,\dots,m}f,\ (a_i\, \breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(4) If $n \equiv 4$, $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \in \beta_{n-1}$, *and* (3.2) *is satisfied, then* $(a_i\, \breve{o}k_i)_{i=1,\dots,m}f \in \beta_n$ *or* $(a_i\, \breve{o}k_i)_{i=\nu+1,\dots,l}g \in \beta_n$.

(5) If $n \equiv 5$, $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(f \vee g) \in \alpha_{n-1}$, *and* (3.2) *is satisfied, then* $(a_i\, \breve{o}k_i)_{i=1,\dots,m}f \in \alpha_n$ *or* $(a_i\, \breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(6) If $n \equiv 6$, $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(f \vee g) \in \beta_{n-1}$, *and* (3.2) *is satisfied, then* $(a_i\, \breve{o}k_i)_{i=1,\dots,m}f,\ (a_i\, \breve{o}k_i)_{i=\nu+1,\dots,l}g \in \beta_n$.

(7) If $n \equiv 7$, $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \alpha_{n-1}$, *and* (3.2) *is satisfied, then* $(a_i\, \breve{o}k_i)_{i=1,\dots,m}f \in \beta_n$ *or* $(a_i\, \breve{o}k_i)_{i=\nu+1,\dots,l}g \in \alpha_n$.

(8) *If* $n \equiv 8$, $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \beta_{n-1}$, *and* (3.2) *is satisfied, then* $(a_i \,\breve{o}k_i)_{i=1,\dots,m} f \in \alpha_n$, $(a_i \,\breve{o}k_i)_{i=\nu+1,\dots,l}\, g \in \beta_n$.

(9) *If* $n \equiv 9$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \alpha_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l} f \in \beta_n$.

(10) *If* $n \equiv 10$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \beta_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l} f \in \alpha_n$.

(11) *If* $n \equiv 11$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a \,\neg\mathfrak{p} k\, f) \in \alpha_{n-1}$ *with* $\mathfrak{p} \in \mathfrak{P}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f^{\Diamond}) \in \alpha_n$.

(12) *If* $n \equiv 12$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a \,\neg\mathfrak{p} k\, f) \in \beta_{n-1}$ *with* $\mathfrak{p} \in \mathfrak{P}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f^{\Diamond}) \in \beta_n$.

(13) *If* $n \equiv 13$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f) \in \alpha_{n-1}$ *with* $\mathfrak{p} \in \mathfrak{P}$ *having the connected components* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ $(m \geq 2)$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p}_j k\, f) \in \alpha_n$ *for some* $j \in \{1, \dots, m\}$.

(14) *If* $n \equiv 14$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f) \in \beta_{n-1}$ *with* $\mathfrak{p} \in \mathfrak{P}$ *having the connected components* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ $(m \geq 2)$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathfrak{p}_j k\, f) \in \beta_n$ *for all* $j \in \{1, \dots, m\}$.

(15) *If* $n \equiv 15$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathbb{P} k\, f) \in \alpha_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{0} k\, f) \in \alpha_n$ *or* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow 0] k\, f) \in \alpha_n$.

(16) *If* $n \equiv 16$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \mathbb{P} k\, f) \in \beta_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{0} k\, f)$, $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow 0] k\, f) \in \beta_n$.

(17) *If* $n \equiv 17$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (pq] k\, f) \in \alpha_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f)$, $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow q] k\, f) \in \alpha_n$.

(18) *If* $n \equiv 18$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (pq] k\, f) \in \beta_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f) \in \beta_n$ *or* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow q] k\, f) \in \beta_n$.

(19) *If* $n \equiv 19$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow q] k\, f) \in \alpha_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{q} k\, f) \in \beta_n$.

(20) *If* $n \equiv 20$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, (\leftarrow q] k\, f) \in \beta_{n-1}$, *then* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{q} k\, f) \in \alpha_n$.

(21) *If* $n \equiv 21$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f) \in \alpha_{n-1}$, *then* $\big(a \sqcap \big(x \,\breve{o}k\, (a_i \,\breve{o}k_i)_{i=1,\dots,l} f\big)\, \Omega x\big)\, \overline{p} \pi\, \mathrm{one} \triangle \in \alpha_n$ *for all* $x \in \mathbb{X}''_{\varepsilon}$.

(22) *If* $n \equiv 22$ *and* $(a_i \,\breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f) \in \beta_{n-1}$, *then* $\big(a \sqcap \big(x \,\breve{o}k\, (a_i \,\breve{o}k_i)_{i=1,\dots,l} f\big)\, \Omega x\big)\, \overline{p} \pi\, \mathrm{one} \triangle \in \beta_n$ *for all* $x \in \mathbb{X}''_{\varepsilon}$.

(23) *If* $n \equiv 23$, $a\, \overline{p} \pi\, \mathrm{one} \triangle \in \alpha_{n-1}$, $b_1\, \overline{q_1} \pi\, \mathrm{one} \triangle, \dots, b_m\, \overline{q_m} \pi\, \mathrm{one} \triangle \in \beta_{n-1}$, *and* $p \geq \sum_{i=1}^{m} q_i$, *then* $x \,\breve{o}\pi\, a \triangle \in \alpha_n$ *and* $x \,\breve{o}\pi\, b_1 \triangle, \dots, x \,\breve{o}\pi\, b_m \triangle \in \beta_n$ *for some* $x \in \mathbb{X}'_{\varepsilon}$.

(24) *If* $n \equiv 24$ *and* $a\, \exists \pi\, \mathrm{one} \triangle \in \beta_{n-1}$, *then* $a_i \,\breve{o}\pi\, a \triangle \in \beta_n$ *for all* $i \in \{1, \dots, n\}$, *where* $A'_{\varepsilon} = \{a_1, a_2, \dots\}$.

(25) *If* $n \equiv 25$ *and* $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in \alpha_{n-1}$, *then* $c\,\breve{o}\pi\,a\triangle, c\,\breve{o}\pi\,b\triangle \in \alpha_n$.

(26) *If* $n \equiv 26$ *and* $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in \beta_{n-1}$, *then* $c\,\breve{o}\pi\,a\triangle \in \beta_n$ *or* $c\,\breve{o}\pi\,b\triangle \in \beta_n$.

(27) *If* $n \equiv 27$ *and* $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in \alpha_{n-1}$, *then* $c\,\breve{o}\pi\,a\triangle \in \alpha_n$ *or* $c\,\breve{o}\pi\,b\triangle \in \alpha_n$.

(28) *If* $n \equiv 28$ *and* $c\,\breve{o}\pi\,(a \sqcup b)\triangle \in \beta_{n-1}$, *then* $c\,\breve{o}\pi\,a\triangle, c\,\breve{o}\pi\,b\triangle \in \beta_n$

(29) *If* $n \equiv 29$ *and* $b\,\breve{o}\pi\,(a^{\square})\triangle \in \alpha_{n-1}$, *then* $b\,\breve{o}\pi\,a\triangle \in \beta_n$.

(30) *If* $n \equiv 30$ *and* $b\,\breve{o}\pi\,(a^{\square})\triangle \in \beta_{n-1}$, *then* $b\,\breve{o}\pi\,a\triangle \in \alpha_n$.

(31) *If* $n \equiv 31$ *and* $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \alpha_{n-1}$, *then* $f(x/a) \in \alpha_n$.

(32) *If* $n \equiv 32$ *and* $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \beta_{n-1}$, *then* $f(x/a) \in \beta_n$.

Proof We will inductively define GSSS's $\alpha_n \to \beta_n$ $(n = 1, 2, \dots)$ starting from the GSSS $\alpha_0 \to \beta_0$. Suppose $n \geq 1$ and the GSSS $\alpha_{n-1} \to \beta_{n-1}$ has been defined. Then we enlarge $\alpha_{n-1} \to \beta_{n-1}$ to a GSSS $\alpha_n \to \beta_n$ by extending α_{n-1} or β_{n-1} or both with a finite number of good sentences.

If $n \equiv 1$, extend α_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f$ such that $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\dots,l}f \in \alpha_{n-1}$ for some $\rho \in \mathfrak{S}_l$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 1 and is a GSSS by Lemma 3.6.

If $n \equiv 2$, extend β_{n-1} with all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}f$ such that $(a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\dots,l}f \in \beta_{n-1}$ for some $\rho \in \mathfrak{S}_l$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 2 and is a GSSS by Lemma 3.6.

If $n \equiv 3$, extend α_{n-1} with the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g$ made of all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g) \in \alpha_{n-1}$ satisfying (3.2). Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 3 and is a GSSS by Lemma 3.7.

If a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \wedge g)$ satisfies (3.2), we call it a $\wedge$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g$ its constituents.

If $n \equiv 4$, let $\{h_1, \dots, h_k\}$ be the set of the $\wedge$-sentences which belong to β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, k)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $\alpha_{n,i-1} \to h'_i\beta_{n,i-1}$ is a GSSS by Lemma 3.7, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 4.

If $n \equiv 6$, extend β_{n-1} with the paired sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g$ made of all the sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g) \in \beta_{n-1}$ satisfying (3.2). Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 6 and is a GSSS by Lemma 3.7.

If a sentence $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(f \vee g)$ satisfies (3.2), we call it a $\vee$-sentence, and call sentences $(a_i\,\breve{o}k_i)_{i=1,\dots,m}f$ and $(a_i\,\breve{o}k_i)_{i=\nu+1,\dots,l}g$ its constituents.

If $n \equiv 5$, let $\{h_1, \dots, h_k\}$ be the set of the $\vee$-sentences contained in α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, k)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $h'_i\alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSS by

Lemma 3.7, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 5.

If $n \equiv 8$, for each sentence $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(f \Rightarrow g) \in \beta_{n-1}$ satisfying (3.2), add $(a_i\, \breve{o} k_i)_{i=1,\dots,m} f$ to α_{n-1} and add $(a_i\, \breve{o} k_i)_{i=\nu+1,\dots,l}\, g$ to β_{n-1}. Let $\alpha_n \to \beta_n$ be the resulting sequent. Then it satisfies the conditions 0, 8 and is a GSSS by Lemma 3.7.

If a sentence $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(f \Rightarrow g)$ satisfies (3.2), we call it a $\Rightarrow$-sentence, and call sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,m} f$ and $(a_i\, \breve{o} k_i)_{i=\nu+1,\dots,l}\, g$ its constituents.

