Red-Black Tree

Red-black trees: Overview

- Red-black trees are a variation of binary search trees to ensure that the tree is ***balanced***.
  - **Balanced search tree:** A search-tree data structure for which a height of O(lg n) is guaranteed when implementing a dynamic set of n items.
- Operations take $O(\lg n)$ time in the worst case.

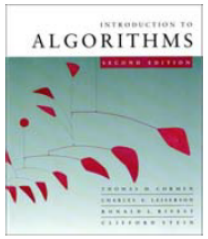
Red-Black Tree

- Binary search tree + 1 bit per node:
  - the attribute *color*, which is either **red** or **black**.
- All other attributes of BST's node are inherited:
  - *key*, *left*, *right*, and *p*.
- All empty trees (leaves) are colored black.

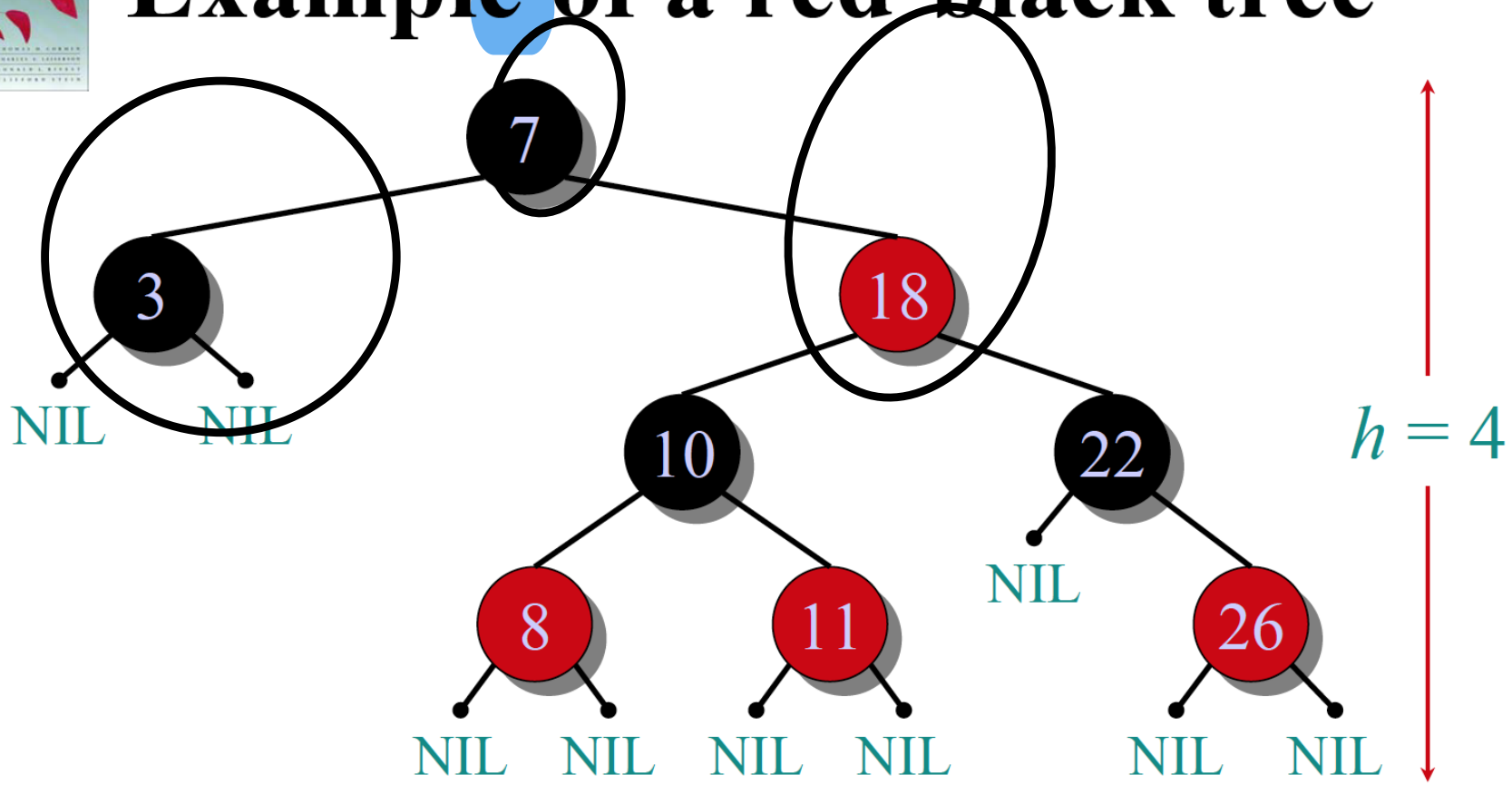
Red-Black properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (*nil*) is black.
4. If a node is red, then both its children are black.

5. For each node, all paths from the node to descendant leaves contain the same number of black nodes.



Example of a red-black tree



Red-Black Tree Height

Consider a node x in an RB tree: The longest descending path from x to a leaf has length $h(x)$, which is at most twice the length of the shortest descending path from x to a leaf.

Proof:

black nodes on any path from x = $bh(x)$ (prop 5)

$\leq$ # nodes on shortest path from x, $s(x)$. (prop 1)

But, there are no consecutive red (prop 4),

and we end with black (prop 3), so $h(x) \leq 2\ bh(x)$.