If $n \equiv 7$, let $\{h_1, \dots, h_k\}$ be the set of the $\Rightarrow$-sentences which belong to α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, k)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,k} \to \beta_{n,k}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, let h_i', h_i'' be the constituents of h_i. Then either $\alpha_{n,i-1} \to h_i' \beta_{n,i-1}$ or $h_i'' \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 3.7, so let $\alpha_{n,i} \to \beta_{n,i}$ be the one which is a GSSS. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 7.

If $n \equiv 9$, extend β_{n-1} with all the sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l} f$ such that $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 9 and is a GSSS by Lemma 3.8.

If $n \equiv 10$, extend α_{n-1} with all the sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l} f$ such that $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(f^{\Diamond}) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 10 and is a GSSS by Lemma 3.8.

If $n \equiv 11$, extend α_{n-1} with all the sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f^{\Diamond})$ such that $\mathfrak{p} \in \mathfrak{P}$ and $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a \neg \mathfrak{p} k\, f) \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 11 and is a GSSS by Lemma 3.9.

If $n \equiv 12$, extend β_{n-1} with all the sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f^{\Diamond})$ such that $\mathfrak{p} \in \mathfrak{P}$ and $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a \neg \mathfrak{p} k\, f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 12 and is a GSSS by Lemma 3.9.

If $n \equiv 14$, extend β_{n-1} with all the sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{q} k\, f)$ such that $\mathfrak{q}$ is a connected component of a disconnected element $\mathfrak{p} \in \mathfrak{P}$ such that $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 14 and is a GSSS by Lemma 3.10.

A sentence $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f)$ with $\mathfrak{p} \in \mathfrak{P}$ disconnected will be called a disconnected sentence, and for each connected component $\mathfrak{q}$ of $\mathfrak{p}$, the sentence $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{q} k\, f)$ will be called a constituent of $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathfrak{p} k\, f)$.

If $n \equiv 13$, let $\{h_1, \dots, h_m\}$ be the set of the disconnected sentences contained in α_{n-1}, inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h_i' of h_i such that $h_i' \alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 3.10, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 13.

If $n \equiv 16$, extend β_{n-1} with all the paired sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \overline{0} k\, f)$ and $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, (\leftarrow 0] k\, f)$ such that $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathbb{P} k\, f) \in \beta_{n-1}$. Then, since $\mathbb{P} = \overline{0} \cup (\leftarrow 0]$, the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 16 and is a GSSS by Lemma 3.10.

We call a sentence $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \mathbb{P} k\, f)$ a $\mathbb{P}$-sentence, and call sentences $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, \overline{0} k\, f)$ and $(a_i\, \breve{o} k_i)_{i=1,\dots,l}(a\, (\leftarrow 0] k\, f)$ its constituents.

If $n \equiv 15$, let $\{h_1, \dots, h_m\}$ be the set of the $\mathbb{P}$-sentences contained in α_{n-1}, inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h_i' of h_i such that $h_i'\alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 3.10, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 15.

If $n \equiv 17$, extend α_{n-1} with all the paired sentences $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f)$ and $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q] k\, f)$ such that $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\,(pq] k\, f) \in \alpha_{n-1}$. Such an extension is possible because the number of those paired sentences is finite by the assumption in Remark 3.1[1]. Since $(pq] = \overline{p} \cap (\leftarrow q]$, the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 17 and is a GSSS by Lemma 3.11.

We call a sentence $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\,(pq] k\, f)$ a proper sentence, and call sentences $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f)$ and $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q] k\, f)$ its constituents.

If $n \equiv 18$, let $\{h_1, \dots, h_m\}$ be the set of the proper sentences contained in β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \dots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h_i' of h_i such that $\alpha_{n,i-1} \to h_i'\beta_{n,i-1}$ is a GSSS by Lemma 3.11, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 18.

If $n \equiv 19$, extend β_{n-1} with all the sentences $(a_i\, \breve{o}k_i)_i(a\, \overline{q} k\, f)$ such that $(a_i\, \breve{o}k_i)_i(a\,(\leftarrow q] k\, f) \in \alpha_{n-1}$. Then, since $(\leftarrow q] = \overline{q}^\circ$, the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 19 and is a GSSS by Lemma 3.12.

If $n \equiv 20$, extend α_{n-1} with all the sentences $(a_i\, \breve{o}k_i)_i(a\, \overline{q} k\, f)$ such that $(a_i\, \breve{o}k_i)_i(a\,(\leftarrow q] k\, f) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 20 and is a GSSS by Lemma 3.12.

If $n \equiv 21$, extend α_{n-1} with all the sentences

$$\left(a \sqcap \left(x\, \breve{o}k\, (a_i\, \breve{o}k_i)_{i=1,\dots,l} f\right) \Omega x\right) \overline{p} \pi\, \mathsf{one} \triangle$$

such that $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f) \in \alpha_{n-1}$ and $x \in \mathbb{X}''_\varepsilon$. Such an extension is possible because $\mathbb{X}''_\varepsilon$ is a finite set by Lemma 3.3. The resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 21 and is a GSSS by Lemma 3.13.

We call a sentence $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f)$ an upper sentence, and call sentences $\left(a \sqcap \left(x\, \breve{o}k\, (a_i\, \breve{o}k_i)_{i=1,\dots,l} f\right) \Omega x\right) \overline{p} \pi\, \mathsf{one} \triangle$ its one-representations.

If $n \equiv 22$, extend β_{n-1} with all the one-representations of the upper sentences contained in β_{n-1}. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 22 and is a GSSS by Lemma 3.13.

If $n \equiv 23$, let $\{T_1, \dots, T_l\}$ be the set of all tuples

$$(a\, \overline{p} \pi\, \mathsf{one} \triangle, b_1\, \overline{q_1} \pi\, \mathsf{one} \triangle, \dots, b_m\, \overline{q_m} \pi\, \mathsf{one} \triangle)$$

of *distinct* sentences $a\, \overline{p} \pi\, \mathsf{one} \triangle \in \alpha_{n-1}$ and $b_1\, \overline{q_1} \pi\, \mathsf{one} \triangle, \dots, b_m\, \overline{q_m} \pi\, \mathsf{one} \triangle \in \beta_{n-1}$ such that $p \geq \sum_{i=1}^m q_i$. The number of such tuples is finite because each tuple consists of distinct sentences. Inductively make GSSS's $\alpha_{n,i} \to$

[1] If we did not exclude intervals $(pq]$ with $p \geq q$, then $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \emptyset k\, f)$ may yield infinitely many sentences $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\, \overline{p} k\, f)$ and $(a_i\, \breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q] k\, f)$ with $p \geq q$.

$\beta_{n,i}$ $(i = 0, 1, \ldots, l)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,l} \to \beta_{n,l}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, suppose T_i is a tuple of $a\,\overline{p}\pi\,\mathrm{one}\triangle \in \alpha_{n-1}$ and $b_1\,\overline{q_1}\pi\,\mathrm{one}\triangle, \ldots, b_m\,\overline{q_m}\pi\,\mathrm{one}\triangle \in \beta_{n-1}$, and take a variable $x \in \mathbb{X}'_\varepsilon$ which does not occur free in the sentences in $\alpha_{n,i-1} \cup \beta_{n,i-1}$ (such a variable exists by Lemma 4.1 of Gomi (09b) because $\mathbb{X}'_\varepsilon$ is enumerable by Lemma 3.3). Then $x\,\breve{o}\pi\,a\triangle, \alpha_{n,i-1} \to x\,\breve{o}\pi\,b_1\triangle, \cdots, x\,\breve{o}\pi\,b_m\triangle, \beta_{n,i-1}$ is a GSSS by Lemma 3.14, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 23.

If $n \equiv 24$, extend β_{n-1} with all the n-tuples $a_1\,\breve{o}\pi\,a\triangle, \ldots, a_n\,\breve{o}\pi\,a\triangle$ such that $a\,\exists\pi\,\mathrm{one}\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 24 and is a GSSS by Lemma 3.15.

If $n \equiv 25$, then extend α_{n-1} with all the paired sentences $c\,\breve{o}\pi\,a\triangle$ and $c\,\breve{o}\pi\,b\triangle$ such that $c\,\breve{o}\pi\,(a \sqcap b)\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 25 and is a GSSS by Lemma 3.16.

We call a sentence $c\,\breve{o}\pi\,(a \sqcap b)\triangle$ a $\sqcap$-sentence, and call sentences $c\,\breve{o}\pi\,a\triangle$ and $c\,\breve{o}\pi\,b\triangle$ its constituents.

If $n \equiv 26$, let $\{h_1, \ldots, h_m\}$ be the set of the $\sqcap$-sentences contained in β_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \ldots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, there is a constituent h'_i of h_i such that $\alpha_{n,i-1} \to h'_i\beta_{n,i-1}$ is a GSSS by Lemma 3.16, so let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 26.

If $n \equiv 28$, extend β_{n-1} with all the paired sentences $c\,\breve{o}\pi\,a\triangle$ and $c\,\breve{o}\pi\,b\triangle$ such that $c\,\breve{o}\pi\,(a \sqcup b) \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 28 and is a GSSS by Lemma 3.16.

We call a sentence $c\,\breve{o}\pi\,(a \sqcup b)\triangle$ a $\sqcup$-sentence, and call sentences $c\,\breve{o}\pi\,a\triangle$ and $c\,\breve{o}\pi\,b\triangle$ its constituents.

If $n \equiv 27$, let $\{h_1, \ldots, h_m\}$ be the set of the $\sqcup$-sentences contained in α_{n-1}, and inductively make GSSS's $\alpha_{n,i} \to \beta_{n,i}$ $(i = 0, 1, \ldots, m)$ as follows, and let $\alpha_n \to \beta_n = \alpha_{n,m} \to \beta_{n,m}$. First, let $\alpha_{n,0} \to \beta_{n,0} = \alpha_{n-1} \to \beta_{n-1}$. Next for $i \geq 1$, since there is a constituent h'_i of h_i such that $h'_i\alpha_{n,i-1} \to \beta_{n,i-1}$ is a GSSS by Lemma 3.16, let it be $\alpha_{n,i} \to \beta_{n,i}$. Then $\alpha_n \to \beta_n$ is a GSSS and satisfies the conditions 0, 27.

If $n \equiv 29$, extend β_{n-1} with all the sentences $b\,\breve{o}\pi\,a\triangle$ such that $b\,\breve{o}\pi\,(a^{\square})\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 29 and is a GSSS by Lemma 3.17.

If $n \equiv 30$, extend α_{n-1} with all the sentences $b\,\breve{o}\pi\,a\triangle$ such that $b\,\breve{o}\pi\,(a^{\square})\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 30 and is a GSSS by Lemma 3.17.

If $n \equiv 31$, extend α_{n-1} with all the sentences $f(x/a)$ such that $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \alpha_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 31 and is a GSSS by Lemma 3.18.

If $n \equiv 32$, extend β_{n-1} with all the sentences $f(x/a)$ such that $a\,\breve{o}\pi\,(f\,\Omega x)\triangle \in \beta_{n-1}$. Then the resulting sequent $\alpha_n \to \beta_n$ satisfies the conditions 0, 32 and is a GSSS by Lemma 3.18.

This completes the inductive definition of the series $(\alpha_n \to \beta_n)_{n=1,2,\ldots}$ of GSSS's which satisfy the above thirty three conditions.

Lemma 3.20 *Define* $\mathsf{P} = \bigcup_{n\geq 0} \alpha_n$ *and* $\mathsf{Q} = \bigcup_{n\geq 0} \beta_n$*. Then* $\mathsf{P}\cup\mathsf{Q}$ *consists of good sentences,* $\mathsf{P}\cap\mathsf{Q} = \emptyset$*, and the following thirty four conditions hold.*

(0) If an element of $\mathbb{P} - \{0\}$ *occurs in* $\mathsf{P}\cup\mathsf{Q}$*, then it also occurs in* $\alpha_0 \cup \beta_0$*.*

(1) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l} f \in \mathsf{P}$*, then* $(a_{\rho i} \,\breve{o} k_{\rho i})_{i=1,\ldots,l} f \in \mathsf{P}$ *for all* $\rho \in \mathfrak{S}_l$*.*

(2) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l} f \in \mathsf{Q}$*, then* $(a_{\rho i} \,\breve{o} k_{\rho i})_{i=1,\ldots,l} f \in \mathsf{Q}$ *for all* $\rho \in \mathfrak{S}_l$*.*

(3) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \wedge g) \in \mathsf{P}$ *satisfies* (3.2)*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{P}$ *and* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{P}$*.*

(4) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \wedge g) \in \mathsf{Q}$ *satisfies* (3.2)*, then either* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{Q}$ *or* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{Q}$*.*

(5) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \vee g) \in \mathsf{P}$ *satisfies* (3.2)*, then either* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{P}$ *or* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{P}$*.*

(6) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \vee g) \in \mathsf{Q}$ *satisfies* (3.2)*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{Q}$ *and* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{Q}$*.*

(7) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in \mathsf{P}$ *satisfies* (3.2)*, then either* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{Q}$ *or* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{P}$*.*

(8) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f \Rightarrow g) \in \mathsf{Q}$ *satisfies* (3.2)*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,m} f \in \mathsf{P}$ *and* $(a_i \,\breve{o} k_i)_{i=\nu+1,\ldots,l}\, g \in \mathsf{Q}$*.*

(9) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f^{\diamond}) \in \mathsf{P}$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l} f \in \mathsf{Q}$*.*

(10) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(f^{\diamond}) \in \mathsf{Q}$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l} f \in \mathsf{P}$*.*

(11) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a \neg\mathfrak{p} k\, f) \in \mathsf{P}$ *with* $\mathfrak{p} \in \mathfrak{P}$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p} k\, f^{\diamond}) \in \mathsf{P}$*.*

(12) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a \neg\mathfrak{p} k\, f) \in \mathsf{Q}$ *with* $\mathfrak{p} \in \mathfrak{P}$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p} k\, f^{\diamond}) \in \mathsf{Q}$*.*

(13) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p} k\, f) \in \mathsf{P}$ *with* $\mathfrak{p} \in \mathfrak{P}$ *having the connected components* $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ $(m \geq 2)$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p}_j k\, f) \in \mathsf{P}$ *for some* $j \in \{1, \ldots, m\}$*.*

(14) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p} k\, f) \in \mathsf{Q}$ *with* $\mathfrak{p} \in \mathfrak{P}$ *having the connected components* $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ $(m \geq 2)$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathfrak{p}_j k\, f) \in \mathsf{Q}$ *for all* $j \in \{1, \ldots, m\}$*.*

(15) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathbb{P} k\, f) \in \mathsf{P}$*, then either* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \overline{0} k\, f) \in \mathsf{P}$ *or* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, (\leftarrow 0] k\, f) \in \mathsf{P}$*.*

(16) If $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \mathbb{P} k\, f) \in \mathsf{Q}$*, then* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, \overline{0} k\, f) \in \mathsf{Q}$ *and* $(a_i \,\breve{o} k_i)_{i=1,\ldots,l}(a\, (\leftarrow 0] k\, f) \in \mathsf{Q}$*.*

(17) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(pq]k\,f)\in P$*, then*
$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)\in P$ *and* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)\in P$.

(18) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(pq]k\,f)\in Q$*, then either*
$(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)\in Q$ *or* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)\in Q$.

(19) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)\in P$*, then* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{q}k\,f)\in Q$.

(20) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,(\leftarrow q]k\,f)\in Q$*, then* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{q}k\,f)\in P$.

(21) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)\in P$*, then*
$\big(a\sqcap\big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi\,\mathrm{one}\triangle\in P$ *for all* $x\in\mathbb{X}''_\varepsilon$.

(22) *If* $(a_i\,\breve{o}k_i)_{i=1,\dots,l}(a\,\overline{p}k\,f)\in Q$*, then*
$\big(a\sqcap\big(x\,\breve{o}k\,(a_i\,\breve{o}k_i)_{i=1,\dots,l}f\big)\,\Omega x\big)\,\overline{p}\pi\,\mathrm{one}\triangle\in Q$ *for all* $x\in\mathbb{X}''_\varepsilon$.

(23) *If* $a\,\overline{p}\pi\,\mathrm{one}\triangle\in P$, $b_1\,\overline{q_1}\pi\,\mathrm{one}\triangle,\dots,b_m\,\overline{q_m}\pi\,\mathrm{one}\triangle\in Q$, *and* $p\ge\sum_{i=1}^m q_i$,
then $x\,\breve{o}\pi\,a\triangle\in P$ *and* $x\,\breve{o}\pi\,b_1\triangle,\cdots,x\,\breve{o}\pi\,b_m\triangle\in Q$ *for some* $x\in\mathbb{X}'_\varepsilon$.

(24) *If* $a\,\exists\pi\,\mathrm{one}\triangle\in Q$*, then* $b\,\breve{o}\pi\,a\triangle\in Q$ *for all elements* $b\in A'_\varepsilon$.

(25) *If* $c\,\breve{o}\pi\,(a\sqcap b)\triangle\in P$*, then* $c\,\breve{o}\pi\,a\triangle\in P$ *and* $c\,\breve{o}\pi\,b\triangle\in P$.

(26) *If* $c\,\breve{o}\pi\,(a\sqcap b)\triangle\in Q$*, then either* $c\,\breve{o}\pi\,a\triangle\in Q$ *or* $c\,\breve{o}\pi\,b\triangle\in Q$.

(27) *If* $c\,\breve{o}\pi\,(a\sqcup b)\triangle\in P$*, then either* $c\,\breve{o}\pi\,a\triangle\in P$ *or* $c\,\breve{o}\pi\,b\triangle\in P$.

(28) *If* $c\,\breve{o}\pi\,(a\sqcup b)\triangle\in Q$*, then* $c\,\breve{o}\pi\,a\triangle\in Q$ *and* $c\,\breve{o}\pi\,b\triangle\in Q$.

(29) *If* $b\,\breve{o}\pi\,(a^\square)\triangle\in P$*, then* $b\,\breve{o}\pi\,a\triangle\in Q$.

(30) *If* $b\,\breve{o}\pi\,(a^\square)\triangle\in Q$*, then* $b\,\breve{o}\pi\,a\triangle\in P$.

(31) *If* $a\,\breve{o}\pi\,(f\,\Omega x)\triangle\in P$*, then* $f(x/a)\in P$.

(32) *If* $a\,\breve{o}\pi\,(f\,\Omega x)\triangle\in Q$*, then* $f(x/a)\in Q$.

Proof If $f\in P$ and $g\in Q$ and $i\in\{1,\dots,32\}$, then since $(\alpha_n\to\beta_n)_{n=0,1,\dots}$ is increasing, there exist infinitely many positive integers n such that $f\in\alpha_n$, $g\in\beta_n$, and $n\equiv i \mod 32$. Therefore by Lemmas 3.5 and 3.19, we see that $P\cup Q$ consists of good sentences, $P\cap Q=\emptyset$, and the first thirty three conditions hold with special attention to (24).

Using the sets P and Q of good sentences in Lemma 3.20, we will construct a MPC world W cognizable by A. First, since A'_ε is enumerable by Lemma 3.4 and in particular non-empty, we may choose A'_ε as the base S of W:

$$S=A'_\varepsilon. \tag{3.3}$$

Then, we may define the basic relation $\exists$ on S by

$$b\,\exists\,a \iff a\,\breve{o}\pi\,b\triangle\notin Q \tag{3.4}$$

for all $a, b \in S$.

In order to define the $\mathbb{P}$-measure $X \mapsto |X|$ on S, first let $\mathbb{P}'$ be the set of the elements of $\mathbb{P}$ which occur in $P \cup Q$. Then $\infty \notin \mathbb{P}'$ by Remark 3.1, and $\mathbb{P}'$ is a finite set by Proposition 1 (2) and Lemma 3.20 (0). Therefore there exists an element $ó \in \mathbb{P}$ such that $0 < ó$ and $p < ó$ for all $p \in \mathbb{P}'$.

Next, for each $a \in G$, we may define

$$S^a = \{s \in S \mid s\,\breve{o}\pi\, a\triangle \notin Q\} \tag{3.5}$$

by virtue of (3.3). In particular for $a \in A'_\varepsilon$, $S^a = \{s \in S \mid a \ni s\}$ by (3.4).