Thus, $h(x) \leq 2\ s(x)$. QED

Red-Black Tree Operations

- **Corollary.**
  - The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR of Binary Search Tree all run in $O(\lg n)$ time on a red-black tree with n nodes.
  - Cause no violation to the red-black properties.
- The operations INSERT and DELETE require modifications to the red-black tree

Insertion into A Red-Black Tree

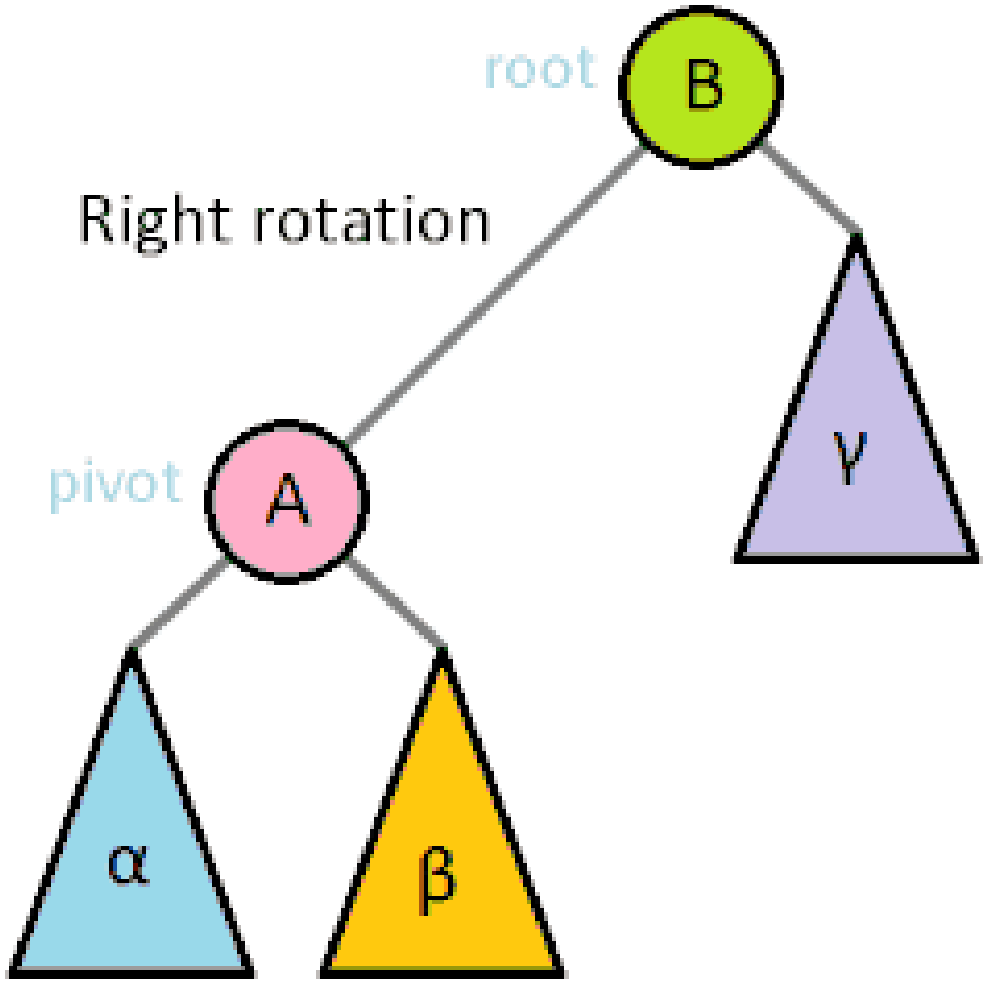
BST-insert x

Color x red

Property 4 may be violated

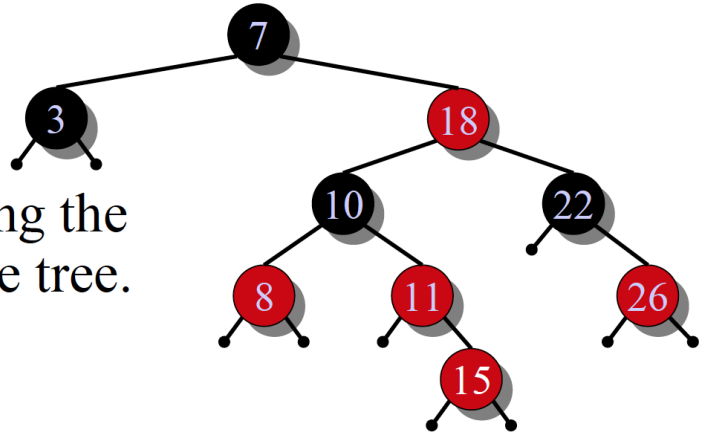
Idea: Move the violation up the tree

Rotation



Example:

- Insert $x = 15$.
- Recolor, moving the violation up the tree.



Example:

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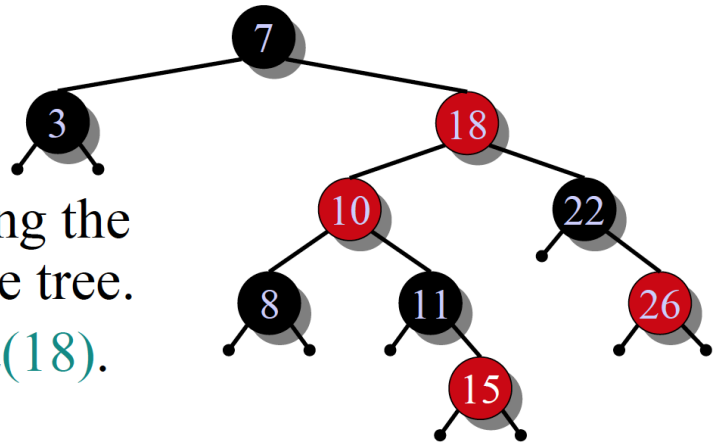
- RIGHT-ROTATE(18).