Next if, for an element $X \in \mathcal{P}S$ and an element $p \in \mathbb{P}$, there exist elements $b_1, \ldots, b_m \in G$ and elements $q_1, \ldots, q_m \in \mathbb{P}$ which satisfy the following three conditions, then we write $X\,R\,p$:

(a) $X \subseteq \bigcup_{i=1}^m S^{b_i}$,

(b) $p = \sum_{i=1}^m q_i$,

(c) $b_i\,\overline{q_i}\pi\,one\triangle \in Q$ $(i = 1, \ldots, m)$,

where if $m = 0$, then $\bigcup_{i=1}^m S^{b_i} = \emptyset$, $\sum_{i=1}^m q_i = 0$ and the condition (c) is vacant. The relation R between $\mathcal{P}S$ and $\mathbb{P}$ thus defined satisfies the following:

(1) $X = \emptyset \iff X\,R\,0$,

(2) $X \subseteq Y$ and $Y\,R\,p \implies X\,R\,p$,

(3) $X\,R\,p$ and $Y\,R\,q \implies (X \cup Y)\,R\,(p + q)$.

That R satisfies (2) and (3) is an immediate consequence of the definition of R. The above remark on (a) (b) (c) with $m = 0$ shows that $\emptyset\,R\,0$. In order to complete the proof of (1), suppose $X\,R\,0$. Then either $X = \emptyset$ or there exist elements $b_1, \ldots, b_m \in G$ and elements $q_1, \ldots, q_m \in \mathbb{P}$ with $m > 0$ which satisfy $X \subseteq \bigcup_{i=1}^m S^{b_i}$, $0 = \sum_{i=1}^m q_i$, and $b_i\,\overline{q_i}\pi\,one\triangle \in Q$ $(i = 1, \ldots, m)$. In the latter case, we have $q_i = 0$, so $b_i \exists\pi\,one\triangle \in Q$, and Lemma 3.20 (24) shows that $s\,\breve{o}\pi\,b_i\triangle \in Q$ for all elements $s \in A'_\varepsilon = S$, hence $S^{b_i} = \emptyset$ $(i = 1, \ldots, m)$, and thus $X = \emptyset$ as desired.

Furthermore, for each element $X \in \mathcal{P}S$, the set $\{p \in \mathbb{P} \mid X\,R\,p\}$ is contained in the subalgebra of $\mathbb{P}$ generated by the finite subset $\mathbb{P}' \cup \{0\}$, and so Theorem 2.1 of Gomi (09c) shows that $\min\bigl(\{p \in \mathbb{P} \mid X\,R\,p\} \cup \{ó\}\bigr)$ exists. Also $0 < ó$. Therefore Theorem 2.2 of Gomi (09c) shows that we may define the $\mathbb{P}$-measure $X \mapsto |X|$ on S by

$$|X| = \min\bigl(\{p \in \mathbb{P} \mid X\,R\,p\} \cup \{ó\}\bigr). \tag{3.6}$$

We have thus defined the basic relation and the $\mathbb{P}$-measure on S. Using these, we may now let $W = (S \to \mathbb{T}) \cup S \cup \bigcup_{O \in \mathcal{P}K} \bigl((O \to S) \to \mathbb{T}\bigr)$ be a MPC world cognizable by A as in §2.3 of Gomi (09c).

Next, we define a specific $\mathbb{C}$-denotation Φ into W by (Φ1)–(Φ3) below. Recall $\mathbb{C} = \mathbb{C}_\delta \amalg \mathbb{C}_\varepsilon \amalg (\mathbb{C} \cap H)$.

(Φ1) For each $a \in \mathbb{C}_\delta$, Φa is the element of $S \to \mathbb{T}$ such that

$$(\Phi a)s = 1 \iff s \in S^a$$

for each $s \in S$. Thus $s \in S^a$ iff $\Phi a \ni s$ for the extended relation $\ni$.

(Φ2) For each $a \in \mathbb{C}_\varepsilon$, since $\mathbb{C}_\varepsilon \subseteq S$ by (3.1) and (3.3), we define $\Phi a = a$.

(Φ3) For each $f \in \mathbb{C} \cap H$, Φf is the element of $(K_f \to S) \to \mathbb{T}$ such that, if $K_f = \{k_1, \ldots, k_n\}$ with $k_1, \ldots, k_n$ distinct, then

$$(\Phi f)\theta = 1 \iff \big((\theta k_i)\, \check{o} k_i\big)_{i=1,\ldots,n} f \notin Q$$

for each $\theta \in K_f \to S$. Since $\theta k_i \in A_\varepsilon$ by (3.3), this definition makes sense. By virtue of Lemma 3.20 (2), this definition does not depend on the numbering $k_1, \ldots, k_n$ of elements of K_f.

Thus Φ is defined and it yields the metadenotation $\Phi^* \in A \to W^{V_{\mathbb{X},W}}$.

Next, we similarly define a specific $\mathbb{X}$-denotation ν into W by (ν1)–(ν3) below. Recall $\mathbb{X} = \mathbb{X}_\delta \amalg \mathbb{X}_\varepsilon \amalg (\mathbb{X} \cap H)$ and $\mathbb{X}_\varepsilon = \mathbb{X}'_\varepsilon \amalg \mathbb{X}''_\varepsilon$ by Lemma 3.3.

(ν1) For each $x \in \mathbb{X}_\delta$, νx is the element of $S \to \mathbb{T}$ such that

$$(\nu x)s = 1 \iff s \in S^x$$

for each $s \in S$. Thus $s \in S^x$ iff $\nu x \ni s$ for the extended relation $\ni$.

(ν2) For each $x \in \mathbb{X}'_\varepsilon$, since $\mathbb{X}'_\varepsilon \subseteq S$ by (3.1) and (3.3), we define $\nu x = x$. For each $x \in \mathbb{X}''_\varepsilon$, we let νx be an arbitrary element of S.

(ν3) For each $f \in \mathbb{X} \cap H$, νf is the element of $(K_f \to S) \to \mathbb{T}$ such that, if $K_f = \{k_1, \ldots, k_n\}$ with $k_1, \ldots, k_n$ distinct, then

$$(\nu f)\theta = 1 \iff \big((\theta k_i)\, \check{o} k_i\big)_{i=1,\ldots,n} f \notin Q$$

for each $\theta \in K_f \to S$. The remark in (Φ3) also applies here.

Thus ν is defined and it yields the projection $\mathrm{pr}_\nu \in W^{V_{\mathbb{X},W}} \to W$. Recall

$$(\Phi^* a)\nu = \begin{cases} \Phi a & \text{if } a \in \mathbb{C}, \\ \nu a & \text{if } a \in \mathbb{X}. \end{cases} \tag{3.7}$$

Lemma 3.21 *The following holds as to the composite* $\mathrm{pr}_\nu \Phi^* \in A \to W$.

(1) If $a \in A'_\varepsilon$, *then* $(\Phi^* a)\nu = a$.

(2) If $a \in \mathbb{S}_\delta \cup A'_\varepsilon$ *and* $s \in S$, *then* $(\Phi^* a)\nu \ni s$ *iff* $s \in S^a$.

Proof The composite induces an $\mathfrak{F}$-homomorphism of A_ε into $W_\varepsilon = S$. (Φ2) shows that $(\Phi^* a)\nu = \Phi a = a$ for $a \in \mathbb{C}_\varepsilon$ and (ν2) shows that $(\Phi^* x)\nu = \nu x = x$ for $x \in \mathbb{X}'_\varepsilon$. Since $A'_\varepsilon = [\mathbb{C}_\varepsilon \cup \mathbb{X}'_\varepsilon]_{\mathfrak{F}}$ by (3.1), (1) holds. (2) for $a \in A'_\varepsilon$ follows from (1) and the remark following (3.5). If $a \in \mathbb{C}_\delta$ then $(\Phi^* a)\nu = \Phi a$, while if $x \in \mathbb{X}_\delta$ then $(\Phi^* x)\nu = \nu x$. Thus (2) for $a \in \mathbb{S}_\delta$ follows from (Φ1) and (ν1).

Let L be the set of the operation symbols of A:

$$\mathsf{L} = \{\breve{o}k, \mathfrak{r}k, \wedge, \vee, \Rightarrow, \Diamond, \triangle, \sqcap, \sqcup, \square, \mathfrak{f}, \Omega x \mid k \in \mathsf{K},\ \mathfrak{r} \in \mathfrak{Q},\ \mathfrak{f} \in \mathfrak{F},\ x \in \mathbb{X}_\varepsilon\}.$$

Lemma 3.22 *There exists a mapping* I *of* $\mathrm{L}\amalg\mathsf{A}$ *into the set* $\mathbb{Z}_{\geq 0}$ *of non-negative integers which satisfies the following conditions.*

(1) If $\lambda \in \mathsf{L}$ *and* $(a_1, \ldots, a_n) \in \mathrm{Dom}\,\lambda$*, then*

$$\mathrm{I}\big(\lambda(a_1, \ldots, a_n)\big) = \mathrm{I}(\lambda) + \mathrm{I}(a_1) + \cdots + \mathrm{I}(a_n).$$

(2) If $a \in \{\breve{o}k, \triangle, \mathfrak{f} \mid k \in \mathsf{K},\ \mathfrak{f} \in \mathfrak{F}\} \amalg \mathbb{S}$*, then* $\mathrm{I}(a) = 0$*.*

(3) If $a \in \{\wedge, \vee, \Rightarrow, \Diamond, \sqcap, \sqcup, \square, \Omega x \mid x \in \mathbb{X}_\varepsilon\}$*, then* $\mathrm{I}(a) = 1$*.*

(4) If $p \in \mathbb{P} - \{\infty\}$*, then* $\mathrm{I}(\overline{p}k) = 4$ *for each* $k \in \mathsf{K}$*.*

(5) If $q \in \mathbb{P} - \{\infty\}$*, then* $\mathrm{I}\big((\leftarrow q]k\big) = 5$ *for each* $k \in \mathsf{K}$*.*

(6) If $\mathfrak{p}$ *is a connected quantifier in* $\mathfrak{P}$ *other than those dealt with in (4) and (5), then* $\mathrm{I}(\mathfrak{p}k) = 6$ *for each* $k \in \mathsf{K}$*.*

(7) If $\mathfrak{p}$ *is a disconnected quantifier in* $\mathfrak{P}$*, then* $\mathrm{I}(\mathfrak{p}k) = 7$ *for each* $k \in \mathsf{K}$*.*

(8) If $\mathfrak{r}$ *is a quantifier in* $\neg\mathfrak{P}$*, then* $\mathrm{I}(\mathfrak{r}k) = 9$ *for each* $k \in \mathsf{K}$*.*

(9) If $a \in \mathsf{A}_\varepsilon$*, then* $\mathrm{I}(a) = 0$*.*

Proof Since $\overline{p} \neq (\leftarrow q]$ for all $p, q \in \mathbb{P}$, we can define $\mathrm{I} \in (\mathrm{L} \amalg \mathbb{S}) \rightarrow \mathbb{Z}_{\geq 0}$ so that the conditions (2)–(8) hold. Theorem 2.2 of Gomi (09b) shows that each element $a \in \mathsf{A} - \mathbb{S}$ has the unique word form $\lambda(a_1, \ldots, a_n)$ and it satisfies $\mathrm{Rank}\, a = 1 + \sum_{i=1}^{n} \mathrm{Rank}\, a_i$. Therefore by induction on the rank, we can extend I to an element of $(\mathrm{L} \amalg \mathsf{A}) \rightarrow \mathbb{Z}_{\geq 0}$ so that the condition (1) also holds. The conditions (1) and (2) imply (9), because $\mathsf{A}_\varepsilon = [\mathbb{S}_\varepsilon]_{\mathfrak{F}}$ by (1.1).