Example:

- Insert $x = 15$.
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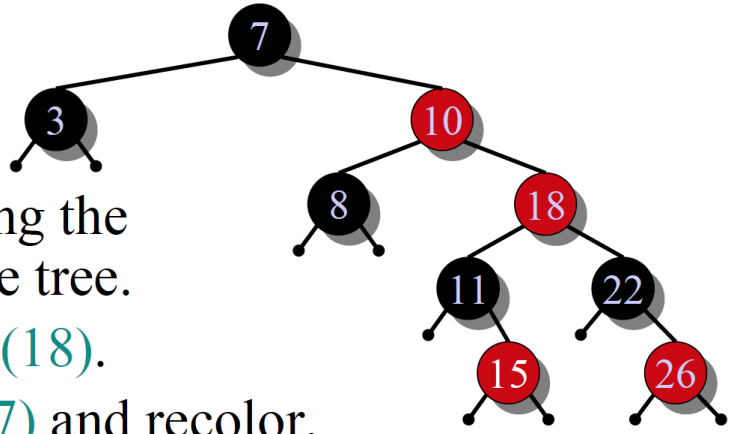
- RIGHT-ROTATE(18).
- LEFT-ROTATE(7) and recolor.



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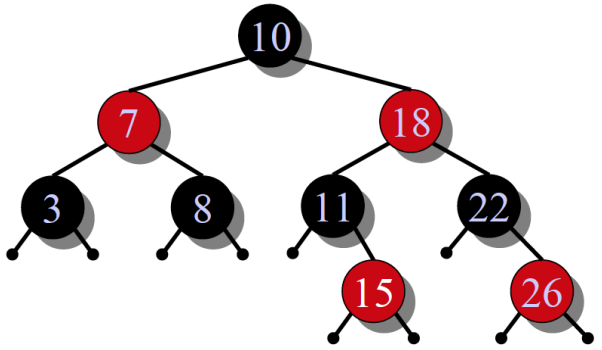
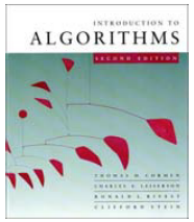
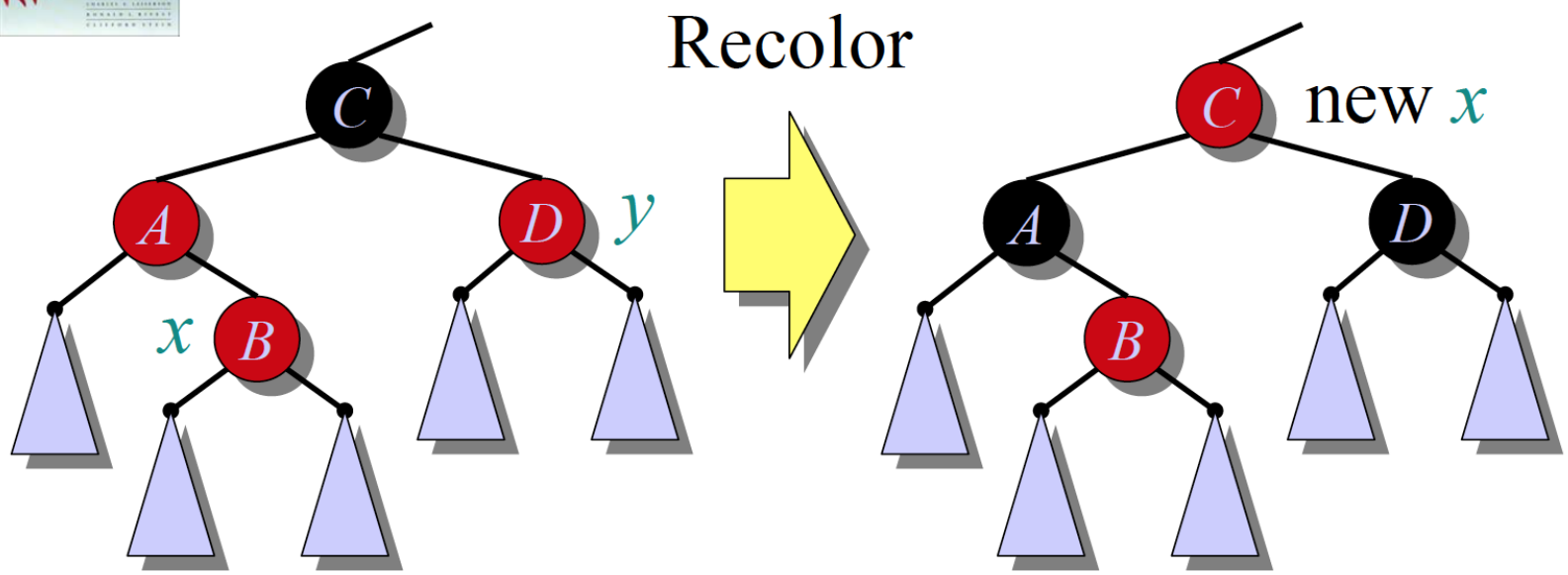