Using Lemma 3.22, we define the mapping J of $\mathrm{L} \amalg \mathsf{A}$ into $\mathbb{Z}_{\geq 0}$ by

$$\mathrm{J}(b) = \begin{cases} \mathrm{I}(a) + 1 & \text{if } b = a\,\overline{p}\pi\,\mathsf{one}\triangle \text{ for some } a \in \mathsf{G} \text{ and } p \in \mathbb{P} - \{\infty\}, \\ \mathrm{I}(b) & \text{otherwise.} \end{cases} \tag{3.8}$$

We call $\mathrm{I}(b)$ and $\mathrm{J}(b)$ the **preindex** and the **index** of b respectively.

Lemma 3.23 *The following holds as to the preindex and the index.*

(1) If $b \in \mathsf{A}$*, then* $\mathrm{I}(b) \geq \mathrm{J}(b)$*.*

(2) If $a \in \mathsf{A},\ x \in \mathbb{X}_\varepsilon$ *and* $b \in \mathsf{A}_\varepsilon$*, then* $\mathrm{I}\big(a(x/b)\big) = \mathrm{I}(a)$*.*

Proof (1) Since $\mathsf{one} = (x_0 \forall\pi x_0 \triangle)\,\Omega x_0$, we have $\mathrm{I}(\mathsf{one}) = \mathrm{I}(x_0) + \mathrm{I}(\forall\pi) + \mathrm{I}(x_0)+\mathrm{I}(\triangle)+\mathrm{I}(\Omega x_0) = 9$, and so $\mathrm{I}(a\,\overline{p}\pi\,\mathsf{one}\triangle) = \mathrm{I}(a)+\mathrm{I}(\overline{p}\pi)+\mathrm{I}(\mathsf{one})+\mathrm{I}(\triangle) = \mathrm{I}(a) + 13 > \mathrm{I}(a) + 1 = \mathrm{J}(a\,\overline{p}\pi\,\mathsf{one}\triangle)$.

(2) We argue by induction on the rank r of a. If $r = 0$, or $a \in \mathbb{S}$, then $a(x/b) \in \mathbb{S}$, hence $\mathrm{I}\big(a(x/b)\big) = 0 = \mathrm{I}(a)$. Suppose $r > 0$ and let $\lambda(a_1, \ldots, a_n)$ be the word form of a. If $\lambda = \Omega x$, then $a(x/b) = a$, so there is nothing to prove. Suppose $\lambda \neq \Omega x$. Then $a(x/b) = \lambda(a_1(x/b), \ldots, a_n(x/b))$, so $\mathrm{I}\big(a(x/b)\big) = \mathrm{I}(\lambda) + \mathrm{I}\big(a_1(x/b)\big) + \cdots + \mathrm{I}\big(a_n(x/b)\big)$. Since $\mathrm{I}\big(a_i(x/b)\big) = \mathrm{I}(a_i)$ $(i = 1, \ldots, n)$ by the induction hypothesis, we conclude that $\mathrm{I}\big(a(x/b)\big) = \mathrm{I}(\lambda) + \mathrm{I}(a_1) + \cdots + \mathrm{I}(a_n) = \mathrm{I}(a)$.

Lemma 3.24 *If* $h \in P$ *then* $(\Phi^* h)\nu = 1$*, while if* $h \in Q$ *then* $(\Phi^* h)\nu = 0$.

Since $\alpha_0 \subseteq P$ and $\beta_0 \subseteq Q$ by the definition of P and Q in Lemma 3.20 and $\alpha_0 \cup \beta_0$ consists of sentences, Lemma 3.24 implies that $\alpha_0 \not\preccurlyeq \beta_0$, which is a contradiction concluding the proof of Theorem 3.

The proof of Lemma 3.24 is long. Let $h \in P \cup Q$. Then Theorem 2.2 of Gomi (09b) shows that there exist elements $a_1, \ldots, a_l \in A_\varepsilon$ $(l \geq 0)$, distinct elements $k_1, \ldots, k_l \in K_h$, and an element $h' \in H$ which satisfy the conditions

$$h = (a_i\,\breve{o}k_i)_{i=1,\ldots,l}h', \qquad h' \notin \textstyle\bigcup_{k\in K} \mathrm{Im}\,\breve{o}k. \tag{3.9}$$

Since h is good by Lemma 3.20, we have

$$a_1, \ldots, a_l \in A'_\varepsilon \tag{3.10}$$

and h' is good by Lemma 3.4. Since h is a sentence by Lemma 3.20, we have

$$K_{h'} = \{k_1, \ldots, k_l\}. \tag{3.11}$$

Since $h' \notin \bigcup_{k\in K} \mathrm{Im}\,\breve{o}k$, either $h' \in \mathbb{S} \cap H$ or h' is in one of the word forms

$$a\,\mathfrak{r}k\,f \;\; (\mathfrak{r} \in \mathfrak{Q}), \qquad f \wedge g, \qquad f \vee g, \qquad f \Rightarrow g, \qquad f^{\Diamond}, \qquad c\triangle. \tag{3.12}$$

If $h' = c\triangle$, then either c is in one of the word forms

$$a \sqcap b, \qquad a \sqcup b, \qquad a^{\square}, \qquad f\,\Omega x. \tag{3.13}$$

or $c \in \mathbb{S}_\delta \cup A_\varepsilon$. If $c \in A_\varepsilon$, then $c \in A'_\varepsilon$ by Lemma 3.4 because h' is good.

We first consider the above two special cases.

The case $h' \in \mathbb{S} \cap H$. Because of (3.11), (3.10) and (3.3), we may define $\theta \in K_{h'} {\to} S$ by $\theta k_i = a_i$ $(i = 1, \ldots, l)$, and then we have

$$\begin{aligned}
(\Phi^* h)\nu &= \big(\Phi^*\big((a_i\,\breve{o}k_i)_i h'\big)\big)\nu && \text{(by (3.9))}\\
&= \big((\Phi^* a_i)\nu\,\breve{o}k_i\big)_i(\Phi^* h')\nu \\
&= (a_i\,\breve{o}k_i)_i(\Phi^* h')\nu && \text{(by (3.10) and Lemma 3.21)}
\end{aligned}$$

$$
\begin{aligned}
&= \big((\theta k_i)\,\check{o}k_i\big)_i(\Phi^* h')\nu && \text{(by the definition of } \theta)\\
&= \big((\Phi^* h')\nu\big)\theta && \text{(by Corollary 3.5.2 of Gomi (09c))}\\
&= \begin{cases} (\Phi h')\theta & \text{if } h' \in \mathbb{C}\cap H,\\ (\nu h')\theta & \text{if } h' \in \mathbb{X}\cap H \end{cases} && \text{(by (3.7))},
\end{aligned}
$$

where the second equality holds because Φ^* and the projection by ν is a homomorphism with respect to the operations $\check{o}k_i$ ($i = 1, \dots, l$). Hence

$$
\begin{aligned}
(\Phi^* h)\nu = 1 &\iff \big((\theta k_i)\,\check{o}k_i\big)_i h' \notin Q && \text{(by } (\Phi 3) \text{ and } (\nu 3))\\
&\iff \big(a_i\,\check{o}k_i\big)_i h' \notin Q && \text{(by the definition of } \theta)\\
&\iff h \notin Q && \text{(by (3.9))}.
\end{aligned}
$$

Therefore if $h \in Q$ then $(\Phi^* h)\nu = 0$, while if $h \in P$ then $(\Phi^* h)\nu = 1$, because $P \cap Q = \emptyset$ by Lemma 3.20.

The case $h' = c\triangle$ with $c \in \mathbb{S}_\delta \cup A'_\varepsilon$. Here $l = 1$ and $k_1 = \pi$ by (3.11), $a_1 \in A'_\varepsilon = S$ by (3.10) and (3.3), and $h = a_1\,\check{o}\pi c\triangle$ by (3.9). Therefore $(\Phi^* h)\nu = (\Phi^* a_1)\nu\,\check{o}\pi\,(\Phi^* c)\nu\triangle = a_1\,\check{o}\pi\,(\Phi^* c)\nu\triangle$ by Lemma 3.21, hence

$$
\begin{aligned}
(\Phi^* h)\nu = 1 &\iff (\Phi^* c)\nu \ni a_1 && \text{(by Theorem 3.9 of Gomi (09c))}\\
&\iff a_1 \in S^c && \text{(by Lemma 3.21)}\\
&\iff a_1\,\check{o}\pi c\triangle \notin Q && \text{(by (3.5))}\\
&\iff h \notin Q.
\end{aligned}
$$

Therefore if $h \in Q$ then $(\Phi^* h)\nu = 0$, while if $h \in P$ then $(\Phi^* h)\nu = 1$.

The general case. We argue by induction on the index $J(h)$ of h.

If $J(h) = 0$, then it follows from (3.8), Lemma 3.22, and the discussion on the word forms of h' that either $h' \in \mathbb{S} \cap H$ or $h' = c\triangle$ for some $c \in \mathbb{S}_\delta \cup A'_\varepsilon$, and in either case, Lemma 3.24 has been proved above. Therefore we assume $J(h) \geq 1$, $h' \notin \mathbb{S} \cap H$, and $h' \neq c\triangle$ for any $c \in \mathbb{S}_\delta \cup A'_\varepsilon$. Then h' is in one of the word forms (3.12) and if $h' = c\triangle$, c is in one of the word forms (3.13). We will consider those cases one by one, redividing them into twelve cases.