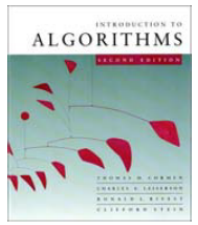


Case 1

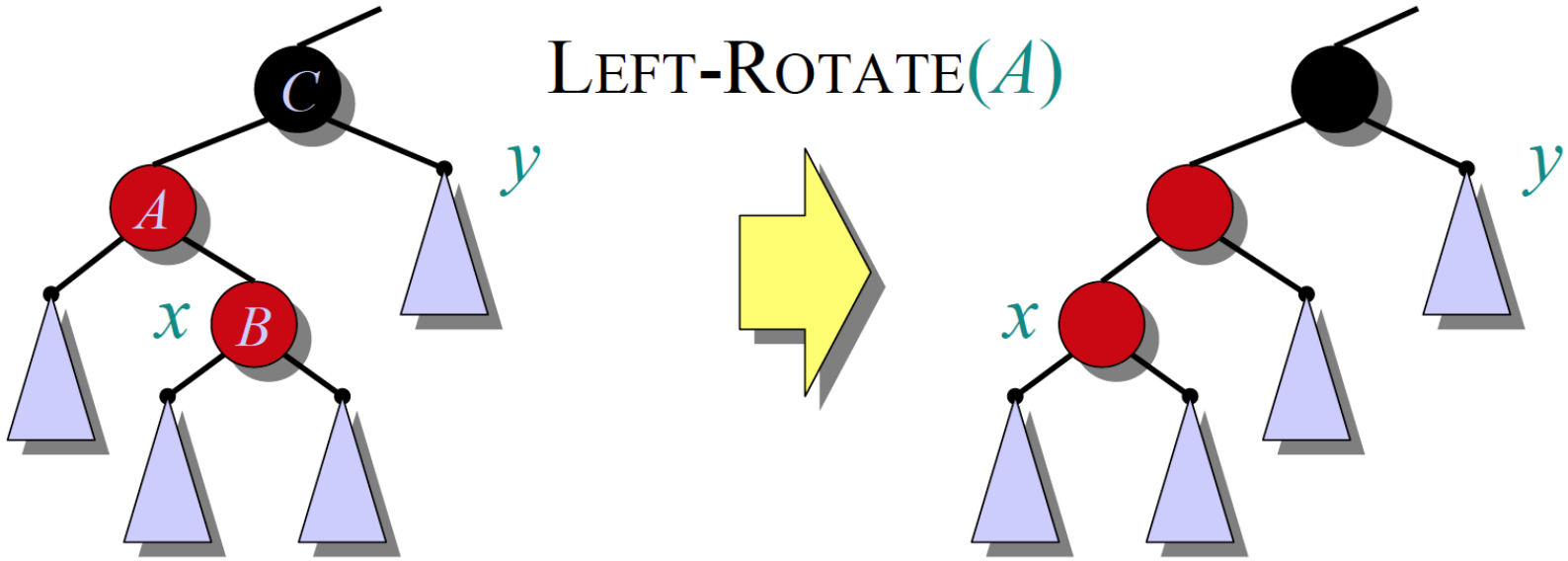


(Or, children of *A* are swapped.)

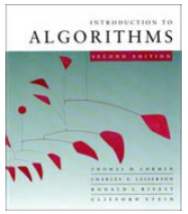
Push *C*'s black onto *A* and *D*, and recurse, since *C*'s parent may be red.



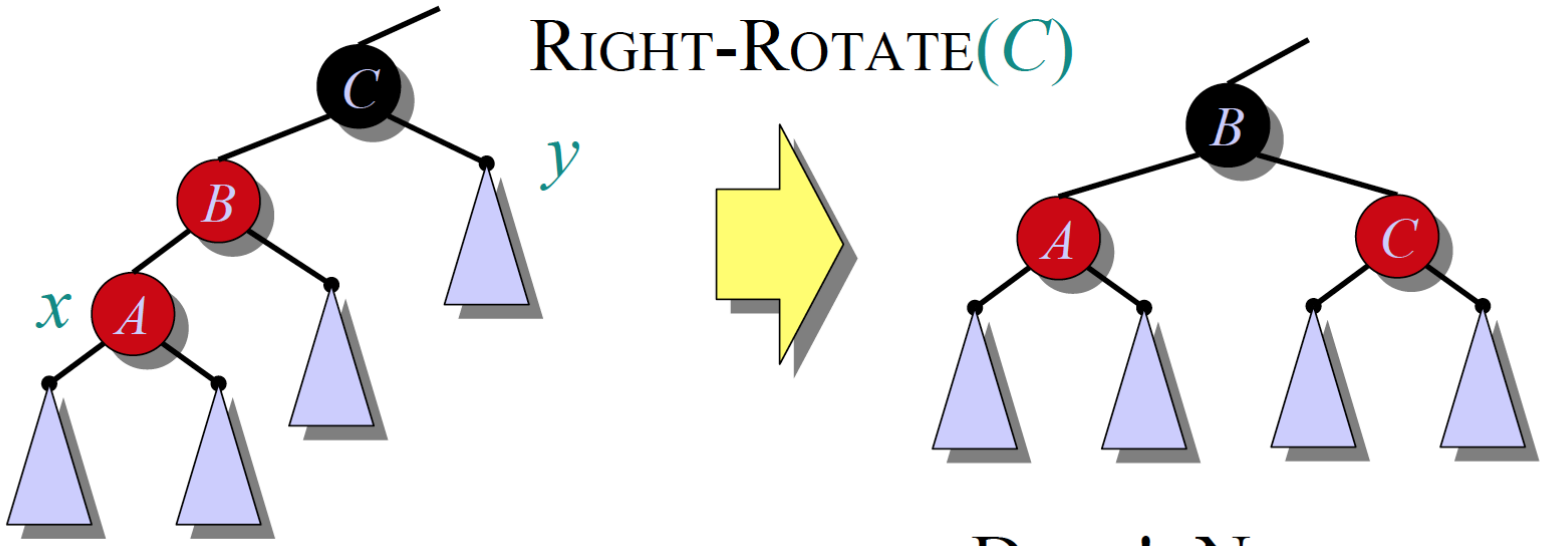
Case 2



Transform to Case 3.