We first consider the case where h' is equal to $a\mathfrak{x}kf$ ($\mathfrak{x} \in \mathfrak{Q}$) on the list (3.12), and further divide it into the three subcases (1) $\mathfrak{x} \in \neg\mathfrak{P}$, (2) $\mathfrak{x} \in \mathfrak{P}$ but $\mathfrak{x}$ is disconnected, and (3) $\mathfrak{x} \in \mathfrak{P}$ and $\mathfrak{x}$ is connected. In the last case, $\mathfrak{x}$ is an interval in one of the four shapes $\mathbb{P}$, $[pq]$ with $p < q \neq \infty$ or $p = q = 0$, $(\leftarrow q]$ with $q \neq \infty$, and $\overline{p} = (p\rightarrow)$ with $p \neq \infty$.

We will often use the following argument. Recalling from Lemma 3.20 that $P \cap Q = \emptyset$, we say that two elements f and g of $P \cup Q$ is equivalent or write $f \sim g$, if either $f, g \in P$ or $f, g \in Q$. Then if $h \sim \grave{h}$ and $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$ and $J(h) > J(\grave{h})$, or if $h \nsim \grave{h}$ and $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ and $J(h) > J(\grave{h})$, then by the induction hypothesis, Lemma 3.24 holds for the h.

Case 1: $h' = a\neg\mathfrak{p}k\,f$ with $\mathfrak{p} \in \mathfrak{P}$. Here $h = (a_i\,\breve{o}k_i)_i(a\neg\mathfrak{p}k\,f)$ by (3.9), and $h \sim \grave{h} = (a_i\,\breve{o}k_i)_i(a\,\mathfrak{p}k\,f^\diamondsuit)$ by Lemma 3.20 (11) (12). Also, $(\Phi^*h)\nu = (\Phi^*\grave{h})\nu$ by Theorem 3.10 of Gomi (09c). Furthermore,

$$\begin{aligned}
J(h) &= I(h) = I(a\neg\mathfrak{p}k\,f) = I(a) + I(\neg\mathfrak{p}k) + I(f) = I(a) + 9 + I(f) \\
&> I(a) + 8 + I(f) = I(a) + 7 + \big(I(f) + 1\big) = I(a) + 7 + I(f^\diamondsuit) \\
&\geq I(a) + I(\mathfrak{p}k) + I(f^\diamondsuit) = I(a\,\mathfrak{p}k\,f^\diamondsuit) = I(\grave{h}) = J(\grave{h})
\end{aligned}$$

by (3.8) and Lemma 3.22. Thus Lemma 3.24 holds for the h.

Case 2: $h' = a\,\mathfrak{p}k\,f$ with $\mathfrak{p} \in \mathfrak{P}$ disconnected. Here $h = (a_i\,\breve{o}k_i)_i(a\,\mathfrak{p}k\,f)$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the connected components of $\mathfrak{p}$ and $h_j = (a_i\,\breve{o}k_i)_i(a\,\mathfrak{p}_j k\,f)$ $(j = 1, \ldots, m)$. Then we have $(\Phi^*h)\nu = (\Phi^*h_1)\nu \vee \cdots \vee (\Phi^*h_m)\nu$ by Theorem 3.11 and Lemma 3.4 of Gomi (09c). Furthermore,

$$\begin{aligned}
J(h) &= I(h) = I(a\,\mathfrak{p}k\,f) = I(a) + I(\mathfrak{p}k) + I(f) = I(a) + 7 + I(f) \\
&> I(a) + 6 + I(f) \geq I(a) + I(\mathfrak{p}_j k) + I(f) = I(a\,\mathfrak{p}_j k\,f) = I(h_j) \geq J(h_j)
\end{aligned}$$

for each $j \in \{1, \ldots, m\}$ by (3.8), Lemmas 3.22 and 3.23. By Lemma 3.20 (13) (14), if $h \in P$ then $h_j \in P$ for some $j \in \{1, \ldots, m\}$, while if $h \in Q$ then $h_j \in Q$ for all $j \in \{1, \ldots, m\}$. Thus Lemma 3.24 holds for the h.

Case 3: $h' = a\,\mathbb{P}k\,f$. Here $h = (a_i\,\breve{o}k_i)_i(a\,\mathbb{P}k\,f)$. Let $h_1 = (a_i\,\breve{o}k_i)_i(a\,\overline{0}k\,f)$ and $h_2 = (a_i\,\breve{o}k_i)_i\big(a\,(\leftarrow 0]k\,f\big)$. Then we have $(\Phi^*h)\nu = (\Phi^*h_1)\nu \vee (\Phi^*h_2)\nu$ as in Case 2, because $\mathbb{P} = \overline{0} \cup (\leftarrow 0]$. Furthermore,

$$\begin{aligned}
J(h) &= I(h) = I(a\,\mathbb{P}k\,f) = I(a) + I(\mathbb{P}k) + I(f) = I(a) + 6 + I(f) \\
&> I(a) + 5 + I(f) \\
&\geq \begin{cases} I(a) + I(\overline{0}k) + I(f) = I(a\,\overline{0}k\,f) = I(h_1) \geq J(h_1), \\ I(a) + I\big((\leftarrow 0]k\big) + I(f) = I\big(a\,(\leftarrow 0]k\,f\big) = I(h_2) = J(h_2) \end{cases}
\end{aligned}$$

by (3.8), Lemmas 3.22 and 3.23. By Lemma 3.20 (15) (16), if $h \in P$ then either h_1 or h_2 belongs to P, while if $h \in Q$ then $h_1, h_2 \in Q$. Thus Lemma 3.24 holds for the h.

Case 4: $h' = a\,(pq]k\,f$ with either $p < q \neq \infty$ or $p = q = 0$ Here $h = (a_i\,\breve{o}k_i)_i\big(a\,(pq]k\,f\big)$. Let $h_1 = (a_i\,\breve{o}k_i)_i(a\,\overline{p}k\,f)$ and $h_2 = (a_i\,\breve{o}k_i)_i\big(a\,(\leftarrow q]k\,f\big)$. Then since $(pq] = \overline{p} \cap (\leftarrow q]$, we have $(\Phi^*h)\nu = (\Phi^*h_1)\nu \wedge (\Phi^*h_2)\nu$ by Theorem 3.11 and Lemma 3.4 of Gomi (09c). Furthermore,

$$\begin{aligned}
J(h) &= I(h) = I\big(a\,(pq]k\,f\big) = I(a) + I\big((pq]k\big) + I(f) = I(a) + 6 + I(f) \\
&> I(a) + 5 + I(f) \\
&\geq \begin{cases} I(a) + I(\overline{p}k) + I(f) = I\big(a\,\overline{p}k\,f\big) = I(h_1) \geq J(h_1), \\ I(a) + I\big((\leftarrow q]k\big) + I(f) = I\big(a\,(\leftarrow q]k\,f\big) = I(h_2) = J(h_2) \end{cases}
\end{aligned}$$

by (3.8), Lemmas 3.22 and 3.23. By Lemma 3.20 (17) (18), if $h \in P$ then $h_1, h_2 \in P$, while if $h \in Q$ then either h_1 or h_2 belongs to Q. Thus Lemma 3.24 holds for the h.

Case 5: $h' = a\,(\leftarrow q]k\,f$ **with** $q \neq \infty$**.** Here $h = (a_i\,\check{o}k_i)_i(a\,(\leftarrow q]k\,f) \not\sim \grave{h} = (a_i\,\check{o}k_i)_i(a\,\overline{q}k\,f)$ by Lemma 3.20 (19) (20). Also, we have $(\Phi^*h)\nu \neq (\Phi^*\grave{h})\nu$ by Theorem 3.10 and Corollary 3.5.3 of Gomi (09c), because $(\leftarrow q] = \overline{q}^\circ$. Furthermore,

$$\begin{aligned} J(h) = I(h) &= I\big(a\,(\leftarrow q]k\,f\big) = I(a) + I\big((\leftarrow q]k\big) + I(f) = I(a) + 5 + I(f) \\ &> I(a) + 4 + I(f) = I(a) + I(\overline{q}k) + I(f) = I(a\,\overline{q}k\,f) = I(\grave{h}) \geq J(\grave{h}) \end{aligned}$$

by (3.8), Lemmas 3.22 and 3.23. Thus Lemma 3.24 holds for the h.

Case 6: $h' = a\,\overline{p}k\,f \neq a\,\overline{p}\pi\,\mathrm{one}\triangle$ **with** $p \neq \infty$**.** Here $h = (a_i\,\check{o}k_i)_i(a\,\overline{p}k\,f)$. In view of Lemma 3.3, let $x \in \mathbb{X}''_\varepsilon$ and $\grave{h} = \big(a \sqcap \big(x\,\check{o}k\,(a_i\,\check{o}k_i)_i f\big)\,\Omega x\big)\,\overline{p}\pi\,\mathrm{one}\triangle$. Then $h \sim \grave{h}$ by Lemma 3.20 (21) (22). Since x does not occur free in $g = (a_i\,\check{o}k_i)_i f$ by Lemma 3.4, we have $h \asymp \grave{h}$ by Lemma 2.10, hence $(\Phi^*h)\nu = (\Phi^*\grave{h})\nu$. Furthermore,

$$\begin{aligned} J(h) = I(h) &= I(a) + I(\overline{p}k) + I(f) = I(a) + 4 + I(f) = I(a) + 4 + I(g) \\ &> I(a) + 3 + I(g) \\ &= I(a) + I(\sqcap) + I(g) + I(\Omega x) + 1 = I\big(a \sqcap (x\,\check{o}k\,g)\,\Omega x\big) + 1 = J(\grave{h}) \end{aligned}$$

by (3.8) and Lemma 3.22. Thus Lemma 3.24 holds for the h.

Case 7: $h' = a\,\overline{p}\pi\,\mathrm{one}\triangle$ **with** $p \neq \infty$**.** Here $h = h' = a\,\overline{p}\pi\,\mathrm{one}\triangle$. Define

$$X = \{s \in S \mid (\Phi^*a)\nu \ni s\}.$$

Then Theorems 3.9 and 3.19 of Gomi (09c) imply

$$(\Phi^*h)\nu = 1 \iff p < |X|.$$

Case 7.1: $h \in P$**.** Here $a\,\overline{p}\pi\,\mathrm{one}\triangle \in P$, so $p \in \mathbb{P}'$ and hence $p < ó$. In order to prove $(\Phi^*h)\nu = 1$ by contradiction, assume $|X| \leq p$. Then $|X| \neq ó$, and so the definition (3.6) of $|X|$ shows that there exist elements $b_1, \ldots, b_m \in G$ and elements $q_1, \ldots, q_m \in \mathbb{P}$ which satisfy the conditions

(i) $X \subseteq \bigcup_{i=1}^m S^{b_i}$,

(ii) $p \geq \sum_{i=1}^m q_i$,

(iii) $b_i\,\overline{q_i}\pi\,\mathrm{one}\triangle \in Q$ $(i = 1, \ldots, m)$, where $m \geq 0$. Therefore by Lemma 3.20 (23), there exists an element $x \in \mathbb{X}'_\varepsilon$ such that $x\,\breve{o}\pi\,a\triangle \in P$ and $x\,\breve{o}\pi\,b_1\triangle, \cdots, x\,\breve{o}\pi\,b_m\triangle \in Q$. Furthermore,

$$J(h) = I(a) + 1 > I(a) = I(x\,\breve{o}\pi\,a\triangle) = J(x\,\breve{o}\pi\,a\triangle)$$

by (3.8) and Lemma 3.22. Therefore $\big(\Phi^*(x\,\breve{o}\pi\,a\triangle)\big)\nu = 1$ by the induction hypothesis. Since $\big(\Phi^*(x\,\breve{o}\pi\,a\triangle)\big)\nu = (\Phi^* x)\nu\,\breve{o}\pi\,(\Phi^* a)\nu\triangle = x\,\breve{o}\pi\,(\Phi^* a)\nu\triangle$ by Lemma 3.21, Theorem 3.9 of Gomi (09c) yields $(\Phi^* a)\nu \ni x$, hence $x \in X$. On the other hand, since $x\,\breve{o}\pi\,b_i\triangle \in Q$, the definition (3.5) of S^{b_i} yields $x \notin S^{b_i}$ for $i = 1, \ldots, m$. This contradicts (i). Thus $(\Phi^* h)\nu = 1$.

Case 7.2: $h \in Q$**.** Here $a\,\overline{p}\pi\,one\triangle \in Q$. Suppose $s \in S - S^a$. Then $s\,\breve{o}\pi\,a\triangle \in Q$ by (3.5). Also $J(h) = I(a) + 1 > I(a) = I(s\,\breve{o}\pi\,a\triangle) = J(s\,\breve{o}\pi\,a\triangle)$ by (3.8) and Lemma 3.22. Therefore $\big(\Phi^*(s\,\breve{o}\pi\,a\triangle)\big)\nu = 0$ by the induction hypothesis, and so $(\Phi^* a)\nu \not\ni s$ by Theorem 3.9 of Gomi (09c) and Lemma 3.21, that is, $s \in S - X$. Thus $X \subseteq S^a$, and so $|X| \leq p$ by (3.6). Therefore $(\Phi^* h)\nu = 0$.

We have exhausted the cases where h' has a word form $a\,\mathfrak{r}k\,f$ ($\mathfrak{r} \in \mathfrak{Q}$).

Case 8: $h' = f \wedge g$ **or** $f \vee g$ **or** $f \Rightarrow g$**.** Let $*$ denote any one of the operations $\wedge, \vee$ and $\Rightarrow$. Then $h = (a_i\,\breve{o}k_i)_{i=1,\ldots,l}(f * g)$. Let $\rho \in \mathfrak{S}_l$ and $h_\rho = (a_{\rho i}\,\breve{o}k_{\rho i})_{i=1,\ldots,l}(f * g)$. Then $h \sim h_\rho$ by Lemma 3.20 (1) (2), $J(h) = J(h_\rho)$ by (3.8) and Lemmas 3.22, and $(\Phi^* h)\nu = (\Phi^* h_\rho)\nu$ by Corollary 3.5.1 of Gomi (09c). Therefore we may assume that (3.2) is satisfied. Let $h_f = (a_i\,\breve{o}k_i)_{i=1,\ldots,m} f$ and $h_g = (a_i\,\breve{o}k_i)_{i=\nu+1,\ldots,l}\, g$. Then $(\Phi^* h)\nu = (\Phi^* h_f)\nu * (\Phi^* h_g)\nu$ by Theorem 3.6 of Gomi (09c). Also, the following holds by Lemma 3.20 (3)–(8).

- When $*$ is $\wedge$, if $h \in P$ then $h_f, h_g \in P$, while if $h \in Q$ then either $h_f \in Q$ or $h_g \in Q$.
- When $*$ is $\vee$, if $h \in P$ then either $h_f \in P$ or $h_g \in P$, while if $h \in Q$ then $h_f, h_g \in Q$.
- When $*$ is $\Rightarrow$, if $h \in P$ then either $h_f \in Q$ or $h_g \in P$, while if $h \in Q$ then $h_f \in P$ and $h_g \in Q$.

Furthermore,

$$J(h) = I(h) = I(f) + I(*) + I(g) = I(f) + 1 + I(g) > \begin{cases} I(f) = I(h_f) \geq J(h_f), \\ I(g) = I(h_g) \geq J(h_g) \end{cases}$$

by (3.8), Lemmas 3.22 and 3.23. Thus Lemma 3.24 holds for the h.

Case 9: $h' = f^\Diamond$**.** Here $h = (a_i\,\breve{o}k_i)_i(f^\Diamond)$, and $h \nsim \grave{h} = (a_i\,\breve{o}k_i)_i f$ by Lemma 3.20 (9) (10). Also, $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ by Corollary 3.5.3 of Gomi (09c). Furthermore,

$$J(h) = I(h) = I(f) + I(\Diamond) = I(f) + 1 > I(f) = I(\grave{h}) \geq J(\grave{h})$$

by (3.8), Lemmas 3.22 and 3.23. Thus Lemma 3.24 holds for the h.

We are left with the case $h' = c\triangle$ with c in one of the word forms (3.13).

Case 10: $h' = (a \sqcap b)\triangle$ **or** $(a \sqcup b)\triangle$**.** Let $*$ denote $\sqcap$ or $\sqcup$. Then $l = 1$, $k_1 = \pi$, and $h = a_1\, \breve{o}\pi\, (a * b)\triangle$. Let $h_a = a_1\, \breve{o}\pi\, a\triangle$ and $h_b = a_1\, \breve{o}\pi\, b\triangle$. Then the following holds by Theorems 3.1 and 3.6 of Gomi (09c) and Lemma 3.20 (25)–(28).

- When $*$ is $\sqcap$, $(\Phi^* h)\nu = (\Phi^* h_a)\nu \wedge (\Phi^* h_b)\nu$, and if $h \in P$ then $h_a, h_b \in P$, while if $h \in Q$ then either $h_a \in Q$ or $h_b \in Q$.
- When $*$ is $\sqcup$, $(\Phi^* h)\nu = (\Phi^* h_a)\nu \vee (\Phi^* h_b)\nu$, and if $h \in P$ then either $h_a \in P$ or $h_b \in P$, while if $h \in Q$ then $h_a, h_b \in Q$.

Furthermore,

$$J(h) = I(h) = I(a) + I(*) + I(b) = I(a) + 1 + I(b) > \begin{cases} I(a) = I(h_a) = J(h_a), \\ I(b) = I(h_b) = J(h_b) \end{cases}$$

by (3.8) and Lemma 3.22. Thus Lemma 3.24 holds for the h.

Case 11: $h' = a^{\square}\triangle$**.** Here $l = 1$, $k_1 = \pi$, $h = a_1\, \breve{o}\pi\, a^{\square}\triangle$, and $h \not\sim \grave{h} = a_1\, \breve{o}\pi\, a\triangle$ by Lemma 3.20 (29) (30). Also, $(\Phi^* h)\nu \neq (\Phi^* \grave{h})\nu$ by Theorem 3.1 and Corollary 3.5.3 of Gomi (09c). Furthermore,

$$J(h) = I(h) = I(a) + I(\square) = I(a) + 1 > I(a) = I(\grave{h}) = J(\grave{h})$$

by (3.8) and Lemma 3.22. Thus Lemma 3.24 holds for the h.

Case 12: $h' = (f\,\Omega x)\triangle$**.** Here $l = 1$, $k_1 = \pi$, $h = a_1\, \breve{o}\pi\, (f\,\Omega x)\triangle$, and $h \sim \grave{h} = f(x/a_1)$ by Lemma 3.20 (31) (32). Also, since x is free from a_1 in f by Lemma 3.4, we have $(\Phi^* h)\nu = (\Phi^* \grave{h})\nu$ by Theorem 3.18 of Gomi (09c). Furthermore,

$$J(h) = I(h) = I(f) + I(\Omega x) = I(f) + 1 > I(f) = I\big(f(x/a_1)\big) = I(\grave{h}) \geq J(\grave{h})$$

by (3.8), Lemmas 3.22 and 3.23. Thus Lemma 3.24 holds for the h.

This completes the proof of Lemma 3.24. Thus we have proved Theorem 3.

4 A predicate deduction system satisfying the law

The purpose of this section is to prove the following theorem, from which Theorem 1 is derived as was shown in §2.

Theorem 4 *The deduction system* $(\wp \cup \&, \nabla)$ *on* H *is* $\mathcal{G}$*-sound and its deduction relation* $\preccurlyeq_{\wp\cup\&,\nabla}$ *is a MPC relation.*

Proof The largest $\mathcal{G}$-logic on H is equal to the partial validity relation $\vDash$ of the predicate logical space $(\mathsf{H}, \mathcal{G})$ by Theorem 6.5 of Gomi (09a), and $\vDash$ is the restriction of the validity relation $\preccurlyeq$ of $(\mathsf{H}, \mathcal{G})$. Also, Theorem 6.6 of Gomi (09a) shows that an element $\mathsf{f} \in \mathsf{H}$ belongs to the core C of $(\mathsf{H}, \mathcal{G})$ iff $\vDash \mathsf{f}$ holds. Therefore by Theorem 7.1 of Gomi (09a), the former assertion means that $\preccurlyeq$ satisfies the three laws

$$\mathsf{f}, \mathsf{f} \Rightarrow \mathsf{g} \preccurlyeq \mathsf{g}, \qquad \mathsf{f}\mathsf{g} \preccurlyeq \mathsf{f} \wedge \mathsf{g}, \qquad \preccurlyeq \mathsf{h}$$

for all elements $\mathsf{f}, \mathsf{g} \in \mathsf{H}$ and $\mathsf{h} \in \nabla$. Since $\preccurlyeq$ is a Boolean relation with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$, it follows from the definition of the Boolean law and Lemma 2.1 that $\preccurlyeq$ satisfies the first and second laws and every Boolean predicate h satisfies $\preccurlyeq \mathsf{h}$.