Case 3



Done! No more violations of RB property 3 are possible.

Runing time

- Go up the tree performing Case 1, which only recolors nodes.
  - O(h) = O(lg n)
- If Case 2 or Case 3 occurs
  - perform 1 or 2 rotations, and terminate.
  - O(1)

➢**Running time:** *O*(lg n) with *O*(1) rotations.

RB-Insert Pseudocode

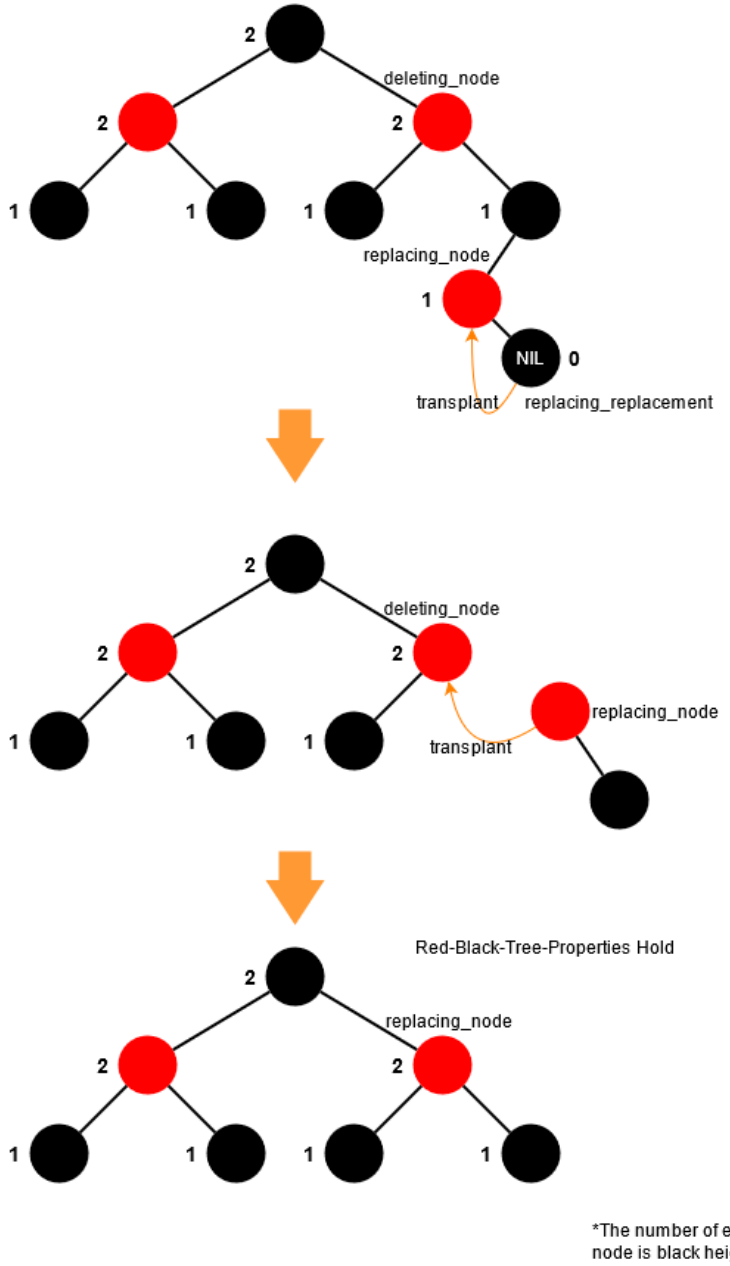
```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED      ▷ only RB property 3 can be violated
    while x ≠ root[T] and color[p[x]] = RED
        do if p[x] = left[p[p[x]]
            then y ← right[p[p[x]]          ▷ y = aunt/uncle of x
                 if color[y] = RED
                  then ⟨Case 1⟩
                  else  if x = right[p[x]]
                          then ⟨Case 2⟩    ▷ Case 2 falls into Case 3
                               ⟨Case 3⟩
            else ⟨"then" clause with "left" and "right" swapped⟩
    color[root[T]] ← BLACK
```

Deletion

- a bit more complicated than inserting a node.
- based on the BST's Tree-Delete
- need to customize the TRANSPLANT so that it applies to a red-black tree

```
RB-TRANSPLANT(T, u, v)
1   if u.p == T.nil
2       T.root = v
3   elseif u == u.p.left
4       u.p.left = v
5   else u.p.right = v
6   v.p = u.p
```

Deletion : Transplant



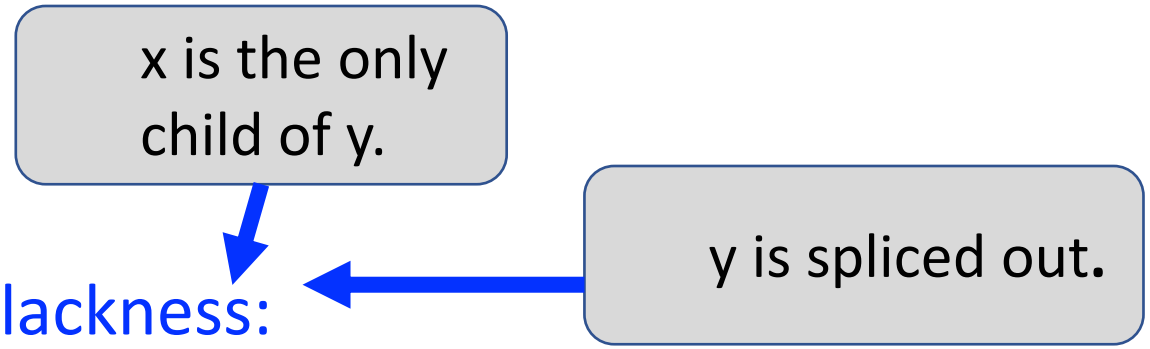
Not over yet!

```
RB-DELETE(T, z)
 1  y = z
 2  y-original-color = y.color
 3  if z.left == T.nil
 4       x = z.right
 5       RB-TRANSPLANT(T, z, z.right)
 6  elseif z.right == T.nil
 7       x = z.left
 8       RB-TRANSPLANT(T, z, z.left)
 9  else y = TREE-MINIMUM(z.right)
10       y-original-color = y.color
11       x = y.right
12       if y.p == z
13            x.p = y
14       else RB-TRANSPLANT(T, y, y.right)
15            y.right = z.right
16            y.right.p = y
17       RB-TRANSPLANT(T, z, y)
18       y.left = z.left
19       y.left.p = y
20       y.color = z.color
21  if y-original-color == BLACK
22       RB-DELETE-FIXUP(T, x)
```

Why RB-DELETE-FIXUP ?