For instance, since $\mathsf{f}^{\Diamond} \preccurlyeq \mathsf{f} \Rightarrow \mathsf{g}$ and $\mathsf{g} \preccurlyeq \mathsf{f} \Rightarrow \mathsf{g}$ by the first and second implication laws, we have $\mathsf{f}^{\Diamond} \vee \mathsf{g} \preccurlyeq \mathsf{f} \Rightarrow \mathsf{g}$, hence $\preccurlyeq (\mathsf{f}^{\Diamond} \vee \mathsf{g}) \Rightarrow (\mathsf{f} \Rightarrow \mathsf{g})$. Also, since $\mathsf{f}, \mathsf{f} \Rightarrow \mathsf{h}, \mathsf{g} \Rightarrow \mathsf{h} \preccurlyeq \mathsf{h}$ and $\mathsf{g}, \mathsf{f} \Rightarrow \mathsf{h}, \mathsf{g} \Rightarrow \mathsf{h} \preccurlyeq \mathsf{h}$ by the cut-implication law, we have $\mathsf{f} \vee \mathsf{g}, \mathsf{f} \Rightarrow \mathsf{h}, \mathsf{g} \Rightarrow \mathsf{h} \preccurlyeq \mathsf{h}$, hence $\preccurlyeq \big((\mathsf{f} \Rightarrow \mathsf{h}) \wedge (\mathsf{g} \Rightarrow \mathsf{h})\big) \Rightarrow \big((\mathsf{f} \vee \mathsf{g}) \Rightarrow \mathsf{h}\big)$. Also, since $\mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow \mathsf{g} \preccurlyeq \mathsf{f}$ and $\mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow \mathsf{g} \preccurlyeq \mathsf{g}$, we have $\mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow \mathsf{g} \preccurlyeq \mathsf{f} \wedge \mathsf{g}$, hence $\preccurlyeq \big((\mathsf{h} \Rightarrow \mathsf{f}) \wedge (\mathsf{h} \Rightarrow \mathsf{g})\big) \Rightarrow \big(\mathsf{h} \Rightarrow (\mathsf{f} \wedge \mathsf{g})\big)$. Also, since

$$\mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow (\mathsf{f} \Rightarrow \mathsf{g}) \preccurlyeq \mathsf{f}, \quad \mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow (\mathsf{f} \Rightarrow \mathsf{g}) \preccurlyeq \mathsf{f} \Rightarrow \mathsf{g}, \quad \mathsf{f}, \mathsf{f} \Rightarrow \mathsf{g} \preccurlyeq \mathsf{g}$$

by the cut-implication law, we have $\mathsf{h}, \mathsf{h} \Rightarrow \mathsf{f}, \mathsf{h} \Rightarrow (\mathsf{f} \Rightarrow \mathsf{g}) \preccurlyeq \mathsf{g}$ by the cut law, hence $\preccurlyeq \big((\mathsf{h} \Rightarrow \mathsf{f}) \wedge \big(\mathsf{h} \Rightarrow (\mathsf{f} \Rightarrow \mathsf{g})\big)\big) \Rightarrow (\mathsf{h} \Rightarrow \mathsf{g})$.

Consequently, C is closed under $\wp$ and $\&$ and contains all Boolean predicates. Since $\preccurlyeq$ satisfies the case+ law, the case− law and the $\forall$+ law, C is also closed under the case+ logic $\perp$, case− logic $\top$ and $\forall$+ logic $\forall$. Therefore in view of (1.3), it remains to show that every proper MPC predicate h satisfies $\preccurlyeq \mathsf{h}$. By virtue of Lemma 2.1, it is a direct consequence of the proved fact that $\preccurlyeq$ satisfies the twenty two laws from the $\mathfrak{Q}, \breve{\mathfrak{o}}$ law to the $\forall-$ law.

Let $\mathsf{R} = \wp \cup \&$. Then the deduction relation $\preccurlyeq_{\mathsf{R},\nabla}$ is Boolean with respect to the operations $\wedge, \vee, \Diamond$ and $\Rightarrow$ by Corollary 5.9.1 of Gomi (09a). Also, since ∇ is closed under $\mathsf{R} \cup \perp \cup \top \cup \forall$ and contains ∂ by (1.3), it follows that $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the twenty five laws which constitute the proper MPC law.

For instance as for the case+ law, assume $\preccurlyeq_{\mathsf{R},\nabla} \mathsf{f}$. Then $\mathsf{f} \in [\nabla]_{\mathsf{R}} = \nabla$, and so since ∇ is closed under $\perp$, we have $\mathfrak{a}\,\breve{\mathfrak{o}}\mathsf{k}\,\mathsf{f} \in \nabla = [\nabla]_{\mathsf{R}}$ for all $\mathfrak{a} \in \mathsf{A}_{\varepsilon}$ and $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, hence $\preccurlyeq_{\mathsf{R},\nabla} \mathfrak{a}\,\breve{\mathfrak{o}}\mathsf{k}\,\mathsf{f}$. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the case+ law. The same argument applies to the case− law and $\forall$+ law as well.

Since $(\mathfrak{one}^{\Box}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f})^{\Diamond} \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}}$ for all $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, and $\mathfrak{p} \in \mathbb{P}$, we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathfrak{one}^{\Box}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f})^{\Diamond}$, hence $\mathfrak{one}^{\Box}\,\overline{\mathfrak{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla}$ by Lemma 2.1. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\mathfrak{one}^{\Box}$ law.

Let $\mathfrak{a} \in \mathsf{G}$, $\mathfrak{b} \in \mathsf{A}_{\varepsilon}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k}, \mathsf{l} \in \mathsf{K}_{\mathsf{f}}$, $\mathsf{k} \neq \mathsf{l}$, and $\mathfrak{x} \in \{\breve{\mathfrak{o}}\} \cup \mathfrak{Q}$. Also, let $\mathfrak{a} \in \mathsf{A}_{\varepsilon}$ in case $\mathfrak{x} = \breve{\mathfrak{o}}$. Then, since $\mathfrak{a}\mathfrak{x}\mathsf{k}(\mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}\,\mathsf{f}) \Rightarrow \mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}(\mathfrak{a}\mathfrak{x}\mathsf{k}\,\mathsf{f}) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}}$, we have $\preccurlyeq_{\mathsf{R},\nabla} \mathfrak{a}\mathfrak{x}\mathsf{k}(\mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}\,\mathsf{f}) \Rightarrow \mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}(\mathfrak{a}\mathfrak{x}\mathsf{k}\,\mathsf{f})$, hence $\mathfrak{a}\mathfrak{x}\mathsf{k}(\mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}\,\mathsf{f}) \preccurlyeq_{\mathsf{R},\nabla} \mathfrak{b}\,\breve{\mathfrak{o}}\mathsf{l}(\mathfrak{a}\mathfrak{x}\mathsf{k}\,\mathsf{f})$

by Lemma 2.1. Similarly we have $\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,(\mathsf{a}\,\mathfrak{r}\mathsf{k}\,\mathsf{f}) \preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\mathfrak{r}\mathsf{k}\,(\mathsf{b}\,\breve{\mathsf{o}}\mathsf{l}\,\mathsf{f})$, and thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\mathfrak{Q}, \breve{\mathsf{o}}$ law. The same argument applies to the remaining laws other than the $\forall, \Rightarrow$ law, the $\forall, \mathfrak{P}$ law, and the $\sqcup, +$ law.

As for the $\forall, \Rightarrow$ law, let $\mathsf{f}, \mathsf{g} \in \mathsf{A}_{\emptyset}$, $\mathsf{x} \in \mathbb{X}_{\varepsilon}$, and assume that x does not occur free in f. Then, since

$$\big(\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} \big(\mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle\big) \Rightarrow \big(\mathsf{f} \Rightarrow \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle\big)$, hence

$$\mathsf{f}, \mathsf{one}\,\forall\pi\,\big((\mathsf{f} \Rightarrow \mathsf{g})\,\Omega\mathsf{x}\big)\triangle \preccurlyeq_{\mathsf{R},\nabla} \mathsf{one}\,\forall\pi\,(\mathsf{g}\,\Omega\mathsf{x})\triangle.$$

by Lemma 2.1. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\forall, \Rightarrow$ law.

As for the $\forall, \mathfrak{P}$ law, let $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, and $\mathsf{p} \in \mathbb{P}$. Then, since

$$(\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle \wedge \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}) \Rightarrow \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$, hence $\mathsf{a}\,\forall\pi\,\mathsf{b}\triangle, \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{b}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}$ by Lemma 2.1. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\forall, \mathfrak{P}$ law.

As for the $\sqcup, +$ law, let $\mathsf{a}, \mathsf{b} \in \mathsf{G}$, $\mathsf{f} \in \mathsf{H}$, $\mathsf{k} \in \mathsf{K}_{\mathsf{f}}$, and $\mathsf{p}, \mathsf{q} \in \mathbb{P}$. Then, since

$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}) \in \partial \subseteq \nabla \subseteq [\nabla]_{\mathsf{R}},$$

we have $\preccurlyeq_{\mathsf{R},\nabla} (\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \Rightarrow (\mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f} \vee \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f})$, hence

$$(\mathsf{a} \sqcup \mathsf{b})\,\overline{\mathsf{p} + \mathsf{q}}\mathsf{k}\,\mathsf{f} \preccurlyeq_{\mathsf{R},\nabla} \mathsf{a}\,\overline{\mathsf{p}}\mathsf{k}\,\mathsf{f}, \mathsf{b}\,\overline{\mathsf{q}}\mathsf{k}\,\mathsf{f}.$$

by Lemma 2.1. Thus $\preccurlyeq_{\mathsf{R},\nabla}$ satisfies the $\sqcup, +$ law.

References

Gomi, K. (2009a) Theory of completeness for logical spaces, Logica Universalis 3, 243–291. (2009b) From logical systems to logical spaces, preprint, http://homepage3.nifty.com/gomiken/english/mathpsy.htm/. (2009c) Case logic for mathematical psychology, preprint, the above URL.

Henkin, L. (1949) The completeness of the first-order functional calculus, Journal of Symbolic Logic 14, 159–166.

Mizumura, Y. (2000) Ronritaikei MCL no kanzensei (The completeness of the logical system MCL), Master's thesis, Graduate School Math. Sci. Univ. Tokyo.

Takaoka, Y. (2009) On existence of models for the logical system MPCL, UTMS Preprint Series.