- If the deleted node is red?
  - **Not a problem** – no RB properties violated

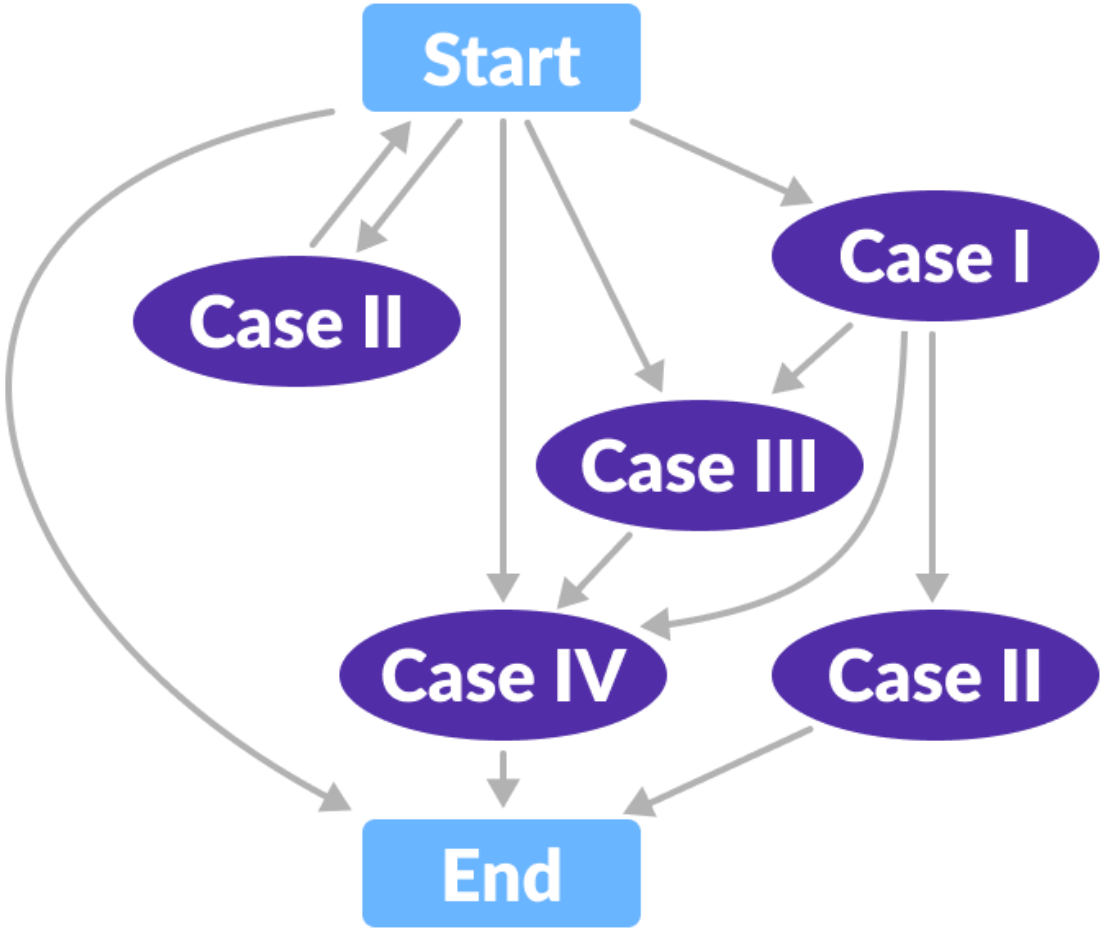
- RB-DELETE



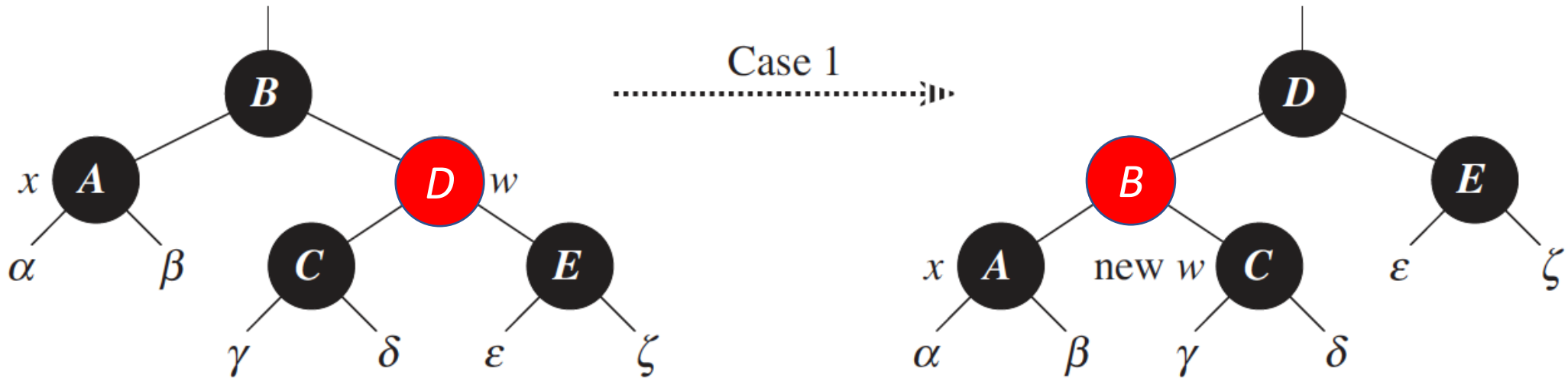
  - To maintain blackness:
    - x takes blackness of y.
    - If x was black, now it is **doubly black** (needs to be resolved)

```
RB-DELETE-FIXUP(T, x)
 1  while x ≠ T.root and x.color == BLACK
 2      if x == x.p.left
 3          w = x.p.right
 4          if w.color == RED
 5              w.color = BLACK                          // case 1
 6              x.p.color = RED                          // case 1
 7              LEFT-ROTATE(T, x.p)                      // case 1
 8              w = x.p.right                            // case 1
 9          if w.left.color == BLACK and w.right.color == BLACK
10              w.color = RED                            // case 2
11              x = x.p                                  // case 2
12          else if w.right.color == BLACK
13                  w.left.color = BLACK                 // case 3
14                  w.color = RED                        // case 3
15                  RIGHT-ROTATE(T, w)                   // case 3
16                  w = x.p.right                        // case 3
17              w.color = x.p.color                      // case 4
18              x.p.color = BLACK                        // case 4
19              w.right.color = BLACK                    // case 4
20              LEFT-ROTATE(T, x.p)                      // case 4
21              x = T.root                               // case 4
22      else (same as then clause with "right" and "left" exchanged)
23  x.color = BLACK
```

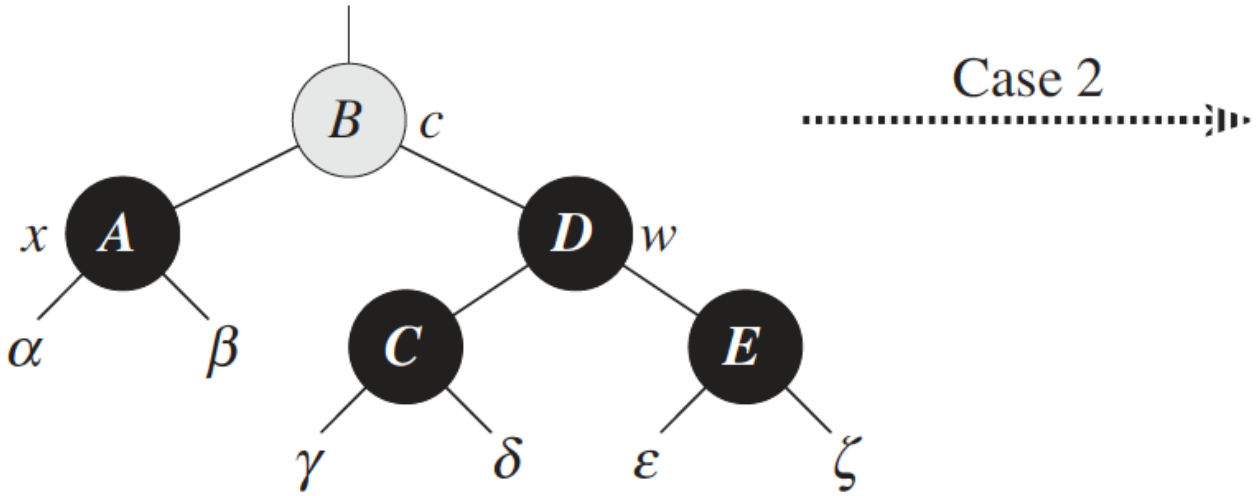
RB-DELETE-FIXUP



Case 1: x's sibling w is red

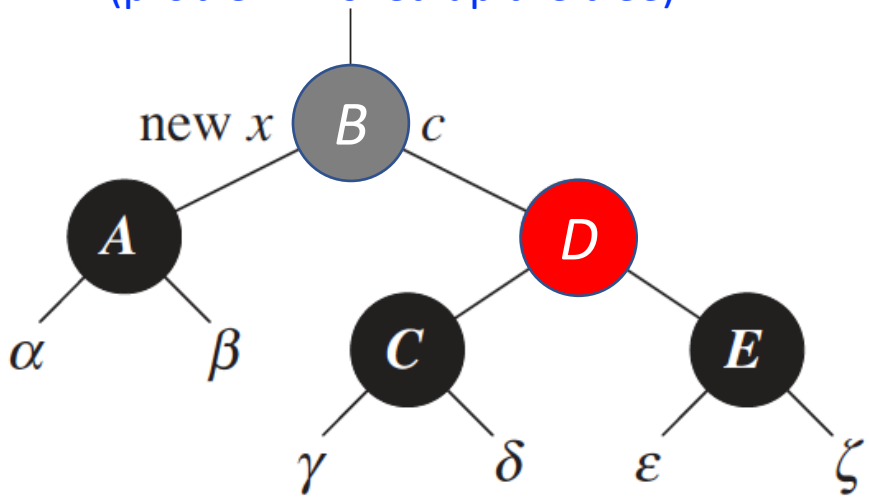


Case 2: x's sibling w is black, and both of w's children are black

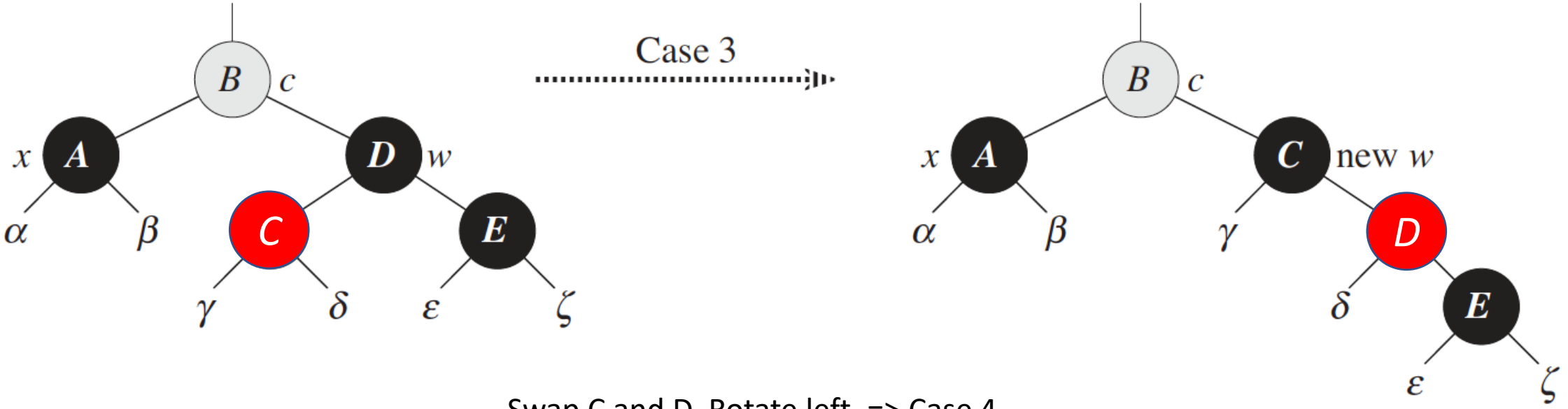


Case 2

If B was red, it becomes black, solved!
If B was black, it becomes doubly black
(problem moved up the tree)



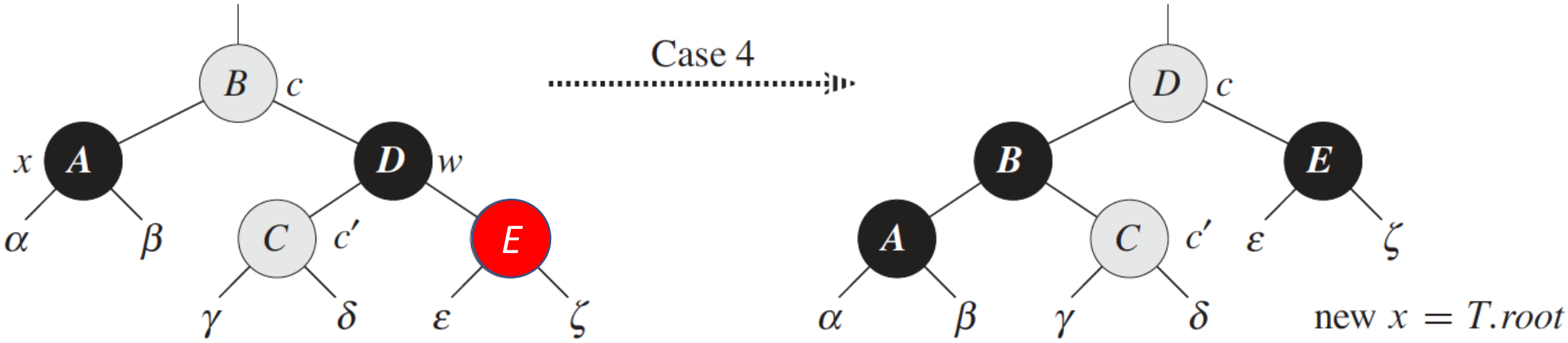
Case 3: x's sibling w is black, w's left child is red, and w's right child is black



Swap C and D. Rotate left. => Case 4

The idea is to align the opposite side of x with redness in between blacknesses.

Case 4: x's sibling w is black, and w's right child is red



As B rotates down to take extra-blackness, D rises to take its place. Then, E's redness allows it to come up to maintain blackness of D.